CAPITAL UNIVERSITY OF SCIENCE AND TECHNOLOGY, ISLAMABAD

MHD Stagnation Point Flow with Cattaneo-Christov Heat Flux and Homogeneous-Heterogeneous Reactions

by

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I dedicate this sincere effort to my dear **Parents** and my elegant **Teachers** who are always source of inspiration for me and their contributions are uncounted.

CERTIFICATE OF APPROVAL

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Abstract

A numerical investigation is performed for the MHD stagnation point flow with Cattaneo-Christov heat flux model and homogeneous-heterogeneous reactions. Investigation of heat and mass transfer on a variably thicked surface is executed for steady, UCM and thermal radiation. An electrically conducting fluid is considered in the presence of non-uniform applied magnetic field. Using suitable similarity transformations, the governing PDEs are transformed into a system of coupled non-linear ODEs. Utilizing the shooting method, the system of ordinary differential equations is solved with the help of the computational software MATLAB to compute the numerical results. The numerical solution obtained for the velocity, temperature and concentration profiles is presented through graphs for different physical parameters. The numerical values of the skin friction, Nusselt and Sherwood numbers have also been presented and analyzed through tables

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Abbreviations

MHD	**M**agneto-hydrodynamics
PDEs	**P**artial **D**ifferential **E**quations
ODEs	**O**rdinary **D**ifferential **E**quations
UCM	**U**pper **C**onvected **M**axwell

Symbols

ρ	**fluid density**
μ	**viscosity**
ν	**kinematic viscosity**
τ	**stress tensor**
λ	**thermal relaxation time**
α	**wall thickness parameter**
τ_w	**wall shear stress**
η	**dimensionless similarity variable**
ψ	**stream function**
u	**velocity component along x direction**
v	**velocity component along y direction**
U_e	**free stream velocity**
U_w	**stretching velocity**
n	**velocity power index**
k	**thermal conductivity**
T	**temperature of fluid**
Sc	**Schmidt number**
δ	**ratio of mass diffusion coefficient**
β	**Deborah number**
S	**thermal stratified parameter**
γ	**thermal relaxation parameter**
α	**wall thickness parameter**
K	**strength of homogeneous reaction parameter**

Ks	**strength of heterogeneous reaction parameter**
Pr	**Prandtl number**
θ	**dimensionless temperature**
N_u	**Nusselt number**
Sh_x	**Sherwood number**
C_{fx}	**Skin friction coefficient**
M	**megnetic parameter**
B	**megnetic field**
T_∞	**free stream temperature**
T_w	**wall temperature**
(x, y)	**cartesian coordinates**
$(a*)$	**concentration of the chemical species A**
$(b*)$	**concentration of the chemical species B**

Chapter 1

Introduction

Magnetohydrodynamics study consists of magnetic properties of electrically conducting fluids. The Swedish Physicist, Alfen [1] introduced the MHD fluid. MHD fluid flow through a heated surface has many important applications in so many engineering scenarios like petroleum industry, MHD power generators and crystal growth etc. Mbeledogu and Ogulu [2] examined the MHD natural convection flow of spining fluid past through a porous sheet. They also observed the impact of heat transfer and radiation as well. Time dependent MHD convective flow through semi finite vertical porous plate was studied by Kesavaiah *et al.* [3]. Modather and chamkha [4] examined the analytical study of MHD heat and mass transfer process on a porous plate. MHD flow of viscous fluid in the presence of transpiration was keenly observed by Mabood *et al.* [5]. Hayat *et al.* [6] exposed the impact of connvective heat transfer in MHD flow of Jeffrey fluid model over a permeable plate. Similarly MHD flow of Maxwell fluid with convective heat transfer was observed by Hayat *et al.* [7].

It is known that the phenomenon of heat transfer occurs between two bodies or within the same body due to a difference of temperature. In various industrial and engineering processes, the characteristics of heat transfer have huge effects on microelectronics, transportation and fuel cells etc. The heat conduction law was suggested by Fourier [8], but it has a limitation that for the temperature field it generates a parabolic energy equation. To resolve this issue in the classical Fourier law of heat conduction, Cattaneo [9] added the thermal relaxation time. After that, Christov [10] changed the Cattaneo law by time derivative in the Maxwell-Cattaneo model with Oldroyd upper-convected derivative to conserve material-invariant formulation. Straughan [11] used the Cattaneo-Christov model just to investigate thermal convection in an incompressible flow. Tibullo and Zampali [12] examined the uniqueness of Cattaneo-Christov heat flux model for flow of an incompressible fluid. Hayat *et al.* [13] numerically investigated the Cattaneo-Christov heat flux model in a visco-elastic flow due to exponentially stretching sheet. Pavlov [14] discussed the MHD flow of an incompressible viscous fluid caused by deformation of flat surface.

MHD with stagnation point flow has always been matter of concern for the researchers for many years. Hayat *et al.* [15] first time brought out the fact about the stagnation point flow with Cattaneo- Christov heat flux and homogeneous- heterogeneous reactions. The effect of MHD and thermal radiation on Maxwell fluid was discussed by Akbar *et al.* [16]. These scientists pointed out the fact that elasticity number became the reason of enhancement in heat transfer rate. Maxwell fluid in a porous medium with its rotation was further explained by Hayat *et al.* [17]. Minsta *et al.* [18] studied the MHD flow of Maxwell fluid and its chemical reaction as well. Wide range of temperature to test the effect of stagnation point flow concentrates was also undertaken time and again. It was proved that there was a severe decrease in temperature particularly for concentration of nano particles. The entropy generation in MHD and slip flow over a rotating penetrable disk with variable properties was investigated by Rashidi *et al.* [19]. Various characteristics of homogeneous- heterogeneous reactions within Jeffrey fluid were observed by Hayat *et al.* [20]. Similarly, Shah *et al.* [21] further studied the MHD effects for heat transfer for the UCM and for the Joule heating simultaneously. Cattaneo-Christov heat flux model was used for this observation. The effects of Cattaneo-Christov heat flux in the flow with variable thickness were highlighted by Hayat *et.al* [22].

In engineering, heat and mass transfer problems with chemical reactions are part and parcel. Homogeneous or heterogeneous is an outcome of any chemical reaction which can further be characterized with certain process including disappearance of evaporation, shifting of impetus and flow in a dessert cooler. A homogeneous reaction occurs with sole entity through specified region whereas a heterogeneous reaction occurs within confined region or space. The reaction rate and the concentration are directly proportional, this kind of reaction is regarded as first order reaction. The diffusion of species with chemical reaction has immense utilities regarding fibrous insulation, pollution studies, synthesis materials and oxidation. Das [23] considered the effects in MHD micropolar flow, heat and mass transfer with thermal radiation and chemical reaction. In MHD, impact of transfer of chemically reactive entities passing over a permeable material was investigated by Kandasamy *et al.* [24]. Afify [25] studied the result when chemically reactive entities were observed in a flow of non Newtonian fluid absorbed the permeable for diffusion. Bhattacharyya and Layek [26] studied the behaviour of chemically reactive solute within MHD process particularly affecting the boundary layer flow over a porous wedge. The MHD flow and mass transfer of an UCM fluid past a permeable shrinking sheet with chemical reaction was examined by Hayat *et al.* [27]. Mansour *et al.* [28] considered the thermal stratification and effects of chemical reaction on MHD through a porous medium over a vertical stretching surface. Bhattacharyya [29] explored solutions for stagnation-point boundary layer flow with chemical reaction past a shrinking /stretching sheet. Relative studies in this field may also be found in [30–41].

1.1 Thesis Contributions

The main purpose of the present study is to perform the numerical analysis for the MHD stagnation point flow with Cattaneo-Christov heat flux and homogeneous-heterogeneous reactions and to examine the effect of different parameters on the velocity, temperature and concentration profiles. The flow governing boundary equations are converted into a set of non-linear ODEs by employing suitable similarity transformations. Utilizing the shooting technique along with the fourth order Runge-Kutta method, the coupled nonlinear ODEs are solved numerically. Graphical results are also presented and discussed to illustrate the solution.

1.2 Thesis Outlines

This thesis has been further organized into four chapters.

- **Chapter 2** comprises of some basic definitions related to fluid dynamics. These concepts are used to describe the flow, heat transfer and the influence of thermophysical properties.
- **Chapter 3** contains a comprehensive review of [15]. A numerical study of incompressible, two dimensional steady fluid flow with convective boundary conditions past a stretching sheet has been performed. The constitutive flow model is solved numerically and the impact of physical parameters concerning the flow model on the dimensionless temperature, velocity and concentration is discussed through graphs and tables.
- **Chapter 4** focuses on the extension of [15]. The obtained system of ODEs are solved numerically after applying proper similarity transformations. Graphs and tables describe the impact of physical parameters. Numerical results of momentum, temperature and concentration have also been computed and discussed.
- **Chapter 5** summarizes up the study and gives the major results obtained from the entire research and suggests some recommendations for the future work.

  All the references used in this study are listed in **Bibliography**.

Chapter 2

Some Basic Definitions and Governing Equations

2.1 Basic Definitions

In this chapter, some fundamental definitions, governing laws and concepts [42] regarding the fluid mechanics will be described. These concepts will be helpful to develop an understanding for the rest of the thesis.

Definition 2.1. (Fluid)
"Fluid is a physical substance that changes regularly under the action of shear stress. It does not depend on how small the shear stress is and repeatedly deforms its shape as long as the shear stress acts."

Definition 2.2. (Fluid Mechanics)
"Fluid mechanics is the branch of engineering that contains the discussion of different properties of fluids and the effect of different forces on them. Fluid mechanics is mainly divided into two branches which are fluid statics and fluid dynamics. Fluid statics describes the properties of the stationary fluids whereas in the fluid dynamics, the flow of moving fluid is discussed."

Definition 2.3. (Pressure)

"The ratio of applied force to the unit area is said to be pressure. It is denoted by P and mathematically, it can be written as

$$P = \frac{F}{A}, \tag{2.1}$$

where F, A denote the applied force and area of the surface, respectively."

Definition 2.4. (Density)

"Density of a material is the ratio of mass to the unit volume. Symbolically it is denoted by ρ and mathematically, it is expressed as

$$\rho = \frac{m}{V}, \tag{2.2}$$

where V and m are the volume of the material and mass of the material, respectively."

Definition 2.5. (Stress)

"Stress is the force acting on the surface of the unit area within the distortable body. Mathematically, it can be written as

$$\sigma = \frac{F}{A}, \tag{2.3}$$

where F is the force and A is the area."

Definition 2.6. (Shear stress)

"Shear stress is the component of stress in which a force acts parallel to the unit surface area."

Definition 2.7. (Normal stress)

"Normal stress is the element of stress in which a force acts normal to the unit surface area."

Definition 2.8. (Viscosity)

"It is the property of the fluid that resists the fluid flow. In other words, a fluid viscosity is that characteristic which measures the amount of resistance to the shear stress. It is denoted by μ and mathematically, it can be written as

$$\text{viscosity}(\mu) = \frac{\text{shear stress}}{\text{shear strain}}.\text{"} \tag{2.4}$$

Definition 2.9. (Kinematic Viscosity)
"The ratio of the dynamic viscosity to the density of fluid is said to be kinematic viscosity. Symbolically, it can be written as ν and mathematically, it is expressed by

$$\nu = \frac{\mu}{\rho}, \tag{2.5}$$

where μ and ρ denote the dynamic viscosity and the density respectively. The dimension of kinematic viscosity is given by $[\frac{L^2}{T}]$."

Definition 2.10. (Magnetohydrodynamics)
"The branch of dynamics which deals with the electrically conducting fluids such as plasma is said to be magnetohydrodynamics."

2.2 Classification of Fluids

Definition 2.11. (Ideal Fluid)
"A fluid, which has zero viscosity, is said to be an ideal fluid. Naturally, ideal fluid is incompressible and does not practically exist ."

Definition 2.12. (Real Fluid)
"A fluid is said to a real fluid if it has a non- zero viscosity. Unlike ideal fluids, it is compressible in nature, e.g. petrol, kerosene, castrol oil."

Definition 2.13. (Newton's Law of Viscosity)
"The shear stress which distorts the fluid component is directly and linearly proportional to the velocity gradient is said to be the Newton's law of viscosity. Mathematically, it can be written as

$$\tau_{xy} \propto \left(\frac{du}{dy}\right), \tag{2.6}$$

$$\tau_{xy} = \mu \frac{du}{dy}, \tag{2.7}$$

where τ_{xy} is the shear stress applied on the fluid, u is the component of the velocity along x-axis and μ is viscosity as the proportionality constant."

2.2.1 Newtonian and Non-Newtonian Fluids

"The fluids, which fulfill Newton's law of viscosity are known as Newtonian fluid. Mathematically,

$$\tau_{xy} = \mu \left(\frac{du}{dy} \right), \tag{2.8}$$

where μ is called the constant of proportionality. The most common example of Newtonian fluids is water. Those fluids, which do not obey the Newton's law of viscosity are known as non-Newtonian fluids. Mathematically

$$\tau_{xy} = k \left(\frac{du}{dy} \right)^n, \tag{2.9}$$

where $n \neq 1$ is the flow behavior index. For $n = 1$ with $k = \mu$ the above equation reduces to the Newton's law of viscosity. Paints, blood, biological fluids and polymer melts etc, are good examples of non-Newtonian fluids."

Definition 2.14. (Laminar Flow)
"A flow in which the particles of the fluid have special path and individual particle does not intersect each other is known as laminar flow. In such flow, the particles move along well-defined path. Laminar flow occurs for the fluids having high viscosity."

Definition 2.15. (Turbulent Flow)
"A flow which has no specific path and moves randomly in any direction is said to be a turbulent flow. Turbulent flow occurs when the fluid is flowing with high speed. If we observe the smoke rising from a cigarette, for the first few centimeters the flow is certainly laminar but later on, the smoke becomes turbulent."

Definition 2.16. (Uniform flow)

"If the velocity of the flow has the same magnitude as well as direction during the motion of the fluid, the the flow is said to be a uniform flow. Mathematically, it can be written as

$$\frac{dV}{ds} = 0, \tag{2.10}$$

where V is the velocity and s is the displacement in any direction."

Definition 2.17. (Non-uniform Flow)

"In non-uniform flow, the velocity is not the same at every point in the fluid at a given instant. Mathematically, it is expressed as

$$\frac{dV}{ds} \neq 0, \tag{2.11}$$

where V is the velocity and s is the displacement."

Definition 2.18. (Internal Flow)

"Internal flows are those where fluids flow through confined spaces, e.g. flow in pipe."

Definition 2.19. (External Flow)

"The flow which is not confined by the solid surface, is known as external flow. The flow of water in the river is an example of the external flow."

Definition 2.20. (Steady Flow)

"The flow, which is independent of time is said to be a steady flow.
Mathematically, it can be written as

$$\frac{d\xi}{dt} = 0, \tag{2.12}$$

where ξ is fluid property."

Definition 2.21. (Unsteady Flow)

"The flow, which depends on time, is known as unsteady flow.

Mathematically, it can be written as

$$\frac{d\xi}{dt} \neq 0, \tag{2.13}$$

where ξ is a fluid property."

Definition 2.22. (Compressible Flow)

"The fluid flow in which the density does not remain constant within the fluid is said to be a compressible flow. Mathematically, it is expressed by

$$\rho(x, y, z, t) \neq c, \quad c\ is \text{ constant."} \tag{2.14}$$

Definition 2.23. (Incompressible Flow)

"The fluid flow in which the density remains constant within the fluid, is called incompressible flow. Mathematically, it can be written as

$$\rho(x, y, z, t) = c, \quad c\ is \text{ constant."} \tag{2.15}$$

2.3 Heat Transfer

Definition 2.24. (Conduction)

"Conduction is the process in which heat is transferred through the material between the objects that are in physical contact. For example: picking up a hot cup of tea."

Definition 2.25. (Convection)

"In this process, the heat transfer occurs due to the bulk fluid motion of molecules or transfer of molecules. Mathematically, it is expressed as

$$q = hA(T_s - T_\infty), \tag{2.16}$$

where h, A, T_s and T_∞ denote the heat transfer coefficient, the area, the temperature of the surface and the temperature away from the surface respectively. It is subdivided into the following three categories."

Definition 2.26. (Forced Convection)

"A method of heat transfer in which the fluid motion is generated by an independent source like a pump or fan, is said to be forced convection."

Definition 2.27. (Natural Convection)

"A method of heat transfer in which the fluid motion is not generated by an independent source is said to be natural convection. In other words, it happens due to the temperature difference which affects the density and buoyancy of the fluid. Natural convection can only occur, when there is a gravitational field and it is also known as free convection. Example: Daily weather."

Definition 2.28. (Mixed Convection)

"It is a combination of both forced convection and natural convection. For example if fluid is moving upward along the moment of the vertical stretching sheet is forced between while in the same phenomena fluid is freely falling due to the gravity which is forced convection. When these two phenomena appear in the same model then such kind of flow is mixed convection."

Definition 2.29. (Radiation)

"Radiation is the process by which heat is transferred directly by electromagnetic radiation. The convection and radiation play a major role in transferring heat in the liquids and gases but in solids convection is totally absent. Thus for solids, conduction plays a major role in heat transfer.

For example, if we place a material object (e.g, a piece of steel) under the sun rays, after a few moments we observe that the material object is heated. Such phenomenon takes place due to radiation. Mathematically, it can be written as

$$q = E\sigma A[(\Delta T)^4], \tag{2.17}$$

where E, σ, $(\Delta T)^4$, A, q are the emissivity of the scheme, the constant of Stephan-Boltzmann ($5.670 \times 10^-8 \frac{W}{m^2K^4}$), the variation of the temperature, the area and the heat transfer respectively."

Definition 2.30. (Thermal Conductivity)

"Thermal conductivity is the property of a substance which measures the ability to conduct heat. Fourier's law of conduction which relates the rate of heat transfer by conduction to the temperature gradient is

$$\frac{dQ}{dt} = -kA\frac{dT}{dx}, \tag{2.18}$$

where A, $\frac{dQ}{dt}$, k, $\frac{dT}{dx}$ are the area, the rate of heat transfer, the thermal conductivity and the temperature gradient respectively. Thermal conductivity of most of the liquids is decreased with an increment in the temperature except water. The SI unit of thermal conductivity is $\frac{Kg.m}{s^3}$ and the dimension of thermal conductivity is $[\frac{ML}{T^3}]$."

Definition 2.31. (Thermal Diffusivity)

"Thermal diffusivity of a substance is defined as the ratio of thermal conductivity (k) of a substance to the product of specific heat at constant pressure (c_p) and density (ρ). It measures the ability of a substance to conduct thermal energy relative to its ability to store thermal energy.

Mathematically, it can be written as

$$\alpha = \frac{k}{\rho c_p}." \tag{2.19}$$

Definition 2.32. (Prandtl Number)

"The ratio of kinematic diffusivity to heat diffusivity is said to be the Prandtl number. It is denoted by Pr and mathematically it can be written as

$$Pr = \frac{\nu}{\alpha} = \frac{\frac{\mu}{\rho}}{\frac{k}{\rho c_p}} = \frac{\mu c_p}{k},$$

where ν, and α denote the momentum diffusivity or kinematic diffusivity and the thermal diffusivity respectively. It controls the relative thickness of the momentum and temperature function. Physical significance of Prandtl number is that it gives the respective thickness of velocity boundary layer and thermal boundary layer. For small Pr heat diffuses very quickly as compared to the momentum."

Definition 2.33. (Grashof Number)

"The ratio of the viscous force and the buoyancy force applied on the fluid is called

Grashof number. It repeatedly occurs in the free convection case. Symbolically, it is denoted by Gr and mathematically it can be written as

$$Gr = \frac{g\beta_0(T_s - T_\infty)\delta^3}{\nu^2}, \tag{2.20}$$

where g, β_o, T_{s_o}, T_∞, δ, ν denote the gravitational acceleration, the coefficient of the volumetric thermal, the surface temperature, the surrounding temperature, the characteristic length and the kinematic viscosity respectively."

Definition 2.34. (Schmidt Number)

"It is defined as the ratio of the momentum diffusivity and the mass diffusivity D_m. It is denoted by Sc and mathematically it can be written as

$$Sc = \frac{\nu}{D_m} = \frac{\mu}{\rho D_m} \tag{2.21}$$

where ν is the kinematic viscosity, D_m is the mass diffusion and μ is the dynamics viscosity."

Definition 2.35. (Reynolds Number)

"Reynold number is specified as the relationship of the inertial force to the viscous force. Inertial forces act upon all masses in a non-inertial frame of reference while viscous forces are the internal fluid flow resistance. It is denoted by Re and mathematically it can be written as

$$Re = \frac{\text{inertial force}}{\text{viscous force}} = \frac{\frac{\rho v^2}{L}}{\frac{\mu v}{L^2}},$$

where v, L and ν denote the fluid velocity, the characteristic length and the kinematic viscosity respectively. For a small Reynold number, the viscous forces are dominant and the flow in this case is characterized as the laminar flow while turbulent flow occurs at high Reynold number due to the dominance of the inertial force."

Definition 2.36. (Nusselt Number)

"It examines the ratio of the convective to the conductive heat transfer through the boundary of the surface. It is a dimensionless number which was first introduced by the German mathematician Nusselt. Heat transfer due to conduction is denoted by $\frac{k\Delta T}{\delta}$ and the heat transfer due to convection is denoted by $h\Delta T$. It is denoted by Nu and mathematically, Nusselt number is expressed by

$$Nu = \frac{h\Delta T}{\frac{k\Delta T}{\delta}} = \frac{h\delta}{k},$$

where h, δ, k denote the coefficient of heat transfer, the characteristic length and the thermal conductivity respectively."

Definition 2.37. (Stagnation point)
"It is a point in a flow field where the fluid velocity is zero. It exists at the surface of objects in the field where fluid is brought to rest by the object. Static pressure is the example of stagnation point."

2.4 Basic Equations

2.4.1 Continuity Equation

"Continuity equation is derived from the law of conservation of mass and mathematically, it is expressed by

$$\frac{\partial\rho}{\partial t} + \nabla.(\rho V) = 0, \tag{2.22}$$

where t is the time. If the fluid is an incompressible, the continuity equation is expressed by

$$\nabla.V = 0." \tag{2.23}$$

2.4.2 Law of Conservation of Momentum

"This law states that the combination of all applied external forces acting on a body is equal to the time rate of change of linear momentum of the body. In vector notation this law can be written as

$$\rho\frac{DV}{Dt} = \text{div}\tau + \rho b, \tag{2.24}$$

$$\tau = -pI + \mu S, \tag{2.25}$$

where S is the tensor and first time it was produced by Rivlin-Erickson.

$$S = \text{grad}V + (\text{grad}V)^t. \tag{2.26}$$

In the above equations, $\frac{D}{Dt}$ denotes the material time derivative or the total derivative, ρ denotes the density, V the velocity field, τ the Cauchy stress tensor, b the body forces, p the pressure and μ the dynamic viscosity.
The Cauchy stress tensor is expressed in the matrix form as

$$\tau = \begin{pmatrix} \sigma_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{pmatrix}, \tag{2.27}$$

where σ_{xx}, σ_{yy} and σ_{zz} are the normal stresses, otherswise the shear stresses. For two-dimensional flow, we have $V = [u(x, y, 0), v(x, y, 0), 0]$ and thus

$$\text{grad}V = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & 0 \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.28}$$

for x component

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right). \tag{2.29}$$

Similarly, the above process is repeated for y component as follows:

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right).\text{''} \tag{2.30}$$

2.4.3 Energy Equation

"The energy equation for a fluid is

$$\rho c_p\Big(\frac{\partial}{\partial t}+V\nabla\Big)T=k\nabla^2T+\tau L+\rho c_p\Big[D_B\nabla C.\nabla T+\frac{DT}{Tm}\nabla T\Big], \tag{2.31}$$

where $(c_p)_f$ denotes the specific heat of the basic fluid, $(c_p)_s$ the specific heat of the material, ρ_f the density of the basic fluid, L the rate of strain tensor, T the temperature of the fluid, D_B the Brownian motion coefficient and D_T the temperature diffusion coefficient and T_m the mean temperature. The expression for the Cauchy stress tensor τ for viscous incompressible fluid is expressed by

$$\tau=-pI+\mu S, \tag{2.32}$$

where S is the tensor, p the pressure, μ the dynamic viscosity and
$\frac{D}{Dt}$ the material time derivative or total derivative,

$$S=gradV+(gradV)^t, \tag{2.33}$$

where τ the strain tensor and can be written as

$$\tau=\begin{pmatrix}\sigma_{xx} & \tau_{yx} & \tau_{zx}\\ \tau_{xy} & \sigma_{yy} & \tau_{zy}\\ \tau_{xz} & \tau_{yz} & \sigma_{zz}\end{pmatrix}.\text{''} \tag{2.34}$$

Chapter 3

The Impact of Cattaneo-Christov Heat Flux Model On the Flow of Maxwell Fluid

3.1 Introduction

In this chapter, a detailed review of [15] has been conducted. The governing flow equations are formulated and then converted into a system of non-linear coupled ODEs by implementing a proper similarity transformation. These converted ODEs are solved numerically by using the shooting method. Finally, the numerical results are discussed at the end of the chapter for various pertinent physical parameters affecting the flow and heat transfer and found to be in excellent agreement with those computed by the MATLAB built-in function bvp4c.

3.2 Mathematical Modeling

Consider a steady, two- dimensional laminar flow of an incompressible UCM fluid over a non-linear stretching surface with variable thickness. The surface is taken at $y = A_1(x+b)^{\frac{1-n}{2}}$. Note that for $n < 1$, the surface is of the uniform thickness.

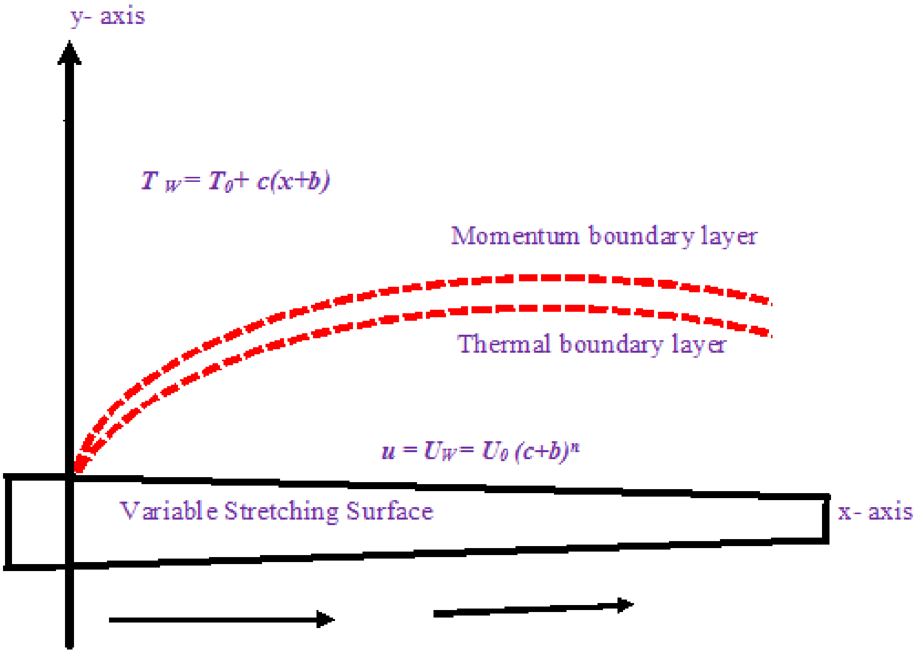

FIGURE 3.1: Geometry of the problem.

Heat flux analysis, in the presence of Cattaneo-Christov heat flux, has been studied. Mass transfer in the presence of chemical reaction has been considered. T_w and T_∞ are the surface and ambient temperatures respectively. The equations of continuity, momentum and the energy are as follows.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{3.1}$$

$$v\frac{\partial u}{\partial y} + u\frac{\partial u}{\partial x} = U_e\frac{dU_e}{dx} + \lambda_1 U_e^2\frac{\partial^2 U_e}{\partial x^2} + \nu\frac{\partial^2 u}{\partial y^2} - \lambda_1\left(2uv\frac{\partial^2 u}{\partial x\partial y} + v^2\frac{\partial^2 u}{\partial y^2} + u^2\frac{\partial^2 u}{\partial x^2}\right), \tag{3.2}$$

$$v\frac{\partial T}{\partial y} + u\frac{\partial T}{\partial x} + \lambda\left(v\frac{\partial v}{\partial y}\frac{\partial T}{\partial y} + u\frac{\partial v}{\partial x}\frac{\partial T}{\partial y} + u\frac{\partial u}{\partial x}\frac{\partial T}{\partial x} + v\frac{\partial u}{\partial y}\frac{\partial T}{\partial x} + 2uv\frac{\partial^2 T}{\partial x\partial y} + v^2\frac{\partial^2 T}{\partial y^2} + u^2\frac{\partial^2 T}{\partial x^2}\right) = \frac{k}{\rho c_p}\frac{\partial^2 T}{\partial y^2}, \tag{3.3}$$

$$u\frac{\partial a^*}{\partial x} + v\frac{\partial a^*}{\partial y} = D_A\frac{\partial^2 a^*}{\partial y^2} - K_1 a^* b^{*2}, \tag{3.4}$$

$$u\frac{\partial b^*}{\partial x} + v\frac{\partial b^*}{\partial y} = D_B\frac{\partial^2 b^*}{\partial y^2} + K_1 a^* b^{*2}. \tag{3.5}$$

The associated boundary conditions for the above system of equations are

$$\left.\begin{aligned} & u = U_w = U_0(x+b)^n, \quad v = 0, \quad T = T_w = T_0 + c(x+b), \\ & D_A \frac{\partial a^*}{\partial y} = K_s a^*, \quad D_B \frac{\partial b^*}{\partial y} = -K_s a^*, \quad at \;\; y = A_1 (x+b)^{\frac{1-n}{2}}. \\ & u \to U_e(x) = U_\infty (x+b)^n, \quad T \to T_\infty = T_0 + d(x+b), \\ & a^* \to a_0, \quad b^* \to 0; \;\; when \;\; y \to \infty. \end{aligned}\right\} \tag{3.6}$$

3.3 Similarity Transformations

To convert the system of governing equations into the dimensionless form, we use the following transformations, where ψ be the stream function satisfying the continuity equation. It is usually written as:

$$\left.\begin{aligned} u &= \frac{\partial \psi}{\partial y}, \\ v &= -\frac{\partial \psi}{\partial x}. \end{aligned}\right\} \tag{3.7}$$

Now introduce the following similarity transformations:

$$\psi = \sqrt{\frac{2}{n+1} \nu U_0 (x+b)^{n+1}} F(\eta),$$

$$G(\eta) = \frac{a^*}{a_0},$$

$$H(\eta) = \frac{b^*}{a_0},$$

$$\eta = \sqrt{\frac{n+1}{2} \frac{U_0}{\nu} (x+b)^{n-1}} y,$$

$$\Theta(\eta) = \frac{T - T_\infty}{T_w - T_0}.$$

The detailed procedure for the conversion of equations (3.1)-(3.5) has been described in the upcoming discussion.

- $$\begin{aligned}\frac{\partial\eta}{\partial x} &= \frac{\partial}{\partial x}\left(\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{n-1}}y\right)\\ &= \sqrt{\frac{n+1}{2}\frac{U_0}{\nu}}y\frac{\partial}{\partial x}(x+b)^{\frac{n-1}{2}}\\ &= \sqrt{\frac{n+1}{2}\frac{U_0}{\nu}}y\left(\frac{n-1}{2}\right)(x+b)^{\frac{n-1}{2}-1}\\ &= \left(\frac{n-1}{2}\right)\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{n-1}}y(x+b)^{-1}\\ &= \left(\frac{n-1}{2}\right)\eta(x+b)^{-1}.\end{aligned}$$
- $$\begin{aligned}\frac{\partial\eta}{\partial y} &= \frac{\partial}{\partial y}\left(\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{n-1}}\;\; y\right)\\ &= \left(\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{n-1}}\right).\end{aligned}$$
- $$\begin{aligned}u &= \frac{\partial\psi}{\partial y}\\ &= \frac{\partial\psi}{\partial\eta}\frac{\partial\eta}{\partial y}\\ &= \left(\frac{\partial}{\partial\eta}\sqrt{\frac{2}{n+1}\nu U_0(x+b)^{n+1}}F(\eta)\right)\left(\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{n-1}}\right)\\ &= \left(\sqrt{(\frac{2}{n+1})\nu U_0(x+b)^{n+1}}\right)F'(\eta)\left(\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{n-1}}\right)\\ &= U_0(x+b)^nF'(\eta).\end{aligned}$$
- $$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial}{\partial x}\left(U_0(x+b)^nF'(\eta)\right)\\ &= \frac{\partial}{\partial x}\left(U_0(x+b)^n\right)F'(\eta)+U_0(x+b)^n\frac{\partial}{\partial x}F'(\eta)\\ &= nU_0(x+b)^{n-1}F'(\eta)+U_0(x+b)^nF''(\eta)\frac{\partial\eta}{\partial x}\\ &= nU_0(x+b)^{n-1}F'(\eta)+U_0(x+b)^nF''(\eta)\left(\frac{n-1}{2}\right)\eta(x+b)^{-1}\\ &= nU_0(x+b)^{n-1}F'(\eta)+\left(\frac{n-1}{2}\right)U_0(x+b)^{n-1}\eta F''(\eta)\\ &= U_0(x+b)^{n-1}\left(nF'(\eta)+\left(\frac{n-1}{2}\right)(\eta)F''\eta\right).\end{aligned}$$

- $$\begin{aligned} v &= -\frac{\partial \psi}{\partial x} \\ &= -\frac{\partial}{\partial x}\left(\sqrt{\frac{2}{n+1}\nu U_0 (x+b)^{n+1}} F(\eta)\right) \\ &= -\left(\sqrt{\frac{2}{n+1}\nu U_0}\right)\frac{\partial}{\partial x}\left((x+b)^{\frac{n+1}{2}} F(\eta)\right) \\ &= -\left(\sqrt{\frac{2}{n+1}\nu U_0}\right)\left(\frac{\partial}{\partial x}(x+b)^{\frac{n+1}{2}} F(\eta) + (x+b)^{\frac{n+1}{2}}\frac{\partial}{\partial x}F(\eta)\right) \\ &= -\left(\sqrt{\frac{2}{n+1}\nu U_0}\right)\left(\frac{n+1}{2}(x+b)^{\frac{n+1}{2}}(x+b)^{-1}F(\eta)\right. \\ &\quad \left. + \frac{n-1}{2}(x+b)^{\frac{n+1}{2}}F'(\eta)(x+b)^{-1}\right) \\ &= -\left(\sqrt{\frac{2}{n+1}\nu U_0}\right)\frac{n+1}{2}(x+b)^{\frac{n+1}{2}}(x+b)^{-1} \\ &\quad \left(F(\eta) + \frac{n-1}{n+1}\eta F'(\eta)\right) \\ &= -\sqrt{\frac{n+1}{2}U_0 \nu (x+b)^{n-1}}\left(F(\eta) + \frac{n-1}{n+1}\eta F'(\eta)\right). \end{aligned}$$

- $$\begin{aligned} \frac{\partial v}{\partial y} &= -\frac{\partial}{\partial y}\left(\sqrt{\frac{n+1}{2}U_0 \nu (x+b)^{n-1}}\left(F(\eta) + \frac{n-1}{n+1}\eta F'(\eta)\right)\right) \\ &= -\sqrt{\frac{n+1}{2}U_0 \nu (x+b)^{n-1}}\left(F'(\eta)\frac{\partial \eta}{\partial y}\right. \\ &\quad \left. + \frac{\partial \eta}{\partial y}\frac{n-1}{n+1}F'(\eta) + \quad \frac{n-1}{n+1}\eta F''(\eta)\frac{\partial \eta}{\partial y}\right) \\ &= -\sqrt{\frac{n+1}{2}U_0 \nu (x+b)^{n-1}}\left(F'(\eta) + \frac{n-1}{n+1}F'(\eta) + \frac{n-1}{n+1}\eta F''(\eta)\right) \\ &\quad \frac{\partial \eta}{\partial y} \\ &= -\left(\sqrt{\frac{n+1}{2}U_0 \nu (x+b)^{n-1}}\right)\left(\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{n-1}}\right) \\ &\quad \left(F'(\eta) + \frac{n-1}{n+1}F'(\eta) + \frac{n-1}{n+1}\eta F''(\eta)\right). \\ &= -\Big(\frac{n+1}{2}\Big)U_0 (x+b)^{n-1}\left(F'(\eta) + \frac{n-1}{n+1}F'(\eta) + \frac{n-1}{n+1}\eta F''(\eta)\right) \\ &= -\Big(\frac{n+1}{2}\Big)U_0 (x+b)^{n-1}\left(\Big(1 + \frac{n-1}{n+1}\Big)F'(\eta) + (\eta)(\frac{n-1}{n+1})F''(\eta)\right) \end{aligned}$$

$$= \quad -U_0(x+b)^{n-1}\left(nF'(\eta)+\frac{n-1}{2}\eta F''(\eta)\right). \tag{3.8}$$

By the choice of the stream function ψ in (3.7), the continuity equation is already satisfied. It can again be verified using (3.8) and (3.9) in (3.1) as follows.

$$\begin{aligned}\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} &= U_0(x+b)^{n-1}\left(nF'(\eta)+\frac{n-1}{2}\eta F''(\eta)\right)\\ &\quad -U_0(x+b)^{n-1}\left(nF'(\eta)+\frac{n-1}{2}\eta F''(\eta)\right)\\ &= 0.\end{aligned}$$

Hence continuity Equation (3.1)is identically satisfied.

Now we include below the procedure for the conversion of (3.2) in the dimensionless form.

- $$\begin{aligned}\frac{\partial u}{\partial y} &= \frac{\partial}{\partial y}\left(U_0(x+b)^n F'(\eta)\right)\\ &= U_0(x+b)^n\frac{\partial F'}{\partial \eta}\frac{\partial \eta}{\partial y}\\ &= U_0(x+b)^n F''(\eta)\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{n-1}}.\end{aligned}$$

- $$\begin{aligned}v\frac{\partial u}{\partial y} &= -\sqrt{\frac{n+1}{2}U_0\nu(x+b)^{n-1}}\left(F(\eta)+\eta\frac{n-1}{n+1}F'(\eta)\right)\\ &\qquad U_0(x+b)^n F''(\eta)\sqrt{(\frac{n+1}{2})\frac{U_0}{\nu}(x+b)^{n-1}}\\ &= -U_0^2(x+b)^{2n-1}\frac{n+1}{2}F''(\eta)\left(F(\eta)+\eta\frac{n-1}{n+1}F'(\eta)\right)\\ &= -U_0^2(x+b)^{2n-1}\left(\frac{n+1}{2}F''(\eta)F(\eta)\right.\\ &\qquad\left.+\frac{n-1}{2}\eta F'(\eta)F''(\eta)\right).\end{aligned} \tag{3.9}$$

- $$\begin{aligned}u\frac{\partial u}{\partial x} &= U_0(x+b)^n F'(\eta)U_0(x+b)^{n-1}\left(nF'(\eta)+\frac{n-1}{2}\eta F''(\eta)\right)\\ &= U_0^2(x+b)^{2n-1}\left(n(F'(\eta))^2+\frac{n-1}{2}\eta F'(\eta)F''(\eta)\right).\end{aligned} \tag{3.10}$$

Using the values of (3.9) and (3.10)), the left side of (3.2) is as follows.

$$
\begin{aligned}
u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= U_0^2(x+b)^{2n-1}\Big(n(F'(\eta))^2 + (\frac{n-1}{2})(\eta)F'(\eta)F''(\eta)\Big)\\
&\quad - U_0^2(x+b)^{2n-1}(\frac{n+1}{2})F''(\eta)\Big(F(\eta) + (\eta)(\frac{n-1}{n+1})F'(\eta)\Big)\\
&= U_0^2(x+b)^{2n-1}\Big(n(F'(\eta))^2(\eta) + (\frac{n-1}{2})(\eta)F'(\eta)F''(\eta)\\
&\quad - (\frac{n+1}{2})F''(\eta)F(\eta) - (\frac{n-1}{2})(\eta)F''(\eta)F'(\eta)\Big)\\
&= U_0^2(x+b)^{2n-1}\Big(n(F'(\eta))^2 - (\frac{n+1}{2})F''(\eta)F(\eta)\Big)\\
&= n\,U_0^2(x+b)^{2n-1}\Big((F'(\eta))^2 - (\frac{n+1}{2n})F''(\eta)F(\eta)\Big).
\end{aligned}
$$

To convert the right side of (3.2) into the dimensionless form, we proceed as follows.

- $$
\begin{aligned}
\frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)\\
&= \frac{\partial}{\partial y}\left(U_0(x+b)^n F''(\eta)\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{n-1}}\right)\\
&= \frac{\partial}{\partial \eta}\left(U_0(x+b)^n F''(\eta)\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{n-1}}\right)\frac{\partial \eta}{\partial y}\\
&= U_0(x+b)^n\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{n-1}}\,\frac{\partial}{\partial \eta}F''(\eta)\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{n-1}}\\
&= \left(\frac{n+1}{2}\right)\frac{U_0^2}{\nu}(x+b)^{2n-1}F'''(\eta). \qquad (3.11)
\end{aligned}
$$
- $$
\begin{aligned}
\frac{\partial^2 u}{\partial x \partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)\\
&= \frac{\partial}{\partial x}\left(U_0(x+b)^n F''(\eta)\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{n-1}}\right)\\
&= \frac{\partial}{\partial x}U_0(x+b)^n F''(\eta)\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{n-1}}\\
&\quad + U_0(x+b)^n\frac{\partial}{\partial x}F''(\eta)\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{n-1}}\\
&\quad + U_0(x+b)^n F''(\eta)\frac{\partial}{\partial x}\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{n-1}}
\end{aligned}
$$

$$= nU_0(x+b)^{n-1}F''(\eta)\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{n-1}}$$
$$+U_0(x+b)^{n}\frac{\partial}{\partial\eta}F''(\eta)\frac{\partial \eta}{\partial x}\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{n-1}}$$
$$+U_0(x+b)^{n}F''(\eta)\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}}\frac{\partial}{\partial x}(x+b)^{\frac{n-1}{2}}$$
$$= nU_0(x+b)^{n-1}F''(\eta)\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{n-1}}$$
$$+U_0(x+b)^{n}F'''(\eta)\Big(\frac{n-1}{2}\Big)(\eta)(x+b)^{n-1}\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{n-1}}$$
$$+\Big(\frac{n-1}{2}\Big)\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}}(x+b)^{\frac{n-1}{2}-1}$$
$$= nU_0(x+b)^{n-1}F''(\eta)\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{n-1}}$$
$$+U_0(x+b)^{n-1}F'''\eta\Big(\frac{n-1}{2}\Big)(\eta)\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{n-1}}$$
$$+\Big(\frac{n-1}{2}\Big)U_0(x+b)^{n}F''(\eta)\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{n-1}}(x+b)^{-1}$$
$$= nU_0(x+b)^{n-1}F''(\eta)\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{n-1}}$$
$$+\frac{n-1}{2}\eta U_0(x+b)^{n-1}F'''(\eta)\sqrt{\frac{n-1}{2}\frac{U_0}{v}(x+b)^{n-1}}$$
$$+\frac{n+1}{2}\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{n-1}F'''(\eta)}$$
$$= U_0(x+b)^{n-1}\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{n-1}}\Bigg(nF''(\eta)+\frac{n-1}{2}F''(\eta)$$
$$+\frac{n-1}{2}\eta F'''(\eta)\Bigg)$$
$$= U_0(x+b)^{n-1}\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{n-1}}\Bigg((n+\frac{n-1}{2})F''(\eta)$$
$$+\frac{n-1}{2}\eta F'''(\eta)\Bigg)$$
$$= U_0(x+b)^{n-1}\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{n-1}}\Bigg((\frac{3n-1}{2})F''(\eta)$$
$$+\frac{n-1}{2}\eta F'''(\eta)\Bigg)$$

- $uv\dfrac{\partial^2 u}{\partial x\partial y} = -U_0(x+b)^{n}F'(\eta)\sqrt{\dfrac{n+1}{2}\nu U_0(x+b)^{n-1}}\left(F(\eta)+\eta\dfrac{n-1}{n+1}F'(\eta)\right)$

$$U_0(x+b)^{n-1}\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{n-1}}\Big((\frac{3n-1}{2})F''(\eta)$$
$$+\Big(\frac{n-1}{2}\Big)\eta F'''(\eta)\Big)$$
$$=-U_0^3(\ x+b)^{3\ n-2}\frac{n+1}{2}F'(\ \eta)\Bigg(F(\ \eta)F''(\ \eta)(\frac{3\ n-1}{2})$$
$$+\Big(\frac{n-1}{2}\Big)(\ \eta)F(\ \eta)F'''(\ \eta)+\ \eta\frac{n-1}{n+1}(\frac{3\ n-1}{2})F'(\ \eta)F''(\ \eta)$$
$$+\ \eta^2\Big(\frac{n-1}{n+1}\Big)\Big(\frac{n-1}{2}\Big)F'''(\ \eta)F'(\ \eta)\Bigg)$$
$$=-U_0^3(\ x+b)^{3\ n-2}\frac{n+1}{2}\Bigg(\frac{3\ n-1}{2}F(\ \eta)F'(\ \eta)F''(\ \eta)$$
$$+\Big(\frac{n-1}{2}\Big)\ \eta F(\ \eta)F'(\ \eta)F'''(\ \eta)+\ \eta\Big(\frac{n-1}{n+1}\Big)\frac{3\ n-1}{2}F''(\ \eta)$$
$$(F'(\ \eta))^2+\ \eta^2\Big(\frac{n-1}{n+1}\Big)\Big(\frac{n-1}{2}\Big)F'''(\ \eta)(F'(\ \eta))^2\Bigg).$$

- $$2uv\frac{\partial^2 u}{\partial x\partial y}=-2\ U_0^3(\ x+b)^{3n-2}\frac{n+1}{2}\Bigg(\frac{3\ n-1}{2}F(\ \eta)F'(\ \eta)F''(\ \eta)$$
$$+\Big(\frac{n-1}{2}\Big)(\eta)F(\ \eta)F'(\ \eta)F'''(\ \eta)+\ \eta\Big(\frac{n-1}{n+1}\Big)\Big(\frac{3n-1}{2}\Big)F''(\ \eta)$$
$$(F'(\ \eta))^2+\ \eta^2\Big(\frac{n-1}{n+1}\Big)\Big(\frac{n-1}{2}\Big)F'''(\ \eta)(F'(\ \eta))^2\Bigg)$$
$$=U_0^3(\ x+b)^{3\ n-2}(\ n+1)\Bigg(\frac{3\ n-1}{2}F(\ \eta)F'(\ \eta)F''(\ \eta)$$
$$+\Big(\frac{n-1}{2}\Big)(\ \eta)F(\eta)F'(\ \eta)F'''(\ \eta)+\ \eta\Big(\frac{n-1}{n+1}\Big)\Big(\frac{3\ n-1}{2}\Big)F''(\ \eta)$$
$$(F'(\ \eta))^2+\ \eta^2\Big(\frac{n-1}{n+1}\Big)\Big(\frac{n-1}{2}\Big)F'''(\ \eta)(F'(\ \eta))^2\Bigg). \tag{3.12}$$

- $$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial\ x}\Big(\frac{\partial u}{\partial x}\Big)$$
$$=\frac{\partial}{\partial x}\Big(\ nU_0(\ x+b)^{\ n-1}F'(\ \eta)+\frac{n-1}{2}U_0(\ x+b)^{\ n-1}(\eta)F''(\ \eta)\Big)$$
$$=\ n\frac{\partial}{\partial x}U_0(\ x+b)^{n-1}F'(\ \eta)+nU_0(x+b)^{\ n-1}\frac{\partial}{\partial\eta}F'(\ \eta)\frac{\partial\ \eta}{\partial x}$$
$$+\Big(\frac{n-1}{2}\Big)\frac{\partial}{\partial x}(\ \eta)F''(\eta)+\Big(\frac{n-1}{2}\Big)U_0(\ x+b)^{\ n-1}\frac{\partial\ \eta}{\partial x}F''(\ \eta)$$
$$+\Big(\frac{n-1}{2}\Big)U_0(\ x+b)^{\ n-1}(\ \eta)\frac{\partial}{\partial\ \eta}F''(\ \eta)\frac{\partial\eta}{\partial x}$$

$$
\begin{aligned}
&= n(n-1)U_0(x+b)^{n-1}F'(\eta) + \left(n\frac{n-1}{2}\right)U_0(x+b)^{n-1}(x+b)^{n-1} \\
&\quad \eta F''(\eta) + \left(\frac{(n-1)^2}{2}\right)U_0(x+b)^{n-2}(\eta)F''(\eta) \\
&\quad + \left(\frac{n-1}{2}\right)^2\left(\frac{n-1}{2}\right)U_0(x+b)^{n-1}(x+b)^{n-1} \\
&\quad \eta F''(\eta) + \left(\frac{n-1}{2}\right)^2\left(\frac{n-1}{2}\right)U_0(x+b)^{n-1}(x+b)^{n-1}(\eta)F'''(\eta) \\
&= n(n-1)U_0(x+b)^{n-2}F' + \left(\frac{n(n-1)}{2}\right)U_0(x+b)^{n-2}\,\eta F''(\eta) \\
&\quad + \left(\frac{n-1}{2}\right)^2(\eta)U_0(x+b)^{n-2}F'' + \left(\frac{(n-1)^2}{2}\right)U_0(x+b)^{n-2}(\eta) \\
&\quad F''(\eta) + \left(\frac{n-1}{2}\right)^2\eta^2U_0(x+b)^{n-2}F'''(\eta) \\
&= U_0(x+b)^{n-2}(n-1)\Bigg(nF' + \frac{n}{2}F''(\eta) + (\eta)\left(\frac{n-1}{4}\right)F''(\eta) \\
&\quad + \left(\frac{n-1}{4}\right)F'''(\eta)\,\eta^2 + \left(\frac{n-1}{2}\right)(\eta)F''(\eta)\Bigg) \\
&= U_0(x+b)^{n-2}(n-1)\Bigg(nF' + \left(\frac{n}{2} + \frac{n-1}{4} + \frac{n-1}{2}\right)(\eta)F''(\eta) \\
&\quad + \left(\frac{n-1}{4}\right)F'''(\eta)\,\eta^2\Bigg) \\
&= U_0(x+b)^{n-2}(n-1)\Bigg(nF' + \left(\frac{5n-3}{4}\right)(\eta)F''(\eta) \\
&\quad + \left(\frac{n-1}{4}\right)F'''(\eta)\,\eta^2\Bigg).
\end{aligned}
$$

$$
\begin{aligned}
\bullet\; u^2\frac{\partial^2 u}{\partial x^2} &= U_0^2(x+b)^{2n}(F'(\eta))^2\Bigg(U_0(x+b)^{n-2}(n-1)\Bigg(nF' \\
&\quad + \left(\frac{5n-3}{4}\right)(\eta)F''(\eta) + \left(\frac{n-1}{4}\right)F'''(\eta)\,\eta^2\Bigg)\Bigg) \\
&= U_0^3(x+b)^{3n-2}(n-1)\Bigg(n(F')^3 + \left(\frac{5n-3}{4}\right)(\eta)F''(\eta)(F')^2 \\
&\quad + \left(\frac{n-1}{4}\right)(F')^2F'''(\eta)(\eta^2)\Bigg). \qquad (3.13)
\end{aligned}
$$

$$
\begin{aligned}
\bullet\; v^2\frac{\partial^2 u}{\partial y^2} &= \left(\frac{n+1}{2}\right)(\nu)U_0(x+b)^{n-1}\left(F(\eta) + (\eta)\frac{n-1}{n+1}F'(\eta)\right)^2 \\
&\quad \left(\frac{n+1}{2}\frac{U_0^2}{\nu}(x+b)^{2n-1}F'''(\eta)\right) \\
&= \left(\frac{n+1}{2}\right)^2U_0^3(x+b)^{3n-2}F'''(\eta)\left(F(\eta) + (\eta)\left(\frac{n-1}{n+1}\right)F'(\eta)\right)^2
\end{aligned}
$$

$$= (\frac{n+1}{2})^2 U_0^3(x+b)^{3\,n-2}\Big((F(\,\eta))^2F'''(\,\eta) + (\,\eta)^2\Big(\frac{n-1}{n+1}\Big)^2 (F'(\,\eta))^2F'''(\,\eta)$$
$$+ 2F(\,\eta)F'(\,\eta)F'''(\,\eta)(\,\eta)\frac{n-1}{n+1}\Big). \tag{3.14}$$

Using (3.12)- (3.14), we get

$$\bullet\; u^2\frac{\partial^2 u}{\partial x^2} + v^2\frac{\partial^2 u}{\partial y^2} + 2uv\frac{\partial^2 u}{\partial xy}$$
$$= U_0^2(x+b)^{3\,n-2}(\,n-1)\Big(\,n(F'(\,\eta))^3 + \Big(\frac{5\,n-3}{4}\Big)(\,\eta)F''(\,\eta)(F'(\,\eta))^2$$
$$+ \Big(\frac{n-1}{4}\Big)(F\,')^2F'''(\,\eta)(\,\eta)^2\Big) + \Big(\frac{n+1}{2}\Big)^2 U_0^3(\,x+b)^{3\,n-2}\Big((F(\,\eta))^2F'''(\,\eta)$$
$$+\ \eta^2\Big(\frac{n-1}{n+1}\Big)^2(F'(\,\eta))^2F'''(\,\eta) + 2F(\,\eta)F'(\,\eta)F'''(\,\eta)(\,\eta)\Big(\frac{n-1}{n+1}\Big)\Big)$$
$$- U_0^3(\,x+b)^{3\,n-2}(n+1)\Big(\frac{3n-1}{2}F(\eta)F'(\,\eta)F''(\,\eta)$$
$$+ \Big(\frac{n-1}{2}\Big)(\,\eta)F(\,\eta)F'(\,\eta)F'''(\,\eta) +\ \eta\Big(\frac{n-1}{n+1}\Big)\Big(\frac{3\,n-1}{2}\Big)F''(\,\eta)(F'(\,\eta))^2$$
$$+\ \eta^2\Big(\frac{n-1}{n+1}\Big)\Big(\frac{n-1}{2}\Big)F'''(\,\eta)(F'(\,\eta))^2\Big)$$
$$= U_0^3(\,x+b)^{3n-2}\Big(\,n(n-1)(F')^3 + (n-1)\Big(\frac{5\,n-3}{4}\Big)(\,\eta)F''(\,\eta)(F')^2$$
$$+ \Big(\frac{(\,n-1)^2}{4}\Big)(F')^2F'''(\,\eta)(\eta)^2 + (\frac{n+1}{2})^2F'''(\eta)F^2(\eta)$$
$$+\ \eta^2\Big(\frac{(\,n+1)^2}{2}\Big)\Big(\frac{n-1}{n+1}\Big)^2(F'(\,\eta))^2F'''(\,\eta)$$
$$+ 2\,\eta\Big(\frac{n-1}{n+1}\Big)\Big(\frac{n+1}{2}\Big)^2F(\,\eta)F'(\eta)F'''$$
$$- (n+1)\Big(\frac{3\,n-1}{2}\Big)F(\,\eta)F'(\eta)F''(\,\eta) - \eta\Big(\frac{n+1}{2}\Big)(\,n-1)F(\,\eta)F'(\,\eta)F'''(\,\eta)$$
$$- \eta\Big(\frac{n+1}{2}\Big)\Big(\frac{n-1}{n+1}\Big)(3\,n-1)(F'(\,\eta))^2F''(\,\eta)$$
$$-\ \eta^2(\,n-1)\Big(\frac{n-1}{n+1}\Big)\Big(\frac{n+1}{2}\Big)(F'(\,\eta))^2F'''(\,\eta)\Big)$$
$$= U_0^3(\,x+b)^{3\,n-2}\Big(n(\,n-1)(F')^3 + \Big(\frac{5\,n-3}{2} - (3n-1)\Big)(\eta)\Big(\frac{n-1}{2}\Big)$$
$$F''(\eta)(F')^2 + \Big(\frac{n+1}{2}\Big)^2F'''(\,\eta)F(\,\eta)^2 + \Big((\frac{(\,n+1)^2}{2})\Big)\Big(\frac{n-1}{n+1}\Big)^2F'''(\,\eta)(F')^2$$
$$+\ \eta(n-1)\Big(\frac{n+1}{2}\Big)F(\eta)F'(\,\eta)F'''(\eta).$$

$$- \Big(\frac{n+1}{2}\Big)(3\ n-1)F(\eta)F'(\ \eta)F''(\ \eta) - \Big(\frac{n+1}{2}\Big)(\eta)(\ n-1)F(\ \eta)F'(\ \eta)F'''$$
$$-\ \eta^2\Big(\frac{(\ n-1)^2}{2}\Big)\Big(F')^2F'''(\ \eta)\Big)U_0^3(\ x+b)^{3\ n-2}\Bigg(n(\ n-1)(F')^3$$
$$-\ \eta\Big(\frac{n-1}{2}\Big)\Big(\frac{n+1}{2}\Big)F''(\ \eta)(F')^2 - (3\ n-1)\Big(\frac{n+1}{2}\Big)F(\ \eta)F'(\ \eta)F''(\ \eta)$$
$$+\Big(\frac{n+1}{2}\Big)^2F(\ \eta)^2F'''(\ \eta)\Bigg)$$
$$= U_0^3(\ x+b)^{3\ n-2}\Big(\frac{n+1}{2}\Big)\Bigg(\frac{2\ n(\ n-1)}{n+1}(F')^3 - (\ \eta)\Big(\frac{n-1}{2}\Big)F''(\ \eta)(F')^2$$
$$-(3\ n-1)F(\ \eta)F'(\ \eta)F''(\ \eta) + \Big(\frac{n+1}{2}\Big)F(\ \eta)^2F'''(\ \eta)\Bigg). \quad (3.15)$$

- $U_e = U_\infty(x+b)^n$
- $\dfrac{dU_e}{dx} = nU_\infty(\ x+b)^{\ n-1}$
- $U_e\dfrac{dU_e}{dx} = U_\infty(\ x+b)^{\ n}nU_\infty(\ x+b)^{\ n-1}$
  $= nU_\infty^2(\ x+b)^{2\ n-1}$
- $\dfrac{\partial U_e}{\partial x} = \dfrac{\partial}{\partial\ x}U_\infty(\ x+b)^{\ n}$
  $= nU_\infty(\ x+b)^{\ n-1}$
- $\dfrac{\partial^2 U_e}{\partial x^2} = \dfrac{\partial}{\partial\ x}\left(\dfrac{\partial U_e}{\partial\ x}\right)$
  $= \dfrac{\partial}{\partial\ x}\left(\ nU_\infty(\ x+b)^{\ n-1}\right)$
  $= n(\ n-1)U_\infty(\ x+b)^{\ n-2}$
- $\lambda_1U_e^2\dfrac{\partial^2U_e}{\partial x^2} = \lambda_1U_\infty^2(\ x+b)^{2\ n}n(\ n-1)U_\infty(\ x+b)^{\ n-2}$
  $= \lambda_1n(\ n-1)U_\infty^3(\ x+b)^{3\ n-2}$
- $U_e\dfrac{dU_e}{dx} + \lambda_1U_e^2\dfrac{\partial^2U_e}{\partial x^2} = nU_\infty^2(x+b)^{2n-1}$
  $+\ \lambda_1n(n-1)U_\infty^3(x+b)^{3n-2} \quad (3.16)$
- $\nu\dfrac{\partial u^2}{\partial y^2} = \nu(\dfrac{n+1}{2})\dfrac{U_0^2}{\nu}(x+b)^{2n-1}F'''(\eta). \quad (3.17)$

Using (3.15) -(3.17) in the right side of (3.2), we get

$$U_e\frac{dU_e}{dx}+\lambda_1 U_e^2\frac{\partial^2 U_e}{\partial x^2}+\nu\frac{\partial^2 u}{\partial y^2}-\lambda_1\left(2uv\frac{\partial^2 u}{\partial x\partial y}+v^2\frac{\partial^2 u}{\partial y^2}+u^2\frac{\partial^2 u}{\partial x^2}\right)$$
$$=nU_\infty^2(x+b)^{2n-1}+\lambda_1 n(n-1)U_\infty^3(x+b)^{3n-2}+(\frac{n+1}{2})U_0^2(x+b)^{2n-1}F'''(\eta)$$
$$-\lambda_1\Bigg(U_0^3(x+b)^{3n-2}(\frac{n+1}{2})\Bigg(n(n-1)(F')^3-(\eta)(\frac{n-1}{2})F''(\eta)(F')^2$$
$$-(3n-1)F(\eta)F'(\eta)F''(\eta)+(\frac{n+1}{2})F(\eta)^2F'''(\eta)\Bigg)\Bigg)$$
$$=n\ U_\infty^2(\ x+b)^{2\ n-1}+\lambda_1\ n(\ n-1)U_\infty^3(\ x+b)^{3\ n-2}+(\frac{n+1}{2})U_0^2(\ x+b)^{2\ n-1}$$
$$\Bigg(F'''(\ \eta)-\ \lambda_1\Bigg(\ U_0(\ x+b)^{\ n-1}\Bigg(n(\ n-1)(\ F')^3-(\ \eta)(\frac{n-1}{2})F''(\ \eta)(F')^2$$
$$-(3\ n-1)F(\ \eta)F'(\ \eta)F''(\ \eta)+(\frac{n+1}{2})F(\ \eta)^2F'''(\ \eta)\Bigg)\Bigg)\Bigg)$$
$$=(\frac{n+1}{2})U_0^2(\ x+b)^{2\ n-1}\Bigg(\Big(\frac{2\ n}{n+1}\Big)\frac{U_\infty^2}{U_0^2}+\Big(\frac{2\ n(\ n-1)}{n+1}\Big)\lambda_1 U_0(\ x+b)^{\ n-1}\frac{U_\infty^3}{U_0^3}$$
$$+F'''(\ \eta)-\lambda_1 U_0(\ x+b)^{\ n-1}\Big(\frac{2\ n(\ n-1)}{n+1}\Big)(F')^3-(\eta)\Big(\frac{n-1}{2}\Big)F''(\ \eta)(F')^2$$
$$-(3\ n-1)F(\ \eta)F'(\ \eta)F''(\ \eta)+\Big(\frac{n+1}{2}\Big)F(\ \eta)^2\Bigg).$$

Hence the dimensionless form of (3.2)becomes

$$nU_0^2(\ x+b)^{2\ n-1}\Bigg((\ F'(\ \eta))^2-\Big(\frac{n+1}{2\ n}\Big)\ F''(\ \eta)F(\ \eta)\Bigg)=\Big(\frac{n+1}{2}\Big)U_0^2(x+b)^{2n-1}$$
$$\Bigg(\Big(\frac{2\ n}{n+1}\Big)\frac{U_\infty^2}{U_0^2}+\Big(\frac{2n(\ n-1)}{n+1}\Big)\lambda_1 U_0(\ x+b)^{\ n-1}\frac{U_\infty^3}{U_0^3}+F'''(\ \eta)$$
$$-\lambda_1 U_0(\ x+b)^{\ n-1}\Bigg(\Big(\frac{2\ n(\ n-1)}{n+1}\Big)(\ F')^3-(\ \eta)\Big(\frac{n-1}{2}\Big)\ F''(\ \eta)(F')^2$$
$$-(3\ n-1)F(\ \eta)\ F'(\ \eta)F''(\ \eta)+\Big(\frac{n+1}{2}\Big)F(\ \eta)^2\Bigg)\Bigg)$$

$$\Rightarrow \left(\frac{2\ n}{n+1}\right)\left((\ F'(\ \eta))^2 - \left(\frac{n+1}{2\ n}\right)\ F''(\ \eta)F(\ \eta)\right) = \left(\frac{2\ n}{n+1}\right)\frac{U_\infty^2}{U_0^2}$$
$$+ 2\frac{n(n-1)}{n+1}\lambda_1 U_0(x+b)^{n-1}\frac{U_\infty^3}{U_0^3} + F'''(\ \eta) - \lambda_1 U_0(x+b)^{n-1}\left(\frac{2n(n-1)}{n+1}(F')^3\right.$$
$$\left. - (\ \eta)(\frac{n-1}{2})F''(\ \eta)(F')^2 - (3\ n-1)F(\ \eta)F'(\ \eta)F''(\ \eta) + (\frac{n+1}{2})F(\ \eta)^2\right)$$
$$\Rightarrow \frac{2\ n}{n+1}(F'(\ \eta))^2 - F''(\ \eta)F(\ \eta) = \frac{2\ n}{n+1}\ A^2 + 2\ \beta\frac{n(\ n-1)}{n+1}A^3$$
$$+ F'''(\ \eta) - \beta\left(\frac{2\ n(\ n-1)}{n+1}(F')^3 \quad \left(\because A = \frac{U_\infty}{U_0}\right)\left(\because \beta = \ \lambda_1 U_0(x+b)n-1\right)\right.$$
$$\left. - (\ \eta)(\frac{n-1}{2})\ F''(\eta)(F')^2 - (3\ n-1)F(\ \eta)F'(\ \eta)\ F''(\ \eta) + (\frac{n+1}{2})F(\ \eta)^2\right)$$
$$\Rightarrow \quad F'''(\ \eta) + \ F''(\ \eta)\ F(\ \eta) - \frac{2\ n}{n+1}(F'(\ \eta))^2 + \frac{2\ n}{n+1}A^2 + \frac{2\ n(\ n-1)}{n+1}\ \beta A^3$$
$$+ \ \beta\left((3\ n-1)F(\ \eta)F'(\ \eta)F''(\ \eta) - \frac{2n(\ n-1)}{n+1}(F'(\ \eta))^3 + (\ \eta)(\frac{n-1}{2})\right.$$
$$\left.(F'(\ \eta))^2F''(\ \eta) - (\frac{n+\ 1}{2})F^2(\ \eta)\ F'''(\ \eta)\right) = 0.$$

Now we include below the procedure for the conversion of (3.3) into the dimensionless form.

- $$\Theta(\ \eta) = \frac{T-\ T_\infty}{T_w -\ T_o}$$

$$\Rightarrow T -\ T_\infty = (\ T_w -\ T_0)\,\Theta(\ \eta)$$
$$\Rightarrow \quad T = (T_w - T_0)\,\Theta(\eta) + T_\infty$$

- $$\frac{\partial T}{\partial\ x} = (\ T_w - T_0)\,\frac{\partial\Theta(\ \eta)}{\partial\ x}$$

$$= (T_w - T_0)\,\frac{\partial\Theta(\ \eta)}{\partial\ \eta}\frac{\partial\ \eta}{\partial\ x}$$
$$= (T_w - T_0)\,\left(\frac{n-1}{2}\right)\ \eta(\ x+b)^{-1}\Theta'(\ \eta)$$
$$= \left(\frac{n-1}{2}\right)(\ T_w - T_0)\ \ \eta(\ x+b)^{-1}\Theta'(\ \eta). \tag{3.18}$$

- $$\begin{aligned}\frac{\partial T}{\partial y} &= \frac{\partial}{\partial\, y}\Big(\,(\,T_w - T_0)\;\;\Theta(\,\eta) + T_\infty\Big)\\ &= (\,T_w - T_0)\,\frac{\partial\,\Theta(\,\eta)}{\partial\,\eta}\frac{\partial\,\eta}{\partial y}\\ &= (\,T_w - T_0)\;\;\Theta'(\,\eta)\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{n-1}} \qquad (3.19)\end{aligned}$$

- $$\begin{aligned}\frac{\partial^2 T}{\partial y^2} &= \frac{\partial}{\partial y}\Big(\frac{\partial T}{\partial\, y}\Big)\\ &= \frac{\partial}{\partial y}\Bigg(\,(\,T_w - T_0)\;\;\Theta'(\,\eta)\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(\,x+b)^{\,n-1}}\Bigg)\\ &= (\,T_w - T_0)\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(\,x+b)^{\,n-1}}\frac{\partial\,\theta}{\partial\,\eta}\frac{\partial\,\eta}{\partial\,y}\\ &= \sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(\,x+b)^{\,n-1}}\,(\,T_w - T_0)\;\;\Theta''(\,\eta)\\ &\quad \sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(\,x+b)^{\,n-1}}\\ &= \frac{n+1}{2}\frac{U_0}{\nu}(\,x+b)^{\,n-1}\,(\,T_w - T_0)\;\;\Theta''(\,\eta)\end{aligned}$$

- $$\begin{aligned}\frac{\partial^2 T}{\partial\,x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right)\\ &= \frac{\partial}{\partial x}\left(\left(\frac{n-1}{2}\right)(\,T_w - T_0)\;\;\eta(x+b)^{-\,1}\Theta'(\,\eta)\right)\\ &= \frac{n-1}{2}\,(\,T_w - T_0)\left(\frac{\partial}{\partial x}(\,x+b)^{-\,1}\eta\;\Theta'(\,\eta) + (\,x+b)^{-\,1}\frac{\partial\,\eta}{\partial x}\;\Theta'(\,\eta)\right.\\ &\quad \left. + (\,x+b)^{-\,1}(\,\eta)\frac{\partial\;\Theta'(\,\eta)}{\partial\eta}\frac{\partial\,\eta}{\partial x}\right)\\ &= \frac{n-1}{2}\,(T_w - T_0)\left(-\,(x+b)^{-2}\eta\Theta'(\eta) + \Big((x+b)^{-1}\Theta'(\eta)\right.\\ &\quad \left. + (x+b)^{-1}\eta\Theta''(\eta)\Big)\frac{\partial\eta}{\partial x}\right)\\ &= \frac{n-1}{2}\,(T_w - T_0)\left(-\,(x+b)^{-2}\eta\Theta'(\eta)\right.\\ &\quad \left. + \Big((x+b)^{-1}\Theta'(\eta) + (x+b)^{-1}\eta\Theta''(\eta)\Big)\frac{n-1}{2}\eta(x+b)^{-1}\right)\end{aligned}$$

$$= \frac{n-1}{2}\left(T_w - T_0\right)\Bigg(- (x+b)^{-2}\eta\Theta'(\eta)$$

$$+ \Big(\frac{n-1}{2}(x+b)^{-2}\Theta'(\eta) + (x+b)^{-2}\frac{n-1}{2}\eta\Theta''(\eta)\Big)\frac{\partial \eta}{\partial x}\Bigg)$$

$$= \frac{n-1}{2}\left(T_w - T_0\right)(x+b)^{-2}\Bigg(- \eta\Theta'(\eta) + \frac{n-1}{2}\eta\Theta'$$

$$+ \frac{n-1}{2}\eta^2\Theta''\Bigg)$$

$$= \frac{n-1}{2}\left(T_w - T_0\right)(x+b)^{-2}\Bigg(\Big(- 1 + \frac{n-1}{2}\Big)\eta\Theta' + \frac{n-1}{2}\eta^2\Theta''\Bigg)$$

$$= \frac{n-1}{2}\left(T_w - T_0\right)(x+b)^{-2}\Bigg(\Big(\frac{n-3}{2}\Big)\eta\Theta' + \frac{n-1}{2}\eta^2\Theta''\Bigg)$$

$$= \Big(\frac{n-1}{2}\Big)^2\left(T_w - T_0\right)(x+b)^{-2}\eta\Bigg(- \Theta' + \eta\Theta''\Bigg). \tag{3.20}$$

- $$u^2\frac{\partial^2 T}{\partial^2 x} = U_0^2(x+b)^{2n}F'^2(\eta)\Bigg(\frac{n-1}{2}\Bigg)^2\left(T_w - T_0\right)(x+b)^{-2}$$

$$\Bigg(\Theta''(\eta)(\eta) + \Theta'(\eta)\Bigg)\eta$$

$$= U_0^2(x+b)^{2n-2}F'^2(\eta)\Bigg(\frac{n-1}{2}\Bigg)^2\left(T_w - T_0\right)\eta\Bigg(- \Theta' + \eta\Theta''\Bigg). \tag{3.21}$$

- $$\frac{\partial^2\, T}{\partial\, x \partial y} = \frac{\partial}{\partial\, y}\left(\frac{\partial\, T}{\partial\, x}\right)$$

$$= \frac{\partial}{\partial\, y}\Bigg(\Big(\frac{n-1}{2}\Big)\Big(\, T_w - T_0\Big)\, \eta(\, x+b)^{-\,1}\,\Theta'(\,\eta)\Bigg)$$

$$= \frac{n-1}{2}\left(\, T_w - T_0\right)(\, x+b)^{-\,1}\,\Theta'(\,\eta)\frac{\partial v \eta}{\partial y}$$

$$+ \frac{n-1}{2}\left(\, T_w - T_0\right)(\, x+b)^{-\,1}(\eta)\,\Theta''\frac{\partial\,\eta}{\partial\, y}$$

$$= \frac{n-1}{2}\left(\, T_w - T_0\right)(\, x+b)^{-\,1}\Bigg(\,\Theta'(\,\eta) + \,\eta\Theta''(\,\eta)\Bigg)\frac{\partial\,\eta}{\partial\, y}$$

$$= \frac{n-1}{2}\left(\, T_w - T_0\right)(\, x+b)^{-\,1}\Bigg(\,\Theta'(\,\eta) + \,\eta\Theta''(\,\eta)\Bigg)$$

$$\sqrt{\frac{n+1}{2}\frac{U_0}{v}(\, x+b)^{\,n-1}}. \tag{3.22}$$

- $$u\frac{\partial T}{\partial x} = U_0(x+b)^{n}F'(\eta)\left(\frac{n-1}{2}\right)(T_w - T_0)\ \eta(x+b)^{-1}\ \Theta'(\eta)$$
$$= \left(\frac{n-1}{2}\right)\eta\,(T_w - T_0)\ U_0(x+b)^{n-1}F'(\eta)\ \Theta'(\eta) \qquad (3.23)$$

- $$v\frac{\partial T}{\partial y} = -\sqrt{\frac{n+1}{2}\frac{U_0}{v}(x+b)^{n-1}}\left(F(\eta) + \eta\frac{n-1}{n+1}F'(\eta)\right)$$
$$(T_w - T_0)\ \Theta'(\eta)\sqrt{\frac{n+1}{2}\frac{U_0}{v}(x+b)^{n-1}}$$
$$= -\left(\frac{n+1}{2}\right)(T_w - T_0)\,U_0(x+b)^{n-1}\ \Theta'(\eta)$$
$$\left(F(\eta) + \eta\frac{n-1}{n+1}F'(\eta)\right)$$

- $$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \left(\frac{n-1}{2}\right)\eta\,(T_w - T_0)\ U_0(x+b)^{n-1}F'(\eta)\ \Theta'(\eta)$$
$$-\left(\frac{n+1}{2}\right)(T_w - T_0)\ U_0(x+b)^{n-1}\ \Theta'(\eta)$$
$$\left(F(\eta) + \eta\frac{n-1}{n+1}F'(\eta)\right)$$
$$= U_0(x+b)^{n-1}\Big(T_w - T_0\Big)\left(\left(\frac{n-1}{2}\right)\eta F'(\eta)\ \Theta'(\eta)\right.$$
$$\left.-\left(\frac{n-1}{2}\right)F(\eta)\Theta'(\eta) - \left(\left(\frac{n-1}{2}\right)\eta F'(\eta)\ \Theta'(\eta)\right)\right)$$
$$= -\left(\frac{n-1}{2}\right)U_0(x+b)^{n-1}\Big(T_w - T_0\Big)F(\eta)\ \Theta'(\eta) \qquad (3.24)$$

- $$uv\frac{\partial^2 T}{\partial x\partial y} = U_0(x+b)^{n}F'(\eta) - \sqrt{(\frac{n+1}{2})U_0\nu(x+b)^{n-1}}$$
$$\left(F(\eta) + \eta\frac{n-1}{n+1}F'(\eta)\right)$$
$$\left(\frac{n-1}{2}\right)\Big(T_w - T_0\Big)(x+b)^{-1}\Big(\Theta'(\eta) + \eta\Theta''(\eta)\Big)$$
$$\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{n-1}}$$
$$= -U_0^2(x+b)^{2n-2}\left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)\Big(T_w - T_0\Big)$$
$$F'(\eta)\Big(\Theta'(\eta) + \eta\Theta''(\eta)\Big)$$
$$\left(F(\eta) + \eta\frac{n-1}{n+1}F'(\eta)\right)$$

$$= -U_0^2(x+b)^{2n-2}\Big(\frac{n^2-1}{4}\Big)\Big(T_w-T_0\Big)F'(\eta)\Bigg(F(\eta)\Theta'(\eta)$$

$$+\eta\frac{n-1}{n+1}F'(\eta)\Theta'(\eta)+F(\eta)\eta\Theta''(\eta)+\eta^2\frac{n-1}{n+1}\Theta''(\eta)F'(\eta)\Bigg)$$

$$= -U_0^2(x+b)^{2n-2}\Big(\frac{n^2-1}{4}\Big)\Big(T_w-T_0\Big)\Bigg(F(\eta)\Theta'(\eta)F'(\eta)$$

$$+\eta\frac{n-1}{n+1}\Big(F'(\eta)\Big)^2\Theta'(\eta)+F(\eta)\eta\Theta''(\eta)F'(\eta)$$

$$+\eta^2\frac{n-1}{n+1}\Theta''(\eta)\Big(F'(\eta)\Big)^2\Bigg)$$

- $$2uv\frac{\partial^2 T}{\partial x\partial y} = U_0^2(x+b)^{2n-2}\Big(\frac{n^2-1}{2}\Big)\Big(T_w-T_0\Big)\Bigg(F(\eta)\Theta'(\eta)F'(\eta)$$

$$+\eta\frac{n-1}{n+1}\Big(F'(\eta)\Big)^2\Theta'(\eta)+F(\eta)\eta\Theta''(\eta)F'(\eta)$$

$$+\eta^2\frac{n-1}{n+1}\Theta''(\eta)\Big(F'(\eta)\Big)^2\Bigg)$$

- $$v^2\frac{\partial^2 T}{\partial y^2} = (\frac{n+1}{2})U_0\nu(x+b)^{n-1}\Bigg(F(\eta)+\eta\frac{n-1}{n+1}F'(\eta)\Bigg)^2$$

$$\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{n-1}(T_w-T_0)\Theta''(\eta)$$

$$=\Big(\frac{n+1}{2}\Big)^2U_0^2(x+b)^{2n-2}(T_w-T_0)\Bigg(F(\eta)$$

$$+\eta\frac{n-1}{n+1}F'(\eta)\Bigg)^2\Theta''(\eta). \tag{3.25}$$

- $$u\frac{\partial u}{\partial x}\frac{\partial T}{\partial x} = U_0(x+b)^nF'(\eta)U_0(x+b)^{n-1}\Bigg(nF'(\eta)$$

$$+\Big(\frac{n-1}{2}\Big)(\eta)F''(\eta)\Bigg)\Big(\frac{n-1}{2}\Big)(T_w-T_0)\eta(x+b)^{-1}\Theta'(\eta)$$

$$=U_0^2(x+b)^{2n-2}(T_w-T_0)\Big(\frac{n-1}{2}\Big)\eta\Theta'(\eta)F'(\eta)$$

$$\Bigg(nF'(\eta)+\Big(\frac{n-1}{2}\Big)(\eta)F''(\eta)\Bigg)$$

$$=U_0^2(x+b)^{2n-2}(T_w-T_0)\Big(\frac{n-1}{2}\Big)\eta$$

$$\Bigg(n(F'(\eta))^2\Theta'(\eta)+\Big(\frac{n-1}{2}\Big)(\eta)F'(\eta)\Theta'(\eta)F''(\eta)\Bigg). \tag{3.26}$$

$$\bullet \quad v\frac{\partial u}{\partial y}\frac{\partial T}{\partial x} = -\sqrt{(\frac{n+1}{2})U_0\nu(x+b)^{n-1}}\left(F(\eta)+\eta\frac{n-1}{n+1}F'(\eta)\right)$$
$$U_0(x+b)^nF''(\eta)\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{n-1}}$$
$$\left(\frac{n-1}{2}\right)(T_w-T_0)\,\eta(x+b)^{-1}\Theta'(\eta)$$
$$= -U_0^2(x+b)^{2n-1}\,(T_w-T_0)\,(\frac{n+1}{2})(\frac{n-1}{2})(\eta)$$
$$\left(F''(\eta)F(\eta)\Theta'(\eta)+\eta\frac{n-1}{n+1}F''(\eta)F'(\eta)\Theta'(\eta)\right)$$

$$\bullet\, u\frac{\partial u}{\partial x}\frac{\partial T}{\partial x}+v\frac{\partial u}{\partial y}\frac{\partial T}{\partial x} = U_0^2(x+b)^{2n-2}\,(T_w-T_0)$$
$$\Big(\frac{n-1}{2}\Big)\eta\bigg(n(F'(\eta))^2\Theta'(\eta)$$
$$+\Big(\frac{n-1}{2}\Big)(\eta)F'(\eta)\Theta'(\eta)F''(\eta)\bigg)$$
$$-U_0^2(x+b)^{2n-1}\,(T_w-T_0)\,(\frac{n+1}{2})(\frac{n-1}{2})(\eta)$$
$$\left(F''(\eta)F(\eta)\Theta'(\eta)+\eta\frac{n-1}{n+1}F''(\eta)F'(\eta)\Theta'(\eta)\right)$$
$$= U_0^2(x+b)^{2n-2}\,(T_w-T_0)\,\frac{n-1}{2}\bigg(n(\eta)(F'(\eta))^2\Theta'(\eta)$$
$$-(\eta)(\frac{n+1}{2})F''(\eta)F(\eta)\Theta'(\eta)\bigg)$$
$$= U_0^2(x+b)^{2n-2}\,(T_w-T_0)\,(\eta)\bigg(n(F'(\eta))^2\Theta'(\eta)$$
$$+\Big(\frac{n-1}{2}\Big)(\eta)F'(\eta)\Theta'(\eta)F''(\eta)-\frac{n+1}{2}F''(\eta)\Theta'(\eta)F(\eta)$$
$$-\eta\frac{n-1}{2}F''(\eta)\Theta'(\eta)F'(\eta)\bigg)$$
$$= U_0^2(x+b)^{2n-2}\,(T_w-T_0)\,(\eta)\frac{n-1}{2}\bigg(n(F'(\eta))^2\Theta'(\eta)$$
$$-\frac{n+1}{2}F''(\eta)\Theta'(\eta)F(\eta)\bigg). \tag{3.27}$$

Using the values in Eq. (3.3),as follows.

$$\Theta''(\eta) + PrF(\eta)\Theta'(\eta) + Pr\gamma\Bigg(\frac{n-3}{2}F(\eta)F'(\eta)\Theta'(\eta) - \frac{n+1}{2}(F(\eta))^2$$
$$\Theta''(\eta)\Bigg) + Pr(S+\theta)\Bigg(\gamma F(\eta)F''(\eta) - \frac{2\,n}{n+1}\gamma(F'(\eta))^2$$
$$-\frac{2}{n+1}F'(\eta)\Bigg) = 0 \tag{3.28}$$

Now we include below the procedure for the conversion of (3.4) into dimensionless for

- $$\begin{aligned}\frac{\partial\, a^*}{\partial x} &= \frac{\partial}{\partial x}\Big(a_0 G(\eta)\Big)\\ &= a_0 G'(\eta)\frac{\partial\,\eta}{\partial x}\\ &= a_0 G'(\eta)\Big(\frac{n-1}{2}\Big)(x+b)^{-1}\,\eta.\end{aligned}$$
- $$\begin{aligned}u\frac{\partial\, a^*}{\partial x} &= \Big(U_0(x+b)^n F'(\eta)\Big)\Big(\frac{n-1}{2}\Big)a_0 G'(\eta)(x+b)^{-1}\,\eta\\ &= \Big(\frac{n-1}{2}\Big)\Big(U_0(x+b)^{\,n}F'(\eta)\Big)(\eta)a_0 G'(\eta)(x+b)^{-1}\\ &= \Big(\frac{n-1}{2}\Big)\Big(U_0(x+b)^{\,n-1}F'(\eta)\Big)(\eta)a_0 G'(\eta).\end{aligned} \tag{3.29}$$
- $$\begin{aligned}\frac{\partial\, a^*}{\partial y} &= \frac{\partial}{\partial y}\Big(a_0 G(\eta)\Big)\\ &= a_0 G'(\eta)\frac{\partial\,\eta}{\partial\, y}\\ &= a_0 G'(\eta)\Big(\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{n-1}}\Big)\end{aligned}$$
- $$\begin{aligned}v\frac{\partial\, a^*}{\partial y} &= -\sqrt{\frac{n+1}{2}U_0\,\nu(x+b)^{\,n-1}}\Big(F(\eta)+\eta\frac{n-1}{n+1}F'\Big)\\ &\quad a_0 G'(\eta)\Big(\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{\,n-1}}\Big)\\ &= -\Big(\frac{n+1}{2}\Big)U_0(x+b)^{\,n-1}a_0 G'(\eta)\Big(F(\eta)+\eta\frac{n-1}{n+1}F'\Big).\end{aligned} \tag{3.30}$$

Using (3.29) and (3.30), the left side of (3.4) becomes

$$
\begin{aligned}
& u\frac{\partial\ a^*}{\partial x}+\ v\frac{\partial\ a^*}{\partial y} \\
&= \Big(\frac{n-1}{2}\Big)\Big(\ U_0(x+\ \ b)^{\ n}F'(\ \eta)\Big)(\ \eta)a_0\ G'\ \eta(\ x+b)^{-\ 1} \\
&\quad -\Big(\frac{n+1}{2}\Big)\ U_0(\ x+b)^{\ n-1}a_0G'(\ \eta)\Big(F(\ \eta)+\eta\frac{n-1}{n+1}F'\Big) \\
&= \Big(\frac{n-1}{2}\Big)\Big(\ U_0(\ x+b)^{\ n-1}F'(\ \eta)\Big)(\ \eta)a_0\ G'(\ \eta) \\
&\quad -\Big(\frac{n+1}{2}\Big)\ U_0(\ x+b)^{\ n-1}a_0G'\eta(F(\ \eta) \\
&\quad -\Big(\frac{n-1}{2}\Big)\ U_0(\ x+b)^{\ n-1}(\eta)a_0G'(\ \eta)F'(\eta) \\
&= -\Big(\frac{n+1}{2}\Big)U_0(\ x+b)^{\ n-1}a_0G'(\eta)(F(\ \eta).
\end{aligned}
$$

To convert the right side of (3.4) into dimensionless form,

we proceed as follows.

- $$
\begin{aligned}
\frac{\partial^2\ a^*}{\partial y^2} &= \frac{\partial}{\partial y}\Big(\frac{\partial\ a^*}{\partial\ y}\Big) \\
&= \frac{\partial}{\partial y}\Big(\ a_0G'(\ \eta)\sqrt{\frac{n+1}{2}\frac{U_0}{v}(\ x+b)^{\ n-1}}\Big) \\
&= \ a_0G''(\ \eta)\frac{\partial\ \eta}{\partial y}\Big(\sqrt{\frac{n+1}{2}\frac{U_0}{v}(x+b)^{\ n-1}}\Big) \\
&= \ a_0G''(\ \eta)\Big(\sqrt{\frac{n+1}{2}\frac{U_0}{v}(\ x+b)^{\ n-1}}\Big) \\
&\quad \Big(\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(\ x+b)^{-\ 1}}\Big) \\
&= \Big(\frac{n+1}{2}\Big)\frac{U_0}{\nu}(\ x+b)^{-\ 1}a_0G''(\ \eta) \qquad (3.31)
\end{aligned}
$$
- $$
D_A\frac{\partial^2\ a^*}{\partial y^2} = D_A\Big(\frac{n+1}{2}\Big)\frac{U_0}{\nu}(\ x+b)^{-\ 1}a_0G''(\ \eta) \qquad (3.32)
$$
- $$
\begin{aligned}
K_1a^*b^{*2} &= \ K_1\Big(\ a_0G(\ \eta)\Big)\Big((\ a_0H(\ \eta)\Big)^2 \\
&= \ K_1a_0^3\ G(\ \eta)H^2(\ \eta) \qquad (3.33)
\end{aligned}
$$

Using (3.32) and (3.33) in the right side of (3.4), we get

$$D_A \frac{\partial^2 a^*}{\partial y^2} - K_1 a^* b^{*2} = D_A \left(\frac{n+1}{2}\right) \frac{U_0}{\nu} (x+b)^{-1} a_0 G''(\eta) - K_1 a_0^3 G(\eta) H^2(\eta)$$

Hence the dimensionless form of (3.4) becomes

$$-\left(\frac{n+1}{2}\right) U_0 (x+b)^{n-1} a_0 F(\eta) G'(\eta) = D_A \left(\frac{n+1}{2}\right) \frac{U_0}{\nu} (x+b)^{-1} a_0 G''(\eta) - K_1 a_0^3 G(\eta) H^2(\eta)$$

$$\Rightarrow -\left(\frac{n+1}{2}\right) U_0 (x+b)^{n-1} a_0 F(\eta) G'(\eta) = \left(\frac{n+1}{2}\right) D_A a_0 \frac{U_0}{\nu} (x+b)^{-1} \left(G''(\eta) - \frac{2\nu K_1 a_0^2}{D_A (n+1) U_0 (x+b)^{n-1}} G(\eta) H^2(\eta))\right)$$

$$\Rightarrow -F(\eta) G'(\eta) = \frac{D_A}{\nu} \left(G''(\eta) - \frac{2\nu K_1 a_0^2}{D_A (n+1) U_0 (x+b)^{n-1}} G(\eta) H^2(\eta)\right)$$

$$\Rightarrow -\frac{\nu}{D_A} F(\eta) G'(\eta) = G''(\eta) - \left(\frac{2}{n+1}\right) \frac{\nu K_1 a_0^2}{D_A (n+1) U_0 (x+b)^{n-1}} G(\eta) H^2(\eta)$$

$$\Rightarrow G''(\eta) + \frac{\nu}{D_A} F(\eta) G'(\eta) - \left(\frac{2}{n+1}\right) \frac{\nu K_1 a_0^2}{D_A (n+1) U_0 (x+b)^{n-1}} G(\eta) H^2(\eta) = 0$$

$$\left(\because U_w = U_0 (x+b)^n\right) \left(\because Sc = \frac{\nu}{D_A}\right) \left(\because K = \frac{K_1 a_0^2}{U_w} (x+b)\right)$$

$$\Rightarrow G''(\eta) + ScF(\eta) G'(\eta) - \left(\frac{2\,ScK}{n+1}\right) G(\eta) H^2(\eta)) = 0.$$

Now we include below the procedure for the conversion of (3.5) into dimensionless form

$$
\begin{aligned}
\bullet\ \frac{\partial b^*}{\partial x} &= \frac{\partial}{\partial x}\Big(a_0 H(\eta)\Big) \\
&= a_0 H'(\eta)\frac{\partial \eta}{\partial x} \\
&= a_0 H'(\eta)\Big(\frac{n-1}{2}\Big)(x+b)^{-1}(\eta) \\
\bullet\ u\frac{\partial b^*}{\partial x} &= \Big(U_0(x+b)^{n}F'(\eta)\Big)\Big(\frac{n-1}{2}\Big)a_0 H'(\eta)(x+b)^{-1}(\eta) \\
&= \Big(\frac{n-1}{2}\Big)\Big(U_0(x+b)^{n}F'(\eta)\Big)(\eta)a_0H'(\eta)(x+b)^{-1} \\
&= \Big(\frac{n-1}{2}\Big)\Big(U_0(x+b)^{n-1}F'(\eta)\Big)(\eta)\ a_0H'(\eta)
\end{aligned}
$$

$$
\begin{aligned}
\bullet\ \frac{\partial b^*}{\partial y} &= \frac{\partial}{\partial y}\Bigg(a_0 H(\eta)\Bigg) \\
&= a_0 H'(\eta)\frac{\partial \eta}{\partial y} \\
&= a_0 H'(\eta)\Big(\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{n-1}}\Big) \\
\bullet\ v\frac{\partial b^*}{\partial y} &= -\sqrt{\frac{n+1}{2}U_0\nu(x+b)^{n-1}}\Bigg(F(\eta)+\eta\frac{n-1}{n+1}F'\Bigg) \\
& a_0H'(\eta)\Big(\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{n-1}}\Big) \\
&= -\Big(\frac{n+1}{2}\Big)U_0(x+b)^{n-1}a_0H'(\eta)\Bigg(F(\eta)+\ \eta\frac{n-1}{n+1}F'\Bigg).
\end{aligned} \tag{3.34}
$$

Using(3.34), the left side of (3.5) becomes

$$
\begin{aligned}
& u\frac{\partial b^*}{\partial x}+v\frac{\partial b^*}{\partial y} \\
&= \Big(\frac{n-1}{2}\Big)\Big(U_0(x+b)^{n}F'(\eta)\Big)(\eta)a_0H'(\eta)(x+b)^{-1} \\
&\quad -\Big(\frac{n+1}{2}\Big)U_0(x+b)^{n-1}a_0H'(\eta)\Bigg(F(\eta)+\ \eta\frac{n-1}{n+1}F'\Bigg) \\
&= \Big(\frac{n-1}{2}\Big)\Big(U_0(x+b)^{n-1}F'(\eta)\Big)(\eta)a_0H'(\eta) \\
&\quad -\Big(\frac{n+1}{2}\Big)U_0(x+b)^{n-1}a_0H'(\eta)(F(\eta)-\Big(\frac{n-1}{2}\Big)U_0(x+b)^{n-1}(\eta)a_0H'(\eta)F'(\eta) \\
&= -\Big(\frac{n+1}{2}\Big)\ U_0(x+b)^{n-1}a_0H'(\eta)(F(\eta).
\end{aligned}
$$

To convert the right side of (3.5) into dimensionless form,we proceed as follows

$$
\begin{aligned}
\bullet\ \frac{\partial^2 b^*}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\partial b^*}{\partial y}\right) \\
&= \frac{\partial}{\partial y}\left(a_0 H'(\eta)\sqrt{\frac{n+1}{2}\frac{U_0}{v}(x+b)^{n-1}}\right) \\
&= a_0 H''(\eta)\frac{\partial \eta}{\partial y}\left(\sqrt{\frac{n+1}{2}\frac{U_0}{v}(x+b)^{n-1}}\right) \\
&= a_0 H''(\eta)\left(\sqrt{\frac{n+1}{2}\frac{U_0}{v}(x+b)^{n-1}}\right)\left(\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{-1}}\right) \\
&= \left(\frac{n+1}{2}\right)\frac{U_0}{\nu}(x+b)^{-1}a_0 H''(\eta) \qquad (3.35)
\end{aligned}
$$

$$
\begin{aligned}
\bullet\ D_B\frac{\partial^2 b^*}{\partial y^2} &= D_B\left(\frac{n+1}{2}\right)\frac{U_0}{\nu}(x+b)^{-1}a_0 H''(\eta) \\
\bullet\ K_1 a^* b^{*2} &= K_1\Big(a_0 G(\eta)\Big)\Big((a_0 H(\eta)\Big)^2 \\
&= K_1 a_0^3 G(\eta) H^2(\eta)
\end{aligned}
$$

Using (3.35) in the right side of (3.5),we get

$$
\begin{aligned}
D_B\frac{\partial^2 b^*}{\partial y^2} + K_1 a^* b^{*2} = {} & D_B\frac{n+1}{2}\frac{U_0}{v}(x+b)^{n-1}a_0 H'' \\
& + K_1 a_0 G(\eta)(a_0)^2(H(\eta))^2
\end{aligned}
$$

Hence the dimensionless form of (3.5) becomes

$$
\begin{aligned}
&-\Big(\frac{n+1}{2}\Big)U_0(\ x+b)^{\ n-1}a_0F(\ \eta)H'(\ \eta)=D_B\Big(\frac{n+1}{2}\Big)\frac{U_0}{\nu}(x+b)^{-1}a_0H''(\eta)\\
&+K_1a_0^3G(\ \eta)H^2(\ \eta)\\
\Rightarrow\quad &-\Big(\frac{n+1}{2}\Big)U_0(\ x+b)^{\ n-1}a_0F(\ \eta)H'(\ \eta)=\Big(\frac{n+1}{2}\Big)D_Ba_0\frac{U_0}{\nu}(\ x+b)^{-\ 1}\\
&\left(H''(\ \eta)+\frac{2\ \nu K_1a_0^2}{D_B(\ n+1)U_0(\ x+b)^{\ n-1}}G(\ \eta)H^2(\eta))\right)\\
\Rightarrow\quad &-F(\eta)H'(\eta)=\frac{D_B}{\nu}\\
&\left(H''(\ \eta)+\frac{2\ \nu K_1a_0^2}{D_B(\ n+1)U_0(\ x+b)^{\ n-1}}G(\ \eta)H^2(\ \eta)\right)\\
\Rightarrow\quad &-\frac{\nu}{D_B}F(\ \eta)H'(\ \eta)=H''(\eta)+\Big(\frac{2}{n+1}\Big)\frac{\nu K_1a_0^2}{D_B(n+1)U_0(x+b)^{n-1}}G(\ \eta)H^2(\eta)\\
\Rightarrow\quad &H''(\ \eta)+\frac{\nu}{D_B}F(\ \eta)H'(\eta)+\Big(\frac{2}{n+1}\Big)\frac{\nu K_1a_0^2}{D_B(n+1)U_0(x+b)^{n-1}}G(\eta)H^2(\eta)\\
&\left(\because\ Sc=\frac{\nu}{D_A}\right)\quad\left(\because K=\frac{K_1a_0^2}{U_w}(\ x+b)\right)\left(\because\delta=\frac{D_B}{D_A}\right)\\
\Rightarrow\quad &H''(\ \eta)+\frac{Sc}{\delta}F(\ \eta)H'(\ \eta)+\left(\frac{2ScK}{(n+1)\ \delta}\right)G(\ \eta)H^2(\ \eta)=0
\end{aligned}
$$

The final dimensionless form of the proposed model, is:

$$
\begin{aligned}
&F'''+\ FF''-\frac{2\ n}{n+1}F'^2+\frac{2\ n}{n+1}A^2+2\beta\frac{n(\ n-1)}{n+1}A^3+\beta(3\ n-1)FF'F''\\
&\quad+\ \beta\left(-\frac{2\ n(\ n-1)}{n+1}\ F'^3+\ \eta\frac{n-1}{2}F'^2\ F''-\frac{n+1}{2}F^2\ F'''\right)=0, \qquad (3.36)\\
&\Theta''+PrF\ \Theta'+Pr\gamma\left(\frac{n-3}{2}FF'\ \Theta'-\frac{n+1}{2}F^2\ \Theta''\right)\\
&\quad+Pr(S+\ \Theta)\left(\ \gamma FF''-\frac{2\ n}{n+1}\gamma F'^2-\frac{2}{n+1}F'\right)=0, \qquad (3.37)\\
&G''-\frac{2ScK}{n+1}GH^2+\ ScFG'=0, \qquad (3.38)\\
&H''+\frac{2ScK}{\delta(n+1)}GH^2+\frac{Sc}{\delta}H'F=0. \qquad (3.39)
\end{aligned}
$$

Here prime represents differentiation with respect to η and $\alpha=A_1\sqrt{\frac{n+1}{2}\frac{U_0}{v}}$ is the wall thickness parameter.

$$F'(\xi) = f(\xi - \alpha) = f(\eta), \Theta'(\xi) = \theta(\xi - \alpha) = \theta(\eta), G'(\xi) = g(\xi - \alpha) = g(\eta) \tag{3.40}$$

let $F(\eta) = f(\eta - \alpha) = f(\xi)$,$\Theta(\eta) = \theta(\eta - \alpha) = \theta(\xi)$, $G(\eta) = g(\eta - \alpha) = g(\xi)$. This change of notations converts the above equations (3.39)-(3.42) into the following form.

$$\begin{aligned} & f''' + ff'' - \frac{2n}{n+1}f'^2 + \frac{2n}{n+1}A^2 + 2\beta\frac{n(n-1)}{n+1}A^3 + \beta(3n-1)ff'f'' \\ & \quad + \beta\left(-\frac{2n(n-1)}{n+1}f'^3 + (\xi+\alpha)\frac{n-1}{2}f'^2f'' - \frac{n+1}{2}f^2f'''\right) = 0, \end{aligned} \tag{3.41}$$

$$\begin{aligned} & \theta'' + Prf\theta' + Pr\gamma\left(\frac{n-3}{2}ff'\theta' - \frac{n+1}{2}f^2\theta''\right) \\ & \quad + Pr(S+\theta)\left(\gamma ff'' - \frac{2n}{n+1}\gamma f'^2 - \frac{2}{n+1}f'\right) = 0, \end{aligned} \tag{3.42}$$

$$g'' - \frac{2ScK}{n+1}gh^2 + Scfg' = 0, \tag{3.43}$$

$$h'' + \frac{2ScK}{\delta(n+1)}hg^2 + \frac{Sc}{\delta}h'f = 0. \tag{3.44}$$

The new form of the associated boudary conditions, is:

$$\left.\begin{aligned} & f(0) = \alpha\frac{1-n}{1+n}, \quad f'(0) = 1, \quad \theta(0) = 1 - S, \\ & g'(0) = \sqrt{\frac{2}{n+1}}Ksg(0), \quad h'(0) = -\frac{1}{\delta}\sqrt{\frac{2}{n+1}}Ksg(0), \\ & f'(\xi) = A, \quad \theta(\xi) = 1, \quad g(\xi) \longrightarrow 1, \quad h(\xi) \longrightarrow 0, \text{as} \quad \xi \longrightarrow \infty, \end{aligned}\right\} \tag{3.45}$$

where $Sc = \frac{\vartheta}{D_A}$ is the Schmidt number, $\delta = \frac{D_B}{D_A}$ the ratio of mass diffusion coefficient, $K = \frac{K_1 a_0^2}{U_w}(x+b)$ the strength of homogeneous parameter, $\beta = \lambda_1 U_0(x+b)^{n-1}$ the Deborah number, $S = \frac{d}{c}$ the thermal stratified parameter, $\gamma = \lambda U_0(x+b)^{n-1}$ the thermal relaxation parameter, α the wall thickness parameter, $Ks = \frac{k_s}{D_A}\sqrt{\frac{\vartheta(x+b)}{U_W}}$ the strength of heterogeneous reaction parameter, $A = \frac{U_\infty}{U_0}$ the velocity ratio parameter and $Pr = \frac{\mu_f C_p}{k}$ the Prandtl number. The diffusion coefficients of chemical species A and B are assumed to be of a comparable size. The argument leads to assume that the diffusion coefficient D_A=D_B, that is, δ=1. Thus

$$\left.\begin{array}{l} \delta = \dfrac{D_B}{D_A}, \quad A = \dfrac{U_\infty}{U_0}, \quad Pr = \dfrac{\mu_f c_p}{k}, \\ \beta = \lambda_1 U_0(\ x + b)^{\ n-1}, \quad \gamma = \lambda U_0(\ x + b)^{\ n-1}, \quad S = \dfrac{d}{c}, \\ S_c = \dfrac{\nu}{D_A}, \quad K_s = \dfrac{k_s}{D_A}\sqrt{\dfrac{\nu(\ x + b)}{U_w}}. \end{array}\right\}$$

Assume that the Diffusion coeeficients D_A and D_B are equal. Thus

$$g(\xi) + h(\xi) = 1.$$

So equation (3.46) gets the following form and (3.47) can be ignored.

$$g'' - \frac{2ScK}{n+1}\, g(1 - \ g)^2 + Scfg' = 0. \tag{3.46}$$

The relevant boundary conditions are:

$$g'(\ 0) = \sqrt{\frac{2}{n+1}} Ksg(\ 0), \quad g(\infty) \longrightarrow \ 1 \quad \text{when} \quad \xi \longrightarrow \infty,$$

3.4 Solution Methodology

In order to solve the system of ordinary differential equations (3.46)-(3.50), the shooting method has been used. Let us use the notations:

$$f = y_1, \theta = y_4, g = y_6.$$

Further denote

$$f' = y_1' \text{ by } y_2, f'' = {y_2}' \text{ by } y_3, \theta' = y_4' \text{ by } y_5 \text{ and } g' = {y_6}' \text{ by } y_7.$$

The system of equations (3.44)-(3.50), can now be written in the form of following first order ODEs:

$$
\begin{aligned}
y_1' &= y_2, \\
y_2' &= y_3, \\
y_3' &= \frac{1}{1-\beta\frac{n+1}{2}y_1^2}\left(-y_1y_3 + \frac{2n}{n+1}y_2^2 - \beta(3n-1)y_1y_2y_3 - \frac{2n}{n+1}A^2 \right. \\
&\quad \left. +\beta\frac{2n(n-1)}{n+1}y_2^3 - (\xi+\alpha)\beta\frac{n-1}{2}y_2^2y_3 - 2\beta\frac{n(n-1)}{n+1}A^3\right), \\
y_4' &= y_5, \\
y_5' &= \frac{1}{1-Pr\gamma\frac{n+1}{2}y_1^2}\left(-Pry_1y_5 - Pr\gamma\frac{n-3}{2}y_1y_2y_5 \right. \\
&\quad \left. -Pr(S+y_4)(\gamma y_1y_3 - \frac{2n}{n+1}\gamma y_2^2 - \frac{2}{n+1}y_2)\right), \\
y_6' &= y_7, \\
y_7' &= -Sc\left(y_1y_7 - \frac{2K}{n+1}y_6(1-y_6)^2\right).
\end{aligned}
$$

The initial conditions for the above ODEs

$$
\begin{aligned}
y_1(0) &= \alpha\frac{1-n}{1+n}, & y_2(0) &= 1, \\
y_3(0) &= s, & y_4(0) &= 1-S, \\
y_5(0) &= t, & y_6(0) &= w \\
y_7(0) &= \sqrt{\frac{2}{n+1}}Ks\ w.
\end{aligned}
$$

The above initial value problem will be solved numerically by the RK-4 method. To get the approximate solution, the domain of the problem has been taken as $[0, \eta_\infty]$ instead of $[0, \infty]$, where η_∞ is an appropriate finite positive real number. In the above system of equations, the missing conditions s, t and w are to be chosen such that

$$
y_2(\eta_\infty, s, t, w) = A, \ \ y_4(\eta_\infty, s, t, w) = 1, \ \ y_6(\eta_\infty, s, t, w) = 1.
$$

To solve the above system of algebraic equations, we use the Newton's method which has the following iterative scheme:

$$\begin{pmatrix} s^{(k+1)} \\ t^{(k+1)} \\ w^{(k+1)} \end{pmatrix} = \begin{pmatrix} s^{(k)} \\ t^{(k)} \\ w^{(k)} \end{pmatrix} - \begin{pmatrix} \frac{\partial y_2}{\partial s} & \frac{\partial y_2}{\partial t} & \frac{\partial y_2}{\partial w} \\ \frac{\partial y_4}{\partial s} & \frac{\partial y_4}{\partial t} & \frac{\partial y_4}{\partial w} \\ \frac{\partial y_6}{\partial s} & \frac{\partial y_6}{\partial t} & \frac{\partial y_6}{\partial w} \end{pmatrix}^{-1}_{(s^{(k)},t^{(k)},w^{(k)})} \begin{pmatrix} y_2^{(k)} \\ y_4^{(k)} \\ y_6^{(k)} \end{pmatrix}_{(s^{(k)},t^{(k)},w^{(k)})}.$$

For further procedure, the following notations have been introduced.

$$\frac{\partial y_1}{\partial s} = y_8, \frac{\partial y_2}{\partial s} = y_9, ..., \frac{\partial y_7}{\partial s} = y_{14},$$

$$\frac{\partial y_1}{\partial w} = y_{22}, \frac{\partial y_2}{\partial w} = y_{23}, ..., \frac{\partial y_7}{\partial w} = y_{28}.$$

As a result of these these new notations, the Newton's iterative scheme gets the form:

$$\begin{pmatrix} s^{(k+1)} \\ t^{(k+1)} \\ w^{(k+1)} \end{pmatrix} = \begin{pmatrix} s^{(k)} \\ t^{(k)} \\ w^{(k)} \end{pmatrix} - \begin{pmatrix} y_9 & y_{16} & y_{23} \\ y_{11} & y_{18} & y_{25} \\ y_{13} & y_{20} & y_{27} \end{pmatrix}^{-1}_{(s^{(k)},t^{(k)},w^{(k)})} \begin{pmatrix} y_2^{(k)} - A \\ y_4^{(k)} - 1 \\ y_6^{(k)} - 1 \end{pmatrix}_{(s^{(k)},t^{(k)},w^{(k)})}. \tag{3.47}$$

Now differentiate the above system of seven first order ODEs with respect to each of the variables s, t and w to have another system of twenty one ODEs.

Writing all these twenty eight ODEs together, we have the the following IVP:

$$
\begin{aligned}
y_1' &= y_2,\\
y_2' &= y_3,\\
y_3' &= \frac{1}{1-\beta(\frac{n+1}{2})y_1^2}\Bigg(-y_1\ y_3 + (\frac{2n}{n+1})y_2^2 - \beta(3n-1)y_1y_2y_3 - \frac{2n}{n+1}A^2\\
&\quad +\beta\frac{2n(n-1)}{n+1}y_2^3 - (\xi+\alpha)\beta\frac{n-1}{2}y_2^2y_3 - 2\beta\frac{n(n-1)}{n+1}A^3\Bigg),\\
y_4' &= y_5,\\
y_5' &= \frac{1}{1-Pr\gamma(\frac{n+1}{2})\ y_1^2}\Bigg(-Pry_1y_5 - Pr\gamma\frac{n-3}{2}y_1\ y_2y_5\\
&\quad -Pr(S+y_4)(\gamma y_1\ y_3 - \frac{2n}{n+1}\gamma y_2^2 - \frac{2}{n+1}y_2)\Bigg),\\
y_6' &= y_7,\\
y_7' &= -Sc\left(y_1y_7 - \frac{2K}{n+1}y_6(1-y_6)^2\right),\\
y_8' &= y_9,\\
y_9' &= y_{10},\\
y_{10}' &= \frac{1}{1-\beta(\frac{n+1}{2})\ 2y_1\ y_8}\Bigg[-y_1y_{10} - y_8y_3 + \frac{2n}{n+1}2y_2y_9 - \beta(3n-1)\\
&\quad (\ y_8\ y_2\ y_3 + y_1\ y_9\ y_3 + y_1\ y_2\ y_{10})\\
&\quad -(\frac{2n}{n+1})A^2 + \beta\frac{2n(n-1)}{n+1}3y_2^2y_9 - (\xi+\alpha)\beta\frac{n-1}{2}\left(2y_2y_9y_3 + y_2^2y_{10}\right)\\
&\quad -2\beta\frac{n(n-1)}{n+1}A^3\Bigg],\\
y_{11}' &= y_{12},\\
y_{12}' &= \frac{1}{1-Pr\ \gamma(\frac{n+1}{2})2y_1y_8}\Bigg[-Pr\left(y_1y_{12}+y_8y_5\right) - Pr\gamma\frac{n-3}{2}\big(y_8y_2y_5 + y_1y_9y_5\\
&\quad + y_1y_2y_{12}\big) - Pr(S+y_{11})(\gamma\left(y_8y_3+y_1y_{10}\right) - \frac{2n}{n+1}\gamma 2y_2y_9 - \frac{2}{n+1}y_9)\Bigg],\\
y_{13}' &= y_{14},\\
y_{14}' &= -Sc\left[(y_1y_{14}+y_8y_7) - \frac{2K}{n+1}\left(y_{13}(1-y_6)^2 + y_6 2(1-y_6)y_{13}\right)\right],\\
y_{15}' &= y_{16},\\
y_{16}' &= y_{17},
\end{aligned}
$$

$$y'_{17} = \frac{1}{1-\beta(\frac{n+1}{2})\ 2y_1\ y_{15}}\Big[-y_1\ y_{17} - y_{15}\ y_3 + \frac{2n}{n+1}2y_2\ y_{16} - \beta(3n-1)(y_{15}\ y_2\ y_3$$
$$+y_1\ y_{16}\ y_3 + y_1\ y_2\ y_{17}) - (\frac{2n}{n+1})A^2 + \ \beta\ \frac{2n(n-1)}{n+1}\ 3\ y_2^2\ y_{16}$$
$$-(\xi+\alpha)\beta\frac{n-1}{2}\left(\ 2y_2\ y_{16}\ y_3 + y_2^2\ y_{17}\right) - 2\beta\ \frac{n(n-1)}{n+1}\ A^3\Big],$$

$$y'_{18} = y_{19},$$

$$y'_{19} = \frac{1}{1-Pr\ \gamma(\frac{n+1}{2})2\ y_1 y_{15}}\Big[-Pr\,(y_1\ y_{19} + y_{15}\ y_5) - Pr\ \gamma\ (\frac{n-3}{2})\ (y_{15}\ y_2\ y_5$$
$$+\ y_1\ y_{16}\ y_5 + y_1\ y_2\ y_{19}) - Pr(S + y_{18})(\gamma\,(y_{15}y_3 + y_1 y_{17}) - \frac{2n}{n+1}\gamma\ 2y_2\ y_{16}$$
$$-\frac{2}{n+1}y_{16})\Big],$$

$$y'_{20} = y_{21},$$

$$y'_{21} = \ -Sc\left[(y_1\ y_{21} + y_{15}\ y_7) - \frac{2K}{n+1}\ \left(y_{20}(1-y_6)^2 + y_6 2(1-y_6)\ y_{20}\right)\right],$$

$$y'_{22} = y_{23},$$

$$y'_{23} = y_{24},$$

$$y'_{24} = \frac{1}{1-\beta\ (\frac{n+1}{2})2y_1\ y_{22}}\Big[-y_1\ y_{24} - y_{22}\ y_3 + \frac{2n}{n+1}2\ y_2\ y_{23} - \beta(3n-1)$$
$$(y_{22}\ y_2\ y_3 + y_1\ y_{23}y_3\ + y_1\ y_2\ y_{24}) - (\frac{2n}{n+1})A^2 + \ \beta\ \frac{2n(n-1)}{n+1}3\ y_2^2\ y_{23}$$
$$-(\xi+\alpha)\beta\frac{n-1}{2}(2y_2\ y_{23}y_3 + y_2^2\ y_{24}) - 2\beta\frac{n(n-1)}{n+1}A^3\Big],$$

$$y'_{25} = y_{26},$$

$$y'_{26} = \frac{1}{1-\beta\ (\frac{n+1}{2})\ 2y_1\ y_{22}}\Big[-y_1\ y_{24} - y_{22}\ y_3 + \frac{2n}{n+1}2y_2\ y_{23} - \beta(3n-1)$$
$$(y_{22}\ y_2\ y_3 + y_1\ y_{23}\ y_3 + y_1\ y_2\ y_{24}) - (\frac{2n}{n+1})A^2 + \beta\frac{2n(n-1)}{n+1}3\ y_2^2\ y_{23}$$
$$-(\xi+\alpha)\beta\frac{n-1}{2}(2y_2\ y_{23}y_3 + y_2^2\ y_{24}) - 2\ \beta\frac{n(n-1)}{n+1}A^3\Big],$$

$$y'_{27} = y_{28},$$

$$y'_{28} = -Sc\left[(y_1\ y_{28} + y_{22}\ y_7) - \frac{2K}{n+1}\left(y_{27}\ (1-y_6)^2 + \ y_6 2(1-y_6)\ y_{27}\right)\right].$$

The coressponding initial conditions are

$$
\begin{aligned}
y_1(0) &= \alpha\frac{1-\ n}{1+n}, & y_2(0) &= 1,\\
y_3(0) &= s, & y_4(0) &= 1-S,\\
y_5(0) &= t, & y_6(0) &= w,\\
y_7(0) &= \sqrt{\frac{2}{n+1}}Ksw, & y_8(0) &= 0,\\
y_9(0) &= 0, & y_{10}(0) &= 1,\\
y_{11}(0) &= 0, & y_{12}(0) &= 0,\\
y_{13}(0) &= 0, & y_{14}(0) &= 0,\\
y_{15}(0) &= 0, & y_{16}(0) &= 0,\\
y_{17}(0) &= 0, & y_{18}(0) &= 0,\\
y_{19}(0) &= 1, & y_{20}(0) &= 0,\\
y_{21}(0) &= 0, & y_{22}(0) &= 0,\\
y_{23}(0) &= 0, & y_{24}(0) &= 0,\\
y_{25}(0) &= 0,, & y_{26}(0) &= 0,\\
y_{27}(0) &= 1,, & y_{28}(0) &= \sqrt{\frac{2}{n+1}}Ks.
\end{aligned}
$$

The fourth order Runge-Kutta method is used to solve the above system of twenty eight equations with initial guesses s, t, w. These guesses are updated by the Newton's scheme (3.51). The iterative process is repeated until the following criteria is met:

$$
max\{|y_2(\eta_\infty) - A|, |y_4(\eta_\infty) - 1|, |y_6(\eta_\infty - 1)|\} < \epsilon,
$$

where $\epsilon > 0$ is the tolerance. For all the calculations in this chapter, we have set $\epsilon = 10^{-6}$.

3.5 Results and Discussion

This section is devoted to the detailed discussions of the numerical solutions of our problem. To examine the effect of different involved physical parameters on the skin friction coefficient, local Nusselt number and Sherwood number, Tables 3.1, 3.2 and 3.3 are prepared. In these tables, a comparison between the present results obtained by shooting method and the MATLAB built in function bvp4c, with those given by Hayat *et al.*[15] has been presented. An excellent agrement is observed between these results, which strengthens the used methodology. Table 3.1 is prepared to analyze the effect of A, α, β on skin friction coefficient. It is observed that, by increasing the velocity ratio parameter A and wall thickness parameter α, skin friction coefficient increases whereas by increasing Deborah number β, the skin coefficient decreases. Table 3.2 is prepared to analyze the effect of γ and Pr. It is observed that, by increasing the thermal relaxation parameter γ, Nusselt number increases whereas for an increment in Prandtl number Pr, Nusselt number decreases. Table 3.3 is prepared to analyze the effect of n, K, Ks and Sc on Nusselt number. It is observed that, by increasing heterogeneous parameter Ks, power-law index n and homogeneous parameter K, Sherwood number increases where as for Schmidt number Sc, Sherwood number is decreased.

The main objective of this section is to analyze the numerical results displayed in the form of tables and graphs. The computations are carried out for the impact of different parameters like, the Schmidt number, the strength of homogeneous parameter, the Deborah number, the thermal stratified parameter, the thickness parameter and the Prandtl number. In Table 3.1, 3.2 and 3.3, are prepared to analyze the effect of different parameters on skin friction and Nusselt number. Figure 3.2 represents the effect of the velocity ratio parameter on the velocity profile. The velocity profile is increased by increasing the velocity ratio parameter A. Figure 3.3 is drawn to inspect the effect of the wall thickness parameter α on the velocity profile. Graph of this figure shows that by the increasing the velocity ratio parameter α, the thickness of the momentum boundary layer and the velocity profile are decreased. Figure 3.4 shows the Deborah number β on the velocity profile $f'(\xi)$. The axial velocity is decreased with an increase in the Deborah number β. As the Deborah number is the ratio of the observation time to the relxation time, a rise in the Deborah number β means an increament in the viscous forces due to which velocity profile decreases.

Figure 3.5 is prepared to analyze the effect of thermal relaxation parameter γ on temperature profile. As the value of the thermal relaxation parameter γ is increased, both the thermal layer thickness and the temperature profile are decreased. Physically, the fluid particles require more time to transfer heat due to an increament in the thermal relaxation parameter γ. Figure 3.6 is plotted to examine the effect of Prandtl number Pr on the temperature profile. Larger value of prandtl number Pr causes a reduction in both boundary layer thickness and the temperature distribution. Figure 3.7 represents the effect of the power law index n on the concentration distribution.

For gradually increasing values of the power-law index n, the concentration profile is decreased. Behavior of Sc on concentration profile is sketched in Figure 3.8. The concentration profile is enhanced for larger values of Schmidt number. Here, the smaller values of Schmidt number correspond to the large diffusivity and so the concentration distribution is decreased. Figure 3.9 is sketched to analyze the influence of the heterogeneous reaction parameter Ks. From this graph, it is clear that the gradually increasing values of Ks decline the concentration profile. Figure 3.10 is prepared to represent the effect of the homogeneous reaction parameter on the concentration profile. Larger value of homogeneous reaction parameter brings about a decrement in the concentration profile.

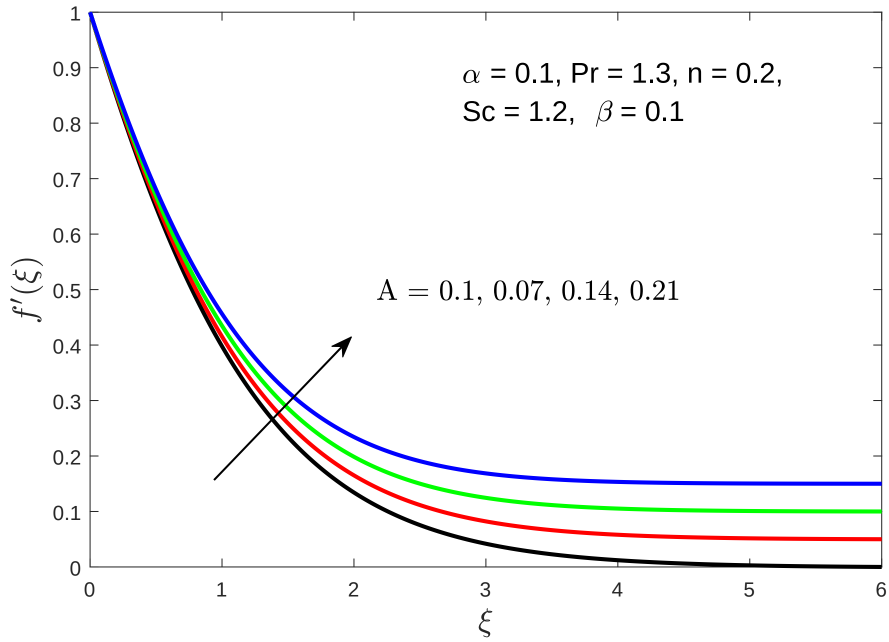

FIGURE 3.2: Effect of A on $f'(\xi)$.

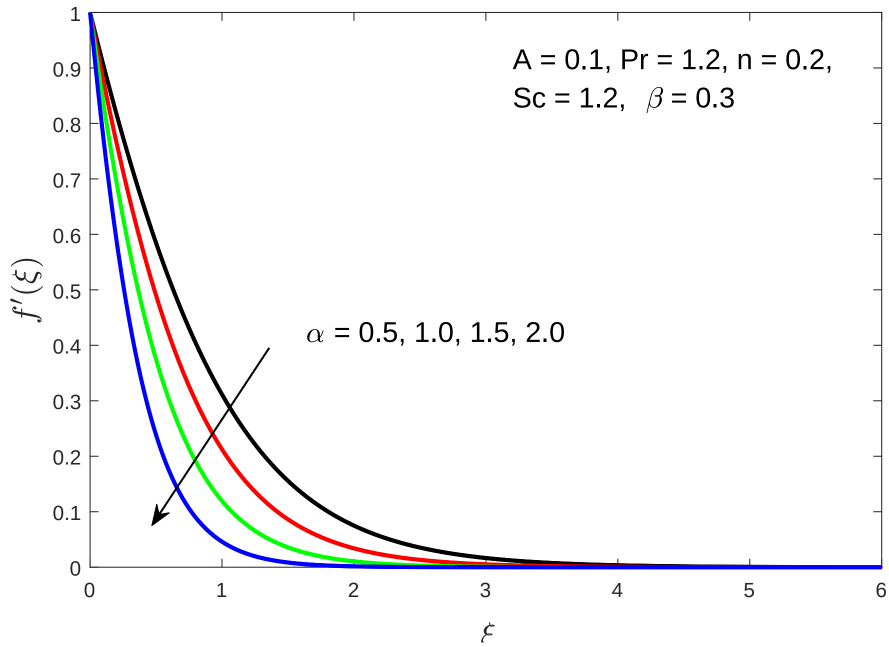

FIGURE 3.3: Effect of α on $f'(\xi)$.

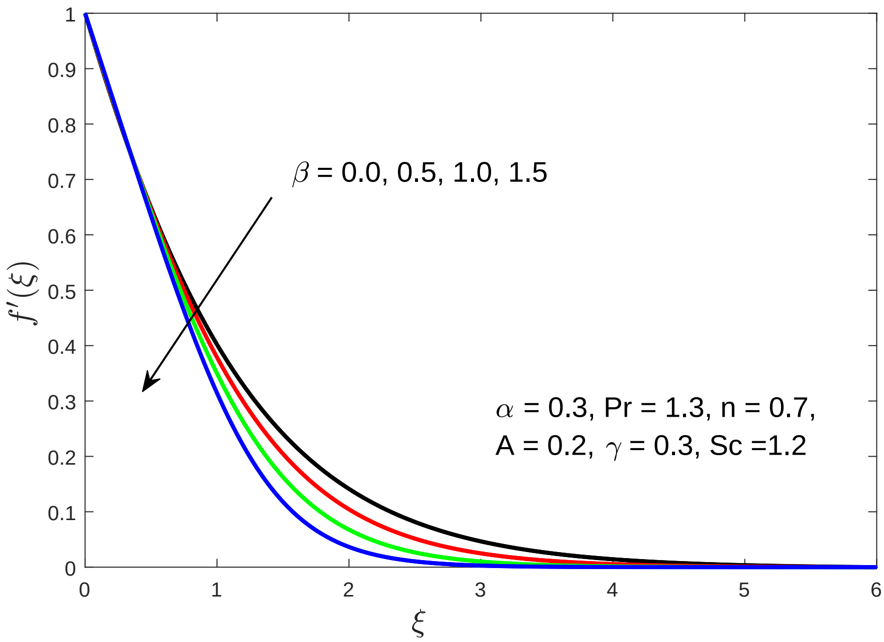

FIGURE 3.4: Effect of β on $f'(\xi)$.

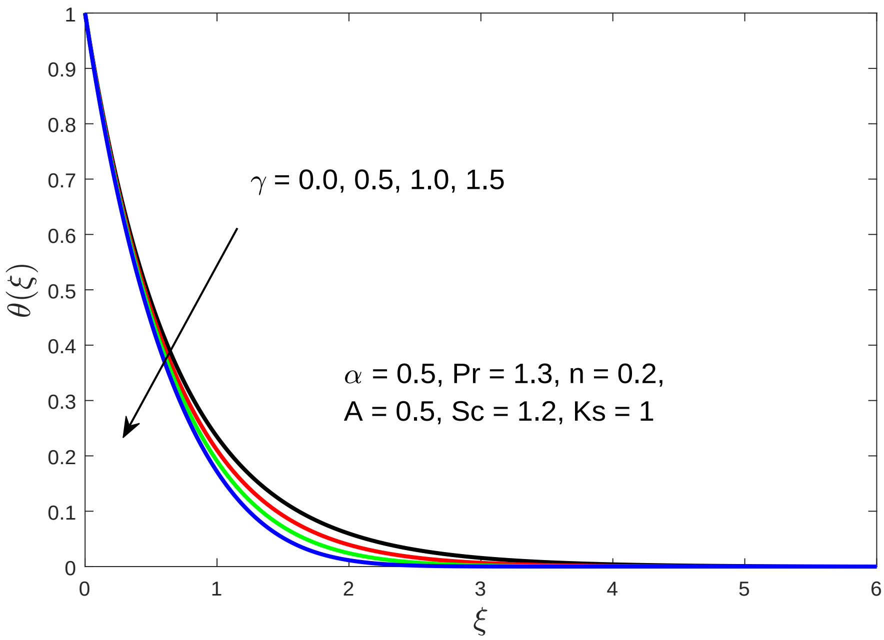

FIGURE 3.5: Effect of γ on $\theta(\xi)$.

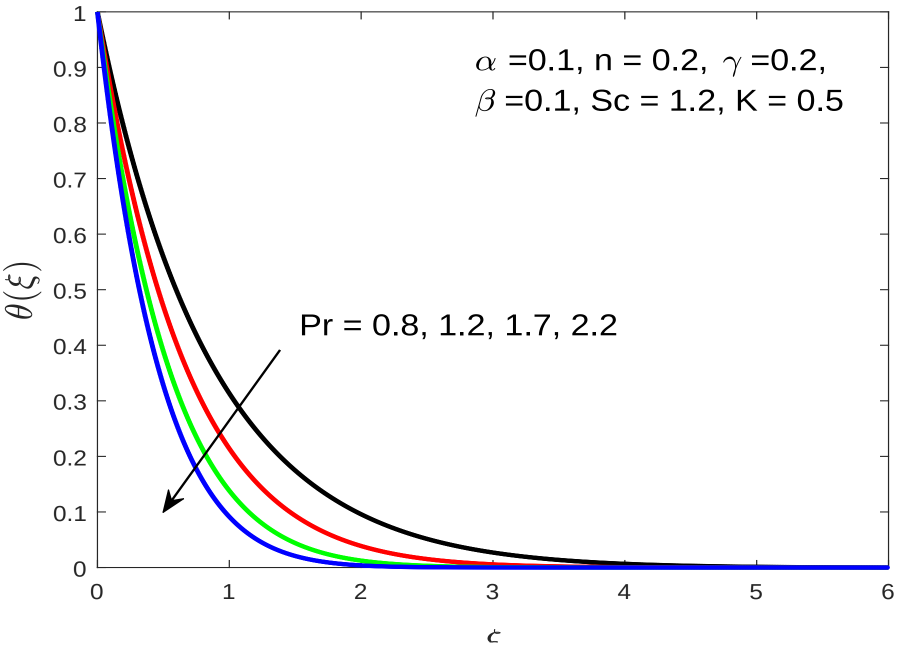

FIGURE 3.6: Effect of Pr on $\theta(\xi)$.

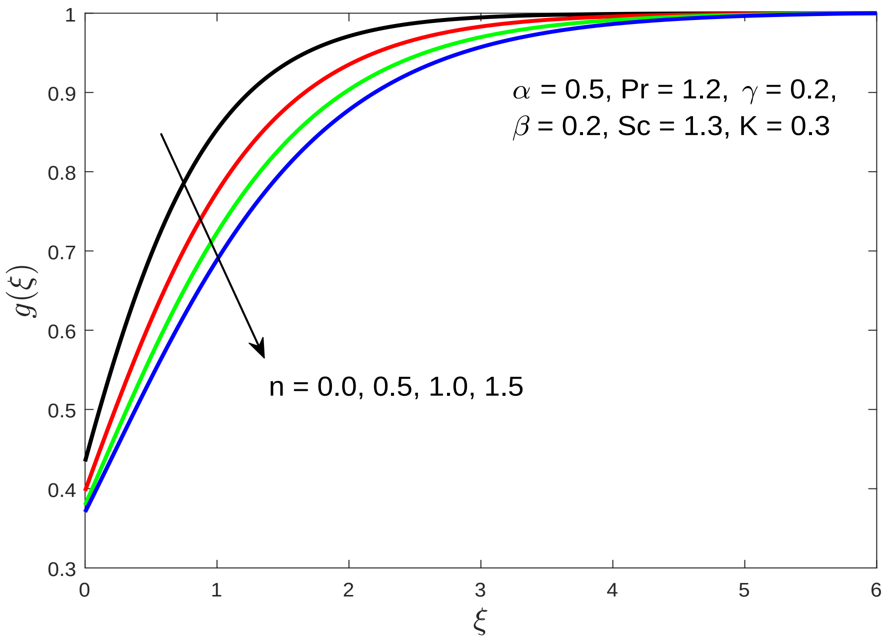

FIGURE 3.7: Effect of n on $g(\xi)$.

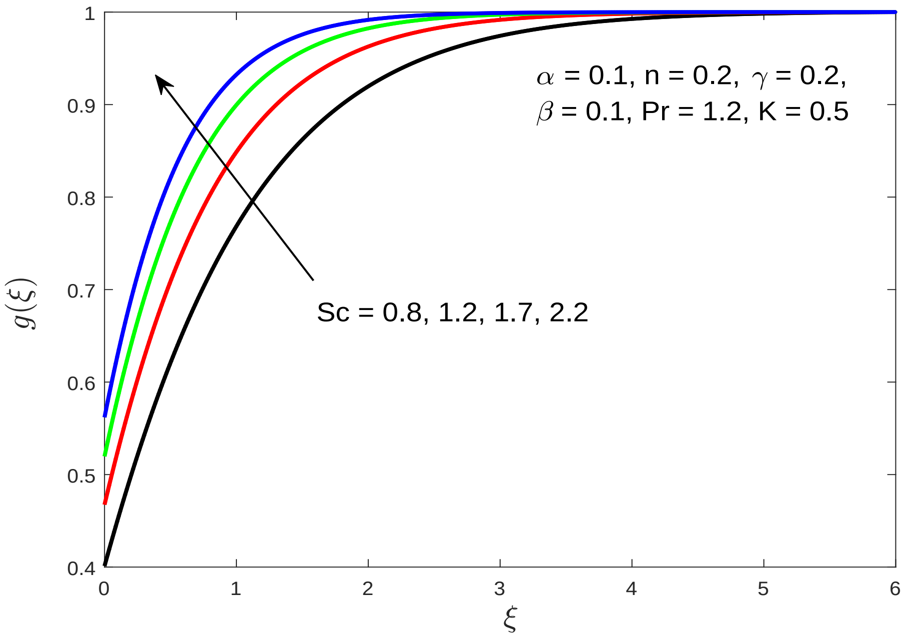

FIGURE 3.8: Effect of Sc on $g(\xi)$.

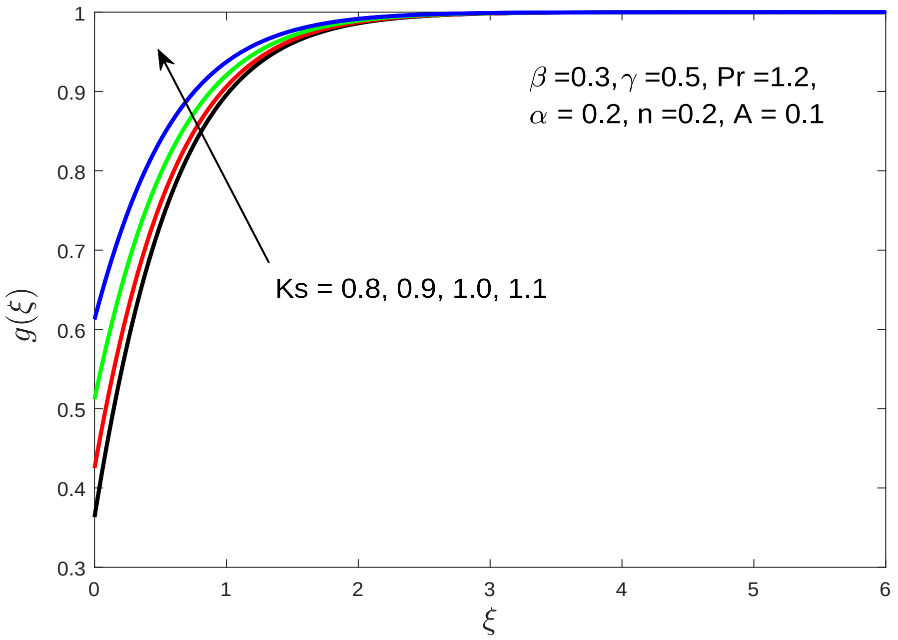

FIGURE 3.9: Effect of Ks on $g(\xi)$.

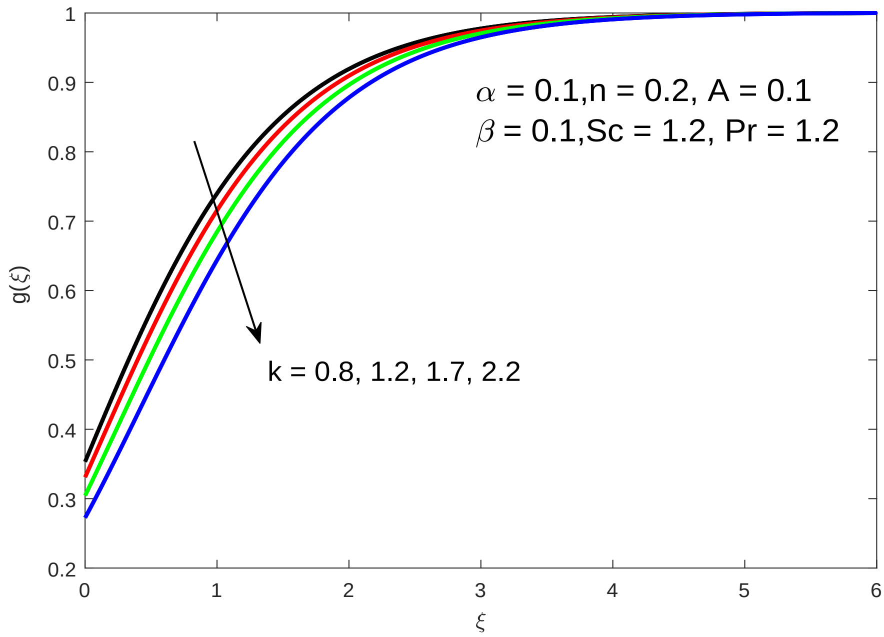

FIGURE 3.10: Effect of k on $g(\xi)$.

TABLE 3.1: Numerical results of $-f''(0)$ for $\gamma = 0.3$, $Pr = 1.2$, $K = 0.5$, $Ks = 1.0$, $n = 0.2$ and $Sc = 1.2$.

			Hayat et al.	Present study	
A	α	β	$-f''(0)$	shooting	bvp4c
0.0	0.5	0.0	1.0500	1.0508	1.0503
0.1			1.0445	1.0446	1.0441
0.2			1.0444	1.0445	1.0440
0.3			1.0440	1.0444	1.0439
	1.0		1.0447	1.0449	1.0444
	1.5		1.0445	1.0448	1.0442
	2.0		1.0443	1.0442	1.0441
		0.5	1.0445	1.0447	1.0445
		1.0	1.0448	1.0449	1.0447
		1.5	1.0449	1.0450	1.0449

Table 3.2: Numerical results of $-\theta'(0)$ for $A = 0.1$, $\beta = 0.1$, $K = 0.5$, $Ks = 1.0$, $\alpha = 0.5$, $n = 0.2$ and $Sc = 1.2$.

		Hayat et al.	Present study	
γ	Pr	$-\theta'(0)$	shooting	bvp4c
0.0	0.8	0.45238	0.45235	0.45232
0.5		0.49606	0.49601	0.49602
1.0		0.49912	0.49909	0.49914
1.5		0.50114	0.50107	0.50106
	1.2	0.49114	0.49114	0.49112
	1.7	0.49109	0.49107	0.49108
	2.2	0.49103	0.49101	0.49103

Table 3.3: Numerical results of $g'(0)$ for $\gamma = 0.3$, $Pr = 1.2$, $\alpha = 0.5$, $A = 0.5$ and $\beta = 0.4$.

				Hayat et al.	Present study	
Sc	Ks	K	n	$-g'(0)$	shooting	bvp4c
0.8	0.8	0.8	0.0	0.28048	0.28046	0.28044
1.2				0.26213	0.26212	0.26211
1.7				0.26199	0.26000	0.26098
2.2				0.25903	0.25001	0.25003
	0.9			0.26104	0.26104	0.26106
	1.0			0.26107	0.26105	0.26109
	1.1			0.26109	0.26116	0.26110
		1.2		0.26104	0.26104	0.26106
		1.7		0.26114	0.26106	0.26107
		2.2		0.26116	0.26108	0.26108
			0.5	0.26101	0.26102	0.26101
			1.0	0.26104	0.26103	0.26106
			1.5	0.26108	0.26104	0.26107

Chapter 4

MHD Stagnation Point Flow Towards a Non-linear Stretching Sheet with Homogeneous and Heterogenous Reactions

4.1 Introduction

In this chapter, a model which is an extension of that discusses in chapter 3 has been analyzed by considering the effect of MHD stagnation point flow towards a non-linear stretching sheet in the presence of Cattaneo-Christov heat flux model and homogeneous -heterogeneous reactions.A steady, incompressible laminar and two dimensional MHD stagnation point flow has been examined with concentration over a stretching sheet.Influence of homogeneous and heterogeneous reactions is also considered.The non-linear partial differential equations of velocity, temperature and concentration are converted into a system of ODEs by employing helpful similarity transformations. By using the shooting technique, numerical solution of these governing ordinary differential equations is obtained. The velocity, temperature and concentration profiles are numerically analyzed by using MATLAB for pertinent variables. The dynamics of various variables of interest are discussed through graphs.

4.2 Mathematical Formulation

Consider the two dimensional MHD, laminar, steady and incompressible stagnation point flow of Maxwell fluid over a non-linear stretching sheet with variable thickness. The geometry of the flow model is given below.

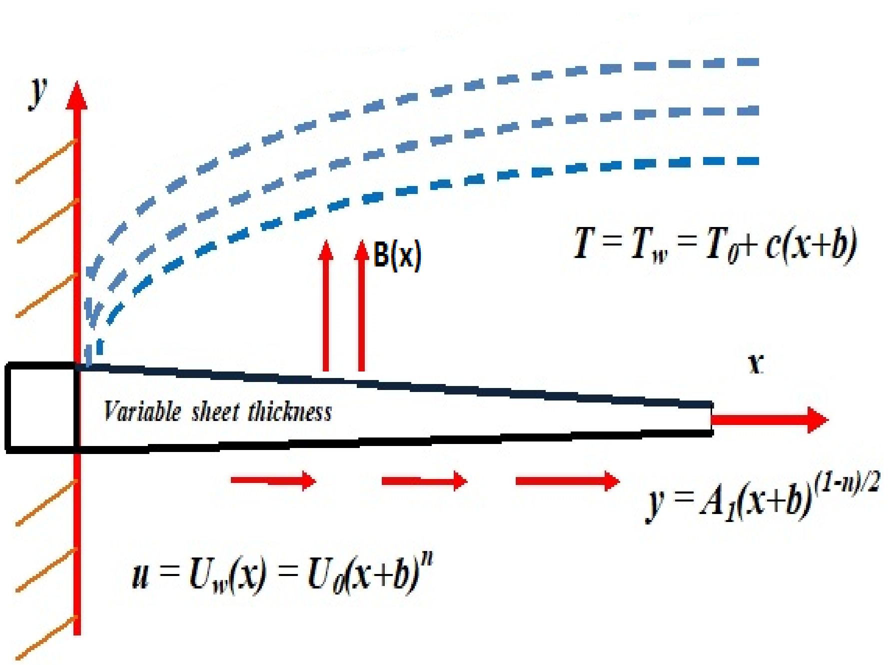

FIGURE 4.1: Geometry of the problem.

Here Cattanneo-Christove model has been considered. A variable magnetic field of strength B_o is applied along y-axis. The induced magnetic field is supposed to be negligible. Influence of homogeneous and heterogeneous reactions is considered. Heat transfer analysis is examined in the presence of thermal radiation. The flow equations based on the conservation principles and the obtained set of PDEs is then converted into non-linear coupled ODEs by employing some reasonable similarity transformations.

$$\frac{\partial\ u}{\partial x}+\frac{\partial\ v}{\partial y}=0, \tag{4.1}$$

$$u\frac{\partial\ u}{\partial x}+v\frac{\partial\ u}{\partial y}=\ v\frac{\partial^2\ u}{\partial\ y^2}+\lambda_1 U_e^2\frac{\partial^2 U_e}{\partial x^2}-\lambda_1\left(\ u^2\frac{\partial^2\ u}{\partial\ x^2}+v^2\frac{\partial^2 u}{\partial y^2}+2\ u\ v\frac{\partial^2\ u}{\partial x\partial y}\right)$$
$$+\ U_e\frac{\partial U_e}{\partial x}-\sigma\frac{B^2(x)}{\rho}(u-\ U_e), \tag{4.2}$$

$$v\frac{\partial\ T}{\partial y}+\ u\frac{\partial\ T}{\partial x}+\lambda\Bigg(\ v\frac{\partial v}{\partial y}\frac{\partial\ T}{\partial y}+\ u\frac{\partial v}{\partial x}\frac{\partial T}{\partial\ y}+u\frac{\partial u}{\partial\ x}\frac{\partial T}{\partial x}+\ v\frac{\partial u}{\partial y}\frac{\partial\ T}{\partial x}$$
$$+2uv\frac{\partial^2\ u}{\partial x\partial y}+\ v^2\frac{\partial^2 T}{\partial y^2}+\ u^2\frac{\partial^2\ T}{\partial x^2}\Bigg)=\frac{k}{\rho c_p}\frac{\partial^2 T}{\partial y^2}, \tag{4.3}$$

$$u\frac{\partial a^*}{\partial x}+v\frac{\partial a*}{\partial y}=D_A\frac{\partial^2 a^*}{\partial y^2}-K_1 a^* b^{*2}, \tag{4.4}$$
$$u\frac{\partial b^*}{\partial x}+v\frac{\partial b^*}{\partial y}=D_B\frac{\partial^2 b^*}{\partial y^2}+K_1 a^* b^{*2}. \tag{4.5}$$

The boundary conditions are

$$\left.\begin{array}{l} u=\ U_w=U_0(\ x+b)^n,\quad v=0,\quad T=\ T_w=T_0+c(\ x+b),\\ D_A\dfrac{\partial a^*}{\partial y}=K_s a^*,\quad D_B\dfrac{\partial b^*}{\partial y}\ =-K_s a^*,\quad \text{at}\ \ y=A_1(x+b)^{\frac{1-n}{2}}.\\ u\to\ U_e(x)=\ \ U_\infty(\ x+b)^n,\quad T\to T_\infty\ =\ T_0+d(\ x+b),\\ a^*\to a_0,\quad b^*\to 0;\quad when\ \ y\to\infty. \end{array}\right\} \tag{4.6}$$

4.3 Similarity Transformations

To convert the system of governing equations into the dimensionless form, the following transformations have been introduced, where ψ be the stream function satisfying the continuity equation. It is usually written as:

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}. \tag{4.7}$$

Now introduce the following similarity transformations :

$$\psi = \sqrt{\frac{2}{n+1}\nu U_0 (x+b)^{n+1}} F(\eta), \quad G(\eta) = \frac{a^*}{a_0}, \quad H(\eta) = \frac{b^*}{a_0}, \tag{4.8}$$

$$\eta = \sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{n-1}}\, y, \quad \Theta(\eta) = \frac{T - T_\infty}{T_w - T_0}. \tag{4.9}$$

The detailed procedure for the conversion of equations (4.1)-(4.5) has been described in the upcoming discussion.

- $$\begin{aligned} \frac{\partial \eta}{\partial x} &= \frac{\partial}{\partial x}\left(\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{n-1}}\, y\right) \\ &= \sqrt{\frac{n+1}{2}\frac{U_0}{\nu}}\, y \frac{\partial}{\partial x}(x+b)^{\frac{n-1}{2}} \\ &= \sqrt{\frac{n+1}{2}\frac{U_0}{\nu}}\, y \left(\frac{n-1}{2}\right)(x+b)^{\frac{n-1}{2}-1} \\ &= \left(\frac{n-1}{2}\right)\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{n-1}}\, y (x+b)^{-1} \\ &= \left(\frac{n-1}{2}\right)\eta (x+b)^{-1}. \end{aligned}$$
- $$\begin{aligned} \frac{\partial \eta}{\partial y} &= \frac{\partial}{\partial y}\left(\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{n-1}}\quad y\right) \\ &= \left(\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{n-1}}\right). \end{aligned}$$
- $$\begin{aligned} u &= \frac{\partial \psi}{\partial y} \\ &= \frac{\partial \psi}{\partial \eta}\frac{\partial \eta}{\partial y} \\ &= \left(\frac{\partial}{\partial \eta}\sqrt{\frac{2}{n+1}\nu U_0 (x+b)^{n+1}} F(\eta)\right)\left(\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{n-1}}\right) \\ &= \left(\sqrt{(\frac{2}{n+1})\nu U_0 (x+b)^{n+1}}\right) F'(\eta)\left(\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{n-1}}\right) \\ &= U_0 (x+b)^n F'(\eta). \end{aligned}$$

- $$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial}{\partial x}\Big(U_0(x+b)^n F'(\eta)\Big)\\ &= \frac{\partial}{\partial x}\Big(U_0(x+b)^n\Big)F'(\eta) + U_0(x+b)^n\frac{\partial}{\partial x}F'(\eta)\\ &= nU_0(x+b)^{n-1}F'(\eta) + U_0(x+b)^n F''(\eta)\frac{\partial \eta}{\partial x}\\ &= nU_0(x+b)^{n-1}F'(\eta) + U_0(x+b)^n F''(\eta)\Big(\frac{n-1}{2}\Big)\eta(x+b)^{-1}\\ &= nU_0(x+b)^{n-1}F'(\eta) + \Big(\frac{n-1}{2}\Big)U_0(x+b)^{n-1}\eta F''(\eta)\\ &= U_0(x+b)^{n-1}\Bigg(nF'(\eta) + \Big(\frac{n-1}{2}\Big)(\eta)F''\eta\Bigg). \qquad (4.10)\end{aligned}$$

- $$\begin{aligned} v &= -\frac{\partial \psi}{\partial x}\\ &= -\frac{\partial}{\partial x}\Bigg(\sqrt{\frac{2}{n+1}\nu U_0(x+b)^{n+1}}F(\eta)\Bigg)\\ &= -\Bigg(\sqrt{\frac{2}{n+1}\nu U_0}\Bigg)\frac{\partial}{\partial x}\Big((x+b)^{\frac{n+1}{2}}F(\eta)\Big)\\ &= -\Bigg(\sqrt{\frac{2}{n+1}\nu U_0}\Bigg)\Bigg(\frac{\partial}{\partial x}(x+b)^{\frac{n+1}{2}}F(\eta) + (x+b)^{\frac{n+1}{2}}\frac{\partial}{\partial x}F(\eta)\Bigg)\\ &= -\Bigg(\sqrt{\frac{2}{n+1}\nu U_0}\Bigg)\Bigg(\frac{n+1}{2}(x+b)^{\frac{n+1}{2}}(x+b)^{-1}F(\eta)\\ &\quad + \frac{n-1}{2}(x+b)^{\frac{n+1}{2}}F'(\eta)(x+b)^{-1}\Bigg)\\ &= -\Bigg(\sqrt{\frac{2}{n+1}\nu U_0}\Bigg)\frac{n+1}{2}(x+b)^{\frac{n+1}{2}}(x+b)^{-1}\\ &\quad \Bigg(F(\eta) + \frac{n-1}{n+1}\eta F'(\eta)\Bigg)\\ &= -\sqrt{\frac{n+1}{2}U_0\nu(x+b)^{n-1}}\Bigg(F(\eta) + \frac{n-1}{n+1}\eta F'(\eta)\Bigg).\end{aligned}$$

- $$\frac{\partial v}{\partial y} = -\frac{\partial}{\partial y}\Bigg(\sqrt{\frac{n+1}{2}U_0\nu(x+b)^{n-1}}\Big(F(\eta) + \frac{n-1}{n+1}\eta F'(\eta)\Big)\Bigg)$$

$$= -\sqrt{\frac{n+1}{2}U_0\nu(x+b)^{n-1}}\Bigg(F'(\eta)\frac{\partial\eta}{\partial y}$$
$$+\frac{\partial\eta}{\partial y}\frac{n-1}{n+1}F'(\eta) + \quad \frac{n-1}{n+1}\eta F''(\eta)\frac{\partial\eta}{\partial y}\Bigg)$$

$$= -\sqrt{\frac{n+1}{2}U_0\nu(x+b)^{n-1}}\Bigg(F'(\eta) + \frac{n-1}{n+1}F'(\eta) + \frac{n-1}{n+1}\eta F''(\eta)\Bigg)\frac{\partial\eta}{\partial y}$$

$$= -\Bigg(\sqrt{\frac{n+1}{2}U_0\nu(x+b)^{n-1}}\Bigg)\Bigg(\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{n-1}}\Bigg)$$
$$\Bigg(F'(\eta) + \frac{n-1}{n+1}F'(\eta) + \frac{n-1}{n+1}\eta F''(\eta)\Bigg).$$

$$= -\Big(\frac{n+1}{2}\Big)U_0(x+b)^{n-1}\Bigg(F'(\eta) + \frac{n-1}{n+1}F'(\eta) + \frac{n-1}{n+1}\eta F''(\eta)\Bigg)$$

$$= -\Big(\frac{n+1}{2}\Big)U_0(x+b)^{n-1}\Bigg(\Big(1+\frac{n-1}{n+1}\Big)F'(\eta) + (\eta)(\frac{n-1}{n+1})F''(\eta)\Bigg)$$
$$= \quad -U_0(x+b)^{n-1}\Bigg(nF'(\eta) + \frac{n-1}{2}\eta F''(\eta)\Bigg). \tag{4.11}$$

Using (4.10) and (4.11) in (4.1),

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = U_0(x+b)^{n-1}\Bigg(nF'(\eta) + \frac{n-1}{2}\eta F''(\eta)\Bigg)$$
$$- U_0(x+b)^{n-1}\Bigg(nF'(\eta) + \frac{n-1}{2}\eta F''(\eta)\Bigg)$$
$$= 0.$$

Hence the continuity equation (4.1) is identically satisfied.

Now we constitute below the procedure for conversion of (4.2) in the dimensionless form .

$$\bullet \quad \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\Big(U_0(x+b)^n F'(\eta)\Big)$$
$$= U_0(x+b)^n \frac{\partial F'}{\partial \eta}\frac{\partial \eta}{\partial y}$$
$$= U_0(x+b)\, F''(\eta)\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{n-1}}.$$

$$\bullet \quad v\frac{\partial u}{\partial y} = -\sqrt{\frac{n+1}{2}U_0\,\nu(x+b)^{n-1}}\left(F(\eta) + \eta\frac{n-1}{n+1}F'(\eta)\right)$$
$$U_0(x+b)^n F''(\eta)\sqrt{(\frac{n+1}{2})\frac{U_0}{\nu}(x+b)^{n-1}}$$
$$= -U_0^2(x+b)^{2\,n-1}\frac{n+1}{2}F''(\eta)\left(F(\eta)+\eta\frac{n-1}{n+1}F'(\eta)\right)$$
$$= \quad -U_0^2(x+b)^{2\,n-1}\left(\frac{n+1}{2}F''(\eta)F(\eta)\right.$$
$$\left. + \frac{n-1}{2}\,\eta F'(\eta)F''(\eta)\right). \tag{4.12}$$

$$\bullet \quad u\frac{\partial u}{\partial x} = U_0(x+b)^n F'(\eta)U_0(x+b)^{n-1}\left(nF'(\eta) + \frac{n-1}{2}\,\eta F''(\eta)\right)$$
$$= U_0^2(x+b)^{2\,n-1}\left(n(F'(\eta))^2 + \frac{n-1}{2}\,\eta F'(\eta)F''(\eta)\right). \tag{4.13}$$

Using (4.12) and (4.13), the left side of (4.2) is as follows.

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U_0^2(x+b)^{2n-1}\left(n(F'(\eta))^2 + (\frac{n-1}{2})(\eta)F'(\eta)F''(\eta)\right)$$
$$- U_0^2(x+b)^{2n-1}(\frac{n+1}{2})F''(\eta)\left(F(\eta)+(\eta)(\frac{n-1}{n+1})F'(\eta)\right)$$
$$= U_0^2(x+b)^{2n-1}\left(n(F'(\eta))^2(\eta) + (\frac{n-1}{2})(\eta)F'(\eta)F''(\eta)\right.$$
$$\left. - (\frac{n+1}{2})F''(\eta)F(\eta) - (\frac{n-1}{2})(\eta)F''(\eta)F'(\eta)\right)$$
$$= U_0^2(x+b)^{2n-1}\left(n(F'(\eta))^2 - (\frac{n+1}{2})F''(\eta)F(\eta)\right)$$
$$= nU_0^2(x+b)^{2n-1}\left((F'(\eta))^2 - (\frac{n+1}{2n})F''(\eta)F(\eta)\right).$$

To convert the right side of (4.2) into the dimensionless form we proceed as follows.

$$
\begin{aligned}
\bullet\ \frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)\\
&= \frac{\partial}{\partial y}\left(U_0(x+b)^{\,n}F''(\eta)\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{\,n-1}}\right)\\
&= \frac{\partial}{\partial \eta}\left(U_0(x+b)^{\,n}F''(\eta)\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{\,n-1}}\right)\frac{\partial \eta}{\partial y}\\
&= U_0(x+b)^{\,n}\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{\,n-1}}\frac{\partial}{\partial \eta}F''(\eta)\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{\,n-1}}\\
&= \left(\frac{n+1}{2}\right)\frac{U_0^2}{\nu}(x+b)^{2\,n-1}F'''(\eta). \qquad (4.14)
\end{aligned}
$$

$$
\begin{aligned}
\bullet\ \frac{\partial^2 u}{\partial x \partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)\\
&= \frac{\partial}{\partial x}\left(U_0(x+b)^{\,n}F''(\eta)\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{\,n-1}}\right)\\
&= \frac{\partial}{\partial x}U_0(x+b)^{\,n}F''(\eta)\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{\,n-1}}\\
&\quad + U_0(x+b)^{\,n}\frac{\partial}{\partial x}F''(\eta)\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{\,n-1}}\\
&\quad + U_0(x+b)^{\,n}F''(\eta)\frac{\partial}{\partial x}\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{\,n-1}}\\
&= nU_0(x+b)^{\,n-1}F''(\eta)\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{\,n-1}}\\
&\quad + U_0(x+b)^{\,n}\frac{\partial}{\partial \eta}F''(\eta)\frac{\partial \eta}{\partial x}\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{\,n-1}}\\
&\quad + U_0(x+b)^{n}F''(\eta)\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}}\frac{\partial}{\partial x}(x+b)^{\frac{n-1}{2}}\\
&= nU_0(x+b)^{\,n-1}F''(\eta)\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{\,n-1}}\\
&\quad + U_0(x+b)^{\,n}F'''(\eta)\left(\frac{n-1}{2}\right)(\eta)(x+b)^{\,n-1}\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{\,n-1}}\\
&\quad + \left(\frac{n-1}{2}\right)\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}}(x+b)^{\frac{n-1}{2}-1}\\
&= nU_0(x+b)^{\,n-1}F''(\eta)\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{\,n-1}}\\
&\quad + U_0(x+b)^{n-1}F'''\,\eta\left(\frac{n-1}{2}\right)(\eta)\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{\,n-1}}\\
&\quad + \left(\frac{n-1}{2}\right)U_0(x+b)^{\,n}F''(\eta)\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{\,n-1}}(x+b)^{-1}
\end{aligned}
$$

$$
\begin{aligned}
&= nU_0(x+b)^{n-1}F''(\eta)\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{n-1}} \\
&+ \frac{n-1}{2}\eta U_0(x+b)^{n-1}F'''(\eta)\sqrt{\frac{n-1}{2}\frac{U_0}{v}(x+b)^{n-1}} \\
&+ \frac{n+1}{2}\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{n-1}F'''(\eta)} \\
&= U_0(x+b)^{n-1}\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{n-1}}\Big(nF''(\eta) + \frac{n-1}{2}F''(\eta) \\
&+ \frac{n-1}{2}\eta F'''(\eta)\Big) \\
&= U_0(x+b)^{n-1}\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{n-1}}\Big((n+\frac{n-1}{2})F''(\eta) \\
&+ \frac{n-1}{2}\eta F'''(\eta)\Big) \\
&= U_0(x+b)^{n-1}\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{n-1}}\Big((\frac{3n-1}{2})F''(\eta) \\
&+ \frac{n-1}{2}\eta F'''(\eta)\Big)
\end{aligned}
$$

- $$
\begin{aligned}
uv\frac{\partial^2 u}{\partial x\partial y} &= -U_0(x+b)^n F'(\eta)\sqrt{\frac{n+1}{2}\nu U_0(x+b)^{n-1}}\Big(F(\eta) + \eta\frac{n-1}{n+1}F'(\eta)\Big) \\
&\quad U_0(x+b)^{n-1}\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{n-1}}\Big((\frac{3n-1}{2})F''(\eta) \\
&+ \Big(\frac{n-1}{2}\Big)\eta F'''(\eta)\Big) \\
&= -U_0^3(x+b)^{3n-2}\frac{n+1}{2}F'(\eta)\Bigg(F(\eta)F''(\eta)(\frac{3n-1}{2}) \\
&+ \Big(\frac{n-1}{2}\Big)(\eta)F(\eta)F'''(\eta) + \eta\frac{n-1}{n+1}(\frac{3n-1}{2})F'(\eta)F''(\eta) \\
&+ \eta^2\Big(\frac{n-1}{n+1}\Big)\Big(\frac{n-1}{2}\Big)F'''(\eta)F'(\eta)\Bigg) \\
&= -U_0^3(x+b)^{3n-2}\frac{n+1}{2}\Big(\frac{3n-1}{2}F(\eta)F'(\eta)F''(\eta) \\
&+ \Big(\frac{n-1}{2}\Big)\eta F(\eta)F'(\eta)F'''(\eta) + \eta\Big(\frac{n-1}{n+1}\Big)\frac{3n-1}{2}F''(\eta)(F'(\eta))^2 \\
&+ \eta^2\Big(\frac{n-1}{n+1}\Big)\Big(\frac{n-1}{2}\Big)F'''(\eta)(F'(\eta))^2\Big).
\end{aligned}
$$

$$\bullet\ 2\,uv\frac{\partial^2 u}{\partial x\partial y} = -2\,U_0^3(x+b)^{3n-2}\frac{n+1}{2}\Bigg(\frac{3n-1}{2}F(\eta)F'(\eta)F''(\eta) + \Big(\frac{n-1}{2}\Big)(\eta)F(\eta)F'(\eta)F'''(\eta) + \eta\Big(\frac{n-1}{n+1}\Big)\Big(\frac{3n-1}{2}\Big)F''(\eta)(F'(\eta))^2 + \eta^2\Big(\frac{n-1}{n+1}\Big)\Big(\frac{n-1}{2}\Big)F'''(\eta)(F'(\eta))^2\Bigg)$$

$$= U_0^3(x+b)^{3n-2}(n+1)\Bigg(\frac{3n-1}{2}F(\eta)F'(\eta)F''(\eta) + \Big(\frac{n-1}{2}\Big)(\eta)F(\eta)F'(\eta)F'''(\eta) + \eta\Big(\frac{n-1}{n+1}\Big)\Big(\frac{3n-1}{2}\Big)F''(\eta)(F'(\eta))^2 + \eta^2\Big(\frac{n-1}{n+1}\Big)\Big(\frac{n-1}{2}\Big)F'''(\eta)(F'(\eta))^2\Bigg). \tag{4.15}$$

$$\bullet\quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\Big(\frac{\partial u}{\partial x}\Big)$$

$$= \frac{\partial}{\partial x}\Bigg(nU_0(x+b)^{n-1}F'(\eta) + \frac{n-1}{2}U_0(x+b)^{n-1}(\eta)F''(\eta)\Bigg)$$

$$= n\frac{\partial}{\partial x}U_0(x+b)^{n-1}F'(\eta) + nU_0(x+b)^{n-1}\frac{\partial}{\partial\eta}F'(\eta)\frac{\partial\eta}{\partial x} + \Big(\frac{n-1}{2}\Big)\frac{\partial}{\partial x}(\eta)F''(\eta) + \Big(\frac{n-1}{2}\Big)U_0(x+b)^{n-1}\frac{\partial\eta}{\partial x}F''(\eta) + \Big(\frac{n-1}{2}\Big)U_0(x+b)^{n-1}(\eta)\frac{\partial}{\partial\eta}F''(\eta)\frac{\partial\eta}{\partial x}$$

$$= n(n-1)U_0(x+b)^{n-1}F'(\eta) + \Big(n\frac{n-1}{2}\Big)U_0(x+b)^{n-1}(x+b)^{n-1}\eta F''(\eta) + \Big(\frac{(n-1)^2}{2}\Big)U_0(x+b)^{n-2}(\eta)F''(\eta) + \Big(\frac{n-1}{2}\Big)^2\Big(\frac{n-1}{2}\Big)U_0(x+b)^{n-1}(x+b)^{n-1}\eta F''(\eta) + \Big(\frac{n-1}{2}\Big)^2\Big(\frac{n-1}{2}\Big)U_0(x+b)^{n-1}(x+b)^{n-1}(\eta)F'''(\eta)$$

$$= n(n-1)U_0(x+b)^{n-2}F' + \Big(\frac{n(n-1)}{2}\Big)U_0(x+b)^{n-2}\,\eta F''(\eta) + \Big(\frac{n-1}{2}\Big)^2(\eta)U_0(x+b)^{n-2}F'' + \Big(\frac{(n-1)^2}{2}\Big)U_0(x+b)^{n-2}(\eta)F''(\eta) + \Big(\frac{n-1}{2}\Big)^2\eta^2U_0(x+b)^{n-2}F'''(\eta)$$

$$= U_0(x+b)^{n-2}(n-1)\Bigg(nF' + \frac{n}{2}F''(\eta) + (\eta)\Big(\frac{n-1}{4}\Big)F''(\eta)$$
$$+ \Big(\frac{n-1}{4}\Big)F'''(\eta)\ \eta^2 + \Big(\frac{n-1}{2}\Big)(\eta)F''(\eta)\Bigg)$$
$$= \ U_0(x+b)^{\ n-2}(n-1)\Bigg(nF' + \Big(\frac{n}{2} + \frac{n-1}{4} + \frac{n-1}{2}\Big)(\eta)F''(\eta)$$
$$+ \Big(\frac{n-1}{4}\Big)F'''(\eta)\ \eta^2\Bigg)$$
$$= U_0(x+b)^{\ n-2}(n-1)\Bigg(nF' + \Big(\frac{5n-3}{4}\Big)(\eta)F''(\eta)$$
$$+ \Big(\frac{n-1}{4}\Big)F'''(\eta)\ \eta^2\Bigg).$$

- $$u^2\frac{\partial^2 u}{\partial x^2} = \ U_0^2(x+b)^{2\ n}(F'(\eta))^2\Bigg(U_0(x+b)^{\ n-2}(n-1)\Bigg(nF'$$
$$+ (\frac{5\ n-3}{4})(\eta)F''(\eta) + \Big(\frac{n-1}{4}\Big)F'''(\eta)\ \eta^2\Bigg)\Bigg)$$
$$= U_0^3(x+b)^{3\ n-2}(n-1)\Bigg(n(F')^3 + \Big(\frac{5\ n-3}{4}\Big)(\eta)F''(\eta)(F')^2$$
$$+ \Big(\frac{n-1}{4}\Big)(F')^2F'''(\eta)(\eta^2)\Bigg). \tag{4.16}$$

- $$v^2\frac{\partial^2 u}{\partial y^2} = (\frac{n+1}{2})(\nu)U_0(x+b)^{\ n-1}\Bigg(F(\eta) + (\eta)\frac{n-1}{n+1}F'(\eta)\Bigg)^2$$
$$\Bigg(\frac{n+1}{2}\frac{U_0^2}{\nu}(x+b)^{2\ n-1}F'''(\eta)\Bigg)$$
$$= (\frac{n+1}{2})^2U_0^3(x+b)^{3\ n-2}F'''(\eta)\ \Bigg(F(\eta) + (\eta)\Big(\frac{n-1}{n+1}\Big)F'(\eta)\Bigg)^2$$
$$= (\frac{n+1}{2})^2U_0^3(x+b)^{3n-2}\Bigg((F(\eta))^2F'''(\eta) + (\eta)^2\Big(\frac{n-1}{n+1}\Big)^2(F'(\eta))^2F'''$$
$$+ 2F(\eta)F'(\eta)F'''(\eta)(\eta)\frac{n-1}{n+1}\Bigg). \tag{4.17}$$

Using (4.15) - (4.17), we get

- $$u^2\frac{\partial^2 u}{\partial\ x^2} + v^2\frac{\partial^2 u}{\partial\ y^2} + 2\ uv\frac{\partial^2\ u}{\partial xy}$$

$$= \ U_0^2(x+b)^{3\ n-2}(n-1)\Bigg(n(F'(\eta))^3 + \Big(\frac{5\ n-3}{4}\Big)(\eta)F''(\eta)(F'(\eta))^2$$
$$+ \Big(\frac{n-1}{4}\Big)(F')^2F'''(\eta)(\eta)^2\Bigg) + \Big(\frac{n+1}{2}\Big)^2\ U_0^3(x+b)^{3\ n-2}\Bigg((F(\eta))^2F'''(\eta)$$

$$+\ \eta^2\Big(\frac{n-1}{n+1}\Big)^2(F'(\eta))^2F'''(\eta)+2F(\eta)F'(\eta)F'''(\eta)(\eta)\Big(\frac{n-1}{n+1}\Big)\Big)$$

$$-U_0^3(x+b)^{3n-2}(n+1)\Bigg(\frac{3n-1}{2}F(\eta)F'(\eta)F''(\eta)$$

$$+\Big(\frac{n-1}{2}\Big)(\eta)F(\eta)F'(\eta)F'''(\eta)$$

$$+\eta\Big(\frac{n-1}{n+1}\Big)\Big(\frac{3n-1}{2}\Big)F''(\eta)(F'(\eta))^2+\eta^2\Big(\frac{n-1}{n+1}\Big)\Big(\frac{n-1}{2}\Big)F'''(\eta)(F'(\eta))^2\Big)$$

$$=U_0^3(x+b)^{3n-2}\Bigg(n(n-1)(F')^3+(n-1)\Big(\frac{5n-3}{4}\Big)(\eta)F''(\eta)(F')^2$$

$$+\Big(\frac{(n-1)^2}{4}\Big)(F')^2F'''(\eta)(\eta)^2+(\frac{n+1}{2})^2F'''(\eta)F^2(\eta)$$

$$+\ \eta^2\Big(\frac{(n+1)^2}{2}\Big)\Big(\frac{n-1}{n+1}\Big)^2(F'(\eta))^2F'''(\eta)$$

$$+2\eta\Big(\frac{n-1}{n+1}\Big)\Big(\frac{n+1}{2}\Big)^2F(\eta)F'(\eta)F'''(\eta)$$

$$-(n+1)\Big(\frac{3n-1}{2}\Big)F(\eta)F'(\eta)F''(\eta)-\eta\Big(\frac{n+1}{2}\Big)(n-1)F(\eta)F'(\eta)F'''(\eta)$$

$$-\ \eta\Big(\frac{n+1}{2}\Big)\Big(\frac{n-1}{n+1}\Big)(3n-1)(F'(\eta))^2F''(\eta)$$

$$-\ \eta^2(n-1)\Big(\frac{n-1}{n+1}\Big)\Big(\frac{n+1}{2}\Big)(F'(\eta))^2F'''(\eta)\Bigg)$$

$$=U_0^3(x+b)^{3n-2}\Bigg(n(n-1)(F')^3+\Big(\frac{5n-3}{2}-(3n-1)\Big)(\eta)\Big(\frac{n-1}{2}\Big)F''(\eta)$$

$$+\Big(\frac{n+1}{2}\Big)^2F'''(\eta)F(\eta)^2+\Big((\frac{(n+1)^2}{2})\Big)\Big(\frac{n-1}{n+1}\Big)^2F'''(\eta)(F')^2$$

$$+\eta(n-1)\Big(\frac{n+1}{2}\Big)F(\eta)F'(\eta)F'''(\eta).$$

$$-\Big(\frac{n+1}{2}\Big)(3n-1)F(\eta)F'(\eta)F''(\eta)-\Big(\frac{n+1}{2}\Big)(\eta)(n-1)F(\eta)F'(\eta)$$

$$-\ \eta^2\Big(\frac{(n-1)^2}{2}\Big)\Big(F')^2F'''(\eta)\Bigg)U_0^3(x+b)^{3n-2}\Bigg(n(n-1)(F')^3$$

$$-\ \eta\Big(\frac{n-1}{2}\Big)\Big(\frac{n+1}{2}\Big)F''(\eta)(F')^2-(3n-1)\Big(\frac{n+1}{2}\Big)F(\eta)F'(\eta)F''(\eta)$$

$$+\Big(\frac{n+1}{2}\Big)^2F(\eta)^2F'''(\eta)\Bigg)$$

$$=U_0^3(x+b)^{3n-2}\Big(\frac{n+1}{2}\Big)\Bigg(\frac{2n(n-1)}{n+1}(F')^3-(\eta)\Big(\frac{n-1}{2}\Big)F''(\eta)(F')^2$$

$$-(3n-1)F(\eta)F'(\eta)F''(\eta)+\Big(\frac{n+1}{2}\Big)F(\eta)^2F'''(\eta)\Bigg). \qquad (4.18)$$

- $U_e = U_\infty(x+b)^n$
- $\dfrac{dU_e}{dx} = nU_\infty(x+b)^{n-1}$
- $U_e\dfrac{dU_e}{dx} = U_\infty(x+b)^n nU_\infty(x+b)^{n-1} = nU_\infty^2(x+b)^{2n-1}$
- $\dfrac{\partial U_e}{\partial x} = \dfrac{\partial}{\partial x}U_\infty(x+b)^n = nU_\infty(x+b)^{n-1}$
- $\dfrac{\partial^2 U_e}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial U_e}{\partial x}\right) = \dfrac{\partial}{\partial x}\left(nU_\infty(x+b)^{n-1}\right) = n(n-1)U_\infty(x+b)^{n-2}$
- $\lambda_1 U_e^2\dfrac{\partial^2 U_e}{\partial x^2} = \lambda_1 U_\infty^2(x+b)^{2n}n(n-1)U_\infty(x+b)^{n-2} = \lambda_1 n(n-1)U_\infty^3(x+b)^{3n-2}$
- $$U_e\frac{dU_e}{dx} + \lambda_1 U_e^2\frac{\partial^2 U_e}{\partial x^2} = nU_\infty^2(x+b)^{2n-1} + \lambda_1 n(n-1)U_\infty^3(x+b)^{3n-2} \tag{4.19}$$
- $$\nu\frac{\partial u^2}{\partial y^2} = \nu(\frac{n+1}{2})\frac{U_0^2}{\nu}(x+b)^{2n-1}F'''(\eta).() \tag{4.20}$$

Using (4.18) -(4.20) in the right side of (4.2), we get

$$\begin{aligned}
&U_e\frac{dU_e}{dx} + \lambda_1 U_e^2\frac{\partial^2 U_e}{\partial x^2} + \nu\frac{\partial^2 u}{\partial y^2} - \lambda_1\left(2\,uv\frac{\partial^2 u}{\partial x\partial y} + v^2\frac{\partial^2 u}{\partial y^2} + u^2\frac{\partial^2 u}{\partial x^2}\right)\\
&= nU_\infty^2(x+b)^{2n-1} + \lambda_1 n(n-1)U_\infty^3(x+b)^{3n-2} + (\frac{n+1}{2})U_0^2(x+b)^{2n-1}F'''(\eta)\\
&\quad - \lambda_1\Bigg(U_0^3(x+b)^{3n-2}(\frac{n+1}{2})\Bigg(n(n-1)(F')^3 - (\eta)(\frac{n-1}{2})F''(\eta)(F')^2\\
&\quad - (3n-1)F(\eta)F'(\eta)F''(\eta) + (\frac{n+1}{2})F(\eta)^2F'''(\eta)\Bigg)\Bigg)
\end{aligned}$$

$$= nU_\infty^2(x+b)^{2n-1} + \lambda_1 n(n-1)U_\infty^3(x+b)^{3n-2} + (\frac{n+1}{2})U_0^2(x+b)^{2n-1}\Bigg(F'''(\eta)$$

$$-\lambda_1\Bigg(U_0(x+b)^{n-1}\Big(n(n-1)(F')^3 - (\eta)(\frac{n-1}{2})F''(\eta)(F')^2$$

$$-(3n-1)F(\eta)F'(\eta)F''(\eta) + (\frac{n+1}{2})F(\eta)^2F'''(\eta)\Big)\Bigg)\Bigg)$$

$$= (\frac{n+1}{2})U_0^2(x+b)^{2n-1}\Bigg(\Big(\frac{2n}{n+1}\Big)\frac{U_\infty^2}{U_0^2} + \Big(\frac{2n(n-1)}{n+1}\Big)\lambda_1 U_0(x+b)^{n-1}\frac{U_\infty^3}{U_0^3}$$

$$+F'''(\eta) - \lambda_1 U_0(x+b)^{n-1}\Big(\frac{2n(n-1)}{n+1}\Big)(F')^3 - (\eta)\Big(\frac{n-1}{2}\Big)F''(\eta)(F')^2$$

$$-(3n-1)F(\eta)F'(\eta)F''(\eta) + \Big(\frac{n+1}{2}\Big)F(\eta)^2\Bigg).$$

- $$\frac{\sigma B_0^2(x)}{\rho}(u - U_e) = \frac{\sigma(x+b)^{n-1}}{\rho}\Big(U_0(x+b)^nF'(\eta) - U_\infty(x+b)^n\Big)$$

$$= \frac{\sigma(x+b)^{2n-1}}{\rho}U_0\Big(F'(\eta) - \frac{U_\infty}{U_0}\Big)$$

$$= \frac{\sigma(x+b)^{2n-1}}{\rho}U_0\Big(F'(\eta) - A\Big) \quad \Big(\because A = \frac{U_\infty}{U_0}\Big)$$

$$= M^2\Big(F'(\eta) - A\Big). \qquad \Big(\because M^2 = \frac{\sigma(x+b)^{2n-1}}{\rho}U_0\Big)$$

Hence the dimensionless form of (4.2) becomes

$$nU_0^2(x+b)^{2n-1}\Bigg((F'(\eta))^2 - \Big(\frac{n+1}{2n}\Big)F''(\eta)F(\eta)\Bigg) = \Big(\frac{n+1}{2}\Big)U_0^2(x+b)^{2n-1}$$

$$\Bigg(\Big(\frac{2n}{n+1}\Big)\frac{U_\infty^2}{U_0^2} + \Big(\frac{2n(n-1)}{n+1}\Big)\lambda_1 U_0(x+b)^{n-1}\frac{U_\infty^3}{U_0^3} + F'''(\eta)$$

$$-\lambda_1 U_0(x+b)^{n-1}\Bigg(\Big(\frac{2n(n-1)}{n+1}\Big)(F')^3 - (\eta)\Big(\frac{n-1}{2}\Big)F''(\eta)(F')^2$$

$$-(3n-1)F(\eta)F'(\eta)F''(\eta) + \Big(\frac{n+1}{2}\Big)F(\eta)^2\Bigg)\Bigg) - M^2\Big(F'(\eta) - A\Big)$$

$$\Rightarrow \quad \Big(\frac{2\,n}{n+1}\Big)\Bigg((F'(\,\eta))^2-\Big(\frac{n+1}{2\,n}\Big)F''(\,\eta)F(\,\eta)\Bigg)=\Big(\frac{2\,n}{n+1}\Big)\frac{U_\infty^2}{U_0^2}$$
$$+\,2\frac{n(\,n-1)}{n+1}\lambda_1U_0(\,x+b)^{\,n-1}\frac{U_\infty^3}{U_0^3}+F'''(\,\eta)-\lambda_1U_0(\,x+b)^{\,n-1}\Bigg(\frac{2\,n(n-1)}{n+1}$$
$$-\,(\,\eta)(\frac{n-1}{2})F''(\,\eta)(F')^2-(3n-1)F(\,\eta)F'(\,\eta)F''(\,\eta)+(\frac{n+1}{2})F(\eta)^2\Bigg)$$
$$-\,M^2\Bigg(F'(\,\eta)-\,A\Bigg)$$

$$\Rightarrow \quad \frac{2\,n}{n+1}(F'(\,\eta))^2-F''(\,\eta)F(\eta)=\frac{2n}{n+1}A^2+2\beta\frac{n(n-1)}{n+1}A^3$$
$$+\,F'''(\,\eta)-\,\beta\Bigg(\frac{2\,n(\,n-1)}{n+1}(F')^3\quad\Big(\because A=\frac{U_\infty}{U_0}\Big)\Big(\because \beta=\lambda_1U_0(x+b)^{\,n-1}\Big)$$
$$-\,(\,\eta)(\frac{n-1}{2})F''(\,\eta)(F')^2-(3n-1)F(\,\eta)F'(\,\eta)F''(\,\eta)+(\frac{n+1}{2})F(\,\eta)^2\Bigg)$$
$$-\,M^2\Bigg(F'(\,\eta)-\,A\Bigg)$$

$$\Rightarrow \quad F'''(\,\eta)+F''(\,\eta)F(\,\eta)-\frac{2\,n}{n+1}(F'(\,\eta))^2+\frac{2\,n}{n+1}A^2+\frac{2\,n(\,n-1)}{n+1}\beta\,A^3$$
$$+\,\beta\Bigg((3n-1)F(\eta)F'(\eta)F''(\eta)-\frac{2n(n-1)}{n+1}(F'(\eta))^3+(\eta)(\frac{n-1}{2})(F'(\eta))^2F''$$
$$-\,(\frac{n+1}{2})F^2(\,\eta)F'''(\,\eta)\Bigg)-\,M^2\Bigg(F'(\,\eta)-\,A\Bigg)=0.$$

Now we include below the procedure for the conversion of (4.3) into the dimensionless form

$$\bullet \quad \Theta(\,\eta)=\frac{T-T_\infty}{T_w-T_o}$$
$$\Rightarrow\; T-T_\infty=(T_w-T_0)\;\Theta(\eta)$$
$$\Rightarrow\quad T=(\,T_w-T_0)\,\Theta(\eta)+T_\infty$$
$$\bullet\quad \frac{\partial\,T}{\partial\,x}=(T_w-T_0)\,\frac{\partial\Theta(\eta)}{\partial x}$$
$$=(T_w-T_0)\,\frac{\partial\,\Theta(\eta)}{\partial\eta}\,\frac{\partial\eta}{\partial\,x}$$
$$=(T_w-T_0)\,\left(\frac{n-1}{2}\right)\eta(x+b)^{-1}\Theta'(\eta)$$
$$=\left(\frac{n-1}{2}\right)(T_w-T_0)\,\eta(x+b)^{-1}\Theta'(\eta) \qquad (4.21)$$

- $$\begin{aligned}\frac{\partial T}{\partial y} &= \frac{\partial}{\partial y}\Bigg((T_w - T_0)\,\Theta(\eta) + T_\infty\Bigg)\\ &= (T_w - T_0)\,\frac{\partial \Theta(\eta)}{\partial \eta}\frac{\partial \eta}{\partial y}\\ &= (T_w - T_0)\,\Theta'(\eta)\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{n-1}} \qquad (4.22)\end{aligned}$$

- $$\begin{aligned}\frac{\partial^2 T}{\partial y^2} &= \frac{\partial}{\partial y}\Big(\frac{\partial T}{\partial y}\Big)\\ &= \frac{\partial}{\partial y}\Bigg((T_w - T_0)\,\Theta'(\eta)\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{n-1}}\Bigg)\\ &= (T_w - T_0)\,\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{n-1}}\frac{\partial \theta}{\partial \eta}\frac{\partial \eta}{\partial y}\\ &= \sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{n-1}}\,(T_w - T_0)\,\Theta''(\eta)\\ &\quad\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{n-1}}\\ &= \frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{n-1}\,(T_w - T_0)\,\Theta''(\eta). \qquad (4.23)\end{aligned}$$

- $$\begin{aligned}\frac{\partial^2 T}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right)\\ &= \frac{\partial}{\partial x}\Bigg(\left(\frac{n-1}{2}\right)(T_w - T_0)\,\eta(x+b)^{-1}\Theta'(\eta)\Bigg)\\ &= \frac{n-1}{2}\,(T_w - T_0)\left(\frac{\partial}{\partial x}(x+b)^{-1}\eta\Theta'(\eta) + (x+b)^{-1}\frac{\partial \eta}{\partial x}\Theta'(\eta)\right.\\ &\quad\left. + (x+b)^{-1}(\eta)\frac{\partial \Theta'(\eta)}{\partial \eta}\frac{\partial \eta}{\partial x}\right)\\ &= \frac{n-1}{2}\,(T_w - T_0)\Bigg(- (x+b)^{-2}\eta\Theta'(\eta) + \Big((x+b)^{-1}\Theta'(\eta)\\ &\quad + (x+b)^{-1}\eta\Theta''(\eta)\Big)\frac{\partial \eta}{\partial x}\Bigg)\\ &= \frac{n-1}{2}\,(T_w - T_0)\Bigg(- (x+b)^{-2}\eta\Theta'(\eta)\\ &\quad + \Big((x+b)^{-1}\Theta'(\eta) + (x+b)^{-1}\eta\Theta''(\eta)\Big)\frac{n-1}{2}\eta(x+b)^{-1}\Bigg)\end{aligned}$$

$$= \frac{n-1}{2}(T_w - T_0)\Bigg(-(x+b)^{-2}\eta\Theta'(\eta)$$
$$+\Big(\frac{n-1}{2}(x+b)^{-2}\Theta'(\eta) + (x+b)^{-2}\frac{n-1}{2}\eta\Theta''(\eta)\Big)\frac{\partial \eta}{\partial x}\Bigg)$$
$$= \frac{n-1}{2}(T_w - T_0)(x+b)^{-2}\Bigg(-\eta\Theta'(\eta) + \frac{n-1}{2}\eta\Theta' + \frac{n-1}{2}\eta^2\Theta''\Bigg)$$
$$= \frac{n-1}{2}(T_w - T_0)(x+b)^{-2}\Bigg(\Big(-1 + \frac{n-1}{2}\Big)\eta\Theta' + \frac{n-1}{2}\eta^2\Theta''\Bigg)$$
$$= \frac{n-1}{2}(T_w - T_0)(x+b)^{-2}\Bigg(\Big(\frac{n-3}{2}\Big)\eta\Theta' + \frac{n-1}{2}\eta^2\Theta''\Bigg)$$
$$= \Big(\frac{n-1}{2}\Big)^2(T_w - T_0)(x+b)^{-2}\eta\Bigg(-\Theta' + \eta\Theta''\Bigg). \tag{4.24}$$

- $$u^2\frac{\partial^2 T}{\partial^2 x} = U_0^2(x+b)^{2n}F'^2(\eta)\Bigg(\frac{n-1}{2}\Bigg)^2(T_w - T_0)(x+b)^{-2}$$
$$\Bigg(\Theta''(\eta)(\eta) + \Theta'(\eta)\Bigg)\eta$$
$$= U_0^2(x+b)^{2n-2}F'^2(\eta)\Bigg(\frac{n-1}{2}\Bigg)^2(T_w - T_0)\,\eta\Bigg(-\Theta' + \eta\Theta''\Bigg). \tag{4.25}$$

- $$\frac{\partial^2 T}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial T}{\partial x}\right)$$
$$= \frac{\partial}{\partial y}\Bigg(\Bigg(\frac{n-1}{2}\Bigg)(T_w - T_0)\,\eta(x+b)^{-1}\Theta'(\eta)\Bigg)$$
$$= \frac{n-1}{2}(T_w - T_0)(x+b)^{-1}\Theta'(\eta)\frac{\partial \eta}{\partial y}$$
$$+\frac{n-1}{2}(T_w - T_0)(x+b)^{-1}(\eta)\Theta''\frac{\partial \eta}{\partial y}$$
$$= \frac{n-1}{2}(T_w - T_0)(x+b)^{-1}\Bigg(\Theta'(\eta) + \eta\Theta''(\eta)\Bigg)\frac{\partial \eta}{\partial y}$$
$$= \frac{n-1}{2}(T_w - T_0)(x+b)^{-1}\Bigg(\Theta'(\eta) + \eta\Theta''(\eta)\Bigg)$$
$$\sqrt{\frac{n+1}{2}\frac{U_0}{v}(x+b)^{n-1}} \tag{4.26}$$

- $$u\frac{\partial T}{\partial x} = U_0(x+b)^n F'(\eta)\left(\frac{n-1}{2}\right)(T_w - T_0)\,\eta(x+b)^{-1}\Theta'(\eta)$$
$$= \left(\frac{n-1}{2}\right)\eta\,(T_w - T_0)\,U_0(x+b)^{n-1}F'(\eta)\Theta'(\eta) \qquad (4.27)$$

- $$v\frac{\partial T}{\partial y} = -\sqrt{\frac{n+1}{2}\frac{U_0}{v}(x+b)^{n-1}}\left(F(\eta)+\eta\frac{n-1}{n+1}F'(\eta)\right)$$
$$(T_w - T_0)\,\Theta'(\eta)\sqrt{\frac{n+1}{2}\frac{U_0}{v}(x+b)^{n-1}}$$
$$= -\left(\frac{n+1}{2}\right)(T_w - T_0)\,U_0(x+b)^{n-1}\Theta'(\eta)$$
$$\left(F(\eta)+\eta\frac{n-1}{n+1}F'(\eta)\right)$$

- $$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \left(\frac{n-1}{2}\right)\eta\,(T_w - T_0)\,U_0(x+b)^{n-1}F'(\eta)\Theta'(\eta)$$
$$-\left(\frac{n+1}{2}\right)(T_w - T_0)\,U_0(x+b)^{n-1}\Theta'(\eta)$$
$$\left(F(\eta)+\eta\frac{n-1}{n+1}F'(\eta)\right)$$
$$= U_0(x+b)^{n-1}\Big(T_w - T_0\Big)\left(\left(\frac{n-1}{2}\right)\eta F'(\eta)\Theta'(\eta)\right.$$
$$\left. -\left(\frac{n-1}{2}\right)F(\eta)\Theta'(\eta) - \left(\left(\frac{n-1}{2}\right)\eta F'(\eta)\Theta'(\eta)\right)\right)$$
$$= -\left(\frac{n-1}{2}\right)U_0(x+b)^{n-1}\Big(T_w - T_0\Big)F(\eta)\Theta'(\eta) \qquad (4.28)$$

- $$uv\frac{\partial^2 T}{\partial x \partial y} = U_0(x+b)^n F'(\eta) - \sqrt{(\frac{n+1}{2})U_0\nu(x+b)^{n-1}}\left(F(\eta)\right.$$
$$\left.+\eta\frac{n-1}{n+1}F'(\eta)\right)\left(\frac{n-1}{2}\right)\Big(T_w - T_0\Big)(x+b)^{-1}\Big(\Theta'(\eta)$$
$$+\eta\Theta''(\eta)\Big)\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{n-1}}$$
$$= -U_0^2(x+b)^{2n-2}\left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)\Big(T_w - T_0\Big)$$
$$F'(\eta)\Big(\Theta'(\eta)+\eta\Theta''(\eta)\Big)\left(F(\eta)+\eta\frac{n-1}{n+1}F'(\eta)\right)$$

$$= -U_0^2(x+b)^{2n-2}\Big(\frac{n^2-1}{4}\Big)\Big(T_w - T_0\Big)F'(\eta)\Bigg(F(\eta)\Theta'(\eta)$$

$$+\eta\frac{n-1}{n+1}F'(\eta)\Theta'(\eta) + F(\eta)\eta\Theta''(\eta) + \eta^2\frac{n-1}{n+1}\Theta''(\eta)F'(\eta)\Bigg)$$

$$= -U_0^2(x+b)^{2n-2}\Big(\frac{n^2-1}{4}\Big)\Big(T_w - T_0\Big)\Bigg(F(\eta)\Theta'(\eta)F'(\eta)$$

$$+\eta\frac{n-1}{n+1}\Big(F'(\eta)\Big)^2\Theta'(\eta) + F(\eta)\eta\Theta''(\eta)F'(\eta)$$

$$+\eta^2\frac{n-1}{n+1}\Theta''(\eta)\Big(F'(\eta)\Big)^2\Bigg)$$

- $2uv\dfrac{\partial^2 T}{\partial x\partial y}$
$$= U_0^2(x+b)^{2n-2}\Big(\frac{n^2-1}{2}\Big)\Big(T_w - T_0\Big)\Bigg(F(\eta)\Theta'(\eta)F'(\eta)$$

$$+\eta\frac{n-1}{n+1}\Big(F'(\eta)\Big)^2\Theta'(\eta) + F(\eta)\eta\Theta''(\eta)F'(\eta)$$

$$+\eta^2\frac{n-1}{n+1}\Theta''(\eta)\Big(F'(\eta)\Big)^2\Bigg)$$

- $v^2\dfrac{\partial^2 T}{\partial y^2}$
$$= (\frac{n+1}{2})U_0\nu(x+b)^{n-1}\Bigg(F(\eta) + \eta\frac{n-1}{n+1}F'(\eta)\Bigg)^2$$

$$\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{n-1}\left(T_w - T_0\right)\Theta''(\eta)$$

$$= \Big(\frac{n+1}{2}\Big)^2 U_0^2(x+b)^{2n-2}\left(T_w - T_0\right)\Bigg(F(\eta)$$

$$+\eta\frac{n-1}{n+1}F'(\eta)\Bigg)^2\Theta''(\eta).$$

- $u\dfrac{\partial u}{\partial x}\dfrac{\partial T}{\partial x}$
$$= U_0(x+b)^n F'(\eta)U_0(x+b)^{n-1}\Bigg(nF'(\eta)$$

$$+\Big(\frac{n-1}{2}\Big)(\eta)F''(\eta)\Bigg)\Bigg(\frac{n-1}{2}\Bigg)\left(T_w - T_0\right)\eta(x+b)^{-1}\Theta'(\eta)$$

$$= U_0^2(x+b)^{2n-2}\left(T_w - T_0\right)\Big(\frac{n-1}{2}\Big)\eta\Theta'(\eta)F'(\eta)$$

$$\Bigg(nF'(\eta) + \Big(\frac{n-1}{2}\Big)(\eta)F''(\eta)\Bigg)$$

$$= U_0^2(x+b)^{2n-2}\left(T_w - T_0\right)\left(\frac{n-1}{2}\right)\eta$$
$$\left(n(F'(\eta))^2\Theta'(\eta) + \left(\frac{n-1}{2}\right)(\eta)F'(\eta)\Theta'(\eta)F''(\eta)\right)$$

- $$v\frac{\partial u}{\partial y}\frac{\partial T}{\partial x} = -\sqrt{(\frac{n+1}{2})U_0\nu(x+b)^{n-1}}\left(F(\eta) + \eta\frac{n-1}{n+1}F'(\eta)\right)$$
$$U_0(x+b)^n F''(\eta)\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{n-1}}$$
$$\left(\frac{n-1}{2}\right)\left(T_w - T_0\right)\eta(x+b)^{-1}\Theta'(\eta)$$
$$= -U_0^2(x+b)^{2n-1}\left(T_w - T_0\right)(\frac{n+1}{2})(\frac{n-1}{2})(\eta)$$
$$\left(F''(\eta)F(\eta)\Theta'(\eta) + \eta\frac{n-1}{n+1}F''(\eta)F'(\eta)\Theta'(\eta)\right)$$

- $$u\frac{\partial u}{\partial x}\frac{\partial T}{\partial x} + v\frac{\partial u}{\partial y}\frac{\partial T}{\partial x} = U_0^2(x+b)^{2n-2}\left(T_w - T_0\right)$$
$$\left(\frac{n-1}{2}\right)\eta\left(n(F'(\eta))^2\Theta'(\eta)\right.$$
$$\left. + \left(\frac{n-1}{2}\right)(\eta)F'(\eta)\Theta'(\eta)F''(\eta)\right)$$
$$- U_0^2(x+b)^{2n-1}\left(T_w - T_0\right)(\frac{n+1}{2})(\frac{n-1}{2})(\eta)$$
$$\left(F''(\eta)F(\eta)\Theta'(\eta) + \eta\frac{n-1}{n+1}F''(\eta)F'(\eta)\Theta'(\eta)\right)$$
$$= U_0^2(x+b)^{2n-2}\left(T_w - T_0\right)\frac{n-1}{2}\left(n(\eta)(F'(\eta))^2\Theta'(\eta)\right.$$
$$\left. - (\eta)(\frac{n+1}{2})F''(\eta)F(\eta)\Theta'(\eta)\right) \tag{4.29}$$

Using the values in Eq. (4.3), as follows.

$$\Theta''(\eta) + PrF(\eta)\,\Theta'(\,\eta) + Pr\gamma\left(\frac{n-3}{2}F(\,\eta)F'(\,\eta)\Theta'(\,\eta) - \frac{n+1}{2}(F(\eta))^2\right.$$
$$\left.\Theta''(\,\eta)\right) + Pr(S+\theta)\left(\gamma F(\eta)F''(\,\eta) - \frac{2n}{n+1}\gamma(F'(\eta))^2\right.$$
$$\left. - \frac{2}{n+1}F'(\,\eta)\right) = 0$$

Now we include below the procedure for the conversion of (4.4) into the dimensionless form.

$$\begin{aligned}
\bullet\ \frac{\partial a^*}{\partial x} &= \frac{\partial}{\partial x}\Big(a_0 G(\eta)\Big) \\
&= a_0 G'(\eta)\frac{\partial \eta}{\partial x} \\
&= a_0 G'(\eta)\Big(\frac{n-1}{2}\Big)(x+b)^{-1}\eta. \\
\bullet\ u\frac{\partial a^*}{\partial x} &= \Big(U_0(x+b)^n F'(\eta)\Big)\Big(\frac{n-1}{2}\Big)a_0 G'(\eta)(x+b)^{-1}\eta \\
&= \Big(\frac{n-1}{2}\Big)\Big(U_0(x+b)^n F'(\eta)\Big)(\eta)a_0 G'(\eta)(x+b)^{-1} \\
&= \Big(\frac{n-1}{2}\Big)\Big(U_0(x+b)^{n-1} F'(\eta)\Big)(\eta)a_0 G'(\eta). \qquad (4.30) \\
\bullet\ \frac{\partial a^*}{\partial y} &= \frac{\partial}{\partial y}\Bigg(a_0 G(\eta)\Bigg) \\
&= a_0 G'(\eta)\frac{\partial \eta}{\partial y} \\
&= a_0 G'(\eta)\Bigg(\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{n-1}}\Bigg) \\
\bullet\ v\frac{\partial a^*}{\partial y} &= -\sqrt{\frac{n+1}{2}U_0\nu(x+b)^{n-1}}\Bigg(F(\eta)+\eta\frac{n-1}{n+1}F'\Bigg) \\
&\quad a_0 G'(\eta)\Bigg(\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{n-1}}\Bigg) \\
&= -\Big(\frac{n+1}{2}\Big)U_0(x+b)^{n-1}a_0 G'(\eta)\Bigg(F(\eta)+\eta\frac{n-1}{n+1}F'\Bigg). \qquad (4.31)
\end{aligned}$$

Using (4.30) and (4.31), the left side of (4.4) becomes

$$\begin{aligned}
& u\frac{\partial a^*}{\partial x} + v\frac{\partial a^*}{\partial y} \\
&= \Big(\frac{n-1}{2}\Big)\Big(U_0(x+b)^n F'(\eta)\Big)(\eta)a_0 G'\,\eta(x+b)^{-1} \\
&\quad - \Big(\frac{n+1}{2}\Big)U_0(x+b)^{n-1}a_0 G'(\eta)\Bigg(F(\eta)+\eta\frac{n-1}{n+1}F'\Bigg) \\
&= \Big(\frac{n-1}{2}\Big)\Big(U_0(x+b)^{n-1} F'(\eta)\Big)(\eta)a_0 G'(\eta) \\
&\quad - \Big(\frac{n+1}{2}\Big)U_0(x+b)^{n-1}a_0 G'\,\eta(F(\eta) - \Big(\frac{n-1}{2}\Big)U_0(x+b)^{n-1}(\eta)a_0 G'(\eta)F'(\eta) \\
&= -\Big(\frac{n+1}{2}\Big)\,U_0(x+b)^{n-1}a_0 G'(\eta)(F(\eta).
\end{aligned}$$

To convert the right side of (4.4) into dimensionless form, we proceed as follows.

$$\bullet \frac{\partial^2 a^*}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial a^*}{\partial y}\right)$$
$$= \frac{\partial}{\partial y}\left(a_0 G'(\eta)\sqrt{\frac{n+1}{2}\frac{U_0}{v}(x+b)^{n-1}}\right)$$
$$= a_0 G''(\eta)\frac{\partial \eta}{\partial y}\left(\sqrt{\frac{n+1}{2}\frac{U_0}{v}(x+b)^{n-1}}\right)$$
$$= a_0 G''(\eta)\left(\sqrt{\frac{n+1}{2}\frac{U_0}{v}(x+b)^{n-1}}\right)\left(\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{-1}}\right)$$
$$= \left(\frac{n+1}{2}\right)\frac{U_0}{\nu}(x+b)^{-1} a_0 G''(\eta) \tag{4.32}$$
$$\bullet D_A\frac{\partial^2 a^*}{\partial y^2} = D_A\left(\frac{n+1}{2}\right)\frac{U_0}{\nu}(x+b)^{-1} a_0 G''(\eta)$$
$$\bullet K_1 a^* b^{*2} = K_1\Big(a_0 G(\eta)\Big)\Big((a_0 H(\eta)\Big)^2$$
$$= K_1 a_0^3 G(\eta) H^2(\eta)$$

Using the right side of (4.4), we get

$$D_A\frac{\partial^2 a^*}{\partial y^2} - K_1 a^* b^{*2} = D_A\left(\frac{n+1}{2}\right)\frac{U_0}{\nu}(x+b)^{-1} a_0 G''(\eta) - K_1 a_0^3 G(\eta) H^2(\eta).$$

Hence the dimensionless form of (4.4) becomes

$$-\left(\frac{n+1}{2}\right)U_0(x+b)^{n-1} a_0 F(\eta)G'(\eta) = D_A\left(\frac{n+1}{2}\right)\frac{U_0}{\nu}(x+b)^{-1} a_0$$
$$- K_1 a_0^3 G(\eta) H^2(\eta)$$
$$\Rightarrow -\left(\frac{n+1}{2}\right) U_0(x+b)^{n-1} a_0 F(\eta)G'(\eta) = \left(\frac{n+1}{2}\right) D_A a_0\frac{U_0}{\nu}(x+b)^{-1}$$
$$\left(G''(\eta) - \frac{2\nu K_1 a_0^2}{D_A(n+1)U_0(x+b)^{n-1}} G(\eta) H^2(\eta))\right)$$
$$\Rightarrow -F(\eta)G'(\eta) = \frac{D_A}{\nu}$$
$$\left(G''(\eta) - \frac{2\nu K_1 a_0^2}{D_A(n+1)U_0(x+b)^{n-1}} G(\eta) H^2(\eta)\right)$$
$$\Rightarrow -\frac{\nu}{D_A}F(\eta)G'(\eta) = G''(\eta) - \left(\frac{2}{n+1}\right)\frac{\nu K_1 a_0^2}{D_A(n+1)U_0(x+b)^{n-1}} G(\eta)$$

$$\Rightarrow \quad G''(\eta) + \frac{\nu}{D_A}F(\eta)G'(\eta) - \left(\frac{2}{n+1}\right)\frac{\nu\ K_1\ a_0^2}{D_A(n+1)U_0(x+b)^{n-1}}G(\eta)H^2(\eta)$$

$$\left(\because\ U_w = U_0(x+b)^n\right)\left(\because Sc = \frac{\nu}{D_A}\right)\ \left(\because\ K = \frac{K_1 a_0^2}{U_w}(x+b)\right)$$

$$\Rightarrow \quad G''(\eta) + ScF(\eta)G'(\eta) - \left(\frac{2Sc\ K}{n+1}\right)G(\eta)H^2(\eta)) = 0.$$

Now we include below the procedure for the conversion of (4.5) into dimensionless form

$$
\begin{aligned}
\bullet\ \frac{\partial b^*}{\partial x} &= \frac{\partial}{\partial x}\Big(a_0 H(\eta)\Big) \\
&= a_0 H'(\eta)\frac{\partial \eta}{\partial x} \\
&= a_0 H'(\eta)\Big(\frac{n-1}{2}\Big)(x+b)^{-1}(\eta) \qquad (4.33) \\
\bullet\ u\frac{\partial b^*}{\partial x} &= \Big(U_0(x+b)^n F'(\eta)\Big)\Big(\frac{n-1}{2}\Big)\ a_0 H'(\eta)(x+b)^{-1}(\eta) \\
&= \Big(\frac{n-1}{2}\Big)\Big(U_0(x+b)^n F'(\eta)\Big)(\eta)\ a_0 H'(\eta)(x+b)^{-1} \\
&= \Big(\frac{n-1}{2}\Big)\Big(U_0(x+b)^{n-1}F'(\eta)\Big)(\eta)a_0 H'(\eta) \qquad (4.34) \\
\bullet\ \frac{\partial b^*}{\partial y} &= \frac{\partial}{\partial y}\Big(a_0 H(\eta)\Big) \\
&= a_0 H'(\eta)\frac{\partial \eta}{\partial y} \\
&= a_0 H'(\eta)\Big(\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{n-1}}\Big) \\
\bullet\ v\frac{\partial b^*}{\partial y} &= -\sqrt{\frac{n+1}{2}\ U_0\nu(x+b)^{n-1}}\Big(F(\eta) + \eta\frac{n-1}{n+1}F'\Big) \\
&\quad a_0 H'(\eta)\Big(\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(x+b)^{n-1}}\Big) \\
&= -\Big(\frac{n+1}{2}\Big)U_0(x+b)^{n-1}a_0 H'(\eta)\Big(F(\eta) + \eta\frac{n-1}{n+1}F'\Big). \qquad (4.35)
\end{aligned}
$$

Using(4.34) and (4.35), the left side of (4.5) becomes

$$
\begin{aligned}
& u\frac{\partial b^*}{\partial\ x} + \ v\frac{\partial\ b^*}{\partial y} \\
&= \Big(\frac{n-1}{2}\Big)\Big(\ U_0(\ x+b)\ ^n F'(\ \eta)\Big)(\eta)a_0 H'(\ \eta)(x+\ b)^{-1} \\
&\quad - \Big(\frac{n+1}{2}\Big)U_0(\ x+b)\ ^{n-1}a_0 H'(\ \eta)\Big(F(\eta)+\ \eta\frac{n-1}{n+1}F'\Big) \\
&= \Big(\frac{n-1}{2}\Big)\Big(\ U_0(\ x+b)\ ^{n-1}F'(\ \eta)\Big)(\ \eta)a_0 H'(\ \eta) \\
&\quad - \Big(\frac{n+1}{2}\Big)U_0(x+b)^{n-1}a_0 H'(\eta)(F(\eta) - \Big(\frac{n-1}{2}\Big)U_0(x+b)^{n-1}(\eta)a_0 H'(\eta)F'(\eta) \\
&= -\Big(\frac{n+1}{2}\Big)U_0(\ x+b)\ ^{n-1}a_0 H'(\ \eta)(F(\ \eta).
\end{aligned}
$$

To convert the right side of (4.5) into the dimensionless form, we proceed as follows

$$
\begin{aligned}
\bullet\ \frac{\partial^2 b^*}{\partial y^2} &= \frac{\partial}{\partial\ y}\Big(\frac{\partial b^*}{\partial\ y}\Big) \\
&= \frac{\partial}{\partial\ y}\Big(a_0 H'(\ \eta)\sqrt{\frac{n+1}{2}\frac{U_0}{v}(\ x+b)\ ^{n-1}}\Big) \\
&= \ a_0 H''(\ \eta)\frac{\partial\ \eta}{\partial y}\Big(\sqrt{\frac{n+1}{2}\frac{U_0}{v}(\ x+b)\ ^{n-1}}\Big) \\
&= \ a_0 H''(\ \eta)\Big(\sqrt{\frac{n+1}{2}\frac{U_0}{v}(\ x+b)\ ^{n-1}}\Big)\Big(\sqrt{\frac{n+1}{2}\frac{U_0}{\nu}(\ x+b)^{-\ 1}}\Big) \\
&= \Big(\frac{n+1}{2}\Big)\frac{U_0}{\nu}(\ x+b)^{-\ 1}a_0 H''(\ \eta).
\end{aligned}
$$

$$
\bullet\ D_{\ B}\frac{\partial^2\ b^*}{\partial y^2} = \ D_B\Big(\frac{n+1}{2}\Big)\frac{U_0}{\nu}(\ x+b)^{-\ 1}a_0 H''(\ \eta) \tag{4.36}
$$

$$
\begin{aligned}
\bullet\ \ K_1 a^* b^{*2} &= \ K_1\Big(a_0 G(\ \eta)\Big)\Big((\ a_0 H(\ \eta)\Big)^2 \\
&= \ K_1 a_0^3 G(\ \eta)H^2(\ \eta).
\end{aligned} \tag{4.37}
$$

Using (4.36) and (4.37) in the right side of (4.5), we get

$$
\begin{aligned}
D_B\frac{\partial^2\ b^*}{\partial y^2} + \ K_1 a^* b^{*2} = D_B\frac{n+1}{2}\frac{U_0}{v}(\ x+b)\ ^{n-1}a_0\ H'' \\
+\ K_1 a_0 G(\ \eta)(a_0)^2(H(\ \eta))^2.
\end{aligned}
$$

Hence the dimensionless form of (4.5) becomes

$$-\left(\frac{n+1}{2}\right) U_0(x+b)^{n-1}a_0F(\eta)H'(\eta) = D_B\left(\frac{n+1}{2}\right)\frac{U_0}{\nu}(x+b)^{-1}a_0$$
$$+\ K_1a_0^3G(\eta)H^2(\eta)$$
$$\Rightarrow\quad -\left(\frac{n+1}{2}\right) U_0(x+b)^{n-1}a_0F(\eta)H'(\eta) = \left(\frac{n+1}{2}\right) D_Ba_0\frac{U_0}{\nu}(x+b)^{-1}$$
$$\left(H''(\eta) + \frac{2\nu\ K_1a_0^2}{D_B(n+1)U_0(x+b)^{n-1}}G(\eta)H^2(\eta))\right)$$
$$\Rightarrow\quad -F(\eta)H'(\eta) = \frac{D_B}{\nu}$$
$$\left(H''(\eta) + \frac{2\nu\ K_1a_0^2}{D_B(n+1)U_0(x+b)^{n-1}}G(\eta)H^2(\eta)\right)$$
$$\Rightarrow\quad -\frac{\nu}{D_B}F(\eta)H'(\eta) = H''(\eta) + \left(\frac{2}{n+1}\right)\frac{\nu\ K_1a_0^2}{D_B(n+1)U_0(x+b)^{n-1}}G(\eta)H^2$$
$$\Rightarrow\quad H''(\eta) + \frac{\nu}{D_B}F(\eta)H'(\eta) + \left(\frac{2}{n+1}\right)\frac{\nu K_1a_0^2}{D_B(n+1)U_0(x+b)^{n-1}}G(\eta)H^2$$
$$\left(\because Sc = \frac{\nu}{D_A}\right)\quad \left(\because K = \frac{K_1a_0^2}{U_w}(x+b)\right)\left(\because \delta = \frac{D_B}{D_A}\right)$$
$$\Rightarrow\quad H''(\eta) + \frac{Sc}{\delta}F(\eta)H'(\eta) + \left(\frac{2Sc\ K}{(n+1)\delta}\right)G(\eta)H^2(\eta) = 0$$

The final dimensionless form of the proposed model, is:

$$F''' + FF'' - \frac{2n}{n+1}F'^2 + \frac{2n}{n+1}A^2 + 2\beta\frac{n(n-1)}{n+1}A^3 + \beta(3n-1)FF'F''$$
$$+\ \beta\left(-\frac{2n(n-1)}{n+1}F'^3 + \eta\frac{n-1}{2}F'^2F'' - \frac{n+1}{2}F^2F'''\right) - M^2\Big(F'(\eta) - A\Big), \tag{4.38}$$

$$\Theta'' + PrF\Theta' + Pr\gamma\left(\frac{n-3}{2}FF'\Theta' - \frac{n+1}{2}F^2\Theta''\right)$$
$$+\ Pr(S+\Theta)\left(\gamma FF'' - \frac{2n}{n+1}\gamma F'^2 - \frac{2}{n+1}F'\right) = 0, \tag{4.39}$$
$$G'' - \frac{2ScK}{n+1}GH^2 + ScFG' = 0, \tag{4.40}$$
$$H'' + \frac{2ScK}{\delta(n+1)}GH^2 + \frac{Sc}{\delta}H'F = 0. \tag{4.41}$$

Here prime represents the differentiation with respect to η.

Consider $F(\eta) = f(\eta-\alpha) = f(\xi)$, $\Theta(\eta) = \theta(\eta-\alpha) = \theta(\xi)$, $G(\eta) =$

$g(\ \eta -\ \alpha) = g(\xi)$. This change of notations converts the above equations (3.39)-(3.42) into the following form.

$$f''' + ff'' - \frac{2\ n}{n+1}f'^2 + \frac{2\ n}{n+1}A^2 + 2\beta\frac{n(\ n-1)}{n+1}A^3 +\ \beta(3\ n-1)ff'f''$$
$$+\ \beta\left(-\frac{2n(n-1)}{n+1}f'^3 + (\xi+\alpha)\frac{n-1}{2}f'^2f'' - \frac{n+1}{2}f^2f'''\right) - M^2\Big(f'(\eta) - A\Big) = 0, \tag{4.42}$$

$$\theta'' + Prf\theta' + Pr\gamma\left(\frac{n-3}{2}ff'\theta' - \frac{n+1}{2}f^2\theta''\right)$$
$$+\,Pr(S+\theta)\left(\gamma ff'' - \frac{2n}{n+1}\gamma f'^2 - \frac{2}{n+1}f'\right) = 0, \tag{4.43}$$

$$g'' - \frac{2ScK}{n+1}gh^2 + Scfg' = 0, \tag{4.44}$$
$$h'' + \frac{2ScK}{\delta(n+1)}hg^2 + \frac{Sc}{\delta}h'f = 0. \tag{4.45}$$

The new form of the associated boudary conditions, is:

$$\left.\begin{aligned} &f(0) = \alpha\frac{1-\ n}{1+n},\quad f'(\ 0) = 1,\quad \theta(0) = 1-S,\\ &g'(0) = \sqrt{\frac{2}{n+1}}Ks\ g(0),\quad h'(0) = -\frac{1}{\delta}\sqrt{\frac{2}{n+1}}Ksg(0),\\ &f'(\xi) = A,\quad \theta(\xi) = 1,\quad g(\xi)\longrightarrow 1,\quad h(\xi)\longrightarrow 0, \text{as}\quad \xi\longrightarrow\infty, \end{aligned}\right\} \tag{4.46}$$

where $Sc = \frac{\vartheta}{D_A}$ is the Schmidt number, $\delta = \frac{D_B}{D_A}$ the ratio of mass diffusion coefficient, $K = \frac{K_1 a_0^2}{U_w}(x+b)$ the strength of homogeneous parameter, $\beta = \lambda_1 U_0(x+\ b)^{n-1}$ the Deborah number, $S = \frac{d}{c}$ the thermal stratified parameter, $\gamma = \lambda U_0(x+\ b)^{n-1}$ the thermal relaxation parameter, α the wall thickness parameter, $Ks = \frac{k_s}{D_A}\sqrt{\frac{\vartheta(x+b)}{U_W}}$ the strength of heterogeneous reaction parameter, $A = \frac{U_\infty}{U_0}$ the velocity ratio parameter and $Pr = \frac{\mu_f C_p}{k}$ the Prandtl number. The diffusion coefficients of chemical species A and B are assumed to be of a comparable size. This argument leads to assume that the diffusion coefficient D_A=D_B, that is, δ=1. Thus

$$\left.\begin{aligned} &\delta = \frac{D_B}{D_A}, \quad A = \frac{U_\infty}{U_0}, \quad Pr = \frac{\mu_f c_p}{k}, \\ &\beta = \lambda_1 U_0(\ x+b)^{n-1}, \quad \gamma = \lambda U_0(\ x+b)^{n-1}, \quad S = \frac{d}{c}, \\ &S_c = \frac{\nu}{D_A}, \quad K_s = \frac{k_s}{D_A}\sqrt{\frac{\nu(\ x+b)}{U_w}}. \end{aligned}\right\}$$

so equation (4.44) gets the following form and (4.45) can be ignored.

$$g'' - \frac{2ScK}{n+1}g(1-g)^2 + Scfg' = 0. \tag{4.47}$$

The relevant boundary conditions are:

$$g'(0) = \sqrt{\frac{2}{vn+1}}Ks\ g(0), \quad g(\infty) \longrightarrow 1 \quad \text{when} \quad \xi \longrightarrow \infty,$$

4.4 Solution Methodology

In order to solve the system of ordinary differential equations (4.42)-(4.47), the shooting method has been used. Let us use the notations:

$$f = y_1, \theta = y_4, g = y_6.$$

Further denote

$$f' = y_1' \text{ by } y_2, \quad f'' = y_2{}' \text{ by } y_3, \quad \theta' = y_4' \text{ by } y_5 \text{ and } g' = y_6{}' \text{ by } y_7.$$

The system of equations (4.42)-(4.47), can now be written in the form of following first order ODEs:

$$
\begin{aligned}
y_1' &= y_2, \\
y_2' &= y_3, \\
y_3' &= \frac{1}{1-\beta\frac{n+1}{2}y_1^2}\Bigg(-y_1y_3 + \frac{2n}{n+1}y_2^2 - \beta(3n-1)y_1y_2y_3 - \frac{2n}{n+1}A^2 \\
&\quad + \beta\frac{2n(n-1)}{n+1}y_2^3 - (\xi+\alpha)\beta\frac{n-1}{2}y_2^2y_3 - 2\beta\frac{n(n-1)}{n+1}A^3 + M^2(y_2 - A)\Bigg), \\
y_4' &= y_5, \\
y_5' &= \frac{1}{1-Pr\gamma\frac{n+1}{2}y_1^2}\Bigg(-Pry_1y_5 - Pr\gamma\frac{n-3}{2}y_1y_2y_5 \\
&\quad - Pr(S+y_4)(\gamma y_1y_3 - \frac{2n}{n+1}\gamma y_2^2 - \frac{2}{n+1}y_2)\Bigg), \\
y_6' &= y_7, \\
y_7' &= -Sc\left(y_1y_7 - \frac{2K}{n+1}y_6(1-y_6)^2\right).
\end{aligned}
$$

The initial conditions for the above ODEs

$$
\begin{aligned}
&y_1(0) = \alpha\frac{1-n}{1+n}, && y_2(0) = 1, \\
&y_3(0) = s, && y_4(0) = 1 - S, \\
&y_5(0) = t, && y_6(0) = w \\
&y_7(0) = \sqrt{\frac{2}{n+1}}Ks\ w.
\end{aligned}
$$

The above initial value problem will be solved numerically by the RK-4 method. To get the approximate solution, the domain of the problem has been taken as [0, η_∞] instead of [0, ∞], where η_∞ is an appropriate finite positive real number. In the above system of equations, the missing conditions s, t and w are to be chosen such that

$$
y_2(\eta_\infty, s, t, w) = A, \;\; y_4(\eta_\infty, s, t, w) = 1, \;\; y_6(\eta_\infty, s, t, w) = 1.
$$

To solve the above system of algebraic equations, we use the Newton's method which has the following iterative scheme:

$$\begin{pmatrix} s^{(k+1)} \\ t^{(k+1)} \\ w^{(k+1)} \end{pmatrix} = \begin{pmatrix} s^{(k)} \\ t^{(k)} \\ w^{(k)} \end{pmatrix} - \begin{pmatrix} \frac{\partial y_2}{\partial s} & \frac{\partial y_2}{\partial t} & \frac{\partial y_2}{\partial w} \\ \frac{\partial y_4}{\partial s} & \frac{\partial y_4}{\partial t} & \frac{\partial y_4}{\partial w} \\ \frac{\partial y_6}{\partial s} & \frac{\partial y_6}{\partial t} & \frac{\partial y_6}{\partial w} \end{pmatrix}^{-1}_{(s^{(k)},t^{(k)},w^{(k)})} \begin{pmatrix} y_2^{(k)} \\ y_4^{(k)} \\ y_6^{(k)} \end{pmatrix}_{(s^{(k)},t^{(k)},w^{(k)})}.$$

For further procedure, the following notations have been introduced.

$$\frac{\partial y_1}{\partial s} = y_8, \frac{\partial y_2}{\partial s} = y_9, ..., \frac{\partial y_7}{\partial s} = y_{14},$$

$$\frac{\partial y_1}{\partial w} = y_{22}, \frac{\partial y_2}{\partial w} = y_{23}, ..., \frac{\partial y_7}{\partial w} = y_{28}.$$

As a result of these these new notations, the Newton's iterative scheme gets the form:

$$\begin{pmatrix} s^{(k+1)} \\ t^{(k+1)} \\ w^{(k+1)} \end{pmatrix} = \begin{pmatrix} s^{(k)} \\ t^{(k)} \\ w^{(k)} \end{pmatrix} - \begin{pmatrix} y_9 & y_{16} & y_{23} \\ y_{11} & y_{18} & y_{25} \\ y_{13} & y_{20} & y_{27} \end{pmatrix}^{-1}_{(s^{(k)},t^{(k)},w^{(k)})} \begin{pmatrix} y_2^{(k)} - A \\ y_4^{(k)} - 1 \\ y_6^{(k)} - 1 \end{pmatrix}_{(s^{(k)},t^{(k)},w^{(k)})}. \tag{4.48}$$

Now differentiate the above system of seven first order ODEs with respect to each of the variables s, t and w to have another system of twenty one ODEs.

Writing all these twenty eight ODEs together, we have the the following IVP:

$$
\begin{aligned}
y_1' &= y_2,\\
y_2' &= y_3,\\
y_3' &= \frac{1}{1-\beta(\frac{n+1}{2})y_1^2}\left(-y_1\ y_3 + (\frac{2n}{n+1})y_2^2 - \beta(3n-1)y_1y_2y_3 - \frac{2n}{n+1}A^2\right.\\
&\qquad \left. +\beta\frac{2n(n-1)}{n+1}y_2^3 - (\xi+\alpha)\beta\frac{n-1}{2}y_2^2y_3 - 2\beta\frac{n(n-1)}{n+1}A^3 + M^2(y_2 - A)\right),\\
y_4' &= y_5,\\
y_5' &= \frac{1}{1-Pr\gamma(\frac{n+1}{2})\ y_1^2}\left(-Pry_1y_5 - Pr\gamma\frac{n-3}{2}y_1\ y_2y_5\right.\\
&\qquad \left. -Pr(S+y_4)(\gamma y_1\ y_3 - \frac{2n}{n+1}\gamma y_2^2 - \frac{2}{n+1}y_2)\right),\\
y_6' &= y_7,\\
y_7' &= -Sc\left(y_1y_7 - \frac{2K}{n+1}y_6(1-y_6)^2\right),\\
y_8' &= y_9,\\
y_9' &= y_{10},\\
y_{10}' &= \frac{1}{1-\beta(\frac{n+1}{2})\ 2y_1\ y_8}\left[-y_1y_{10} - y_8y_3 + \frac{2n}{n+1}2y_2y_9 - \beta(3n-1)\right.\\
&\qquad (\ y_8\ y_2\ y_3 + y_1\ y_9\ y_3 + y_1\ y_2\ y_{10})\\
&\qquad -(\frac{2n}{n+1})A^2 + \beta\frac{2n(n-1)}{n+1}3y_2^2y_9 - (\xi+\alpha)\beta\frac{n-1}{2}\left(2y_2y_9y_3 + y_2^2y_{10}\right)\\
&\qquad \left. -2\beta\frac{n(n-1)}{n+1}A^3 + M^2y_9\right],\\
y_{11}' &= y_{12},\\
y_{12}' &= \frac{1}{1-Pr\ \gamma(\frac{n+1}{2})2y_1y_8}\left[-Pr\left(y_1y_{12} + y_8y_5\right) - Pr\gamma\frac{n-3}{2}\big(y_8y_2y_5 + y_1y_9y_5\right.\\
&\qquad \left. + y_1y_2y_{12}\big) - Pr(S+y_{11})(\gamma\left(y_8y_3 + y_1y_{10}\right) - \frac{2n}{n+1}\gamma 2y_2y_9 - \frac{2}{n+1}y_9)\right],\\
y_{13}' &= y_{14},\\
y_{14}' &= -Sc\left[(y_1y_{14} + y_8y_7) - \frac{2K}{n+1}\left(y_{13}(1-y_6)^2 + y_6 2(1-y_6)y_{13}\right)\right],\\
y_{15}' &= y_{16},\\
y_{16}' &= y_{17},
\end{aligned}
$$

$$y'_{17} = \frac{1}{1-\beta(\frac{n+1}{2})\, 2y_1\, y_{15}} \Big[-y_1\, y_{17} - y_{15}\, y_3 + \frac{2n}{n+1} 2y_2\, y_{16} - \beta(3n-1)(y_{15}\, y_2\, y_3$$
$$+ y_1\, y_{16}\, y_3 + y_1\, y_2\, y_{17}) - (\frac{2n}{n+1})A^2 + \beta\, \frac{2n(n-1)}{n+1}\, 3\, y_2^2\, y_{16}$$
$$-(\xi+\alpha)\beta\frac{n-1}{2}\left(2y_2\, y_{16}\, y_3 + y_2^2\, y_{17}\right) - 2\beta\, \frac{n(n-1)}{n+1}\, A^3 + M^2 y_{16}\Big],$$

$$y'_{18} = y_{19},$$

$$y'_{19} = \frac{1}{1-Pr\,\gamma(\frac{n+1}{2})2\, y_1 y_{15}} \Big[-Pr\,(y_1\, y_{19} + y_{15}\, y_5) - Pr\,\gamma\,(\frac{n-3}{2})\,(y_{15}\, y_2\, y_5$$
$$+ y_1\, y_{16}\, y_5 + y_1\, y_2\, y_{19}) - Pr(S + y_{18})(\gamma\,(y_{15} y_3 + y_1 y_{17}) - \frac{2n}{n+1}\gamma\, 2y_2\, y_{16}$$
$$-\frac{2}{n+1} y_{16})\Big],$$

$$y'_{20} = y_{21},$$

$$y'_{21} = -Sc\left[(y_1\, y_{21} + y_{15}\, y_7) - \frac{2K}{n+1}\,\left(y_{20}(1-y_6)^2 + y_6 2(1-y_6)\, y_{20}\right)\right],$$

$$y'_{22} = y_{23},$$

$$y'_{23} = y_{24},$$

$$y'_{24} = \frac{1}{1-\beta\,(\frac{n+1}{2})2y_1\, y_{22}} \Big[-y_1\, y_{24} - y_{22}\, y_3 + \frac{2n}{n+1} 2\, y_2\, y_{23} - \beta(3n-1)$$
$$(y_{22}\, y_2\, y_3 + y_1\, y_{23} y_3 + y_1\, y_2\, y_{24}) - (\frac{2n}{n+1})A^2 + \beta\, \frac{2n(n-1)}{n+1} 3\, y_2^2\, y_{23}$$
$$-(\xi+\alpha)\beta\frac{n-1}{2}(2y_2\, y_{23} y_3 + y_2^2\, y_{24}) - 2\beta\frac{n(n-1)}{n+1} A^3 + M^2 y_{23}\Big],$$

$$y'_{25} = y_{26},$$

$$y'_{26} = \frac{1}{1-\beta\,(\frac{n+1}{2})\, 2y_1\, y_{22}} \Big[-y_1\, y_{24} - y_{22}\, y_3 + \frac{2n}{n+1} 2y_2\, y_{23} - \beta(3n-1)$$
$$(y_{22}\, y_2\, y_3 + y_1\, y_{23}\, y_3 + y_1\, y_2\, y_{24}) - (\frac{2n}{n+1})A^2 + \beta\frac{2n(n-1)}{n+1} 3\, y_2^2\, y_{23}$$
$$-(\xi+\alpha)\beta\frac{n-1}{2}(2y_2\, y_{23} y_3 + y_2^2\, y_{24}) - 2\,\beta\frac{n(n-1)}{n+1} A^3\Big],$$

$$y'_{27} = y_{28},$$

$$y'_{28} = -Sc\left[(y_1\, y_{28} + y_{22}\, y_7) - \frac{2K}{n+1}\left(y_{27}\,(1-y_6)^2 + y_6 2(1-y_6)\, y_{27}\right)\right].$$

The coressponding initial conditions are

$$
\begin{aligned}
&y_1(0) = \alpha\frac{1-n}{1+n}, && y_2(0) = 1,\\
&y_3(0) = s, && y_4(0) = 1-S,\\
&y_5(0) = t, && y_6(0) = w,\\
&y_7(0) = \sqrt{\frac{2}{n+1}}Ksw, && y_8(0) = 0,\\
&y_9(0) = 0, && y_{10}(0) = 1,\\
&y_{11}(0) = 0, && y_{12}(0) = 0,\\
&y_{13}(0) = 0, && y_{14}(0) = 0,\\
&y_{15}(0) = 0, && y_{16}(0) = 0,\\
&y_{17}(0) = 0, && y_{18}(0) = 0,\\
&y_{19}(0) = 1, && y_{20}(0) = 0,\\
&y_{21}(0) = 0, && y_{22}(0) = 0,\\
&y_{23}(0) = 0, && y_{24}(0) = 0,\\
&y_{25}(0) = 0,, && y_{26}(0) = 0,\\
&y_{27}(0) = 1,, && y_{28}(0) = \sqrt{\frac{2}{n+1}}Ks.
\end{aligned}
$$

The fourth order Runge-Kutta method is used to solve the above system of twenty eight equations with initial guesses s, t, w. These guesses are updated by the Newton's scheme (3.51). The iterative process is repeated until the following criteria is met:

$$max\{|y_2(\eta_\infty) - A|, |y_4(\eta_\infty) - 1|, |y_6(\eta_\infty - 1)|\} < \epsilon,$$

where $\epsilon > 0$ is the tolerance. For all the calculations in this chapter, we have set $\epsilon = 10^{-6}$.

4.5 Results and Discussion

In order to evaluate the solution of the given system the dimensionless velocity profile, temperature and concentration profile for different parameters are sketched. Figure 4.2 is sketched to study the behavior of velocities ratio parameter A on velocity distribution. Velocity profile is increased by increasing velocity parameter A. Behaviour of wall thickness parameter on velocity profile is shown in Figure 4.3. The dimensionless velocity profile decreases for the increment in the value of wall thickness parameter. It is due to fact that on increasing the wall thickness, stretching velocity profile is decreased which results a reduction in the velocity profile and its boundary layer thickness. Figure 4.4 reflect the effect of the Deborah number β on velocity profile. As Deborah number is the ratio of the fluid relaxation time to its characteristic time scale. When the shear stress is applied to a fluid, the time in which it gains its equilibrium position is called the relaxation time. This time is higher for the fluids having high velocity. So, an increase in the Deborah number cause an increase in the velocity of fluid due to which profile decreases. Figure 4.5 represents the effect of power index n on the velocity profile curves of this graph indicates that velocity profile is decreasing near the surface and increases away from the surface. Figure 4.6 shows the behaviour of M on velocity profile it is noticed The effects of magnetic field are to reduce the velocity profile. Because of the application of transverse magnetic field in an electrically conducting fluid, a resistive force similar to a drag force is produced, which is Lorentz force. The presence of Lorentz force retards the force on the velocity field. To view the effect of velocity ratio parameter on the temperature profile Figure 4.7 is presented. It is noticed that by increasing the value of the velocity ratio parameter A temperature profile decreases. Behaviour of the wall thickness parameter on the temperature profile is shown in Figure 4.8. It is noticed that by increasing the wall thickness. The temperature distribution and thermal boundary layer thickness is decreased. It is due to fact by increasing the wall thickness parameter less amount of heat i transferred. The temperature distribution increases for different values of the Deborah number and illustrated in Figure 4.9 It is studied that the elastic force enhance the heat transfer in upper convected Maxwell fluid which result an increase of mass transfer. Thermal relaxation parameter on temperature profile is shown in Figure 4.10 from the figure, it is clearly observed that the distribution of temperature is a decreasing function of thermal relaxation parameter. By increasing the thermal relaxation parameter γ, particles within the material requires more time to transfer heat to its nearby particle, which causes reduction in temperature distribution and boundary layer thickness. Figure 4.11 shows the behaviour of power law index n on temperature profile, by increasing the power law index n the temperature fluid flow increases in the stretching sheet. Figure 4.12 is prepared to analyze the effect of Prandtl number on temperature profile, by increasing the Prandtl number reduction of temperature profile is observed. Figure 4.13 is plotted to visualize the effect of Magnetohydrodynamics on temperature profile. An increase on temperature profiles is shown. Because of the presence of Lorentz force retards the force on the velocity field and This force has the tendency to slow down the fluid motion and the resistance offered to the flow. Therefore, it is possible for the increase in the temperature. It is manifest from the Figure 4.14 that the temperature profile and related boundary layer thickness is increased with the increase in the small parameter ϵ associated with temperature. An increase in thermal conductivity means increase in kinetic energy of the fluid which cause an increase in temperature.

Figure 4.15 is prepared to observe the effect of the velocity ratio parameter A on concentration profile θ. It is observed that the concentration profile decreases with the increase in the velocity ratio parameter. Effect of the wall thickness parameter on the concentration profile is shown in Figure 4.16, when the wall thickness parameter α increased, concentration profile increased. The concentration boundary layer thickness reduces due to the conversion of species that occurs as a result of chemical reaction and hence the concentration boundary layer thickness decreased. Figure 4.17, is prepared to represent the effect of homogeneous reaction parameter on concentration profile that the concentration boundary layer of the reactants is increased near the surface and away from the surface, the homogeneous reaction has no effect on the concentration of the reactants. Figure 4.18 is sketched to analyze the behaviour of heterogeneous reaction parameter Ks.

From the graph of this figure it is clear that gradually increasing value of Ks decline the concentration profile. The effect of power law index on concentration profile is displayed in Figure 4.19. An increment in power law index n causes a decrement in concentration profile. Figure 4.20 is drawn to illustrates the behavior of Schmidt number on concentration distribution, greater values of Schmidt number Sc represents the lower mass diffusivity.

Due to this effect a decline in concentration profile is noticed. Figure 4.21 depicts the effect of MHD on concentration profile which shows that by enhancing M, concentration profile reduces.

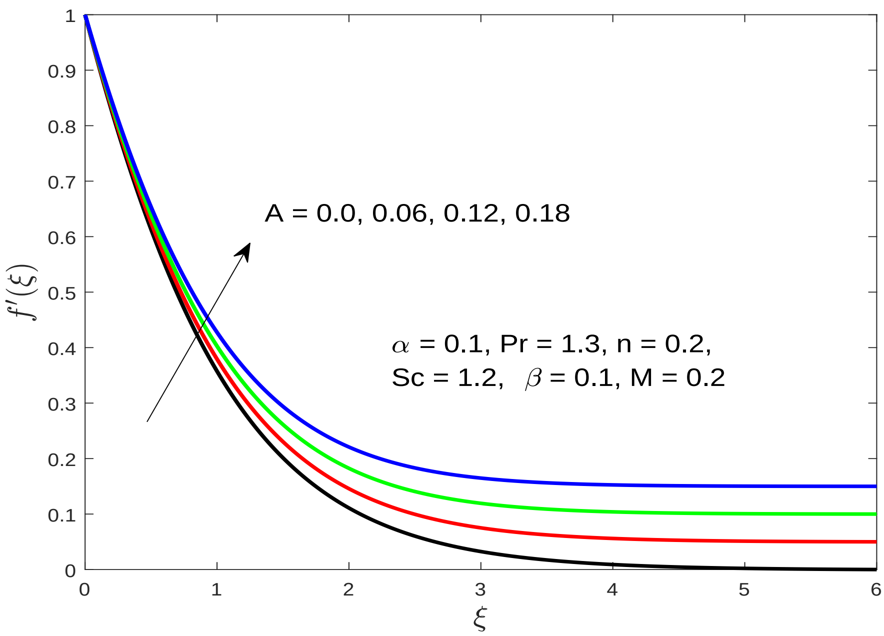

FIGURE 4.2: Effect of A on $f'(\xi)$.

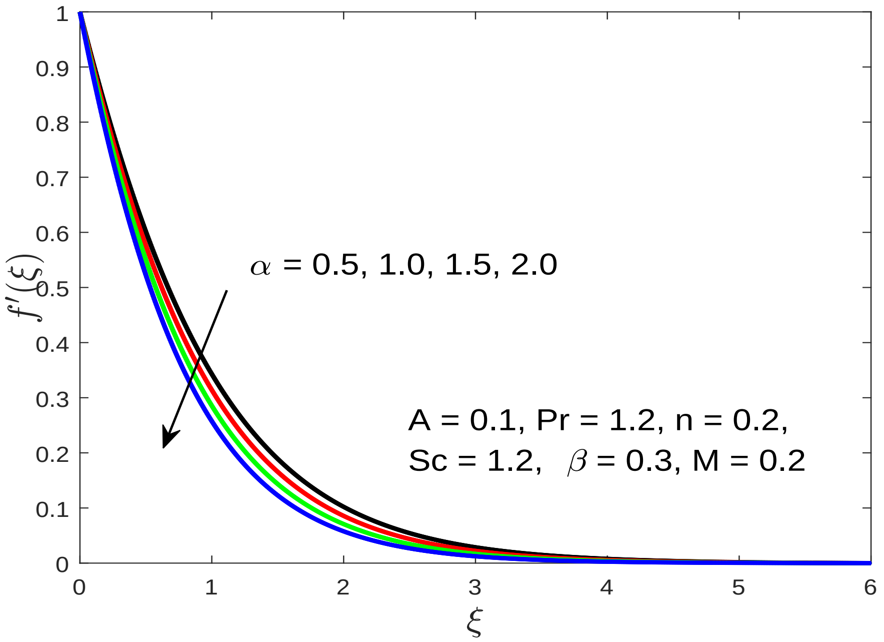

FIGURE 4.3: Effect of α on $f'(\xi)$.

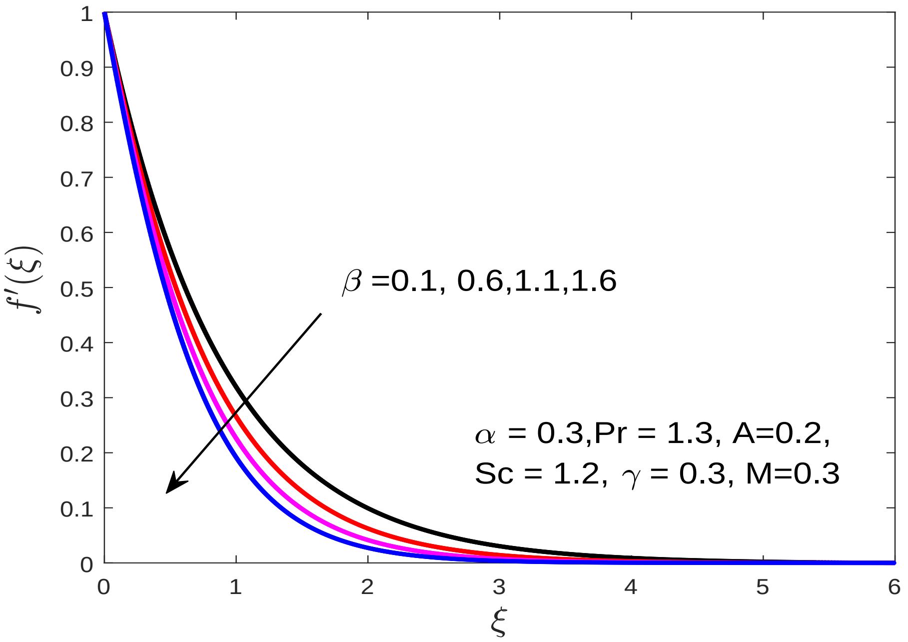

FIGURE 4.4: Effect of β on $f'(\xi)$.

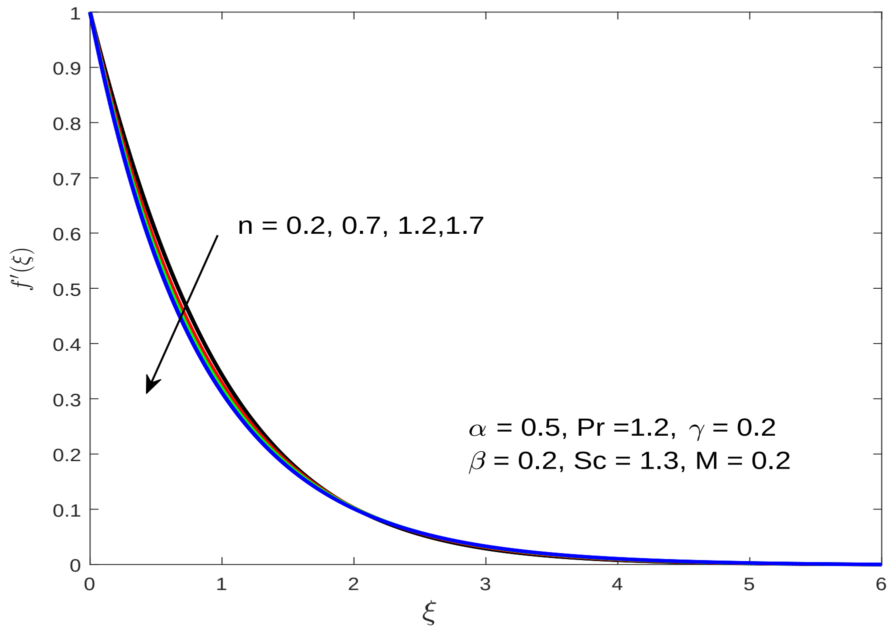

FIGURE 4.5: Effect of n on $f'(\xi)$.

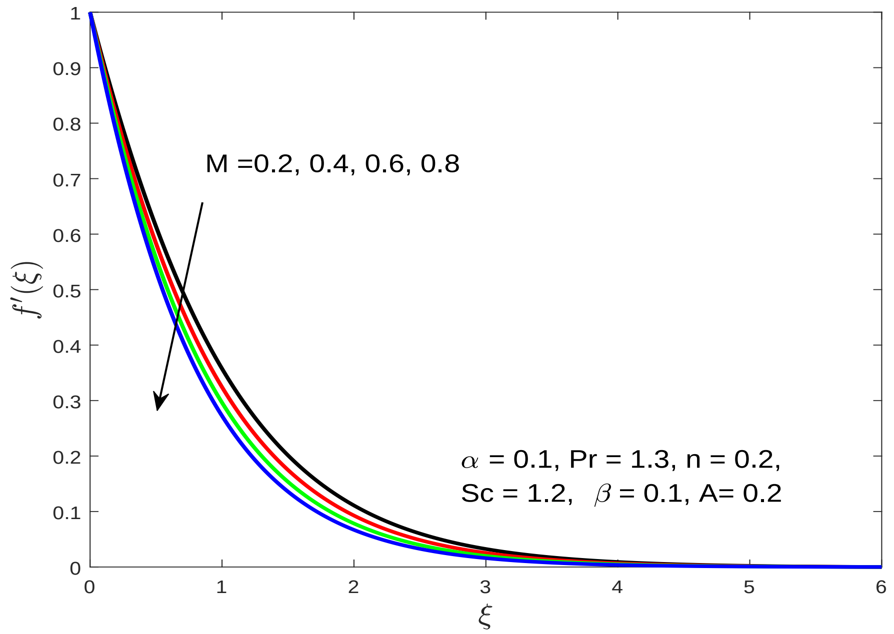

FIGURE 4.6: Effect of M on $f'(\xi)$.

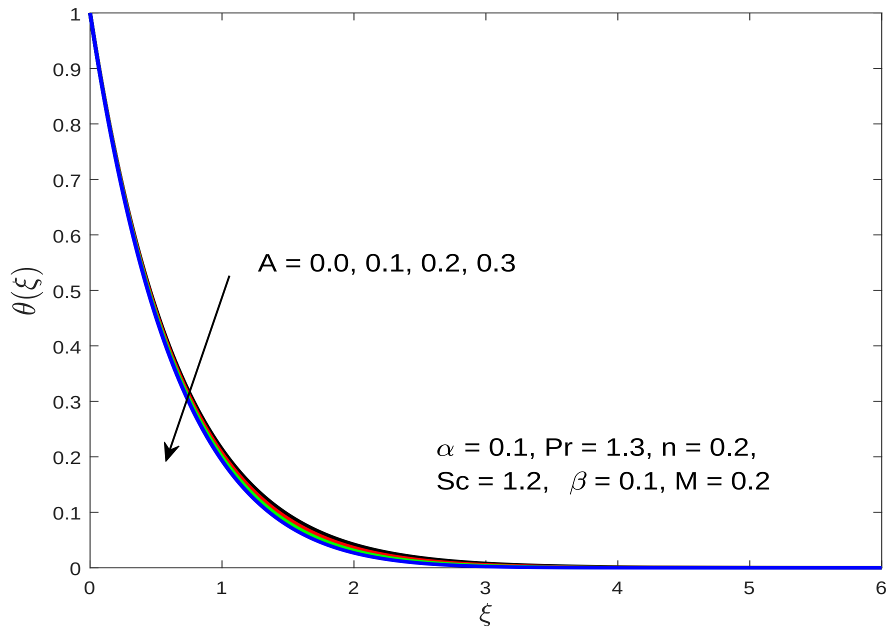

FIGURE 4.7: Effect of A on $\theta(\xi)$.

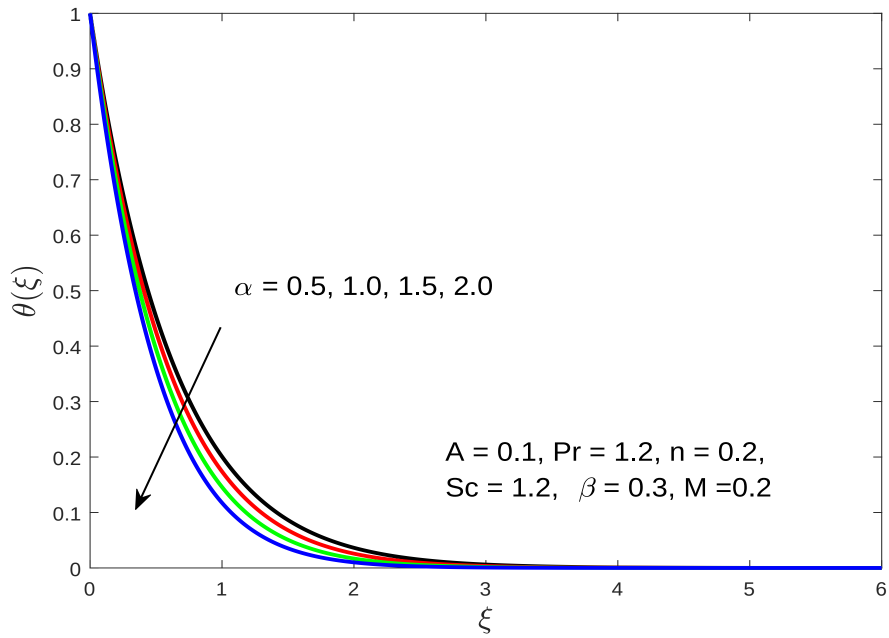

FIGURE 4.8: Effect of α on $\theta(\xi)$.

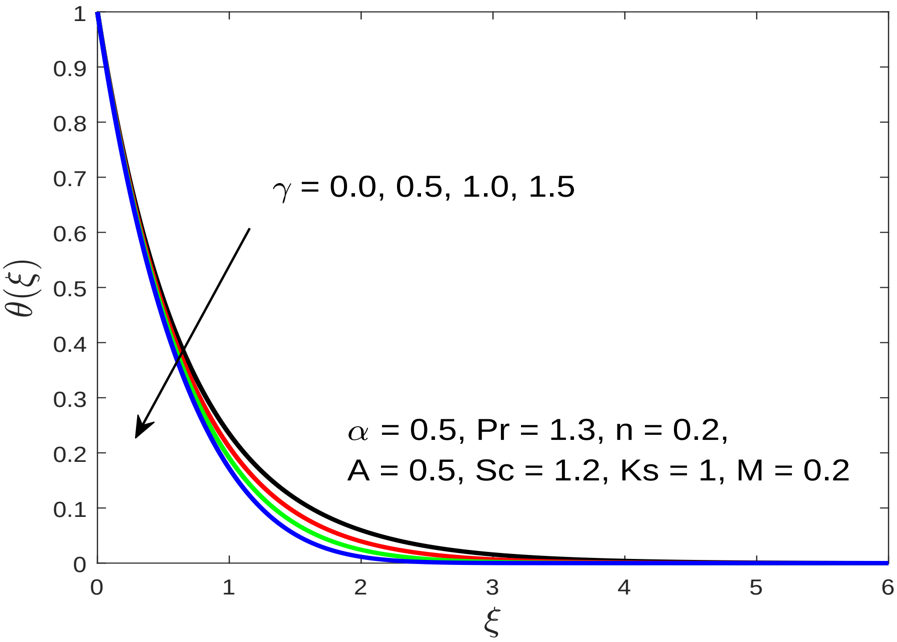

FIGURE 4.9: Effect of γ on $\theta(\xi)$.

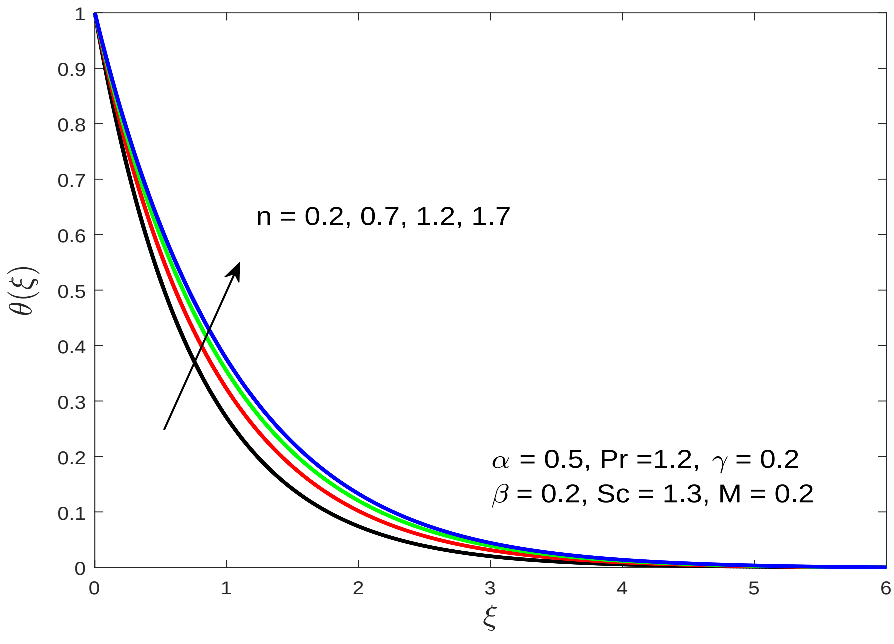

FIGURE 4.10: Effect of n on $\theta(\xi)$.

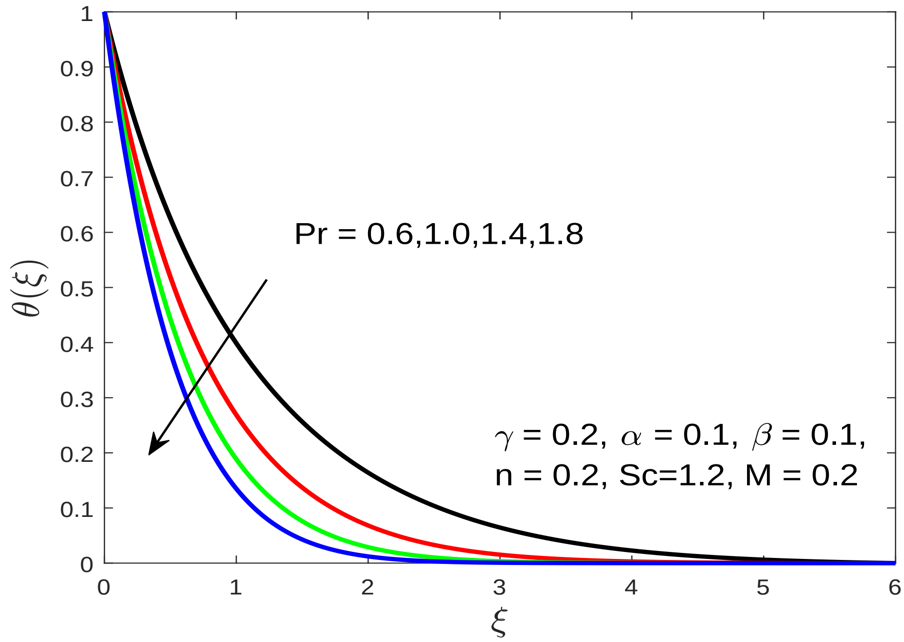

FIGURE 4.11: Effect of Pr on $\theta(\xi)$.

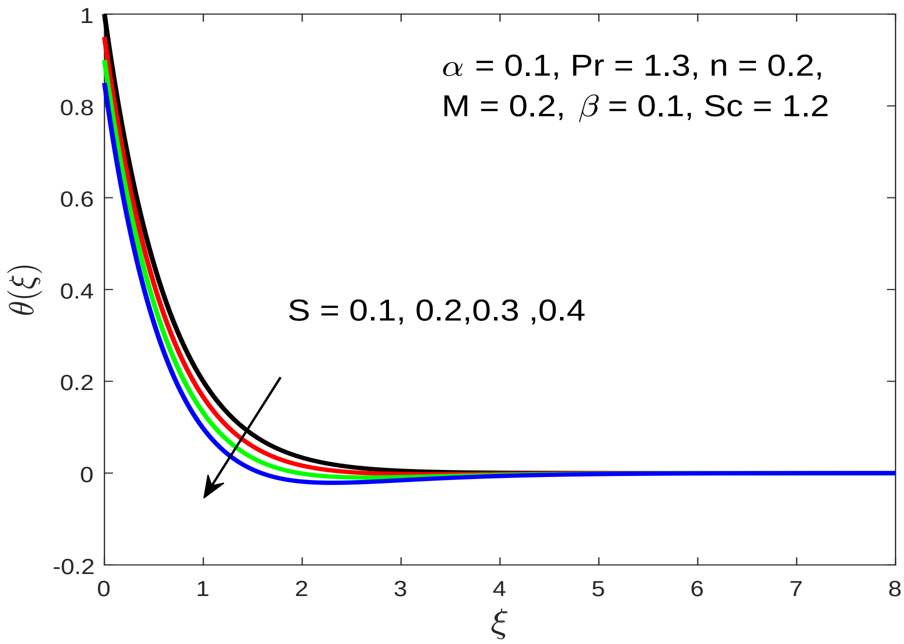

FIGURE 4.12: Effect of S on $\theta(\xi)$.

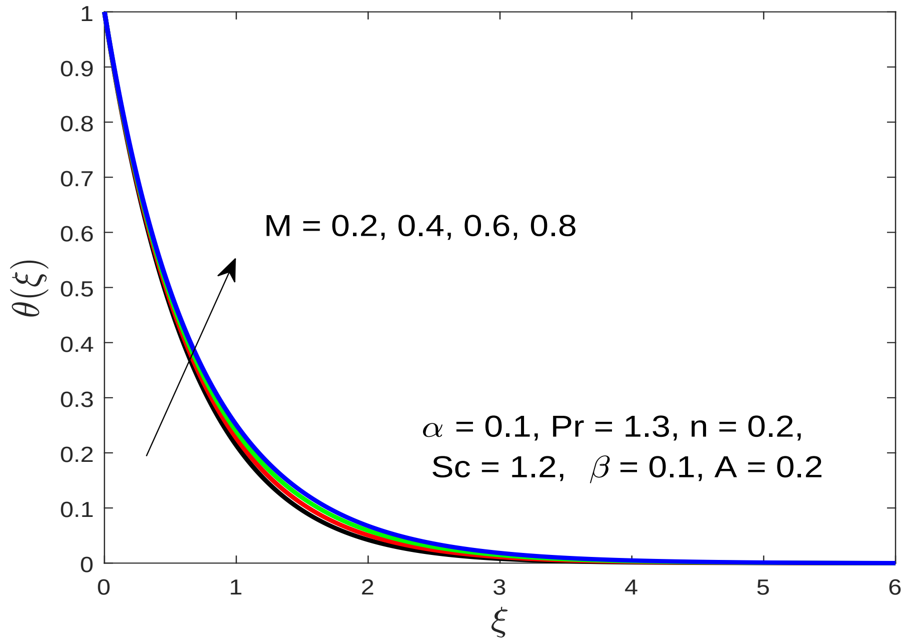

FIGURE 4.13: Effect of M on $\theta(\xi)$.

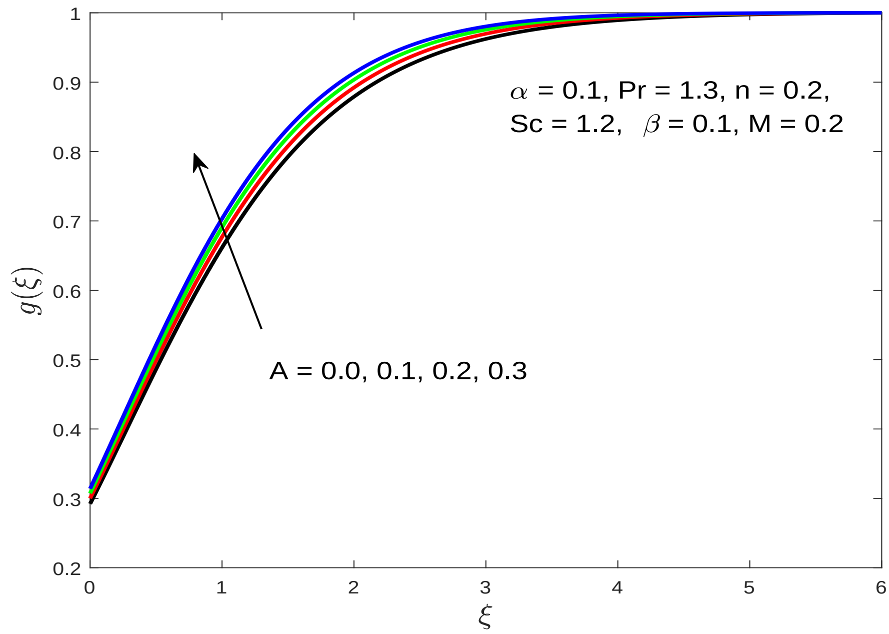

FIGURE 4.14: Effect of A on $g(\xi)$.

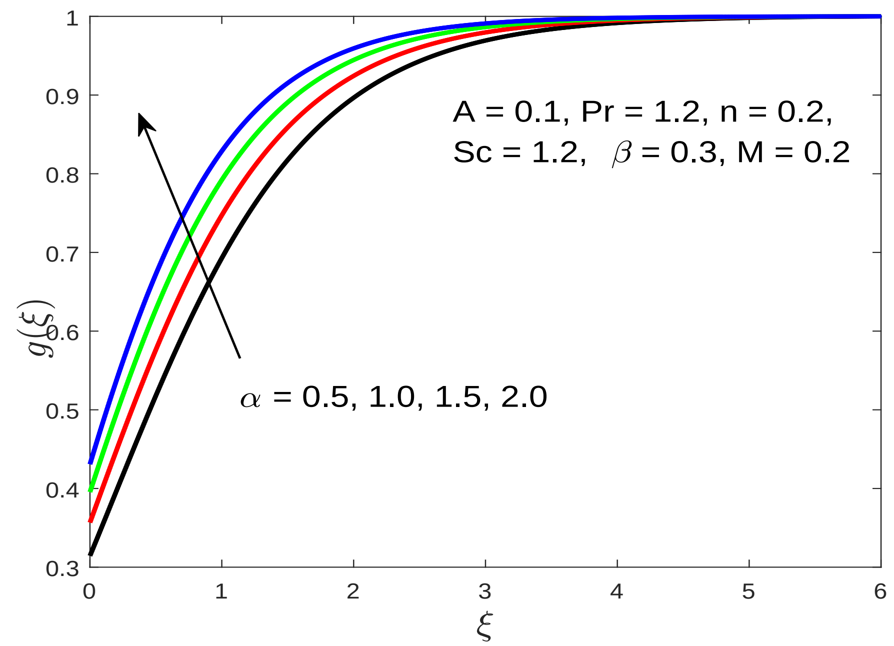

FIGURE 4.15: Effect of α on $g(\xi)$.

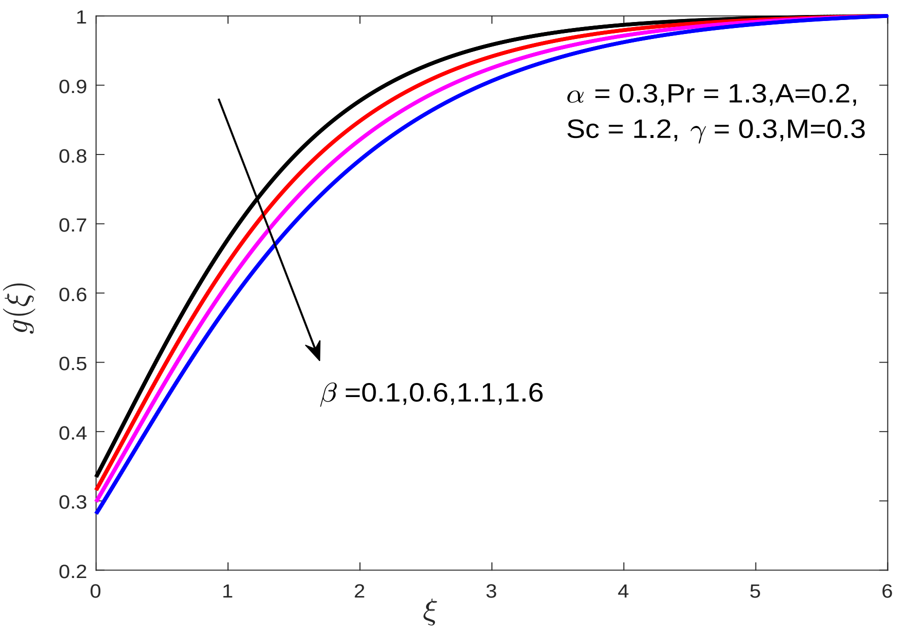

FIGURE 4.16: Effect of β on $g(\xi)$.

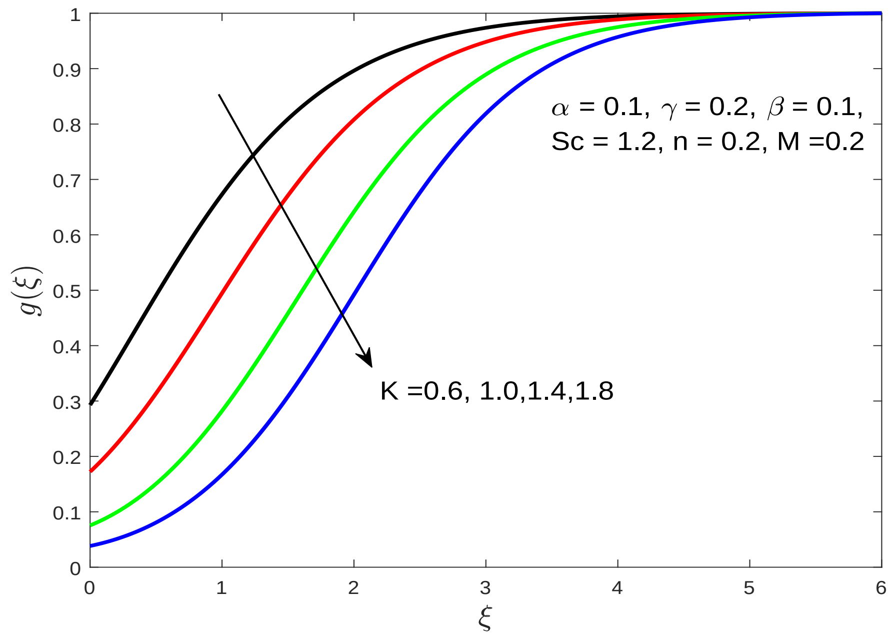

FIGURE 4.17: Effect of K on $g(\xi)$.

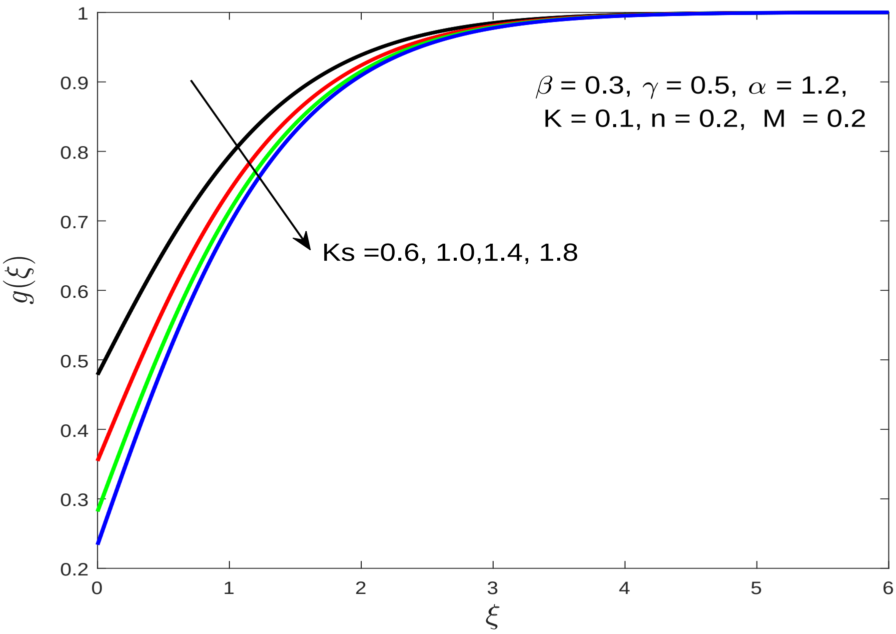

FIGURE 4.18: Effect of Ks on $g(\xi)$.

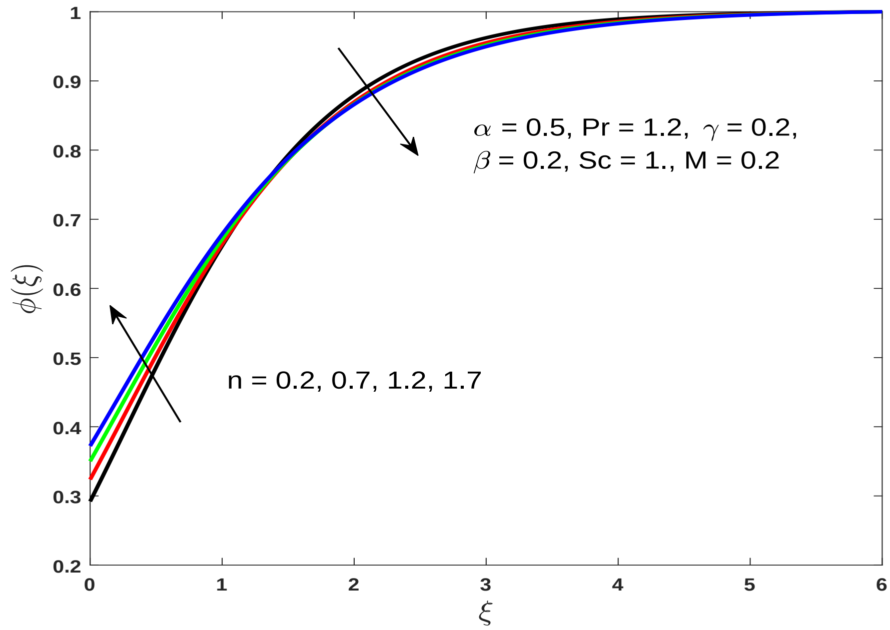

FIGURE 4.19: Effect of n on $g(\xi)$.

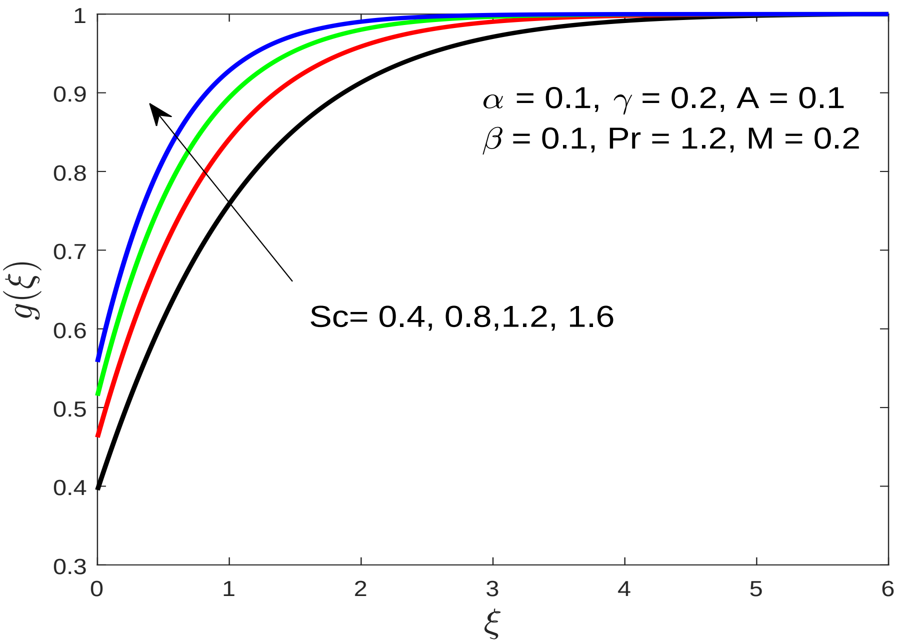

FIGURE 4.20: Effect of Sc on $g(\xi)$.

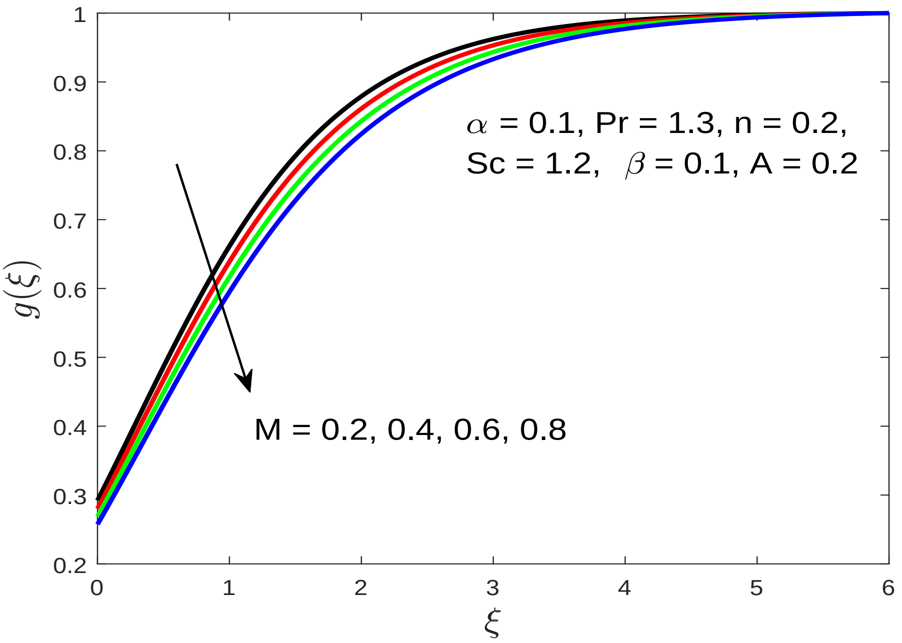

FIGURE 4.21: Effect of M on $g(\xi)$.

TABLE 4.1: Numerical results of $f''(0)$ for $A = 0.1$, $\beta = 0.1$, $K = 0.5$, $Ks = 1.0$, $\alpha = 0.5$, $M = 0.2$ and $Sc = 1.2$..

				$-f''(0)$	
A	α	β	n	shooting	bvp4c
0.3	0.1	0.2	0.2	0.99546	0.99545
0.5				1.09714	1.09712
0.7				1.19627	1.19627
0.9				1.29293	1.29296
	0.2			1.09257	1.09255
	0.3			0.99546	0.99544
	0.4			0.89939	0.89940
		0.4		0.89446	0.89442
		0.6		0.84410	0.84411
		0.8		0.69607	0.69609
			0.4	1.01709	1.01708
			0.6	1.04002	1.04001
			0.8	1.06179	1.06180

TABLE 4.2: Numerical results of $-\theta''(0)$ for $A = 0.1$, $\beta = 0.1$, $K = 0.5$, $Ks = 1.0$, $\alpha = 0.5$, $M = 0.2$ and $Sc = 1.2$..

							$-\theta''(0)$	
A	α	β	γ	M	n	Pr	shooting	bvp4c
0.3	0.1	0.1	0.1	0.5	1.0	2.0	1.135504	1.135503
0.5							1.182832	1.182833
0.7							1.229482	1.229482
	0.2						1.077498	1.077499
	0.5						0.798767	0.798768
	1.0						0.044135	0.044139
		0.2					1.077498	1.077499
		0.3					1.000720	1.000719
		0.4					0.906847	0.906848
			0.3				1.221272	1.221271
			0.5				1.299390	1.299391
			1.2				1.543654	1.543652
				0.2			1.135504	1.135503
				0.4			1.135404	1.135403
				0.6			1.133991	1.133992
					0.9		1.135504	1.135503
					0.7		1.134404	1.134403
					0.5		1.133204	1.133203
						1.5	1.135504	1.135503
						1.0	1.133991	1.133992
						0.5	1.133901	1.133902

Table 4.3: Numerical results of $g'(0)$ for $\gamma = 0.3$, $Pr = 1.2$, $\alpha = 0.5$, $A = 0.5$, $M = 0.2$ and $\beta = 0.4$.

							$-g'(0)$	
A	α	K	Ks	M	n	Sc	shooting	bvp4c
0.3	0.1	0.1	0.1	0.5	1.0	2.0	0.6050838	0.6050837
0.5							0.5912941	0.5912943
0.7							0.5712940	0.5712942
	0.2						0.636992	0.636993
	0.5						0.745345	0.745348
	1.0						1.100055	1.100060
		0.2					0.636992	0.636993
		0.3					0.670242	0.670241
		0.4					0.729453	0.729455
			0.3				0.590773	0.590773
			0.5				0.600664	0.600664
			1.2				0.675825	0.675823
				0.2			0.580542	0.580542
				0.4			0.596509	0.596509
				0.6			0.634370	0.634373
					0.9		0.562295	0.562295
					0.7		0.481124	0.481124
					0.5		0.282923	0.282923
						1.5	0.605083	0.605083
						1.0	0.608772	0.608772
						0.5	0.609172	0.609172

Chapter 5

Conclusion

In the present research work, the MHD Stagnation point flow with Cattaneo-Christov heat flux and homogeneous-heterogeneous reactions is studied. The governing nonlinear partial differential equations (PDEs) are converted into ordinary differential equations (ODEs) by means of the similarity transformation. The numerical solution of these ordinary differential equations (ODEs) is obtained by using the shooting technique. A numerical correlation has shown for different physical parameters influencing flow and heat transfer and found to be in excellent agreement with MATLAB built-in function bvp4c. The impact of different physical parameters such as velocity ratio parameter A, wall thickness parameter α, the Deborah number β, the Prandtl number Pr, Schmidt number Sc, the power law index n, thermal stratified parameter S and thermal relaxation parameter γ on velocity, temperature and concentration profiles are presented graphically and discussed. Some of the main conclusions of this investigation are:

- The velocity and concentration profile is enhanced while the temperature profile is diminished as the velocity ratio parameter A is increased.
- The velocity and temperature profile are found to reduce while the concentration profile is enhanced for the gradually mounting values of the wall thickness parameter α.

- The temperature profile is increased, while the axial velocity is decreased with an increase in Deborah number β.

- On temperature profile, the Prandtl number Pr, the thermal stratified parameter S and the thermal relaxation parameter γ have a decreasing effect whereas a rise in the power-law index n, and the MHD parameter M causes an increase in the temperature profile.

- The concentration profile decreases as each of the Schmidt number Sc, the MHD M, the Deborah number β , the power-law index n is increased whereas an increment in the heterogeneous reaction parameter ks causes a decrement in the concentration profile.

  Future Recommendations:

  There is a possibility of extention by considering the stagnation point flow towards a nonlinear vertical stretching sheet in the presence of Cattaneo-Christov heat flux model, and homogeneous - heterogeneous reactions and second order velocity slip .

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