CAPITAL UNIVERSITY OF SCIENCE AND TECHNOLOGY, ISLAMABAD

On the Planar Central Configurations of Rhomboidal and Triangular Four- and Five-Body Problems

by

Raz Muhammad

A thesis submitted in partial fulfillment for the degree of Master of Philosophy

in the

Faculty of Computing
Department of Mathematics

2019

To my parents and teachers

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Acknowledgements

First and foremost I would like to pay my cordial gratitude to the almighty **Allah** who created us as a human being with the great boon of intellect. I would like to pay my humble gratitude to the Allah almighty, for blessing us with the holy prophet **Hazrat Mohammad (Sallallahu Alaihy Wa'alihi wassalam)** for whom the whole universe is being created. He (Sallallahu Alaihy Wa'alaihi wassalam) removed the ignorance from the society and brought us out of darkness. Thanks again to that Monorealistic power for granting me with a strength and courage whereby I dedicatly completed my MPhil thesis with positive and significant result.

I owe honour, reverence and indebtedness to my acomplished supervisor and mentor **Dr. Abdul Rehman Kashif** whose affectionate guidence, authentic supervision, keen interest and ingenuity was a source of inspiration for commencement, advancement and completion of the present study. I would like to acknowledge CUST to provide me such a favourable environment to conduct this research.

Thanks to my seniors especially Ateeq-ur-Rehman and Hazrat Bilal for helping me. Thanks to my beloved parents, brothers, sister and kind persons. Thanks to my father Miraj Muhammad, my great mother, brothers Sada Muhammad and Wisal Muhammad and my dear sister for their ever encouragement and overall support during my entire university life. May almighty Allah show his blessing and prosperity on all those who assisted me during the complition of this thesis.

Abstract

We study central configuration of a set of symmetric planar five-body problems where in the first case, four of the masses effectively form a rhombus, with the fifth mass placed anywhere on the axis of symmetry. We construct expressions for mass ratios and identify regions in the phase space where it is possible to choose positive masses which will make the central configuration and in the second case five masses will form a triangle, with one of the masses moved up on the axis of symmetry. As a particular case, we also discuss a four-body configuration with a zero central mass.

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Abbreviations

2BPs	Two-Body Ptoblems
3BPs	Three-Body Problems
CCs	Central Configurations
5BPs	Five-Body Problems
SI	System International
Ms	Mass of the Sun
N	Numerator
D	Denominator
4BPs	Four-Body Problems
R	Region

Symbols

symbol	name	unit
G	Universal gravitational constant	$m^3 kg^{-1} s^{-2}$
F	Gravitational force	Newton
r	Distance	Meter
P	Linear momentum	$kgms^{-1}$
L	Angular momentum	$kgm^2 s^{-1}$
m_i	Point masses	kg
Δ_{ijk}	Area of the triangle	unit square
μ_i	Ratio of the masses	
$\mathbb{R}$	Real number	
$\ni$	Such that	
$\forall$	For all	
$\in$	Belongs to	
R_{μ_i}	Region of μ_i	
μ_i	Ratio of the masses	

Chapter 1

Introduction

In mechanics, the n-body problem is the problem which predicts the individual motion of a group of celestial bodies interacting with each other gravitationally. Motivation behind the solution of these kind of problems is to learn about the motions of the Moon, Sun, planets etc. Mathematicians and astronomers are attracted towards n-body problems in 17th century. The statement of the problem is "What would be the orbit, if we are given n celestial objects interacting with each others under the gravitational forces." Newton solved two body-problem (2BP) by his universal law of gravitational. The problem has no significant solution if $n \geq 3$, although it may gives us a particular solutions if we have a restricted n-body problem. In the last four centuries, mathematicians and astronomers continued work on the n-body problems. First of all, Kepler, in 17th century, defining the elliptical trajectories of the planets around the sun in his laws of planetary motion. ***Philosophiae Naturalis Principia Mathematica*** [1], one of the most important works in the history of science, in which Newton derived and formulated Kepler's law. As a special case, the law for two particles when they are interacting with each others by gravitational force is:

$$\mathbf{F} = G\frac{m_1 m_2}{r^2}\mathbf{r}, \tag{1.1}$$

where the two masses m_1 and m_2 are apart from each others by r and G, is the universal gravitational constant. After the justification of Kepler's laws, Newton turned his attention to comparatively more complex systems. Although, after a lot of struggle, he was unable throughout his life to get any breakthrough in three-body problems (3BPs). After twenty years the death of Newton, Alexis Clairaut succeeded in presenting an approximation for the 3BP. After some small adjustment, his work accounted for the perigee of the moon, which was the aim of Newton. He won the St. Petersburg Academy prize in 1752. When Halley's comet passed by earth in 1759, the value of his approximations was amply to demonstrate its motion. He himself take off the margin of error which he predicted in his equations, within a month.

Leonhard Euler also work on the 3BP at the same time. The extremely influential work of Henri Poincaré on 3BP has end the classical period of work. King Oscar II of Sweden, in the late 19th century setup an award for solving the n-body problem (a more general form of the problem with n rather than 3 masses) on the recommendation of Karl Weierstrass, Gsta Mittag-Leffler, and Charles Hermite. The statement was as follow: [2]

"*Given a system of arbitrarily many point masses that attract each other according to Newtons law, under the assumption that no two points ever collide, try to find a representation of the coordinates of each point as a series in a variable that is some known function of time and for all of whose values the series converges uniformly*".

Many eminent mathematicians and astronomers like Carl Gustav Jacob Jacobi, Lagrange and Euler working on it in the 19th century. Until 1991, the general solution to the problem was remained unsolved, when a Professor in the University of Arizona, Qiudong Wang published "The global solution of n-body Problem" [3]. However his work meets the requirements of King Oscars problem, Wang himself would characterize his result as a tricky, simple and useless answer while praising the publications the Poincaré did complete [4].

1.1 Central Configurations

A *central configuration* (CC) is a special arrangement of point masses interacting by Newton's law of gravitation with the following property *"the gravitational acceleration vector produced on each mass by all others should point toward the center of mass and proportional to the distance to the center of mass"*. CCs play an important role in the study of the Newtonian n-body problem. For example, they lead to the only explicit solutions of the equations of motion, they govern the behavior of solutions near collisions, and they influence the topology of the integral manifolds.

As it is known that roughly 67% of our galaxy's stars are included in multi-stellar systems, that's why understanding the four-body problem (4BP) and five-body problem (5BP) is very valuable. CC is valuable for understanding the gravitational n-body problems [5–7]. They can also be helpful in finding solutions (explicit homographic) of the equations of motion and periodic solutions [8]. CCs are also helpful in understanding that type of solutions which is near the collisions and the energy level sets that holds the CC to find the topology of the integral manifolds. In [9] Simmons and Bakker gave analysis (linear stability) of a rhomboidal 4BP and show that collisions (isolated binary) can be regularized at origin. For regularization of binary collisions, Pérez-Chavela and Lacomba [10] had earlier analyzed the same kind of problem. They study it's escape and capture orbits in [11]. For equal masses, Yan [12] studied the existence and linear stability of periodic orbits of the same model. For rhomboidal 4BP, to find the regions of stability Ji, Liao, and Liu [13] uses the Poincaré section. [14–16] are some other work related to rhomboidal 4BP.

Most of the periodic orbits in a rhomboidal 5BP when 2 pairs of masses are put on the edges of a rhombus and fifth stationary object is put at the center which is also the center of mass of the system, calculated by Corbera and Llibre [17]. Shoaib et al. in [18] obtained CC regions for the same problem. Different features of the restricted rhomboidal 5BP which has 4 positive objects on the edges of a rhombus and 5^{th} infinitesimal object is placed in the plane of the 4 objects are studied by Marchesin and Vidal [19] and Kulesza et al. [20].

1.2 Thesis Contribution

We assume symmetric 5BP, where on the axis of symmetry, there are three different collinear masses. The other 2 masses are placed symmetrically on each side as shown in Figure 3.2. CCs regions are derived for 4BP and 5BP. The possible CCs regions are first obtained analytically and explored numerically as well.

1.3 Dissertation Outlines

This dissertation is divided into four chapters.

In **Chapter 1** the introduction of the problem and aim of this research is shortly discussed.

In **Chapter 2**, we revisit the basic definitions related to celestial mechanics, Newton's laws of motion and Kepler's laws of planetary motion. In the next portion of this chapter, we discuss the 2BP and n-body problem briefly.

In **Chapter 3**, the paper [21] is reviewed comprehensively.

In **chapter 4**, we conclude the thesis.

Chapter 2

Preliminaries

In this chapter, we need to recall the basic definitions, concepts, terminologies and laws from existing literature [22, 23], related to our research work.

2.1 Basic Definitions

Definition 2.1.1.(Motion)

"Motion is the action used to change the location or position of an object with respect to the surroundings over time."

Definition 2.1.2. (Mechanics)

"Mechanics is a branch of physics concerned with motion or change in position of physical objects. It is sometimes further subdivided into:

1. **Kinematics**, which is concerned with the geometry of the motion,

2. **Dynamics**, which is concerned with the physical causes of the motion,

3. **Statics**, which is concerned with conditions under which no motion is apparent."

Definition 2.1.3. (Scalar)

"Various quantities of physics, such as length, mass and time, requires for their specification a single real number (apart from units of measurement which are decided upon in advance). Such quantities are called **Scalars** and the real number is called the magnitude of the quantity."

Definition 2.1.4. (Vector)
"Other quantities of physics, such as displacement, velocity, momentum, force etc require for their specification a direction as well as magnitude. Such quantities are called **Vectors**."

Definition 2.1.5. (Field)
"A field is a physical quantity associated with every point of spacetime. The physical quantity may be either in vector form, scalar form or tensor form."

Definition 2.1.6. (Scalar field)
"If at every point in a region, a scalar function has a defined value, the region is called a scalar field. i.e.,

$$f : \mathbb{R}^3 \to \mathbb{R},$$

e.g. temperature and pressure fields around the earth."

Definition 2.1.7. (Vector field)
"If at every point in a region, a vector function has a defined value, the region is called a vector field.

$$V : \mathbb{R}^3 \to \mathbb{R}^3,$$

e.g. tangent vector around a smooth curve."

Definition 2.1.8. (Conservative vector field)
"A vector field $\mathbf{V}$ is conservative if and only if there exists a continuously differentiable scalar field f such that $\mathbf{V} = -\nabla f$ or equivalently if and only if

$$\nabla \times \mathbf{V} = \texttt{Curl}\mathbf{V} = \mathbf{0}.\text{''}$$

Definition 2.1.9. (Tensor field)
"If at every point in a region, a tensor function has a defined value, the region is called a tensor field. e.g. Riemann curvature tensor, stress-energy-momentum tensor, electromagnetic tensor."
Definition 2.1.10. (Uniform force field)
"A force field which has constant magnitude and direction is called a uniform or constant force field. If the direction of the field is taken as negative z direction and magnitude is constant $F_0 > 0$, then the force field is given by

$$\mathbf{F} = -F_0\hat{\mathbf{k}}.\text{''}$$

Definition 2.1.11. (Central force)
"Suppose that a force acting on a particle of mass m such that
(a) it is always directed from m toward or away from a fixed point O,
(b) its magnitude depends only on the distance r from O.
then we call the force a central force or central force field with O as the center of force. In symbols $\mathbf{F}$ is a central force if and only if

$$\mathbf{F} = f(r)\mathbf{r}_1 = f(r)\frac{\mathbf{r}}{r},$$

where $\mathbf{r}_1 = \frac{\mathbf{r}}{r}$ is a unit vector in the direction of $\mathbf{r}$. The central force is one of attraction towards O or repulsion from O according as *f(r)* < 0 or *f(r)* > 0 respectively."

Definition 2.1.12. (Degree of freedom)
"The number of coordinates required to specify the position of a system of one or more particles is called number of degree of freedom of the system.
Example: A particle moving freely in space requires 3 coordinates, e.g. (x, y, z), to specify its position. Thus the number of degree of freedom is 3."

Definition 2.1.13. (Center of mass)

"Let $r_1, r_2, ..., r_n$ be the position vector of a system of n particles of masses $m_1, m_2, ...m_n$ respectively. The center of mass or centroid of the system of particles is defined as that point having position vector

$$\hat{\mathbf{r}} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + ... + m_n\mathbf{r}_n}{m_1 + m_2 + ... + m_n} = \frac{1}{\mathbf{M}}\sum_{\nu=1}^{n} m_\nu \mathbf{r}_\nu,$$

where

$$\mathbf{M} = \sum_{\nu=1}^{n} m_\nu,$$

is the total mass of the system."

Definition 2.1.14. (Center of gravity)

"If a system of particles is in a uniform gravitational field, the center of mass is sometimes called the center of gravity."

Definition 2.1.15. (Torque)

"If a particle with a position vector $\mathbf{r}$ moves in a force field $\mathbf{F}$, we define $\boldsymbol{\tau}$ as torque or moment of the force as

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}.$$

The magnitude of $\boldsymbol{\tau}$ is

$$\tau = rF\sin\theta.$$

The magnitude of torque is a measure of the turning effect produced on the particle by the force."

Definition 2.1.16. (Momentum)

"The linear momentum $\mathbf{p}$ of an object with mass m and velocity $\mathbf{v}$ is defined as:

$$\mathbf{p} = m\mathbf{v}.$$

Under certain circumstances the linear momentum of a system is conserved. The linear momentum of a particle is related to the net force acting on that object:

$$\mathbf{F} = m\mathbf{a} = m\frac{d\mathbf{v}}{dt} = \frac{d}{dt}(m\mathbf{v}) = \frac{d\mathbf{p}}{dt}.$$

The rate of change of linear momentum of a particle is equal to the net force acting on the object, and is pointed in the direction of the force. If the net force acting on an object is zero, its linear momentum is constant (conservation of linear momentum). The total linear momentum $\mathbf{p}$ of a system of particles is defined as the vector sum of the individual linear momentum.

$$\mathbf{p} = \sum_{1}^{n} \mathbf{p}_i.\text{''}$$

Definition 2.1.17. (Point-like particle)
"A point-like particle is an idealization of particles mostly used in different fields of physics. Its defining features is the lacks of spatial extension:being zero-dimensional, it does not take up space. A point-like particle is an appropriate representation of an object whose structure, size and shape is irrelevant in a given context. e.g., from far away, a finite-size mass (object) will look like a point-like particle."

Definition 2.1.18. (Angular momentum)
"Angular momentum for a point-like particle of mass m with linear momentum $\mathbf{p}$ about a point O, defined by the equation

$$\mathbf{L} = \mathbf{r} \times \mathbf{p},$$

where $\mathbf{r}$ is the vector from the point O to the particle. The torque about the point O acting on the particle is equal to the rate of change of the angular momentum about the point O of the particle i.e.,

$$\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt}.\text{''}$$

Definition 2.1.19. (Holonomic and non-holonomic constraints)

"The limitations on the motion are often called constraints. If the constraint condition can be expressed as an equation

$$\phi(\mathbf{r}_1, \mathbf{r}_2, ...\mathbf{r}_n, t) = \mathbf{0}$$

connecting the position vector of the particles and the time, then the constraints are called *holonomic*.
If it cannot be so expressed it is called *non-holonomic*."

Definition 2.1.20. (Inertial frame of reference)
"A frame of reference that remains at rest or moves with constant velocity with respect to other frames of reference is called inertial frame of reference. Actually, an unaccelerated frame of reference is an inertial frame of reference. In this frame of reference a body does not acted upon by external forces. Newton's laws of motion are valid in all inertial frames of reference. All inertial frames of reference are equivalent."

2.2 Kepler's Laws of Planetary Motion

"Kepler's three laws of planetary motion can be described as follows:

1. All planets are moving in an elliptical path with sun at one focus.
2. The radius vector drawn from the sun to a planet sweeps out equal areas in equal time intervals.
3. The cube of the semi major axis of the planetary orbits are proportional to the square of the planets periods of revolution. Mathematically, Kepler's third law can be written as:

$$T^2 = \left(\frac{4\pi^2}{GM_s}\right) r^3,$$

where T is the time period, r is the semi major axis, M_s is the mass of sun and G is the universal gravitational constant."

2.3 Newton's Laws of Motion

"The following three laws of motion given by Newton are considered the axioms of mechanics:

1. **First law of motion**

   Every particle persists in a state of rest or of uniform motion in a straight line unless acted upon by a force.

2. **Second law of motion**

   If $\mathbf{F}$ is the external force acting on a particle of mass m which as a reaction is moving with velocity $\mathbf{v}$, then

   $$\mathbf{F} = \frac{d}{dt}(m\mathbf{v}) = \frac{d\mathbf{P}}{dt}.$$

   If m is independent of time this becomes

   $$\mathbf{F} = m\frac{d\mathbf{v}}{dt} = m\mathbf{a},$$

   where $\mathbf{a}$ is the acceleration of the particle.

3. **Third law of motion**

   For every action, there is an equal and opposite reaction."

2.3.1 Newton's Universal Law of Gravitation

"Every particle of matter in the universe attracts every other particle of matter with a force which is directly proportional to the product of the masses and inversely proportional to the square of the distance between them. Hence, for any two particles separated by a distance r, the magnitude of the gravitational force $\mathbf{F}$ is:

$$\mathbf{F} = G\frac{m_1 m_2}{r^2}\mathbf{r}$$

where G is universal gravitational constant. Its numerical value in SI units is $6.67408 \times 10^{-11} m^3 kg^{-1} s^{-2}$."

2.4 Two Body Problem

"The two-body problem , first studied and solved by Newton, states: Suppose that the positions and velocities of two massive bodies moving under their mutual gravitational force are given at any time t, then what should their position and velocities be for any other time t, if the masses are known? Example include a planet orbiting around a star (Earth-Sun, Moon-Earth), two stars orbiting around each other, satellite orbiting around orbit. The two-body problem is very important because of the following facts:

1. It is the only gravitational problem in celestial mechanics, apart from rather restricted solutions of three body problem, for which we have a complete and general solution.
2. A wide range of practical orbital motion problems can be treated as approximate two-body problems.
3. The two-body solution may be used to provide approximate orbital parameters and predictions or serve as a starting point for the generation of analytical solutions valid to higher orders of accuracy."

2.4.1 The Solution to the Two-Body Problem

"The governing law for the two-body is Newton's universal gravitational law:

$$\mathbf{F} = G\frac{m_1 m_2}{r^3}\mathbf{r}, \tag{2.1}$$

for two masses m_1 and m_2 separated by a distance of $\mathbf{r}$, and G the universal gravitational constant. The aim here is to determine the path of the particles for any time t, if the initial positions and velocities are known. In Figure 2.1, the force of attraction $\mathbf{F}_1$ is directed along r towards m, while the force $\mathbf{F}_2$ on M is in opposite direction. By Newton's third law,

$$\mathbf{F}_1 = -\mathbf{F}_2. \tag{2.2}$$

From Figure 2.1,

$$\mathbf{F}_1 = G\frac{mM}{r^3}\mathbf{r}. \tag{2.3}$$

Using Newton's second law of motion and by equation (2.1) and (2.2), the equation of motion of the particles under their mutual gravitational attractions are given by

$$m\ddot{\mathbf{r}}_1 = m\frac{d^2\mathbf{r}_1}{dt^2} = G\frac{mM}{r^3}\mathbf{r}, \tag{2.4}$$

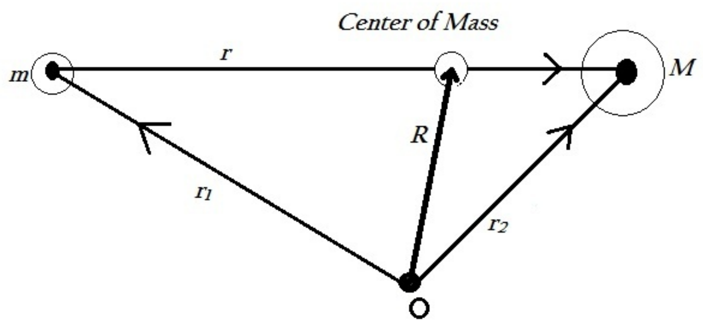


Figure 2.1: Center of mass of two body system

$$M\ddot{\mathbf{r}}_2 = M\frac{d^2\mathbf{r}_2}{dt^2} = -G\frac{mM}{r^3}\mathbf{r}, \tag{2.5}$$

where $\mathbf{r}_1$ and $\mathbf{r}_2$ be the position vectors from the reference O as shown in Figure 2.1. Adding equation (2.4) and (2.5), we get:

$$m\ddot{\mathbf{r}}_1 + M\ddot{\mathbf{r}}_2 = \mathbf{0}, \tag{2.6}$$

integrating above equations yields:

$$m\dot{\mathbf{r}}_1 + M\dot{\mathbf{r}}_2 = \mathbf{c}_1, \tag{2.7}$$

that the total linear momentum of the system i.e. $m\mathbf{v}_m + M\mathbf{v}_M = \mathbf{c}_1$ is a constant. Again integrating equation (2.7) implies:

$$m\mathbf{r}_1 + M\mathbf{r}_2 = \mathbf{c}_1 t + \mathbf{c}_2, \tag{2.8}$$

where $\mathbf{c}_1$ and $\mathbf{c}_2$ are constant vectors.
Using the definition of center of mass in 2BP, $\mathbf{R}$ is defined as:

$$\begin{aligned}(m+M)\mathbf{R} &= m\mathbf{r}_1 + M\mathbf{r}_2,\\ M_t\mathbf{R} &= m\mathbf{r}_1 + M\mathbf{r}_2,\end{aligned} \tag{2.9}$$

where $M_t = m + M$. Taking the derivative of equation (2.9) and comparing with equation (2.21), we get

$$M_t\dot{\mathbf{R}} = \mathbf{c}_1 \quad \Rightarrow \quad \dot{\mathbf{R}} = \frac{\mathbf{c}_1}{M_t} = constant$$

show that $\dot{\mathbf{R}} = \mathbf{v}_c$ (velocity of center of mass) is constant.
Subtracting the equations (2.4) and (2.5) gives:

$$\ddot{\mathbf{r}}_1 - \ddot{\mathbf{r}}_2 = \frac{GM}{r^3}r + \frac{Gm}{r^3}\mathbf{r}, \tag{2.10}$$

$$\ddot{\mathbf{r}}_1 - \ddot{\mathbf{r}}_2 = G(m+M)\frac{\mathbf{r}}{r^3}$$

$$\Rightarrow \ddot{\mathbf{r}} = \mu\frac{\mathbf{r}}{r^3}$$

$$\Rightarrow \ddot{\mathbf{r}} + \mu\frac{\mathbf{r}}{r^3} = \mathbf{0}, \tag{2.11}$$

where $\mu = G(m+M)$ is defined as reduced mass and $\mathbf{r}_1 - \mathbf{r}_2 = -\mathbf{r}$, see Figure 2.1. Taking the cross product of $\mathbf{r}$ with equation (2.11) we obtain:

$$\mathbf{r} \times \mu\ddot{\mathbf{r}} + \frac{\mu^2}{r^3}\mathbf{r} \times \mathbf{r} = \mathbf{0}$$
$$\Rightarrow \mathbf{r} \times \ddot{\mathbf{r}} = \mathbf{0}, \tag{2.12}$$

integrating above equation yields:

$$\mathbf{r} \times \dot{\mathbf{r}} = \mathbf{L}, \tag{2.13}$$

where $\mathbf{L}$ is a constant vector. We may write equation (2.12),

$$\Rightarrow \mathbf{r} \times \mu\ddot{\mathbf{r}} = \mathbf{0},$$
$$\Rightarrow \mathbf{r} \times \mathbf{F} = \mathbf{0}, \tag{2.14}$$

where $\mathbf{F} = \mu\ddot{\mathbf{r}} = \mu\mathbf{a}$ (μ is reduced mass i.e. constant).

From Chapter 2, by the definition of torque and angular momentum:

$$\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt} = \mathbf{r} \times \mathbf{F}, \tag{2.15}$$

comparing equations (2.14) and (2.15), we get:

$$\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt} = \mathbf{r} \times \mathbf{F} = \mathbf{0},$$
$$\frac{d\mathbf{L}}{dt} = \mathbf{0}$$

$$\Rightarrow \mathbf{L} = \text{constant},$$

i.e. angular momentum of the system is constant.

Radial and transverse components of velocity and acceleration:

If polar coordinates r and θ are taken in this plane as in Figure 2.2, the velocity components along and perpendicular to the radius vector joining m to M

are $\dot{r}$ and $r\dot{\theta}$, then,

$$\dot{\mathbf{r}} = \frac{d\mathbf{r}}{dt} = \dot{r}\mathbf{i} + r\dot{\theta}\mathbf{j}, \tag{2.16}$$

where **i** and **j** are unit vectors along and perpendicular to the radius vector. Hence, by equations (2.13) and (2.16),

$$\mathbf{r} \times (\dot{r}\mathbf{i} + r\dot{\theta}\mathbf{j}) = r^2\dot{\theta}\mathbf{k} = L\mathbf{k}, \tag{2.17}$$

where **k** is a unit vector perpendicular to the plane of the orbit. We may then write

$$r^2\dot{\theta} = L, \tag{2.18}$$

where the constant L is seen to be twice the rate of description of area by the radius vector. This is the mathematical form of Kepler's second law.

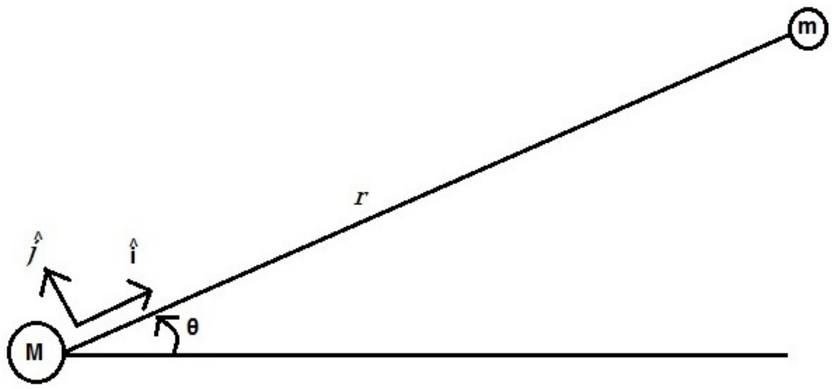


Figure 2.2: Radial and transverse components of velocity and acceleration

Now taking the scalar product of $\dot{\mathbf{r}}$ with equation (2.11), we get:

$$\dot{\mathbf{r}}.\frac{d^2\mathbf{r}}{dt^2} + \mu\frac{\dot{\mathbf{r}}.\mathbf{r}}{r^3} = 0,$$

which may be integrated to give:

$$\begin{aligned} \frac{1}{2}\dot{\mathbf{r}}.\dot{\mathbf{r}} - \frac{mu}{r} &= C, \\ \frac{1}{2}v^2 - \frac{\mu}{r} &= C, \end{aligned} \tag{2.19}$$

where C is a constant. This is th energy conservation form of the system. The quantity C is not the total energy; $\frac{1}{2}\mu^2$ is related to the kinetic energy and $\frac{-mu}{r}$ to the potential energy of the system i.e. total energy is conserved.
Recall that from celestial mechanics, components of acceleration vector along and perpendicular to the radius vector (see Figure 2.2):

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{i}} + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\hat{\mathbf{j}},$$

using above equation in (2.11), we get

$$\ddot{r} - r\dot{\theta}^2 = -\frac{\mu}{r^2}, \tag{2.20}$$

$$\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0. \tag{2.21}$$

Integrating equation (2.21) gives the angular momentum integral:

$$r^2\dot{\theta} = L, \tag{2.22}$$

making the usual substitution of

$$u = \frac{1}{r}, \tag{2.23}$$

and eliminating the time between equation (2.20) and (2.22), implies:

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{L^2}. \tag{2.24}$$

The general solution of above equation is :

$$u = \frac{\mu}{L^2} + A\cos(\theta - \theta_0), \tag{2.25}$$

where A and θ_0 are two constants of integration. Substitute $u = \frac{1}{r}$ in above equation:

$$\frac{1}{r} = \frac{\mu}{L^2} + A\cos(\theta - \theta_0)$$

$$\Rightarrow r = \frac{\frac{L^2}{\mu}}{1 + \frac{L^2 A}{\mu}\cos(\theta - \theta_0)},$$

is the polar form of the equation of the conic and may be written as:

$$r = \frac{p}{1 + e\cos(\theta - \theta_0)},$$

where

$$p = \frac{L^2}{\mu},$$
$$e = \frac{AL^2}{\mu}.$$

Eccentricity e classifies the trajectory of one celestial body around another. Thus:
(i) If $0 < e < 1$ then the orbit is elliptical,
(ii) If $e = 1$ then the orbit is a parabolic,
(iii) If $e > 1$ then the orbit is a hyperbolic.
Hence the solution of the two-body problem is a conic, includes Kepler's first law as a special case."

2.5 The Equations of Motion in the n-Body Problem

The 2BP deals much of the important work in astrodynamics, but sometimes we need to model the real world by including other bodies. The next logical step, then, is to drive formulas for 3BP. A further generalization of three body problem is n-body problem. In general, solving general differential equations of motions in n-body problem requires a fixed number of integration constants. Consider a simple gravity problem in which we have constant acceleration over time, $a(t) = a_0$. If we integrate this equation, we obtain the velocity, $v(t) = a_0 t + v_0$. Integrating once more provides, $r(t) = r_0 + v_0 t + \frac{1}{2}a_0 t^2$. To complete the solution, we must know the initial conditions. This example is a straight froward analytical solution using the initial values, or a function of the time and constants of integration, called integrals of the motion. Unfortunately, this isn't always the simple case. When initial conditions alone don't provide a solution, integrals of the motion can reduce the order of differential equations, also called the degrees of freedom of the dynamical system. Ideally, if the number of integrals equals the order of differential equations, we can reduce it to order zero. These integrals are constant functions of the initial conditions, as well as the position and velocity of at any time, hence the term constants of the motion.

For the n-body problem, a system of $3n$ second order differential equations, we need $6n$ integrals of motion for a complete solution. Conservation of linear momentum provides six, conservation of energy one, and conservation of total angular momentum three, for a total of ten. There are no laws analogous to Kepler's first two laws to obtain additional constants, thus we are left with a system of order $6n - 10$ for $n \geq 3$.

These equations for n bodies, $n \geq 3$, defy all attempts at closed-form solutions. H. Brun, in 1887, showed that there were no other algebraic integrals. Although Poincaré later generalized Brun's work, we still have only the ten known integrals. They give us insight into the motions within the three body and n-body problems. Conservation of total linear momentum assumes no external forces are on the system.

First, here we set up the equations of motions of n massive particles of masses $m_i (i = 1, 2...n)$ whose radius vectors from an unaccelerated point O are $\mathbf{r}_i$ while their mutual radius vectors are given by r_{ij} where

$$\mathbf{r}_{ij} = \mathbf{r}_j - \mathbf{r}_i \tag{2.26}$$

From Newton's laws of motion and the law of gravitation,

$$m_i \ddot{r}_i = G \sum_{j=1}^{n} \frac{m_i m_j}{r_{ij}^3} r_{ij}, \qquad (j \neq i, i = 1, 2, ...n) \tag{2.27}$$

here we note that $\mathbf{r}_{ij}$ implies that the vector between m_i and m_j is directed for m_i to m_j, thus

$$\mathbf{r}_{ij} = -\mathbf{r}_{ji} \tag{2.28}$$

The set of equations (2.27) are the required equation of motion for n-body problem, G being the constant of gravitation.

Chapter 3

On the Planar Central Configurations of Rhomboidal and Triangular Four- and Five-Body Problems

3.1 Introduction

In this review [21] research work, we set up a 5BP having 2 bodies are symmetrically put on each side of the axis of symmetry. Other 3 collinear different objects on axis of symmetry. We discussed 2 main cases. First, 4 of our objects effectively making a rhombus and the 5th one is placed on axis of symmetry (See Figure 3.1 and 3.2). In second case, all five objects will make an equilateral triangle, with one of the object moved up symmetrically. Since it is a special set up, we study configuration for four bodies as well having a zero mass object in center. We obtain the region of possible CCs for four- and five-body analytically and numerically.

3.2 Problem Formulation

Suppose that $n-1$ point masses $(m_0, m_1, ..., m_{n-1})$, $m_i > 0$, $i = 0, 1, 2, ...n-1$, $r_i, i = 0, 1, 2, ...n-1$ are the position vectors of $n-1$ masses, and the distance between any two masses are $r_{ij}, i, j = 0, 1, 2, ...n-1$.

By using the symbols of [24], it is obvious that this system will makes a planar CC (non-collinear) if the following equation holds:

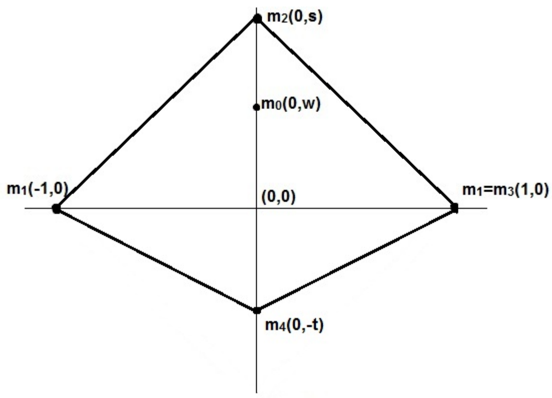


Figure 3.1: Rhomboidal five-body configurations

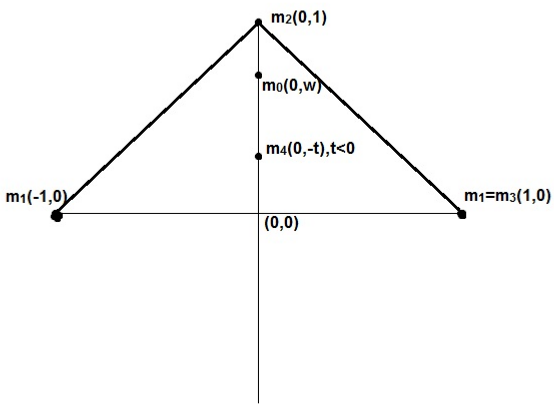


Figure 3.2: Triangular five-body configurations

$$f_{ij} = \sum_{k=0, k\neq i,j}^{n-1} m_k (R_{ik} - R_{jk})\Delta_{ijk} = 0, \tag{3.1}$$

where $R_{ij} = \frac{1}{r_{ij}^3}$ and $\Delta_{ijk} = (r_i - r_j) \wedge (r_i - r_k)$. The Δ_{ijk} shows area of the triangle obtained by $(r_i - r_j)$ and $(r_i - r_k)$. For $n = 5$, from equation (3.1) we will get the non-collinear general 5BP with 10 CC equations given below.

$$f_{01} = m_2(R_{02} - R_{12})\Delta_{012} + m_3(R_{03} - R_{13})\Delta_{013} + m_4(R_{04} - R_{14})\Delta_{014}, \quad (3.2)$$

$$f_{02} = m_1(R_{01} - R_{21})\Delta_{021} + m_3(R_{03} - R_{23})\Delta_{023} + m_4(R_{04} - R_{24}\Delta_{024}), \quad (3.3)$$

$$f_{03} = m_1(R_{01} - R_{31})\Delta_{031} + m_2(R_{02} - R_{32})\Delta_{032} + m_4(R_{04} - R_{34})\Delta_{034}, \quad (3.4)$$

$$f_{04} = m_1(R_{01} - R_{41})\Delta_{041} + m_2(R_{02} - R_{42})\Delta_{042} + m_3(R_{03} - R_{43})\Delta_{043}, \quad (3.5)$$

$$f_{12} = m_0(R_{10} - R_{20})\Delta_{120} + m_3(R_{13} - R_{23})\Delta_{123} + m_4(R_{14} - R_{24})\Delta_{124}, \quad (3.6)$$

$$f_{13} = m_0(R_{10} - R_{30})\Delta_{130} + m_2(R_{12} - R_{32})\Delta_{132} + m_4(R_{14} - R_{34})\Delta_{134}, \quad (3.7)$$

$$f_{14} = m_0(R_{10} - R_{40})\Delta_{140} + m_2(R_{12} - R_{42})\Delta_{142} + m_3(R_{13} - R_{43})\Delta_{143}, \quad (3.8)$$

$$f_{23} = m_0(R_{20} - R_{30})\Delta_{230} + m_1(R_{21} - R_{31})\Delta_{231} + m_4(R_{24} - R_{34})\Delta_{234}, \quad (3.9)$$

$$f_{24} = m_0(R_{20} - R_{40})\Delta_{240} + m_1(R_{21} - R_{41})\Delta_{241} + m_3(R_{23} - R_{43})\Delta_{243}, \quad (3.10)$$

$$f_{34} = m_0(R_{30} - R_{40})\Delta_{340} + m_1(R_{31} - R_{41})\Delta_{341} + m_2(R_{32} - R_{42})\Delta_{342}. \quad (3.11)$$

Lemma 3.2.1

The *Dziobek* equations [25–27] for a 5BP with the following position vectors $\mathbf{r}_0 = (0, w), \mathbf{r}_1 = (-1, 0), \mathbf{r}_2 = (0, s), \mathbf{r}_3 = (1, 0), \mathbf{r}_4 = (0, -t)$ (Figures 3.1 and 3.2), where s, t, w belongs to the set of real numbers are:

$$f_{01} = m_1(R_{03} - R_{13})\Delta_{013} + m_2(R_{02} - R_{12})\Delta_{012} + m_4(R_{04} - R_{14})\Delta_{014} = 0, \quad (3.12)$$

$$f_{12} = m_0(R_{10} - R_{20})\Delta_{120} + m_1(R_{13} - R_{23})\Delta_{123} + m_4(R_{14} - R_{24})\Delta_{124} = 0, \quad (3.13)$$

$$f_{14} = m_0(R_{10} - R_{40})\Delta_{140} + m_1(R_{13} - R_{43})\Delta_{143} + m_2(R_{12} - R_{42})\Delta_{142} = 0. \quad (3.14)$$

Proof

From the expression of R_{ij}, Δ_{ijk} and $\mathbf{r}_i$ $(i = 0, 1, 2, 3, 4)$ we get

$$R_{01} = R_{03} = \frac{1}{(1 + w^2)^{\frac{3}{2}}},$$

$$
\begin{aligned}
R_{02} &= \frac{1}{\mid s - w \mid^3}, \\
R_{13} &= \frac{1}{8}, \\
R_{12} = R_{23} &= \frac{1}{(1+s^2)^{\frac{3}{2}}}, \\
R_{24} &= \frac{1}{(s+t)^3}, \\
R_{14} = R_{34} &= \frac{1}{(1+t^2)^{\frac{3}{2}}}, \\
R_{04} &= \frac{1}{\mid w + t \mid^3}.
\end{aligned}
\tag{3.15}
$$

By using the following symmetries we find Δ_{ijk}, where $i, j, k = 1, 2, ..., 5$,

$$
\begin{aligned}
&\Delta_{ijk} = -\Delta_{jik} = -\Delta_{ikj} = -\Delta_{kji}, \\
&\Delta_{ijk} = \Delta_{jki} = \Delta_{kij}, \\
&\Delta_{ijk} = 0, \ if \ i = j \ or \ i = k \ or \ j = k,
\end{aligned}
$$

and

$$
\begin{aligned}
&\Delta_{043} = \Delta_{014} = t + w, \\
&\Delta_{013} = 2w, \quad \Delta_{132} = 2s, \\
&\Delta_{324} = \Delta_{142} = t + s, \\
&\Delta_{143} = 2t, \quad \Delta_{024} = 0.
\end{aligned}
\tag{3.16}
$$

Substituting $m_1 = m_3$ in Equations (3.2)-(3.11), and applying the corresponding relations from (3.15) and (3.16) we see that f_{01} and f_{03} are identical (Equations (3.2) and (3.4)). In the same way f_{12} is identical to f_{23} (Equations (3.6) and (3.9)), and f_{14} is identical to f_{34} (Equation (3.8) and (3.11)). When we put $R_{10} = R_{30}$, $R_{12} = R_{32}$ and $R_{14} = R_{34}$. This implies that $f_{13} = 0$. Similarly if we substitute $\Delta_{024} = \Delta_{042} = \Delta_{240} = 0$ in Equation (3.3), (3.5) and (3.10) gives

$$
f_{02} = ((R_{01} - R_{21})\Delta_{021} + (R_{03} - R_{23})\Delta_{023})m_1, \tag{3.17}
$$

$$f_{04} = ((R_{01} - R_{41})\Delta_{041} + (R_{03} - R_{43})\Delta_{043})m_1, \tag{3.18}$$

$$f_{24} = ((R_{21} - R_{41})\Delta_{241} + (R_{23} - R_{43})\Delta_{243})m_1. \tag{3.19}$$

Using equation (3.15) and (3.16) in (3.17) - (3.20) we get $f_{02} = f_{04} = f_{24} = 0$. We have shown that $f_{01} = f_{03}, \quad f_{12} = f_{23}, \quad f_{14} = f_{34}, \quad f_{02} = f_{04} = f_{24} = f_{13} = 0$. consequently, $f_{01}, \quad f_{12}$ and f_{14} are only three essential equations for the general 5BP (according to (3.12) - (3.14)). Hence the proof.

Theorem 3.2.2

Suppose that 2 pairs of same objects (masses) on vertices of rhomboidal shape and a 5th object is on axis of symmetry. All masses having $\boldsymbol{r}_0 = (0, w)$, $\boldsymbol{r}_1 = (-1, 0)$, $\boldsymbol{r}_2 = (0, t)$, $\boldsymbol{r}_3 = (1, 0)$, $\boldsymbol{r}_4 = (0, -t)$ *are position vectors. where* t *is a positive real number and* $w \in \mathbb{R} - \{0\}$. Then no CC exists.

Proof

First of all we will substitute $m_2 = m_4$ in Lemma 1's *Dziobek* equations and then write in the following form:

$$\begin{bmatrix} 0 & B_0 & C_0 \\ A_1 & B_1 & C_1 \\ A_2 & -B_1 & -C_1 \end{bmatrix} \begin{bmatrix} m_0 \\ m_1 \\ m_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

where

$$B_0 = (R_{03} - R_{13})\Delta_{013},$$

$$B_1 = (R_{13} - R_{23})\Delta_{123},$$

$$A_1 = (R_{10} - R_{20})\Delta_{120},$$

$$A_2 = (R_{10} - R_{40})\Delta_{140},$$

$$C_1 = (R_{14} - R_{24})\Delta_{124},$$

$$C_0 = (R_{02} - R_{12})\Delta_{012} + (R_{04} - R_{14})\Delta_{014}.$$

For the non-trivial solution of the above system of equations, the augmented matrix's determinant have to be zero. Summing up the 2^{nd} row of the augmented matrix with the 3^{rd} row, the row 3 reduce to $\{A_1 + A_2, 0, 0\}$. Therefore, $f_{A_1A_2} = A_1 + A_2 = 0$ This yields the presence of the solution which is non-trivial (In last section we will discuss the case for $m_0 = 0$ in detail). After simplification, the remaining two equations are:

$$\mu_0 = \frac{m_0}{m_2} = \frac{B_0C_1 - B_1C_0}{B_0A_2} \quad and \quad \mu_1 = \frac{m_1}{m_2} = -\frac{C_0}{B_0}. \tag{3.20}$$

Hence they will form a CC w.r.t the geometric constraint

$$\begin{aligned} f_{A_1A_2} &= (R_{01} - R_{02})\Delta_{012} + (R_{01} - R_{04})\Delta_{014} \\ &= \frac{2w}{(w+1)^{\frac{3}{2}}} + \frac{t-w}{|t-w|^3} - \frac{t+w}{|t+w|^3} = 0. \end{aligned} \tag{3.21}$$

Requirements for positive solution are:

$$f_{A_1A_2} = 0, \quad \mu_0 > 0, \quad \mu_1 > 0.$$

We have the following three cases:

(a) If t is greater than the absolute of w, $w \in \mathbb{R}\backslash\{0\}$, then the above equation will be a polynomial in w and t such that $t \neq \pm w$.

$$G_1(w) = 2t^4w - 4t^2w^3 + 4tw(w^2+1)^{\frac{3}{2}} + 2w^5 = 0. \tag{3.22}$$

The polynomial $G_1(w)$ has two complex roots and two real negative roots, which is not of our interest as we need $t > 0$.

(b) If t is less than w and w greater than zero, then $f_{A_1A_2} = 0$ will be a polynomial in w and $t \ni t \neq \pm w \neq 0$.

$$G_2(w) = t^4w - t^2(2w^3 + (1+w^2)^{\frac{3}{2}}) + w^2(w^3 - (1+w^2)^{\frac{3}{2}}) = 0. \tag{3.23}$$

Since the above polynomial $G_2(w)$ is quadratic in t^2. $G_2(w)$ has the four roots given below.

$$t(w) = \pm\sqrt{\frac{(w^2+1)^{\frac{3}{2}}}{2w} + w^2 \pm \frac{\sqrt{(w^2+1)^{\frac{3}{2}} + 8w^3(w^2+1)^{\frac{3}{2}}}}{2w}}.$$

The function below

$$f_1(w) = \frac{(w^2+1)^{\frac{3}{2}}}{2w} + w^2 - \frac{\sqrt{(w^2+1)^3 + 8w^3(w^2+1)^{\frac{3}{2}}}}{2w}$$

is negative $\forall$ values of w (See Figure 3.3(a)).
Similarly

$$f_2(w) = \frac{(w^2+1)^{\frac{3}{2}}}{2w} + w^2 + \frac{\sqrt{(w^2+1)^3 + 8w^3(w^2+1)^{\frac{3}{2}}}}{2w}$$

is positive for $w > 0$ (See Figure 3.3(b)). Now $G_2(w)$ has one negative and one positive root. Only the positive roots are interesting.
So finally we conclude that for presence of CCs, the necessary condition is satisfied at

$$t_1(w) = \sqrt{f_2(w)}.$$

If we look at Figure 3.4, it is quite clear that $\mu_0(w) > 0$ when w is less than $\sqrt{3}$. Similarly $\mu_1(w)$ is greater than 0 when w is greater than $\sqrt{3}$. Hence there is no region where both μ_0, μ_1 are positive (Figure 3.5).

(c) If t is less than the absolute of w and w is less than zero, then $f_{A_1A_2} = 0$ will be a polynomial in w and t such that $t \neq \pm w \neq 0$.

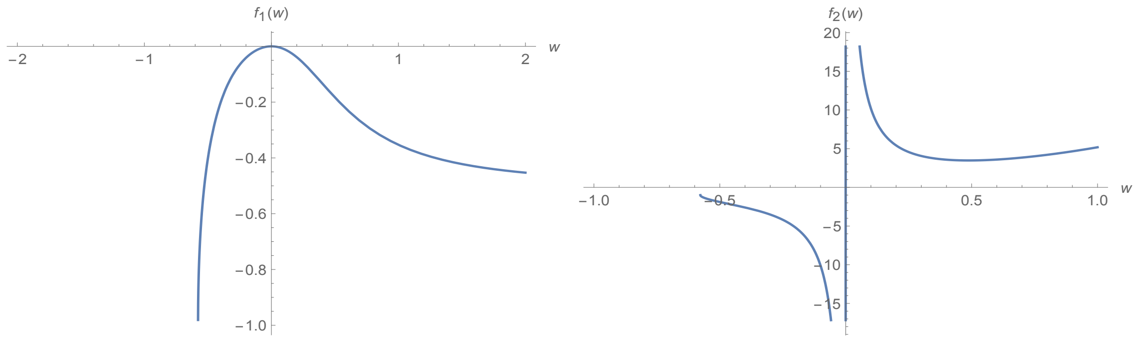
$$G_3(w) = t^4 w + t^2((w^2+1)^{\frac{3}{2}} - 2w^3) + w^2(w^3 + (w^2+1)^{\frac{3}{2}}) = 0. \quad (3.24)$$

The above polynomial $G_3(w)$ is quadratic in t^2 and has four real and different roots as a functions of w which are given below:

$$t_2(w) = -\sqrt{f_3(w)}, \qquad t_3(w) = \sqrt{f_3(w)},$$

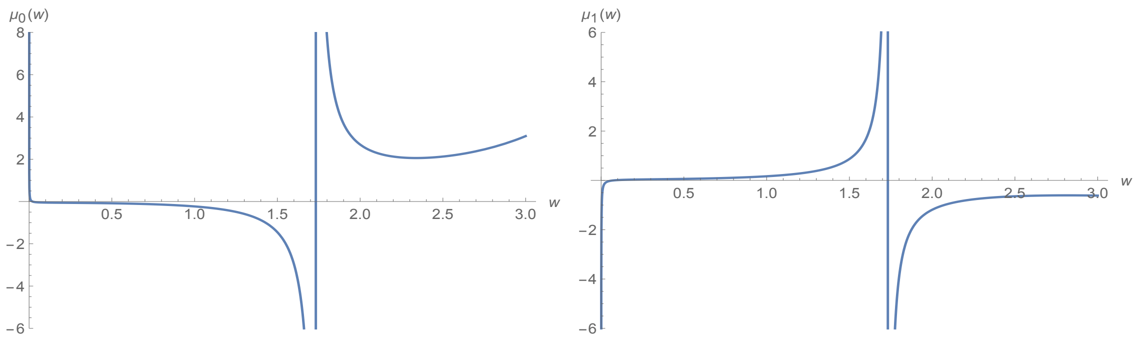
$$t_4(w) = -\sqrt{f_4(w)}, \qquad t_5(w) = \sqrt{f_4(w)},$$

where



(a) (b)

Figure 3.3: (a) $f_1(w)$, (b) $f_2(w)$



(a) (b)

Figure 3.4: (a) $\mu_0(w)$ whenever $t = t_1(w)$, (b) $\mu_1(w)$ whenever $t = t_1(w)$

$$f_3(w) = -\frac{(w^2+1)^{\frac{3}{2}}}{2w} + w^2 - \frac{\sqrt{(w^2+1)^{\frac{3}{2}}(1-8w^3)}}{2w},$$

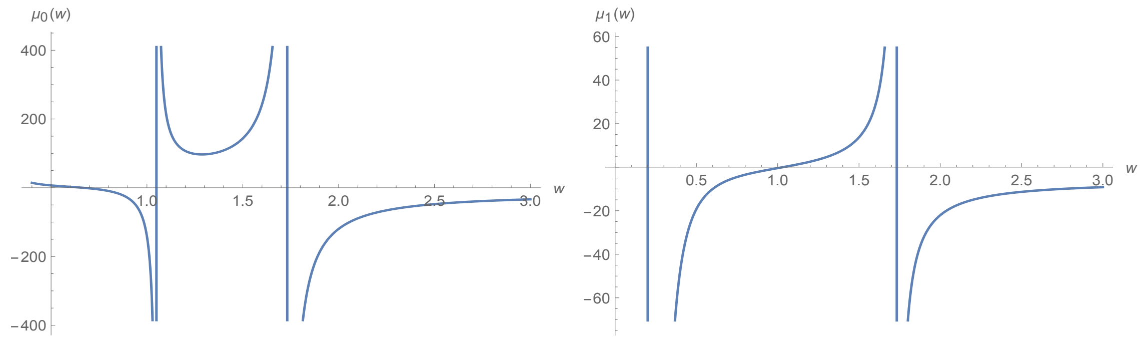
$$f_4(w) = -\frac{(w^2+1)^{\frac{3}{2}}}{2w} + w^2 = \frac{\sqrt{(w^2+1)^{\frac{3}{2}}(1-8w^3)}}{2w}.$$

As we see in Figure 3.6, that $t_3(w)$ and $t_5(w)$ are positive roots, and the remaining two roots are negative.

It is clear from Figure 3.6 that $t_3(w) > |w| \;\; \forall w < 0$. For this special case we have an extra constraint of $t < |w|$, so this root is completely invalid. Here the positive root $t_5(w)$ will be less than the absolute of w for $w < 0$ when it is defined in these intervals $(-4, -1.8)$ and $(-0.4, 0)$.

When $w \in (-0.2, 0)$, the mass ratio μ_1 will be positive as B_0 and C_0 have opposite signs in that interval; see Figure 3.7(a). A_2B_0 and $B_0C_1 - B_1C_0$ have opposite signs in the same interval so μ_0 is not positive, no CCs at $t = t_5(w)$ when $w \in (-0.4, 0)$.

For $w < -1.8, C_0$ is function of w (increasing) with the absolute minimum 0, occurring when $w \to \infty$. So $C_0 > 0 \;\; \forall w < -1.8$. Similarly, B_0 is positive as well. Hence $\mu_1 < 0$ for $w < -1.8$ (see Figure 3.7(b)). Hence the proof.



(a) (b)

Figure 3.5: (a) $\mu_0(w)$ whenever $t = -t_1(w)$, (b) $\mu_1(w)$ whenever $t = -t_1(w)$

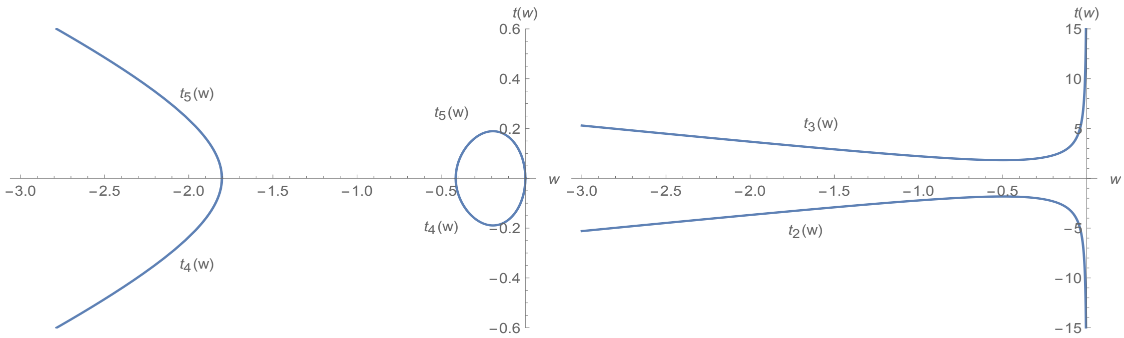


Figure 3.6: Roots of $G_3(w)$

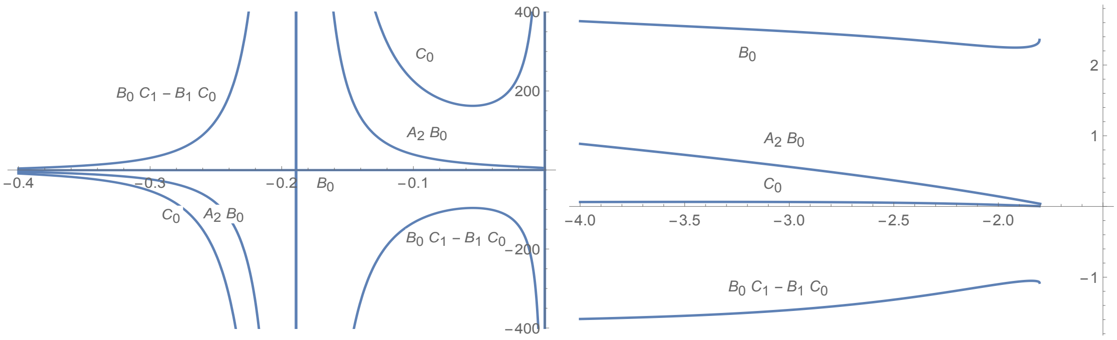


Figure 3.7: The curves B_0, C_0, A_2B_0 and $B_0C_1 - B_1C_0$ at $t = t_5(w) \ni$ (a) $w \in (-0.4, 0)$, (b) $w \in (-4, -1.8)$

Theorem 3.2.3

Let $r_1 = (-1, 0), r_2 = (0, s), \mathbf{r}_3 = (1, 0), \mathbf{r}_4 = (0, -t),\ t, s \in \mathbb{R}^2$. Then the central configurations $\mathbf{r} = (\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4)$ make a CCs in the following region

$$\begin{aligned} R_{\mu_2\mu_4}(s,t) = \{(s,t)|(&0.268 < t \leq 0.577 \wedge \frac{1-t^2}{2t} < s < \sqrt{3}) \\ &\vee (0.577 < t < \sqrt{3} \wedge \sqrt{t^2+1} - t < s < \sqrt{3})\}. \end{aligned} \tag{3.25}$$

Proof

After putting m_0 equal to zero in Lemma 3.2.1, the system of *Dziobek* equations gets the form:

$$\begin{aligned} (R_{13} - R_{23})\Delta_{123}m_1 + (R_{14} - R_{24})\Delta_{124}m_4 = 0, \\ (R_{13} - R_{14})\Delta_{134}m_1 + (R_{12} - R_{24})\Delta_{124}m_2 = 0, \end{aligned} \tag{3.26}$$

where

$$\begin{aligned} &R_{23} = (1+s^2)^{-\frac{3}{2}}, \quad R_{14} = (1_t^2)^{-\frac{3}{2}}, \quad R_{24} = (t+s)^{-3}, \\ &\Delta_{124} = -(s+t), \qquad \Delta_{134} = -2t, \qquad \Delta_{123} = -2s. \end{aligned}$$

Now we will define $\mu_2 = \frac{m_2}{m_1}$ and $\mu_4 = \frac{m_4}{m_1}$, as the above system is under determined. After solving equation (3.26) we get:

$$\mu_2 = \frac{f_5(t)(s^2+1)^{\frac{3}{2}} t(s+t)^2}{4(t^2+1)^{\frac{3}{2}} f_6(s,t)}, \tag{3.27}$$

$$\mu_4 = \frac{s f_7(s)(t^2+1)^{\frac{3}{2}} (s+t)^2}{4(s^2+1)^{\frac{3}{2}} f_8(s,t)}, \tag{3.28}$$

where

$$f_5(t) = \left((t^2+1)^{\frac{3}{2}} - 8\right), f_6(s,t) = \left((s^2+1)^{\frac{3}{2}} - (s+t)^3\right),$$
$$f_7(s) = \left((s^2+1)^{\frac{3}{2}} - 8\right), f_8(s,t) = \left((t^2+1)^{\frac{3}{2}} - (s+t)^3\right).$$

To obtain CC regions, in which all masses are positive, we are going to find regions in st-plane, in which the mass ratios in both cases are positive. Since we know that $f_5(t)$ is monotonically increasing function of the variable t, $\forall\ t > 0$, with only one zero at $t = \sqrt{3}$, so it is quite simple to look at $f_5(t)$, which is less than zero when t is less than $\sqrt{3}$. Solving $f_6(s,t) = 0$ for t, it will gives us $t(s) = -s + \sqrt{s^2+1}$. It is quite simple to investigate that $f_6(s,t) < 0$ when $t(s) > -s + \sqrt{s^2+1}$. $f_5(t)$ and $f_6(s,t)$ can't be positive simultaneously, therefore $\mu_2 > 0$ in

$$R_{\mu_2} = \left\{(s,t)|t > 0 \wedge s > 0 \wedge \sqrt{s^2+1} - s < t < \sqrt{3}\right\}. \tag{3.29}$$

When $s = \sqrt{3}$, $f_7(s) = 0$, and $f_8(s,t) = 0$, when
$s(t) = -t + \sqrt{t^2+1}$. Hence, $\mu_4 > 0$ in

$$R_{\mu_4} = \left\{(s,t)|s > 0 \wedge t > 0 \wedge \sqrt{t^2+1} - t < s < \sqrt{3}\right\}. \tag{3.30}$$

The intersection of the above two equations is $R_{\mu_2\mu_4}$ which is given by Equation (3.25) (see Figure 3.8).

Corollary 3.2.4

Let us suppose that t is less than zero in the arrangement of Theorem 3.2.3, assuring a four-body triangular configuration, then (m_1, m_2, m_3, m_4) will make a

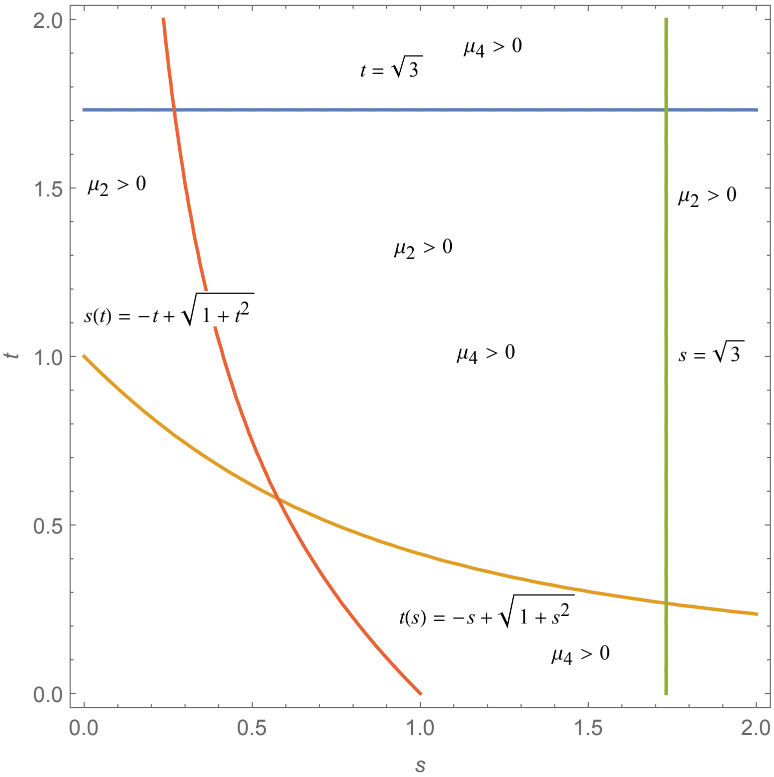


Figure 3.8: CC region for rhomboidal 4BP

triangular CC in the region given below.

$$\begin{aligned} TR_{t-}(s,t) = (s,t)|(-3.73 < t < -\sqrt{3} \wedge h_1(t) < s < \sqrt{3}) \vee (-\sqrt{3} < t \leq -1 \\ \wedge (0 < s < h_1(t) \vee \sqrt{3} < s < h_2(t))) \vee (-1 < t < -\frac{1}{\sqrt{3}} \\ \wedge \sqrt{3} < s < h_2(t)) \vee (-\frac{1}{\sqrt{3}} < t < 0 \wedge h_2(t) < s < \sqrt{3}), \end{aligned} \tag{3.31}$$

where

$$h_1(t) = \frac{1-t^2}{2t}, \quad h_2(t) = \sqrt{t^2+t} - t. \tag{3.32}$$

Proof

One follows the same process as in Theorem 3.2.3, for the proof of this Corollary. That's why we give only an outline and left the detail for the readers who are interested.

Utilizing the like process as we did in Theorem 3.2.3, we can easily show that

$m_2 \geq 0$ in the following region:

$$\begin{aligned} TR_{m_2}(s,t) = \{(s,t)|(0 < s < \frac{1}{\sqrt{3}} \wedge (-\sqrt{3} < t < H(s) \vee -s < t < 0)) \\ \vee (\frac{1}{\sqrt{3}} < s < \sqrt{3} \wedge (H(s) < t < -\sqrt{3} \vee -s < t < 0)) \\ \vee (s > \sqrt{3} \wedge (H(s) < t < -s \vee -\sqrt{3} < t < 0))\}, \end{aligned} \tag{3.33}$$

where

$$H(s) = -\sqrt{s^2+1} - s. \tag{3.34}$$

Similarly, $m_4 \geq 0$ in the region given below

$$\begin{aligned} TR_{m_4}(s,t) = \ & (t < -\sqrt{3} \wedge (0 < s < \sqrt{3} \vee -t < s < h_2(t))) \\ & \vee \left(-\sqrt{3} < t < -\frac{1}{\sqrt{3}} \wedge (0 < s < -t \vee \sqrt{3} < s < h_2(t))\right) \\ & \vee \left(-\frac{1}{\sqrt{3}} < t < 0 \wedge (0 < s < -t \vee h_2(t) < s < \sqrt{3})\right). \end{aligned} \tag{3.35}$$

The intersection of $TR_{m_4}(s,t)$ and TR_{m_2} give $TR_{t_-}(s,t)$.

Theorem 3.2.5

Let $\mathbf{r}_0 = (0,w)$, $\mathbf{r}_1 = (-1,0)$, $\mathbf{r}_2 = (0,s)$, $\mathbf{r}_3 = (1,0)$, $\mathbf{r}_4 = (0,-t)$, in which $w,t,s \in \mathbb{R}$.

(***a***) The arrangement $\mathbf{r} = (\mathbf{r}_0, \mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4)$ will make a CC with

$$\begin{aligned} \mu_0 = \frac{m_0}{m_1} = \frac{A_1C_2C_3 - (B_1B_3C_2 + B_2C_1C_3)}{A_2C_1C_3 + A_3B_1C_2}, \\ \mu_2 = \frac{m_2}{m_1} = \frac{A_3B_2C_1 - (A_1A_3C_2 + A_2B_3C_1)}{A_2C_1C_3 + A_3B_1C_2}, \\ \mu_4 = \frac{m_4}{m_1} = \frac{A_2B_1B_3 - (A_1A_2C_3 + A_3B_1B_2)}{A_2C_1C_3 + A_3B_1C_2}, \end{aligned} \tag{3.36}$$

where

$$\begin{aligned} A_1 = (R_{03} - R_{13})\Delta_{013}, \quad A_2 = (R_{10} - R_{20})\Delta_{120}, \quad A_3 = (R_{10} - R_{40})\Delta_{140}, \\ B_1 = (R_{02} - R_{12})\Delta_{012}, \quad B_2 = (R_{13} - R_{23})\Delta_{123}, \quad B_3 = (R_{13} - R_{43})\Delta_{143}, \\ C_1 = (R_{04} - R_{14})\Delta_{014}, \quad C_2 = (R_{14} - R_{24})\Delta_{124}, \quad C_3 = (R_{12} - R_{42})\Delta_{142}. \end{aligned}$$

(***b***) The masses ratios $\mu_0 > 0, \mu_2 > 0$ and $\mu_4 > 0$ form a CCs in

$$R(t,w) = R_{\mu_0} \cap R_{\mu_2} \cap R_{\mu_4}, \tag{3.37}$$

Where equation (3.40), (3.45) and (3.47) are respectively, shows us R_{μ_0}, R_{μ_2} *and* R_{μ_4}.

Proof of Theorem 3.2.5(a)

In Lemma 3.2.1, equation (3.12) - (3.14) describe the CCs of triangular or rhomboidal 5BPs. Let $\mu_0 = \frac{m_0}{m_1}$, $\mu_2 = \frac{m_2}{m_1}$, $\mu_4 = \frac{m_4}{m_1}$ and rewrite Lemma 3.2.1's equations as:

$$A_1 m_1 + B_1 m_2 + C_1 m_4 = 0,$$
$$A_2 m_0 + B_2 m_1 + C_3 m_2 = 0,$$
$$A_3 m_0 + B_3 m_1 + C_3 m_2 = 0,$$

where

$$A_1 = (R_{03} - R_{13})\Delta_{013}, \quad A_2 = (R_{10} - R_{20})\Delta_{120}, \quad A_3 = (R_{10} - R_{40})\Delta_{140},$$
$$B_1 = (R_{02} - R_{12})\Delta_{012}, \quad B_2 = (R_{13} - R_{23})\Delta_{123}, \quad B_3 = (R_{13} - R_{43})\Delta_{143},$$
$$C_1 = (R_{04} - R_{14})\Delta_{014}, \quad C_2 = (R_{14} - R_{24})\Delta_{124}, \quad C_3 = (R_{12} - R_{42})\Delta_{142}.$$

After dividing the above three equations by m_1, then write in matrix form, this will gets the form:

$$\begin{bmatrix} 0 & B_1 & C_1 \\ A_2 & 0 & C_2 \\ A_3 & C_3 & 0 \end{bmatrix} \begin{bmatrix} \mu_0 \\ \mu_2 \\ \mu_4 \end{bmatrix} = - \begin{bmatrix} A_1 \\ B_2 \\ B_3 \end{bmatrix}. \tag{3.38}$$

After doing a number of row operations, we get the following simultaneous solution:

$$\begin{aligned} \mu_0 &= \frac{A_1C_2C_3 - (B_1B_3C_2 + B_2C_1C_3)}{A_2C_1C_3 + A_3B_1C_2}, \\ \mu_2 &= \frac{A_3B_2C_1 - (A_1A_3C_2 + A_2B_3C_1)}{A_2C_1C_3 + A_3B_1C_2}, \\ \mu_4 &= \frac{A_2B_1B_3 - (A_1A_2C_3 + A_3B_1B_2)}{A_2C_1C_3 + A_3B_1C_2}. \end{aligned} \tag{3.39}$$

Hence the proof.

Proof of Theorem 3.2.5(b)

By substituting $s = 1$ in Lemma 3.2.1, we can generate the values of R_{ij} and Δ_{ijk}.

Lemma 3.2.6

The function

$$\mu_0 = \frac{A_1C_2C_3 - (B_1B_3C_2 + B_2C_1C_3)}{A_2C_1C_3 + A_3B_1C_2}$$

attains a positive value in the region

$$R_{\mu_0} = (R_D \cap R_{N_{\mu_0}}) \cup (R_D^c \cap R_{N_{\mu_0}}^c), \tag{3.40}$$

Where $R_{N_{\mu_0}}^c$ and R_D^c represent the complements of the regions $R_{N_{\mu_0}}$ and R_D.

Proof

Let $N_{\mu_0} = A_1C_2C_3 - (B_1B_3C_2 + B_2C_1C_3)$ and $D = A_2C_1C_3 + A_3B_1C_2$. For the positive value of μ_0, D and N_{μ_0} must have the same signs. The denominator, in following two cases should be positive:

(i) When both the factors $A_2C_1C_3$ and $A_3B_1C_2$ are positive.

(ii) At least one of the factors is not negative, $\ni$ the positive portion is greater than the absolute of negative portion.

In the following region, both of the factors will be positive

$$\begin{aligned} R_{da}(t,w) = (t,w)|(0 < w \leq 0.41 \wedge d_1(w) < t < 1) \vee (0.41 < w < 0.58 \\ \wedge\, d_1 < t < 0.5 \cdot d_2(w)) \vee (w > 2.41 \wedge 0.41 < t < 1), \end{aligned} \tag{3.41}$$

where

$$d_1(w) = \sqrt{w^2+1} - w, \quad d_2(w) = \frac{(1-w^2)}{w}. \tag{3.42}$$

Similarly, when

1. $A_2C_1C_3 > 0$ and $A_3B_1C_2 < 0$ then $A_2C_1C_3 > |A_3B_1C_2|$ in the region given below.

$$R_{db}(t,w) = \{(t,w)|\,(0.4 < t \leq 0.58 \wedge 0.6 \cdot d_2(t) < w < 1) \vee (0.58 < t < 1 \wedge d_1(t) < w < 1)\}$$

2. $A_2C_1C_3 < 0$ and $A_3B_1C_2 > 0$ then $A_3B_1C_2 > |A_2C_1C_3|$ in the regions given below

$$R_{dc}(t,w) = \{(t,w)|(w \leq -1.49 \wedge 0 < t < 0.23) \vee (-1.49 < w < -1 \wedge 0 < t < 0.2 \cdot d_2(w)) \vee (1 < w < 2.41 \wedge 0.41 < t \leq 1)\}.$$

Thus, the denominator $(D) \geq 0$ in the following region (Figure 3.9(a)):

$$R_D = R_{da}(t,w) \cup R_{db}(t,w) \cup R_{dc}(t,w). \tag{3.43}$$

Similarly the nominator N_{μ_0} is positive in the region given below (Figure 3.9(b)):

$$R_{N_{\mu_0}} = R_{cN_{\mu_0}}(t,w) \cup R_{dN_{\mu_0}}(t,w). \tag{3.44}$$

Hence, in the intersection of R_D and $R_{N_{\mu_0}}$, $\mu_0 > 0$, and the intersection of their complements. Region R_{μ_0} is represented in Figure 3.9(c). Hence the proof.

Lemma 3.2.7

The function

$$\mu_2 = \frac{A_3B_2C_1 - (A_1A_3C_2 + A_2B_3C_1)}{A_2C_1C_3 + A_3B_1C_2}$$

obtain positive values in following region

$$R_{\mu_2} = (R_D \cap R_{N_{\mu_2}}) \cup (R_D^c \cap R_{N_{\mu_2}}^c), \tag{3.45}$$

where the complements of the regions $R_{N_{\mu_2}}$ and R_D are $R^c_{N_{\mu_2}}$ and R_D, respectively.

Proof

Let $N_{\mu_2} = A_3B_2C_1 - (A_1A_3C_2 + A_2B_3C_1)$. For the positive value of μ_2, numerator

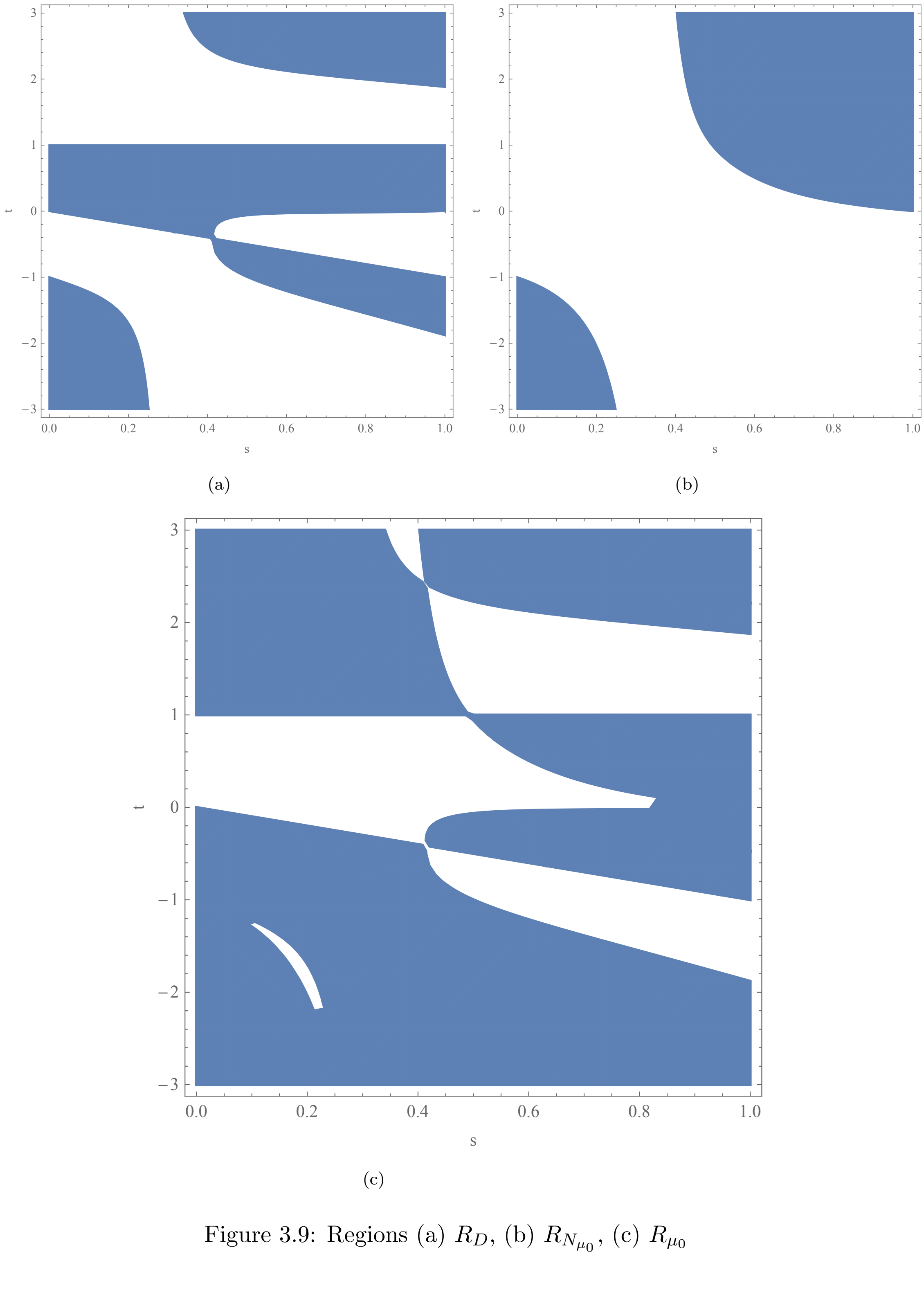
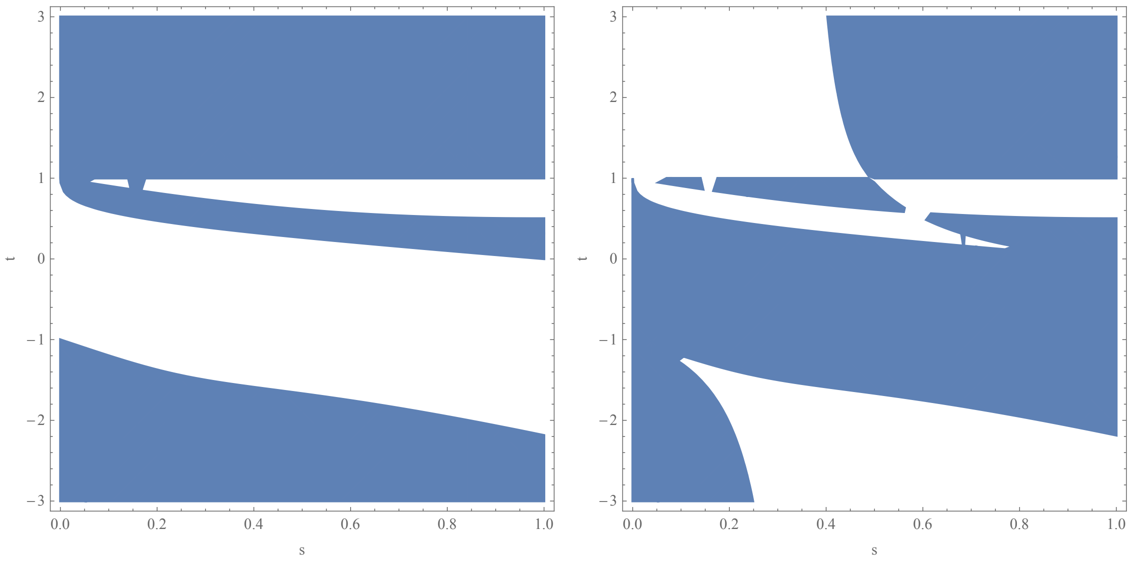


Figure 3.9: Regions (a) R_D, (b) $R_{N_{\mu_0}}$, (c) R_{μ_0}

N_μ and denominator D must have the same sign. We already discuss D in detail in Lemma 3.2.6. Now we will find the region where N_{μ_2} is positive.We can easily prove that $N_{\mu_2}(t, w) > 0$ in

$$R_{N_{\mu_2}} = R_{aN_{\mu_2}}(t, w) \cup R_{bN_{\mu_2}}(t, w) \cup R_{cN_{\mu_2}}(t, w). \tag{3.46}$$

In Figure 3.10(a), we have shown the region $R_{N_{\mu_2}}$. Consequently, $\mu_2 > 0$ in the intersection of $R_{N_{\mu_2}}$ and D, as well as the complement's intersection. Similarly the region R_{μ_2} is displayed in Figure 3.10(b). Hence the proof.



(a) (b)

Figure 3.10: Regions (a) $R_{N_{\mu_2}}$ (colored) , (b) R_{μ_2} (colored)

Lemma 3.2.8

The function

$$\mu_4 = \frac{A_2B_1B_3 - (A_3B_1B_2 + A_1A_2C_3)}{A_2C_1C_3 + A_3B_1C_2}$$

gets the positive values in the following region

$$R_{\mu_4} = (R_D \cap R_{N_{\mu_4}}) \cup (R_D^c \cap R_{N_{\mu_4}}^c). \tag{3.47}$$

Proof

Let $N_{\mu_4} = A_2B_1B_3 - (A_3B_1B_2 + A_1A_2C_3)$. For the positive values of μ_4, the signs of the numerator N_{μ_4} and denominator (D) must be same. In Lemma 3.2.6, we have already given a detail analysis of the denominator of μ_4. Now we will find

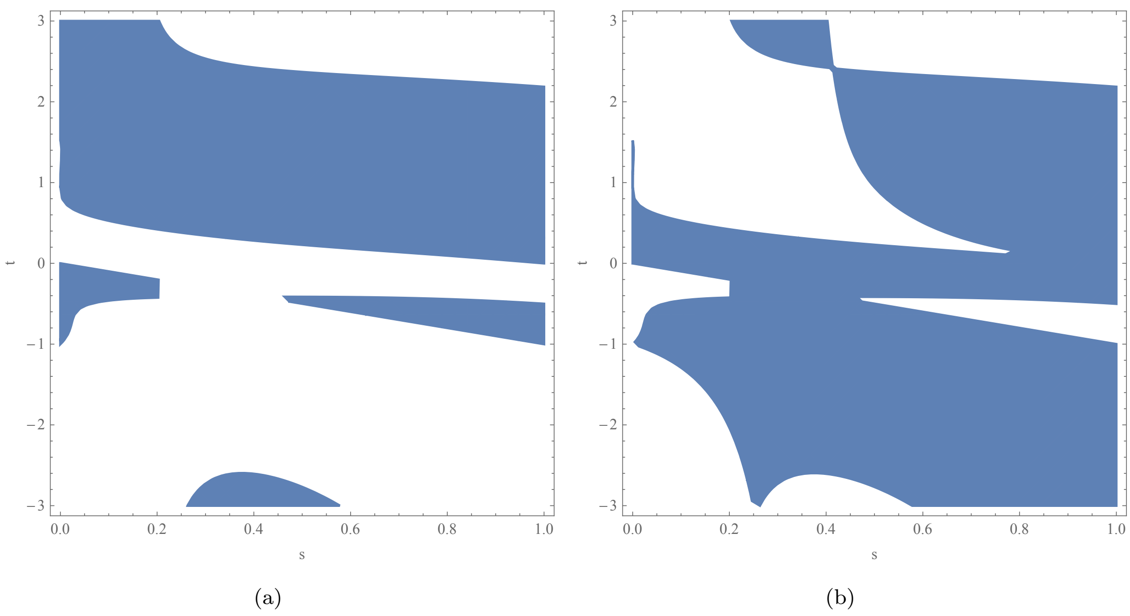


Figure 3.11: Regions (a) $R_{N_{\mu_4}}$ (colored), (b) R_{μ_4} (colored)

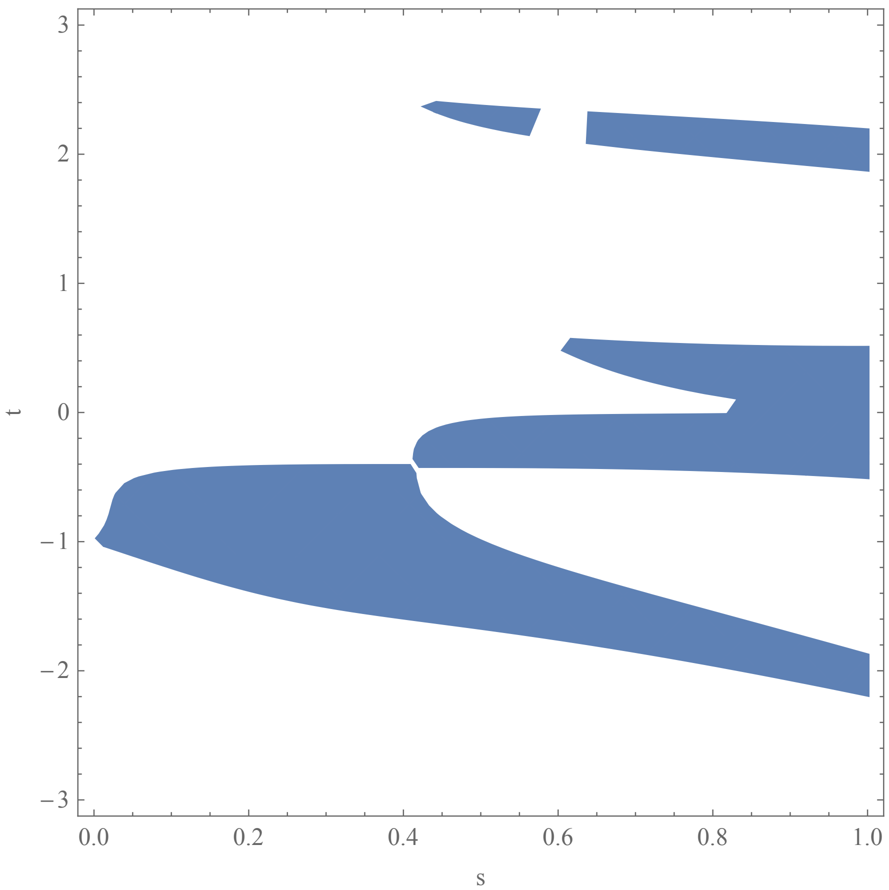


Figure 3.12: CC Region ($R(t, w)$) for the rhomboidal 5BP

the region where $N_{\mu_4} \geq 0$. It can be easily prove that $N_{\mu_4}(t, w)$ is positive in (see

Figure 3.11(a)):

$$R_{N_{\mu_4}} = R_{aN_{\mu_4}}(t, w) \cup R_{bN_{\mu_4}}(t, w) \cup R_{cN_{\mu_4}}(t, w). \tag{3.48}$$

Thus μ_4 is greater than zero in the intersection of $R_{N_{\mu_4}}$ and R_D as shown in Figure 3.11(b). which is the complete proof.

For all positive masses the region of CC is obtained by:

$$R(t, w) = R_{\mu_0} \cap R_{\mu_2} \cap R_{\mu_4}. \tag{3.49}$$

In Figure 3.12, we showed the region $R(t, w)$. This is the complete proof of Theorem 4b. For better understanding the behavior of the complex central configuration regions in Figure 3.12, some of the related examples are shown in Figures 3.13 and 3.14.

In Lemma 3.2.1, he supposed variables $s = 1$ and $t \in (-1, 0)$ will give a five-body triangular configurations (Figure 3.2).

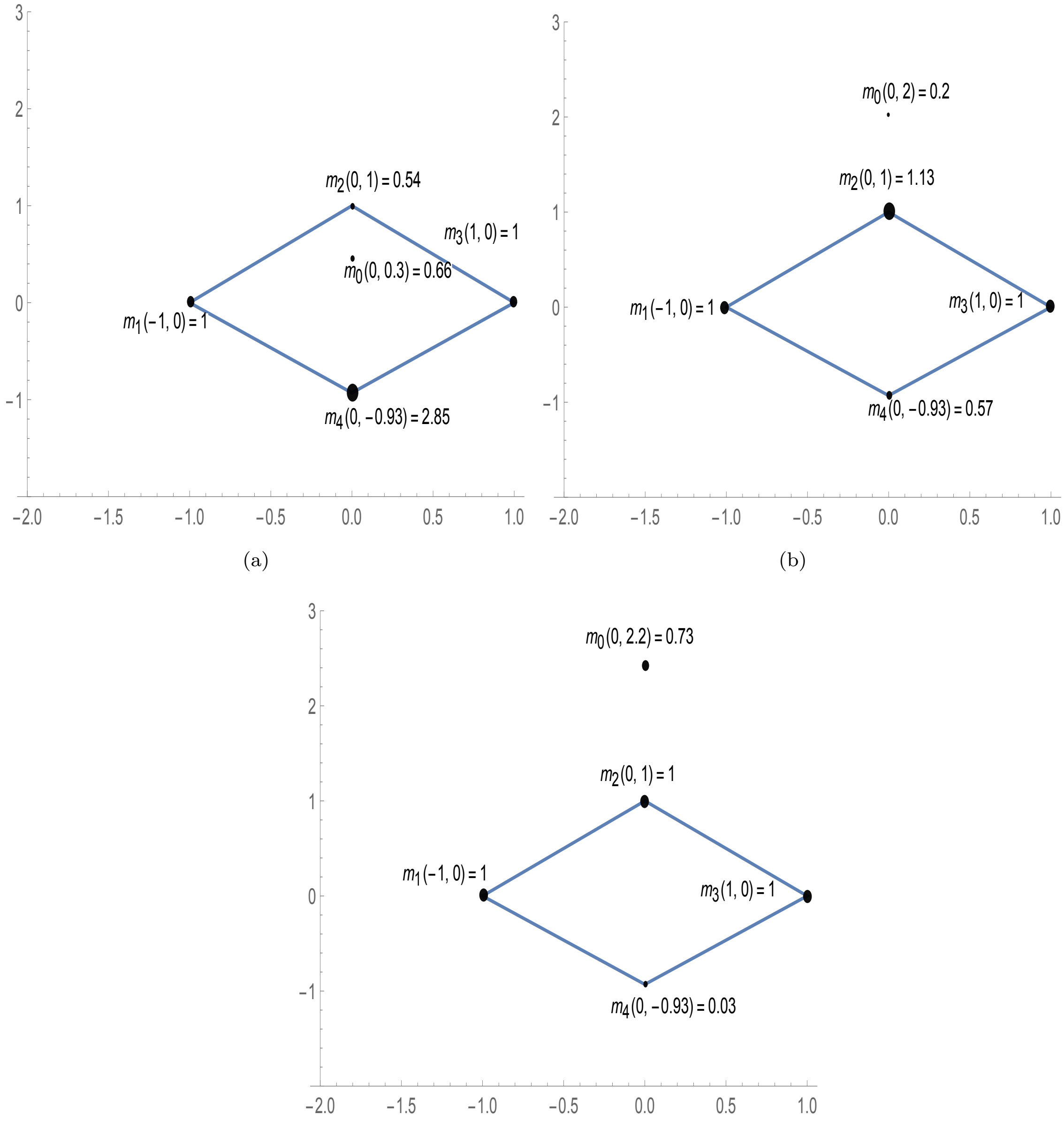
Corollary 3.2.9

In the setup of Theorem 3.2.4, let us suppose that $t \in (-1, 0)$, assure a triangular 5-body settlement for w greater than zero. The configuration $(m_0, m_1, m_2, m_1, m_4)$ will make a CC in the region given below:

$$R_{t_-}(t, w) = R_{m_0}(t, w) \cap R_{m_2}(t, w) \cap R_{m_4}(t, w). \tag{3.50}$$

Proof

Let us suppose that $t < 0$ and solve Equation 3.41 as we did in Theorem 3.2.4 to get the following region of CCs given in Figure 3.2 (isosceles triangular 5BP). We are giving an outline of the proof and left the detail for the readers who are interested. When $w > 0$, the denominator will always be negative. That's why, μ_0, μ_2 and μ_4 will be greater or equal to zero if the corresponding numerators (N) negative. So, the regions of CC where μ_0, μ_2 and μ_4 are respectively positive are:



(a) (b) (c)

Figure 3.13: Examples of rhombus type five-body central configurations when w is positive

$$R_{m_0} = (-1 < t < -w \wedge 0 < w < 1) \vee (1 < w < 1.73 \wedge (-1 < t < d_2(w) \vee \frac{0.5}{w} - 0.5w < t < 0)) \vee 1.5 < w < p,$$

$$R_{m_2} = (0 < w < 1 \wedge (-1 < t < -w \vee -w < t < 0)) \vee (-1 < t < d_2(w) \wedge 1.73205 < w < 2.41421) \vee t > d_2(w),$$

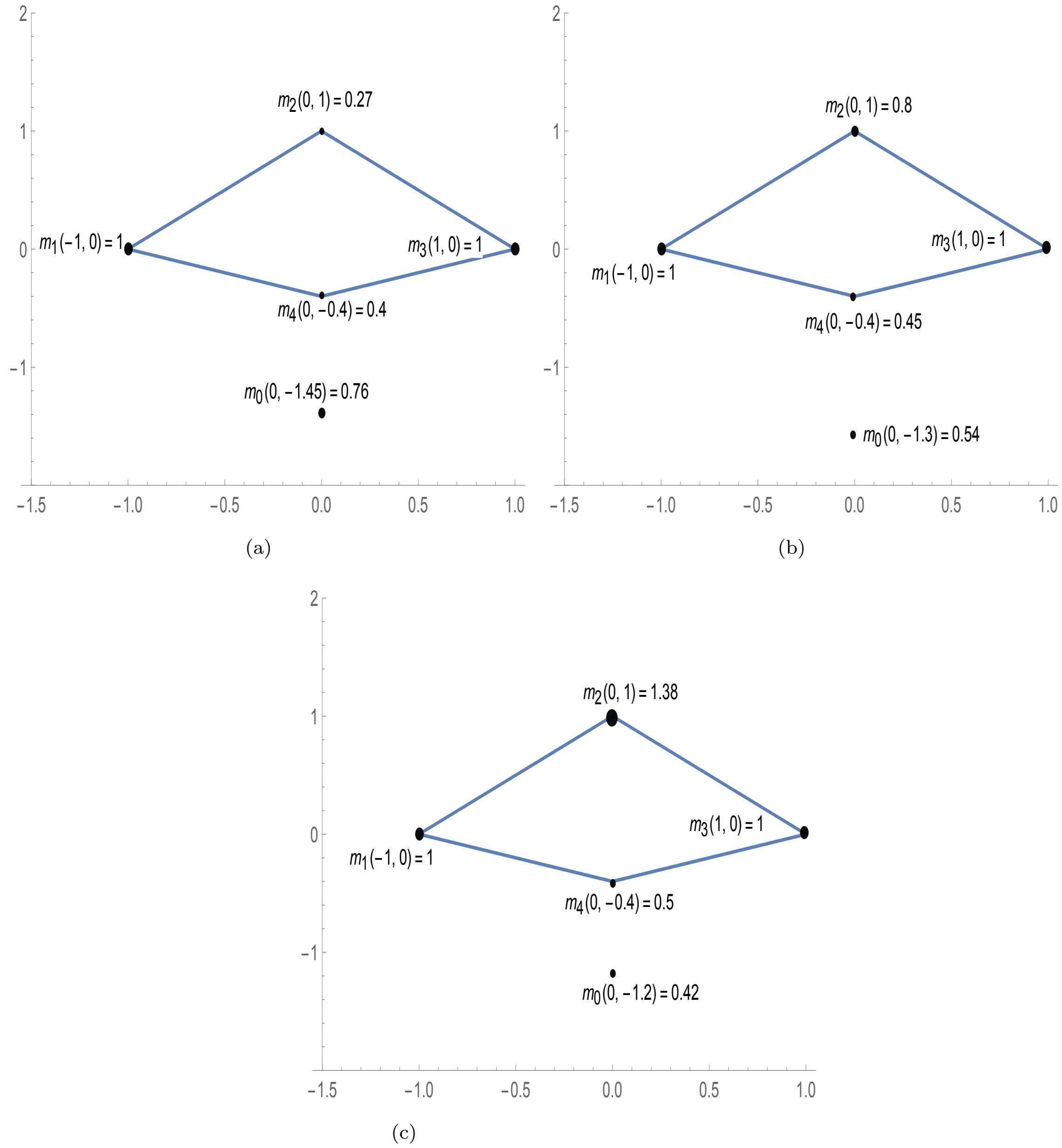


Figure 3.14: Examples of rhomboidal 5-body central configurations when w is less than zero.

$$R_{m_4} = (0 < w < 1 \wedge -w < t < 0) \vee (1 < w < 1.73205 \wedge -1 < t < 0) \vee (1.73 < w < q) \vee (1.6 < w < r),$$

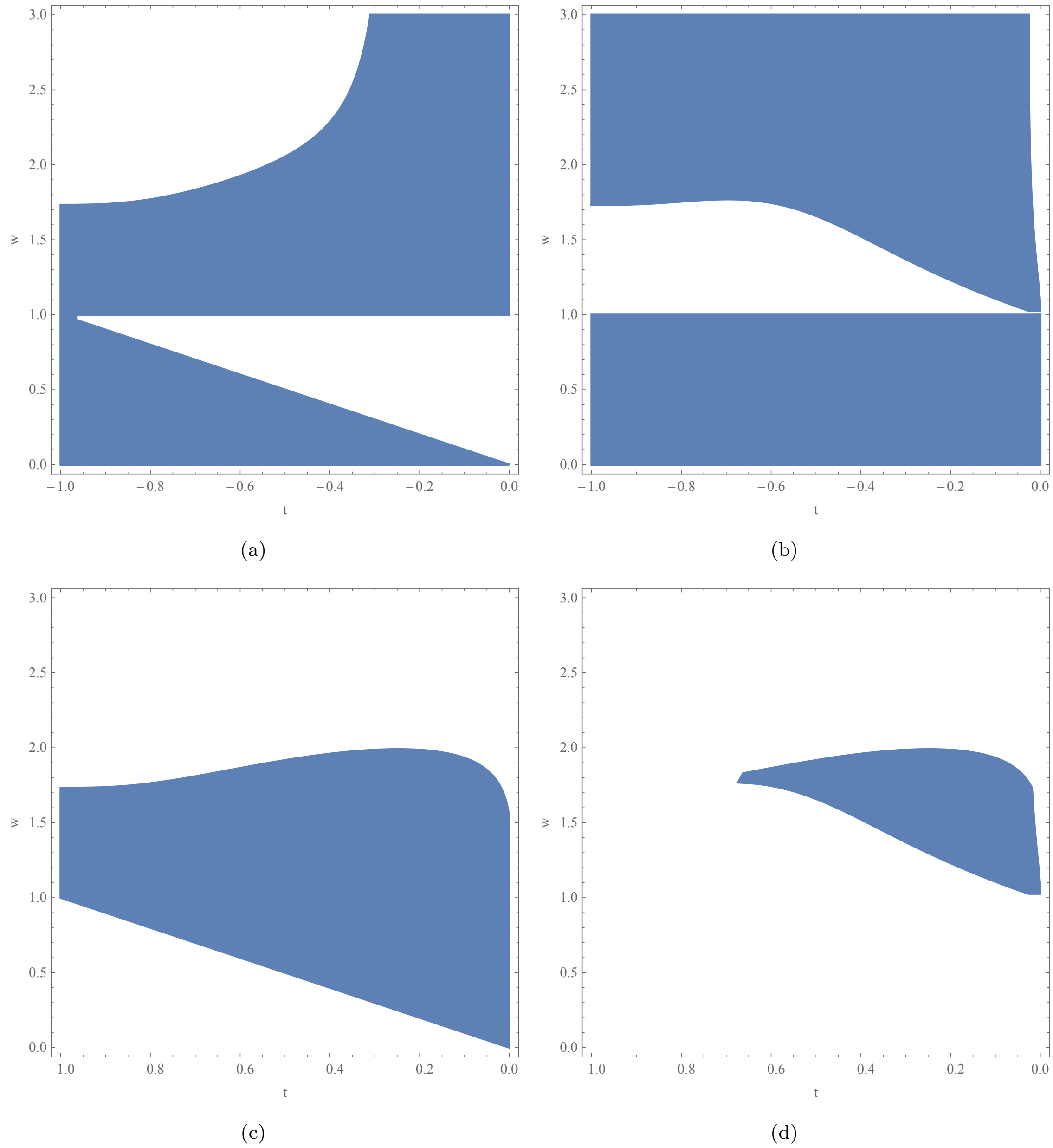


Figure 3.15: CC regions for triangular 5BP
(a) $m_0 > 0, R_{m_0}(w,t)$, (b) $m_2 > 0, R_{m_2}(t,w)$, (c) $m_4 > 0, R_{m_4(t,w)}$,
(d) $m_i > 0, i = 0, 2, 4, R_{t_-}(w,t)$

where

$$p(t) = 84.13t^5 + 301.01t^4 + 424.28t^3 + 295.98t^2 + 103.56t + 16.68,$$

$$q(t) = -3.91t^3 - 7.57t^2 - 4.22t + 1.22,$$

$$r(t) = 305.66t^3 + 52t^2 + 0.25t + 1.76.$$

For triangular 5BP, the CC region is

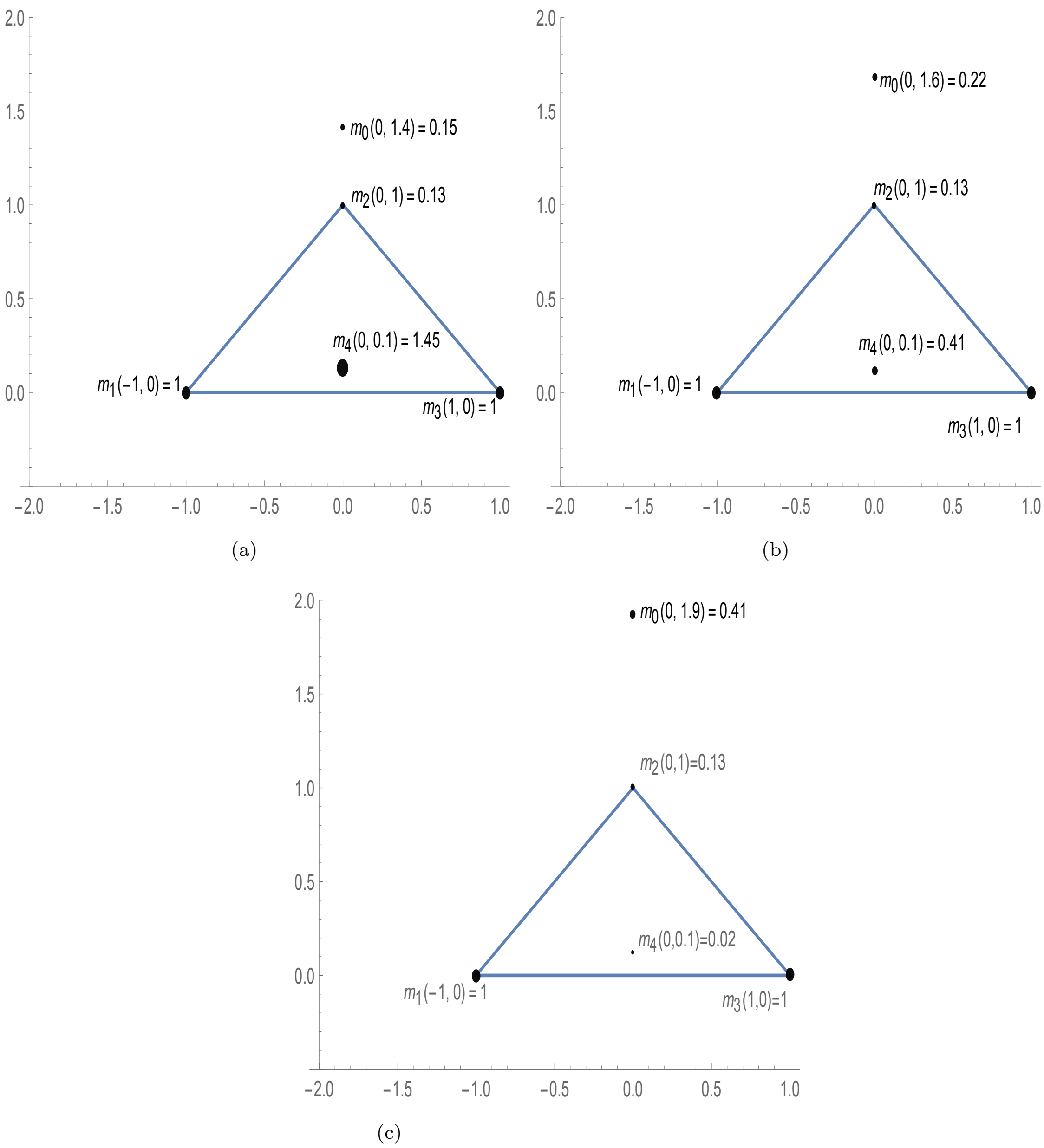


Figure 3.16: Examples of triangular 5-body CCs.

$$R_{t_-}(w,t) = R_{m_0}(w,t) \cap R_{m_2}(w,t) \cap R_{m_4}(w,t). \tag{3.51}$$

The above region is shown in Figure 3.15(d) alongside R_{m_0} (Figure 3.15(a)), R_{m_2} (Figure 3.15(b)) and R_{m_4} (Figure 3.15(c)). The CC region in

Figure 3.15 corresponds to the solutions (triangular) of 5BP. Now by picking up $s = \sqrt{3}$, the CC in Figure 3.2 will become a triangle (equilateral). For this triangle, the CC can be found in the like way as we did in Corollary 3.2.9 or Theorem 3.2.4. To see the actual location of the objects in case of triangular 5-body CCs, have a look at Figure 3.16. It is not possible analytically to find the central configuration regions in fully general cases (not fixing $s = 1$). Still, we obtained most of the solution in the current analysis.

Chapter 4

Conclusions

We studied the CC of different types of symmetric triangular and rhomboidal 5BP. Different cases related to the problem were discussed and it was shown the existence and non existence of CC related to each case. We formed expressions for mass ratios and derived regions of CC for positive masses in each case. The existence of continuous family of CCs is shown to exist for triangle and rhombus of various sizes.

We reached at a conclusion that no CCs exists when one mass is put anywhere except the origin on the axis of symmetry of a rhombus, and 2 pairs of objects are put on vertices (symmetric about the axes). Regions of CCs are obtained for all other cases, comprise of equilateral and isosceles triangular 5BPs and rhomboidal 4- and 5BPs, using analytical techniques. For the completion of analytical results, we numerically explored these regions.

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