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Fixed Point Theorems in Fuzzy Abstract Spaces

by

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Fixed Point Theorems in Fuzzy Abstract Spaces

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TO

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CERTIFICATE OF APPROVAL

This is to certify that the research work presented in the thesis, entitled "**Fixed Point Theorems in Fuzzy Abstract Spaces**" was conducted under the supervision of **Dr. Rashid Ali**. No part of this thesis has been submitted anywhere else for any other degree. This thesis is submitted to the **Department of Mathematics, Capital University of Science and Technology** in partial fulfillment of the requirements for the degree of Doctor in Philosophy in the field of **Mathematics**. The open defence of the thesis was conducted on **February 09, 2022**.

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List of Publications

It is certified that following publication(s) have been made out of the research work that has been carried out for this thesis:-

1. **F. Mehmood**, R. Ali, and N. Hussain, "Contractions in fuzzy rectangular b-metric spaces with application", *Journal of Intelligent & Fuzzy Systems*, vol. 37, no. 1, pp.1275–1285, 2019.

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Abstract

In this dissertation, the notion of fuzzy rectangular-b-metric space is introduced, which generalizes the notions of a fuzzy metric space and a fuzzy b-metric space. Well known fixed point theorems are established in the setting of fuzzy rectangular b-metric spaces and illustrated by an example. Also by introducing the concept of extended fuzzy b-metric space, a Banach-type fixed point theorem in the setting of this more general class of fuzzy metric spaces is proved. The notion of Hausdorff extended fuzzy b-metric space is also studied and certain fixed point results for some multivalued contractions in the setting of G-complete extended fuzzy b-metric space are also established. Some examples are furnished which illustrate main results. As applications of main results, fixed point results involving fuzzy integral inequalities and fuzzy integral inclusion are established. These results extend and generalize many existing results in literature.

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Abbreviations

Acronym	**W**hat (it) **S**tands **F**or
BMS	b-metic space
EFBMS	extended fuzzy b-metric space
FBMS	fuzzy b-metric space
FMS	fuzzy metric space
FRBMS	fuzzy rectangular b-metric space
FRMS	fuzzy rectangular metric space
HEFBMS	Housedorrf extended fuzzy b-metric space
HFBMS	Housedorrf fuzzy b-metric space
RBMS	rectangular b-metric space
RMS	rectangular metric space

Chapter 1

Introduction

The study of fixed point and its existence has a remarkable contribution in many branches of applied and pure mathematics and provides very valuable and effective tools in mathematics to solve problems in linear and non linear analysis. In addition this theory is a nice fusion of topology, geometry and analysis which has significant importance in various branches of mathematics and other applied sciences. In particular, fixed point theorems are used to find the successive approximations for the presence and uniqueness of all equation solution and the same notions are associated with many famous mathematicians like Banach, Lipschitz, Fredrick, Picard, Peano, Fredholm and Cauchy.

1.1 Background

In 1886, the first step was taken in fixed point theory by Poincare [1]. Afterwards in 1911, the solution of the equation $f(\xi) = \xi$ was obtained by Brouwer [2] in his result stated as; any continuous function on compact convex set to itself has a fixed point ζ_0 such that, $f(\xi_0) = \xi_0$. Later on Kakutani [3] generalized Brouwer fixed point theorem on set valued function and also extended and generalized work of [3] for n-dimensional counter parts of a sphere and a square.

Fréchet [4] was the first mathemation who gave the concept of a metric space in 1906. The study of metric fixed point theory was initiated in 1922 by the remarkable work of a Polish mathematician Stefan Banach [5]. The work of Banach is considered as one of the most valuable and adaptable consequence in the literature of this theory. His famous result is known as the Banach Contraction Principle (BCP) [5]. It is stated as:
A self mapping $\Gamma : W \longrightarrow W$ on a complete metric space (W, d) has a unique fixed point, if for $k \in [0, 1)$, we have

$$d(\Gamma\xi, \Gamma\varrho) \leq kd(\xi, \varrho) \qquad \text{for all} \quad \xi, \varrho \in W. \tag{1.1}$$

Later on, the BCP has been prolonged and widespread in different ways by changing the structure of contraction mapping or by changing the structure of underlying space for instance Rakotch [6] introduced a contractive condition, in which a monotonic decreasing function $\alpha\colon [0, \infty) \longrightarrow [0, 1]$ replaces the constant k, of inequality (1.1) as

$$d(\Gamma\xi, \Gamma\varrho) \leq \alpha(t)d(\xi, \varrho) \qquad \text{for all} \qquad \xi, \varrho \in W.$$

The above contractive condition coincides with (1.1) if $\alpha(t)$ has a constant value in $(0, 1)$ for all t. We refer the work presented in [7–13] for studying different contractive conditions and related fixed point results. A comprehensive comparison of various contractive conditions is done by Rhoades [14].
On the other hand, to establish BCP in a more general structure, the notion of metric space was generalized by Bakhtin [15] in 1989 by presenting the concept of b-metric space (BMS). Later on, the same concept was further investigated by Czerwick [16] to establish different results in BMS. The study of BMS holds a prominent place in fixed point theory with multiple aspects. Many mathematicians led the foundation to improve fixed point theory in BMS [17–24].
An other prominent generalization of metric space is the idea of rectangular metric spaces (RMS). This generalization was given by Branciari [25] in 2000 and proved some related results. Similarly in 2015, George et. al [26] introduced the notion

of rectangular b-metric spaces (RBMS) and constructed some fixed point results. Later on, the theory was further developed by many mathematicians by proving a number of fixed point results in RBMS. For instance see [27–29].
Another milestone was achieved by Kamran et al. [30] in 2017 by introducing the concept of extended b-metric space (EBMS) which is a generalization of BMS. Authors changed the triangular inequality of b-metric by introducing a function $\alpha : W \times W \longrightarrow [1, \infty)$ as follows

$$d_\alpha(\xi, \vartheta) \leq \alpha(\xi, \vartheta)\Big[d_\alpha(\xi, \varrho) + d_\alpha(\varrho, \vartheta)\Big] \quad \forall \quad \xi, \varrho, \vartheta \in W,$$

where d_α is the extended b-metric on W as defined in Definition 2.1.2. The study of EBMS became an exciting subject for many authors [31–36].

1.2 Fuzzy Fixed Point Theory

The foundation of fuzzy mathematics was laid by Zadeh [37] in 1965 with the introduction of notion of a fuzzy set to extend the classical notion of a set. Unlike the ordinary set, the element of a fuzzy set are as a degree of membership which is assigned with the help of a membership function with values in the closed interval $[0, 1]$. Fuzziness is a completely different notion than probability. Probability defines the objective uncertainty derived from a huge number of observations. Fuzziness explains the subjective sense of the uncertainty. The success of fuzzy set theory in solving control problems derives from its ability to manage certain conditions that the classical control theory has trouble dealing with, but undefined, complex, non-linear structures are managed by fuzzy sets [38–40]. In the current rapidly evolving fields of artificial intelligence and neural networks, fuzzy set theory is becoming an ever more important tool [41, 42]. This provides entirely new opportunities in chemical engineering for application of fuzzy sets [43].
The significant applications in various fields such as remote sensing, data mining, pattern recognition have made distance measures important [44]. But because of the presence of vagueness, a logical problem emerges when the distance is measured

in an imprecise context. In order to make a global decision [45], there are many cases where understanding, experience and expertise [46] need to be combined with the information available. The crisp number is converted into a fuzzy number in these situations. Although vagueness in a fuzzy number is unavoidable, it is more insightful than a precise number.

Later on, many ideas of mathematics are extended by using the idea of fuzzy sets and membership function [47, 48]. In this context, in 1975, Kramosil and Michálek [49] introduced the concept of fuzzy metric spaces (FMS) which could be considered as a reformulation, in the fuzzy context, of the notion metric space. Later on, a new idea was presented by George and Veeramani [50] in the form of GV-FMS (Definition 2.2.4), which strengthen the notion of KM-FMS (Definition 2.2.3). The concepts of both notions have no close relation with each other but to some extent their characteristics have links, for example, the GV-FMS properties can be defined for KM-FMS and vice versa. The above relation leads the term FMS to anyone of them and we can relate it with anyone of them.

In 1988, Grabiec [51] extended BCP to FMS in the sense of KM-FMS and initiated the fuzzy fixed point theory. The author stated the famous BCP in fuzzy setting as; Let $(W, F, *)$ be a complete FMS such that

$$\lim_{t\to\infty} F(\xi, \varrho, t) = 1.$$

If $\Gamma : W \to W$ is a self mapping satisfying

$$F(\Gamma\xi, \Gamma\varrho, kt) \geq F(\xi, \varrho, t) \tag{1.2}$$

for all $\xi, \varrho \in W$ and $k \in (0, 1)$, then Γ has a unique fixed point.
In 2002, Gregori and Sapena [52] introduced a fuzzy contractive mapping in FMS and established a fixed point result as:
Let $(W, F, *)$ be a complete FMS in which fuzzy contractive sequences are Cauchy and $\Gamma : W \to W$ be a fuzzy contractive mapping such that

$$\frac{1}{F(\Gamma(\xi),\Gamma(\varrho),t)}-1\leq k\left(\frac{1}{F(\xi,\varrho,t)}-1\right) \tag{1.3}$$

for all $\xi,\varrho\in W$ and $k\in(0,1)$, then Γ has a unique fixed point.

Gupta et al. [53] proved two fixed point results using rational inequality and proved the existence of of fixed point by integral equation in 2013. Many more contractions and fixed point results, in fuzzy context, are proved by many authors in different ways for instance see [54–76].

In 2015, Hussain et al. [77] related the parametric b-metric and fuzzy b- metric spaces and proved some results. In 2016, Nădăban [78] studied the concept of FBMS and proved some results. In 2017 Shahzad et al. [79] proved some fixed point results for multivalued mapping in Hausdorff fuzzy metric space (HFMS) using rational inequality and to strengthen the results they established an application for the existence of solution of integral equation. For more interesting results for multivalued mapping in HFMS see [80–84].

1.3 Thesis Contribution

In 2015, George et al. [26] introduced the concept of RBMS which generalized the concepts of RMS, BMS and metric space. Inspired by the concept of RBMS, the idea of Fuzzy rectangular b-metric spaces is presented and BCP is established in this new defined space. By defining the notion of Γ-orbitally upper semi continuous function in FRBMS, the result of Hicks and Rhoads [85] is proved. By sudying the notion of Geraghty-type contraction the result of Roshan et al. [28] is proved in this new space. To strengthen the main result an application for the existence of solution of integral equation is also established. These results are the generalization of many existing results in the literature [26, 28, 51]. The published form of these results is available in [86] as;

" Contractions in fuzzy rectangular b-metric spaces with application"

In 2017, Kamran et al. [30] defined the notion of extended b-metric spaces (EBMS) and established some fixed point results. In this dissertation, motivated by Kamran et al. [30], the concept of extended fuzzy b-metric spaces is introduced (EFBMS) and established some fixed point results which are extension of many previous results in the literature of FMS, for instance see [51–53]. Also some results in EFBMS for Geraghty-type contraction and for multivalued mappings are established which generalize the results of [79]. All these results are illustrated by an example and an application of integral inclusion. The published form of some of these results can be seen in [87] as " Extended fuzzy b-metric spaces."

1.4 Organization of Thesis

The rest of the thesis is organized as follows.

- In Chapter 2, some basic definitions of abstract spaces and examples are stated.
- In Chapter 3, the notion of FRBMS is introduced and some examples are established and by extending the results of Banach [5], Hicks [85] and Roshan et. al [28] some results are established in FRBMS. To strengthen the results, an application of integral equation is established. Last part of chapter comprises a brief conclusion of our work. All the work of this chapter is published in [86]
- In Chapter 4, the notion of EFBMS is introduced, and illustrated by an example. The well known BCP [5] is established in this new space and an example illustrated the theorem. Moreover the results of [28, 53, 87, 88] are extended in the setting of EFBMS using Geraghty-type Contraction. The work of this chapter is published in [87].
- In Chapter 5, the notions of Hausdorff fuzzy b- metric space (HFBMS) and Hausdorff extended fuzzy b- metric spaces (HEFBMS) are introduced and affirmed by some examples. Also by extending the results of Banach [5] ,

Gupta et al. [53] and Roshan et. al [28] some fixed point results are established. Similarly the study is strengthened by providing some applications of the obtained results. All the findings of this chapter are submitted for possible publication.

Chapter 2

Preliminaries

The purpose of this chapter is to review the fundamental concepts, results and examples which are the basis for other chapters. In the first section, various types of metric spaces are discussed. The second section consists of various types of FMS.

2.1 Generalization of Metric Space

In Euclidean spaces, the concept of the distance between the points generalises to a more general notion of a distance between two points of an arbitrary nonempty set W, known as a metric on W. With this more general notion, Fréchet [4] was the first mathematician who presented the concept of a metric space in 1906. Many authors have generalized the notion of metric space in different ways. In 1989, Bakhtin [15] used the real number $b \geq 1$ in triangular inequality and introduced the concept of a BMS.

Definition 2.1.1.
"Let W be a non-empty set and 'b' be any real number such that $b \geq 1$. A function $d_b \colon W \times W \longrightarrow \mathbb{R}$ is called b-metric space if it satisfies the following properties for all $\xi, \varrho, \vartheta \in W$.

bM1 $d_b(\xi, \varrho) \geq 0$,

bM2 $d_b(\xi, \varrho) = 0$ if and only if $\xi = \varrho$,

bM3 $d_b(\xi, \varrho) = d_b(\varrho, \xi)$ for all $\xi, \varrho \in W$,

bM4 $d_b(\xi, \varrho) \leq b\Big[d_b(\xi, \vartheta) + d_b(\vartheta, \varrho)\Big]$.

The pair (W, d_b) is called a b**-metric space (BMS)** [15] ."

Example 2.1.1. Let $W = \{0, 1, 2\}$. Consider a mapping $d_b : W \times W \longrightarrow [0, \infty)$ defined as

$$d_b(\xi, \eta) = (\xi - \eta)^2.$$

Then (W, d_b) is a BMS with coefficient $b = 2$.

Example 2.1.2.

Let (W, d) be a metric space. Define $d_1 : W \times W \to R$ by

$$d_1(\xi, \varrho) = \eta_1 d(\xi, \varrho) + ad(\xi, \varrho)^{\eta_2}.$$

For some $\eta_2 > 1, \eta_1 \geq 0, a > 0$ and for $\xi, \varrho \in W$.

Hence d_1 is not a metric on W but (W, d_1) is a BMS with $b = 2^{\eta_2 - 1}$.

Let ϑ be any arbitrary element of W then we have

$$\begin{aligned} d_1(\xi, \varrho) &= \eta_1 d(\xi, \varrho) + ad(\xi, \varrho)^{\eta_2} \\ &\leq \eta_1[d(\xi, \vartheta) + d(\vartheta, \varrho] + a[d(\xi, \vartheta) + d(\vartheta, \varrho)]^{\eta_2} \\ &\leq \eta_1[d(\xi, \vartheta) + d(\vartheta, \varrho)] + 2^{\eta_2 - 1}a[d(\xi, \vartheta)^{\eta_2} + d(\vartheta, \varrho)^{\eta_2}] \\ &\leq 2^{\eta_2 - 1}[d_1(\xi, \vartheta) + d_1(\xi, \varrho)] \end{aligned}$$

$$d_1(\xi, \varrho) \leq 2^{\eta_2 - 1}[d_1(\xi, \vartheta) + d_1(\xi, \varrho)].$$

If $a, b \in \mathbb{R}^+$ and $\eta_2 > 1$ then the above inequality follows by

$$\left(\frac{\xi + \varrho}{2}\right)^{\eta_2} \leq \frac{\xi^{\eta_2} + \varrho^{\eta_2}}{2}.$$

To generalize the notion of BMS, Kamran et al. [30] came with the notion of an extended b-metric space. They use a function $\alpha(\xi, \varrho)$ depending upon the values of ξ and ϱ in the triangular inequality.

Definition 2.1.2.

"Let W be a non-empty set and $\alpha : W \times W \longrightarrow [1, \infty)$ be a mapping. A function $d_\alpha : W \times W \longrightarrow \mathbb{R}$ is called an **extended b-metric** if for all $\xi, \varrho, \vartheta \in W$, it satisfies the following conditions

EBM1 $d_\alpha(\xi, \varrho) \geq 0,$

EBM2 $d_\alpha(\xi, \varrho) = 0, \text{ if and only if } \xi = \varrho,$

EBM3 $d_\alpha(\xi, \varrho) = d_\alpha(\varrho, \xi),$

EBM4 $d_\alpha(\xi, \vartheta) \leq \alpha(\xi, \vartheta)\Big[d_\theta(\xi, \varrho) + d_\alpha(\xi, \vartheta)\Big].$

The pair (W, d_α) is called an **Extended b-metric Space(EBMS)** [30]."

Remark 2.1.3.

Class of EBMS is larger than the classes of BMS and metric spaces as by setting $\alpha(\xi, \varrho) = b$, the above definition coincides with the definition of BMS and by setting $\alpha(\xi, \varrho) = 1$, the above definition coincides with the definition of metric spaces.

Hence every metric space is a BMS and every BMS is EBMS but converse is not true.

Example 2.1.3.

Consider $W = \{1, 2, 3\}$. Define mappings $d_\alpha : W \times W \longrightarrow [0, \infty)$ by

$$d_\alpha(\xi, \varrho) = (\xi - \varrho)^2$$

and $\alpha : W \times W \longrightarrow [1, \infty)$ as

$$\alpha(\xi, \varrho) = \xi + \varrho + 2,$$

then (W, d_α) is an EBMS.

EBM1 and EBM2 are trivial. We need to prove only EBM3, which follows from the fact that

$$(\xi - \varrho)^2 = (\varrho - \xi)^2$$

for all $\xi, \varrho \in W$.

To verify EBM4, note that

$$\begin{aligned} 1 = d_\alpha(1,2) &\leq \alpha(1,2)[d_\alpha(1,3) + d_\alpha(3,2)] \\ &\leq (1+2+2)[4+1] \\ &\leq (5)[5] \\ &= 25. \end{aligned}$$

Similarly we have,

$$\begin{aligned} 1 = d_\alpha(2,3) &\leq \alpha(2,3)[d_\alpha(2,1) + d_\alpha(1,3)] \\ &\leq (2+3+2)[1+4] \\ &\leq (7)[5] \\ &= 35. \end{aligned}$$

and

$$\begin{aligned} 4 = d_\alpha(3,1) &\leq \alpha(3,1)[d_\alpha(3,2) + d_\alpha(2,1)] \\ &\leq (3+1+2)[1+1] \\ &\leq (6)[2] \\ &= 12. \end{aligned}$$

Therefore, for all $\xi, \varrho, \vartheta \in W$,

$$d_\alpha(\xi, \vartheta) \leq \alpha(\xi, \vartheta)[d_\alpha(\xi, \varrho) + d_\alpha(\xi, \vartheta)].$$

Since all the conditions are satisfied. Thus (W, d_α) is an EBMS.

Example 2.1.4.

Consider $W = \{1, 2, 3\}$. Define $\alpha : W \times W \longrightarrow [1, \infty)$ and $d_\alpha : W \times W \longrightarrow [0, \infty)$ as

$$\alpha(\xi, \varrho) = \xi + \varrho + 1,$$

Now

$$d_\alpha(1,1) = d_\alpha(2,2) = d_\alpha(3,3) = 0,$$

$$d_\alpha(1,2) = d_\alpha(2,1) = 80,$$

$$d_\alpha(1,3) = d_\alpha(3,1) = 1000,$$

$$d_\alpha(2,3) = d_\alpha(3,2) = 600,$$

$$d_\alpha(1,1) = d_\alpha(2,2) = d_\alpha(3,3) = 0.$$

EBM1 and EBM2 are trivial. We need to prove only EBM4, for all $\xi, \varrho \in W$, note that

$$\begin{aligned} 80 = d_\alpha(1,2) &\leq \alpha(1,2)[d_\alpha(1,3) + d_\alpha(3,2)] \\ &= (4)[1000 + 600] \\ &= (4)[1600] = 6400. \end{aligned}$$

Similarly we have,

$$\begin{aligned} 1000 = d_\alpha(1,3) &\leq \alpha(1,3)[d_\alpha(1,2) + d_\alpha(2,3)] \\ &= (5)[80 + 600] \\ &= (5)[680] = 3400. \end{aligned}$$

And

$$\begin{aligned} 600 = d_\alpha(2,3) &\leq \alpha(2,3)[d_\alpha(2,1) + d_\alpha(1,3)] \\ &= (6)[80 + 1000] \\ &= (6)[1080] = 6480. \end{aligned}$$

Therefore, for all $\xi, \varrho, \vartheta \in W$,

$$d_\alpha(\xi, \vartheta) \leq \alpha(\xi, \vartheta)[d_\alpha(\xi, \varrho) + d_\alpha(\xi, \vartheta)]$$

Since all the conditions are satisfied, and (W, d_α) is an EBMS but it is not a metric space. Further, note that, taking $b = 7$, (W, d_α) becomes a BMS.

The class of EBMS is larger than that of BMS, as shown in the following example.

Example 2.1.5.
Let $W = [0, +\infty)$. Define a mapping $d_\alpha : W \times W \longrightarrow [0, \infty)$ as

$$d_\alpha(\xi, \varrho) = \begin{cases} (\xi + \varrho), & \text{if } \xi \neq \varrho \\ 0 & \text{if } \ \xi = \varrho \end{cases}$$

Define $\alpha : W \times W \longrightarrow [1, \infty)$ as $\alpha(\xi, \varrho) = 1 + \xi + \varrho$, then (W, d_α) is an EBMS which is not a BMS.
EBM1 and EBM2 are trivial. We need to prove only EBM3.

(i) If $\xi = \varrho$ then EBM3 is clear.

(ii) If $\xi \neq \varrho, \xi = \eta$, then

$$\begin{aligned} \alpha(\xi, \varrho)[d_\alpha(\xi, \eta) + d_\alpha(\eta, \varrho)] &= (1 + \xi + \varrho)[0 + (\eta + \varrho)] \\ &= (1 + \xi + \varrho)(\xi + \varrho) \\ &\geq (\xi + \varrho) \\ &= d_\alpha(\xi, \varrho). \end{aligned}$$

(iii) If $\xi \neq \varrho, \varrho = \eta$, then

$$\begin{aligned} \alpha(\xi, \varrho)[d_\alpha(\xi, \eta) + d_\alpha(\eta, \varrho)] &= (1 + \xi + \varrho)[(\xi + \eta) + 0] \\ &= (1 + \xi + \varrho)(\xi + \varrho) \\ &\geq (\xi + \varrho) = d_\alpha(\xi, \varrho). \end{aligned}$$

(iv) If $\xi \neq \varrho, \varrho \neq \eta, \xi \neq \eta$, then

$$\begin{aligned}\alpha(\xi,\varrho)[d_\alpha(\xi,\eta)+d_\alpha(\eta,\varrho)] &= (1+\xi+\varrho)[(\xi+\eta)+(\eta+\varrho)] \\ &\geq (\xi+2\eta+\varrho) \\ &\geq (\xi+\varrho) = d_\alpha(\xi,\varrho).\end{aligned}$$

It follows from above all cases that (W, d_α) is an EBMS which is not a BMS.

Definition 2.1.4.

"Let (W, d_α) be an extended b-metric space. A sequence $\{\xi_n\}$ in W is said to be **Convergent Sequence** if for every $\varepsilon > 0$ there exists an $N = N(\varepsilon) \in \mathbb{N}$ such that,

$$d_\alpha(\xi_n, \xi) < \varepsilon \qquad \text{for all } n \geq N.$$

In this case, we write

$$\lim_{n \longrightarrow \infty} \xi_n = \xi\text{" [30].}$$

Definition 2.1.5.

"A sequence $\{\xi_n\}$ W is said to be a **(Cauchy Sequence)**, if for every $\varepsilon > 0$ there exists $N = N(\varepsilon) \in \mathbb{N}$ such that,

$$d_\alpha(\xi_m, \xi_n) < \varepsilon \qquad \text{for every } m, n \geq N.$$

An extended b-metric space (W, d_α) [30] is complete if every **Cauchy sequence** in W is convergent."

An other wellknown generalization of metric space is the notion of rectangular metric space (RMS) introduced by Branciari [25] in 2000. By generalizing both the notions of BMS and RMS, George et al. [26] presented the concept of rectangular b-metric space in 2015 as follows.

Definition 2.1.6.

"Let W be a non-empty set and b be any real number such that $b \geq 1$. A function $d_b : W \times W \longrightarrow [0, \infty)$ is called a rectangular b-metric on W if it satisfies the following conditions for all $\xi, \varrho, \eta, \vartheta \in W$.

RbM1 $d_b(\xi, \varrho) \geq 0$,

RbM2 $d_b(\xi, \varrho) = 0$ if and only if $\xi = \varrho$,

RbM3 $d_b(\xi, \varrho) = d_b(\varrho, \xi)$,

RbM4 $d_b(\xi, \vartheta) \leq b(d_b(\xi, \varrho) + d_b(\varrho, \eta) + d_b(\eta, \vartheta))$.

The pair (W, d_b) is called **rectangular b-metric space"**[26]**.**

Example 2.1.6.

Let $W = \mathbb{N}$ and a function $d_b : W \times W \to W$ be defined by

$$d_b(\xi, \varrho) = \begin{cases} 0, & \text{if } \xi = \varrho \\ \eta, & \text{if } \xi \text{ or } \varrho \notin \{1, 2\}, \xi \neq \varrho \\ 4\eta, & \text{if } \xi, \varrho \in \{1, 2\}, \xi \neq \varrho \end{cases}$$

where $\eta > 0$ is a constant. Then (W, d_b) is a RBMS with coefficient $b = 2$ and is not a RMS as,

$$\begin{aligned} d_b(1, 2) &= 4\eta > 3\eta \\ &= d_b(1, 3) + d_b(3, 4) + d_b(4, 2). \end{aligned}$$

Definition 2.1.7.

Let $\Gamma : W \to W$ be self mapping an element $\xi \in W$ is called fixed point of Γ if $\Gamma\xi = \xi$.

Example 2.1.7.

Following are the examples of fixed points.

1. The mapping $\Gamma : W \to W$ defined by

$$\Gamma(\xi) = \xi^2 + 2$$

has no fixed point in $W = \mathbb{R}$.

Geometrically it means that the graph of $I(\xi) = \xi$ never intersects the graph of $\Gamma(\xi) = \xi^2 + 2$ (see figure 2.1) .

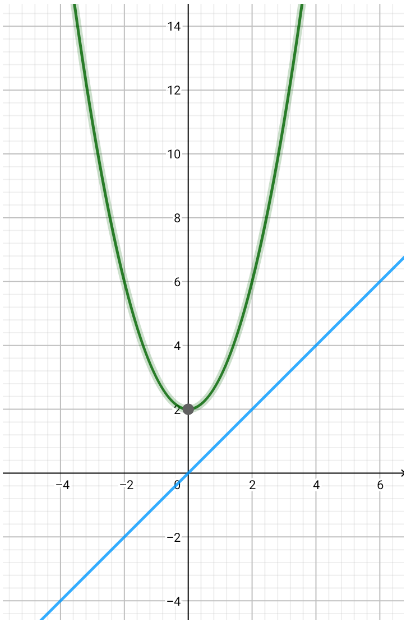

FIGURE 2.1: A mappig having no fixed point.

2. Let $W = \mathbb{R}$ and defined $\Gamma : W \to W$ by

$$\Gamma\xi = \xi^2 \quad \forall\, \xi \in \mathbb{R}.$$

It is clear from Figure 2.2 that 0 and 1 are two fixed points.
Geometrically it means that the graph of $I(\xi) = \xi$ intersects the graph of $\Gamma(\xi) = \xi^2$ at two points 0 and 1.

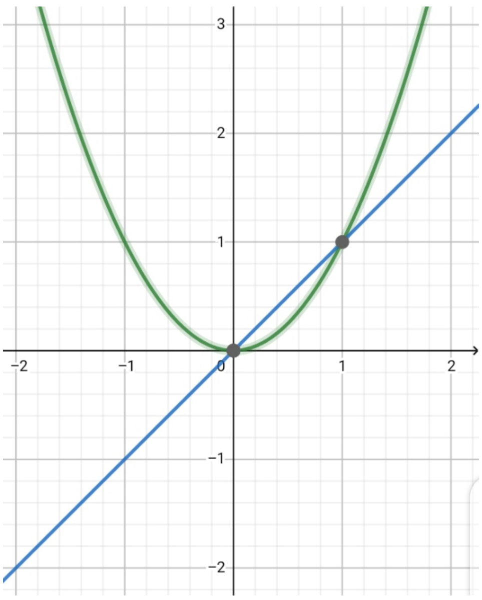

FIGURE 2.2: A mapping having two fixed points.

3. $\Gamma : W \to W$ given by

$$\Gamma(\xi) = \xi^3$$

has three fixed points *i.e;* $\xi = -1, 0, 1$ (see Figure 2.2).

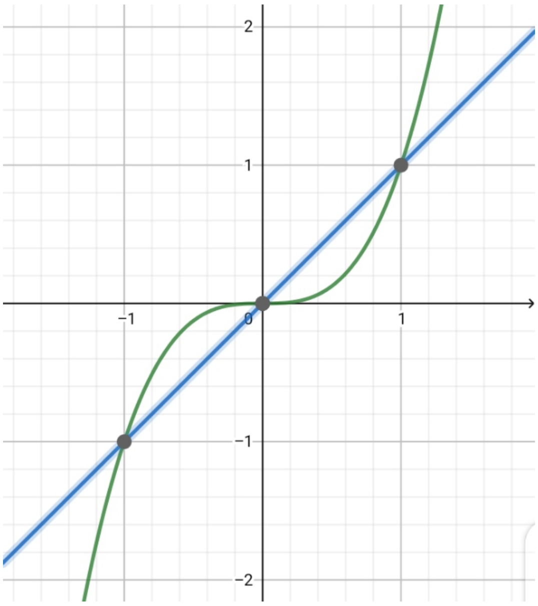

FIGURE 2.3: A mapping having three fixed points.

2.2 Fuzzy Metric Space

In 1965, the concept of fuzzy set was introduced by L.A. Zadeh [37].

Definition 2.2.1.

Let W be any set. A fuzzy set $f_{\mathcal{A}}$ on W is a function from domain W and values in $[0, 1]$.

Example 2.2.1.

Consider $W = \{a, b, c, d\}$ and $f_{\mathcal{A}} : W \to [0, 1]$ defined as

$$f_{\mathcal{A}}(a) = 0,$$
$$f_{\mathcal{A}}(b) = 0.5,$$
$$f_{\mathcal{A}}(c) = 0.2,$$
$$f_{\mathcal{A}}(d) = 1.$$

Then $f_{\mathcal{A}}$ is a fuzzy set on W.

This fuzzy set can also be written as follows:

$$f_{\mathcal{A}} = \{(a, 0), (b, 0.5), (c, 0.2), (d, 1)\}$$

To define FMS, the notion of t-norm is required. Triangular norms are important methods in fuzzy logic [89] for understanding the conjunction and, consequently, for the intersection of fuzzy sets [37]. However, they are fascinating mathematical objects on their own. Based on some ideas proposed in [90], triangular norms, as we use them today, were first implemented in the sense of probabilistic metric spaces [91, 92]. They also play an important role in decision-making [93, 94], both in statistics [95] and in non-additive measures [96] and cooperative games theories. Some parameterized families of t-norms [97] turn out to be solutions of well-known functional equations.

Definition 2.2.2.

"A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is called a **continuous t-norm** [91], if it satisfies the following conditions:

1. $*$ is associative and commutative,
2. $*$ is continuous,
3. $\xi * 1 = \xi$ for all $\xi \in [0, 1]$ and
4. $\xi * \varrho \leq \vartheta * \eta$ wherever $\xi \leq \vartheta$ and $\varrho \leq \eta$ for all $\xi, \varrho, \vartheta, \eta \in [0, 1]$."

Example 2.2.2.

Define $* : [0, 1] \times [0, 1] \to [0, 1]$ by

1. $\xi * \varrho = \xi\varrho$ for $\xi, \varrho \in [0, 1]$, then $*$ is a continuous t-norm which is known as product norm.
2. $\xi * \varrho = \xi \wedge \varrho = \min\{\xi, \varrho\}$ for all $\xi, \varrho \in [0, 1]$, then $\wedge$ satisfies all conditions of Definition 2.2.2 and hence it is a t-norm known as minimum t-norm.

3. $\xi * \varrho = \xi *_L \varrho = \max\{\xi + \varrho - 1, 0\}$ is also a continuous t-norm.

In 1975 Kramosil and Michálek [49] combined the idea of fuzzy set and t-norm to define FMS.

Definition 2.2.3.
"A 3-tuple $(W, F, *)$ is said to be a **fuzzy metric space** (FMS), if W is an arbitrary set, $*$ is a continuous t-norm and F is a fuzzy set on $W \times W \times [0, \infty)$ satisfying the following conditions:

KFM1: $F(\xi, \varrho, 0) = 0$

KFM2: $F(\xi, \varrho, t) = 1, \forall\, t > 0$ if and only if $\xi = \varrho$

KFM3: $F(\xi, \varrho, t) = F(\varrho, \xi, t)$

KFM4: $F(\xi, \vartheta, t + s) \geq F(\xi, \varrho, t) * F(\varrho, \vartheta, s)\ \forall\, t, s \geq 0$

KFM5: If $F(\xi, \varrho, .) \colon [0, \infty) \to [0, 1]$ is left continuous, $\xi, \varrho, \vartheta \in W$ and $t, s > 0$" [49].

Example 2.2.3.
Let (W, d) be a metric space. We define

$$\xi * \varrho = \xi\varrho$$

for all $\xi, \varrho \in [0, 1]$. Let F be fuzzy set on $W \times W \times (0, \infty)$ defined as follows:

$$F(\xi, \varrho, t) = \frac{t}{t + d(\xi, \varrho)},$$

then $(W, F, *)$ is a FMS. This F is known as standard fuzzy metric on W induced by the metric d on W.

In [50], George and Veeramani notice that $F(\xi, \varrho, t) = 1$ means ξ and ϱ are exactly same, that is, $\xi = \varrho$ and $F(\xi, \varrho, t) = 0$ is identified with ∞. This understanding allows George and Veeramani to modify Definition 2.2.3 as below

Definition 2.2.4.

"A 3-tuple $(W, F, *)$ is said to be a **fuzzy metric space**, if W is an arbitrary set, $*$ is a continuous t-norm and F is a fuzzy set on $W \times W \times (0, \infty)$ satisfying the following conditions:

GFM1: $F(\xi, \varrho, t) > 0$;

GFM2: $F(\xi, \varrho, t) = 1$ if and only if $\xi = \varrho$ for all $t > 0$;

GFM3: $F(\xi, \varrho, t) = F(\varrho, \xi, t)$;

GFM4: $F(\xi, \vartheta, t + s) \geq F(\xi, \varrho, t) * F(\varrho, \vartheta, s)$;

GFM5: $F(\xi, \varrho, .) \colon (0, \infty) \to [0, 1]$ is continuous , $\xi, \varrho, \vartheta \in W$ and $t, s > 0$" [50].

The property (GFM1) is justified as in the case of a metric space (W, d), as d does not take the value ∞, in the same way in FMS, F cannot take the value 0. The property (GFM2) gives the concept that the degree of nearness of ξ and ϱ is 1, only when $\xi = \varrho$, and then $F(\xi, \xi, t) = 1$ for each $t > 0$ and for each $\xi \in W$. That is, in this fuzzy theory, values 0 and ∞ are associated with 1 and 0 respectively in the traditional theory of metric spaces. Properties (GFM3) and (GFM4) are fuzzy forms of both symmetry and triangular inequality respectively.

Finally, the function $t \mapsto F(\xi, \varrho, t)$ in (GFM5), for fixed ξ and ϱ, is continuous without any restriction for F as $t \to \infty$.

Example 2.2.4.

Let (W, d) be a metric space and t-norm $*$ be the product norm. For $k, m, n \in \mathbb{R}^+$, define $F : W \times W \times (0, \infty) \to [0, 1]$ by

$$F(\xi,\ \varrho,\ t) = \frac{kt^n}{kt^n + md(\xi,\ \varrho)}$$

for all $\xi, \varrho \in W$ and $t > 0$. Then $(W, F, *)$ is a FMS.

Example 2.2.5.

Let (W, d) be a metric space and t-norm $*$ be the product norm. Define a mapping

$F : W \times W \times (0, \infty) \to [0, 1]$ by

$$F(\xi, \varrho, t) = e^{-\dfrac{|\xi - \varrho|}{t}},$$

then it is simple to demonstrate that $(W, F, *)$ is a FMS.

In fact, all other properties are obvious. To prove GFM4, notice that,

$$\begin{aligned}
F(\xi, \vartheta, t+s) &= e^{-\dfrac{|\xi - \vartheta|}{t+s}} \\
&= e^{-\dfrac{|\xi - \varrho + \varrho - \vartheta|}{t+s}} \\
&\geqslant e^{-\dfrac{|\xi - \varrho| + |\varrho - \vartheta|}{t+s}} \\
&= e^{-\dfrac{|\xi - \varrho|}{t+s}} . e^{-\dfrac{|\varrho - \vartheta|}{t+s}} \\
&\geq e^{-\dfrac{|\xi - \varrho|}{t}} . e^{-\dfrac{|\varrho - \vartheta|}{s}} \\
&= F(\xi, \varrho, t) * F(\varrho, \vartheta, s).
\end{aligned}$$

To prove the BCP [5] and Edelstien [8] results FMS, Grabiec [51] introduced the notion of convergent sequence, Cauchy sequence and completeness in FMS as follows:

Definition 2.2.5.

"Let $(W, F, *)$ be a fuzzy metric space. A sequence $\{\xi_n\}$ in W is convergent to $\xi \in W$ if

$$\lim_{n\to\infty} F(\xi_n, \xi, t) = 1 \text{ for each } t > 0.$$

A sequence $\{\xi_n\}$ in W is Cauchy if

$$\lim_{n\to\infty} F(\xi_n, \xi_{n+m}, t) = 1$$

for each $t > 0$ and $m > 0$.

A fuzzy metric space in which every Cauchy sequence is convergent is called complete"[51].

Remark 2.2.6.

Throughout the thesis, the convergent sequence, the Cauchy sequence and the completeness in the sense of by Grabiec [51](Definition 2.2.5) will respectively be termed as G-convergent sequence, G-Cauchy sequence and G-completeness.

In 1988, Grabiec [51] proved the BCP and Edelstien fixed point theorem in FMS which are are stated below.

Theorem 2.2.7.

"Let $(W, F, *)$ be a complete fuzzy metric space such that $\lim_{t\to\infty} F(\xi, \varrho, t) = 1$ for all $\xi, \varrho \in W$. Let $\Gamma : W \to W$ be a mapping satisfying

$$F(\Gamma\xi, \Gamma\varrho, kt) \geq F(\xi, \varrho, t)$$

$\forall\, \xi, \varrho \in W$ where $k \in (0, 1)$. Then Γ has a unique fixed point."

Theorem 2.2.8.

"Let $(W, F, *)$ be a compact fuzzy metric space with $F(\xi, \varrho, .)$ continuous for all $\xi, \varrho \in W$. Let $\Gamma : W \to W$ be a mapping satisfying

$$F(\Gamma\xi, \Gamma\varrho, t) > F(\xi, \varrho, t)$$

for all $\xi \neq \varrho$ and $t > 0$. Then Γ has a unique fixed point."

In [50], George and Veeramani noted that the set of real number $\mathbb{R}$ failed to be complete in the setting of Definition 2.2.5 by Grabiec. This is illustrated by the following example.

Example 2.2.6.

Let $W = \mathbb{R}$. Define

$$F(\xi, \varrho, t) = \begin{cases} \dfrac{t}{t + d_b(\xi, \varrho)} & \text{if } t > 0 \\ 0 & \text{if } t = 0 \end{cases},$$

for $\xi, \varrho \in W, t \geq 0$, then F is a fuzzy metric on $\mathbb{R}$.

Let

$$S_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \text{ for } n \in \mathbb{N},$$

then

$$F(S_{n+p}, S_n, t) = \frac{t}{t + \frac{1}{n+1} + \ldots + \frac{1}{n+p}}$$
$$\to 1 \text{ as } n \to \infty \text{ for } p > 0.$$

Hence $\{S_n\}$ is G-Cauchy sequence in FMS.

If $\mathbb{R}$ is fuzzy complete then there exists $\xi \in \mathbb{R}$ such that

$$F(S_n, \xi, t) \to 1 \text{ as } n \to \infty.$$

From this it follows that

$$\frac{t}{t + |S_n - \xi|} \to 1 \text{ as } n \to \infty$$

$$\Rightarrow |S_n - \xi| \to 0 \text{ as } n \to \infty$$

and so $S_n \to \xi \in \mathbb{R}$ which is not true.

Hence for $\mathbb{R}$ to be a complete FMS, George and Veeramani [50] redefined the Cauchy sequence as follows.

Definition 2.2.9.

"Let $(W, F, *)$ be a fuzzy metric space and $\{\xi_n\}$ be a sequence in W. Then $\{\xi_n\}$ is called a Cauchy sequence if there exists $n_0 \in \mathbb{N}$ such that

$$F(\xi_n, \xi_m, t) > 1 - \epsilon, \; \forall \; m, n \geq n_0, \quad \epsilon \in (0, 1) \quad \text{and} \quad t > 0.$$

The sequence $\{\xi_n\}$ is called convergent and converges to ξ if there exists $n_0 \in \mathbb{N}$ such that

$$F(\xi_n, \xi, t) > 1 - \epsilon, \; \forall \quad n \geq n_0 \quad \textit{and} \quad \epsilon \in (0, 1), \quad t > 0.$$

We say that the space $(W, F, *)$ is complete if every Cauchy sequence in W is convergent to some $\xi \in W$”[50].

In literature, the convergent sequence, Cauchy sequence and completeness defined in Definition 2.2.9 are termed as M-convergent sequence, M-Cauchy sequence and M-completeness respectively. Both the sequences G-Cauchy sequence and M-Cauchy sequence are not equivalent because the notion of G-Cauchy sequence is weaker than M-Cauchy sequence.

Song [98] noted that, due to Vasuki [99] and Grabiec [51], the criteria of certain fixed point theorems are incomplete and the proof of the theorems is incorrect. To justify this, Song emphasized that the Definition 2.2.5 and Definition 2.2.9 are similar. Song said that in a FMS $\{\xi_n\}$ is a Cauchy sequence if and only if

$$F(\xi_n, \xi_{n+q}, t) \to 1 (\forall t > 0) \text{ as } n \to \infty$$

uniformly on $q \in \mathbb{N}$, where $\mathbb{N}$ is the set of all positive integers, otherwise Cauchy sequence defined in [51] is incorrect. This can easily be seen in the illustration below.

In fact, if $\{\xi_n\}$ is a convergent sequence in FMS, we can assume $\xi_n \to \xi_0$ $(n \to +\infty)$, without the loss of the generality, then we have

$$F(\xi_{n+p}, \xi_n, t) \geq F(\xi_{n+p}, \xi_0, \frac{t}{2}) * F(\xi_0, \xi_n, \frac{t}{2}) \to 1, \ (n \to \infty)$$

for any $t > 0$. Thus, with respect to $q \in \mathbb{N}$, $\lim_{n\to\infty} F(\xi_{n+q}, \xi_n, t) = 1$ $(t > 0)$, uniformly, where $\mathbb{N}$ is the set of all natural numbers. This implies that if we consider the Cauchy sequence in FMS in accordance with Vasuki 's views, there is no complete FMS. Vasuki in [55] reported that Song in [98] assumed the concept of the Cauchy sequence in FMS should be specified in a particular way. It is true that Grabiec's definition of Cauchy sequence for FMS is weaker than Song's one. George and Veeramani noted that the description of the Cauchy sequence by Grabiec is weaker and therefore it is necessary to change that definition to obtain better results in FMS. In [50] some results are obtained in this connection. Definitions 2.2.5 and 2.2.9 are obviously not equivalent. Example 2.2.6 demonstrates the statement given above.

Some results in [51] and [99] are proved for complete FMS, in the sense of Definition 2.2.5 . Let us call this complete FMS as a G-complete FMS (refer [52]). Grabiec should used world weak Cauchy sequence and weak FMS instead of Cauchy sequence and FMS. Perhaps because of this usage, Song [98] feels it is appropriate to correct the sense of the Cauchy sequence as given in Grabiec [51]. It is clear from the definitions that every complete FMS need not be a G-complete FMS. Some how Song [98] skipped this argument in his article. Once we understand that the Cauchy sequences given in Definitions 2.2.5 and 2.2.9 are distinct, then it is easy to see that fixed point theorems proved are different for complete FMS and G-complete FMS.

2.2.1 Some Generalization of Fuzzy Metric Space

To generalize the idea of BMS in fuzzy settings, Nădăban [78] introduced the notion of FBMS in 2016. As EBMS is more general form of metric space and BMS, like wise EFBMS generalizes FMS and FBMS. In [87] the notion of EFBMS is introduced in 2017. Chugh and Kumar [100] generalized the notion RMS in fuzzy settings by introducing the idea of fuzzy rectangular metric spaces(FRMS) in 2002.

In this section we recall the notion of FBMS introduced by Nădăban [78]. We illustrate the definition by an example and also show that FBMS need not to be FMS. The concept of convergence sequence, Cauchy sequence and completeness in FBMS in the sense of Grabiec is also included in this section. Following Kramosil and Michálek [49], Nădăban [78] defined FBMS as follows.

Definition 2.2.10.

"Let W be a non empty set, let $b \geq 1$ be a given real number and $*$ be a continuous t-norm. A fuzzy set F_b on $W \times W \times [0, \infty)$ is called **fuzzy b-metric** if for all $\xi, \varrho, \vartheta \in W$, the following conditions hold:

FBM1: $F_b(\xi, \varrho, 0) = 0$

FBM2: $F_b(\xi, \varrho, t) = 1, \forall\, t > 0$ if and only if $\xi = \varrho$

FBM3: $F_b(\xi, \varrho, t) = F_b(\varrho, \xi, t),\ \forall\, t \geq 0$

FBM4: $F_b(\xi, \vartheta, b(t+s)) \geq F_b(\xi, \varrho, t) * F_b(\varrho, \vartheta, s),\ \forall\, t, s \geq 0$

FBM5: If $F_b(\xi, \varrho, .)\colon [0, \infty) \to [0, 1]$ is left continuous and $\lim_{t\to\infty} F_b(\xi, \varrho, t) = 1$.

Then the triplet $W, F_b, *$ is called **fuzzy b-metric space** (FBMS)"[78].

Remark 2.2.11.

The class of FBMS is larger than that of FMS. Setting $b = 1$, the above definition coincides with Definition 2.2.3 of FMS.

The following example shows that FBMS is not a FMS.

Example 2.2.7.

Let $W = \mathbb{R}$. Consider a function $F_b\colon W \times W \times [0, \infty) \to [0, 1]$ defined by

$$F_b(\xi, \varrho, t) = e^{-\dfrac{|\xi - \varrho|^m}{t}}$$

for all $\xi, \varrho \in \mathbb{R}$ and for a real number $m \geq 1$. Then $(W, F_b, *)$ is a FBMS with the product norm $*$.

The properties from FBM1 to FBM3 and FBM5 are obvious. We only prove FBM4.

Now define a function

$$f(x) = x^m \quad \text{for} \quad x > 0$$

for $\xi, \varrho, \vartheta \in W$, we have

$$\left(\frac{\xi + \varrho}{2}\right)^m \leq \frac{\xi^m + \varrho^m}{2}$$
$$\frac{(\xi + \varrho)^m}{2^m} \leq \frac{\xi^m + \varrho^m}{2}$$
$$(\xi + \varrho)^m \leq 2^{m-1}(\xi^m + \varrho^m)$$

Now for $\xi, \varrho, \vartheta \in X$ and $t_1, t_2 > 0$, we have

$$\begin{aligned}
\frac{|\xi-\vartheta|^m}{t_1+t_2} &= \frac{|\xi-\varrho+\varrho-\vartheta|^m}{t_1+t_2} \\
&\leq 2^{m-1}\left[\frac{|\xi-\varrho|^m}{t_1+t_2}\right] + 2^{m-1}\left[\frac{|\varrho-\vartheta|^m}{t_1+t_2}\right] \\
&\leq 2^{m-1}\left[\frac{|\xi-\varrho|^m}{t_1}\right] + 2^{m-1}\left[\frac{|\varrho-\vartheta|^m}{t_2}\right] \\
&= \left[\frac{|\xi-\varrho|^m}{\frac{t_1}{2^{m-1}}}\right] + \left[\frac{|\varrho-\vartheta|^m}{\frac{t_2}{2^{m-1}}}\right].
\end{aligned}$$

Now,

$$\begin{aligned}
F_b(\xi,\vartheta,t_1+t_2) &= e^{-\frac{|\xi-\vartheta|^m}{t_1+t_2}} \\
&\geq e^{-\left[\frac{|\xi-\varrho|^m}{\frac{t_1}{2^{m-1}}}+\frac{|\varrho-\vartheta|^m}{\frac{t_2}{2^{m-1}}}\right]} \\
&\geq e^{-\left[\frac{|\xi-\varrho|^m}{\frac{t_1}{2^{m-1}}}\right]}.e^{-\left[\frac{|\varrho-\vartheta|^m}{\frac{t_2}{2^{m-1}}}\right]} \\
&= F_b\left(\xi,\varrho,\frac{t_1}{2^{m-1}}\right) * F_b\left(\varrho,\vartheta,\frac{t_2}{2^{m-1}}\right)
\end{aligned}$$

$\Rightarrow (W, F_b, *)$ is a FBMS with $b = 2^{m-1}$.

In the above example, if we take $m = 2$, then $(W, F_b, *)$ is not a FMS.

For the generalization of FMS, Chugh and Kumar [100] gave the idea of fuzzy rectangular metric spaces(FRMS) in 2002 as follows.

Definition 2.2.12.

"A 3-tuple $(W, F_r, *)$ is said to be a **fuzzy rectangular metric space** (FRMS) if W is a non empty set, $*$ is a continuous t-norm and F_r is a fuzzy set on $W \times W \times [0, \infty)$ satisfying the following conditions for all $\xi, \varrho, \vartheta \in W$ and $t, s > 0$

FRM-1 $F_r(\xi, \varrho, t) > 0$

FRM-2 $F_r(\xi, \varrho, t) = 1$ if and only if $\xi = \varrho$

FRM-3 $F_r(\xi, \varrho, t) = F_r(\xi, \varrho, t)$

FRM-4 $F_r(\xi, \vartheta, t+s+w) \geq F_r(\xi, \varrho, t) * F_r(\varrho, \eta, s) * F_r(\eta, \vartheta, w)$ $\forall$ distinct $\varrho, \eta \in W \setminus \{\xi, \vartheta\}$.

FRM-5 $F_r(\xi, \varrho, .)\colon [0, \infty) \to [0, 1]$ is left continuous, and $\lim_{t\to\infty} F_r(\xi, \varrho, t) = 1$."

In [101], FRMS is also called as fuzzy generalized metric space.

Example 2.2.8.

Let $W = \mathbb{R}$. A function $F_r\colon W \times W \times [0, \infty) \to [0, 1]$ defined by

$$F_r(\xi, \varrho, t) = \begin{cases} e^{\dfrac{-|\xi-\varrho|}{t}} & \text{if } t > 0 \\ 0 & \text{if } t = 0. \end{cases}$$

for all $\xi, \varrho \in \mathbb{R}$. Then $(W, F_r, *)$ is a FRMS.

In fact, all other properties are obvious. To prove FRM-4, notice that

$$\begin{aligned} F_r(\xi, \vartheta, t+s+w) &= e^{-\dfrac{|\xi-\vartheta|}{t+s+w}} \\ &= e^{-\dfrac{|\xi-\varrho+\varrho-\eta+\eta-\vartheta|}{t+s+w}} \\ &\geqslant e^{-\dfrac{|\xi-\varrho|+|\varrho-\eta|+|\eta-\vartheta|}{t+s+w}} \\ &= e^{-\dfrac{|\xi-\varrho|}{t+s+w}}.e^{-\dfrac{|\varrho-\eta|}{t+s+w}}.e^{-\dfrac{|\eta-\vartheta|}{t+s+w}} \\ &\geq e^{-\dfrac{|\xi-\varrho|}{t}}.e^{-\dfrac{|\varrho-\eta|}{s}}.e^{-\dfrac{|\eta-\vartheta|}{w}} \\ &= F_r(\xi, \varrho, t) * F_r(\varrho, \eta, s) * F_r(\eta, \vartheta, w). \end{aligned}$$

Chapter 3

Fuzzy Rectangular b-Metric Space

Motivated by the notion of FRMS and FBMS, first the notion of fuzzy rectangular b-metric space (FRBMS) is introduced in Section 3.1. A counter example shows that FRBMS need not be a FRMS. In Section 3.2, BCP is established in FRBMS. Fixed point theorem of Hicks and Rhoads [85] in the setting of FRBMS is establised in Section 3.2. At the end of this section, an example is established that illustrates Theorem 3.2.2. In Section 3.2.2, a fixed point result analogue to [28, Theorem 1] is established in the setting of G-complete FRBMS. In Section 3.3, an application related to main result is established and in the last section, a brief conclusion of chapter is given.

3.1 Fuzzy Rectangular b-Metric Space

Following the notion of FBMS [78], we now generalize Definition 2.2.12 by introducing the idea of FRBMS:

Definition 3.1.1.

Let W be a nonempty set, $*$ be a continuous t-norm and F_{rb} be a fuzzy set on $W \times W \times [0, \infty)$. A triplet $(W, F_{rb}, *)$ is said to be a FRBMS if there is $b \geq 1$ and the fuzzy set F_{rb} satisfies the following conditions for all $\vartheta, \varrho, \vartheta \in W$ and $t, s, w > 0$,

FRBM-1 $F_{rb}(\vartheta, \varrho, 0) = 0$

FRBM-2 $F_{rb}(\xi, \varrho, t) = 1$ if and only if $\xi = \varrho$

FRBM-3 $F_{rb}(\xi, \varrho, t) = F_{rb}(\varrho, \xi, t)$

FRBM-4 $F_{rb}(\xi, \vartheta, b(t + s + w)) \geq F_{rb}(\xi, \varrho, t) * F_{rb}(\varrho, \eta, s) * F_{rb}(\eta, \vartheta, w)$ for all distinct $\varrho, \eta \in X \setminus \{\xi, \vartheta\}$

FRBM-5 $F_{rb}(\xi, \varrho, .)\colon [0, \infty) \to [0, 1]$ is left continuous and $\lim_{t\to\infty} F_{rb}(\xi, \varrho, t) = 1$.

Remark 3.1.2.

Taking $b = 1$, the above definition coincides with Definition 2.2.12 of FRMS.

A FRBMS can be induced by RBMS as shown in the following example.

Example 3.1.1.

Let (W, d_b) be a RBMS and define $F_{rb}\colon W \times W \times [0, \infty) \to [0, 1]$ by

$$F_{rb}(\xi, \varrho, t) = \begin{cases} \dfrac{t}{t + d_b(\xi, \varrho)} & \text{if } t > 0 \\ 0 & \text{if } t = 0. \end{cases} \tag{3.1}$$

Choose continuous t-norm $*$ as $a * b = \min\{a, b\}$. Then F_{rb} satisfies all the conditions given in Definition 3.1.1. Infact, all properties follow immediately from (3.1) and to prove **FRBM-4**, we proceed as follows:

Let $\xi, \varrho, \vartheta \in W$ and $t, s, w > 0$. Without restraining the generality, assume that

$$F_{rb}(\xi, \varrho, t) \leq F_{rb}(\varrho, \eta, s),$$

and

$$F_{rb}(\xi, \varrho, t) \leq F_{rb}(\eta, \vartheta, w),$$

which implies that

$$\frac{t}{t + d_b(\xi, \varrho)} \leq \frac{s}{s + d_b(\varrho, \eta)}$$

and

$$\frac{t}{t + d_b(\xi, \varrho)} \leq \frac{w}{w + d_b(\eta, \vartheta)}.$$

Hence we have

$$td_b(\varrho, \eta) \leq sd_b(\xi, \varrho),$$

and

$$td_b(\eta, \vartheta) \leq wd_b(\xi, \varrho)$$

$$\Rightarrow (s+w)d_b(\xi, \varrho) \geq t(d_b(\varrho, \eta) + d_b(\eta, \vartheta)). \tag{3.2}$$

Note that

$$\begin{aligned} F_{rb}(\xi, \vartheta, b(t+s+w)) &\geq F_{rb}(\xi, \varrho, t) \\ \Leftrightarrow \frac{b(t+s+w)}{b(t+s+w) + d_b(\xi, \vartheta)} &\geq \frac{t}{t + d_b(\xi, \varrho)} \\ \Leftrightarrow \frac{b(t+s+w)}{b(t+s+w) + b[d_b(\xi, \varrho) + d_b(\varrho, \eta) + d_b(\eta, \vartheta)]} &\geq \frac{t}{t + d_b(\xi, \varrho)} \\ \Leftrightarrow \frac{t+s+w}{t+s+w + d_b(\xi, \varrho) + d_b(\varrho, \eta) + d_b(\eta, \vartheta)} &\geq \frac{t}{t + d_b(\xi, \varrho)} \end{aligned}$$

$$\Leftrightarrow (s+w)d_b(\xi, \varrho) \geq t(d_b(\varrho, \eta) + d_b(\eta, \vartheta)),$$

which is the same as (3.2). Hence it follows that

$$F_{rb}(\xi, \vartheta, b(t+s+w)) \geq F_{rb}(\xi, \varrho, t) * F_{rb}(\varrho, \eta, s) * F_{rb}(\eta, \vartheta, w).$$

Therefore, $(W, F_{rb}, *)$ is a FRBMS.

The following example shows that a FRBMS need not be a FRMS.

Example 3.1.2.

Let $W = \mathbb{N}$ and choose t-norm $*$ as $a * b = ab$. Define a mapping F_{rb} on $W \times W \times [0, \infty)$ by

$$F_{rb}(\xi, \varrho, t) = e^{-\dfrac{(\xi - \varrho)^2}{t}}$$

for all $\xi, \varrho \in W$ and $t > 0$. Then $(W, F_{rb}, *)$ is a FRBMS with coefficient $b = 3$ but it is not a FRMS.

The properties FRBM-1 to FRBM-3 and FRBM-5 of Definition 3.1 are obvious. We prove only FRBM-4.

Note that for all $\xi, \varrho, \vartheta, \eta \in W$, we have

$$\begin{aligned}(\xi-\vartheta)^2 &= (\xi-\varrho+\varrho-\eta+\eta-\vartheta)^2\\ \Rightarrow (\xi-\vartheta)^2 &\le 3\left\{(\xi-\varrho)^2+(\varrho-\eta)^2+(\eta-\vartheta)^2\right\}.\end{aligned}$$

$\forall\, t, s, w > 0$, it follows that

$$\begin{aligned}F_{rb}(\xi,\vartheta,t+s+w) &= e^{-\dfrac{(\xi-\vartheta)^2}{t+s+w}}\\ &\ge e^{-\dfrac{3\left\{(\xi-\varrho)^2+(\varrho-\eta)^2+(\eta-\vartheta)^2\right\}}{t+s+w}}\\ &= e^{-\dfrac{3\{(\xi-\varrho)^2\}}{t+s+w}}.e^{-\dfrac{3\{(\varrho-\eta)^2\}}{t+s+w}}.e^{-\dfrac{3\{(\eta-\vartheta)^2\}}{t+s+w}}\\ &\ge e^{-\dfrac{3\{(\xi-\varrho)^2\}}{t}}.e^{-\dfrac{3\{(\varrho-\eta)^2\}}{s}}.e^{-\dfrac{3\{(\eta-\vartheta)^2\}}{w}}\\ &= e^{-\dfrac{\{(\xi-\varrho)^2\}}{\frac{t}{3}}}.e^{-\dfrac{\{(\varrho-\eta)^2\}}{\frac{s}{3}}}.e^{-\dfrac{\{(\eta-\vartheta)^2\}}{\frac{w}{3}}}\\ &= F_{rb}\left(\xi,\varrho,\frac{t}{3}\right) * F_{rb}\left(\varrho,\eta,\frac{s}{3}\right) * F_{rb}\left(\eta,\vartheta,\frac{w}{3}\right).\end{aligned}$$

Hence $(W, F_{rb}, *)$ is a FRBMS and not a FRMS.

Remark 3.1.3.

It follows from the above example that the class of FRBMS is larger than the class of FRMS.

Following Grabiec [94], the notion of G-convergent sequence, G-Cauchy sequence and G-completeness in FRBMS naturally as follows:

Definition 3.1.4.

Let $(W, F_{rb}, *)$ be a FRBMS and let $\{\xi_n\}$ in W be any sequence. Then

1. $\{\xi_n\}$ is G-convergent sequence if there exits $\xi \in W$ such that

$$\lim_{n\to\infty} F_{rb}(\xi_n, \xi, t) = 1,\ \forall\, t > 0.$$

2. $\{\xi_n\}$ in W is a G-Cauchy sequence if

$$\lim_{n\to\infty} F_{rb}(\xi_n, \xi_{n+q}, t) = 1.$$

Following are the examples of Definition 3.1.4.

Example 3.1.3.

Let $W = \mathbb{R}$. Define

$$F_{rb}(\xi, \varrho, t) = \begin{cases} \dfrac{t}{t + (\xi - \varrho)^2} & \text{if } t > 0 \\ 0 & \text{if } t = 0 \end{cases}$$

$\forall\ \xi, \varrho \in W$, then F_{rb} is a G-complete FRBMS.
Let $\{\xi_n\} = \dfrac{1}{n^2}\ \forall\, n \in \mathbb{N}$ be a sequence in W, then $\{\xi_n\}$ converges to 0.
Now

$$\begin{aligned}\lim_{n\to\infty} F_{rb}(\xi_n, 0, t) &= \lim_{n\to\infty} \frac{t}{t + \frac{1}{n^2}} \\ &= 1.\end{aligned}$$

Hence $\{\xi_n\}$ is a G-convergent sequence

Example 3.1.4.

Consider again the FRBMS given in Example 3.1.3. Let the sequence $\{\xi_n\}$ be given by

$$\xi_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \quad \forall\, n \in \mathbb{N}.$$

Now for all $q \in \mathbb{N}$, we have

$$\begin{aligned}F_{rb}(\xi_{n+q}, \xi_n, t) &= \frac{t}{t + \frac{1}{(n+1)^2} + \cdots + \frac{1}{(n+q)^2}} \\ &\to 1 \quad \text{as } n \to \infty.\end{aligned}$$

Hence $\{\xi_n\}$ is a G-Cauchy sequence.

3.2 Fixed Point Results in FRBMS

Before establishing the BCP in the setting of FRBMS, following [53], we need the following useful property of a FRBMS.

Lemma 3.2.1.
Let $(W, F_{rb}, *)$ be a G-complete FRBMS and

$$F_{rb}(\xi, \varrho, kt) \geq F_{rb}\left(\xi, \varrho, t\right)$$

for all $\xi, \varrho \in W,\ k \in (0,1)$ and $t > 0$ then $\xi = \varrho$.

The famous BCP [5] was established for FMS by Grabeic [94], following Grabeic, we established this result in the setting of FRBMS as follows.

Theorem 3.2.2.
Let $(W, F_{rb}, *)$ be a G-complete FRBMS with $b \geq 1$ and let $\Gamma\colon W \to W$ be a mapping satisfying

$$F_{rb}(\Gamma\xi, \Gamma\varrho, kt) \geq F_{rb}(\xi, \varrho, t) \tag{3.3}$$

for all $\xi, \varrho \in W, \quad k \in \left(0, \dfrac{1}{b}\right)$. Then Γ has a unique fixed point.

Proof.
Fix an arbitrary point $a_0 \in W$ and for $n = 0, 1, 2, \cdots$, start an iterative process

$$a_{r+1} = \Gamma a_r.$$

Successively applying Inequality (3.3), we get for all $n, t > 0$,

$$F_{rb}(a_r, a_{r+1}, t) \geq F_{rb}\left(a_0, a_1, \frac{t}{k^r}\right). \tag{3.4}$$

Since $(W, F_{rb}, *)$ is a FRBMS, so for the sequence $\{a_r\}$, writing $t = \dfrac{t}{3} + \dfrac{t}{3} + \dfrac{t}{3}$ and using the rectangular inequality given in FRBM-4 on $F_{rb}(a_r, a_{r+p}, t)$, one can prove that

$$\lim_{r\to\infty} F_{rb}(a_r, a_{r+p}, t) = 1 \text{ for all } t > 0.$$

In fact, the proof follows from the following two cases:

Case-1:

If p is odd say $p = 2m + 1$ where $m \in \mathbb{N}$, we have

$$\begin{aligned}
F_{rb}(a_r,&a_{r+2m+1}, t) \\
&\geq F_{rb}\left(a_r, a_{r+1}, \frac{t}{3b}\right) * F_{rb}\left(a_{r+1}, a_{r+2}, \frac{t}{3b}\right) * F_{rb}\left(a_{r+2}, a_{r+2m+1}, \frac{t}{3b}\right) \\
&\geq F_{rb}\left(a_r, a_{r+1}, \frac{t}{3b}\right) * F_{rb}\left(a_{r+1}, a_{r+2}, \frac{t}{3b}\right) * F_{rb}\left(a_{r+2}, a_{r+3}, \frac{t}{(3b)^2}\right) * \\
&\quad F_{rb}\left(a_{r+3}, a_{r+4}, \frac{t}{(3b)^2}\right) * F_{rb}\left(a_{r+4}, a_{r+2m+1}, \frac{t}{(3b)^2}\right) \\
&\geq F_{rb}\left(a_r, a_{r+1}, \frac{t}{3b}\right) * F_{rb}\left(a_{r+1}, a_{r+2}, \frac{t}{3b}\right) * F_{rb}\left(a_{r+2}, a_{r+3}, \frac{t}{(3b)^2}\right) * \\
&\quad F_{rb}\left(a_{r+3}, a_{r+4}, \frac{t}{(3b)^2}\right) * F_{rb}\left(a_{r+4}, a_{r+5}, \frac{t}{(3b)^3}\right) * \\
&\quad F_{rb}\left(a_{r+5}, a_{r+2m+1}, \frac{t}{(3b)^3}\right) \\
&\geq F_{rb}\left(a_r, a_{r+1}, \frac{t}{3b}\right) * F_{rb}\left(a_{r+1}, a_{r+2}, \frac{t}{3b}\right) * F_{rb}\left(a_{r+2}, a_{r+3}, \frac{t}{(3b)^2}\right) * \\
&\quad F_{rb}\left(a_{r+3}, a_{r+4}, \frac{t}{(3b)^2}\right) * F_{rb}\left(a_{r+4}, a_{r+5}, \frac{t}{(3b)^3}\right) * F_{rb}\left(a_{r+4}, a_{r+5}, \frac{t}{(3b)^3}\right) \\
&\quad * \cdots * F_{rb}\left(a_{r+2m}, a_{r+2m+1}, \frac{t}{(3b)^m}\right).
\end{aligned}$$

Using the contraction (3.4) on the above inequality we get

$$\begin{aligned}
&F_{rb}(a_r, a_{r+2m+1}, t) \\
&\geq F_{rb}\left(a_0, a_1, \frac{t}{3bk^r}\right) * F_{rb}\left(a_0, a_1, \frac{t}{3bk^{r+1}}\right) * F_{rb}\left(a_0, a_1, \frac{t}{(3b)^2k^{r+2}}\right) * \\
&\quad F_{rb}\left(a_0, a_1, \frac{t}{(3b)^2k^{r+3}}\right) * F_{rb}\left(a_0, a_1, \frac{t}{(3b)^3k^{r+4}}\right) * F_{rb}\left(a_0, a_1, \frac{t}{(3b)^3k^{r+5}}\right) \\
&\quad * \cdots * F_{rb}\left(a_0, a_1, \frac{t}{(3b)^mk^{r+2m}}\right) \\
&\geq F_{rb}\left(a_0, a_1, \frac{t}{(3b)k^r}\right) * F_{rb}\left(a_0, a_1, \frac{t}{(3bk)k^r}\right) * F_{rb}\left(a_0, a_1, \frac{t}{(3bk)^2k^r}\right) \\
&\quad * F_{rb}\left(a_0, a_1, \frac{t}{(3bk)^2k^{r+1}}\right) * F_{rb}\left(a_0, a_1, \frac{t}{(3bk)^3k^{r+1}}\right) \\
&\quad * F_{rb}\left(a_0, a_1, \frac{t}{(3bk)^3k^{r+2}}\right) * \cdots * F_{rb}\left(a_0, a_1, \frac{t}{(3bk)^mk^{r+m}}\right).
\end{aligned}$$

Case-2:

If p is even say $p = 2m;\ m \in \mathbb{N}$, then we have

$$\begin{aligned}
&F_{rb}(a_r, a_{r+2m}, t) \\
&\geq F_{rb}\left(a_r, a_{r+1}, \frac{t}{3b}\right) * F_{rb}\left(a_{r+1}, a_{r+2}, \frac{t}{3b}\right) * F_{rb}\left(a_{r+2}, a_{r+2m}, \frac{t}{3b}\right) \\
&\geq F_{rb}\left(a_r, a_{r+1}, \frac{t}{3b}\right) * F_{rb}\left(a_{r+1}, a_{r+2}, \frac{t}{3b}\right) * F_{rb}\left(a_{r+2}, a_{r+3}, \frac{t}{(3b)^2}\right) \\
&\quad * F_{rb}\left(a_{r+3}, a_{r+4}, \frac{t}{(3b)^2}\right) \\
&\geq F_{rb}\left(a_r, a_{r+1}, \frac{t}{3b}\right) * F_{rb}\left(a_{r+1}, a_{r+2}, \frac{t}{3b}\right) * F_{rb}\left(a_{r+2}, a_{r+3}, \frac{t}{(3b)^2}\right) * \\
&\quad F_{rb}\left(a_{r+3}, a_{r+4}, \frac{t}{(3b)^2}\right) * F_{rb}\left(a_{r+4}, a_{r+2m}, \frac{t}{(3b)^2}\right) * F_{rb}\left(a_{r+4}, a_{r+5}, \frac{t}{(3b)^3}\right) \\
&* F_{rb}\left(a_{r+5}, a_{r+2m}, \frac{t}{(3b)^3}\right)
\end{aligned}$$

$$\begin{aligned}
&F_{rb}(a_r, a_{r+2m}, t) \\
&\geq F_{rb}\left(a_r, a_{r+1}, \frac{t}{3b}\right) * F_{rb}\left(a_{r+1}, a_{r+2}, \frac{t}{3b}\right) * F_{rb}\left(a_{r+2}, a_{r+3}, \frac{t}{(3b)^2}\right) * \\
&\quad F_{rb}\left(a_{r+3}, a_{r+4}, \frac{t}{(3b)^2}\right) * F_{rb}\left(a_{r+4}, a_{r+2m}, \frac{t}{(3b)^2}\right) * F_{rb}\left(a_{r+4}, a_{r+5}, \frac{t}{(3b)^3}\right) \\
&* F_{rb}\left(a_{r+5}, a_{r+6}, \frac{t}{(3b)^3}\right) * F_{rb}\left(a_{r+6}, a_{r+7}, \frac{t}{(3b)^4}\right) * F_{rb}\left(a_{r+7}, a_{r+2m}, \frac{t}{(3b)^4}\right)
\end{aligned}$$

$$\begin{aligned}
&F_{rb}(a_r, a_{r+2m}, t) \\
&\geq F_{rb}\left(a_r, a_{r+1}, \frac{t}{3b}\right) * F_{rb}\left(a_{r+1}, a_{r+2}, \frac{t}{3b}\right) * F_{rb}\left(a_{r+2}, a_{r+3}, \frac{t}{(3b)^2}\right) * \\
&\quad F_{rb}\left(a_{r+3}, a_{r+4}, \frac{t}{(3b)^2}\right) * F_{rb}\left(a_{r+4}, a_{r+5}, \frac{t}{(3b)^3}\right) * F_{rb}\left(a_{r+5}, a_{r+6}, \frac{t}{(3b)^3}\right) \\
&\quad * F_{rb}\left(a_{r+6}, a_{r+7}, \frac{t}{(3b)^4}\right) * F_{rb}\left(a_{r+7}, a_{r+8}, \frac{t}{(3b)^4}\right) * \cdots * \\
&\quad F_{rb}\left(a_{r+2m-4}, a_{r+2m-3}, \frac{t}{(3b)^{m-1}}\right) * F_{rb}\left(a_{r+2m-3}, a_{r+2m-2}, \frac{t}{(3b)^{m-1}}\right) * \\
&\quad F_{rb}\left(a_{r+2m-2}, a_{r+2m}, \frac{t}{(3b)^{m-1}}\right)
\end{aligned}$$

$$
\begin{aligned}
& F_{rb}(a_r, a_{r+2m}, t) \\
& \geq F_{rb}\left(a_0, a_1, \frac{t}{3bk^r}\right) * F_{rb}\left(a_0, a_1, \frac{t}{3bk^{r+1}}\right) * F_{rb}\left(a_0, a_1, \frac{t}{(3b)^2k^{r+2}}\right) * \\
& \quad F_{rb}\left(a_0, a_1, \frac{t}{(3b)^2k^{r+3}}\right) * F_{rb}\left(a_0, a_1, \frac{t}{(3b)^3k^{r+4}}\right) * F_{rb}\left(a_0, a_1, \frac{t}{(3b)^3k^{r+5}}\right) \\
& \quad * F_{rb}\left(a_0, a_1, \frac{t}{(3b)^4k^{r+6}}\right) * F_{rb}\left(a_0, a_1, \frac{t}{(3b)^4k^{r+7}}\right) * \cdots * \\
& \quad F_{rb}\left(a_0, a_2, \frac{t}{(3b)^{m-1}k^{n+2m-2}}\right) \\
& \geq F_{rb}\left(a_0, a_1, \frac{t}{(3b)k^r}\right) * F_{rb}\left(a_0, a_1, \frac{t}{(3bk)k^r}\right) * F_{rb}\left(a_0, a_1, \frac{t}{(3bk)^2k^r}\right) * \\
& \quad F_{rb}\left(a_0, a_1, \frac{t}{(3bk)^2k^{r+1}}\right) * F_{rb}\left(a_0, a_1, \frac{t}{(3bk)^3k^{r+1}}\right) * F_{rb}\left(a_0, a_1, \frac{t}{(3bk)^3k^{r+2}}\right) \\
& \quad * \cdots * F_{rb}\left(a_0, a_2, \frac{t}{(3bk)^{m-1}k^{r+m-1}}\right)
\end{aligned}
$$

Therefore, from Case 1 and Case 2, together with FRBM-5 it follows that for all $p \in \mathbb{N}$, we have

$$
\begin{aligned}
\lim_{r\to\infty} F_{rb}(a_r, a_{r+p}, t) &\geq 1 * 1 * \cdots * 1 \\
&= 1.
\end{aligned}
$$

So $\{a_r\}$ is a G-Cauchy sequence and by the G-completeness of $(W, F_{rb}, *)$, $\{a_r\}$ is also convergent sequence. That is, there exists $\vartheta \in W$ such that

$$\lim_{r\to\infty} a_r = \vartheta.$$

We now show that ϑ is a fixed point of Γ.

$$
\begin{aligned}
F_{rb}(\vartheta, \Gamma\vartheta, t) &\geq F_{rb}\left(\vartheta, a_r, \frac{t}{3b}\right) * F_{rb}\left(a_r, a_{r+1}, \frac{t}{3b}\right) * F_{rb}\left(a_{r+1}, \Gamma\vartheta, \frac{t}{3b}\right) \\
&\geq F_{rb}\left(\vartheta, a_r, \frac{t}{3b}\right) * F_{rb}\left(\Gamma a_{r-1}, \Gamma a_r, \frac{t}{3b}\right) * F_{rb}\left(\Gamma a_r, \Gamma\vartheta, \frac{t}{3b}\right) \\
&\geq F_{rb}\left(\vartheta, a_r, \frac{t}{3b}\right) * F_{rb}\left(a_{r-1}, a_r, \frac{t}{3bk}\right) * F_{rb}\left(a_r, \vartheta, \frac{t}{3bk}\right) \\
&\longrightarrow 1 * 1 * 1 = 1 \ \text{ as } r \to \infty,
\end{aligned}
$$

which shows that $\Gamma\vartheta = \vartheta$ is a fixed point.

Uniqueness:

To prove the uniqueness, take another fixed point η of Γ *i.e;* $\Gamma\eta = \eta$.

Now

$$\begin{aligned}
F_{rb}(\eta, \vartheta, t) &= F_{rb}(\Gamma\eta, \Gamma\vartheta, t) \\
&\geq F_{rb}\left(\eta, \vartheta, \frac{t}{k}\right) \\
&= F_{rb}\left(\Gamma\eta, \Gamma\vartheta, \frac{t}{k}\right) \\
&\geq F_{rb}\left(\eta, \vartheta, \frac{t}{k^2}\right) \\
&\geq F_{rb}\left(\eta, \vartheta, \frac{t}{k^3}\right) \\
&\vdots \\
&\geq F_{rb}\left(\eta, \vartheta, \frac{t}{k^{r-1}}\right) \\
&\geq F_{rb}\left(\eta, \vartheta, \frac{t}{k^r}\right) \longrightarrow 1 \quad \text{as} \quad r \longrightarrow \infty.
\end{aligned}$$

Thus $\vartheta = \eta$. Hence the fixed point is unique. □

Corollary 3.2.3. (Banach Contraction Theorem in FRMS)

Let $(W, F_r, *)$ be a G-complete FRMS. Let $\Gamma\colon W \to W$ be a mapping satisfying

$$F_r(\Gamma\xi, \Gamma\varrho, kt) \geq F_r(\xi, \varrho, t)$$

for all $\xi, \varrho \in W, \quad 0 < k < 1$. Then Γ has a unique fixed point.

Proof.

The result can be proved by taking $b = 1$ in Theorem 3.2.2. □

Remark 3.2.4.

As every FMS is also FRBMS so the result of [51] follows from Theorem 3.2.2.

Theorem 3.2.2 is illustrated by the following example.

Example 3.2.1.

Let $W = [0,1]$ and define $F_{rb}\colon W \times W \times [0,\infty) \to [0,1]$ by

$$F_{rb}(\xi,\varrho,t) = \begin{cases} e^{-\frac{(\xi-\varrho)^2}{t}} & \text{if } t > 0 \\ 0 & \text{if } t = 0. \end{cases}$$

Same as in Example 3.1.2, $(W, F_{rb}, *)$ is a G-complete FRBMS with the coefficient $b = 3$. Let $k \in (0,1)$ and define $\Gamma : W \to W$ by

$$\Gamma(\xi) = \sqrt{k}\xi.$$

Now for all $t > 0$ we have

$$\begin{aligned} F_{rb}(\Gamma\xi, \Gamma\varrho, kt) &= F_{rb}(\sqrt{k}\xi, \sqrt{k}\varrho, kt) \\ &= e^{-\dfrac{(\sqrt{k}\xi - \sqrt{k}\varrho)^2}{kt}} \\ &= e^{-\dfrac{(\xi-\varrho)^2}{t}} \\ &= F_{rb}(\xi,\varrho,t). \end{aligned}$$

Since all the requirements of Theorem 3.2.2 are met so $\xi = 0 \in [0,1]$ is a unique fixed point of Γ.

3.2.1 Contractive Type Mapping in FRBMS

In this section, the notion of Γ-orbitally upper semi continuous function in FRBMS is introduced and established the fixed point theorem of Hicks and Rhoades [85] in the setting of FRBMS.
The following definitions are useful for the construction of the main result.

Definition 3.2.5.

Let $\Gamma\colon W \to W$ a mapping and $\xi \in W$, the **orbit** of ξ with respect to Γ is defined as the following sequence of points $\mathcal{O}_\Gamma(\xi) = \{\xi, \Gamma\xi, \Gamma^2\xi, \ldots\}$.

Example 3.2.2.

Take $W = [-2, 2] \times [-2, 2]$ and define $\Gamma\colon W \to W$ by

$$\Gamma a = \Gamma(\xi, \varrho) = \begin{cases} \left(\dfrac{\xi}{2}, \dfrac{\varrho}{2}\right) & \text{if } \xi, \varrho \geq 0 \\ (2, 0) & \text{otherwise} . \end{cases}$$

Clearly Γ is discontinuous at $(0, 0) \in W$.

Taking $a = (\xi, \varrho) \in W$ such that $0 < \xi, \varrho < 1$,

$$\begin{aligned} \Gamma(a) &= \left(\frac{\xi}{2}, \frac{\varrho}{2}\right) \\ &= \frac{a}{2} \end{aligned}$$

$$\begin{aligned} \Gamma^2(a) &= \Gamma(\Gamma(a)) \\ &= \Gamma\left(\frac{\xi}{2}, \frac{\varrho}{2}\right) \\ &= \left(\frac{\xi}{4}, \frac{\varrho}{4}\right) \\ &= \frac{a}{4} \end{aligned}$$

$$\begin{aligned} \Gamma^3(a) &= \Gamma(\Gamma^2(a)) \\ &= \Gamma\left(\frac{\xi}{4}, \frac{\varrho}{4}\right) \\ &= \left(\frac{\xi}{8}, \frac{\varrho}{8}\right) \\ &= \frac{a}{8} \end{aligned}$$

Similarly

$$\Gamma^4(a) = \frac{a}{16}$$

then

$$\mathcal{O}_\Gamma(a) = \left\{a, \frac{a}{2}, \frac{a}{4}, \ldots\right\}.$$

Definition 3.2.6.

Let $\Gamma\colon W \to W$ be a self mapping and for $\xi_0 \in W$,

$$\mathcal{O}_\Gamma(\xi_0) = \{\xi_0,\ \Gamma\xi_0,\ \Gamma^2\xi_0, \cdots\}$$

be the orbit of ξ_0. A function $F\colon W \to [0,1]$ is said to be Γ-orbitally upper semi continuous at $\varrho \in W$ if for any $\{\xi_n\} \subset \mathcal{O}_\Gamma(\xi_0)$ and $\xi_n \to \varrho$,

$$\Rightarrow F(\varrho) \geq \limsup_{n\to\infty} F(\xi_n).$$

The next example demonstrate the above definition.

Example 3.2.3.

Consider the set $W = [0,2]$. Let the self map Γ defined on W by

$$\Gamma\xi = \frac{1}{2}\xi^2.$$

Choose an element $\xi_0 = \dfrac{1}{2}$ in W, then we have

$$\mathcal{O}_\Gamma(\xi_0) = \mathcal{O}_\Gamma\left(\frac{1}{2}\right) = \left\{\frac{1}{2}, \frac{1}{2^3}, \frac{1}{2^7}, \dots\right\}.$$

Clearly, for any sequence $\{\xi_n\} \subset \mathcal{O}_\Gamma(\frac{1}{2})$, we have $\xi_n \to 0$.

Consider a function $F : W \to [0,1]$ given by

$$F(\xi) = \begin{cases} 1 & \text{if } \xi = 0 \\ \sqrt{2\xi - \xi^2} & \text{if } 0 < \xi \leq 2. \end{cases}$$

Now $F(0) = 1$ and $\xi_n \to \varrho = 0$,

implies that

$$\begin{aligned} F(0) = 1 > 0 &= \limsup_{n\to\infty} F(\xi_n) \\ &= \limsup_{n\to\infty} \sqrt{2\xi_n - \xi_n^2}. \end{aligned}$$

It follows that F is Γ-orbitally upper semi-continuous at $\varrho = 0$.

Below is the fixed point theorem of Hicks and Rhoads [85] in the setting of FRBMS.

Theorem 3.2.7.

Let $(W, F_{rb}, *)$ be a G-complete FRBMS. Let $\Gamma\colon W \to W$ be a mapping and there exists $a_0 \in W$ such that

$$F_{rb}(\Gamma\xi, \Gamma^2\xi, kt) \geq F_{rb}(\xi, \Gamma\xi, t) \tag{3.5}$$

for each $\xi \in \mathcal{O}_\Gamma(a_0)$, where $k \in (0, \frac{1}{3b})$. Then $\Gamma^r a_0 \to a \in W$.

Furthermore a will be fixed point of Γ if and only if

$$F(\xi) = F_{rb}(\xi, \Gamma\xi, t)$$

is Γ-orbitally upper semi continuous at a_0.

Proof.

For $a_0 \in W$, we define an iterative scheme $\{a_r\}$ by

$$a_r = \Gamma^r a_0.$$

With $a_1 = \Gamma a_0$ and successive application of (3.5) we get

$$\begin{aligned} F_{rb}(\Gamma^r a_0, \Gamma^{r+1} a_0, kt) &= F_{rb}(a_r, a_{r+1}, kt) \\ &\geq F_{rb}\left(a_{r-1}, a_r, \frac{t}{k}\right). \\ &\vdots \\ &\geq F_{rb}\left(a_0, a_1, \frac{t}{k^r}\right) \end{aligned} \tag{3.6}$$

For any $p \in \mathbb{N}$, we have

$$F_{rb}(\Gamma^r a_0, \Gamma^{r+p} a_0, t) = F_{rb}(a_r, a_{r+p}, t).$$

As in the proof of Theorem 3.2.2, starting with $F_{rb}(a_r, a_{r+p}, t)$ together with (3.6) we get for all $p \in \mathbb{N}$

$$\begin{aligned}\lim_{r\to\infty} F_{rb}(\Gamma^r a_0, \Gamma^{r+p} a_0, t) &\geq 1 * 1 * \cdots * 1 \\ &= 1.\end{aligned}$$

So $\{\Gamma^r a_0\}$ is G-Cauchy sequence and by the G-completeness of $(W, F_{rb}, *)$, $\{\Gamma^r a_0\}$ is also convergent sequence. That is, there is a point $a \in W$ such that

$$a_r = \Gamma^r a_0 \to a \in W.$$

Suppose that that F is upper semi continuous at $a \in W$ then

$$\begin{aligned}F_{rb}(a, \Gamma a, t) &\geq \lim_{r\to\infty} \sup F_{rb}(\Gamma^r a_0, \Gamma^{r+1} a_0, t) \\ &\geq \lim_{r\to\infty} \sup F_{rb}\left(\Gamma^{r-1} a_0, \Gamma^r a_0, \frac{t}{k}\right) \\ &\vdots \\ &\geq \lim_{r\to\infty} \sup F_{rb}\left(a_0, a_1, \frac{t}{k^r}\right) \to 1.\end{aligned}$$

So, we have

$$a = \Gamma a.$$

Conversely, suppose $a = \Gamma a$ and $\xi \in \mathcal{O}_\Gamma(\xi)$ with $a_r \to a$, then

$$\begin{aligned}F(a) =& F_{rb}(a, \Gamma a, t) \\ &= 1 \\ &\geq \lim_{r\to\infty} \sup F(a_r) \\ &= F_{rb}(\Gamma^r a_0, \Gamma^{r+1} a_0, t).\end{aligned}$$

□

The following corollary becomes an immediate consequence of Theorem 3.2.7 by setting $b = 1$.

Corollary 3.2.8.

Let $(W, F_r, *)$ be a G-complete FRMS. Let $\Gamma\colon W \to W$ be a mapping satisfying

$$F_r(\Gamma\xi, \Gamma^2\xi, kt) \geq F_r(\xi, \Gamma\xi, t) \tag{3.7}$$

for each $\xi \in \mathcal{O}_\Gamma(a_0)$ for $a_0 \in W$ and $0 < k < 1$. Here $a_r = \Gamma^r a_0 \quad (r \in \mathbb{N})$, then $\Gamma^r a_0 \to \xi \in W$.

Furthermore ξ will be fixed point of Γ if and only if $F(\xi) = F_r(\xi, \Gamma\xi, t)$ is Γ-orbitally upper semi continuous at ξ.

3.2.2 Geraghty Type Contraction

Following [28], for a real number $b > 1$, let $\mathfrak{F}_b$ denotes the class of all functions $\beta\colon [0, \infty) \to [0, \frac{1}{b})$ satisfying the following condition:

$$\limsup_{n\to\infty} \beta(t_n) = \frac{1}{b} \quad \text{implies} \quad \lim_{n\to\infty} t_n = 0$$

In the fuzzy setting, the class of Geraghty-type contraction is modified as follows:

$$\mathfrak{F}_b = \left\{\beta : [0, \infty) \to [0, \frac{1}{b});\ \limsup_{n\to\infty} \beta(t_n) = \frac{1}{b} \ \text{ implies } \ \lim_{n\to\infty} t_n = 1\right\} \tag{3.8}$$

A fixed point result, analogue to [28, Theorem 1], is established in the setting of G-complete FRBMS.

Theorem 3.2.9.

Let $(W, F_{rb}, *)$ be a G-complete FRBMS with $b \geq 1$ and $\Gamma\colon W \to W$ be a mapping satisfying $F_{rb}(\Gamma\xi, \Gamma\varrho, \beta(F_{rb}(\xi, \varrho, t))t) \geq \delta(\xi, \varrho, t) \quad \forall\, \xi, \varrho \in W$ and for some $\beta \in \mathfrak{F}_b$, where

$$\delta(\xi, \varrho, t) = \min\left\{\frac{F_{rb}(\xi, \Gamma\varrho, t)\,[1 + F_{rb}(\varrho, \Gamma\varrho, t)]}{1 + F_{rb}(T\xi, \Gamma\varrho, t)}, \frac{F_{rb}(\varrho, \Gamma\varrho, t)\,[1 + F_{rb}(\xi, \Gamma\xi, t)]}{1 + F_{rb}(\xi, \varrho, t)}, \right.$$
$$\left. \frac{F_{rb}(\xi, \Gamma\xi, t)\,[2 + F_{rb}(\xi, \Gamma\varrho, t)]}{1 + F_{rb}(\xi, \Gamma\varrho, t) + F_{rb}(\varrho, \Gamma\xi, t)}, F_{rb}(\xi, \varrho, t)\right\}.$$

Then Γ has a unique fixed point.

Proof.

For any arbitrary point $a_0 \in W$, we choose a sequence $\{a_r\}$ in W.

Start with iterative process $a_{r+1} = \Gamma a_r$.

For all $r, t > 0$, we have

$$F_{rb}(a_r, a_{r+1}, t) = F_{rb}(\Gamma a_{r-1}, \Gamma a_r, t) \geq \delta\left(a_{r-1}, a_r, \frac{t}{\beta(F_{rb}(a_{r-1}, a_r, t))}\right). \quad (3.9)$$

Now,

$$\delta\left(a_{r-1}, a_r, \frac{t}{\beta(F_{rb}(a_{r-1}, a_r, t))}\right)$$

$$= \min\left\{ \frac{F_{rb}(a_{r-1}, \Gamma a_{r-1}, \frac{t}{\beta(F_{rb}(a_{r-1}, a_r, t))}) \left[1 + F_{rb}(a_r, \Gamma a_r, \frac{t}{\beta(F_{rb}(a_{r-1}, a_r, t))})\right]}{1 + F_{rb}(\Gamma a_{r-1}, \Gamma a_r, \frac{t}{\beta(F_{rb}(a_{r-1}, a_r, t))})}, \right.$$

$$\frac{F_{rb}(a_r, \Gamma a_r, \frac{t}{\beta(F_{rb}(a_{r-1}, a_r, t))}) \left[1 + F_{rb}(a_{r-1}, \Gamma a_{r-1}, \frac{t}{\beta(F_{rb}(a_{r-1}, a_r, t))})\right]}{1 + F_{rb}(a_{r-1}, a_r, \frac{t}{\beta(F_{rb}(a_{r-1}, a_r, t))})},$$

$$\frac{F_{rb}(a_{r-1}, \Gamma a_{r-1}, \frac{t}{\beta(F_{rb}(a_{r-1}, a_r, t))}) \left[2 + F_{rb}(a_{r-1}, \Gamma a_r, \frac{t}{\beta(F_{rb}(a_{r-1}, a_r, t))})\right]}{1 + F_{rb}(a_{r-1}, \Gamma a_r, \frac{t}{\beta(F_{rb}(a_{r-1}, a_r, t))}) + F_{rb}(a_r, \Gamma a_{r-1}, \frac{t}{\beta(F_{rb}(a_{r-1}, a_r, t))})},$$

$$\left. F_{rb}(a_{r-1}, a_r, \frac{t}{\beta(F_{rb}(a_{r-1}, a_r, t))}) \right\}.$$

$$= \min\left\{ \frac{F_{rb}(a_{r-1}, a_r, \frac{t}{\beta(F_{rb}(a_{r-1}, a_r, t))}) \left[1 + F_{rb}(a_r, a_{r+1}, \frac{t}{\beta(F_{rb}(a_{r-1}, a_r, t))})\right]}{1 + F_{rb}(a_r, a_{r+1}, \frac{t}{\beta(F_{rb}(a_{r-1}, a_r, t))})}, \right.$$

$$\frac{F_{rb}(a_r, a_{r+1}, \frac{t}{\beta(F_{rb}(a_{r-1}, a_r, t))}) \left[1 + F_{rb}(a_{r-1}, a_r, \frac{t}{\beta(F_{rb}(a_{r-1}, a_r, t))})\right]}{1 + F_{rb}(a_{r-1}, a_r, \frac{t}{\beta(F_{rb}(a_{r-1}, a_r, t))})},$$

$$\frac{F_{rb}(a_{r-1}, a_r, \frac{t}{\beta(F_{rb}(a_{r-1}, a_r, t))}) \left[2 + F_{rb}(a_{r-1}, a_{r+1}, \frac{t}{\beta(F_{rb}(a_{r-1}, a_r, t))})\right]}{1 + F_{rb}(a_{r-1}, a_{r+1}, \frac{t}{\beta(F_{rb}(a_{r-1}, a_r, t))}) + F_{rb}(a_r, a_r, \frac{t}{\beta(F_{rb}(a_{r-1}, a_r, t))})},$$

$$\left. F_{rb}(a_{r-1}, a_r, \frac{t}{\beta(F_{rb}(a_{r-1}, a_r, t))}) \right\}.$$

$$\delta\left(a_{r-1},a_r,\frac{t}{\beta(F_{rb}(a_{r-1},a_r,t))}\right)$$
$$=\min\Bigg\{\frac{F_{rb}(a_{r-1},a_r,\frac{t}{\beta(F_{rb}(a_{r-1},a_r,t))})\left[1+F_{rb}(a_r,a_{r+1},\frac{t}{\beta(F_{rb}(a_{r-1},a_r,t))})\right]}{1+F_{rb}(a_r,a_{r+1},\frac{t}{\beta(F_{rb}(a_{r-1},a_r,t))})},$$
$$\frac{F_{rb}(a_r,a_{r+1},\frac{t}{\beta(F_{rb}(a_{r-1},a_r,t))})\left[1+F_{rb}(a_{r-1},a_r,\frac{t}{\beta(F_{rb}(a_{r-1},a_r,t))})\right]}{1+F_{rb}(a_{r-1},a_r,\frac{t}{\beta(F_{rb}(a_{r-1},a_r,t))})},$$
$$\frac{F_{rb}(a_{r-1},a_r,\frac{t}{\beta(F_{rb}(a_{r-1},a_r,t))})\left[2+F_{rb}(a_{r-1},a_{r+1},\frac{t}{\beta(F_{rb}(a_{r-1},a_r,t))})\right]}{1+F_{rb}(a_{r-1},a_{r+1},\frac{t}{\beta(F_{rb}(a_{r-1},a_r,t))})+1},$$
$$F_{rb}(a_{r-1},a_r,\frac{t}{\beta(F_{rb}(a_{r-1},a_r,t))})\Bigg\}.$$
$$=\min\left\{F_{rb}\left(a_{r-1},a_r,\frac{t}{\beta(F_{rb}(a_{r-1},a_r,t))}\right),F_{rb}\left(a_r,a_{r+1},\frac{t}{\beta(F_{rb}(a_{r-1},a_r,t))}\right)\right\} \tag{3.10}$$

If

$$\min\left\{F_{rb}\left(a_{r-1},a_r,\frac{t}{\beta(F_{rb}(a_{r-1},a_r,t))}\right),F_{rb}\left(a_r,a_{r+1},\frac{t}{\beta(F_{rb}(a_r,a_{r+1},t))}\right)\right\}$$
$$=F_{rb}\left(a_r,a_{r+1},\frac{t}{\beta(F_{rb}(a_{r-1},a_r,t))}\right),$$

then from (3.10),

$$F_{rb}(a_r,a_{r+1},t)\geq F_{rb}\left(a_r,a_{r+1},\frac{t}{\beta(F_{rb}(a_{r-1},a_r,t))}\right).$$

Since $\beta\in\mathfrak{F}_b$, so by Lemma 3.2.1, there is nothing to prove .

If

$$\min\left\{F_{rb}\left(a_{r-1},a_r,\frac{t}{\beta(F_{rb}(a_{r-1},a_r,t))}\right),F_{rb}\left(a_r,a_{r+1},\frac{t}{\beta(F_{rb}(a_{r-1},a_r,t))}\right)\right\}$$
$$=F_{rb}\left(a_{r-1},a_r,\frac{t}{\beta(F_{rb}(a_{r-1},a_r,t))}\right),$$

then from (3.10),

$$F_{rb}(a_r, a_{r+1}, t) \geq F_{rb}\left(a_{r-1}, a_r, \frac{t}{\beta(F_{rb}(a_{r-1}, a_r, t))}\right).$$

Continuing in this way, it follows that

$$\begin{aligned} &F_{rb}(a_r, a_{r+1}, t) \\ &\quad \geq F_{rb}\left(a_0, a_1, \frac{t}{\beta(F_{rb}(a_{r-1}, a_r, t). \cdots \beta(F_{rb}(a_0, a_1, t)}\right). \end{aligned} \tag{3.11}$$

Since $(X, F_{rb}, *)$ is a FRBMS, for the sequence $\{a_r\}$, writing

$$t = \frac{t}{3} + \frac{t}{3} + \frac{t}{3}$$

and using the rectangular inequality given in FRBM-4, on $F_{rb}(a_r, a_{r+p}, t)$ in the following two cases.

Case-1:

If p is odd say $p = 2m + 1$ where $m \in \mathbb{N}$, we have

$$\begin{aligned} &F_{rb}(a_r, a_{r+2m+1}, t) \\ &\quad \geq F_{rb}\left(a_r, a_{r+1}, \frac{t}{3b}\right) * F_{rb}\left(a_{r+1}, a_{r+2}, \frac{t}{3b}\right) \\ &\quad * F_{rb}\left(a_{r+2}, a_{r+3}, \frac{t}{(3b)^2}\right) * F_{rb}\left(a_{r+3}, a_{r+4}, \frac{t}{(3b)^2}\right) \\ &\quad * F_{rb}\left(a_{r+4}, a_{r+5}, \frac{t}{(3b)^3}\right) * F_{rb}\left(a_{r+5}, a_{r+6}, \frac{t}{(3b)^3}\right) \\ &\quad * F_{rb}\left(a_{r+6}, a_{r+7}, \frac{t}{(3b)^4}\right) * F_{rb}\left(a_{r+7}, a_{r+8}, \frac{t}{(3b)^4}\right) \\ &\quad * F_{rb}\left(a_{r+8}, a_{r+9}, \frac{t}{(3b)^5}\right) * F_{rb}\left(a_{r+9}, a_{r+10}, \frac{t}{(3b)^5}\right) \\ &\quad * F_{rb}\left(a_{r+10}, a_{r+11}, \frac{t}{(3b)^6}\right) * F_{rb}\left(a_{r+11}, a_{r+12}, \frac{t}{(3b)^6}\right) \\ &\quad \vdots \\ &\quad * F_{rb}\left(a_{r+2m-1}, a_{r+2m}, \frac{t}{(3b)^m}\right) * F_{rb}\left(a_{r+2m}, a_{r+2m+1}, \frac{t}{(3b)^m}\right). \end{aligned}$$

Using the contraction (3.11) on the above inequality the following will be obtained.

$$
\begin{aligned}
&F_{rb}(a_r, a_{r+2m+1}, t)\\
&\geq F_{rb}\left(a_0, a_1, \frac{t}{3b\beta(F_{rb}(a_{r-1}, a_r, t)\cdots\beta(F_{rb}(a_0, a_1, t)}\right)\\
&* \quad F_{rb}\left(a_0, a_1, \frac{t}{3b\beta(F_{rb}(a_r, a_{r+1}, t)\cdots\beta(F_{rb}(a_0, a_1, t)}\right)\\
&* \quad F_{rb}\left(a_0, a_1, \frac{t}{(3b)^2\beta(F_{rb}(a_{r+1}, a_{r+2}, t)\cdots\beta(F_{rb}(a_0, a_1, t)}\right)\\
&* \quad F_{rb}\left(a_0, a_1, \frac{t}{(3b)^2\beta(F_{rb}(a_{r+2}, a_{r+3}, t)\cdots\beta(F_{rb}(a_0, a_1, t)}\right)\\
&* \quad F_{rb}\left(a_0, a_1, \frac{t}{(3b)^3\beta(F_{rb}(a_{r+3}, a_{r+4}, t)\cdots\beta(F_{rb}(a_0, a_1, t)}\right)\\
&* \quad F_{rb}\left(a_0, a_1, \frac{t}{(3b)^3\beta(F_{rb}(a_{r+4}, a_{r+5}, t)\cdots\beta(F_{rb}(a_0, a_1, t)}\right)\\
&* \quad F_{rb}\left(a_0, a_1, \frac{t}{(3b)^4\beta(F_{rb}(a_{r+5}, a_{r+6}, t)\cdots\beta(F_{rb}(a_0, a_1, t)}\right)\\
&* \cdots * \quad F_{rb}\left(a_0, a_1, \frac{t}{(3b)^m\beta(F_{rb}(a_{r+2m-1}, a_{r+2m}, t)\cdots\beta(F_{rb}(a_0, a_1, t)}\right)\\
&\geq F_{rb}\left(a_0, a_1, \frac{t}{(3b)\frac{1}{b^r}}\right) * F_{rb}\left(a_0, a_1, \frac{t}{(3b)\frac{1}{b^{r+1}}}\right)\\
&* \quad F_{rb}\left(a_0, a_1, \frac{t}{(3b)^2\frac{1}{b^{r+2}}}\right) * F_{rb}\left(a_0, a_1, \frac{t}{(3b)^2\frac{1}{b^{r+3}}}\right)\\
&* \quad F_{rb}\left(a_0, a_1, \frac{t}{(3b)^3\frac{1}{b^{r+4}}}\right) * F_{rb}\left(a_0, a_1, \frac{t}{(3b)^3\frac{1}{b^{r+5}}}\right)\\
&* \quad F_{rb}\left(a_0, a_1, \frac{t}{(3b)^4\frac{1}{b^{r+6}}}\right) * F_{rb}\left(a_0, a_1, \frac{t}{(3b)^4\frac{1}{b^{r+7}}}\right)\\
&* \quad F_{rb}\left(a_0, a_1, \frac{t}{(3b)^5\dfrac{1}{b^{r+8}}}\right) * \cdots * \quad F_{rb}\left(a_0, a_1, \frac{t}{(3b)^m\dfrac{1}{b^{r+2m-1}}}\right).\\
&\geq F_{rb}\left(a_0, a_1, \frac{b^r t}{(3b)}\right) * F_{rb}\left(a_0, a_1, \frac{b^{r+1}t}{(3b)}\right) * F_{rb}\left(a_0, a_1, \frac{b^{r+2}t}{(3b)^2}\right) *\\
&\quad F_{rb}\left(a_0, a_1, \frac{b^{r+3}t}{(3b)^2}\right) * F_{rb}\left(a_0, a_1, \frac{b^{r+4}t}{(3b)^3}\right) * F_{rb}\left(a_0, a_1, \frac{b^{r+5}t}{(3b)^3}\right)\\
&\quad F_{rb}\left(a_0, a_1, \frac{b^{r+3}t}{(3b)^4}\right) * F_{rb}\left(a_0, a_1, \frac{b^{r+6}t}{(3b)^4}\right) * F_{rb}\left(a_0, a_1, \frac{b^{r+7}t}{(3b)^5}\right)\\
&\quad F_{rb}\left(a_0, a_1, \frac{b^{r+8}t}{(3b)^5}\right) * F_{rb}\left(a_0, a_1, \frac{b^{r+9}t}{(3b)^6}\right) * F_{rb}\left(a_0, a_1, \frac{b^{r+10}t}{(3b)^6}\right)\\
&\quad * F_{rb}\left(a_0, a_1, \frac{b^{r+11}t}{(3b)^7}\right) * \cdots * \quad F_{rb}\left(a_0, a_1, \frac{b^{r+2m-1}t}{(3b)^m}\right)
\end{aligned}
$$

$$F_{rb}(a_r, a_{r+2m+1}, t) \geq F_{rb}\left(a_0, a_1, \frac{b^{r-1}t}{3}\right) * F_{rb}\left(a_0, a_1, \frac{b^r t}{3}\right) * F_{rb}\left(a_0, a_1, \frac{b^r t}{(3)^2}\right) * F_{rb}\left(a_0, a_1, \frac{b^{r+1}t}{(3)^2}\right) * F_{rb}\left(a_0, a_1, \frac{b^{r+1}t}{(3)^3}\right) * F_{rb}\left(a_0, a_1, \frac{b^{r+2}t}{(3)^3}\right) * F_{rb}\left(a_0, a_1, \frac{b^{r+2}t}{(3)^4}\right) * \cdots * F_{rb}\left(a_0, a_1, \frac{b^{r+m-1}t}{(3)^m}\right).$$

Case-2:

If p is even say $p = 2m$; $m \in \mathbb{N}$, then we have

$$\begin{aligned}
&F_{rb}(a_r, a_{r+2m}, t)\\
&\geq F_{rb}\left(a_r, a_{r+1}, \frac{t}{3b}\right) * F_{rb}\left(a_{r+1}, a_{r+2}, \frac{t}{3b}\right) * F_{rb}\left(a_{r+2}, a_{r+3}, \frac{t}{(3b)^2}\right) *\\
&\quad F_{rb}\left(a_{r+3}, a_{r+4}, \frac{t}{(3b)^2}\right) * F_{rb}\left(a_{r+4}, a_{r+5}, \frac{t}{(3b)^3}\right) * F_{rb}\left(a_{r+5}, a_{r+6}, \frac{t}{(3b)^3}\right) *\\
&\quad F_{rb}\left(a_{r+6}, a_{r+7}, \frac{t}{(3b)^4}\right) * \cdots * F_{rb}\left(a_{r+2m-4}, a_{r+2m-3}, \frac{t}{(3b)^{m-1}}\right) *\\
&\quad F_{rb}\left(a_{r+2m-3}, a_{r+2m-1}, \frac{t}{(3b)^{m-1}}\right) * F_{rb}\left(a_{r+2m-2}, a_{r+2m}, \frac{t}{(3b)^{m-1}}\right)\\
&\geq F_{rb}\left(a_0, a_1, \frac{t}{3b\beta(F_{rb}(a_{r-1}, a_r, t) \cdots \beta(F_{rb}(a_0, a_1, t)}\right) *\\
&\quad F_{rb}\left(a_0, a_1, \frac{t}{3b\beta(F_{rb}(a_r, a_{r+1}, t) \cdots \beta(F_{rb}(a_0, a_1, t)}\right) *\\
&\quad F_{rb}\left(a_0, a_1, \frac{t}{(3b)^2\beta(F_{rb}(a_{r+1}, a_{r+2}, t) \cdots \beta(F_{rb}(a_0, a_1, t)}\right) *\\
&\quad F_{rb}\left(a_0, a_1, \frac{t}{(3b)^2\beta(F_{rb}(a_{r+2}, a_{r+3}, t) \cdots \beta(F_{rb}(a_0, a_1, t)}\right) *\\
&\quad F_{rb}\left(a_0, a_1, \frac{t}{(3b)^3\beta(F_{rb}(a_{r+3}, a_{r+4}, t) \cdots \beta(F_{rb}(a_0, a_1, t)}\right) *\\
&\quad F_{rb}\left(a_0, a_1, \frac{t}{(3b)^3\beta(F_{rb}(a_{r+4}, a_{r+5}, t) \cdots \beta(F_{rb}(a_0, a_1, t)}\right) *\\
&\quad F_{rb}\left(a_0, a_1, \frac{t}{(3b)^4\beta(F_{rb}(a_{r+5}, a_{r+6}, t) \cdots \beta(F_{rb}(a_0, a_1, t)}\right) *\\
&\quad F_{rb}\left(a_0, a_1, \frac{t}{(3b)^4\beta(F_{rb}(a_{r+6}, a_{r+7}, t) \cdots \beta(F_{rb}(a_0, a_1, t)}\right) *\\
&\quad F_{rb}\left(a_0, a_1, \frac{t}{(3b)^5\beta(F_{rb}(a_{r+7}, a_{r+8}, t) \cdots \beta(F_{rb}(a_0, a_1, t)}\right) *\\
&\quad F_{rb}\left(a_0, a_1, \frac{t}{(3b)^5\beta(F_{rb}(a_{r+8}, a_{r+9}, t) \cdots \beta(F_{rb}(a_0, a_1, t)}\right) * \cdots *\\
&\quad F_{rb}\left(a_0, a_2, \frac{t}{(3b)^{m-1}\beta(F_{rb}(a_{r+2m-1}, a_{r+2m-3}, t) \cdots \beta(F_{rb}(a_0, a_1, t)}\right)
\end{aligned}$$

$$
\begin{aligned}
&F_{rb}(a_r, a_{r+2m}, t)\\
&\quad \geq F_{rb}\left(a_0, a_1, \frac{t}{3b\frac{1}{b^{r-1}}}\right) * F_{rb}\left(a_0, a_1, \frac{t}{3b\frac{1}{b^{r}}}\right) * F_{rb}\left(a_0, a_1, \frac{t}{(3b)^2\frac{1}{b^{r+1}}}\right)\\
&\quad * F_{rb}\left(a_0, a_1, \frac{t}{(3b)^2\frac{1}{b^{r+2}}}\right) * F_{rb}\left(a_0, a_1, \frac{t}{(3b)^3\frac{1}{b^{r+3}}}\right) * F_{rb}\left(a_0, a_1, \frac{t}{(3b)^3\frac{1}{b^{r+4}}}\right)\\
&\quad * \cdots * \quad F_{rb}\left(a_0, a_2, \frac{t}{(3b)^{m-1}\frac{1}{b^{r+m-3}}}\right)\\
&\quad \geq F_{rb}\left(a_0, a_1, \frac{b^{r-1}t}{(3b)}\right) * F_{rb}\left(a_0, a_1, \frac{b^{r}t}{(3b)}\right) * F_{rb}\left(a_0, a_1, \frac{b^{r+1}t}{(3b)^2}\right)\\
&\quad\quad * F_{rb}\left(a_0, a_1, \frac{b^{r+2}t}{(3b)^2}\right) * F_{rb}\left(a_0, a_1, \frac{b^{r+3}t}{(3b)^3}\right) * \cdots * \quad F_{rb}\left(a_0, a_2, \frac{b^{n+2m-2}t}{(3b)^{m-1}}\right)\\
&\quad \geq F_{rb}\left(a_0, a_1, \frac{b^{r-2}t}{3}\right) * F_{rb}\left(a_0, a_1, \frac{b^{r-1}t}{3}\right) * F_{rb}\left(a_0, a_1, \frac{b^{r-1}t}{(3)^2}\right)\\
&\quad * F_{rb}\left(a_0, a_1, \frac{b^{r}t}{(3)^2}\right) * F_{rb}\left(a_0, a_1, \frac{b^{r}t}{(3)^3}\right) * F_{rb}\left(a_0, a_1, \frac{b^{r+1}t}{(3)^3}\right)\\
&\quad * \cdots * \quad F_{rb}\left(a_0, a_2, \frac{b^{n+m-3}t}{(3)^{m-1}}\right).
\end{aligned}
$$

Therefore, from Case-1 and Case-2 together with FRBM-5, it follows that for all $p \in \mathbb{N}$, we have

$$
\begin{aligned}
\lim_{r\to\infty} F_{rb}(a_r, a_{r+p}, t) &\geq 1 * 1 * \cdots * 1\\
&= 1.
\end{aligned}
$$

This shows that $\{a_r\}$ is G-Cauchy sequence. Since $(W, F_{rb}, *)$ is a G-complete FRBMS so there exists $\vartheta \in W$ such that

$$\lim_{r\to\infty} a_r = \vartheta.$$

We now show that ϑ is fixed point of Γ.

$$F_{rb}(\vartheta, \Gamma\vartheta, t) \geq F_{rb}\left(\vartheta, a_r, \frac{t}{3b}\right) * F_{rb}\left(a_r, a_{r+1}, \frac{t}{3b}\right) * F_{rb}\left(a_{r+1}, T\vartheta, \frac{t}{3b}\right).$$

Using $a_r = \Gamma a_{r-1}$, we get

$$\begin{aligned}
&F_{rb}(\vartheta, \Gamma\vartheta, t)\\
&\quad \geq F_{rb}\left(\vartheta, a_r, \frac{t}{3b}\right) * F_{rb}\left(\Gamma a_{r-1}, \Gamma a_r, \frac{t}{3b}\right) * F_{rb}\left(\Gamma a_r, \mathfrak{T}u, \frac{t}{3b}\right)\\
&\quad \geq F_{rb}\left(u, a_r, \frac{t}{3b}\right) * F_{rb}\left(a_{r-1}, a_r, \frac{t}{3b\beta(F_{rb}(a_{r-1}, a_r, t)}\right) *\\
&\qquad F_{rb}\left(a_r, \vartheta, \frac{t}{3b\beta(F_{rb}(a_r, \vartheta, t)}\right)\\
&\quad \longrightarrow 1 * 1 * 1 = 1 \ \text{ as } r \to \infty.
\end{aligned}$$

Which shows that $\Gamma\vartheta = \vartheta$ is a fixed point.

Uniqueness:

Let η be another fixed point of Γ such that

$$\Gamma\eta = \eta$$

for some $\eta \in W$, then

$$\begin{aligned}
F_{rb}(\eta, \vartheta, t) &= F_{rb}(\Gamma\eta, \Gamma\vartheta, t)\\
&\geq F_{rb}\left(\eta, \vartheta, \frac{t}{\beta(F_{rb}(\eta, \vartheta, t)}\right)\\
&= F_{rb}\left(\Gamma\eta, \Gamma\vartheta, \frac{t}{\beta(F_{rb}(\eta, \vartheta, t)}\right)\\
&\geq F_{rb}\left(\eta, \vartheta, \frac{t}{(\beta(F_{rb}(\eta, \vartheta, t)))^2}\right)\\
&\vdots\\
&\geq F_{rb}\left(\eta, \vartheta, \frac{t}{(\beta(F_{rb}(\eta, \vartheta, t)))^r}\right)\\
&= F_{rb}\left(\eta, \vartheta, b^r t\right)\\
&\longrightarrow 1 \quad \text{as} \quad r \longrightarrow \infty
\end{aligned}$$

Thus $\quad \vartheta = \eta$. Hence the fixed point is unique. □

Following is the immediate consequence of the Theorem 3.2.9.

Corollary 3.2.10.

Let $(W, F_r, *)$ be a G-complete FRMS and let $\Gamma\colon W \to W$ be a mapping satisfying

$$F_r(\Gamma\xi, \Gamma\varrho, \beta(F_r(\xi, \varrho, t))t) \geq \delta(\xi, \varrho, t),$$

for all $\xi, \varrho \in W$ and for some $\beta \in \mathfrak{F}_b$,

where

$$\delta(\xi, \varrho, t) = \min\left\{ \frac{F_r(\xi, \Gamma\xi, t)\,[1 + F_r(\varrho, \Gamma\varrho, t)]}{1 + F_r(\Gamma\xi, \Gamma\varrho, t)}, \frac{F_r(\varrho, \Gamma\varrho, t)\,[1 + F_r(\xi, \Gamma\xi, t)]}{1 + F_r(\xi, \varrho, t)}, \right.$$
$$\left. \frac{F_r(\xi, \Gamma\xi, t)\,[2 + F_r(\xi, \Gamma\varrho, t)]}{1 + F_r(\xi, \Gamma\varrho, t) + F_r(\varrho, \Gamma\xi, t)}, F_r(\xi, \varrho, t) \right\}.$$

Then Γ has a unique fixed point.

Proof. The result follows from Theorem 3.2.10 by taking $b = 1$. □

Remark 3.2.11.

The same result can be obtained in FRMS by taking $b = 1$ in Theorem 3.2.10.

3.3 Application

Fixed point theorems for operators in (ordered) metric spaces are widely investigated and have found various applications in differential and integral equations (see [102, 103] and references therein). Inspired by Mishra et al. [104], an application of main fixed point result stated in Theorem 3.2.2 is presented here.

In particular, we show the existence of the solution of an integral equation of the form

$$\xi(s) = g(s) + \int_0^s F(s, r, \xi(r))dr, \tag{3.12}$$

$\forall\ s \in [0, a]$ where $a > 0$.

Let the space of all continuous functions defined on $[0, a]$ is $C([0, a], \mathbb{R})$ equipped with the product t-norm $a * b = ab$ for all $a, b \in [0, 1]$ and define the G-complete fuzzy rectangular b-metric on $C([0,a],\mathbb{R})$ by

$$F_{rb}(\xi,\varrho,t) = e^{-\frac{\sup_{s\in[0,a]}|\xi(s)-\varrho(s)|^2}{t}},$$

for all $t>0$ and $\xi,\varrho\in C([0,a],\mathbb{R})$.

The following theorem proves the existence of a solution of the integral equation (3.12).

Theorem 3.3.1.

Let $\Gamma : C([0,a],\mathbb{R}) \to C([0,a],\mathbb{R})$ be the integral operator given by

$$\Gamma(\xi(s)) = g(s) + \int_0^s F(s,r,\xi(r))dr, \quad g\in C([0,a],\mathbb{R}),$$

where $F\in C([0,a]\times[0,a]\times\mathbb{R},\ \mathbb{R})$ satisfies the following condition:

(i) there exists $f:[0,a]\times[0,a]\to[0,+\infty]$ such that

$$\text{for all} \quad r,s\in[0,a], \quad f(s,r)\in L^1([0,a],\mathbb{R})$$

(ii) $\forall\ \xi,\varrho\in C([0,a],\mathbb{R})$, we have

$$|F(s,r,\xi(r)) - F(s,r,\varrho(r))|^2 \le f^2(s,r)|\xi(r)-\varrho(r)|^2,$$

where

$$\sup_{s\in[0,a]}\int_0^s f^2(s,r)dr \le k < 1.$$

Then the integral equation has the solution $\xi^*\in C([0,a],\mathbb{R})$.

Proof.

For all $\xi,\varrho\in C([0,a],\mathbb{R})$, we have

$$F_{rb}(\Gamma(\xi(s)),\Gamma(\varrho(s)),kt) = e^{-\frac{\sup_{s\in[0,a]}|\Gamma(\xi(s))-\Gamma(\varrho(s))|^2}{kt}}$$

$$
\begin{aligned}
F_{rb}(\Gamma(\xi(s)), \Gamma(\varrho(s)), kt) &\geq e^{-\frac{\sup\limits_{s\in[0,a]}\int_0^s |F(s,r,\xi(r)) - F(s,r,\varrho(r))|^2 dr}{kt}} \\
&\geq e^{-\frac{\sup\limits_{s\in[0,a]}\int_0^s f^2(s,r)|\xi(r) - \varrho(r)|^2 dr}{kt}} \\
&\geq e^{-\frac{|\xi(r) - \varrho(r)|^2 \sup\limits_{s\in[0,a]}\int_0^s f^2(s,r) dr}{kt}} \\
&\geq e^{-\frac{k|\xi(r) - \varrho(r)|^2}{kt}} \\
&= e^{-\frac{|\xi(r) - \varrho(r)|^2}{t}} \\
&\geq e^{-\frac{\sup\limits_{r\in[0,a]}|\xi(r) - \varrho(r)|^2}{t}} \\
&= F_{rb}(\xi, \varrho, t).
\end{aligned}
$$

Hence $\xi^* \in C([0,a], \mathbb{R})$ is a fixed point of Γ, which is the solution of integral equation (3.12). □

3.4 Conclusion

In the present Chapter, the famous BCP for FRBMS have been proved and illustrated by an example. In this way, the main result of Grabeic [51] becomes the special case of Theorem 3.2.2. Moreover, by restricting the contraction mapping to the elements in the orbit of a point in FRBMS, an analogue of the fixed point theorem of Hicks and Rhoads [85] have also been proved in the setting of FRBMS. By studying the Geragthy-type contraction in FRBMS, the result of [28] is also established. To support these results an application is presented in the end of the chapter. Thus these results are more general than the existing results in the fuzzy set theory. Fuzzy set theory has been found to be useful not only in decision making problems arising in physical and social sciences but also has application in multi-attribute decision making. For a combined study of fuzzy set theory and rough set theory and its application in making an optimal decision see the recent work presented in [105–108] and the references therein. The structure of a FMS might also be useful to solve fixed point problems related to some sort of distance between the programs to measure, for instance, the complexity of an algorithm. The results of this chapter are published in [86].

Chapter 4

Extended Fuzzy b-Metric Space

The work presented is infact an extension of the ideas of EBMS (Definition 2.1.2) and FBMS (Definition 2.2.10) and related results. In this chapter, the notion of EFBMS was introduced and illustrated by an example. The well known BCP and the fixed point results of [52] are proved in the setting of EFBMS. The main result of this chapter is illustrated by an example. Moreover the results of [28, 53, 87, 88] are extended in the setting of EFBMS in Section 4.2. For the published form of this chapter see [87].

4.1 Extended Fuzzy b-Metric Space

Using the idea of Kamran et al. [30], the notion of EFBMS is defined as follows:

Definition 4.1.1.
Let W be a non empty set, α is a mapping on $W \times W$ in to $[1, \infty)$ and $*$ be a continuous t-norm. A fuzzy set F_α on $W \times W \times [0, \infty)$ is called extended fuzzy b-metric on W if for all $\xi, \varrho, \vartheta \in W$, the following conditions hold.

EFBM1 : $F_\alpha(\xi, \varrho, 0) = 0$

EFBM2 : $F_\alpha(\xi, \varrho, t) = 1, \forall\, t > 0 \iff \xi = \varrho$

EFBM3 : $F_\alpha(\xi, \varrho, t) = F_\alpha(\varrho, \xi, t)$

EFBM4 : $F_\alpha(\xi, \vartheta, \alpha(\xi, \vartheta)(t+s)) \geq F_\alpha(\xi, \varrho, t) * F_\alpha(\varrho, \vartheta, s) \ \forall \ t, s \geq 0$

EFBM5 : $F_\alpha(\xi, \varrho, .) \colon [0, \infty) \to [0, 1]$ is left continuous, and $\lim_{t\to\infty} F_\alpha(\xi, \varrho, t) = 1$.

Then $(W, F_\alpha, *)$ is an EFBMS.

Remark 4.1.2.

Setting $\alpha(\xi, \varrho) = b$ for some $b \geq 1$ then Definition 2.2.10 becomes a special case of the above definition of EFBMS. That is, every EFBMS is a FBMS with $\alpha(\xi, \varrho) = b$ and is a FMS when $\alpha(\xi, \varrho) = 1$ for all ξ, ϱ in $W \times W$.

The following example illustrates Definition 4.1.1.

Example 4.1.1.

Let $W = \{1, 2, 3\}$ and define $d_\alpha : W \times W \to \mathbb{R}$ by

$$d_\alpha(\xi, \varrho) = (\xi - \varrho)^2.$$

Then (X, d_α) is an EBMS.

Define a mapping $\alpha \colon W \times W \to [1, \infty)$ by

$$\alpha(\xi, \varrho) = 1 + \xi + \varrho$$

Let $F_\alpha \colon W \times W \times [0, \infty) \to [0, 1]$ be defined by

$$F_\alpha(\xi, \varrho, t) = \begin{cases} \dfrac{t}{t + d_\alpha(\xi, \varrho)} & \text{if } t > 0 \\ 0 & \text{if } t = 0, \end{cases}$$

and take the continuous t-norm $* = \wedge$, that is, $t_1 * t_2 = t_1 \wedge t_2 = \min\{t_1, t_2\}$.

To show that $(X, F_\alpha, \wedge)$ is a EFBMS. Note that,

$$d_\alpha(1, 1) = d_\alpha(2, 2) = d_\alpha(3, 3) = 0$$

$$d_\alpha(1, 2) = d_\alpha(2, 1) = 1,$$

$$d_\alpha(2,3) = d(3,2) = 1,$$

$$d_\alpha(1,3) = d_\alpha(3,1) = 4.$$

Also

$$\alpha(1,1) = 3,\ \alpha(2,2) = 5,\ \alpha(3,3) = 7$$

$$\alpha(1,2) = \alpha(2,1) = 4,$$

$$\alpha(2,3) = \alpha(3,2) = 6,$$

$$\alpha(1,3) = \alpha(3,1) = 5.$$

To prove that $(W, F_\alpha, \wedge)$ is an EFBMS, the conditions EFBM1, EFBM2, EFBM3 and EFBM5 of Definition 4.1.1 are trivially true. To prove the property EFBM4 for all $\xi, \varrho \in W$, first note that

$$F_\alpha(\xi, \vartheta, \alpha(\xi,\vartheta)(t+s)) = \frac{\alpha(\xi,\vartheta)(t+s)}{\alpha(\xi,\vartheta)(t+s) + d(\xi,\vartheta)}.$$

For $\xi = 1, \vartheta = 2$

$$\begin{aligned} F_\alpha(1,2,\alpha(1,2)(t+s)) &= \frac{\alpha(1,2)(t+s)}{\alpha(1,2)(t+s) + d(1,2)} \\ &= \frac{4(t+s)}{4(t+s)+1} \\ &= 1 - \frac{1}{4(t+s)+1} \end{aligned}$$

$$\begin{aligned} F_\alpha(1,3,t) &= \frac{t}{t + d(1,3)} \\ &= \frac{t}{t+4} = 1 - \frac{4}{t+4} \end{aligned}$$

and

$$\begin{aligned} F_\alpha(3,2,s) &= \frac{s}{s + d(3,2)} \\ &= \frac{s}{s+1} = 1 - \frac{1}{s+1}. \end{aligned}$$

We have, for all $t, s > 0$,

$$\begin{aligned}
16(t+s)+4 &> t+4 \\
\frac{1}{16(t+s)+4} &< \frac{1}{t+4} \\
\frac{4}{16(t+s)+4} &< \frac{4}{t+4} \\
\frac{-1}{4(t+s)+1} &> \frac{-4}{t+4} \\
1-\frac{1}{4(t+s)+1} &> 1-\frac{4}{t+4}
\end{aligned}$$

and

$$\begin{aligned}
4(t+s)+1 &> s+1 \\
\frac{1}{4(t+s)+1} &< \frac{1}{s+1} \\
-\frac{1}{4(t+s)+1} &> -\frac{1}{s+1} \\
1-\frac{1}{4(t+s)+1} &> 1-\frac{1}{s+1}.
\end{aligned}$$

It, therefore, follows that

$$F_\alpha(1,2,\alpha(1,2)(t+s)) \geq \min\{F_\alpha(1,3,t), F_\alpha(3,2,s)\}$$

$$\Rightarrow \quad F_\alpha(1,2,\alpha(1,2)(t+s)) \geq F_\alpha(1,3,t) * F_\alpha(3,2,s).$$

For $\xi = 2, \vartheta = 3$

$$\begin{aligned}
F_\alpha(2,3,\alpha(2,3)(t+s)) &= \frac{\alpha(2,3)(t+s)}{\alpha(2,3)(t+s)+d(2,3)} \\
&= \frac{6(t+s)}{6(t+s)+1} \\
&= 1-\frac{1}{6(t+s)+1}
\end{aligned}$$

$$\begin{aligned}
F_\alpha(2,1,t) &= \frac{t}{t+d(2,1)} \\
F_\alpha(2,1,t) &= \frac{t}{t+1}
\end{aligned}$$

$$F_\alpha(2,1,t) = 1 - \frac{1}{t+1},$$

and

$$\begin{aligned}F_\alpha(1,3,s) &= \frac{s}{s+d(1,3)}\\ &= \frac{s}{s+4} = 1 - \frac{4}{s+4}.\end{aligned}$$

Again for all $t, s > 0$,

$$\begin{aligned}24(t+s)+4 &> s+4\\ \frac{1}{24(t+s)+4} &< \frac{1}{s+4}\\ \frac{4}{24(t+s)+4} &< \frac{4}{s+4}\\ \frac{-1}{6(t+s)+1} &> \frac{-4}{t+4}\\ 1 - \frac{1}{6(t+s)+1} &> 1 - \frac{4}{t+4}\end{aligned}$$

and

$$\begin{aligned}6(t+s)+1 &> t+1\\ \frac{1}{6(t+s)+1} &< \frac{1}{t+1}\\ -\frac{1}{6(t+s)+1} &> -\frac{1}{t+1}\\ 1 - \frac{1}{6(t+s)+1} &> 1 - \frac{1}{t+1}.\end{aligned}$$

Hence

$$\begin{aligned}& F_\alpha(2,3,\alpha(2,3)(t+s)) \geq \min\{F_\alpha(2,1,t), F_\alpha(1,3,s)\}\\ \Rightarrow\quad & F_\alpha(2,3,\alpha(2,3)(t+s)) \geq F_\alpha(2,1,t) * F_\alpha(1,3,s).\end{aligned}$$

For $\xi = 1, \vartheta = 3$

$$F_\alpha(1,3,\alpha(1,3)(t+s)) = \frac{\alpha(1,3)(t+s)}{\alpha(1,3)(t+s)+d(1,3)}$$

$$F_\alpha(1,3,\alpha(1,3)(t+s)) = \frac{5(t+s)}{5(t+s)+4}$$
$$= 1 - \frac{4}{5(t+s)+4}$$

$$F_\alpha(1,2,t) = \frac{t}{t+d(1,2)}$$
$$= \frac{t}{t+1}$$
$$= 1 - \frac{1}{t+1},$$

and

$$F_\alpha(2,3,s) = \frac{s}{s+d(2,3)}$$
$$= \frac{s}{s+1} = 1 - \frac{1}{s+1}.$$

As before, for all $t, s > 0$, one can show that

$$1 - \frac{4}{5(t+s)+4} > 1 - \frac{1}{s+1}$$

$$1 - \frac{4}{5(t+s)+4} > 1 - \frac{1}{t+1}.$$

Therefore, we get

$$F_\alpha(2,3,\alpha(2,3)(t+s)) \geq \min\{F_\alpha(2,1,t), F_\alpha(1,3,s)\}$$
$$\Rightarrow \quad F_\alpha(2,3,\alpha(2,3)(t+s)) \geq F_\alpha(2,1,t) * F_\alpha(1,3,s).$$

Hence for all $\xi, \varrho, \vartheta \in W$

$$F_\alpha(\xi,\vartheta,\alpha(\xi,\vartheta)(t+s)) \geq F_\alpha(\xi,\varrho,t) * F_\alpha(\varrho,\vartheta,s).$$

Therefore $(W, F_\alpha, *)$ is an EFBMS.

The concept of G-convergent sequence, G-Cauchy sequence and G-completeness can be generalized naturally in the setting of EFBMS as follows:

Definition 4.1.3. Let $(W, F_\alpha, *)$ be an EFBMS.

1. A sequence $\{\xi_n\}$ in W will be G-convergent if there exits $\xi \in W$ such that

$$\lim_{r\to\infty} F_\alpha(\xi_n, \xi, t) = 1,$$

   for all $t > 0$.

2. A sequence $\{\xi_n\}$ in W is said to be a G-Cauchy sequence if $\forall\, t > 0$, we have

$$\lim_{r\to\infty} F_\alpha(\xi_n, \xi_{n+q}, t) = 1$$

   for $t > 0$ and $q > 0$.

The work of Grabiec[51] can now be established for EFBMS as follows:

Theorem 4.1.4. (Banach Contraction Theorem for EFBMS)
Let $(W, F_\alpha, *)$ be a G-complete EFBMS and $\alpha \colon W \times W \to [1, \infty)$ be a mapping.
Let $\Gamma \colon W \to W$ be a mapping satisfying

$$F_\alpha(\Gamma\xi, \Gamma\varrho, kt) \geq F_\alpha(\xi, \varrho, t). \tag{4.1}$$

for all $\xi, \varrho \in W$ and $k \in (0, 1)$. Further, for an arbitrary $a_0 \in W$, and $r, q \in \mathbb{N}$, we have

$$\alpha(a_r, a_{r+q}) \leq \frac{1}{k},$$

where $a_r = \Gamma^r a_0$. Then Γ has a unique fixed point.

Proof.
Let $a_0 \in W$ and generate $\{a_r\}$ by $a_r = \Gamma^r a_0 \quad (r \in \mathbb{N})$.
First, note that for all $r, t > 0$, by the application of (4.1), it follows that,

$$F_\alpha(a_r, a_{r+1}, kt) = F_\alpha(\Gamma a_{r-1}, \Gamma a_r, kt)$$

$$\begin{aligned}
F_\alpha(a_r, a_{r+1}, kt) &\geq F_\alpha(a_{r-1}, a_r, t) \\
&\geq F_\alpha\left(a_{r-2}, a_{r-1}, \frac{t}{k}\right) \\
&\geq F_\alpha\left(a_{r-3}, a_{r-2}, \frac{t}{k^2}\right) \\
&\vdots \\
&\geq F_\alpha\left(a_0, a_1, \frac{t}{k^{r-1}}\right).
\end{aligned}$$

So

$$F_\alpha(a_r, a_{r+1}, kt) \geq F_\alpha\left(a_0, a_1, \frac{t}{k^{r-1}}\right). \tag{4.2}$$

For any $q \in \mathbb{N}$, writing $t = \frac{t}{2} + \frac{t}{2}$ and using EFBM4 repeatedly,

$$\begin{aligned}
&F_\alpha(a_r, a_{r+q}, t) \\
&\quad \geq F_\alpha\left(a_r, a_{r+1}, \frac{t}{2\alpha(a_r, a_{r+q})}\right) * F_\alpha\left(a_{r+1}, a_{r+2}, \frac{t}{2^2\alpha(a_r, a_{r+q})\alpha(a_{r+1}, a_{r+q})}\right) \\
&\quad * F_\alpha\left(a_{r+2}, a_{r+3}, \frac{t}{2^3\alpha(a_r, a_{r+q})\alpha(a_{r+1}, a_{r+q})\alpha(a_{r+2}, a_{r+q})}\right) \\
&\quad * F_\alpha\left(a_{r+2}, a_{r+3}, \frac{t}{2^4\alpha(a_r, a_{r+q})\alpha(a_{r+1}, a_{r+q})\alpha(a_{r+2}, a_{r+q})\alpha(a_{r+3}, a_{r+q})}\right) * \ldots * \\
&\quad F_\alpha\left(a_{r+q-1}, a_{r+q}, \frac{t}{2^q\alpha(a_r, a_{r+q})\alpha(a_{r+1}, a_{r+q})\alpha(a_{r+2}, a_{r+q})\ldots\alpha(a_{r+q-1}, a_{r+q})}\right).
\end{aligned}$$

Using the contraction (4.4) and EFBM5, it follows that

$$\begin{aligned}
&F_\alpha(a_r, a_{r+q}, t) \\
&\quad \geq F_\alpha\left(a_0, a_1, \frac{t}{2\alpha(a_r, a_{r+q})k^r}\right) * F_\alpha\left(a_0, a_1, \frac{t}{2^2\alpha(a_r, a_{r+q})\alpha(a_{r+1}, a_{r+q})k^{r+1}}\right) \\
&\quad * F_\alpha\left(a_0, a_1, \frac{t}{2^3\alpha(a_r, a_{r+q})\alpha(a_{r+1}, a_{r+q})\alpha(a_{r+2}, a_{r+q})k^{r+3}}\right) * \ldots * \\
&\quad F_\alpha\left(a_0, a_1, \frac{t}{2^q\alpha(a_r, a_{r+q})\alpha(a_{r+1}, a_{r+q})\alpha(a_{r+2}, a_{r+q})\ldots\alpha(a_{r+q-1}, a_{r+q})k^{r+q}}\right).
\end{aligned}$$

Since for all $r, q \in \mathbb{N}$, we have

$$\frac{1}{\alpha(a_r, a_{r+q})} \leq k < 1.$$

Taking limit as $r \to \infty$, we get

$$\lim_{r\to\infty} F_\alpha(a_r, a_{r+q}, t) = 1 * 1 * \ldots * 1 = 1.$$

Hence $\{a_r\}$ is G-Cauchy sequence. Since $(W, F_\alpha, *)$ be a complete EFBMS so there exists $p_1 \in W$ such that

$$\lim_{r\to\infty} a_r = p_1.$$

We want to show that p is fixed point of Γ.

$$\begin{aligned}
F_\alpha(\Gamma p_1, p_1, t) &\geq F_\alpha\left(\Gamma p_1, \Gamma a_r, \frac{t}{2\alpha(\Gamma p_1, p_1)}\right) * F_\alpha\left(\Gamma a_r, p_1, \frac{t}{2\alpha(\Gamma p_1, p_1)}\right)\\
&\geq F_\alpha\left(p_1, a_r, \frac{t}{2k\alpha(\Gamma p_1, p_1)}\right) * F_\alpha\left(a_{r+1}, a_r, \frac{t}{2\alpha(a_{r+1}, a_r)}\right)\\
&\longrightarrow 1 * 1 = 1,
\end{aligned}$$

which shows that $\Gamma p_1 = p_1$ is a fixed point.

Uniqueness:

Let p_2 be the an other fixed point of Γ such that $\Gamma p_2 = p_2$ for some $p_2 \in W$, then

$$\begin{aligned}
F_\alpha(p_2, p_1, t) &= F_\alpha(\Gamma p_2, \Gamma p_1, t)\\
&\geq F_\alpha\left(p_2, p_1, \frac{t}{k}\right)\\
&= F_\alpha\left(\Gamma p_2, \Gamma p_1, \frac{t}{k}\right)\\
&\geq F_\alpha\left(p_2, p_1, \frac{t}{k^2}\right)\\
&\vdots\\
&\geq F_\alpha\left(p_2, p_1, \frac{t}{k^r}\right) \longrightarrow 1 \quad \text{as} \quad r \to \infty
\end{aligned}$$

Thus $p_1 = p_2$. Hence fixed point is unique. □

Remark 4.1.5.

The special cases of Theorem 5.3.1 can be obtained for FBMS and FMS by setting $\alpha(\xi, \varrho) = b \geq 1$ and $\alpha(\xi, \varrho) = 1$ respectively.

The following example illustrates the above theorem.

Example 4.1.2.

Let $W = [0, 1]$ and define a mapping $\alpha\colon W \times W \to [1, \infty)$ by

$$\alpha(\xi, \varrho) = 1 + \xi + \varrho.$$

Let

$$F_\alpha(\xi, \varrho, t) = \begin{cases} \left(\dfrac{1}{t}\right)^{(\xi-\varrho)^2} & \text{if} \quad t \neq 0 \\ 0 & \text{if} \quad t = 0. \end{cases}$$

One can easily prove that $(W, F_\alpha, *)$ is a G-complete EFBMS.

To prove that EFBMS is G-complete, let $\{\xi_n\}$ be any G-Cauchy sequence in W, then $\forall\, t > 0$, we have

$$\lim_{p\to\infty} F_\alpha(\xi_p, \xi_{p+q}, t) = 1 \quad \forall\, p, q > 0,$$

$$\begin{aligned} \Rightarrow \lim_{p\to\infty} \left(\frac{1}{t}\right)^{(\xi_p-\xi_{p+q})^2} &= 1 \\ \Rightarrow \lim_{p\to\infty} (\xi_p - \xi_{p+q}) &= 0 \\ \Rightarrow \xi_p &= \xi_{p+q} \\ &= \xi \end{aligned}$$

Thus $\{\xi_p\} = \xi_1, \xi_2, \xi_3, , \xi_{p-1}, \xi, \xi, \ldots$

This shows that the Cauchy sequence $\{\xi_p\}$ tends to a definite point ξ as $p \to \infty$ *i.e*, $\{\xi_p\}$ converges to $\xi \in W$. This implies that $(W, F_\alpha, *)$ is a G-complete.

Let $\Gamma\colon W \to W$ be a mapping defined by $\Gamma(\xi) = 1 - \xi$.

Now for all $t > 0$ and $k \in (0, 1)$, we have

$$F_\alpha(\Gamma\xi, \Gamma\varrho, kt) = F_\alpha(1 - \xi, 1 - \varrho, kt)$$

$$= \left(\frac{1}{kt}\right)^{(1-\xi-1+\varrho)^2}$$
$$= \left(\frac{1}{kt}\right)^{(\varrho-\xi)^2}$$
$$= \left(\frac{1}{kt}\right)^{(\xi-\varrho)^2}.$$

Since

$$k < 1$$
$$\Rightarrow \quad kt < t$$
$$\Rightarrow \frac{1}{kt} > \frac{1}{t}$$
$$\Rightarrow \left(\frac{1}{kt}\right)^{(\xi-\varrho)^2} > \left(\frac{1}{t}\right)^{(\xi-\varrho)^2}.$$

So we have

$$\Rightarrow F_\alpha(\Gamma\xi, \Gamma\varrho, kt) > F_\alpha(\xi, \varrho, t).$$

Also $\xi = \dfrac{1}{2}$ is a unique fixed point of Γ and $\dfrac{1}{2} \in [0, 1]$.
The following result is the extension of the result of Gregori and Sapena [52] in the setting of EFBMS.

Theorem 4.1.6.
Let $(W, F_\alpha, *)$ be a G-complete EFBMS and $\alpha\colon W \times W \to [1, \infty)$ be a mapping. Let $\Gamma\colon W \to W$ be a mapping satisfying

$$\frac{1}{F_\alpha(\Gamma\xi, \Gamma\varrho, t)} - 1 \leq k\left(\frac{1}{F_\alpha(\xi, \varrho, t)} - 1\right) \quad \forall\, \xi, \varrho \in W \quad \text{and } k \in (0, 1). \tag{4.3}$$

Further, for an arbitrary $a_0 \in W$, and $r, q \in \mathbb{N}$, we have $\alpha(a_r, a_{r+q}) \leq \dfrac{1}{k}$, where $a_r = \Gamma^r a_0$. Then Γ has a unique fixed point.

Proof.
Let $a_0 \in W$, generate a sequence $\{a_r\}$ by

$$a_r = \Gamma^n a_0 \quad (r \in \mathbb{N}).$$

First, note that for all $n, t > 0$, by the contractive condition (4.3), it follows that,

$$
\begin{aligned}
\frac{1}{F_\alpha(a_r, a_{r+1}, t)} - 1 &= \frac{1}{F_\alpha(\Gamma(a_{r-1}), \Gamma(a_r), t))} - 1 \\
&\le k\left(\frac{1}{F_\alpha(a_{r-1}, a_r, t)} - 1\right) \\
&\le k^2\left(\frac{1}{F_\alpha(a_{r-2}, a_{r-1}, t)} - 1\right) \\
&\le k^3\left(\frac{1}{F_\alpha(a_{r-3}, a_{r-2}, t)} - 1\right) \\
&\le k^4\left(\frac{1}{F_\alpha(a_{r-4}, a_{r-3}, t)} - 1\right) \\
&\vdots \\
&\le k^r\left(\frac{1}{F_\alpha(a_0, a_1, t)} - 1\right).
\end{aligned}
$$

So

$$
\frac{1}{F_\alpha(a_r, a_{r+1}, t)} - 1 \le k^r\left(\frac{1}{F_\alpha(a_0, a_1, t)} - 1\right). \tag{4.4}
$$

For any $q \in \mathbb{N}$, writing $t = \frac{t}{2} + \frac{t}{2}$ and using EFBM4 repeatedly,

$$
\begin{aligned}
&F_\alpha(a_r, a_{r+q}, t) \\
&\ge F_\alpha\left(a_r, a_{r+1}, \frac{t}{2\alpha(a_r, a_{r+q})}\right) * F_\alpha\left(a_{r+1}, a_{r+2}, \frac{t}{2^2\alpha(a_r, a_{r+q})\alpha(a_{r+1}, a_{r+q})}\right) \\
&* F_\alpha\left(a_{r+2}, a_{r+3}, \frac{t}{2^3\alpha(a_r, a_{r+q})\alpha(a_{r+1}, a_{r+q})\alpha(a_{r+2}, a_{r+q})}\right) * \ldots * \\
&F_\alpha\left(a_{r+q-1}, a_{r+q}, \frac{t}{2^q\alpha(a_r, a_{r+q})\alpha(a_{r+1}, a_{r+q})\alpha(a_{r+2}, a_{r+q})\ldots\alpha(a_{r+q-1}, a_{r+q})}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{F_\alpha(a_r, a_{r+q}, t)} - 1 \\
&\le \frac{1}{F_\alpha\left(a_r, a_{r+1}, \frac{t}{2\alpha(a_r, a_{r+q})}\right)} - 1 * \frac{1}{F_\alpha\left(a_{r+1}, a_{r+2}, \frac{t}{2^2\alpha(a_r, a_{r+q})\alpha(a_{r+1}, a_{r+q})}\right)} - 1 \\
&* \frac{1}{F_\alpha\left(a_{r+2}, a_{r+3}, \frac{t}{2^3\alpha(a_r, a_{r+q})\alpha(a_{r+1}, a_{r+q})\alpha(a_{r+2}, a_{r+q})}\right)} - 1 * \ldots * \\
&\frac{1}{F_\alpha\left(a_{r+q-1}, a_{r+q}, \frac{t}{2^q\alpha(a_r, a_{r+q})\alpha(a_{r+1}, a_{r+q})\alpha(a_{r+2}, a_{r+q})\ldots\alpha(a_{r+q-1}, a_{r+q})}\right)} - 1.
\end{aligned}
$$

Using the contraction (4.4) and EFBM5, it follows that

$$\begin{aligned}
&\frac{1}{F_\alpha(a_r, a_{r+q}, t)} - 1 \\
&\leq k^r \left(\frac{1}{F_\alpha\left(a_0, a_1, \frac{t}{2\alpha(a_r,a_{r+q})}\right)} - 1 \right) * k^{r+1} \left(\frac{1}{F_\alpha\left(a_0, a_1, \frac{t}{2^2\alpha(a_r,a_{r+q})\alpha(a_{r+1},a_{r+q})}\right)} - 1 \right) \\
&* k^{r+2} \left(\frac{1}{F_\alpha\left(a_0, a_0, \frac{t}{2^3\alpha(a_r,a_{r+q})\alpha(a_{r+1},a_{r+q})\alpha(a_{r+2},a_{r+q})}\right)} - 1 \right) \\
&* k^{r+3} \left(\frac{1}{F_\alpha\left(a_0, a_0, \frac{t}{2^4\alpha(a_r,a_{r+q})\alpha(a_{r+1},a_{r+q})\alpha(a_{r+2},a_{r+q})\alpha(a_{r+3},a_{r+q})}\right)} - 1 \right) \\
&\vdots \\
&* k^{r+q-1} \left(\frac{1}{F_\alpha\left(a_0, a_1, \frac{t}{2^q\alpha(a_r,a_{r+q})\alpha(a_{r+1},a_{r+q})\alpha(a_{r+2},a_{r+q})\ldots\alpha(a_{r+q-1},a_{r+q})}\right)} - 1 \right).
\end{aligned}$$

Since $0 < k < 1$. Taking limit as $r \to \infty$, we get

$$\lim_{r\to\infty} \left(\frac{1}{F_\alpha(a_r, a_{r+q}, t)} - 1 \right) = 0 * 0 * \ldots * 0$$

$$\lim_{r\to\infty} F_\alpha(a_r, a_{r+q}, t) = 1$$

Hence $\{a_r\}$ is G-Cauchy sequence. Since $(W, F_\alpha, *)$ be a complete EFBMS so there exists $p_1 \in W$ such that

$$\lim_{r\to\infty} a_r = p_1.$$

We want to show that p is fixed point of Γ.

$$\frac{1}{F_\alpha(\Gamma p_1, \Gamma a_r, t)} - 1 \leqslant k \left[\frac{1}{F_\alpha(p_1, a_r, t)} - 1 \right] \to 0 \text{ as } r \to \infty$$

$$\lim_{r\to\infty} F_\alpha(\Gamma p_1, \Gamma a_r, t) = 1$$

$$\lim_{r\to\infty} (\Gamma a_r) = \Gamma p_1$$

$$\lim_{r\to\infty} (a_{r+1}) = \Gamma p_1$$

$$\Gamma p_1 = p_1,$$

which shows that p_1 is a fixed point.

Uniqueness:

Let p_2 be the an other fixed point of Γ such that $\Gamma p_2 = p_2$ for some $p_2 \in W$, then

$$\begin{aligned}\frac{1}{F_\alpha(p_2,p_1,t)} - 1 &= \frac{1}{F_\alpha(\Gamma p_2,\Gamma p_1,t)} - 1\\ &\leqslant k\left(\frac{1}{F_\alpha(p_2,p_1,t)} - 1\right)\\ &= k\left(\frac{1}{F_\alpha(\Gamma p_2,\Gamma p_1,t)} - 1\right)\\ &\leqslant k^2\left(\frac{1}{F_\alpha(p_2,p_1,t)} - 1\right)\\ &\leqslant k^{r-1}\left(\frac{1}{F_\alpha(p_2,p_1,t)} - 1\right)\\ &\leqslant k^{r}\left(\frac{1}{F_\alpha(p_2,p_1,t)} - 1\right)\\ &\longrightarrow 0 \quad \text{as} \quad r \to \infty.\end{aligned}$$

Thus $F_\alpha(p_2,p_1,t) = 1$ and $p_1 = p_2$.

Hence fixed point is unique. □

Corollary 4.1.7.

Let $(W, F_b, *)$ be a G-complete FBMS and $b \geq 1$. Let $\Gamma\colon W \to W$ be a mapping satisfying

$$\frac{1}{F_b(\Gamma\xi,\Gamma\varrho,t)} - 1 \leq k\left(\frac{1}{F_b(\xi,\varrho,t)} - 1\right)$$

$\forall\, \xi, \varrho \in W$. Further, for an arbitrary $a_0 \in W$ and $r \in \mathbb{N}$, we have $bk < 1$, where $a_r = \Gamma^r a_0$. Then Γ has a unique fixed point.

Remark 4.1.8.

By taking $\alpha(\xi,\varrho) = 1$ in Theorem 4.1.6, the result of [52] is obtained in FMS.

4.2 Geraghty-type Contraction in Extended Fuzzy b-Metric Space

In this Section, certain fixed point results for Geraghty-type contraction are established in the setting of G-complete EFBMS. An application of the results obtained is also presented. These results are the extension and generalization of the existing results in literature.

Following lemma will be used in upcoming results

Lemma 4.2.1.

Let $(W, F_\alpha, *)$ be a G-complete EFBMS and

$$F_\alpha(\xi, \varrho, kt) \geq F_\alpha\,(\xi, \varrho, t)$$

for all $\xi, \varrho \in X,\ k \in (0,1)$ and $t > 0$ then $\xi = \varrho$.

Recall that

$$\mathfrak{F}_b = \left\{\beta : [0,\infty) \to [0, \frac{1}{b});\ \limsup_{r\to\infty} \beta(t_r) = \frac{1}{b} \ \text{ implies } \ \lim_{r\to\infty} t_r = 0\right\} \tag{4.5}$$

In the fuzzy setting, the class of Geraghty-type contraction is modified as follows:

$$\mathfrak{F}_b = \left\{\beta : [0,\infty) \to [0, \frac{1}{b});\ \limsup_{r\to\infty} \beta(t_r) = \frac{1}{b} \ \text{ implies } \ \lim_{r\to\infty} t_r = 1\right\} \tag{4.6}$$

We now establish a fixed point result, analogue to [87, Theorem 1], in the setting of G-complete EFBMS, as follows:

Theorem 4.2.2.

Let $(W, F_\alpha, *)$ be a G-complete EFBMS and $\alpha\colon W \times W \to [1,\infty)$ be a mapping. Let $\Gamma\colon W \to W$ be a mapping satisfying

$$F_\alpha(\Gamma\xi, \Gamma\varrho, \beta(F_\alpha(\xi, \varrho, t))t) \geq F_\alpha(\xi, \varrho, t) \tag{4.7}$$

$\forall\, \xi, \varrho \in W,\ \beta \in \mathfrak{F}_b$ and $\beta(\xi, \varrho, t)\alpha(\xi, \varrho) < 1$. Then Γ has a unique fixed point.

Proof.

Let $a_0 \in W$, generate a sequence $\{a_r\}$ by

$$a_r = \Gamma^r a_0 \quad (r \in \mathbb{N}).$$

First, note that for all $r, t > 0$, by (4.7), we have

$$\begin{aligned}
&F_\alpha(a_r, a_{r+1}, t) = F_\alpha(\Gamma a_{r-1}, \Gamma a_r, t)\\
&\geq F_\alpha(a_{r-1}, a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))})\\
&\geq F_\alpha\left(a_{r-2}, a_{r-1}, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t)).\beta(F_\alpha(a_{r-2}, a_{r-1}, t))}\right)\\
&\geq F_\alpha\left(a_{r-3}, a_{r-2}, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t)).\beta(F_\alpha(a_{r-2}, a_{r-1}, t)).\beta(F_\alpha(a_{r-3}, a_{r-2}, t)}\right)\\
&\geq \ldots \geq F_\alpha\left(a_1, a_2, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t)).\beta(F_\alpha(a_{r-2}, a_{r-1}, t))\ldots\beta(F_\alpha(a_1, a_2, t))}\right)\\
&\geq F_\alpha\left(a_0, a_1, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t)).\beta(F_\alpha(a_{r-2}, a_{r-1}, t))\ldots\beta(F_\alpha(a_0, a_1, t))}\right).
\end{aligned}$$

So, we have

$$\begin{aligned}
&F_\alpha(a_r, a_{r+1}, t)\\
&\geq F_\alpha\left(a_0, a_1, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t)).\beta(F_\alpha(a_{r-2}, a_{r-1}, t))\ldots\beta(F_\alpha(a_0, a_1, t))}\right) \quad (4.8)
\end{aligned}$$

For any $q \in \mathbb{N}$, writing $t = \frac{t}{q} + \frac{t}{q} + \ldots + \frac{t}{q}$ and using $[EFBM4]$ repeatedly,

$$\begin{aligned}
&F_\alpha(a_r, a_{r+q}, t)\\
&\geq F_\alpha\left(a_r, a_{r+1}, \frac{t}{q\alpha(a_r, a_{r+q})}\right) * F_\alpha\left(a_{r+1}, a_{r+2}, \frac{t}{q\alpha(a_r, a_{r+q})\alpha(a_{r+1}, a_{r+q})}\right)\\
&* F_\alpha\left(a_{r+2}, a_{r+3}, \frac{t}{q\alpha(a_r, a_{r+q})\alpha(a_{r+1}, a_{r+q})\alpha(a_{r+2}, a_{r+q})}\right)\\
&F_\alpha\left(a_{r+3}, a_{r+4}, \frac{t}{q\alpha(a_r, a_{r+q})\alpha(a_{r+1}, a_{r+q})\alpha(a_{r+2}, a_{r+q})\alpha(a_{r+3}, a_{r+q})}\right)\\
&F_\alpha\left(a_{r+4}, a_{r+5}, \frac{t}{q\alpha(a_r, a_{r+q})\alpha(a_{r+1}, a_{r+q})\ldots\alpha(a_{r+4}, a_{r+q})}\right)\\
&F_\alpha\left(a_{r+5}, a_{r+6}, \frac{t}{q\alpha(a_r, a_{r+q})\alpha(a_{r+1}, a_{r+q})\ldots\alpha(a_{r+5}, a_{r+q})}\right)\\
&* F_\alpha\left(a_{r+6}, a_{r+7}, \frac{t}{q\alpha(a_r, a_{r+q})\alpha(a_{r+1}, a_{r+q})\ldots\alpha(a_{r+6}, a_{r+q})}\right) * \ldots *\\
&F_\alpha\left(a_{r+q-2}, a_{r+q-1}, \frac{t}{q\alpha(a_r, a_{r+q})\alpha(a_{r+1}, a_{r+q})\alpha(a_{r+2}, a_{r+q})\ldots\alpha(a_{r+q-2}, a_{r+q})}\right)\\
&* F_\alpha\left(a_{r+q-1}, a_{r+q}, \frac{t}{q\alpha(a_r, a_{r+q})\alpha(a_{r+1}, a_{r+q})\alpha(a_{r+2}, a_{r+q})\ldots\alpha(a_{r+q-1}, a_{r+q})}\right).
\end{aligned}$$

Using (4.8) and $[EFBM5]$, we get

$$\begin{aligned}
&F_\alpha(a_r, a_{r+q}, t)\\
&\geq F_\alpha\left(a_0, a_1, \frac{t}{q\alpha(a_r, a_{r+q})\beta(F_\alpha(a_{r-1}, a_r, t))\ldots\beta(F_\alpha(a_0, a_1, t))}\right)\\
&* F_\alpha\left(a_0, a_1, \frac{t}{q\alpha(a_r, a_{r+q})\alpha(a_{r+1}, a_{r+q})\beta(a_r, a_{r+1}, t).\ldots\beta(F_\alpha(a_0, a_1, t))}\right)\\
&* \ldots * F_\alpha\left(a_0, a_1, \frac{t}{q\alpha(a_r, a_{r+q})\ldots\alpha(a_{r+q-1}, a_{r+q})\ldots\beta(F_\alpha(a_0, a_1, t))}\right)
\end{aligned}$$

$$\begin{aligned}
&F_\alpha(a_r, a_{r+q}, t)\\
&\geq F_\alpha\left(a_0, a_1, \frac{b^{r-1}t}{q\alpha(a_r, a_{r+q})\beta(F_\alpha(a_{r-1}, a_r, t))}\right)\\
&* F_\alpha\left(a_0, a_1, \frac{b^{r-1}t}{q\alpha(a_r, a_{r+q})\alpha(a_{r+1}, a_{r+q})\beta(F_\alpha(a_r, a_{r+1}, t)).\beta(F_\alpha(a_{r-1}, a_r, t))}\right)\\
&* \ldots * F_\alpha\left(a_0, a_1, \frac{b^{r-1}t}{q\alpha(a_r, a_{r+q})\ldots\alpha(a_{r+q-1}, a_{r+q})\ldots\beta(F_\alpha(a_{r-1}, a_r, t))}\right).
\end{aligned}$$

Since for all $r, q \in \mathbb{N}$, we have

$$\alpha(a_r, a_{r+q})\beta(F_\alpha(a_{r-1}, a_r, t)) < 1,$$

taking limit as $r \to \infty$, we get

$$\begin{aligned}
\lim_{r\to\infty} F_\alpha(a_r, a_{r+q}, t) &= 1 * 1 * \ldots * 1\\
&= 1.
\end{aligned}$$

Hence $\{a_r\}$ is G-Cauchy sequence. Since $(W, F_\alpha, *)$ be a G-complete EFBMS so there exists $p_1 \in W$ such that

$$\lim_{r\to\infty} a_r = p_1.$$

We want to show that p is fixed point of Γ.

$$\begin{aligned}
F_\alpha(\Gamma p_1, p_1, t) &\geq F_\alpha\left(\Gamma p_1, \Gamma a_r, \frac{t}{2\alpha(\Gamma p_1, p_1)}\right) * F_\alpha\left(\Gamma a_r, p_1, \frac{t}{2\alpha(\Gamma p_1, p_1)}\right)\\
&\geq F_\alpha\left(p_1, a_r, \frac{t}{2\beta(p_1, a_r, t)\alpha(\Gamma p_1, p_1)}\right) * F_\alpha\left(a_{r+1}, a_r, \frac{t}{2\alpha(a_{r+1}, a_r)}\right)\\
&\longrightarrow 1 * 1 = 1.
\end{aligned}$$

Which shows that $\Gamma p_1 = p_1$ is a fixed point.

Uniqueness:

Assume $\Gamma p_2 = p_2$ for some $p_2 \in W$, then

$$\begin{aligned}
F_\alpha(p_2, p_1, t) &= F_\alpha(\Gamma p_2, \Gamma p_1, t) \\
&\geq F_\alpha\left(p_2, p_1, \frac{t}{\beta(F_\alpha(p_2, p_1, t))}\right) \\
&= F_\alpha\left(\Gamma p_2, \Gamma p_1, \frac{t}{\beta(F_\alpha(p_2, p_1, t))}\right) \\
&\geq F_\alpha\left(p_2, p_1, \frac{t}{\beta(F_\alpha(p_2, p_1, t))^2}\right) \\
&= F_\alpha\left(\Gamma p_2, \Gamma p_1, \frac{t}{\beta(F_\alpha(p_2, p_1, t))^2}\right) \\
&\geq F_\alpha\left(p_2, p_1, \frac{t}{\beta(F_\alpha(p_2, p_1, t))^3}\right) \\
&\vdots \\
&\geq F_\alpha\left(p_2, p_1, \frac{t}{\beta(F_\alpha(p_2, p_1, t))^r}\right) \\
&= F_\alpha\left(p_2, p_1, b^r t\right) \\
&\longrightarrow 1 \quad \text{as} \quad r \to \infty
\end{aligned}$$

Thus $\quad p_1 = p_2$.

Hence fixed point is unique. □

Following example illustrates Theorem 4.2.2.

Example 4.2.1. Let $W = \{0, 1, 2\}$ and define a mapping $\alpha\colon W \times W \to [1, \infty)$ by

$$\alpha(\xi, \varrho) = 1 + \xi + \varrho$$

and let

$$F_\alpha(\xi, \varrho, t) = \frac{t}{t + (\xi - \varrho)^2}.$$

Then it is easy to verify that $(W, F_\alpha, *)$ is a G-complete EFBMS.

Define a mapping $\Gamma\colon W \to W$ such that

$$\Gamma(\xi) = \frac{\sqrt{\beta(F_\alpha(\xi, \varrho, t))}\xi}{1 + \xi}.$$

Where $\beta(F_\alpha(\xi, \varrho, t)$ is defined in eq (4.6). Now for all $t > 0$, we have

$$\begin{aligned}
&F_\alpha(\Gamma\xi, \Gamma\varrho, \beta(F_\alpha(\xi, \varrho, t))t) \\
&\qquad = \frac{\beta(F_\alpha(\xi, \varrho, t))t}{\beta(F_\alpha(\xi, \varrho, t))t + \left(\frac{\sqrt{\beta(F_\alpha(\xi, \varrho, t))}\xi}{1+\xi} - \frac{\sqrt{\beta(F_\alpha(\xi, \varrho, t))}\varrho}{1+\varrho}\right)^2} \\
&\qquad = \frac{\beta(F_\alpha(\xi, \varrho, t))t}{\beta(F_\alpha(\xi, \varrho, t))t + \beta(F_\alpha(\xi, \varrho, t))\left(\frac{\xi}{1+\xi} - \frac{\varrho}{1+\varrho}\right)^2} \\
&\qquad = \frac{t}{t + \frac{(\xi-\varrho)^2}{(1+\xi)^2(1+\xi)^2}}.
\end{aligned}$$

Since

$$\begin{aligned}
\frac{(\xi-\varrho)^2}{(1+\xi)^2(1+\xi)^2} &\leq (\xi-\varrho)^2 \\
t + \frac{(\xi-\varrho)^2}{(1+\xi)^2(1+\xi)^2} &\leq t + (\xi-\varrho)^2 \\
\frac{1}{t + \frac{(\xi-\varrho)^2}{(1+\xi)^2(1+\xi)^2}} &\geq \frac{1}{t + (\xi-\varrho)^2} \\
\frac{t}{t + \frac{(\xi-\varrho)^2}{(1+\xi)^2(1+\xi)^2}} &\geq \frac{t}{t + (\xi-\varrho)^2}.
\end{aligned}$$

This implies that

$$F_\alpha(\Gamma\xi, \Gamma\varrho, \beta(F_\alpha(\xi, \varrho, t))t) \geq F_\alpha(\xi, \varrho, t).$$

Also $\xi = 0$ is a unique fixed point of Γ.

Following are the immediate consequences of Theorem 4.2.2.

Corollary 4.2.3.

Let $(W, F_b, *)$ be a G-complete FBMS with $b \geqslant 1$. Let $\Gamma\colon W \to W$ be a mapping satisfying

$$F_b(\Gamma\xi, \Gamma\varrho, \beta(F_\alpha(\xi, \varrho, t))t) \geq F_b(\xi, \varrho, t)$$

$\forall\, \xi, \varrho \in W,\ \beta \in \mathfrak{F}_b$ and $\beta(F_b(\xi, \varrho, t))b < 1$. Then Γ has a unique fixed point.

Corollary 4.2.4.

Let $(W, F, *)$ be a G-complete FMS. Let $\Gamma\colon W \to W$ be a mapping satisfying

$$F(\Gamma\xi, \Gamma\varrho, \beta(F(\xi, \varrho, t))t) \geq F(\xi, \varrho, t)$$

$\forall\, \xi, \varrho \in W$ and $\beta \in \mathfrak{F}_b$ with $b = 1$. Then Γ has a unique fixed point.

Theorem 4.2.5.

Let $(W, F_\alpha, *)$ be a G-complete EFBMS and $\alpha\colon W \times W \to [1, \infty)$ be a mapping. Let $\Gamma\colon W \to W$ be a mapping satisfying the condition

$$\begin{aligned} F_\alpha(\Gamma\xi, \Gamma\varrho, &\beta(F_\alpha(\xi, \varrho, t))t) \\ &\geq \min\Big\{F_\alpha(\Gamma\xi, \Gamma\varrho, t), F_\alpha(\xi, \Gamma\xi, t), F_\alpha(\varrho, \Gamma\varrho, t), F_\alpha(\xi, \varrho, t)\Big\} \end{aligned} \tag{4.9}$$

for all $\xi, \varrho \in W,\ \beta \in \mathfrak{F}_b$ and $\alpha(\xi, \varrho)\beta(F_\alpha(\xi, \varrho, t)) < 1$. Then T has a unique fixed point.

Proof.

Starting in the same way as in Theorem 4.2.2, we have

$$\begin{aligned} &F_\alpha(a_r, a_{r+1}, t) = F_\alpha(\Gamma a_{r-1}, \Gamma a_r, t) \\ &\geq \min\Bigg\{F_\alpha\left(\Gamma a_{r-1}, \Gamma a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right), F_\alpha\left(a_{r-1}, \Gamma a_{r-1}, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right), \\ &\qquad F_\alpha\left(a_r, \Gamma a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right), F_\alpha\left(a_{r-1}, a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)\Bigg\} \\ &\geq \min\Bigg\{F_\alpha\left(a_r, a_{r+1}, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right), F_\alpha\left(a_{r-1}, a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right), \\ &\qquad F_\alpha\left(a_r, a_{r+1}, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right), F_\alpha\left(a_{r-1}, a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)\Bigg\} \\ &= \min\left\{F_\alpha\left(a_r, a_{r+1}, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right), F_\alpha\left(a_{r-1}, a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)\right\}. \\ &= \min\left\{F_\alpha\left(a_r, a_{r+1}, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right), F_\alpha\left(a_{r-1}, a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)\right\}. \end{aligned} \tag{4.10}$$

If

$$\min\left\{F_\alpha\left(a_r, a_{r+1}, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right), F_\alpha\left(a_{r-1}, a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)\right\}$$
$$= F_\alpha(a_r, a_{r+1}, t),$$

then (4.10) implies

$$F_\alpha(a_r, a_{r+1}, t) \geq F_\alpha\left(a_r, a_{r+1}, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)$$

Therefore, there is nothing to prove by Lemma 4.2.1.

If

$$\min\left\{F_\alpha\left(a_r, a_{r+1}, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right), F_\alpha\left(a_{r-1}, a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)\right\}$$
$$= F_\alpha\left(a_{r-1}, a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right),$$

then from (4.10), we have

$$F_\alpha(a_r, a_{r+1}, t) \geq F_\alpha\left(a_{r-1}, \xi_n, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)$$
$$\geq \ldots \geq F_\alpha\left(a_0, a_1, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t)).\beta(F_\alpha(a_{r-2}, a_{r-1}, t)) \ldots \beta(F_\alpha(a_0, a_1, t))}\right).$$

By adopting the same procedure after Inequality (4.8) as in Theorem 4.2.2 we can complete the proof. □

Following are the immediate consequences of Theorem 4.2.5.

Corollary 4.2.6.

Let $(W, F_b, *)$ be a G-complete FBMS with $b \geq 1$. Let $\Gamma\colon W \to W$ be a mapping satisfying the condition

$$F_b(\Gamma\xi, \Gamma\varrho, \beta(F_b(\xi, \varrho, t))t)$$
$$\geq \min\left\{F_b(\Gamma\xi, \Gamma\varrho, t), F_b(\xi, \Gamma\xi, t), F_b(\varrho, \Gamma\varrho, t), F_b(\xi, \varrho, t)\right\}$$

for all $\xi, \varrho \in W,\ \beta \in \mathfrak{F}_b$ and $\beta(F_b(\xi,\varrho,t))b < 1$. Then Γ has a unique fixed point.

Corollary 4.2.7.

Let $(W, F, *)$ be a G-complete FMS. Let $\Gamma\colon W \to W$ be a mapping satisfying the condition

$$\begin{aligned} F(\Gamma\xi, \Gamma\varrho, &\beta(F(\xi,\varrho,t))t) \\ &\geq \min\left\{ F(\Gamma\xi,\Gamma\varrho,t), F(\xi,\Gamma\xi,t), F(\varrho,\Gamma\varrho,t), F(\xi,\varrho,t) \right\} \end{aligned}$$

for all $\xi, \varrho \in W$ and $\beta \in \mathfrak{F}_b$ with $b = 1$. Then Γ has a unique fixed point.

Following result is an extention of the main result of Gupta et al. [53].

Theorem 4.2.8.

Let $(W, F_\alpha, *)$ be a G-complete EFBMS and $\alpha\colon W \times W \to [1,\infty)$ be a mapping. Let $\Gamma\colon W \to W$ be a mapping satisfying

$$\begin{aligned} F_\alpha(\Gamma\xi, \Gamma\varrho, &\beta(F_\alpha(\xi,\varrho,t))t) \\ &\geq \min\left\{ \frac{F_\alpha(\varrho,\Gamma_\varrho,t)\,[1 + F_\alpha(\xi,\Gamma_\xi,t)]}{1 + F_\alpha(\xi,\varrho,t)}, F_\alpha(\xi,\varrho,t) \right\} \end{aligned} \tag{4.11}$$

$\forall\ \xi, \varrho \in W,\ \beta \in \mathfrak{F}_b$ and $\beta(F_\alpha(\xi,\varrho,t))\alpha(\xi,\varrho) < 1$. Then Γ has a unique fixed point.

Proof.

Starting in the same way as in Theorem 4.2.2 we have

$$F_\alpha(a_r, a_{r+1}, t) = F_\alpha(\Gamma a_{r-1}, \Gamma a_r, t)$$

$$\geq \min\left\{ \frac{F_\alpha\left(a_r, \Gamma_{a_r}, \dfrac{t}{\beta(F_\alpha(a_{r-1},a_r,t))}\right)\left[1 + F_\alpha\left(a_{r-1}, \Gamma_{a_{r-1}}, \dfrac{t}{\beta(F_\alpha(a_{r-1},a_r,t))}\right)\right]}{1 + F_\alpha\left(a_{r-1}, a_r, \dfrac{t}{\beta(F_\alpha(a_{r-1},a_r,t))}\right)}, \right.$$
$$\left. F_\alpha\left(a_{r-1}, a_r, \frac{t}{\beta(F_\alpha(a_{r-1},a_r,t))}\right) \right\}$$

$$F_\alpha(a_r, a_{r+1}, t) \geq \min\left\{ \frac{F_\alpha\left(a_r, a_{r+1}, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)\left[1 + F_\alpha\left(a_{r-1}, a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)\right]}{1 + F_\alpha\left(a_{r-1}, a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)}, F_\alpha\left(a_{r-1}, a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right) \right\}$$

$$F_\alpha(a_r, a_{r+1}, t) \geq \min\left\{ F_\alpha\left(a_r, a_{r+1}, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right), F_\alpha\left(a_{r-1}, a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right) \right\}. \tag{4.12}$$

If

$$\min\left\{ F_\alpha\left(a_r, a_{r+1}, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right), F_\alpha\left(a_{r-1}, a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right) \right\} = F_\alpha\left(a_r, a_{r+1}, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right).$$

Then (4.12) implies

$$F_\alpha(a_r, a_{r+1}, t) \geq F_\alpha\left(a_r, a_{r+1}, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right).$$

Then nothing to prove by lemma 4.2.1.

If

$$\min\left\{ F_\alpha\left(a_r, a_{r+1}, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right), F_\alpha\left(a_{r-1}, a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right) \right\} = F_\alpha\left(a_{r-1}, a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)$$

Then from (4.12) we have

$$F_\alpha(a_r, a_{r+1}, t) \geq F_\alpha\left(a_{r-1}, a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)$$

Continuing in this way we have,

$$\begin{aligned}&F_\alpha(a_r, a_{r+1}, t)\\&\geq F_\alpha\left(a_0, a_1, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t).\beta(F_\alpha(a_{r-2}, a_{r-1}, t)\ldots\beta(F_\alpha(a_0, a_1, t)}\right)\end{aligned}$$

By adopting the same procedure after Inequality (4.8) as in Theorem 4.2.2 we can complete the proof. □

Following are the immediate consequences of theorem 4.2.8.

Corollary 4.2.9.

Let $(W, F_b, *)$ be a G-complete FBMS with $b \geq 1$. Let $\Gamma\colon W \to W$ be a mapping satisfying

$$\begin{aligned}&F_b(\Gamma\xi, \Gamma\varrho, \beta(F_b(\xi, \varrho, t))t)\\&\qquad\geq \min\left\{\frac{F_b(\varrho, \Gamma_\varrho, t)\,[1 + F_b(\xi, \Gamma_\xi, t)]}{1 + F_b(\xi, \varrho, t)}, F_b(\xi, \varrho, t)\right\}\end{aligned}$$

$\forall\, \xi, \varrho \in W,\ \beta \in \mathfrak{F}_b$ and $\beta(F_b(\xi, \varrho, t))b < 1$. Then Γ has a unique fixed point.

Corollary 4.2.10.

Let $(W, F, *)$ be a G-complete FMS. Let $\Gamma\colon W \to W$ be a mapping satisfying

$$\begin{aligned}&F(\Gamma\xi, \Gamma\varrho, \beta(F(\xi, \varrho, t)t)\\&\qquad\geq \min\left\{\frac{F(\varrho, \Gamma_\varrho, t)\,[1 + F(\xi, \Gamma_\xi, t)]}{1 + F(\xi, \varrho, t)}, F(\xi, \varrho, t)\right\}\end{aligned}$$

$\forall\, \xi, \varrho \in W$, and $\beta \in \mathfrak{F}_b$ with $b = 1$. Then Γ has a unique fixed point.

Theorem 4.2.11.

Let $(W, F_\alpha, *)$ be a G-complete EFBMS and $\alpha\colon W \times W \to [1, \infty)$ be a mapping. Let $\Gamma\colon W \to W$ be a mapping satisfying

$$\begin{aligned}&F_\alpha(\Gamma\xi, \Gamma\varrho, \beta(F_\alpha(\xi, \varrho, t))t)\\&\qquad\geq \min\left\{\frac{F_\alpha(\varrho, \Gamma_\varrho, t)\,[1 + F_\alpha(\xi, \Gamma_\xi, t) + F_\alpha(\varrho, \Gamma_\xi, t)]}{2 + F_\alpha(\xi, \varrho, t)}, F_\alpha(\xi, \varrho, t)\right\} \quad (4.13)\end{aligned}$$

$\forall\, \xi, \varrho \in W,\ \beta \in \mathfrak{F}_b$ and $\beta(F_\alpha(\xi, \varrho, t))\alpha(\xi, \varrho) < 1$. Then Γ has a unique fixed point.

Proof.

For any arbitrary point $a_0 \in W$, we choose a sequence $\{a_r\}$ in W and start with iterative process

$$a_{r+1} = \Gamma a_r.$$

For all $r, t > 0$, we have

$$F_\alpha(a_r, a_{r+1}, t) = F_\alpha(\Gamma a_{r-1}, \Gamma a_r, t)$$

$$\geq \min\left\{ \frac{F_\alpha\left(a_r, ga_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)\left[1 + F_\alpha\left(a_{r-1}, ga_{r-1}, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right) + F_\alpha\left(a_r, ga_{r-1}, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)\right]}{2 + F_\alpha\left(a_{r-1}, a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)}, F_\alpha\left(a_{r-1}, a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)\right\}$$

$$= \min\left\{ \frac{F_\alpha\left(a_r, a_{r+1}, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)\left[1 + F_\alpha\left(a_{r-1}, a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right) + F_\alpha\left(a_r, a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)\right]}{2 + F_\alpha\left(a_{r-1}, a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)}, F_\alpha\left(a_{r-1}, a_r, \frac{t}{k}\right)\right\}$$

$$= \min\left\{ \frac{F_\alpha\left(a_r, a_{r+1}, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)\left[1 + F_\alpha\left(a_{r-1}, a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right) + 1\right]}{2 + F_\alpha\left(a_{r-1}, a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)}, F_\alpha\left(a_{r-1}, a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)\right\}$$

$$= \min\left\{F_\alpha\left(a_r, a_{r+1}, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right),\ F_\alpha\left(a_{r-1}, a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)\right\} \quad (4.14)$$

If

$$\min\left\{F_\alpha\left(a_r, a_{r+1}, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right), F_\alpha\left(a_{r-1}, a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)\right\}$$

$$= F_\alpha\left(a_r, a_{r+1}, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right),$$

then from (4.14),

$$F_\alpha(a_r, a_{r+1}, t) \geq F_\alpha\left(a_r, a_{r+1}, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)$$

so there is nothing to prove by Lemma 4.2.1.

If

$$\min\left\{F_\alpha\left(a_r, a_{r+1}, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right), F_\alpha\left(a_{r-1}, a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)\right\}$$
$$= F_\alpha\left(a_{r-1}, a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)$$

then from (4.14),

$$F_\alpha(a_r, a_{r+1}, t) \geq F_\alpha\left(a_{r-1}, a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)$$

Continuing in this way we get,

$$F_\alpha(a_r, a_{r+1}, t) \geq F_\alpha\left(a_0, a_1, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t)).\beta(F_\alpha(a_{r-2}, a_{r-1}, t)) \ldots \beta(F_\alpha(a_0, a_1, t))}\right) \tag{4.15}$$

By adopting the same procedure after inequality (4.8) of Theorem 4.2.2 we can complete the proof. □

Following are immediate consequences of Theorem 4.2.11.

Corollary 4.2.12.

Let $(W, F_b, *)$ be a G-complete FBMS with $b \geq 1$. Let $\Gamma\colon W \to W$ be a mapping satisfying

$$F_b(\Gamma\xi, \Gamma\varrho, \beta(F_b(\xi, \varrho, t)t)) \geq \min\left\{\frac{F_b(\varrho, \Gamma_\varrho, t)\left[1 + F_b(\xi, \Gamma_\xi, t) + F_b(\varrho, \Gamma_\xi, t)\right]}{2 + F_b(\xi, \varrho, t)}, F_b(\xi, \varrho, t)\right\}$$

$\forall\, \xi, \varrho \in W,\ \beta \in \mathfrak{F}_b$ and $\beta(F_b(\xi, \varrho, t))b < 1$. Then Γ has a unique fixed point.

Corollary 4.2.13.

Let $(W, F, *)$ be a G-complete FMS. Let $\Gamma\colon W \to W$ be a mapping satisfying

$$\begin{aligned} F(\Gamma\xi, \Gamma\varrho, &\beta(F(\xi, \varrho, t))t) \\ &\geq \min\left\{\frac{F(\varrho, \Gamma_{\varrho}, t)\,[1 + F(\xi, \Gamma_{\xi}, t) + F(\varrho, \Gamma_{\xi}, t)]}{2 + F(\xi, \varrho, t)}, F(\xi, \varrho, t)\right\} \end{aligned}$$

$\forall\, \xi, \varrho \in W$, and $\beta \in \mathfrak{F}_b$ with $b = 1$ Then Γ has a unique fixed point.

The following result is the extention of the main result of Roshan et.al [28].

Theorem 4.2.14.

Let $(W, F_\alpha, *)$ be a G-complete EFBMS and $\alpha\colon W \times W \to [1, \infty)$ be a mapping. Let $\Gamma\colon W \to W$ be a mapping satisfying

$$F_\alpha(\Gamma\xi, \Gamma\varrho, \beta(F_\alpha(\xi, \varrho, t))t) \geq \delta(\xi, \varrho, t) \tag{4.16}$$

$\forall\, \xi, \varrho \in W,\ \beta \in \mathfrak{F}_b$ and $\beta(F_\alpha(\xi, \varrho, t))\alpha(\xi, \varrho) < 1$,

where

$$\begin{aligned} \delta(\xi, \varrho, t) = \min\Bigg\{ &\frac{F_\alpha(\xi, \Gamma\xi, t)\,[1 + F_\alpha(\varrho, \Gamma\varrho, t)]}{1 + F_\alpha(\Gamma\xi, \Gamma\varrho, t)}, \frac{F_\alpha(\varrho, \Gamma\varrho, t)\,[1 + F_\alpha(\xi, \Gamma\xi, t)]}{1 + F_\alpha(\xi, \varrho, t)}, \\ &\frac{F_\alpha(\xi, \Gamma\xi, t)\,[2 + F_\alpha(\xi, \Gamma\varrho, t)]}{1 + F_\alpha(\xi, \Gamma\varrho, t) + F_\alpha(\varrho, \Gamma\xi, t)}, F_\alpha(\xi, \varrho, t)\Bigg\}. \end{aligned}$$

Then Γ has a unique fixed point.

Proof.

For any arbitrary point $a_0 \in W$, we choose a sequence $\{a_r\}$ in W and start with iterative process $a_{r+1} = \Gamma a_r$. For all $r, t > 0$, we have

$$\begin{aligned} F_\alpha(a_r, a_{r+1}, t) &= F_\alpha(\Gamma a_{r-1}, \Gamma a_r, t) \\ &\geq \delta\left(a_{r-1}, a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right). \end{aligned} \tag{4.17}$$

Now

$$\delta\left(a_{r-1}, a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)$$

$$= \min\left\{\frac{F_\alpha\left(a_{r-1}, \Gamma a_{r-1}, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)\left[1 + F_\alpha\left(a_r, \Gamma a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)\right]}{1 + F_\alpha\left(\Gamma a_{r-1}, \Gamma a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)},\right.$$

$$\frac{F_\alpha\left(a_r, \Gamma a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)\left[1 + F_\alpha\left(a_{r-1}, \Gamma a_{r-1}, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)\right]}{1 + F_\alpha\left(a_{r-1}, a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)},$$

$$\frac{F_\alpha\left(a_{r-1}, \Gamma a_{r-1}, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)\left[2 + F_\alpha\left(a_{r-1}, \Gamma a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)\right]}{1 + F_\alpha\left(a_{r-1}, \Gamma a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right) + F_\alpha\left(a_r, \Gamma a_{r-1}, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)},$$

$$\left.F_\alpha\left(a_{r-1}, a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)\right\}$$

$$\delta\left(a_{r-1}, a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)$$

$$= \min\left\{\frac{F_\alpha\left(a_{r-1}, a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)\left[1 + F_\alpha\left(a_r, a_{r+1}, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)\right]}{1 + F_\alpha\left(a_r, a_{r+1}, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)},\right.$$

$$\frac{F_\alpha\left(a_r, a_{r+1}, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)\left[1 + F_\alpha\left(a_{r-1}, a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)\right]}{1 + F_\alpha\left(a_{r-1}, a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)},$$

$$\frac{F_\alpha\left(a_{r-1}, a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)\left[2 + F_\alpha\left(a_{r-1}, a_{r+1}, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)\right]}{1 + F_\alpha\left(a_{r-1}, a_{r+1}, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right) + F_\alpha\left(a_r, a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)},$$

$$\left.F_\alpha\left(a_{r-1}, a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)\right\}$$

$$\delta\left(a_{r-1},a_r,\frac{t}{\beta(F_\alpha(a_{r-1},a_r,t))}\right)$$

$$=\min\left\{\frac{F_\alpha\left(a_{r-1},a_r,\frac{t}{\beta(F_\alpha(a_{r-1},a_r,t))}\right)\left[1+F_\alpha\left(a_r,a_{r+1},\frac{t}{\beta(F_\alpha(a_{r-1},a_r,t))}\right)\right]}{1+F_\alpha\left(a_r,a_{r+1},\frac{t}{\beta(F_\alpha(a_{r-1},a_r,t))}\right)},\right.$$

$$\frac{F_\alpha\left(a_r,a_{r+1},\frac{t}{\beta(F_\alpha(a_{r-1},a_r,t))}\right)\left[1+F_\alpha\left(a_{r-1},a_r,\frac{t}{\beta(F_\alpha(a_{r-1},a_r,t))}\right)\right]}{1+F_\alpha\left(a_{r-1},a_r,\frac{t}{\beta(F_\alpha(a_{r-1},a_r,t))}\right)},$$

$$\frac{F_\alpha(a_{r-1},a_r,\frac{t}{\beta(F_\alpha(a_{r-1},a_r,t))})\left[2+F_\alpha\left(a_{r-1},a_{r+1},\frac{t}{\beta(F_\alpha(a_{r-1},a_r,t))}\right)\right]}{1+F_\alpha\left(a_{r-1},a_{r+1},\frac{t}{\beta(F_\alpha(a_{r-1},a_r,t))}\right)+1},$$

$$\left.F_\alpha\left(a_{r-1},a_r,\frac{t}{\beta(F_\alpha(a_{r-1},a_r,t))}\right)\right\}.$$

So

$$\delta\left(a_{r-1},a_r,\frac{t}{\beta(F_\alpha(a_{r-1},a_r,t))}\right)$$
$$=\min\left\{F_\alpha\left(a_{r-1},a_r,\frac{t}{\beta(F_\alpha(a_{r-1},a_r,t))}\right),F_\alpha\left(a_r,a_{r+1},\frac{t}{\beta(F_\alpha(a_{r-1},a_r,t))}\right)\right\}.$$

If

$$\min\left\{F_\alpha\left(a_{r-1},a_r,\frac{t}{\beta(F_\alpha(a_{r-1},a_r,t))}\right),F_\alpha\left(a_r,a_{r+1},\frac{t}{\beta(F_\alpha(a_r,a_{r+1},t))}\right)\right\}$$
$$=F_\alpha\left(a_r,a_{r+1},\frac{t}{\beta(F_\alpha(a_{r-1},a_r,t))}\right).$$

then from (4.17),

$$M_b(a_r,a_{r+1},t)\geq F_\alpha\left(a_r,a_{r+1},\frac{t}{\beta(F_\alpha(a_{r-1},a_r,t))}\right).$$

Since $\beta \in \mathfrak{F}_{\mathfrak{b}}$. Therefore, there is nothing to prove by Lemma 4.2.1.

If

$$\min\left\{ F_\alpha\left(a_{r-1}, a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right), F_\alpha\left(a_r, a_{r+1}, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)\right\}$$
$$= F_\alpha\left(a_{r-1}, a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)$$

then from (4.17),

$$F_\alpha(a_r, a_{r+1}, t) \geq F_\alpha\left(a_{r-1}, a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)$$

Continuing in this way, we have

$$F_\alpha(a_r, a_{r+1}, t)$$
$$\geq F_\alpha\left(a_0, a_1, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t).\beta(F_\alpha(a_{r-2}, a_{r-1}, t)\dots\beta(F_\alpha(a_0, a_1, t)}\right)$$

By adopting the same procedure after Inequality (4.8) as in Theorem 4.2.2 we can complete the proof. □

Remark 4.2.15.

The special cases of Theorem 4.2.14 can be obtained for FBMS and FMS by setting $\alpha(\xi, \varrho) = b \geq 1$ and $\alpha(\xi, \varrho) = 1$ respectively.

Theorem 4.2.16.

Let $(W, F_\alpha, *)$ be a G-complete EFBMS and $\alpha\colon W \times W \to [1, \infty)$ be a mapping. Let $\Gamma\colon W \to W$ be a mapping satisfying the condition

$$F_\alpha(\Gamma\xi, \Gamma\varrho, \beta(F_\alpha(\xi, \varrho, t))t) \geq \frac{\delta(\xi, \varrho, t)}{\max\{F_\alpha(\xi, \Gamma\xi, t), F_\alpha(\varrho, T\varrho, t)\}}, \tag{4.18}$$

where

$$\delta(\xi, \varrho, t) = \min\left\{ F_\alpha(\Gamma\xi, \Gamma\varrho, t).F_\alpha(\xi, \varrho, t), F_\alpha(\xi, \Gamma\xi, t).F_\alpha(\varrho, \Gamma\varrho, t)\right\}$$

for all $\xi, \varrho \in W$, $\beta \in \mathfrak{F}_b$ and $\alpha(\xi, \varrho)\beta(F_\alpha(\xi, \varrho, t)) < 1$. Then Γ has a unique fixed point.

Proof.

Starting in same way as in Theorem 4.2.2 we have

$$F_\alpha(a_r, a_{r+1}, t) = F_\alpha(\Gamma a_{r-1}, \Gamma a_r, t)$$
$$\geq \frac{\delta\left(a_r, a_{r+1}, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)}{\max\left\{F_\alpha\left(a_{r-1}, \Gamma a_{r-1}, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right), F_\alpha\left(a_r, \Gamma a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)\right\}} \tag{4.19}$$

Now,

$$\delta\left(a_r, a_{r+1}, \frac{t}{\beta(a_{r-1}, a_r, t)}\right)$$
$$= \min\left\{F_\alpha\left(\Gamma a_{r-1}, \Gamma a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right).F_\alpha\left(a_{r-1}, a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)\right.$$
$$\left., F_\alpha\left(a_{r-1}, \Gamma a_{r-1}, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right).F_\alpha\left(a_r, \Gamma a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)\right\}$$
$$= \min\left\{F_\alpha\left(a_r, a_{r+1}, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right).F_\alpha\left(a_{r-1}, a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)\right.$$
$$\left., F_\alpha\left(a_{r-1}, a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right).F_\alpha\left(a_r, a_{r+1}, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)\right\}$$
$$= F_\alpha\left(a_{r-1}, a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right).F_\alpha\left(a_r, a_{r+1}, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right) \tag{4.20}$$

Using (4.20) in (4.19), we get

$$F_\alpha(a_r, a_{r+1}, t)$$
$$\geq \frac{F_\alpha\left(a_r, a_{r+1}, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right).F_\alpha\left(a_{r-1}, a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)}{\max\left\{F_\alpha\left(a_{r-1}, a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right), F_\alpha\left(a_r, a_{r+1}, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)\right\}} \tag{4.21}$$

If

$$\max\left\{F_\alpha\left(a_r, a_{r+1}, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right), F_\alpha\left(a_{r-1}, a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)\right\} = F_\alpha\left(a_{r-1}, a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)$$

then (4.21) implies

$$F_\alpha\left(a_r, a_{r+1}, kt) \geq F_\alpha(a_r, a_{r+1}, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)$$

Therefore, there is nothing to prove by Lemma 4.2.1.

If

$$\max\left\{F_\alpha\left(a_r, a_{r+1}, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right), F_\alpha\left(a_{r-1}, a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)\right\} = F_\alpha\left(a_r, a_{r+1}, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)$$

then from (4.21), we have

$$\begin{aligned}
&F_\alpha(a_r, a_{r+1}, t) \\
&\quad\geq F_\alpha\left(a_{r-1}, a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right) \\
&\quad\geq F_\alpha\left(a_{r-2}, a_{r-1}, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))\beta(F_\alpha(a_{r-2}, a_{r-1}, t))}\right) \\
&\quad\vdots \\
&\quad\geq F_\alpha\left(a_0, a_1, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t)).\beta(F_\alpha(a_{r-2}, a_{r-1}, t))\ldots\beta(F_\alpha(a_0, a_1, t))}\right)
\end{aligned}$$

By adopting the same procedure used in Theorem (4.2.2) after Inequality (4.8), we can complete the proof. □

Remark 4.2.17.

The same result can be obtained in FBMS and FMS by setting $\alpha(\xi, \varrho) = b \geq 1$ and $\alpha(\xi, \varrho) = 1$ in Theorem 4.2.16 respectively.

Theorem 4.2.18.

Let $(W, F_\alpha, *)$ be a G-complete EFBMS and $\alpha\colon W \times W \to [1, \infty)$ be a mapping.

Let $\Gamma\colon W \to W$ mapping satisfying the condition

$$F_\alpha(\Gamma\xi, \Gamma\varrho, \beta(F_\alpha(\xi,\varrho,t))t) \geq \lambda(\xi,\varrho,t) * \gamma(\xi,\varrho,t), \tag{4.22}$$

where,

$$\left.\begin{cases} \lambda(\xi,\varrho,t) & = \min\{F_\alpha(\Gamma\xi,\Gamma\varrho,t), F_\alpha(\xi,\Gamma\xi,t), F_\alpha(\varrho,\Gamma\varrho,t), F_\alpha(\xi,\varrho,t)\} \\ \gamma(\xi,\varrho,t) & = \max\{F_\alpha(\xi,\Gamma\varrho,t), F_\alpha(\Gamma\xi,\varrho,t)\} \end{cases}\right\} \tag{4.23}$$

for all $\xi,\varrho \in W$, $\beta \in \mathfrak{F}_b$ and $\alpha(\xi,\varrho)\beta(F_\alpha(\xi,\varrho,t)) < 1$. Then Γ has a unique fixed point, where $a * b = \min(a,b)$

Proof.

Starting in the same way as in Theorem 4.2.2, we have

$$\begin{aligned} F_\alpha(a_r,a_{r+1},t) &= F_\alpha(\Gamma a_{r-1},\Gamma a_r,t) \\ &\geq \lambda\left(a_{r-1},a_r,\frac{t}{\beta(a_{r-1},a_r,t)}\right) * \gamma\left(a_{r-1},a_r,\frac{t}{\beta(a_{r-1},a_r,t)}\right). \end{aligned} \tag{4.24}$$

Now

$$\begin{aligned} &\lambda\left(a_{r-1},a_r,\frac{t}{\beta(a_{r-1},a_r,t)}\right) \\ &= \min\Bigg\{F_\alpha\left(\Gamma a_{r-1},\Gamma a_r,\frac{t}{\beta(F_\alpha(a_{r-1},a_r,t))}\right), F_\alpha\left(a_{r-1},\Gamma a_{r-1},\frac{t}{\beta(a_{r-1},a_r,t)}\right) \\ &\qquad, F_\alpha\left(a_r,\Gamma a_r,\frac{t}{\beta(F_\alpha(a_{r-1},a_r,t))}\right), F_\alpha\left(a_{r-1},a_r,\frac{t}{\beta(F_\alpha(a_{r-1},a_r,t))}\right)\Bigg\} \\ &= \min\Bigg\{F_\alpha\left(a_r,a_{r+1},\frac{t}{\beta(F_\alpha(a_{r-1},a_r,t))}\right), F_\alpha\left(a_{r-1},a_r,\frac{t}{\beta(F_\alpha(a_{r-1},a_r,t))}\right) \\ &\qquad, F_\alpha\left(a_r,a_{r+1},\frac{t}{\beta(F_\alpha(a_{r-1},a_r,t))}\right), F_\alpha\left(a_{r-1},a_r,\frac{t}{\beta(F_\alpha(a_{r-1},a_r,t))}\right)\Bigg\} \\ &= \min\left\{F_\alpha\left(a_r,a_{r+1},\frac{t}{\beta(F_\alpha(a_{r-1},a_r,t))}\right), F_\alpha\left(a_{r-1},a_r,\frac{t}{\beta(F_\alpha(a_{r-1},a_r,t))}\right)\right\} \end{aligned}$$

$$\lambda\left(a_{r-1},a_{r-1},\frac{t}{\beta(a_{r-1},a_r,t)}\right)$$
$$=\min\left\{F_\alpha\left(a_r,a_{r+1},\frac{t}{\beta(F_\alpha(a_{r-1},a_r,t))}\right),F_\alpha\left(a_{r-1},a_r,\frac{t}{\beta(F_\alpha(a_{r-1},a_r,t))}\right)\right\} \tag{4.25}$$

$$\gamma\left(a_{r-1},a_r,\frac{t}{\beta(a_{r-1},a_r,t)}\right)$$
$$=\max\left\{F_\alpha\left(a_{r-1},\Gamma a_r,\frac{t}{\beta(F_\alpha(a_{r-1},a_r,t))}\right),F_\alpha\left(\Gamma a_{r-1},a_r,\frac{t}{\beta(F_\alpha(a_{r-1},a_r,t))}\right)\right\}$$
$$=\max\left\{F_\alpha\left(a_{r-1},a_{r+1},\frac{t}{\beta(F_\alpha(a_{r-1},a_r,t))}\right),F_\alpha\left(a_r,a_r,\frac{t}{\beta(F_\alpha(a_{r-1},a_r,t))}\right)\right\}$$
$$=\max\left\{F_\alpha\left(a_{r-1},a_{r+1},\frac{t}{\beta(F_\alpha(a_{r-1},a_r,t))}\right),1\right\}$$

$$\gamma\left(a_{r-1},a_r,\frac{t}{\beta(F_\alpha(a_{r-1},a_r,t))}\right)=1. \tag{4.26}$$

Using (4.25) and (4.26) in (4.24) we have

$$F_\alpha(a_r,a_{r+1},t)$$
$$\geq\min\left\{F_\alpha\left(a_r,a_{r+1},\frac{t}{\beta(F_\alpha(a_{r-1},a_r,t))}\right),F_\alpha\left(a_{r-1},a_r,\frac{t}{\beta(F_\alpha(a_{r-1},a_r,t))}\right)\right\}*1$$
$$\geq\min\left\{F_\alpha\left(a_r,a_{r+1},\frac{t}{\beta(F_\alpha(a_{r-1},a_r,t))}\right),F_\alpha\left(a_{r-1},a_r,\frac{t}{\beta(F_\alpha(a_{r-1},a_r,t))}\right)\right\}. \tag{4.27}$$

If

$$\min\left\{F_\alpha\left(a_r,a_{r+1},\frac{t}{\beta(F_\alpha(a_{r-1},a_r,t))}\right),F_\alpha\left(a_{r-1},a_r,\frac{t}{\beta(F_\alpha(a_{r-1},a_r,t))}\right)\right\}$$
$$=F_\alpha\left(a_r,a_{r+1},\frac{t}{\beta(F_\alpha(a_{r-1},a_r,t))}\right),$$

then (4.27) implies

$$F_\alpha(a_r,a_{r+1},t)\geq F_\alpha\left(a_r,a_{r+1},\frac{t}{\beta(F_\alpha(a_{r-1},a_r,t))}\right).$$

Therefore, there is nothing to prove by Lemma 4.2.1.

If

$$\min\left\{F_\alpha\left(a_r, a_{r+1}, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right), F_\alpha\left(a_{r-1}, a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)\right\}$$
$$= F_\alpha\left(a_{r-1}, a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)$$

then from (4.27), we have

$$F_\alpha(a_r, a_{r+1}, t) \geq F_\alpha\left(a_{r-1}, a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right).$$

Continuing in this way we will get

$$\begin{aligned}
&F_\alpha(a_r, a_{r+1}, t)\\
&\quad\geq F_\alpha\left(a_{r-1}, a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)\\
&\quad\geq F_\alpha\left(a_{r-2}, a_{r-1}, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))\beta(F_\alpha(a_{r-2}, a_{r-1}, t))}\right)\\
&\quad\vdots\\
&\quad\geq F_\alpha\left(a_0, a_1, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t)).\beta(F_\alpha(a_{r-2}, a_{r-1}, t))\ldots\beta(F_\alpha(a_0, a_1, t))}\right).
\end{aligned}$$

By the same procedure used in Theorem (4.2.2) after Inequality (4.8), we can complete the proof. □

Remark 4.2.19.

The special cases of Theorem 4.2.18 can be obtained for FBMS and FMS by setting

$$\alpha(\xi, \varrho) = b \geq 1 \quad \text{and} \quad \alpha(\xi, \varrho) = 1.$$

Theorem 4.2.20.

Let $(W, F_\alpha, *)$ be a G-complete EFBMS and $\alpha\colon W \times W \to [1, \infty)$ be a mapping. Let $\Gamma\colon W \to W$ be a mapping satisfying the condition

$$F_\alpha(\Gamma\xi, \Gamma\varrho, \beta(F_\alpha(\xi, \varrho, t))t) \geq \frac{\lambda(\xi, \varrho, t) * \gamma(\xi, \varrho, t)}{\delta(\xi, \varrho, t)}, \tag{4.28}$$

where

$$\begin{cases} \lambda(\xi,\varrho,t) &= \min\{F_\alpha(\Gamma\xi,\Gamma\varrho,t).F_\alpha(\xi,\varrho,t),F_\alpha(\xi,\Gamma\xi,t).F_\alpha(\varrho,\Gamma\varrho,t)\} \\ \gamma(\xi,\varrho,t) &= \max\{F_\alpha(\xi,\Gamma\xi,t).F_\alpha(\xi,\Gamma\varrho,t),(F_\alpha(\varrho,\Gamma\xi,t))^2\} \\ \delta(\xi,\varrho,t) &= \max\{F_\alpha(\xi,\Gamma\xi,t),F_\alpha(\varrho,\Gamma\varrho,t)\} \end{cases} \tag{4.29}$$

$\forall\, \xi,\varrho \in W,\ \beta \in \mathfrak{F}_b$ and $\alpha(\xi,\varrho)\beta(F_\alpha(\xi,\varrho,t)) < 1$. Then Γ has a unique fixed point.

Proof.

In the same way as in Theorem 4.2.2, we have

$$\begin{aligned} F_\alpha(a_r,a_{r+1},t) &= F_\alpha(\Gamma a_{r-1},\Gamma a_r,t) \\ &\geq \frac{\lambda\left(a_{r-1},a_r,\dfrac{t}{\beta(F_\alpha(a_{r-1},a_r,t))}\right)*\gamma\left(a_{r-1},a_r,\dfrac{t}{\beta(F_\alpha(a_{r-1},a_r,t))}\right)}{\delta\left(\xi,\varrho,\dfrac{t}{\beta(F_\alpha(a_{r-1},a_r,t))}\right)}. \end{aligned} \tag{4.30}$$

$$\begin{aligned} &\lambda\left(a_{r-1},a_r,\frac{t}{\beta(F_\alpha(a_{r-1},a_r,t))}\right) \\ &= \min\Bigg\{F_\alpha\left(\Gamma a_{r-1},\Gamma a_r,\frac{t}{\beta(F_\alpha(a_{r-1},a_r,t))}\right).F_\alpha\left(a_{r-1},a_r,\frac{t}{\beta(F_\alpha(a_{r-1},a_r,t))}\right), \\ &\qquad F_\alpha\left(a_{r-1},\Gamma a_{r-1},\frac{t}{\beta(F_\alpha(a_{r-1},a_r,t))}\right).F_\alpha(a_r,\Gamma a_r,\frac{t}{\beta(F_\alpha(a_{r-1},a_r,t))})\Bigg\} \\ &= \min\Bigg\{F_\alpha\left(a_r,a_{r+1},\frac{t}{\beta(F_\alpha(a_{r-1},a_r,t))}\right).F_\alpha\left(a_{r-1},a_r,\frac{t}{\beta(F_\alpha(a_{r-1},a_r,t))}\right), \\ &\qquad F_\alpha\left(a_{r-1},a_r,\frac{t}{\beta(F_\alpha(a_{r-1},a_r,t))}\right).F_\alpha\left(a_r,a_{r+1},\frac{t}{\beta(F_\alpha(a_{r-1},a_r,t))}\right)\Bigg\} \end{aligned}$$

$$\begin{aligned} &\lambda\Big(a_{r-1},a_r,\frac{t}{\beta(F_\alpha(a_{r-1},a_r,t))}\Big) \\ &\qquad = F_\alpha\left(a_r,a_{r+1},\frac{t}{\beta(F_\alpha(a_{r-1},a_r,t))}\right).F_\alpha\left(a_{r-1},a_r,\frac{t}{\beta(F_\alpha(a_{r-1},a_r,t))}\right) \end{aligned} \tag{4.31}$$

$$\begin{aligned}
&\gamma\left(a_{r-1}, a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right) \\
&= \max\left\{F_\alpha\left(a_{r-1}, \Gamma a_{r-1}, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right).F_\alpha\left(a_{r-1}, \Gamma a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right),\right. \\
&\qquad \left.(F_\alpha\left(a_r, \Gamma a_{r-1}, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))})\right)^2\right\} \\
&= \max\left\{F_\alpha\left(a_{r-1}, a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right).F_\alpha\left(a_{r-1}, a_{r+1}, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right),\right. \\
&\qquad \left.(F_\alpha\left(a_r, a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))})\right)^2\right\} \\
&= \max\left\{F_\alpha\left(a_{r-1}, a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right).F_\alpha\left(a_{r-1}, a_{r+1}, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right), 1\right\} \\
&= 1.
\end{aligned}$$

So, we have

$$\gamma\left(a_{r-1}, a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right) = 1. \tag{4.32}$$

$$\begin{aligned}
&\delta\left(a_{r-1}, a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right) \\
&= \max\left\{F_\alpha\left(a_{r-1}, ga_{r-1}, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right), F_\alpha\left(a_r, ga_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)\right\} \\
&= \max\left\{F_\alpha\left(a_{r-1}, a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right), F_\alpha\left(a_r, a_{r+1}, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)\right\}
\end{aligned} \tag{4.33}$$

Using (4.31), (4.32) and (4.33) in (4.30), we have

$$\begin{aligned}
&F_\alpha\Big(a_r, a_{r+1}, t\Big) \\
&\geq \frac{F_\alpha\left(a_r, a_{r+1}, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right).F_\alpha\left(a_{r-1}, a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)}{\max\left\{F_\alpha\left(a_{r-1}, a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right), F_\alpha\left(a_r, a_{r+1}, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)\right\}}.
\end{aligned} \tag{4.34}$$

If

$$\max\left\{F_\alpha\left(a_r, a_{r+1}, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right), F_\alpha\left(a_{r-1}, a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)\right\}$$
$$= F_\alpha\left(a_{r-1}, a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)$$

then (4.34) implies

$$F_\alpha(a_r, a_{r+1}, t) \geq F_\alpha\left(a_r, a_{r+1}, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right).$$

Therefore, there is nothing to prove by Lemma 4.2.1.

If

$$\max\left\{F_\alpha\left(a_r, a_{r+1}, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right), F_\alpha\left(a_{r-1}, a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)\right\}$$
$$= F_\alpha\left(a_r, a_{r+1}, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right)$$

then from (4.34), we have

$$F_\alpha(a_r, a_{r+1}, t) \geq F_\alpha\left(a_{r-1}, a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right).$$

Continuing in this way we will get

$$\begin{aligned} &F_\alpha(a_r, a_{r+1}, t) \\ &\quad \geq F_\alpha\left(a_{r-1}, a_r, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t))}\right) \\ &\quad \geq \ldots \geq F_\alpha\left(a_0, a_1, \frac{t}{\beta(F_\alpha(a_{r-1}, a_r, t)).\beta(F_\alpha(a_{r-2}, a_{r-1}, t)) \ldots \beta(F_\alpha(a_0, a_1, t))}\right) \end{aligned}$$

By adopting the same procedure used in Theorem (4.2.2) after Inequality (4.8), we can complete the proof.

□

Remark 4.2.21.

The special cases of Theorem 4.2.20 can be obtained for FBMS and FMS by setting $\alpha(\xi, \varrho) = b \geq 1$ and $\alpha(\xi, \varrho) = 1$ respectively.

4.3 Application

Fixed point theory has shown to be an effective method for determining the presence of solutions to various types of integral and differential equations, (see [102, 103] and references therein). We now show an application of Theorem 4.2.2. As an application of the fixed point result, non-linear integral equation has been explored for the existence of a solution.

Consider $W = C[0, I]$, the set of real valued continuous functions on $[0, I]$ and define a G-complete EFBM $F_\alpha \colon W \times W \times [0, \infty) \to [0, 1]$ by

$$F_\alpha(a_1, a_2, t) = \begin{cases} e^{-\frac{\sup\limits_{r\in[0,I]} |a_1(r) - a_2(r)|^2}{t}} & \text{if } t > 0 \\ 0 & \text{if } t = 0 \end{cases}$$

with

$$\alpha(a_1, a_2) = 1 + a_1 + a_2.$$

Consider the integral equation

$$a_1(s) = f(s) + \int_0^I h(s, r)F(s, r, a_1(r))dr, \tag{4.35}$$

where, $I > 0$, $f \colon [0, I] \to \mathbb{R}$, $h \colon [0, I] \times [0, I] \to \mathbb{R}$, and $F \colon [0, I] \times [0, I] \times R \to \mathbb{R}$ are continuous functions.

Theorem 4.3.1.

Suppose that the following conditions holds:

(i) for all $s, r \in [0, I]$, $a_1, a_2 \in W$ and $\beta \in \mathfrak{F}_b$, we have

$$|F(s, r, a_1(r)) - F(s, r, a_2(r))| < \sqrt{\beta(F_\alpha(a_1, a_2, t))}|a_1(r) - a_2(r)|$$

(ii) for all $s, r \in [0, I]$,

$$\sup_{r\in[0,I]} \int_0^I (h(s, r))^2 dr \le \frac{1}{I}.$$

Then the integral equation (4.35) has a solution in W.

Proof.

Define $\Gamma : W \to W$ by

$$\Gamma a_1(s) = f(s) + \int_0^I h(s,r)F(s,r,a_1(r))dr,$$

$\forall\ a_1 \in W$, and $s, r \in [0, I]$.

Now for all $a_1, a_2 \in W$ and by using Conditions (i) and (ii), we have

$$\begin{aligned}
F_\alpha(\Gamma a_1, \Gamma a_2, \beta(F_\alpha(a_1,a_2,t))t) &= e^{-\frac{\sup_{r\in[0,I]} |\Gamma a_1(r) - \Gamma a_2(r)|^2}{\beta(F_\alpha(a_1,a_2,t))t}} \\
&= e^{-\frac{\sup_{r\in[0,I]} |\int_0^I h(s,r)F(s,r,a_1(r))dr - \int_0^I h(s,r)F(s,r,a_2(r))dr|^2}{\beta(F_\alpha(a_1,a_2,t))t}} \\
&= e^{-\frac{\sup_{r\in[0,I]} |\int_0^I h(s,r)\{F(s,r,a_1(r)) - F(s,r,a_2(r)\}dr|^2}{\beta(F_\alpha(a_1,a_2,t))t}} \\
&\geq e^{-\frac{\sup_{r\in[0,I]} |\int_0^I (h(s,r))^2 dr| \int_0^I |F(s,r,a_1(r)) - F(s,r,a_2(r)|^2 dr}{\beta(F_\alpha(a_1,a_2,t))t}} \\
&\geq e^{-\frac{\sup_{r\in[0,I]} \frac{1}{I}\int_0^I \{\sqrt{\beta(F_\alpha(a_1,a_2,t))}|a_1(r) - a_2(r)|\}^2 dr}{\beta(F_\alpha(a_1,a_2,t))t}} \\
&\geq e^{-\frac{\sup_{r\in[0,I]} \beta(F_\alpha(a_1,a_2,t))|a_1(r) - a_2(r)|^2}{\beta(F_\alpha(a_1,a_2,t))t}} \\
&\geq e^{-\frac{\sup_{r\in[0,I]} |a_1(r) - a_2(r)|^2}{t}} \\
&= F_\alpha(a_1,a_2,t)
\end{aligned}$$

$$\Rightarrow \quad F_\alpha(\Gamma a_1, \Gamma a_2, \beta(F_\alpha(a_1,a_2,t))t) \geq F_\alpha(a_1,a_2,t).$$

Hence Γ has a fixed point. □

Chapter 5

Multivalued Mapping in Fuzzy Abstract Space

In this chapter, fixed point results for multivalued mappings in fuzzy abstract spaces are investigated. These results are more general than the existing results in literature. To support the results, an application for the existence of the solution of integral inclusion is established.

5.1 Introduction

In 1928, von Neumann in [109] initiated the study of fixed point for multivalued mappings. The development of geometric fixed point theory for multivalued mapping was implimented with the work of Nadler [110]. He combined the ideas of multivalued mapping and Lipschitz mapping and used the concept of Hausdorff metric to establish the multivalued contraction principle, usually referred as Nadler's contraction mapping principle. Several researches generalized the concept of Nadler's contraction mapping principle [80–84]. The notion of hausdorff fuzzy metric on compact set is introduced by J. Rodríguez-López and S. Romaguera in [111] and recently studied by Shahzad et.al [79] to established fixed point theorems for multivalued mappings in complete FMS.

The notions of Hausdorff fuzzy b-metric space(HFBMS) and Hausdorff extended fuzzy b-metric spaces are defined by using the idea of FBM and EFBM space and some fixed point results for multivalued mappings in HFBMS and HEFBMS are proved.

Recall that a mapping Γ is said to be multivalued mapping on W if Γ is a function from W to the power set of $\mathcal{Y}$, where W and $\mathcal{Y}$ are nonempty sets, the multivalued mapping is denoted by $\Gamma : W \longrightarrow P(\mathcal{Y})$, here $P(\mathcal{Y})$ denote the power set of $\mathcal{Y}$.

Example 5.1.1.

Let $\mathcal{W} = [0,1]$ and $P(\mathcal{W}) = \{\mathcal{B} \subset \mathcal{W} : \mathcal{B} \neq \emptyset\}$. Define a mapping $\Gamma : \mathcal{W} \to P(\mathcal{W})$ and $S : \mathcal{W} \to P(\mathcal{W})$ by

$$\Gamma(\xi) = [0, \xi]$$

also

$$S(\xi) = \begin{cases} \{0\} & \text{if } \xi = 0 \\ \left\{0, \dfrac{\xi}{2}\right\} & \text{otherwise} \end{cases}$$

Example 5.1.2.

Let $W = [0,1]$ and $g : [0,1] \to [0,1]$ defined by

$$g(\xi) = \begin{cases} \dfrac{1}{2}\xi + \dfrac{1}{2} & \text{if } \ 0 \leq \xi \leq \dfrac{1}{2} \\ -\dfrac{1}{2\xi} + 1 & \text{if } \ \dfrac{1}{2} \leq \xi \leq 1 \end{cases}$$

Define a mapping $\Gamma : W \to P(W)$ by

$$\Gamma(\xi) = \{0\} \cup \{\{g(\xi)\}\}$$

for each $\xi \in W$. Then Γ is multivalued mapping.

The fuzzy distance in [79] is defined as

Definition 5.1.1.

Let $(W, F, *)$ be a FMS and $\mathcal{B}$ be any non empty subset of W then, the fuzzy distance $\mathcal{F}$ of an element $\varrho_1 \in W$ and the subset $\mathcal{B} \subset W$ is defined as

$$\mathcal{F}(\varrho_1, \mathcal{B}, t) = \sup\{F(\varrho_1, \varrho_2, t) \colon \varrho_2 \in \mathcal{B}\} \; \forall \, t > 0.$$

Note that $\mathcal{F}(\varrho_1, \mathcal{B}, t) = \mathcal{F}(\mathcal{B}, \varrho_1, t)$.

Definition 5.1.2.

Let $(W, F, *)$ be a FMS. Define a function $\mathcal{H}_{\mathcal{F}}$ on $\hat{C}_0(W) \times \hat{C}_0(W) \times (0, \infty)$ by,

$$\mathcal{H}_{\mathcal{F}}(\mathcal{A}, \mathcal{B}, t) = \min\{ \inf_{\varrho_1 \in \mathcal{A}} \mathcal{F}(\varrho_1, \mathcal{B}, t), \inf_{\varrho_2 \in \mathcal{B}} \mathcal{F}(\mathcal{A}, \varrho_2, t)\}$$

for all $\mathcal{A}, \mathcal{B} \in \hat{C}_0(W)$ and $t > 0$, where $\hat{C}_0(W)$ is the collection of all nonempty compact subsets of W.

Definition 5.1.3.

Let $(W, F, *)$ be a FMS. Then

$$\mathcal{B}(\xi, \eta, t) = \{\varrho \in W : F(\xi, \varrho, t) < 1 - \eta\}$$

and

$$\mathcal{B}(\xi, \eta, t) = \{\varrho \in W : F(\xi, \varrho, t) \geq 1 - \eta\}$$

are open and closed balls respectively in FMS with radius η; $0 < \eta < 1$ and center $\xi \in W$.

5.2 Multivalued Map in Fuzzy b-Metric Space

The Definitions 5.1.1 and 5.1.2 can easily be extended in FBMS as follows

$$\mathcal{F}_b(\varrho_1, \mathcal{B}, t) = \sup \left\{ F_b(\varrho_1, \varrho_2, t) \colon \varrho_2 \in \mathcal{B} \; \forall \, t > 0, \right\}$$

and

$$\mathcal{H}_{\mathcal{F}_b}(\mathcal{A}, \mathcal{B}, t) = \min \left\{ \inf_{\varrho_1 \in \mathcal{A}} \mathcal{F}_b(\varrho_1, \mathcal{B}, t), \inf_{\varrho_2 \in \mathcal{B}} \mathcal{F}_b(\mathcal{A}, \varrho_2, t) \; \forall \, t > 0. \right\}$$

Example 5.2.1.

Let $(W, F_b, *)$ be a FBMS and $W = \{1, 2, 3\}$. Define a function $F_b : W \times W \times [0, \infty) \to [0, 1]$ by

$$F_b(\xi, \varrho, t) = \frac{t}{t + (\xi - \varrho)^2}.$$

Take $\mathcal{B} = \{2, 3\}$ and choose $\varrho_1 = 1$ then

$$\begin{aligned}
\mathcal{F}_b(1, \mathcal{B}, t) &= F_b(1, \{2, 3\}, t) \\
&= \sup\{F_b(1, 2, t), F_b(1, 3, t)\} \\
&= \sup\left\{\frac{t}{t+1}, \frac{t}{t+4}\right\} \\
&= \frac{t}{t+1}.
\end{aligned}$$

Similarly,

$$\mathcal{F}_b(2, \mathcal{B}, t) = \mathcal{F}_b(3, \mathcal{B}, t) = 1$$

Example 5.2.2.

Let $(W, F_b, *)$ be a FBMS and $W = \{0, 1, 2\}$. Define a function $F_b : W \times W \times [0, \infty) \to [0, 1]$ by

$$F_b(\xi, \varrho, t) = \frac{t}{t + (\xi - \varrho)^2}.$$

Let $\mathcal{A} = \{0, 1\}$ and $B = \{1, 2\}$ be the subsets of W. Now

$$\mathcal{H}_{\mathcal{F}_b}(\mathcal{A}, \mathcal{B}, t) = \min\left\{ \inf_{\varrho_1 \in \mathcal{A}} \mathcal{F}_b(\varrho_1, \mathcal{B}, t), \inf_{\varrho_2 \in \mathcal{B}} \mathcal{F}_b(\mathcal{A}, \varrho_2, t)\right\} \tag{5.1}$$

For each $\varrho_1 \in \mathcal{A}$ and $\mathcal{B} = \{2, 3\}$, we have

$$\begin{aligned}
\inf_{\varrho_1 \in \mathcal{A}} \mathcal{F}_b(\varrho_1, \mathcal{B}, t) &= \inf\Big\{\mathcal{F}_b(0, \{1, 2\}, t), \mathcal{F}_b(1, \{1, 2\}, t)\Big\} \\
&= \inf\Big\{ \sup\{F_b(0, 1, t), F_b(0, 2, t)\}, \sup\{F_b(1, 1, t), F_b(1, 2, t)\}\Big\} \\
&= \inf\left\{\sup\{\frac{t}{t+1}, \frac{t}{t+2}\}, \sup\{1, \frac{t}{t+1}\}\right\} \\
&= \inf\left\{\frac{t}{t+1}, 1\right\} \\
&= \frac{t}{t+1}
\end{aligned}$$

Similarly,

$$\begin{aligned}
\inf_{\varrho_2\in\mathcal{B}} \mathcal{F}_b(\mathcal{A},\varrho_2,t) &= \inf\{\mathcal{F}_b(\{0,1\},1,t),\mathcal{F}_b(\{0,1\},2,t)\}\\
&= \inf\{\sup\{F_b(0,1,t),F_b(1,1,t)\},\sup\{F_b(0,2,t),F_b(1,2,t)\}\}\\
&= \inf\left\{\sup\{\frac{t}{t+1},1\},\sup\{\frac{t}{t+2},\frac{t}{t+1}\}\right\}\\
&= \inf\left\{1,\frac{t}{t+1}\right\}\\
&= \frac{t}{t+1}
\end{aligned}$$

So from (5.1), we conclude that

$$\begin{aligned}
\mathcal{H}_{\mathcal{F}_b}(\mathcal{A},\mathcal{B},t) &= \min\{\inf_{\varrho_1\in\mathcal{A}} \mathcal{F}_b(\varrho_1,\mathcal{B},\alpha),\inf_{\varrho_2\in\mathcal{B}} \mathcal{F}_b(\mathcal{A},\varrho_2,t)\}\\
&= \min\left\{\frac{t}{t+1},\frac{t}{t+1}\right\} = \frac{t}{t+1}
\end{aligned}$$

The following lemmas will be used in the proof of upcoming theorems.

Lemma 5.2.1.

If $\mathcal{A}\in C\mathcal{B}(W)$, then $\xi\in\mathcal{A}$ if and only if

$$\mathcal{F}_b(\mathcal{A},\xi,t)=1\ \forall\ t>0,$$

where $C\mathcal{B}(W)$ is closed bounded subset of W.

Proof.

Since

$$\mathcal{F}_b(\mathcal{A},\xi,t)=\sup\{F_b(\varrho,\xi,t)\colon\ \varrho\in\mathcal{A}\}=1,$$

there exists a sequence $\{\varrho_n\}\subset\mathcal{A}$ such that

$$F_b(\xi,\varrho_n,t)>1-\frac{1}{n}.$$

Letting $n\to\infty$, we get $\varrho_n\to\xi$.

From $\mathcal{A}\in C\mathcal{B}(W)$, it follows that $\xi\in\mathcal{A}$.

Conversely, if $\xi \in \mathcal{A}$, we have

$$\mathcal{F}_b(\mathcal{A}, \xi, t) = \sup\{F_b(\xi, \varrho, t) \colon \varrho \in \mathcal{A}\}$$
$$> F_b(\xi, \xi, t) = 1$$

□

Again following [104], the following fact follows from $FBM5$.

Lemma 5.2.2.

Let $(W, F_b, *)$ be a G-complete FBMS. If for two elements $\xi \in W$ and for a number $k < 1$,

$$F_b(\xi, \varrho, kt) \geq F_b\,(\xi, \varrho, t)$$

then $\xi = \varrho$.

Lemma 5.2.3.

Let $(W, F_b, *)$ be a G-complete FBMS, such that $(\hat{C}_0, \mathcal{H}_{\mathcal{F}_b}, *)$ is a HFBM on $\hat{C}_0$. Then for all $A, B \in \hat{C}_0$ and for each $\xi \in \mathcal{A}$ and for $t > 0$, there exists an element $\varrho_\xi \in \mathcal{B}$ satisfying $\mathcal{F}_b(\xi, \mathcal{B}, t) = F_b(\xi, \varrho_\xi, t)$, then

$$\mathcal{H}_{\mathcal{F}_b}(\mathcal{A}, \mathcal{B}, t) \leq F_b(\xi, \varrho_\xi, t).$$

Proof.

If

$$\mathcal{H}_{\mathcal{F}_b}(\mathcal{A}, \mathcal{B}, t) = \inf_{\xi \in \mathcal{A}} \mathcal{F}_b(\xi, \mathcal{B}, t).$$

then

$$\mathcal{H}_{\mathcal{F}_b}(\mathcal{A}, \mathcal{B}, t) \leq \mathcal{F}_b(\xi, \mathcal{B}, t).$$

Since for each $\xi \in \mathcal{A}$ there exists $\varrho_\xi \in \mathcal{B}$ satisfying

$$\mathcal{F}_b(\xi, \mathcal{B}, t) = F_b(t, \varrho_\xi, t).$$

Hence

$$\mathcal{H}_{\mathcal{F}_b}(\mathcal{A}, \mathcal{B}, t) \leq F_b(\xi, \varrho_\xi, t).$$

Now if

$$\mathcal{H}_{\mathcal{F}_b}(\mathcal{A}, \mathcal{B}, t) = \inf_{\varrho \in \mathcal{B}} \mathcal{F}_b(\mathcal{A}, \varrho, t)$$

$$\leq \inf_{\xi \in \mathcal{A}} \mathcal{F}_b(\xi, \mathcal{A}, t)$$

$$\leq \mathcal{F}_b(\xi, \mathcal{B}, t) = F_b(\xi, \varrho_\xi, t)$$

This implies

$$\mathcal{H}_{\mathcal{F}_b}(\mathcal{A}, \mathcal{B}, t) \leq F_b(\xi, \varrho_\xi, t)$$

for some $\varrho_\xi \in \mathcal{B}$. Hence in both cases result is proved. □

Theorem 5.2.4.

Let $(W, F_b, *)$ be a G-complete FBMS with $b \geqslant 1$ and $\mathcal{H}_{\mathcal{F}_b}$ be a HFBM. Let $S\colon W \to \hat{C}_0(W)$ be a multivalued mapping satisfying

$$\mathcal{H}_{\mathcal{F}_b}(S\xi, S\varrho, kt) \geq F_b(\xi, \varrho, t) \tag{5.2}$$

$\forall\ \xi, \varrho \in W,\ k \in (0, 1)$ and $bk < 1$, then S has a fixed point.

Proof.

For $a_0 \in W$, we choose a sequence $\{a_r\}$ in W as follows:

Let $a_1 \in W$ such that $a_1 \in Sa_0$ by using Lemma 5.2.3, we can choose $a_2 \in Sa_1$ such that

$$F_b(a_1, a_2, t) \geqslant \mathcal{H}_{\mathcal{F}_b}(Sa_0, Sa_1, t)$$

for all $t > 0$.

By induction, we have $a_{r+1} \in Sa_r$ satisfying

$$F_b(a_r, a_{r+1}, t) \geqslant \mathcal{H}_{\mathcal{F}_b}(Sa_{r-1}, Sa_r, t)\ \forall\ r\ \in \mathbb{N}.$$

Using (5.2) together with Lemma 5.2.3, we have

$$\begin{aligned} F_b(a_r, a_{r+1}, t) &\geq \mathcal{H}_{\mathcal{F}_b}(Sa_{r-1}, Sa_r, t) \\ &\geq F_b\left(a_{r-1}, a_r, \frac{t}{k}\right) \\ &\geq \mathcal{H}_{\mathcal{F}_b}\left(Sa_{r-2}, Sa_{r-1}, \frac{t}{k}\right) \\ &\geq F_b\left(a_{r-2}, a_{r-1}, \frac{t}{k^2}\right) \\ &\vdots \\ &\geq F_b\left(a_1, a_2, \frac{t}{k^{r-1}}\right) \\ &\geq \mathcal{H}_{\mathcal{F}_b}\left(Sa_0, Sa_1, \frac{t}{k^{r-1}}\right) \\ &\geq F_b\left(a_0, a_1, \frac{t}{k^r}\right). \end{aligned} \tag{5.3}$$

For any $q \in \mathbb{N}$, writing

$$q(\frac{t}{q}) = \frac{t}{q} + \frac{t}{q} + \ldots + \frac{t}{q}$$

and using $[FBM4]$ repeatedly,

$$\begin{aligned} &F_b(a_r, a_{r+q}, t) \\ &\quad \geq F_b\left(a_r, a_{r+1}, \frac{t}{qb}\right) * F_b\left(a_{r+1}, a_{r+2}, \frac{t}{qb^2}\right) * F_b\left(a_{r+2}, a_{r+3}, \frac{t}{qb^3}\right) \\ &\quad * \ldots * F_b\left(a_{r+q-1}, a_{r+q}, \frac{t}{qb^q}\right). \end{aligned}$$

Using (5.3) and $[FBM5]$, we get

$$\begin{aligned} &F_b(a_r, a_{r+q}, t) \\ &\quad \geq F_b\left(a_0, a_1, \frac{t}{qbk^r}\right) * F_b\left(a_0, a_1, \frac{t}{qb^2k^{r+1}}\right) * F_b\left(a_0, a_1, \frac{t}{qb^3k^{r+2}}\right) \\ &\quad * \ldots * F_b\left(a_0, a_1, \frac{t}{qb^qk^{r+q-1}}\right) \\ &\quad \geq F_b\left(a_0, a_1, \frac{t}{q(bk)k^{r-1}}\right) * F_b\left(a_0, a_1, \frac{t}{q(bk)^2k^{r-1}}\right) \\ &\quad * F_b\left(a_0, a_1, \frac{t}{q(bk)^3k^{r-1}}\right) * \ldots * F_b\left(a_0, a_1, \frac{t}{q(bk)^qk^{r-1}}\right). \end{aligned}$$

As $r, q \in \mathbb{N}$ and $bk < 1$, taking limit as $r \to \infty$, we get

$$\begin{aligned}\lim_{r\to\infty} F_b(a_r, a_{r+q}, t) &= 1 * 1 * \ldots * 1 \\ &= 1.\end{aligned}$$

Hence $\{a_r\}$ is G-Cauchy sequence. Therefore, G-completeness of W implies that there exists $z \in W$ such that

$$\begin{aligned}F_b(z, Sz, t) &\geq F_b\left(z, a_{r+1}, \frac{t}{2b}\right) * F_b\left(a_{r+1}, Sz, \frac{t}{2b}\right) \\ &\geq F_b\left(z, a_{r+1}, \frac{t}{2b}\right) * \mathcal{H}_{\mathcal{F}_b}\left(Sa_r, Sz, \frac{t}{2b}\right) \\ &\geq F_b\left(z, a_{r+1}, \frac{t}{2b}\right) * F_b\left(a_r, z, \frac{t}{2bk}\right) \longrightarrow 1 \quad \text{as} \quad r \to \infty.\end{aligned}$$

By Lemma 5.2.1 $z \in Sz$. Hence z is fixed point for S. □

Example 5.2.3.

Let $W = [0, 1]$ and define a mapping $F_b \colon W \times W \times (0, \infty) \to [0, 1]$ by

$$F_b(\xi, \varrho, t) = \frac{t}{t + (\xi - \varrho)^2}.$$

Then $(W, F_b, *)$ is a G-complete fuzzy b-metric space with $b = 2$.

For $k \in (0, 1)$, define a mapping $S \colon W \to \hat{C}_0(W)$ by

$$S(\xi) = \begin{cases} \{0\} & \text{if } \xi = 0 \\ \{0, \frac{\sqrt{k}\xi}{2}\} & \text{otherwise} \end{cases}$$

For $\xi = \varrho$,

$$\mathcal{H}_{\mathcal{F}_b}(S\xi, S\varrho, kt) = 1 = F_b(\xi, \varrho, t).$$

For $\xi \neq \varrho$, we have the following cases.

For $\xi = 0$ and $\varrho \in (0,1]$, we have

$$\begin{aligned}
\mathcal{H}_{\mathcal{F}_b}(S(0), S(\varrho), kt) &= \min\left\{ \inf_{a\in S(0)} \mathcal{F}_b(a, S(\varrho), kt), \inf_{b\in S(\varrho)} \mathcal{F}_b(S(0), b, kt)\right\} \\
&= \min\left\{ \inf_{a\in S(0)} \mathcal{F}_b\left(a, \left\{0, \frac{\sqrt{k}\varrho}{2}\right\}, kt\right), \inf_{b\in S(\varrho)} \mathcal{F}_b\left(\{0\}, b, kt\right)\right\} \\
&= \min\left\{ \inf\left\{ \mathcal{F}_b\left(0, \{0, \frac{\sqrt{k}\varrho}{2}\}, kt\right)\right\}, \right. \\
&\qquad \left. \inf\left\{ \mathcal{F}_b\left(\{0\}, 0, kt\right), \mathcal{F}_b\left(\{0\}, \frac{\sqrt{k}\varrho}{2}, kt\right)\right\}\right\} \\
&= \min\left\{ \inf\left\{ \sup\left\{ \mathcal{F}_b(0,0,kt), \mathcal{F}_b(0, \frac{\sqrt{k}\varrho}{2}, kt)\right\}\right\}, \right. \\
&\qquad \left. \inf\left\{ \mathcal{F}_b(0,0,kt), \mathcal{F}_b(0, \frac{\sqrt{k}y}{2}, kt)\right\}\right\} \\
&= \min\left\{ \inf\left\{ \sup\left\{ 1, \frac{t}{t+\frac{\varrho^2}{4}}\right\}\right\}, \inf\left\{ 1, \frac{t}{t+\frac{\varrho^2}{4}}\right\}\right\} \\
&= \min\left\{ \inf\{1\}, \frac{t}{t+\frac{\varrho^2}{4}}\right\} \\
&= \min\{1, \frac{t}{\alpha+\frac{\varrho^2}{4}}\} \\
&= \frac{t}{t+\frac{\varrho^2}{4}}.
\end{aligned}$$

It follows that

$$\mathcal{H}_{\mathcal{F}_b}(S(0), S(\varrho), kt) > F_b(0, \varrho, t) = \frac{t}{t+\varrho^2}.$$

For ξ and $\varrho \in (0,1]$, an easy calculation, with either possibility of supremum and infimum, yields:

$$\begin{aligned}
\mathcal{H}_{\mathcal{F}_b}(S(\xi), S(\varrho), kt) &= \min\left\{ \sup\left\{ \frac{t}{t+\frac{\xi^2}{4}}, \frac{t}{t+\frac{(\xi-\varrho)^2}{4}}\right\}, \sup\left\{ \frac{t}{t+\frac{\varrho^2}{4}}, \frac{t}{t+\frac{(\xi-\varrho)^2}{4}}\right\}\right\} \\
&\geq \frac{t}{t+\dfrac{(\xi-\varrho)^2}{4}} \\
&> \frac{t}{t+(\xi-\varrho)^2} \\
&= F_b(\xi, \varrho, t).
\end{aligned}$$

Thus for all cases, we have

$$\mathcal{H}_{\mathcal{F}_b}(S\xi, S\varrho, kt) \geq F_b(\xi, \varrho, t)$$

Hence all the conditions of Theorem 5.2.4 are satisfied and 0 is a fixed point of S.

Theorem 5.2.5.

Let $(W, F_b, *)$ be a G-complete FBMS with $b \geqslant 1$ and $\mathcal{H}_{\mathcal{F}_b}$ be a HFBM space. Let $S\colon W \to \hat{C}_0(W)$ be a multivalued mapping satisfying

$$\mathcal{H}_{\mathcal{F}_b}(S\xi, S\varrho, kt) \geq \min\left\{\frac{\mathcal{F}_b(\varrho, S\varrho, t)\,[1 + \mathcal{F}_b(\xi, S\varrho, t)]}{1 + F_b(\xi, \varrho, t)}, F_b(\xi, \varrho, t)\right\} \tag{5.4}$$

$\forall\, \xi, \varrho \in W,\ k \in (0, 1)$ and $bk < 1$, then S has a fixed point.

Proof.

In the same way as in Theorem 5.2.4 for $a_0 \in W$, we choose a sequence $\{a_r\}$ in W as follows:

Let $a_1 \in W$ such that $a_1 \in Sa_0$. By using Lemma 5.2.3, we can choose $a_2 \in Sa_1$ such that

$$F_b(a_1, a_2, t) \geqslant \mathcal{H}_{\mathcal{F}_b}(Sa_0, Sa_1, t) \quad \forall\, t > 0.$$

By induction, we have $a_{r+1} \in Sa_r$ satisfying

$$F_b(a_r, a_{r+1}, t) \geqslant \mathcal{H}_{\mathcal{F}_b}(Sa_{r-1}, Sa_r, t) \quad \forall\, r\ \in \mathbb{N}.$$

Now by (5.4) together with Lemma 5.2.3, we have

$$F_b(a_r, a_{r+1}, t) \geq \mathcal{H}_{\mathcal{F}_b}(Sa_{r-1}, Sa_r, t)$$

$$F_b(a_r, a_{r+1}, t) \geq \min\left\{\frac{\mathcal{F}_b\left(a_r, Sa_r, \frac{t}{k}\right)\left[1 + \mathcal{F}_b\left(a_{r-1}, Sa_{r-1}, \frac{t}{k}\right)\right]}{1 + F_b\left(a_{r-1}, a_r, \frac{t}{k}\right)},\right.$$
$$\left. F_b\left(a_{r-1}, a_r, \frac{t}{k}\right)\right\}$$

$$F_b(a_r, a_{r+1}, t) \geq \min \left\{ \frac{F_b\left(a_r, a_{r+1}, \frac{t}{k}\right)\left[1 + F_b\left(a_{r-1}, a_r, \frac{t}{k}\right)\right]}{1 + F_b\left(a_{r-1}, a_r, \frac{t}{k}\right)}, F_b\left(a_{r-1}, a_r, \frac{t}{k}\right) \right\}$$

$$F_b(a_r, a_{r+1}, t) \geq \min \left\{ F_b\left(a_r, a_{r+1}, \frac{t}{k}\right), F_b\left(a_{r-1}, a_r, \frac{t}{k}\right) \right\}. \tag{5.5}$$

If

$$\min \left\{ F_b\left(a_r, a_{r+1}, \frac{t}{k}\right), F_b\left(a_{r-1}, a_r, \frac{t}{k}\right) \right\} = F_b\left(a_r, a_{r+1}, \frac{t}{k}\right).$$

Then (5.5) implies

$$F_b(a_r, a_{r+1}, t) \geq F_b\left(a_r, a_{r+1}, \frac{t}{k}\right).$$

Then nothing to prove by lemma 5.2.2

If

$$\min \left\{ F_b\left(a_r, a_{r+1}, \frac{t}{k}\right), F_b\left(a_{r-1}, a_r, \frac{t}{k}\right) \right\} = F_b\left(a_{r-1}, a_r, \frac{t}{k}\right).$$

Then from (5.5) we have

$$\begin{aligned} F_b(a_r, a_{r+1}, t) &\geq F_b\left(a_{r-1}, a_r, \frac{t}{k}\right) \\ &\geq F_b\left(a_{r-2}, a_{r-1}, \frac{t}{k^2}\right) \\ &\geq F_b\left(a_{r-3}, a_{r-2}, \frac{t}{k^3}\right) \\ &\vdots \\ &\geqslant F_b\left(a_0, a_1, \frac{t}{k^r}\right). \end{aligned}$$

By adopting the same procedure as in Theorem 5.2.4 after inequality (5.3) we can complete the proof. □

Remark 5.2.6.

By taking $b = 1$ in Theorem 5.2.5, we get the main result of [79].

Theorem 5.2.7.

Let $(W, F_b, *)$ be a G-complete FBMS with $b \geqslant 1$ and $\mathcal{H}_{\mathcal{F}_b}$ be a HFBM. Let $S\colon W \to \hat{C}_0(W)$ be a multivalued mapping satisfying

$$\mathcal{H}_{\mathcal{F}_b}(S\xi, S\varrho, kt) \geq \min\left\{\frac{\mathcal{F}_b(\varrho, S\varrho, t)\left[1 + \mathcal{F}_b(\xi, S\xi, t) + \mathcal{F}_b(\varrho, S\xi, t)\right]}{2 + F_b(\xi, \varrho, t)}, F_b(\xi, \varrho, t)\right\} \tag{5.6}$$

$\forall\, \xi, \varrho \in W,\ k \in (0, 1)$ and $bk < 1$, then S has a fixed point.

Proof.

Starting same way as in Theorem 5.2.4 we have

$$F_b(a_1, a_2, t) \geqslant \mathcal{H}_{\mathcal{F}_b}(Sa_0, Sa_1, t) \quad \forall\, t > 0.$$

By induction, we have $a_{r+1} \in Sa_r$ satisfying

$$F_b(a_r, a_{r+1}, t) \geqslant \mathcal{H}_{\mathcal{F}_b}(Sa_{r-1}, Sa_r, t) \quad \forall\, r\ \in \mathbb{N}.$$

Now by (5.6) together with Lemma 5.2.3, we have

$$\begin{aligned}
F_b(a_r, a_{r+1}, t) &\geq \mathcal{H}_{\mathcal{F}_b}(Sa_{r-1}, Sa_r, t)\\
&\geq \min\left\{\frac{\mathcal{F}_b\left(a_r, Sa_r, \frac{t}{k}\right)\left[1 + \mathcal{F}_b\left(a_{r-1}, Sa_{r-1}, \frac{t}{k}\right) + \mathcal{F}_b\left(a_r, Sa_{r-1}, \frac{t}{k}\right)\right]}{2 + F_b\left(a_{r-1}, a_r, \frac{t}{k}\right)},\right.\\
&\qquad\qquad \left. F_b\left(a_{r-1}, a_r, \frac{t}{k}\right)\right\}\\
&\geq \min\left\{\frac{F_b\left(a_r, a_{r+1}, \frac{t}{k}\right)\left[1 + F_b\left(a_{r-1}, a_r, \frac{t}{k}\right) + F_b\left(a_r, a_r, \frac{t}{k}\right)\right]}{2 + F_b\left(a_{r-1}, a_r, \frac{t}{k}\right)},\right.\\
&\qquad\qquad \left. F_b\left(a_{r-1}, a_r, \frac{t}{k}\right)\right\}\\
&\geq \min\left\{\frac{F_b\left(a_r, a_{r+1}, \frac{t}{k}\right)\left[1 + F_b\left(a_{r-1}, a_r, \frac{t}{k}\right) + 1\right]}{2 + F_b\left(a_{r-1}, a_r, \frac{t}{k}\right)}, F_b\left(a_{r-1}, a_r, \frac{t}{k}\right)\right\}
\end{aligned}$$

$$\geq \min\left\{\frac{F_b\left(a_r, a_{r+1}, \frac{t}{k}\right)\left[2 + F_b\left(a_{r-1}, a_r, \frac{t}{k}\right)\right]}{2 + F_b\left(a_{r-1}, a_r, \frac{t}{k}\right)}, F_b\left(a_{r-1}, a_r, \frac{t}{k}\right)\right\}$$

$$\geq \min\left\{F_b\left(a_r, a_{r+1}, \frac{t}{k}), F_b(a_{r-1}, a_r, \frac{t}{k}\right)\right\}.$$

So, we get

$$F_b(a_r, a_{r+1}, t) \geq \min\left\{F_b\left(a_r, a_{r+1}, \frac{t}{k}), F_b(a_{r-1}, a_r, \frac{t}{k}\right)\right\}. \tag{5.7}$$

If

$$\min\left\{F_b\left(a_r, a_{r+1}, \frac{t}{k}\right), F_b\left(a_{r-1}, a_r, \frac{t}{k}\right)\right\} = F_b\left(a_r, a_{r+1}, \frac{t}{k}\right).$$

Then (5.7) implies

$$F_b(a_r, a_{r+1}, t) \geq F_b\left(a_r, a_{r+1}, \frac{t}{k}\right).$$

Then nothing to prove by Lemma 5.2.2.

If

$$\min\left\{F_b\left(a_r, a_{r+1}, \frac{t}{k}\right), F_b\left(a_{r-1}, a_r, \frac{t}{k}\right)\right\} = F_b\left(a_{r-1}, a_r, \frac{t}{k}\right).$$

Then from (5.7) we have

$$\begin{aligned} F_b(a_r, a_{r+1}, t) &\geq F_b\left(a_{r-1}, a_r, \frac{t}{k}\right) \\ &\geq F_b\left(a_{r-2}, a_{r-1}, \frac{t}{k^2}\right) \\ &\geqslant F_b\left(a_{r-3}, a_{r-2}, \frac{t}{k^3}\right) \\ &\vdots \\ &\geqslant F_b\left(a_0, a_1, \frac{t}{k^r}\right). \end{aligned}$$

By adopting the same procedure as in Theorem 5.2.4 after inequality (5.3) we can complete the proof. □

Next the corollary of Theorem 5.2.7 is given.

Corollary 5.2.8.

Let $(W, F, *)$ be a G-complete FMS and $\mathcal{H}_{\mathcal{F}}$ be a HFM. Let $S\colon W \to \hat{C}_0(W)$ be a multivalued mapping satisfying

$$\mathcal{H}_{\mathcal{F}}(S\xi, S\varrho, kt) \geq \min\left\{\frac{\mathcal{F}(\varrho, S\varrho, t)\,[1+\mathcal{F}(\xi, S\xi, t)+\mathcal{F}(\varrho, S\xi, t)]}{2+F(\xi,\varrho,t)}, F(\xi,\varrho,t)\right\}$$

$\forall\, \xi, \varrho \in W,\ k \in (0,1)$. Then S has a fixed point.

Proof.

Taking $b = 1$ in Theorem 5.2.7, one can complete the proof. □

Theorem 5.2.9.

Let $(W, F_b, *)$ be a G-complete FBMS with $b \geqslant 1$ and $\mathcal{H}_{\mathcal{F}_b}$ be a HFBM. Let $S\colon W \to \hat{C}_0(W)$ be a multivalued mapping satisfying

$$\begin{aligned}&\mathcal{H}_{\mathcal{F}_b}(S\xi, S\varrho, kt)\\ &\quad\geq \min\left\{\frac{\mathcal{F}_b(\xi, S\xi, t)\,[1+\mathcal{F}_b(\varrho, S\varrho, t)]}{1+\mathcal{F}_b(S\xi, S\varrho, t)}, \frac{\mathcal{F}_b(\xi, S\varrho, t)\,[1+\mathcal{F}_b(\xi, S\xi, t)]}{1+F_b(\xi,\varrho,t)},\right.\\ &\qquad\left.\frac{\mathcal{F}_b(\xi, S\xi, t)\,[2+\mathcal{F}_b(\xi, S\varrho, t)]}{1+F_b(\xi, S\varrho, t)+\mathcal{F}_b(\varrho, S\xi, t)}, F_b(\xi,\varrho,t)\right\}\end{aligned} \tag{5.8}$$

$\forall\, \xi, \varrho \in W,\ k \in (0,1)$ and $bk < 1$. Then S has a fixed point.

Proof.

For $a_0 \in W$, we choose a sequence $\{x_n\}$ in W as follows;

Let $a_1 \in W$ such that $a_1 \in Sa_0$. By using Lemma 5.2.3, we can choose $a_2 \in Sa_1$ such that

$$F_b(a_1, a_2, t) \geqslant \mathcal{H}_{\mathcal{F}_b}(Sa_0, Sa_1, t) \quad \forall t > 0.$$

By induction we have $a_{r+1} \in Sa_r$ satisfying

$$F_b(a_r, a_{r+1}, t) \geqslant \mathcal{H}_{\mathcal{F}_b}(Sa_{r-1}, Sa_r, t) \quad \forall\, r\ \in \mathbb{N}.$$

Now by (5.8) together with 5.2.3 we have

$$F_b(a_r, a_{r+1}, t) \geq \mathcal{H}_{\mathcal{F}_b}(Sa_{r-1}, Sa_r, t)$$

$$\geq \min\left\{ \frac{\mathcal{F}_b\left(a_{r-1}, Sa_{r-1}, \frac{t}{k}\right)\left[1+\mathcal{F}_b\left(a_r, S_{a_r}, \frac{t}{k}\right)\right]}{1+\mathcal{F}_b\left(Sa_{r-1}, Sa_r, \frac{t}{k}\right)}, \frac{\mathcal{F}_b\left(a_r, Sa_r, \frac{t}{k}\right)\left[1+\mathcal{F}_b\left(a_{r-1}, Sa_{r-1}, \frac{t}{k}\right)\right]}{1+F_b\left(a_{r-1}, a_r, \frac{t}{k}\right)}, \frac{\mathcal{F}_b\left(a_{r-1}, Sa_{r-1}, \frac{t}{k}\right)\left[2+\mathcal{F}_b\left(a_{r-1}, Sa_r, \frac{t}{k}\right)\right]}{1+\mathcal{F}_b\left(a_{r-1}, Sa_r, \frac{t}{k}\right)+\mathcal{F}_b\left(a_r, Sa_{r-1}, \frac{t}{k}\right)}, F_b\left(a_{r-1}, a_r, \frac{t}{k}\right) \right\}$$

$$\geq \min\left\{ \frac{F_b\left(a_{r-1}, a_r, \frac{t}{k}\right)\left[1+F_b\left(a_r, a_{r+1}, \frac{t}{k}\right)\right]}{1+F_b\left(a_r, a_{r+1}, \frac{t}{k}\right)}, \frac{F_b\left(a_r, a_{r+1}, \frac{t}{k}\right)\left[1+F_b\left(a_{r-1}, a_r, \frac{t}{k}\right)\right]}{1+F_b\left(a_{r-1}, a_r, \frac{t}{k}\right)}, \frac{F_b\left(a_{r-1}, a_r, \frac{t}{k}\right)\left[2+F_b\left(a_{r-1}, a_{r+1}, \frac{t}{k}\right)\right]}{1+F_b\left(a_{r-1}, a_{r+1}, \frac{t}{k}\right)+F_b\left(a_r, a_r, \frac{t}{k}\right)}, F_b\left(a_{r-1}, a_r, \frac{t}{k}\right) \right\}$$

$$\geq \min\left\{F_b\left(a_r, a_{r+1}, \frac{t}{k}\right), F_b\left(a_{r-1}, a_r, \frac{t}{k}\right)\right\}.$$

So, we have

$$F_b(a_r, a_{r+1}, t) \geq \min\left\{F_b\left(a_r, a_{r+1}, \frac{t}{k}\right), F_b\left(a_{r-1}, a_r, \frac{t}{k}\right)\right\}. \tag{5.9}$$

If

$$\min\left\{F_b\left(a_r, a_{r+1}, \frac{t}{k}\right), F_b\left(a_{r-1}, a_r, \frac{t}{k}\right)\right\} = F_b\left(a_r, a_{r+1}, \frac{t}{k}\right).$$

Then (5.9) implies

$$F_b(a_r, a_{r+1}, t) \geq F_b\left(a_r, a_{r+1}, \frac{t}{k}\right).$$

Then nothing to prove by Lemma 5.2.2.

If

$$\min\left\{F_b(\left(a_r, a_{r+1}, \frac{t}{k}\right), F_b\left(a_{r-1}, a_r, \frac{t}{k}\right)\right\} = F_b\left(a_{r-1}, a_r, \frac{t}{k}\right).$$

Then from (5.9) we have

$$\begin{aligned} F_b(a_r, a_{r+1}, t) &\geq F_b\left(a_{r-1}, a_r, \frac{t}{k}\right) \\ &\geq F_b\left(a_{r-2}, a_{r-1}, \frac{t}{k^2}\right) \\ &\vdots \\ &\geqslant F_b\left(a_0, a_1, \frac{t}{k^r}\right). \end{aligned}$$

By adopting the same procedure as in Theorem 5.2.4 after inequality (5.3) we can complete the proof. □

An immediate consequence of the above result is stated in the following corollary.

Corollary 5.2.10.

Let $(W, F, *)$ be a G-complete FMS and $\mathcal{H}_{\mathcal{F}}$ be a HFM. Let $S\colon W \to \hat{C}_0(W)$ be a multivalued mapping satisfying

$$\mathcal{H}_{\mathcal{F}}(S\xi, S\varrho, kt) \geq \min\left\{\frac{\mathcal{F}(\xi, S\xi, t)\,[1 + \mathcal{F}(\varrho, S\varrho, t)]}{1 + \mathcal{F}(S\xi, S\varrho, t)}, \frac{\mathcal{F}(\varrho, S\varrho, t)\,[1 + \mathcal{F}(\xi, S_{\xi}, t)]}{1 + F(\xi, \varrho, t)}, \right. \\ \left. \frac{\mathcal{F}(\xi, S\xi, t)\,[2 + \mathcal{F}(\xi, S\varrho, t)]}{1 + \mathcal{F}(\xi, S\varrho, t) + \mathcal{F}(\varrho, S\xi, t)}, F(\xi, \varrho, t)\right\}$$

$\forall\, \xi, \varrho \in W,\ k \in (0, 1)$. Then S has a fixed point.

Proof.

Taking $b = 1$ in Theorem 5.2.9, one can complete the proof. □

5.3 Consequences

In this section, we establish some fixed point theorems involving integral inequalities as consequences of our results. Define a function $\Omega\colon [0,\infty) \to [0,\infty)$ by

$$\Omega(s) = \int_0^s \psi(s)ds \quad \forall \;\; s > 0, \tag{5.10}$$

where $\Omega(s)$ is non-decreasing and continuous function.

Moreover $\psi(s) > 0$ for $s > 0$ and $\psi(s) = 0$ iff $s = 0$.

Theorem 5.3.1.

Let $(W, F_b, *)$ be a complete FBMS and $\mathcal{H}_{\mathcal{F}_b}$ be a HFBM. Let $S\colon W \to \hat{C}_0(W)$ be a multivalued mapping satisfying

$$\int_0^{\mathcal{H}_{\mathcal{F}_b}(S\xi,S\varrho,kt)} \psi(t)dt \geq \int_0^{F_b(\xi,\varrho,t)} \psi(t)dt \tag{5.11}$$

$\forall\, \xi, \varrho \in W,\;\; k \in (0,1)$ and $bk < 1$. Then S has a fixed point.

Proof.

Taking (5.10) in account, (5.11) implies that

$$\Omega(\mathcal{H}_{\mathcal{F}_b}(S\xi, S\varrho, kt)) \geqslant \Omega(F_b(\xi, \varrho, t)).$$

Since Ω is continuous and non-decreasing, so we have

$$\mathcal{H}_{\mathcal{F}_b}(S\xi, S\varrho, kt) \geqslant F_b(\xi, \varrho, t).$$

The rest of the proof follows immediately from Theorem 5.2.4. □

A more general form of Theorem 5.3.1 can be stated an immediate consequence of Theorem 5.2.7 as follows

Theorem 5.3.2.

Let $(W, F_b, *)$ be a complete FBMS and $\mathcal{H}_{\mathcal{F}_b}$ be a HFBM. Let $S\colon W \to \hat{C}_0(W)$ be a multivalued mapping satisfying

$$\int_0^{\mathcal{H}_{\mathcal{F}_b}(S\xi,S\varrho,kt)} \psi(t)dt \geq \int_0^{\beta(\xi,\varrho,t)} \psi(t)dt, \tag{5.12}$$

where

$$\gamma(\xi,\varrho,t) = \min\left\{\frac{\mathcal{F}_b(\varrho,S\varrho,t)\left[1+\mathcal{F}_b(\xi,S\xi,t)+\mathcal{F}_b(\varrho,S\xi,t)\right]}{2+F_b(\xi,\varrho,t)}, F_b(\xi,\varrho,t)\right\}$$

$\forall\, \xi,\varrho \in W,\ k \in (0,1)$ and $bk < 1$, then S has a fixed point.

Proof.
Note that if

$$\gamma(\xi,\varrho,t) = F_b(\xi,\varrho,t),$$

then the above theorem follows from Theorem 5.3.1. □

5.4 Application

Nonlinear integral equations in abstract spaces arise in different fields of physical sciences, engineering, biology, and applied mathematics [112, 113]. The theory of nonlinear integral equations in abstract spaces is a fast growing field with important applications to a number of areas of analysis as well as other branches of science [114]. Fixed point theory is a valuable tool for the analysis of the existence of the solution of different kinds of inclusions such as [114, 115]. Many authors provided the solution of different integeral inclusion in this context, for instance see [116–120].

In this section, Volterra-Type integral inclusion is applied on Theorem 5.2.4.
Consider $W = C([0,1],\mathbb{R})$ be the space of all real valued continuous functions defined on $[0,1]$ and define the G-complete fuzzy b-metric on W by

$$F_b(\xi,\varrho,t) = e^{-\frac{\sup\limits_{u\in[0,1]}|\xi(u)-\varrho(u)|^2}{t}}$$

for all $t > 0$ and $\xi, \varrho \in W$.

First, recall the following definition and theorem, known as Michael's Selection Theorem [121].

Definition 5.4.1. "If W and Y are two spaces,and S is a function from W to the subsets of Y, then a selection for S is a continuous $f : W \to Y$ such that $f(x) \in S(x)$ for every $x \in W$".

Theorem 5.4.2.
"If W is paracompact space, then every lower semi-continuous function S to the non-empty, closed, convex subsets of a Banach space Y admits a selection."

Consider the integral inclusion

$$\xi(u) \in \int_0^u G(u, v, \xi(v))dv + h(u) \quad \text{for all } \; u, v \in [0, 1] \text{ and } h, \xi \in C([0, 1]. \tag{5.13}$$

where $G\colon [0, 1] \times [0, 1] \times \mathbb{R} \to P_{cv}(\mathbb{R})$ is multivalued continuous functions.
For the above integral inclusion, we define a multivalued operator $S : W \to \hat{C}_0(W)$ by

$$S(\xi(u)) = \left\{ w \in W : w \in \int_0^u G(u, v, \xi(v))dv + h(u), \quad u \in [0, 1] \right\}$$

The next result proves the existence of a solution of the integral inclusion (5.13).

Theorem 5.4.3.
Let $S : W \to \hat{C}_0(W)$ be the multivalued integral operator given by

$$S(\xi(u)) = \left\{ w \in W : w \in \int_0^u G(u, v, \xi(v))dv + h(u), \quad u \in [0, 1] \right\}$$

Suppose the following conditions are satisfied.

1. $G\colon [0, 1] \times [0, 1] \times \mathbb{R} \to P_{cv}(\mathbb{R})$ is such that $G(u, v, \xi(v))$ is lower semi-continuous in $[0, 1] \times [0, 1]$.

2. For all $u, v \in [0, 1]$, $f(u, v) \in W$ and for all $\xi, \varrho \in W$, we have

$$|G(u, v, \xi(v)) - G(u, v, \varrho(v))|^2 \le f^2(u, v)|\xi(v) - \varrho(v)|^2,$$

where $f\colon [0,1] \to [0,\infty)$ is continuous.

3. There exists $0 < k < 1$ such that

$$\sup_{u\in[0,1]} \int_0^u f^2(u,v)dv \leq k.$$

Then the integral inclusion (5.13) has the solution in W.

Proof.

For

$$G\colon [0,1] \times [0,1] \times \mathbb{R} \to P_{cv}(\mathbb{R})$$

it follows from Michael's selection theorem that there exists a continuous operator

$$G_i\colon [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}$$

such that $G_i(u,v,\xi(v)) \in G(u,v,\xi(v))$ for all $u,v \in [0,1]$. It follows that

$$\xi(u) \in \int_0^u G_i(u,v,\xi(v))dv + h(u)) \in S(\xi(u))$$

hence $S(\xi(u)) \neq \emptyset$ and closed. Moreover, since $h(u)$ is continuous on $[0,1]$, and G is continuous, their ranges are bounded. This means that $S(\xi(u))$ is bounded and $S(\xi(u)) \in \hat{C}_0(W)$ Let $q,r \in W$ there exist $q(u) \in S(\xi(u))$ and $r(u) \in S(\varrho(u))$ such that

$$q(\xi(u)) = \left\{ w \in W : w \in \int_0^u G_i(u,v,\xi(v))dv + h(u), \quad u \in [0,1] \right\}$$

and

$$r(\varrho(u)) = \left\{ w \in W : w \in \int_0^u G_i(u,v,\varrho(v))dv + h(u), \quad u \in [0,1] \right\}$$

It follows from item 2 that

$$|G_i(u,v,\xi(v)) - G_i(u,v,\varrho(v))|^2 \leq f^2(u,v)|\xi(v) - \varrho(v)|^2$$

Now

$$
\begin{aligned}
e^{-\frac{\sup\limits_{t\in[0,1]} |q(u)-r(u))|^2}{kt}} &\geq e^{-\frac{\sup\limits_{u\in[0,1]} \int_0^u |G_i(u,v,\xi(v))-G_i(u,v,\varrho(v))|^2 dv}{kt}} \\
&\geq e^{-\frac{\sup\limits_{u\in[0,1]} \int_0^u f^2(u,v)|\xi(v)-\varrho(v)|^2 dv}{kt}} \\
&\geq e^{-\frac{|\xi(v)-\varrho(v)|^2 \sup\limits_{u\in[0,1]} \int_0^u f^2(u,v) dv}{kt}} \\
&\geq e^{-\frac{k|\xi(v)-\varrho(v)|^2}{kt}} \\
&= e^{-\frac{|\xi(v)-\varrho(v)|^2}{t}} \\
&\geq e^{-\frac{\sup\limits_{v\in[0,1]} |\xi(v)-\varrho(v)|^2}{t}} \\
&= F_b(\xi,\varrho,t)
\end{aligned}
$$

So, we have

$$F_b(q,r,kt) \geq F_b(\xi,\varrho,t)$$

By interchanging the roll of ξ and ϱ, we reach to

$$\mathcal{H}_{\mathcal{F}_b}(S\xi,S\varrho,kt) \geq F_b(\xi,\varrho,t)$$

Hence S has a fixed point in W which is the solution of integral inclusion (5.13). □

5.5 Multivalued Mapping in Extended Fuzzy b-Metric Space

In this section, generalized form of the results of Section 5.2 are presented by using the idea of Hausdorff fuzzy metric in EFBM. The notion of HFMS in Definition 5.1.2 of [79], can be extended naturally for HEFBM space on $\hat{C}_0$ as follows:

Definition 5.5.1.

Let $(W, F_\alpha, *)$ be a EFBMS with $\alpha\colon W \times W \to [1, \infty)$. Define a function $\mathcal{H}_{\mathcal{F}_\alpha}$ on $\hat{C}_0(W) \times \hat{C}_0(W) \times (0, \infty)$ by,

$$\mathcal{H}_{\mathcal{F}_\alpha}(\mathcal{A}, \mathcal{B}, t) = \min\left\{ \inf_{\xi\in\mathcal{A}} \mathcal{F}_\alpha(\xi, \mathcal{B}, t), \inf_{\varrho\in\mathcal{B}} \mathcal{F}_\alpha(\mathcal{A}, \varrho, t)\right\}$$

for all $\mathcal{A}, \mathcal{B} \in \hat{C}_0(W)$, $t > 0$ and $\hat{C}_0(W)$ is the collection of all nonempty compact subsets of W, where $\mathcal{F}_\alpha$ is defined in the same way as in Definition 5.1.1.

That is,

$$\mathcal{F}_\alpha(\xi, \mathcal{B}, t) = \sup\left\{F_\alpha(\xi, \varrho, t)\colon \varrho \in \mathcal{B}\right\}.$$

Lemma 5.2.1 to 5.2.3 extended naturally for EFBMS. That is where ever needed, these Lemmas will be considered in the setting of EFBMS.

Theorem 5.2.4 can now be stated in more general setting as follows:

Theorem 5.5.2.

Let $(W, F_\alpha, *)$ be a G-complete EFBMS with $\alpha(\xi, \varrho) \geqslant 1$ and $\mathcal{H}_{\mathcal{F}_\alpha}$ be a HEFBM. Let $S\colon W \to \hat{C}_0(W)$ be a multivalued mapping satisfying

$$\mathcal{H}_{\mathcal{F}_\alpha}(S\xi, S\varrho, kt) \geq F_\alpha(\xi, \varrho, t) \tag{5.14}$$

$\forall\, \xi, \varrho \in W,\ k \in (0, 1)$ and $k\alpha(\xi, \varrho) < 1$. Then S has a fixed point.

Proof.

For $a_0 \in W$, we choose a sequence $\{a_r\}$ in W as follows:

Let $a_1 \in W$ such that $a_1 \in Sa_0$ by using Lemma 5.2.3 we can choose $a_2 \in Sa_1$ such that

$$F_\alpha(a_1, a_2, t) \geqslant \mathcal{H}_{\mathcal{F}_\alpha}(Sa_0, Sa_1, t)\ \forall\, t > 0.$$

By induction we have $a_{r+1} \in Sa_r$ satisfying

$$F_\alpha(a_r, a_{r+1}, t) \geqslant \mathcal{H}_{\mathcal{F}_\alpha}(Sa_{r-1}, Sa_r, t)\ \forall\, r\ \in \mathbb{N}.$$

Now by (5.14) together with Lemma 5.2.3, we have

$$\begin{aligned}
F_\alpha(a_r, a_{r+1}, t) &\geq \mathcal{H}_{\mathcal{F}_\alpha}(Sa_{r-1}, Sa_r, t) \\
&\geq \mathcal{H}_{\mathcal{F}_\alpha}(Sa_{r-1}, Sa_r, t) \\
&\geq F_\alpha\left(a_{r-1}, a_r, \frac{t}{k}\right) \\
&\geq \mathcal{H}_{\mathcal{F}_\alpha}\left(Sa_{r-2}, Sa_{r-1}, \frac{t}{k}\right) \\
&\geq F_\alpha\left(a_{r-2}, a_{r-1}, \frac{t}{k^2}\right) \\
&\geq \mathcal{H}_{\mathcal{F}_\alpha}\left(Sa_{r-3}, Sa_{r-2}, \frac{t}{k^2}\right) \\
&\geq F_\alpha\left(a_{r-2}, a_{r-2}, \frac{t}{k^3}\right) \\
&\vdots \\
&\geq \mathcal{H}_{\mathcal{F}_\alpha}\left(Sa_1, Sa_2, \frac{t}{k^{r-2}}\right) \\
&\geq F_\alpha\left(a_1, a_2, \frac{t}{k^{r-1}}\right) \\
&\geq \mathcal{H}_{\mathcal{F}_\alpha}\left(Sa_0, Sa_1, \frac{t}{k^{r-1}}\right) \\
&\geq F_\alpha\left(a_0, a_1, \frac{t}{k^r}\right). \qquad (5.15)
\end{aligned}$$

For any $q \in \mathbb{N}$, writing

$$q\left(\frac{t}{q}\right) = \frac{t}{q} + \frac{t}{q} + \ldots + \frac{t}{q}$$

and using $[EFBM4]$ repeatedly,

$$\begin{aligned}
&F_\alpha(a_r, a_{r+q}, t) \\
&\quad \geq F_\alpha\left(a_r, a_{r+1}, \frac{t}{q\alpha(a_r, a_{r+q})}\right) * F_\alpha\left(a_{r+1}, a_{r+2}, \frac{t}{q\alpha(a_r, a_{r+q})\alpha(a_{r+1}, a_{r+q})}\right) \\
&\quad * F_\alpha\left(a_{r+2}, a_{r+3}, \frac{t}{q\alpha(a_r, a_{r+q})\alpha(a_{r+1}, a_{r+q})\alpha(x_{n+2}, x_{n+q})}\right) \\
&\quad * \ldots * \\
&\quad F_\alpha\left(a_{r+q-1}, a_{r+q}, \frac{t}{q\alpha(a_r, a_{r+q})\alpha(a_{r+1}, a_{r+q})\alpha(a_{r+2}, a_{r+q}) \ldots \alpha(a_{r+q-1}, a_{r+q})}\right).
\end{aligned}$$

Using the contraction (5.15) and $[EFBM5]$, it follows that

$$\begin{aligned}
&F_\alpha(a_r, a_{r+q}, t) \\
&\geq F_\alpha\left(a_0, a_1, \frac{t}{q\alpha(a_r, a_{r+q})k^r}\right) * F_\alpha\left(a_0, a_1, \frac{t}{q\alpha(a_r, a_{r+q})\alpha(a_{r+1}, a_{r+q})k^{r+1}}\right) \\
&* F_\alpha\left(a_0, a_1, \frac{t}{q\alpha(a_r, a_{r+q})\alpha(a_{r+1}, a_{r+q})\alpha(a_{r+2}, a_{r+q})k^{r+3}}\right) * \ldots * \\
&F_\alpha\left(a_0, a_1, \frac{t}{q\alpha(a_r, a_{r+q})\alpha(a_{r+1}, a_{r+q})\alpha(a_{r+2}, a_{r+q}) \ldots \alpha(a_{r+q-1}, a_{r+q})k^{r+q}}\right).
\end{aligned}$$

Since for all $r, q \in \mathbb{N}$, and $\alpha(a_r, a_{r+q})k < 1$, we have

$$\lim_{r \to \infty} F_\alpha(a_r, a_{r+q}, t) = 1 * 1 * \ldots * 1 = 1.$$

This shows that $\{a_r\}$ is G-Cauchy sequence.

Now it is claimed that z is a fixed point of S

$$\begin{aligned}
F_\alpha(z, Sz, t) &\geq F_\alpha\left(z, a_{r+1}, \frac{t}{2}\right) * F_\alpha\left(a_{r+1}, Sz, \frac{t}{2}\right) \\
&\geq F_\alpha\left(z, a_{r+1}, \frac{t}{2}\right) * \mathcal{H}_{\mathcal{F}_\alpha}\left(Sa_r, Sz, \frac{t}{2}\right) \\
&\geq F_\alpha\left(z, a_{r+1}, \frac{t}{2}\right) * F_\alpha\left(a_r, z, \frac{t}{2k}\right) \\
&\longrightarrow 1 \quad \text{as} \quad r \to \infty.
\end{aligned}$$

By Lemma 5.2.1 $z \in Sz$.

Hence z is fixed point for S. □

Following are some results which can be proved in similar way as Theorem 5.5.2.

Theorem 5.5.3. Let $(W, F_\alpha, *)$ be a G-complete EFBMS with $\alpha(x, y) \geqslant 1$ and $\mathcal{H}_{\mathcal{F}_\alpha}$ be a HEFBM. Let $S\colon W \to \hat{C}_0(W)$ be a multivalued mapping satisfying

$$\begin{aligned}
&\mathcal{H}_{\mathcal{F}_\alpha}(S\xi, S\varrho, kt) \\
&\qquad \geq \min\left\{\frac{\mathcal{F}_\alpha(\varrho, S\varrho, t)\,[1 + \mathcal{F}_\alpha(\xi, S\xi, t)]}{1 + F_\alpha(\xi, \varrho, t)}, F_\alpha(\xi, \varrho, t)\right\}
\end{aligned}$$

$\forall\, \xi, \varrho \in W,\ k\alpha(\xi, \varrho) < 1$. Then S has a fixed point.

Theorem 5.5.4. Let $(W, F_\alpha, *)$ be a G-complete EFBMS with $\alpha(\xi,\varrho) \geqslant 1$ and $\mathcal{H}_{\mathcal{F}_\alpha}$ be a HEFBM. Let $S\colon W \to \hat{C}_0(W)$ be a multivalued mapping satisfying

$$\begin{aligned}&\mathcal{H}_{\mathcal{F}_\alpha}(S\xi, S\varrho, kt)\\ &\qquad \geq \min\left\{\frac{\mathcal{F}_\alpha(\varrho, S\varrho, t)\left[1+\mathcal{F}_\alpha(\xi, S\xi, t)+\mathcal{F}_\alpha(\varrho, S\xi, t)\right]}{2+F_\alpha(\xi,\varrho,t)}, F_\alpha(\xi,\varrho,t)\right\}\end{aligned}$$

$\forall\, \xi,\varrho \in W,\ k\alpha(\xi,\varrho) < 1$. Then S has a fixed point.

Theorem 5.5.5.

Let $(W, F_\alpha, *)$ be a G-complete EFBM with $\alpha(\xi,\varrho) \geqslant 1$ and $\mathcal{H}_{\mathcal{F}_\alpha}$ be a HEFBM. Let $S\colon W \to \hat{C}_0(W)$ be a multivalued mapping satisfying

$$\begin{aligned}&\mathcal{H}_{\mathcal{F}_\alpha}(S\xi, S\varrho, kt)\\ &\qquad \geq \min\left\{\frac{\mathcal{F}_\alpha(\xi, S\xi, t)\left[1+\mathcal{F}_\alpha(\varrho, S\varrho, t)\right]}{1+\mathcal{F}_\alpha(S\xi, S\varrho, t)}, \frac{\mathcal{F}_\alpha(\varrho, S\varrho, t)\left[1+\mathcal{F}_\alpha(\xi, S\xi, t)\right]}{1+F_\alpha(\xi,\varrho,t)},\right.\\ &\qquad\qquad \left.\frac{\mathcal{F}_\alpha(\xi, S\xi, t)\left[2+\mathcal{F}_\alpha(\xi, S\varrho, t)\right]}{1+F_\alpha(\xi, S\varrho, t)+\mathcal{F}_\alpha(\varrho, S\xi, t)}, F_\alpha(\xi,\varrho,t)\right\}\end{aligned} \tag{5.16}$$

$\forall\, \xi,\varrho \in W,\ k\alpha(\xi,\varrho) < 1$. Then S has a fixed point.

Chapter 6

Conclusion and Future Work

This research deals with function spaces in general and fixed point theory in particular. This thesis, mainly focused on the generalizations of certain fixed theorems available in the literature of fixed point theory, which include generalizations of fixed results in FBMS, FRBMS, EFBMS. The applications of fixed point theory is to seek unique solution of linear algebraic, differential and integral equations reduced to functional equations. In Chapter 3, the famous BCP for FRBMS is established and an example is furnished to illustrate the theorem. In this way, the main result of Grabiec [51] is generalized. At the same time, by restricting the contraction mapping to the elements in the orbit of a point in FRBMS, an analogue of the fixed point theorem of Hicks and Rhoads [85] in the setting of FRBMS is proved. By using Geraghty type contraction, the result of [28] is also established. For authenticity of results an application of integral equation is furnished in the end of chapter. In Chapter 4, BCP for EFBMS is establised and illustrated by an example which generalized the main result of Grabiec [51]. In the next section of this chapter some fixed point results using Geraghty type contraction are proved. An example demonstrate the result. Further to strengthen the result, an application to study the existence of the solution of Voltera type integral equation is provided.

In Chapter **??**, BCP and an analogue of the fixed point theorem of Gupta [53] and Roshan et. al [28] in the setting of both HFBMS and HEFBMS are established for multivalued mapping and an example is furnished to illustrate the results. An application of Voltera type integral inclusion is also presented in the end of this chapter. Thus these results are more general than the existing literature in the fuzzy mathematics. Theory of fuzzy mathematics is extended by investigating the fuzzy distances in the form of FBMS, EFBMS and FRBMS. In future one can try to generalize many rsults of metric spaces, BMS like [13, 18, 19, 122] in the setting of EFBMS and FRBMS, and further investigate these results for multivalued mappings.

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