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# Fixed Point Theorems in Fuzzy $b$-metric Spaces

by

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# Fixed Point Theorems in Fuzzy $b$-metric Spaces

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*Dedicated*

*To*

*my loving and caring Parents,*

*my wife*

*and*

*my beloved daughters*

*Ghulam Ayesha, Noor Fatima*

*&*

*Sammarah Zainab*

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# CERTIFICATE OF APPROVAL

This is to certify that the research work presented in the thesis, entitled "**Fixed Point Theorems in Fuzzy b-metric Spaces**" was conducted under the supervision of **Dr. Rashid Ali**. No part of this thesis has been submitted anywhere else for any other degree. This thesis is submitted to the **Department of Mathematics, Capital University of Science and Technology** in partial fulfillment of the requirements for the degree of Doctor in Philosophy in the field of **Mathematics**. The open defence of the thesis was conducted on **May 13, 2022**.

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It is certified that following publications have been made out of the research work that has been carried out in this thesis:-

1. **M. S. Ashraf**, R. Ali and N. Hussain, "Geraghty type contractions in fuzzy $b$-metric spaces with application to integral equations," *Filomat*, vol. 34, no. 9 , pp.3083-3098, 2020.
2. **M. S. Ashraf**, R. Ali and N. Hussain, "New Fuzzy Fixed Point Results in Generalized Fuzzy Metric Spaces with Application to Integral Equations," *IEEE Access*, vol. 8, pp. 91653-91660, 2020.

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# *Abstract*

In this dissertation, we study the concept of fuzzy $b$-metric space which is the generalization of fuzzy metric spaces and $b$-metric spaces. The Banach contraction principle is extended in the setting of fuzzy $b$-metric spaces and this result has been illustrated by an example. The notion of $g$-orbitally upper semi continuous function is also introduced in fuzzy metric space and the fixed point result of Hicks and Rhoads is generalized in the setting of fuzzy $b$-metric space. Some fixed point results are also proved by introducing a novel and rational contraction and using a control function in fuzzy $b$-metric spaces. Some applications are also highlighted as consequences of our results. This idea is further used to prove some new fixed point results and some common fixed point results for Geraghty-type contraction in $G$-complete fuzzy $b$-metric spaces. Further, the notion of generalized fuzzy metric space is introduced. Many topological spaces like fuzzy metric spaces, fuzzy $b$-metric spaces and dislocated fuzzy metric spaces have been generalized by this new generalized fuzzy metric space. It is also proved that the class of generalized fuzzy metric spaces contains the classes of fuzzy metric spaces, fuzzy $b$-metric spaces and dislocated fuzzy metric spaces as proper sub-classes. The Banach contraction principle and Ćirić's quasi-contraction theorem are demonstrated in the setting of generalized fuzzy metric space. As consequences of our results, we obtain Jleli and Samet's and many other author's recent results as corollaries. We also present an application related to our main result for nonlinear integral equation.

# Contents

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# Abbreviations

| **Acronym** | **W**hat (it) **S**tands **F**or |
|---|---|
| BCP | Banach Contraction Principle |
| $b$MS | $b$-Metric Space |
| DMS | Dislocated Metric Space |
| F$b$MS | Fuzzy $b$-Metric Space |
| FDMS | Fuzzy Dislocated Metric Space |
| FMS | Fuzzy Metric Space |
| GFMS | Generalized Fuzzy Metric Space |
| GMS | Generalized Metric Space |
| MS | Metric Space |

# Chapter 1

# Introduction

Mathematics is an important branch of science and is further divided into many branches, each of which has its own significance according to its applications. One of the most significant field of mathematics is referred as functional analysis. It can be used to solve a variety of problems including both linear and non-linear differential equations. It has numerous applications in the field of numerical analysis, error estimation of polynomial interpolation and finite difference method, see for instance [1–5].

Fixed point theory is an important and valuable concept in functional analysis. This certainly enhances the importance and significance of functional analysis due to its wide use in solving the different types of linear and non-linear problems. The concept of fixed point has a wide variety of applications in different scientific fields such as mathematical economics, game theory, optimization theory, approximation theory etc, for instance [6–9].

Poincare [10] was the first to work on fixed point theory in 1886. After that, the equation $f(\varrho) = \varrho$ was taken into consideration by Brouwer [11] and he found the solution of this equation by proving a fixed point theorem in 1912. He also contributed to prove fixed point results for the shapes like a square and a sphere etc. This work was further extended and generalized by Kuktani [12] for $n$-dimensional counter parts of a sphere and a square.

## 1.1 Metric Fixed Point Theory

In 1906, Fréchet [13] presented the concept of metric spaces. Throughout the analysis of a variety of mathematical disciplines like functional analysis, topology and non-linear analysis, the theory of metric spaces is applied as a basic tool. In the same time, an important concept was appeared in the field of fixed point theory that is, Banach Contraction Principle (BCP) which played very important role in solving non-linear problems. This famous and well known result was introduced in 1922 by Banach [14]. He demonstrated that on a complete metric space, every contraction mapping has a unique fixed point. That is,

If $(\mathcal{S}, d)$ is a complete metric space and $T : \mathcal{S} \longrightarrow \mathcal{S}$ is a self map on $\mathcal{S}$. Then there exists an $\alpha \in [0, 1)$, such that

$$d(T\varrho, T\eta) \leq \alpha d(\varrho, \eta)$$

for all $\varrho, \eta \in \mathcal{S}$, then $T$ has a unique fixed point.

Picard [15] introduced the iterative process that was used to prove the Banach Contraction theorem. One can observe that the successive approximation method for finding the existence and uniqueness of differential equation solutions is infact the start and origin of fixed point theory and contraction principle. The frequent use of BCP in this domain by various researchers has resulted in significant advancement in fixed point theory. An important reason behind this development is the keen interest of the mathematicians as it provides a guarantee for the existence of solution of non-linear problems but it also gives the guarantee of uniqueness of solution. In the recent few decades, the field of fixed point has seen many advances and generalisations of metric spaces. The work in this direction is further subdivided into two categories.
In first category, the fixed point theorems are obtained by extending the contraction conditions and hence generalizing the BCP.
In the second category, researchers established fixed point theorems by changing the underlying space.

Many authors worked on different generalized metric spaces and proved Banach Contraction Principle by using different contractions. See for instance [16–22].

In 1989, Bakhtin [23] proposed the novel idea of $b$-metric space ($b$MS). This extension may rightly be called first generalization of metric space. Czerwik [24], further explored the notion of $b$MS by using the contraction conditions in $b$-metric space and generalized Banach contraction principle in this space. Many mathematicians played a leading role for the strong foundation of fixed point theory in $b$MS, proving a number of interesting fixed point results in $b$-metric space and its several extensions, as in [25–28].

In 1998, Czerwik [29] demonstrated a result in $b$-metric spaces for nonlinear single-value contraction maps. For set-value quasi-contraction maps, Aydi et al. [30] established a fixed point result in 2012. They also provided a common fixed point theorem [31] satisfying a weak $\phi$-contraction for single and multi-valued maps in $b$MS that generalizes many well known fixed point results in the current literature. Alghamdi et al. [26] presented the concept of a $b$-metric-like space in 2013 and represented the fixed point's existence and uniqueness in both $b$-metric-like spaces and a partially ordered $b$-metric-like spaces. Shukla [32] introduced the definition of a partial $b$-metric space in 2014 as a generalisation of a partial metric space and a $b$MS and proved the Banach contraction principle and Kannan-type fixed point result in the setting of partial $b$MS. Recently, Faraji et al. [33] generalized the BCP and proved some common fixed point theorems in complete $b$MS by using Geraghty type contractive mappings [34].

Dislocated metric space (DMS) is another interesting notion and generalization of metric space. In 2000, the idea of dislocated metric space was presented by Hitzler and Seda [35], which was further investigated by various researchers, see [36–38]. The fascinating aspect of Hitzler and Seda's concept of DMS is that a point's self distance does not have to be zero. The well-known Banach contraction theorem [14] was also expanded in this new framework. A fascinating element of dislocated metric space is that it is employed in topology, logical programming, and electrical engineering, see [36, 39, 40].

Jleli and Samet [41] defined generalised metric space in 2015 as a new generalisation of metric spaces. It includes MS, $b$MS and DMS as examples of topological spaces. They defined $k$-contraction for a self mapping $\mathcal{S}$ in generalized metric space

$$D(f\varrho, f\eta) \leq kD(\varrho, \eta),$$

for every $(\varrho, \eta) \in \mathcal{S} \times \mathcal{S}$ and also defined $k$-quasi contraction for a self mapping $\mathcal{S}$ in generalized metric space as

$$D(f\varrho, f\eta) \leq k \max\Big\{D(\varrho, \eta), D(\varrho, T\varrho), D(\eta, T\eta), D(\varrho, T\eta), D(\eta, T\varrho)\Big\},$$

for $k \in (0, 1)$ and for every $(\varrho, \eta) \in \mathcal{S} \times \mathcal{S}$.
The authors also expanded some famous fixed point theorems such as BCP [14], Ćirić's fixed point theorem [16] and a fixed point result for Ran and Reurings [42]. Due to the work of Jleli and Samet [41], Senapati et al. [43] studied the interesting generalization of standard MS, $b$MS and DMS in 2016. They modified the result for Ćirić quasi contraction type mappings and also used D-admissible mappings to prove the same result. Moreover, their work establishes two fixed point results for Wardowski type contraction and rational contraction mappings. Further, Tanusri Senapati and Lakshmi Kanta [43] investigated coupled fixed point results in newly discovered JS-metric spaces in 2016. They provided a broader edition of several coupled fixed point results. In the context of JS-metric spaces proposed by Jleli and Samet, Karapinar et al. [44] introduced two classes of Meir–Keeler type contractions in 2018 and established a fixed-point result for each class. In 2019, Senapati et al. [45] developed a few non unique fixed point or periodic point results in the setting of JS-metric spaces involving ZD-contraction and generalized Meir–Keeler contraction.

## 1.2 Fuzzy Fixed Point Theory

With the introduction of fuzzy sets in 1965, Zadeh [46] set the groundwork for fuzzy mathematics. A fuzzy set is a collection of items that has a range of membership grades. A membership function, which assigns a grade of membership to each object ranging from 0 to 1, characterises such a set. In the sense of fuzzy sets, the notions of inclusion, union, intersection, complement, relation, convexity, and their various properties are defined.

In recent studies of the logical and set-theoretical foundations of mathematics, the adjective “fuzzy” appears to be a common and frequently used one. The key explanation for this rapid growth, in our view, is simple to comprehend. The world around us is filled with uncertainty, the information we gather from the atmosphere, the concepts we employ, and the data derived from our observations or measurements are all, in general, ambiguous and incorrect. As a result, each formal representation of the real world or any part of it is merely an approximation and idealization of the actual state in any case. Fuzzy sets, fuzzy languages and other related concepts make it possible to manage and research the above-mentioned degree of uncertainty in a strictly mathematical and formal manner. A brief overview of the most important findings and implementations relating to the concept of fuzzy set and related concepts is given in [47].

Kramosil and Michálek [48] presented the idea of FMS in 1975 in order to describe it in a natural and intuitively justifiable way. After that, the definition is compared to statistical metric space, and both are shown to be similar in certain sense. The goal of this concept is to apply the idea of fuzziness to traditional notions of metric and metric spaces and compare the results with those obtained from other, probabilistic statistical generalisations of metric spaces.

The BCP was extended in FMS in the context of Kramosil and Michálek by Grabiec [49] in 1998, as

Let $(\mathcal{S}, M, *)$ be a complete fuzzy metric space such that

$$\lim_{t\to\infty} F(\varrho, \eta, t) = 1 \text{ for all } \varrho, \eta \in \mathcal{S}.$$

Let $T : \mathcal{S} \to \mathcal{S}$ be a mapping satisfying

$$M(T\varrho, T\eta, kt) \geq M(\varrho, \eta, t)$$

for all $\varrho, \eta \in \mathcal{S}$ and $k \in (0, 1)$. Then $T$ has a unique fixed point.

Grabiec [49] also established Edelstein contraction theorem in FMS as

Let $(\mathcal{S}, M, *)$ be a compact fuzzy metric space with $M(\varrho, \eta, .)$ continuous for all $\varrho, \eta, \in \mathcal{S}$. Let $T : \mathcal{S} \to \mathcal{S}$ be a mapping satisfying

$$M(T\varrho, T\eta, t) \geq F(\varrho, \eta, t)$$

for all $\varrho \neq \eta$ and $t > 0$. Then $T$ has a unique fixed point.

With the support of continuous $t$-norms, George and Veeramani [50] updated the definition of FMS in 1994. The authors also proved some well-known metric space results, such as Baire's theorem for fuzzy metric spaces and established a Hausdorff topology.

Grogri and Sapena [51] developed a fuzzy contractive mapping in 2002 and proved fixed point theorems for complete FMS in George and Veeramani's context as well as Kramosil and Michálek's FMS that are complete in the sense of Grabiec [49]. Sushil Sharma [52] developed few common fixed point results in FMS for six mappings in 2002. Their key results in fuzzy metric spaces, probabilistic metric spaces and uniform spaces expanded, generalised and fuzzified some previously known results. In 2005, Razani [53] described fuzzy $\epsilon$-contractive mapping and demonstrated some fixed point results. In 2007, Mihet [54] used the mapping given by Razani [53] and proved some results. The author also affirmatively responded to an open question posed by Razani in 2005. The separation axiom in the description of a FMS defined as George and Veeramani, plays a crucial role in the proofs of theorems.

Abbas et al. [55] presented the concept of $\Psi$-weak contraction in FMS in 2011 and demonstrated some effects. The result of this article generalized the result of Gregori and Sapena [51]. Gopal et al. [56] introduced cyclic weak $\phi$ contractions in FMS in 2012 and used them to prove some results on fixed point existence and uniqueness in fuzzy metric spaces. In addition to presenting examples, several related results are also illustrated. Gupta et al. [57] proved two FP results using rational inequality and provide an integral application in 2013. Dey and Saha [58] define $T$-contraction in FMS in 2014. The existence of a fixed point of mapping satisfying a general contractive condition in a complete fuzzy metric space was defined in this article by the author. In particular, $T$-Banach contraction principle by Beiranvand et al. [2] in FMS is an analogue of this result. Many more contractions and fixed point results are proved by many authors in different ways, for instance, see [56, 59–83]

In 2015, Hussain et al. [84] established a significant relationship between parametric $b$-metric and fuzzy $b$-metric and deduced some fixed point theorem in triangular partially ordered F$b$MS. In computer science, metric spaces and their numerous generalisations are widely used. This is why, in 2016, Nădăban [85] presented and researched the definition of F$b$MS, generalising both the concept of FMS introduced by Kramosil and Michálek [48] as well as the concept of $b$MS. While on the other hand, Nădăban [85] defined fuzzy quasi $b$-metric space, which builds on Gregori and Romaguera's idea of fuzzy quasi-metric space [86] and also established a decomposition theorem in this space. Abbas et al. [87] introduced $\psi$-contraction and monotone $\psi$-contraction correspondences in F$b$MS and for these contractive mappings, few fixed point results were obtained. Some current results in FMS and F$b$MS are generalised by these results. Mehmood et al. [88] presented the idea of extended F$b$MS as a generalisation of F$b$MS and in the context of this more general class of F$b$MS, they proved BCP. The results of this article are the extension and generalization of the existing results in literature.

## 1.3 Thesis Contribution

Nădăban [85] explored the concepts of F$b$MS and a fuzzy-quasi pseudo metric space, developing some interesting findings in these spaces and concluding his work by proposing to extend the BCP to F$b$MS, which could be useful in solving fixed point problems in mathematics, engineering, and computer science. As a result, he laid the groundwork for establishing some fixed point results in these areas.

The aim of this research is to look into potential work and issues suggested by Nădăban [85] at the end of his work. For fuzzy metric spaces, Grabiec [49] proved the Banach fixed point theorem [14]. As inspired by Grabiec, we develope BCP in the setting of F$b$MS. The Hicks and Rhoades fixed point theorem [21] is established in the setting of F$b$MS by defining $g$-orbitally upper semi continuous function. We also use a control function to prove some fixed point results, analogous to [57, Theorem 1] for $G$-complete F$b$MS and illustrated the theorems by providing an example. At the end, several applications are made based on our results.

Faraji [33], recently used Geraghty type contractions [34] to prove a few fixed point theorems in $b$MS. We prove some new fixed point results for Geraghty-type contraction in $G$-complete F$b$MS. Particularly, the extension of main result of Grabiec [49] is established in our first theorem. Second result is the extention of the main result of Faraji et al. [33] and other results are the generlization of the results of Alsulami et al. [89] in the setting of $G$-complete F$b$MS. We establish some common fixed point results for Geraghty-type contraction in the setting of $G$-complete F$b$MS. Our first two results are the extensions of the main results of Faraji [33] and third result is the generalization of the results of Gupta et. al. [57] in the setting of $G$-complete F$b$MS. We also establish some common fixed point results for Geraghty-type contraction which are generalization of the results of Faraji [33] and the results of Gupta et. al. [57] in the setting of $G$-complete F$b$MS. We furnish an example to illustrate our main result and also present an application of the results obtained.

In addition, the definition of a generalised fuzzy metric space (GFMS) is introduced and an example of a GFMS has been provided to demonstrate its description. We also show that the classes of FMS, F$b$MS, and DFMS are proper sub-classes of the class of GFMS. In the context of GFMS, we have also proven the BCP [14] and Ćirić's quasi-contraction theorem [16]. As consequences of our results, we obtain many author's recent results as corollaries. In the end, we give an example which illustrates our main result. Some fixed point theorems are also proved by applying our results.

## 1.4 Organization of Thesis

The rest of the thesis is arranged in the following way.

- In Chapter 2, the basic concepts, outcomes and examples are updated on which other chapters are based. In the first section, we discuss various kinds of metric spaces, such as $b$MS, DMS and GMS. The FMS and F$b$MS were covered in the second section. The final section includes several fixed point results defined on FMS.

- In Chapter 3, we investigate the definition of F$b$MS and demonstrate the BCP [14] in the context of F$b$MS using an example. In addition, we introduce the definition of a $g$-orbitally upper semi continuous function in FMS and prove Hicks and Rhoades's fixed point theorem [21] in F$b$MS. A control function in F$b$MS is used to prove certain fixed point results using a novel and logical contraction. As a result of our observations, some applications are highlighted as well. A brief conclusion to our work is provided at the end of the chapter.

- In Chapter 4, Geraghty-type contraction in $G$-complete F$b$MS is used to prove some new fixed point theorems, using the concept of F$b$MS. In particular, our first theorem establishes the extension of Grabiec's main result [49]. The second result is an extension of Faraji et al [33] key's result and the other results are the generalization of results of Alsulami et al. [89] in the setting of $G$-complete F$b$MS. There is also a demonstration of how our main result for the existence of solutions to non linear integral equations can be implemented. This work is published in the following journal article.

  **M. S. Ashraf**, R. Ali and N. Hussain, "Geraghty type contractions in fuzzy $b$-metric spaces with application to integral equations," *Filomat*, vol. 34, no. 9 , pp.3083-3098, 2020.

  In rest of chapter, some common fixed point theorems for Geraghty-type contraction in the setting of $G$-complete F$b$MS are presented. Our two results are the extensions of the main results of Faraji [33] and third result is the generalization of the results of Gupta et. al. [57] in the setting of $G$-complete F$b$MS. We furnish an example to illustrate our main result and also present an application of the results obtained. This work is submitted for possible publication.

"Some Common Fixed Point Results in Fuzzy $b$-Metric Spaces Using Geraghty Type Contraction," Submitted, 2021.

- In Chapter 5, the notion of generalized fuzzy metric space GFMS is defined. This new GFMS has generalised a wide variety of topological spaces, including FMS, F$b$MS, and DFMS. An example of a GFMS has been provided to demonstrate its description. We also showed that the classes of FMS, F$b$MS, and DFMS are proper sub-classes of the class of GFMS. The BCP [14] and Ćirić's quasi-contraction theorem [16] have also proved in the context of GFMS . As a consequence of our observations, we receive several author's recent findings as corollaries.

  Finally, we include an example that illustrates our main result. Our results are also used to prove certain fixed point theorems in existing literature. The work of this chapter is published in the following journal article.

  **M. S. Ashraf**, R. Ali and N. Hussain, "New Fuzzy Fixed Point Results in Generalized Fuzzy Metric Spaces With Application to Integral Equations," *IEEE Access*, vol. 8, pp. 91653-91660, 2020.

# Chapter 2

# Preliminaries

The aim of this chapter is to update the basic concepts, outcomes and examples on which other chapters are based. We start with the concepts of different types of metric spaces *i.e,* spaces equipped with a $b$-metric, dislocated metric and generalized metric etc. In the second section, fuzzy metric spaces ($FMS$) and fuzzy $b$-metric spaces ($FbMS$) are discussed. The last section consists of some fixed point theorems for mappings defined on $FbMS$.

## 2.1 Metric Spaces

In this section, we shall concern with the basic definitions and examples of metric spaces along with the elementary concepts. Metric space ($MS$) is one of the most important topic in pure mathematics. It is the generalization of usual concept of distance between two points in Euclidean spaces. It has certain properties and some very interesting concepts. In fact, metric spaces and topological spaces are frequently used in scientific researches now a days. In 1906, Fréchet [13] gave the notion of $MS$ as follows:

**Definition 2.1.1.**

"Let $\mathcal{S}$ be a nonempty set. A function $d : \mathcal{S} \times \mathcal{S} \longrightarrow \mathbb{R}$ is called a metric on $\mathcal{S}$ if it satisfies the following conditions for all $\varrho, \eta, \zeta \in \mathcal{S}$,

$M1$: $d(\varrho, \eta) \geq 0$,

$M2$: $d(\varrho, \eta) = 0$ if and only if $\varrho = \eta$,

$M3$: $d(\varrho, \eta) = d(\eta, \varrho)$,

$M4$: $d(\varrho, \eta) \leq d(\varrho, \zeta) + d(\zeta, \eta)$.

The pair $(\mathcal{S}, d)$ is called a **metric space**" [13].

**Example 2.1.2.**

Let $\mathcal{S} = \mathbb{R}$. A mapping $d \colon \mathcal{S} \times \mathcal{S} \longrightarrow \mathbb{R}$ defined by

$$d(\varrho, \eta) = \frac{|\varrho - \eta|}{1 + |\varrho - \eta|} \qquad \forall \;\; \varrho, \eta \in \mathcal{S},$$

is a metric on $\mathcal{S}$ and the pair $(\mathcal{S}, d)$ is a metric space.

**Example 2.1.3.**

Let $\mathcal{S} = C[a, b]$ represents the collection of all real-valued continuous functions defined on $[a, b]$. The function $d \colon \mathcal{S} \times \mathcal{S} \to \mathbb{R}$ given by

$$d(g, h) = \max_{t \in [a,b]} |g(t) - h(t)|, \quad g, h \in C[a, b],$$

is a metric on $\mathcal{S}$ and $(\mathcal{S}, d)$ is a metric space.

**Example 2.1.4.**

Let $\mathcal{S}$ represents the collection of all real-valued continuous functions specified on $[0, 1]$. The function $d \colon \mathcal{S} \times \mathcal{S} \to \mathbb{R}$ given by

$$d(g, h) = \int_0^1 |g(t) - h(t)| dt \quad \forall \;\; g(t), h(t) \in \mathcal{S},$$

is a metric on $\mathcal{S}$ and $(\mathcal{S}, d)$ is a metric space.

In the following, we state the Archimedean property of $\mathbb{R}$, which is frequently used in the convergence problems.

**Archimedean Property**

If $\epsilon > 0$ is any real number then for any real number $\varrho$, there is a positive integer $n_0$ such that $n_0 \epsilon > \varrho$.

**Definition 2.1.5.**

"A sequence $\{\varrho_n\}$ in a metric space $\mathcal{S}$ is said to be **convergent** to a point $\varrho \in \mathcal{S}$ if for every $\varepsilon > 0$ there a positive integer an $N = N(\varepsilon)$ such that,

$$d(\varrho_n, \varrho) < \varepsilon \qquad \text{for all } n \geq N.$$

$\varrho$ is called the limit of sequence $\{\varrho_n\}$ and we write

$$\lim_{n \longrightarrow \infty} d(\varrho_n, \varrho) = 0\text{''} [90].$$

**Example 2.1.6.**

Consider $\mathcal{S} = \mathbb{R}$ with usual metric and a sequence $\{\varrho_n\} = \left\{\frac{1}{n}\right\}$. For any real number $\epsilon > 0$ and $n \in \mathbb{N}$,

$$\begin{aligned} d(\varrho_n, 0) &= |\varrho_n - 0| \\ &= \left|\frac{1}{n} - 0\right| \\ &= \frac{1}{n}. \end{aligned} \tag{2.1}$$

By Archimedean Property, there exists a positive integer $n_0$ for any $\epsilon > 0$ such that

$$\begin{aligned} & n_0 \ \epsilon > 1 \\ \Rightarrow \quad & \frac{1}{n_0} < \epsilon \\ \Rightarrow \quad & \frac{1}{n} \leq \frac{1}{n_0} < \epsilon \quad \forall \quad n \geq n_0. \end{aligned} \tag{2.2}$$

Combining (2.1) and (2.2), we have

$$d(\varrho_n, 0) < \epsilon \quad \forall \quad n \geq n_0.$$

This shows that the given sequence $\{\varrho_n\} = \left\{\frac{1}{n}\right\}$ converges to 0.

**Definition 2.1.7.**

"A sequence $\{\varrho_n\}$ in a metric space $\mathcal{S}$ is said to be a **Cauchy sequence** if for every $\varepsilon > 0$, there is positive integer $N = N(\varepsilon)$ such that

$$d(\varrho_m, \varrho_n) < \varepsilon,$$

for every $m, n \geq N$" [90].

**Definition 2.1.8.**

"A metric space $(\mathcal{S}, d)$ is said to be **complete** if every Cauchy sequence in $\mathcal{S}$ has a limit in $\mathcal{S}$" [90].

**Remark 2.1.9.**

Any convergent sequence is a Cauchy sequence in a metric space but the converse is not valid. In other words, every Cauchy sequence in a metric space need not to be convergent.

**Definition 2.1.10.**

"Let $(\mathcal{S}, d)$ be a metric space. A mapping $T : \mathcal{S} \to \mathcal{S}$ is called **contraction** if there exists $0 \leq \alpha < 1$ such that

$$d(T\varrho, T\eta) \leq \alpha d(\varrho, \eta),$$

for $\varrho, \eta \in \mathcal{S}$" [90].

**Example 2.1.11.**

Consider $\mathcal{S} = \mathbb{R}$ and the usual metric space $(\mathbb{R}, d)$ defined by

$$d(\varrho, \eta) = |\varrho - \eta|.$$

Then a mapping $T \colon \mathbb{R} \to \mathbb{R}$ defined by

$$T(\varrho) = \frac{\varrho}{a} + b,$$

is a contraction for all $a > 1$.

In fact,

$$d(T\varrho, T\eta) = d\left(\frac{\varrho}{a} + b, \frac{\eta}{a} + b\right),$$

which implies

$$\begin{aligned} d(T\varrho, T\eta) &= \left|\frac{\varrho}{a} + b - \frac{\eta}{a} - b\right| \\ &= \left|\frac{\varrho}{a} - \frac{\eta}{a}\right| \\ &= \frac{1}{a}\left|\varrho - \eta\right| \\ &= \frac{1}{a} d(\varrho, \eta). \end{aligned} \tag{2.3}$$

From (2.3), it is clear that $\alpha = \frac{1}{a}$ such that $0 < \alpha < 1$ for all $a > 1$. Hence $T$ is a contraction.

**Definition 2.1.12.**

Consider a metric space $(\mathcal{S}, d)$ and $\mathrm{T} : \mathcal{S} \to \mathcal{S}$ a self map. A point $\varrho \in \mathcal{S}$ is called a **fixed point** of $T$ if

$$\mathrm{T}\varrho = \varrho.$$

Generally a point that does not move by a given transformation is a fixed point of that transformation.

Geometrically, if $\eta = \mathrm{T}\varrho$ is a real valued function in $\mathbb{R}^2$ then by fixed point of T, it means that the points where the graph of T intersect with line $\mathrm{T}\varrho = \varrho$. Thus a mapping T may or may not have fixed point. Further, fixed point may not be unique, see Figure 2.3 and Figure 2.4.

Following are the examples of fixed points.

**Example 2.1.13.**

1. Let $\mathcal{S} = \mathbb{R}$. A mapping $\mathrm{T} : \mathcal{S} \to \mathcal{S}$ defined by

$$\mathrm{T}\varrho = \varrho + 1,$$

has no fixed point.

Geometrically, it means that the graph of $\mathrm{T}\varrho = \varrho + 1$ never intersects the graph of $\mathrm{T}\varrho = \varrho$, as shown in Figure 2.1 .

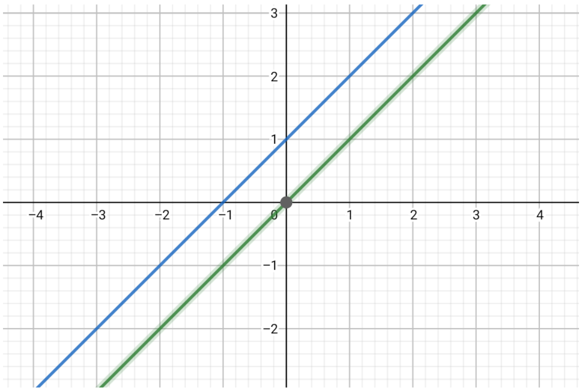


Figure 2.1: A graph having no fixed point.

2. Let $\mathcal{S} = \mathbb{R}$. A mapping $\mathrm{T} : \mathcal{S} \to \mathcal{S}$ defined by

$$\mathrm{T}\varrho = 2\varrho + 1,$$

has exactly one fixed point, as shown in Figure 2.2.

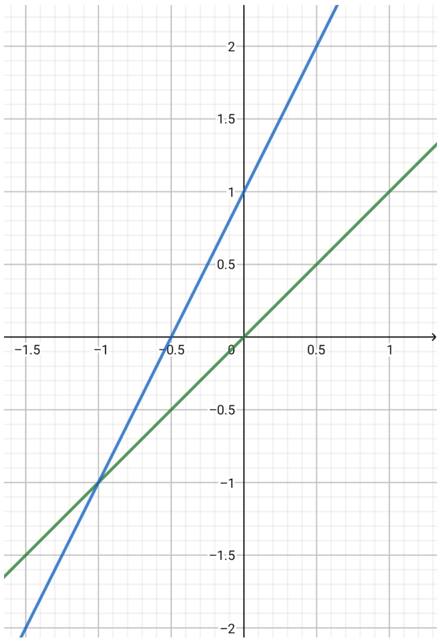


Figure 2.2: A graph having one fixed point.

3. Let $\mathcal{S} = \mathbb{R}$. A mapping $\mathrm{T} : \mathcal{S} \to \mathcal{S}$ defined by

$$\mathrm{T}\varrho = \varrho^2 - 2,$$

has exactly two fixed points, that is $-1$ and $2$, as shown in Figure 2.3.

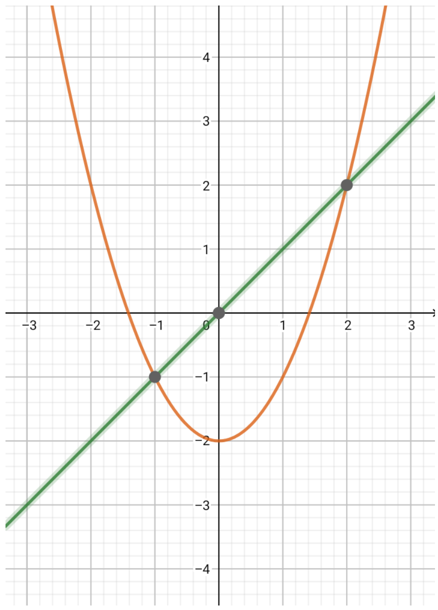

FIGURE 2.3: A graph having two fixed points.

4. Let $\mathcal{S} = \mathbb{R}$. A mapping $\mathrm{T} : \mathcal{S} \to \mathcal{S}$ defined by

$$\mathrm{T}\varrho = \varrho^3 - 3\varrho,$$

has exactly three fixed points, that is $-2, 0$ and $2$, as shown in Figure 2.4.

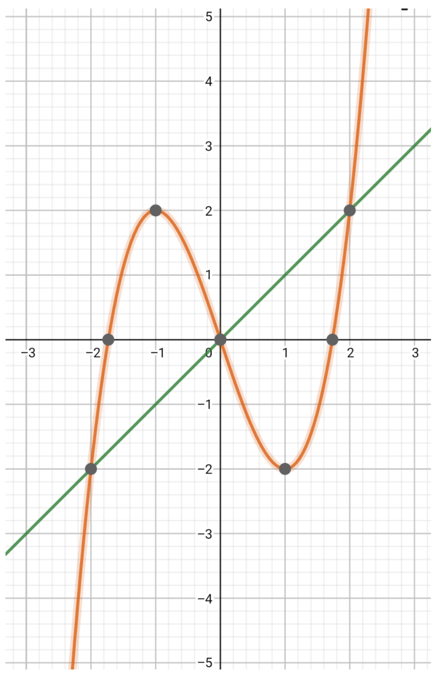

FIGURE 2.4: A graph having three fixed points.

## 2.2 $b$-Metric Spaces

Bakhtin [23] introduced the notion of $b$-metric spaces ($bMS$) in 1989 by using the real number $b \geq 1$ in triangular inequality. Later on, the concept of $bMS$ was further investigated by Czerwik [24] by generalizing the the well-known BCP and also presented various fixed point results in $bMS$. The extension of fixed point theorems in $bMS$ was explored by a several researchers, including Aydi [30], Boriceanu [91], Bota [92], Kir [93], Păcurar [94].

The definition of $bMS$ is as follows:

**Definition 2.2.1.**
"Let $\mathcal{S}$ be a non-empty set. For any real number $b \geq 1$, a function $d_b \colon \mathcal{S} \times \mathcal{S} \longrightarrow \mathbb{R}$ is called $b$-metric if it satisfies the following properties for all $\varrho, \eta, \zeta \in \mathcal{S}$.

$BM1$**:** $d_b(\varrho, \eta) \geq 0$,

$BM2$**:** $d_b(\varrho, \eta) = 0$ if and only if $\varrho = \eta$,

$BM3$**:** $d_b(\varrho, \eta) = d_b(\eta, \varrho) \quad$ for all $\varrho, \eta \in \mathcal{S}$,

$BM4$**:** $d_b(\varrho, \eta) \leq b\Big[d_b(\varrho, \zeta) + d_b(\zeta, \varrho)\Big]$.

The pair $(\mathcal{S}, d_b)$ is called a $b$-metric space"[23].

**Remark 2.2.2.**
Note that, by setting $b = 1$ in Definition 2.2.1, it coincides with the concept of metric spaces (Definition 2.1.1). Therefore, the class of $bMS$ is larger than that of $MS$.

Following example shows that every $b$-metric space need not to be metric space.

**Example 2.2.3.**
Let $\mathcal{S} = \mathbb{R}$. A mapping $d_b \colon \mathcal{S} \times \mathcal{S} \longrightarrow \mathbb{R}$ defined by

$$d_b(\varrho, \eta) = (\varrho - \eta)^2 \quad \forall \quad \varrho, \eta \in X,$$

is a $b$-metric on $\mathbb{R}$. The pair $(\mathcal{S}, d_b)$ is a $bMS$.

The properties $BM1$, $BM2$, $BM3$ of Definition 2.2.1 are obvious. Here we prove only $BM4$. For all $\varrho, \eta, \zeta \in \mathcal{S}$,

$$\begin{aligned} d_b(\varrho, \eta) &= (\varrho - \eta)^2 \\ &= (\varrho - \zeta + \zeta - \eta)^2 \\ &\leq 2\left[(\varrho - \zeta)^2 + (\zeta - \eta)^2\right] \\ &= 2\left[d_b(\varrho, \zeta) + d_b(\zeta, \eta)\right]. \end{aligned}$$

Hence $d_b$ is a $b$-metric with $b = 2$ but not a metric.

**Example 2.2.4.**

Let $\mathcal{S} = \{0, 1, 2\}$ and a $b$-metric $d_b$ on $\mathcal{S}$ is defined by

$$d_b(2, 0) = d_b(0, 2) = \lambda \geq 2$$

$$d_b(1, 0) = d_b(0, 1) = d_b(1, 2) = d_b(2, 1) = 1.$$

$$d_b(0, 0) = d_b(1, 1) = d_b(2, 2) = 0.$$

Then,

$$d_b(\varrho, \eta) \leq \frac{\lambda}{2}\Big[d_b(\varrho, \zeta) + d_b(\zeta, \eta)\Big] \quad \forall\ \varrho, \eta, \zeta \in \mathcal{S}.$$

Thus $(\mathcal{S}, d_b)$ is a $bMS$ and for $\lambda > 2$, it is not a metric space.

## 2.3 Dislocated Metric Space

Hitzler and Seda [35] first proposed the definition of dislocated metric space $DMS$ in 2000 as follows:

**Definition 2.3.1.**

"Let $\mathcal{S}$ be a nonempty set and $D : \mathcal{S} \times \mathcal{S} \to [0, +\infty)$ be a given mapping. We say that $D$ is a dislocated metric on $\mathcal{S}$, if it satisfies the following conditions:

$DM1$: $D(\varrho, \eta) = 0 \ \Rightarrow\ \varrho = \eta$,

$DM2$**:** $D(\varrho,\eta) = D(\eta,\varrho)$ for all $\varrho,\eta \in \mathcal{S}$,

$DM3$**:** $D(\varrho,\eta) \leq D(\varrho,\zeta) + D(\zeta,\eta)$.

The pair $(\mathcal{S}, D)$ is called a dislocated metric space."

## 2.4 Generalized Metric Space

Jleli and Samet [41] proposed a fascinating generalisation of metric spaces in 2015 which is known as generalized metric space ($GMS$), from which various well-known structures such as standard $MS$, $bMS$ and $DMS$ etc. can easily be derived. The definition of a $GMS$ is as follows:

**Definition 2.4.1.**
"Consider a nonempty set $\mathcal{S}$ and a mapping $\mathcal{D} : \mathcal{S}\times\mathcal{S} \to [0,\infty)$ . For all $\varrho \in \mathcal{S}$, define a set

$$C(\mathcal{D},\mathcal{S},\varrho) = \left\{\{\varrho_n\} \subset \mathcal{S} : \lim_{n\to\infty} \mathcal{D}(\varrho_n,\varrho) = 0\right\},$$

then $\mathcal{D}$ is said to be a generalized metric on $\mathcal{S}$ if for every $(\varrho_1,\varrho_2) \in \mathcal{S}\times\mathcal{S}$, the following conditions hold:

$GM1$**:** $\mathcal{D}(\varrho_1,\varrho_2) = 0 \quad \Rightarrow \quad \varrho_1 = \varrho_2$,

$GM2$**:** $\mathcal{D}(\varrho_1,\varrho_2) = \mathcal{D}(\varrho_2,\varrho_1)$,

$GM3$**:** there exists $c > 0$ so that if $\{\varrho_n\} \in C(\mathcal{D},\mathcal{S},\varrho_1)$, then

$$\mathcal{D}(\varrho_1,\varrho_2) \leq c \limsup_{n\to\infty} \mathcal{D}(\varrho_n,\varrho_2),$$

then the pair $(\mathcal{S},\mathcal{D})$ is called a $GMS$" [41].

**Remark 2.4.2.**
In case, if $C(\mathcal{D},\mathcal{S},\varrho)$ is empty for each $\varrho \in \mathcal{S}$ then $(\mathcal{S},\mathcal{D})$ is a $GMS$ if and only if $GM1$ and $GM2$ are satisfied [41].

**Proposition 2.4.3.**

$C(\mathcal{D}, \mathcal{S}, \varrho)$ is a non-empty if and only if $\mathcal{D}(\varrho, \varrho) = 0$ [95].

*Proof.*

If $C(\mathcal{D}, \mathcal{S}, \varrho)$ is a non-empty *i.e,* $C(\mathcal{D}, \mathcal{S}, \varrho) \neq \phi$, then there is a sequence $\{\varrho_n\}$ in $\mathcal{S}$ such as

$$\lim_{n\to\infty} \mathcal{D}(\varrho_n, \varrho) = 0.$$

Using $GM3$, we get

$$\mathcal{D}(\varrho, \varrho) \leq c \limsup_{n\to\infty} \mathcal{D}(\varrho_n, \varrho) = 0.$$

Conversely, suppose that $\mathcal{D}(\varrho, \varrho) = 0$, then a sequence $\{\varrho_n\}$ in $\mathcal{S}$ defined by $\varrho_n = \varrho$ $\forall\, n \in \mathbb{N}$ converges to $\varrho$. Thus

$$C(\mathcal{D}, \mathcal{S}, \varrho) \neq \phi.$$

□

Following example demonstrates that $GMS$ need not to be a metric space.

**Example 2.4.4.**

Let $\mathcal{S} = [0, 1]$. Define a mapping $\mathcal{D}\colon \mathcal{S} \times \mathcal{S} \to [0, +\infty]$ as :

$$\begin{cases} \mathcal{D}(\varrho_1, \varrho_2) = \varrho_1 + \varrho_2 & \text{if } \varrho_1 \neq 0 \text{ and } \ \varrho_2 \neq 0 \\ \mathcal{D}(0, \varrho_1) = \mathcal{D}(\varrho_1, 0) = \dfrac{\varrho_1}{2} & \text{for all } \varrho_1 \in \mathcal{S}. \end{cases} \tag{2.4}$$

With this $\mathcal{D}$, the pair $(\mathcal{S}, \mathcal{D})$ is a $GMS$ as shown below.

The Conditions $GM1$ and $GM2$ are trivially satisfied. We need to verify $GM3$ only for those elements of $\mathcal{S}$ such that

$$\mathcal{D}(\varrho_1, \varrho_1) = 0 \quad \Rightarrow \quad \varrho_1 = 0 \quad \forall\, \varrho_1 \in \mathcal{S}.$$

Define the set

$$C(\mathcal{S}, \mathcal{D}, 0) = \left\{ \{\varrho_n\} \subset \mathcal{S} : \lim_{n\to\infty} \mathcal{D}(\varrho_n, 0) = 0 \right\}.$$

Using $\varrho_n$ and $\varrho_2$ in (2.4), we get

$$\mathcal{D}(\varrho_n, \varrho_2) = \begin{cases} \varrho_n + \varrho_2 & \text{if} \quad \varrho_n \neq 0 \quad \forall\, n \in \mathbb{N} \\ \dfrac{\varrho_2}{2} & \text{if} \quad \varrho_n = 0. \end{cases}$$

Clearly,

$$\begin{aligned} & \frac{\varrho_2}{2} \leq \varrho_n + \varrho_2 = \mathcal{D}(\varrho_n, \varrho_2) \\ \Rightarrow \quad & \mathcal{D}(0, \varrho_2) = \frac{\varrho_2}{2} \leq \mathcal{D}(\varrho_n, \varrho_2) \\ \Rightarrow \quad & \mathcal{D}(0, \varrho_2) \leq \limsup_{n \to \infty} \mathcal{D}(\varrho_n, \varrho_2). \end{aligned}$$

Thus $(\mathcal{S}, \mathcal{D})$ is a $GMS$. But it is not a metric space because there is an absurdity in triangular-inequality.

If $\varrho_1, \varrho_2 \in \mathcal{S} \setminus \{0\}$, then we get,

$$\mathcal{D}(\varrho_1, \varrho_2) = \varrho_1 + \varrho_2$$

$$\begin{aligned} \Rightarrow \mathcal{D}(\varrho_1, 0) + \mathcal{D}(0, \varrho_2) &= \frac{\varrho_1}{2} + \frac{\varrho_2}{2} \\ &= \frac{\varrho_1 + \varrho_2}{2}. \end{aligned}$$

Hence

$$\begin{aligned} & \varrho_1 + \varrho_2 > \frac{\varrho_1 + \varrho_2}{2} \\ \Rightarrow\ & \mathcal{D}(\varrho_1, \varrho_2) > \mathcal{D}(\varrho_1, 0) + \mathcal{D}(0, \varrho_2). \end{aligned}$$

Hence every $GMS$ is not a metric space.

**Definition 2.4.5.**

"Let $(\mathcal{S}, \mathcal{D})$ be a $GMS$. A sequence $\{\varrho_n\}$ in $\mathcal{S}$ is **$\mathcal{D}$-convergent** to an element $\varrho$ in $\mathcal{S}$ if $\{\varrho_n\} \in C(\mathcal{S}, \mathcal{D}, \varrho)$" [41].

**Definition 2.4.6.**

Let $(\mathcal{S}, \mathcal{D})$ be a $GMS$ and $\{\varrho_n\}$ is a sequence in $\mathcal{S}$. Then for $\varrho \in \mathcal{S}$, $\{\varrho_n\}$ is called **$\mathcal{D}$-Cauchy sequence** if $\lim_{m,n \to \infty} \mathcal{D}(\varrho_n, \varrho_{n+m}) = 0$.

**Definition 2.4.7.**

A $GMS$ $(\mathcal{S}, \mathcal{D})$ is called **$\mathcal{D}$-complete** if each $\mathcal{D}$-Cauchy sequence in $\mathcal{S}$ is $\mathcal{D}$-convergent to an element $\varrho \in \mathcal{S}$.

**Remark 2.4.8.**

In a $GMS$, it is possible that a $\mathcal{D}$-convergent sequence is not necessarily a $\mathcal{D}$-Cauchy sequence.

We demonstrate in the following example that a $\mathcal{D}$-convergent sequence in a $GMS$ might not be a $\mathcal{D}$-Cauchy sequence.

**Example 2.4.9.**

Let $\mathcal{S} = \mathbb{R}^+ \cup \{0, \infty\}$. Define $\mathcal{D} : \mathcal{S} \times \mathcal{S} \to [0, \infty]$ by

$$\mathcal{D}(\varrho, \eta) = \begin{cases} \varrho + \eta & \text{either } \varrho \text{ or } \eta \text{ is } 0 \\ 1 + \varrho + \eta & \text{otherwise,} \end{cases}$$

then $(\mathcal{S}, \mathcal{D})$ is a $GMS$.

We now prove that any $\mathcal{D}$-convergent sequence is not necessarily a $\mathcal{D}$-Cauchy sequence in a $GMS$.

Consider a sequence $\{\varrho_n\}$ as

$$\varrho_n = \frac{1}{n} \quad \forall\, n \in \mathbb{N}.$$

Now

$$\begin{aligned} \lim_{n\to\infty} \mathcal{D}(\varrho_n, 0) &= \lim_{n\to\infty} \left(\frac{1}{n} + 0\right) \\ &= \lim_{n\to\infty} \left(\frac{1}{n}\right) \\ &= 0. \end{aligned}$$

$\Rightarrow$ $\{\varrho_n\}$ is $\mathcal{D}$-convergent to 0.

Note that

$$\lim_{n,m\to\infty} \mathcal{D}(\varrho_n, \varrho_{n+m}) = \lim_{n,m\to\infty} (1 + \varrho_n + \varrho_{n+m})$$

$$= \lim_{n,m\to\infty}\left(1+\frac{1}{n}+\frac{1}{n+m}\right)$$
$$= 1+0+0$$
$$= 1 \neq 0.$$

Hence $\{\varrho_n\}$ is not a $\mathcal{D}$-Cauchy sequence.

**Proposition 2.4.10.**

Consider a $GMS$ $(\mathcal{S}, D)$ and $\{\varrho_n\}$ is a sequence in $\mathcal{S}$. For each $(\varrho_1, \varrho_2) \in \mathcal{S} \times \mathcal{S}$, if $\{\varrho_n\}$ $\mathcal{D}$-converges to $\varrho_1$ and $\{\varrho_n\}$ $\mathcal{D}$-converges to $\varrho_2$, then

$$\varrho_1 = \varrho_2.$$

*Proof.*

Using the Condition $GM3$, we have

$$\mathcal{D}(\varrho_1, \varrho_2) \leq c \limsup_{n\to\infty} \mathcal{D}(\varrho_n, \varrho_2) = 0,$$

which implies that

$$\varrho_1 = \varrho_2.$$

□

**Proposition 2.4.11.**

Every $b$-metric space is a $GMS$ [41].

*Proof.*

Let $d_b$ be a $b$-metric on $\mathcal{S}$. We have just to proof that $d_b$ satisfies the property $GM3$.

Let $\varrho_1 \in \mathcal{S}$ and $\{\varrho_n\} \in C(D, \mathcal{S}, \varrho_1)$. For every $\varrho_2 \in \mathcal{S}$ and by using the Condition $BM4$, we have

$$D(\varrho_1, \varrho_2) \leq b\Big[D(\varrho_1, \varrho_n) + D(\varrho_n, \varrho_2)\Big],$$

$$D(\varrho_1, \varrho_2) \leq bD(\varrho_1, \varrho_n) + bD(\varrho_n, \varrho_2).$$

Thus we have

$$D(\varrho_1, \varrho_2) \leq b \limsup_{n\to\infty} D(\varrho_n, \varrho_2).$$

The Condition $GM3$ then satisfied with $c = b$. □

## 2.5 Fuzzy Metric Spaces

Zadeh [46] was the first to present the idea of a fuzzy set in 1965.

**Definition 2.5.1.**
"A **fuzzy set** $A$ in $\mathcal{S}$ is characterized by a membership(characteristic) function $f_A(s)$ which associates a real number in the interval $[0, 1]$ to each point in $\mathcal{S}$."

The grade of membership of $s$ in $A$ is represented by the value of $f_A(s)$. As a result, the value of $f_A(s)$ closer to unity means the membership grade of $s$ in $A$ is higher, and $f_A(s) = 0$ indicates that $s$ is not in $A$. When a set $A$ is in the conventional sense, its membership function has only two possible values, 0 and 1 with $f_A(s) = 1$ or 0, indicating whether $s$ belongs to $A$ or not. Fuzzy sets enable one to work in uncertain and ambiguous situations and solve ill-posed problems with incomplete information.

**Example 2.5.2.**
It is assumed that a person wants to purchase a cheap car. By the aggregate of prices, "Cheap" is taken as a fuzzy set which depends on the condition, model and the purchasing power of the buyer. The interpretation of the fuzzy membership function of "Cheap" car can be seen from the following figure.

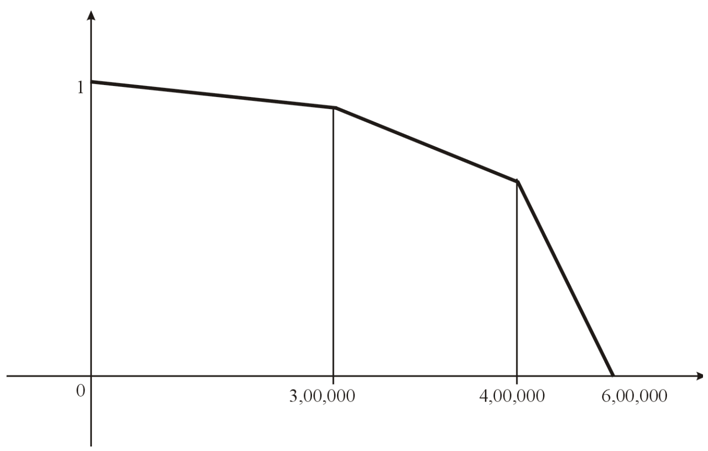


FIGURE 2.5: Membership function of "Cheap".

1. The cost of car less than Rs. $300,000$ can be assumed as cheap and there is no significant difference in the prices for purchaser.

2. If the prices vary from Rs. 300,000 to Rs. 400,000, then there is a weak preference for the car.

3. If the prices vary from Rs. 400,000 to Rs. 600,000, then there is a more weak preference for the car.

4. If the cost of car is above than Rs. 600,000, then it is considered as out of the range.

**Definition 2.5.3.**
"A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is called a **continuous *t*-norm** if it satisfies the following conditions:

**1.** $*$ is associative and commutative,

**2.** $*$ is continuous,

**3.** $\varrho * 1 = \varrho$ for all $\varrho \in [0,1]$, and

**4.** $\varrho * \eta \leq \zeta * \delta$ wherever $\varrho \leq \zeta$ and $\eta \leq \delta$

for all $\varrho, \eta, \zeta, \delta \in [0,1]$" [96].

**Example 2.5.4.**
Define a mapping $* : [0,1] \times [0,1] \to [0,1]$ by

**1.** $\varrho * \eta = \varrho\eta$ for $\varrho, \eta \in [0,1]$.

It is then obvious that $*$ is a continuous $t$-norm, known as product norm.

**2.** $\varrho * \eta = \varrho \wedge \eta = \min\{\varrho, \eta\}$ for all $\varrho, \eta \in [0,1]$.

Then $\wedge$ satisfies all conditions of Definition 2.5.3 and hence it is a $t$-norm, known as minimum $t$-norm.

**3.** $\varrho * \eta = \varrho *_L \eta = \max\{\varrho + \eta - 1, 0\}$ for $\varrho, \eta \in [0,1]$.

It is also continuous $t$-norm and known as maximum $t$-norm.

Kramosil and Michálek [48] combined the idea of fuzzy set and $t$-norm, in 1975 to define fuzzy metric spaces ($FMS$).

**Definition 2.5.5.**

"A 3-tuple $(\mathcal{S}, M, *)$ is said to be a **fuzzy metric space** if $\mathcal{S}$ is an arbitrary set, $*$ is a continuous $t$-norm and $M$ is a fuzzy set on $\mathcal{S} \times \mathcal{S} \times [0, \infty)$ satisfying the following conditions:

$FM1$: $M(\varrho, \eta, 0) = 0$

$FM2$: $M(\varrho, \eta, t) = 1, \forall\, t > 0$ if and only if $\varrho = \eta$

$FM3$: $M(\varrho, \eta, t) = M(\eta, \varrho, t)$

$FM4$: $M(\varrho, \zeta, t + s) \geq M(\varrho, \eta, t) * M(\eta, \zeta, s)\ \forall\, t, s \geq 0$

$FM5$: If $M(\varrho, \eta, .) \colon [0, \infty) \to [0, 1]$ is left continuous.

for all $\varrho, \eta, \zeta \in \mathcal{S}$ and $t, s > 0$" [48].

**Example 2.5.6.**

If $(\mathcal{S}, d)$ is a metric space and a function $g \colon \mathbb{R}^+ \to \mathbb{R}^+$, which is increasing and continuous. A function $M : \mathcal{S} \times \mathcal{S} \times [0, \infty) \to [0, 1]$ defined by

$$M(\varrho, \eta, t) = e^{-\dfrac{d(\varrho, \eta)}{g(t)}}. \tag{2.5}$$

Then $(\mathcal{S}, M, *)$ is a $FMS$, where $*$ is taken as product norm *i.e,* $\varrho * \eta = \varrho.\eta$.

If $g$ is taken as an identity function as a special case, *i.e,* $g(t) = t$, then 2.5 becomes

$$M(\varrho, \eta, t) = e^{-\dfrac{d(\varrho, \eta)}{t}}.$$

In this particular case, $(M, .)$ and $(M, \wedge)$ are a fuzzy metrics on $\mathcal{S}$.

However, by letting $g$ to be a constant function, *i.e,* $g(t) = k > 0$, 2.5 becomes

$$M(\varrho, \eta, t) = e^{-\dfrac{d(\varrho, \eta)}{k}}.$$

In this case, $(\mathcal{S}, M, .)$ is a $FMS$.

**Example 2.5.7.**

Consider a metric space $(\mathcal{S}, d)$ and a function $M : \mathcal{S} \times \mathcal{S} \times [0, \infty) \to [0, 1]$ defined by

$$M(\varrho, \eta, t) = \frac{t}{t + d(\varrho, \eta)},$$

then $(\mathcal{S}, M, *)$ is a $FMS$, where $*$ is taken as product norm *i.e,* $\varrho * \eta = \varrho.\eta$. The metric $d$ on $\mathcal{S}$ induces $M$, that is referred to as standard fuzzy metric on $\mathcal{S}$.

**Example 2.5.8.**

Consider a bounded metric space $(\mathcal{S}, d)$ and suppose $d(\varrho, \eta) < k \ \forall \ \varrho, \eta \in \mathcal{S}$. Let $M : \mathcal{S} \times \mathcal{S} \times [0, \infty) \to [0, 1]$ be a function defined by

$$M(\varrho, \eta, t) = 1 - \frac{d(\varrho, \eta)}{g(t)},$$

where $g : R^+ \longrightarrow ]k, +\infty[$ be an increasing continuous function. Then $(\mathcal{S}, M, *)$ is a $FMS$.

In 1994, to define a Hausdorff topology on $FMS$, George and Veeramani [50] modified the idea of the $FMS$ given in Definition 2.5.5. The reason for a modified definition of $FMS$ follows from the remark given by George and Veeramani.

**Remark 2.5.9.**

"$M(\varrho, \eta, t)$ can be thought of as the degree of nearness between $\varrho$ and $\eta$ with respect to $t$. We identify $\varrho = \eta$ with $M(\varrho, \eta, t) = 1$, for $t > 0$ and $M(\varrho, \eta, t) = 0$ with $\infty$. In this context, above definition is modified in order to introduce a Hausdorff topology on the fuzzy metric space"[50].

**Definition 2.5.10.**

"A 3-tuple $(\mathcal{S}, M, *)$ is said to be a fuzzy metric space if $\mathcal{S}$ is an arbitrary set, $*$ is a continuous t-norm and $M$ is a fuzzy set on $\mathcal{S} \times \mathcal{S} \times (0, \infty)$ satisfying the following conditions:

$GvFM1$**:** $M(\varrho, \eta, t) > 0$;

$GvFM2$**:** $M(\varrho, \eta, t) = 1$ if and only if $\varrho = \eta$ for all $t > 0$;

$GvFM3$**:** $M(\varrho, \eta, t) = M(\eta, \varrho, t)$;

$GvFM4$**:** $M(\varrho, \zeta, t + s) \geq M(\varrho, \eta, t) * M(\eta, \zeta, s)$;

$GvFM5$**:** $M(\varrho, \eta, .) \colon (0, \infty) \to [0, 1]$ is continuous,

for all $\varrho, \eta, \zeta \in \mathcal{S}$ and $t, s > 0$." [50]

**Example 2.5.11.**

Consider $(\mathcal{S}, d)$ is a metric space and a function $g \colon \mathbb{R}^+ \to \mathbb{R}^+$ is increasing and continuous. Define $M : \mathcal{S} \times \mathcal{S} \times (0, \infty) \to [0, 1]$ by

$$M(\varrho, \eta, t) = \frac{g(t)}{g(t) + \lambda d(\varrho, \eta)}, \tag{2.6}$$

for all $\varrho, \eta \in \mathcal{S}$ , $\lambda \in \mathbb{R}^+$. Then $(\mathcal{S}, M, *)$ is a $FMS$, where $t$-norm $*$ is the product norm.

Particularly, if we consider $g(t) = t^n$ for $n \in \mathbb{N}$ and $\lambda = 1$ in (2.6), then we have

$$M(\varrho, \eta, t) = \frac{t^n}{t^n + d(\varrho, \eta)}.$$

In this case $(\mathcal{S}, M, *)$ is a $FMS$.

If we take $n = 1$ in above equation, then we get standard fuzzy metric space as stated in Example 2.5.7, that is,

$$M(\varrho, \eta, t) = \frac{t}{t + d(\varrho, \eta)}.$$

Grabiec [49], in 1988, introduced the ideas of convergent sequence and Cauchy sequence in order to prove the BCP [14] in the setting of $FMS$ as follows:

**Definition 2.5.12.**

"Let $(\mathcal{S}, M, *)$ be a fuzzy metric space. A sequence $\{\varrho_n\}$ in $\mathcal{S}$ is said to be **convergent** (or converges to $\varrho \in \mathcal{S}$) if $\lim_{n\to\infty} M(\varrho_n, \varrho, t) = 1$ for each $t > 0$" [49].

**Definition 2.5.13.**

"A sequence $\{\varrho_n\}$ in $\mathcal{S}$ is **Cauchy sequence** if $\lim_{n\to\infty} M(\varrho_n, \varrho_{n+m}, t) = 1$ for each $t > 0$ and $m > 0$" [49].

"If every Cauchy sequence is convergent then it is called complete $FMS$" [49].

**Remark 2.5.14.**

The convergent sequence, Cauchy sequence and the completeness given by Grabiec [49] in Definition 2.5.12 and Definition 2.5.13 will be called as $G$-convergent sequence, $G$-Cauchy sequence and $G$-completeness in the rest of the thesis.

## 2.6 Fuzzy $b$-Metric Spaces

As a generalization of metric space, $FMS$ have been introduced by Kramosil and Michálek [48] in 1975 by using continuous $t$-norms. After that, many authors generalized $FMS$ in different forms. To extend the notion of $bMS$ in fuzzy settings, Nădăban [85] presented the definition of fuzzy $b$-metric spaces ($FbMS$) in 2016. As extended $b$-metric space is more general form of metric space and $bMS$, like wise extended $FbMS$ generalizes $FMS$ and $FbMS$ which was given by Mehmood et al. [88] in 2017.

In this section we recall the notion of $FbMS$ introduced by Nădăban [85]. We illustrate the definition by an example and also show that $FbMS$ needs not to be $FMS$. The concept of convergence sequence, Cauchy sequence and completeness in $FbMS$ in the sense of Grabiec is also included in this section.

Following Kramosil and Michálek [48], Nădăban [85] defined $FbMS$ as follows:

**Definition 2.6.1.**

"Let $\mathcal{S}$ be a non empty set, let $b \geq 1$ be a given real number and $*$ be a continuous $t$-norm. A fuzzy set $M_b$ on $\mathcal{S} \times \mathcal{S} \times [0, \infty)$ is called **fuzzy $b$-metric** if for all $\varrho, \eta, \zeta \in \mathcal{S}$, the following conditions hold:

$FBM1$**:** $M_b(\varrho, \eta, 0) = 0$;

$FBM2$**:** $M_b(\varrho, \eta, t) = 1, \forall\, t > 0$ if and only if $\varrho = \eta$ ;

$FBM3$**:** $M_b(\varrho, \eta, t) = M_b(\eta, \varrho, t), \ \forall\, t \geq 0$;

$FBM4$**:** $M_b(\varrho, \zeta, b(t+s)) \geq M_b(\varrho, \eta, t) * M_b(\eta, \zeta, s)$, $\forall\, t, s \geq 0$;

$FBM5$**:** $M_b(\varrho, \eta, .)\colon [0, \infty) \to [0, 1]$ is left continuous and $\lim_{t\to\infty} M_b(\varrho, \eta, t) = 1$".

**Remark 2.6.2.**

The class of $FbMS$ is larger than that of $FMS$. Setting $b = 1$, the above definition coincides with $FMS$.

**Example 2.6.3.**

Let $\mathcal{S} = \mathbb{R}$. For a real number $b \geq 1$, define a function $M_b\colon \mathcal{S} \times \mathcal{S} \times [0, \infty) \to [0, 1]$ by

$$M_b(\varrho, \eta, t) = e^{-\dfrac{(\varrho-\eta)^2}{t}} \qquad \forall\, \varrho, \eta \in \mathbb{R}.$$

Then $(\mathcal{S}, M_b, *)$ is a $FbMS$ with the product norm $*$.

We only prove property $FBM4$ of Definition 2.6.1 because other properties are obvious.

For $\varrho, \eta, \zeta \in \mathcal{S}$ and $t_1, t_2 > 0$, we have

$$\begin{aligned}
M_b(\varrho, \zeta, t_1 + t_2) &= e^{-\dfrac{(\varrho-\zeta)^2}{t_1+t_2}} \\
&\geq e^{-\dfrac{2((\varrho-\eta)^2 + (\eta-\zeta)^2)}{t_1+t_2}} \\
&= e^{-\dfrac{2(\varrho-\eta)^2}{t_1+t_2}} . e^{-\dfrac{2(\eta-\zeta)^2}{t_1+t_2}} \\
&\geq e^{-\dfrac{2(\varrho-\eta)^2}{t_1}} . e^{-\dfrac{2(\eta-\zeta)^2}{t_2}} \\
&= e^{-\dfrac{(\varrho-\eta)^2}{\frac{t_1}{2}}} . e^{-\dfrac{(\eta-\zeta)^2}{\frac{t_2}{2}}} \\
&= M_b\left(\varrho, \eta, \frac{t_1}{2}\right) * M_b\left(\eta, \zeta, \frac{t_2}{2}\right).
\end{aligned}$$

Thus

$$M_b(\varrho, \zeta, t_1 + t_2) \geq M_b\left(\varrho, \eta, \frac{t_1}{2}\right) * M_b\left(\eta, \zeta, \frac{t_2}{2}\right).$$

Hence $(\mathcal{S}, M_b, *)$ is a $FbMS$.

The example below demonstrates that every $FbMS$ needs not to be a $FMS$.

**Example 2.6.4.**

Let $\mathcal{S} = \mathbb{R}$. For a real number $b \geq 1$, a function $M_b\colon \mathcal{S} \times \mathcal{S} \times [0,\infty) \to [0,1]$ defined by

$$M_b(\varrho,\eta,t) = e^{-\dfrac{d_b(\varrho,\eta)}{t}} \quad \forall\ \varrho,\eta \in \mathbb{R}.$$

Then $(\mathcal{S}, M_b, *)$ is a $FbMS$ with the product norm $*$.

We prove the property $FBM4$ only of Definition 2.6.1, because other properties are obvious.

For $\varrho,\eta,\zeta \in \mathcal{S}$ and $t_1, t_2 > 0$, we have

$$\begin{aligned}
M_b(\varrho,\zeta,t_1+t_2) &= e^{-\dfrac{d_b(\varrho,\zeta)}{t_1+t_2}} \\
&\geq e^{-\dfrac{b(d_b(\varrho,\eta)+d_b(\eta,\zeta))}{t_1+t_2}} \\
&= e^{-\dfrac{b(d_b(\varrho,\eta))}{t_1+t_2}} . e^{-\dfrac{b(d_b(\eta,\zeta))}{t_1+t_2}} \\
&\geq e^{-\dfrac{b(d_b(\varrho,\eta))}{t_1}} . e^{-\dfrac{b(d_b(\eta,\zeta))}{t_2}} \\
&\geq e^{-\dfrac{d_b(\varrho,\eta)}{\frac{t_1}{b}}} . e^{-\dfrac{d_b(\eta,\zeta)}{\frac{t_2}{b}}} \\
&= M_b\left(\varrho,\eta,\frac{t_1}{b}\right) * M_b\left(\eta,\zeta,\frac{t_2}{b}\right).
\end{aligned}$$

$\Rightarrow (\mathcal{S}, M_b, *)$ is a $FbMS$.

The following are the definitions of $G$-convergence sequence, $G$-Cauchy sequence, and $G$-completeness in $FbMS$.

**Definition 2.6.5.**

Let $(\mathcal{S}, M_b, *)$ be a $FbMS$. A sequence $\{\varrho_n\}$ in $\mathcal{S}$ is $G$-**convergent** to $\varrho \in \mathcal{S}$ if

$$\lim_{n\to\infty} M_b(\varrho_n,\varrho,t) = 1,$$

for each $t > 0$.

**Example 2.6.6.**

Let $\mathcal{S} = [0, 1]$ and a mapping $M_b \colon \mathcal{S} \times \mathcal{S} \times [0, \infty) \to [0, 1]$ defined by

$$M_b(\varrho, \eta, t) = \begin{cases} e^{-\dfrac{(\varrho - \eta)^2}{t}} & \text{if } t > 0 \\ 0 & \text{if } t = 0. \end{cases}$$

Then $(\mathcal{S}, M_b, *)$ is a $FbMS$ with $b = 2$.

Consider a sequence $\{\varrho_n\}$ in $\mathcal{S}$ such that

$$\varrho_n = \frac{1}{n} \quad \forall \quad n \in \mathbb{N},$$

then clearly $\{\varrho_n\}$ $G$-converges to 0, which is as follows:

$$\begin{aligned} \lim_{n\to\infty} M_b(\varrho_n, 0, t) &= \lim_{n\to\infty} e^{-\dfrac{(\varrho_n - 0)^2}{t}} \\ &= \lim_{n\to\infty} e^{-\dfrac{(\varrho_n)^2}{t}} \\ &= \lim_{n\to\infty} e^{-\dfrac{\frac{1}{n^2}}{t}} \\ &= \lim_{n\to\infty} e^{-\dfrac{1}{n^2 t}} = 1. \end{aligned}$$

Thus the sequence $\{\varrho_n\}$ is $G$-convergent.

**Definition 2.6.7.**

A sequence $\{\varrho_n\}$ in $\mathcal{S}$ is $G$-**Cauchy** if

$$\lim_{n\to\infty} M_b(\varrho_n, \varrho_{n+m}, t) = 1,$$

for each $t > 0$ and $m > 0$.

**Example 2.6.8.**

Consider the fuzzy $b$-metric space given in Example 2.6.6 and a sequence $\{\varrho_n\}$ with

$$\varrho_n = \frac{1}{n} \quad \forall \quad n \in \mathbb{N}.$$

Now for all $m \in \mathbb{N}$, we have

$$\begin{aligned}\lim_{n\to\infty} M_b(\varrho_n, \varrho_{n+m}, t) &= \lim_{n\to\infty} e^{-\dfrac{(\varrho_n - \varrho_{n+m})^2}{t}}\\ &= \lim_{n\to\infty} e^{-\dfrac{\left(\dfrac{1}{n} - \dfrac{1}{n+m}\right)^2}{t}}\\ &= 1.\end{aligned}$$

Hence $\{\varrho_n\}$ is a $G$-Cauchy sequence.

A $G$-complete $FbMS$ is one where every $G$-Cauchy sequence is $G$-convergent.

**Definition 2.6.9.**

Let $(\mathcal{S}, M_b, *)$ be a $FbMS$. $M_b$ is said to be continuous on $\mathcal{S} \times \mathcal{S} \times [0, \infty)$ if

$$\lim_{n\to\infty} M_b(\varrho_n, \eta_n, t_n) = M_b(\varrho, \eta, t),$$

whenever $\{\varrho_n\}$ converges to $\varrho$, $\{\eta_n\}$ converges to $\eta$ and $\{t_n\}$ converges to $t$, that is

$$\lim_{n\to\infty} M_b(\varrho_n, \varrho, t) = 1,$$

$$\lim_{n\to\infty} M_b(\eta_n, \eta, t) = 1,$$

$$\lim_{n\to\infty} M_b(\varrho, \eta, t_n) = M_b(\varrho, \eta, t).$$

In general, every fuzzy $b$-metric is not continuous as demonstrated by the following example.

**Example 2.6.10.**

Let $\mathcal{S} = [0, \infty)$. A function $M_b \colon \mathcal{S} \times \mathcal{S} \times [0, \infty) \to [0, 1]$ defined by

$$M_b(\varrho, \eta, t) = e^{-\dfrac{d_b(\varrho, \eta)}{t}} \quad \forall\ \varrho, \eta \in \mathcal{S},$$

taking product norm *i.e* $\varrho * \eta = \varrho\eta$, where the $b$-metric $d_b$ is taken as

$$d_b(\varrho,\eta)=\begin{cases}0 & \text{if} \quad \varrho=\eta,\\ 2|\varrho-\eta| & \text{if} \quad \varrho,\eta\in[0,1),\\ \frac{1}{2}|\varrho-\eta| & \text{otherwise.}\end{cases}$$

We prove the property $FBM4$ only of Definition 2.6.1, because other properties are obvious. For all $\varrho,\eta,\zeta\in\mathcal{S}$ and $t_1,t_2>0$,

$$\begin{aligned}M_b(\varrho,\zeta,t_1+t_2)&=e^{-\dfrac{d_b(\varrho,\zeta)}{t_1+t_2}}\\ &\geq e^{-\dfrac{b(d_b(\varrho,\eta)+d_b(\eta,\zeta))}{t_1+t_2}}\\ &=e^{-\dfrac{b(d_b(\varrho,\eta))}{t_1+t_2}}.e^{-\dfrac{b(d_b(\eta,\zeta))}{t_1+t_2}}\\ &\geq e^{-\dfrac{b(d_b(\varrho,\eta))}{t_1}}.e^{-\dfrac{b(d_b(\eta,\zeta))}{t_2}}\\ &\geq e^{-\dfrac{d_b(\varrho,\eta)}{\frac{t_1}{b}}}.e^{-\dfrac{d_b(\eta,\zeta)}{\frac{t_2}{b}}}\\ &=M_b\left(\varrho,\eta,\frac{t_1}{b}\right)*M_b\left(\eta,\zeta,\frac{t_2}{b}\right).\end{aligned}$$

$\Rightarrow(\mathcal{S},M_b,*)$ is a $FbMS$.

It is important to note that $M_b$ is not continuous.

Consider

$$\begin{aligned}\lim_{n\to\infty}M_b\left(1,1-\frac{1}{n},t\right)&=\lim_{n\to\infty}e^{-\dfrac{\frac{1}{2}\left|1-1+\frac{1}{n}\right|}{t}}\\ &=\lim_{n\to\infty}e^{-\dfrac{1}{2nt}}\\ &=1=M_b(1,1,t).\end{aligned}$$

This implies that

$$\lim_{n\to\infty}M_b\left(1,1-\frac{1}{n},t\right)=M_b(1,1,t).$$

Also, as

$$\begin{aligned}
\lim_{n\to\infty} M_b\left(0, 1-\frac{1}{n}, t\right) &= \lim_{n\to\infty} e^{-\frac{2\left|0-1+\frac{1}{n}\right|}{t}} \\
&= e^{-\frac{2}{t}} \\
&\neq e^{\frac{1}{2t}} = M_b(0,1,t).
\end{aligned}$$

$$\Rightarrow \qquad \lim_{n\to\infty} M_b\left(0, 1-\frac{1}{n}, t\right) \neq M_b(0,1,t).$$

Hence fuzzy $b$-metric $M_b$ is not continuous.

## 2.7 Fixed point Theorems in Fuzzy Metric Space

Gupta et.al [57] developed two fixed point results using rational inequality and presented an integral application of their results in 2013. Here, we state the results of [57] in the sense of $G$-Cauchy sequence which we reformulate in $FbMS$ in Chapter 3.

**Theorem 2.7.1.**

Consider $(\mathcal{S}, M, *)$ is a complete $FMS$. A mapping $\mathrm{T} : \mathcal{S} \to \mathcal{S}$ satisfying

$$\lim_{t\to\infty} M(\varrho, \eta, t) = 1$$

and

$$M(\mathrm{T}\varrho, \mathrm{T}\eta, kt) \geq \lambda(\varrho, \eta, t),$$

where

$$\lambda(\varrho, \eta, t) = \min\left\{\frac{M(\eta, \mathrm{T}\eta, t)\left[1 + M(\varrho, \mathrm{T}\varrho, t)\right]}{1 + M(\varrho, \eta, t)}, M(\varrho, \eta, t)\right\},$$

for all $\varrho, \eta \in \mathcal{S}$ and $k \in (0,1)$. Then T has a unique fixed point [57].

**Definition 2.7.1.**

Consider a set $\Phi$ which consists of all continuous functions $\phi : [0,1] \to [0,1]$ such that

$$\phi(0) = 0, \quad \phi(1) = 1, \quad \text{and} \quad \phi(\alpha) > \alpha,$$

for all $\alpha \in (0,1)$ [57].

**Theorem 2.7.2.**

Consider $(\mathcal{S}, M, *)$ is a complete $FMS$. A mapping $\mathrm{T} : \mathcal{S} \to \mathcal{S}$ satisfying

$$\lim_{t\to\infty} M(\varrho, \eta, t) = 1,$$

and

$$M(\mathrm{T}\varrho, \mathrm{T}\eta, kt) \geq \phi(\lambda(\varrho, \eta, t)),$$

where

$$\lambda(\varrho, \eta, t) = \min\left\{\frac{M(\eta, \mathrm{T}\eta, t)\,[1 + M(\varrho, \mathrm{T}\varrho, t)]}{1 + M(\varrho, \eta, t)}, M(\varrho, \eta, t)\right\}$$

for all $\varrho, \eta \in \mathcal{S}$ , $k \in (0,1)$, $\phi \in \Phi$. Then T has a unique fixed point [57].

To provide integral application of the result, they first defined the following notion.

**Definition 2.7.2.**

" Define $\Psi : [0, \infty) \to [0, \infty)$, as

$$\Psi(t) = \int_0^t \phi(t)dt \quad \forall \;\; t > 0$$

be a non-decreasing and continuous function. Moreover for each $\epsilon > 0, \phi(\epsilon) > 0$. Also implies that $\phi(t) = 0$ if and only if $t = 0$" [57].

**Theorem 2.7.3.**

Consider $(\mathcal{S}, M, *)$ is a complete $FMS$. A mapping $\mathrm{T} : \mathcal{S} \to \mathcal{S}$ satisfying

$$\lim_{t\to\infty} M(\varrho, \eta, t) = 1,$$

and

$$\int_0^{M(\mathrm{T}\varrho, \mathrm{T}\eta, kt)} \varphi(t)dt \geq \int_0^{\lambda(\varrho, \eta, t)} \varphi(t)dt,$$

where

$$\lambda(\varrho,\eta,t)=\min\left\{\frac{M(\eta,\mathrm{T}\eta,t)\,[1+M(\varrho,\mathrm{T}\varrho,t)]}{1+M(\varrho,\eta,t)},M(\varrho,\eta,t)\right\},$$

for all $\varrho,\eta\in\mathcal{S}$, $k\in(0,1)$ and $\varphi\in\Psi$. Then T has a unique fixed point [57].

**Theorem 2.7.4.**

Consider $(\mathcal{S},M,*)$ is a complete $FMS$. A mapping $\mathrm{T}:\mathcal{S}\to\mathcal{S}$ satisfying

$$\lim_{t\to\infty}M(\varrho,\eta,t)=1$$

and

$$\int_0^{M(\mathrm{T}\varrho,\mathrm{T}\eta,kt)}\varphi(t)dt\geq\phi\left\{\int_0^{\lambda(\varrho,\eta,t)}\varphi(t)dt\right\},$$

where

$$\lambda(\varrho,\eta,t)=\min\left\{\frac{M(\eta,\mathrm{T}\eta,t)\,[1+M(\varrho,\mathrm{T}\varrho,t)]}{1+M(\varrho,\eta,t)},M(\varrho,\eta,t)\right\}$$

for all $\varrho,\eta\in\mathcal{S}$ , $\varphi\in\Psi$ , $k\in(0,1)$ and$\phi\in\Phi$. Then T has a unique fixed point [57]."

# Chapter 3

# Fixed Point Theorems in Fuzzy $b$-metric Spaces

In this chapter, the concept of $FbMS$ is studied which is the generalization of $FMS$. The Banach contraction principle is extended in the settings of $FbMS$ and furnished an example to illustrate the result. A new contraction has been introduced in $FbMS$ and certain fixed point results by using a control function are proved. Some applications are also highlighted as consequences of our results.

## 3.1 Fixed Point Results in $FbMS$

The BCP for $FMS$ is proved in [49]. This result has been established in the setting of $FbMS$ as follows:

**Theorem 3.1.1.**
Let $(\mathcal{S}, M_b, *)$ be a $G$-complete $FbMS$ with $b \geq 1$. A mapping $g\colon \mathcal{S} \to \mathcal{S}$ satisfies

$$M_b(g\varrho, g\eta, kt) \geq M_b(\varrho, \eta, t), \tag{3.1}$$

for all $\varrho, \eta \in \mathcal{S}$ and $k \in \left[0, \frac{1}{b}\right)$. Then $g$ has a unique fixed point.

*Proof.*

Let $\varrho_0 \in \mathcal{S}$. For $n \in \mathbb{N}$, define $\varrho_n = g^n \varrho_0$.

Then, note that, for all $n, t > 0$,

$$\begin{aligned} M_b(\varrho_n, \varrho_{n+1}, t) &\geq M_b\left(\varrho_{n-1}, \varrho_n, \frac{t}{k}\right) \\ &\geq M_b\left(\varrho_{n-2}, \varrho_{n-1}, \frac{t}{k^2}\right) \\ &\vdots \\ &\geq M_b\left(\varrho_0, \varrho_1, \frac{t}{k^n}\right). \end{aligned}$$

For any $\ell \in \mathbb{N}$,

$$\begin{aligned} & M_b(\varrho_n, \varrho_{n+\ell}, t) \\ & \geq M_b\left(\varrho_n, \varrho_{n+1}, \frac{t}{2b}\right) * M_b\left(\varrho_{n+1}, \varrho_{n+2}, \frac{t}{(2b)^2}\right) * \ldots * M_b\left(\varrho_{n+\ell-1}, \varrho_{n+\ell}, \frac{t}{(2b)^\ell}\right) \\ & \geq M_b\left(\varrho_0, \varrho_1, \frac{t}{2bk^n}\right) * M_b\left(\varrho_0, \varrho_1, \frac{t}{(2b)^2 k^{n+1}}\right) * \ldots * M_b\left(\varrho_0, \varrho_1, \frac{t}{(2b)^\ell k^{n+\ell-1}}\right) \\ & = M_b\left(\varrho_0, \varrho_1, \frac{t}{(2bk)k^{n-1}}\right) * M_b\left(\varrho_0, \varrho_1, \frac{t}{(2bk)^2 k^{n-1}}\right) * \ldots * \\ & \quad M_b\left(\varrho_0, \varrho_1, \frac{t}{(2bk)^\ell k^{n-1}}\right) \\ & \geq 1 * 1 * \ldots * 1 = 1. \end{aligned}$$

So, we have

$$\lim_{n\to\infty} M_b(\varrho_n, \varrho_{n+\ell}, t) = 1.$$

Hence $\{\varrho_n\}$ is a $G$-Cauchy sequence.

As $(\mathcal{S}, M_b, *)$ be a $G$-complete $FbMS$, so there is a point $\varrho \in \mathcal{S}$ such that

$$\lim_{n\to\infty} \varrho_n = \varrho.$$

Now, we like to prove that $\varrho$ is a fixed point of $g$.

$$M_b(g\varrho, \varrho, t) \geq M_b\left(g\varrho, g\varrho_n, \frac{t}{2b}\right) * M_b\left(\varrho_{n+1}, \varrho, \frac{t}{2b}\right)$$

$$\geq M_b\left(\varrho, \varrho_n, \frac{t}{2bk}\right) * M_b\left(\varrho_{n+1}, \varrho, \frac{t}{2b}\right)$$
$$\longrightarrow 1 * 1 = 1,$$

which shows that $g\varrho = \varrho$. Thus $\varrho$ is a fixed point.

**Uniqueness**

For some $\zeta \in \mathcal{S}$, assume that $g\zeta = \zeta$, then

$$\begin{aligned}
M_b(\zeta, \varrho, t) &= M_b(g\zeta, g\varrho, t) \\
&\geq M_b\left(\zeta, \varrho, \frac{t}{k}\right) \\
&= M_b\left(g\zeta, g\varrho, \frac{t}{k}\right) \\
&\geq M_b\left(\zeta, \varrho, \frac{t}{k^2}\right) \\
&\vdots \\
&\geq M_b\left(\zeta, \varrho, \frac{t}{k^n}\right) \\
&\longrightarrow 1 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

Thus $\quad \varrho = \zeta$. This proves that fixed point is unique. □

**Remark 3.1.1.**

The main result of Grabiec [49] becomes the special case of Theorem 3.1.1 by setting $b = 1$ .

The following example illustrates Theorem 3.1.1.

**Example 3.1.2.**

Let $\mathcal{S} = [0, 1]$. A function $M_b\colon \mathcal{S} \times \mathcal{S} \times [0, \infty) \to [0, 1]$ defined by

$$M_b(\varrho, \eta, t) = \begin{cases} \left(\dfrac{1}{t}\right)^{(\varrho-\eta)^2} & \text{for} \quad t > 0 \\ 0 & \text{if} \quad t = 0, \end{cases} \tag{3.2}$$

for all $\varrho, \eta \in \mathcal{S}$ with the product norm $*$ and a real number $b \geq 1$. Then $(\mathcal{S}, M_b, *)$ is a $G$-complete $FbMS$.

Define a function $g\colon \mathcal{S} \to \mathcal{S}$ such as $g\varrho = 1 - \varrho$.

Now

$$\begin{aligned} M_b(g\varrho, g\eta, kt) &= M_b(1-\varrho, 1-\eta, kt) \\ &= \left(\frac{1}{kt}\right)^{(1-\varrho-1+\eta)^2} \\ &= \left(\frac{1}{kt}\right)^{(\eta-\varrho)^2} \\ &= \left(\frac{1}{kt}\right)^{(\varrho-\eta)^2}. \end{aligned}$$

As,

$$\begin{aligned} k &< 1 \\ \Rightarrow \quad kt &< t \\ \Rightarrow \quad \frac{1}{kt} &> \frac{1}{t} \\ \Rightarrow \left(\frac{1}{kt}\right)^{(\varrho-\eta)^2} &> \left(\frac{1}{t}\right)^{(\varrho-\eta)^2}. \end{aligned}$$

Thus

$$M_b(g\varrho, g\eta, kt) > M_b(\varrho, \eta, t).$$

So, the conditions of Theorem 3.1.1 are satisfied.

Note that $g\varrho = 1 - \varrho$,

then $g\varrho = \varrho$ implies that

$$\begin{aligned} 1 - \varrho &= \varrho \\ \Rightarrow \quad \varrho &= \frac{1}{2}. \end{aligned}$$

Thus $\varrho = \frac{1}{2} \in [0, 1]$ is the unique fixed point.

To establish the fixed point result of Hicks and Rhoades [21] in $FbMS$, the notion of a $g$-orbitally upper semi continuous function is needed.

**Definition 3.1.3.**

Consider a self map $g\colon \mathcal{S} \to \mathcal{S}$. The orbit of an element $\varrho_0 \in \mathcal{S}$ is given by

$$\mathcal{O}(\varrho_0) = \Big\{\varrho_0,\ g\varrho_0,\ g^2\varrho_0, \dots\Big\}.$$

A function $G\colon \mathcal{S} \to [0,1]$ is called $g$-orbitally upper semi continuous at $\varrho \in \mathcal{S}$ if $\{\varrho_n\} \subset \mathcal{O}(\varrho_0)$ and $\{\varrho_n\} \to \varrho$,

$$\Rightarrow G(\varrho) \geq \limsup_{n\to\infty} G(\varrho_n).$$

**Example 3.1.4.**

Take a set $\mathcal{S} = [0,2]$. Define a self map $g\colon \mathcal{S} \to \mathcal{S}$ by

$$g\varrho = \frac{1}{2}\varrho^2.$$

Choose an element $\varrho_0 = \frac{1}{2}$ in $\mathcal{S}$, then we have

$$\mathcal{O}(\varrho_0) = \mathcal{O}\left(\frac{1}{2}\right) = \left\{\frac{1}{2}, \frac{1}{2^3}, \frac{1}{2^7} \cdots\right\}.$$

Then for any sequence $\{\varrho_n\}$ in $\mathcal{O}\left(\frac{1}{2}\right)$, we have $\varrho_n \to 0$.
Define a function $G\colon \mathcal{S} \to [0,1]$ by

$$G(\varrho) = \begin{cases} 1 & \text{if} \quad \varrho = 0 \\ \sqrt{2\varrho - \varrho^2} & \text{if} \quad 0 < \varrho \leq 2. \end{cases}$$

It is clear that $G(0) = 1$ and $\varrho_n \to \varrho = 0$, so we have

$$\begin{aligned} G(0) = 1 > 0 &= \limsup_{n\to\infty} G(\varrho_n) \\ &= \limsup_{n\to\infty} \sqrt{2\varrho_n - \varrho_n^2} \\ &= \limsup_{n\to\infty} \sqrt{\frac{1}{2^{n-1}} - \frac{1}{2^{2n}}}. \end{aligned}$$

Hence $G$ is $g$-orbitally upper semi continuous at $\varrho = 0$.

**Theorem 3.1.2.**

Let $(\mathcal{S}, M_b, *)$ be a $G$-complete $FbMS$ and for some $\varrho_0 \in \mathcal{S}$, a self map $g \colon \mathcal{S} \to \mathcal{S}$ satisfies

$$M_b(g\varrho, g^2\varrho, kt) \geq M_b(\varrho, g\varrho, t), \tag{3.3}$$

for each $\varrho \in \mathcal{O}(\varrho_0)$ and $k \in [0, \frac{1}{b})$. If $\varrho_n = g^n\varrho_0$ $(n \in \mathbb{N})$ then $g^n\varrho_0 \to \varrho \in \mathcal{S}$.

Furthermore, $g$ has a fixed point $\varrho$ if and only if $G(\varrho) = M_b(\varrho, g\varrho, t)$ is $g$-orbitally upper semi continuous at $\varrho$.

*Proof.*

For $\varrho_0 \in \mathcal{S}$, establish an iterative procedure $\{\varrho_n\}$ as

$$\varrho_0, \quad g\varrho_0 = \varrho_1, \quad g^2\varrho_0 = g\varrho_1 = \varrho_2, \quad \ldots, \quad g^n\varrho_0 = \varrho_n.$$

Applying (3.3) successively, we get

$$M_b(g^n\varrho_0, g^{n+1}\varrho_0, kt) = M_b(\varrho_n, \varrho_{n+1}, kt) \geq M_b\left(\varrho_0, \varrho_1, \frac{t}{k^n}\right).$$

For any $\ell \in \mathbb{N}$,

$$\begin{aligned}
&M_b(g^n\varrho_0, g^{n+\ell}\varrho_0, t) = M_b(\varrho_n, \varrho_{n+\ell}, t)\\
&\geq M_b\left(\varrho_n, \varrho_{n+1}, \frac{t}{2b}\right) * M_b\left(\varrho_{n+1}, \varrho_{n+2}, \frac{t}{(2b)^2}\right) * \ldots * M_b\left(\varrho_{n+\ell-1}, \varrho_{n+\ell}, \frac{t}{(2b)^\ell}\right)\\
&\geq M_b\left(\varrho_0, \varrho_1, \frac{t}{2bk^n}\right) * M_b\left(\varrho_0, \varrho_1, \frac{t}{(2b)^2k^{n+1}}\right) * \ldots * M_b\left(\varrho_0, \varrho_1, \frac{t}{(2b)^\ell k^{n+\ell-1}}\right)\\
&= M_b\left(\varrho_0, \varrho_1, \frac{t}{(2bk)k^{n-1}}\right) * M_b\left(\varrho_0, \varrho_1, \frac{t}{(2bk)^2k^{n-1}}\right) * \ldots *\\
&\quad M_b\left(\varrho_0, \varrho_1, \frac{t}{(2bk)^\ell k^{n-1}}\right)\\
&\geq 1 * 1 * \ldots * 1 = 1.
\end{aligned}$$

$$\Rightarrow \quad \lim_{n\to\infty} M_b(g^n\varrho_0, g^{n+\ell}\varrho_0, t) = 1.$$

Hence $\{\varrho_n\}$ is $G$-Cauchy sequence. As $(\mathcal{S}, M_b, *)$ is a $G$-complete $FbMS$ so there is an element $\varrho \in \mathcal{S}$ such as

$$\varrho_n = g^n\varrho_0 \to \varrho \in \mathcal{S}.$$

Let $G$ be upper semi continuous at $\varrho \in \mathcal{S}$, then

$$\begin{aligned} M_b(\varrho, g\varrho, t) &\geq \lim_{n\to\infty} \sup M_b(g^n\varrho_0, g^{n+1}\varrho_0, t) \\ &\geq \limsup_{n\to\infty} M_b\left(\varrho_0, \varrho_1, \frac{t}{k^n}\right) \\ &= 1. \end{aligned}$$

So, we have

$$\varrho = g\varrho.$$

Suppose, on the other hand, $\varrho = g\varrho$ and $\varrho \in \mathcal{O}(\varrho)$ with $\varrho_n \to \varrho$, then

$$\begin{aligned} G(\varrho) &= M_b(\varrho, g\varrho, t) \\ &= 1 \geq \limsup_{n\to\infty} G(\varrho_n) \\ &= M_b(g^n\varrho_0, g^{n+1}\varrho_0, t). \end{aligned}$$

□

Theorem 3.1.2 is illustrated by the following example .

**Example 3.1.5.**

Let $\mathcal{S} = [0, 1]$. A function $M_b\colon \mathcal{S} \times \mathcal{S} \times [0, \infty) \to [0, 1]$ defined by

$$M_b(\varrho, \eta, t) = \begin{cases} \left(\frac{1}{t}\right)^{(\varrho-\eta)^2} & \text{for} \quad t > 0 \\ 0 & \text{if} \quad t = 0, \end{cases} \tag{3.4}$$

for all $\varrho, \eta \in \mathcal{S}$ with the product norm $*$ and a real number $b \geq 1$. Then $(\mathcal{S}, M_b, *)$ is a $G$-complete $FbMS$. Define a self map $g\colon \mathcal{S} \to \mathcal{S}$ by

$$g\varrho = 1 - \varrho.$$

Choose an element $\varrho_0 = \frac{1}{2}$ in $\mathcal{S}$, then we have

$$\mathcal{O}(\varrho_0) = \mathcal{O}\left(\frac{1}{2}\right) = \left\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \cdots\right\}.$$

Then for any sequence $\{\varrho_n\}$ in $\mathcal{O}\left(\frac{1}{2}\right)$, we have $\varrho_n \to \frac{1}{2}$.
Define a function $G\colon \mathcal{S} \to [0,1]$ by

$$G(\varrho) = \begin{cases} 1 & \text{if} \quad \varrho = \frac{1}{2}, \\ \frac{1}{2} - \varrho & \text{otherwise.} \end{cases}$$

It is clear that $G\left(\frac{1}{2}\right) = 1$ and $\varrho_n \to \varrho = \frac{1}{2}$, so we have

$$\begin{aligned} G\left(\frac{1}{2}\right) &= 1 \\ &> 0 = \limsup_{n\to\infty} G(\varrho_n) \\ &= \limsup_{n\to\infty} \left(\frac{1}{2} - \varrho_n\right). \end{aligned}$$

Hence $G$ is $g$-orbitally upper semi continuous at $\varrho = \frac{1}{2}$.

Now,

$$\begin{aligned} M_b(g\varrho, g^2\varrho, kt) &= M_b\left(1-\varrho, \varrho, kt\right) \\ &= \left(\frac{1}{kt}\right)^{(2\varrho-1)^2} \\ &> \left(\frac{1}{t}\right)^{(2\varrho-1)^2} \\ &= M_b(\varrho, g\varrho, t). \end{aligned}$$

So, the conditions of Theorem 3.1.1 are satisfied.

Note that $g(\varrho) = 1 - \varrho$. Thus $\varrho = \frac{1}{2} \in [0,1]$ is the fixed point.

**Lemma 3.1.6.**

Let $(\mathcal{S}, M_b, *)$ be a complete $FbMS$. If

$$M_b(\varrho, \eta, kt) \geq M_b\left(\varrho, \eta, t\right),$$

$\forall\ \varrho, \eta \in \mathcal{S}$ ,$t > 0$ and $k \in (0,1)$, then $\varrho = \eta$.

*Proof.*

It follows from $FBM5$ that

$$\lim_{t\to\infty} M_b(\varrho,\eta,t)=1, \qquad \Rightarrow M_b(\varrho,\eta,kt)=1.$$

It follows from $FBM2$ that $\varrho=\eta$. □

Now, we establish some fixed point results, analogue to [57, Theorem 1] for $G$-complete $FbMS$.

**Theorem 3.1.3.**

Let $(\mathcal{S},M_b,*)$ be a $G$-complete $FbMS$ with $b\geq 1$. Let a mapping $g\colon\mathcal{S}\to\mathcal{S}$ satisfies

$$M_b(g\varrho,g\eta,kt)\geq \min\left\{\frac{M_b(\eta,g\eta,t)\,[1+M_b(\varrho,g\varrho,t)+M_b(\eta,g\varrho,t)]}{2+M_b(\varrho,\eta,t)},M_b(\varrho,\eta,t)\right\}, \tag{3.5}$$

$\forall\ \varrho,\eta\in\mathcal{S},\ k\in[0,\frac{1}{b})$. Then $g$ has a unique fixed point.

*Proof.*

For $\varrho_0\in\mathcal{S}$, choose a sequence $\{\varrho_n\}$ in $\mathcal{S}$ such as $g\varrho_n=\varrho_{n+1}$.

We, now show that the sequence $\{\varrho_n\}$ is a Cauchy sequence. Consider

$$\begin{aligned}
M_b(\varrho_n,\varrho_{n+1},t)&=M_b(g\varrho_{n-1},g\varrho_n,t)\\
&\geq \min\left\{\frac{M_b\left(\varrho_n,g\varrho_n,\frac{t}{k}\right)\left[1+M_b\left(\varrho_{n-1},g\varrho_{n-1},\frac{t}{k}\right)+M_b\left(\varrho_n,g\varrho_{n-1},\frac{t}{k}\right)\right]}{2+M_b\left(\varrho_{n-1},\varrho_n,\frac{t}{k}\right)},\right.\\
&\qquad\qquad\left. M_b\left(\varrho_{n-1},\varrho_n,\frac{t}{k}\right)\right\}\\
&=\min\left\{\frac{M_b\left(\varrho_n,\varrho_{n+1},\frac{t}{k}\right)\left[1+M_b\left(\varrho_{n-1},\varrho_n,\frac{t}{k}\right)+M_b\left(\varrho_n,\varrho_n,\frac{t}{k}\right)\right]}{2+M_b\left(\varrho_{n-1},\varrho_n,\frac{t}{k}\right)},\right.\\
&\qquad\qquad\left. M_b\left(\varrho_{n-1},\varrho_n,\frac{t}{k}\right)\right\}
\end{aligned}$$

$$= \min\left\{\frac{M_b\left(\varrho_n,\varrho_{n+1},\frac{t}{k}\right)\left[1+M_b\left(\varrho_{n-1},\varrho_n,\frac{t}{k}\right)+1\right]}{2+M_b\left(\varrho_{n-1},\varrho_n,\frac{t}{k}\right)},\ M_b\left(\varrho_{n-1},\varrho_n,\frac{t}{k}\right)\right\}$$
$$= \min\left\{M_b\left(\varrho_n,\varrho_{n+1},\frac{t}{k}\right),\ M_b\left(\varrho_{n-1},\varrho_n,\frac{t}{k}\right)\right\}.$$

So, we have

$$M_b(\varrho_n,\varrho_{n+1},t) \geq \min\left\{M_b\left(\varrho_n,\varrho_{n+1},\frac{t}{k}\right),\ M_b\left(\varrho_{n-1},\varrho_n,\frac{t}{k}\right)\right\}. \tag{3.6}$$

Here arises the following two cases:

**Case-1:**

If

$$\min\left\{M_b\left(\varrho_n,\varrho_{n+1},\frac{t}{k}\right),\ M_b\left(\varrho_{n-1},\varrho_n,\frac{t}{k}\right)\right\} = M_b\left(\varrho_n,\varrho_{n+1},\frac{t}{k}\right),$$

then from (3.6),

$$M_b(\varrho_n,\varrho_{n+1},t) \geq M_b\left(\varrho_n,\varrho_{n+1},\frac{t}{k}\right),$$

which is proved by Lemma 3.1.6.

**Case-2:**

If

$$\min\left\{M_b\left(\varrho_n,\varrho_{n+1},\frac{t}{k}\right),\ M_b\left(\varrho_{n-1},\varrho_n,\frac{t}{k}\right)\right\} = M_b\left(\varrho_{n-1},\varrho_n,\frac{t}{k}\right),$$

then from (3.6),

$$\begin{aligned}
M_b(\varrho_n,\varrho_{n+1},t) &\geq M_b\left(\varrho_{n-1},\varrho_n,\frac{t}{k}\right)\\
&= M_b\left(g\varrho_{n-2},g\varrho_{n-1},\frac{t}{k}\right)\\
&\geq M_b\left(\varrho_{n-2},\varrho_{n-1},\frac{t}{k^2}\right)\\
&\geq M_b\left(\varrho_{n-3},\varrho_{n-2},\frac{t}{k^3}\right)\\
&\vdots\\
&\geq M_b\left(\varrho_0,\varrho_1,\frac{t}{k^n}\right).
\end{aligned}$$

So, we get

$$M_b(\varrho_n, \varrho_{n+1}, t) \geq M_b\left(\varrho_0, \varrho_1, \frac{t}{k^n}\right). \tag{3.7}$$

For any $q \in \mathbb{N}$,

$$\begin{aligned}
&M_b(\varrho_n, \varrho_{n+q}, t) \\
&\geq M_b\left(\varrho_n, \varrho_{n+1}, \frac{t}{2b}\right) * M_b\left(\varrho_{n+1}, \varrho_{n+2}, \frac{t}{(2b)^2}\right) * \ldots * M_b\left(\varrho_{n+q-1}, \varrho_{n+q}, \frac{t}{(2b)^q}\right) \\
&\geq M_b\left(\varrho_0, \varrho_1, \frac{t}{2bk^n}\right) * M_b\left(\varrho_0, \varrho_1, \frac{t}{(2b)^2k^{n+1}}\right) * \ldots * M_b\left(\varrho_0, \varrho_1, \frac{t}{(2b)^qk^{n+q-1}}\right) \\
&= M_b\left(\varrho_0, \varrho_1, \frac{t}{(2bk)k^{n-1}}\right) * M_b\left(\varrho_0, \varrho_1, \frac{t}{(2bk)^2k^{n-1}}\right) * \ldots * \\
&\quad M_b\left(\varrho_0, \varrho_1, \frac{t}{(2bk)^qk^{n-1}}\right),
\end{aligned}$$

when $n \to \infty$ then we get

$$\lim_{n\to\infty} M_b(\varrho_n, \varrho_{n+q}, t) = 1.$$

Hence $\{\varrho_n\}$ is a $G$-Cauchy sequence.

Since $(\mathcal{S}, M_b, *)$ is a $G$-complete $FbMS$, so there is an element $\varrho \in \mathcal{S}$ such as

$$\lim_{n\to\infty} \varrho_n = \varrho.$$

Now, we prove that $\varrho$ is fixed point of $g$.

$$\begin{aligned}
M_b(g\varrho, \varrho, t) &\geq M_b\left(g\varrho, \varrho_{n+1}, t\right) * M_b\left(\varrho_{n+1}, \varrho, t\right) \\
&= M_b(g\varrho, g\varrho_n, t) * M_b(\varrho_{n+1}, \varrho_n, t).
\end{aligned} \tag{3.8}$$

Consider,

$$M_b(g\varrho, g\varrho_n, t) \geq \min\left\{ \frac{M_b\left(\varrho_n, g\varrho_n, \frac{t}{k}\right)\left[1 + M_b\left(\varrho_n, g\varrho_n, \frac{t}{k}\right) + M_b\left(\varrho_n, g\varrho, \frac{t}{k}\right)\right]}{2 + M_b\left(\varrho, \varrho_n, \frac{t}{k}\right)}, \right.$$
$$\left. M_b\left(\varrho, \varrho_n, \frac{t}{k}\right) \right\}$$

$$= \min\left\{\frac{M_b\left(\varrho_n, \varrho_{n+1}, \frac{t}{k}\right)\left[1 + M_b\left(\varrho_n, \varrho_{n+1}, \frac{t}{k}\right) + M_b\left(\varrho_n, g\varrho_n, \frac{t}{k}\right)\right]}{2 + M_b\left(\varrho_n, \varrho_n, \frac{t}{k}\right)}, \right.$$
$$\left. M_b\left(\varrho_n, \varrho_n, \frac{t}{k}\right)\right\}$$

$$= \min\left\{\frac{M_b\left(\varrho_n, \varrho_{n+1}, \frac{t}{k}\right)\left[1 + M_b\left(\varrho_n, \varrho_{n+1}, \frac{t}{k}\right) + M_b\left(\varrho_n, \varrho_{n+1}, \frac{t}{k}\right)\right]}{2+1}, 1\right\}$$

$$= \min\left\{\frac{M_b\left(\varrho_n, \varrho_{n+1}, \frac{t}{k}\right)\left[1 + M_b\left(\varrho_n, \varrho_{n+1}, \frac{t}{k}\right) + M_b\left(\varrho_n, \varrho_{n+1}, \frac{t}{k}\right)\right]}{3}, 1\right\}.$$

So, from (3.8), we get

$$M_b(g\varrho, \varrho, t)$$
$$\geq \min\left\{\frac{M_b\left(\varrho_n, \varrho_{n+1}, \frac{t}{k}\right)\left[1 + M_b\left(\varrho_n, \varrho_{n+1}, \frac{t}{k}\right) + M_b\left(\varrho_n, \varrho_{n+1}, \frac{t}{k}\right)\right]}{3}, 1\right\}$$
$$* M_b(\varrho_{n+1}, \varrho_n, t).$$

Letting $n \to \infty$, we get

$$\begin{aligned} M_b(g\varrho, \varrho, t) &= \min\left\{\frac{1[1+1+1]}{2+1}, 1\right\} * 1 \\ &= \min\{1, 1\} * 1 \\ &= 1 * 1 \\ &= 1. \end{aligned}$$

Thus

$$g\varrho = \varrho.$$

Hence $g$ has a fixed point $\varrho$.

**Uniqueness**

For some $\zeta \in \mathcal{S}$, assume $g\zeta = \zeta$, then

$$
\begin{aligned}
&M_b(\zeta, \varrho, t) = M_b(g\zeta, g\varrho, t)\\
&\geq \min\left\{\frac{M_b\left(\varrho, g\varrho, \frac{t}{k}\right)\left[1 + M_b\left(\zeta, g\zeta, \frac{t}{k}\right) + M_b\left(\varrho, g\zeta, \frac{t}{k}\right)\right]}{2 + M_b\left(\zeta, u, \frac{t}{k}\right)}, M_b\left(\zeta, \varrho, \frac{t}{k}\right)\right\}\\
&= \min\left\{\frac{M_b\left(\varrho, \varrho, \frac{t}{k}\right)\left[1 + M_b\left(\zeta, \zeta, \frac{t}{k}\right) + M_b\left(\varrho, \zeta, \frac{t}{k}\right)\right]}{2 + M_b\left(\zeta, \varrho, \frac{t}{k}\right)}, M_b\left(\zeta, \varrho, \frac{t}{k}\right)\right\}\\
&= \min\left\{\frac{1\left[1 + 1 + M_b\left(\zeta, \varrho, \frac{t}{k}\right)\right]}{2 + M_b\left(\zeta, \varrho, \frac{t}{k}\right)}, M_b\left(\zeta, \varrho, \frac{t}{k}\right)\right\}\\
&= \min\left\{1, M_b\left(\zeta, \varrho, \frac{t}{k}\right)\right\}.
\end{aligned}
$$

If

$$\min\left\{1, M_b\left(\zeta, \varrho, \frac{t}{k}\right)\right\} = 1,$$

then

$$M_b(\zeta, \varrho, t) = 1.$$

Thus

$$\varrho = \zeta.$$

If

$$\min\left\{1, M_b\left(\zeta, \varrho, \frac{t}{k}\right)\right\} = M_b\left(\zeta, \varrho, \frac{t}{k}\right),$$

then

$$M_b(\zeta, \varrho, t) \geq M_b\left(\zeta, \varrho, \frac{t}{k}\right).$$

So, from Lemma 3.1.6,

$$\varrho = \zeta.$$

Hence fixed point is unique. □

**Remark 3.1.7.**

Taking

$$\min\left\{\frac{M_b(\eta, g\eta, t)\,[1 + M_b(\varrho, g\varrho, t) + M_b(\eta, g\varrho, t)]}{2 + M_b(\varrho, \eta, t)}, M_b(\varrho, \eta, t)\right\} = M_b(\varrho, \eta, t)$$

in Theorem 3.1.3, then we get Theorem 3.1.1.

Following is the consequence of Theorem 3.1.3.

**Corollary 3.1.8.**

Let $(\mathcal{S}, M, *)$ be a $G$-complete $FMS$. Let a mapping $g\colon \mathcal{S} \to \mathcal{S}$ such as

$$M(g\varrho, g\eta, kt) \geq \min\left\{\frac{M(\eta, g\eta, t)\,[1 + M(\varrho, g\varrho, t) + M(\eta, g\varrho, t)]}{2 + M(\varrho, \eta, t)}, M(\varrho, \eta, t)\right\},$$

$\forall\ \varrho, \eta \in \mathcal{S}$ , $k \in (0, 1)$. Then $g$ has a unique fixed point.

Following example is furnished to illustrate Theorem 3.1.3.

**Example 3.1.9.**

Let $\mathcal{S} = \{0, 1, 2\}$. A function $M_b\colon \mathcal{S} \times \mathcal{S} \times [0, \infty) \to [0, 1]$ defined by

$$M_b(\varrho, \eta, t) = \frac{t}{t + (\varrho - \eta)^2}, \qquad \forall\ \varrho, \eta \in \mathcal{S}.$$

Then $(\mathcal{S}, M_b, *)$ is a $G$-complete $FbMS$. Define a mapping $g\colon \mathcal{S} \to \mathcal{S}$ such as

$$g\varrho = \sqrt{k}\ \varrho.$$

Now, for all $t > 0$ , we have

$$M_b(g\varrho, g\eta, kt) \geq \min\left\{\frac{M_b(\eta, g\eta, t)\,[1 + M_b(\varrho, g\varrho, t) + M_b(\eta, g\varrho, t)]}{2 + M_b(\varrho, \eta, t)}, M_b(\varrho, \eta, t)\right\}$$

$$M_b(\sqrt{k}\varrho, \sqrt{k}\eta, kt) \geq \min\left\{\frac{M_b(\eta, \sqrt{k}\eta, t)\left[1 + M_b(\varrho, \sqrt{k}\varrho, t) + M_b(\eta, \sqrt{k}\varrho, t)\right]}{2 + M_b(\varrho, \eta, t)},\right.$$

$$\left. M_b(\varrho, \eta, t)\right\}$$

$$\frac{kt}{kt+(\sqrt{k}\varrho-\sqrt{k}\eta)^2}$$
$$\geq \min\left\{\frac{\frac{t}{t+(\eta-\sqrt{k}\eta)^2}\left[1+\frac{t}{t+(\varrho-\sqrt{k}\varrho)^2}+\frac{t}{t+(\eta-\sqrt{k}\varrho)^2}\right]}{2+\frac{t}{t+(\varrho-\eta)^2}}, \frac{t}{t+(\varrho-\eta)^2}\right\}$$

$$\frac{t}{t+(\varrho-\eta)^2} \geq \min\left\{\frac{\frac{t}{t+\eta^2(1-\sqrt{k})^2}\left[1+\frac{t}{t+\varrho^2(1-\sqrt{k})^2}+\frac{t}{t+(\eta-\sqrt{k}\varrho)^2}\right]}{2+\frac{t}{t+(\varrho-\eta)^2}}, \frac{t}{t+(\varrho-\eta)^2}\right\}. \tag{3.9}$$

If

$$\min\left\{\frac{\frac{t}{t+\eta^2(1-\sqrt{k})^2}\left[1+\frac{t}{t+\varrho^2(1-\sqrt{k})^2}+\frac{t}{t+(\eta-\sqrt{k}\varrho)^2}\right]}{2+\frac{t}{t+(\varrho-\eta)^2}}, \frac{t}{t+(\varrho-\eta)^2}\right\}$$

$$= \frac{\frac{t}{t+\eta^2(1-\sqrt{k})^2}\left[1+\frac{t}{t+\varrho^2(1-\sqrt{k})^2}+\frac{t}{t+(\eta-\sqrt{k}\varrho)^2}\right]}{2+\frac{t}{t+(\varrho-\eta)^2}},$$

then from (3.9)

$$\frac{t}{t+(\varrho-\eta)^2} \geq \frac{\frac{t}{t+\eta^2(1-\sqrt{k})^2}\left[1+\frac{t}{t+\varrho^2(1-\sqrt{k})^2}+\frac{t}{t+(\eta-\sqrt{k}\varrho)^2}\right]}{2+\frac{t}{t+(\varrho-\eta)^2}}.$$

This implies that

$$M_b(g\varrho, g\eta, kt)$$
$$\geq \min\left\{\frac{M_b(\eta, g\eta, t)\left[1+M_b(\varrho, g\varrho, t)+M_b(\eta, g\varrho, t)\right]}{2+M_b(\varrho, \eta, t)}, M_b(\varrho, \eta, t)\right\}.$$

If

$$\min\left\{\frac{\frac{t}{t+\eta^2(1-\sqrt{k})^2}\left[1+\frac{t}{t+\varrho^2(1-\sqrt{k})^2}+\frac{t}{t+(\eta-\sqrt{k}\varrho)^2}\right]}{2+\frac{t}{t+(\varrho-\eta)^2}}, \frac{t}{t+(\varrho-\eta)^2}\right\} = \frac{t}{t+(\varrho-\eta)^2},$$

then from (3.9)

$$\frac{t}{t+(\varrho-\eta)^2} \geq \frac{t}{t+(\varrho-\eta)^2}.$$

This implies that

$$M_b(g\varrho, g\eta, kt) \geq \min\left\{\frac{M_b(\eta, g\eta, t)\left[1+M_b(\varrho, g\varrho, t)+M_b(\eta, g\varrho, t)\right]}{2+M_b(\varrho, \eta, t)}, M_b(\varrho, \eta, t)\right\}.$$

Thus all the conditions of Theorem 3.1.3 are satisfied. As

$$g\varrho = \sqrt{k}\,\varrho \quad \Rightarrow \quad g0 = 0.$$

Thus $g$ has a unique fixed point $\varrho = 0$.

To prove the following theorem, we use a control function $\vartheta\colon [0,1] \to [0,1]$ which is continuous and non-decreasing such as

1. $\vartheta(0) = 0,\ \vartheta(1) = 1$.
2. $\vartheta(a) > a$ for $0 < a < 1$.

**Theorem 3.1.4.**

Let $(\mathcal{S}, M_b, *)$ be a $G$-complete $FbMS$ with $b \geq 1$. Consider a continuous function $\vartheta\colon [0,1] \to [0,1]$ and a mapping $g\colon \mathcal{S} \to \mathcal{S}$ satisfying

$$M_b(g\varrho, g\eta, kt) \geq \vartheta\left\{\min\left\{\frac{M_b(\eta, g\eta, t)\left[1+M_b(\varrho, g\varrho, t)+M_b(\eta, g\varrho, t)\right]}{2+M_b(\varrho, \eta, t)}, M_b(\varrho, \eta, t)\right\}\right\} \tag{3.10}$$

$\forall\ \varrho, \eta \in \mathcal{S},\ k \in (0, \frac{1}{b})$. Then $g$ has a unique fixed point.

*Proof.*

Since $\vartheta$ is a continuous function and $\vartheta(a) > a$ for $0 < a < 1$, then from (3.10), we get

$$\begin{aligned} & M_b(g\varrho, g\eta, kt) \\ & \quad \geq \vartheta\left\{\min\left\{\frac{M_b(\eta, g\eta, t)\,[1 + M_b(\varrho, g\varrho, t) + M_b(\eta, g\varrho, t)]}{2 + M_b(\varrho, \eta, t)}, M_b(\varrho, \eta, t)\right\}\right\} \\ & \quad \geq \min\left\{\frac{M_b(\eta, g\eta, t)\,[1 + M_b(\varrho, g\varrho, t) + M_b(\eta, g\varrho, t)]}{2 + M_b(\varrho, \eta, t)}, M_b(\varrho, \eta, t)\right\}. \end{aligned}$$

Now, by using Theorem 3.1.3, we get our required result. □

Following is the consequence of Theorem 3.1.4.

**Corollary 3.1.10.**

Let $(X, M, *)$ be a $G$-complete $FMS$ and a continuous function $\vartheta\colon [0,1] \to [0,1]$. A self map $g\colon \mathcal{S} \to \mathcal{S}$ satisfies

$$\begin{aligned} & M(g\varrho, g\eta, kt) \\ & \quad \geq \vartheta\left\{\min\left\{\frac{M(\eta, g\eta, t)\,[1 + M(\varrho, g\varrho, t) + M(\eta, g\varrho, t)]}{2 + M(\varrho, \eta, t)}, M(\varrho, \eta, t)\right\}\right\}, \end{aligned}$$

$\forall\ \varrho, \eta \in \mathcal{S},\ k \in (0,1)$. Then $g$ has a unique fixed point.

**Theorem 3.1.5.**

Let $(\mathcal{S}, M_b, *)$ be a $G$-complete $FbMS$ with $b \geq 1$. Let a mapping $g\colon \mathcal{S} \to \mathcal{S}$ satisfies

$$\begin{aligned} & M_b(g\varrho, g\eta, kt) \\ & \quad \geq \min\Bigg\{\frac{M_b(\varrho, g\varrho, t)\,[1 + M_b(\eta, g\eta, t)]}{1 + M_b(g\varrho, g\eta, t)}, \frac{M_b(\eta, g\eta, t)\,[1 + M_b(\varrho, g\varrho, t)]}{1 + M_b(\varrho, \eta, t)}, \\ & \qquad \frac{M_b(\varrho, g\varrho, t)\,[2 + M_b(\varrho, g\eta, t)]}{1 + M_b(\varrho, g\eta, t) + M_b(\eta, g\varrho, t)}, M_b(\varrho, \eta, t)\Bigg\}, \end{aligned} \tag{3.11}$$

for all $\varrho, \eta \in \mathcal{S}$ and for some $k \in (0, \frac{1}{b})$. Then $g$ has a unique fixed point.

*Proof.*

For any $\varrho_0 \in \mathcal{S}$, choose an iterative sequence $\{\varrho_n\}$ in $\mathcal{S}$ such as

$$\varrho_{n+1} = g\varrho_n.$$

Now,

$$\begin{aligned}
&M_b(\varrho_n, \varrho_{n+1}, t) = M_b(g\varrho_{n-1}, g\varrho_n, t)\\
&\geq \min\left\{ \frac{M_b\left(\varrho_{n-1}, T\varrho_{n-1}, \frac{t}{k}\right)\left[1 + M_b\left(\varrho_n, T\varrho_n, \frac{t}{k}\right)\right]}{1 + M_b\left(T\varrho_{n-1}, T\varrho_n, \frac{t}{k}\right)}, \right.\\
&\quad \frac{M_b\left(\varrho_n, T\varrho_n, \frac{t}{k}\right)\left[1 + M_b\left(\varrho_{n-1}, T\varrho_{n-1}, \frac{t}{k}\right)\right]}{1 + M_b\left(\varrho_{n-1}, \varrho_n, \frac{t}{k}\right)},\\
&\quad \left. \frac{M_b\left(\varrho_{n-1}, T\varrho_{n-1}, \frac{t}{k}\right)\left[2 + M_b\left(\varrho_{n-1}, T\varrho_n, \frac{t}{k}\right)\right]}{1 + M_b\left(\varrho_{n-1}, T\varrho_n, \frac{t}{k}\right) + M_b\left(\varrho_n, T\varrho_{n-1}, \frac{t}{k}\right)}, M_b\left(\varrho_{n-1}, \varrho_n, \frac{t}{k}\right) \right\}
\end{aligned}$$

$$\begin{aligned}
&= \min\left\{ \frac{M_b\left(\varrho_{n-1}, \varrho_n, \frac{t}{k}\right)\left[1 + M_b\left(\varrho_n, \varrho_{n+1}, \frac{t}{k}\right)\right]}{1 + M_b\left(\varrho_n, \varrho_{n+1}, \frac{t}{k}\right)}, \right.\\
&\quad \frac{M_b\left(\varrho_n, \varrho_{n+1}, \frac{t}{k}\right)\left[1 + M_b\left(\varrho_{n-1}, \varrho_n, \frac{t}{k}\right)\right]}{1 + M_b\left(\varrho_{n-1}, \varrho_n, \frac{t}{k}\right)},\\
&\quad \left. \frac{M_b\left(\varrho_{n-1}, \varrho_n, \frac{t}{k}\right)\left[2 + M_b\left(\varrho_{n-1}, \varrho_{n+1}, \frac{t}{k}\right)\right]}{1 + M_b\left(\varrho_{n-1}, \varrho_{n+1}, \frac{t}{k}\right) + M_b\left(\varrho_n, \varrho_n, \frac{t}{k}\right)}, M_b\left(\varrho_{n-1}, \varrho_n, \frac{t}{k}\right) \right\}\\
&= \min\left\{ M_b\left(\varrho_{n-1}, \varrho_n, \frac{t}{k}\right), M_b\left(\varrho_n, \varrho_{n+1}, \frac{t}{k}\right), \right.\\
&\quad \left. \frac{M_b\left(\varrho_{n-1}, \varrho_n, \frac{t}{k}\right)\left[2 + M_b\left(\varrho_{n-1}, \varrho_{n+1}, \frac{t}{k}\right)\right]}{1 + M_b\left(\varrho_{n-1}, \varrho_{n+1}, \frac{t}{k}\right) + 1}, M_b\left(\varrho_{n-1}, \varrho_n, \frac{t}{k}\right) \right\}
\end{aligned}$$

$$= \min\left\{M_b\left(\varrho_{n-1}, \varrho_n, \frac{t}{k}\right), M_b\left(\varrho_n, \varrho_{n+1}, \frac{t}{k}\right), M_b\left(\varrho_{n-1}, \varrho_n, \frac{t}{k}\right), M_b\left(\varrho_{n-1}, \varrho_n, \frac{t}{k}\right)\right\}$$

$$= \min\left\{M_b\left(\varrho_{n-1}, \varrho_n, \frac{t}{k}\right), M_b\left(\varrho_n, \varrho_{n+1}, \frac{t}{k}\right)\right\}.$$

So, we have

$$M_b(\varrho_n, \varrho_{n+1}, t) \geq \min\left\{M_b\left(\varrho_{n-1}, \varrho_n, \frac{t}{k}\right), M_b\left(\varrho_n, \varrho_{n+1}, \frac{t}{k}\right)\right\}. \tag{3.12}$$

Following two cases arise: **Case-1:**

If

$$\min\left\{M_b\left(\varrho_{n-1}, \varrho_n, \frac{t}{k}\right), M_b\left(\varrho_n, \varrho_{n+1}, \frac{t}{k}\right)\right\} = M_b\left(\varrho_n, \varrho_{n+1}, \frac{t}{k}\right),$$

then from (3.12),

$$M_b(\varrho_n, \varrho_{n+1}, t) \geq M_b\left(\varrho_n, \varrho_{n+1}, \frac{t}{k}\right).$$

So, there is nothing to prove by Lemma 3.1.6.

**Case-2:**

If

$$\min\left\{M_b\left(\varrho_{n-1}, \varrho_n, \frac{t}{k}\right), M_b\left(\varrho_n, \varrho_{n+1}, \frac{t}{k}\right)\right\} = M_b\left(\varrho_{n-1}, \varrho_n, \frac{t}{k}\right),$$

then from (3.12),

$$M_b(\varrho_n, \varrho_{n+1}, t) \geq M_b\left(\varrho_{n-1}, \varrho_n, \frac{t}{k}\right).$$

Continuing in this way, we have

$$M_b(\varrho_n, \varrho_{n+1}, t) \geq M_b\left(\varrho_0, \varrho_1, \frac{t}{k^n}\right).$$

One can complete the proof using the same procedure after inequality (3.7) as in Theorem 3.1.3. □

**Corollary 3.1.11.**

Let $(\mathcal{S}, M, *)$ be a $G$-complete $FMS$. Let a mapping $g\colon \mathcal{S} \to \mathcal{S}$ satisfies

$$M(g\varrho, g\eta, kt) \geq \min\left\{ \frac{M(\varrho, g\varrho, t)\,[1 + M(\eta, g\eta, t)]}{1 + M(g\varrho, g\eta, t)}, \frac{M(\eta, g\eta, t)\,[1 + M(\varrho, g\varrho, t)]}{1 + M(\varrho, \eta, t)}, \right.$$

$$\left. \frac{M(\varrho, g\varrho, t)\,[2 + M(\varrho, g\eta, t)]}{1 + M(\varrho, g\eta, t) + M(\eta, g\varrho, t)}, M(\varrho, \eta, t) \right\},$$

for all $\varrho, \eta \in \mathcal{S}$ and for some $k \in (0, 1)$. Then $g$ has a unique fixed point.

**Theorem 3.1.6.**

Let $(\mathcal{S}, M_b, *)$ be a $G$-complete $FbMS$ with $b \geq 1$. Consider a continuous function $\vartheta\colon [0, 1] \to [0, 1]$ and a mapping $g\colon \mathcal{S} \to \mathcal{S}$ satisfying

$$\begin{aligned} M_b(g\varrho, g\eta, kt) \\ \geq \vartheta\Bigg\{ \min\Bigg\{ \frac{M_b(\varrho, g\varrho, t)\,[1 + M_b(\eta, g\eta, t)]}{1 + M_b(g\varrho, g\eta, t)}, \frac{M_b(\eta, g\eta, t)\,[1 + M_b(\varrho, g\varrho, t)]}{1 + M_b(\varrho, \eta, t)}, \\ \frac{M_b(\varrho, g\varrho, t)\,[2 + M_b(\varrho, g\eta, t)]}{1 + M_b(\varrho, g\eta, t) + M_b(\eta, g\varrho, t)}, M_b(\varrho, \eta, t) \Bigg\} \Bigg\}, \end{aligned} \tag{3.13}$$

for all $\varrho, \eta \in \mathcal{S},\ k \in (0, \frac{1}{b})$. Then $g$ has a unique fixed point.

*Proof.*

Since $\vartheta$ is a continuous function and $\vartheta(a) > a$ for $0 < a < 1$, then from (3.13), we get

$$\begin{aligned} M_b(g\varrho, &g\eta, kt) \\ &\geq \vartheta\Bigg\{ \min\Bigg\{ \frac{M_b(\varrho, g\varrho, t)\,[1 + M_b(\eta, g\eta, t)]}{1 + M_b(g\varrho, g\eta, t)}, \frac{M_b(\eta, g\eta, t)\,[1 + M_b(\varrho, g\varrho, t)]}{1 + M_b(\varrho, \eta, t)}, \\ &\qquad \frac{M_b(\varrho, g\varrho, t)\,[2 + M_b(\varrho, g\eta, t)]}{1 + M_b(\varrho, g\eta, t) + M_b(\eta, g\varrho, t)}, \quad M_b(\varrho, \eta, t) \Bigg\} \Bigg\} \end{aligned}$$

$$\geq \min\left\{\frac{M_b(\varrho,g\varrho,t)\,[1+M_b(\eta,g\eta,t)]}{1+M_b(g\varrho,g\eta,t)}, \frac{M_b(\eta,g\eta,t)\,[1+M_b(\varrho,g\varrho,t)]}{1+M_b(\varrho,\eta,t)}, \right.$$

$$\left. \frac{M_b(\varrho,g\varrho,t)\,[2+M_b(\varrho,g\eta,t)]}{1+M_b(\varrho,g\eta,t)+M_b(\eta,g\varrho,t)}, M_b(\varrho,\eta,t)\right\}.$$

Now, by using Theorem 3.1.5, we get our required result. □

Following is the consequence of Theorem 3.1.6.

**Corollary 3.1.12.**

Let $(\mathcal{S}, M, *)$ be a $G$-complete $FMS$ and a continuous function $\vartheta\colon [0,1] \to [0,1]$. A self mapping $g\colon \mathcal{S} \to \mathcal{S}$ satisfies

$$M(g\varrho,g\eta,kt) \geq \vartheta\left\{\min\left\{\frac{M(\varrho,g\varrho,t)\,[1+M(\eta,g\eta,t)]}{1+M(g\varrho,g\eta,t)}, \frac{M(\eta,g\eta,t)\,[1+M(\varrho,g\varrho,t)]}{1+M(\varrho,\eta,t)}, \right.\right.$$
$$\left.\left. \frac{M(\varrho,g\varrho,t)\,[2+M(\varrho,g\eta,t)]}{1+M(\varrho,g\eta,t)+M(\eta,g\varrho,t)}, M(\varrho,\eta,t)\right\}\right\},$$

for all $\varrho,\eta \in \mathcal{S},\ k \in (0,1)$. Then $g$ has a unique fixed point.

## 3.2 Application

In this section, some applications are given regarding to our results. Define a function $\Omega\colon [0,\infty) \to [0,\infty)$ by

$$\Omega(t) = \int_0^t \psi(t)dt \quad \forall\ t > 0, \tag{3.14}$$

where $\psi(t)$ is a continuous and non-decreasing function. Moreover, $\psi(0) = 0$ and $\psi(\epsilon) > 0$ for all $\epsilon > 0$ .

**Theorem 3.2.1.**

Let $(\mathcal{S}, M_b, *)$ be a $G$-complete $FbMS$ with $b \geq 1$. Let a mapping $g\colon \mathcal{S} \to \mathcal{S}$ satisfies

$$\int_0^{M_b(g\varrho,g\eta,kt)} \psi(t)dt \geq \int_0^{\alpha(\varrho,\eta,t)} \psi(t)dt, \tag{3.15}$$

where,

$$\alpha(\varrho,\eta,t)=\min\left\{\frac{M_b(\eta,g\eta,t)\left[1+M_b(\varrho,g\varrho,t)+M_b(\eta,g\varrho,t)\right]}{2+M_b(\varrho,\eta,t)},M_b(\varrho,\eta,t)\right\},$$

for all $\varrho,\eta\in\mathcal{S},\ k\in\left(0,\frac{1}{b}\right)$. Then $g$ has a unique fixed point.

*Proof.*

In account of (3.14), (3.15) implies

$$\Omega(M_b(g\varrho,g\eta,kt))\geq\Omega(\alpha(\varrho,\eta,t)).$$

Since $\Omega$ is non-decreasing and continuous, so it follows that

$$M_b(g\varrho,g\eta,kt)\geq\alpha(\varrho,\eta,t).$$

Hence, the rest of the proof follows from Theorem 3.1.3. □

**Theorem 3.2.2.**

Let $(\mathcal{S},M_b,*)$ be a $G$-complete $FbMS$ with $b\geq 1$. Let a mapping $g\colon\mathcal{S}\to\mathcal{S}$ satisfies

$$\int_0^{M_b(g\varrho,g\eta,kt)}\psi(t)dt\geq\vartheta\left\{\int_0^{\alpha(\varrho,\eta,t)}\psi(t)dt\right\},\tag{3.16}$$

where,

$$\alpha(\varrho,\eta,t)=\min\left\{\frac{M_b(\eta,g\eta,t)\left[1+M_b(\varrho,g\varrho,t)+M_b(\eta,g\varrho,t)\right]}{2+M_b(\varrho,\eta,t)},M_b(\varrho,\eta,t)\right\}$$

for all $\varrho,\eta\in\mathcal{S},\ k\in(0,\frac{1}{b}),\ \psi\in\Omega$. Then $g$ has a unique fixed point.

*Proof.*

In account of (3.14), (3.16) implies

$$\Omega(M_b(g\varrho,g\eta,kt))\geq\vartheta(\Omega(\alpha(\varrho,\eta,t))).$$

As we have defined in Theorem 3.1.4 that for $0 < a < 1$ , $\vartheta(a) > a$.

So, we have

$$\begin{aligned}\Omega(M_b(g\varrho, g\eta, kt)) &\geq \vartheta(\Omega(\alpha(\varrho,\eta,t))) \\ &\geq \Omega(\alpha(\varrho,\eta,t)).\end{aligned}$$

Since $\Omega$ is non-decreasing and continuous, so, we have

$$M_b(g\varrho, g\eta, kt) \geq \alpha(\varrho,\eta,t).$$

The rest of proof follows immediately from Theorem 3.1.3. □

**Theorem 3.2.3.**

Let $(\mathcal{S}, M_b, *)$ be a $G$-complete $FbMS$ with $b \geq 1$. Let a mapping $g\colon \mathcal{S} \to \mathcal{S}$ satisfies

$$\int_0^{M_b(g\varrho,g\eta,kt)} \psi(t)dt \geq \int_0^{\alpha(\varrho,\eta,t)} \psi(t)dt, \tag{3.17}$$

where,

$$\begin{aligned}\alpha(\varrho,\eta,t) = \min\Bigg\{ &\frac{M_b(\varrho, g\varrho, t)\,[1 + M_b(\eta, g\eta, t)]}{1 + M_b(g\varrho, g\eta, t)}, \frac{M_b(\eta, g\eta, t)\,[1 + M_b(\varrho, g\varrho, t)]}{1 + M_b(\varrho,\eta,t)}, \\ &\frac{M_b(\varrho, g\varrho, t)\,[2 + M_b(\varrho, g\eta, t)]}{1 + M_b(\varrho, g\eta, t) + M_b(\eta, g\varrho, t)}, M_b(\varrho,\eta,t)\Bigg\},\end{aligned}$$

for all $\varrho, \eta \in \mathcal{S},\ k \in (0, \frac{1}{b})$. Then $g$ has a unique fixed point.

*Proof.*

In account of (3.14), (3.17) implies

$$\Omega(M_b(g\varrho, g\eta, kt)) \geq \Omega(\alpha(\varrho,\eta,t)).$$

Since $\Omega$ is non-decreasing and continuous, so we have

$$M_b(g\varrho, g\eta, kt) \geq \alpha(\varrho,\eta,t).$$

By using Theorem 3.1.5, we get the required result. □

**Theorem 3.2.4.**

Let $(\mathcal{S}, M_b, *)$ be a $G$-complete $FbMS$ with $b \geq 1$. Let a mapping $g\colon \mathcal{S} \to \mathcal{S}$ satisfies

$$\int_0^{M_b(g\varrho, g\eta, kt)} \psi(t)dt \geq \vartheta\left\{\int_0^{\alpha(\varrho,\eta,t)} \psi(t)dt\right\}, \tag{3.18}$$

where

$$\alpha(\varrho,\eta,t) = \min\left\{\frac{M_b(\varrho, g\varrho, t)\,[1 + M_b(\eta, g\eta, t)]}{1 + M_b(g\varrho, g\eta, t)}, \frac{M_b(\eta, g\eta, t)\,[1 + M_b(\varrho, g\varrho, t)]}{1 + M_b(\varrho, \eta, t)},\right.$$
$$\left.\frac{M_b(\varrho, g\varrho, t)\,[2 + M_b(\varrho, g\eta, t)]}{1 + M_b(\varrho, g\eta, t) + M_b(\eta, g\varrho, t)}, M_b(\varrho, \eta, t)\right\},$$

for all $\varrho, \eta \in \mathcal{S},\ k \in (0, \frac{1}{b}),\ \psi \in \Omega$. Then $g$ has a unique fixed point.

*Proof.*

The proof follows immediately from Theorem 3.1.6. □

## 3.3 Conclusion

Recently, Nădăban [85] highlighted the properties and usefulness of fuzzy Euclidean normed spaces and $FbMS$ in solving problems in various sciences. The author has prepared a ground to extend the theory of fixed points in these spaces. In the conclusion of [85], the author suggested to establish certain fixed point theorem in $FbMS$. In this chapter, the version of BCP in the setting of $FbMS$ has been established and an example is provided to demonstrate our main result. The main Grabiec's result [49] has been generalised in this way. Furthermore, an analogue of Hicks and Rhoad's fixed-point theorem [21] in the setting of $FbMS$ has been proved by restricting the contraction mapping to the elements in the orbit of a point in $FbMS$. Our results may be of interest for the readers/researchers in some specialized area of computer science and information technology like computational intelligence method and advance decision support systems to solve problems with fixed points involving some kind of distance between programmes in order to assess the complexity of an algorithm.

# Chapter 4

# Geraghty Type Contractions in Fuzzy $b$-metric Spaces

In 1973, Banach contraction principle was generalized by Geraghty [34]. After that various authors proved many fixed point results by using Geraghty type contractive mappings. Recently, Faraji [33] provided certain fixed point results in $b$-metric spaces by using Geraghty type contractions.

In Section 4.1 of this chapter, some interesting fixed point results for Geraghty-type contraction in $G$-complete $FbMS$ are established. Particularly, the extension of main result of Grabiec [49] is established in the first theorem. Second result is the extention of the main result of Faraji et.al [33] and other results are the generlization of the results of Alsulami et.al [89] in the setting of $G$-complete fuzzy $b$-metric space. These results are illustrated by examples. Finally the Section 4.1 is concluded by an application of these results.

Section 4.3 consists some common fixed point results for Geraghty-type contraction in $G$-complete $FbMS$. The first two results are the extensions of the main results of Faraji [33] and third result is the generalization of the results of Gupta et. al. [57] in the setting of $G$-complete $FbMS$. An example is provided to illustrate our main result and an application is also presented of the results obtained.

## 4.1 Geraghty Type Contraction

Following [84], let the class of all functions $\beta\colon [0,\infty) \to [0, \frac{1}{b})$, for $b > 1$ be denoted by $F_b$ and satisfying the condition

$$\beta(\varrho_n) \to \frac{1}{b} \text{ as } n \to \infty \quad \Rightarrow \quad \varrho_n \to 0 \text{ as } n \to \infty.$$

That is,

$$F_b = \left\{\beta\colon [0,\infty) \to [0, \frac{1}{b}) \Big| \lim_{n\to\infty} \beta(\varrho_n) = \frac{1}{b} \quad \Rightarrow \quad \lim_{n\to\infty} \varrho_n = 0\right\}.$$

For example, the function $\beta\colon [0,\infty) \to [0, \frac{1}{b})$, for $b > 1$ defined by

$$\beta(\varrho_n) = \frac{1}{b} - \varrho_n,$$

where the sequence $\{\varrho_n\}$ is taken as

$$\varrho_n = \frac{1}{b^n}.$$

Note that

$$\begin{aligned}\lim_{n\to\infty} \beta(\varrho_n) &= \lim_{n\to\infty} \left(\frac{1}{b} - \frac{1}{b^n}\right) \\ &= \frac{1}{b}.\end{aligned}$$

Also

$$\lim_{n\to\infty} \varrho_n = \lim_{n\to\infty} \frac{1}{b^n} = 0.$$

**Lemma 4.1.1.**

Let $(\mathcal{S}, M_b, *)$ be a $G$-complete $FbMS$ and

$$M_b\left(\varrho, \eta, \beta(M(\varrho, \eta, t))t\right) \geq M_b\left(\varrho, \eta, t\right),$$

for all $\varrho, \eta \in \mathcal{S},\ t > 0$ and $\beta \in F_b$, then $\varrho = \eta$.

*Proof.*

The proof follows from Lemma 3.1.6. □

In the context of $G$-complete $FbMS$, we now develop the following fixed point result, analogous to [49, Theorem 1].

**Theorem 4.1.1.**

Let $(\mathcal{S}, M_b, *)$ be a $G$-complete $FbMS$. Let a mapping $g\colon \mathcal{S} \to \mathcal{S}$ satisfies

$$M_b(g\varrho, g\eta, \beta(M_b(\varrho, \eta, t))t) \geq M_b(\varrho, \eta, t), \tag{4.1}$$

for all $\varrho, \eta \in \mathcal{S}$ and $\beta \in F_b$. Then $g$ has a unique fixed point.

*Proof.*

Let $\varrho_0 \in \mathcal{S}$ and choose an iterative sequence $\{\varrho_n\}$; $(n \in \mathbb{N})$ such as

$$\varrho_n = g^n \varrho_0.$$

To begin, keep in mind that the contractive condition (4.1) is applied repeatedly, we have for all $n \in \mathbb{N}$ and $t > 0$,

$$\begin{aligned}
M_b(\varrho_n, \varrho_{n+1}, t) &= M_b(g\varrho_{n-1}, g\varrho_n, t) \\
&\geq M_b\left(\varrho_{n-1}, \varrho_n, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right) \\
&\geq M_b\left(\varrho_{n-2}, \varrho_{n-1}, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t)).\beta(M_b(\varrho_{n-2}, \varrho_{n-1}, t))}\right) \\
&\vdots \\
&\geq M_b\left(\varrho_0, \varrho_1, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t)).\beta(M_b(\varrho_{n-2}, \varrho_{n-1}, t)).\ldots.\beta(M_b(\varrho_0, \varrho_1, t))}\right)
\end{aligned}$$

So, we have

$$\begin{aligned}
&M_b(\varrho_n, \varrho_{n+1}, t) \\
&\geq M_b\left(\varrho_0, \varrho_1, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t)).\beta(M_b(\varrho_{n-2}, \varrho_{n-1}, t))\ldots\beta(M_b(\varrho_0, \varrho_1, t))}\right).
\end{aligned} \tag{4.2}$$

For any $q \in \mathbb{N}$, letting $t = \frac{t}{q} + \frac{t}{q} + \ldots + \frac{t}{q}$ and using $FBM4$ repeatedly,

$$\begin{aligned}
&M_b(\varrho_n, \varrho_{n+p}, t)\\
&\quad \geq M_b\left(\varrho_n, \varrho_{n+1}, \frac{t}{qb}\right) * M_b\left(\varrho_{n+1}, \varrho_{n+2}, \frac{t}{qb^2}\right) * \ldots * M_b\left(\varrho_{n+p-1}, \varrho_{n+p}, \frac{t}{qb^{n+p}}\right).
\end{aligned}$$

Using (4.2) in above equation to get,

$$\begin{aligned}
&M_b(\varrho_n, \varrho_{n+p}, t)\\
&\geq M_b\left(\varrho_0, \varrho_1, \frac{t}{qb\beta(M_b(\varrho_{n-1}, \varrho_n, t)).\beta(M_b(\varrho_{n-2}, \varrho_{n-1}, t)) \ldots \beta(M_b(\varrho_0, \varrho_1, t))}\right)\\
&* M_b\left(\varrho_0, \varrho_1, \frac{t}{qb^2\beta(M_b(\varrho_n, \varrho_{n+1}, t)).\beta(M_b(\varrho_{n-1}, \varrho_n, t)) \ldots \beta(M_b(\varrho_0, \varrho_1, t))}\right)\\
&* \ldots * M_b\left(\varrho_0, \varrho_1, \frac{t}{qb^{n+p}\beta(M_b(\varrho_{n+q}, \varrho_{n+q-1}, t)) \ldots \beta(M_b(\varrho_0, \varrho_1, t))}\right)\\
&\geq M_b\left(\varrho_0, \varrho_1, \frac{b^{n-1}t}{q}\right) * M_b\left(\varrho_0, \varrho_1, \frac{b^{n-1}t}{q}\right) * \ldots * M_b\left(\varrho_0, \varrho_1, \frac{b^{n-1}t}{q}\right).
\end{aligned}$$

Taking limit as $n \to \infty$, we get

$$\lim_{n\to\infty} M_b(\varrho_n, \varrho_{n+p}, t) = 1 * 1 * \ldots * 1 = 1.$$

This shows that $\{\varrho_n\}$ is a $G$-Cauchy sequence. So, by $G$-completeness of the space $(\mathcal{S}, M_b, *)$, there is an element $\varrho \in \mathcal{S}$ such as

$$\lim_{n\to\infty} \varrho_n = \varrho.$$

We continue as follows to prove that $\varrho$ is a fixed point of $g$.

$$\begin{aligned}
M_b(g\varrho, \varrho, t) &\geq M_b\left(g\varrho, g\varrho_n, \frac{t}{2b}\right) * M_b\left(g\varrho_n, \varrho, \frac{t}{2b}\right)\\
&\geq M_b\left(\varrho, \varrho_n, \frac{t}{2b\beta((M_b(\varrho, \varrho_n, t))}\right) * M_b\left(\varrho_{n+1}, \varrho_n, \frac{t}{2b}\right)\\
&= 1 * 1 = 1.
\end{aligned}$$

That is,

$$M_b(g\varrho, \varrho, t) = 1.$$

This proves that

$$g\varrho = \varrho.$$

Thus $\varrho$ is a fixed point.

**Uniqueness:**

Suppose that $g$ has another fixed point $\zeta \in \mathcal{S}$. That is, $g\zeta = \zeta$, then

$$\begin{aligned} M_b(\zeta, \varrho, t) &= M_b(g\zeta, g\varrho, t) \\ &\geq M_b\left(\zeta, \varrho, \frac{t}{\beta(M_b(\zeta, \varrho, t))}\right) \\ &= M_b\left(g\zeta, g\varrho, \frac{t}{\beta(M_b(\zeta, \varrho, t))}\right) \\ &\geq M_b\left(\zeta, \varrho, \frac{t}{\beta(M_b(\zeta, \varrho, t))^2}\right) \\ &\vdots \\ &\geq M_b\left(\zeta, \varrho, \frac{t}{\beta(M_b(\zeta, \varrho, t))^n}\right) \\ &= M_b\left(\zeta, \varrho, b^n t\right) \\ &\longrightarrow 1 \,\text{as}\, n \to \infty. \end{aligned}$$

Thus $\quad \varrho = \zeta.$

Hence $g$ has unique fixed point. □

Theorem 4.1.1 is illustrated with the following example.

**Example 4.1.2.**

Let $\mathcal{S} = [0, 1]$. Define a function $M_b \colon \mathcal{S} \times \mathcal{S} \times [0, \infty) \to [0, 1]$ by

$$M_b(\varrho, \eta, t) = \frac{t}{t + (\varrho - \eta)^2}.$$

One can easily show that $(\mathcal{S}, M_b, *)$ is a $G$-complete $FbMS$ with $b = 2$.

Consider a mapping $g \colon \mathcal{S} \to \mathcal{S}$ defined by

$$g(\varrho) = \frac{\varrho}{2(1 + \varrho)}.$$

Moreover, set the map $\beta\colon [0,1] \to [0, \frac{1}{2})$ as $\beta(\varrho) = \frac{1}{4}$. Clearly $\beta \in F_2$.

Now, for all $\varrho, \eta \in \mathcal{S}$ and $t > 0$, we have

$$\begin{aligned} M_b(g\varrho, g\eta, \beta(M_b(\varrho,\eta,t))t) &= \frac{\beta(M_b(\varrho,\eta,t))t}{\beta(M_b(\varrho,\eta,t))t + \left(\frac{\varrho}{4(1+\varrho)} - \frac{\eta}{4(1+\eta)}\right)^2} \\ &= \frac{\frac{1}{4}t}{\frac{1}{4}t + \left(\frac{\varrho}{4(1+\varrho)} - \frac{\eta}{4(1+\eta)}\right)^2} \\ &= \frac{\frac{1}{4}t}{\frac{1}{4}t + \frac{1}{4}\left(\frac{\varrho}{1+\varrho} - \frac{\eta}{1+\eta}\right)^2} \\ &= \frac{t}{t + \frac{(\varrho-\eta)^2}{(1+\varrho)^2(1+\eta)^2}}. \end{aligned}$$

Since

$$\begin{aligned} \frac{(\varrho-\eta)^2}{(1+\varrho)^2(1+\eta)^2} &\leq (\varrho-\eta)^2 \\ t + \frac{(\varrho-\eta)^2}{(1+\varrho)^2(1+\eta)^2} &\leq t + (\varrho-\eta)^2 \\ \frac{1}{t + \frac{(\varrho-\eta)^2}{(1+\varrho)^2(1+\eta)^2}} &\geq \frac{1}{t+(\varrho-\eta)^2} \\ \frac{t}{t + \frac{(\varrho-\eta)^2}{(1+\varrho)^2(1+\eta)^2}} &\geq \frac{t}{t+(\varrho-\eta)^2}. \end{aligned}$$

This implies that

$$M_b(g\varrho, g\eta, \beta(\varrho,\eta,t)t) \geq M_b(\varrho,\eta,t).$$

Hence all the conditions of Theorem 4.1.1 are satisfied and $g0 = 0$ is unique fixed point of $g$.

**Remark 4.1.3.**

1. Taking $\beta(\varrho) = k \ \forall \ \varrho \in [0,1]$ for some $k \in [0, \frac{1}{b})$, then Theorem 4.1.1 becomes the well-known BCP for $FbMS$ established in Theorem 3.1.1.

2. Similarly, by setting $b = 1$ and $\beta(\varrho) = k$ in Theorem 4.1.1, the main result of Grabiec [49, Theorem 5] is obtained.

Now, we extend the main result of Faraji et al. [33] in our setting as follows;

**Theorem 4.1.2.**

Let $(\mathcal{S}, M_b, *)$ be a $G$-complete $FbMS$. Let a mapping $g \colon \mathcal{S} \to \mathcal{S}$ satisfies

$$M_b(g\varrho, g\eta, \beta(M_b(\varrho,\eta,t))t) \geq \min\Big\{M_b(\varrho,\eta,t), M_b(\varrho, g\varrho, t), M_b(\eta, g\eta, t), \Big(M_b(\varrho, g\eta, 2bt) * M_b(\eta, g\varrho, 2bt)\Big)\Big\}, \tag{4.3}$$

for all $\varrho, \eta \in \mathcal{S}$ and $\beta \in F_b$. Then $g$ has a unique fixed point.

*Proof.*

Starting the same way as in Theorem 4.1.1, we have

$$\begin{aligned}
&M_b(\varrho_n, \varrho_{n+1}, t) = M_b(g\varrho_{n-1}, g\varrho_n, t)\\
&\geq \min\Bigg\{M_b\left(\varrho_{n-1}, \varrho_n, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right),\\
&\quad M_b\left(\varrho_{n-1}, g\varrho_{n-1}, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right), M_b\left(\varrho_n, g\varrho_n, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right),\\
&\quad \left(M_b\left(\varrho_{n-1}, g\varrho_n, \frac{2bt}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right) * M_b\left(\varrho_n, g\varrho_{n-1}, \frac{2bt}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right)\right)\Bigg\}
\end{aligned}$$

$$\begin{aligned}
&\geq \min\Bigg\{M_b\left(\varrho_{n-1}, \varrho_n, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right),\\
&\quad M_b\left(\varrho_{n-1}, \varrho_n, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right), M_b\left(\varrho_n, \varrho_{n+1}, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right),\\
&\quad \left(M_b\left(\varrho_{n-1}, \varrho_{n+1}, \frac{2bt}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right) * M_b\left(\varrho_n, \varrho_n, \frac{2bt}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right)\right)\Bigg\}
\end{aligned}$$

$$\begin{aligned}
&\geq \min\Bigg\{M_b\left(\varrho_{n-1}, \varrho_n, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right), M_b\left(\varrho_{n-1}, \varrho_n, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right),\\
&\quad M_b\left(\varrho_n, \varrho_{n+1}, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right), \left(M_b\left(\varrho_{n-1}, \varrho_{n+1}, \frac{2bt}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right) * 1\right)\Bigg\}
\end{aligned}$$

$$\geq \min\Bigg\{M_b\left(\varrho_{n-1},\varrho_n,\frac{t}{\beta(M_b(\varrho_{n-1},\varrho_n,t))}\right),M_b\left(\varrho_{n-1},\varrho_n,\frac{t}{\beta(M_b(\varrho_{n-1},\varrho_n,t))}\right),$$
$$M_b\left(\varrho_n,\varrho_{n+1},\frac{t}{\beta(M_b(\varrho_{n-1},\varrho_n,t))}\right),M_b\left(\varrho_{n-1},\varrho_{n+1},\frac{2bt}{\beta(M_b(\varrho_{n-1},\varrho_n,t))}\right)\Bigg\}$$
$$\geq \min\Bigg\{M_b\left(\varrho_{n-1},\varrho_n,\frac{t}{\beta(M_b(\varrho_{n-1},\varrho_n,t))}\right),M_b\left(\varrho_{n-1},\varrho_n,\frac{t}{\beta(M_b(\varrho_{n-1},\varrho_n,t))}\right),$$
$$M_b\left(\varrho_n,\varrho_{n+1},\frac{t}{\beta(M_b(\varrho_{n-1},\varrho_n,t))}\right),\left(M_b\left(\varrho_{n-1},\varrho_n,\frac{t}{\beta(M_b(\varrho_{n-1},\varrho_n,t))}\right)*\right.$$
$$\left.M_b\left(\varrho_n,\varrho_{n+1},\frac{t}{\beta(M_b(\varrho_{n-1},\varrho_n,t))}\right)\right)\Bigg\}.$$

So, we have

$$M_b(\varrho_n,\varrho_{n+1},t)$$
$$\geq \min\Bigg\{M_b\left(\varrho_{n-1},\varrho_n,\frac{t}{\beta(M_b(\varrho_{n-1},\varrho_n,t))}\right),M_b\left(\varrho_n,\varrho_{n+1},\frac{t}{\beta(M_b(\varrho_{n-1},\varrho_n,t))}\right),$$
$$\left(M_b\left(\varrho_{n-1},\varrho_n,\frac{t}{\beta(M_b(\varrho_{n-1},\varrho_n,t))}\right)*M_b\left(\varrho_n,\varrho_{n+1},\frac{t}{\beta(M_b(\varrho_{n-1},\varrho_n,t))}\right)\right)\Bigg\}. \tag{4.4}$$

If

$$\min\left\{M_b\left(\varrho_n,\varrho_{n+1},\frac{t}{\beta(M_b(\varrho_{n-1},\varrho_n,t))}\right),M_b\left(\varrho_{n-1},\varrho_n,\frac{t}{\beta(M_b(\varrho_{n-1},\varrho_n,t))}\right)\right\}$$
$$=M_b\left(\varrho_n,\varrho_{n+1},\frac{t}{\beta(M_b(\varrho_{n-1},\varrho_n,t))}\right),$$

then (4.4) implies that

$$M_b(\varrho_n,\varrho_{n+1},t)\geq M_b\left(\varrho_n,\varrho_{n+1},\frac{t}{\beta(M_b(\varrho_{n-1},\varrho_n,t))}\right),$$

and hence there is nothing to prove by Lemma 4.1.1.

If

$$\min\left\{M_b\left(\varrho_n,\varrho_{n+1},\frac{t}{\beta(M_b(\varrho_{n-1},\varrho_n,t))}\right),M_b\left(\varrho_{n-1},\varrho_n,\frac{t}{\beta(M_b(\varrho_{n-1},\varrho_n,t))}\right)\right\}$$
$$=M_b\left(\varrho_{n-1},\varrho_n,\frac{t}{\beta(M_b(\varrho_{n-1},\varrho_n,t))}\right),$$

then from (4.4), we have

$$M_b(\varrho_n, \varrho_{n+1}, t) \geq M_b\left(\varrho_{n-1}, \varrho_n, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right)$$
$$\geq M_b\left(\varrho_{n-2}, \varrho_{n-1}, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t)).\beta(M_b(\varrho_{n-2}, \varrho_{n-1}, t))}\right)$$
$$\geq M_b\left(\varrho_{n-3}, \varrho_{n-2}, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t)).\beta(M_b(\varrho_{n-2}, \varrho_{n-1}, t)).\beta(M_b(\varrho_{n-3}, \varrho_{n-2}, t))}\right)$$

In a similar way, we can get

$$M_b(\varrho_n, \varrho_{n+1}, t) \geq M_b\left(\varrho_0, \varrho_1, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t)).\beta(M_b(\varrho_{n-2}, \varrho_{n-1}, t)) \ldots \beta(M_b(\varrho_0, \varrho_1, t))}\right). \tag{4.5}$$

One can now complete the proof by following the same procedure as used after inequality (4.2) of Theorem 4.1.1. □

An immediate consequence of Theorem 4.1.2 is given below.

**Corollary 4.1.4.**
Let $(\mathcal{S}, M, *)$ be a $G$-complete $FMS$ and a mapping $g\colon \mathcal{S} \to \mathcal{S}$ satisfies

$$M(g\varrho, g\eta, \beta(M(\varrho, \eta, t))t) \geq \min\left\{M(\varrho, \eta, t), M(\varrho, g\varrho, t), M(\eta, g\eta, t), \Big(M(\varrho, g\eta, 2t) * M(\eta, g\varrho, 2t)\Big)\right\},$$

for all $\varrho, \eta \in \mathcal{S}$ and $\beta \in F_1$. Then $g$ has a unique fixed point.

**Theorem 4.1.3.**
Let $(\mathcal{S}, M_b, *)$ be a $G$-complete $FbMS$. Let a mapping $g\colon \mathcal{S} \to \mathcal{S}$ satisfies

$$M_b(g\varrho, g\eta, \beta(M_b(\varrho, \eta, t))t) \geq \min\left\{M_b(g\varrho, g\eta, t), M_b(\varrho, g\varrho, t), M_b(\eta, g\eta, t), M_b(\varrho, \eta, t)\right\} \tag{4.6}$$

for all $\varrho, \eta \in \mathcal{S}$ and $\beta \in F_b$. Then $g$ has a unique fixed point.

*Proof.*

Starting in the same way as in Theorem 4.1.1, we have

$$
\begin{aligned}
&M_b(\varrho_n, \varrho_{n+1}, t) = M_b(g\varrho_{n-1}, g\varrho_n, t)\\
&\geq \min\Bigg\{ M_b\left(g\varrho_{n-1}, g\varrho_n, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right), M_b\left(\varrho_{n-1}, g\varrho_{n-1}, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right),\\
&\qquad M_b\left(\varrho_n, g\varrho_n, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right), M_b\left(\varrho_{n-1}, \varrho_n, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right)\Bigg\}\\
&\geq \min\Bigg\{ M_b\left(\varrho_n, \varrho_{n+1}, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right), M_b\left(\varrho_{n-1}, \varrho_n, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right),\\
&\qquad M_b\left(\varrho_n, \varrho_{n+1}, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right), M_b\left(\varrho_{n-1}, \varrho_n, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right)\Bigg\}\\
&\geq \min\Bigg\{ M_b\left(\varrho_n, \varrho_{n+1}, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right), M_b\left(\varrho_{n-1}, \varrho_n, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right)\Bigg\}.
\end{aligned}
\tag{4.7}
$$

By adopting the same procedure as in Theorem 4.1.2 after inequality (4.4), we can complete the proof. □

**Remark 4.1.5.**

If we take

$$
\min\Bigg\{ M_b(g\varrho, g\eta, t), M_b(\varrho, g\varrho, t), M_b(\eta, g\eta, t), M_b(\varrho, \eta, t) \Bigg\} = M_b(\varrho, \eta, t)
$$

in Theorem 4.1.3, then it reduces to Theorem 4.1.1.

The immediate consequence of Theorem 4.1.3 is as follows:

**Corollary 4.1.6.**

Let $(\mathcal{S}, M, *)$ be a $G$-complete $FMS$ . Let $g$ be a self map on $\mathcal{S}$ satisfying the condition

$$
M(g\varrho, g\eta, \beta(M(\varrho, \eta, t))t) \geq \min\Bigg\{ M(g\varrho, g\eta, t), M(\varrho, g\varrho, t), M(\eta, g\eta, t), M(\varrho, \eta, t) \Bigg\},
$$

for all $\varrho, \eta \in \mathcal{S}$ and $\beta \in F_1$. Then $T$ has a unique fixed point.

**Theorem 4.1.4.**

Let $(\mathcal{S}, M_b, *)$ be a $G$-complete $FbMS$. Let a mapping $g\colon \mathcal{S} \to \mathcal{S}$ satisfies

$$M_b(g\varrho, g\eta, \beta(M_b(\varrho, \eta, t))t) \geq \frac{\alpha(\varrho, \eta, t)}{\max\left\{M_b(\varrho, g\varrho, t), M_b(\eta, g\eta, t)\right\}}, \tag{4.8}$$

where

$$\alpha(\varrho, \eta, t) = \min\left\{M_b(g\varrho, g\eta, t).M_b(\varrho, \eta, t), M_b(\varrho, g\varrho, t).M_b(\eta, g\eta, t)\right\},$$

for all $\varrho, \eta \in \mathcal{S}$ and $\beta \in F_b$. Then $g$ has a unique fixed point.

*Proof.*

Starting in same way as in Theorem 4.1.1, we have

$$\begin{aligned} &M_b(\varrho_n, \varrho_{n+1}, t) = M_b(g\varrho_{n-1}, g\varrho_n, t) \\ &\geq \frac{\alpha\left(\varrho_n, \varrho_{n+1}, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right)}{\max\left\{M_b\left(\varrho_{n-1}, g\varrho_{n-1}, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right), M_b\left(\varrho_n, g\varrho_n, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right)\right\}}. \end{aligned} \tag{4.9}$$

Now,

$$\begin{aligned} &\alpha\left(\varrho_n, \varrho_{n+1}, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right) \\ &\quad = \min\left\{M_b\left(g\varrho_{n-1}, g\varrho_n, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right).M_b\left(\varrho_{n-1}, \varrho_n, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right),\right. \\ &\qquad\qquad \left. M_b\left(\varrho_{n-1}, g\varrho_{n-1}, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right).M_b\left(\varrho_n, g\varrho_n, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right)\right\} \\ &\quad = \min\left\{M_b\left(\varrho_n, \varrho_{n+1}, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right).M_b\left(\varrho_{n-1}, \varrho_n, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right),\right. \\ &\qquad\qquad \left. M_b\left(\varrho_{n-1}, \varrho_n, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right).M_b\left(\varrho_n, \varrho_{n+1}, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right)\right\} \\ &\quad = M_b\left(\varrho_{n-1}, \varrho_n, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right).M_b\left(\varrho_n, \varrho_{n+1}, \frac{t}{\beta(\varrho_{n-1}, \varrho_n, t)}\right). \end{aligned} \tag{4.10}$$

Using (4.10) in (4.9), we get

$$M_b(\varrho_n,\varrho_{n+1},t) \geq \frac{M_b\left(\varrho_n,\varrho_{n+1},\frac{t}{\beta(M_b(\varrho_{n-1},\varrho_n,t))}\right).M_b\left(\varrho_{n-1},\varrho_n,\frac{t}{\beta(M_b(\varrho_{n-1},\varrho_n,t))}\right)}{\max\left\{M_b\left(\varrho_{n-1},\varrho_n,\frac{t}{\beta(M_b(\varrho_{n-1},\varrho_n,t))}\right),M_b\left(\varrho_n,\varrho_{n+1},\frac{t}{\beta(M_b(\varrho_{n-1},\varrho_n,t))}\right)\right\}}. \tag{4.11}$$

If

$$\max\left\{M_b\left(\varrho_n,\varrho_{n+1},\frac{t}{\beta(M_b(\varrho_{n-1},\varrho_n,t))}\right),M_b\left(\varrho_{n-1},\varrho_n,\frac{t}{\beta(M_b(\varrho_{n-1},\varrho_n,t))}\right)\right\} = M_b\left(\varrho_{n-1},\varrho_n,\frac{t}{\beta(M_b(\varrho_{n-1},\varrho_n,t))}\right),$$

then (4.11) implies

$$M_b(\varrho_n,\varrho_{n+1},t) \geq M_b\left(\varrho_n,\varrho_{n+1},\frac{t}{\beta(M_b(\varrho_{n-1},\varrho_n,t))}\right).$$

So, there is nothing to prove by Lemma 4.1.1.

If

$$\max\left\{M_b\left(\varrho_n,\varrho_{n+1},\frac{t}{\beta(M_b(\varrho_{n-1},\varrho_n,t))}\right),M_b\left(\varrho_{n-1},\varrho_n,\frac{t}{\beta(M_b(\varrho_{n-1},\varrho_n,t))}\right)\right\} = M_b\left(\varrho_n,\varrho_{n+1},\frac{t}{\beta(M_b(\varrho_{n-1},\varrho_n,t))}\right),$$

then from (4.11), we have

$$\begin{aligned}
&M_b(\varrho_n,\varrho_{n+1},t) \geq M_b\left(\varrho_{n-1},\varrho_n,\frac{t}{\beta(M_b(\varrho_{n-1},\varrho_n,t))}\right)\\
&\geq M_b\left(\varrho_{n-2},\varrho_{n-1},\frac{t}{\beta(M_b(\varrho_{n-1},\varrho_n,t)).\beta(M_b(\varrho_{n-2},\varrho_{n-1},t))}\right)\\
&\geq M_b\left(\varrho_{n-3},\varrho_{n-2},\frac{t}{\beta(M_b(\varrho_{n-1},\varrho_n,t)).\beta(M_b(\varrho_{n-2},\varrho_{n-1},t)).\beta(M_b(\varrho_{n-3},\varrho_{n-2},t))}\right)\\
&\vdots\\
&\geq M_b\left(\varrho_0,\varrho_1,\frac{t}{\beta(M_b(\varrho_{n-1},\varrho_n,t)).\beta(M_b(\varrho_{n-2},\varrho_{n-1},t))\ldots\beta(M_b(\varrho_0,\varrho_1,t))}\right).
\end{aligned}$$

By following the same procedure as in Theorem 4.1.1 after inequality (4.2), we can complete the proof. □

The immediate consequence of Theorem 4.1.4 is as follows.

**Corollary 4.1.7.**

Let $(\mathcal{S}, M, *)$ be a $G$-complete $FMS$ . Let $g$ be a self map on $X$ satisfying the condition

$$M(g\varrho, g\eta, \beta(M(\varrho,\eta,t))t) \geq \frac{\alpha(\varrho,\eta,t)}{\max\Big\{M(\varrho, g\varrho, t), M(\eta, g\eta, t)\Big\}},$$

where

$$\alpha(\varrho,\eta,t) = \min\left\{M(g\varrho, g\eta, t).M(\varrho,\eta,t), M(\varrho, g\varrho, t).M(\eta, g\eta, t)\right\},$$

for all $\varrho, \eta \in \mathcal{S}$ and $\beta \in F_1$. Then $g$ has a unique fixed point.

Following result is the generalization of Theorem 2.3 of Alsulami et al. [89] in the setting of $FbMS$.

**Theorem 4.1.5.**

Let $(\mathcal{S}, M_b, *)$ be a $G$-complete $FbMS$. Let a mapping $g\colon \mathcal{S} \to \mathcal{S}$ satisfies

$$M_b(g\varrho, g\eta, \beta(M_b(\varrho,\eta,t))t) \geq \lambda(\varrho,\eta,t) * \gamma(\varrho,\eta,t), \tag{4.12}$$

where,

$$\left\{\begin{aligned} \lambda(\varrho,\eta,t) &= \min\Big\{M_b(g\varrho, g\eta, t), M_b(\varrho, g\varrho, t), M_b(\eta, g\eta, t), M_b(\varrho,\eta,t)\Big\} \\ \gamma(\varrho,\eta,t) &= \max\Big\{M_b(\varrho, g\eta, t), M_b(g\varrho, \eta, t)\Big\} \end{aligned}\right\} \tag{4.13}$$

for all $\varrho, \eta \in \mathcal{S}$ and $\beta \in F_b$. Then $g$ has a unique fixed point, where $a*b = \min(a, b)$.

*Proof.*

Starting in same way as in Theorem 4.1.1, we have

$$\begin{aligned} M_b(\varrho_n, \varrho_{n+1}, t) &= M_b(g\varrho_{n-1}, g\varrho_n, t) \\ &\geq \lambda\left(\varrho_{n-1}, \varrho_n, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right) * \gamma\left(\varrho_{n-1}, \varrho_n, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right). \end{aligned} \tag{4.14}$$

Now,

$$\begin{aligned} &\lambda\Big(\varrho_{n-1}, \varrho_n, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\Big) \\ &= \min\Bigg\{ M_b\left(g\varrho_{n-1}, g\varrho_n, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right), M_b\left(\varrho_{n-1}, g\varrho_{n-1}, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right), \\ &\qquad M_b\left(\varrho_n, g\varrho_n, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right), M_b\left(\varrho_{n-1}, \varrho_n, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right)\Bigg\} \\ &= \min\Bigg\{ M_b\left(\varrho_n, \varrho_{n+1}, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t)})\right), M_b\left(\varrho_{n-1}, \varrho_n, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right), \\ &\qquad M_b\left(\varrho_n, \varrho_{n+1}, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right), M_b\left(\varrho_{n-1}, \varrho_n, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right)\Bigg\} \\ &= \min\left\{ M_b\left(\varrho_n, \varrho_{n+1}, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right), M_b\left(\varrho_{n-1}, \varrho_n, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right)\right\}. \end{aligned} \tag{4.15}$$

Also

$$\begin{aligned} &\gamma\Big(\varrho_{n-1}, \varrho_n, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\Big) \\ &= \max\left\{ M_b\left(\varrho_{n-1}, g\varrho_n, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right), M_b\left(g\varrho_{n-1}, \varrho_n, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right)\right\} \\ &= \max\left\{ M_b\left(\varrho_{n-1}, \varrho_{n+1}, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right), M_b\left(\varrho_n, \varrho_n, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right)\right\} \\ &= \max\left\{ M_b\left(\varrho_{n-1}, \varrho_{n+1}, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right), 1\right\} \\ &= 1. \end{aligned}$$

$$\Rightarrow \qquad \gamma\Big(\varrho_{n-1}, \varrho_n, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t)}\Big) = 1. \tag{4.16}$$

Using (4.15) and (4.16) in (4.14), we have

$$
\begin{aligned}
&M_b(\varrho_n, \varrho_{n+1}, t)\\
&\geq \min\left\{M_b\left(\varrho_n, \varrho_{n+1}, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right), M_b\left(\varrho_{n-1}, \varrho_n, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right)\right\} * 1\\
&\geq \min\left\{M_b\left(\varrho_n, \varrho_{n+1}, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right), M_b\left(\varrho_{n-1}, \varrho_n, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right)\right\}.
\end{aligned} \tag{4.17}
$$

By using the the same procedure as in Theorem 4.1.2 after inequality (4.4), we can complete the proof. □

Following is the immediate consequence of Theorem 4.1.5.

**Corollary 4.1.8.**
Let $(\mathcal{S}, M, *)$ be a $G$-complete $FMS$. Let $g$ be a self map on $X$ satisfying the condition

$$M(g\varrho, g\eta, \beta(M(\varrho, \eta, t))t) \geq \lambda(\varrho, \eta, t) * \gamma(\varrho, \eta, t),$$

where

$$
\left\{
\begin{aligned}
\lambda(\varrho, \eta, t) &= \min\left\{M(g\varrho, g\eta, t), M(\varrho, g\varrho, t), M(\eta, g\eta, t), M(\varrho, \eta, t)\right\}\\
\gamma(\varrho, \eta, t) &= \max\left\{M(\varrho, g\eta, t), M(g\varrho, \eta, t)\right\},
\end{aligned}
\right\}
$$

for all $\varrho, \eta \in \mathcal{S}$ and $\beta \in F_1$. Then $g$ has a unique fixed point, where $a*b = \min(a, b)$.

Now, we establish Theorem 2.10 of Alsulami et al. [89] in the setting of $FbMS$.

**Theorem 4.1.6.**
Let $(\mathcal{S}, M_b, *)$ be a $G$-complete $FbMS$. Let a mapping $g\colon \mathcal{S} \to \mathcal{S}$ satisfies

$$M_b(g\varrho, g\eta, \beta(M_b(\varrho, \eta, t))t) \geq \frac{\lambda(\varrho, \eta, t) * \gamma(\varrho, \eta, t)}{\alpha(\varrho, \eta, t)}, \tag{4.18}$$

where

$$\begin{cases} \lambda(\varrho,\eta,t) & = \min\Big\{M_b(g\varrho,g\eta,t).M_b(\varrho,\eta,t), M_b(\varrho,g\varrho,t).M_b(\eta,g\eta,t)\Big\}, \\ \gamma(\varrho,\eta,t) & = \max\Big\{M_b(\varrho,g\varrho,t).M_b(\varrho,g\eta,t),(M_b(\eta,g\varrho,t))^2\Big\}, \\ \alpha(\varrho,\eta,t) & = \max\Big\{M_b(\varrho,g\varrho,t),M_b(\eta,g\eta,t)\Big\}, \end{cases} \tag{4.19}$$

for all $\varrho,\eta\in\mathcal{S}$ and $\beta\in F_b$. Then $g$ has a unique fixed point.

*Proof.*

In the same way as in Theorem 4.1.1, we have

$$M_b(\varrho_n,\varrho_{n+1},t) = M_b(g\varrho_{n-1},g\varrho_n,t)$$

$$\geq \frac{\lambda\left(\varrho_{n-1},\varrho_n,\frac{t}{\beta(M_b(\varrho_{n-1},\varrho_n,t))}\right)*\gamma\left(\varrho_{n-1},\varrho_n,\frac{t}{\beta(M_b(\varrho_{n-1},\varrho_n,t))}\right)}{\alpha\left(\varrho_{n-1},\varrho_n,\frac{t}{\beta(M_b(\varrho_{n-1},\varrho_n,t))}\right)}. \tag{4.20}$$

Now,

$$\begin{aligned} &\lambda\Big(\varrho_{n-1},\varrho_n,\frac{t}{\beta(M_b(\varrho_{n-1},\varrho_n,t))}\Big) \\ &= \min\Big\{M_b\left(g\varrho_{n-1},g\varrho_n,\frac{t}{\beta(M_b(\varrho_{n-1},\varrho_n,t))}\right).M_b\left(\varrho_{n-1},\varrho_n,\frac{t}{\beta(M_b(\varrho_{n-1},\varrho_n,t))}\right), \\ &\qquad M_b\left(\varrho_{n-1},g\varrho_{n-1},\frac{t}{\beta(M_b(\varrho_{n-1},\varrho_n,t))}\right).M_b\left(\varrho_n,g\varrho_n,\frac{t}{\beta(M_b(\varrho_{n-1},\varrho_n,t))}\right)\Big\} \\ &= \min\Big\{M_b\left(\varrho_n,\varrho_{n+1},\frac{t}{\beta(M_b(\varrho_{n-1},\varrho_n,t))}\right).M_b\left(\varrho_{n-1},\varrho_n,\frac{t}{\beta(M_b(\varrho_{n-1},\varrho_n,t))}\right), \\ &\qquad M_b\left(\varrho_{n-1},\varrho_n,\frac{t}{\beta(M_b(\varrho_{n-1},\varrho_n,t))}\right).M_b\left(\varrho_n,\varrho_{n+1},\frac{t}{\beta(M_b(\varrho_{n-1},\varrho_n,t))}\right)\Big\} \\ &= M_b\left(\varrho_n,\varrho_{n+1},\frac{t}{\beta(M_b(\varrho_{n-1},\varrho_n,t))}\right).M_b\left(\varrho_{n-1},\varrho_n,\frac{t}{\beta(M_b(\varrho_{n-1},\varrho_n,t))}\right) \end{aligned}$$

$$= M_b\left(\varrho_n,\varrho_{n+1},\frac{t}{\beta(M_b(\varrho_{n-1},\varrho_n,t))}\right).M_b\left(\varrho_{n-1},\varrho_n,\frac{t}{\beta(M_b(\varrho_{n-1},\varrho_n,t))}\right) \tag{4.21}$$

and

$$\begin{aligned}
&\gamma\Big(\varrho_{n-1}, \varrho_n, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\Big) \\
&= \max\Bigg\{ M_b\left(\varrho_{n-1}, g\varrho_{n-1}, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right).M_b\left(\varrho_{n-1}, g\varrho_n, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right), \\
&\qquad \left(M_b\Big(\varrho_n, g\varrho_{n-1}, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\Big)\right)^2\Bigg\} \\
&= \max\Bigg\{ M_b\left(\varrho_{n-1}, \varrho_n, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right).M_b\left(\varrho_{n-1}, \varrho_{n+1}, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right), \\
&\qquad \left(M_b\Big(\varrho_n, \varrho_n, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\Big)\right)^2\Bigg\} \\
&= \max\left\{ M_b\left(\varrho_{n-1}, \varrho_n, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right).M_b\left(\varrho_{n-1}, \varrho_{n+1}, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right), 1\right\} \\
&= 1. \\
&\Rightarrow \qquad \gamma\Big(\varrho_{n-1}, \varrho_n, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\Big) = 1. \qquad (4.22)
\end{aligned}$$

Also

$$\begin{aligned}
&\alpha\Big(\varrho_{n-1}, \varrho_n, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\Big) \\
&= \max\left\{ M_b\left(\varrho_{n-1}, g\varrho_{n-1}, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right), M_b\left(\varrho_n, g\varrho_n, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right)\right\} \\
&= \max\left\{ M_b\left(\varrho_{n-1}, \varrho_n, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right), M_b\left(\varrho_n, \varrho_{n+1}, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right)\right\} \\
&= \max\left\{ M_b\left(\varrho_{n-1}, \varrho_n, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right), M_b\left(\varrho_n, \varrho_{n+1}, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right)\right\}. \qquad (4.23)
\end{aligned}$$

Using (4.21), (4.22) and (4.23) in (4.20), we have

$$\begin{aligned}
&M_b(\varrho_n, \varrho_{n+1}, t) \\
&\geq \frac{M_b\left(\varrho_n, \varrho_{n+1}, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right).M_b\left(\varrho_{n-1}, \varrho_n, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right)}{\max\left\{ M_b\left(\varrho_{n-1}, \varrho_n, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right), M_b\left(\varrho_n, \varrho_{n+1}, \frac{t}{\beta(M_b(\varrho_{n-1}, \varrho_n, t))}\right)\right\}}. \qquad (4.24)
\end{aligned}$$

If

$$\max\left\{M_b\left(\varrho_n,\varrho_{n+1},\frac{t}{\beta(M_b(\varrho_{n-1},\varrho_n,t))}\right),M_b\left(\varrho_{n-1},\varrho_n,\frac{t}{\beta(M_b(\varrho_{n-1},\varrho_n,t))}\right)\right\}$$
$$=M_b\left(\varrho_{n-1},\varrho_n,\frac{t}{\beta(M_b(\varrho_{n-1},\varrho_n,t))}\right),$$

then (4.24) implies

$$M_b(\varrho_n,\varrho_{n+1},t)\geq M_b\left(\varrho_n,\varrho_{n+1},\frac{t}{\beta(M_b(\varrho_{n-1},\varrho_n,t))}\right),$$

there is nothing to prove by Lemma 4.1.1.

If

$$\max\left\{M_b\left(\varrho_n,\varrho_{n+1},\frac{t}{\beta(M_b(\varrho_{n-1},\varrho_n,t))}\right),M_b\left(\varrho_{n-1},\varrho_n,\frac{t}{\beta(M_b(\varrho_{n-1},\varrho_n,t))}\right)\right\}$$
$$=M_b\left(\varrho_n,\varrho_{n+1},\frac{t}{\beta(M_b(\varrho_{n-1},\varrho_n,t))}\right),$$

then from (4.24), we have

$$M_b(\varrho_n,\varrho_{n+1},t)\geq M_b\left(\varrho_{n-1},\varrho_n,\frac{t}{\beta(M_b(\varrho_{n-1},\varrho_n,t))}\right).$$

In the similar way, we get

$$\begin{aligned}
&M_b(\varrho_n,\varrho_{n+1},t)\geq M_b\left(\varrho_{n-1},\varrho_n,\frac{t}{\beta(M_b(\varrho_{n-1},\varrho_n,t))}\right)\\
&\geq M_b\left(\varrho_{n-2},\varrho_{n-1},\frac{t}{\beta(M_b(\varrho_{n-1},\varrho_n,t)).\beta(M_b(\varrho_{n-2},\varrho_{n-1},t))}\right)\\
&\geq M_b\left(\varrho_{n-3},\varrho_{n-2},\frac{t}{\beta(M_b(\varrho_{n-1},\varrho_n,t)).\beta(M_b(\varrho_{n-2},\varrho_{n-1},t)).\beta(M_b(\varrho_{n-3},\varrho_{n-2},t))}\right)\\
&\vdots\\
&\geq M_b\left(\varrho_0,\varrho_1,\frac{t}{\beta(M_b(\varrho_{n-1},\varrho_n,t)).\beta(M_b(\varrho_{n-2},\varrho_{n-1},t))\ldots\beta(M_b(\varrho_0,\varrho_1,t))}\right).
\end{aligned}$$

Continuing the same way as in Theorem 4.1.1 after inequality (4.2), one can complete the proof. □

The immediate consequence of Theorem 4.1.6 is as follows:

**Corollary 4.1.9.**

Consider $(\mathcal{S}, M, *)$ is a $G$-complete $FMS$ . Let $g$ be a self map on $\mathcal{S}$ satisfies the condition

$$M(g\varrho, g\eta, \beta(M(\varrho,\eta,t))t) \geq \frac{\lambda(\varrho,\eta,t) * \gamma(\varrho,\eta,t)}{\alpha(\varrho,\eta,t)},$$

where

$$\left\{\begin{array}{ll} \lambda(\varrho,\eta,t) & = \min\Big\{M(g\varrho, g\eta, t).M(\varrho,\eta,t), M(\varrho, g\varrho, t).M(\eta, g\eta, t)\Big\}, \\ \gamma(\varrho,\eta,t) & = \max\Big\{M(\varrho, g\varrho, t).M(\varrho, g\eta, t), (M(\eta, g\varrho, t))^2\Big\}, \\ \alpha(\varrho,\eta,t) & = \max\Big\{M(\varrho, g\varrho, t), M(\eta, g\eta, t)\Big\}, \end{array}\right\}$$

for all $\varrho, \eta \in \mathcal{S}$ and $\beta \in F_1$. Then $g$ has a unique fixed point.

## 4.2 Application

Fixed point theory turns out to be an important tool for studying the existence and uniqueness problems for the soloution of various types of of integral and differential equations, for instance see [33, 97, 98].

In this section, a particular non-linear integral equation has been studied for the existence of the solution as an application of fixed point results established in the previous section.

Let $\mathcal{S}$ denote the set of real-valued continuous functions on $[0, J]$ $i.e$, $\mathcal{S} = C[0, J]$. Define $M_b \colon \mathcal{S} \times \mathcal{S} \times [0, \infty) \to [0, 1]$ by

$$M_b(\varrho,\eta,t) = \begin{cases} e^{-\frac{\sup\limits_{s\in[0,I]} |\varrho(s)-\eta(s)|^2}{t}} & \text{if } t > 0, \\ 0 & \text{if } t = 0. \end{cases}$$

Obviously, $(\mathcal{S}, M_b, *)$ a $G$-complete $FbMS$.

Consider the following equation

$$\varrho(t) = \lambda(t) + \int_0^J \varsigma(t,s)\omega(t,s,\varrho(s))ds, \tag{4.25}$$

where $J > 0$ and $\lambda\colon [0,J] \to \mathbb{R}$, $\varsigma\colon [0,J]\times[0,J] \to \mathbb{R}$ and $\omega\colon [0,J]\times[0,J]\times\mathbb{R} \to \mathbb{R}$ are continuous functions.

**Theorem 4.2.1.**

Suppose that the following conditions hold:

(i) for all $t, s \in [0,J]$, $\varrho, \eta \in \mathcal{S}$ and $\beta \in F_b$, where

$$F_b = \left\{\beta\colon [0,\infty) \to [0,\frac{1}{b})\middle|\ \lim_{n\to\infty}\beta(t_n) = \frac{1}{b} \quad \Rightarrow \quad \lim_{n\to\infty} t_n = 0\right\},$$

we have

$$\Big|\omega(t,s,\varrho(s)) - \omega(t,s,\eta(s))\Big| < \sqrt{\beta(M_b(\varrho,\eta,t))}\,\Big|\varrho(s) - \eta(s)\Big|.$$

(ii) for all $t, s \in [0,J]$ ,

$$\sup_{s\in[0,I]}\int_0^J (\varsigma(t,s))^2 ds \le \frac{1}{J}.$$

Then the integeral equation (4.25) has a solution in $\mathcal{S}$.

*Proof.*

Let a mapping $g : \mathcal{S} \to \mathcal{S}$ defined by

$$g\varrho(t) = \lambda(t) + \int_0^J \varsigma(t,s)\omega(t,s,\varrho(s))ds,$$

for all $\varrho \in \mathcal{S}$, and $t, s \in [0,J]$.

From (i) and (ii), we may write, for all $\varrho, \eta \in \mathcal{S}$

$$M_b(g\varrho, g\eta, \beta(M_b(\varrho,\eta,t))t) = e^{-\frac{\sup_{s\in[0,J]}\Big|g\varrho(t) - g\eta(t)\Big|^2}{\beta(M_b(\varrho,\eta,t))t}}$$

$$
\begin{aligned}
M_b(g\varrho, g\eta, &\beta(M_b(\varrho,\eta,t))t) \\
&= e^{-\dfrac{\sup\limits_{s\in[0,J]}\left|\int_0^J \varsigma(t,s)\omega(t,s,\varrho(s))ds - \int_0^J \varsigma(t,s)\omega(t,s,\eta(s))ds\right|^2}{\beta(M_b(\varrho,\eta,t))t}} \\
&= e^{-\dfrac{\sup\limits_{s\in[0,J]}\left|\int_0^J \varsigma(t,s)\{\omega(t,s,\varrho(s)) - \omega(t,s,\eta(s)\}ds\right|^2}{\beta(M_b(\varrho,\eta,t))t}} \\
&\geq e^{-\dfrac{\sup\limits_{s\in[0,J]}\left|\int_0^J (\varsigma(t,s))^2 ds\right| \int_0^J \left|\omega(t,s,\varrho(s)) - \omega(t,s,\eta(s)\right|^2 ds}{\beta(M_b(\varrho,\eta,t))t}} \\
&\geq e^{-\dfrac{\sup\limits_{s\in[0,J]}\frac{1}{I}\int_0^J \left\{\sqrt{\beta(M_b(\varrho,\eta,t))}\left|\varrho(s)-\eta(s)\right|\right\}^2 ds}{\beta(M_b(\varrho,\eta,t))t}} \\
&\geq e^{-\dfrac{\sup\limits_{s\in[0,J]}\beta(M_b(\varrho,\eta,t))\left|\varrho(s)-\eta(s)\right|^2}{\beta(M_b(\varrho,\eta,t))t}} \\
&= e^{-\dfrac{\sup\limits_{s\in[0,J]}\left|\varrho(s)-\eta(s)\right|^2}{t}} \\
&= M_b(\varrho,\eta,t)
\end{aligned}
$$

$$\Rightarrow \quad M_b(g\varrho, g\eta, \beta(M_b(\varrho,\eta,t))t) \geq M_b(\varrho,\eta,t).$$

Thus, the conditions of Theorem 4.1.1 are fulfilled therefore $g$ has a fixed point. As a result, there is a solution to the integral equation (4.25). □

## 4.3 Some Common Fixed Point Results in Fuzzy $b$-Metric Spaces

In this section, some common fixed point results are proved for Geraghty-type contraction in $G$-complete $FbMS$. The first two results are the extensions of the main results of Faraji [33] and the third result is the generalization of the results of Gupta et al. [57] in the setting of $G$-complete $FbMS$. In addition, an example is provided to demonstrate our main result. Furthermore, an application of the acquired results is presented.

Now, we established a common fixed point result for Geraghty type contraction in $FbMS$.

**Theorem 4.3.1.**
Let $(\mathcal{S}, M_b, *)$ be a $G$-complete $FbMS$. Let $g, h\colon \mathcal{S} \to \mathcal{S}$ be mappings satisfying

$$M_b(g\varrho, h\eta, \beta(\lambda(\varrho, \eta, t))t) \geq \lambda(\varrho, \eta, t), \tag{4.26}$$

where

$$\lambda(\varrho, \eta, t) = \min\left\{ M_b(\varrho, g\varrho, t), M_b(\eta, h\eta, t) \right\},$$

for all $\varrho, \eta \in \mathcal{S}$ and $\beta \in F_b$. If $g$ or $h$ is continuous then $g$ and $h$ has a unique common fixed point.

*Proof.*
Let $\varrho_0 \in \mathcal{S}$ and start with iterative sequences $\varrho_{2n+1} = g\varrho_{2n}$ and $\varrho_{2n+2} = h\varrho_{2n+1}$.

$$\begin{aligned} M_b(\varrho_{2n+1}, \varrho_{2n+2}, t) &= M_b(g\varrho_{2n}, h\varrho_{2n+1}, t) \\ &\geq \lambda\left( \varrho_{2n}, \varrho_{2n+1}, \frac{t}{\beta(\lambda(\varrho_{2n}, \varrho_{2n+1}, t))} \right). \end{aligned} \tag{4.27}$$

Now,

$$\begin{aligned} \lambda(\varrho_{2n}, \varrho_{2n+1}, t) &= \min\left\{ M_b(\varrho_{2n}, g\varrho_{2n}, t), M_b(\varrho_{2n+1}, h\varrho_{2n+1}, t) \right\} \\ &= \min\left\{ M_b(\varrho_{2n}, \varrho_{2n+1}, t), M_b(\varrho_{2n+1}, \varrho_{2n+2}, t) \right\}. \end{aligned}$$

If

$$\min\left\{ M_b(\varrho_{2n}, \varrho_{2n+1}, t), M_b(\varrho_{2n+1}, \varrho_{2n+2}, t) \right\} = M_b(\varrho_{2n+1}, \varrho_{2n+2}, t),$$

then

$$\lambda(\varrho_{2n}, \varrho_{2n+1}, t) = M_b(\varrho_{2n+1}, \varrho_{2n+2}, t).$$

(4.27) implies

$$M_b(\varrho_{2n+1}, \varrho_{2n+2}, t) \geq M_b\left(\varrho_{2n+1}, \varrho_{2n+2}, \frac{t}{\beta(M_b(\varrho_{2n}, \varrho_{2n+1}, t))}\right),$$

there is nothing to prove by Lemma 4.1.1.

If

$$\min\left\{M_b(\varrho_{2n}, \varrho_{2n+1}, t), M_b(\varrho_{2n+1}, \varrho_{2n+2}, t)\right\} = M_b(\varrho_{2n}, \varrho_{2n+1}, t),$$

then

$$\lambda(\varrho_{2n}, \varrho_{2n+1}, t) = M_b(\varrho_{2n}, \varrho_{2n+1}, t).$$

(4.27) implies

$$\begin{aligned} & M_b(\varrho_{2n+1}, \varrho_{2n+2}, t) \\ & \quad \geq M_b\left(\varrho_{2n}, \varrho_{2n+1}, \frac{t}{\beta(M(\varrho_{2n}, \varrho_{2n+1}, t))}\right) \\ & \quad \geq M_b\left(\varrho_{2n-1}, \varrho_{2n}, \frac{t}{\beta(M(\varrho_{2n}, \varrho_{2n+1}, t)).\beta(M(\varrho_{2n-1}, \varrho_{2n}, t))}\right) \\ & \quad \vdots \\ & \quad \geq M_b\left(\varrho_0, \varrho_1, \frac{t}{\beta(M(\varrho_{2n}, \varrho_{2n+1}, t)).\beta(M(\varrho_{2n-1}, \varrho_{2n}, t)) \ldots \beta(M(\varrho_0, \varrho_1, t))}\right). \end{aligned} \tag{4.28}$$

For any $q \in \mathbb{N}$ and using $FBM4$ repeatedly, we get

$$\begin{aligned} & M_b(\varrho_{2n}, \varrho_{2n+p}, t) \\ & \geq M_b\left(\varrho_{2n}, \varrho_{2n+1}, \frac{t}{qb}\right) * M_b\left(\varrho_{2n+1}, \varrho_{2n+2}, \frac{t}{qb^2}\right) * \ldots * M_b\left(\varrho_{2n+p-1}, \varrho_{2n+p}, \frac{t}{qb^p}\right). \end{aligned}$$

Using (4.28), we get

$$\begin{aligned} & M_b(\varrho_{2n}, \varrho_{2n+p}, t) \\ & \geq M_b\left(\varrho_0, \varrho_1, \frac{t}{qb.\beta(M(\varrho_{2n-1}, \varrho_{2n}, t)).\beta(M(\varrho_{2n-2}, \varrho_{2n-1}, t)) \ldots \beta(M(\varrho_0, \varrho_1, t))}\right) * \\ & M_b\left(\varrho_0, \varrho_1, \frac{t}{qb^2.\beta(M(\varrho_{2n}, \varrho_{2n+1}, t)).\beta(M(\varrho_{2n-1}, \varrho_{2n}, t)) \ldots \beta(M(\varrho_0, \varrho_1, t))}\right) * \ldots * \\ & M_b\left(\varrho_0, \varrho_1, \frac{t}{qb^p.\beta(M(\varrho_{2n+p-1}, \varrho_{2n+p}, t)).\beta(M(\varrho_{2n+p-2}, \varrho_{2n+p-1}, t)) \ldots \beta(M(\varrho_0, \varrho_1, t))}\right). \end{aligned}$$

$$\begin{aligned} &M_b(\varrho_{2n}, \varrho_{2n+p}, t) \\ &\quad \geq M_b\left(\varrho_0, \varrho_1, \frac{b^{2n-1}t}{q}\right) * M_b\left(\varrho_0, \varrho_1, \frac{b^{2n-1}t}{q}\right) * \ldots * M_b\left(\varrho_0, \varrho_1, \frac{b^{2n-1}t}{q}\right), \end{aligned}$$

when $n \to \infty$ then we get

$$\lim_{n\to\infty} M_b(\varrho_{2n}, \varrho_{2n+p}, t) = 1.$$

Thus $\{\varrho_n\}$ is a $G$-Cauchy sequence. Since $(\mathcal{S}, M_b, *)$ is $G$-complete $FbMS$ so there is some $\varrho \in \mathcal{S}$ such as

$$\lim_{n\to\infty} \varrho_n = \varrho.$$

Now, we prove that $\varrho$ is fixed point of $g$.

If $g$ is continuous then

$$\begin{aligned} M_b(g\varrho, \varrho, t) &\geq M_b\left(g\varrho, g\varrho_n, \frac{t}{2b}\right) * M_b\left(g\varrho_n, \varrho, \frac{t}{2b}\right) \\ &\geq M_b\left(g\varrho, g\varrho, \frac{t}{2b}\right) * M_b\left(\varrho_{n+1}, \varrho_n, \frac{t}{2b}\right) \\ &= 1 * 1 = 1 \quad \text{as} \quad n \to \infty. \end{aligned}$$

This implies that $\quad g\varrho = \varrho.$

So, $\varrho$ is a fixed point of $g$.

Now, from (4.26),

$$M_b(\varrho, h\varrho, \beta(\lambda(\varrho, \varrho, t))t) = M_b(g\varrho, h\varrho, \beta(\lambda(\varrho, \varrho, t))t) \geq \lambda(\varrho, \varrho, t), \tag{4.29}$$

where

$$\begin{aligned} \lambda(\varrho, \varrho, t) &= \min\left\{M_b(\varrho, g\varrho, t), M_b(\varrho, h\varrho, t)\right\} \\ &= \min\left\{1, M_b(\varrho, h\varrho, t)\right\} \\ &= M_b(\varrho, h\varrho, t). \end{aligned}$$

(4.29) implies

$$M_b(\varrho, h\varrho, \beta(\lambda(\varrho, \varrho, t))t) \geq M_b(\varrho, h\varrho, t).$$

So, by Lemma 4.1.1, $h\varrho = \varrho$, thus $\varrho$ is fixed point of $h$.

Now, if $h$ is continuous then one can show that $g$ and $h$ have a common fixed point using the above procedure.

**Uniqueness**

Assume that $\delta$ be any other common fixed point of $g$ and $h$ *i.e,* $\delta = g\delta = h\delta$, then from (4.26),

$$M_b(\varrho, \delta, \beta(\lambda(\varrho, \delta, t))t) = M_b(g\varrho, h\delta, \beta(\lambda(\varrho, \delta, t))t) \geq \lambda(\varrho, \delta, t), \tag{4.30}$$

where

$$\lambda(\varrho, \delta, t) = \min\left\{ M_b(\varrho, g\varrho, t), M_b(\delta, h\delta, t) \right\} = \min\left\{1, 1\right\} = 1.$$

Thus $\varrho = \delta$. Hence $g$ and $h$ have unique common fixed point. □

Following are immediate consequences of Theorem 4.3.1.

**Corollary 4.3.1.**

Let $(\mathcal{S}, M, *)$ be a $G$-complete $FMS$. Let the mappings $g, h\colon \mathcal{S} \to \mathcal{S}$ satisfy

$$M(g\varrho, h\eta, \beta(\lambda(\varrho, \eta, t))t) \geq \lambda(\varrho, \eta, t),$$

where

$$\lambda(\varrho, \eta, t) = \min\left\{ M(\varrho, \eta, t), M(\varrho, g\varrho, t), M(\eta, h\eta, t) \right\},$$

for all $\varrho, \eta \in \mathcal{S}$ and $\beta \in F_1$. If $g$ or $h$ is continuous then $g$ and $h$ have a unique common fixed point.

By taking $g = h$ in Theorem 4.3.1, the following result can be obtained.

**Corollary 4.3.2.**

Let $(\mathcal{S}, M_b, *)$ be a $G$-complete $FbMS$. Let $g\colon \mathcal{S} \to \mathcal{S}$ be a mapping satisfying

$$M_b(g\varrho, g\eta, \beta(\lambda(\varrho, \eta, t))t) \geq \lambda(\varrho, \eta, t),$$

where

$$\lambda(\varrho, \eta, t) = \min\left\{ M_b(\varrho, g\varrho, t), M_b(\eta, g\eta, t) \right\},$$

for all $\varrho, \eta \in \mathcal{S}$ and $\beta \in F_b$. If $g$ is continuous then $g$ has a fixed point.

Analogous to [33, Theorem 4], we now establish the following common fixed point theorem for $G$-complete $FbMS$.

**Theorem 4.3.2.**

Let $(\mathcal{S}, M_b, *)$ be a $G$-complete $FbMS$. Let $g, h \colon \mathcal{S} \to \mathcal{S}$ be maps satisfying the condition

$$M_b(g\varrho, h\eta, \beta(\lambda(\varrho, \eta, t))t) \geq \lambda(\varrho, \eta, t), \tag{4.31}$$

where

$$\lambda(\varrho, \eta, t) = \min\left\{ M_b(\varrho, \eta, t), M_b(\varrho, g\varrho, t), M_b(\eta, h\eta, t) \right\},$$

for all $\varrho, \eta \in \mathcal{S}$ and $\beta \in F_b$. If $g$ or $h$ is continuous then $g$ and $h$ have only one common fixed point.

*Proof.*

Let $\varrho_0 \in \mathcal{S}$. Starting with iterative sequences $\varrho_{2n+1} = g\varrho_{2n}$ and $\varrho_{2n+2} = h\varrho_{2n+1}$.

$$M_b(\varrho_{2n+1}, \varrho_{2n+2}, t) = M_b(g\varrho_{2n}, h\varrho_{2n+1}, t) \geq \lambda\left( \varrho_{2n}, \varrho_{2n+1}, \frac{t}{\beta(\lambda(\varrho_{2n}, \varrho_{2n+1}, t))} \right). \tag{4.32}$$

As

$$\begin{aligned}
\lambda(\varrho_{2n}, \varrho_{2n+1}, t) &= \min\left\{ M_b(\varrho_{2n}, \varrho_{2n+1}, t), M_b(\varrho_{2n}, g\varrho_{2n}, t), M_b(\varrho_{2n+1}, h\varrho_{2n+1}, t) \right\} \\
&= \min\left\{ M_b(\varrho_{2n}, \varrho_{2n+1}, t), M_b(\varrho_{2n}, \varrho_{2n+1}, t), M_b(\varrho_{2n+1}, \varrho_{2n+2}, t) \right\} \\
&= \min\left\{ M_b(\varrho_{2n}, \varrho_{2n+1}, t), M_b(\varrho_{2n+1}, \varrho_{2n+2}, t) \right\}.
\end{aligned}$$

If

$$\min\left\{ M_b(\varrho_{2n}, \varrho_{2n+1}, t), M_b(\varrho_{2n+1}, \varrho_{2n+2}, t) \right\} = M_b(\varrho_{2n+1}, \varrho_{2n+2}, t),$$

then $\lambda(\varrho_{2n}, \varrho_{2n+1}, t) = M_b(\varrho_{2n+1}, \varrho_{2n+2}, t).$

(4.32) implies

$$M_b(\varrho_{2n+1}, \varrho_{2n+2}, t) \geq M_b\left(\varrho_{2n+1}, \varrho_{2n+2}, \frac{t}{\beta(M_b(\varrho_{2n}, \varrho_{2n+1}, t))}\right),$$

there is nothing to prove by Lemma 4.1.1.

If

$$\min\left\{M_b(\varrho_{2n}, \varrho_{2n+1}, t), M_b(\varrho_{2n+1}, \varrho_{2n+2}, t)\right\} = M_b(\varrho_{2n}, \varrho_{2n+1}, t),$$

then $\lambda(\varrho_{2n}, \varrho_{2n+1}, t) = M_b(\varrho_{2n}, \varrho_{2n+1}, t).$

(4.32) implies

$$\begin{aligned} M_b(\varrho_{2n+1}, \varrho_{2n+2}, t) &\geq M_b\left(\varrho_{2n}, \varrho_{2n+1}, \frac{t}{\beta(M(\varrho_{2n}, \varrho_{2n+1}, t))}\right) \\ &\geq M_b\left(\varrho_{2n-1}, \varrho_{2n}, \frac{t}{\beta(M(\varrho_{2n}, \varrho_{2n+1}, t)).\beta(M(\varrho_{2n-1}, \varrho_{2n}, t))}\right) \\ &\vdots \\ &\geq M_b\left(\varrho_0, \varrho_1, \frac{t}{\beta(M(\varrho_{2n}, \varrho_{2n+1}, t)).\beta(M(\varrho_{2n-1}, \varrho_{2n}, t))\ldots\beta(M(\varrho_0, \varrho_1, t))}\right). \end{aligned} \tag{4.33}$$

For any $q \in \mathbb{N}$, and using $FBM4$ repeatedly,

$$\begin{aligned} &M_b(\varrho_{2n}, \varrho_{2n+p}, t) \\ &\geq M_b\left(\varrho_{2n}, \varrho_{2n+1}, \frac{t}{qb}\right) * M_b\left(\varrho_{2n+1}, \varrho_{2n+2}, \frac{t}{qb^2}\right) * \ldots * M_b\left(\varrho_{2n+p-1}, \varrho_{2n+p}, \frac{t}{qb^p}\right). \end{aligned}$$

Using (4.33), we get

$$\begin{aligned} &M_b(\varrho_{2n}, \varrho_{2n+p}, t) \\ &\geq M_b\left(\varrho_0, \varrho_1, \frac{t}{qb.\beta(M(\varrho_{2n-1}, \varrho_{2n}, t)).\beta(M(\varrho_{2n-2}, \varrho_{2n-1}, t))\ldots\beta(M(\varrho_0, \varrho_1, t))}\right) \\ &* M_b\left(\varrho_0, \varrho_1, \frac{t}{qb^2.\beta(M(\varrho_{2n}, \varrho_{2n+1}, t)).\beta(M(\varrho_{2n-1}, \varrho_{2n}, t))\ldots\beta(M(\varrho_0, \varrho_1, t))}\right) \\ &\vdots \\ &* M_b\left(\varrho_0, \varrho_1, \frac{t}{qb^p.\beta(M(\varrho_{2n+p-1}, \varrho_{2n+p}, t)).\beta(M(\varrho_{2n+p-2}, \varrho_{2n+p-1}, t))\ldots\beta(M(\varrho_0, \varrho_1, t))}\right). \end{aligned}$$

$$
\begin{aligned}
&M_b(\varrho_{2n}, \varrho_{2n+p}, t) \\
&\qquad \geq M_b\left(\varrho_0, \varrho_1, \frac{b^{2n-1}t}{q}\right) * M_b\left(\varrho_0, \varrho_1, \frac{b^{2n-1}t}{q}\right) * \ldots * M_b\left(\varrho_0, \varrho_1, \frac{b^{2n-1}t}{q}\right),
\end{aligned}
$$

when $n \to \infty$ then we get

$$\lim_{n\to\infty} M_b(\varrho_{2n}, \varrho_{2n+p}, t) = 1.$$

Thus $\{\varrho_n\}$ is a $G$-Cauchy sequence. Since $(\mathcal{S}, M_b, *)$ is a $G$-complete $FbMS$, so, there is some $\varrho \in \mathcal{S}$ such that

$$\lim_{n\to\infty} \varrho_n = \varrho.$$

Now, we show that $\varrho$ is fixed point of $g$.

If $g$ is continuous then

$$
\begin{aligned}
M_b(g\varrho, \varrho, t) &\geq M_b\left(g\varrho, g\varrho_n, \frac{t}{2b}\right) * M_b\left(g\varrho_n, \varrho, \frac{t}{2b}\right) \\
&\geq M_b\left(g\varrho, g\varrho, \frac{t}{2b}\right) * M_b\left(\varrho_{n+1}, \varrho_n, \frac{t}{2b}\right) \\
&= 1 * 1 = 1 \quad \text{as} \quad n \to \infty.
\end{aligned}
$$

This implies that $g\varrho = \varrho$, thus $\varrho$ is a fixed point of $g$.

Now, from (4.31)

$$M_b(\varrho, h\varrho, \beta(\lambda(\varrho, \varrho, t))t) = M_b(g\varrho, h\varrho, \beta(\lambda(\varrho, \varrho, t))t) \geq \lambda(\varrho, \varrho, t), \tag{4.34}$$

where

$$
\begin{aligned}
\lambda(\varrho, \varrho, t) &= \min\left\{M_b(\varrho, \varrho, t), M_b(\varrho, g\varrho, t), M_b(\varrho, h\varrho, t)\right\} \\
&= \min\left\{1, 1, M_b(\varrho, h\varrho, t)\right\} \\
&= M_b(\varrho, h\varrho, t).
\end{aligned}
$$

(4.34) implies

$$M_b(\varrho, h\varrho, \beta(\lambda(\varrho, \varrho, t))t) \geq M_b(\varrho, h\varrho, t).$$

So, by Lemma 4.1.1, $h\varrho = \varrho$. Thus $\varrho$ is fixed point of $h$.

Now, if $h$ is continuous then one can show that $g$ and $h$ have a common fixed point using the same above procedure.

**Uniqueness**

Let $\delta = g\delta = h\delta$ be any other common fixed point of $g$ and $h$, then from (4.31),

$$M_b(\varrho, \delta, \beta(\lambda(\varrho, \delta, t))t) = M_b(g\varrho, h\delta, \beta(\lambda(\varrho, \delta, t))t) \geq \lambda(\varrho, \delta, t), \tag{4.35}$$

where

$$\begin{aligned} \lambda(\varrho, \delta, t) &= \min\left\{ M_b(\varrho, \delta, t), M_b(\varrho, g\varrho, t), M_b(\delta, h\delta, t) \right\} \\ &= \min\left\{ M_b(\varrho, \delta, t), 1, 1 \right\} \\ &= M_b(\varrho, \delta, t). \end{aligned}$$

(4.35) implies that

$$M_b(\varrho, \delta, \beta(\lambda(\varrho, \delta, t))t) \geq M_b(\varrho, \delta, t).$$

So, by Lemma 3.1.6, $\varrho = \delta$.

Hence $g$ and $h$ have unique common fixed point. □

Following are immediate consequences of Theorem 4.3.1.

**Corollary 4.3.3.**

Let $(\mathcal{S}, M, *)$ be a $G$-complete $FMS$. Let $g, h \colon \mathcal{S} \to \mathcal{S}$ be mappings satisfying

$$M(g\varrho, h\eta, \beta(\lambda(\varrho, \eta, t))t) \geq \lambda(\varrho, \eta, t),$$

where

$$\lambda(\varrho, \eta, t) = \min\left\{ M(\varrho, \eta, t), M(\varrho, g\varrho, t), M(\eta, h\eta, t) \right\},$$

for all $\varrho, \eta \in \mathcal{S}$ and $\beta \in F_1$. If $g$ or $h$ is continuous then $g$ and $h$ have a unique common fixed point.

By taking $g = h$ in Theorem 3.1.2, the following result can be obtained.

**Corollary 4.3.4.**

Let $(\mathcal{S}, M_b, *)$ be a $G$-complete $FbMS$. Let a mapping $g\colon \mathcal{S} \to \mathcal{S}$ satisfies

$$M_b(g\varrho, g\eta, \beta(\lambda(\varrho, \eta, t))t) \geq \lambda(\varrho, \eta, t),$$

where

$$\lambda(\varrho, \eta, t) = \min\left\{ M_b(\varrho, \eta, t), M_b(\varrho, g\varrho, t), M_b(\eta, g\eta, t) \right\},$$

for all $\varrho, \eta \in \mathcal{S}$, and $\beta \in F_b$. If $g$ is continuous then $g$ has a unique fixed point.

Now, we prove a common fixed point result that is analogous to [57, Theorem 1] for $G$-complete $FbMS$ using Geraghty type contraction.

**Theorem 4.3.3.**

Let $(\mathcal{S}, M_b, *)$ be a $G$-complete $FbMS$. Let $g, h\colon \mathcal{S} \to \mathcal{S}$ be mappings satisfying the condition

$$M_b(g\varrho, h\eta, \beta(\lambda(\varrho, \eta, t))t) \geq \lambda(\varrho, \eta, t), \tag{4.36}$$

where

$$\lambda(\varrho, \eta, t) = \min\left\{ \frac{M_b(\eta, h\eta, t)\,[1 + M_b(\varrho, g\varrho, t)]}{1 + M_b(\varrho, \eta, t)}, M_b(\varrho, \eta, t) \right\},$$

for all $\varrho, \eta \in \mathcal{S}$, $t > 0$ and $\beta \in F_b$. If $g$ or $h$ is continuous then $g$ and $h$ have a common fixed point.

*Proof.*

Let $\varrho_0 \in \mathcal{S}$. Starting with iterative sequences

$$\varrho_{2n+1} = g\varrho_{2n} \quad \text{and} \quad \varrho_{2n+2} = h\varrho_{2n+1}.$$

$$\begin{aligned} M_b(\varrho_{2n+1}, \varrho_{2n+2}, t) &= M_b(g\varrho_{2n}, h\varrho_{2n+1}, t) \\ &\geq \lambda\left(\varrho_{2n}, \varrho_{2n+1}, \frac{t}{\beta(\lambda(\varrho_{2n}, \varrho_{2n+1}, t))}\right). \end{aligned} \tag{4.37}$$

As

$$\begin{aligned}
&\lambda(\varrho_{2n}, \varrho_{2n+1}, t)\\
&= \min\left\{\frac{M_b(\varrho_{2n+1}, h\varrho_{2n+1}, t)\,[1 + M_b(\varrho_{2n}, g\varrho_{2n}, t)]}{1 + M_b(\varrho_{2n}, \varrho_{2n+1}, t)}, M_b(\varrho_{2n}, \varrho_{2n+1}, t)\right\}\\
&= \min\left\{\frac{M_b(\varrho_{2n+1}, \varrho_{2n+2}, t)\,[1 + M_b(\varrho_{2n}, \varrho_{2n+1}, t)]}{1 + M_b(\varrho_{2n}, \varrho_{2n+1}, t)}, M_b(\varrho_{2n}, \varrho_{2n+1}, t)\right\}\\
&= \min\left\{M_b(\varrho_{2n}, \varrho_{2n+1}, t), M_b(\varrho_{2n+1}, \varrho_{2n+2}, t)\right\}.
\end{aligned}$$

If

$$\min\left\{M_b(\varrho_{2n}, \varrho_{2n+1}, t), M_b(\varrho_{2n+1}, \varrho_{2n+2}, t)\right\} = M_b(\varrho_{2n+1}, \varrho_{2n+2}, t),$$

then $\qquad \lambda(\varrho_{2n}, \varrho_{2n+1}, t) = M_b(\varrho_{2n+1}, \varrho_{2n+2}, t).$

(4.37) implies

$$M_b(\varrho_{2n+1}, \varrho_{2n+2}, t) \geq M_b\left(\varrho_{2n+1}, \varrho_{2n+2}, \frac{t}{\beta(M_b(\varrho_{2n}, \varrho_{2n+1}, t))}\right),$$

then there is nothing to prove by Lemma 4.1.1.

If

$$\min\left\{M_b(\varrho_{2n}, \varrho_{2n+1}, t), M_b(\varrho_{2n+1}, \varrho_{2n+2}, t)\right\} = M_b(\varrho_{2n}, \varrho_{2n+1}, t),$$

then $\qquad \lambda(\varrho_{2n}, \varrho_{2n+1}, t) = M_b(\varrho_{2n}, \varrho_{2n+1}, t).$

(4.37) implies

$$\begin{aligned}
&M_b(\varrho_{2n+1}, \varrho_{2n+2}, t) \geq M_b\left(\varrho_{2n}, \varrho_{2n+1}, \frac{t}{\beta(M(\varrho_{2n}, \varrho_{2n+1}, t))}\right)\\
&\geq M_b\left(\varrho_{2n-1}, \varrho_{2n}, \frac{t}{\beta(M(\varrho_{2n}, \varrho_{2n+1}, t)).\beta(M(\varrho_{2n-1}, \varrho_{2n}, t))}\right)\\
&\geq M_b\Bigg(\varrho_{2n-2}, \varrho_{2n-1},\\
&\qquad \frac{t}{\beta(M(\varrho_{2n}, \varrho_{2n+1}, t)).\beta(M(\varrho_{2n-1}, \varrho_{2n}, t)).\beta(M(\varrho_{2n-2}, \varrho_{2n-1}, t))}\Bigg)\\
&\vdots\\
&\geq M_b\left(\varrho_0, \varrho_1, \frac{t}{\beta(M(\varrho_{2n}, \varrho_{2n+1}, t)).\beta(M(\varrho_{2n-1}, \varrho_{2n}, t))\ldots\beta(M(\varrho_0, \varrho_1, t))}\right). \qquad (4.38)
\end{aligned}$$

For any $q \in \mathbb{N}$, and using $FBM4$ repeatedly,

$$M_b(\varrho_{2n}, \varrho_{2n+p}, t)$$
$$\geq M_b\left(\varrho_{2n}, \varrho_{2n+1}, \frac{t}{qb}\right) * M_b\left(\varrho_{2n+1}, \varrho_{2n+2}, \frac{t}{qb^2}\right) * \ldots * M_b\left(\varrho_{2n+p-1}, \varrho_{2n+p}, \frac{t}{qb^p}\right).$$

Using (4.38), we get

$$M_b(\varrho_{2n}, \varrho_{2n+p}, t)$$
$$\geq M_b\left(\varrho_0, \varrho_1, \frac{t}{qb.\beta(M(\varrho_{2n-1}, \varrho_{2n}, t)).\beta(M(\varrho_{2n-2}, \varrho_{2n-1}, t)) \ldots \beta(M(\varrho_0, \varrho_1, t))}\right)$$
$$* M_b\left(\varrho_0, \varrho_1, \frac{t}{qb^2.\beta(M(\varrho_{2n}, \varrho_{2n+1}, t)).\beta(M(\varrho_{2n-1}, \varrho_{2n}, t)) \ldots \beta(M(\varrho_0, \varrho_1, t))}\right)$$
$$\vdots$$
$$* M_b\left(\varrho_0, \varrho_1, \frac{t}{qb^p.\beta(M(\varrho_{2n+p-1}, \varrho_{2n+p}, t)).\beta(M(\varrho_{2n+p-2}, \varrho_{2n+p-1}, t)) \ldots \beta(M(\varrho_0, \varrho_1, t))}\right)$$

$$M_b(\varrho_{2n}, \varrho_{2n+p}, t)$$
$$\geq M_b\left(\varrho_0, \varrho_1, \frac{b^{2n-1}t}{q}\right) * M_b\left(\varrho_0, \varrho_1, \frac{b^{2n-1}t}{q}\right) * \ldots * M_b\left(\varrho_0, \varrho_1, \frac{b^{2n-1}t}{q}\right),$$

when $n \to \infty$ then we get

$$\lim_{n\to\infty} M_b(\varrho_{2n}, \varrho_{2n+p}, t) = 1.$$

Thus $\{\varrho_n\}$ is a $G$-Cauchy sequence. Since $(\mathcal{S}, M_b, *)$ is a $G$-complete $FbMS$, so, there is some $\varrho \in \mathcal{S}$ such that

$$\lim_{n\to\infty} \varrho_n = \varrho.$$

We prove that $\varrho$ is fixed point of $g$.

If $g$ is continuous then

$$\begin{aligned} M_b(g\varrho, \varrho, t) &\geq M_b\left(g\varrho, g\varrho_n, \frac{t}{2b}\right) * M_b\left(g\varrho_n, \varrho, \frac{t}{2b}\right) \\ &\geq M_b\left(g\varrho, g\varrho, \frac{t}{2b}\right) * M_b\left(\varrho_{n+1}, \varrho_n, \frac{t}{2b}\right) \\ &= 1 * 1 = 1 \quad \text{as} \quad n \to \infty. \end{aligned}$$

This shows that $g\varrho = \varrho$, so $\varrho$ is a fixed point of $g$.

Now, from (4.36),

$$\begin{aligned} M_b(\varrho, h\varrho, \beta(\lambda(\varrho, \varrho, t))t) &= M_b(g\varrho, h\varrho, \beta(\lambda(\varrho, \varrho, t))t) \\ &\geq \lambda(\varrho, \varrho, t), \end{aligned} \tag{4.39}$$

where

$$\begin{aligned} \lambda(\varrho, \varrho, t) &= \min\left\{\frac{M_b(\varrho, h\varrho, t)\,[1 + M_b(\varrho, g\varrho, t)]}{1 + M_b(\varrho, \varrho, t)}, M_b(\varrho, \varrho, t)\right\} \\ &= \min\left\{\frac{M_b(\varrho, h\varrho, t)\,[1 + 1]}{1 + 1}, 1\right\} \\ &= \min\left\{M_b(\varrho, h\varrho, t), 1\right\} \\ &= M_b(\varrho, h\varrho, t). \end{aligned}$$

(4.39) implies

$$M_b(\varrho, h\varrho, \beta(\lambda(\varrho, \varrho, t))t) \geq M_b(\varrho, h\varrho, t).$$

So, by Lemma 4.1.1, $h\varrho = \varrho$.

Thus $\varrho$ is fixed point of $h$. □

Following are immediate consequences of Theorem 4.3.3.

**Corollary 4.3.5.**

Let $(\mathcal{S}, M, *)$ be a $G$-complete $FMS$. Let $g, h\colon \mathcal{S} \to \mathcal{S}$ be the mappings satisfying

$$M(g\varrho, h\eta, \beta(\lambda(\varrho, \eta, t))t) \geq \lambda(\varrho, \eta, t),$$

where

$$\lambda(\varrho, \eta, t) = \min\left\{\frac{M(\eta, g\eta, t)\,[1 + M(\varrho, g\varrho, t)]}{1 + M(\varrho, \eta, t)}, M(\varrho, \eta, t)\right\},$$

for all $\varrho, \eta \in \mathcal{S}$ and $\beta \in F_1$. If $g$ or $h$ is continuous then $g$ and $h$ have only one common fixed point.

By letting $g = h$ in Theorem 4.3.3, we obtain the following result.

**Corollary 4.3.6.**

Let $(\mathcal{S}, M_b, *)$ be a $G$-complete $FbMS$. Let a mapping $g\colon \mathcal{S} \to \mathcal{S}$ satisfies

$$M_b(g\varrho, g\eta, \beta(\lambda(\varrho, \eta, t))t) \geq \lambda(\varrho, \eta, t),$$

where

$$\lambda(\varrho, \eta, t) = \min\left\{\frac{M_b(\eta, g\eta, t)\,[1 + M_b(\varrho, g\varrho, t)]}{1 + M_b(\varrho, \eta, t)}, M_b(\varrho, \eta, t)\right\},$$

for all $\varrho, \eta \in \mathcal{S}$ and $\beta \in F_b$. If $g$ is continuous then $g$ has only one fixed point.

The example given below illustrates Theorem 4.3.2.

**Example 4.3.7.**

Let $\mathcal{S} = [0, 1]$. A mapping $M_b\colon \mathcal{S} \times \mathcal{S} \times [0, \infty) \to [0, 1]$ defined by

$$M_b(\varrho, \eta, t) = \frac{t}{t + (\varrho - \eta)^2}. \tag{4.40}$$

Then $(\mathcal{S}, M_b, *)$ is a $G$-complete $FbMS$. Define mappings $g, h\colon \mathcal{S} \to \mathcal{S}$ such that $g\varrho = \dfrac{\varrho}{2}$ and $h\eta = \dfrac{\eta}{3}$. Taking $\beta(t) = \dfrac{1}{3}$, for all $t > 0$, we have

$$\begin{aligned} M_b(g\varrho, h\eta, \beta(\lambda(\varrho, \eta, t))t) &= M_b\left(\frac{\varrho}{2}, \frac{\eta}{3}, \frac{t}{3}\right) \\ &= \frac{t}{t + \dfrac{(3\varrho - 2\eta)^2}{12}}. \end{aligned} \tag{4.41}$$

Now,

$$\begin{aligned} M_b(\varrho, g\varrho, t) &= M_b\left(\varrho, \frac{\varrho}{2}, t\right) \\ &= \frac{t}{t + \dfrac{\varrho^2}{4}}, \end{aligned} \tag{4.42}$$

and

$$\begin{aligned} M_b(\eta, h\eta, t) &= M_b\left(\eta, \frac{\eta}{3}, t\right) \\ &= \frac{t}{t + \dfrac{4\eta^2}{9}}. \end{aligned} \tag{4.43}$$

If $\varrho < \eta$, say $\varrho = \frac{1}{3}$ and $\eta = \frac{1}{2}$,
then from (4.41), we have

$$M_b(g\varrho, h\eta, \beta(\lambda(\varrho, \eta, t))t) = 1.$$

Also from (4.40), (4.42) and (4.43), we have

$$M_b(\varrho, \eta, t) = \frac{t}{t + 0.027},$$

$$M_b(\varrho, g\varrho, t) = \frac{t}{t + 0.028},$$

and

$$M_b(\eta, h\eta, t) = \frac{t}{t + 0.111}.$$

Since

$$1 > \frac{t}{t + 0.111} = \min\left\{\frac{t}{t + 0.027}, \frac{t}{t + 0.028}, \frac{t}{t + 0.111}\right\},$$

so, we have

$$M_b(g\varrho, h\eta, \beta(\lambda(\varrho, \eta, t))t) > \min\left\{M_b(\varrho, \eta, t), M_b(\varrho, g\varrho, t), M_b(\eta, h\eta, t)\right\}.$$

If $\varrho > \eta$, say $\varrho = \frac{2}{3}$ and $\eta = \frac{1}{2}$,
then from (4.41), we have

$$M_b(g\varrho, h\eta, \beta(\lambda(\varrho, \eta, t))t) = \frac{t}{t + 0.0833}.$$

Also from (4.40), (4.42) and (4.43), we have

$$M_b(\varrho, \eta, t) = \frac{t}{t + 0.111},$$
$$M_b(\varrho, g\varrho, t) = \frac{t}{t + 0.111},$$

and

$$M_b(\eta, h\eta, t) = \frac{t}{t + 0.111}.$$

Since

$$\frac{t}{t+.0833} > \frac{t}{t+0.111} = \min\left\{\frac{t}{t+0.111}, \frac{t}{t+0.111}, \frac{t}{t+0.111}\right\},$$

so, we have

$$M_b(g\varrho, h\eta, \beta(\lambda(\varrho,\eta,t))t) > \min\left\{M_b(\varrho,\eta,t), M_b(\varrho, g\varrho, t), M_b(\eta, h\eta, t)\right\}.$$

If $\varrho = \eta$, then from (4.41), we have

$$M_b(g\varrho, h\eta, \beta(\lambda(\varrho,\eta,t))t) = \frac{t}{t+0.0833\varrho^2}.$$

Also from (4.40), (4.42) and (4.43), we have

$$M_b(\varrho,\eta,t) = 1,$$

$$M_b(\varrho, g\varrho, t) = \frac{t}{t+0.25\varrho^2},$$

and

$$M_b(\eta, h\eta, t) = \frac{t}{t+0.444\varrho^2}.$$

Since

$$\frac{t}{t+.0833\varrho^2} > \frac{t}{t+0.25\varrho^2} = \min\left\{1, \frac{t}{t+0.25\varrho^2}, \frac{t}{t+0.444\varrho^2}\right\},$$

so, we have

$$M_b(g\varrho, h\eta, \beta(\lambda(\varrho,\eta,t))t) > \min\left\{M_b(\varrho,\eta,t), M_b(\varrho, g\varrho, t), M_b(\eta, h\eta, t)\right\}.$$

Hence all the axioms of theorem 4.3.2 are fulfilled.

Note that

$$g0 = 0 \quad \text{and} \quad h0 = 0.$$

So, 0 is the only common fixed point of $g$ and $h$.

## 4.4 Application

Fixed point theory turns out to be a powerful tool for the study of existence of the solution of various kind of integral and differential equations, for instance see [27, 84, 98].

In this section, a particular non-linear integral equation has been studied for the existence of the solution as an application of fixed point results presented in last section.

Consider all the real valued continuous functions on the interval $[0, J]$ as $\mathcal{S} = [0, J]$ and $M_b \colon \mathcal{S} \times \mathcal{S} \times [0, \infty) \to [0, 1]$ defined by

$$M_b(\varrho, \eta, t) = \begin{cases} 0 & \text{if} \quad t = 0, \\ e^{-\frac{\sup_{s \in [0,J]} |\varrho(s) - \eta(s)|^2}{t}} & \text{if} \quad t > 0, \end{cases}$$

then it is obvious that $(\mathcal{S}, M_b, *)$ be a $G$-complete $FbMS$.

Consider

$$\varrho(t) = \lambda(t) + \int_0^J \varsigma(t, s)\omega(t, s, \varrho(s))ds, \tag{4.44}$$

where $J > 0$, $\lambda \colon [0, J] \to \mathbb{R}$, $\varsigma \colon [0, J] \times [0, J] \to \mathbb{R}$ and$\omega \colon [0, J] \times [0, J] \times R \to \mathbb{R}$ are continuous functions.

**Theorem 4.4.1.**

Suppose that $\forall\, t, s \in [0, J]$, $\varrho, \eta \in \mathcal{S}$ and $\beta \in F_b$, the following conditions hold:

(i) $\Big|\omega(t, s, \varrho(s)) - \omega(t, s, \eta(s))\Big| < \sqrt{\beta(\lambda(\varrho, \eta, t))}\, \Big|\varrho(s) - \eta(s)\Big|$,

(ii) $\sup_{s \in [0,J]} \int_0^J (\varsigma(t, s))^2 ds \leq \dfrac{1}{I}$.

Then the equation (4.44) has a solution $\varrho^* \in \mathcal{S}$.

*Proof.*

Let $g, h : \mathcal{S} \to \mathcal{S}$ be the integral operators defined by

$$g\varrho(t) = \lambda(t) + \int_0^J \varsigma(t, s)\omega(t, s, \varrho(s))ds, \quad \varrho \in \mathcal{S}, \text{ and } \ t, s \in [0, J]$$

and

$$h\eta(t) = \lambda(t) + \int_0^J \varsigma(t,s)F(t,s,\eta(s))ds, \quad \varrho \in \mathcal{S}, \text{ and } t,s \in [0,J]$$

For all $\varrho, \eta \in \mathcal{S}$ and by using Conditions (i) and (ii), we have

$$M_b(g\varrho, h\eta, \beta(\lambda(\varrho,\eta,t))t) = e^{-\frac{\sup_{s\in[0,J]}\left|g\varrho(t) - h\eta(t)\right|^2}{\beta(\lambda(\varrho,\eta,t))t}}$$

$$= e^{-\frac{\sup_{s\in[0,J]}\left|\int_0^J \varsigma(t,s)\omega(t,s,\varrho(s))ds - \int_0^J \varsigma(t,s)\omega(t,s,\eta(s))ds\right|^2}{\beta(\lambda(\varrho,\eta,t))t}}$$

$$= e^{-\frac{\sup_{s\in[0,J]}\left|\int_0^J \varsigma(t,s)\{\omega(t,s,\varrho(s)) - \omega(t,s,\eta(s)\}ds\right|^2}{\beta(\lambda(\varrho,\eta,t))t}}$$

$$\geq e^{-\frac{\sup_{s\in[0,J]}\left|\int_0^J (\varsigma(t,s))^2 ds\right| \int_0^J \left|\omega(t,s,\varrho(s)) - \omega(t,s,\eta(s)\right|^2 ds}{\beta(\lambda(\varrho,\eta,t))t}}$$

$$\geq e^{-\frac{\sup_{s\in[0,J]}\frac{1}{I}\int_0^J \left\{\sqrt{\beta(\lambda(\varrho,\eta,t))}\left|\varrho(s) - \eta(s)\right|\right\}^2 ds}{\beta(\lambda(\varrho,\eta,t))t}}$$

$$\geq e^{-\frac{\sup_{s\in[0,J]}\beta(\lambda(\varrho,\eta,t))\left|\varrho(s) - \eta(s)\right|^2}{\beta(\lambda(\varrho,\eta,t))t}}$$

$$= e^{-\frac{\sup_{s\in[0,J]}\left|\varrho(s) - \eta(s)\right|^2}{t}}$$

$$= M_b(\varrho,\eta,t).$$

So, we have

$$M_b(T\varrho, S\eta, \beta(\lambda(\varrho,\eta,t))t) \geq M_b(\varrho,\eta,t).$$

Hence $\varrho^* \in \mathcal{S}$ is a common fixed point of $g$ and $h$, so the equation (4.44)has a solution. □

## 4.5 Conclusion

In this chapter, by using Geraphty type contraction, the main result of Grabeic [49] has been generalized for $G$-complete $FbMS$. The result is illustrated by an example. Moreover, we have also established analogue of the main results of Faraji et al. [33] and Alsulami et al. [89] in $G$-complete $FbMS$. The existence problem for the solution of a nonlinear integral equation is also presented as an application of our main result. Further, the presented corollaries indicate that the theorems established in this work generalize many existing results in the literature. We have also established common fixed point result, analogue to [33, Theorem 4] for Geraghty type contraction in $G$-complete $FbMS$. We have furnished an example to demonstrate our second result. Moreover, we have established fixed-point theorems analogous of Gupta et al. [57] in $G$-complete $FbMS$. Our results may be of interest for the readers/researchers in some specialized area of computer science.

# Chapter 5

# Fixed Point Results in Generalized Fuzzy Metric Spaces

In this chapter, the definition of a generalised fuzzy metric space $GFMS$ is introduced. Many topological spaces like $FMS$, $FbMS$ and $DFMS$ have been generalized by this new space. An example of a generalised fuzzy metric space has been provided to demonstrate its definition. It is also proven that the class of $GFMS$ contains the classes of $FMS$, $FbMS$ and $DFMS$ as proper sub-classes. The well known BCP[14] and Ćirić's quasi-contraction theorem [16] have been established in $GFMS$. As a result of our findings, several authors recent results are obtained as corollaries. In the end, an example is presented to illustrate the main result. Some fixed point theorems are also proved by applying our results.

## 5.1 Generalized Fuzzy Metric Spaces

Motivated by Jleli et al. [41] and following George and Veeramani [50], we now introduce the notion of a generalized fuzzy metric spaces as follows:

**Definition 5.1.1.**

Consider a non empty set $\mathcal{S}$ and a mapping $\mathcal{G} : \mathcal{S} \times \mathcal{S} \times (0, \infty) \to [0, 1]$. Define a set

$$C(\mathcal{G},\mathcal{S},\varrho) = \left\{ \{\varrho_n\} \subset \mathcal{S} : \lim_{n\to\infty} \mathcal{G}(\varrho_n, \varrho, t) = 1 \; \forall \; t > 0 \right\},$$

for every $\varrho \in \mathcal{S}$ then $\mathcal{G}$ is said to be a generalized fuzzy metric if for all $\varrho, \eta \in \mathcal{S}$ and $t > 0$ , it satisfies the following conditions:

$GFM1$: $\mathcal{G}(\varrho, \eta, t) > 0$;

$GFM2$: $\mathcal{G}(\varrho, \eta, t) = 1 \quad \Rightarrow \quad \varrho = \eta$;

$GFM3$: $\mathcal{G}(\varrho, \eta, t) = \mathcal{G}(\eta, \varrho, t)$;

$GFM4$: there exists $c \geq 1$ such that if $\{\varrho_n\} \in C(\mathcal{G}, \mathcal{S}, \varrho)$ then

$$\mathcal{G}(\varrho, \eta, t) \geq \limsup_{n\to\infty} \mathcal{G}\left(\varrho_n, \eta, \frac{t}{c}\right);$$

$GFM5$: $\mathcal{G}(\varrho, \eta, .) \colon (0, \infty) \to [0, 1]$ is continuous and $\lim_{t\to\infty} \mathcal{G}(\varrho, \eta, t) = 1$.

Then $(\mathcal{G}, \mathcal{S}, *)$ is called a $GFMS$.

The above definition is illustrated by the following example.

**Example 5.1.2.**

Consider a generalized metric space $(\mathcal{S}, \mathcal{D})$. Define a mapping $\mathcal{G} : \mathcal{S} \times \mathcal{S} \times (0, \infty) \to [0, 1]$ by

$$\mathcal{G}(\varrho, \eta, t) = e^{-\dfrac{\mathcal{D}(\varrho, \eta)}{t}} \tag{5.1}$$

and

$$C(\mathcal{G},\mathcal{S},\varrho) = \left\{ \{\varrho_n\} \subset \mathcal{S} : \lim_{n\to\infty} \mathcal{G}(\varrho_n, \varrho, t) = 1 \right\},$$

for every $\varrho \in \mathcal{S}$ and $t > 0$. Then $(\mathcal{G}, \mathcal{S}, *)$ is $GFMS$, where the $t$-norm " $*$ " is taken as product norm *i.e,* $\varrho * \eta = \varrho\eta$.

We only prove that $\mathcal{G}$ satisfies $GFM4$ of Definition 5.1.1.

Let $\varrho, \eta \in \mathcal{S}$ and $\{\varrho_n\} \in C(\mathcal{D}, \mathcal{S}, \varrho)$. Since $\mathcal{D}$ is generalized metric, so, from condition $(\mathcal{D}_3)$ of Definition 2.1 and from equation (5.1), it is clear that $\{\varrho_n\}$ also belongs to $C(\mathcal{G}, \mathcal{S}, \varrho)$. It follows that

$$
\begin{aligned}
\mathcal{G}(\varrho, \eta, t) &= e^{-\dfrac{\mathcal{D}(\varrho, \eta)}{t}} \\
&\geq e^{-\dfrac{c \limsup\limits_{n\to\infty} \mathcal{D}(\varrho_n, \eta)}{t}} \\
&= e^{-\dfrac{\limsup\limits_{n\to\infty} \mathcal{D}(\varrho_n, \eta)}{\frac{t}{c}}} \\
&= \limsup_{n\to\infty} e^{-\dfrac{\mathcal{D}(\varrho_n, \eta)}{\frac{t}{c}}} \\
&= \limsup_{n\to\infty} \mathcal{G}\left(\varrho_n, \eta, \frac{t}{c}\right) \\
\Rightarrow \quad \mathcal{G}(\varrho, \eta, t) &\geq \limsup_{n\to\infty} \mathcal{G}\left(\varrho_n, \eta, \frac{t}{c}\right).
\end{aligned}
$$

**Proposition 5.1.3.**

The set $C(\mathcal{G}, \mathcal{S}, \varrho)$ is non empty if and only if $\mathcal{G}(\varrho, \varrho, t) = 1$.

*Proof.*

If $C(\mathcal{G}, \mathcal{S}, \varrho) \neq \phi$ then there is a sequence $\{\varrho_n\} \subset \mathcal{S}$ such as

$$\lim_{n\to\infty} \mathcal{G}(\varrho_n, \varrho, t) = 1 \text{ for all } t > 0.$$

Using $GFM4$ of Definition 5.1.1, we get

$$\mathcal{G}(\varrho, \varrho, t) \geq \limsup_{n\to\infty} \mathcal{G}\left(\varrho_n, \varrho, \frac{t}{c}\right),$$

it follows that

$$\mathcal{G}(\varrho, \varrho, t) = 1.$$

Conversely, if

$$\mathcal{G}(\varrho, \varrho, t) = 1,$$

then we can take the sequence $\{\varrho_n\} \subset \mathcal{S}$ such that for all $n \in \mathbb{N}$, $\varrho_n = \varrho$, we have

$$\lim_{n\to\infty} \mathcal{G}(\varrho_n, \varrho, t) = 1.$$

Hence

$$C(\mathcal{G}, \mathcal{S}, \varrho) \neq \phi.$$

□

**Remark 5.1.4.**

It is worth mentioning that the class of $GFMS$ is larger than the class of $FMS$ which is elaborated in the following example.

**Example 5.1.5.**

Let $\mathcal{S} = [0, 1]$. Define a set

$$C(\mathcal{G}, \mathcal{S}, \varrho) = \left\{ \{\varrho_n\} \subset \mathcal{S} : \lim_{n\to\infty} \mathcal{G}(\varrho_n, \varrho, t) = 1 \right\},$$

for every $\varrho \in \mathcal{S}$ and $t > 0$, where $\mathcal{G} : \mathcal{S} \times \mathcal{S} \times (0, \infty) \to [0, 1]$ is defined by

$$\mathcal{G}(\varrho, \eta, t) = \begin{cases} e^{-\dfrac{\varrho+\eta}{t}} & \text{if} \quad \varrho \neq 0 \text{ and } \eta \neq 0, \\ e^{-\dfrac{\varrho}{2t}} & \text{if} \quad \eta = 0, \\ e^{-\dfrac{\eta}{2t}} & \text{if} \quad \varrho = 0, \end{cases}$$

for all $\varrho, \eta \in \mathcal{S}$ then $(\mathcal{G}, \mathcal{S}, *)$ is $GFMS$, where " $*$ " is the product $t$-norm *i.e,* $\varrho * \eta = \varrho\eta$.

In view of Proposition 5.1.3, we need to verify $GFM4$ only for those elements $\varrho \in \mathcal{S}$ such that

$$\mathcal{G}(\varrho, \varrho, t) = 1,$$

which implies that

$$\varrho = 0.$$

Let $\{\varrho_n\} \subset \mathcal{S}$ be a sequence such that

$$\lim_{n\to\infty} \mathcal{G}(\varrho_n, 0, t) = 1.$$

For every $n \in \mathbb{N}$ and $\eta \in \mathcal{S}$, we have:

$$\mathcal{G}(\varrho_n, \eta, t) = \begin{cases} e^{-\dfrac{\varrho_n + \eta}{t}} & \text{if } \varrho_n \neq 0 \text{ and } \eta \neq 0, \\ e^{-\dfrac{\eta}{2t}} & \text{if } \varrho_n = 0. \end{cases}$$

Since

$$\eta \leq \varrho_n + \eta \quad \text{and} \quad t > 0,$$

therefore,

$$\frac{\eta}{2t} < \frac{\varrho_n + \eta}{t}$$
$$-\frac{\eta}{2t} > -\frac{\varrho_n + \eta}{t}$$
$$e^{-\dfrac{\eta}{2t}} > e^{-\dfrac{\varrho_n + \eta}{t}},$$

then

$$\begin{aligned} \mathcal{G}(0, \eta, t) &= e^{-\dfrac{\eta}{2t}} \\ &\geq \limsup_{n \to \infty} e^{-\dfrac{\varrho_n + \eta}{t}} \\ &= \limsup_{n \to \infty} \mathcal{G}(\varrho_n, \eta, t), \end{aligned}$$

which implies that

$$\mathcal{G}(0, \eta, t) \geq \limsup_{n \to \infty} \mathcal{G}\left(\varrho_n, \eta, \frac{t}{c}\right); \quad \text{where } c = 1.$$

Hence $(\mathcal{G}, \mathcal{S}, *)$ is $GFMS$ but it is not a $FMS$ because the condition $FM4$ of Definition 2.5.5 does not hold.

Since

$$\mathcal{G}(\varrho, \eta, t) = e^{-\dfrac{\varrho + \eta}{t}}$$

and

$$\begin{aligned} \mathcal{G}\left(\varrho, 0, \frac{t}{2}\right) * \mathcal{G}\left(0, \eta, \frac{t}{2}\right) &= e^{-\dfrac{\varrho}{4t}} * e^{-\dfrac{\eta}{4t}} \\ &= e^{-\dfrac{\varrho + \eta}{4t}}. \end{aligned}$$

Thus

$$\mathcal{G}(\varrho,\eta,t) \leq \mathcal{G}\left(\varrho,0,\frac{t}{2}\right) * \mathcal{G}\left(0,\eta,\frac{t}{2}\right).$$

This implies that, $FM4$ of Definition 2.5.5 is not satisfied. Hence every $GFMS$ is not a $FMS$.

The following propositions show that $FMS$ and $FbMS$ are $GFMS$.

**Proposition 5.1.6.**

Every fuzzy metric space $(M,\mathcal{S},*)$ is a $GFMS$.

*Proof.*

We only prove that $M$ satisfies the property $GFM4$ of Definition 5.1.1.

Now, for $\varrho,\eta \in \mathcal{S}$ and $\{\varrho_n\} \in C(\mathcal{G},\mathcal{S},\varrho)$, using $FM4$ of Definition 2.5.5.

$$\begin{aligned} M(\varrho,\eta,t) &\geq M\left(\varrho,\varrho_n,\frac{t}{2}\right) * M\left(\varrho_n,\eta,\frac{t}{2}\right) \\ &= 1 * \limsup_{n\to\infty} M\left(\varrho_n,\eta,\frac{t}{2}\right) \\ &= \limsup_{n\to\infty} M\left(\varrho_n,\eta,\frac{t}{2}\right) \\ \Rightarrow \quad M(\varrho,\eta,t) &\geq \limsup_{n\to\infty} M\left(\varrho_n,\eta,\frac{t}{c}\right); \quad \text{where} \quad c=2. \end{aligned}$$

□

**Proposition 5.1.7.**

Every fuzzy $b$-metric space $(M_b,\mathcal{S},*)$ is a $GFMS$.

*Proof.*

We only prove that $M_b$ satisfies the property $GFM4$ of Definition 5.1.1.

Now, for $\varrho,\eta \in \mathcal{S}, b \geq 1$ and $\{\varrho_n\} \in C(\mathcal{G},\mathcal{S},\varrho)$, using $FBM4$ of Definition 2.6.1.

$$\begin{aligned} M_b(\varrho,\eta,t) &\geq M_b\left(\varrho,\varrho_n,\frac{t}{2b}\right) * M_b\left(\varrho_n,\eta,\frac{t}{2b}\right) \\ &= 1 * \limsup_{n\to\infty} M_b\left(\varrho_n,\eta,\frac{t}{2b}\right) \\ &= \limsup_{n\to\infty} M_b\left(\varrho_n,\eta,\frac{t}{2b}\right). \end{aligned}$$

Thus

$$M_b(\varrho, \eta, t) = \limsup_{n\to\infty} M_b\left(\varrho_n, \eta, \frac{t}{c}\right); \quad \text{where} \quad c = 2b.$$

□

**Proposition 5.1.8.**

Every dislocated fuzzy metric space $(M_d, X, *)$ is a $GFMS$.

*Proof.*

The proof follows immediately from Proposition 5.1.6. □

**Definition 5.1.9.**

Let $(\mathcal{G}, \mathcal{S}, *)$ be a $GFMS$. A sequence $\{\varrho_n\}$ in $\mathcal{S}$ is called $\mathcal{G}$-convergent sequence if for $\varrho \in \mathcal{S}$,

$$\{\varrho_n\} \in C(\mathcal{G}, \mathcal{S}, \varrho).$$

Following Grabiec [49], the notion of a Cauchy sequence in generalized fuzzy metric spaces can be extended as follows:

**Definition 5.1.10.**

Let $(\mathcal{G}, \mathcal{S}, *)$ be a $GFMS$. A sequence $\{\varrho_n\}$ in $\mathcal{S}$ is called $\mathcal{G}$-Cauchy sequence if

$$\lim_{n,m\to\infty} \mathcal{G}(\varrho_n, \varrho_{n+m}, t) = 1,$$

for all $t > 0$.

**Definition 5.1.11.**

A $GFMS$ in which every $\mathcal{G}$-Cauchy sequence is $\mathcal{G}$-convergent is called a $\mathcal{G}$-complete $GFMS$.

**Remark 5.1.12.**

In generalized fuzzy metric space, a $\mathcal{G}$-convergent sequence may not to be a $\mathcal{G}$-Cauchy sequence.

We now construct an example to show that a $\mathcal{G}$-convergent sequence in $GFMS$ may not be a $\mathcal{G}$-Cauchy sequence.

**Example 5.1.13.**

Let $\mathcal{S} = \mathbb{R}^+ \cup \{0\}$. Define a set

$$C(\mathcal{G}, \mathcal{S}, \varrho) = \left\{ \{\varrho_n\} \subset \mathcal{S} : \lim_{n\to\infty} \mathcal{G}(\varrho_n, \varrho, t) = 1 \right\},$$

for every $\varrho \in \mathcal{S}$ and $t > 0$, where $\mathcal{G} : \mathcal{S} \times \mathcal{S} \times (0, \infty) \to [0, 1]$ is defined by

$$\mathcal{G}(\varrho, \eta, t) = \begin{cases} e^{-\dfrac{\varrho + \eta}{t}} & \text{if atleast one of } \varrho \text{ or } \eta \text{ is } 0, \\ e^{-\dfrac{1 + \varrho + \eta}{t}} & \text{otherwise,} \end{cases}$$

then $(\mathcal{G}, \mathcal{S}, *)$ is $GFMS$, where $*$ is the product $t$-norm *i.e,* $\varrho * \eta = \varrho\eta$.

Consider a sequence $\{\varrho_n\}$ as $\varrho_n = \dfrac{1}{n}$, for all $n \in \mathbb{N}$.

$$\begin{aligned} \lim_{n\to\infty} \mathcal{G}(\varrho_n, 0, t) &= \lim_{n\to\infty} e^{-\dfrac{\varrho_n}{t}} \\ &= \lim_{n\to\infty} e^{-\dfrac{1}{nt}} = 1. \end{aligned}$$

$\Rightarrow$ $\{\varrho_n\}$ is $\mathcal{G}$-convergent to 0.

Note that

$$\begin{aligned} \lim_{n,m\to\infty} \mathcal{G}(\varrho_n, \varrho_{n+m}, t) &= \lim_{n,m\to\infty} e^{-\dfrac{1 + \varrho_n + \varrho_{n+m}}{t}} \\ &= \lim_{n,m\to\infty} e^{-\dfrac{1 + \dfrac{1}{n} + \dfrac{1}{n+m}}{t}} \\ &= \lim_{n,m\to\infty} e^{-\dfrac{1}{t}} . e^{-\dfrac{1}{nt}} . e^{-\dfrac{1}{(n+m)t}} \\ &= e^{-\dfrac{1}{t}} .1.1 \\ &= e^{-\dfrac{1}{t}} \neq 1. \end{aligned}$$

Hence $\{\varrho_n\}$ is not a $\mathcal{G}$-Cauchy sequence.

**Proposition 5.1.14.**

Let $(\mathcal{G}, \mathcal{S}, *)$ be a $GFMS$, $\{\varrho_n\}$ be a sequence in $\mathcal{S}$ and $(\varrho, \eta) \in \mathcal{S} \times \mathcal{S}$. If $\{\varrho_n\}$ $\mathcal{G}$-converges to $\varrho$ and $\{\varrho_n\}$ $\mathcal{G}$-converges to $\eta$ then $\varrho = \eta$.

*Proof.*

Using the property $GFM4$ of Definition 3.1, we have

$$\begin{aligned}\mathcal{G}(\varrho, \eta, t) &\geq \limsup_{n\to\infty} \mathcal{G}\left(\varrho_n, \eta, \frac{t}{c}\right)\\ &= 1.\end{aligned}$$

So, we have $\varrho = \eta$. □

**Definition 5.1.15.**

Let $(\mathcal{G}, \mathcal{S}, *)$ be a $GFMS$. A self mapping $g\colon \mathcal{S} \to \mathcal{S}$ is a fuzzy $k$-contraction if for all $\varrho, \eta \in \mathcal{S}$ and for some $k \in (0, 1)$, we have

$$\mathcal{G}(g(\varrho), g(\eta), kt) \geq \mathcal{G}(\varrho, \eta, t) \quad \forall\, t > 0.$$

**Proposition 5.1.16.**

Let $(\mathcal{G}, \mathcal{S}, *)$ be a $GFMS$ and $g$ be a fuzzy $k$-contraction. If any fixed point $\varrho$ of $g$ satisfies $\mathcal{G}(\varrho, \varrho, t) > 0$ then $\mathcal{G}(\varrho, \varrho, t) = 1$.

*Proof.*

Let $\varrho \in \mathcal{S}$ be a fixed point of $g$. As $g$ is a fuzzy $k$-contraction, so

$$\begin{aligned}\mathcal{G}(\varrho, \varrho, t) &= \mathcal{G}(g(\varrho), g(\varrho), t))\\ &\geq \mathcal{G}\left(\varrho, \varrho, \frac{t}{k}\right)\\ &\geq \mathcal{G}\left(\varrho, \varrho, \frac{t}{k^2}\right)\\ &\vdots\\ &\geq \mathcal{G}\left(\varrho, \varrho, \frac{t}{k^n}\right)\\ &= 1 \text{ as } n \to \infty.\end{aligned}$$

So, we have

$$\mathcal{G}(\varrho, \varrho, t) = 1.$$

□

We now establish the BCP in the setting of $GFMS$ as follows:

**Theorem 5.1.1.**

Let $(\mathcal{G}, \mathcal{S}, *)$ be a $\mathcal{G}$-complete $GFMS$ and $g$ be a fuzzy $k$-contraction. If there exists $\varrho_0 \in \mathcal{S}$ such that $\delta(\mathcal{G}, g, \varrho_0, t) > 0$, where

$$\delta(\mathcal{G}, g, \varrho_0, t) = \sup\left\{\mathcal{G}\left(g^i(\varrho_0), g^j(\varrho_0), t\right); i, j \in \mathbb{N}, t > 0\right\}.$$

Then $\{g^n(\varrho_0)\}$ is convergent to a unique fixed point of $g$.

*Proof.*

Since $g$ is a fuzzy $k$-contraction. So, $\forall\, i, j \in \mathbb{N}$,

$$\mathcal{G}\left(g^{n+i}(\varrho_0), g^{n+j}(\varrho_0), t\right) \geq \mathcal{G}\left(g^{n-1+i}(\varrho_0), g^{n-1+j}(\varrho_0), \frac{t}{k}\right)$$

$$\sup\left\{\mathcal{G}\left(g^{n+i}(\varrho_0), g^{n+j}(\varrho_0), t\right)\right\} \geq \sup\left\{\mathcal{G}\left(g^{n-1+i}(\varrho_0), g^{n-1+j}(\varrho_0), \frac{t}{k}\right)\right\}$$

$$\delta\left(\mathcal{G}, g, g^n(\varrho_0), t\right) \geq \delta\left(\mathcal{G}, g, g^{n-1}(\varrho_0), \frac{t}{k}\right).$$

For every $n > 0$, we get

$$\delta\left(\mathcal{G}, g, g^n(\varrho_0), t\right) \geq \delta\left(\mathcal{G}, g, \varrho_0, \frac{t}{k^n}\right). \tag{5.2}$$

For every $m, n \in \mathbb{N}$, we use (5.2) to obtain

$$\begin{aligned}\mathcal{G}\left(g^n(\varrho_0), g^{n+m}(\varrho_0), t\right) &\geq \delta\left(\mathcal{G}, g, g^n(\varrho_0), \frac{t}{k}\right)\\ &\vdots\\ &\geq \delta\left(\mathcal{G}, g, \varrho_0, \frac{t}{k^n}\right).\end{aligned}$$

Since

$$\delta\left(\mathcal{G}, g, \varrho_0, \frac{t}{k^n}\right) > 0 \quad \text{and} \quad k \in (0,1),$$

then

$$\lim_{n,m\to\infty} \mathcal{G}\left(g^n(\varrho_0), g^{n+m}(\varrho_0), t\right) = 1,$$

which shows that $\{g^n(\varrho_0)\}$ is a $G$-Cauchy sequence.

As $(\mathcal{G}, \mathcal{S}, *)$ is a $\mathcal{G}$-complete $GFMS$, therefore, there is a point $\varrho \in \mathcal{S}$ such as $\{g^n(\varrho_0)\}$ converges to $\varrho$.

$$\begin{aligned} \mathcal{G}\left(g^{n+1}(\varrho_0), g(\varrho), t\right) &\geq \mathcal{G}\left(g^n(\varrho_0), \varrho, \frac{t}{k}\right) \\ &\vdots \\ &\geq \mathcal{G}\left(g(\varrho_0), \varrho, \frac{t}{k^n}\right). \end{aligned}$$

when $n \to \infty$, we get

$$\lim_{n\to\infty} \mathcal{G}\left(g^{n+1}(\varrho_0), g(\varrho), t\right) = 1,$$

which shows that $g^n(\varrho_0)$ converges to $g(\varrho)$.

Using Proposition 3.5, we have

$$g(\varrho) = \varrho,$$

that is $\varrho$ is fixed point of $g$.

**Uniqueness:**

Let $g$ have another fixed point $\delta \in \mathcal{S}$ such that

$$M(\varrho, \delta, t) > 0.$$

Since $g$ is a fuzzy $k$-contraction, so we have

$$\begin{aligned} \mathcal{G}(\varrho, \delta, t) &= \mathcal{G}(g(\varrho), g(\delta), t) \\ &\geq \mathcal{G}\left(\varrho, \delta, \frac{t}{k}\right) \end{aligned}$$

$$= \mathcal{G}\left(g(\varrho), g(\delta), \frac{t}{k}\right)$$
$$\geq \mathcal{G}\left(\varrho, \delta, \frac{t}{k^2}\right)$$
$$\vdots$$
$$\geq \mathcal{G}\left(\varrho, \delta, \frac{t}{k^n}\right)$$
$$= 1 \text{ as } n \to \infty,$$

thus

$$\delta = \varrho.$$

□

The main result of [49] is an immediate consequence of Theorem 5.1.1 and Proposition 5.1.6.

**Corollary 5.1.17.**

Let $(M, \mathcal{S}, *)$ be a $G$-complete $FMS$ and the self map $g : \mathcal{S} \to \mathcal{S}$ be such that for some $k \in (0, 1)$,

$$M(g(\varrho), g(\eta), kt) \geq M(\varrho, \eta, t),$$

for all $\varrho, \eta \in \mathcal{S}$ and $t > 0$. If there is $\varrho_0 \in \mathcal{S}$ such as

$$\sup\left\{M\left(g^i(\varrho_0), g^j(\varrho_0), t\right); i, j \in \mathbb{N}, t > 0\right\} > 0.$$

Then the sequence $\{g^n(\varrho_0)\}$ is $g$-convergent to a unique fixed point of $g$.

Similarly, the Banach contraction theorem for fuzzy $b$-metric space becomes an immediate consequence of Theorem 5.1.1 by using Proposition 5.1.7.

**Corollary 5.1.18.**

Let $(M_b, \mathcal{S}, *)$ be a complete $FbMS$ and the self-map $g$ on $\mathcal{S}$ be such that for some $k \in \left(0, \frac{1}{b}\right)$ and $t > 0$,

$$M_b(g(\varrho), g(\eta), kt) \geq M_b(\varrho, \eta, t),$$

for all $\varrho, \eta \in \mathcal{S}$. If there is $\varrho_0 \in \mathcal{S}$ such as

$$\sup\left\{M_b\left(g^i(\varrho_0), g^j(\varrho_0), t\right); i, j \in \mathbb{N}, t > 0\right\} > 0.$$

Then $\{g^n(\varrho_0)\}$ is $g$-convergent to a unique fixed point of $g$.

The next result follows from Theorem 5.1.1 and Proposition 5.1.8.

**Corollary 5.1.19.**

Let $(M_d, \mathcal{S}, *)$ be a $g$-complete $DFMS$ and $g$ be a self mapping on $\mathcal{S}$ . Suppose there is $k \in (0,1)$ such that

$$M_d(g(\varrho), g(\eta), kt) \geq M_d(\varrho, \eta, t),$$

for all $\varrho, \eta \in \mathcal{S}$. If there is $\varrho_0 \in \mathcal{S}$ such as

$$\sup\left\{M_d\left(g^i(\varrho_0), g^j(\varrho_0), t\right); i, j \in \mathbb{N}, t > 0\right\} > 0.$$

Then $\{g^n(\varrho_0)\}$ is $g$-convergent to a unique fixed point of $g$.

Theorem 5.1.1 is illustrated in the following example.

**Example 5.1.20.**

Let $\mathcal{S} = [0,1]$ and define $\mathcal{G} \colon \mathcal{S} \times \mathcal{S} \times (0, \infty) \to [0,1]$ by

$$\mathcal{G}(\varrho, \eta, t) = e^{-\dfrac{|\varrho - \eta|}{t}},$$

for all $\varrho, \eta \in \mathcal{S}$ and $t > 0$. Then $(\mathcal{G}, \mathcal{S}, *)$ is $\mathcal{G}$-complete $GFMS$.

For $k \in (0,1)$, we define $g : \mathcal{S} \to \mathcal{S}$ by

$$g(\varrho) = \frac{k\varrho}{n} \quad \text{for} \quad n \geq 1,$$

we have

$$\mathcal{G}(g\varrho, g\eta, kt) = \mathcal{G}\left(\frac{k\varrho}{n}, \frac{k\eta}{n}, kt\right)$$

$$= e^{-\frac{\left|\frac{k\varrho}{n} - \frac{k\eta}{n}\right|}{kt}}$$
$$= e^{-\frac{k|\varrho-\eta|}{knt}}$$
$$\geq e^{-\frac{|\varrho-\eta|}{t}} = \mathcal{G}(\varrho,\eta,t).$$
$$\Rightarrow \quad \mathcal{G}(g\varrho, g\eta, kt) \geq \mathcal{G}(\varrho,\eta,t).$$

Thus $g$ is a fuzzy $k$-contraction.

Further, for $\varrho = 0 \in \mathcal{S}$, we have

$$\delta(\mathcal{G}, g, 0, t) = \sup\left\{\mathcal{G}\left(g^i(0), g^j(0), t\right); i, j \in \mathbb{N}, t > 0\right\}$$
$$= \sup\left\{e^{-\frac{|g^i(0)-g^j(0)|}{t}}\right\} = 1 > 0.$$

As a result, all axioms of Theorem 5.1.1 are fulfilled and $\varrho = 0 \in [0,1]$ is a unique fixed point of $g$.

To establish Ćirić's fixed point theorem [16] in $GFMS$, we introduce the notion of quasi-contraction type mapping in $GFMS$ as follows:

**Definition 5.1.21.**

Let $(\mathcal{G}, \mathcal{S}, *)$ be a $GFMS$. A mapping $g\colon \mathcal{S} \to \mathcal{S}$ is called a fuzzy $k$-quasi contraction if for all $\varrho, \eta \in \mathcal{S}, t > 0$ and for some $k \in (0,1)$, we have

$$\mathcal{G}(g(\varrho), g(\eta), kt) \geq \min\left\{\mathcal{G}(\varrho,\eta,t), \mathcal{G}(\varrho, g\varrho, t), \mathcal{G}(\eta, g\eta, t), \mathcal{G}(\varrho, g\eta, t), \mathcal{G}(\eta, g\varrho, t)\right\}.$$

The next result shows that Proposition 5.1.16 also holds for fuzzy $k$-quasi contraction.

**Proposition 5.1.22.**

Let $(\mathcal{G}, \mathcal{S}, *)$ be a $GFMS$ and $g$ be a fuzzy $k$-quasi contraction. If any fixed point $\varrho \in \mathcal{S}$ of $g$ satisfies $\mathcal{G}(\varrho, \varrho, t) \geq 0$ then $\mathcal{G}(\varrho, \varrho, t) = 1$.

*Proof.*

Let $\varrho \in \mathcal{S}$ be a fixed point of $g$. Since $g$ is a fuzzy $k$-quasi contraction so

$$\begin{aligned}
\mathcal{G}(\varrho,\varrho,t) &= \mathcal{G}(g(\varrho),g(\varrho),t)\\
&\geq \min\left\{\mathcal{G}\left(\varrho,\varrho,\frac{t}{k}\right),\mathcal{G}\left(\varrho,g\varrho,\frac{t}{k}\right),\mathcal{G}\left(\varrho,g\varrho,\frac{t}{k}\right),\mathcal{G}\left(\varrho,g\varrho,\frac{t}{k}\right),\right.\\
&\qquad\left.\mathcal{G}\left(\varrho,g\varrho,\frac{t}{k}\right)\right\}\\
&= \min\left\{\mathcal{G}\left(\varrho,\varrho,\frac{t}{k}\right),\mathcal{G}\left(\varrho,\varrho,\frac{t}{k}\right),\mathcal{G}\left(\varrho,\varrho,\frac{t}{k}\right),\mathcal{G}\left(\varrho,\varrho,\frac{t}{k}\right),\right.\\
&\qquad\left.\mathcal{G}\left(\varrho,\varrho,\frac{t}{k}\right)\right\}\\
&= \mathcal{G}\left(\varrho,\varrho,\frac{t}{k}\right)\\
&\geq \mathcal{G}\left(\varrho,\varrho,\frac{t}{k^2}\right)\\
&\vdots\\
&\geq \mathcal{G}\left(\varrho,\varrho,\frac{t}{k^n}\right)\\
&= 1 \text{ as } n\to\infty.
\end{aligned}$$

So, we have

$$\mathcal{G}(\varrho,\varrho,t) = 1.$$

□

Now, we prove one of the main results of [41] in the setting of $GFMS$.

**Theorem 5.1.2.**

Let $(\mathcal{G},\mathcal{S},*)$ be a $\mathcal{G}$-complete $GFMS$ and $g\colon \mathcal{S}\to\mathcal{S}$ be a fuzzy $k$-quasi contraction. If there exists $\varrho_0 \in \mathcal{S}$ such that

$$\delta(\mathcal{G},g,\varrho_0,t) > 0,$$

then $\{g^n(\varrho_0)\}$ is $\mathcal{G}$-convergent to a fixed point $\varrho$ of $g$. Further, if $g$ has any other fixed point $\delta$ such as $\mathcal{G}(\varrho,\delta,t) > 0$ then $\varrho = \delta$.

*Proof.*

Since $g$ is a fuzzy $k$-quasi contraction, so, $\forall\, i, j \in \mathbb{N}$, we have

$$\begin{aligned}
&\mathcal{G}(g^{n+i}(\varrho_0), g^{n+j}(\varrho_0), t)\\
&\geq \min\Bigg\{\mathcal{G}\left(g^{n-1+i}(\varrho_0), g^{n-1+j}(\varrho_0), \frac{t}{k}\right), \mathcal{G}\left(g^{n-1+i}(\varrho_0), g^{n+i}(\varrho_0), \frac{t}{k}\right),\\
&\quad \mathcal{G}\left(g^{n-1+i}(\varrho_0), g^{n+j}(\varrho_0), \frac{t}{k}\right), \mathcal{G}\left(g^{n-1+j}(\varrho_0), g^{n+j}(\varrho_0), \frac{t}{k}\right),\\
&\quad \mathcal{G}\left(g^{n-1+j}(\varrho_0), g^{n+i}(\varrho_0), \frac{t}{k}\right)\Bigg\}.
\end{aligned}$$

$$\begin{aligned}
&\sup \mathcal{G}\Big(g^{n+i}(\varrho_0), g^{n+j}(\varrho_0), t\Big)\\
&\geq \min\Bigg\{\sup \mathcal{G}\left(g^{n-1+i}(\varrho_0), g^{n-1+j}(\varrho_0), \frac{t}{k}\right), \sup \mathcal{G}\left(g^{n-1+i}(\varrho_0), g^{n+i}(\varrho_0), \frac{t}{k}\right),\\
&\quad \sup \mathcal{G}\left(g^{n-1+i}(\varrho_0), g^{n+j}(\varrho_0), \frac{t}{k}\right), \sup \mathcal{G}\left(g^{n-1+j}(\varrho_0), g^{n+j}(\varrho_0), \frac{t}{k}\right),\\
&\quad \sup \mathcal{G}\left(g^{n-1+j}(\varrho_0), g^{n+i}(\varrho_0), \frac{t}{k}\right)\Bigg\}.
\end{aligned}$$

$$\begin{aligned}
&\delta(\mathcal{G}, g, g^{n}(\varrho_0), t)\\
&\geq \min\Bigg\{\delta\left(\mathcal{G}, g, g^{n-1}(\varrho_0), \frac{t}{k}\right), \delta\left(\mathcal{G}, g, g^{n}(\varrho_0), \frac{t}{k}\right), \delta\left(\mathcal{G}, g, g^{n}(\varrho_0), \frac{t}{k}\right),\\
&\quad \delta\left(\mathcal{G}, g, g^{n}(\varrho_0), \frac{t}{k}\right), \delta\left(\mathcal{G}, g, g^{n}(\varrho_0), \frac{t}{k}\right)\Bigg\}\\
&= \min\left\{\delta\left(\mathcal{G}, g, g^{n-1}(\varrho_0), \frac{t}{k}\right), \delta\left(\mathcal{G}, g, g^{n}(\varrho_0), \frac{t}{k}\right)\right\}.
\end{aligned}$$

This implies that

$$\delta\left(\mathcal{G}, g, g^{n}(\varrho_0), t\right) \geq \delta\left(\mathcal{G}, g, g^{n-1}(\varrho_0), \frac{t}{k}\right).$$

for all $n > 0$, it follows that ,

$$\delta\left(\mathcal{G}, g, g^{n}(\varrho_0), t\right) \geq \delta\left(\mathcal{G}, g, \varrho_0, \frac{t}{k^n}\right).$$

For every $n, m \in \mathbb{N}$, using the above inequality, we get

$$\begin{aligned}\mathcal{G}\left(g^n(\varrho_0), g^{n+m}(\varrho_0), t\right) &\geq \delta\left(\mathcal{G}, g, g^n(\varrho_0), t\right)\\ &\vdots\\ &\geq \delta\left(\mathcal{G}, g, \varrho_0, \frac{t}{k^n}\right). \end{aligned} \tag{5.3}$$

As

$$\delta\left(\mathcal{G}, g, \varrho_0, \frac{t}{k^n}\right) > 0 \quad \text{and} \quad k \in (0,1),$$

so,

$$\lim_{n,m\to\infty} \mathcal{G}\left(g^n(\varrho_0), g^{n+m}(\varrho_0), t\right) = 1,$$

which implies that $\{g^n(\varrho_0)\}$ is a $G$-Cauchy sequence. Since $(\mathcal{G}, \mathcal{S}, *)$ is a $\mathcal{G}$-complete $GFMS$, so, there is some $\varrho \in \mathcal{S}$ such as $\{g^n(\varrho_0)\}$ converges to $\varrho$.

$$\begin{aligned}&\mathcal{G}(g^{n+1}(\varrho_0), g(\varrho), t)\\ &\quad\geq \min\left\{\mathcal{G}\left(g^n(\varrho_0), \varrho, \frac{t}{k}\right), \mathcal{G}\left(g^n(\varrho_0), g^{n+1}(\varrho_0), \frac{t}{k}\right), \mathcal{G}\left(\varrho, g\varrho, \frac{t}{k}\right),\right.\\ &\qquad\qquad \left.\mathcal{G}\left(g^n(\varrho_0), g\varrho, \frac{t}{k}\right), \mathcal{G}\left(\varrho, g^{n+1}(\varrho_0), \frac{t}{k}\right)\right\}\\ &\quad= \mathcal{G}\left(g^n(\varrho_0), \varrho, \frac{t}{k}\right).\end{aligned}$$

So, we have

$$\begin{aligned}\mathcal{G}\left(g^{n+1}(\varrho_0), g(\varrho), t\right) &\geq \mathcal{G}\left(g^n(\varrho_0), \varrho, \frac{t}{k}\right)\\ &\vdots\\ &\geq \mathcal{G}\left(g(\varrho_0), \varrho, \frac{t}{k^n}\right),\end{aligned}$$

when $n \to \infty$, then we get

$$\lim_{n\to\infty} \mathcal{G}\left(g^{n+1}(\varrho_0), g(\varrho), t\right) = 1,$$

which shows that $g(\varrho)$ is fixed point of $g^n(\varrho_0)$.

By using Proposition 3.5, we have

$$g(\varrho) = \varrho.$$

This implies that $\varrho$ is fixed point of $g$.

**Uniqueness**

On the other hand, suppose $g$ has any other fixed point $\delta \in \mathcal{S}$ such as

$$\mathcal{G}(\varrho, \delta, t) > 0.$$

Since $g$ is a fuzzy $k$-quasi contraction, so we have

$$\begin{aligned}
\mathcal{G}(\varrho, \delta, t) &= \mathcal{G}(g(\varrho), g(\delta), t) \\
&\geq \min\left\{\mathcal{G}\left(\varrho, \delta, \frac{t}{k}\right), \mathcal{G}\left(\varrho, g\varrho, \frac{t}{k}\right), \mathcal{G}\left(\delta, g\delta, \frac{t}{k}\right), \mathcal{G}\left(\varrho, g\delta, \frac{t}{k}\right), \mathcal{G}\left(\delta, g\varrho, \frac{t}{k}\right)\right\} \\
&= \min\left\{\mathcal{G}\left(\varrho, \delta, \frac{t}{k}\right), \mathcal{G}\left(\varrho, \varrho, \frac{t}{k}\right), \mathcal{G}\left(\delta, \delta, \frac{t}{k}\right), \mathcal{G}\left(\varrho, \delta, \frac{t}{k}\right), \mathcal{G}\left(\delta, \varrho, \frac{t}{k}\right)\right\} \\
&= \min\left\{\mathcal{G}\left(\varrho, \delta, \frac{t}{k}\right), 1, 1, \mathcal{G}\left(\varrho, \delta, \frac{t}{k}\right), \mathcal{G}\left(\delta, \varrho, \frac{t}{k}\right)\right\} \\
&= \mathcal{G}\left(\varrho, \delta, \frac{t}{k}\right).
\end{aligned}$$

So,

$$\begin{aligned}
\mathcal{G}(\varrho, \delta, t) &\geq \mathcal{G}\left(\varrho, \delta, \frac{t}{k}\right) \\
&\geq \mathcal{G}\left(\varrho, \delta, \frac{t}{k^2}\right) \\
&\vdots \\
&\geq \mathcal{G}\left(\varrho, \delta, \frac{t}{k^n}\right) \\
&= 1 \text{ as } n \to \infty.
\end{aligned}$$

So, we have

$$\mathcal{G}(\varrho, \delta, t) = 1.$$

Thus

$$\delta = \varrho.$$

□

**Remark 5.1.23.**

As consequences of Theorem 5.1.2, as before (see Corollary 5.1.17–5.1.19), one can obtain fixed point theorems for Ćirić's quasi-contractions in $FMS$, $FbMS$ and $DFMS$.

## 5.2 Application:

Fixed point theory turns out to be a powerful tool for the study of existence of the solution of various kind of integral and differential equations, for instance see [27, 28, 84, 98]. In this section, a particular non-linear integral equation has been studied for the existence of the solution as an application of our main result proved in Theorem 5.1.1. Consider $\mathcal{S} = C[0, J]$, the class of all real valued continuous functions defined on $[0, J]$. Define a $\mathcal{G}$-complete generalized fuzzy metric $\mathcal{G}\colon \mathcal{S} \times \mathcal{S} \times (0, \infty) \to [0, 1]$ by

$$\mathcal{G}(\varrho, \eta, t) = e^{-\dfrac{\sup\limits_{s\in[0,J]} |\varrho(s) - \eta(s)|}{t}}$$

for all $\varrho, \eta \in \mathcal{S}$ and $t > 0$. Taking the following integral equation

$$\varrho(t) = \lambda(t) + \int_0^J \varsigma(t, s)\omega(t, s, \varrho(s))ds, \tag{5.4}$$

where $J > 0$, $\lambda\colon [0, J] \to \mathbb{R}$, $\varsigma\colon [0, J] \times [0, J] \to \mathbb{R}$, $\omega\colon [0, J] \times [0, J] \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

**Theorem 5.2.1.**

Let $(\mathcal{G}, \mathcal{S}, *)$ be a $\mathcal{G}$-complete $GFMS$ defined above. Let $g : \mathcal{S} \to \mathcal{S}$ be the integral operator defined by

$$g(\varrho(t)) = \lambda(t) + \int_0^J \varsigma(t,s)\, \omega(t,s,\varrho(s))ds$$

for all $\varrho \in \mathcal{S}$, and $t, s \in [0, J]$.

Suppose that the following conditions are satisfied:

(i) For all $t, s \in [0, J]$ and $\varrho, \eta \in \mathcal{S}$, we have

$$|\omega(t,s,\varrho(s)) - \omega(t,s,\eta(s))| < |\varrho(s) - \eta(s)|.$$

(ii) For all $t, s \in [0, J]$ ,

$$\sup_{s\in[0,J]} \left| \int_0^J (\varsigma(t,s))ds \right| \leq k < 1.$$

Then the integral equation (5.4) has a solution $\varrho^* \in \mathcal{S}$.

*Proof.* Note that for $\varrho^* \in \mathcal{S}$, we have

$$\begin{aligned} \delta(\mathcal{G}, g, \varrho^*, t) &= \sup\left\{ \mathcal{G}\left(g^i(\varrho^*), g^j(\varrho^*), t\right); i, j \in \mathbb{N}, t > 0 \right\} \\ &= \sup\left\{ e^{-\frac{\sup_{s\in[0,J]} |g^i(\varrho^*(s)) - g^j(\varrho^*(s))|}{t}} \right\} > 0. \end{aligned}$$

We only have to show that $g$ is fuzzy $k$-contraction. For all $\varrho, \eta \in \mathcal{S}$ ,

$$\begin{aligned} \mathcal{G}(g\varrho, g\eta, kt) &= e^{-\frac{\sup_{s\in[0,J]} |g\varrho(t) - g\eta(t)|}{kt}} \\ &= e^{-\frac{\sup_{s\in[0,J]} \left| \int_0^J \varsigma(t,s)\, \omega(t,s,\varrho(s))ds - \int_0^J \varsigma(t,s)\, \omega(t,s,\eta(s))ds \right|}{kt}} \\ &= e^{-\frac{\sup_{s\in[0,J]} \left| \int_0^J \varsigma(t,s)\{\, \omega(t,s,\varrho(s)) - \omega(t,s,\eta(s)\}ds \right|}{kt}} \end{aligned}$$

$$\geq e^{-\frac{\sup_{s\in[0,J]}\left|\int_0^J (\varsigma(t,s))ds\right| \int_0^J |\omega(t,s,\varrho(s)) - \omega(t,s,\eta(s)|ds}{kt}}$$

$$\geq e^{-\frac{k\int_0^J |\varrho(s)-\eta(s)|ds}{kt}}$$

$$\geq e^{-\frac{k\sup_{s\in[0,J]}|\varrho(s)-\eta(s)|}{kt}}$$

$$= e^{-\frac{\sup_{s\in[0,J]}|\varrho(s)-\eta(s)|}{t}} = \mathcal{G}(\varrho,\eta,t).$$

So, we have

$$\mathcal{G}(g\varrho, g\eta, kt) \geq \mathcal{G}(\varrho,\eta,t).$$

Since all the axioms of Theorem 5.1.1 are fulfilled and hence $g$ has a fixed point. So, the integral equation (5.4) has a solution. □

The following result is an immediate consequence of Theorem 5.2.1 and Corollary 5.1.17.

**Corollary 5.2.1.**

Let $(M, \mathcal{S}, *)$ be a $G$-complete $FMS$. Let $g : \mathcal{S} \to \mathcal{S}$ be the integral operator defined by

$$g(\varrho(t)) = \lambda(t) + \int_0^J \varsigma(t,s)\,\omega(t,s,\varrho(s))ds,$$

for all $\varrho \in \mathcal{S}$, and $t, s \in [0, J]$. Suppose that the following conditions are satisfied:

(i) For all $t, s \in [0, J]$ and $\varrho, \eta \in \mathcal{S}$, we have

$$|\omega(t,s,\varrho(s)) - \omega(t,s,\eta(s))| < |\varrho(s) - \eta(s)|.$$

(ii) For all $t, s \in [0, J]$ ,

$$\sup_{s\in[0,J]}\left|\int_0^J (\varsigma(t,s))ds\right| \leq k < 1.$$

Then the equation (5.4) has a solution $\varrho^* \in \mathcal{S}$.

Similar results for $G$-complete $FbMS$ and $G$-complete $DFMS$ follow immediately from Theorem 5.2.1.

## 5.3 Conclusion

In this chapter, the notion of $GFMS$ have been introduced which is a proper generalization of class of $FMS$. We have also defined fuzzy $k$-contraction and fuzzy $k$-quasi contraction. Using these contractions, we have established certain new fixed point results. Various fixed point theorems in other fuzzy abstract spaces can be seen as immediate consequences of our main results. The work presented here is likely to provide a ground to the researchers to do work in different structures by using these contractions and related fixed point results.

# Chapter 6

# Conclusion and Future Work

The main focus of this research is to study the notion of $FbMS$ and $GFMS$ to extend the theory of fixed point for these fuzzy abstract spaces. Several fixed point theorems are presented that generalise and unify a number of related results in the literature under various contractive assumptions. Some important examples are presented to demonstrate the main results. Recently, Nădăban [85] highlighted the properties and usefulness of fuzzy Euclidean normed spaces and $FbMS$ in solving problems in various sciences. The author has prepared a ground to extend the theory of fixed points in these spaces. In the conclusion of [85], the author suggested to prove certain fixed point theorem in $FbMS$.

- In Chapter 3 of this dissertation, the well known BCP [14] and the main result of Hicks and Rhoad's [21] are established in the setting of $FbMS$ by restricting the contraction mapping to the elements in the orbit of a point in $FbMS$ and these results are illustrated by an example. These obtained results generalize and unify the main result of Grabiec [49] as well as the result of Hicks and Rhoad's [21] in the literature of fuzzy metric spaces. In addition, the results of [57] are proven utilising the rational contraction and the control functions in the $FbMS$. An example is also presented to illustrate the theorems. There is also an application for elaborating the validity of the obtained results.

- In Chapter 4, the concept of Geraghty type contraction is used to generalize the main result of Grabeic [49] in the setting of $G$-complete $FbMS$. An example is given to validate the obtaiend result. Moreover, analogue of the main results of Faraji et al. [33] and Alsulami et al. [89] are also established in $G$-complete $FbMS$. The existence problem for the solution of a non linear integral equation is also presented as an application of our main result. Further, corollaries indicate that the theorems established in this work generalize many existing results in the literature.

  We have also established some common fixed point results analogues to [33] using Geraghty type contraction in $G$-complete $FbMS$. Moreover, an analogues of the fixed-point theorems of Gupta et.al [57] are proved in $G$-complete $FbMS$. An application to explore the existence of the solution of an integral equation is also presented to support the results.

- In Chapter 5, the notion of $GFMS$ is introduced which generalizes $FMS$, $FbMS$ and $DFMS$. The notions of fuzzy $k$-contraction and fuzzy $k$-quasi contraction are also defined. Using these contractions, certain new fixed point results are established. These results are more general than the existing results in the literature and generalize many existing results in fuzzy abstract spaces. The present research provides a ground to the researchers to do work in different structures by using these contractions and related fixed point results.

# Future Work

In future, in the context of *i.e.* new notion $GFMS$, various contraction and expansion maps presented in this work, can be applied in different abstract spaces such as Modular $FMS$, $FbMS$, Hausdorf $FMS$ etc. as well as for different kinds of contraction and expansion mappings like weakly commuting mappings, compatible mappings etc. and hence can be generalized in many ways. Moreover, these new concepts can provide a wide range for applications in numerous areas such as

Chemistry, Health Sciences, Economics etc. The study of $GFMS$ presented in Chapter 5 can also be extended by exploring common coupled fixed point results.

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