Synchronization Control for Master-Slave Systems Subject to Nonlinearities and Time Delay

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DEDICATION

To my beloved Mother, for supporting me spiritually throughout my life and to respected Brother Muhammad Iqbal and my beloved wife.

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ABSTRACT

Synchronization is a fundamental nonlinear phenomena observed in diverse natural systems. This dissertation contributes to the problem of synchronization of nonlinear master-slave systems under the restraints of time-delays and parametric uncertainties. We investigate the synchronization phenomena in both non-identical and identical nonlinear master-slave systems by application of a control system. To investigate the problem of synchronization for non-identical nonlinear master-slave systems, firstly, a novel mutually Lipschitz condition is proposed. Secondly, an algebraic Riccati inequality based control methodology is formulated and, further, a less conservative LMI-based robust control strategy by virtue of proposed mutually Lipschitz condition and Lyapunov stability theory is established for synchronization of two dissimilar nonlinear master-slave systems. Additionally, a novel robust adaptive control scheme for synchronization of nonlinear master-slave systems is developed that ensures a low gain controller through adaptive cancellation of the unknown mismatch in nonlinearities.

Novel frameworks comprising of delay-dependent and delay-range-dependent synchronization schemes are established. The input nonlinearity is transformed into linear time-varying parameters belonging to a known range. Using the linear parameter varying (LPV) approach, applying the information of delay range, exploiting the triple-integral-based Lyapunov-Krasovskii (LK) functional and utilizing the bounds on nonlinear dynamics, nonlinear matrix inequalities for designing a simple delay-range-dependent state feedback control for synchronization of the master and the slave systems is derived. In contrast to the conventional adaptive approaches, the proposed approach is simple in design and implementation and is capable to synchronize nonlinear oscillators under input delays in addition to the slope-restricted nonlinearity. Further, time-delays are treated using an advanced delay-range-dependent approach, which is adequate to synchronize nonlinear systems with either large or small delays. Furthermore, the resultant approach is applicable to the input nonlinearity, without using any adaptation law, owing to the utilization of LPV approach. In the end, numerical simulation results are adorned for the testimony of the proposed synchronization schemes of nonlinear master-slave systems.

LIST OF PUBLICATIONS

Journals Publications

[1] **M. Riaz**, M. Rehan, K.-S. Hong, M. Ashraf, and H. U. Rashid, "Static and adaptive feedback control for synchronization of different chaotic oscillators with mutually Lipschitz nonlinearities," *Chinese Physics B*, vol. 23, pp. 110502-110509, 2014. **(I.F = 1.603)**

[2] **M. Riaz**, M. Rehan, and M. Ashraf, "Synchronization of nonlinear master - slave systems under input delay and slope-restricted input nonlinearity," *Complexity*, vol. DOI: 10.1002/cplx.21734, 2015. **(I.F = 3.514)**

Conference Publication

[3] **M. Riaz**, M. Ashraf, K.-S. Hong, M. Rehan, and G.-H. Kim, "Adaptive synchronization of two different chaotic systems containing mutually Lipschitz nonlinearities subject to state time-delays," *presented at 15th International Conference on Control, Automation and Systems (ICCAS),* BEXCO, Busan, Korea, pp. 60-65, 2015.

Chapter 1

INTRODUCTION

1.1 Overview

All physical systems are nonlinear by nature. In order to attain the better understanding about the dynamical behavior of the different nonlinear systems, an interesting and important phenomenon is to investigate the synchronization between these dynamical systems. Synchronization, observed as naturally occurring process, has significant impact in diverse areas of engineering, sciences and even in the social life. Synchronization of two or more nonlinear systems is a process of adjustment of the given motion properties, either naturally through strong coupling effects or forced through application of an appropriate control law. Figure 1.1, illustrates the basic model for synchronization of two coupled systems, two pendulum clocks are coupled through a beam or wall, which offer mutual synchronization of two pendulum clocks.

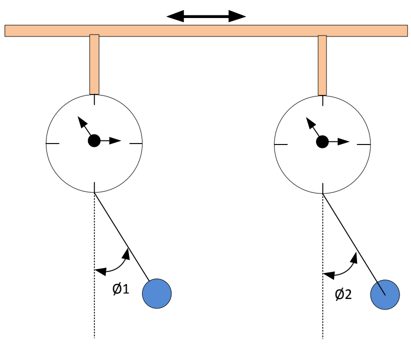


Figure 1.1: Synchronization of two coupled pendulum clocks

Synchronization of nonlinear systems is an attractive area among the researchers of different disciplines due to its numerous applications in the fields of engineering and technology. Noteworthy efforts by researchers have been devoted to investigate the problem of synchronization of nonlinear systems. To address the problem of synchronization of nonlinear systems demands the investigation of different dynamical parameters associated with nonlinear systems such as input delays, output delays, state delays, dead zone, input saturation, unknown dynamics, slope-restricted input nonlinearity, time-varying delay and external disturbances, which have strong influence on synchronization. These dynamical properties of nonlinear systems urge to be investigated along with nonlinear systems, because their impact on the performance of nonlinear systems cannot be ignored. In different nonlinear systems, different parameters can be source of instability and degrade the closed-loop performance of nonlinear systems. Figure 1.2, demonstrates the basic model for synchronization of nonlinear master-slave systems through an appropriate controller. Convergence of error is assured by selection of a suitable control signal $u(t)$.

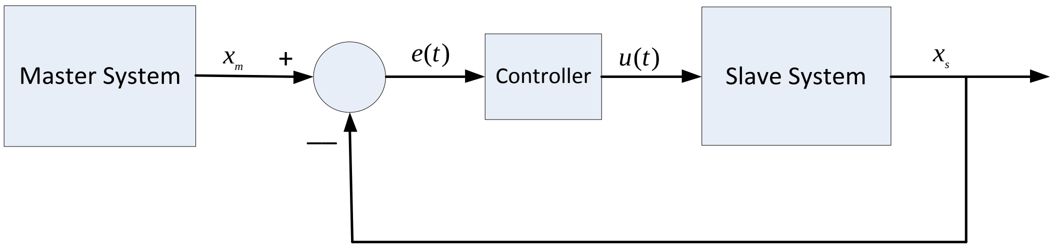


Figure 1.2: Elementary block diagram for synchronization through controller

Various control techniques [1-15], have been established for synchronization of nonlinear systems, some popular techniques are, adaptive control strategy [1], robust approach [2-3], robust adaptive [4-5], adaptive sliding mode [6], feedback linearization method [7-8], observer based synchronization methodology [9-11], fuzzy logic control technique [12-14] and H_∞ synchronization scheme [15]. A suitable synchronization technique can be applied to attain coherency among nonlinear systems, depending upon the nature and applications of nonlinear systems. For example, adaptive control technique is preferred to address the synchronization of nonlinear systems for slowly varying unknown parameters, whereas, the exponential synchronization technique can be opted for the fast varying parametric uncertainties.

The results of the synchronization of nonlinear systems have diverse applications in applied sciences, engineering and technology [16-40]. The common application of synchronization of nonlinear systems are such as wireless sensor networks [16-18], energy management systems [19-22], gyros systems [23-25], biomedical entities [26-29], unmanned air vehicles [30-32], secure communication networks [33-34], synchronization of multiple robots [35-37], automotive systems [38] and mechanical systems [39-40]. The recent work on the synchronization of nonlinear systems is focused on synchronization of micro-grid system for energy resources management, synchronization approach for formation control of multiple mobile robots, complete synchronization of gyro systems for navigations system, chaotic synchronization of communication networks for secure communication and synchronization of biomedical entities. The problem of synchronization of nonlinear systems appears as an attractive area due to it such a diverse application.

1.2 Motivation

Motivation of the thesis is to address the problem of synchronization of nonlinear master-slave systems in the presence of parametric uncertainties and evolve some novel control strategies to synchronize the systems that offer the desired performance of the closed-loop nonlinear systems. Research work is intended to design control methodologies that will be simple in design compared to the existing techniques, straight-forwardly implementable, robust in approach and adaptive for varying parametric uncertainties.

Synchronization of nonlinear systems contains diverse area of application in almost every field of life. It is quite difficult to discuss all the application areas in this short section, however some active research areas and applied examples of the synchronization are described.

- **Synchronization of Energy Systems**

  Electricity can be produced from different energy resources such as hydro power plants, nuclear power plants, thermal units, wind power and solar systems. A major challenge for the power system is to integrate and synchronize itself with centralized computing, communication and control mechanism, up to an appropriate level for correct system operation. The energy resource systems are a class of complex nonlinear systems. With the continuous development of economy, the issue of energy supply and demand has been paid more and more attention to in recent years. The problem of control and synchronization for the energy resource system has attracted increasing attention because of its potential importance in actual applications and these days, research is mainly focused on the synchronization of micro-grids [19-22], [41-46]. Figure 1.3, shows the block diagram of a microgrid configuration. Microgrids are smaller systems, usually of medium or small voltage, that include distributed generators (e.g., small hydro turbines, diesel generators, solar panels) and storage devices (e.g., energy capacitors, batteries, flywheels). Synchronization between multiple energy resources connected at single grid is necessary for correct operation.

Figure 1.3: Block diagram of synchronization of micro-grid [47]

Some recent research work in this field is described as follows. In [41], the authors describe the frequency synchronization of multiple isolated microgrids through the synchronization scheme of multi-agent systems and proposed a decentralized controller on the basis of consensus algorithm. Synchronization for smart grid for wind-solar hybrid systems by applying phase lock loop (PLL) method has been described in [43] and proposed the synchronization scheme for conventional grid and renewable energy grid for working as a single system. The work is focused on synchronization of phase and frequency of both grids. The problem of synchronization of renewable energy grid in the normal operation and as well under the fault condition like highly distorted voltages and unbalanced grid faults has been investigated by [44]. A novel PLL-based fast and accurate robust synchronization scheme [45] has been accomplished that improve the power quality of the renewable energy systems. In [46], the problem of chaos synchronization of the energy resource demand-supply system has been investigated and an adaptive control law for asymptotic convergence of synchronization error has been proposed.

- **Secure Communication**

In modern era, communication has great utility in our daily routine life such as online banking, internet, online shopping, messaging, audio and video chat etc. To maintain the privacy, accuracy and security in communication systems is a challenging task. To attain the secure communication, different cryptography techniques have been developed. Since the dynamic behavior of chaotic system is highly sensitive to the initial conditions, a considerable attention has been paid to the application of chaotic system to secure communication [33-34], [48-52]. A block diagram of chaotic synchronization based secure communication network is shown in Figure 1.4. Input message signal is modulated with a chaotic carrier signal generated by chaotic oscillator at transmitter and similarly output signal is recovered by chaotic demodulation process at the receiver.

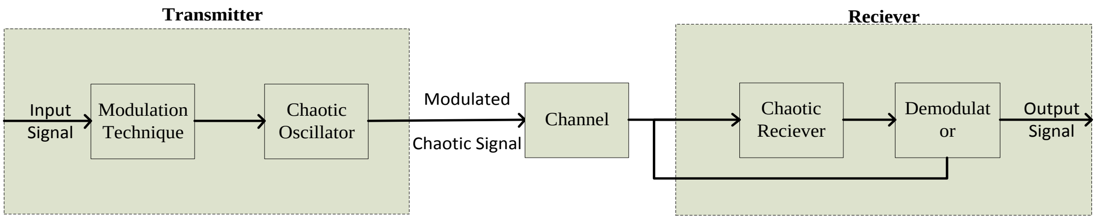


Figure 1.4: Block diagram of secure communication network

In [48], the problem of synchronization for secure communication has been investigated and a feedback control mechanism for synchronization has been developed. Message encryption by N-shift cipher and public key before transmission technique was introduced to improve the secure communication. In [49-51], role of synchronization in digital communication systems and chaos in secure digital communication have been described. In [52], the synchronization problem for hyper chaotic Chen systems have been investigated through feedback linearization and adaptive control techniques. The proposed synchronization scheme applied to obtain the secure communication.

- **Synchronization of Gyro Systems**

Gyro systems are usually employed to sense the angular motion of different moving objects. Gyro systems are complex and attractive dynamical nonlinear systems. Recently, the synchronization problem of two chaotic gyros has been widely investigated due to their great utility in aeronautical systems, navigational purpose, areas of secure communications, attitude control of long duration spacecraft, and signal processing in optical gyros. Plate-form stabilization can also employed by gyro systems. Different control methodologies have been applied for synchronization of gyro systems over the past few years [23-25], [53-55]. The article [53] described the master-slave synchronization scheme for gyro systems through adaptive fuzzy sliding mode controller for robust synchronization subject to uncertainties and disturbances in the gyro systems. The work in [54], proposes the feedback control based synchronization scheme for two nonlinear chaotic gyro systems with and without noise. In [55], the sliding mode control based synchronization and anti-synchronization of dissipative gyros system with input nonlinearity has been investigated.

The study of identical and non-identical nonlinear systems and different parameters associated with the dynamics of the nonlinear systems is one of the most attractive subject. The parameters like external disturbances, input nonlinearity and time-varying delays are of great importance due to their effects on the performance of closed-loop nonlinear systems. Delay is an important parameter that cannot be avoided in many physical systems. To control diverse effects of time-delays on the linear and nonlinear systems, several authors carried out the research [56-60] and develop some control strategies like delay-independent, delay-dependent and delay-range-dependent for stability analysis and synchronization of time-delay systems. Incorporation of slope-restricted nonlinearities is important in studying synchronization controller synthesis for nonlinear systems under uncertain inputs [61-66]. Sliding mode control strategies for nonlinear gyroscopes and unified second order complex oscillatory systems with input nonlinearities are explored in [61]. In [62-63], adaptive control and H_∞ control strategies for achieving coherent behavior of two uncertain systems under unknown dynamics and perturbations are formulated. Some advanced studies concerning robust, sliding mode and adaptive controller design for synchronization of general forms of two different nonlinear or chaotic systems under uncertainties and perturbations have been taken into account in the articles [64-66]. To the best of our cognizance, delay-range-dependent synchronization of the nonlinear master-slave systems under slope-restricted input nonlinearities and time-varying input time-delay has not been reported in the previous studies. So, it can be opted for research work and addressed using linear parameter varying (LPV) method, which is less conservative approach for design, analysis and synthesis of feedback controllers for nonlinear systems. In the past few years, LPV approach has received more attention as it has been successfully applied to the gain scheduling, H_∞ controller design [67-68], system identification, and state feedback control [69-70].

Numerous control methodologies such as adaptive, evolutionary, intelligent, optimal and robust, based on neural networks, state-feedback and fuzzy logic, for synchronization of the identical nonlinear systems have been investigated in the literature to attain asymptotic (or exponential) stability, finite-time stability, robustness, disturbance rejection, desired steady state performance, improved transient response, and noise handling (see, for example, [71-75]). Control strategies have been applied to cope with different circumstances, like input saturation, slope bounds, time-delays and unknown dynamics, and to deal with different dynamics, like Lure oscillators, Rössler systems, Chua's circuits, FitzHugh-Nagumo networks and Lipschitz structures [4], [76-79].

The problem of synchronization of non-identical nonlinear systems is lacking in the literature, whereas it has a great potential in diverse application. A few of the exceptional research works on synchronization of unlike dynamical systems are mentioned at this juncture [80-86]. In [80-82], adaptive control schemes are developed to cope with synchronization of two different chaotic oscillators by formulating adaptation laws for unknown parameters. Sliding mode control strategies for synchronizing the distinct chaotic systems under disturbances, slope-restricted input nonlinearity and different types of uncertainties have been addressed [83-84]. Recently, a control methodology has been developed for chaos synchronization of non-identical fractional-order systems with different number of states [85]. Adaptive synchronization approach of a unified chaotic oscillator and a cellular network, to develop an asymmetric image cryptosystem, is provided in the recent work [86] by utilizing Lyapunov stability theory.

The existing control strategies are complex enough to implement for synchronization of the different nonlinear systems. Moreover, further research is needed to classify various types of different nonlinear systems depending upon their dynamical characteristics and to design a simple synchronization controllers derived from these dynamical properties. Also not many control schemes for synchronization of the different nonlinear identities have been developed; therefore, this problem requires substantial research attention of the scientific community.

1.3 Problem Statement

The purpose of the study is to develop appropriate synchronization schemes for two nonlinear systems working according to master-slave principal. That address

- Synchronization of non-identical nonlinear master-slave systems subject to external disturbances and state delay.
- Synchronization of nonlinear master-slave systems under input delay and slope restricted input nonlinearity.

1.4 Contributions

The main contributions of the dissertation are described as follows.

1. A novel mutually Lipschitz condition is provided. The provided mutually Lipschitz condition is more general than the traditional Lipschitz condition [87-89] and also it is useful to derive the sufficient conditions for synchronization of nonlinear systems. The proposed mutually Lipschitz condition is advantageous for synchronization of non-identical nonlinear systems in contrast to the conventional Lipschitz condition, which is often used for synchronization of similar nonlinear systems [66], [90-91] and is inapplicable for distinctive systems.
2. State-feedback control law is designed by virtue of proposed mutually Lipschitz condition to attain the robust synchronization of Lipschitz non-identical nonlinear (chaotic) systems via LMIs. The proposed control law is simple in design and implementation, and straightforward for the optimal results, compared to the traditional control strategies for the synchronization different nonlinear systems [80-86]. Further, a novel adaptive control strategy for synchronization of non-identical nonlinear systems is proposed.
3. A novel LPV based treatment for the slope-restricted input nonlinearity is provided that in contrast to the previous adaptive approaches, can be applied to formulate a simple constrained synchronization controller for nonlinear systems. To the best of our knowledge, delay-range-dependent feedback synchronization

approach under input time-delay and slope-restricted input nonlinearity has been devised for the first time.

1.5 Organization of the Dissertation

This dissertation contains eight main Chapters as follows. Chapter 2 provides a short account of essential notions and concept to the reader and also makes the report self-sufficient. Different control laws, methods, tools and techniques are revisited, which will be frequently recalled in the rest of the dissertation, such as Schur complement, change of variable method, congruence transformation and Lipschitz nonlinearities. It begins with the historical background of synchronization followed by the comprehensive literature review on the existing techniques for synchronization of nonlinear systems. In Section 3, classification of the synchronization for nonlinear systems is described and a short note on the chaotic system is outlined. Next section contains the detailed literature review on time-delay nonlinear systems. Different methodologies developed to investigate the problem of synchronization of nonlinear systems under the parametric constraints are revisited to gain the better understanding of the opted problem. In the end, a numerical simulation example of synchronization is illustrated.

In Chapter 3, to derive the sufficient conditions for synchronization of different nonlinear systems, firstly, a novel mutually Lipschitz condition is proposed and proofs of its properties are also outlined. The accomplished mutually Lipschitz condition is more general than the traditional Lipschitz condition and also favorable to design a simple controller for synchronization of different nonlinear systems. State-feedback control law is proposed for synchronization of different dynamical master-slave systems, which led to derive an algebraic Riccati inequality based approach. To address the bottle-necks of algebraic Riccati inequality based synchronization approach, an advance linear matrix inequality (LMI) based synchronization scheme for non-identical nonlinear master-slave systems is entrenched. In the end, numerical simulations are illustrated to justify the proposed synchronization criterion.

Chapter 4 is focused on the robust and robust adaptive control strategies for synchronization of nonlinear master-slave systems. Compared to the Chapter 3, at this juncture two different nonlinear systems subject to external disturbances are considered and uncomplicated and implementable state feedback control law is proposed to attain the coherent behavior of master-slave systems. It is pointed-out that robust synchronization results into a high gain controller, which is never, be a smart choice for physical systems. Therefore, to design feasible control law, robust adaptive synchronization scheme is administered. Furthermore, the robust adaptive synchronization methodology is extended for time-delay system and novel delay-independent synchronization strategy is provided. In the end, the performance of the both robust and robust adaptive synchronization schemes is justified by couples of numerical simulation examples.

Chapter 5 extends to the detail of delay-dependent synchronization scheme for nonlinear master-slave systems under the constraints of time-varying input delay and slope-restricted input nonlinearity. To design a computationally uncomplicated controller in the presence of input nonlinearity, LPV approach is inferred. An LMI-based delay-dependent synchronization criterion is derived in Theorem 5.1, which is employed the analysis of proposed simple state-feedback control law. However, it founds that the scheme has limitation to determine the controller gain matrix. So, an advanced synchronization condition is derived in Theorem 5.2, which is less conservative and advantageous of computing the controller gain matrix. The numerical simulation results of two chaotic gyro systems are illustrated at the end of the chapter to witness the proposed synchronization scheme.

Chapter 6 presents a novel prospective of delay-range-dependent synchronization of nonlinear time-delay systems. Delay is an important parameter of nonlinear systems and unavoidable in various circumstances. In the literature it seems that conventional techniques for synchronization of nonlinear time-delay systems were implied through adaptive control, but determining the controller gain is computationally complex under the constraints of time-varying delay. So a simple state-feedback control law is proposed to provide the delay-range-dependent synchronization of nonlinear master-slave systems subject to time-varying input delay and slope-restricted input nonlinearity. Convex optimization is preferred to deal with complex nonlinearities. In the end, a numerical simulation is illustrated to justify the proposed synchronization strategy.

Chapter 7 demonstrates a frame work for synchronization of nonlinear master-slave systems under multiple delays in overlapping condition. Most commonly occurring delay in physical systems are state delays and input delays, both are incorporated in the dynamics of the master-slave systems. By utilizing the zero order hold technique, a simple state feedback control law is proposed that extends to derive an LMI-based synchronization scheme for nonlinear time-delay systems subject to multiple delays in overlapping scenario.

Finally, Chapter 8 completes the thesis with an overall summary of the carried-out research work and contributions highlights. Furthermore, some remarks and suggestions are provided for future work in this research extent.

Chapter 2

THEORETICAL BACKGROUND

2.1 Overview

In the previous chapter, a brief introduction on synchronization of nonlinear systems related to the selected research topic was detailed, to expose the scope of the problem. Research rationale and contribution of the dissertation was also explained briefly. In this chapter, theoretical background is described to make the dissertation self-sufficient and for the clarity of the readers. Some basic definitions, control laws, tools, techniques followed by the detailed literature review are outlined, these contents will be helpful to carry-out the research work on the topic of synchronization control for master-slave systems subject to nonlinearities and time-delay. Described identities will be frequently recalled in the rest of the dissertation.

Different existing tools and techniques for stability analysis of linear and nonlinear systems are revisited that can be implied to retrieve the desired results of synchronization. Chaos and nonlinear chaotic systems are also described, as synchronization of nonlinear (chaotic) systems, observed in naturally occurred processes, has a significant impact on biological, chemical, physical, engineering and biomedical systems [4], [70-72]. An appropriate control law is enforced to resolve the chaos synchronization dilemma for diverse applications such as secure communication, aerospace engineering, information processing, image processing, optics and medical therapies [4], [72], [92-94].

The study of time-delay nonlinear systems attracts a great number of researchers due to its disparate influence on the performance of nonlinear systems. Time-delay systems are classified into two categories on the basis of stability analysis criterion, delay-independent approach and delay-dependent approach. A considerable work of the dissertation is committed to time-delay nonlinear systems, so existing research work on time-delay systems and its classifications are described in the later part of this Chapter.

Other important control tools like Lyapunov stability theory, role of LMIs for stability analysis and synchronization of nonlinear systems, Jensen's Inequality, congruence transformation, change of variable methods, Schur complement and Lipschitz nonlinearities are also presented. In the end, a numerical simulation example of hyper chaotic master-slave systems is illustrated.

2.2 Synchronization (Historical Aspects)

Dutch researcher Christiaan Huygens was probably the first scientist who observed and described the synchronization phenomena in seventeenth century. In 1658, Christiaan Huygens investigated the synchronization between two weekly coupled pendulum clocks [95]. Despite the study of the first synchronization phenomena, the actual work on synchronization of nonlinear systems was started late in 1920, when W. H. Reck, and J. H. Vincent investigated the synchronization properties for electrical circuits and applied it for coherency of triode generator. After few years in 1927, Balthasar Vander Pol extended the efforts of W. H. Reck and J. H. Vincent by obtaining the theoretical and practical results for synchronization of triode generators by an external input signal of slightly different frequency [95]. This study got meaningful attention due to its physical importance and practical significance for radio communication networks. Remember, those days, triode generator was the basic element of the radio communication systems.

Modern nonlinear dynamics revived in 1990s, when different new dynamical properties of nonlinear systems were explored and innovative work of numerical methods were recognized for controllability and stability analysis of the dynamical nonlinear systems. Peccora and Carrol [96] gives the idea of synchronization of nonlinear (chaotic) systems, by investigating the properties of two nonlinear systems and described that two nonlinear systems can be synchronized by linking them with a common signal. After the inspirational work of Peccora and Carrol, on synchronization of dynamical systems, this problem attracted a great number of researchers from different fields of engineering and sciences. Considerable research work has been carried out to investigate the synchronization phenomena in different nonlinear systems and different control strategies have been developed.

Research work on synchronization of nonlinear systems is briefly revisited as follows. Since after the pioneer work on synchronization of two identical nonlinear systems, namely, response and drive systems [96], the problem of synchronization of nonlinear systems has been extensively studied in both theoretical and practical systems. The study of synchronization is evolved with the dynamical parameters of nonlinear systems such as time-delay, external disturbances, input saturation, unknown parameters and dead zone etc. To address the synchronization of time-delay nonlinear systems delay-independent, delay-dependent, delay-range-dependent and delay-derivative-dependent techniques have been developed [56-60]. Effect of disturbance on the controllability and stability of nonlinear systems has been investigated to attain the robust synchronization in [2-5]. A brief literature is revisited to explore the existing studies on the synchronization.

2.2.1 Control Strategies Based Study on Synchronization

The modern techniques for synchronization of nonlinear systems are based on the different control law design strategies. Various control techniques for synchronization of nonlinear systems like feedback control, sliding mode control, active control, observer-based control, adaptive technique, back stepping method and H_∞ method have been extensively reported in the literature [1-15], [97-103]. In [97], modern control technique of input-output feedback linearization has been implied to obtain the synchronization of identical nonlinear systems namely multi-agent systems by application of distributed control system.

Synchronization problem for nonlinear chaotic systems, known as sensitive to the initial conditions has been addressed by the observer-based technique for discrete-time and continuous-time nonlinear systems [98]. A novel delay-dependent synchronization criterion has been proposed for the master and slave systems subject to the unknown channel time-delay, to attain the synchronization based on adaptive control technique. The proposed techniques are useful for real-time implementation and delay estimation [59], [99]. In [100], the problem of synchronization with external disturbance using fault-tolerant dissipative method has been investigated and an innovative technique, namely, mixed fuzzy delayed feedback dissipativity-based synchronization technique, for nonlinear chaotic systems has been proposed. A delay-dependent H_∞ synchronization of the master and slave systems subject to time-delay and nonlinear uncertainties has been addressed using sliding mode control technique in [24], [101]. The article [102] provides a self-tuning approach for synchronization of two FIR filters called the master and the slave systems, by determining the coefficient of these filters. The problem of synchronization of chaotic systems subject to multiple time-delays has been addressed by a simple adaptive feedback control technique and also characteristics like the uncertainty and disturbance along with time-delay are considered for the robust lag synchronization [103].

2.2.2 Application Based Study on Synchronization

Synchronization is important phenomena for many biological, physical and chemical systems. A great deal of research on the synchronization of nonlinear systems is focused for different applied areas and results are specific to the applications. Some considerable applications have been reported in the literature such as, synchronization of energy systems for efficient utilization of energy resources, synchronization of communication networks for secure communication, synchronization of gyros systems for aeronautical operation, synchronization of actuators and sensors for stable functioning of robots, biomedicine for human health care and also synchronization has been observed in chemical and automobile industries [16-40], [99], [104-106].

Synchronization in energy sector has a key role, especially in the present scenario, when developing countries are suffering with shortage of electric supply and looking for efficient energy management system to overcome the losses and to improve the performance by managing the limited resources. At a grid, multiple energy sources are connected and need to be operated in a synchronized manner. Synchronization of frequency, phase and voltage of multiple sources is required. The work of [19] is focused to maintain the same frequency of all energy sources with communication infrastructure using consensus based approach. In article [104], the author addresses the problem of synchronization of energy systems having mismatch in parameters and proposed a linear feedback control technique.

The article [105] shows the application of synchronization in chemical and mechanical systems, where two coupled pipes conveying pulsating fluid are synchronized using motion equations. In [106], synchronization of communication networks subject to time-delay has been investigated. Lyapunov stability theory along with linear matrix inequalities methods has been implied to design a control law for maximum sampling rate through optimization technique. Modern vehicles are complex nonlinear systems, having mechanical and electrical components. To provide smooth transmission, all its peripherals are needed to be work in synchronized manner. The synchronization analysis of the automobile vehicles has been addressed using experimental data analysis in [99].

2.2.3 Parametric Based Study on Synchronization

In the literature, as different techniques and application based studies have been reported, similarly, different parameters of nonlinear systems are important and discussed in the literature quite often. In the dynamical model of nonlinear systems, some parameters can be known and some others can be unknown. The unknown parameters can be estimated through identification and estimation techniques. In the same aspect, some parameters of the nonlinear systems are of constant and others are varying that compose the complexities in the system model. The problem of synchronization of nonlinear systems can be categorized as identical and non-identical nonlinear systems. The other distribution can be defined on the basis of two or more systems, namely synchronization of master-slave systems (response-drive systems) and synchronization of networks.

Synchronization schemes developed for nonlinear systems considering the different dynamical parameters such as disturbances, uncertainties time-delays, nonlinear dynamics, input/output nonlinearities, unknown parameters, input saturation, and slope restriction have been reported in the literature [4], [6-7], [19], [59], [61-66], [71], [84], [97-111]. Disturbances occurring in the nonlinear systems have been extensively studied by the researchers, to observe their effect on the performance of nonlinear systems and proposed various approaches to minimize the effect of disturbances on nonlinear systems [84]. Delay is another important parameter of nonlinear systems, which is unavoidable in many physical systems. Since last few years, time-delay systems have been reported quite extensively [56-60], [59], [103], [106], [109-111]. Both types of delays, constant and varying delay, appearing in physical systems have been addressed using advance control techniques for synchronization of nonlinear systems. Nonlinearities and saturation observed at the input or at the output of the nonlinear systems, different control techniques have been developed to minimize their effect on performance [60], [84]. The problem of synchronization of nonlinear systems subject to unknown parameters has also been investigated by different control strategies in [79-84]. LMI-based robust adaptive technique, to estimate the values of unknown parameters and to ensure the stability of nonlinear system, which leads to obtain the synchronization of such nonlinear systems, has been addressed in [4]. The role of different dynamical parameters of nonlinear systems on synchronization such as input nonlinearity, nonlinear perturbation, input and output saturation, dead zone, slope-restricted input nonlinearity and modeling uncertainties are quite versatile. These issues have been widely investigated and reported in the literature [61-67], [70], [72-74], [77-79].

In this modern era, as science and technology is growing faster, more challenging problems are faced. The complexity level of nonlinear systems is increasing day by day, which makes problem of synchronization of nonlinear systems, a more challenging task. Many unknown parameters of nonlinear systems are available to investigate and to provide the solutions to improve the performance of closed-loop nonlinear systems. So, still plenty of room is available for the researcher of control community to address the problem of synchronization of nonlinear systems. These days, the problem of synchronization of time-delay nonlinear systems is a hot topic due to its significance in physical systems. The problem of synchronization of similar nonlinear systems has been widely reported in the literature, whereas synchronization of different nonlinear systems is lacking in the literature.

2.3 Classification of Synchronization

Synchronization is defined as "adjustment of rhythms of oscillating objects due to their weak interaction" [95]. To achieve the synchronization between the master and the slave systems, different techniques and methodologies like sliding mode, fuzzy logic, state-feedback and neural networks are proposed with advantage over each other. Synchronization phenomena can be classified into different categories as follows.

- **Complete Synchronization:** The error trajectory of two systems, if converge exactly to the origin, then synchronization between these two systems are called as complete synchronization. Complete synchronization is usually practicable in coupled systems having identical nonlinear parameters [112]. Mathematically complete synchronization can be expressed as

$$X(t) = Y(t)\,. \qquad (Eq\ 2.1)$$

Eq. 2.1 shows that two similar systems will follow the exact trajectory of each other. Complete synchronization can be attained without any control input or with unity controller gain. This type of synchronization mentions to the identical evolution of the interacting systems [96].

- **Generalized Synchronization:** Synchronization between the states of two systems by a functional relation is defined as generalized synchronization. Complete synchronization can be achieved from generalized synchronization by selecting the static parameter as unity.

$$Y(t) = F(X(t))\,, \qquad (Eq\ 2.2)$$

where F is some constant parameter, a suitable value of F, synchronize the slave system to the master system, by converging error trajectory to the origin. Generalized synchronization has been observed between different types of coupled systems, which follow the above relation of Eq. 2.2 [112].

- **Lag Synchronization:** The phenomenon of synchronization, where the states of the slave system lag the states of the master system with a time delay $\tau > 0$ is known as lag synchronization [113] , Mathematically represented as follows

$$Y(t) = (X(t-\tau))\,. \qquad (Eq\ 2.3)$$

- **Anticipatory synchronization** [113]: Anticipatory synchronization is defined as the states of the drive system anticipate the states of the master system with a time delay $\tau > 0$.
- **Phase synchronization**: Phase of the response system converges to the phase of drive system, irrespective of their amplitude that may remain uncorrelated systems will be known as under phase synchronization [113].

2.4 Chaos and Nonlinear Chaotic Systems

Chaos is the science of surprises. Chaos theory is the study of dynamical systems that are highly sensitive to the initial conditions. Very small difference in initial conditions of two systems results different reaction from each other. Chaos can be seen in many physical systems such as electrical circuits, oscillating chemical reactions, fluid dynamics, and planetary bodies orbiting each other. Chaos is naturally occurring phenomenon and the system that exhibit the chaos is known as chaotic system. Chaotic systems are complex nonlinear systems. Response of the chaotic systems is usually unpredictable; however it could be in a bounded region. A sound research work has been focused on the synchronization of chaotic nonlinear system due to its numerous applications in the fields of engineering, sciences and even in social life. Various real-time systems show the chaotic behavior, some popular examples includes double compound pendulum, human heart beat, human brain, stock market, global weather etc. [95].

Stability analysis and attaining synchronization of chaotic systems are challenging tasks for research due to the complex nonlinear and unpredictable behavior of these systems. Typical configuration to achieve the goal of chaos synchronization is by implementation of an appropriate control law to synchronize the slave system with the master system. Results of chaos synchronization are utilized in biological, social, chemical, physical, energy and many other systems, some common applications include gyro systems, secure communication network, cryptography, image processing, and harmonic oscillators [22-24], [33-34], [50], [114-116].

2.5 Time-Delay Nonlinear Systems

Delay is important parameter of linear and a nonlinear system, which is an undesirable phenomenon occurring in many physical systems. It can be a source of instability and degradation of the performance of closed-loop performance of the systems. Different kinds of delays are encounters in many engineering and industrial systems like deterministic and stochastic, constant and varying, known and unknown delays etc. To control the diverse effects of time-delays on the linear and nonlinear systems, several researchers have investigated the problem of stability analysis and synchronization of time-delay systems by considering the different delays such as exponential delays, communication delays, random delays, time-varying delays, stochastic and multiple delays, and proposed various feedback control strategies [7], [11], [16-17], [30-32], [38], [48], [56-60], [91], [103], [110-111], [117-118].

Time-delay systems can be classified into two categories on the basis of the stability analysis criteria, as follows.

- **Delay-independent approach:** It is tends to be more conservative especially for small interval of delay, as this approach is independent of delay range. Delay independent system has advantage that such systems are stable over the entire range of delay.
- **Delay-dependent approach:** The other well-known approach for stability analysis of time-delay system is delay-dependent approach, which is less conservative compared to the delay-independent approach. Delay-dependent stability technique is further classified as delay-dependent and delay-range-dependent approach. Stability of delay-dependent time-delay systems can be addressed using two approaches, a frequency domain approach and time domain approach. Lyapunov functional is powerful tool to deal with time domain approach. The research work is focused on time domain approach to deal with the time-delay nonlinear systems to get the advantage of Lyapunov stability theory.

Delay-dependent stability of time-delay systems is a hot topic in research. In [111], improved delay dependent stability criteria for the time-varying delay is specific interval is obtained, by implying the Lyapunov functional and LMI-based approach. In this paper, Shao considered the system with time varying delay as

$$\dot{x}(t) = Ax(t) + A_1 x(t - d(t)), \qquad (Eq\ 2.4)$$

$$x(t) = \phi(t), \quad t \in [-h_2, 0],$$

where $d(t)$ is a continuous time-varying delay that satisfies the condition as $0 \le h_1 \le d(t) \le h_2$, lower and upper bounds of the delay interval are represented by h_1 and h_2, respectively. In this thesis, the work of Shao on delay dependent stability analysis [111], [117], has been extended for synchronization of nonlinear systems.

2.6 Role of LMIs in Control

Linear matrix inequalities (LMIs) are matrix inequalities, which contains linear set of matrix variables. LMI was first reported in seventies; however development of sophisticated numerical algorithms, such as semi-definite programming [119-120], during the past 15-20 years, offers efficient utilization of LMI for solving control problems. There are several features of LMI that attract the control researchers to transform their problem into LMI format. Some important features of LMI method are as follows:

- LMI methods solve the problems which involve several matrix variables.
- LMI methods are flexible that allow to transform various problem as LMI problem.
- In several cases, LMI methods are helpful to minimize the restrictions associated with traditional control methods and offer more general scenario.
- LMI methods can be applied, where other traditional control methods fail or struggle to find the solution.

LMI based control approaches are computationally fast, reliable and also used for the optimization of control problems. Flexibility of LMI provides much wider scope for controller design. It allows the efficient consideration of H_2 and H_∞ constraints for performance, robustness and robust performance in a single controller. It also has advantage of flexibility to impose the different rules and methods to transform the complexities into simple and solvable format. Moreover, multiple LMIs can also be transformed into a single LMI. Some other inequalities like algebraic Riccati inequalities, Lyapunov functional and bilinear matrix inequalities (BMIs) can be transformed into LMIs to get the solution of complex problems [119-120].

An LMI has the form

$$F(x) = F_O + \sum_{i=1}^{m} x_i F_i > 0 \,, \qquad (Eq\ 2.5)$$

where $x \in R^m$ and $F_i \in R^{m \times n}$. The inequality means that matrix $F(x)$ is considered a positive-definite $F(x) > 0$. Usually the variable x is composed of one or more matrices, whose column has been stacked as vectors, such as

$$F(x) = F(X_1, X_2, X_3, ...X_n) \,. \qquad (Eq\ 2.6)$$

Eq. 2.5 is known as a strict LMI and the other case $F(x) \geq 0$ positive semi-definite known as non-strict LMI. The LMI methods have great potential in the field of control engineering. It can be implied for optimization of control problem, stability analysis, gain scheduling and synthesis of controller for robustness of the system.

Although there are various control problems that can be cast as LMI problem, however a considerable number of problems need to be transformed into the LMIs format. There are number of identities and rules available to transform such problems into LMI problems, known as LMI tricks. Some of the useful rules that will be frequently recalled in this report to are described here.

- **Change of variable method:** Many control problems contain the different nonlinearities that cannot be transformed as LMI problems. To transform such nonlinearities in LMI format are renamed by some other variables. To demonstrate the method, let us consider a state-feedback controller synthesis problem that holds following inequality.

  $$A^T P + PA + F^T B^T P + PBF < 0 . \qquad (Eq\ 2.7)$$

  Eq. 2.7 is not in the bilinear matrix inequality (BMI) form, as F and P are nonlinear terms in the product form. Multiplying the either side by $Q = P^{-1}$, it gives

  $$QA^T + AQ + QF^T B^T + BFQ < 0 . \qquad (Eq\ 2.8)$$

  Now defining a second new variable $L = FQ$, we obtain the LMI form as below.

  $$QA^T + AQ + L^T B^T + BL < 0 . \qquad (Eq\ 2.9)$$

- **Congruence transformation:** Congruence transformation is a useful way to convert bilinear terms in the linear form. Congruence transformation is applied by pre and post multiplication of a full rank real matrix.
- **Schur complement:** Schur complement is a handy method to transform convex nonlinear inequalities that are quadratic in nature and appears regularly in control problems into an LMI [121]. Schur complement states that following mathematical statement are equivalent for any given matrix

  $$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{12}^T & \Theta_{22} \end{bmatrix}, \qquad (Eq\ 2.10)$$

  if $\Theta_{11} = \Theta_{11}^T$ and $\Theta_{22} = \Theta_{22}^T$, the following conditions

  $\Theta < 0$, $\Theta_{22} < 0$ and $\Theta_{11} - \Theta_{12}\Theta_{22}^{-1}\Theta_{12}^T < 0$ are equivalent.

2.7 Lipschitz Nonlinearities

Lot of work has already been established by the researchers to address the controllability and stability analysis of nonlinear time-delay systems. Conventional strategies to deal with the nonlinearities are to transform these nonlinearities into any of the approximate linear models and apply existing classical control techniques of linear system.

To define the traditional Lipschitz condition, let us have a nonlinear function $f(x)$, which satisfies a Lipschitz condition on the interval $[a,b]$, if there exists a constant Ω, such that

$$|f(x_m) - f(x_s)| \le \Omega |x_m - x_s|, \qquad (Eq\ 2.11)$$

where Ω is called the Lipschitz constant, dependent on both f and the interval $[a,b]$. Lipschitz condition is classified as locally Lipschitz and globally Lipschitz conditions.

2.8 Simulation Example

To describe the concept of synchronization of master-slave systems, a numerical simulation example is illustrated. Two dynamical nonlinear hyper chaotic systems described in [72], namely master and slave systems are considered.

Dynamics model of the master system is described as

$$\dot{x}_{m1}(t) = z_1(x_{m2}(t) - x_{m1}(t) - g(x_{m1}(t))), \qquad (Eq\ 2.12a)$$

$$\dot{x}_{m2}(t) = x_{m1}(t) - x_{m2}(t) + x_{m3}(t), \qquad (Eq\ 2.12b)$$

$$\dot{x}_{m3}(t) = -z_2(x_{m2}(t) - x_{m4}(t)), \qquad (Eq\ 2.12c)$$

$$\dot{x}_{m4}(t) = -z_3(x_{m3}(t) + x_{m5}(t)), \qquad (Eq\ 2.12d)$$

$$\dot{x}_{m5}(t) = -z_5(x_{m4}(t) - z_4 x_{m5}(t)). \qquad (Eq\ 2.12e)$$

Similarly dynamical model of the slave system is described as

$$\dot{x}_{s1}(t) = z_1(x_{s2}(t) - x_{s1}(t) - g(x_{s1}(t))) + u_1(t), \qquad (Eq\ 2.13a)$$

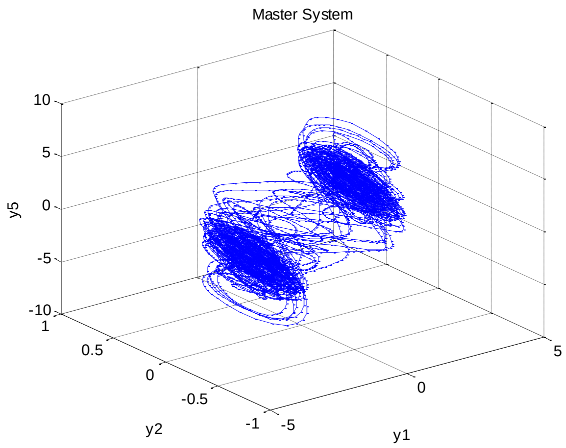
$$\dot{x}_{s2}(t) = x_{s1}(t) - x_{s2}(t) + x_{s3}(t), \qquad (Eq\ 2.13b)$$

$$\dot{x}_{s3}(t) = -z_2(x_{s2}(t) - x_{s4}(t)), \tag{Eq 2.13c}$$

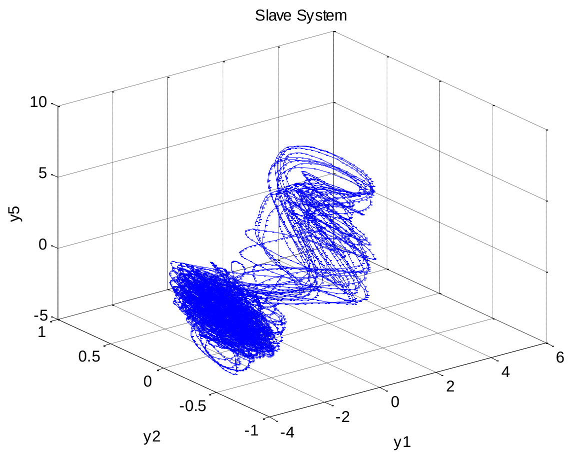
$$\dot{x}_{s4}(t) = -z_3(x_{s3}(t) + x_{s5}(t)), \tag{Eq 2.13d}$$

$$\dot{x}_{s5}(t) = -z_5(x_{s4}(t) - z_4 x_{s5}(t)), \tag{Eq 2.13e}$$

where $x_m(t)$ and $x_s(t)$ represents the states of the master and slave systems respectively. Difference between the states of master and slave system is defined as an synchronization error, such as $e_1(t) = x_{m1}(t) - x_{s1}(t)$ represent the synchronization error between first state of master and slave system. $u(t)$ is the control input applied to the slave system to synchronize it with the master system. The results obtained using proportional controller, are shown as follows.



(a)



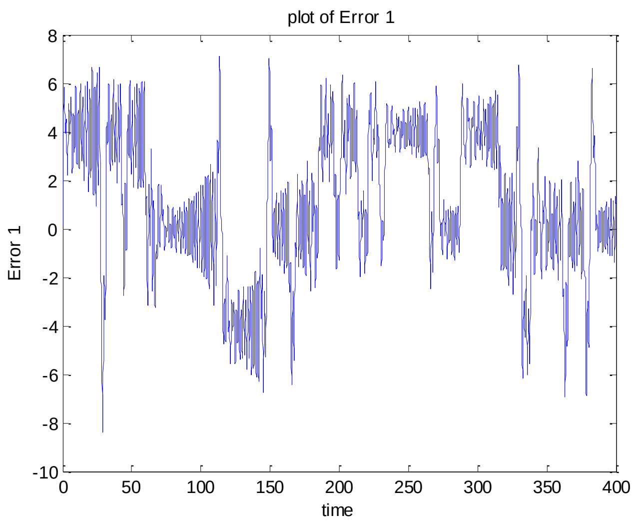
(b)

Fig 2.1: Phase portrait of the hyper chaotic systems, (a) master systems, (b) slave system

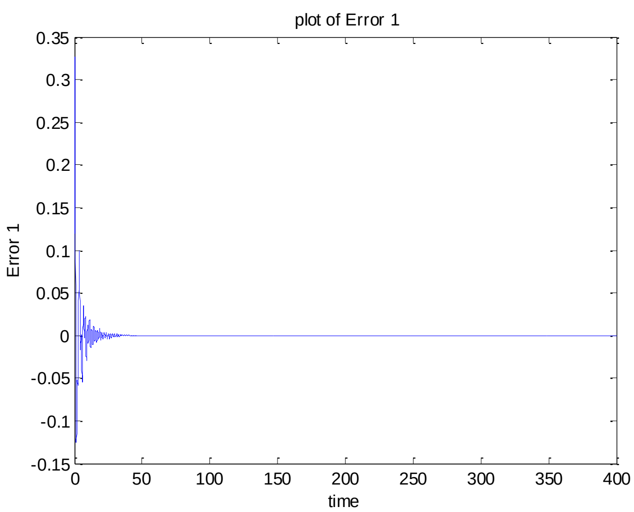
Figure 2.1(a), shows the phase portrait of the master system (hyper chaotic system), without controller. Similarly, Figure 2.1(b) shows the phase portrait of the slave system without controller. Different initial conditions for the master and slave systems are selected randomly, as follows

$x_{m0} = [\ 0.01\ \ 0.12\ \ 0.10\ \ 0.1\ \ 0.1\ ]$,

$x_{s0} = [\ -0.11\ \ -0.1\ \ -0.1\ \ -0.1\ \ -0.1\ ]$.



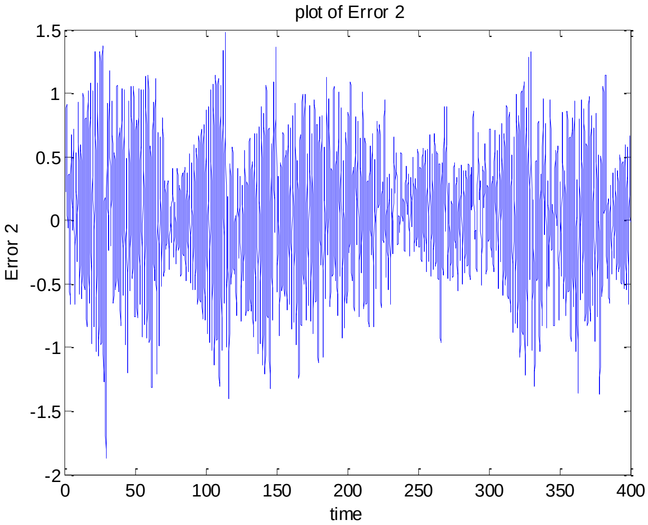
(a)



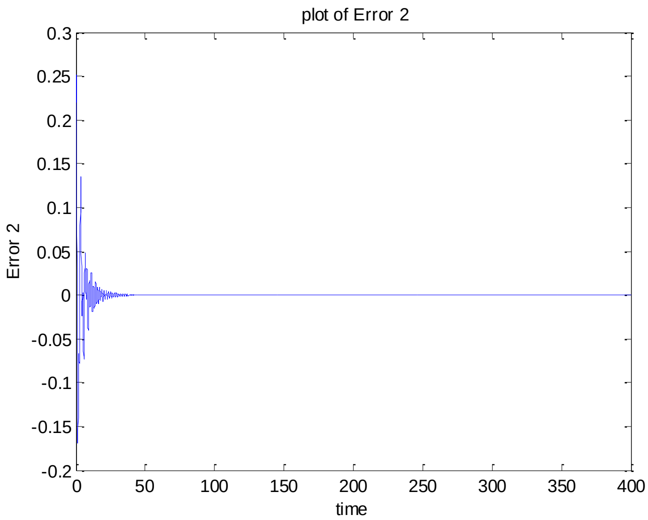
(b)

Fig. 2.2: Error between x_{m1} and x_{s1}, (a) without controller and (b) with controller

Figure 2.2 shows the response of error between first state of master and slave systems $e_1(t) = x_{m1}(t) - x_{s1}(t)$ without and with control input. Figure 2.2(a) represents the error, the response shows that error trajectory is not converging to the origin in the absence of control law, whereas error is converging to the zero in Figure 2.2(b), when control signal is activated.



(a)



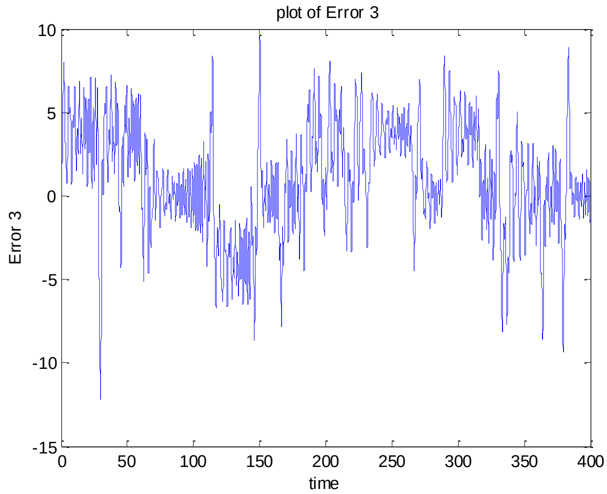
(b)

Fig. 2.3: Error between x_{m2} and x_{s2}, (a) without controller and (b) with controller

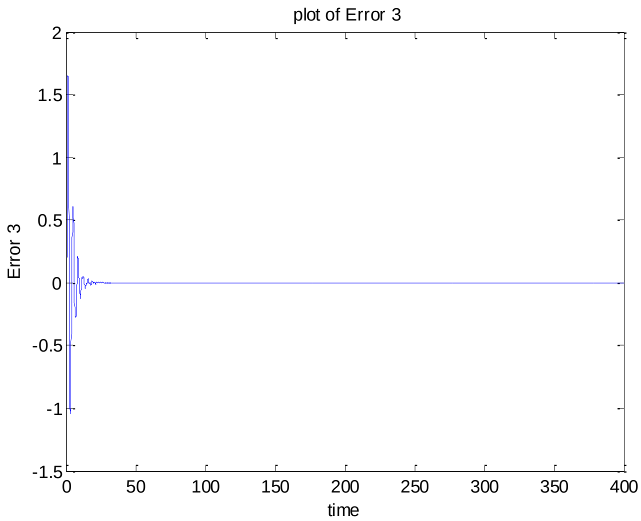
Figure 2.3 shows the response of error between second state of master and slave systems $e_2(t) = x_{m2}(t) - x_{s2}(t)$ without and with control input. Figure 2.3(a) represents the error, the response shows that error trajectory is not converging to the origin in the absence of control law, whereas error is converging to the zero in Figure 2.3(b), when control signal is activated.

Similarly, Figure 2.4 shows the response of error between fifth state of master and slave systems $e_5(t) = x_{m5}(t) - x_{s5}(t)$ without and with control input. Figure 2.4(a) shows that error trajectory is not converging to the origin in the absence of control law, whereas error is converging to the origin in Figure 2.4(b), when control signal is activated.

Simulation results demonstrate that synchronization between master and slave systems can be established through an appropriate control law.



(a)



(b)

Fig. 2.4: Error between x_{m5} and x_{s5}, (a) without controller and (b) with controller

2.9 Summary

This chapter provides the foundation of the dissertation. Different laws, tools, techniques and methods imperative to carry-out the research work are recalled. A brief literature review on the historical aspects of synchronization is provided. Existing control techniques for synchronization of nonlinear systems are revisited to explore the previous research work on this topic. Classification of synchronization is also mentioned in this Chapter. An overview of nonlinear systems and nonlinear chaotic systems is provided. Chaotic systems are complex nonlinear systems known as sensitive to the initial conditions. Nonlinear systems are complex systems and synchronization of such systems have a great importance in different fields of engineering, information sciences, optical systems, communication network and physical systems. To attain the master-slave synchronization of nonlinear systems, various existing techniques and methods can be implied. The identities like Schur complement, Lyapunov functional, Jensen's inequality, time-delay systems, traditional Lipschitz condition and LMI-tools are described, which will be recalled frequently in the rest of the dissertation.

Chapter 3

SYNCHRONIZATION OF MUTUALLY LIPSCHITZ NONLINEAR SYSTEMS

3.1 Overview

Problem of synchronization of nonlinear systems is of great importance, due to its diverse applications in biological systems, chemical processes, energy systems and communication networks [16-40]. In the literature, a vigorous research has been established on synchronization of identical nonlinear systems [2-10]. There are different control methodologies like optimal, robust and adaptive for synchronization of identical nonlinear systems, considering the parameters like time-delay, disturbance, saturation and unknown dynamics have been extensively reported in the literature [61-64], [66-67], [72-73].

However, the problem of synchronization of non-identical nonlinear systems is lacking in the literature and these days, it is attaining a significant importance from the researchers. A few of the exceptional research works on synchronization of unlike dynamical systems are mentioned at this juncture [80-86]. In [80-82], adaptive control schemes are developed to cope with synchronization of two different chaotic oscillators by formulating adaptation laws for unknown parameters. Sliding mode control strategies for synchronizing the distinct chaotic systems under disturbances, slope-restricted input nonlinearity and different types of uncertainties have been addressed [83-84].

In this Chapter, problem of synchronization of non-identical nonlinear master-slave systems is considered. Key idea of synchronization is illustrated in Figure 3.1, It shows that difference between the states of the master and slave systems called error, is fed to the controller block, and controller generate an appropriate signal that fed to the slave system, that synchronize the slave system to the master system by converging the error to the origin.

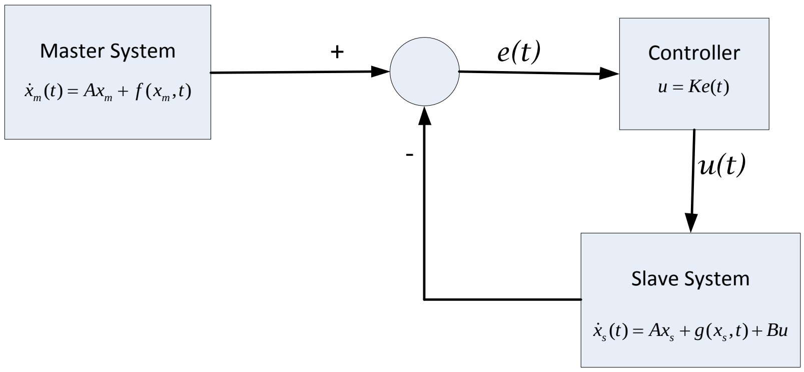


Figure 3.1: Block diagram for synchronization of nonlinear master-slave systems

The problem is formulated for a class of nonlinear systems. To simplify the design, nonlinearities in nonlinear systems are considered to be Lipschitz and for such Lipschitz systems, a novel mutually Lipschitz condition is proposed. Proposed mutually Lipschitz condition is more general than the traditional Lipschitz condition and powerful tool for designing a control law for synchronization of distinctive nonlinear systems. Traditional Lipschitz condition can be used for synchronization of similar nonlinear systems, whereas, it is inapplicable for synchronization of non-identical nonlinear systems.

Proof of proposed novel mutually Lipschitz condition along with it properties and their proofs is given, properties can be useful for investigating the nonlinear parameters. A simple quadratic Lyapunov function along with a mutually Lipschitz condition is used to design a simple state feedback control law for synchronization. After some mathematical treatment, an algebraic Riccati equation based control methodology is derived, which is demonstrated in Theorem 3.1. Congruence transformation, change of variable method and Schur compliment are applied on algebraic Riccati equation to formulate a linear matrix inequality (LMI) based approach for synchronization. LMI-based approach is more powerful tool and less conservative. The proposed scheme of synchronization using mutually Lipschitz condition is novel and provides a state-feedback controller deign, which is simple in design and implementation compared to the existing techniques for synchronization of non-identical nonlinear systems [80-86].

This Chapter is organized as follows. Next section contains the problem formulation. In Section 3, novel mutually Lipschitz condition is provided along with its properties followed by the proof of the properties of the mutually Lipschitz condition. Controller structure for synchronization is provided in section 5. An algebraic Riccati inequality based approach for synchronization is established in section 6. In section 7, an advance LMI-based synchronization approach is derived. In the end, a numerical simulation example to witness the proposed synchronization scheme is provided followed by the concluding remarks.

3.2 Systems Description

To formulate the problem for synchronization of non-identical nonlinear systems, two different nonlinear systems are considered, named as master system and slave system. Nonlinearities considered in the master and slave systems are different from each other. Dynamical model for nonlinear master system is described by

$$\dot{x}_m(t) = Ax_m + f(x_m, t), \;\; x_m(0) = x_{m0}, \qquad \textit{(Eq 3.1)}$$

where $x_m \in R^n$ denotes the state of the master system. $A \in R^{n\times n}$ is a linear known matrix with constant entries. Vector $f(x_m, t) \in R^n$ represents the nonlinearities in master system. $x_m(0) = x_{m0}$ is the initial condition for the master system.

Similarly dynamical model for the nonlinear slave system is described by:

$$\dot{x}_s(t) = Ax_s + g(x_s, t) + Bu, \; x_s(0) = x_{s0}, \qquad \textit{(Eq 3.2)}$$

where $x_s \in R^n$ denotes the state of the slave system. Vector $g(x_s, t) \in R^n$ represents the nonlinearities in slave system, which shows that the nonlinearities in the slave system are different compared to master system. $x_s(0) = x_{s0}$ represents the initial condition of nonlinear slave systems. $A \in R^{n\times n}$ is linear known matrix and is similar to a master system, it can be different for master and slave system, however for simplicity similar matrix is considered. $B \in R^{n\times m}$ is a linear matrix with known constant entries and $u \in R^m$ is the control input applied to the slave system to synchronize it with a master system.

Master and slave systems are considered to be Lipschitz. By virtue of Lipschitz condition, complex nonlinearities can be transformed into an equivalent upper bound of linear function. There are several techniques available to handle the linear functions which can be utilized for synchronization of such nonlinear functions. Traditional Lipschitz condition is useful for synchronization of identical nonlinear systems. However, it is inapplicable for synchronization of non-identical nonlinear systems. To address the problem of synchronization of different nonlinear systems a novel mutually Lipschitz condition is proposed herein.

3.3 Mutually Lipschitz Condition

Nonlinearities in master and slave systems are considered to be Lipschitz. Further these Lipschitz nonlinearities are assumed to be mutually Lipschitz. Mutually Lipschitz condition is more general than the conventional Lipschitz condition. Proposed mutually Lipschitz condition is an advance and novel condition that provides the advantage of controller synthesis for non-identical nonlinear systems, over the traditional Lipschitz condition.

Two different nonlinear functions $f(x_m,t)$ and $g(x_s,t)$ are said to be mutually Lipschitz, if satisfy the following mathematical inequality

$$\|f(x_m,t)-g(x_s,t)\|^2 \le l_{\max}^2 \|x_m - x_s\|^2 + \phi_{\max}, \qquad (Eq\ 3.3)$$

where $l_{\max}$ is a scalar quantity called the mutually Lipschitz constant. Similarly, $\phi_{\max}$ is a scalar quantity. Dimensions of $x(t)$ and $f(x,t)$ are considered to be the similar $(n=p)$, however for general case, different dimensions of $x(t)$ and $f(x,t)$ can be considered. The condition provided in Eq. 3.3 is called mutually Lipschitz condition.

3.3.1 Globally Mutually Lipschitz Nonlinearities

The nonlinear vector functions $f(x_m,t)$ and $g(x_s,t)$ for all states of x_m and x_s are said to be globally mutually Lipschitz, if following mathematical relationship holds among them,

$$\|f(x_m,t)-g(x_s,t)\|^2 \le l_{\max}^2 \|x_m - x_s\|^2 + \phi_{\max},\ x_m, x_m \in R^p \qquad (Eq\ 3.4)$$

3.3.2 Locally Mutually Lipschitz Nonlinearities

The nonlinear vector functions $f(x_m,t)$ and $g(x_s,t)$ are said to be locally mutually Lipschitz, if the mutually Lipschitz condition provided in Eq. 3.3 is satisfied for specific region, such that all $x_m, x_s \in \Omega \subset R^p$ with scalars $l_{\max} \ge 0$ and $\phi_{\max} \ge 0$.

Now, as mutually Lipschitz condition is defined, there are two questions needs to be answered. First the identification of nonlinear system that can be treated by mutually Lipschitz nonlinearities and secondly, computation of unknown parameters defined in mutually Lipschitz condition provided in Eq. 3.3. These concerned are addressed in the next section by providing the properties of mutually Lipschitz conditions along with their detailed proofs.

3.4 Properties of Mutually Lipschitz Condition

Properties for the mutually Lipschitz condition are provided herein, which can be useful for identification of nonlinear systems, that can be treated with mutually Lipschitz condition. Unknown dynamical parameters can also be computed by virtue of these properties. Different properties of the mutually Lipschitz condition are described as follows.

1. If nonlinear vector functions $f(x_m,t)\in R^n$ and $g(x_s,t)\in R^n$ for all $x_m,x_s\in R^p$ belong to Lipschitz nonlinearities with Lipschitz constants l_f and l_g, respectively, then following inequalities are satisfied:

$$\| f(x_m,t)-g(x_s,t)\|^2 \le (1+\varepsilon)\, l_f^2 \|x_m-x_s\|^2 + \left(1+\varepsilon^{-1}\right)\delta_1, \qquad (Eq\ 3.5)$$

$$\| f(x_m,t)-g(x_s,t)\|^2 \le (1+\varepsilon)\, l_g^2 \|x_m-x_s\|^2 + \left(1+\varepsilon^{-1}\right)\delta_2, \qquad (Eq\ 3.6)$$

where

$\varepsilon =$ any positive scalar ,

$\delta_1 = \| f(x_s,t)-g(x_s,t)\|^2$,

$\delta_2 = \| f(x_m,t)-g(x_m,t)\|^2$.

2. If two nonlinear functions $f(x_m,t)\in R^n$ and $g(x_s,t)\in R^n$ for all $x_m,x_s\in R^p$ belonging to Lipschitz nonlinearities are mutually Lipschitz.
3. Proposed mutually Lipschitz condition of Eq. 3.3 is more general than the traditional Lipschitz condition. It can be easily verified using the following condition that if two different nonlinearities are supposed to be similar, mathematically $f(x,t)=g(x,t)$ and by choosing $\phi_{\max}=0$, than the mutually

Lipschitz condition of Eq. 3.3 reduces to the conventional Lipschitz condition as below.

$$\| f(x_m,t) - f(x_s,t)\|^2 \le l_{\max}^2 \|x_m - x_s\|^2 , \qquad \textit{(Eq 3.7)}$$

where $l_{\max} \ge 0$ is a Lipschitz constant.

3.4.1 Proof of Inequalities in Property 1

There are two inequalities provided in first property of mutually Lipschitz condition. Inequality of Eq. 3.5 can be derived as follows. Taking the left hand side of Eq. 3.5, and by adding-subtracting the nonlinear function $f(x_s,t)$, it reveals

$$\| f(x_m,t) - g(x_s,t) + f(x_s,t) - f(x_s,t)\|^2 . \qquad \textit{(Eq 3.8)}$$

Rearranging it, we obtain

$$\|\{ f(x_m,t) - f(x_s,t)\} + \{ f(x_s,t) - g(x_s,t)\}\|^2 . \qquad \textit{(Eq 3.9)}$$

It can be expand as square of two functions as below,

$$\begin{aligned}\|\{ f(x_m,t) - f(x_s,t)\} + \{ f(x_s,t) - g(x_s,t)\}\|^2 &= \| f(x_m,t) - f(x_s,t)\|^2 \\ &+ \| f(x_s,t) - g(x_s,t)\|^2 \\ &+ 2(f(x_m,t) - f(x_s,t)) \\ &\times(f(x_s,t) - g(x_s,t)).\end{aligned} \qquad \textit{(Eq 3.10)}$$

Now, introducing mathematical identity $2ab \le a^2 + b^2$, and extending this inequality by introducing a scalar function k. It can also be rewritten as

$$2ab \le (\kappa)a^2 + (\kappa^{-1})b^2 , \qquad \textit{(Eq 3.11)}$$

where introduction of the scalar function k does not affect the health of the equation, but become useful to derive the desired inequality. Using this mathematical identity into Eq. 3.10, we have

$$\begin{aligned}\|\{ f(x_m,t) - f(x_s,t)\} + \{ f(x_s,t) - g(x_s,t)\}\|^2 &\le \| f(x_m,t) - f(x_s,t)\|^2 \\ &+ \| f(x_s,t) - g(x_s,t)\|^2 \\ &+ \varepsilon \| f(x_m,t) - f(x_s,t)\|^2 \\ &+ \varepsilon^{-1} \| f(x_s,t) - g(x_s,t)\|^2 .\end{aligned} \qquad \textit{(Eq 3.12)}$$

Now for the further simplification, traditional Lipschitz condition of Eq. 3.7, is employed and Eq. 3.12, transformed as

$$\begin{aligned}\left\|\{f(x_m,t)-f(x_s,t)\}+\{f(x_s,t)-g(x_s,t)\}\right\|^2 \le\; & l_f^2\left\|x_m-x_s\right\|^2+\varepsilon l_f^2\left\|x_m-x_s\right\|^2\\ & +\varepsilon^{-1}\left\|f(x_s,t)-g(x_s,t)\right\|^2\\ & +\left\|f(x_s,t)-g(x_s,t)\right\|^2.\end{aligned} \qquad (Eq\ 3.13)$$

Rearranging it, we get

$$\begin{aligned}\left\|\{f(x_m,t)-f(x_s,t)\}+\{f(x_s,t)-g(x_s,t)\}\right\|^2 \le\; & l_f^2\left\|x_m-x_s\right\|^2(1+\varepsilon)\\ & +(1+\varepsilon^{-1})\left\|f(x_s,t)-g(x_s,t)\right\|^2.\end{aligned} \qquad (Eq\ 3.14)$$

For further simplification, introducing $\delta_1=\left\|f(x_s,t)-g(x_s,t)\right\|^2$, it gives

$$\begin{aligned}\left\|\{f(x_m,t)-f(x_s,t)\}+\{f(x_s,t)-g(x_s,t)\}\right\|^2 \le\; & l_f^2\left\|x_m-x_s\right\|^2(1+\varepsilon)\\ & +(1+\varepsilon^{-1})\delta_1.\end{aligned} \qquad (Eq\ 3.15)$$

It completes the proof of proposed inequality.

Using the same approach, proof of inequality of Eq. 3.6 can be provided as below.

Taking the left hand side of Eq. 3.6 and by adding-subtracting the nonlinear function $g(x_m,t)$, it reveals

$$\left\|f(x_m,t)-g(x_s,t)+g(x_m,t)-g(x_m,t)\right\|^2. \qquad (Eq\ 3.16)$$

Rearranging it, we have

$$\left\|\{g(x_m,t)-g(x_s,t)\}+\{f(x_m,t)-g(x_m,t)\}\right\|^2. \qquad (Eq\ 3.17)$$

It can be expanded as square of two functions as under

$$\begin{aligned}\left\|\{g(x_m,t)-g(x_s,t)\}+\{f(x_m,t)-g(x_m,t)\}\right\|^2 =\; & \left\|g(x_m,t)-g(x_s,t)\right\|^2\\ & +\left\|f(x_m,t)-g(x_m,t)\right\|^2\\ & +2(g(x_m,t)-g(x_s,t))\\ & \times(f(x_m,t)-g(x_m,t)).\end{aligned} \qquad (Eq\ 3.18)$$

Using identity defined in Eq. 3.12 for $2(g(x_m,t)-g(x_s,t))(f(x_m,t)-g(x_m,t))$ into Eq. 3.18, it yields

$$\left\|\{g(x_m,t)-g(x_s,t)\}+\{f(x_m,t)-g(x_m,t)\}\right\|^2 \le \left\|g(x_m,t)-g(x_s,t)\right\|^2 + \left\|f(x_m,t)-g(x_m,t)\right\|^2 + \varepsilon\left\|g(x_m,t)-g(x_s,t)\right\|^2 + \varepsilon^{-1}\left\|f(x_m,t)-g(x_m,t)\right\|^2. \quad \textit{(Eq 3.19)}$$

Now for the further simplification, traditional Lipschitz condition of Eq. 3.7 is employed and Eq. 3.19 is transformed as follows

$$\left\|\{g(x_m,t)-g(x_s,t)\}+\{f(x_m,t)-g(x_m,t)\}\right\|^2 \le l_g^2\left\|x_m-x_s\right\|^2 + \varepsilon l_g^2\left\|x_m-x_s\right\|^2 + \varepsilon^{-1}\left\|f(x_m,t)-g(x_m,t)\right\|^2 + \left\|f(x_m,t)-g(x_m,t)\right\|^2. \quad \textit{(Eq 3.20)}$$

Rearranging it, we get

$$\left\|\{g(x_m,t)-g(x_s,t)\}+\{f(x_m,t)-g(x_m,t)\}\right\|^2 \le l_g^2\left\|x_m-x_s\right\|^2(1+\varepsilon)+(1+\varepsilon^{-1}) \times\left\|f(x_m,t)-g(x_m,t)\right\|^2. \quad \textit{(Eq 3.21)}$$

For simplicity introducing $\delta_2=\left\|f(x_m,t)-g(x_m,t)\right\|^2$, it gives

$$\left\|\{g(x_m,t)-g(x_s,t)\}+\{f(x_m,t)-g(x_m,t)\}\right\|^2 \le l_g^2\left\|x_m-x_s\right\|^2(1+\varepsilon) +(1+\varepsilon^{-1})\delta_2. \quad \textit{(Eq 3.22)}$$

It completes the proof of the proposed inequality.

3.4.2 Proof of Property 2

Two nonlinear functions $f(x_m,t)\in R^n$ and $g(x_s,t)\in R^n$ for all $x_m,x_s\in R^p$ belongs to Lipschitz nonlinearities are mutually Lipschitz. It can be proven mathematically by comparing mutually Lipschitz conditions provided in Eq. 3.3 and inequality defined in Eq. 3.5, rewritten as

$$\left\|f(x_m,t)-g(x_s,t)\right\|^2 \le l_{\max}^2\left\|x_m-x_s\right\|^2+\phi_{\max}, \quad \textit{(Eq 3.23)}$$

$$\left\|f(x_m,t)-g(x_s,t)\right\|^2 \le \left(1+\varepsilon\right)l_f^2\left\|x_m-x_s\right\|^2+\left(1+\varepsilon^{-1}\right)\delta_1. \quad \textit{(Eq 3.24)}$$

Left hand side of the both inequalities is the similar, so by comparing right hand side of these equations, we obtain

$$l_{\max}^2 \|x_m - x_s\|^2 + \phi_{\max} = (1+\varepsilon) l_f^2 \|x_m - x_s\|^2 + \left(1+\varepsilon^{-1}\right)\delta_1. \qquad \textit{(Eq 3.25)}$$

It can be separated into two parts as under

$$l_{\max}^2 \|x_m - x_s\|^2 = (1+\varepsilon) l_f^2 \|x_m - x_s\|^2, \qquad \textit{(Eq 3.26)}$$

and

$$\phi_{\max} = \left(1+\varepsilon^{-1}\right)\max\delta_1. \qquad \textit{(Eq 3.27)}$$

First part of the equation can be solved by simple mathematics

$$l_{\max}^2 = (1+\varepsilon) l_f^2. \qquad \textit{(Eq 3.28)}$$

It can be further simplified as

$$l_{\max} = \sqrt{(1+\varepsilon) l_f^2}, \qquad \textit{(Eq 3.29)}$$

$l_{\max}$ represents the Lipschitz constant of the nonlinear function.

Similarly by comparing mutually Lipschitz condition of Eq. 3.3 with Eq. 3.6, we get

$$\| f(x_m,t) - g(x_s,t)\|^2 \le l_{\max}^2 \|x_m - x_s\|^2 + \phi_{\max}. \qquad \textit{(Eq 3.30)}$$

$$\| f(x_m,t) - g(x_s,t)\|^2 \le (1+\varepsilon) l_g^2 \|x_m - x_s\|^2 + \left(1+\varepsilon^{-1}\right)\delta_2. \qquad \textit{(Eq 3.31)}$$

By comparing right hand side of both equations,

$$l_{\max}^2 \|x_m - x_s\|^2 + \phi_{\max} = (1+\varepsilon) l_g^2 \|x_m - x_s\|^2 + \left(1+\varepsilon^{-1}\right)\delta_2. \qquad \textit{(Eq 3.32)}$$

$$l_{\max}^2 = (1+\varepsilon) l_g^2. \qquad \textit{(Eq 3.33)}$$

It can be further simplified as

$$l_{\max} = \sqrt{(1+\varepsilon) l_g^2}. \qquad \textit{(Eq 3.34)}$$

$$\phi_{\max} = \left(1+\varepsilon^{-1}\right)\max(\delta_2). \qquad \textit{(Eq 3.35)}$$

3.5 Controller Design

Different controls strategies have been developed for the synchronization of non-identical nonlinear systems such as adaptive and sliding mode. Existing techniques are computational complex. So, by virtue of proposed mutually Lipschitz condition, a simple static feedback control law is proposed for synchronization of two mutually Lipschitz nonlinear systems. The proposed controller is uncomplicated in design, easy to implement and provides the optimal results. Structure of the proposed static state feedback controller is selected as

$$u = Ke(t)\,, \tag{Eq 3.36}$$

where $u \in R^{m}$ represents the controller input, $K \in R^{m \times n}$ is the controller gain and $e(t)$ denotes the error between the states of master and slave systems. Using proposed controller of Eq. 3.36 dynamics of the slave system of Eq. 3.2 can be rewritten as

$$\dot{x}_s(t) = Ax_s + g(x_s, t) + BKe(t). \tag{Eq 3.37}$$

Difference between the states of the master system and slave system is defined as an error $e(t) = x_m(t) - x_s(t)$, and taking its time derivative. It yields

$$\dot{e}(t) = \dot{x}_m(t) - \dot{x}_s(t)\ . \tag{Eq 3.38}$$

Incorporating the dynamics of master system of Eq. 3.1 and slave system with control input of Eq 3.37, along with the error definition, it implies

$$\dot{e}(t) = Ae(t) + f(x_m, t) - g(x_s, t) - BKe(t)\ . \tag{Eq 3.39}$$

It can be further simplified as below

$$\dot{e}(t) = \left(A - BK\right)e(t) + f(x_m, t) - g(x_s, t)\,. \tag{Eq 3.40}$$

To attain the synchronization between master and slave systems, algebraic Riccati based approach is provided.

3.6 Theorem 3.1-(Algebraic Riccati Based Approach)

Consider a nonlinear master and slave systems of Eq. 3.1 and Eq. 3.2, respectively, that satisfy the mutually Lipschitz condition provided in Eq. 3.3 and suppose there exist a positive scalar λ along with positive-definite symmetric matrix P , such that following inequality holds

$$A^T P + PA - K^T B^T P - PBK + P^2 + l_{\max}^2 I + \lambda I < 0\,. \tag{Eq 3.41}$$

Then proposed controller of Eq. 3.36 guarantee the uniformly ultimately synchronization of nonlinear master-slave systems by converging the error in following region:

$$\|e\|^2 \le (\phi_{\max})/\lambda . \qquad \textit{(Eq 3.42)}$$

Selection of appropriate λ provides efficient error convergence.

3.6.1 Proof of Theorem 3.1

Now to ensure the synchronization between master and slave systems, error trajectory should converge to the origin. To show the error convergence, Lyapunov stability theory is utilized and a quadratic Lyapunov function is constructed for convergence. To ensure the error convergence, quadratic Lyapunov function should be positive-definite or its derivate should be negative definite. To provide the proof of Theorem 3.1, a simple quadratic Lyapunov function is constructed as

$$V(t,e) = e^T(t)Pe(t) . \qquad \textit{(Eq 3.43)}$$

Taking time derivative of energy function, it reveals

$$\dot{V}(t,e) = \dot{e}^T(t)Pe(t) + e^T(t)P\dot{e}(t) . \qquad \textit{(Eq 3.44)}$$

Now by incorporating the values of error derivative $\dot{e}(t)$ from Eq. 3.40 into Eq. 3.44, it implies

$$\begin{aligned}\dot{V}(t,e) = {} & [(A-BK)e(t) + f(x_m,t) - g(x_s,t)]^T Pe(t) \\ & + e^T(t)P(A-BK)e(t) + f(x_m,t) - g(x_s,t).\end{aligned} \qquad \textit{(Eq 3.45)}$$

Rearranging it, we obtain

$$\begin{aligned}\dot{V}(t,e) = {} & e^T(t)A^T Pe(t) - e^T(t)K^T B^T Pe(t) + f(x_m,t) - g(x_s,t))^T Pe(t) \\ & + e^T(t)PAe(t) - e^T(t)PBKe(t) + e^T(t)P(f(x_m,t) - g(x_s,t)).\end{aligned} \qquad \textit{(Eq 3.46)}$$

It can be written as

$$\begin{aligned}\dot{V}(t,e) = {} & e^T[A^T P - K^T B^T P + PA - PBK]e + (f(x_m,t) - g(x_s,t))^T Pe \\ & + e^T P(f(x_m,t) - g(x_s,t)).\end{aligned} \qquad \textit{(Eq 3.47)}$$

Introducing the mathematical identity $2A^T B \le A^T A + B^T B$, and using for $(f(x_m,t) - g(x_s,t))^T Pe$ and $e^T P(f(x_m,t) - g(x_s,t))$, it can be expand as

$$2\left(f(x_m,t)-g(x_s,t)\right)^T Pe \le \left(f(x_m,t)-g(x_s,t)\right)^T \left(f(x_m,t)-g(x_s,t)\right) \\ +e^T PPe. \quad \textit{(Eq 3.48)}$$

Using Eq. 3.48 into Eq. 3.47, it implies

$$\dot{V}(t,e) \le e^T[A^T P - K^T B^T P + PA - PBK]e + e^T P^2 e \\ +(f(x_m,t)-g(x_s,t))^T (f(x_m,t)-g(x_s,t)). \quad \textit{(Eq 3.49)}$$

Rearranging it, we obtain

$$\dot{V}(t,e) \le e^T[A^T P - K^T B^T P + PA - PBK + P^2]e \\ +(f(x_m,t)-g(x_s,t))^T (f(x_m,t)-g(x_s,t)). \quad \textit{(Eq 3.50)}$$

Now, as nonlinear functions $f(x_m,t)$ and $g(x_s,t)$ are assumed to be mutually Lipschitz. Then by application of proposed mutually Lipschitz condition of Eq. 3.3, relationship between the nonlinear functions can be describe as follows

$$(f(x_m,t)-g(x_s,t))^T (f(x_m,t)-g(x_s,t)) \le l_{\max}^2 (\sqrt{(x_m-x_s)(x_m-x_s)^T})^2 \\ +\Phi_{\max}. \quad \textit{(Eq 3.51)}$$

Using the identity provided in Eq. 3.51 into Eq. 3.50, it implies

$$\dot{V} \le e^T[A^T P - K^T B^T P + PA - PBK + P^2]e + e^T l_{\max}^2 e + \Phi_{\max}. \quad \textit{(Eq 3.52)}$$

It can be further simplified as below

$$\dot{V} \le e^T[A^T P - K^T B^T P + PA - PBK + P^2 + l_{\max}^2]e + \Phi_{\max}. \quad \textit{(Eq 3.53)}$$

Now introducing the positive scalar function λ, and also it is assumed that inequality $A^T P + PA - K^T B^T P - PBK + P^2 + l_{\max}^2 I < -\lambda I$, reveals. This completes the proof of algebraic Riccati based inequality of Theorem 3.1.

The algebraic Riccati inequality based approach for synchronization by asymptotic error convergence provided in Theorem 3.1 is quite simple and useful, however the proposed methodology has some issues regarding selection of unknown parameters. The challenges of algebraic Riccati based approach are outlined.

1. There is no procedure available for computation of unknown parameters, e.g., P and K. So selection of matrices P and K is a difficult task.

2. To minimize the error $e(t)$, high value of the scalar parameter λ is required, so the proposed approach may not offer optimization of the parameter λ to achieve the maximum rejection of unwanted disturbances.

The concerns raised on algebraic Riccati based approach of Theorem 3.1 for synchronization are of great significance, to address these issues, an advance linear matrix inequality (LMI) based approach is provided in Theorem 3.2.

3.7 Theorem 3.2-(LMI Based Approach)

Let the master and slave systems of Eq. 3.1 and Eq. 3.2, respectively, that satisfy the mutually Lipschitz condition provided in Eq. 3.3. Suppose their exist a positive scalar μ along with positive-definite symmetric matrix $Q \in R^{n\times n}$ and matrix $M \in R^{m\times n}$. Than by solving the optimization $\min\ \mu$

$$Q > 0, \quad \mu > \mu_o \geq 0, \qquad \textit{(Eq 3.54)}$$

such that following LMI holds

$$\begin{bmatrix} QA^T + AQ - M^T B^T - BM + I & Ql_{\max} & Q \\ * & -I & 0 \\ * & * & -\mu I \end{bmatrix} < 0. \qquad \textit{(Eq 3.55)}$$

Using this LMI, parameter K of control law can be obtained by solving $K = MQ^{-1}$. Further, the proposed control law $u = Ke(t)$ ensures uniformly ultimately bounded synchronization of nonlinear non-identical master-slave systems in the region

$$\|e\|^2 \leq \mu\left(\phi_{\max}\right). \qquad \textit{(Eq 3.56)}$$

3.7.1 Proof of Theorem 3.2

To provide the proof of Theorem 3.2, some LMI tricks are applied on algebraic Riccati based inequality of Eq. 3.41 to transform it into linear mathematical inequality form of Eq. 3.55. Incorporating the congruence transformation by pre and post multiplication of P^{-1} to the algebraic Riccati inequality of Eq. 3.41, we obtain

$$\begin{aligned} &P^{-1}A^T PP^{-1} + P^{-1}PAP^{-1} - P^{-1}K^T B^T PP^{-1} - P^{-1}PBKP^{-1} \\ &+P^{-1}PPP^{-1} + P^{-1}l_{\max}^2 IP^{-1} + P^{-1}\lambda IP^{-1} < 0. \end{aligned} \qquad \textit{(Eq 3.57)}$$

Hence, $P^{-1}P = I$, it can be simplified as

$$P^{-1}A^T + AP^{-1} - P^{-1}K^T B^T - P^{-1}PBKP^{-1} + I$$
$$+P^{-1}l_{\max}^2 IP^{-1} + P^{-1}\lambda IP^{-1} < 0. \quad \textit{(Eq 3.58)}$$

Now applying change of variable method on Eq. 3.58, by $P^{-1} = Q$ and $\lambda^{-1} = \mu$, it yields

$$QA^T + AQ - QK^T B^T - BKQ + I + Ql_{\max}^2 IQ + Q\mu^{-1}IQ < 0. \quad \textit{(Eq 3.59)}$$

Substituting $M = KQ$ for reducing two unknown parameters into single identity, it reveals

$$QA^T + AQ - M^T B^T - BM + I + l_{\max}^2 Q^2 + \mu^{-1}Q^2 < 0. \quad \textit{(Eq 3.60)}$$

Now applying Schur complement to above inequality, an LMI of form Eq. 3.55 can be obtained. This completes the proof of Theorem 3.2.

It is seems that by the application of the proposed mutually Lipschitz condition, a simple state-feedback control law is provided for the synchronization of two different nonlinear master-slave systems. Mutually Lipschitz condition provides a significant feature, that a simple control law can be designed, even if nonlinear functions of master-salve systems are unknown. This can be made using the knowledge of $l_{\max}$, as this information is sufficient to propose a control law for synchronization of different nonlinear systems.

3.8 Simulation Results

Couple of numerical examples are illustrated, to witness the proposed mutually Lipschitz condition and its effectiveness for the synchronization of non-identical nonlinear systems observing different Lipschitz nonlinearities.

3.8.1 Synchronization of Two Different Chua’s Systems

Consider the following uncertain and different chaotic Chua’s systems described in [66]. Parameters of Chua system used as master system and slave system are different from each other, to make them non-identical systems.

Equation of motion of Chua's system used as master system is presented by following set of equations

$$\dot{x}_{m1} = a_{11}x_1 + a_{12}x_2 + \alpha_1(|x_m + \alpha_2| + |x_m + \alpha_3|), \quad \textit{(Eq 3.61a)}$$

$$\dot{x}_{m2} = a_{21}x_1 - a_{22}x_2 + a_{23}x_3, \quad \textit{(Eq 3.61b)}$$

$$\dot{x}_{m3} = a_{32}x_2. \quad \textit{(Eq 3.61c)}$$

Equation of motion of Chua's system used as slave system is presented by following set of equations

$$\dot{x}_{s1} = a_{11}x_1 + a_{12}x_2 + \beta_1(|x_m + \beta_2| + |x_m + \beta_3|), \quad \textit{(Eq 3.62a)}$$

$$\dot{x}_{s2} = a_{21}x_1 - a_{22}x_2 + a_{23}x_3, \quad \textit{(Eq 3.62b)}$$

$$\dot{x}_{s3} = a_{32}x_2. \quad \textit{(Eq 3.62c)}$$

Nonlinear functions in master and slave systems can be written in the form of matrices as under

$$f(x_m, t) = \begin{bmatrix} \alpha_1\left(|x_m + \alpha_2| + |x_m - \alpha_3|\right) \\ 0 \\ 0 \end{bmatrix}, \quad \textit{(Eq 3.63)}$$

$$g(x_s, t) = \begin{bmatrix} \beta_1\left(|x_s + \beta_2| + |x_s - \beta_3|\right) \\ 0 \\ 0 \end{bmatrix}. \quad \textit{(Eq 3.64)}$$

Clearly, the functions $f(x_m, t)$ and $g(x_s, t)$ are different from each other and, further, their parameters α_i and β_i, for $i = 1, 2, 3$, are assumed to be unknown and bounded such that $\alpha_1, \beta_1 \in [0 \quad 2]$ and $\alpha_2, \alpha_3, \beta_2, \beta_3 \in [0 \quad 1.2]$. Using bounded constraints parametric values of Chua's system considered as master system are $\alpha_1 = 1.9286$, $\alpha_2 = 1$ and $\alpha_3 = 1.1$. Similarly parametric values of Chua's system considered as slave system are $\beta_1 = 1.8482$, $\beta_2 = 1.1$ and $\beta_3 = 1$. Let a linear matrix with known entries selected as

$$A = \begin{bmatrix} -2.548 & 9.1 & 0 \\ 1 & -1 & 1 \\ 0 & 14.2 & 0 \end{bmatrix}. \quad \textit{(Eq 3.65)}$$

To determine the controller gain matrix, first LMI feasibility is ensured using robust control toolbox of MATLAB and then using $K = MQ^{-1}$, by selecting $\mu_o = 0.01$ and $L_{\max} = diag(3,0,0)$, feedback gain matrix is computed as

$$K = \begin{bmatrix} 52.02 & 0.17 & -7.95 \\ 12.90 & 42.23 & 2.67 \\ 10.59 & -15.87 & 42.23 \end{bmatrix}. \qquad \textit{(Eq 3.66)}$$

The proposed controller is applied to attain the synchronization between master system and slave system, which are Chua's systems with different parameters

Figure 3.2, shows the phase portraits of different Chua's systems without controller. Phase portraits are different from each other due to low mismatch of parameters between both systems. Figure 3.2(a), shows the 3-D phase portrait of master system without controller and Figure 3.2(b), shows the 3-D phase portrait of slave system without controller.

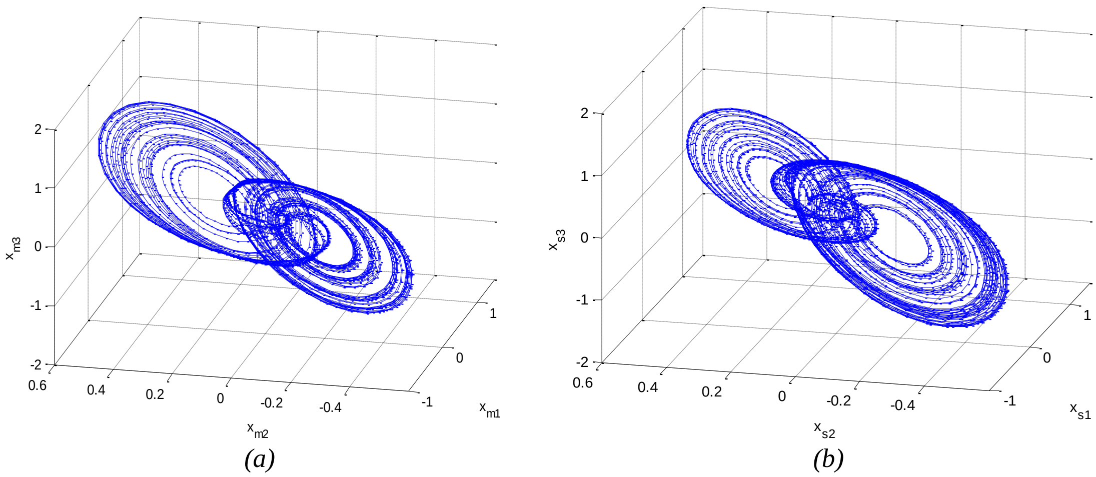


Fig. 3.2: 3-D Phase portrait of Chua system with low mismatch of parameter without Controller, (a) master system, (b) slave system

When control signal is activated, response of master and slave systems is shown in Figure 3.3. Figure 3.3(a) shows the phase portrait of Chua system used as master system with low mismatch of parameter with state feedback controller and Figure 3.3(b) shows the phase portrait of Chua's system used as slave system with low mismatch of parameter with state feedback controller. Response of the slave system is following the trajectory of the master system.

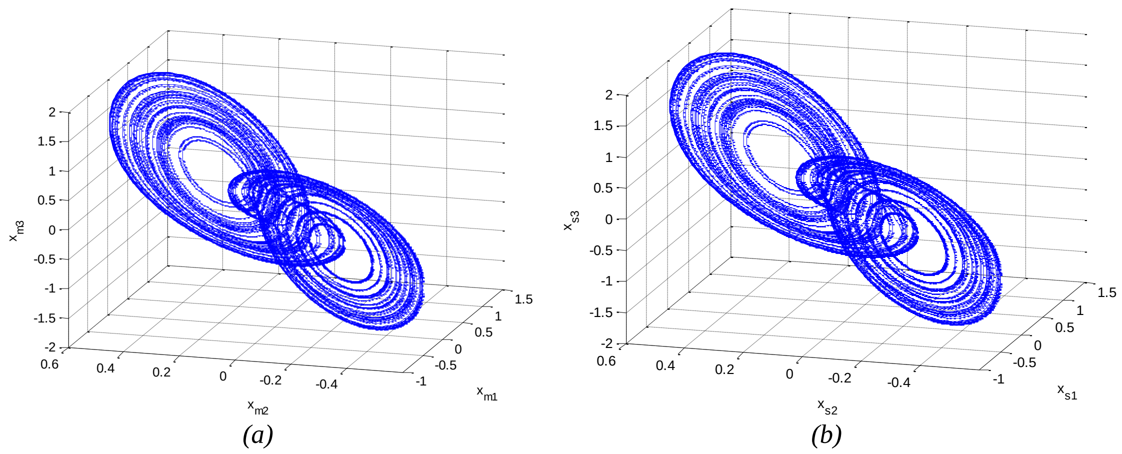


Fig. 3.3: Phase portrait of Chua system with low mismatch of parameter with state feedback controller (a) master system, (b) slave system

Error responses between different states of the master system and the slave system without controller and with static state feedback controller are shown in Figure 3.4, Figure 3.5 and Figure 3.6. Figure 3.4(a) shows the error response $e_1(t) = x_{m1}(t) - x_{s1}(t)$ between first state of the master and the slave systems without controller, which shows the oscillatory response. Whereas, Figure 3.4(b) show the error $e_1(t) = x_{m1}(t) - x_{s1}(t)$ between first state of the master and the slave systems in the presence of control input. The error plot validates convergence of the synchronization errors to a small region near origin.

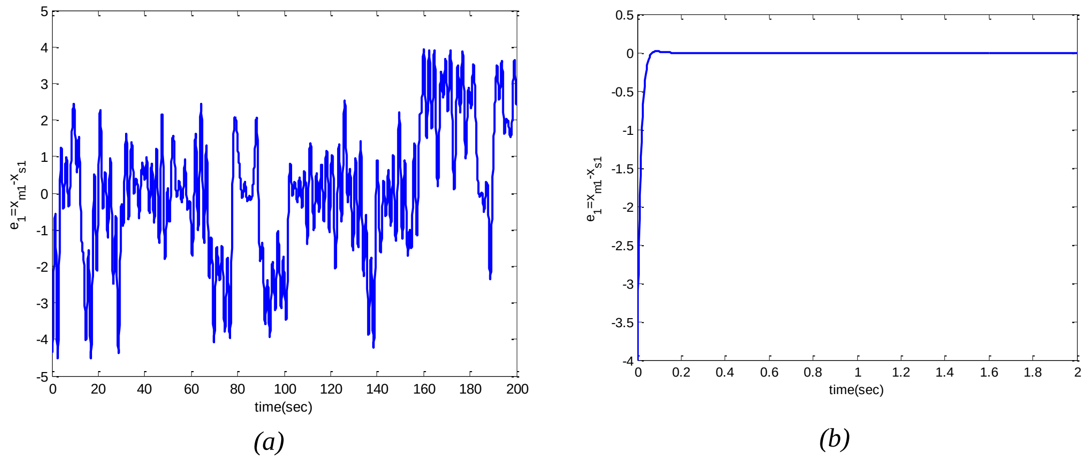
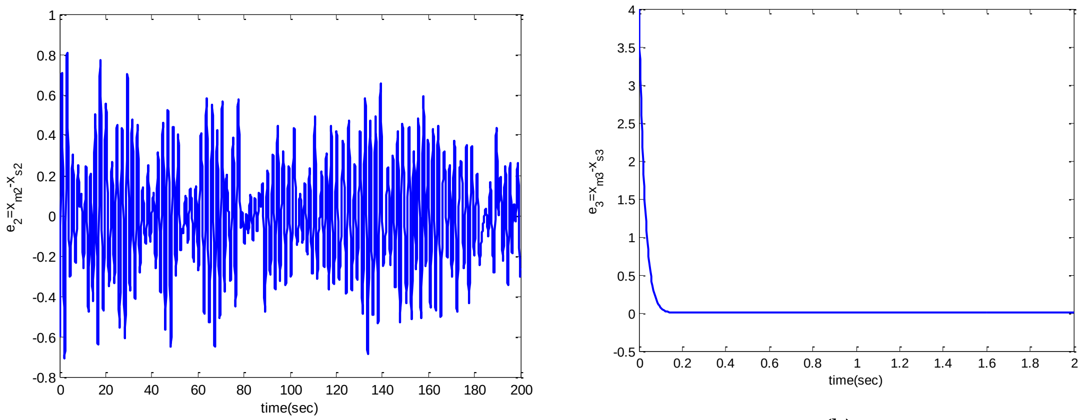


Fig. 3.4: Phase portrait of Chua system with low mismatch of parameter with state feedback controller (a) master system, (b) slave system

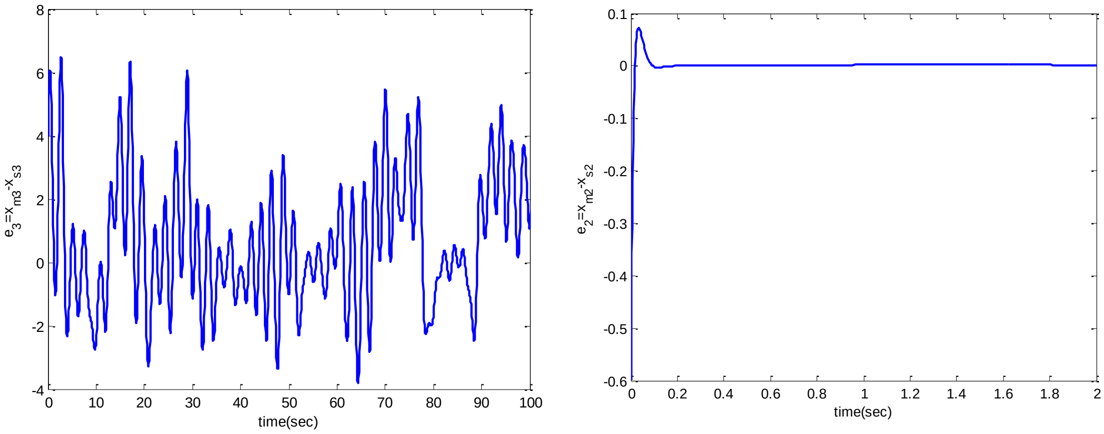
Similarly Figures 3.5 shows the error response between second state of master system and slave system $e_2(t) = x_{m2}(t) - x_{s2}(t)$ with and without controller. Which shows that without controller, error response is oscillatory and not converging to the reference, on the other hand error converges to the origin as controller is activated.



(a) *(b)*

Figure 3.5: Synchronization error plot for $e_2(t)$*, (a) without controller, (b) with controller*

Similarly Figures 3.6(a) and 3.6(b) shows the error response between the third state of the master system and the slave system $e_3(t) = x_{m3}(t) - x_{s3}(t)$ in the absence and presence of state feedback controller, respectively, which shows that without controller, error response is oscillatory and not converging to the reference, on the other hand error converges to the origin as controller is activated.



(a) *(b)*

Figure 3.6: Synchronization error plot for $e_3(t)$*, (a) without controller, (b) with controller*

3.8.2 Synchronization of Modified Chua's and Rossler Systems

In the first example, two nonlinear systems with low mismatch were synchronized. Now to witness the worth of proposed controller two non-identical systems are considered. Modified Chua's system is considered as the master system and a Rössler system is considered as the slave system [122]. The dynamics of the master and slave systems are described in matrices form as follows

$$A = \begin{bmatrix} -2.548 & 9.1 & 0 \\ 1 & -1 & 1 \\ 0 & 14.2 & 0 \end{bmatrix}, \qquad \textit{(Eq 3.67)}$$

$$f(x_m,t) = \begin{bmatrix} 10\left(x_{m2} - \dfrac{2x_{m1}^3 - x_{m1}}{7} \right) \\ x_{m1} - x_{m2} + x_{m3} \\ -\dfrac{100}{7} x_{m2} \end{bmatrix}, \qquad \textit{(Eq 3.68)}$$

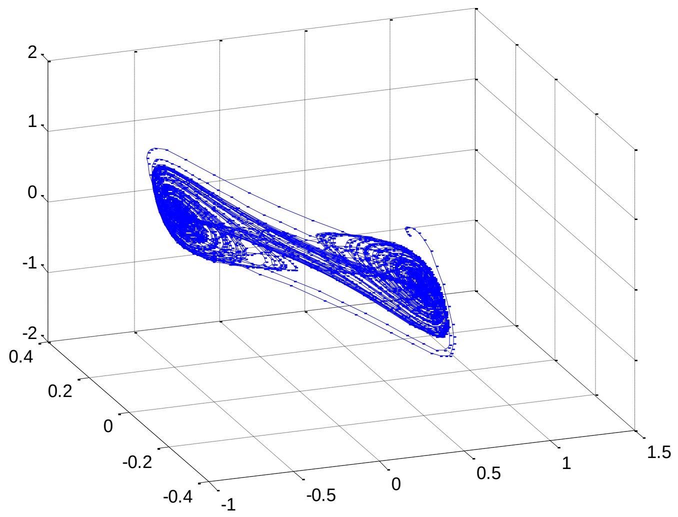
$$g(x_s,t) = \begin{bmatrix} -x_{s2} - x_{s3} \\ x_{s1} + 0.2x_{s2} \\ 0.2 + x_{s3}\left(x_{s1} + 5.7\right) \end{bmatrix}, \qquad \textit{(Eq 3.69)}$$

The Lipschitz constant l_f is calculated numerically by determining supremum of the maximum eigen values of $\left(\partial f(x,t)/\partial x\right)^T \left(\partial f(x,t)/\partial x\right)$ for $x \in \left[-2 \quad 2\right]$. To determine the value of mutually Lipschitz constant, parameter $\varepsilon = 0.1$ is selected, and the resultant Lipschitz constant is $l_{\max} = 23.72$. Now feedback controller gain matrix is determined by solving the linear matrix inequality and then by solving $K = MQ^{-1}$.

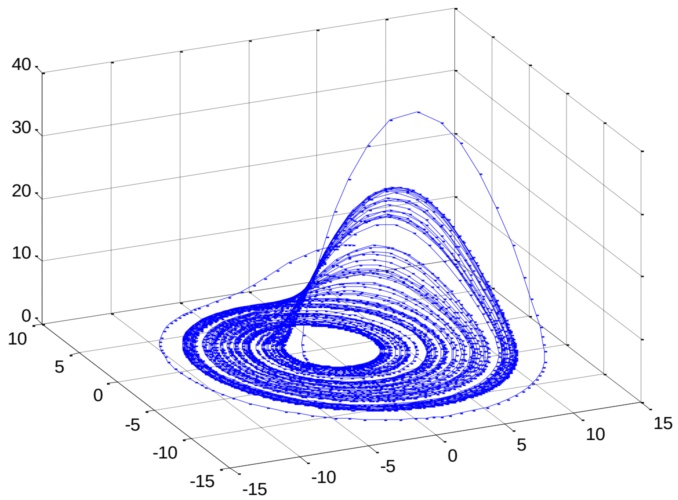
$$K = \begin{bmatrix} 44.45 & 0 & 0 \\ 0 & 44.45 & 0 \\ 0 & 0 & 44.45 \end{bmatrix}. \qquad \textit{(Eq 3.70)}$$

Figures 3.7(a) and 3.7(b) shows the phase portraits of Chua's system and the phase portrait of Rossler system, selected as master system and slave system, respectively, when no control input is applied. Response of master system and slave system show the typical phase portrait of Chua's systems and Rossler system, and it reflects unsynchronized behavior.

Figures 3.8(a) and 3.8(b) show the chaotic behaviors of the master and the slave systems, respectively, in the presence of controller. Responses show that when control input is applied to the slave system (Rossler system), its response is changed and follow the trajectory of master system (Chua circuit).



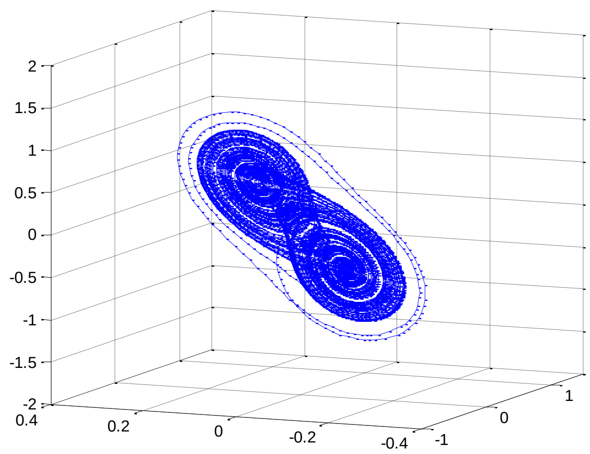
(a)



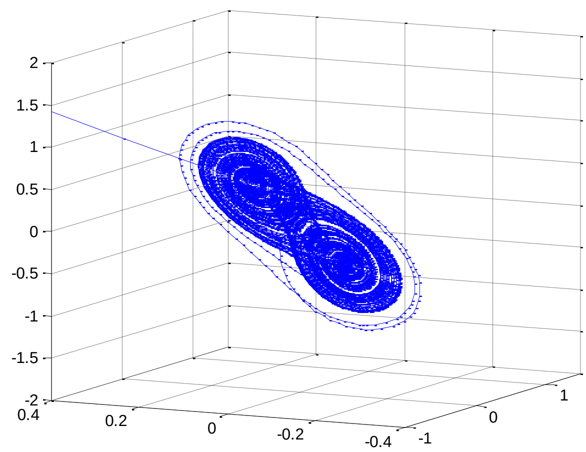
(b)

Fig. 3.7: Phase portraits of the modified Chua systems used as master system and Rössler system used as slave system, respectively, without controller

Figures 3.9(a), 3.9(b) and 3.9(c) show the synchronization error plots for different states of master and slave systems. Figure 3.9 (a) shows the error $e_1(t) = x_{m1}(t) - x_{s1}(t)$ for the first state of master and slave systems. Plot demonstrates the error response without control signal and in the presence of control input. When $t < 200\sec$, slave system is running without control input, response of the error is oscillatory, which is not converging to the origin. It reflects that there is no synchronization between master and slave systems. After time $t \geq 200\sec$, controller is incorporated, it shows that dynamics of errors converges to the origin.

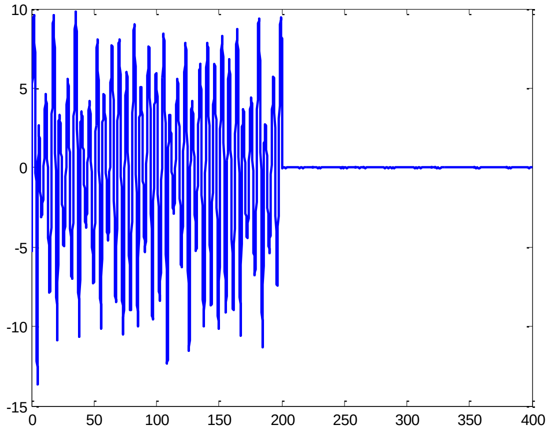


(a)

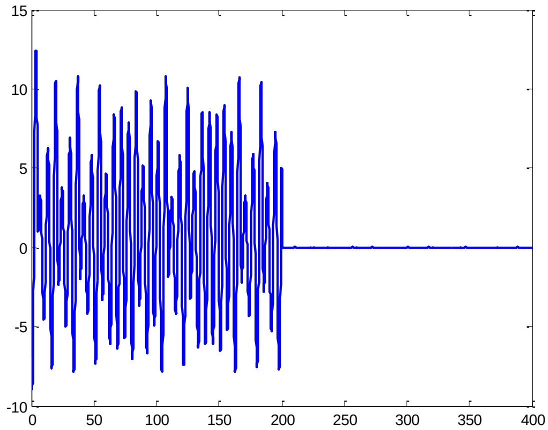


(b)

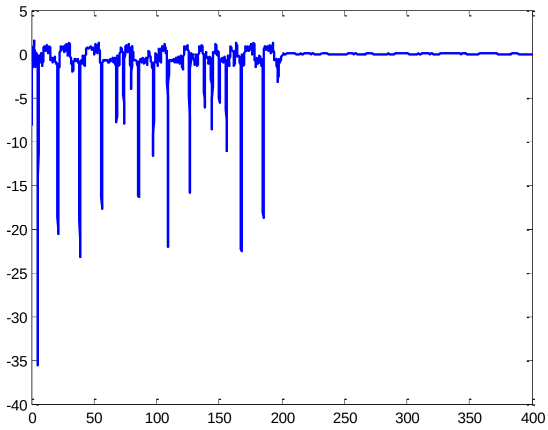
Fig. 3.8: Phase portraits of the modified Chua systems used as master system and Rössler system used as slave system, respectively, with proposed controller



(a)



(b)



(c)

Fig. 3.9: Synchronization error plot using proposed static state feedback controller,
(a) $e_1(t) = x_{m1}(t) - x_{s1}(t)$, *(b)* $e_2(t) = x_{m2}(t) - x_{s2}(t)$, *(c)* $e_3(t) = x_{m3}(t) - x_{s3}(t)$

3.9 Summary

In this chapter, problem of synchronization of two nonlinear systems having different nonlinearities is addressed. Nonlinear functions are considered to be Lipschitz and these Lipschitz nonlinear systems are considered mutually Lipschitz. For the analysis of nonlinear system having different nonlinearities, a novel mutually Lipschitz condition is provided. Proposed mutually Lipschitz condition is more general than traditional Lipschitz condition. Mutually Lipschitz condition is effective for the analysis of the properties of the nonlinear systems having non-identical nonlinearities. Moreover this condition provides the advantage of uncomplicated state feedback controller design for the synchronization of nonlinear master-slave systems.

To analyze the proposed controller for synchronization two distinct nonlinear systems, by virtue of mutually Lipschitz condition, a quadratic Lyapunov function is selected. By applying control stability theory an algebraic Riccati equation is obtained, which is useful for design and implementation of different control strategies for synchronization of nonlinear systems. This algebraic Riccati inequality based approach is further extended to derive an advanced linear matrix inequality (LMI) based control technique for synthesis of controller. LMI based technique is useful for simple and implementable controller design. It provides the optimization of control parameters. It also provides the advantage of maximum mismatch rejection of nonlinearities between master-slave systems. In the end, two different simulation examples are illustrated to guarantee the effectiveness of proposed control law. Results of numerical simulation are obtained for two Chua's systems with low mismatch of parameters and in second example two different nonlinear circuits, modified Chua and Rossler systems are synchronized.

Chapter 4

ROBUST AND ROBUST ADAPTIVE SYNCHRONIZATION

4.1 Overview

In Chapter 3, the problem of synchronization of non-identical nonlinear master-slave systems was discussed and a novel mutually Lipschitz condition was derived, which is useful identity to derive the sufficient conditions for synchronization of non-identical nonlinear systems. A simple state-feedback control law was provided to obtain the synchronize behavior for dissimilar nonlinear master-slave systems.

In this Chapter, the problem of synchronization of non-identical nonlinear systems under the constraints of disturbances for low mismatch of nonlinearities and large mismatch of nonlinearities is investigated. To obtain the robust synchronization for low mismatch of nonlinearities, a simple state feedback controller based robust technique is proposed. Then for cancellation of nonlinearities and for optimal controller gain, robust adaptive technique is proposed, which is further extended for time-delay nonlinear systems. Nonlinear systems are considered to be mutually Lipschitz, to get the advantage of proposed mutually Lipschitz condition for synchronization of non-identical nonlinear systems.

The parameters like time-delay, uncertainties and saturation have a great significance for stability analysis of nonlinear systems. Delay is unavoidable in many physical systems and becomes source of instability in nonlinear systems. Disturbances are another source that degrades the performance of nonlinear systems. Keeping in view the importance of time-delays and uncertainties, the problem is formulated by considering state delays and disturbances in the dynamics of the nonlinear systems that makes the problem more challenging and attractive. Objective is to design an efficient, simple and computationally uncomplicated controller that enforces the error to converge in a sphere to attain the synchronization of time-delay nonlinear systems.

A simple state-feedback control law is derived for robust synchronization by incorporating a Lyapunov stability theory along with bounded disturbances and mutually Lipschitz condition. Initially, Lyapunov stability theory is implied to derive an algebraic Riccati based inequality, which can be utilized for the controller synthesis. However the approach is computationally complex, so algebraic Riccati inequality based approach is transformed into to an advance and less conservative LMI-based robust synchronization approach by implying the change of variable methods, congruence transformation and by virtue of Schur complement. LMI-based approach for robust synchronization is advantageous that offer a simple controller and also effective against external disturbances. Compared to the existing techniques of non-identical nonlinear systems [80-86], proposed synchronization scheme is straight forward and uncomplicated in design and implementation.

Furthermore, a novel adaptive control technique is established, which provides the advantage of designing a controller of suitable gain by cancellation of mismatch between nonlinearities in master and slave systems. It should be emphasized that the proposed novel robust adaptive synchronization scheme works efficiently for nonlinear systems with entirely different dynamics and against uncertainties. The value of unknown parameters can be computed by utilizing the mutually Lipschitz condition. Proposed robust adaptive control technique is extended for time-delay nonlinear systems and a novel delay-independent synchronization methodology for nonlinear master-slave systems subject to external disturbances and state-delays is developed. In the end, couples of numerical simulation examples for low mismatch and large mismatch of nonlinearities are illustrated to witness the proposed synchronization schemes.

Synchronization of dissimilar nonlinear systems with external disturbances and time-delays is of great significance, due to its applications in aerospace engineering, secure communication and medical systems. Different techniques have been developed for synchronization of identical nonlinear systems, e.g., adaptive control strategy, sliding mode control and observer based technique [1-15]. Control strategies have been applied to address the synchronization of nonlinear systems with other constraints like time-delay, uncertainties, saturation and input nonlinearities [56-64]. However, robust synchronization of dissimilar nonlinear (chaotic) systems is lacking in the literature.

There are few exceptional examples of synchronization of dissimilar nonlinear systems [80-86]. In [81], the problem of different chaotic systems with unknown parameter using adaptive control technique has been addressed. The work of [83], provides a control strategy for synchronization of different chaotic fractional order systems. However, these schemes are computationally complex to determine the controller gain and also difficult for real time implementation.

Chapter 4 is organized as follows, in the next section dynamical model of different nonlinear systems along with couples of assumptions is described. State-feedback controller based robust control synchronization scheme is derived in section 3, whereas, section 4 gives a novel adaptive control law based robust adaptive synchronization scheme for large mismatch of nonlinearities among master and slave systems. Proposed robust adaptive synchronization scheme is extended to the time-delay nonlinear system in section 5. Couples of numerical simulation examples are illustrated in section 6 to show the effectiveness of proposed synchronization strategies and in the end, entire chapter is summarized.

4.2 Systems Description and Preliminaries

Synchronization of non-identical nonlinear master-slave systems subject to external disturbances is considered. Dynamics of the master and slave systems are described as follows

$$\dot{x}_m(t) = Ax_m + f(x_m, t) + d_m, \quad x_m(0) = x_{m0}, \qquad \textit{(Eq 4.1)}$$

$$\dot{x}_s(t) = Ax_s + g(x_s, t) + d_s + Bu, \; x_s(0) = x_{s0}, \qquad \textit{(Eq 4.2)}$$

where $x_m \in R^n$ and $x_s \in R^n$ represent the states of the master and slave systems, respectively. $A \in R^{n\times n}$ is a linear known matrix with constant entries. Vector $f(x_m, t) \in R^n$ and $g(x_s, t) \in R^n$ represents the nonlinearities in master and slave systems, respectively. Vector functions $f(x_m, t)$ and $g(x_s, t)$ reflect that nonlinearities in master system and slave system are different from each other. Disturbances in master and slave systems are denoted by $d_m \in R^n$ and $d_s \in R^n$, respectively. $x_m(0) = x_{m0}$ and $x_s(0) = x_{s0}$ are the initial conditions of nonlinear master and slave systems, respectively. $B \in R^{n\times m}$ is a linear matrix with known constant entries and $u \in R^m$ is the control input applied to the slave system to synchronize it with a master system.

Disturbances are important dynamical parameter of the nonlinear systems, which cannot be ignored, source of instability and degradation of performance of nonlinear systems. So, disturbances are incorporated in the dynamical model of nonlinear master and slave systems for robust synchronization.

4.2.1 Assumptions

To obtain the main results, following assumptions are adopted.

1. Disturbances modeled in the master system (d_m) and in slave system (d_s) are bounded such that:

   $$\left| d_m - d_s \right|^2 \le d_{\max} . \qquad \textit{(Eq 4.3)}$$

   The stated condition can be verified if both disturbances d_m and d_s are bounded in Euclidean norm sense

2. The nonlinear master-salve systems having Lipschitz nonlinearities and these Lipschitz nonlinearities are considered to be mutually Lipschitz.

   $$\| f(x_m, t) - g(x_s, t) \|^2 \le l_{\max}^2 \| x_m - x_s \|^2 + \phi_{\max} . \qquad \textit{(Eq 4.4)}$$

   Mutually Lipschitz condition is more general that traditional Lipschitz condition and useful to design a control law for synchronization of non-identical nonlinear master-slave systems.

4.3 Robust Control Methodology

Synchronization of different nonlinear systems is considered through robust control methodology. Dynamics of the nonlinear systems are considered to be Lipschitz. Mutually Lipschitz condition along with Lyapunov stability theory can be implied to attain the robust synchronization of Lipschitz nonlinear systems.

Figure 4.1 shows the basic structure of the proposed robust synchronization scheme. Difference between states of the master and slave systems (error) is fed to the state-feedback controller, which provides an appropriate controller gain to synchronize the slave system with the master system.

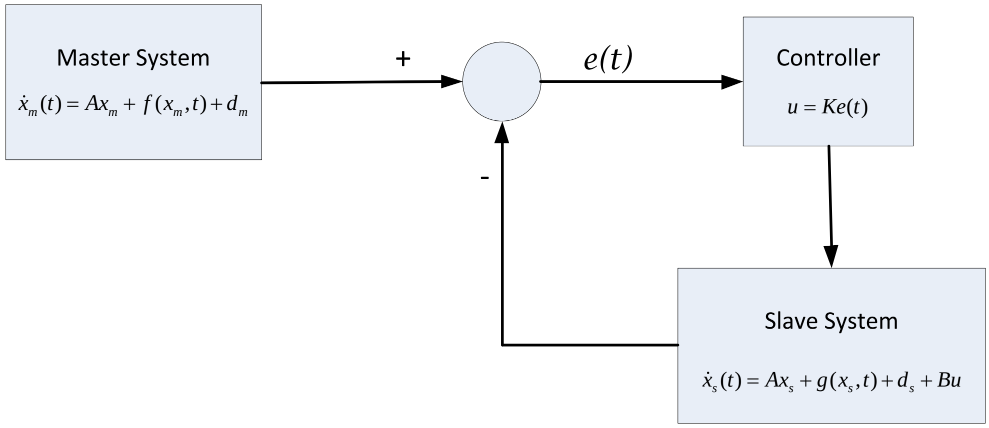


Figure 4.1: Block diagram for robust synchronization of master-slave systems

Structure of the proposed static state-feedback controller is selected as

$$u = Ke(t), \tag{Eq 4.5}$$

where u represents the controller input, $K \in R^{m\times n}$ is the controller gain matrix and $e(t)$ represents the error between master and slave systems. Now by incorporating the proposed controller of Eq. 4.5, dynamics of the slave system of Eq. 4.2 can be rewritten as

$$\dot{x}_s(t) = Ax_s + g(x_s, t) + d_s + BKe(t). \tag{Eq 4.6}$$

Difference between the master and the slave systems is defined as an error $e(t) = x_m(t) - x_s(t)$ and taking its time derivative

$$\dot{e}(t) = \dot{x}_m(t) - \dot{x}_s(t). \tag{Eq 4.7}$$

Subtracting Eq. 4.6 from Eq. 4.1 and incorporating the error definition, we get

$$\dot{e}(t) = Ae + f(x_m, t) - g(x_s, t) + (d_m - d_s) - BKe. \tag{Eq 4. 8}$$

It can be further simplified as

$$\dot{e}(t) = \left(A - BK\right)e + f(x_m, t) - g(x_s, t) + (d_m - d_s). \tag{Eq 4.9}$$

Now to analyze the performance of controller for synchronization between master and slave systems, matrix inequality based approach is provided for the mutually Lipschitz nonlinear master-slave systems.

4.3.1 Theorem 4.1

Consider the nonlinear master system of Eq. 4.1 and the slave system of Eq. 4.2 that satisfies the mutually Lipschitz condition of Eq. 4.4 and assumption stated in Eq. 4.3. Suppose there exist a positive-definite symmetric matrix $Q \in R^{n\times n}$, a matrix $M \in R^{m\times n}$ and a positive scalar μ. By solving the optimization $\min \mu$

$$Q > 0\,,\;\; \mu > \mu_o \geq 0\,. \qquad \textit{(Eq 4.10)}$$

Following linear matrix inequality (LMI) holds

$$\begin{bmatrix} QA^T + AQ - M^T B^T - BM + I & Ql_{\max} & Q \\ * & -I & 0 \\ * & * & -\mu I \end{bmatrix} < 0\,. \qquad \textit{(Eq 4.11)}$$

Then by solving the linear matrix inequality of Eq. 4.11, value of controller K can be obtained by solving $K = MQ^{-1}$. Furthermore, the proposed state-feedback control law of Eq. 4.5, ensures uniformly ultimately bounded synchronization of nonlinear non-identical master-slave systems through convergence of error $e(t) = x_m(t) - x_s(t)$ in the following region

$$\|e\|^2 \leq \left(\phi_{\max} + d_{\max}\right)/\lambda\,. \qquad \textit{(Eq 4.12)}$$

Error convergence depends on the appropriate selection of parameter λ.

4.3.2 Proof of Theorem 4.1

To provide the proof of Theorem 4.1, a simple quadratic Lyapunov function is selected as

$$V(t,e) = e^T(t)Pe(t)\,. \qquad \textit{(Eq 4.13)}$$

Taking the time derivative of constructed Lyapunov function and also incorporating the dynamics of error derived in Eq. 4.9, it depicts

$$\begin{aligned} \dot{V}(t,e) = {} & [\left(A - BK\right)e + f(x_m,t) - g(x_s,t) + (d_m - d_s)]^T Pe \\ & + e^T P[\left(A - BK\right)e + f(x_m,t) - g(x_s,t) + (d_m - d_s)]. \end{aligned} \qquad \textit{(Eq 4.14)}$$

Rearranging it, we get

$$\begin{aligned}\dot{V}(t,e) &= e^T A^T Pe - e^T K^T B^T Pe + e^T PAe - e^T PBKe \\ &\quad +(f(x_m,t) - g(x_s,t))^T Pe + (d_m - d_s)^T Pe \\ &\quad +e^T P(f(x_m,t) - g(x_s,t)) + e^T P(d_m - d_s).\end{aligned} \qquad \textit{(Eq 4.15)}$$

It can be further simplified as

$$\begin{aligned}\dot{V}(t,e) &= e^T [A^T P - K^T B^T P + PA - PBK]e + (f(x_m,t) - g(x_s,t))^T Pe \\ &\quad +(d_m - d_s)^T Pe + e^T P(f(x_m,t) - g(x_s,t)) + e^T P(d_m - d_s).\end{aligned} \qquad \textit{(Eq 4.16)}$$

Introducing the mathematical identity

$$2A^T B \le A^T A + B^T B. \qquad \textit{(Eq 4.17)}$$

Using mathematical identity defined in Eq. 4.17 for $\left(f(x_m,t) - g(x_s,t)\right)^T Pe$ and $(d_m - d_s)^T Pe$, it gives

$$\begin{aligned}\left(f(x_m,t) - g(x_s,t)\right)^T Pe \le \frac{1}{2}&\left(f(x_m,t) - g(x_s,t)\right)^T \\ &\left(f(x_m,t) - g(x_s,t)\right) + e^T PPe\end{aligned} \qquad \textit{(Eq 4.18)}$$

and similarly

$$(d_m - d_s)^T Pe \le \frac{1}{2}(d_m - d_s)^T (d_m - d_s) + e^T PPe. \qquad \textit{(Eq 4.19)}$$

Using results of Eq. 4.18 and Eq. 4.19 into Eq 4.16, it implies

$$\begin{aligned}\dot{V}(t,e) &\le e^T [A^T P - K^T B^T P + PA - PBK + 2P^2]e \\ &\quad +(f(x_m,t) - g(x_s,t))^T (f(x_m,t) - g(x_s,t)) \\ &\quad +(d_m - d_s)^T (d_m - d_s).\end{aligned} \qquad \textit{(Eq 4.20)}$$

Now, as nonlinear functions $f(x_m,t)$ and $g(x_s,t)$ are assumed to be mutually Lipschitz, then by the application of proposed mutually Lipschitz condition provided in Eq. 4.4, the relationship between these nonlinear functions can also be describe as under

$$\begin{aligned}(f(x_m,t) - g(x_s,t))^T (f(x_m,t) - g(x_s,t)) &\le \\ l_{\max}^2 (\sqrt{(x_m - x_s)(x_m - x_s)^T})^2 + \Phi_{\max}.\end{aligned} \qquad \textit{(Eq 4.21)}$$

Incorporating the assumption 1 about bounded disturbances of Eq. 4.3 and using relationship between nonlinear functions established in Eq. 4.21 into Eq. 4.20, it yields

$$\begin{aligned}\dot{V}(t,e) \le\; & e^T[A^TP - K^TB^TP + PA - PBK + 2P^2 + l_{\max}^2 I]e \\ & + \Phi_{\max} + d_{\max}.\end{aligned} \qquad \textit{(Eq 4.22)}$$

Introducing a positive scalar function λ and assuming, we have

$$A^TP + PA - K^TB^TP - PBK + 2P^2 + l_{\max}^2 I + \lambda I < 0. \qquad \textit{(Eq 4.23)}$$

The Eq. 4.23 is known as algebraic Riccati Inequality. It is used for computation of controller gain, but at the same time computation of unknown matrices K and P by this inequality is quite difficult task. Therefore, an advance linear matrix inequality (LMI) based approach is provided for controller synthesis for synchronization of nonlinear systems. Algebraic Riccati inequality of Eq. 4.23 can be transformed into an LMI by applying congruence transformation, change of variable method and Schur complement.

Pre- and post- multiplication of P^{-1}, Eq. 4.23 implies

$$\begin{aligned}& P^{-1}A^TPP^{-1} + P^{-1}PAP^{-1} - P^{-1}K^TB^TPP^{-1} - P^{-1}PBKP^{-1} \\ & + 2P^{-1}PPP^{-1} + P^{-1}l_{\max}^2 IP^{-1} + P^{-1}\lambda IP^{-1} < 0.\end{aligned} \qquad \textit{(Eq 4.24)}$$

Hence, $P^{-1}P = I$, results

$$\begin{aligned}& P^{-1}A^T + AP^{-1} - P^{-1}K^TB^T - P^{-1}PBKP^{-1} + 2I \\ & + P^{-1}l_{\max}^2 IP^{-1} + P^{-1}\lambda IP^{-1} < 0.\end{aligned} \qquad \textit{(Eq 4.25)}$$

Now applying change of variable methods by using $P^{-1} = Q$, $\lambda^{-1} = \mu$, and $M = KQ$, to transform the bilinear matrix inequality into LMI, it yields

$$QA^T + AQ - M^TB^T - BM + 2I + l_{\max}^2 Q^2 + \mu^{-1}Q^2 < 0. \qquad \textit{(Eq 4.26)}$$

LMI of Eq. 4.11 can be obtained by applying Schur complement [120] on Eq. 4.26. It completes the proof of Theorem 4.1.

It shows that by virtue of mutually Lipschitz condition, a simple state feedback control law is proposed in Theorem 4.1. Proposed control law is useful for the synchronization of two non-identical nonlinear systems. Furthermore, robustness of the resultant control methodology against disturbances can be obtained by selection of appropriate value of the $\min \mu$. Another significant feature of the proposed synchronization scheme is that a controller gain can be easily computed, even if information of nonlinearities in master system and slave system are unknown. It can be obtained by the knowledge of Lipschitz constant $l_{\max}$.

The control methodologies developed for the synchronization of non-identical nonlinear systems are [80-86], to considerable extent, computationally intricate in their implementation. On the other hand proposed state-feedback control law is straightforward, uncomplicated and produce implementable optimal results via linear matrix inequality.

The proposed robust synchronization scheme is effective in design and implementation for low mismatch of nonlinearities. However synchronization of nonlinear systems having large nonlinearities $f(x_m, t)$ and $g(x_s, t)$, higher value of $\Phi_{\max}$ is desired, that offer a high value controller gain. Such a high gain controller can be problematic for many practical systems. For example a system containing actuators, it may reach to saturation when a high gain controller is implied, similarly if noise is present in the measurement system. To address the problem of synchronization to address the large mismatch of nonlinearities, a robust adaptive control technique is developed and provided in the next section.

4.4 Robust Adaptive Control Methodology

State-feedback controller scheme for synchronization of non-identical nonlinear master-slave systems is simple, straightforward and effective for small differences of nonlinearities. It has limitation of controller gain to deal with large mismatch of nonlinearities. To deal with this issue, an adaptive synchronization strategy is developed. Adaptive technique is based on adaptive cancellation of unknown term, which is handy to accommodate large mismatch of nonlinearities and achieve synchronization of different nonlinear master-slave systems. Figure 4.2, shows the basic structure of the robust adaptive synchronization methodology.

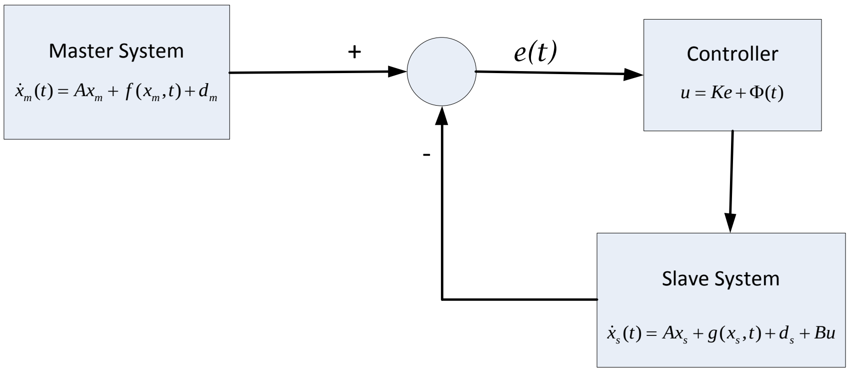


Figure 4.2: Block diagram of robust adaptive synchronization

The structure of the proposed adaptive controller is as follows

$$u = Ke + \Phi(t)\,, \qquad \text{(Eq 4.27)}$$

where $\Phi(t)$ is an adaptive parameter, which is additional to the state-feedback controller of Theorem 4.1. This additional term is used for compensation of large mismatch of nonlinearities between two nonlinear systems and also useful to deal with the disturbances occurring in master system and slave system.

Incorporating the proposed adaptation law of Eq. 4.27 into the error dynamics of Eq. 4.9, it yields

$$\dot{e}(t) = \left(A - BK\right)e + f(x_m,t) - g(x_s,t) - B\Phi + (d_m - d_s)\,. \qquad \text{(Eq 4.28)}$$

4.4.1 Theorem 4.2

Consider a mutually Lipschitz non-identical nonlinear master-slave systems described in Eq. 4.1 and Eq. 4.2 satisfying assumptions provided in Eq. 4.3, Eq. 4.4 and suppose that there exist a scalar function μ along with a matrix $M \in R^{m\times n}$ and a symmetric matrix $Q \in R^{n\times n}$, such that the LMIs

$$Q > 0\,, \mu > 0\,. \qquad \text{(Eq 4.29)}$$

and the LMI of Eq. 4.11 are satisfied. Then proposed adaptive control law of Eq. 4.27 exists with following adaptation law

$$\dot{\Phi} = B^T Pe - \frac{\left(\phi_{\max} + d_{\max}\right)\Phi}{\left\|\Phi\right\|^2 + \sigma}\,, \qquad \text{(Eq 4.30)}$$

where scalar function σ is introduced to avoid the singularity and its value can be infinitesimally small, such that the synchronization error converges to the origin. The controller gain matrix K of the adaptive control law of Eq. 4.27 can be computed by solving $K = MQ^{-1}$.

4.4.2 Proof of Theorem 4.2

Synchronization scheme using the adaption law can be established by introducing following Lyapunov function.

$$V(t,e) = e^T Pe + \Phi^T \Phi, \tag{Eq 4.31}$$

where Φ represents the adaptive parameter. Taking the time derivative of constructed Lyapunov function, it depicts

$$\dot{V}(t,e) = \dot{e}^T Pe + e^T P\dot{e} + \dot{\Phi}^T \Phi + \Phi^T \dot{\Phi}. \tag{Eq 4.32}$$

Incorporating error dynamics of Eq. 4.28, it gives

$$\begin{aligned}\dot{V}(t,e) = &[\left(A - BK\right)e + f(x_m,t) - g(x_s,t) - B\Phi + (d_m - d_s)]^T Pe \\ &+ e^T P[\left(A - BK\right)e + f(x_m,t) - g(x_s,t) - B\Phi + (d_m - d_s)] \\ &+ \dot{\Phi}^T \Phi + \Phi^T \dot{\Phi}.\end{aligned} \tag{Eq 4.33}$$

It can be further simplified as under

$$\begin{aligned}\dot{V}(t,e) = &e^T \left(A^T P + PA - K^T B^T P - PBK\right)e + \left(f(t,x_m) - g(t,x_s)\right)^T \\ &\times Pe + e^T P\left(f(t,x_m) - g(t,x_s)\right) - e^T PB\Phi - \Phi^T B^T Pe \\ &+ \left(d_m - d_s\right)^T Pe + e^T P\left(d_m - d_s\right) + \Phi^T \dot{\Phi} + \dot{\Phi}^T \Phi.\end{aligned} \tag{Eq 4.34}$$

Incorporating the mathematical identities derived in Eq. 4.18 and Eq. 4.19, we obtain

$$\begin{aligned}\dot{V}(t,e) \le &e^T \left(A^T P + PA - K^T B^T P - PBK + 2P^2\right)e + \Phi^T \dot{\Phi} + \dot{\Phi}^T \Phi \\ &+ \left(f(t,x_m) - g(t,x_s)\right)^T \left(f(t,x_m) - g(t,x_s)\right) \\ &- e^T PB\Phi - \Phi^T B^T Pe + \left(d_m - d_s\right)^T \left(d_m - d_s\right).\end{aligned} \tag{Eq 4.35}$$

Applying assumptions of Eq. 4.3 and Eq. 4.4, it implies

$$\begin{aligned}\dot{V}(t,e) \le &e^T \left(A^T P + PA - K^T B^T P - PBK + 2P^2 + l_{\max}^2 I\right)e \\ &+ \phi_{\max} + d_{\max} - e^T PB\Phi - \Phi^T B^T Pe + \Phi^T \dot{\Phi} + \dot{\Phi}^T \Phi.\end{aligned} \tag{Eq 4.36}$$

By exploring the proposed adaptive control law, it reveals

$$\dot{V}(t,e) \le e^T \left(A^T P + PA - K^T B^T P - PBK + 2P^2 + l_{\max}^2 I \right) e + \phi_{\max} + d_{\max} - \frac{\left(\phi_{\max} + d_{\max}\right) \Phi^T \Phi}{\|\Phi\|^2 + \sigma}. \qquad \textit{(Eq 4.37)}$$

The scalar parameter σ is assumed to be positive and infinitesimally small, it depicts

$$\dot{V}(t,e) \le e^T \left(A^T P + PA - K^T B^T P - PBK + 2P^2 + l_{\max}^2 I \right) e. \qquad \textit{(Eq 4.38)}$$

To achieve the objective of error convergence, it is required that an algebraic Riccati inequality of Eq. 4.23 must hold, which further reveals after some mathematical treatment in the form of LMI of Eq. 4.11 must be satisfied. It completes the proof of Theorem 4.2. □

An appropriate small positive value of the scalar parameter σ can be selected for practical implementation rather than infinitesimally small number. In that case, results obtained in Eq. 4.38 entails

$$\dot{V}(t,e) \le e^T \left(A^T P + PA - K^T B^T P - PBK + 2P^2 + l_{\max}^2 I \right) e + \frac{\left(\phi_{\max} + d_{\max}\right) \sigma}{\|\Phi\|^2 + \sigma}. \qquad \textit{(Eq 4.39)}$$

It can be further simplified by incorporating the algebraic Riccati inequality of Eq. 4.23, it implies

$$\dot{V}(t,e) < -\lambda \|e\|^2 + \frac{\left(\phi_{\max} + d_{\max}\right) \sigma}{\|\Phi\|^2 + \sigma}. \qquad \textit{(Eq 4.40)}$$

Hence the synchronization error can be derived from Eq. 4.38, which remains uniformly ultimately bounded as

$$\|e\|^2 \le \frac{\left(\phi_{\max} + d_{\max}\right) \sigma}{\lambda \left(\|\Phi\|^2 + \sigma \right)}. \qquad \textit{(Eq 4.41)}$$

The result derived for synchronization error is rearranged as

$$\left(\phi_{\max} + d_{\max}\right) \sigma \Big/ \left(\|\Phi\|^2 + \sigma \right) \le \phi_{\max} + d_{\max}. \qquad \textit{(Eq 4.42)}$$

This shows that the region of sphere for convergence of synchronization error is comparably smaller than robust control strategy provided in Theorem 4.1. Adaptive control law provides the advantage of suitable controller gain. Proposed robust adaptive control scheme in Theorem 4.2 for synchronization of different nonlinear master-salve systems is also advantageous because it can handle large mismatch of nonlinearities by adaptive cancellation of unknown terms and disturbances by utilizing their bounds. It is worth noting that the adaptive parameter Φ introduced in control law for cancellation of unknown terms, does not enhance complexity for computation of controller gain. Compared to the existing conventional adaptive techniques like [80-81], the requirement of invertible input matrix B is not necessary. So it can be concluded that the developed robust adaptive technique is more effective and less conservative compared to the existing techniques.

4.5 Robust Adaptive Control Methodology Extended for Time-Delay Systems

Delay is an important parameter for theoretical analysis and for the same time for practical systems. In practical systems, the necessary part actuators, sensors and propagation of signals are also source of delay. Study of time-delay nonlinear systems attracts the researcher of different fields especially to the control community. Delay can be modeled in different ways, which depends on its existence in nonlinear systems like input delay, state delay and output delay etc.

State delay is a substantial part of real-time nonlinear systems. Any real-time system consists of multiple stages, and there are number of sources of delays like actuators, sensors, transportation delay and input and output delay. A considerable amount of delay is incorporated by state delay. Disturbances are another fundamental parameter, which degrade the performance of nonlinear systems. The identities that affect the performance of nonlinear systems are also source of instability for synchronized nonlinear systems. The problem is designed by incorporating the different nonlinearities, state delays and disturbances to design a robust synchronization technique.

Dynamics of the master-slave system under the constraints of external disturbances and state-delays are described as follows:

$$\dot{x}_m(t) = Ax_m(t) + A_1 x_m(t-\tau) + f(x_m(t)) + g(x_m(t-\tau)) + d_m, \qquad (Eq\ 4.43)$$

$$\dot{x}_s(t) = Ax_s(t) + A_1 x_s(t-\tau) + f(x_s(t)) + g(x_s(t-\tau)) + d_s + Bu, \qquad (Eq\ 4.44)$$

where $x_m(t-\tau) \in R^n$ and $x_s(t-\tau) \in R^n$ represent the delayed state with time-delay τ of the master and slave systems, respectively. The vector functions $f(x_m) \in R^n$ and $g(x_m(t-\tau)) \in R^n$, represent time-varying nonlinearities in the master system without delay and with delay, respectively. The vector functions $f(x_s) \in R^n$ and $g(x_s(t-\tau)) \in R^n$, represents a time-varying nonlinearities in the slave system without delay and with delay, respectively.

Assumption about mutually Lipschitz nonlinearities provided in Eq. 4.4 can be extended for delayed nonlinear function as follows. The nonlinear functions in the master system and in the slave system subject to time-delay $g(x_m(t-\tau))$ and $g(x_s(t-\tau))$ are also said to be mutually Lipschitz, if

$$\begin{aligned}\left\| g(x_m(t-\tau)) - g(x_s(t-\tau)) \right\|^2 \le l_{\max(g)}^2 &\left\| x_m(t-\tau) - x_s(t-\tau) \right\|^2 \\ &+ \Phi_{\max(g)} ,\end{aligned} \qquad (Eq\ 4.45)$$

where $l_{\max(g)}$ and $\Phi_{\max(g)}$ are positive scalar functions.

Now, by subtracting Eq. 4.44 from Eq. 4.43, the synchronization error dynamics can be written as

$$\begin{aligned}\dot{e}(t) = Ae(t) + A_1 e(t-\tau) &+ (f(x_m(t)) - f(x_s(t))) \\ &+ (g(x_m(t-\tau)) - g(x_s(t-\tau))) + (d_m - d_s) - Bu.\end{aligned} \qquad (Eq\ 4.46)$$

4.5.1 Controller Design

The dynamics of the proposed adaptive controller is selected as

$$u = Ke(t) + \Psi(t), \qquad (Eq\ 4.47)$$

where $K \in R^{m \times n}$ is a controller gain matrix, $e(t)$ represents the error and $\Psi(t) \in R^m$ represents the adaptive parameter, which is useful to compensate mismatch between nonlinearities and disturbances. Hence by incorporating the dynamics of the proposed control law and error definition into Eq. 4.46, it depicts

$$\begin{aligned}\dot{e}(t) &= (A-BK)e(t) + A_1 e(t-\tau) + (f(x_m(t)) - f(x_s(t))) \\ &\quad + (g(x_m(t-\tau)) - g(x_s(t-\tau))) + (d_m - d_s) - B\Psi(t).\end{aligned} \qquad \textit{(Eq 4.48)}$$

This section is intended towards the synthesis of a robust adaptive controller to attain the synchronization between the nonlinear, non-identical master-slave systems subject to states delays.

4.5.2 Theorem 4.3

Let the master system and the slave system with dynamics in Eq. 4.43 and Eq. 4.44, respectively, along with assumptions of Eq. 4.4 and Eq. 4.45 are supposed to be synchronized, if there exist positive-definite matrices $X \in R^{n\times n}$, $Z \in R^{n\times n}$ and $G \in R^{m\times n}$, such that following linear matrix inequality (LMI) holds

$$\begin{bmatrix} \Gamma & A_1 X & Xl_{\max(f)} & 0 \\ * & -\bar{Z} & -I & Xl_{\max(g)} \\ * & * & 0 & -I \\ * & * & * & 0 \end{bmatrix} < 0 \ , \qquad \textit{(Eq 4.49)}$$

where * represents the symmetric terms of the LMI and

$\Gamma = XA^T + AX - G^T B^T - BG + 3I + \bar{Z}$.

There exists a controller provided in Eq. 4.46 with following adaptation law

$$\dot{\Psi} = B^T Pe - \frac{\left(\Phi_{\max(f)} + \Phi_{\max(g)} + d_{\max}\right)\Psi}{2\|\Psi\|^2 + \alpha} \ , \qquad \textit{(Eq 4.50)}$$

where α is a scalar function, which is introduced to escape from singularity and assumed to be infinitesimally small.

4.5.3 Proof of Theorem 4.3

Theorem 4.3 is baptized as robust adaptive approach for synchronization of complex nonlinear systems having state delays. To provide the proof of Theorem 4.3, Lyapunov stability theory along with some arithmetic treatment is inferred. The Lyapunov function is selected as under

$$V(t,e) = e^T(t)Pe(t) + \Psi^T\Psi + \int_{t-\tau}^{t} e^T(t)Z\,\mathrm{e}(t)dt \ . \qquad \textit{(Eq 4.51)}$$

Taking the time-derivative of constructed Lyapunov function

$$\dot{V}(t,e) = \dot{e}^T(t)Pe(t) + e^T(t)P\dot{e}(t) + \Psi^T\dot{\Psi} + \dot{\Psi}^T\Psi \\ +e^T(t)Ze(t) - e^T(t-\tau)Ze(t-\tau) \ . \qquad \textit{(Eq 4.52)}$$

Incorporating the value of $\dot{e}(t)$ from Eq. 4.48 into Eq. 5.52 and rearranging it

$$\begin{aligned}\dot{V}(t,e) = {} & e^T(t)(A^TP - K^TB^TP + PA - PBK + Z)e(t) + e^T(t)(PA_1) \\ & \times e(t-\tau) + e^T(t-\tau)(A_1^TP)e(t) - e^T(t-\tau)Ze(t-\tau) - e^T(t) \\ & \times PB\Psi - \Psi^TB^TPe(t) + \Psi^T\dot{\Psi} + \dot{\Psi}^T\Psi + \left(f(x_m(t)) - f_s(x_s(t))\right)^T \\ & \times Pe(t) + \left(g(x_m(t-\tau)) - g(x_s(t-\tau))\right)^T Pe(t) + e^T(t)P \\ & \times\left(f(x_m(t)) - f(x_s(t))\right) + e^T(t)P\left(g(x_m(t-\tau)) - g(x_s(t-\tau))\right) \\ & + e^T(t)P\left(d_m - d_s\right) + \left(d_m - d_s\right)^T Pe(t).\end{aligned} \qquad \textit{(Eq 4.53)}$$

Using the mathematical identity defined in Eq. 4.17, following inequality holds

$$\begin{aligned}\left(g(x_m(t-\tau)) - g(x_s(t-\tau))\right)^T Pe \le {} & \frac{1}{2}\left(g(x_m(t-\tau)) - g(x_s(t-\tau))\right)^T \\ & \times\left(g(x_m(t-\tau)) - g(x_s(t-\tau))\right) \\ & + e^T(t)PPe(t) \ .\end{aligned} \qquad \textit{(Eq 4.54)}$$

Now, by incorporating the inequalities of Eq. 4.18, Eq. 4.19 and Eq. 4.54 into Eq. 4.53, it yields

$$\begin{aligned}\dot{V}(t,e) \le {} & e^T(A^TP - K^TB^TP + PA - PBK + Z + 3P^2)e + e^T(t)\left(PA_1\right) \\ & \times e(t-\tau) + e^T(t-\tau)(A_1^TP)e(t) - e^T(t-\tau)Ze(t-\tau) - e^T(t)PB\Psi \\ & -\Psi^TB^TPe(t) + \Psi^T\dot{\Psi} + \dot{\Psi}^T\Psi + \left(f(x_m(t)) - f(x_s(t))\right)^T \\ & \times\left(f(x_m(t)) - f(x_s(t))\right) + \left(g(x_m(t-\tau)) - g(x_s(t-\tau))\right)^T \\ & \times\left(g(x_m(t-\tau)) - g(x_s(t-\tau))\right) + \left(d_m - d_s\right)^T\left(d_m - d_s\right) \ .\end{aligned} \qquad \textit{(Eq 4.55)}$$

Applying assumptions provided in Eq. 4.3, Eq. 4.4 and Eq. 4.45 into Eq. 4.55, it reveals

$$\begin{aligned}\dot{V}(t,e) \le {} & e^T(\mathrm{t})(A^TP - K^TB^TP + PA - PBK + Z + 3P^2 + l^2{}_{\max(f)}I)e(\mathrm{t}) \\ & + e^T(t)\left(PA_1\right)e(t-\tau) + e^T(t-\tau)(A_1^TP)e(t) - e^T(t-\tau)Ze(t-\tau) \\ & + e^T(t-\tau)(l^2{}_{\max(g)}I)e^T(t-\tau) - e^T(t)PB\Psi - \Psi^TB^TPe(t) \\ & + \Psi^T\dot{\Psi} + \dot{\Psi}^T\Psi + \Phi_{\max(f)} + \Phi_{\max(g)} + d_{\max} \ .\end{aligned} \qquad \textit{(Eq 4.56)}$$

Incorporating the proposed adaptive control law of Eq. 4.46 into Eq. 4.56, it depicts

$$
\begin{aligned}
\dot{V}(t,e) \le\ & e^T(\mathrm{t})(A^T P - K^T B^T P + PA - PBK + Z + 3P^2 + l^2_{\max(f)} I)e(\mathrm{t}) \\
& + e^T(t)\left(PA_1\right)e(t-\tau) + e^T(t-\tau)(A_1^T P)e(t) + \Phi_{\max(f)} \\
& - e^T(t-\tau)(Z + l^2_{\max(g)} I)e(t-\tau) + \Phi_{\max(g)} + d_{\max} \\
& - \frac{2\left(\Phi_{\max(f)} + \Phi_{\max(g)} + d_{\max}\right)\Psi^2}{2\|\Psi\|^2 + \alpha}\ ,
\end{aligned}
\qquad \textit{(Eq 4.57)}
$$

where parameter α is defined as a positive scalar and its value is assumed to be infinitesimally small, inferring its value, gives

$$
\begin{aligned}
\dot{V}(t,e) \le\ & e^T(t)(PA_1)e(t-\tau) + e^T(t-\tau)(A_1^T P)e(t) + e^T(t-\tau) \\
& \times(l^2_{\max(g)} I - Z)e(t-\tau) + e^T(\mathrm{t})(A^T P - K^T B^T P + PA \\
& - PBK + Z + 3P^2 + l^2_{\max(f)} I)e(\mathrm{t})\ .
\end{aligned}
\qquad \textit{(Eq 4.58)}
$$

It can be rearranged in LMI format as follows

$$
\begin{bmatrix} \Pi & PA_1 \\ * & l^2_{\max(g)} I - Z \end{bmatrix} < 0\,, \qquad \textit{(Eq 4.59)}
$$

where

$$
\Pi = A^T P + PA - K^T B^T P - PBK + 3P^2 + Z + l^2_{\max(f)} I\ .
$$

The result derived in Eq. 4.59 is not exactly the LMI. In fact it is a bilinear matrix inequality. To transform this BMI into LMI, congruence transformation, change of variable method and Schur complement are applied.

Pre and post multiplication of P^{-1} and incorporating the $P^{-1}P = PP^{-1} = I$, and for further simplification, change of variable method is implied for $X = P^{-1}$ and $G = KX$. It results into the following LMI

$$
\begin{bmatrix} \hat{\Pi} & A_1 X \\ * & X l^2_{\max(g)} IX - XZX \end{bmatrix} < 0\ , \qquad \textit{(Eq 4.60)}
$$

where

$$
\hat{\Pi} = XA^T + AX - G^T B^T - BG + 3I + XZX + Xl^2_{\max(f)} IX\ .
$$

Applying Schur complement [120] on Eq. 4.60, the LMI of Eq. 4.49 can be obtained. This completes the proof of Theorem 4.3. □

LMI-based an advanced approach is established for synchronization of non-identical nonlinear systems. An adaptive control law is proposed to deal with different types of nonlinearities, state delays and disturbances in nonlinear systems. Adaptive parameter defined in proposed adaptive control law is exploited for cancellation of mismatch in nonlinearities. Also it is advantageous to compensate the disturbances.

4.6 Simulation Results

There are two numerical examples illustrated to witness the proposed scheme for the synchronization of different nonlinear systems. In the first example synchronization of nonlinear systems with low mismatch of nonlinearities is considered. Whereas in the second example, to show the worth of proposed synchronization criteria, two different nonlinear master-slave systems are selected to achieve the synchronization between them.

4.6.1 Simulation of Two Different Chua's Systems

Chua's circuits [66] are considered as nonlinear master and slave systems for low mismatch of nonlinearities, parameters of master and slave systems are selected slightly different to each other. Nonlinear function of the master system (Eq. 4.1) and the slave system (Eq. 4.2), can be represented in the dynamical form of Chua's system as follows

$$f(x_m,t)=\begin{bmatrix}\alpha_1\left(|x_m+\alpha_2|+|x_m-\alpha_3|\right)\\ 0\\ 0\end{bmatrix}, \qquad \textit{(Eq 4.61)}$$

$$g(x_s,t)=\begin{bmatrix}\beta_1\left(|x_s+\beta_2|+|x_s-\beta_3|\right)\\ 0\\ 0\end{bmatrix}. \qquad \textit{(Eq 4.62)}$$

Obviously, the nonlinear functions $f(x_m,t)$ and $g(x_s,t)$ are dissimilar, also values assigned to the parameters α and β are different to each other, which are $\alpha_1=1.9286$, $\alpha_2=1.0$ and $\alpha_3=1.1$. Similarly $\beta_1=1.8482$, $\beta_2=1.1$ and $\beta_3=1$. The scalar parameter $\mu_o=0.01$ and Lipschitz constant matrix $L_{\max}=diag(3,0,0)$. Disturbances in master and slave systems are selected as

$$d_m = \begin{bmatrix} 0.8\sin 70t \\ 0.15\sin 90t \\ 1.2\sin 130t \end{bmatrix}, \quad d_s = -\begin{bmatrix} 0.7\sin 75t \\ 0.12\sin 100t \\ 1.1\sin 138t \end{bmatrix}. \qquad \textit{(Eq 4.63)}$$

The linear known matrix A is selected as

$$A = \begin{bmatrix} -2.548 & 9.1 & 0 \\ 1 & -1 & 1 \\ 0 & 14.2 & 0 \end{bmatrix}. \qquad \textit{(Eq 4.64)}$$

The feedback controller gain matrix is computed by solving the LMI derived in Theorem 4.1, which provides

$$K = \begin{bmatrix} 53.04 & 8.3 & -10.35 \\ 2.07 & 43 & 2.38 \\ 13.78 & -15.58 & 44 \end{bmatrix}. \qquad \textit{(Eq 4.65)}$$

The proposed robust controller of Eq. 4.5 is applied for synchronization of different Chua's circuits. Figure 4.3(a) shows the error between first state of master system and slave system $e_1(t) = x_{m1}(t) - x_{s1}(t)$, which shows that error converges to very small region around the origin. Similarly, Figures 4.3(b) and 4.3(c) show the synchronization error $e_2(t) = x_{m2}(t) - x_{s2}(t)$ and $e_3(t) = x_{m3}(t) - x_{s3}(t)$, respectively. Responses reflect that states errors are also converging to the zero. Hence, the proposed strategy for synchronization of non-identical nonlinear systems developed in Theorem 4.1 is applicable for low mismatch of nonlinearities.

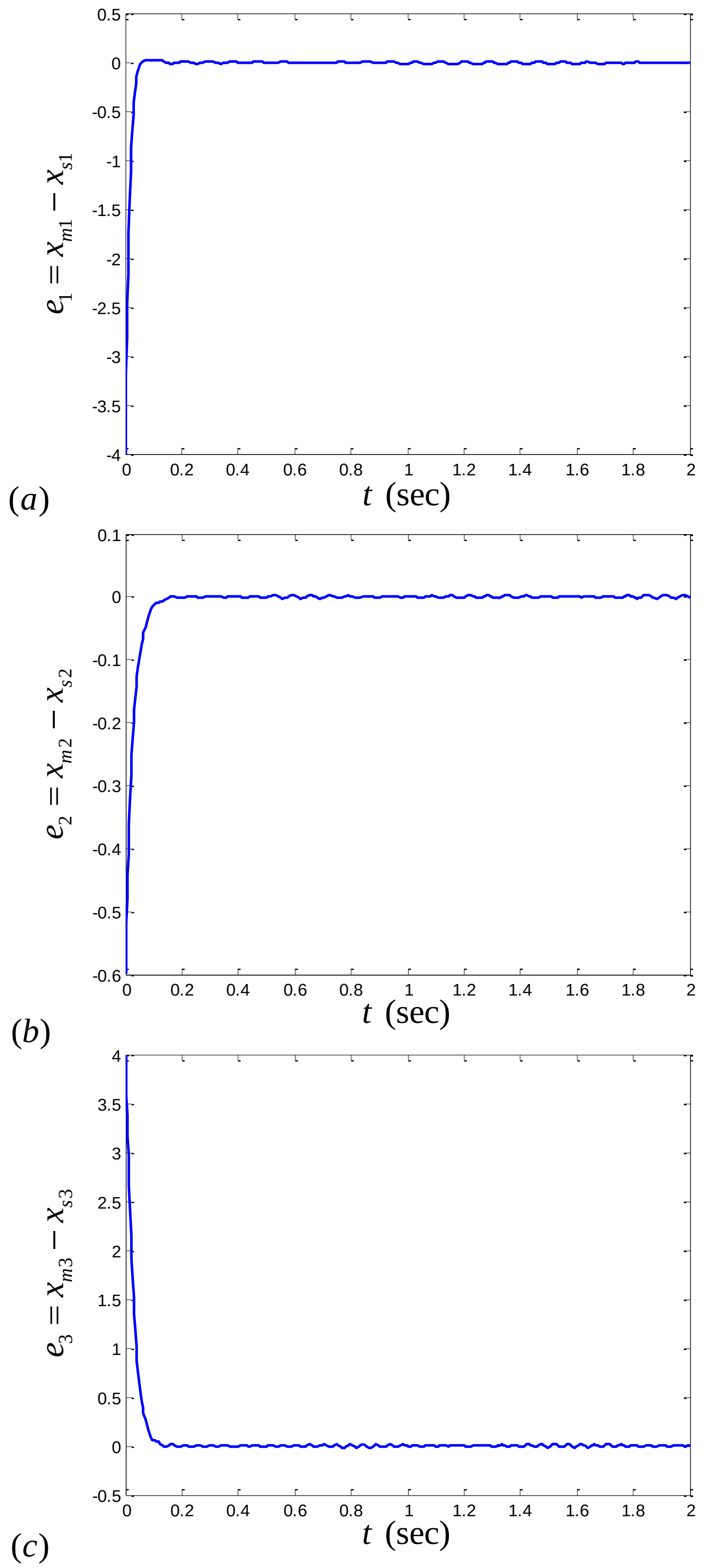


Fig. 4.3: Synchronization error plots for the two different Chua's circuits in Example 1

To show the effectiveness of proposed robust and robust adaptive synchronization scheme of Theorem 4.1 and Theorem 4.2, respectively, for two different nonlinear systems, example 2 is provided.

4.6.2 Simulation of Modified Chua's System and Rossler System

To guarantee the proposed state feedback control law, dynamics for of the master system described in Eq. 4.1 is implied by a modified Chua's system and dynamics of the slave system of Eq. 4.2 is implied by a Rössler system [122], given by

$$f(x_m,t)=\begin{bmatrix}10\left(x_{m2}-\dfrac{2x_{m1}^3-x_{m1}}{7}\right)\\ x_{m1}-x_{m2}+x_{m3}\\ -\dfrac{100}{7}x_{m2}\end{bmatrix}, \qquad \textit{(Eq 4.66)}$$

$$g(x_s,t)=\begin{bmatrix}-x_{s2}-x_{s3}\\ x_{s1}+0.2x_{s2}\\ 0.2+x_{s3}\left(x_{s1}+5.7\right)\end{bmatrix}. \qquad \textit{(Eq 4.67)}$$

The Lipschitz constant parameter l_f is computed mathematically by determining supremum of the maximum eigen values of $\left(\partial f(t,x)/\partial x\right)^T\left(\partial f(t,x)/\partial x\right)$ for $x\in\left[-2 \quad 2\right]$. Scalar parameter $\varepsilon=0.1$ is selected and using it mutually Lipschitz constant is determined to be $l_{\max}=23.72$. Controller gain matrix K is computed by solving the linear matrix inequality of Theorem 4.1 as

$$K=\begin{bmatrix}77.61 & 0 & 0\\ 0 & 77.61 & 0\\ 0 & 0 & 77.61\end{bmatrix}. \qquad \textit{(Eq 4.68)}$$

Initially when no control signal is applied to the slave system for synchronization, the phase portraits of Chua's circuit and Rossler system implied as master system and slave system, respectively are shown in Figures 4.4(a) and 4.4(b), respectively. Phase portraits show unsynchronized behavior. Now when controller is activated, responses of the master and slave system are shown in Figures 4.5(a) and 4.5(b), respectively. Clearly it can be seen that in the presence of control input trajectory of Rossler system (slave system), follows the trajectory of Chua's system (master system).

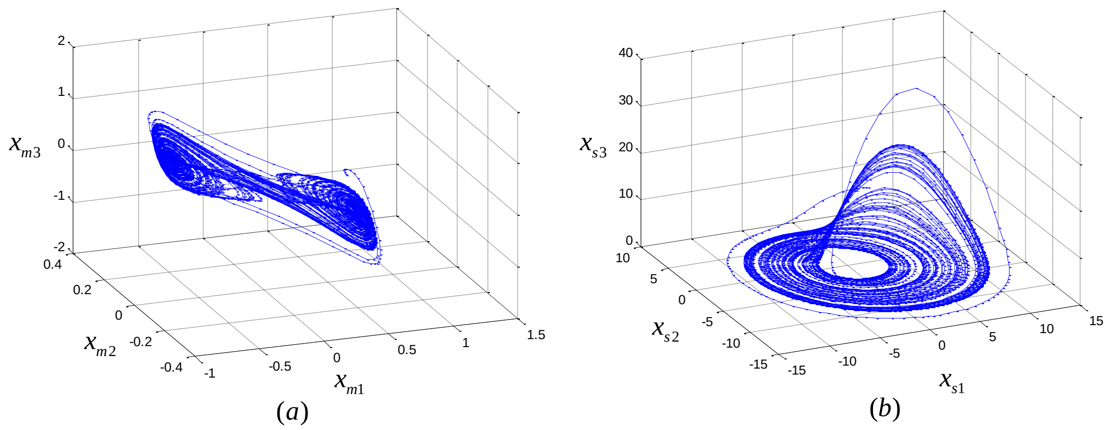


Fig. 4.4: Phase portraits of the chaotic master-slave systems without controller: (a) phase portrait of the modified Chua's circuit; (b) phase portrait of the Rössler system.

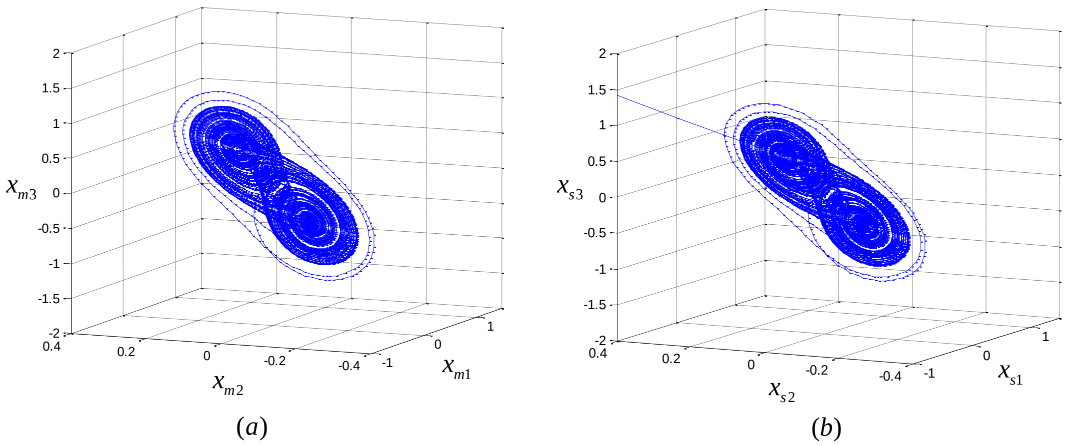


Fig. 4.5: Phase portraits of the chaotic master-slave systems with proposed controller: (a) phase portrait of the modified Chua's circuit; (b) phase portrait of the Rössler system.

Figure 4.6 shows the time series plots of the synchronization errors. Figures 4.6(a), 4.6(b) and 4.6(c) demonstrate the response of $e_1(t)$, $e_2(t)$ and $e_3(t)$, respectively. Initially for time $t < 200\,\text{sec}$, control signal is not activated to the slave system and response of the synchronization error is oscillatory, which is not converging to the origin. After time $t \geq 200\,\text{sec}$, controller is incorporated and it can be observed that the synchronization errors converge to zero and synchronization is obtained between master and slave system.

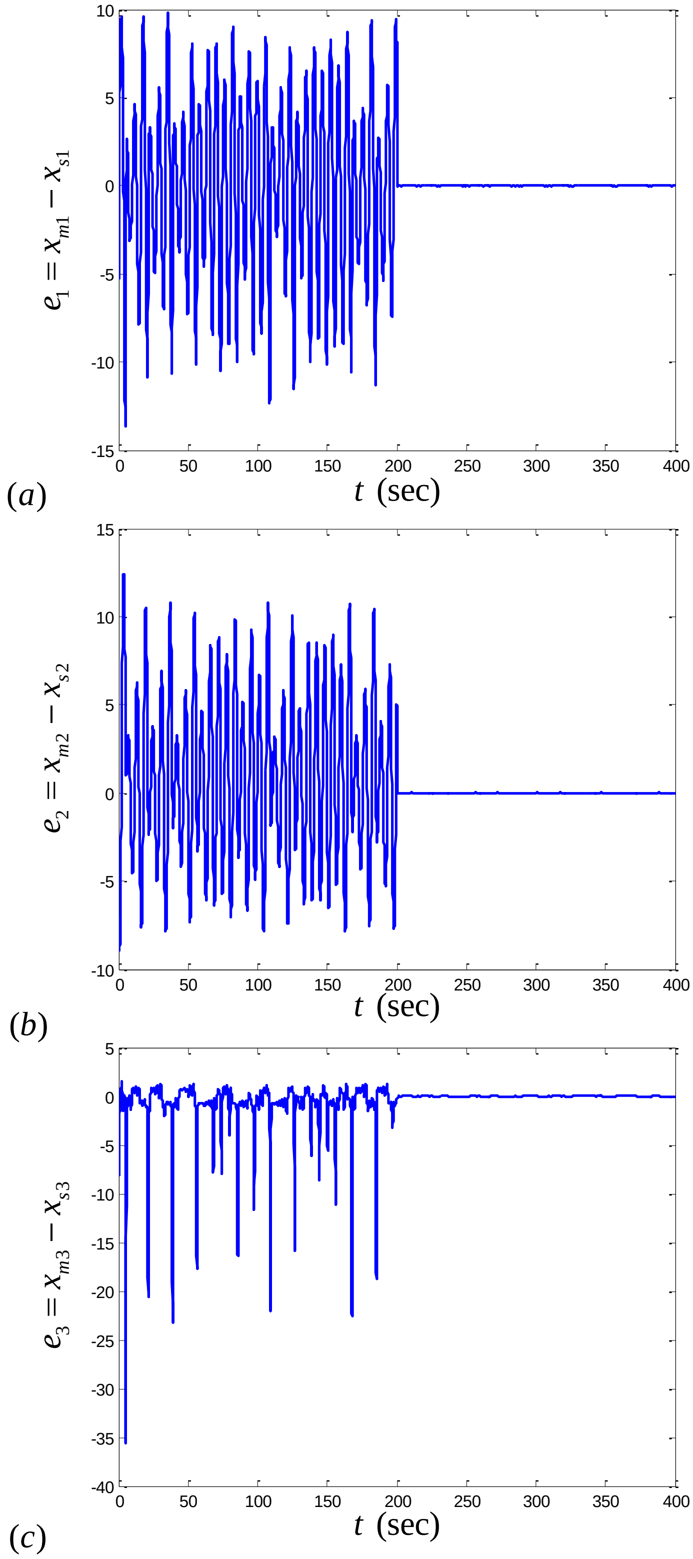


Fig. 4.6. Synchronization error plots for the Chua's circuit and the Rössler system using the proposed static state feedback controller.

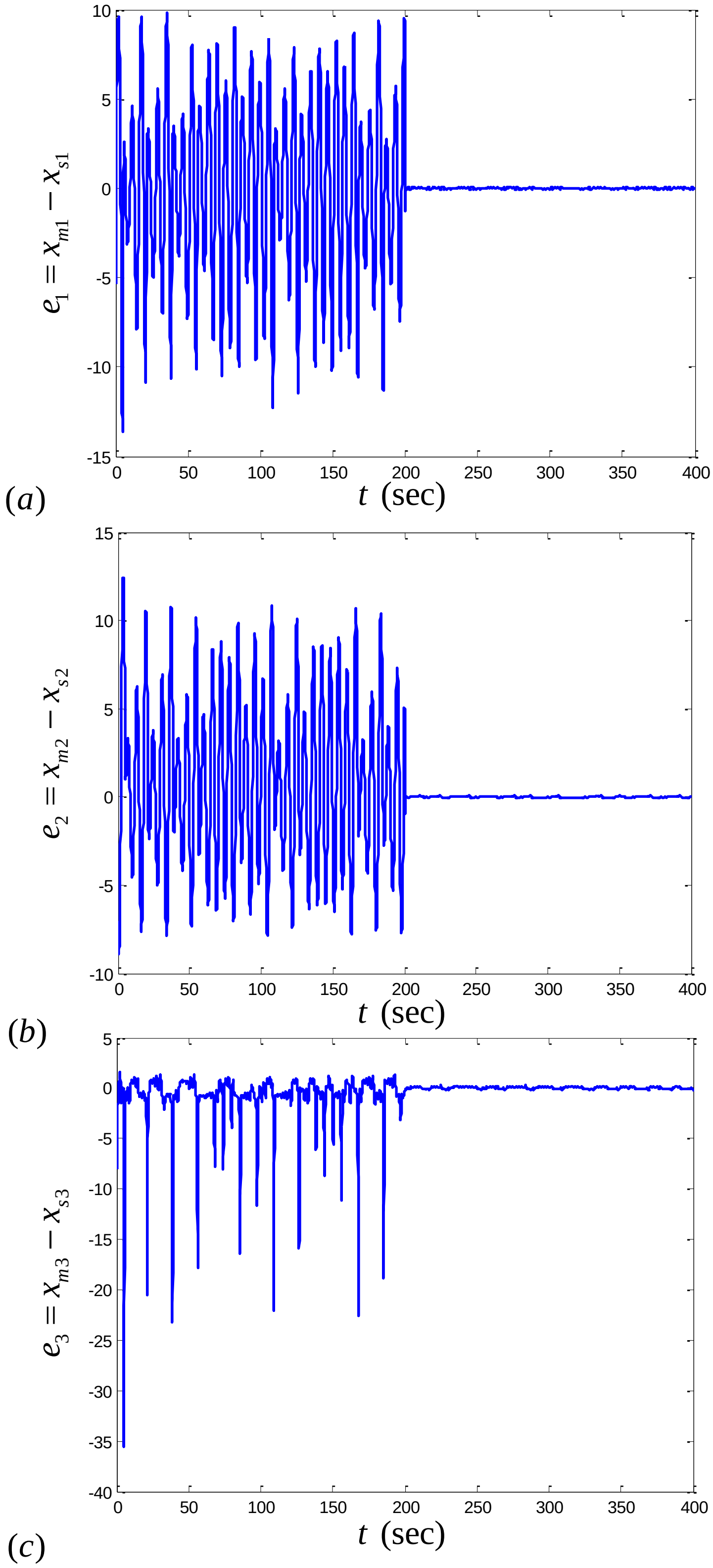


Fig. 4.7. Synchronization error plots for the Chua's circuit and the Rössler system in Example 2 using the proposed adaptive control strategy.

Now to show the effectiveness of proposed adaptive control technique, $l_{\max}=2$ is determined. The controller gain is computed through adaptive control scheme provided in Theorem 4.2. The controller gain is acquired as

$$K=\begin{bmatrix}18.96 & 0 & 0\\ 0 & 18.96 & 0\\ 0 & 0 & 18.96\end{bmatrix}. \qquad \textit{(Eq 4.69)}$$

The value of the controller gain matrix of Eq. 4.69 computed by robust adaptive technique is about four-times lower than the controller gain of Eq. 4.68 computed by robust control technique, which is effective and implementable low gain controller for physical systems. The results of the proposed adaptive control strategy for $\sigma=0.05$ are plotted in Figure 4.7. The controller is activated at time $t=200\sec$ as in earlier case, to provide the comparison between two different approaches of synchronization. It is of notable mention that the synchronization errors $e_1(t)=x_{m1}(t)-x_{s1}(t)$, $e_2(t)=x_{m2}(t)-x_{s2}(t)$ and $e_3(t)=x_{m3}(t)-x_{s3}(t)$ shown in Figures 4.7 are converging in a bounded region with a similar performance as for the case of state-feedback controller approach with advantage of low controller gain.

4.7 Summary

In this Chapter, problem of synchronization of nonlinear master-slave systems with disturbance is addressed through robust and robust adaptive methodologies. Dynamics of the nonlinear systems are supposed to satisfy the mutually Lipschitz condition, which provides the advantage to derive the sufficient conditions for synchronization of different nonlinear systems. Uncertainties are always there in nonlinear systems, so disturbance is incorporated in the dynamics of the master and slave systems. A simple state-feedback control law is proposed for robust synchronization of master-slave systems. To analyze the performance of the proposed controller for robust synchronization, using Lyapunov stability theory, an algebraic Riccati inequality based strategy is developed, which is further enhanced to derive an advanced LMI-based approach for synchronization.

LMI-based controller design and synthesis technique is efficient to achieve optimal results for the synchronization against low mismatch of nonlinearities and disturbance in nonlinear systems. However, the proposed state-feedback controller is not efficient for large mismatch of nonlinearities and its performance is poor against different complex nonlinear systems. To address the two different nonlinear master-slave systems, a novel adaptive control technique is established, which provides the advantage of cancellation of mismatch of nonlinearities among nonlinear systems and ensures the robust adaptive synchronization. Adaptive control scheme of synchronization is a powerful technique for synchronization of non-identical nonlinear systems, it is also useful to find the unknown and uncertain parameters of such nonlinear systems.

Robust adaptive synchronization scheme is extended for time-delay nonlinear systems. The problem is reformulated by considering the state delays in the dynamics of nonlinear master-slave systems. A new adaptation law is introduced for synchronization of time-delay nonlinear systems. To compensate the external disturbances and state-delays, adaptive treatment is useful technique. The proposed synchronization scheme is novel and less conservative than the existing techniques.

In the end, two examples of numerical simulations are illustrated to show the effectiveness of proposed synchronization techniques. Highly complex nonlinear systems (chaotic) are selected for synchronization. In the first example, static feedback control law is applied to show synchronization against low mismatch and disturbance in the master and the slave systems. In the second numerical example, modified Chua’s system and Rossler systems are selected as master system and slave system, respectively. Slave system is derived to master system using state-feedback controller and then using adaptive based control technique. The controller gain computed through robust adaptive scheme is one fourth of the controller gain computed by the robust synchronization scheme. It shows that controller gain is appropriate for real-time implementation using adaptive control strategy of synchronization for different nonlinear systems.

Chapter 5

DELAY-DEPENDENT SYNCHRONIZATION

5.1 Overview

In last couples of chapters, the problem of synchronization for non-identical nonlinear master-slave systems was addressed. In chapter 3, feedback control law based strategy was developed for class of nonlinear systems by implying the mutually Lipschitz condition. Whereas, robust and robust adaptive synchronization schemes for different nonlinear master and slave systems were described in chapter 4. Furthermore, robust adaptive synchronization scheme was extended for time-delay nonlinear systems by considering the state-delays in the dynamics of master and slave systems and delay-independent synchronization methodology was developed.

In this chapter, the problem synchronization of nonlinear master-slave systems under input time-delay and slope-restricted input nonlinearity is considered and delay-dependent synchronization criterion for synchronization of time-delay nonlinear master-slave systems is proposed. Problem is formulated by considering the input delay and slope-restricted input nonlinearity. Linear parameter varying (LPV) approach [68-70] is employed to transform the input nonlinearity into linear time-varying parameters of known range. LPV-approach is a valuable technique for synthesis of controller, where the input nonlinearity is embedded in some varying parameters that depend on the systems states.

A triple integral based Lyapunov-Krasovskii (LK) functional [118] is inferred, and control stability theory is applied to derive the linear matrix inequalities (LMIs), which leads to design a simple state-feedback control law for delay-dependent synchronization of nonlinear systems. Proposed state-feedback control law is simple in design and implementation, compared to the existing adaptive control techniques [23], [55], [63]-[66]. Proposed simple state-feedback controller based delay-dependent synchronization technique is novel. In the end, a numerical example of nonlinear gyro systems is illustrated to witness the proposed scheme of synchronization. The basic structure of the proposed delay-dependent synchronization scheme for master-slave systems is demonstrated in Figure 5.1 below.

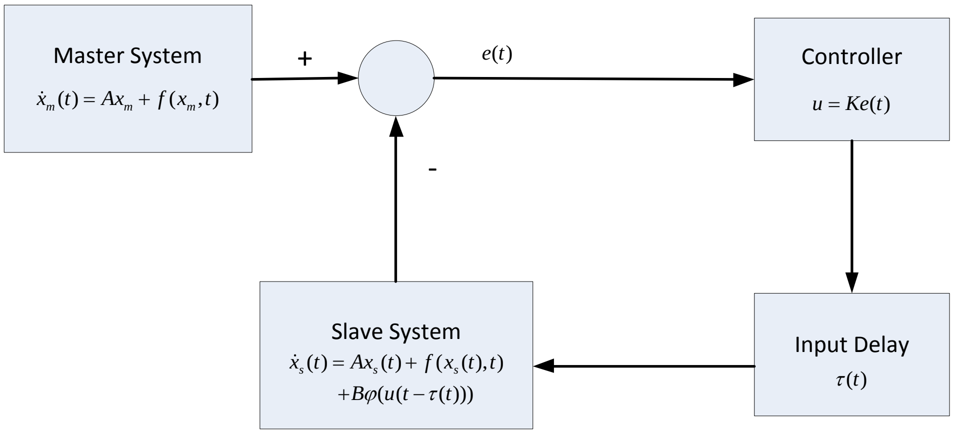


Figure 5.1: Block diagram of delay-dependent synchronization

Time-delays are frequently encountered in many physical systems including chemical process, electrical systems, pneumatic control, hydraulic networks and medical systems etc. Especially, existence of delay in electrical systems exhibits manifold, such as, delay generated between input and output of the electronics circuits, delay in communication networks between transmitter and receiver, delay appears during the propagation of electrical signals and similarly, energy systems also encountered the transmission delay.

In some cases the delay can be ignored because it does not affect the performance of the system or process. However, in many physical systems, presence of small or considerable amount of delay cannot be ignored depending upon the nature of the application of systems, such systems are considered sensitive to the delay. To compensate the effect of delay in these systems, delayed systems model is incorporated. Delayed system model provides the facility of designing an appropriate control law to compensate the delay effects and improves the performance of the system.

Recently, a sound research trend is observed on the stability analysis and synchronization of nonlinear time-delay systems [7], [11], [56-60], [103], [110] which shows the importance of delay dynamics of nonlinear systems. The research is focused to analyze the effect of different kinds of delays on the stability of the nonlinear systems and to provide the solution for the compensation of such delays. Nonlinear time-delay systems can be classified into two categories for stability analysis, which are delay-independent and delay-dependent stability criteria. Different stability criteria have been described in the literature for stability analysis and synchronization of nonlinear time-delay systems.

This chapter is organized as follows. In the next section, problem of synchronization of master-slave systems under the delay constraints is formulated and preliminaries like delay-dependent approach and LPV-method are recalled. Sections 3, provide the controller design. Section 4 and 5 are devoted for attaining main results in the form of LMIs by implying the proposed control law. The numerical simulation results to witness the proposed synchronization scheme are illustrated in section 6. In the end some concluding remarks about this chapter are furbished.

5.2 Systems Description and Preliminaries

The problem of synchronization problem of two nonlinear systems with input delay and slope-restricted input nonlinearity is considered. Nonlinear systems are characterized as master and slave systems. Dynamics of the nonlinear master system is describe as follows

$$\dot{x}_m(t) = Ax_m(t) + f(x_m(t), t), \qquad \textit{(Eq 5.1)}$$

where $x_m \in R^n$ denotes the state of the master system and $A \in R^{n\times n}$ is a linear known matrix with constant entries. Time-varying nonlinear vector function of master system is represented by $f(x_m(t), t) \in R^n$. Similarly, dynamics of the nonlinear slave system is describe as

$$\dot{x}_s(t) = Ax_s(t) + f(x_s(t), t) + B\varphi(u(t-\tau(t))), \qquad \textit{(Eq 5.2)}$$

where $x_s \in R^n$ denotes the state of the slave system. $A \in R^{n\times n}$ is a linear matrix similar to the master system. Time-varying nonlinear vector function of slave system is represented by $f(x_s, t) \in R^n$. Control input provided to the slave systems is denoted by $u \in R^p$ and $B \in R^{n\times p}$ represents the input linear matrix with known entries. $\varphi(u) \in R^p$ represents the continuous-time slope-restricted input nonlinearity. $\tau(t)$ denotes the time-varying input delay.

An appropriate control law is needed to obtain the synchronization between the master and the slave systems, which ensures the convergence of the synchronization error to the origin. Defining the difference between the states of the master system and the slave system as an error $e(t) = x_m(t) - x_s(t)$ and now taking its time derivative as

$$\dot{e}(t) = \dot{x}_m(t) - \dot{x}_s(t) \,. \qquad \textit{(Eq 5.3)}$$

Now subtracting Eq. 5.2 from Eq. 5.1 and using the error definition, error dynamics can be given as

$$\dot{e}(t) = Ae(t) - B\varphi(u(t-\tau(t))) + f(x_m(t),t) - f(x_s(t),t) \,. \qquad \textit{(Eq 5.4)}$$

To simplify the error dynamics of Eq. 5.4, nonlinearities are defined as $\Psi(x_m, x_s) = f(x_m(t),t) - f(x_s(t),t)$. Error system of Eq. 5.4 reveals

$$\dot{e}(t) = Ae(t) - B\varphi(u(t-\tau(t))) + \Psi(x_m, x_s) \,. \qquad \textit{(Eq 5.5)}$$

5.2.1 Delay-Dependent Approach

Problem of synchronization of nonlinear time-delay system has a considerable significance in both theoretical and practical systems. When the amount of delay is considered for the stability analysis, the approach is called delay-dependent. Delay-dependent approach is less conservative compared to the delay-independent approach in which delay is considered without information about size of the delay.

In this chapter, delay-dependent criterion is derived for synchronization of nonlinear master and slave systems. In the dynamics of slave system, $\tau(t)$ represents the input delay, which is differentiate-able function with respect to time and it is assumed to satisfy the following mathematical conditions,

$$0 \le \tau(t) \le \tau^* \qquad \textit{(Eq 5.6)}$$

and also delay derivative bound as

$$\dot{\tau}(t) \le \eta \,. \qquad \textit{(Eq 5.7)}$$

Delay bound can be assigned from zero to some constant numerical values of the delay. Now couples of assumptions and Lemmas are provided, which will be useful to obtain the main results for synchronization of nonlinear time-delay master and slave systems.

5.2.2 Assumptions

To simplify the problem and for obtaining sufficient conditions for synchronization of nonlinear systems, following assumptions about input nonlinearity and nonlinear function are described herein.

1. It is assumed that input nonlinearity $\varphi(u(t))$ is a continuous function, which satisfies the following mathematical condition

   $$l_m u(t) \leq \varphi(u) \leq l_M u(t), \qquad \textit{(Eq 5.8)}$$

   where $l_M = diag(l_{M,1}, l_{M,2}, \ldots, l_{M,p}) > 0$ and $l_m = diag(l_{m,1}, l_{m,2}, \ldots, l_{m,p}) > 0$.

2. Nonlinearities in master and slave systems are assumed to be Lipschitz and these nonlinearities $f(x(t),t)$ are assumed to be of continuous nature, validating the Lipschitz condition as follows

   $$\| f(x_m(t),t) - f(x_s(t),t)\| \leq \Omega \| x_m(t) - x_s(t)\|, \qquad \textit{(Eq 5.9)}$$

   where $\Omega > 0$ is the Lipschitz constant, for all $x_m, x_s \in R^n$, and $\|\cdot\|$ represents the Euclidian norm.

5.2.3 Lemmas

Following two Lemmas are provided herein, useful to derive the main results.

1. For a constant matrix $Y = Y^T > 0$ and scalars $\tau^* \geq 0$, the following inequality holds [123]:

   $$-\tau^* \int_{t-\tau^*}^{t} e^T(\vartheta) Y e(\vartheta) d\vartheta \leq -\int_{t-\tau^*}^{t} e^T(\vartheta) d\vartheta Y \int_{t-\tau^*}^{t} e(\vartheta) d\vartheta, \qquad \textit{(Eq 5.10)}$$

   where τ^* represents the upper limit of delay bound whereas lower bound is assumed to be zero, given that the concerned integral terms are well-defined ([117-118]). In a similar fashion, we can write the integral inequality for $\tilde{\tau} = 0.5\tau^{*2}$

$$-\tilde{\tau}\int_{-\tau^*}^{0}\int_{t+\alpha}^{t} e^T(\vartheta)Ye(\vartheta)d\vartheta d\alpha \le -\int_{-\tau^*}^{0}\int_{t+\alpha}^{t} e^T(\vartheta)d\vartheta d\alpha Y\int_{-\tau^*}^{0}\int_{t+\alpha}^{t} e(\vartheta)d\vartheta d\alpha. \qquad \textit{(Eq 5.11)}$$

2. For a given matrix

$$\phi = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{12}^T & \phi_{22} \end{bmatrix}, \qquad \textit{(Eq 5.12)}$$

if $\phi_{11} = \phi_{11}^T$ and $\phi_{22} = \phi_{22}^T$, the following conditions are equivalent (see, for instance, [109] and [124]):

(a) $\phi < 0$,

(b) $\phi_{22} < 0$, and $\phi_{11} - \phi_{12}\phi_{22}^{-1}\phi_{12}^T < 0$.

The condition is well known as Schur complement [120].

The provided Lemmas of Jensen's inequality and Schur complement will be useful to derive the sufficient condition for synchronization.

5.2.4 LPV-Approach

Linear parameter varying (LPV) technique is useful to deal with the nonlinear function. Using this approach, nonlinearities are transformed into linear parameters in terms of exogenous values and also it provides the advantage of designing a computationally uncomplicated controller.

The delayed input nonlinear function $\varphi(u(t-\tau(t)))$, using the Assumption 1 of Eq. 5.8, can be transformed as linear parameter varying (LPV) function as under

$$\varphi(u(t-\tau(t))) = \wp(t)u(t-\tau(t)), \qquad \textit{(Eq 5.13)}$$

where $\wp(t) \in R^p$ represents a time-varying diagonal matrix. It verifies the following mathematical equation,

$$l_m \le \wp(t) \le l_M . \qquad \textit{(Eq 5.14)}$$

Now defining $q_p(i) = \begin{bmatrix} 0_{1\times(i-1)} & 1 & 0_{1\times(p-i)} \end{bmatrix}^T$, the relation $l_m \le \wp(t) \le l_M$ can be expressed as

$$\wp(t)=\sum_{i=1}^{p}\sum_{j=1}^{p}\wp_{ij}(t)q_p(i)q_p^T(j), \qquad \textit{(Eq 5.15)}$$

for each $\wp_{ij}(t)$ applicable

$$l_{m,i}\le \wp_{ii}(t)\le l_{M,i}\,,\, i=1,2,...,p\,, \qquad \textit{(Eq 5.16)}$$

$$\wp_{ij}(t)=0\,,\; i=1,2,...,p\,,\; j=1,2,...,p\,,\;\forall i\ne j\,, \qquad \textit{(Eq 5.17)}$$

where $\wp(t)$ represents the time-varying diagonal matrix function and it belongs to the following set

$$\mathbb{Q}=\left\{\Upsilon\in R^{p\times p}:\Upsilon_{ij}=0, i\ne j, \Upsilon_{ii}\in\left\{l_{m,i},l_{M,i}\right\}\right\}. \qquad \textit{(Eq 5.18)}$$

Now by incorporating LPV realization, the dynamics of error system of Eq. 5.5 is rewritten as

$$\dot{e}(t)=Ae(t)-B\Upsilon u(t-\tau)+\Psi(x_m,x_s)\,,\,\forall\Upsilon\in\mathbb{Q}\,. \qquad \textit{(Eq 5.19)}$$

This LPV-based method is very suitable for robust controller design in the presence of input delay, which makes it superior than the orthodox adaptive controller design technique [61-64], as it allows simple controller design by reducing the adaptive terms.

5.3 Controller Design

An argument vector is defined as

$$\begin{aligned}\lambda(t)=col\Big\{&e(t),e(t-\tau(t)),e(t-\tau^*),\dot{e}(t-\tau^*),\\ &\int_{t-\tau(t)}^{t}e(\vartheta)d\vartheta,\int_{t-\tau^*}^{t-\tau(t)}e(\vartheta)d\vartheta,\Psi(x_m,x_s)\Big\}.\end{aligned} \qquad \textit{(Eq 5.20)}$$

Let $\Im_i(i=1,2,3,...,7)$ represents a matrix by replacing ith entry of $n\times 7n$ zero matrix with an identity matrix, for example fourth entry of the argument matrix can be represented as $\Im_4^T=\begin{bmatrix}0 & 0 & 0 & I & 0 & 0 & 0\end{bmatrix}$. There are seven elements in this argument matrix $(\lambda(t))$. For instance, $\Im_3\lambda(t)=e(t-\tau^*)$, which is third entry of the argument matrix $\lambda(t)$.

Now, a simple state-feedback control law for synchronization is proposed. Structure of the proposed control law is as follows

$$u(t) = Ke(t), \tag{Eq 5.21}$$

where $K \in R^{m\times n}$ represents the state-feedback controller gain matrix.

By incorporating controller design of Eq. 5.21 into Eq. 5.19, we get

$$\dot{e}(t) = Ae(t) - B\Upsilon Ke(t-\tau) + \Psi(x_m, x_s), \ \forall \Upsilon \in \mathbb{Q}. \tag{Eq 5.22}$$

Following Theorems are derived by application of proposed state-feedback control law $u(t) = Ke(t)$ to ensure the convergence of synchronization error $e(t)$ to the origin.

5.4 Theorem 5.1

Let the master system described in Eq. 5.1 and the slave system in Eq. 5.2 satisfy the assumptions provided in Eq. 5.8, Eq. 5.9, input delay bounds of Eq. 5.6 and delay derivative bound provided in Eq. 5.7 along with $\tau^* > 0$. Suppose there exist symmetric matrices $P > 0$, $Q > 0$, $S > 0$, $H_i > 0$ and $Z_j > 0$ of appropriate dimensions, for $i = 1,2,3,$ and $j = 1,2$, such that the following LMIs, for a given matrix $K \in R^{m\times n}$, are satisfied

$$\varphi = \begin{bmatrix} \aleph_{11} & -PB\Upsilon K & 0 & 0 & \tau^* S & \tau^* S & P & A^T H_3 & \tau^* A^T Z_1 & \tilde{\tau} A^T S \\ * & \aleph_{22} & 2Z_1 & 0 & 0 & 0 & 0 & -EH_3 & -\tau^* EZ_1 & -\tilde{\tau} ES \\ * & * & \aleph_{33} & Q & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -H_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & \aleph_{55} & -S & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \aleph_{66} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & -I & H_3 & \tau^* Z_1 & \tilde{\tau} S \\ * & * & * & * & * & * & * & -H_3 & 0 & 0 \\ * & * & * & * & * & * & * & * & -Z_1 & 0 \\ * & * & * & * & * & * & * & * & * & -S \end{bmatrix}$$

$$<0, \quad \forall \Upsilon \in \mathbb{Q}, \tag{Eq 5.23}$$

$$\hat{\varphi}=\begin{bmatrix} \aleph_{11} & -PB\Upsilon K & 0 & 0 & \tau^{*}S & \tau^{*}S & P & A^{T}H_{3} & \tau^{*}A^{T}Z_{1} & \tilde{\tau}A^{T}S \\ * & \aleph_{22} & Z_{1} & 0 & 0 & 0 & 0 & -EH_{3} & -\tau^{*}EZ_{1} & -\tilde{\tau}ES \\ * & * & \hat{\aleph}_{33} & Q & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -H_{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & \hat{\aleph}_{55} & -S & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \hat{\aleph}_{66} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & -I & H_{3} & \tau^{*}Z_{1} & \tilde{\tau}S \\ * & * & * & * & * & * & * & -H_{3} & 0 & 0 \\ * & * & * & * & * & * & * & * & -Z_{1} & 0 \\ * & * & * & * & * & * & * & * & * & -S \end{bmatrix}$$

$$<0, \quad \forall \Upsilon \in \mathbb{Q}, \qquad \textit{(Eq 5.24)}$$

where

$\aleph_{11} = PA + A^{T}P + H_{1} + H_{2} + \tau^{*2}Z_{2} - \tau^{*2}S + \Omega^{T}\Omega,$

$\aleph_{22} = -3Z_{1} - (1-\eta)H_{2},$

$\aleph_{33} = -2Z_{1} - H_{1},$

$\aleph_{55} = -Z_{2} - S,$

$\aleph_{66} = -2Z_{2} - S,$

$\hat{\aleph}_{33} = -Z_{1} - H_{1},$

$\hat{\aleph}_{55} = -2Z_{2} - S,$

$\hat{\aleph}_{66} = -Z_{2} - S,$

$\mathrm{E} = K^{T}\Upsilon^{T}B^{T}$

and * is representing the symmetric terms in LMIs.

Than provided state-feedback controller of Eq. 5.21, ensures the asymptotic synchronization of the master and slave systems under input delay and slope-restricted input nonlinearity.

5.4.1 Proof of Theorem 5.1

Lyapunov stability theory is used as standard tool for stability analysis of nonlinear time-delay systems. There are two popular methods exists for stability analysis of time-delay systems, the Krasovskii method for Lyapunov functional and Razumikhin method for Lyapunov functional. A positive-definite Lyapunov- Krasovskii functional is constructed to provide the proof of Theorem 5.1, given by

$$\begin{aligned} V(e_t) &= e^T(t)Pe(t) + e^T(t-\tau^*)Qe(t-\tau^*) + \int_{t-\tau^*}^{t} e^T(\vartheta)H_1 e(\vartheta)d\vartheta \\ &+\int_{t-\tau(t)}^{t} e^T(\vartheta)H_2 e(\vartheta)d\vartheta + \int_{t-\tau^*}^{t} \dot{e}^T(\vartheta)H_3\dot{e}(\vartheta)d\vartheta + \int_{-\tau^*}^{0}\int_{t+\alpha}^{t}\tau^* \\ &\times \dot{e}^T(\vartheta)Z_1\dot{e}(\vartheta)d\vartheta d\alpha + \int_{-\tau^*}^{0}\int_{t+\alpha}^{t}\tau^* e^T(\vartheta)Z_2 e(\vartheta)d\vartheta d\alpha \\ &+\int_{-\tau^*}^{0}\int_{\alpha}^{0}\int_{t+\gamma}^{t}\tilde{\tau}\dot{e}^T(\vartheta)S\dot{e}(\vartheta)d\vartheta d\gamma d\alpha. \end{aligned} \qquad \textit{(Eq 5.25)}$$

Evaluating the time-derivative of LK functional, it gives

$$\begin{aligned} \dot{V}(e_t) &= \dot{e}^T(t)Pe(t) + e^T(t)P\dot{e}(t) + \dot{e}^T(t-\tau^*)Qe(t-\tau^*) + e^T(t-\tau^*) \\ &\times Q\dot{e}(t-\tau^*) + e^T(t)H_1e(t) - e^T(t-\tau^*)H_1e(t-\tau^*) + e^T(t)H_2 \\ &\times e(t) - (1-\dot{\tau})e^T(t-\tau(t))H_2e(t-\tau(t)) + \dot{e}^T(t)H_3\dot{e}(t) - \dot{e}^T \\ &\times (t-\tau^*)H_3\dot{e}(t-\tau^*) + \tau^{*2}\dot{e}^T(t)Z_1\dot{e}(t) - \int_{t-\tau^*}^{t}\tau^*\dot{e}^T(\vartheta)Z_1\dot{e}(\vartheta)d\vartheta \\ &+\tau^{*2}e^T(t)Z_2e(t) - \int_{t-\tau^*}^{t}\tau^* e^T(\vartheta)Z_2 e(\vartheta)d\vartheta + \tilde{\tau}^2\dot{e}^T(t)S\dot{e}(t) \\ &-\tilde{\tau}\int_{-\tau^*}^{0}\int_{t+\alpha}^{t}\dot{e}^T(\vartheta)S\dot{e}(\vartheta)d\vartheta d\alpha. \end{aligned} \qquad \textit{(Eq 5.26)}$$

To simplify, let us define

$$M = H_3 + \tau^{*2}Z_1 + \tilde{\tau}^2 S\ . \qquad \textit{(Eq 5.27)}$$

Using the relationship of Eq. 5.27 and incorporating the delay derivative bound $\dot{\tau}(t)$ defined in Eq. 5.7 into Eq. 5.26, it reveals

$$\begin{aligned} \dot{V}(e_t) &\le e^T(t)(H_1 + H_2 + \tau^{*2}Z_2)e(t) + \dot{e}^T(t)Pe(t) + e^T(t)P\dot{e}(t) \\ &+\dot{e}^T M\dot{e}(t) + \dot{e}^T(t-\tau^*)Q(t-\tau^*) + e^T(t-\tau^*)Q\dot{e}(t-\tau^*) \\ &-e^T(t-\tau^*)H_1e(t-\tau^*) - (1-\eta)e^T(t-\tau(t))H_2e(t-\tau(t)) \\ &-\dot{e}^T(t-\tau^*)H_3\dot{e}(t-\tau^*) - \int_{t-\tau^*}^{t}\tau^*\dot{e}^T(\vartheta)Z_1\dot{e}(\vartheta)d\vartheta \\ &-\int_{t-\tau^*}^{t}\tau^* e^T(\vartheta)Z_2e(\vartheta)d\vartheta - \tilde{\tau}\int_{-\tau^*}^{0}\int_{t+\alpha}^{t}\dot{e}^T(\vartheta)S\dot{e}(\vartheta)d\vartheta d\alpha. \end{aligned} \qquad \textit{(Eq 5.28)}$$

Incorporating the Jensen’s inequality provided in Lemma 1, integral terms can be solved as follows

$$\begin{aligned} -\tilde{\tau}\int_{-\tau^*}^{0}\int_{t+\alpha}^{t}\dot{e}^T(\vartheta)S\dot{e}(\vartheta)d\vartheta d\alpha &\le -\lambda^T(t)[(\tau^*\Im_1^T - \Im_5^T - \Im_6^T)S \\ &\times(\tau^*\Im_1 - \Im_5 - \Im_6)]\lambda(t). \end{aligned} \qquad \textit{(Eq 5.29)}$$

Similarly

$$-\int_{t-\tau^*}^{t}\tau^*\dot{e}^T(\vartheta)Z_1\dot{e}(\vartheta)d\vartheta=-\int_{t-\tau^*}^{t-d(t)}\tau^*\dot{e}^T(\vartheta)Z_1\dot{e}(\vartheta)d\vartheta \\ -\int_{t-d(t)}^{t}\tau^*\dot{e}^T(\vartheta)Z_1\dot{e}(\vartheta)d\vartheta. \qquad \textit{(Eq 5.30)}$$

Incorporating the inequalities $-\tau^*\le-(\tau^*-d(t))$ and $-\tau^*\le(d(t))$, furthermore assigning $\chi=(d(t))/\tau^*$, leads to

$$-\int_{t-\tau^*}^{t}\tau^*\dot{e}^T(\vartheta)Z_1\dot{e}(\vartheta)d\vartheta\le-\tilde{\lambda}^T(t)(\Im_2^T-\Im_3^T)Z_1(\Im_2-\Im_3)\tilde{\lambda}(t) \\ +\tilde{\lambda}^T(t)(\Im_2^T)Z_1(\Im_2)\tilde{\lambda}(t)-\chi\tilde{\lambda}^T(t)(\Im_2^T-\Im_3^T) \\ \times Z_1(\Im_2-\Im_3)\tilde{\lambda}(t)+(1-\chi)\tilde{\lambda}^T(t)(\Im_2^T)Z_1(\Im_2)\tilde{\lambda}(t) \qquad \textit{(Eq 5.31)}$$

and similarly

$$-\int_{t-\tau^*}^{t}\tau^*e^T(\vartheta)Z_2e(\vartheta)d\vartheta\le-\tilde{\lambda}^T(t)[\Im_5^TZ_2\Im_5+\Im_6^TZ_2\Im_6]\tilde{\lambda}(t) \\ -\chi\tilde{\lambda}^T(t)(\Im_6^TZ_2\Im_6)\tilde{\lambda}(t)-(1-\chi)\tilde{\lambda}^T(t) \\ \times(\Im_5^TZ_2\Im_5)\tilde{\lambda}(t). \qquad \textit{(Eq 5.32)}$$

Hence Eq. 5.29 to Eq. 5.32 holds, it is implicit to obtain

$$\dot{V}(e_t)\le e^T(t)P\dot{e}(t)+\dot{e}^T(t)Pe(t)+\dot{e}^T(t)M\dot{e}(t)+\tilde{\lambda}^T(t)[\Im_3^TQ\Im_4 \\ +\Im_4^TQ\Im_3+\Im_1^T(H_1+H_2+\tau^{*2}Z_2)\Im_1-(1-\eta)\Im_2^TH_2\Im_2 \\ -\Im_3^TH_1\Im_3-\Im_4^TH_3\Im_4-(\Im_2^T-\Im_3^T)Z_1(\Im_2-\Im_3)+\Im_2^TZ_1\Im_2 \\ -\chi(\Im_2^T-\Im_3^T)Z_1(\Im_2-\Im_3)+(1-\chi)(\Im_2^T)Z_1(\Im_2) \\ -\Im_5^TZ_2\Im_5-\Im_6^TZ_2\Im_6-\chi\Im_6^TZ_2\Im_6-(1-\chi)\Im_5^TZ_2\Im_5 \\ -(\tau^*\Im_1^T-\Im_5^T-\Im_6^T)S(\tau^*\Im_1-\Im_5-\Im_6)]\tilde{\lambda}(t). \qquad \textit{(Eq 5.33)}$$

Incorporating the error dynamics $\dot{e}(t)$ from Eq. 5.22 into Eq. 5.33, it yields

$$\dot{V}(e_t)\le e^T(t)P[Ae(t)-B\Upsilon Ke(t-\tau)+\Psi(x_m,x_s)]+[Ae(t)-B\Upsilon K \\ \times e(t-\tau)+\Psi(x_m,x_s)]^TPe(t)+[Ae(t)-B\Upsilon Ke(t-\tau)+\Psi(x_m,x_s)]^T \\ \times M[Ae(t)-B\Upsilon Ke(t-\tau)+\Psi(x_m,x_s)]+\tilde{\lambda}^T(t)[\Im_3^TQ\Im_4+\Im_4^TQ\Im_3 \\ +\Im_1^T(H_1+H_2+\tau^{*2}Z_2)\Im_1-(1-\eta)\Im_2^TH_2\Im_2-\Im_3^TH_1\Im_3-\Im_4^TH_3\Im_4 \\ -(\Im_2^T-\Im_3^T)Z_1(\Im_2-\Im_3)+\Im_2^TZ_1\Im_2-\chi(\Im_2^T-\Im_3^T)Z_1(\Im_2-\Im_3) \\ +(1-\chi)(\Im_2^T)Z_1(\Im_2)-\Im_5^TZ_2\Im_5-\Im_6^TZ_2\Im_6-\chi\Im_6^TZ_2\Im_6-(1-\chi) \\ \times\Im_5^TZ_2\Im_5-(\tau^*\Im_1^T-\Im_5^T-\Im_6^T)S(\tau^*\Im_1-\Im_5-\Im_6)]\tilde{\lambda}(t). \qquad \textit{(Eq 5.34)}$$

Applying Lipschitz condition provided in Eq. 5.9, it depicts

$$\begin{aligned}\dot{V}(e_t) \le \lambda^T(t)[&\Im_1^T(A^TP+PA)\Im_1 - \Im_1^T PB\Upsilon K\Im_2 + \Im_1^T P\Im_7 - \Im_2^T K^T \\ &\times\Upsilon^T B^T P\Im_1 + \Im_7^T P\Im_1 + A_c^T M A_c + \Im_3^T Q\Im_4 + \Im_4^T Q\Im_3 + \Im_1^T \\ &\times(H_1 + H_2 + \tau^{*2}Z_2)\Im_1 - (1-\eta)\Im_2^T H_2\Im_2 - \Im_3^T H_1\Im_3 - \Im_4^T H_3\Im_4 \\ &-(\Im_2^T - \Im_3^T)Z_1(\Im_2 - \Im_3) + \Im_2^T Z_1\Im_2 - \chi(\Im_2^T - \Im_3^T)Z_1(\Im_2 - \Im_3) \\ &+(1-\chi)(\Im_2^T)Z_1(\Im_2) - \Im_5^T Z_2\Im_5 - \Im_6^T Z_2\Im_6 - \chi\Im_6^T Z_2\Im_6 - (1-\chi) \\ &\times\Im_5^T Z_2\Im_5 - (\tau^*\Im_1^T - \Im_5^T - \Im_6^T)S(\tau^*\Im_1 - \Im_5 - \Im_6)]\lambda(t),\end{aligned} \qquad \textit{(Eq 5.35)}$$

where A_c is defined as

$$A_c = \begin{bmatrix} A & -B\Upsilon K & 0 & 0 & 0 & 0 & I \end{bmatrix}.$$

It can be rearranged by applying some algebra rules, as under

$$\dot{V}(e_t) \le \lambda^T(t)[\chi\varphi + (1-\chi)\hat{\varphi}]\lambda(t), \ \forall \Upsilon \in \mathbb{Q}, \qquad \textit{(Eq 5.36)}$$

where

$$\begin{aligned}\varphi = \lambda^T(t)[&(A_c^T M A_c) + \Im_1^T(PA)\Im_1 - \Im_1^T(PB\Upsilon K)\Im_2 + \Im_1^T P\Im_7 + \Im_1^T \\ &\times(A^TP)\Im_1 - \Im_2^T(B\Upsilon K)^T P\Im_1 + \Im_7^T P\Im_1 + \Im_3^T Q\Im_4 + \Im_4^T Q\Im_3 + \Im_1^T \\ &\times(\tau^{*2}Z_2)\Im_1 - (1-\eta)\Im_2^T H_2\Im_2 - \Im_3^T H_1\Im_3 - \Im_4^T H_3\Im_4 - 2(\Im_2^T - \Im_3^T) \\ &\times Z_1(\Im_2 - \Im_3) - \Im_2^T Z_1\Im_2 - \Im_5^T Z_2\Im_5 - 2\Im_6^T Z_2\Im_6 - (\tau^*\Im_1^T - \Im_5^T - \Im_6^T) \\ &\times S(\tau^*\Im_1 - \Im_5 - \Im_6)]\lambda(t)\end{aligned} \qquad \textit{(Eq 5.37)}$$

and

$$\begin{aligned}\hat{\varphi} = \lambda^T(t)[&(A_c^T M A_c) + \Im_1^T(PA)\Im_1 - \Im_1^T(PB\Upsilon K)\Im_2 + \Im_1^T P\Im_7 + \Im_1^T \\ &\times(A^TP)\Im_1 - \Im_2^T(B\Upsilon K)^T P\Im_1 + \Im_7^T P\Im_1 + \Im_3^T Q\Im_4 + \Im_4^T Q\Im_3 + \Im_1^T \\ &\times(\tau^{*2}Z_2)\Im_1 - (1-\eta)\Im_2^T H_2\Im_2 - \Im_3^T H_1\Im_3 - \Im_4^T H_3\Im_4 - (\Im_2^T - \Im_3^T) \\ &\times Z_1(\Im_2 - \Im_3) - 2\Im_2^T Z_1\Im_2 - 2\Im_5^T Z_2\Im_5 - \Im_6^T Z_2\Im_6 - (\tau^*\Im_1^T - \Im_5^T - \Im_6^T) \\ &\times S(\tau^*\Im_1 - \Im_5 - \Im_6)]\lambda(t).\end{aligned} \qquad \textit{(Eq 5.38)}$$

Rearranging it

$$\begin{aligned}\varphi = \lambda^T(t)[&(A_c^T M A_c) + \Im_1^T(A^TP + PA + H_1 + H_2 + \tau^{*2}Z_2 - \tau^{*2}S)\Im_1 \\ &-\Im_1^T(PB\Upsilon K)\Im_2 + \Im_1^T P\Im_7 + \tau^*\Im_1^T S\Im_5 + \tau^*\Im_1^T S\Im_6 - \Im_2^T(B\Upsilon K)^T P\Im_1 \\ &-(1-\eta)\Im_2^T(H_2 + 3Z_1)\Im_2 + 2\Im_2^T Z_1\Im_3 + 2\Im_3^T Z_1\Im_2 - \Im_3^T(2Z_1 + H_1)\Im_3 \\ &+\Im_3^T Q\Im_4 + \Im_4^T Q\Im_3 - \Im_4^T H_3\Im_4 + \tau^*\Im_5^T S\Im_1 - \Im_5^T(Z_2 + S)\Im_5 - \Im_5^T S\Im_6 \\ &+\tau^*\Im_6^T S\Im_1 - \Im_6^T S\Im_5 - \Im_6^T(2Z_2 - S)\Im_6 + \Im_7^T P\Im_1]\lambda(t)\end{aligned} \qquad \textit{(Eq 5.39)}$$

and

$$
\begin{aligned}
\hat{\varphi} = \lambda^T(t)[(A_c^T M A_c) &+ \Im_1^T(A^T P + PA + H_1 + H_2 + \tau^{*2} Z_2 - \tau^{*2} S)\Im_1 \\
&- \Im_1^T(PB\Upsilon K)\Im_2 + \tau^* \Im_1^T S \Im_5 + \tau^* \Im_1^T S \Im_6 + \Im_1^T P \Im_7 - \Im_2^T (B\Upsilon K)^T P \Im_1 \\
&- \Im_2^T(3Z_1 + (1-\eta)H_2)\Im_2 + \Im_2^T Z_1 \Im_3 + \Im_3^T Z_1 \Im_2 - \Im_3^T(Z_1 + H_1)\Im_3 \\
&+ \Im_3^T Q \Im_4 + \Im_4^T Q \Im_3 - \Im_4^T H_3 \Im_4 + \tau^* \Im_5^T S \Im_1 - \Im_5^T(2Z_2 + S)\Im_5 \\
&- \Im_5^T S \Im_6 + \tau^* \Im_6^T S \Im_1 - \Im_6^T S \Im_5 - \Im_6^T(Z_2 + S)\Im_6 + \Im_7^T P \Im_1]\lambda(t).
\end{aligned} \qquad \textit{(Eq 5.40)}
$$

Eq. 5.39 and Eq. 5.40 can be rearranged in the LMI format as below

$$
\varphi = \begin{bmatrix}
\aleph_{11} & -PB\Upsilon K & 0 & 0 & \tau^* S & \tau^* S & P \\
* & \aleph_{22} & 2Z_1 & 0 & 0 & 0 & 0 \\
* & * & \aleph_{33} & Q & 0 & 0 & 0 \\
* & * & * & -H_3 & 0 & 0 & 0 \\
* & * & * & * & \aleph_{55} & -S & 0 \\
* & * & * & * & * & \aleph_{66} & 0 \\
* & * & * & * & * & * & -I
\end{bmatrix} + \qquad \textit{(Eq 5.41)}
$$

$$
[A \quad -B\Upsilon K \quad 0 \quad 0 \quad 0 \quad 0 \quad I]^T M [A \quad -B\Upsilon K \quad 0 \quad 0 \quad 0 \quad 0 \quad I]
$$

and

$$
\hat{\varphi} = \begin{bmatrix}
\aleph_{11} & -PB\Upsilon K & 0 & 0 & \tau^* S & \tau^* S & P \\
* & \aleph_{22} & Z_1 & 0 & 0 & 0 & 0 \\
* & * & \hat{\aleph}_{33} & Q & 0 & 0 & 0 \\
* & * & * & -H_3 & 0 & 0 & 0 \\
* & * & * & * & \hat{\aleph}_{55} & -S & 0 \\
* & * & * & * & * & \hat{\aleph}_{66} & 0 \\
* & * & * & * & * & * & -I
\end{bmatrix} + \qquad \textit{(Eq 5.42)}
$$

$$
[A \quad -B\Upsilon K \quad 0 \quad 0 \quad 0 \quad 0 \quad I]^T M [A \quad -B\Upsilon K \quad 0 \quad 0 \quad 0 \quad 0 \quad I].
$$

To ensure the asymptotic stability, following condition is essential to be true

$\dot{V}(e_t) < 0$, or $\chi\varphi + (1-\chi)\hat{\varphi} < 0$, $0 \le \chi \le 1$,

$\varphi < 0$ and $\hat{\varphi} < 0$.

Applying the Schur complement provided in the Lemma 2, on Eq. 5.41 and Eq. 5.42, the LMIs of Eq. 5.23 and Eq. 5.24 can be obtained. This completes the proof of Theorem 5.1. □

Synchronization of nonlinear time-delay master and slave systems described in Eq. 5.1 and Eq. 5.2, respectively, can be accomplished using Theorem 5.1. To attain the desired results by implying Theorem 5.1, information of the controller gain may be known prior or selected arbitrarily. However, this is not the case for all the time that controller information is readily available. Theorem 5.2 provided herein has advantage of computing the value of unknown controller gain to achieve the synchronization of nonlinear time-delay systems.

5.5 Theorem 5.2

Let the master system described in Eq. 5.1 and the slave system in Eq. 5.2 satisfy the assumptions provided in Eq. 5.8, Eq. 5.9, input delay bounds of Eq. 5.6, and delay derivative bound provided in Eq. 5.7 along with $\tau^* > 0$. Suppose there exist symmetric matrices $X > 0$, $\bar{S} > 0$, $\bar{Q} > 0$, $\bar{H}_i > 0$, $\bar{Z}_j > 0$ and a matrix G of appropriate dimensions, for $i = 1, 2, 3,$ and $j = 1, 2$, such that the sets of linear matrix inequalities are satisfied

$$\Gamma = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ * & \Gamma_{22} \end{bmatrix}, \forall \Upsilon \in \mathbb{Q}, \qquad \textit{(Eq 5.43)}$$

$$\hat{\Gamma} = \begin{bmatrix} \hat{\Gamma}_{11} & \Gamma_{12} \\ * & \Gamma_{22} \end{bmatrix}, \forall \Upsilon \in \mathbb{Q}, \qquad \textit{(Eq 5.44)}$$

where

$$\Gamma_{11} = \begin{bmatrix} \Sigma_{11} & -B\Upsilon G & 0 & 0 & \tau^*\bar{S} & \tau^*\bar{S} & I \\ * & \Sigma_{22} & 2\bar{Z}_1 & 0 & 0 & 0 & 0 \\ * & * & -2\bar{Z}_1 - \bar{H}_1 & \bar{Q} & 0 & 0 & 0 \\ * & * & * & -\bar{H}_3 & 0 & 0 & 0 \\ * & * & * & * & -\bar{Z}_2 - \bar{S} & -\bar{S} & 0 \\ * & * & * & * & * & -2\bar{Z}_2 - \bar{S} & 0 \\ * & * & * & * & * & * & -I \end{bmatrix},$$

$$\Gamma_{12}=\begin{bmatrix} XA^T & \tau^* XA^T & \tilde{\tau} XA^T & X\Omega^T \\ -G^T\Upsilon^T B^T & -\tau^* G^T\Upsilon^T B^T & -\tilde{\tau} G^T\Upsilon^T B^T & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ I & \tau^* I & \tilde{\tau} I & 0 \end{bmatrix},$$

$\Gamma_{22}=-diag(X\bar{H}_3^{-1}, X\bar{Z}_1^{-1}X, X\bar{S}^{-1}X, I)$, and

$$\hat{\Gamma}_{11}=\begin{bmatrix} \Sigma_{11} & -B\Upsilon G & 0 & 0 & \tau^*\bar{S} & \tau^*\bar{S} & I \\ * & \Sigma_{22} & \bar{Z}_1 & 0 & 0 & 0 & 0 \\ * & * & -\bar{Z}_1-\bar{H}_1 & \bar{Q} & 0 & 0 & 0 \\ * & * & * & -\bar{H}_3 & 0 & 0 & 0 \\ * & * & * & * & -2\bar{Z}_2-\bar{S} & -\bar{S} & 0 \\ * & * & * & * & * & -\bar{Z}_2-\bar{S} & 0 \\ * & * & * & * & * & * & -I \end{bmatrix},$$

where

$\Sigma_{11}=AX+XA^T+\bar{H}_1+\bar{H}_2+\tau^{*2}\bar{Z}_2-\tau^{*2}\bar{S}$,

$\Sigma_{22}=-3\bar{Z}_1-(1-\eta)\bar{H}_2$.

Then there exists a reliable state-feedback controller of Eq. 5.21 that guarantees asymptotic synchronization of the master and the slave systems under input delay and slope-restricted input nonlinearity. The controller gain matrix can be computed by solving $K=GX^{-1}$.

5.5.1 Proof of Theorem 5.2

To provide the proof of Theorem 5.2, explicit mathematical treatment is integrated on the inequalities established in Theorem 5.1. Pre-and post-multiplication of $diag(\sigma_1,\sigma_2,\sigma_3)$, to the inequalities obtained in Eq. 5.23 and Eq. 5.24, where

$\sigma_1=diag(P^{-1},P^{-1},P^{-1},P^{-1},P^{-1},P^{-1})$,

$\sigma_2=diag(I)$,

$\sigma_3=diag(H_3^{-1},Z_1^{-1},S^{-1})$, and also applying the mathematical identity $P^{-1}P = PP^{-1} = I$, we obtain following set of LMIs,

$$\Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ * & \Pi_{22} \end{bmatrix},\ \forall \Upsilon \in \mathbb{Q}, \qquad \textit{(Eq 5.45)}$$

$$\hat{\Pi} = \begin{bmatrix} \hat{\Pi}_{11} & \Pi_{12} \\ * & \Pi_{22} \end{bmatrix},\ \forall \Upsilon \in \mathbb{Q}, \qquad \textit{(Eq 5.46)}$$

where

$$\Pi_{11} = \begin{bmatrix} \nabla_{11} & -B\Upsilon KP^{-1} & 0 & 0 & \tau^* P^{-1}SP^{-1} & \tau^* P^{-1}SP^{-1} & I \\ * & \nabla_{22} & 2P^{-1}Z_1P^{-1} & 0 & 0 & 0 & 0 \\ * & * & \nabla_{33} & P^{-1}QP^{-1} & 0 & 0 & 0 \\ * & * & * & -P^{-1}H_3P^{-1} & 0 & 0 & 0 \\ * & * & * & * & \nabla_{55} & -P^{-1}SP^{-1} & 0 \\ * & * & * & * & * & \nabla_{66} & 0 \\ * & * & * & * & * & * & -I \end{bmatrix},$$

$$\Pi_{12} = \begin{bmatrix} A^TP^{-1} & \tau^* A^TP^{-1} & \tilde{\tau} A^TP^{-1} \\ -P^{-1}E & -\tau^* P^{-1}E & -\tilde{\tau} P^{-1}E \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ I & \tau^* I & \tilde{\tau} I \end{bmatrix},$$

$\Pi_{22} = -diag(P^{-1}PH_3^{-1}PP^{-1}, P^{-1}PZ_1^{-1}PP^{-1}, P^{-1}PS^{-1}PP^{-1})$, and

$$\hat{\Pi}_{11} = \begin{bmatrix} \nabla_{11} & -B\Upsilon KP^{-1} & 0 & 0 & \tau^* P^{-1}SP^{-1} & \tau^* P^{-1}SP^{-1} & I \\ * & \nabla_{22} & P^{-1}Z_1P^{-1} & 0 & 0 & 0 & 0 \\ * & * & \hat{\nabla}_{33} & P^{-1}QP^{-1} & 0 & 0 & 0 \\ * & * & * & -P^{-1}H_3P^{-1} & 0 & 0 & 0 \\ * & * & * & * & \hat{\nabla}_{55} & -P^{-1}SP^{-1} & 0 \\ * & * & * & * & * & \hat{\nabla}_{66} & 0 \\ * & * & * & * & * & * & -I \end{bmatrix}.$$

Elements of the matrices are as

$$\nabla_{11} = AP^{-1} + P^{-1}A^T + \tau^{*2}P^{-1}Z_2P^{-1} + P^{-1}H_1P^{-1} + P^{-1}H_2P^{-1} - \tau^{*2}P^{-1}SP^{-1} + P^{-1}\Omega^T\Omega P^{-1},$$

$$\nabla_{22} = -3P^{-1}Z_1P^{-1} - (1-\eta)P^{-1}H_2P^{-1},$$

$$\nabla_{33} = -2P^{-1}Z_1P^{-1} - P^{-1}H_1P^{-1},$$

$$\nabla_{55} = -P^{-1}Z_2P^{-1} - P^{-1}SP^{-1},$$

$$\nabla_{66} = -2P^{-1}Z_2P^{-1} - P^{-1}SP^{-1},$$

$$\hat{\nabla}_{33} = -P^{-1}Z_1P^{-1} - P^{-1}H_1P^{-1},$$

$$\hat{\nabla}_{55} = -2P^{-1}Z_2P^{-1} - P^{-1}SP^{-1},$$

$$\hat{\nabla}_{66} = -P^{-1}Z_2P^{-1} - P^{-1}SP^{-1}.$$

Now applying the change of variable transformations to simplify the matrix inequality terms given by $P^{-1} = X$, $\bar{Q} = P^{-1}QP^{-1}$, $\bar{H}_i = P^{-1}H_iP^{-1}$, $\bar{Z}_j = P^{-1}Z_jP^{-1}$, and $\bar{S} = P^{-1}SP^{-1}$, it reveals

$$\Xi = \begin{bmatrix} \Xi_{11} & \Xi_{12} \\ * & \Xi_{22} \end{bmatrix}, \ \forall \Upsilon \in \mathbb{Q}, \tag{Eq 5.47}$$

$$\hat{\Xi} = \begin{bmatrix} \hat{\Xi}_{11} & \Xi_{12} \\ * & \Xi_{22} \end{bmatrix}, \ \forall \Upsilon \in \mathbb{Q}, \tag{Eq 5.48}$$

where

$$\Xi_{11} = \begin{bmatrix} \tilde{\Sigma}_{11} & -B\Upsilon KX & 0 & 0 & \tau^*\bar{S} & \tau^*\bar{S} & I \\ * & \Sigma_{22} & 2\bar{Z}_1 & 0 & 0 & 0 & 0 \\ * & * & -2\bar{Z}_1 - \bar{H}_1 & \bar{Q} & 0 & 0 & 0 \\ * & * & * & -\bar{H}_3 & 0 & 0 & 0 \\ * & * & * & * & -\bar{Z}_2 - \bar{S} & -\bar{S} & 0 \\ * & * & * & * & * & -2\bar{Z}_2 - \bar{S} & 0 \\ * & * & * & * & * & * & -I \end{bmatrix},$$

$$\Xi_{12} = \begin{bmatrix} A^T X & \tau^* A^T X & \tilde{\tau} A^T X \\ -XK^T \Upsilon^T B^T & -\tau^* XK^T \Upsilon^T B^T & -\tilde{\tau} XK^T \Upsilon^T B^T \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ I & \tau^* I & \tilde{\tau} I \end{bmatrix},$$

$\Xi_{22} = -diag(X\bar{H}_3^{-1}X, X\bar{Z}_1^{-1}X, X\bar{S}^{-1}X)$, and

$$\hat{\Xi}_{11} = \begin{bmatrix} \tilde{\Sigma}_{11} & -B\Upsilon KX & 0 & 0 & \tau^*\bar{S} & \tau^*\bar{S} & I \\ * & \Sigma_{22} & \bar{Z}_1 & 0 & 0 & 0 & 0 \\ * & * & -\bar{Z}_1 - \bar{H}_1 & \bar{Q} & 0 & 0 & 0 \\ * & * & * & -\bar{H}_3 & 0 & 0 & 0 \\ * & * & * & * & -2\bar{Z}_2 - \bar{S} & -\bar{S} & 0 \\ * & * & * & * & * & -\bar{Z}_2 - \bar{S} & 0 \\ * & * & * & * & * & * & -I \end{bmatrix},$$

for

$\tilde{\Sigma}_{11} = AX + XA^T + \bar{H}_1 + \bar{H}_2 + \tau^{*2}\bar{Z}_2 - \tau^{*2}\bar{S} + X\Omega^T\Omega X$.

Now incorporating $G = KX$ and applying schur complement provided in Lemma on Eq. 5.47 and Eq. 5.48, the LMIs of Eq. 5.43 and Eq. 5.44 can be obtained, it completes the proof of Theorem 5.2.

It is seen that Theorem 5.1 provides the delay-dependent synchronization scheme for nonlinear time-delay systems, if the information of the controller is known. Now the controller gain matrix can be computed by solving the set of LMIs derived in Theorem 5.2 and then by solving $K = GX^{-1}$. The synchronization scheme accomplished in Theorem 5.2 is more effective and pragmatic compared to the synchronization criterion provided in Theorem 5.1. Using the controller gain computed through Theorem 5.2 synchronization of the master and the slave systems subject to the unknown slope-restricted input nonlinearity can be achieved quite comfortably.

Delay-dependent synchronization scheme under slope-restricted input nonlinearity owing to the input delay is innovative and effective then existing synchronization techniques [23], [55], [63]-[66]. The LPV-approach is employed to transform the complex input nonlinearities into LPV-realization, which provides the advantage of uncomplicated and straightforward controller design compared to the existing complex controller design techniques like adaptive controller design strategies [59], [61], [71], [73], [80]. The delay-dependent synchronization methodology developed herein allows the design of simple state-feedback control law, by relaxing the adaptive terms in the presence of additional constraints like input nonlinearity and input delay.

5.6 Simulation Results

To demonstrate the effectiveness of the proposed synchronization criteria, a numerical example for synchronization of gyros systems is provided. Gyro systems are selected due to their applications in the field of aerospace engineering. In literature, synchronization of gyro systems is discussed using different control techniques, such as adaptive terminal sliding mode, PID control approach, adaptive fuzzy control scheme [23-25], [53-55]. But synchronization of chaotic gyro systems subject to time-delay and slope-restricted input nonlinearity is lacking in the literature.

Gyro system is represented by its motion equation (see [53-54]) as under

$$\ddot{\theta}+C_1\dot{\theta}+C_2\dot{\theta}^3+\alpha^2\frac{(1-\cos\theta)^2}{\sin^3\theta}-\beta\sin\theta=-f\sin\omega t\sin\theta, \qquad \textit{(Eq 5.49)}$$

where $f\sin\omega t\sin\theta$ represents the parametric excitation, $C_1\dot{\theta}$ and $C_2\dot{\theta}^3$ are the linear and nonlinear terms, respectively. Nonlinear part of the gyro system is represented by $\alpha^2\frac{(1-\cos\theta)^2}{\sin^3\theta}-\beta\sin\theta$. To derive the normalized state equations, let us incorporate $x_1=\theta$ and $y_1=\dot{\theta}$, the dynamical equation of gyro transformed as

$$\dot{x}_1=y_1, \qquad \textit{(Eq 5.50)}$$

$$\dot{y}_1=-C_1y_1-C_2y_1^3-\alpha^2\frac{(1-\cos x_1)^2}{\sin^3 x_1}+\beta\sin x_1-f\sin\omega t\sin x_1. \qquad \textit{(Eq 5.51)}$$

By selecting the values of different parameter $C_1=0.5$, $C_2=0.05$, $\alpha=10$, $l_{\max}\geq 0$ $f=35.5$ and $\omega=2$, the gyro system can be represented for Eq. 5.1 and Eq. 5.2 with

$$f(x)=\begin{bmatrix} 0 \\ -0.05-100\dfrac{(1-\cos x_1)^2}{\sin^3 x_1}+(1-35.5\sin 2t) \end{bmatrix}, \qquad \textit{(Eq 5.52)}$$

where $f(x)$ represents the nonlinear part. Nonlinearities in master and slave systems are considered to be similar, so $f(x)$ is representing the $f(x_m(t))$ and $f(x_s(t))$.

Robust control toolbox of MATLAB is used to solve the linear matrix inequalities (LMIs) derived in Theorems 5.1 and 5.2. To ensure the feasibility of designed LMIs, known linear matrices for the master and slave systems A and B are selected as below

$$A=\begin{bmatrix} 0 & 2 \\ 0 & -0.8 \end{bmatrix}, \qquad \textit{(Eq 5.53)}$$

$$B=\begin{bmatrix} 0 \\ 1.1 \end{bmatrix}. \qquad \textit{(Eq 5.54)}$$

Other parameters like information of time-delay, Lipschitz constant and LPV constraints are selected as follows

Time-delay (τ^*) = 0.35,

Time-delay ($\tilde{\tau}$) = 0.04,

Lipschitz constant (Ω) = 0.3,

Derivative of time-delay (η) = 0.15,

Lower bound of LPV parameter (L_m) = 0.9,

Upper bound of LPV parameter (L_M) = 1.1.

In this numerical example, input nonlinearities with time-delay $t-0.1-0.01\sin(0.01t)$ is incorporated and by applying the conditions derived in Theorem 5.2, controller gain matrix K and matrix P are computed as below:

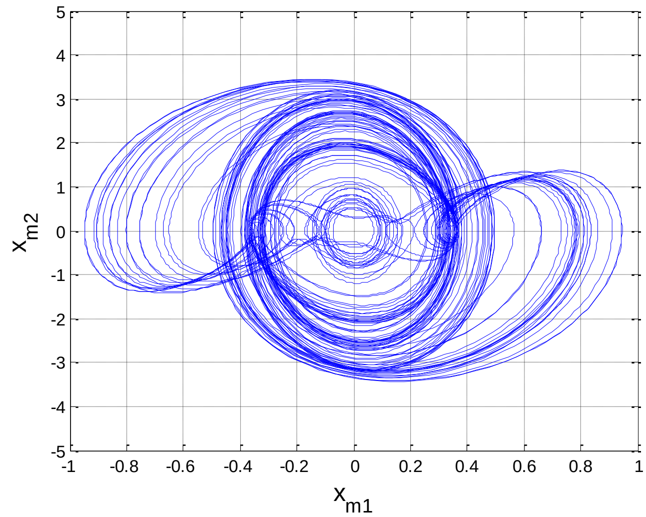
$$K=\begin{bmatrix} 0.985 & 1.264 \end{bmatrix}, \qquad \textit{(Eq 5.55)}$$

$$P=\begin{bmatrix} 0.2867 & 0.2021 \\ 0.2021 & 0.4443 \end{bmatrix}. \qquad \textit{(Eq 5.56)}$$

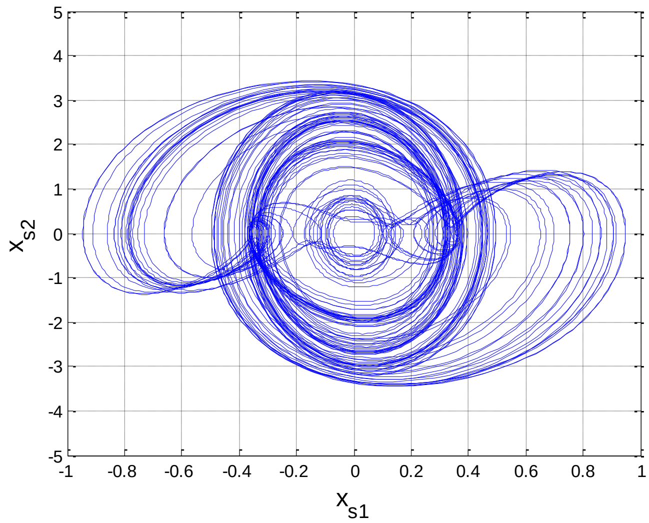
To observe the behavior of chaotic master and slave gyro systems, MATLAB solver ddex23 is used for simulating of time-delay differential equations with time-varying initial conditions as:

$$\begin{aligned} &x(0)= 0.4+0.4\sin(0.1t), \\ &x(1)=-2-0.02\sin(0.1t), \\ &y(0)=-0.4-0.04\sin(0.1t), \\ &y(1)=2+0.02\sin(0.1t). \end{aligned} \qquad \textit{(Eq 5.57)}$$

Figure 5.2 shows phase portrait of the master-slave gyro systems without controller.



(a)



(b)

Figure 5.2: Phase portrait of the gyro systems without controller, (a) master system, (b) slave system.

Figure 5.3 shows the synchronization error response between two states of master and slave gyro systems without controller, where $e_1(t) = x_{m1}(t) - x_{s1}(t)$ in solid line is the error between first state of the master and slave system and $e_2(t) = x_{m2}(t) - x_{s2}(t)$ in dotted line is the error between second state of the master and slave systems, respectively. When no control signal is applied, plots show there is no synchronization between master and slave systems. In the Figure 5.3, oscillation and irregularities appear in both error states reflecting unsynchronized behavior by showing that error dynamics are not converging to the origin.

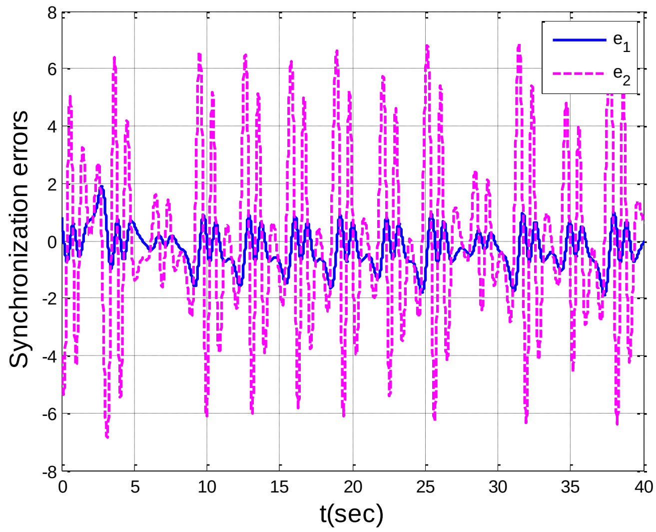
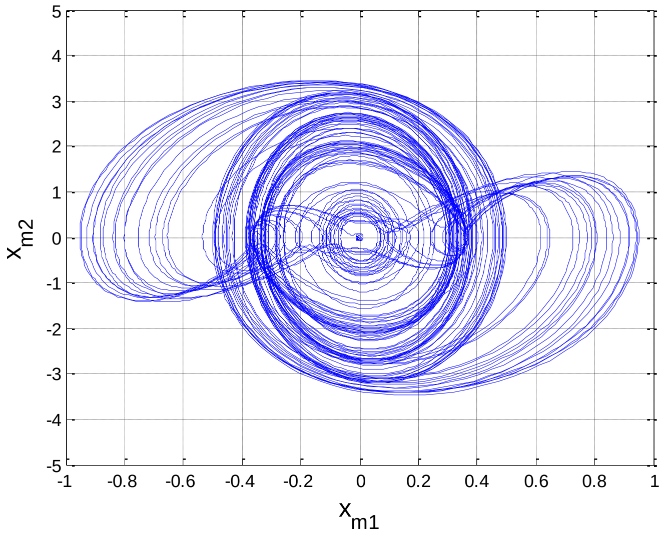
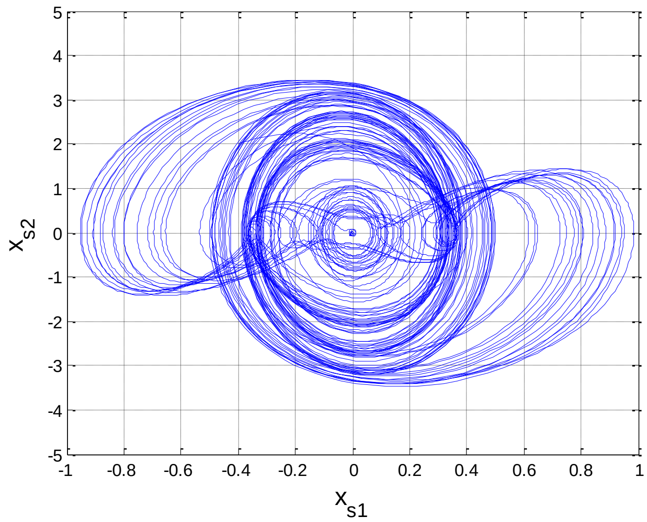


Figure 5.3: Synchronization errors e_1 and e_2 without controller

Figure 5.4 shows the phase portrait of the master and slave gyro systems, when control signal is applied for the synchronization. Phase portrait of the slave system in Figure 5.4(b) is following the trajectory of the master system of Figure 5.4(a), that shows the synchronized behavior among master and slave systems in the presence of proposed state-feedback control law. Figure 5.5 shows synchronization error plots between two states of master and slave systems, where $e_1(t)$ in solid line and $e_2(t)$ in dotted line. It shows that error converges to zero and in small amount of time synchronization between master and slave systems is established.



(a)



(b)

Figure 5.4: Phase portrait of the gyro systems with controller, (a) master system, (b) slave system.

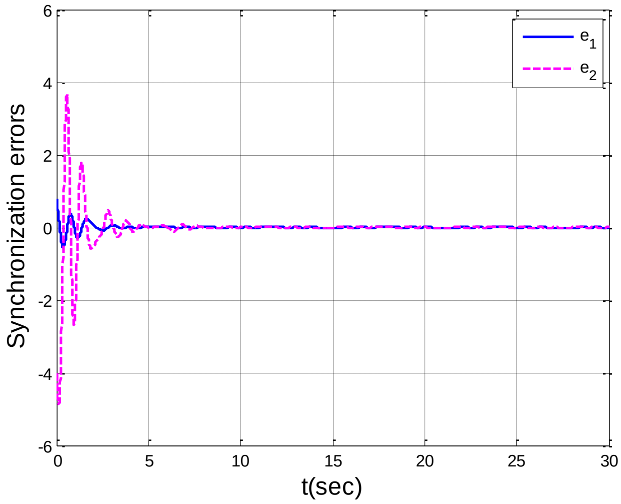


Figure 5.5: Synchronization errors $e_1(t)$ *and* $e_2(t)$ *with controller.*

5.7 Summary

In this chapter, synchronization problem for response and drive systems subject to time-varying input delay and slope-restricted input nonlinearity is investigated. To attain the delay-dependent synchronization between the master and the slave systems, Lyapunov stability theory for time-delay systems is exploited. An advance LPV approach is inferred to transform complex input nonlinearity into simple LPV realization, which is helpful to derive sufficient conditions for synchronization of such nonlinearities. Different tools and techniques are implied like Schur complement, delay-dependent stability criteria, Jensen's inequality, delay derivative bounds and linear parameter varying approach. A triple integral based Lyapunov-Krasovskii functional is constructed for nonlinear time-delay systems and an advance LMI-based synchronization methodology is developed.

LMI-based approach for synchronization under the constraints like time-delay and slope-restricted input nonlinearity is provided by designing a simple state-feedback control law that ensures the asymptotic convergence of synchronization error to the origin. Proposed control law is uncomplicated in design and straight forward for implementation compared to existing techniques like adaptive control and sliding mode techniques. The proposed technique is also capable of handling dynamics with time-varying input delay. Moreover proposed scheme is useful for both small and large interval of input delay. Time-varying input delay is treated using LPV approach and advanced delay-dependent technique is provided.

Proposed technique for synchronization of nonlinear master and slave systems is simple, reliable and less conservative. In future, it can be extended for state delay, output delay and to handle the saturation in nonlinear systems. In the end, a numerical example of gyro systems is illustrated to show the effectiveness of proposed synchronization technique.

Chapter 6

DELAY-RANGE-DEPENDENT SYNCHRONIZATION

6.1 Overview

In Chapter 5, a delay-dependent synchronization criterion was proposed for nonlinear time-delay systems considering the slope-restricted input nonlinearities and input time-delay. A simple state-feedback control law was design for convergence of the error to the origin and attains the asymptotic synchronization of time-delay nonlinear systems.

In this chapter, a less conservative delay-range-dependent synchronization scheme for small and large input delay intervals is derived. A triple integral based Lyapunov-Krasovskii (LK) functional is constructed to derive the sufficient conditions for synchronization of nonlinear time-delay systems. Input time-delay nonlinearity is transformed into an equivalent LPV-realization that leads to design a simple state feedback control law. Then Jensen’s inequality and rigorous algebraic manipulation are incorporated to derive a set of LMIs to compute a controller gain for synchronization. Couples of Theorems are derived to provide the synchronization conditions by assuming the value of controller gain matrix in first Theorem, whereas second Theorem provides the advantage of computing the controller gain matrix according to the varying time-delay. To deal with complex nonlinear terms, cone complementary linearization is provided that can simplify these nonlinear identities and provides simple solution. In the end, a numerical simulation is illustrated to show the effectiveness of proposed synchronization scheme.

A simple state-feedback controller based delay-range-dependent synchronization of nonlinear master and slave systems, is proposed by utilizing the LPV approach. Conventional techniques for synchronization of nonlinear systems implied adaptive control, but demonstrating of controller gain is computationally complex, under the constraints of time-varying delay. Here LPV approach is adopted to handle the input nonlinearity and input varying time-delay by transforming these nonlinearities into a LPV realization. LPV approach provides the advantage of dealing with input nonlinearities and provides a simple controller design by relaxing the adaptive terms, compared to the existing works on synchronization of nonlinear time-delay systems under input slope-restricted nonlinearity.

These days, investigation of the problem of synchronization of nonlinear time-delay systems is popular among the research communities of different disciplines due to its numerous applications in the field of engineering and sciences. The common applications include synchronization of micro-grid systems, biomedical systems, chemical processes and synchronization of multiple robots [16-40]. A meaningful research is devoted to address the synchronization of nonlinear systems under different constraints like disturbance, time-delay, uncertainties, saturation and dead zone etc. [5-7], [9], [11], [15], [21], [23], [25], [56-66], [77]. These parameters are source of instability and degradation of performance of nonlinear systems. Therefore, different strategies such as PID control [24], sliding mode control [6], [12], adaptive control [1], [4], [5], observer-based methodology [9]-[11] and linear feedback controller [7], [8], [22], [57] are developed to address the problem of synchronization subject to different parameters. However, still challenging tasks are available for researcher to investigate and propose the solutions to improve the performance of the nonlinear systems.

Synchronization of time-delay nonlinear systems is considered by different authors [56-60], by inferring the various control strategies such as delay-dependent, delay-independent and delay-range-dependent. Incorporation of slope-restricted nonlinearities is important in studying synchronization controller synthesis for nonlinear systems under uncertain inputs [23], [55], [62]-[66]. Sliding mode control strategies for nonlinear gyroscopes and unified second order complex oscillatory systems with input nonlinearities are explored in [23] and [55]. In [62]-[63], adaptive control and H∞ control strategies for achieving coherent behavior of two uncertain systems under unknown dynamics and perturbations are formulated. Some advanced studies concerning robust, sliding mode and adaptive controller design for synchronization of general forms of two different nonlinear or chaotic systems under uncertainties and perturbations have been taken into account in [64]-[66]. However to the best of our knowledge, feedback based controller design strategy of delay-range-dependent synchronization of nonlinear master and slave systems under the constraints of input time-delay and slope-restricted input nonlinearities has not been reported so far in the literature.

This Chapter is organized as follows; the next Section is comprised of dynamics of the nonlinear master and slave systems along with necessary assumptions and Lemmas. Section 3 is about the controller design. Theorem 6.1 having set of LMIs is derived in Section 4 and Theorem 6.2 along with its proof is provided in Section 5. Cone complementary linearization is discussed in Section 6 and in the end a numerical example of gyro systems is provided.

6.2 Systems Description and Preliminaries

Delay-range-dependent synchronization of nonlinear master and slave systems with input delay and slope-restricted input nonlinearity is considered in this Chapter. Dynamics of the master and slave systems are similar as considered in previous Chapter, whereas a less conservative delay-range-dependent technique is provided rather than delay-dependent technique. Master and slave systems are described as:

$$\dot{x}_m(t) = Ax_m(t) + f(x_m(t), t), \qquad \textit{(Eq 6.1)}$$

$$\dot{x}_s(t) = Ax_s(t) + f(x_s(t), t) + B\varphi(u(t-\tau(t))), \qquad \textit{(Eq 6.2)}$$

where $x_m \in R^n$ and $x_s \in R^n$ represents the states of the master and slave systems, respectively. $A \in R^{n\times n}$ and $B \in R^{n\times p}$ represents the linear matrix with known constant entries and similar in both master and slave systems. Vectors $f(x_m(t), t) \in R^n$ and $f(x_s(t), t) \in R^n$ represent time-varying continuous nonlinearities in master system and slave system, respectively. $u \in R^p$ denotes the control input and $\varphi(u) \in R^p$ represents the continuous time slope-restricted input nonlinearity. $\tau(t)$ denotes the time-varying input delay, which is differentiate-able function with respect to time. It is assumed that time-varying delay satisfy the following inequality

$$0 < \tau_1 \le \tau(t) \le \tau_2 \qquad \textit{(Eq 6.3)}$$

and condition posed on delay derivative as

$$\dot{\tau}(t) \le \eta \,. \qquad \textit{(Eq 6.4)}$$

The condition provided in Eq. 6.3 reflects the delay is varying in specific range and condition of Eq. 6.4 described the delay derivative bound.

Objectives of the study are to design an appropriate control law that provides synchronized behavior among the states of the master and slave systems by converging the synchronization error to a sphere. To resolve the synchronization dilemma, let us define the difference between the master system and slave system as an error $e(t) = x_m(t) - x_s(t)$ and taking its time-derivative, it reveals

$$\dot{e}(t) = \dot{x}_m(t) - \dot{x}_s(t) \,. \qquad \textit{(Eq 6.5)}$$

Now incorporating the values of $\dot{x}_m(t)$ and $\dot{x}_s(t)$ in Eq. 6.5, error dynamics is depicts

$$\dot{e}(t) = Ax_m(t) - Ax_s(t) - B\varphi(u(t-\tau(t))) + f(x_m(t),t) - f(x_s(t),t) \,. \qquad \textit{(Eq 6.6)}$$

Incorporating the error definition $e(t) = x_m(t) - x_s(t)$, it yields

$$\dot{e}(t) = Ae(t) - B\varphi(u(t-\tau(t))) + f(x_m(t),t) - f(x_s(t),t) \,. \qquad \textit{(Eq 6.7)}$$

To further simplify the error dynamics of Eq. 6.7, difference of nonlinearities of master and slave systems can be written as $\Psi(x_m, x_s) = f(x_m(t),t) - f(x_s(t),t)$. It gives

$$\dot{e}(t) = Ae(t) - B\varphi(u(t-\tau(t))) + \Psi(x_m, x_s) \,. \qquad \textit{(Eq 6.8)}$$

Input time-delay and slope-restricted input nonlinearity is considered in dynamical systems described in Eq. 6.1 and Eq. 6.2. There are few research works and control techniques developed for synchronization of nonlinear systems under slope-restricted input nonlinearity [5]-[6], [63-66], however input time-delay is not considered in traditional research. Input time-delays are unavoidable in most of the physical systems. For example an actuator in real-time system may be placed far away from the system which causes to produce the input time-delay in the systems dynamics. So, ignoring the input time-delay in dynamics of such nonlinear systems may lead to non-coherent behavior as well as cause the performance degradation of overall closed-loop system.

6.2.1 Assumptions

To attain the sufficient conditions for synchronization of nonlinear time-delay systems and simplifying the problem, following assumptions for input nonlinearity and nonlinear function are described.

3. It is assumed that input nonlinearity $\varphi(u)$ is a continuous function, which satisfies the following mathematical condition

$$l_m u(t) \leq \varphi(u) \leq l_M u(t), \qquad \textit{(Eq 6.9)}$$

where $l_M = diag(l_{M,1}, l_{M,2}, \ldots, l_{M,p}) > 0$ and $l_m = diag(l_{m,1}, l_{m,2}, \ldots, l_{m,p}) > 0$.

4. Nonlinearities $f(x(t),t)$ in master and slave systems are assumed to be Lipschitz and of continuous nature, validating the following mathematical inequality called Lipschitz condition

$$\| f(x_m(t),t) - f(x_s(t),t)\| \leq \Omega \| x_m(t) - x_s(t)\|, \qquad \textit{(Eq 6.10)}$$

where $\Omega > 0$ is the Lipschitz constant, for all $x_m, x_s \in R^n$, and $\|\cdot\|$ represents the Euclidian norm.

Detailed derivation of assumption provided in Eq. 6.9 can be seen from [68-69]. The assumptions can be used to provide a road map for controller design for synchronization of nonlinear system subject to input nonlinearity.

6.2.2 Lemmas

Following two Lemmas are provided, helpful to derive the main Theorems.

1. For a constant matrix $Z = Z^T > 0$ and scalars $\tau_2 \geq \tau_1 \geq 0$, the following inequality holds:

$$-\tau_{12} \int_{t-\tau_2}^{t-\tau_1} e^T(\vartheta) Z e(\vartheta) d\vartheta \leq -\int_{t-\tau_2}^{t-\tau_1} e^T(\vartheta) d\vartheta Z \int_{t-\tau_2}^{t-\tau_1} e(\vartheta) d\vartheta, \qquad \textit{(Eq 6.11)}$$

where $\tau_{12} = \tau_2 - \tau_1$, given that the concerned integral terms are well-defined ([117]-[118]). In a similar fashion, we can write the integral inequality for $\tilde{\tau} = 0.5(\tau_2^2 - \tau_1^2)$ as

$$-\tilde{\tau}\int_{-\tau_2}^{-\tau_1}\int_{t+\alpha}^{t} e^T(\vartheta)Ze(\vartheta)d\vartheta d\alpha \le -\int_{-\tau_2}^{-\tau_1}\int_{t+\alpha}^{t} e^T(\vartheta)d\vartheta d\alpha Z\int_{-\tau_2}^{-\tau_1}\int_{t+\alpha}^{t} e(\vartheta)d\vartheta d\alpha. \quad \textit{(Eq 6.12)}$$

3. For a given matrix

$$\phi = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{12}^T & \phi_{22} \end{bmatrix}, \quad \textit{(Eq 6.13)}$$

if $\phi_{11} = \phi_{11}^T$ and $\phi_{11} = \phi_{11}^T$, the following conditions are equivalent (see, for instance, [121]):

(a) $\phi < 0$,

(b) $\phi_{22} < 0$, and $\phi_{11} - \phi_{12}\phi_{22}^{-1}\phi_{12}^T < 0$.

The key idea is to develop a novel synchronization scheme for given master system of Eq. 6.1 and slave system of Eq. 6.2, subject to slope-restricted input nonlinearity and under the constraints of input time-delay. Coherent behavior between master and slave systems can be accomplished by choosing a suitable controller $u(t)$ that ensures the convergence of error dynamics as $\lim_{t\to\infty} \| e(t) \| = 0$.

6.3 Controller Design

The function $\varphi(u(t-\tau(t)))$ can be represented as an LPV function. LPV approach is discussed in detail earlier Chapter under sub section 6.2. By applying the same LPV realization, input nonlinearity $\varphi(u(t-\tau(t)))$ is obtained as below

$$\varphi(u(t-\tau(t))) = \wp(t)u(t-\tau(t)), \qquad l_m \le \wp(t) \le l_M . \quad \textit{(Eq 6.14)}$$

The time-varying diagonal matrix function $\wp(t)$ belongs to the set

$$\mathbb{Q} = \left\{ \Upsilon \in R^{p\times p} : \Upsilon_{ij} = 0, i \ne j, \Upsilon_{ii} \in \left\{ l_{m,i}, l_{M,i} \right\} \right\}. \quad \textit{(Eq 6.15)}$$

The input nonlinearity $\varphi(u(t-\tau(t)))$ has been transformed into an equivalent LPV realization. By virtue of LPV technique the error dynamics of Eq. 6.8 can be rewritten into an equivalent LPV representation as

$$\dot{e}(t) = Ae(t) - B\Upsilon u(t-\tau(t)) + \Psi(x_m, x_s), \ \forall \Upsilon \in \mathbb{Q} . \quad \textit{(Eq 6.16)}$$

LPV approach [69-70], is useful to handle the input nonlinearity $\varphi(u(t-\tau(t)))$, and also it provides the advantage of simple controller design by relaxing the adaptive terms in the presence of additional constraints like input time-delay and slope-restricted input nonlinearity.

Defining an argument vector as

$$\lambda(t) = col\{e(t), e(t-\tau(t)), e(t-\tau_1), e(t-\tau_2), \dot{e}(t-\tau_1), \dot{e}(t-\tau_2), \int_{t-\tau_1}^{t} e(\vartheta)d\vartheta, \int_{t-\tau(t)}^{t-\tau_1} e(\vartheta)d\vartheta, \int_{t-\tau_2}^{t-\tau(t)} e(\vartheta)d\vartheta, \Psi(x_m, x_s)\}. \quad \textit{(Eq 6.17)}$$

Let $\Im_i (i = 1, 2, 3, ..., 10)$ characterizes a matrix by swapping its *i*th term of $n \times 10n$ zero matrix along with an identity matrix, for example, to represent the fifth entry in argument matrix form, $\Im_5^T = [0 \quad 0 \quad 0 \quad 0 \quad I \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]$. Argument matrix $\lambda(t)$ contains ten elements and to represent the different entries of argument matrix, $\Im_i (i = 1, 2, 3, ..., 10)$ can be used. For example fifth entry of this argument matrix is $\Im_5 \lambda(t) = \dot{e}(t-\tau_1(t))$. A simple state feedback control law is proposed to attain the synchronization of nonlinear systems. Dynamics of the proposed state feedback control law is given by

$$u(t) = Ke(t), \quad \textit{(Eq 6.18)}$$

where $K \in R^{p \times n}$ represents a feedback controller gain matrix with constant entries, which can be computed by LMI based technique. Now by incorporating the control law into Eq 6.16, it gives

$$\dot{e}(t) = Ae(t) - B\Upsilon Ke(t-\tau(t)) + \Psi(x_m, x_s), \ \forall \Upsilon \in \mathbb{Q}. \quad \textit{(Eq 6.19)}$$

Following theorems are derived by virtue of proposed state feedback control law, $u(t) = Ke(t)$, that guarantee the convergence of synchronization error $e(t)$ to the origin.

6.4 Theorem 6.1

Let the master system and the slave system, described in Eq. 6.1 and Eq. 6.2, respectively, satisfy the assumptions provided in Eq. 6.9, Eq. 6.10, input delay range bounds provided in Eq. 6.3 and delay derivative bound of Eq. 6.4. Suppose there exist symmetric matrices $P > 0$, $S_i > 0$, $Q_i > 0$, $H_j > 0$ and $Z_k > 0$ of appropriate dimensions, for $i=1,2,\ \ j=1,2,3,4,5$, and $k=1,2,3,4$, such that the following sets of LMIs holds

$$\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ * & \Phi_{22} \end{bmatrix} < 0, \forall \Upsilon \in \mathbb{Q}, \qquad \textit{(Eq 6.20)}$$

$$\hat{\Phi} = \begin{bmatrix} \hat{\Phi}_{11} & \Phi_{12} \\ * & \Phi_{22} \end{bmatrix} < 0, \forall \Upsilon \in \mathbb{Q}, \qquad \textit{(Eq 6.21)}$$

for a given matrix $K \in R^{p\times n}$,

where

$$\Phi_{11} = \begin{bmatrix} \aleph_{11} & -PB\Upsilon K & Z_1 & 0 & 0 & 0 & \tau_1 S_1 & \tau_{12} S_2 & \tau_{12} S_2 & P \\ * & \aleph_{22} & Z_2 & 2Z_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & \aleph_{33} & 0 & Q_1 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \aleph_{44} & 0 & Q_2 & 0 & 0 & 0 & 0 \\ * & * & * & * & \aleph_{55} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -H_5 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & \aleph_{77} & 0 & 0 & 0 \\ * & * & * & * & * & * & * & \aleph_{88} & -S_2 & 0 \\ * & * & * & * & * & * & * & * & \aleph_{99} & 0 \\ * & * & * & * & * & * & * & * & * & -I \end{bmatrix},$$

$$\Phi_{12} = \begin{bmatrix} A^T H_4 & \tau_1 A^T Z_1 & \tau_{12} A^T Z_2 & \frac{\tau_1^2}{2} A^T S_1 & \tilde{\tau} A^T S_2 \\ -\mathrm{E} H_4 & -\tau_1 \mathrm{E} Z_1 & -\tau_{12} \mathrm{E} Z_2 & -\frac{\tau_1^2}{2} \mathrm{E} S_1 & -\tilde{\tau} \mathrm{E} S_2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ H_4 & \tau_1 Z_1 & \tau_{12} Z_2 & \frac{\tau_1^2}{2} S_1 & \tilde{\tau} S_2 \end{bmatrix},$$

$\Phi_{22} = -diag(H_4, Z_1, Z_2, S_1, S_2)$, and

$$\hat{\Phi}_{11} = \begin{bmatrix} \aleph_{11} & -PB\Upsilon K & Z_1 & 0 & 0 & 0 & \tau_1 S_1 & \tau_{12} S_2 & \tau_{12} S_2 & P \\ * & \aleph_{22} & 2Z_2 & Z_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & \hat{\aleph}_{33} & 0 & Q_1 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \hat{\aleph}_{44} & 0 & Q_2 & 0 & 0 & 0 & 0 \\ * & * & * & * & \aleph_{55} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -H_5 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & \aleph_{77} & 0 & 0 & 0 \\ * & * & * & * & * & * & * & \hat{\aleph}_{88} & -S_2 & 0 \\ * & * & * & * & * & * & * & * & \hat{\aleph}_{99} & 0 \\ * & * & * & * & * & * & * & * & * & -I \end{bmatrix},$$

$$\aleph_{11} = H_1 + \tau_1^2 Z_3 + \tau_{12}^2 Z_4 - Z_1 - \tau_1^2 S_1 - \tau_{12}^2 S_2 + A^T P + PA + \Omega^T \Omega,$$
$$\aleph_{22} = -3Z_2 - (1-\eta) H_3,$$
$$\aleph_{33} = -Z_1 - Z_2 - H_1 + H_2 + H_3,$$
$$\aleph_{44} = -2Z_2 - H_2,$$
$$\aleph_{55} = -H_4 + H_5,$$
$$\aleph_{77} = -S_1 - Z_3,$$
$$\aleph_{88} = -Z_4 - S_2,$$
$$\aleph_{99} = -2Z_4 - S_2,$$
$$\mathrm{E} = K^T \Upsilon^T B^T,$$
$$\hat{\aleph}_{33} = -Z_1 - 2Z_2 - H_1 + H_2 + H_3,$$
$$\hat{\aleph}_{44} = -Z_2 - H_2,$$
$$\hat{\aleph}_{88} = -2Z_4 - S_2,$$
$$\hat{\aleph}_{99} = -Z_4 - S_2$$

and * is used for the symmetric terms of matrix inequalities. Then, provided state feedback controller of Eq. 6.18 ensures the asymptotic synchronization of the master and the slave systems under input delay and slope-restricted input nonlinearity.

6.4.1 Proof of Theorem 6.1

A triple integral based positive-definite Lyapunov-Krasovskii functional is constructed, to provide the proof of Theorem 6.1, given by

$$
\begin{aligned}
V(e_t) &= e^T(t)Pe(t) + e^T(t-\tau_1)Q_1e(t-\tau_1) + e^T(t-\tau_2)Q_2e(t-\tau_2) \\
&+\int_{t-\tau_1}^{t} e^T(\vartheta)H_1e(\vartheta)d\vartheta + \int_{t-\tau_2}^{t-\tau_1} e^T(\vartheta)H_2e(\vartheta)d\vartheta + \int_{t-\tau(t)}^{t-\tau_1} e^T(\vartheta)H_3 \\
&\times e(\vartheta)d\vartheta + \int_{t-\tau_1}^{t} \dot{e}^T(\vartheta)H_4\dot{e}(\vartheta)d\vartheta + \int_{t-\tau_2}^{t-\tau_1} \dot{e}^T(\vartheta)H_5\dot{e}(\vartheta)d\vartheta \\
&+\int_{-\tau_1}^{0}\int_{t+\alpha}^{t} \tau_1\dot{e}^T(\vartheta)Z_1\dot{e}(\vartheta)d\vartheta d\alpha + \int_{-\tau_2}^{-\tau_1}\int_{t+\alpha}^{t} \tau_{12}\dot{e}^T(\vartheta)Z_2\dot{e}(\vartheta)d\vartheta d\alpha \qquad \textit{(Eq6.22)} \\
&+\int_{-\tau_1}^{0}\int_{t+\alpha}^{t} \tau_1 e^T(\vartheta)Z_3e(\vartheta)d\vartheta d\alpha + \int_{-\tau_2}^{-\tau_1}\int_{t+\alpha}^{t} \tau_{12}e^T(\vartheta)Z_4e(\vartheta)d\vartheta d\alpha \\
&+\int_{-\tau_1}^{0}\int_{\alpha}^{0}\int_{t+\gamma}^{t} \frac{\tau_1^2}{2}\dot{e}^T(\vartheta)S_1\dot{e}(\vartheta)d\vartheta d\gamma d\alpha \\
&+\int_{-\tau_2}^{-\tau_1}\int_{\alpha}^{0}\int_{t+\gamma}^{t} \tilde{\tau}\dot{e}^T(\vartheta)S_2\dot{e}(\vartheta)d\vartheta d\gamma d\alpha.
\end{aligned}
$$

Taking the time derivative of Eq.6.22, we obtain

$$
\begin{aligned}
\dot{V}(e_t) &= \dot{e}^T(t)Pe(t) + e^T(t)P\dot{e}(t) + \dot{e}^T(t-\tau_1)Q_1e(t-\tau_1) + e^T(t-\tau_1)Q_1 \\
&\times\dot{e}(t-\tau_1) + \dot{e}^T(t-\tau_2)Q_2e(t-\tau_2) + e^T(t-\tau_2)Q_2\dot{e}(t-\tau_2) + e^T(t) \\
&\times H_1e(t) - e^T(t-\tau_1)H_1e(t-\tau_1) + e^T(t-\tau_1)H_2e(t-\tau_1) - e^T(t-\tau_2) \\
&\times H_2e(t-\tau_2) + e^T(t-\tau_1)H_3e(t-\tau_1) - (1-\dot{\tau})e^T(t-\tau(t))H_3e(t-\tau(t)) \\
&+\dot{e}^T(t)H_4\dot{e}(t) - \dot{e}^T(t-\tau_1)H_4\dot{e}(t-\tau_1) + \dot{e}^T(t-\tau_1)H_5\dot{e}(t-\tau_1) \\
&-\dot{e}^T(t-\tau_2)H_5\dot{e}(t-\tau_2) + \tau_1^2\dot{e}^T(t)Z_1\dot{e}(t) - \int_{t-\tau_1}^{t}\tau_1\dot{e}^T(\vartheta)Z_1\dot{e}(\vartheta)d\vartheta \qquad \textit{(Eq 6.23)} \\
&+\tau_{12}^2\dot{e}^T(t)Z_2\dot{e}(t) - \int_{t-\tau_2}^{t-\tau_1}\tau_{12}\dot{e}^T(\vartheta)Z_2\dot{e}(\vartheta)d\vartheta + \tau_1^2e^T(t)Z_3e(t) \\
&-\int_{t-\tau_1}^{t}\tau_1e^T(\vartheta)Z_3e(\vartheta)d\vartheta + \tau_{12}^2e^T(t)Z_4e(t) - \int_{t-\tau_2}^{t-\tau_1}\tau_{12}e^T(\vartheta)Z_4e(\vartheta)d\vartheta \\
&+\frac{\tau_1^4}{4}\dot{e}^T(t)S_1\dot{e}(t) - \frac{\tau_1^2}{2}\int_{-\tau_1}^{0}\int_{t+\alpha}^{t}\dot{e}^T(\vartheta)S_1\dot{e}(\vartheta)d\vartheta d\alpha \\
&+\tilde{\tau}^2\dot{e}^T(t)S_2\dot{e}(t) - \tilde{\tau}\int_{-\tau_2}^{\tau_1}\int_{t+\alpha}^{t}\dot{e}^T(\vartheta)S_2\dot{e}(\vartheta)d\vartheta d\alpha.
\end{aligned}
$$

To simplify, let us define

$$
\hat{M} = H_4 + \tau_1^2Z_1 + \tau_{12}^2Z_2 + \frac{\tau_1^4}{4}S_1 + \tilde{\tau}^2S_2. \qquad \textit{(Eq 6.24)}
$$

Now incorporating the Eq. 6.24 and also using time-delay derivative bound $\dot{\tau}(t) \le \eta$, Eq. 6.23 reveals

$$\begin{aligned}\dot{V}(e_t) \le\; & e^T(t)P\dot{e}(t) + \dot{e}^T(t)Pe(t) + \dot{e}^T(t)\hat{M}\dot{e}(t) + \dot{e}^T(t-\tau_1)Q_1 e(t-\tau_1) \\ & + e^T(t-\tau_1)Q_1\dot{e}(t-\tau_1) + \dot{e}^T(t-\tau_2)Q_2 e(t-\tau_2) + e^T(t-\tau_2)Q_2\dot{e}(t-\tau_2) \\ & + e^T(t)(H_1 + \tau_1^2 Z_3 + \tau_{12}^2 Z_4)e(t) - e^T(t-\tau_1)(H_1 - H_2 - H_3)e(t-\tau_1) \\ & - e^T(t-\tau_2)H_2 e(t-\tau_2) - (1-\eta)e^T(t-\tau(t))H_3 e(t-\tau(t)) \\ & - \dot{e}^T(t-\tau_1)(H_4 - H_5)\dot{e}(t-\tau_1) - \dot{e}^T(t-\tau_2)H_5\dot{e}(t-\tau_2) \\ & - \int_{t-\tau_1}^{t} \tau_1 \dot{e}^T(\vartheta)Z_1\dot{e}(\vartheta)d\vartheta - \int_{t-\tau_2}^{t-\tau_1} \tau_{12}\dot{e}^T(\vartheta)Z_2\dot{e}(\vartheta)d\vartheta \\ & - \int_{t-\tau_1}^{t} \tau_1 e^T(\vartheta)Z_3 e(\vartheta)d\vartheta - \int_{t-\tau_2}^{t-\tau_1} \tau_{12} e^T(\vartheta)Z_4 e(\vartheta)d\vartheta \\ & - \frac{\tau_1^2}{2}\int_{-\tau_1}^{0}\int_{t+\alpha}^{t} \dot{e}^T(\vartheta)S_1\dot{e}(\vartheta)d\vartheta d\alpha - \tilde{\tau}\int_{-\tau_2}^{\tau_1}\int_{t+\alpha}^{t} \dot{e}^T(\vartheta)S_2\dot{e}(\vartheta)d\vartheta d\alpha.\end{aligned} \qquad \textit{(Eq 6.25)}$$

Now by applying the Jensen's inequality defined in Lemma 1, following inequalities can be obtained.

$$-\tau_1\int_{t-\tau_1}^{t} \dot{e}^T(\vartheta)Z_1\dot{e}(\vartheta)d\vartheta \le -\lambda^T(t)[(\Im_1^T - \Im_3^T)Z_1(\Im_1 - \Im_3)]\lambda(t)\,, \qquad \textit{(Eq 6.26)}$$

$$-\tau_1\int_{t-\tau_1}^{t} e^T(\vartheta)Z_3 e(\vartheta)d\vartheta \le -\lambda^T(t)\Im_7^T Z_3 \Im_7 \lambda(t)\,, \qquad \textit{(Eq 6.27)}$$

$$-\frac{\tau_1^2}{2}\int_{-\tau_1}^{0}\int_{t+\alpha}^{t} \dot{e}^T(\vartheta)S_1\dot{e}(\vartheta)d\vartheta d\alpha \le -\lambda^T(t)[(\tau_1\Im_1^T - \Im_7^T)S_1(\tau_1\Im_1 - \Im_7)]\lambda(t)\,, \qquad \textit{(Eq 6.28)}$$

$$\begin{aligned}-\tilde{\tau}\int_{-\tau_2}^{-\tau_1}\int_{t+\alpha}^{t} \dot{e}^T(\vartheta)S_2\dot{e}(\vartheta)d\vartheta d\alpha \le\; & -\lambda^T(t)[(\tau_{12}\Im_1^T - \Im_8^T - \Im_9^T)S_2 \\ & \times(\tau_{12}\Im_1 - \Im_8 - \Im_9)]\lambda(t).\end{aligned} \qquad \textit{(Eq 6.29)}$$

For further simplification, we have

$$\begin{aligned}-\int_{t-\tau_2}^{t-\tau_1} \tau_{12}\dot{e}^T(\vartheta)Z_2\dot{e}(\vartheta)d\vartheta = & -\int_{t-\tau_2}^{t-d(t)} \tau_{12}\dot{e}^T(\vartheta)Z_2\dot{e}(\vartheta)d\vartheta \\ & -\int_{t-d(t)}^{t-\tau_1} \tau_{12}\dot{e}^T(\vartheta)Z_2\dot{e}(\vartheta)d\vartheta.\end{aligned} \qquad \textit{(Eq 6.30)}$$

By defining the inequalities $-\tau_{12} \le -(\tau_2 - d(t))$ and $-\tau_{12} \le (d(t) - \tau_1)$ and further, assigning $\chi = (d(t) - \tau_1)/\tau_{12}$ give

$$\begin{aligned}-\int_{t-\tau_2}^{t-\tau_1} \tau_{12}\dot{e}^T(\vartheta)Z_2\dot{e}(\vartheta)d\vartheta \le\; & -\lambda^T(t)(\Im_2^T - \Im_4^T)Z_2(\Im_2 - \Im_4)\lambda(t) \\ & -\lambda^T(t)(\Im_3^T - \Im_2^T)Z_2(\Im_3 - \Im_2)\lambda(t) \\ & -\chi\lambda^T(t)(\Im_2^T - \Im_4^T)Z_2(\Im_2 - \Im_4)\lambda(t) \\ & -(1-\chi)\lambda^T(t)(\Im_3^T - \Im_2^T)Z_2(\Im_3 - \Im_2)\lambda(t),\end{aligned} \qquad \textit{(Eq 6.31)}$$

$$-\int_{t-\tau_2}^{t-\tau_1}\tau_{12}e^T(\vartheta)Z_4e(\vartheta)d\vartheta \le -\lambda^T(t)[\Im_8^T Z_4\Im_8+\Im_9^T Z_4\Im_9]\lambda(t)$$
$$-\chi\lambda^T(t)(\Im_9^T Z_4\Im_9)\lambda(t)-(1-\chi)\lambda^T(t) \quad \textit{(Eq 6.32)}$$
$$\times(\Im_8^T Z_4\Im_8)\lambda(t).$$

Now incorporating the inequalities provided in Eq 6.26 to Eq. 6.32 into Eq. 6.25 and further simplification give

$$\begin{aligned}\dot V(e_t) \le\ & e^T(t)P\dot e(t)+\dot e^T(t)Pe(t)+\dot e^T(t)\hat M\dot e(t)+\lambda^T(t)[\Im_3^T Q_1\Im_5\\
&+\Im_5^T Q_1\Im_3+\Im_4^T Q_2\Im_6+\Im_6^T Q_2\Im_4+\Im_1^T(H_1+\tau_1^2Z_3+\tau_{12}^2Z_4)\Im_1\\
&-(1-\eta)\Im_2^T H_3\Im_2-\Im_3^T(H_1-H_2-H_3)\Im_3-\Im_4^T H_2\Im_4-\Im_5^T\\
&\times(H_4-H_5)\Im_5-\Im_6^T H_5\Im_6-(\Im_1^T-\Im_3^T)Z_1(\Im_1-\Im_3)-(\Im_2^T-\Im_4^T)\\
&\times Z_2(\Im_2-\Im_4)-(\Im_3^T-\Im_2^T)Z_2(\Im_3-\Im_2)-\chi(\Im_2^T-\Im_4^T)Z_2(\Im_2-\Im_4)\\
&-(1-\chi)(\Im_3^T-\Im_2^T)Z_2(\Im_3-\Im_2)-\Im_7^T Z_3\Im_7-\Im_8^T Z_4\Im_8-\Im_9^T Z_4\Im_9\\
&-\chi\Im_9^T Z_4\Im_9-(1-\chi)\Im_8^T Z_4\Im_8-(\tau_1\Im_1^T-\Im_7^T)S_1(\tau_1\Im_1-\Im_7)\\
&-(\tau_{12}\Im_1^T-\Im_8^T-\Im_9^T)S_2(\tau_{12}\Im_1-\Im_8-\Im_9)]\lambda(t).\end{aligned} \quad \textit{(Eq 6.33)}$$

Let us introduce the error dynamics $\dot e(t)=Ae(t)-B\Upsilon Ke(t-\tau(t))+\Psi(x_m,x_s)$, it yields

$$\begin{aligned}\dot V(e_t) \le\ & e^T(t)P[Ae(t)-B\Upsilon Ke(t-\tau)+\Psi(x_m,x_s)]+[Ae(t)-B\Upsilon\\
&\times Ke(t-\tau)+\Psi(x_m,x_s)]^T Pe(t)+[Ae(t)-B\Upsilon Ke(t-\tau)\\
&+\Psi(x_m,x_s)]^T\hat M[Ae(t)-B\Upsilon Ke(t-\tau)+\Psi(x_m,x_s)]\\
&+\lambda^T(t)[\Im_3^T Q_1\Im_5+\Im_5^T Q_1\Im_3+\Im_4^T Q_2\Im_6+\Im_6^T Q_2\Im_4+\Im_1^T(H_1\\
&+\tau_1^2Z_3+\tau_{12}^2Z_4)\Im_1-(1-\eta)\Im_2^T H_3\Im_2-\Im_3^T(H_1-H_2-H_3)\Im_3\\
&-\Im_4^T H_2\Im_4-\Im_5^T(H_4-H_5)\Im_5-\Im_6^T H_5\Im_6-(\Im_1^T-\Im_3^T)Z_1\\
&\times(\Im_1-\Im_3)-(\Im_2^T-\Im_4^T)Z_2(\Im_2-\Im_4)-(\Im_3^T-\Im_2^T)Z_2(\Im_3-\Im_2)\\
&-\chi(\Im_2^T-\Im_4^T)Z_2(\Im_2-\Im_4)-(1-\chi)(\Im_3^T-\Im_2^T)Z_2(\Im_3-\Im_2)\\
&-\Im_7^T Z_3\Im_7-\Im_8^T Z_4\Im_8-\Im_9^T Z_4\Im_9-\chi\Im_9^T Z_4\Im_9-(1-\chi)\Im_8^T Z_4\Im_8\\
&-(\tau_1\Im_1^T-\Im_7^T)\times S_1(\tau_1\Im_1-\Im_7)-(\tau_{12}\Im_1^T-\Im_8^T-\Im_9^T)S_2(\tau_{12}\Im_1\\
&-\Im_8-\Im_9)]\lambda(t).\end{aligned} \quad \textit{(Eq 6.34)}$$

For further simplification, we define

$$\hat A_c=\begin{bmatrix}A & -B\Upsilon K & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I\end{bmatrix}. \quad \textit{(Eq 6.35)}$$

Using the definition of Eq. 6.35 and applying the Lipschitz condition described in Assumption 2, we have

$$
\begin{aligned}
\dot{V}(e_t) \le \; & \lambda^T(t)[(\hat{A}_c^T \hat{M}\hat{A}_c) + \Im_1^T(PA)\Im_1 - \Im_1^T(PB\Upsilon K)\Im_2 + \Im_1^T P\Im_{10} \\
& + \Im_1^T(A^T P)\Im_1 - \Im_2^T(B\Upsilon K)^T P\Im_1 + \Im_{10}^T P\Im_1 + \Im_3^T Q_1\Im_5 + \Im_5^T Q_1\Im_3 \\
& + \Im_4^T Q_2\Im_6 + \Im_6^T Q_2\Im_4 + \Im_1^T(H_1 + \tau_1^2 Z_3 + \tau_{12}^2 Z_4)\Im_1 - (1-\eta)\Im_2^T H_3\Im_2 \\
& - \Im_3^T(H_1 - H_2 - H_3)\Im_3 - \Im_4^T H_2\Im_4 - \Im_5^T(H_4 - H_5)\Im_5 - \Im_6^T H_5\Im_6 \\
& - (\Im_1^T - \Im_3^T)Z_1(\Im_1 - \Im_3) - (\Im_2^T - \Im_4^T)Z_2(\Im_2 - \Im_4) - (\Im_3^T - \Im_2^T) \\
& \times Z_2(\Im_3 - \Im_2) - \chi(\Im_2^T - \Im_4^T)Z_2(\Im_2 - \Im_4) - (1-\chi)(\Im_3^T - \Im_2^T)Z_2 \\
& \times(\Im_3 - \Im_2) - \Im_7^T Z_3\Im_7 - \Im_8^T Z_4\Im_8 - \Im_9^T Z_4\Im_9 - \chi\Im_9^T Z_4\Im_9 - (1-\chi) \\
& \times\Im_8^T Z_4\Im_8 - (\tau_1\Im_1^T - \Im_7^T)S_1(\tau_1\Im_1 - \Im_7) - (\tau_{12}\Im_1^T - \Im_8^T - \Im_9^T) \\
& \times S_2(\tau_{12}\Im_1 - \Im_8 - \Im_9)]\lambda(t).
\end{aligned} \qquad \textit{(Eq 6.36)}
$$

Eq. 6.36 can be rearranged as below

$$
\begin{aligned}
\dot{V}(e_t) \le \; & \lambda^T(t)[(\hat{A}_c^T \hat{M}\hat{A}_c) + \Im_1^T(A^T P + PA + H_1 - Z_1 + \tau_1^2 Z_3 + \tau_{12}^2 Z_4 \\
& - \tau_1^2 S_1 - \tau_{12}^2 S_2)\Im_1 - \Im_1^T(PB\Upsilon K)\Im_2 + \Im_1^T Z_1\Im_3 + \tau_1\Im_1^T S_1\Im_7 \\
& + \tau_{12}\Im_1^T S_2\Im_8 + \tau_{12}\Im_1^T S_2\Im_9 + \Im_1^T P\Im_{10} - \Im_2^T(B\Upsilon K)^T P\Im_1 \\
& - \Im_2^T(2Z_2 + \chi Z_2 + (1-\chi)Z_2 + (1-\eta)H_3)\Im_2 + \Im_2^T Z_2\Im_3 \\
& + \Im_2^T(Z_2 + \chi Z_2)\Im_4 + \Im_3^T Z_1\Im_1 + \Im_3^T(Z_2 + (1-\chi)Z_2)\Im_2 \\
& - \Im_3^T(H_1 - H_2 - H_3 + Z_1 + Z_2 + (1-\chi)Z_2)\Im_3 + \Im_3^T Q_1\Im_5 \\
& + \Im_4^T(Z_2 + \chi Z_2)\Im_2 - \Im_4^T(H_2 + Z_2 + \chi Z_2)\Im_4 + \Im_4^T Q_2\Im_6 \\
& + \Im_5^T Q_1\Im_3 - \Im_5^T(H_4 - H_5)\Im_5 + \Im_6^T Q_2\Im_4 - \Im_6^T H_5\Im_6 \\
& + \tau_1\Im_7^T S_1\Im_1 - \Im_7^T(Z_3 + S_1)\Im_7 + \tau_{12}\Im_8^T S_2\Im_1 \\
& - \Im_8^T(Z_4 + (1-\chi)Z_4 + S_2)\Im_8 - \Im_8^T S_2\Im_9 + \tau_{12}\Im_9^T S_2\Im_1 \\
& - \Im_9^T S_2\Im_8 - \Im_9^T(Z_4 + \chi Z_4 + S_2)\Im_9 + \Im_{10}^T P\Im_1]\lambda(t),
\end{aligned} \qquad \textit{(Eq 6.37)}
$$

Eq. 6.37 can be rewritten in simple form as

$$
\dot{V}(e_t) \le \lambda^T(t)[\chi\varphi + (1-\chi)\hat{\varphi}]\lambda(t), \; \forall \Upsilon \in \mathbb{Q}, \qquad \textit{(Eq 6.38)}
$$

where

$$
\begin{aligned}
\varphi = \; & \lambda^T(t)[(\hat{A}_c^T \hat{M}\hat{A}_c) + \Im_1^T(A^T P + PA + H_1 + \tau_1^2 Z_3 + \tau_{12}^2 Z_4)\Im_1 \\
& - \Im_1^T(PB\Upsilon K)\Im_2 + \Im_1^T P\Im_{10} - \Im_2^T(B\Upsilon K)^T P\Im_1 + \Im_{10}^T P\Im_1 \\
& + \Im_3^T Q_1\Im_5 + \Im_5^T Q_1\Im_3 + \Im_4^T Q_2\Im_6 + \Im_6^T Q_2\Im_4 - \Im_7^T Z_3\Im_7 \\
& - (1-\eta)\Im_2^T H_3\Im_2 - \Im_3^T(H_1 - H_2 - H_3)\Im_3 - \Im_4^T H_2\Im_4 \\
& - \Im_5^T(H_4 - H_5)\Im_5 - \Im_6^T H_5\Im_6 - (\Im_1^T - \Im_3^T)Z_1(\Im_1 - \Im_3) \\
& - 2(\Im_2^T - \Im_4^T)Z_2(\Im_2 - \Im_4) - (\Im_3^T - \Im_2^T)Z_2(\Im_3 - \Im_2) \\
& - \Im_8^T Z_4\Im_8 - 2\Im_9^T Z_4\Im_9 - (\tau_1\Im_1^T - \Im_7^T)S_1(\tau_1\Im_1 - \Im_7) \\
& - (\tau_{12}\Im_1^T - \Im_8^T - \Im_9^T)S_2(\tau_{12}\Im_1 - \Im_8 - \Im_9)]\lambda(t),
\end{aligned}
$$

$$
\begin{aligned}
\hat{\varphi} = {} & \lambda^T(t)[(\hat{A}_c^T \hat{M}\hat{A}_c) + \Im_1^T(A^T P + PA + H_1 + \tau_1^2 Z_3 + \tau_{12}^2 Z_4)\Im_1 \\
& -\Im_1^T(PB\Upsilon K)\Im_2 + \Im_1^T P\Im_{10} - \Im_2^T(B\Upsilon K)^T P\Im_1 + \Im_{10}^T P\Im_1 \\
& +\Im_3^T Q_1\Im_5 + \Im_5^T Q_1\Im_3 + \Im_4^T Q_2\Im_6 + \Im_6^T Q_2\Im_4 - (1-\eta)\Im_2^T H_3\Im_2 \\
& -\Im_3^T(H_1 - H_2 - H_3)\Im_3 - \Im_4^T H_2\Im_4 - \Im_5^T(H_4 - H_5)\Im_5 \\
& -\Im_6^T H_5\Im_6 - (\Im_1^T - \Im_3^T)Z_1(\Im_1 - \Im_3) - (\Im_2^T - \Im_4^T)Z_2(\Im_2 - \Im_4) \\
& -2(\Im_3^T - \Im_2^T)Z_2(\Im_3 - \Im_2) - \Im_7^T Z_3\Im_7 - 2\Im_8^T Z_4\Im_8 \\
& -\Im_9^T Z_4\Im_9 - (\tau_1\Im_1^T - \Im_7^T)S_1(\tau_1\Im_1 - \Im_7) \\
& -(\tau_{12}\Im_1^T - \Im_8^T - \Im_9^T)S_2(\tau_{12}\Im_1 - \Im_8 - \Im_9)]\lambda(t).
\end{aligned}
$$

To ensure the asymptotic stability, following mathematical condition is required to be true

$\dot{V}(e_t) < 0$, or $\chi\varphi + (1-\chi)\hat{\varphi} < 0$, for $0 \le \chi \le 1$,

$\varphi < 0$, and $\hat{\varphi} < 0$.

By implementation of Schur complement provided in Lemma 2 to Eq. 6.38, the set of LMIs of Eq. 6.21 and Eq. 6.22 can be obtained.

Theorem 6.1 is applicable for the synchronization of the master and slave systems defined in Eq. 6.1 and Eq. 6.2, if the controller values are known prior or through selection of suitable controller gain matrix according to the other parameters. Selection of controller gain matrix also depends on the upper and lower limit of input time-delay, which makes it difficult to select the appropriate values for controller. However, a good approach is to compute the value of controller gain matrix, by solving the set of LMIs according to the rest of the variables of the system. To make the synchronization scheme more general controller gain matrix is determined in Theorem 6.2 to obtain the synchronize behavior between master and slave systems.

6.5 Theorem 6.2

Let the master system and the slave system, described in Eq. 6.1 and Eq. 6.2, respectively, satisfy the assumptions provided in Eq. 6.9, Eq. 6.10, and also satisfy the input delay bound provided in Eq. 6.3 along with delay derivative bound of Eq. 6.4. Suppose there exist symmetric matrices $X > 0$, $\bar{Q}_i > 0$, $\bar{H}_j > 0$, $\bar{Z}_k > 0$, $\bar{S}_i > 0$ and a matrix G of suitable dimensions, for $i = 1, 2,$ $j = 1, 2, 3, 4, 5$, and $k = 1, 2, 3, 4$, such that the sets of following matrix inequalities holds:

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ * & \Theta_{22} \end{bmatrix} < 0\,, \forall \Upsilon \in \mathbb{Q}\,, \qquad \textit{(Eq 6.39)}$$

$$\hat{\Theta} = \begin{bmatrix} \hat{\Theta}_{11} & \Theta_{12} \\ * & \Theta_{22} \end{bmatrix} < 0\,, \forall \Upsilon \in \mathbb{Q}\,, \qquad \textit{(Eq 6.40)}$$

where

$$\Theta_{11} = \begin{bmatrix} \Sigma_{11} & -B\Upsilon G & \bar{Z}_1 & 0 & 0 & 0 & \tau_1\bar{S}_1 & \tau_{12}\bar{S}_2 & \tau_{12}\bar{S}_2 & I \\ * & \Sigma_{22} & \bar{Z}_2 & 2\bar{Z}_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & \Sigma_{33} & 0 & \bar{Q}_1 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \Sigma_{44} & 0 & \bar{Q}_2 & 0 & 0 & 0 & 0 \\ * & * & * & * & \Sigma_{55} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -\bar{H}_5 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & \Sigma_{77} & 0 & 0 & 0 \\ * & * & * & * & * & * & * & \Sigma_{88} & -\bar{S}_2 & 0 \\ * & * & * & * & * & * & * & * & \Sigma_{99} & 0 \\ * & * & * & * & * & * & * & * & * & -I \end{bmatrix},$$

$$\Theta_{12} = \begin{bmatrix} XA^T & \tau_1 XA^T & \tau_{12} XA^T & \frac{\tau_1^2}{2} XA^T & \tilde{\tau} XA^T & X\Omega^T \\ \Lambda_1 & \Lambda_2 & \Lambda_3 & \Lambda_4 & \Lambda_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ I & \tau_1 I & \tau_{12} I & \frac{\tau_1^2}{2} I & \tilde{\tau} I & 0 \end{bmatrix},$$

$\Theta_{22} = -diag(X\bar{H}_4^{-1}X\,, X\bar{Z}_1^{-1}X_1, X\bar{Z}_2^{-1}X\,, X\bar{S}_1^{-1}X\,, X\bar{S}_2^{-1}X\,, I)$, and

$$\hat{\Theta}_{11} = \begin{bmatrix} \Sigma_{11} & -B\Upsilon G & \bar{Z}_1 & 0 & 0 & 0 & \tau_1\bar{S}_1 & \tau_{12}\bar{S}_2 & \tau_{12}\bar{S}_2 & I \\ * & \Sigma_{22} & 2\bar{Z}_2 & \bar{Z}_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & \hat{\Sigma}_{33} & 0 & \bar{Q}_1 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \hat{\Sigma}_{44} & 0 & \bar{Q}_2 & 0 & 0 & 0 & 0 \\ * & * & * & * & \Sigma_{55} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -\bar{H}_5 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & \Sigma_{77} & 0 & 0 & 0 \\ * & * & * & * & * & * & * & \hat{\Sigma}_{88} & -\bar{S}_2 & 0 \\ * & * & * & * & * & * & * & * & \hat{\Sigma}_{99} & 0 \\ * & * & * & * & * & * & * & * & * & -I \end{bmatrix},$$

$\Sigma_{11} = \bar{H}_1 - \bar{Z}_1 + \tau_1^2\bar{Z}_3 + \tau_{12}^2\bar{Z}_4 - \tau_1^2\bar{S}_1 - \tau_{12}^2\bar{S}_2 + XA^T + AX$,

$\Sigma_{22} = -(1-\eta)\bar{H}_3 - 3\bar{Z}_2$,

$\Sigma_{33} = -\bar{H}_1 + \bar{H}_2 + \bar{H}_3 - \bar{Z}_1 - \bar{Z}_2$,

$\Sigma_{44} = -\bar{H}_2 - 2\bar{Z}_2$,

$\Sigma_{55} = -\bar{H}_4 + \bar{H}_5$,

$\Sigma_{77} = -\bar{Z}_3 - \bar{S}_1$,

$\Sigma_{88} = -\bar{Z}_4 - \bar{S}_2$,

$\Sigma_{99} = -2\bar{Z}_4 - \bar{S}_2$,

$\Lambda_1 = -G^T B^T \Upsilon^T$,

$\Lambda_2 = -\tau_1 G^T B^T \Upsilon^T$,

$\Lambda_3 = -\tau_{12} G^T B^T \Upsilon^T$,

$\Lambda_4 = -\dfrac{\tau_1^2}{2} G^T B^T \Upsilon^T$,

$\Lambda_5 = -\tilde{\tau} G^T B^T \Upsilon^T$,

$\hat{\Sigma}_{33} = -\bar{H}_1 + \bar{H}_2 + \bar{H}_3 - \bar{Z}_1 - 2\bar{Z}_2$,

$\hat{\Sigma}_{44} = -\bar{H}_2 - \bar{Z}_2$,

$\hat{\Sigma}_{88} = -2\bar{Z}_4 - \bar{S}_2$, and

$\hat{\Sigma}_{99} = -\bar{Z}_4 - \bar{S}_2$.

If set of LMIs hold, then synchronization of the master and the slave systems under input delay and slope-restricted input nonlinearity can be ensure by application of proposed state feedback control law. Theorem 6.2 provides the advantage to compute the controller gain matrix by solving $K = GX^{-1}$.

6.5.1 Proof of Theorem 6.2

Pre and post multiplication of the matrix $diag(\sigma_1, \sigma_2, \sigma_3)$ to the inequalities obtained: in Eq. 6.20 and Eq. 6.21, where

$\sigma_1 = diag(P^{-1}, P^{-1}, P^{-1}, P^{-1}, P^{-1}, P^{-1}, P^{-1}, P^{-1}, P^{-1})$,

$\sigma_2 = diag(I)$,

$\sigma_3 = diag(H_4^{-1}, Z_1^{-1}, Z_2^{-1}, S_1^{-1}, S_2^{-1})$,

and also using $P^{-1}P = PP^{-1} = I$, following set of LMIs are obtained,

$$\Gamma = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ * & \Gamma_{22} \end{bmatrix} < 0, \forall \Upsilon \in \mathbb{Q}, \qquad \textit{(Eq 6.41)}$$

$$\hat{\Gamma} = \begin{bmatrix} \hat{\Gamma}_{11} & \Gamma_{12} \\ * & \Gamma_{22} \end{bmatrix} < 0, \forall \Upsilon \in \mathbb{Q}, \qquad \textit{(Eq 6.42)}$$

for a given matrix $K \in R^{p\times n}$, are satisfied, where

$$\Gamma_{11} = \begin{bmatrix} \Delta_{11} & \Delta_{12} & \Delta_{13} & 0 & 0 & 0 & \Delta_{17} & \Delta_{18} & \Delta_{19} & P^{-1} \\ * & \Delta_{22} & \Delta_{23} & \Delta_{24} & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & \Delta_{33} & 0 & \Delta_{35} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \Delta_{44} & 0 & \Delta_{46} & 0 & 0 & 0 & 0 \\ * & * & * & * & \Delta_{55} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \Delta_{66} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & \Delta_{77} & 0 & 0 & 0 \\ * & * & * & * & * & * & * & \Delta_{88} & \Delta_{89} & 0 \\ * & * & * & * & * & * & * & * & \Delta_{99} & 0 \\ * & * & * & * & * & * & * & * & * & -I \end{bmatrix},$$

$$\Gamma_{12} = \begin{bmatrix} A^T P^{-1} & \tau_1 A^T P^{-1} & \tau_{12} A^T P^{-1} & \frac{\tau_1^2}{2} A^T P^{-1} & \tilde{\tau} A^T P^{-1} \\ -P^{-1}\mathrm{E} & -\tau_1 P^{-1}\mathrm{E} & -\tau_{12} P^{-1}\mathrm{E} & -\frac{\tau_1^2}{2} P^{-1}\mathrm{E} & -\tilde{\tau} P^{-1}\mathrm{E} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ I & \tau_1 I & \tau_{12} I & \frac{\tau_1^2}{2} I & \tilde{\tau} I \end{bmatrix},$$

$\Gamma_{22} = -diag(P^{-1}H_4P^{-1}, P^{-1}Z_1P^{-1}, P^{-1}Z_2P^{-1}, P^{-1}S_1P^{-1}, P^{-1}S_2P^{-1})$, and

$$\hat{\Gamma}_{11} = \begin{bmatrix} \Delta_{11} & \Delta_{12} & \Delta_{13} & 0 & 0 & 0 & \Delta_{17} & \Delta_{18} & \Delta_{19} & P^{-1} \\ * & \Delta_{22} & \Delta_{23} & \Delta_{24} & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & \hat{\Delta}_{33} & 0 & \Delta_{35} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \hat{\Delta}_{44} & 0 & \Delta_{46} & 0 & 0 & 0 & 0 \\ * & * & * & * & \Delta_{55} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \Delta_{66} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & \Delta_{77} & 0 & 0 & 0 \\ * & * & * & * & * & * & * & \hat{\Delta}_{88} & \Delta_{89} & 0 \\ * & * & * & * & * & * & * & * & \hat{\Delta}_{99} & 0 \\ * & * & * & * & * & * & * & * & * & -I \end{bmatrix},$$

for the entries

$$\Delta_{11} = P^{-1}H_1P^{-1} + \tau_1^2 P^{-1}Z_3P^{-1} + \tau_{12}^2 P^{-1}Z_4P^{-1} - P^{-1}Z_1P^{-1} - \tau_1^2 P^{-1}S_1P^{-1}$$
$$-\tau_{12}^2 P^{-1}S_2P^{-1} + A^TP^{-1} + P^{-1}A + P^{-1}\Omega^T\Omega P^{-1},$$

$$\Delta_{12} = -B\Upsilon KP^{-1},$$

$$\Delta_{13} = P^{-1}Z_1P^{-1},$$

$$\Delta_{17} = \tau_1 P^{-1}S_1P^{-1},$$

$$\Delta_{18} = \tau_{12} P^{-1}S_2P^{-1},$$

$$\Delta_{19} = \tau_{12} P^{-1}S_2P^{-1},$$

$$\Delta_{22} = -3P^{-1}Z_2P^{-1} - (1-\eta)P^{-1}H_3P^{-1},$$

$\Delta_{23} = P^{-1}Z_2P^{-1},$

$\Delta_{24} = 2P^{-1}Z_2P^{-1},$

$\Delta_{33} = -P^{-1}Z_1P^{-1} - P^{-1}Z_2P^{-1} - P^{-1}H_1P^{-1} + P^{-1}H_2P^{-1} + P^{-1}H_3P^{-1},$

$\Delta_{35} = P^{-1}Q_1P^{-1},$

$\Delta_{44} = -2P^{-1}Z_2P^{-1} - P^{-1}H_2P^{-1},$

$f(x\Delta_{46} = P^{-1}Q_2P^{-1},)$

$\Delta_{55} = -P^{-1}H_4P^{-1} + P^{-1}H_5P^{-1},$

$\Delta_{66} = -P^{-1}H_5P^{-1},$

$\Delta_{77} = -P^{-1}S_1P^{-1} - P^{-1}Z_3P^{-1},$

$\Delta_{88} = -P^{-1}Z_4P^{-1} - P^{-1}S_2P^{-1},$

$\Delta_{89} = -P^{-1}S_2P^{-1},$

$\Delta_{99} = -2P^{-1}Z_4P^{-1} - P^{-1}S_2P^{-1},$

$\hat{\Delta}_{33} = -P^{-1}Z_1P^{-1} - 2P^{-1}Z_2P^{-1} - P^{-1}H_1P^{-1} + P^{-1}H_2P^{-1} + P^{-1}H_3P^{-1},$
$\hat{\Delta}_{44} = -P^{-1}Z_2P^{-1} - P^{-1}H_2P^{-1},$
$\hat{\Delta}_{88} = -2P^{-1}Z_4P^{-1} - P^{-1}S_2P^{-1},$
$\hat{\Delta}_{99} = -P^{-1}Z_4P^{-1} - P^{-1}S_2P^{-1}.$

Now applying the change of variable method, bilinear matrix inequalities (BMIs) are transformed into LMIs. The set of LMIs is obtained by interchanging $P^{-1} = X$, $\bar{H}_i = P^{-1}H_iP^{-1}$, $\bar{Q}_i = P^{-1}Q_iP^{-1}$, $\bar{Z}_i = P^{-1}Z_iP^{-1}$, and $\bar{S}_i = P^{-1}S_iP^{-1}$.

$$\Xi = \begin{bmatrix} \Xi_{11} & \Xi_{12} \\ * & \Xi_{22} \end{bmatrix}, \forall \Upsilon \in \mathbb{Q}, \qquad \textit{(Eq 6.43)}$$

$$\hat{\Xi} = \begin{bmatrix} \hat{\Xi}_{11} & \Xi_{12} \\ * & \Xi_{22} \end{bmatrix}, \forall \Upsilon \in \mathbb{Q}, \qquad \textit{(Eq 6.44)}$$

where

$$\Xi_{11}=\begin{bmatrix} \Sigma_1 & -B\Upsilon KX & \bar{Z}_1 & 0 & 0 & 0 & \tau_1\bar{S}_1 & \tau_{12}\bar{S}_2 & \tau_{12}\bar{S}_2 & I \\ * & \Sigma_{22} & \bar{Z}_2 & 2\bar{Z}_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & \Sigma_{33} & 0 & \bar{Q}_1 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \Sigma_{44} & 0 & \bar{Q}_2 & 0 & 0 & 0 & 0 \\ * & * & * & * & \Sigma_{55} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -\bar{H}_5 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & \Sigma_{77} & 0 & 0 & 0 \\ * & * & * & * & * & * & * & \Sigma_{88} & -\bar{S}_2 & 0 \\ * & * & * & * & * & * & * & * & \Sigma_{99} & 0 \\ * & * & * & * & * & * & * & * & * & -I \end{bmatrix},$$

$$\Xi_{12}=\begin{bmatrix} XA^T & \tau_1XA^T & \tau_{12}XA^T & \frac{\tau_1^2}{2}XA^T & \tilde{\tau}XA^T \\ -X\mathrm{E} & -\tau_1X\mathrm{E} & -\tau_{12}X\mathrm{E} & -\frac{\tau_1^2}{2}X\mathrm{E} & -\tilde{\tau}X\mathrm{E} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ I & \tau_1I & \tau_{12}I & \frac{\tau_1^2}{2}I & \tilde{\tau}I \end{bmatrix},$$

$\Xi_{22}=-diag(X\bar{H}_4^{-1}X, X\bar{Z}_1^{-1}X_1, X\bar{Z}_2^{-1}X, X\bar{S}_1^{-1}X, X\bar{S}_2^{-1}X)$, and

$$\hat{\Xi}_{11}=\begin{bmatrix} \Sigma_1 & -B\Upsilon KX & \bar{Z}_1 & 0 & 0 & 0 & \tau_1\bar{S}_1 & \tau_{12}\bar{S}_2 & \tau_{12}\bar{S}_2 & I \\ * & \Sigma_{22} & 2\bar{Z}_2 & \bar{Z}_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & \hat{\Sigma}_{33} & 0 & \bar{Q}_1 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \hat{\Sigma}_{44} & 0 & \bar{Q}_2 & 0 & 0 & 0 & 0 \\ * & * & * & * & \Sigma_{55} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -\bar{H}_5 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & \Sigma_{77} & 0 & 0 & 0 \\ * & * & * & * & * & * & * & \hat{\Sigma}_{88} & -\bar{S}_2 & 0 \\ * & * & * & * & * & * & * & * & \hat{\Sigma}_{99} & 0 \\ * & * & * & * & * & * & * & * & * & -I \end{bmatrix},$$

$\sum_1 = \sum_{11} + X\Omega^T\Omega X$.

Now by incorporating the Schur complement defined in Lemma 2 and $G = KX$, Eq. 6.43 and Eq. 6.44 can be transformed into set of LMIs given in Eq. 6.39 and Eq. 6.40, which completes the proof of Theorem 6.2. □

Theorem 6.1 offers a synchronization strategy by selection of an appropriate controller gain matrix. Value of the controller gain is difficult to determine as it also depends on the range of input time-delay. So in case of variation in upper and lower bound of input time-delay, it also requires the change in controller gain matrix accordingly. Theorem 6.2 is more effective as compared to the Theorem 6.1, as it allows the computation of the controller gain matrix rather than its selection. The controller gain computed by Theorem 6.2 provides the effective delay-range-dependent synchronization scheme for nonlinear master and slave systems under constraints like input time-delay and unknown slope-restricted input nonlinearity.

The traditional delay-range-dependent synchronization schemes were design for varying delays within specific range [7], [106], [109]. However, provided scheme of synchronization is less conservative, it is more practical and adaptable approach. It is applicable for the case if lower bound of the delay is considered to be zero.

There are some nonlinear terms like $diag(X\bar{H}_4^{-1}X, X\bar{Z}_1^{-1}X_1, X\bar{Z}_2^{-1}X, X\bar{S}_1^{-1}X, X\bar{S}_2^{-1}X)$ in the inequalities of Theorem 6.2. To determine the controller gain matrix in the presence of such nonlinear terms is quite challenging. These nonlinear constraints can be solved by applying the cone complementary linearization technique provided in the next section.

6.6 Cone Complementary Linearization

It is desired to compute the controller gain matrix from set of LMIs provided in Theorem 6.2, but computation of controller gain matrix is quite difficult in presence of nonlinear terms such as $\Xi_{22} = -diag(X\bar{H}_4^{-1}X, X\bar{Z}_1^{-1}X_1, X\bar{Z}_2^{-1}X, X\bar{S}_1^{-1}X, X\bar{S}_2^{-1}X)$. To handle these diagonal nonlinear terms of Ξ_{22}, cone complementary linearization technique is implied, which is very useful to simplify these complex nonlinear terms, hence nonlinear terms can be transformed as

$diag(-N,-T_1,-T_2,-V_1,-V_2)$,

where

$N = X\bar{H}_4^{-1}X$,

$T_1 = X\bar{Z}_1^{-1}X_1$,

$T_2 = X\bar{Z}_2^{-1}X$,

$V_1 = X\bar{S}_1^{-1}X$ and

$V_2 = X\bar{S}_2^{-1}X$.

Now by utilizing this transformation, nonlinear terms from the set of LMIs of Eq. 6.39 and Eq. 6.40 can be easily simplified and solvable by minimizing

$$\text{Trace}\left\{X\bar{X} + H_4\bar{H}_4 + N\bar{N} + Z_1\bar{Z}_1 + Z_2\bar{Z}_2 + T_1\bar{T}_1 + T_2\bar{T}_2 + S_1\bar{S}_1 + S_2\bar{S}_2 + V_1\bar{V}_1 + V_2\bar{V}_2\right\}$$

subject to

$$\begin{bmatrix} X & I \\ * & \bar{X} \end{bmatrix} \ge 0, \begin{bmatrix} H_4 & I \\ * & \bar{H}_4 \end{bmatrix} \ge 0, \; f(x)\begin{bmatrix} N & I \\ * & \bar{N} \end{bmatrix} \ge 0, \qquad \textit{(Eq 6.45)}$$

$$\begin{bmatrix} Z_i & I \\ * & \bar{Z}_i \end{bmatrix} \ge 0, \begin{bmatrix} T_i & I \\ * & \bar{T}_i \end{bmatrix} \ge 0, \qquad \textit{(Eq 6.46)}$$

$$\begin{bmatrix} S_i & I \\ * & \bar{S}_i \end{bmatrix} \ge 0, \begin{bmatrix} V_i & I \\ * & \bar{V}_i \end{bmatrix} \ge 0, \qquad \textit{(Eq 6.47)}$$

and the inequalities of Theorem 6.2, where terms $\bar{Z}_i, \bar{H}_4$, $\bar{N}$, $\bar{T}_i$, $\bar{S}_i$, $\bar{V}_i$ and $\bar{X}$ are used to represent the inverse of matrices Z_i, H_4, N, T_i, S_i, V_i and X, respectively, for $i = 1, 2$. As $N = X\bar{H}_4^{-1}X$, $T_i = X\bar{Z}_i^{-1}X$ and $V_i = X\bar{S}_i^{-1}X$. Therefore, supplementary constraints can be given as

$$\begin{bmatrix} X\bar{Z}_i^{-1}X & I \\ * & \bar{T}_i \end{bmatrix} \ge 0, \begin{bmatrix} X\bar{H}_4^{-1}X & I \\ * & \bar{N} \end{bmatrix} \ge 0, \begin{bmatrix} X\bar{S}_i^{-1}X & I \\ * & \bar{V}_i \end{bmatrix} \ge 0. \qquad \textit{(Eq 6.48)}$$

Now applying the congruence transformation by using $diag(\bar{X}, I)$ and involving $\bar{H}_4 = H_4^{-1}$, $\bar{Z}_i = Z_i^{-1}$, and $\bar{S}_i = S_i^{-1}$, $\forall i = 1, 2$, the inequalities of Eq. 6.48 are given as below

$$\begin{bmatrix} Z_i & \bar{X} \\ * & \bar{T}_i \end{bmatrix} \geq 0, \begin{bmatrix} H_4 & \bar{X} \\ * & \bar{N} \end{bmatrix} \geq 0, \begin{bmatrix} S_i & \bar{X} \\ * & \bar{V}_i \end{bmatrix} \geq 0, \; f(x) \; \forall i = 1, 2. \qquad \textit{(Eq 6.49)}$$

Hence, by applying the cone complementary linearization the nonlinear constraints of Theorem 6.2 can be easily solved using LMIs (see [125]):

$$\begin{aligned} \min \text{ Trace } \{ & X\bar{X} + H_4\bar{H}_4 + 0.5N\bar{N} + 0.5XH_4X\bar{N} + Z_1\bar{Z}_1 + Z_2\bar{Z}_2 \\ & + 0.5T_1\bar{T} + 0.5XZ_1X\bar{T} + 0.5T_2\bar{T}_2 + 0.5XZ_2X\bar{T}_2 + S_1\bar{S}_1 \\ & + S_2\bar{S}_2 + 0.5V_1\bar{V}_1 + 0.5XS_1X\bar{V}_1 + 0.5V_2\bar{V}_2 + 0.5XS_2X\bar{V} \}, \end{aligned} \qquad \textit{(Eq 6.50)}$$

subject to Eq. 6.45, Eq. 6.46, Eq. 6.47 and Eq. 6.49 and inequalities of Theorem 6.2.

The set of LMIs of Theorem 6.2 contains the nonlinear constraints; however, these nonlinear constraints are transformed into nonlinear objective function of Eq. 6.50 as an optimization problem, which can be solved using LMIs by the virtue of the cone complementary linearization algorithm (see details in [125]-[126] and references therein).

6.7 Simulation Results

To witness the proposed delay-range-dependent synchronization strategy for nonlinear master and slave systems, a numerical example of gyros systems is illustrated. Dynamics of Gyro systems are considered due to their applications in the field of aerospace engineering.

The motion equation of the chaotic gyro system ([62] and [127]-[128]) is described as

$$\ddot{\theta} + C_1\dot{\theta} + C_2\dot{\theta}^3 + \alpha^2 \frac{(1-\cos\theta)^2}{\sin^3\theta} - \beta\sin\theta = -f\sin\omega t\sin\theta, \qquad \textit{(Eq 6.51)}$$

where $f\sin\omega t\sin\theta$ represents the parametric excitation, $C_1\dot{\theta}$ and $C_2\dot{\theta}^3$ are the linear and nonlinear terms, respectively. Nonlinear part of the gyro system is represented by $\alpha^2 \frac{(1-\cos\theta)^2}{\sin^3\theta} - \beta\sin\theta$. To derive the normalized state equations, incorporating $x_1 = \theta$ and $y_1 = \dot{\theta}$, the dynamical equation of gyro system is transformed as

$$\dot{x}_1 = y_1, \qquad \textit{(Eq 6.52)}$$

$$\dot{y}_1 = -C_1y_1 - C_2y_1^3 - \alpha^2 \frac{(1-\cos x_1)^2}{\sin^3 x_1} + \beta\sin x_1 - f\sin\omega t\sin x_1. \qquad \textit{(Eq 6.53)}$$

By selecting, $C_1 = 0.5$, $C_2 = 0.05$, $\alpha = 10$, $f = 35.5$, and $\omega = 2$ the gyro system can be represented by Eq. 6.1 with

$$f(x) = \begin{bmatrix} 0 \\ -0.05 - 100\dfrac{(1-\cos x_1)^2}{\sin^3 x_1} + (1 - 35.5\sin 2t) \end{bmatrix},$$

(Eq 6.54)

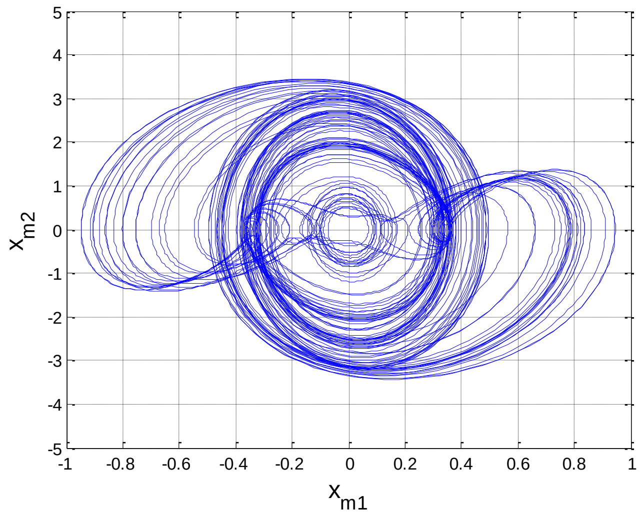
where $f(x)$ represents the nonlinear function of master system and slave system.

Robust control toolbox of MATLAB is utilized to solve the LMIs derived in Theorem 6.1 and Theorem 6.2. Lower bound for input time-delay τ_1, upper bound for input time-delay τ_{12}, and time-delay $\tilde{\tau}$ are taken to be 0.15, 0.35 and 0.05, respectively; whereas Lipschitz constant $\Omega = 0.3$ and delay derivative bound $\eta = 0.3$ are selected. The lower and upper bounds for slope-restricted input nonlinearity are selected as $l_m = 0.9$ and $l_M = 1.1$, respectively. The input time-delay of $0.25 + 0.09\sin 0.1t$ is incorporated in this numerical example. By applying the condition derived in Theorem 6.2, controller gain matrix is computed as $K = \begin{bmatrix} 1.312 & 1.874 \end{bmatrix}$. To observe the behavior of chaotic master and slave gyro systems, MATLAB solver ddex23 is used for implementation of time-delay differential equations with time-varying initial conditions for the master and slave systems as below

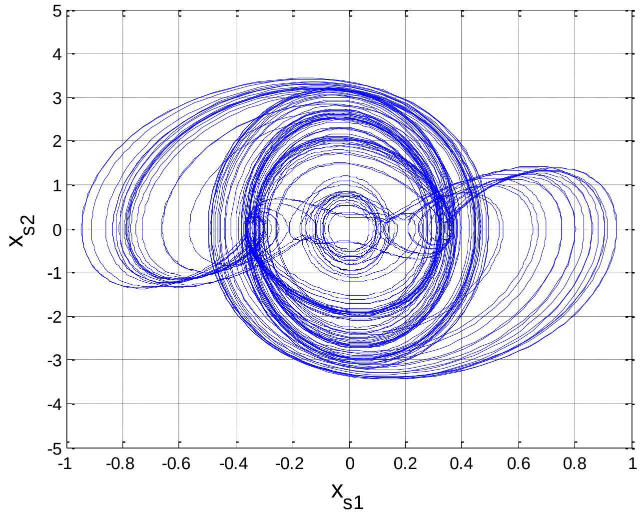
$$\begin{aligned} x_{m1}(t) &= 0.5 + 0.05\sin 0.1t, \\ x_{m2}(t) &= -1 - 0.01\sin 0.1t \end{aligned} \qquad \textit{(Eq 6.55)}$$

$$\begin{aligned} x_{s1}(t) &= -0.5 - 0.05\sin 0.1t, \\ x_{s2}(t) &= 1 + 0.01\sin 0.1t. \end{aligned} \qquad \textit{(Eq 6.56)}$$

Figure 6.1 shows the behavior of the master and slave systems (Gyro system) in the absence of controller input. Figure 6.2 shows the synchronization error plot between two states of master and slave systems, where $e_1(t) = x_{m1}(t) - x_{s1}(t)$ in solid line is the error between first state of the master and slave systems and $e_2(t) = x_{m2}(t) - x_{s2}(t)$ in dotted line is the error between second state of the master and slave systems. Plots show unsynchronized behavior among master and slave systems, as there are oscillation and irregularities for both error states with controller.



(a)



(b)

Figure 6.1: Behavior of the slave gyro system without control law, (a) master system, (b) slave system.

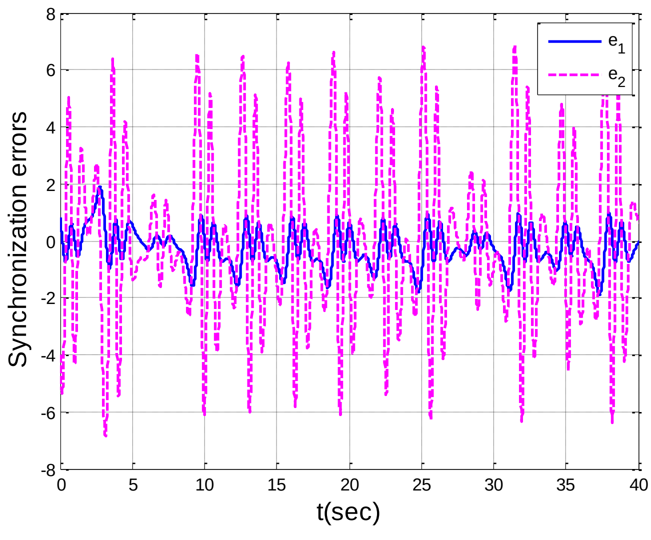
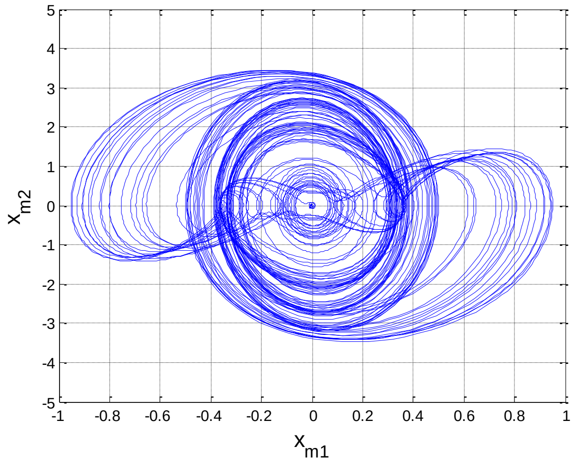
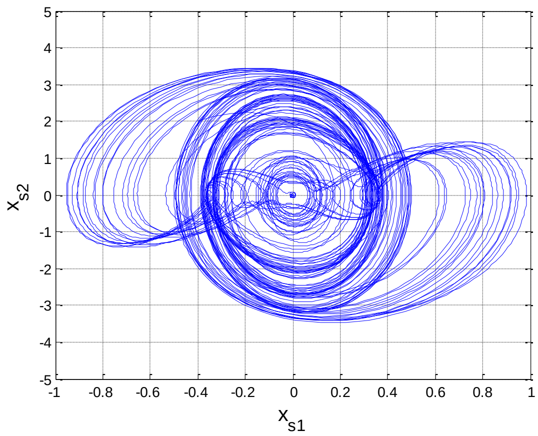


Figure 6.2: Synchronization error between master and slave systems without controller

Figures 6.3(a) and 6.3(b) shows the phase portraits of the master gyro system and slave gyro system when controller is activated for synchronization, respectively.



(a)

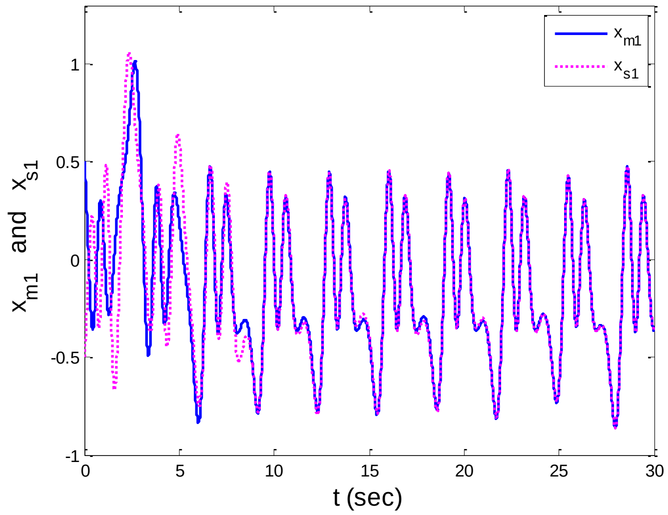


(b)

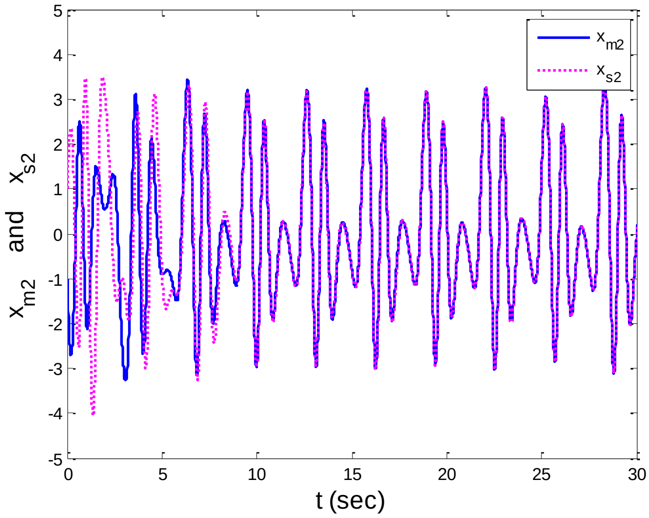
Figure 6.3: Behavior of the master-slave gyro systems with control law.

Figures 6.4(a) and 6.4(b) shows the time series of synchronization error plot between first and second state of the master-slave gyro systems, respectively, in the presence of control input. Time series plots show that states of the slave systems (dotted line) following the trajectory of master systems (solid line).

Figure 6.5 shows the synchronization error plots $e_1(t) = x_{m1}(t) - x_{s1}(t)$ in solid line and $e_2(t) = x_{m2}(t) - x_{s2}(t)$ in dotted line in the presence of state feedback controller. Response of the error trajectories shows that both errors are converging to the origin as controller is activated and synchronization between master and slave systems is achieved.



(a)



(b)

Figure 6.4: Time series plot for the first and second of the master and slave gyro systems.

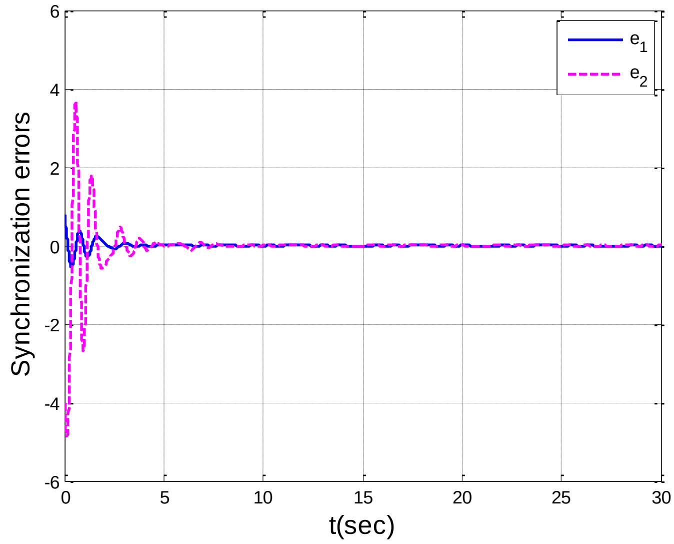


Figure 6.5: Synchronization errors e_1 and e_2 of the master and the slave gyro system with controller.

6.8 Summary

In this Chapter delay-range-dependent synchronization of nonlinear master and slave systems is considered under the constraints of time-varying input delay and slope restricted input nonlinearity. Different mathematical tools and strategies such as LPV approach, delay derivative bound conditions, delay range expressions, Jensen's inequality, triple integral based Lyapunov-Krasovskii functional, and LMI theory and convex handling technique are implied to obtain the sufficient conditions for synchronization of time-delay nonlinear master and slave systems. The provided design conditions guarantee the asymptotic convergence of the synchronization error to the origin. Compared to the traditional adaptive control and sliding mode synchronization schemes, the proposed state feedback control law based synchronization criteria is uncomplicated in designing and easy for implementation in the presence of time-varying input time-delay.

Furthermore, the proposed scheme is capable to deal with the short time-delay interval and large time-delay interval for synchronization of the nonlinear master and slave systems because the input delay has been treated using an innovative delay-range-dependent methodology. Hence, the resultant synchronization schemes are reliable, implementable and less conservative in contrast to the available techniques. In future, the proposed synchronization scheme can be extended for other complex problem like output delays, state delays, input saturation, adaptation of unknown parameters, robustness against perturbations, fast convergence rate and handling of parametric uncertainties. In the end, to witness the proposed synchronization criterion a numerical simulation example of chaotic gyro systems was illustrated.

Chapter 7

SYNCHRONIZATION OF NONLINEAR SYSTEMS UNDER MUTIPLE DELAYS

7.1 Overview

In the last two Chapters, the problem of synchronization of complex nonlinear time-delay master-slave systems was investigated. Delay-dependent and delay-range-dependent synchronization approaches are provided to deal with the time-varying input delay in the dynamics of the nonlinear master-slave systems.

In this Chapter, the problem of synchronization of nonlinear systems subject to multiple overlapping delays is considered. The problem is formulated by considering the state-delays in master and slave systems. Furthermore, time-varying input delay is incorporated in the dynamics of slave system for broader scope of the problem. Zero order hold technique is implied to design a feedback controller. To describe the multiple and overlapping delays, binary logic based four possible cases of overlapping are considered. Lyapunov Krasovskii functional is constructed by incorporating the four different cases of overlapping delay and after rigorous algebra efforts, an LMI based approach is developed to provide the necessary conditions for synthesis of proposed controller for synchronization of complex nonlinear mater-slave systems subject to overlapping delays. Basic structure of the proposed synchronization strategy is shown in Figure 7.1 as below.

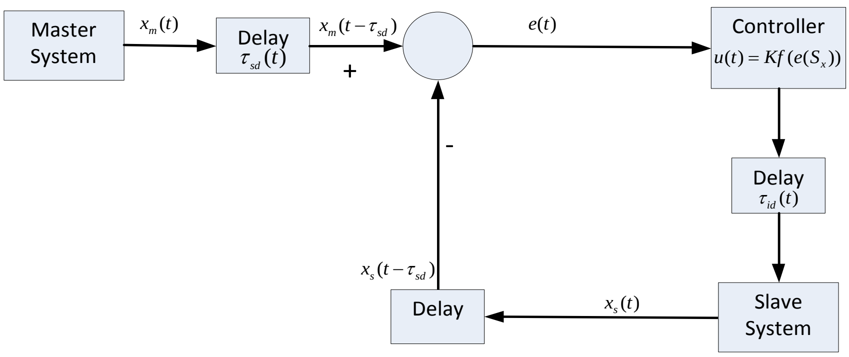


Figure 7.1: Block diagram of synchronization of nonlinear systems with multiple delays

The multiple time-varying delays frequently appear in many physical systems such as, communication networks, neural networks, power transmission lines, chemical processes and mechanical systems, etc. Delay is an important parameter of nonlinear systems, because ignoring the effects of time-delays results instability and degradation of the performance of nonlinear systems. The problem of stability analysis and synchronization of nonlinear systems with multiple time-delays received considerable attention over the years from researcher of control community [129-138]. In [136], the problem of exponential synchronization under the constraints of multiple time-varying delays has been investigated. The problem of output tracking control for class of switched nonlinear systems considering the multiple delay and LMI based synchronization approach for complex nonlinear systems by free weighting matrix technique have been addressed in [133], [138].

During the last few years, the problem of stability analysis of dynamical nonlinear systems subject to multiple delays such as input delays, output delays and states delays are frequently reported in the literature. However, the problem of synchronization of nonlinear master-slave systems subject to multiple and overlapping delays is lacking in the literature. The proposed technique is novel, as it ensures the synchronization of nonlinear master-slave systems subject to external disturbances, state delays and state delayed nonlinearities. The approach provides the advances multiple delays handling using overlapping concept. Proposed delay-range-dependent synchronization scheme is less conservative compared to the existing weighted matrix techniques.

This Chapter is organized as follows. Next section is about the problem formulation and preliminaries needed to derive the main results. In section 3, structure of the controller is described followed by the main results of proposed synchronization criteria. In the end, a numerical simulation example of two gyro systems considered as master and slave systems has been provided to witness the proposed synchronization scheme.

7.2 Systems Description and Preliminaries

Synchronization of nonlinear time-delay systems under the constraints of multiple overlapping delays and disturbances is considered.

Dynamics of the master system is described as follows

$$\begin{aligned}\dot{x}_m(t) &= A_o x_m(t) + A_1 x_m\left(t-\tau_{sd}(t)\right) + g_o\left(x_m(t),t\right) \\ &\quad + g_1\left(x_m\left(t-\tau_{sd}(t)\right),t\right),\end{aligned} \qquad \textit{(Eq 7.1)}$$

where $x_m(t) \in R^n$ and $x_m(t-\tau_{sd}(t)) \in R^n$ represents the state vectors of the master system, without and with state delay, respectively. Linear components in the dynamics of the master system of constant and known entries are represented by $A_o \in R^{n\times n}$ and $A_1 \in R^{n\times n}$. Nonlinear functions in the master system are represented by $g_o\left(x_m(t),t\right) \in R^n$ and $g_1(x_m(t-\tau_{sd}(t)),t) \in R^n$. $\tau_{sd}(t)$ represents the time-varying state delay. Similarly dynamics of the nonlinear time-delay slave system is described as

$$\begin{aligned}\dot{x}_s(t) &= A_o x_s(t) + A_1 x_s\left(t-\tau_{sd}(t)\right) + g_o\left(x_s(t),t\right) \\ &\quad + g_1(x_s(t-\tau_{sd}(t)),t) + Bu(t),\end{aligned} \qquad \textit{(Eq 7.2)}$$

where $x_s(t) \in R^n$ and $x_s(t-\tau_{sd}(t)) \in R^n$ represents the state vectors of the slave system, without and with state delay, respectively. Linear components in the dynamics of the slave system of constant and known entries are represented by $A_o \in R^{n\times n}$, $A_1 \in R^{n\times n}$ and $B \in R^{n\times n}$. Nonlinear functions in the slave system are represented by $g_o\left(x_s(t),t\right) \in R^n$ and $g_1(x_s(t-\tau_{sd}(t)),t) \in R^n$. $u \in R^p$ is the control input applied to the slave system to synchronize it with a master system.

7.2.1 Assumptions

1. It is assumed that nonlinearities in master and slave systems, satisfies the following inequalities known as Lipschitz conditions.

$$\left\|g_o\left(x_m(t),t\right) - g_o\left(x_s(t),t\right)\right\| \le \left\|\Omega_1\left(x_m(t)-x_s(t)\right)\right\|, \qquad \textit{(Eq 7.3)}$$

$$\left\|g_1\left(x_m(t-\tau_{sd}),t\right) - g_1\left(x_s(t-\tau_{sd}),t\right)\right\| \le \left\|\Omega_2\left(x_m(t-\tau_{sd})-x_s(t-\tau_{sd})\right)\right\|, \qquad \textit{(Eq 7.4)}$$

   where $\Omega_1 > 0$ and $\Omega_2 > 0$ are constant valued functions called Lipschitz constants.

2. Time-varying state delay in the master and slave systems is restricted in the interval, that satisfy following condition

$$0 \le \tau_{11} \le \tau_{sd}(t) \le \tau_{12}, \qquad \text{(Eq 7.5)}$$

where delay bounds τ_{12} and τ_{11} represents the upper and lower limit of the interval for state delay, respectively.

Difference between the states of the master system and slave system is defined as synchronization error $e(t) = x_m(t) - x_s(t)$, and taking it time derivative, we get

$$\dot{e}(t) = \dot{x}_m(t) - \dot{x}_s(t) \qquad \text{(Eq 7.6)}$$

Subtracting Eq. 7.2 from Eq. 7.1, error system reveals

$$\begin{aligned} \dot{e}(t) = {} & A_o e(t) + A_1 e\left(t - \tau_1(t)\right) + g_o\left(x_m(t), t\right) - g_o\left(x_s(t), t\right) \\ & + g_1\left(x_m\left(t - \tau_{sd}(t)\right), t\right) - g_1\left(x_s\left(t - \tau_{sd}(t)\right), t\right) - Bu. \end{aligned} \qquad \text{(Eq 7.7)}$$

The key idea is to synchronize the slave system to the master system by convergence of the error system to the origin, mathematically

$$\lim_{t \to 0} \left\| e(t) \right\| = 0. \qquad \text{(Eq 7.8)}$$

Clearly it shows that slave system of Eq. 7.2 synchronize to the master systems of Eq. 7.1, if error system of Eq. 7.7 is globally asymptotically stable.

7.3 Controller Design

Synchronization of nonlinear time-delay systems is under consideration. Zero order hold scheme is utilized to design a simple state feedback control law in the presence of clock driven sampler and an event driven quantizer. The proposed state-feedback control law is given by

$$u(t) = Kf\left(e(S_x)\right), \qquad \text{(Eq 7.9)}$$

where $K \in R^{m \times n}$ represents the state feedback controller gain matrix. S_x represents the sampling instant of the sampler with $x = 1, 2, 3 \ldots \infty$. $f(.)$ is called the logarithmic quantizer and its sets can be described as below [139].

$$u_j = \{\mu_i = \rho^i \mu_0 : i = 0,\ \pm 1,\ \pm 2,\ \ldots.\} \cup \{0\},\ \mu_0 = 0, \qquad \text{(Eq 7.10)}$$

where $\rho \in (0,1)$ and furthermore for the logarithmic quantizer, the associated quantizer are described as below

$$f(y) = \begin{cases} \rho^i \mu_0, & \text{if } \frac{1}{1+\delta}\rho^i \mu_0 < y \le \frac{1}{1-\delta}\rho^i \mu_0, \\ 0, & \text{if y=0,} \\ -f(-y) & \text{if y<0,} \end{cases} \qquad \textit{(Eq 7.11)}$$

where

$$\delta = \frac{1+\rho}{1-\rho}.$$

Now utilizing the logarithmic quantizer and zero order hold technique [139], state feedback control is transformed as

$$u(t) = Kf(e(t-\tau_{id})), \qquad \textit{(Eq 7.12)}$$

where τ_{id} is the time varying input delay and its bounds can be described by $0 \le \tau_{21} \le \tau_{id} \le \tau_{22}$.

Using the state feedback control law of Eq. 7.12 into the error systems of Eq. 7.7, we obtain

$$\begin{aligned} \dot{e}(t) = {} & A_o e(t) + A_1 e\left(t-\tau_{sd}(t)\right) + g_o\left(x_m(t),t\right) - g_o\left(x_s(t),t\right) \\ & + g_1\left(x_m\left(t-\tau_{sd}(t)\right),t\right) - g_1\left(x_s\left(t-\tau_{sd}(t)\right),t\right) - B(Kf(e(t-\tau_{id})). \end{aligned} \qquad \textit{(Eq 7.13)}$$

Rearranging it, we get

$$\begin{aligned} \dot{e}(t) = {} & A_o e(t) + A_1 e\left(t-\tau_{sd}(t)\right) - BKf(e(t-\tau_{id})) + g_o\left(x_m(t),t\right) \\ & - g_o\left(x_s(t),t\right) + g_1\left(x_m\left(t-\tau_{sd}(t)\right),t\right) - g_1\left(x_s\left(t-\tau_{sd}(t)\right),t\right). \end{aligned} \qquad \textit{(Eq 7.14)}$$

To simplify the complex error dynamics, let define the nonlinear functions as

$$\Psi\left(t, x_m, x_s\right) = g_o\left(x_m(t),t\right) - g_o\left(x_s(t),t\right) \text{ and} \qquad \textit{(Eq 7.15)}$$

$$\Phi\left(t, \tau_{sd}, x_m, x_s\right) = g_1\left(x_m\left(t-\tau_{sd}(t)\right),t\right) - g_1\left(x_s\left(t-\tau_{sd}(t)\right),t\right). \qquad \textit{(Eq 7.16)}$$

Incorporating the Eq. 7.15 and Eq. 7.16 into Eq. 7.14, it implies

$$\dot{e}(t) = A_o e(t) + A_1 e\left(t - \tau_{sd}(t)\right) - BKf\left(e(t - \tau_{id})\right) + \Psi\left(t, x_m, x_s\right) \\ \quad + \Phi\left(t, \tau_{sd}, x_m, x_s\right), \qquad \textit{(Eq 7.17)}$$

The simplified form of the synchronization error system of Eq. 7.17 is suitable to derive the sufficient conditions for synthesis of the proposed controller. To attain the required conditions, an LMI-based approach is established for analysis of the controller for the synchronization of nonlinear systems subject to multiple overlapping delays.

7.4 Theorem 7.1

Let the master system of Eq. 7.1 and the slave system of Eq. 7.2, satisfies the assumptions provided in Eq. 7.3, Eq. 7.4 and the delay bound of Eq. 7.5, than slave system will be synchronized to the master system, if there exist symmetric matrices $P > 0$, $H_{ij} > 0$ and $Z_{ik} > 0$ of appropriate dimensions, for $i = 1, 2$, $j = 1, 2$, and $k = 1, 2, 3, 4$, such that following LMI holds

$$\Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ * & \Pi_{22} \end{bmatrix} < 0, \qquad \textit{(Eq 7.18)}$$

$$\Pi_{11} = \begin{bmatrix} \Phi_{11} & 2PA_1 & \sigma H_{11} & \Phi_{14} & -2PBK & \hat{\delta} H_{21} & 0 & 2P & 2P \\ * & \Phi_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & \Phi_{33} & \sigma H_{12} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \Phi_{44} & 0 & \Phi_{46} & 0 & 0 & 0 \\ * & * & * & * & \Phi_{55} + \Lambda^2 Y & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \Phi_{66} & \hat{\delta} H_{22} & 0 & 0 \\ * & * & * & * & * & * & \Phi_{77} & 0 & 0 \\ * & * & * & * & * & * & * & -I & 0 \\ * & * & * & * & * & * & * & 0 & -I \end{bmatrix},$$

$$
\Pi_{12} = \begin{bmatrix}
\sigma\tau_{11}A_o{}^T H_{11} & \sigma\hat{\tau}_1 A_o{}^T H_{12} & \hat{\sigma}\tau_{12}A_o{}^T H_{11} & \hat{\sigma}\hat{\tau}_2 A_o{}^T H_{12} & \delta\tau_{12}A_o{}^T H_{21} & \delta\hat{\tau}_2 A_o{}^T H_{22} & \hat{\delta}\tau_{21}A_o{}^T H_{21} & \hat{\delta}\hat{\tau}_3 A_o{}^T H_{22} & -PBK \\
\sigma\tau_{11}A_1{}^T H_{11} & \sigma\hat{\tau}_1 A_1{}^T H_{12} & \hat{\sigma}\tau_{12}A_1{}^T H_{11} & \hat{\sigma}\hat{\tau}_2 A_1{}^T H_{12} & \delta\tau_{12}A_1{}^T H_{21} & \delta\hat{\tau}_2 A_1{}^T H_{22} & \hat{\delta}\tau_{21}A_1{}^T H_{21} & \hat{\delta}\hat{\tau}_3 A_1{}^T H_{22} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\sigma\tau_{11}\hat{K}H_{11} & -\sigma\hat{\tau}_1\hat{K}H_{12} & -\hat{\sigma}\tau_{12}\hat{K}H_{11} & -\hat{\sigma}\hat{\tau}_2\hat{K}H_{12} & -\delta\tau_{12}\hat{K}H_{21} & -\delta\hat{\tau}_2\hat{K}H_{22} & -\hat{\delta}\tau_{21}\hat{K}H_{21} & -\hat{\delta}\hat{\tau}_3\hat{K}H_{22} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\sigma\tau_{11}H_{11} & -\sigma\hat{\tau}_1 H_{12} & -\hat{\sigma}\tau_{12}H_{11} & -\hat{\sigma}\hat{\tau}_2 H_{12} & -\delta\tau_{12}H_{21} & -\delta\hat{\tau}_2 H_{22} & -\hat{\delta}\tau_{21}H_{21} & -\hat{\delta}\hat{\tau}_3 H_{22} & 0 \\
\sigma\tau_{11}H_{11} & -\sigma\hat{\tau}_1 H_{12} & -\hat{\sigma}\tau_{12}H_{11} & -\hat{\sigma}\hat{\tau}_2 H_{12} & -\delta\tau_{12}H_{21} & -\delta\hat{\tau}_2 H_{22} & -\hat{\delta}\tau_{21}H_{21} & -\hat{\delta}\hat{\tau}_3 H_{22} & 0
\end{bmatrix},
$$

and

$$
\Pi_{22} = \begin{bmatrix}
-H_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sigma\tau_{11}H_{11}BK \\
* & -H_{12} & 0 & 0 & 0 & 0 & 0 & 0 & -\sigma\hat{\tau}_1 H_{12}BK \\
* & * & -H_{11} & 0 & 0 & 0 & 0 & 0 & -\hat{\sigma}\tau_{12}H_{11}BK \\
* & * & * & -H_{12} & 0 & 0 & 0 & 0 & -\hat{\sigma}\hat{\tau}_2 H_{12}BK \\
* & * & * & * & -H_{21} & 0 & 0 & 0 & -\delta\tau_{12}H_{21}BK \\
* & * & * & * & * & -H_{22} & 0 & 0 & -\delta\hat{\tau}_2 H_{22}BK \\
* & * & * & * & * & * & -H_{21} & 0 & -\hat{\delta}\tau_{21}H_{21}BK \\
* & * & * & * & * & * & * & -H_{22} & -\hat{\delta}\hat{\tau}_3 H_{22}BK \\
* & * & * & * & * & * & * & * & -Y
\end{bmatrix}
$$

The entries of the LMI, are

$\hat{\tau}_1 = \tau_{12} - \tau_{11}$,

$\hat{\tau}_2 = \tau_{21} - \tau_{12}$,

$\hat{\tau}_3 = \tau_{22} - \tau_{21}$,

$\hat{\sigma} = 1 - \sigma$,

$\hat{\delta} = 1 - \delta$,

$\hat{K} = K^T B^T$

$$\Phi_{11} = 2PA_o + Z_{11} + Z_{12} + Z_{13} + Z_{21} + Z_{22} + Z_{23} - \sigma H_{11} - \hat{\sigma} H_{11} - \delta H_{21} - \hat{\delta} H_{21} + \Omega_1^T \Omega_1,$$

$$\Phi_{14} = \hat{\sigma} H_{11} + \delta H_{21},$$

$$\Phi_{22} = -(1-\eta_1) Z_{13} - (1-\eta_1)\hat{\sigma} Z_{14} - \delta(1-\eta_1) Z_{14} + \Omega_2^T \Omega_2,$$

$$\Phi_{33} = \sigma(Z_{14} - H_{11} - H_{12}) - Z_{12}$$

$$\Phi_{44} = \sigma H_{12} - Z_{22} + \hat{\sigma}(Z_{14} - H_{11} - H_{12}) + \delta(Z_{24} - H_{21} - H_{22}),$$

$$\Phi_{46} = \hat{\sigma} H_{12} + \delta H_{22}$$

$$\Phi_{55} = -(1-\eta_2) Z_{23} - \delta(1-\eta_2) Z_{24} - \hat{\delta}(1-\eta_2) Z_{24},$$

$$\Phi_{66} = -Z_{11} - \hat{\delta} H_{12} - \delta H_{22} + \hat{\delta}(Z_{24} - H_{21} - H_{22}),$$

$$\Phi_{77} = -Z_{21} - \hat{\delta} H_{22}$$

and * is used for the symmetric terms of LMI. There exists a reliable controller of the form Eq. 7.12 that guarantees asymptotic synchronization of the master and the slave systems subject to multiple and overlapping delays.

7.4.1 Proof of Theorem 7.1

To provide the proof of the Theorem 7.1, let we define four different cases of overlapping delay, using the binary logic:

Case I: $\sigma = 1$ and $\delta = 1$,

Case II: $\sigma = 1$ and $\delta = 0$,

Case III: $\sigma = 0$ and $\delta = 1$,

Case IV: $\sigma = 0$ and $\delta = 0$.

Using the definition of these four cases of delay parameters, let construct a Lyapunov-Krasovskii (LK) functional [140-141] that satisfy the statement of Theorem 7.1, LK functional is given by

$$V(e,t) = V_1(t) + \sigma V_2(t) + (1-\sigma) V_3(t) + \delta V_4(t) + (1-\delta) V_5(t), \qquad \textit{(Eq 7.19)}$$

where

$$V_1(e,t)=e^T(t)Pe(t)+\int_{t-\tau_{21}}^{t}e^T(\omega)Z_{11}e(\omega)d\omega+\int_{t-\tau_{22}}^{t}e^T(\omega)Z_{21}e(\omega)d\omega$$
$$+\int_{t-\tau_{11}}^{t}e^T(\omega)Z_{12}e(\omega)d\omega+\int_{t-\tau_{12}}^{t}e^T(\omega)Z_{22}e(\omega)d\omega \qquad \textit{(Eq 7.20a)}$$
$$+\int_{t-\tau_{sd}(t)}^{t}e^T(\omega)Z_{13}e(\omega)d\omega+\int_{t-\tau_{id}(t)}^{t}e^T(\omega)Z_{23}e(\omega)d\omega,$$

$$V_2(e,t)=\int_{t-\tau_{sd}(t)}^{t-\tau_{11}}e^T(\omega)Z_{14}e(\omega)d\omega+\tau_{11}\int_{t-\tau_{11}}^{t}\int_{s}^{t}\dot{e}^T(\theta)H_{11}\dot{e}(\theta)dsd\theta$$
$$+(\tau_{12}-\tau_{11})\int_{t-\tau_{12}}^{t-\tau_{11}}\int_{s}^{t}\dot{e}^T(\theta)H_{12}\dot{e}(\theta)d\theta ds, \qquad \textit{(Eq 7.20b)}$$

$$V_3(e,t)=\int_{t-\tau_{sd}(t)}^{t-\tau_{12}}e^T(\omega)Z_{14}e(\omega)d\omega+\tau_{12}\int_{t-\tau_{12}}^{t}\int_{s}^{t}\dot{e}^T(\theta)H_{11}\dot{e}(\theta)dsd\theta$$
$$+(\tau_{21}-\tau_{12})\int_{t-\tau_{21}}^{t-\tau_{12}}\int_{s}^{t}\dot{e}^T(\theta)H_{12}\dot{e}(\theta)d\theta ds, \qquad \textit{(Eq 7.20c)}$$

$$V_4(e,t)=\int_{t-\tau_{id}(t)}^{t-\tau_{12}}e^T(\omega)Z_{24}e(\omega)d\omega+\tau_{12}\int_{t-\tau_{12}}^{t}\int_{s}^{t}\dot{e}^T(\theta)H_{21}\dot{e}(\theta)d\theta ds$$
$$+(\tau_{21}-\tau_{12})\int_{t-\tau_{21}}^{t-\tau_{12}}\int_{s}^{t}\dot{e}^T(\theta)H_{22}\dot{e}(\theta)d\theta ds \qquad \textit{(Eq 7.20d)}$$

and

$$V_5(e,t)=\int_{t-\tau_{id}(t)}^{t-\tau_{21}}e^T(\omega)Z_{24}e(\omega)d\omega+\tau_{21}\int_{t-\tau_{21}}^{t}\int_{s}^{t}\dot{e}^T(\theta)H_{21}\dot{e}(\theta)d\theta ds$$
$$+(\tau_{22}-\tau_{21})\int_{t-\tau_{22}}^{t-\tau_{21}}\int_{s}^{t}\dot{e}^T(\theta)H_{22}\dot{e}(\theta)d\theta ds. \qquad \textit{(Eq 7.20e)}$$

Taking the time-derivative of constructed LK functional and rearranging it, reveals

$$\begin{aligned}\dot{V}_1(e,t)&=2e^T(t)P\dot{e}(t)+e^T(t)[Z_{11}+Z_{12}+Z_{13}+Z_{21}+Z_{22}+Z_{23}]e(t)\\&-e^T(t-\tau_{21})Z_{11}e(t-\tau_{21})-e^T(t-\tau_{11})Z_{12}e(t-\tau_{11})\\&-(1-\dot{\tau}_{sd}(t))e^T(t-\tau_{sd})Z_{13}e(t-\tau_{sd})-e^T(t-\tau_{22})Z_{21}e(t-\tau_{22})\\&-e^T(t-\tau_{12})Z_{22}e(t-\tau_{12})-(t-\dot{\tau}_{id}(t))e^T(t-\tau_{id})Z_{23}e(t-\tau_{id}),\end{aligned} \qquad \textit{(Eq 7.21a)}$$

$$\begin{aligned}\dot{V}_2(e,t)&=e^T(t-\tau_{11})Z_{14}e(t-\tau_{11})-(1-\dot{\tau}_{sd}(t))e^T(t-\tau_{sd})Z_{14}e(t-\tau_{sd})\\&+\dot{e}^T(t)\{\tau_{11}^2H_{11}+(\tau_{12}-\tau_{11})^2H_{12}\}\dot{e}(t)-e^T(t)H_{11}e(t)\\&+e^T(t)H_{11}e(t-\tau_{11})+e^T(t-\tau_{11})H_{11}e(t)-e^T(t-\tau_{11})\\&\times H_{11}e(t-\tau_{11})-e^T(t-\tau_{11})H_{12}e(t-\tau_{11})+e^T(t-\tau_{11})\\&\times H_{12}e(t-\tau_{12})+e^T(t-\tau_{12})H_{12}e(t-\tau_{11})\\&-e^T(t-\tau_{12})H_{12}e(t-\tau_{12}),\end{aligned} \qquad \textit{(Eq 7.21b)}$$

$$\begin{aligned}\dot{V}_3(e,t) &= e^T(t-\tau_{12})Z_{14}e(t-\tau_{12}) - (1-\dot{\tau}_{sd}(t))e^T(t-\tau_{sd})Z_{14}e(t-\tau_{sd}) \\ &\quad +\dot{e}^T(t)\{\tau_{12}^2 H_{11} + (\tau_{21}-\tau_{12})^2 H_{12}\}\dot{e}(t) - e^T(t)H_{11}e(t) + e^T(t) \\ &\quad \times H_{11}e(t-\tau_{12}) + e^T(t-\tau_{12})H_{11}e(t) - e^T(t-\tau_{12})H_{11}e(t-\tau_{12}) \\ &\quad -e^T(t-\tau_{12})H_{12}e(t-\tau_{12}) + e^T(t-\tau_{12})H_{12}e(t-\tau_{21}) \\ &\quad +e^T(t-\tau_{21})H_{12}e(t-\tau_{12}) - e^T(t-\tau_{21})H_{12}e(t-\tau_{21}),\end{aligned} \qquad \textit{(Eq 7.21c)}$$

$$\begin{aligned}\dot{V}_4(e,t) &= e^T(t-\tau_{12})Z_{24}e(t-\tau_{12}) - (1-\dot{\tau}_{id}(t))e^T(t-\tau_{id})Z_{24}e(t-\tau_{id}) \\ &\quad +\dot{e}^T(t)\{\tau_{12}^2 H_{21} + (\tau_{21}-\tau_{12})^2 H_{22}\}\dot{e}(t) - e^T(t)H_{21}e(t) + e^T(t) \\ &\quad \times H_{21}e(t-\tau_{12}) + e^T(t-\tau_{12})H_{21}e(t) - e^T(t-\tau_{12})H_{21}e(t-\tau_{12}) \\ &\quad -e^T(t-\tau_{12})H_{22}e(t-\tau_{12}) + e^T(t-\tau_{12})H_{22}e(t-\tau_{21}) \\ &\quad +e^T(t-\tau_{21})H_{22}e(t-\tau_{12}) - e^T(t-\tau_{21})H_{22}e(t-\tau_{21}),\end{aligned} \qquad \textit{(Eq 7.21d)}$$

$$\begin{aligned}\dot{V}_5(e,t) &= e^T(t-\tau_{21})Z_{24}e(t-\tau_{21}) - (1-\dot{\tau}_{id}(t))e^T(t-\tau_{id})Z_{24}e(t-\tau_{id}) \\ &\quad +\dot{e}^T(t)\{\tau_{21}^2 H_{21} + (\tau_{22}-\tau_{21})^2 H_{22}\}\dot{e}(t) - e^T(t)H_{21}e(t) + e^T(t) \\ &\quad \times H_{21}e(t-\tau_{21}) + e^T(t-\tau_{21})H_{21}e(t) - e^T(t-\tau_{21})H_{21}e(t-\tau_{21}) \\ &\quad -e^T(t-\tau_{21})H_{22}e(t-\tau_{21}) + e^T(t-\tau_{21})H_{22}e(t-\tau_{22}) \\ &\quad +e^T(t-\tau_{22})H_{22}e(t-\tau_{21}) - e^T(t-\tau_{22})H_{22}e(t-\tau_{22}).\end{aligned} \qquad \textit{(Eq 7.21e)}$$

Incorporating the $\dot{e}(t)$ of Eq. 7.17 into the Eq. 7.21, it yields

$$\begin{aligned}\dot{V}_1(e,t) &= 2e^T(t)P(A_o e(t) + A_1 e\left(t-\tau_{sd}(t)\right) - BK(e(t-\tau_{id}(t))) \\ &\quad +\Psi\left(t,x_m,x_s\right) + \Phi\left(t,\tau_{sd},x_m,x_s\right)) - e^T(t-\tau_{21})Z_{11}e(t-\tau_{21}) \\ &\quad +e^T(t)[Z_{11}+Z_{12}+Z_{13}+Z_{21}+Z_{22}+Z_{23}]e(t) \\ &\quad -e^T(t-\tau_{11})Z_{12}e(t-\tau_{11}) - (1-\dot{\tau}_{sd}(t))e^T(t-\tau_{sd})Z_{13}e(t-\tau_{sd}) \\ &\quad -e^T(t-\tau_{22})Z_{21}e(t-\tau_{22}) - e^T(t-\tau_{12})Z_{22}e(t-\tau_{12}) \\ &\quad -(t-\dot{\tau}_{id}(t))e^T(t-\tau_{id})Z_{23}e(t-\tau_{id}),\end{aligned} \qquad \textit{(Eq 7.22a)}$$

$$\begin{aligned}\dot{V}_2(e,t) &= -e^T(t)H_{11}e(t) + e^T(t)H_{11}e(t-\tau_{11}) + e^T(t-\tau_{11})[Z_{14}-H_{11} \\ &\quad -H_{12}]e(t-\tau_{11}) - (1-\dot{\tau}_{sd}(t))e^T(t-\tau_{sd})Z_{14}e(t-\tau_{sd}) \\ &\quad +e^T(t-\tau_{11})H_{11}e(t) + [A_o e(t) + A_1 e\left(t-\tau_{sd}(t)\right) - BK \\ &\quad \times(e(t-\tau_{id}(t))) + \Psi\left(t,x_m,x_s\right) + \Phi\left(t,\tau_{sd},x_m,x_s\right)]^T\{\tau_{11}^2 H_{11} \\ &\quad +(\tau_{12}-\tau_{11})^2 H_{12}\}[A_o e(t) + A_1 e\left(t-\tau_{sd}(t)\right) - BK(e(t-\tau_{id}(t))) \\ &\quad +\Psi\left(t,x_m,x_s\right) + \Phi\left(t,\tau_{sd},x_m,x_s\right)] + e^T(t-\tau_{11})H_{12}e(t-\tau_{12}) \\ &\quad +e^T(t-\tau_{12})H_{12}e(t-\tau_{11}) - e^T(t-\tau_{12})H_{12}e(t-\tau_{12}),\end{aligned} \qquad \textit{(Eq 7.22b)}$$

$$\begin{aligned}\dot{V}_3(e,t) &= -e^T(t)H_{11}e(t) + e^T(t)H_{11}e(t-\tau_{12}) + e^T(t-\tau_{12})H_{11}e(t) \\ &+ e^T(t-\tau_{12})[Z_{14} - H_{11} - H_{12}]e(t-\tau_{12}) - (1-\dot{\tau}_{sd}(t)) \\ &\times e^T(t-\tau_{sd})Z_{14}e(t-\tau_{sd}) + [A_o e(t) + A_1 e\left(t-\tau_{sd}(t)\right) - BK \\ &\times (e(t-\tau_{id}(t))) + \Psi\left(t, x_m, x_s\right) + \Phi\left(t, \tau_{sd}, x_m, x_s\right)]^T\{\tau_{12}^2 H_{11} \\ &+ (\tau_{21}-\tau_{12})^2 H_{12}\}[A_o e(t) + A_1 e\left(t-\tau_{sd}(t)\right) - BK(e(t-\tau_{id}(t))) \\ &+ \Psi\left(t, x_m, x_s\right) + \Phi\left(t, \tau_{sd}, x_m, x_s\right)] + e^T(t-\tau_{12})H_{12}e(t-\tau_{21}) \\ &+ e^T(t-\tau_{21})H_{12}e(t-\tau_{12}) - e^T(t-\tau_{21})H_{12}e(t-\tau_{21}),\end{aligned} \quad \text{(Eq 7.22c)}$$

$$\begin{aligned}\dot{V}_4(e,t) &= -e^T(t)H_{21}e(t) + e^T(t)H_{21}e(t-\tau_{12}) + e^T(t-\tau_{12})H_{21}e(t) \\ &+ e^T(t-\tau_{12})[Z_{24} - H_{21} - H_{22}]e(t-\tau_{12}) - (1-\dot{\tau}_{id}(t))e^T(t-\tau_{id}) \\ &\times Z_{24}e(t-\tau_{id}) + [A_o e(t) + A_1 e\left(t-\tau_{sd}(t)\right) - BK(e(t-\tau_{id}(t))) \\ &+ \Psi\left(t, x_m, x_s\right) + \Phi\left(t, \tau_{sd}, x_m, x_s\right)]^T\{\tau_{12}^2 H_{21} + (\tau_{21}-\tau_{12})^2 H_{22}\} \\ &\times [A_o e(t) + A_1 e\left(t-\tau_{sd}(t)\right) - BK(e(t-\tau_{id}(t))) + \Psi\left(t, x_m, x_s\right) \\ &+ \Phi\left(t, \tau_{sd}, x_m, x_s\right)] + e^T(t-\tau_{12})H_{22}e(t-\tau_{21}) + e^T(t-\tau_{21}) \\ &\times H_{22}e(t-\tau_{12}) - e^T(t-\tau_{21})H_{22}e(t-\tau_{21}),\end{aligned} \quad \text{(Eq 7.22d)}$$

$$\begin{aligned}\dot{V}_5(e,t) &= -e^T(t)H_{21}e(t) + e^T(t)H_{21}e(t-\tau_{21}) + e^T(t-\tau_{21})H_{21}e(t) \\ &+ e^T(t-\tau_{21})[Z_{24} - H_{21} - H_{22}]e(t-\tau_{21}) - (1-\dot{\tau}_{id}(t))e^T(t-\tau_{id}) \\ &\times Z_{24}e(t-\tau_{id}) + [A_o e(t) + A_1 e\left(t-\tau_{sd}(t)\right) - BK(e(t-\tau_{id}(t))) \\ &+ \Psi\left(t, x_m, x_s\right) + \Phi\left(t, \tau_{sd}, x_m, x_s\right)]^T\{\tau_{21}^2 H_{21} + (\tau_{22}-\tau_{21})^2 H_{22}\} \\ &\times [A_o e(t) + A_1 e\left(t-\tau_{sd}(t)\right) - BK(e(t-\tau_{id}(t))) + \Psi\left(t, x_m, x_s\right) \\ &+ \Phi\left(t, \tau_{sd}, x_m, x_s\right)] + e^T(t-\tau_{21})H_{22}e(t-\tau_{22}) + e^T(t-\tau_{22}) \\ &\times H_{22}e(t-\tau_{21}) - e^T(t-\tau_{22})H_{22}e(t-\tau_{22}).\end{aligned} \quad \text{(Eq 7.22e)}$$

It is assumed delay derivatives bounded by following inequalities

$$\dot{\tau}_{sd}(t) \le \eta_1, \quad \text{(Eq 7.23)}$$

$$\dot{\tau}_{id}(t) \le \eta_2. \quad \text{(Eq 7.24)}$$

Defining an argument matrix as

$$\begin{aligned}\lambda^T(t) = \Big\{&e^T(t), e^T(t-\tau_{sd}), e^T(t-\tau_{11}), e^T(t-\tau_{12}), e^T(t-\tau_{id}) \\ &e^T(t-\tau_{21}), e^T(t-\tau_{22}), \Psi^T(t, x_m, x_s), \Phi^T(t, \tau_{sd}, x_m, x_s)\Big\}.\end{aligned} \quad \text{(Eq 7.25)}$$

Let $\Im_i (i = 1, 2, 3, ..., 9)$ represents a matrix by replacing *i*th entry of $n \times 9n$ zero matrix with an identity matrix, for example second entry of the argument matrix can be represented as $\Im_2^T = \begin{bmatrix} 0 & I & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. There are none elements in in this argument matrix $\lambda(t)$. For instance, $\Im_5 \lambda(t) = e^T(t-\tau_{id})$, which is fifth entry of the argument matrix $\lambda(t)$. Using the argument matrix to simplify the main results and applying the delay derivative bounds of Eq. 7.23 and Eq. 7.24 along with the assumptions of Eq. 7.3, Eq. 7.4 and Eq. 7.5, it implies

$$\dot{V}(e,t) = \dot{V}_1(t) + \sigma \dot{V}_2(t) + (1-\sigma)\dot{V}_3(t) + \delta \dot{V}_4(t) + (1-\delta)\dot{V}_5(t) \le 0 . \qquad \textit{(Eq 7.26)}$$

It can be written in simplified form as below

$$\dot{V}(e,t) \le \lambda^T(t)(\Phi + A_c^T \Gamma A_c)\lambda(t), \qquad \textit{(Eq 7.27)}$$

where

$$A_c = \begin{bmatrix} A_o & A_1 & 0 & 0 & -BK(1+\Lambda) & 0 & 0 & I & I \end{bmatrix},$$

$$\begin{aligned}\Gamma = (&\sigma\tau_{11}^2 H_{11} + \sigma(\tau_{12}-\tau_{11})^2 H_{12} + (1-\sigma)\tau_{12}^2 H_{11} + (1-\sigma)(\tau_{21}-\tau_{12})^2 H_{12} \\ &+\delta\tau_{12}^2 H_{21} + \delta(\tau_{21}-\tau_{12})^2 H_{22} + (1-\delta)\tau_{21}^2 H_{21} + (1-\delta)(\tau_{22}-\tau_{21})^2 H_{22},\end{aligned}$$

and

$$\Phi = \begin{bmatrix} \Phi_{11} & 2PA_1 & \sigma H_{11} & \hat{\sigma} H_{11} + \delta H_{21} & -PBK(1+\Lambda) & \hat{\delta} H_{21} & 0 & 2P & 2P \\ * & \Phi_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & \Phi_{33} & \sigma H_{12} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \Phi_{44} & 0 & \Phi_{46} & 0 & 0 & 0 \\ * & * & * & * & \Phi_{55} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \Phi_{66} & \hat{\delta} H_{22} & 0 & 0 \\ * & * & * & * & * & * & -Z_{21} - \hat{\delta} H_{22} & 0 & 0 \\ * & * & * & * & * & * & * & -I & 0 \\ * & * & * & * & * & * & * & 0 & -I \end{bmatrix}.$$

Schur complement is applied to Eq. 7.27, it reveals

$$\dot{V}(e,t) \le \lambda^T(t)\Theta\lambda(t), \qquad \textit{(Eq 7.28)}$$

where

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ * & \Theta_{22} \end{bmatrix},$$

$$
\Theta_{11} = \begin{bmatrix}
\Phi_{11} & 2PA_1 & \sigma H_{11} & \Phi_{14} & -PBK(1+\Lambda) & \hat{\delta} H_{21} & 0 & 2P & 2P \\
* & \Phi_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & \Phi_{33} & \sigma H_{12} & 0 & 0 & 0 & 0 & 0 \\
* & * & * & \Phi_{44} & 0 & \Phi_{46} & 0 & 0 & 0 \\
* & * & * & * & \Phi_{55} & 0 & 0 & 0 & 0 \\
* & * & * & * & * & \Phi_{66} & \hat{\delta} H_{22} & 0 & 0 \\
* & * & * & * & * & * & \Phi_{77} & 0 & 0 \\
* & * & * & * & * & * & * & -I & 0 \\
* & * & * & * & * & * & * & 0 & -I
\end{bmatrix},
$$

$$
\Theta_{12} = \begin{bmatrix}
\sigma\tau_{11} A_o^T H_{11} & \sigma\hat{\tau}_1 A_o^T H_{12} & \hat{\sigma}\tau_{12} A_o^T H_{11} & \hat{\sigma}\hat{\tau}_2 A_o^T H_{12} & \delta\tau_{12} A_o^T H_{21} & \delta\hat{\tau}_2 A_o^T H_{22} & \hat{\delta}\tau_{21} A_o^T H_{21} & \hat{\delta}\hat{\tau}_3 A_o^T H_{22} \\
\sigma\tau_{11} A_1^T H_{11} & \sigma\hat{\tau}_1 A_1^T H_{12} & \tau_{12} A_1^T H_{11} & \hat{\sigma}\hat{\tau}_2 A_1^T H_{12} & \delta\tau_{12} A_1^T H_{21} & \delta\hat{\tau}_2 A_1^T H_{22} & \hat{\delta}\tau_{21} A_1^T H_{21} & \hat{\delta}\hat{\tau}_3 A_1^T H_{22} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\sigma\tau_{11}\varsigma\hat{K} H_{11} & -\sigma\hat{\tau}_1\varsigma\hat{K} H_{12} & -\hat{\sigma}\tau_{12}\varsigma\hat{K} H_{11} & -\hat{\sigma}\hat{\tau}_2\varsigma\hat{K} H_{12} & -\delta\tau_{12}\varsigma\hat{K} H_{21} & -\delta\hat{\tau}_2\varsigma\hat{K} H_{22} & -\hat{\delta}\tau_{21}\varsigma\hat{K} H_{21} & -\hat{\delta}\hat{\tau}_3\varsigma\hat{K} H_{22} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\sigma\tau_{11} H_{11} & \sigma\hat{\tau}_1 H_{12} & \hat{\sigma}\tau_{12} H_{11} & \hat{\sigma}\hat{\tau}_2 H_{12} & \delta\tau_{12} H_{21} & \delta\hat{\tau}_2 H_{22} & \hat{\delta}\tau_{21} H_{21} & \hat{\delta}\hat{\tau}_3 H_{22} \\
\sigma\tau_{11} H_{11} & \sigma\hat{\tau}_1 H_{12} & \hat{\sigma}\tau_{12} H_{11} & \hat{\sigma}\hat{\tau}_2 H_{12} & \delta\tau_{12} H_{21} & \delta\hat{\tau}_2 H_{22} & \hat{\delta}\tau_{21} H_{21} & \hat{\delta}\hat{\tau}_3 H_{22}
\end{bmatrix}
$$

and

$$\Theta_{22} = -diag(H_{11}, H_{12}, H_{11}, H_{12}, H_{21}, H_{22}, H_{21}, H_{22}),$$

with $\varsigma = (1+\Lambda)$

Rewriting in the following form

$$\Sigma_1 + \Sigma_2\Sigma_3 + \Sigma_3^T\Sigma_2^T < 0, \qquad \text{(Eq 7.29)}$$

where

$$\Sigma_1 = \begin{bmatrix} \Sigma_{11} & \Theta_{12} \\ * & \Theta_{22} \end{bmatrix},$$

$$\Sigma_{11} = \begin{bmatrix} \Phi_{11} & 2PA_1 & \sigma H_{11} & \Phi_{14} & -PBK & \hat{\delta} H_{21} & 0 & 2P & 2P \\ * & \Phi_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & \Phi_{33} & \sigma H_{12} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \Phi_{44} & 0 & \Phi_{46} & 0 & 0 & 0 \\ * & * & * & * & \Phi_{55} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \Phi_{66} & \hat{\delta} H_{22} & 0 & 0 \\ * & * & * & * & * & * & \Phi_{77} & 0 & 0 \\ * & * & * & * & * & * & * & -I & 0 \\ * & * & * & * & * & * & * & 0 & -I \end{bmatrix},$$

$$\begin{aligned} \Sigma_2 = (&-K^T B^T P \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \\ &-\sigma\tau_{11} K^T B^T H_{11} \quad -\sigma\hat{\tau}_1 K^T B^T H_{12} \quad -\hat{\sigma}\tau_{12} K^T B^T H_{11} \quad -\hat{\sigma}\hat{\tau}_2 K^T B^T H_{12} \\ &-\delta\tau_{12} K^T B^T H_{21} \quad -\delta\hat{\tau}_2 K^T B^T H_{22} \quad -\hat{\delta}\tau_{21} K^T B^T H_{21} \quad -\hat{\delta}\hat{\tau}_3 K^T B^T H_{22}), \end{aligned}$$

and

$$\Sigma_3 = (0 \quad 0 \quad 0 \quad 0 \quad \Lambda(t) \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0),$$

The inequality of Eq. 7.29 is satisfied if $\Sigma_1 + \Sigma_2 Y^{-1} \Sigma_2^T + \Sigma_3^T Y \Sigma_3 < 0$ and $Y > 0$ holds. Using schur complement, LMI of Eq. 7.18 can be derived. It completes the proof of Theorem 7.1.

7.5 Simulation Results

To witness the proposed delay-range-dependent synchronization criteria for nonlinear master-slave systems subject to time-varying input delay and state delay in overlapping contrast, a numerical example of gyros systems is illustrated. Dynamical gyro systems are extensively used in many engineering applications especially in aerospace engineering for aeronautical operation. The normalized motion of equation for dynamical gyro system is described by [54]

$$\ddot{\theta} + C_1\dot{\theta} + C_2\dot{\theta}^3 + h(\theta) - \beta\sin\theta = -f\sin\omega t\sin\theta, \qquad \text{(Eq 7.30)}$$

where

$$h(\theta)=\alpha^2\frac{(1-\cos\theta)^2}{\sin^3\theta}$$

and $f\sin\omega t\sin\theta$ represents the parametric excitation, $C_1\dot{\theta}$ and $C_2\dot{\theta}^3$ are the linear and nonlinear terms, respectively. The variable θ is rotational angle. Nonlinear part of the gyro system is represented by $\alpha^2\frac{(1-\cos\theta)^2}{\sin^3\theta}-\beta\sin\theta$. Normalized state equations can be derived by selecting

$x_1=\theta$ and $x_2=\frac{d\theta}{dt}$, the dynamical equation of gyro transformed as

$$\dot{x}_1=x_2, \qquad (Eq\ 7.31)$$

$$\dot{x}_2=-C_1y_1-C_2y_1^3-\alpha^2\frac{(1-\cos x_1)^2}{\sin^3 x_1}+\beta\sin x_1-f\sin\omega t\sin x_1. \qquad (Eq\ 7.32)$$

System parameters are selected as follows, $C_1=0.5$, $C_2=0.05$, $\alpha=10$, $\beta=1.0$, $\omega=2$ and $f=35.5$. Lower bound (τ_{21}) and upper bounds (τ_{22}) for input time-delay (τ_{id}) are selected 0.93 and 2.02, respectively. Similarly Lower bound (τ_{11}) and upper bounds (τ_{12}) for input time-delay (τ_{sd}) are selected 0.10 and 0.30, respectively. Whereas Lipschitz constants $\Omega_1=0.3$, $\Omega_2=0.4$ and delay derivative bounds $\eta_1=0.3$ and $\eta_2=0.42$ are selected. By applying the condition derived in Theorem 7.1, controller gain matrix is computed as $K=[1.504 \quad 2.015]$. Initial conditions for the master systems and slave systems are as follows

$$\begin{aligned}x_{m1}(t)&=0.6+0.06\sin 0.2t,\\ x_{m2}(t)&=-0.8-0.02\sin 0.1t,\end{aligned} \qquad (Eq\ 7.33)$$

$$\begin{aligned}x_{s1}(t)&=-0.05-0.01\sin 0.09t,\\ x_{s2}(t)&=1.1+0.013\sin 0.15t.\end{aligned} \qquad (Eq\ 7.34)$$

Figures 7.2(a) and 7.2(b), shows the phase portrait of master gyro system and slave gyro system in the absence of controller. It can be seen that response is not similar to each other of the master and slave systems under slightly different initially condition. When control signal is not applied to the slave systems, synchronization errors

$e_1(t) = x_{m1}(t) - x_{s1}(t)$ and $e_2(t) = x_{m2}(t) - x_2(t)$ are shown in Figure 7.3, which shows that both error states are not converging to the origin and oscillating around it, results unsynchronized behavior among master and slave systems.

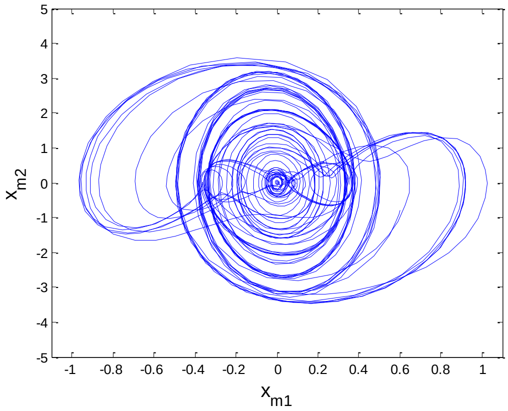


Figure 7.2(a): Phase portrait of master gyro system with multiple delays without controller

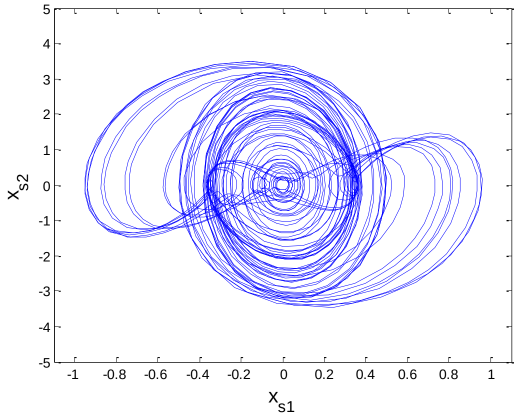


Figure 7.2(b): Phase portrait of slave gyro system with multiple delays without controller

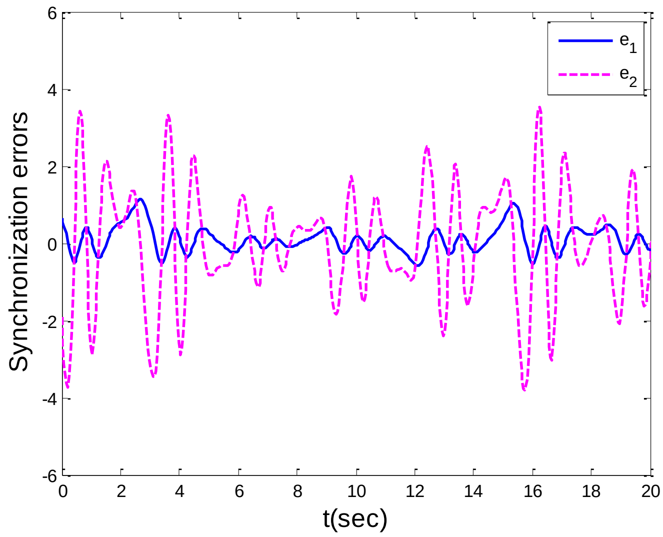


Figure 7.3: Synchronization errors $e_1(t)$ and $e_2(t)$ without controller

Figures 7.4(a) and 7.4(b), shows the phase portrait of master gyro system and slave gyro system in the presence of state-feedback controller. It can be seen that trajectory of the slave systems is following the trajectory of master systems under different initially condition. Figure 7.5, shows the errors $e_1(t)$ and $e_2(t)$, when controller is activated. Both error states are converging to the origin and synchronization between master and slave systems is established.

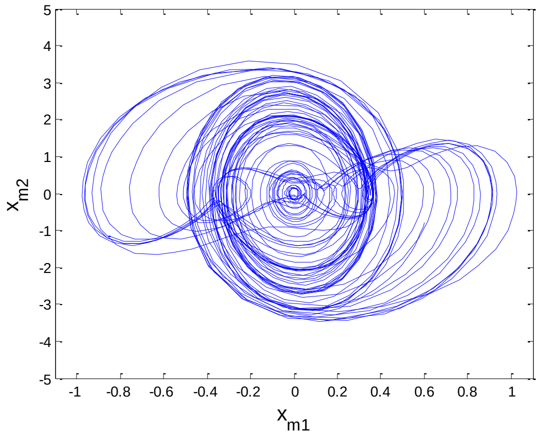


Figure 7.4(a): Phase portrait of master gyro system with multiple delays with controller

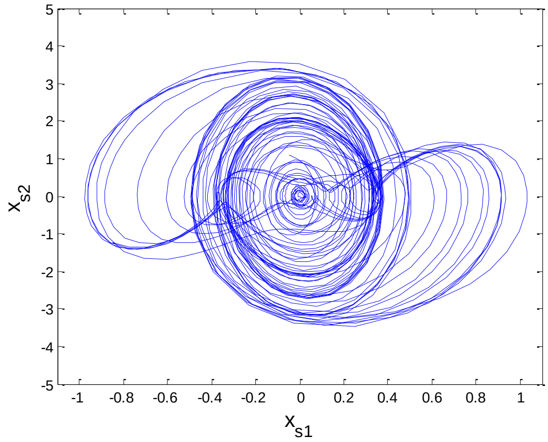


Figure 7.4(b): Phase portrait of slave gyro system with multiple delays with controller

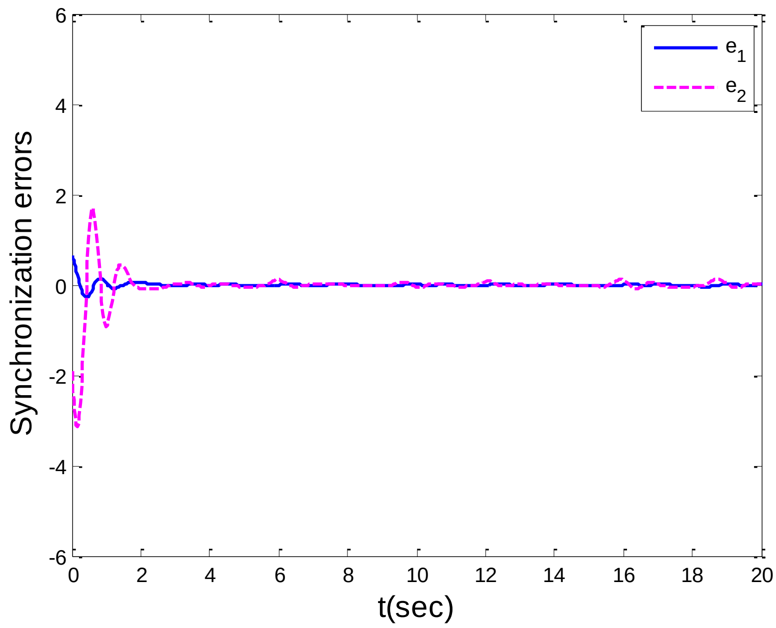


Figure 7.5: Synchronization errors $e_1(t)$ and $e_2(t)$ with controller

7.6 Summary

The problem of synchronization of multiple delays includes time-varying input delay and state delays with overlapping constraints has been investigated in this chapter and delay-range-dependent synchronization criterion is provided. Time varying input delay and state delays are considered in the intervals and partitions in the non-uniform subintervals. Problem has been designed by considering the four possible cases of overlapping delays on the basis of binary logic.

Zero order hold technique is implied to derive a simple state feedback control law. Furthermore, an advanced LMI conditions are established via improved Lyapunov-Krasovskii functional for multiple and overlapping delays. In the end, to ensure the feasibility and effectiveness of the proposed synchronization methodology has been demonstrated by a numerical simulation example of two dynamical nonlinear gyro systems.

Chapter 8

CONCLUSIONS AND FUTURE WORK

A brief conclusion of thesis is outlined in this Chapter. Moreover, some future research proposals are suggested for the researchers interested to work in the area of synchronization of nonlinear systems.

8.1 Conclusions

In this thesis, the problem of synchronization of nonlinear master-slave systems under the restraints of time-delays and parametric uncertainties was investigated for both distinctive and identical nonlinear systems by designing appropriate control laws.

Firstly, this thesis addressed uncomplicated static state feedback and adaptive dynamic controller synthesis methodologies for synchronization of the different chaotic oscillators containing mutually Lipschitz nonlinearities. An algebraic Riccati inequality based and an LMI-based formulations were provided to compute the proposed controller gain matrix for synchronization of the mutually Lipschitz nonlinear systems. The proposed design conditions, developed using a quadratic Lyapunov function, the mutually Lipschitz condition and the uniformly ultimately bounded stability theory, offer robustness against disturbances and dynamical perturbations. An adaptive schema was investigated for synchronization of the chaotic systems in order to cancel the nonlinearities arising from mismatch between dynamics of the systems. The proposed control schemes, uncomplicated to design as well as to implement, can be applied for synchronization of different chaotic oscillators with unknown dynamics.

Secondly, the problem of synchronization of the drive-response systems is investigated subject to slope-restricted input nonlinearity as well as time-varying input delay. To obtain the synchronization conditions for the master-slave systems, different tools and techniques like Jensen's inequality, delay range expression, delay-derivative-bounds, triple-integral-based LK functional, LMI-tools, LPV method, and cone complementary linearization were employed. The proposed design conditions ensure asymptotic convergence of the synchronization error to the origin. In contrast to the conventional sliding mode and adaptive control techniques, the proposed technique is simple in design, straightforward in implementation, and capable of handling input time-delay. Moreover, the resultant scheme is capable to incorporate both small and large time-delay in an interval for synchronization of the nonlinear systems because the input delay was treated using an advanced delay-range-dependent approach. Hence, the proposed chaos synchronization schemes are practical, reliable and less conservative compared to the existing methods. In future, the proposed synchronization techniques can be extended for other complexities like output delays, state delays, input saturation, adaptation of unknown parameters, robustness against perturbations, fast convergence rate, and handling of parametric uncertainties. In the end, a numerical simulation examples of were illustrated to witness the effectiveness of the proposed synchronization control schemes.

8.2 Future Work

Future research proposals on the basis of the carried-out research works are suggested for the researchers interested to work in the area of synchronization of nonlinear systems as follows.

- The problem of synchronization of nonlinear systems can be extended for frequency, phase and amplitude synchronization of tied grid elements.
- In this thesis, the research work was focused on robust and robust adaptive synchronization of non-identical nonlinear systems, which can be excelled for time-delay systems by considering the input and output delays.
- Delay-dependent and delay-range-dependent synchronization criteria for non-identical nonlinear systems can be carried out in future.
- The problem of synchronization of nonlinear master-slave systems under the constraints of time-varying input delay and slope restricted input nonlinearity can be extended for other complexities like output delays, state delays, input saturation, adaptation of unknown parameters, robustness against perturbations, fast convergence rate, and handling of parametric uncertainties.
- The problem of synchronization of time-delay nonlinear systems under the effects of multiple and overlapping delay can also be worked out for the robust and the

robust adaptive synchronization methodologies to deals with the parametric uncertainties, external disturbances, saturation, and unknown parameters etc.

- Synchronization of two systems was considered in this thesis. This work can be extended for multiple systems, called multi-agents, to attain a consensus or formation. Further communication constraints and coupling terms can also be studied for synchronization analysis.

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