CAPITAL UNIVERSITY OF SCIENCE AND TECHNOLOGY, ISLAMABAD

Synchronization and Anti-synchronization of Financial Chaotic Systems using Sliding Mode Control Technique

by

Jawad Ali

A thesis submitted in partial fulfillment for the degree of Master of Science

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Department of Electrical Engineering

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This thesis is dedicated to my Parents and Family.

For their endless love, support, care and encouragement.

CERTIFICATE OF APPROVAL

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Abstract

In this work a complete synchronization and anti-synchronization of financial chaotic systems are presented. The proposed control strategies are based on first order sliding mode and adaptive integral sliding mode for complete synchronization and anti-synchronization of financial chaotic system. In first case the system parameters are supposed to be known and first order sliding mode control is used for synchronization and anti-synchronization. In second case the system parameters are supposed to unknown and adaptive integral sliding mode control is used to adopt the unknown parameters of the system for synchronization and anti-synchronization. To employ the adaptive integral sliding mode control, the error system is transformed into a special structure containing a nominal part and some unknown terms. Then the error system is stabilized using integral sliding mode control. The stabilizing controller for the error system is constructed which consists of the nominal control plus compensator control. The compensator controller and the adapted laws are derived on the basis of Lyapunov stability theory.

The proposed control strategies are verified for the following chaotic systems: 3D Financial Chaotic System and Identical 4D Hyperchaotic Financial System to achieved the complete synchronization and anti-synchronization together with the improved performance.

Contents

List of Figures

Abbreviations

AISMC	Adaptive Integral Sliding Mode Control
CS	Complete Synchronization
HOSMC	Higher Order Sliding Mode Control
MRAC	Model-Reference Adaptive Control
MIMO	Multi-Input Multi-Output
RP	Reaching Phase
SS	Sliding Surface
SOSMC	Second Order Sliding Mode Control
SMC	Sliding Mode Control
STC	Self Tuning Control
VSC	Variable Structure Control
3D	3 Dimensional
4D	4 Dimensional

Chapter 1

Introduction

1.1 Introduction

In literature chaos has not a general definition but there are some properties of chaotic systems which identify the chaotic behavior of systems. Sensitivity to initial conditions is the most common property of these systems. Any non-linear systems can have the ability to show chaotic behavior if it have at least 3-Dimensional system for autonomous system and 2-Dimensional system for non-autonomous system [1]. Any system can be chaotic, identified by its lyapunov exponent [2]. For 3D system, the system show a chaotic behavior if its lyapunov exponents are:

- First lyapunov exponent must be positive
- Second lyapunov exponent must be negative
- Third lyapunov exponent must be zero

So, in a third order dynamical system, the sign of the Lyapunov exponent could be positive, negative and zero to show chaotic behavior [3].
For 4D system, the system show a chaotic behavior if its lyapunov exponents are:

- First and second lyapunov exponents must be positive
- Third lyapunov exponent must be zero

- Fourth lyapunov exponent must be negative

So, in a forth order dynamical system, the sign of the Lyapunov exponent could be positive, negative and zero to show chaotic behavior [4].
All physical system have nonlinear dynamics and most of them show a chaotic behavior. For better understanding of regarding dynamical behavior of nonlinear systems, a crucial circumstance is investigated synchronization between the nonlinear physical systems. In many naturally processes synchronization has been observed and show a significant impact on everyday life including science, technology and social life. Synchronization play an important role among researchers because of its diverse applications in various fields. In literature, the issue of synchronization of nonlinear systems have been extensively studied. Many times, the parameters of nonlinear chaotic systems are unknown, the estimation of unknown parameters is crucial. Estimated laws have a strong effect on synchronization and anti-synchronization of chaotic systems, their influence on effecting of nonlinear systems cannot be prevented. In many non-linear chaotic systems, wrong value of unknown parameters could be supply uncertainty and disturb the closed loop performance of system. Figure (1.1) and (1.2) shows the block diagrams for master-slave systems and synchronization of nonlinear master-slave systems via an appropriate controller respectively. Appropriate control signal ensured the convergence of error dynamics. The proposed work presents a robust sliding mode

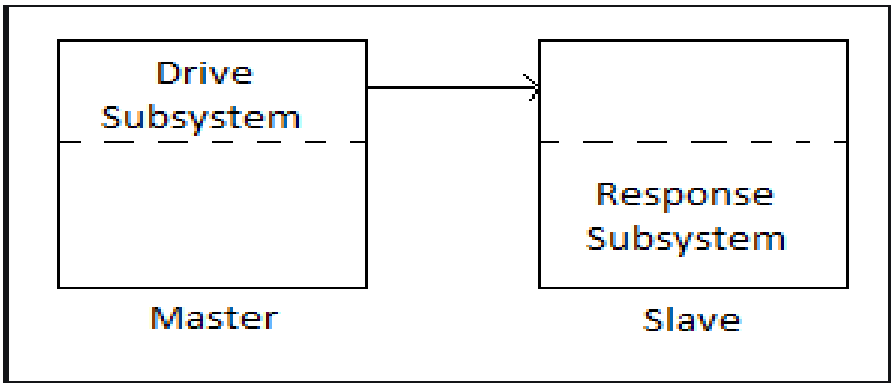


Figure 1.1: Master and Slave Systems

control for synchronization and anti-synchronization of 3D and 4D financial chaotic systems. A sliding manifold is chosen to design a sliding mode control and synchronization and anti-synchronization is achieved in the attending of uncertainty. The simulation results is done in MATLAB. The error system is asymptotically stable and converges to origin.

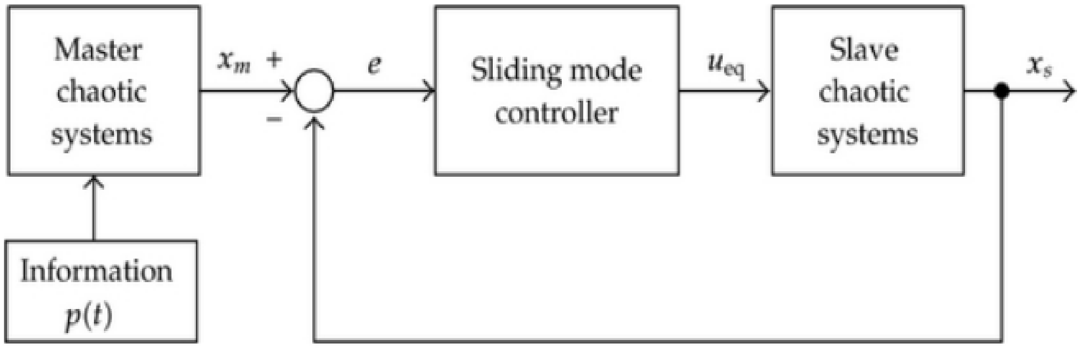


FIGURE 1.2: Block diagram of synchronization using controller

1.2 Overview

Synchronization and anti-synchronization of 3D and 4D financial chaotic systems is the rudimentary determination of this work. We need to stabilize the dynamics of error system (difference of master and slave system) for any initial condition. The technique used in this work is SMC. In first case, parameters of systems is considered to be known and First Order SMC is applied to accomplish synchronization and anti-synchronization. In second case, parameters of systems is considered to be unknown and AISMC is concerned. Adaptive laws are designed via Lyapunov stability theory and convergence of error system is ensured in the presence of external disturbance to verify the robustness.

1.3 Motivation

In last decade, the interests in synchronization and anti-synchronization of nonlinear systems has been increased. Chaotic financial systems have a broad range of applications in different fields including cryptography, Geophysics, biology, electrical engineering, robotics and so on. Due to diverse applications of chaotic systems, it is very difficult to avoid contact with chaotic behaviors. Control problem of chaotic systems is very difficult because of its sensitive nature. It is actually tough to talk about all the appliance domain throughout this short portion, but, some vigorous research areas and applied examples within the synchronization are described. Figure (1.3) shows chaotic behaviors in different phenomenon.

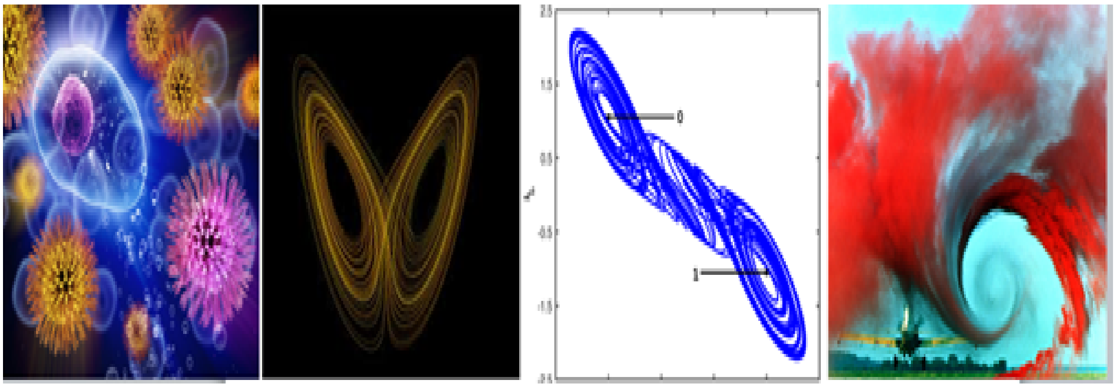


FIGURE 1.3: Chaotic Behavior

1.4 Thesis Objective

The intention of this research work is to develop suitable synchronization and anti-synchronization strategies for 3D and 4D nonlinear financial chaotic systems working in accordance with master-slave principal that addresses

- Chaos Synchronization of 3D and 4D financial chaotic systems.
- Anti-Synchronization of 3D and 4D financial chaotic systems.

1.5 Application of Proposed Work

Since we are dealing with the financial chaotic systems, these systems have very diverse applications in different fields. This research thesis can help scientific society in the different fields which are as following.

- Secure Communication
- Encryption
- Stock Exchange
- Contagious Diseases
- Economic Forecasting
- Power Grid
- Control of irregular devices and systems

1.6 Thesis Organization

This thesis has 5 chapters. After a brief introduction The rest of this thesis is organized as follows:

Chapter 2: Literature survey

This chapter provides the available literature published regarding the synchronization and anti-synchronization of financial chaotic systems. Base available on literature review, a more effective control strategies are proposed for 3D and 4D financial chaotic systems.

Chapter 3: Proposed Control Algorithms for Complete Synchronization and Anti-Synchronization

This chapter provides the proposed robust control technique for synchronization and anti-synchronization of financial chaotic systems. Adaptive sliding mode is designed to investigate problem of synchronization and anti-synchronization of non-linear chaotic financial systems considered the known and unknown parameters, finally using lyapunov function to substantiate the stability of preferred control technique.

Chapter 4: Applications of Proposed Algorithm

This chapter presents a simulations and results. The efficiency of proposed control technique is applied to different financial chaotic systems such as synchronization of financial chaotic systems and complete synchronization of 4D hyper chaotic financial systems

Chapter 5: Conclusion and Future work

This chapter summarize the thesis and draws assumption. The significance of the proposed work is emphasized. Future directions have also been set for further work.

Chapter 2

Literature Survey

2.1 Introduction

This chapter give a review of chaotic systems, synchronization, anti-synchronization, SMC and AISMC and its technological tendencies are given in literature.

2.2 Chaos

Chaos theory is definitely the branch of mathematics, it's the study of apparently unpredictable behavior in systems governed by deterministic laws. A dynamical strategy is addressed chaotic when this satisfies following properties given in [5], which popularly termed as butterfly effect.

- Boundness
- Infinite recurrence
- Sensitive reliance on initial conditions

Chaos is referred to as the fact that dynamical system which does not repeat itself, despite this system is governed by deterministic equations [6]. Period and the frequency are accustomed to identify chaotic signals, while phase-plane and correlation are accustomed to identify the attractor and randomness during the chaotic system. The attractor is section of a state space that there won't be exit paths.

That is obviously, points which get close enough to the attractor remain close even after being slightly disturbed. A single state which is known as equilibrium state occurs in attractor, as well as a cycle of states referred to as a limit cycle [6]. For chaotic systems, the attractor probably wouldn't fix to one particular but explores all a state space surrounding the attractor forever without repeating.

Applying the mathematics of chaos are highly diverse, including study regarding turbulent flow of fluids, swirling smoke from cigarette, population dynamics, chemical reactions, communication engineering, plasma physics, along with the motion of groups and clusters of stars. Apart from irregular performance of actual-world systems, chaos is also invoked for making clear properties such as real trajectories shown in a particular state space or sojourn times during trajectories in exacting areas of state space [7]. The nature of scientific details whilst in the literature on chaos is carefully under-discussed that will put it gently.

Figure (2.1) show the trajectory in the Lorenz attractor from the phase plane, depicting the stretching and folding properties [8], which is seen when plotting the phase plane.

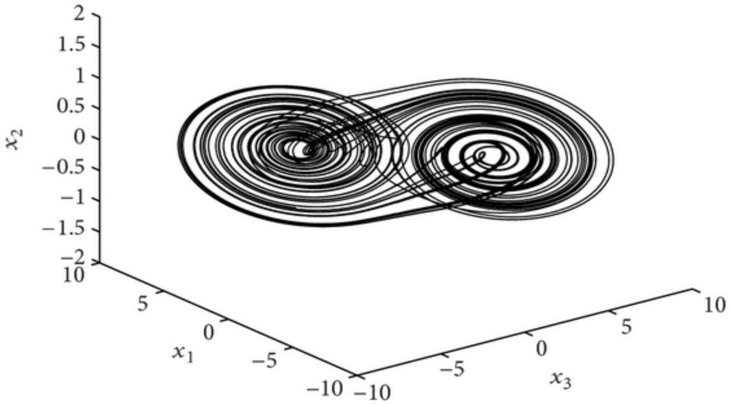


Figure 2.1: The phase portrait of x_1, x_2, x_3

In weather model Lorentz discovered a chaotic phenomenon in [6]. Subsequently, In 1976 Rossler discovered a chaotic system [9]. Chaos theory has applications in a variety of fields of science and engineering like oscillators, dynamos, Tokamak systems, chemical reactions, neural networks, neurology, biology, electrical circuits crypto systems, memristors random bit generator etc.

Quite a lot of reality phenomena exhibit non-linear behavior, whereas others are typically nonlinear. In many systems different chaotic orders occurs including

Swirling smoke from cigarettes, randomly dribbling water through faucet, a waving flag in wind and biological populations [5].

Initially a mathematical type of chaos was first discovered by Lorenz in 1963 [10]. After Lorenz various popular chaotic systems are suggested by Rikitake in [7], Rossler [11], Shimizu-Morioka [12], Chua [13], Rucklidge [14], Sprott [9] and Chen [15]. In parallel while using the developments, chaos and chaotic systems are utilized in lots of scientific disciplines including engineering, computing, communication, medicine, biology, management-finance and electronics [16]. Numerous novel chaotic and hyperchaotic systems shows a different dynamical behaviors have studied in literature [17, 18].

A hyperchaotic attractor is normally considered chaotic behavior with at the least two positive Lyapunov exponents, coupled with one null exponent throughout the flow one negative exponent to ensure the boundless of the perfect solution, so minimum dimension to the hyperchaotic system is 4.

Recently, there seemed to be great involvement with research on hyperchaotic systems and applications in secure communications, data encryption, etc. The earliest 4D hyperchaotic strategy is discovered by O.E. Rossler in 1976. This figure (2.2) is taken from [18].

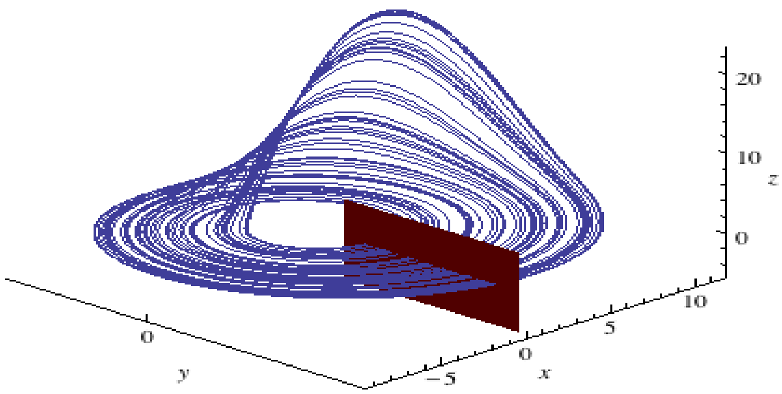


FIGURE 2.2: First Rossler Hyperchaos

Recently, the generation of hyper chaos together with hyperchaotic circuit realization have attracted researchers increasing attention. The hyperchaotic system has a minimum of two positive Lyapunov exponents, indicating the fact that dynamics are enlarged in lots of direction at same time. For virtually any autonomous

continuous system, the dimension in the hyperchaotic attractor must be at least four, however, for that chaotic attractor, three-dimension is enough and has just a certain positive Lyapunov exponent. Therefore, in comparison with ordinary chaotic system, hyperchaotic system has more difficult and richer dynamics so as to be superior used in a number of chaos needed fields.
This thesis presents control technique for chaotic financial systems. The core determination in the work is usually to introduce the most recent control technique for chaotic financial systems. A first order SMC technique introduced for known, while AISMC technique with unknown parameters of chaotic financial system.

2.3 Synchronization of Chaotic Systems

Synchronization of chaos identifies an operation wherein two or many chaotic systems either identical or non identical rearrange a specific property with the motion to a standard behavior due to coupling or towards forcing (periodical or noisy). While chaos synchronization most prone to accomplish, despite the fact that chaotic subsystems may be different with exact same initial conditions, and their outcomes often diverge from another.
Synchronization processes occur in each and every field of life, which play a crucial role in many different contexts, the inclination of just living entities, including animals to humans, to synchronize jointly may be known as the commonest tendency throughout the universe. Many fireflies synchronously illuminate, while geese fly at an identical speed in formation. Applause at concert halls merges carryout a harmonized sound eventually, along with the menstrual periods of women who closely interrelate for a long time also synchronize. 1000s of cardiac pacemaker cells during the heart fire in synchronization to sustain life. Inanimate objects, similar to particles and planets, synchronize as well. Lasers are produced when trillions of atoms oscillating synchronized emit photons the identical phase and frequency. Moreover, either side belonging to the moon could be displayed since orbital and rotational periods belonging to the moon are synchronized because of the gravitational pull between the planet earth and moon.
In the previous couple of decades, there was clearly considerable interest concerning synchronization of chaotic and hyperchaotic systems. Regarding their seminal paper in 1990, Pecora and Carroll [13] initiated a way to synchronize two identical

chaotic systems and says it turned out feasible for many chaotic systems being completely synchronized. Later, chaos synchronization are actually utilized for numerous fields including physics [19], chemistry [20], ecology [21], secure communications [22], cardiology [16], robotics [23], complex dynamical networks and so on.

Previously, various control techniques are proposed for synchronization and anti-synchronization financial chaotic and various non-linear chaotic systems e.g, complete synchronization [11, 24, 25], lag synchronization [12, 26], anticipated synchronization [14], phase synchronization [9], project synchronization [16, 27], generalized synchronization [18], mixed synchronization [28] and passivity based synchronization [29]. As a precise case of complete synchronization and anti-synchronization is attained if driven and response meet that they are quite same. It has been confirmed numerically and experimentally, the fact that coupled chaotic systems can attain anti-synchronization [30]. Recently, control methods are accustomed to anti-synchronize identical or non-identical chaotic systems and derive sufficient anti-synchronization conditions, e.g, observer control [31], linear feedback control [32], back-stepping control [33], adaptive control [34], SMC [35], non-linear control [36], $H\infty$ control [37], etc.

Throughout this thesis, the most recent control scheme relative to the adaptive sliding mode control for chaotic synchronization of two chaotic financial systems is used. The SMC method used for the basic attributes of fast response, easy realization, and good transient performance in addition to its insensitivity to parameter variations and external disturbances.

Research work at synchronization of non-linear systems is briefly revisited as follows. Since as the pioneer utilize synchronization of two non-linear systems, namely, master and slave systems [12], the contest of synchronization of non-linear systems are generally extensively studied within theoretical and practical systems. Study of synchronization is evolved making use of dynamical parameters of nonlinear systems similar to unknown parameters etc.

2.3.1 Types of Synchronization

Some main types of synchronization are discussed below:

- **Complete Synchronization:**
  Signify when master and slave meet for being exactly same is named a Complete synchronization.

$$\lim_{t\to+\infty} \|e(t)\| = \lim_{t\to+\infty} \|y(t) - x(t)\| = 0 \tag{2.1}$$

- **Generalized Synchronization:**
  Signify synchronization within states of two systems utilizing a functional relation is named generalized synchronization.

$$\lim_{t\to+\infty} \|e(t)\| = \lim_{t\to+\infty} \|y(t) - D(x(t))\| = 0 \tag{2.2}$$

- **Phase Synchronization:**
  Phase Synchronization mean that when they have bounded phase difference and uncorrelated amplitude. $\lim_{t\to\infty} \|\phi_1(t) - \phi_2(t)\|$=0 where, $\phi_1(t)$ and $\phi_2(t)$ indicates the phases of two coupled oscillators.

- **Lag Synchronization:**
  Signify when dynamics is explained delay differential equations. Actually during this one of the many oscillators follows of other.
  $\lim_{t\to\infty} \|X_1(t) - X_2(t-\tau)\|$=0, Where τ is delay.

- **Projective Synchronization:**
  In this the states of master $X(t)$ and slave system $Y(t)$ synchronize with respect to scaling factor α. i.e.

$$\lim_{t\to+\infty} \|e(t)\| = \lim_{t\to+\infty} \|y(t) - \alpha x(t)\| = 0. \tag{2.3}$$

2.3.1.1 Complete Synchronization (CS)

The trajectories belonging to the master along with slave systems converge in becoming precisely the same. This is actually the foremost and an effective way of synchronization [11]. This is situated coupled somehow the identical systems and well referred being identical synchronization.
Chaotic systems are dynamical systems that shows synchronization, this can essential feature that it is very sensitive to initial conditions [12]. This means that, two identical chaotic systems starting at nearly exactly the same initial points

in phase space develop onto trajectories which become uncorrelated through the entire time.

Chaos synchronization problem have been served using control design techniques where synchronization is perhaps addressed for the tracking or will probably be stabilization problem. We applied stabilization control techniques rather than tracking issues. Applying these stabilization control techniques to dynamical systems which is known as error system, a controller might be designed which renders the stabilization from the error trajectories to the origin. These dynamical error strategy is constructed from main distinction between the master and slave systems. We consider this to be particular synchronization as complete synchronization (CS) as shown in Figure (2.3).

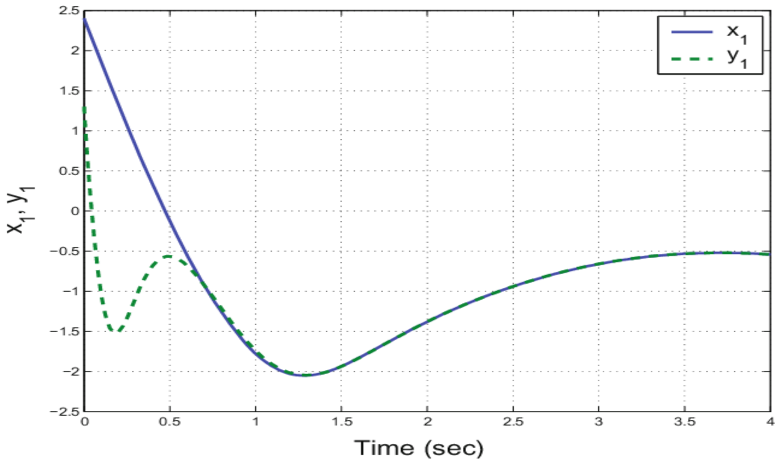


FIGURE 2.3: Complete Synchronization of x_1 and y_1

Two continuous-time chaotic systems:

$$\begin{aligned} \dot{x} &= F(x(t)) \\ \dot{y} &= H(y(t)) \end{aligned} \tag{2.4}$$

called complete synchronization if obey the following condition:

$$\lim_{t\to\infty} \|e(t)\| = \lim_{t\to\infty} \|y(t) - x(t)\| = 0 \tag{2.5}$$

2.4 Sliding Mode Control

SMC is robust nonlinear control design technique with inherent robustness properties in case of uncertainties and parametric variations. SMC inherent a discontinuous term for robustness. SMC structure is obviously more desirable to develop and use. Along concentrating on the same to the control systems method, also, it is utilized towards disturbance estimation and rejection.

SMC is special class of variable structure control system [11]. The particular fundamental notion of SMC is described in [38]. SMC are generally contain two phases the initial one is reaching phase and another is sliding phase [15]. In that order belonging to the system are going to be reduce in sliding phase. Reaching phase signify that system states are force to come along specific sliding manifold in the finite time while sliding phase mean that after the states are reached to manifold it is going to slide towards origin in this particular sliding surface.

2.5 Adaptive Sliding Mode Control

In tangible world many non-linear systems being controlled have constant or time varying parameters which increase the risk for uncertainty. For illustration, robot manipulators may carry objects with unknown inertial parameters. Power system might go through large variations in loading conditions. Fire-fighting aircraft are affected considerable mass changes once they load and unload wide range of water. To regulate these kind of systems adaptive control strategy is preferred. The fundamental reasoning behind adaptive control approach is always estimate the values of the varying parameters of plant. Adaptive control is developed for both linear and non-linear systems. Two main approaches are around for designing adaptive control.

1. Model-Reference Adaptive Control (MRAC)
2. Self-Tuning Controllers (STC)

2.5.1 Model-Reference Adaptive Control

Fig (2.4) represents the block diagram of MARC. It consists of 4 parts: a plant containing unknown parameters, a reference model for specifying the required output, a feedback control for adjusting the values of parameters and adaptation mechanism for updating the values of adjustable parameters.

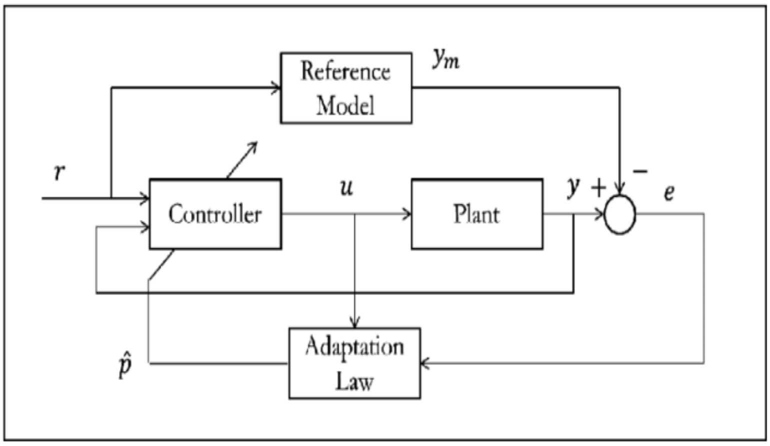


FIGURE 2.4: The block diagram model-reference adaptive control

2.5.2 Self-Tuning Controllers

In other control approaches the parameters of controller is computed from that regarding plant, in case your parameters are unknown, then that parameters are replaced by their estimated values which is available from estimator. A controller thus design by coupling a control with estimator is addressed as self-tunning controller. Fig (2.5) represents the block diagram of this kind of adaptive controller.

2.6 Integral Sliding Mode Control

Integral sliding mode efforts to reject uncertainties and could possibly help to circumvent chattering [39]. ISMC has no reaching phase. It implies that sliding is carried out in initially instant. In integral its order dynamics will be exactly same while in normal sliding mode its order is reduce in in sliding phase [39]. The

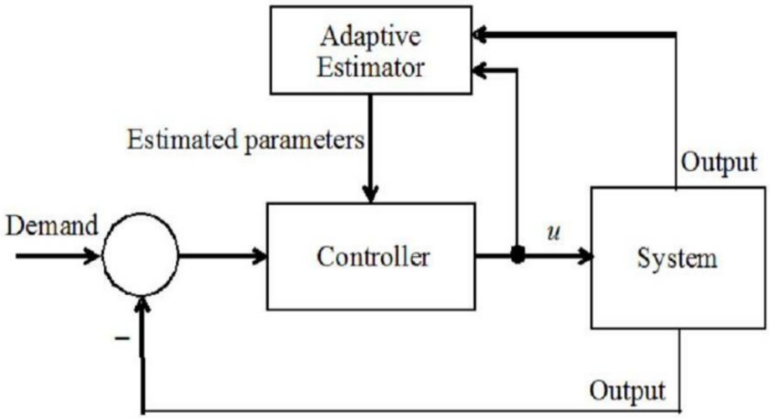


FIGURE 2.5: The block diagram self-tuning controller

fundamental introduction of integral sliding mode is discussed in below section. Look at the following nonlinear system with state space description.

$$\dot{x} = g(x,t) + C(x,t)u \tag{2.6}$$

where $x \in R^n$ represents the state vector and $u \in R$ represents the control which appears linearly in system representation.
The system operation under u_o may have the following form

$$\dot{x_o} = g(x_0,t) + C(x_o,t)u_o \tag{2.7}$$

So the system 2.6 become

$$\dot{x} = g(x,t) + C(x,t)u + \zeta(x,t) \tag{2.8}$$

where $\zeta(x,t)$ is the perturbations caused by uncertainty in dynamics which is often resulting from parameter variations and external disturbances.
The prospective is to design a control law which meets $x(t) = x_0(t)$ from the primary time instant $x(0) = x_0(0)$. The required control law is in the type

$$u = u_o + u_1 \tag{2.9}$$

where u_o is a perfect control and u_1 is defined as that will reject the perturbation tern $\zeta(x,t)$. By putting equation 2.9 in 2.8, yields

$$\dot{x} = g(x,t) + C(x,t)u_o + C(x,t)u_1 + \zeta(x,t) \tag{2.10}$$

Sliding surface is define as [40]

$$\sigma(x) = \sigma_o(x) + z \tag{2.11}$$

The initial term whilst in the right hand side of 2.11 indicates the contribution of conventional sliding surface along with second term is definitely the integral term which can be usually that should be determined from the subsequent analysis. Time derivative of 2.11 of the dynamics of 2.10, takes the form

$$\dot{\sigma} = \bigtriangledown\sigma_o[g(x,t) + C(x,t)u_o + C(x,t)u_1 + \zeta(x,t)] + \dot{z} \tag{2.12}$$

By choosing integral term dynamics

$$\begin{aligned} \dot{z} &= \frac{\partial\sigma_o(x,t)}{\partial x}(g(x,t) + C(x,t)u_o) \\ z(0) &= -\sigma_o x(0) \end{aligned} \tag{2.13}$$

Hence condition $z(0)$ select such that it fulfill the requirement $\sigma(0) = 0$.
For achieving the congruence condition $x(t) = x_o(t)$, altering the procedure of the equivalent control method [41].
The expression of u_1eq mentioned below

$$u_1eq = -\delta \tag{2.14}$$

For Verification of this condition $u_1eq = -\delta$, leads to the forth coming state equations which force the motion of the system in sliding mode.

$$\dot{x} = (g(x,t) + C(x,t)u_o) \tag{2.15}$$

By enforcing the sliding mode along with the integral sliding surface 2.11 the discontinuous control function u_1 in 2.9 are mentioned below

$$u_1 = -M(x)sign(\sigma) \tag{2.16}$$

2.7 Sliding Manifold

Towards employment of SMC, at start up a switching surface delineation is needed. The switching surface may be called in the form of sliding surface. That the sliding surface is established, then a aforementioned two phases consist of devote particular order. Reaching phase is accomplished first, and in addition it is responsible for the attractiveness of system states through initial condition for any switching surface. When reaching phased is attained, and also system will lie upon the sliding surface, then sliding phase constantly in place, in addition to system's stats glides into the equilibrium point utilizing a discontinuous control action (which also ensures robustness). Figure (2.6) shows the reaching phase (RP), sliding mode (SM) and sliding surface (SS) inside pictorial way.

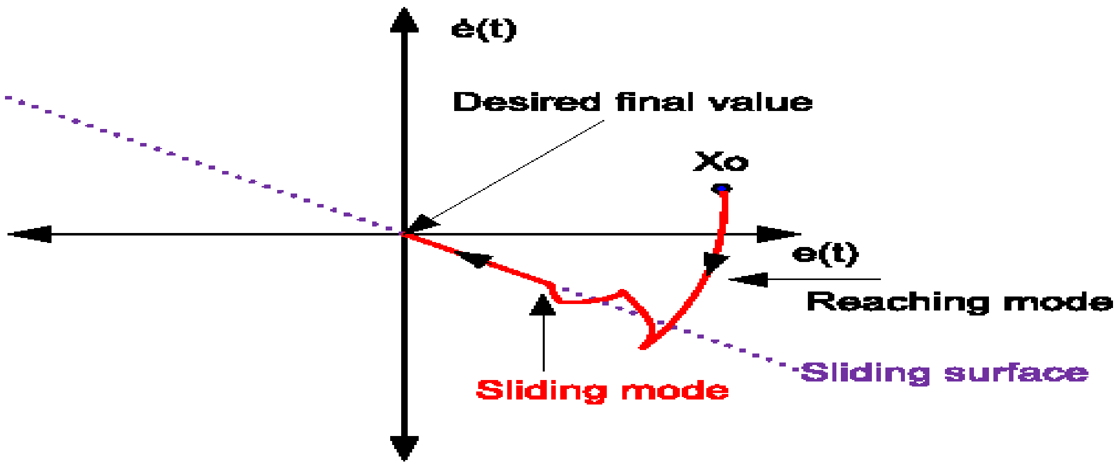


FIGURE 2.6: The Sliding Phase, Reaching Phase, Sliding Surface.

2.8 Chattering Phenomenon

As a consequence of discontinuous control, chattering will produce during the system as manifest in Figure (2.7) , which is recognized as dangerous to the system's mechanical and electromechanical chunks. Like chattering has considerable detrimental effect in real-world solicitations. That phenomenon often how you can considerable undesirable oscillations that reduce the achievement on the system. Avoiding chattering effect, differing solutions these challenge are proposed. Modern design scheme using the estimation of sliding variable was presented. The strategy according to narrating function point of view originated for chattering research in the structure during the inclusion of this un-modeled dynamics. An alternate way to diminish chattering effect is perhaps HOSM control techniques.

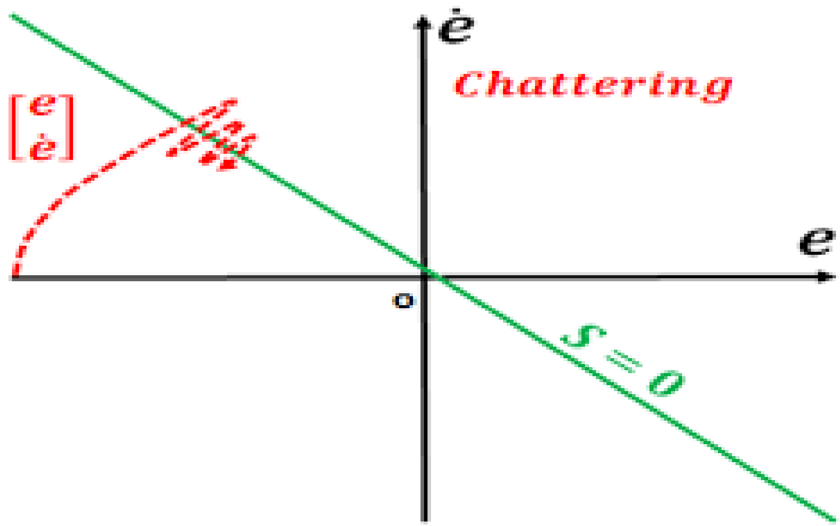


FIGURE 2.7: The Chattering Effect.

Chapter 3

Proposed Control Algorithms for Complete Synchronization and Anti-Synchronization

In this particular chapter sliding mode control technique (SMC) are offered to attain Complete Synchronization and Anti-Synchronization between two financial chaotic systems. Two cases are believed to be, first one with known parameters by using first order SMC and 2nd one with unknown parameters by using AISMC of financial chaotic systems.

3.1 Controller Designing Based on First Order Sliding Mode Control

Consider the two chaotic systems:

$$\begin{aligned} \dot{x} &= a(x) + A(x)\beta \\ \dot{y} &= b(y) + B(y)\phi + u \end{aligned} \tag{3.1}$$

where $x = [x_1, x_2, ..., x_n]^T \in R^n$ and $y = [y_1, y_2, ..., y_n]^T \in R^n$ represents state vectors of master and slave systems (3.1) respectively. $\beta \in \Re^p$ and $\phi \in \Re^q$ represents real vectors for known parameters. $A(x) \in R^{n\times p}$ and $B(y) \in R^{n\times q}$ are matrices. $a(x) \in R^n$ and $b(y) \in R^m$ represents vectors of nonlinear functions, and

$u(x, y) \in R^m$ represents control vector.
Error defined as:

$$e = y - qx \tag{3.2}$$

For synchronization we consider $q = 1$ and for anti-synchronization we consider $q = -1$.
Then error dynamics is:

$$\dot{e} = \dot{y} - q\dot{x} = b(y) + B(y)\phi + u - q\{a(x) + A(x)\beta\} \tag{3.3}$$

Now design u , such that error system (3.13) becomes asymptotically stable for complete synchronization.

3.1.1 Synchronization and anti-synchronization with known parameters

Examine the following chaotic system:

$$\begin{aligned} \dot{x} &= a(x) + A(x)\beta \\ \dot{y} &= b(y) + B(y)\phi + u \end{aligned} \tag{3.4}$$

By taking error as:

$$e = y - qx \tag{3.5}$$

where error is $e = [e_1, e_2, ..., e_n]^T \in R^n$ now for error dynamics by taking derivative of error:

$$\dot{e} = \dot{y} - q\dot{x} = b(y) + B(y)\phi + u - q\{a(x) + A(x)\beta\} \tag{3.6}$$

If we chose

$$u = -b(y) - B(y)\phi + q\{a(x) + A(x)\beta\} + ee \tag{3.7}$$

where, $ee = [e_2, e_3, ..., e_n \ v]^T$ and put u in 3.6
while v is a new input and system dynamics (3.6) will become

$$\begin{aligned}
\dot{e}_1 &= e_2 \\
\dot{e}_2 &= e_3 \\
&\vdots \\
\dot{e}_n &= v
\end{aligned} \tag{3.8}$$

After that define the Hurwitz sliding surface system (3.8) as:

$$\begin{aligned}
\sigma &= e_1 + \sum_{i=2}^{n-1} c_i e_i + e_n \\
\dot{\sigma} &= \dot{e_1} + \sum_{i=2}^{n-1} c_i \dot{e_i} + \dot{e_n} \\
\dot{\sigma} &= e_2 + \sum_{i=2}^{n-1} c_i e_{i+1} + v
\end{aligned} \tag{3.9}$$

By choosing $v = -e_2 - \sum_{i=2}^{n-1} c_i e_{i+1} - ksign(\sigma) - k\sigma$ we have, $\dot{\sigma} = -ksign(\sigma) - k\sigma$. Therefore we can say that system (3.8) is asymptotically stable.
For this we conclude that, $\sigma \to 0$, consequently $(e_1, e_2, ..., e_n) \to 0$.

3.2 Controller Designing Based on Adaptive Integral Sliding Mode Control

Examine the chaotic system having external disturbance:

$$\begin{aligned}
\dot{x} &= a(x) + A(x)\beta \\
\dot{y} &= b(y) + B(y)\phi + hv + u \\
\dot{v} &= kf(y) - jf(v)
\end{aligned} \tag{3.10}$$

where $x = [x_1, x_2, ..., x_n]^T \in R^n$ and $y = [y_1, y_2, ..., y_n]^T \in R^n$ represents state vectors of master and slave systems (3.10) respectively, while $\dot{v}$ in (3.10) represents the vector of time varying external disturbances. $\beta \in \Re^p$ and $\phi \in \Re^q$ represents real vectors for known parameters. $A(x) \in R^{n\times p}$ and $B(y) \in R^{n\times q}$ are matrices. $a(x) \in R^n$ and $b(y) \in R^m$ represents vectors of nonlinear functions, and $u(x, y) \in R^m$ represents control vector. The external disturbance has upper bound

v_o.

$$v \leq v_o \tag{3.11}$$

Now define error as:

$$e = y - qx \tag{3.12}$$

For synchronization we consider $q = 1$ and for anti-synchronization we consider $q = -1$.
For error dynamics by taking derivative of error signal:

$$\dot{e} = \dot{y} - q\dot{x} = b(y) + B(y)\phi + hv + u - q\{a(x) + A(x)\beta\} \tag{3.13}$$

Now design u , such that error system (3.13) becomes asymptotically stable for complete synchronization.

3.2.1 Synchronization and anti-synchronization with unknown parameters

Permit $\hat{\beta}, \hat{\phi}$ be estimate of β, ϕ respectively, $\tilde{\phi} = \phi - \hat{\phi}$ and $\tilde{\beta} = \beta - \hat{\beta}$ be error in estimating β, ϕ.
so the equation (3.1) becomes,

$$\begin{aligned} \dot{x} &= a(x) + A(x)\tilde{\theta} + A(x)\hat{\beta} \\ \dot{y} &= b(y) + B(y)\tilde{\phi} + B(y)\hat{\phi} + hv + u \end{aligned} \tag{3.14}$$

Define error signal,

$$e = y - qx \tag{3.15}$$

Now for error dynamics by taking derivative of equation (3.15):

$$\dot{e} = \dot{y} - q\dot{x} = b(y) + B(y)\hat{\phi} + B(y)\tilde{\phi} + hv + u - q\{a(x) + A(x)\hat{\beta} + A(x)\tilde{\beta}\} \tag{3.16}$$

If we choose

$$u = -b(y) - B(y)\hat{\phi} + q\{a(x) + A(x)\hat{\beta}\} + ee \tag{3.17}$$

By putting value of u so equation (3.16) becomes

$$\dot{e} = ee + B(y)\tilde{\phi} - qA(x)\tilde{\beta} + hv \tag{3.18}$$

$$[\dot{e_1}, \dot{e_2}, ..., \dot{e_n}]^T = [e_2, e_3, ..., e_n \; v]^T + B(y)\tilde{\phi} - qA(x)\tilde{\beta} + hv \tag{3.19}$$

By using AISMC, we choose first the nominal system for (3.19) as:

$$\begin{aligned} \dot{e}_1 &= e_2 \\ \dot{e}_2 &= e_3 \\ &\vdots \\ \dot{e}_n &= v_0 \end{aligned} \tag{3.20}$$

By defining the hurwitz sliding manifold for 3.20 as:

$$\begin{aligned} \sigma_0 &= e_1 + \sum_{i=2}^{n-1} c_i e_i + e_n \\ \dot{\sigma}_0 &= \dot{e_1} + \sum_{i=2}^{n-1} c_i \dot{e_i} + \dot{e_n} \\ \dot{\sigma}_0 &= e_2 + \sum_{i=2}^{n-1} c_i e_{i+1} + v_0 \end{aligned} \tag{3.21}$$

If we choose $v_0 = -e_2 - \sum_{i=2}^{n-1} c_i e_{i+1} - k\sigma_0$ we have, $\dot{\sigma}_0 = -k\sigma_0$. Therefore we can say that system (3.20) is asymptotically stable.
By choosing the integral sliding surface for the system (3.19)
$\sigma = \sigma_0 + z$
Where, z is integral term calculated later, to keep away from the reaching phase, choose z(0) in such a way $z(0) = -\sigma(0)$. choose $v = v_0 + v_s$ spot, v_0 is a nominal input and v_s is discontinuous term calculated later.
Where $C = [1 \; c_1 \; c_2, ..., c_{n-1} \; 1]$ is chosen in this way that σ become Hurwitz polynomial.

Then,

$\sigma = Ce + z$

By taking derivative,

$\dot{\sigma} = C\dot{e} + \dot{z}$

By putting $\dot{e}$ value we have,

$\dot{\sigma} = C[ee + B(y)\tilde{\phi} - qA(x)\tilde{\beta} + hv] + \dot{z}$

$\dot{\sigma} = e_2 + \sum_{i=2}^{n-1} c_i e_i + v_0 + v_s + CB(y)\tilde{\phi} - qCA(x)\tilde{\beta} + Chv + \dot{z}$

Now define Lyapunov function:

$V = \frac{1}{2}\sigma^2 + \frac{1}{2}\tilde{\beta}^T\tilde{\beta} + \frac{1}{2}\tilde{\phi}^T\tilde{\phi}$

In this we sketch the adaptive law for $\tilde{\beta}, \hat{\beta}, \tilde{\phi}, \hat{\phi}$ and determine v_s in this a way that $\dot{V} < 0$.

Examine a lyapunov function $V = \frac{1}{2}\sigma^2 + \frac{1}{2}\tilde{\beta}^T\tilde{\beta} + \frac{1}{2}\tilde{\phi}^T\tilde{\phi}$.Then $\dot{v} < 0$ if the adaptives laws for $\tilde{\beta}, \hat{\beta}, \tilde{\phi}, \hat{\phi}$ and the utility of v_s are selected as:

$\dot{z} = -e_2 - \sum_{i=2}^{n-1} c_i e_{i+1} - v_0$

$v_s = -ksign(\sigma) - k\sigma$

$\dot{\tilde{\beta}} = \sigma q A^T(x) C^T - k_1\tilde{\beta}$

$\dot{\tilde{\phi}} = -\sigma B^T(y) C^T - k_2\tilde{\phi}$ where, $k, k_2, k_3 > 0$

$\dot{\hat{\beta}}$=-$\dot{\tilde{\beta}}$

$\dot{\hat{\phi}}$=-$\dot{\tilde{\phi}}$

Proof:

$V = \frac{1}{2}\sigma^2 + \frac{1}{2}\tilde{\beta}^T\tilde{\beta} + \frac{1}{2}\tilde{\phi}^T\tilde{\phi}$

By taking derivative we have:

$\dot{V} = \sigma\dot{\sigma} + \tilde{\beta}^T\ \dot{\tilde{\beta}} + \tilde{\phi}^T\ \dot{\tilde{\phi}}$

$= \sigma\{e_2 + \sum_{i=2}^{n-1} c_i e_i + v_0 + v_s + CB(y)\tilde{\phi} - qCA(x)\tilde{\beta} + Chv + \dot{z}\} + \tilde{\beta}^T\ \dot{\tilde{\beta}} + \tilde{\phi}^T\ \dot{\tilde{\phi}}$

$= \sigma\{e_2 + \sum_{i=2}^{n-1} c_i e_i + v_0 + v_s + Chv + \dot{z}\} + \tilde{\beta}^T\{\dot{\tilde{\beta}} - q\sigma A^T(x)C^T\} + \tilde{\phi}^T\{\dot{\tilde{\phi}} + \sigma B^T(s)C^T\}$

By using

$\dot{z} = -e_2 - \sum_{i=2}^{n-1} c_i e_{i+1} - v_0$

$v_s = -ksign(\sigma) - k\sigma$

$\dot{\tilde{\beta}} = \sigma q A^T(x) C^T - k_1\tilde{\beta}$

$\dot{\tilde{\phi}} = -\sigma B^T(y) C^T - k_2\tilde{\phi}$ where, $k, k_2, k_3 > 0$

We have

$\dot{V} = -k\sigma^2 + \sigma[Chv_o - ksign(\sigma)] - k_1\tilde{\beta}^T\tilde{\beta} - k_2\tilde{\phi}^T\tilde{\phi}$

If $k > Chv_o$ then we can wind up that $\sigma, \tilde{\beta}, \tilde{\phi} \to 0$ that $\sigma \to 0$, consequently $(e_1, e_2, ..., e_n) \to 0$.

Chapter 4

Applications of Proposed Algorithm

Introduction
In this chapter, numerical examples of chaotic finance systems are contemplate to verify the Suggested control strategy.

4.1 Numerical Example 1

4.1.1 3D Financial Chaotic System

In non-linear systems, researchers are struggling to use the notion of non-linear dynamics, specially the chaos theory, to analyze the complexness of economic and financial systems recently [42–45]. That Strotz et al. have performed the pioneering effort in this field [46], a number of economics chaotic models have already been preferred, for instance Kaldorian model [47], the IS-LM model [48], the hyperchaotic finance system [4], and various non-linear dynamical models [49–51].
It is recognized that economic chaotic systems are certainly troubled by external disturbances esteemed from environmental involvement [52–55] which may accompany the disrupting of economic and financial chaotic systems and may cause undesirable outcomes. It is significant to evaluate the global stabilization of economic and financial chaotic systems to underneath the inclusion of external

disturbance. Few outcomes are actually revealed with regard to robust stabilization of complex systems [56, 57]. Previously decades, complete synchronization of chaotic systems has attracted numerous attention, complete because synchronization could possibly get the essential outcomes faster. Behind schedule, while using the magnify within the research, miscellaneous complete synchronization methods were debated. Just like, in [24, 30], the authors have explored the function of complete synchronization of chaotic systems, in addition to the scaling function adopted to get constant or unity. In [11, 24, 25], the authors debated function complete synchronization of chaotic financial systems. Currently, many research efforts stated above are dedicated to examine the presetting scaling function in numerical specimen. Whereas the complications faced by the authors about the definite integral scaling function are dealt to possible extend but they've got rarely been explored, which is still unsolved. Inspired by the prevailing works, we are going to acquire complete synchronization and anti-synchronization criteria to the financial chaotic systems.
Appraise the Chaotic financial system [28] as the master system mentioned here under

$$
\begin{aligned}
\dot{x_1} &= x_3 + (x_2 - a)x_1 \\
\dot{x_2} &= 1 - bx_2 - x_1^2 \\
\dot{x_3} &= -x_1 - cx_3
\end{aligned}
\tag{4.1}
$$

where x_1 is the interest rate, x_2 is the investment demand, and x_3 is the price index. $a > 0$ represents the saving amount, $b > 0$ represents the cost per investment and $c > 0$ represents the elasticity of demand of commercial markets.
and the dynamics of slave system

$$
\begin{aligned}
\dot{y_1} &= y_3 + (y_2 - a)y_1 + u_1 \\
\dot{y_2} &= 1 - by_2 - y_1^2 + u_2 \\
\dot{y_3} &= -y_1 - cy_3 + u_3
\end{aligned}
\tag{4.2}
$$

The system parameters are $a = 0.9$, $b = 0.2$ and $c = 1.5$, with these parameters system 4.1 exhibits chaotic behavior.

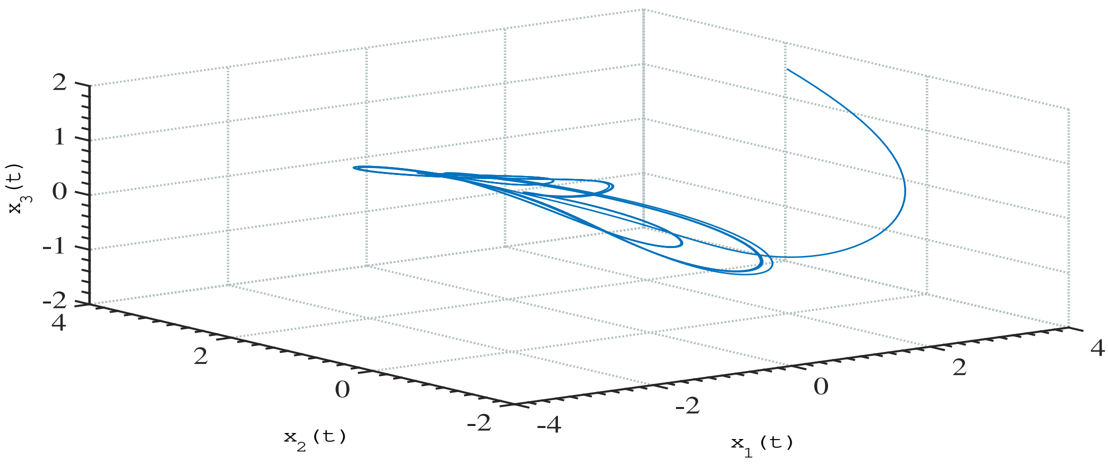


FIGURE 4.1: Phase portrait of 3 Dimensional Chaotic Financial system.

4.2 Numerical Example 2

4.2.1 Identical 4D Hyperchaotic Financial Systems

The financial Chaotic systems has attracted a substantial amount of attraction from researchers in recent years. The financial systems are involved with the existence [43]. Chaos look when economic crisis happens. The 2007 global economic crisis shows arsenic intoxication the chaos. The dynamical behaviors in the system are more intricate simply because they have many positive Lyapunov exponent and are generally expanded in many direction. So a highly effective and rapid control method is incredibly essential for government for taking safety measures when chaotic phenomenon appears. The dynamics of the financial system do a significant role while in the continuing build out of economic system. Although the, dynamics of any financial system relies on multiple input variables in an incredibly complex and nonlinear fashion. Financial system, even though deterministic, can exhibits chaotic behavior. Since a chaotic system might be more responsive to small errors and alterations in parameters, their synchronization is vital originating from a control reason for view.

In the work we present synchronization and anti-synchronization of 3D financial chaotic system and 4D hyperchaotic financial systems.The response strategy is taken like a perturbed system by some bounded external disturbances. Two cases are believed to be:

1. **System Parameters are Known:** In this case the synchronization and anti-synchronization is achieved using first order sliding mode control.

2. **System Parameters are Unknown:** In this case the adaptive integral sliding mode control is used to achieve the synchronization and anti- synchronization, and to estimate the system parameters.

In 2012, a new hyperchaotic finance system was suggested. The model is expressed by the admirers 4D hyperchaotic financial system [4]:
Master system is given below

$$\begin{aligned} \dot{x_1} &= x_3 + (x_2 - a)x_1 + x_4 \\ \dot{x_2} &= 1 - bx_2 - x_1^2 \\ \dot{x_3} &= -x_1 - cx_3 \\ \dot{x_4} &= -dx_1x_2 - kx_4 \end{aligned} \tag{4.3}$$

where x_1 represents the interest rate, x_2 represents the investment demand, x_3 represents the price index, and x_4 represents the average profit margins. $a > 0$ represents the saving amount, $b > 0$ represents the cost per investment, $c > 0$ represents the elasticity of demand of commercial markets and $d, k > 0$ represents some system's parameters.
and the slave system

$$\begin{aligned} \dot{y_1} &= y_3 + (y_2 - a)y_1 + y_4 + u_1 \\ \dot{y_2} &= 1 - by_2 - y_1^2 + u_2 \\ \dot{y_3} &= -y_1 - cy_3 + u_3 \\ \dot{y_4} &= -dy_1y_2 - ky_4 + u_4 \end{aligned} \tag{4.4}$$

The system parameters are $a = 0.9,\ b = 0.2,\ c = 1.5,\ d = 0.2,\ k = 0.17$, with these parameters system (4.3) exhibits chaotic behavior.

4.3 First Order Sliding Mode Control

In this section first order SMC is dispensed for synchronization and anti-synchronization of chaotic financial system.

4.3.1 3 Dimensional System

By defining error signals:

$$\begin{aligned} e_1 &= y_1 - qx_1 \\ e_2 &= y_2 - qx_2 \\ e_3 &= y_3 - qx_3 \end{aligned} \tag{4.5}$$

For synchronization put $q = 1$ and for anti-synchronization put $q = -1$.
By taking the derivative of error signals we get error dynamics:

$$\begin{aligned} \dot{e_1} &= \dot{y_1} - q\dot{x_1} = y_3 + (y_2 - a)y_1 - q(x_3 + (x_2 - a)x_1) + u_1 \\ \dot{e_2} &= \dot{y_2} - q\dot{x_2} = 1 - by_2 - y_1^2 - q(1 - bx_2 - x_1^2) + u_2 \\ \dot{e_3} &= \dot{y_3} - q\dot{x_3} = -y_1 - cy_3 - q(-x_1 - cx_3) + u_3 \end{aligned} \tag{4.6}$$

By choosing

$$\begin{aligned} u_1 &= -y_3 - (y_2 - a)y_1 + q(x_3 + (x_2 - a)x_1) + e_2 \\ u_2 &= -1 + by_2 + y_1^2 + q(1 - bx_2 - x_1^2) + e_3 \\ u_3 &= y_1 + cy_3 + q(-x_1 - cx_3) + v \end{aligned} \tag{4.7}$$

In 4.7 v represents the new input, which can be mentioned below:

$$\begin{aligned} \dot{e_1} &= e_2 \\ \dot{e_2} &= e_3 \\ \dot{e_3} &= v \end{aligned} \tag{4.8}$$

Defining the Hurwitz sliding surface for system 4.6 as:
$\sigma = (1 + \frac{d}{dt})^2 e_1$
$\sigma = e_1 + 2e_2 + e_3$
By taking the derivative
$\dot{\sigma} = \dot{e_1} + 2\dot{e_2} + \dot{e_3}$
By putting values we have
$\dot{\sigma} = e_2 + 2e_3 + v$
By choosing $v = -e_2 - 2e_3 - ksign(\sigma) - k\sigma$
By putting value of v we get
$\dot{\sigma} = -ksign(\sigma) - k\sigma$
So we can say that error system 4.6 is asymptotically stable.
In simulations, the initial conditions are chosen as given in system $x(0) = [3, 1, 2]^T$, $y(0) = [-2, 3, -1]^T$. The parameters values are: $a = 0.9, b = 0.2, c = 1.5$.

Consider a Lyapunov function:

$V = 0.5\sigma^2$

Hence by taking the derivative,

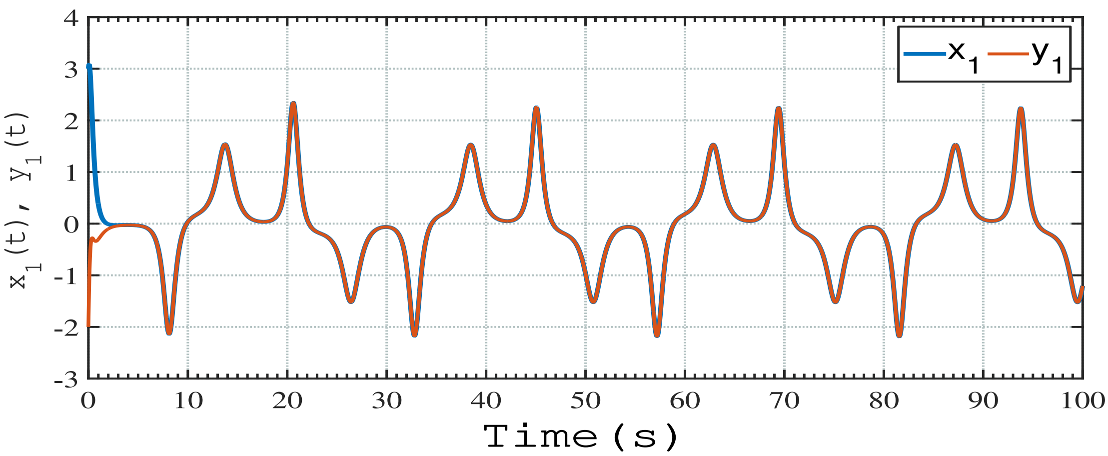
$\dot{V} = \sigma\dot{\sigma}$

$\dot{V} = \sigma(-ksign(\sigma) - k\sigma) = -|k| - k\sigma^2$

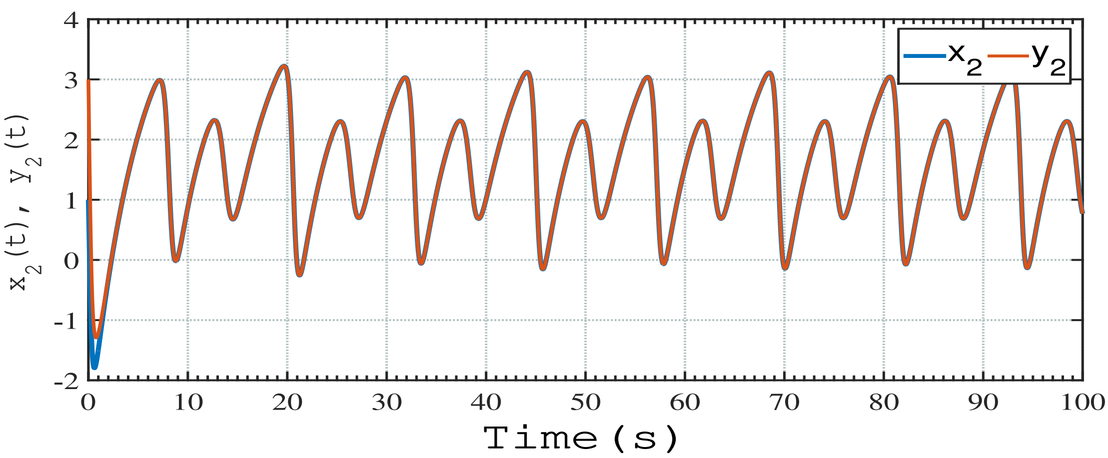
From this we can say that $\sigma \to 0$, since σ is Hurwitz therefore $e_i \to 0, i = 1, .., 3$, therefore the systems 4.8 is asymptotically stable.

4.3.2 Synchronization of 3 Dimensional Chaotic Financial System

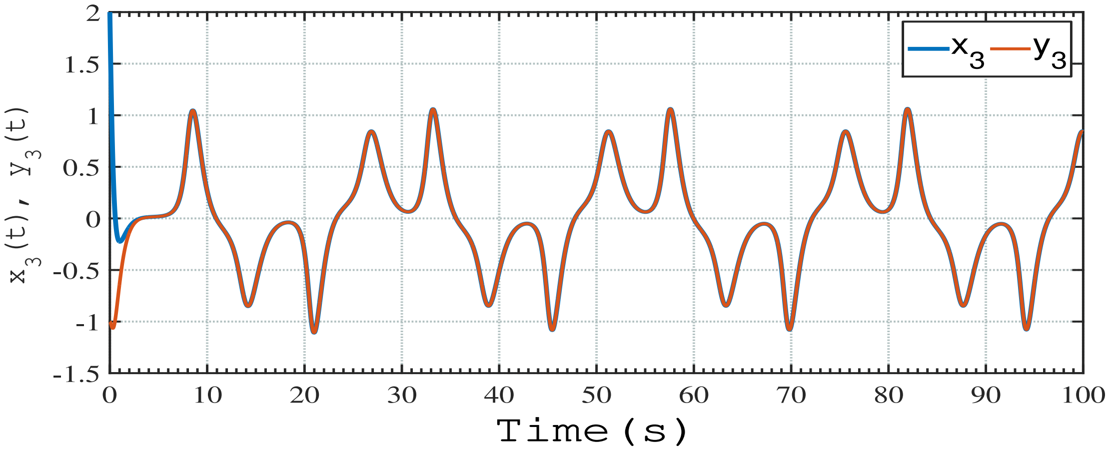
For Synchronization set q=1 in eq (4.5):



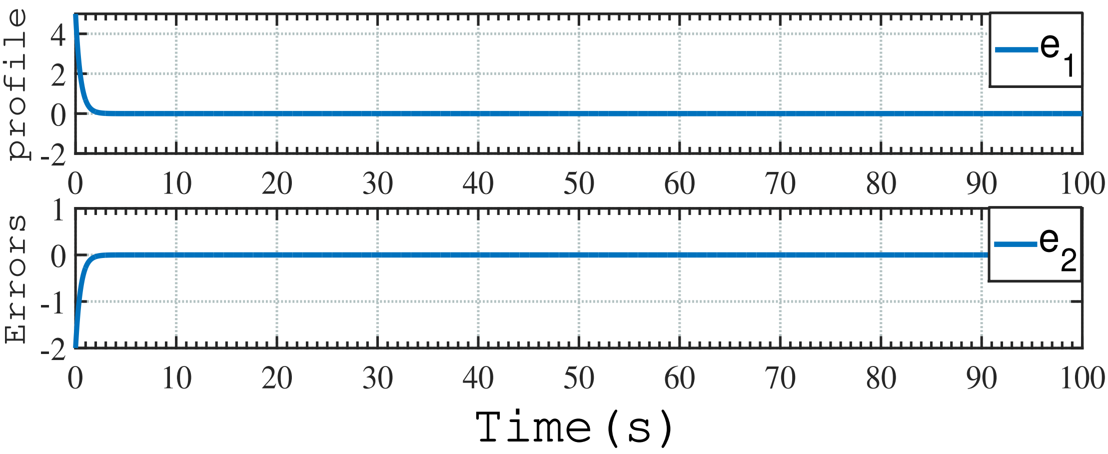
(a)



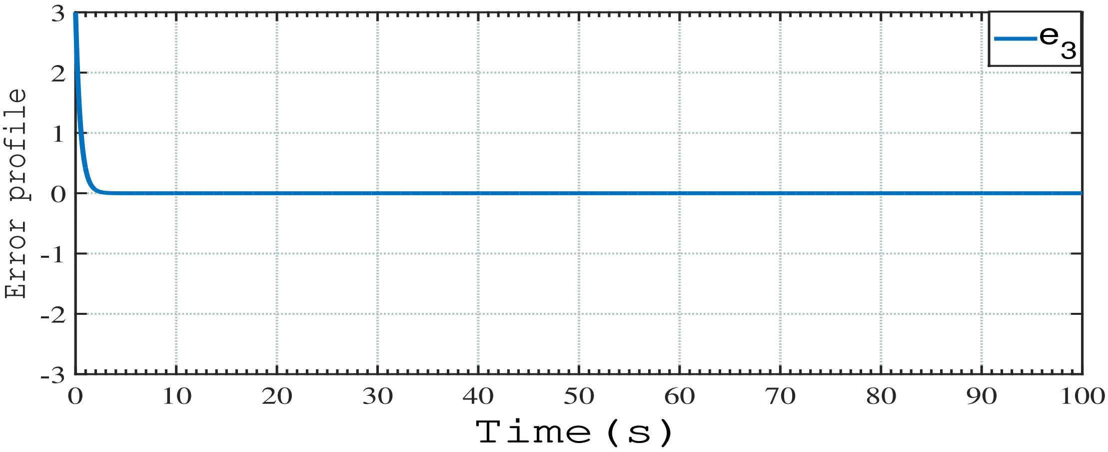
(b)


x3(t), y3(t)
Time(s)
x3
y3

(c)

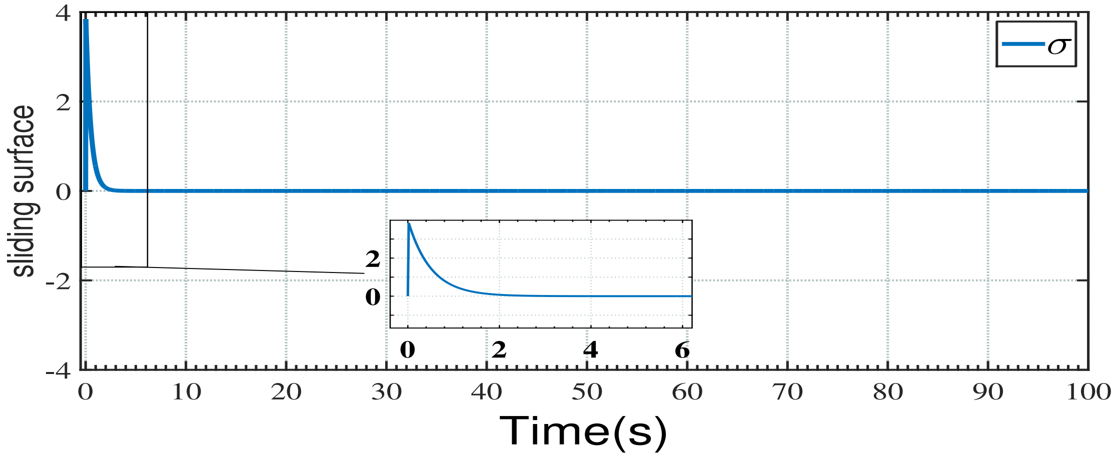

Errors profile
Time(s)
e1
e2

(d)

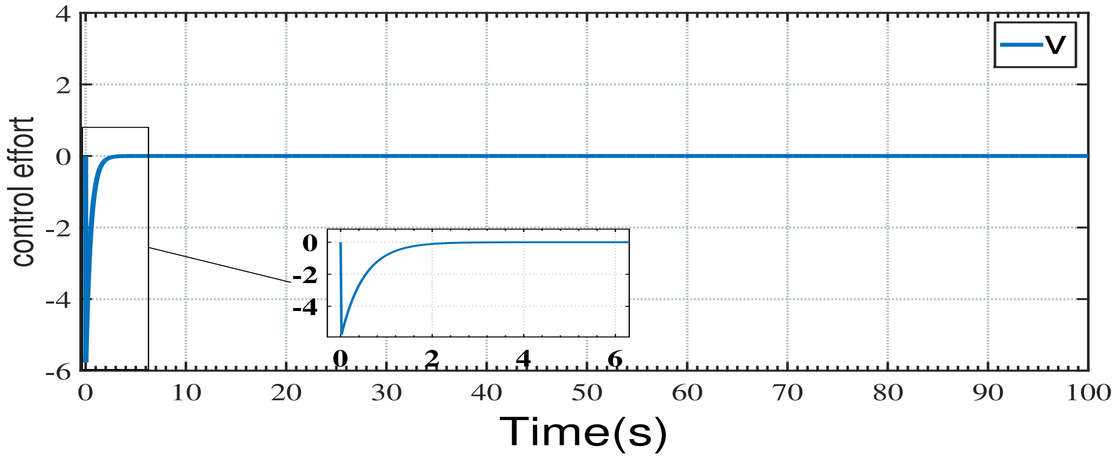

Error profile
Time(s)
e3

(e)

FIGURE 4.2: *Synchronization of 3D Financial Chaotic System, (a) Synchronization of interest rate corresponding to initial condition* $[x_1(0), y_1(0) = (3, -2)]$, *(b) Synchronization of investment demand corresponding to initial condition* $[x_2(0), y_2(0) = (1, 3)]$, *(c) Synchronization of price index corresponding to initial condition* $[x_3(0), y_3(0) = (2, -1)]$, *(d) and (e) Time history of the errors* e_1, e_2 *and* e_3



(a)

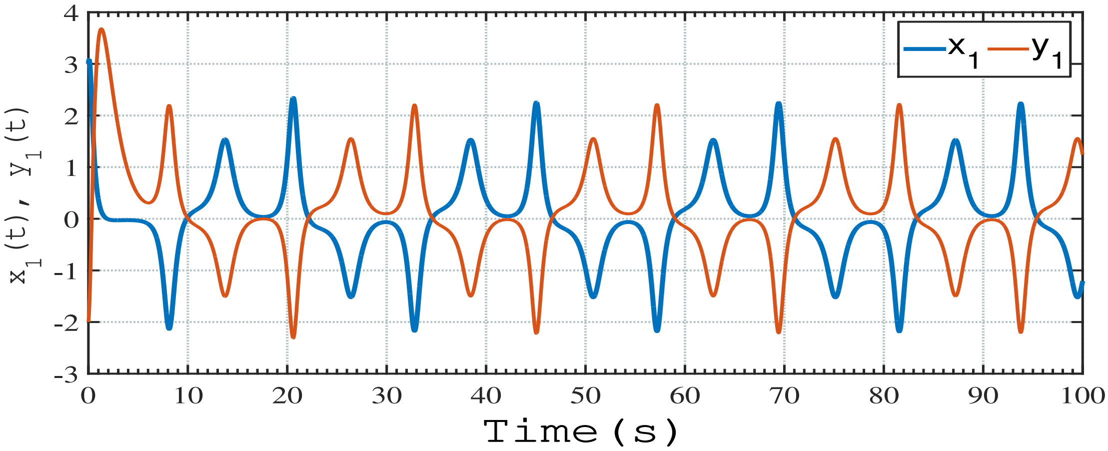


(b)

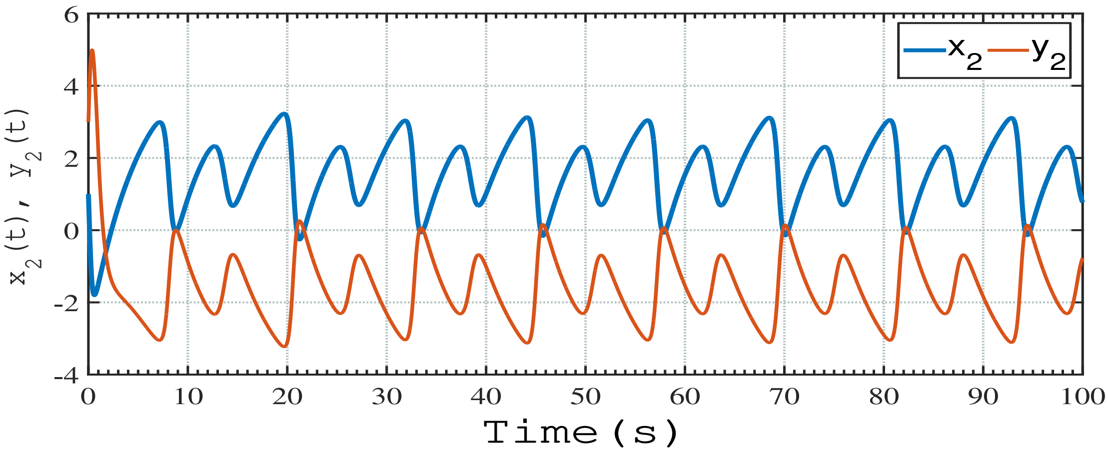
FIGURE 4.3: *Synchronization of 3D Financial Chaotic System, (a) Sliding manifold* σ *(b) Control effort* v

4.3.3 Anti-Synchronization of 3 Dimensional Chaotic Financial System

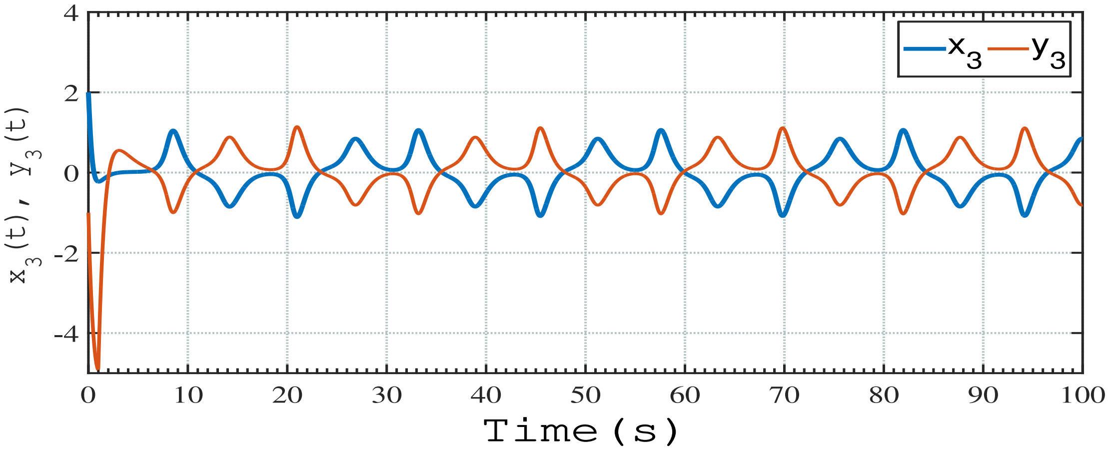
For anti-synchronization set q= -1 in eq (4.5):



(a)

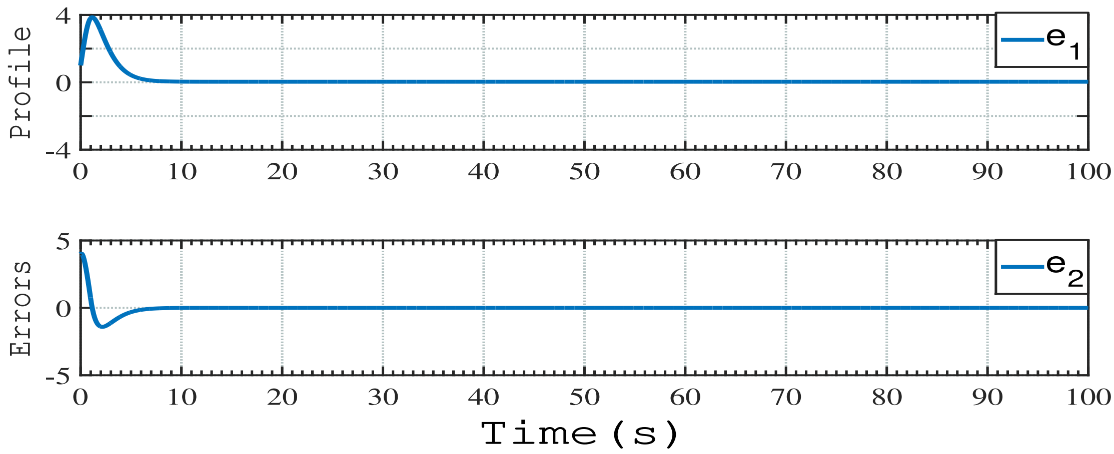


(b)



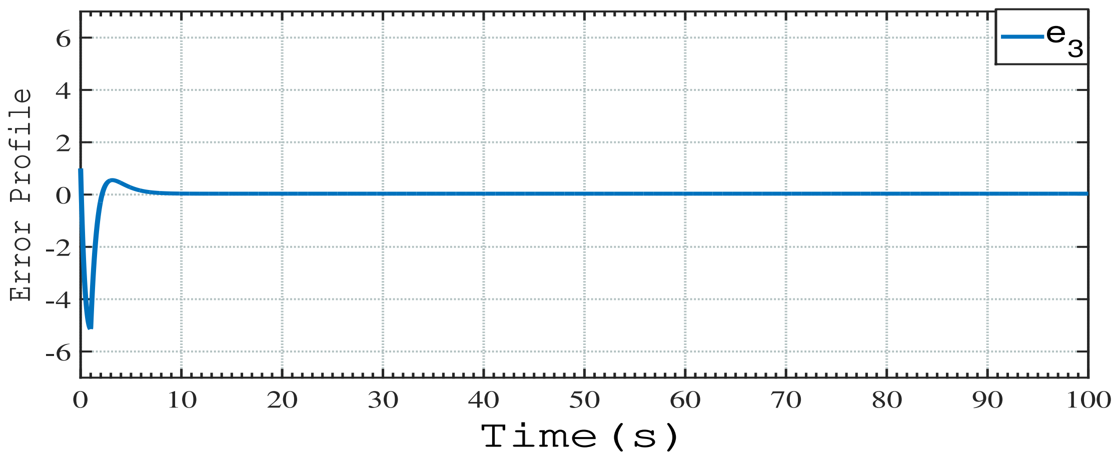
x3
y3
x3(t), y3(t)
Time(s)

(c)



e1
e2
Profile
Errors
Time(s)

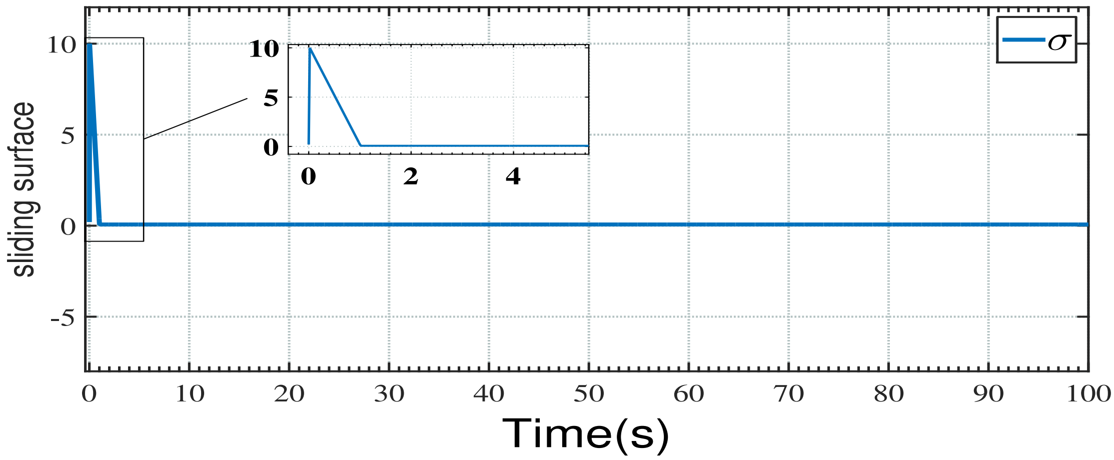
(d)



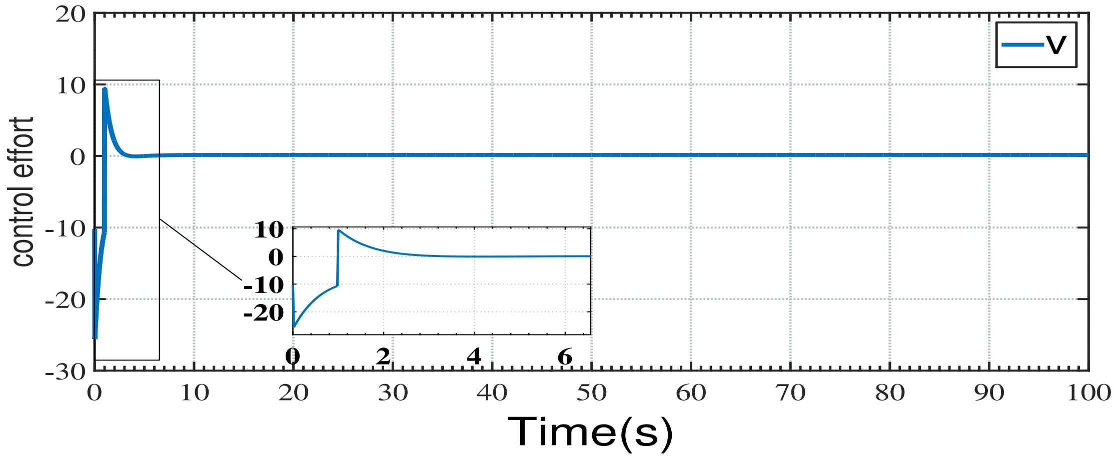
e3
Error Profile
Time(s)

(e)

FIGURE 4.4: *Anti-Synchronization of 3D Financial Chaotic System, (a) Anti-Synchronization of interest rate corresponding to initial condition* $[x_1(0), y_1(0) = (3, -2)]$, *(b) Anti-Synchronization of investment demand corresponding to initial condition* $[x_2(0), y_2(0) = (1, 3)]$, *(c) Anti-Synchronization of price index corresponding to initial condition* $[x_3(0), y_3(0) = (2, -1)]$, *(d) and (e) Time history of the errors* e_1, e_2 *and* e_3



(a)



(b)

FIGURE 4.5: *Anti-Synchronization of 3D Financial Chaotic System, (a) Sliding manifold* σ *(b) Control effort* v

4.3.4 4 Dimensional System

Now by defining error signals:

$$
\begin{aligned}
e_1 &= y_1 - qx_1 \\
e_2 &= y_2 - qx_2 \\
e_3 &= y_3 - qx_3 \\
e_4 &= y_4 - qx_4
\end{aligned} \tag{4.9}
$$

For synchronization select $q = 1$ and for anti-synchronization select $q = -1$
By taking derivative of error signals error dynamics becomes:

$$
\begin{aligned}
\dot{e_1} &= \dot{y_1} - q\dot{x_1} = (y_3 + (y_2 - a)y_1 + y_4) + u_1 - q(x_3 + (x_2 - a)x_1 + x_4) \\
\dot{e_2} &= \dot{y_2} - q\dot{x_2} = (1 - by_2 - y_1^2) + u_2 - q(1 - bx_2 - x_1^2) \\
\dot{e_3} &= \dot{y_3} - q\dot{x_3} = (-y_1 - cy_3) + u_3 - q(-x_1 - cx_3) \\
\dot{e_4} &= \dot{y_4} - q\dot{x_4} = (-dy_1y_2 - ky_4) + u_4 - q(-dx_1x_2 - kx_4)
\end{aligned} \tag{4.10}
$$

By choosing:

$$
\begin{aligned}
u_1 &= -(y_3 + (y_2 - a)y_1 + y_4) + q(x_3 + (x_2 - a)x_1 + x_4) + e_1 \\
u_2 &= -(1 - by_2 - y_1^2) + q(1 - bx_2 - x_1^2) + e_2 \\
u_3 &= -(-y_1 - cy_3) + q(-x_1 - cx_3) + e_3 \\
u_4 &= -(-dy_1y_2 - ky_4) + q(-dx_1x_2 - kx_4) + v
\end{aligned} \tag{4.11}
$$

In 4.11 v represents the new input, which can be mentioned below:

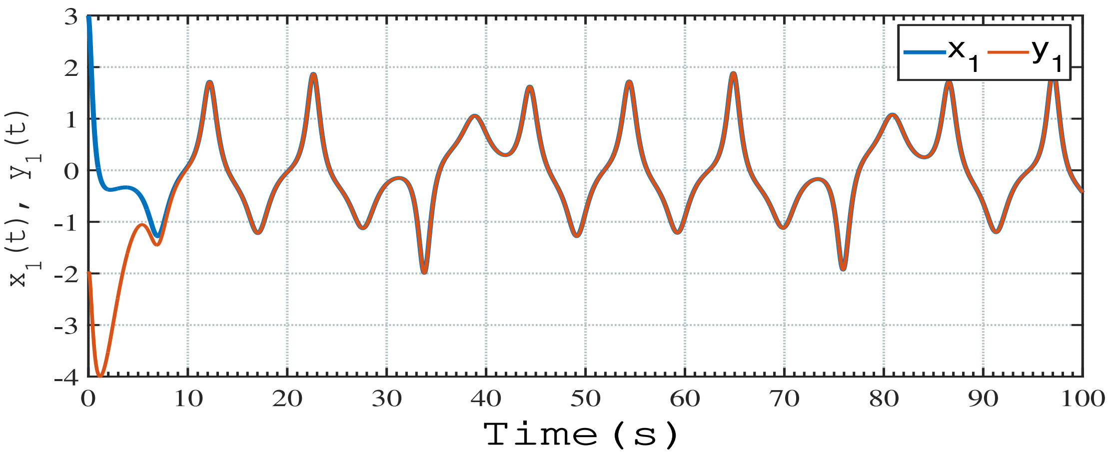
$$
\begin{aligned}
\dot{e_1} &= e_2 \\
\dot{e_2} &= e_3 \\
\dot{e_3} &= e_4 \\
\dot{e_4} &= v
\end{aligned} \tag{4.12}
$$

Defining the Hurwitz sliding surface for 4.10 as:
$\sigma = (1 + \frac{d}{dt})^3 e_1$
$\sigma = e_1 + 3e_2 + 3e_3 + e_4$
By taking derivative we have:
$\dot{\sigma} = \dot{e_1} + 3\dot{e_2} + 3\dot{e_3} + \dot{e_4}$
Putting values:
$\dot{\sigma} = e_2 + 3e_3 + 3e_4 + v$

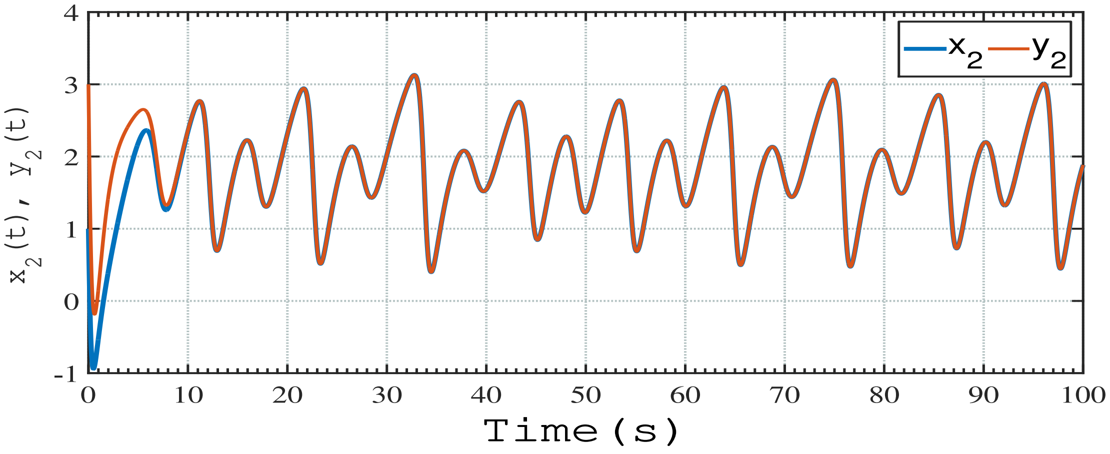
If we choose $v = -e_2 - 3e_3 - 3e_4 - k\sigma$
By putting v value $\dot{\sigma} = -k\sigma$.
So we can say that error system 4.10 is asymptotically stable.
In simulations, the initial conditions are chosen as given in system
$x(0) = [3, 1, 2, -3]^T$, $y(0) = [-2, 3, -1, -4]^T$.
The parameters values are: $a = 0.9$, $b = 0.2$, $c = 1.5$, $d = 0.2$, $k = 0.17$.
Define a Lyapunov function:
$V = 0.5\sigma^2$
By taking derivative:
$\dot{V} = \sigma\dot{\sigma}$
$\dot{V} = \sigma(-k\sigma) = -k\sigma^2$
From this we can say that $\sigma \to 0$, since σ is Hurwitz therefore $e_i \to 0$,$i = 1, .., 4$, therefore the systems 4.12 is asymptotically stable.

4.3.5 Synchronization of 4D Hyperchaotic Financial System

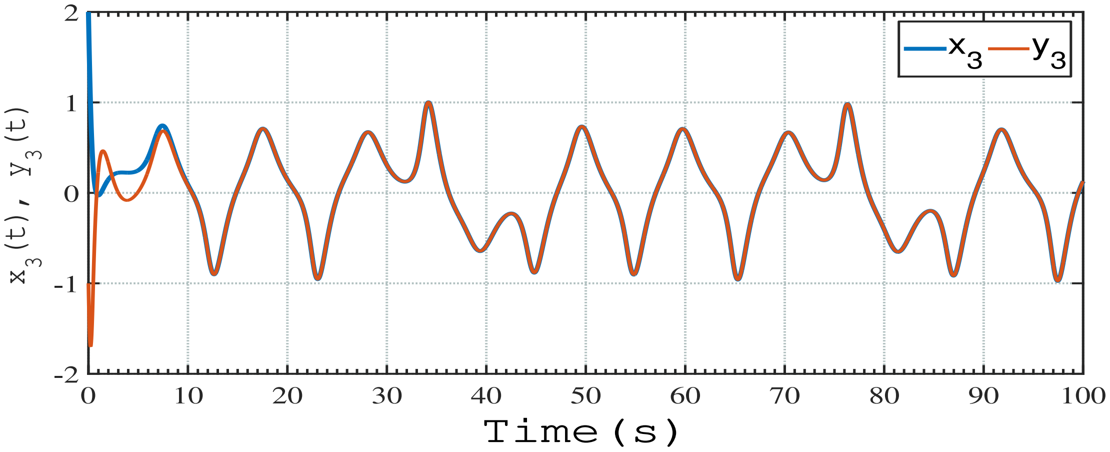
For Synchronization set q= 1 in eq (4.18):



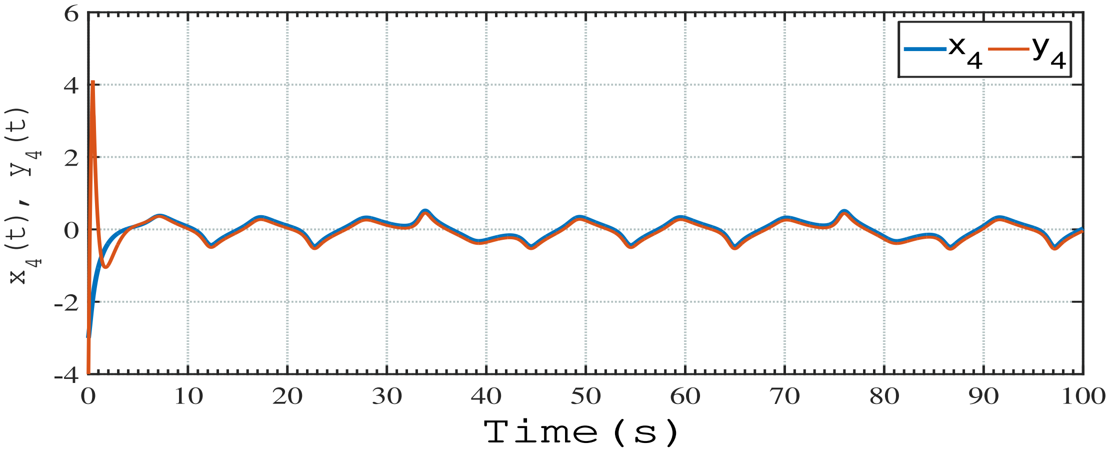
(a)



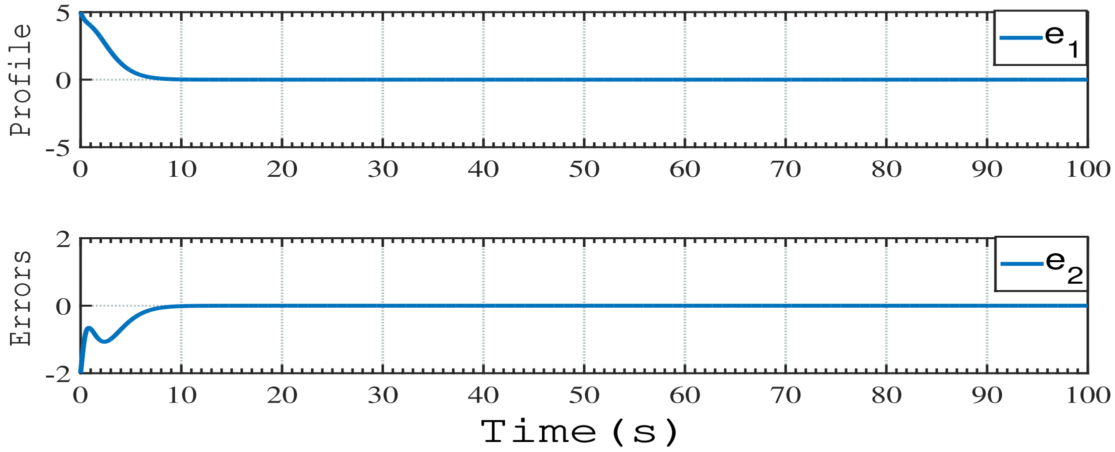
(b)



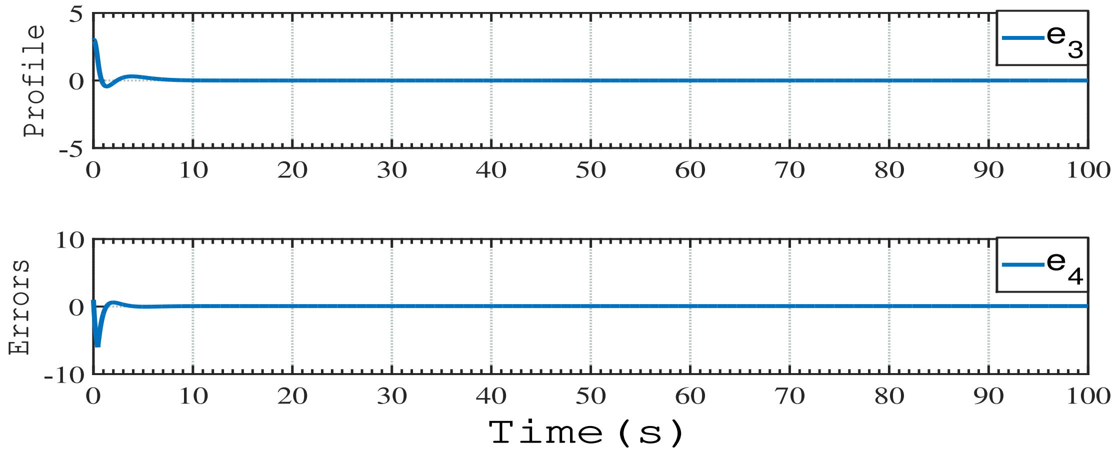
(c)



(d)

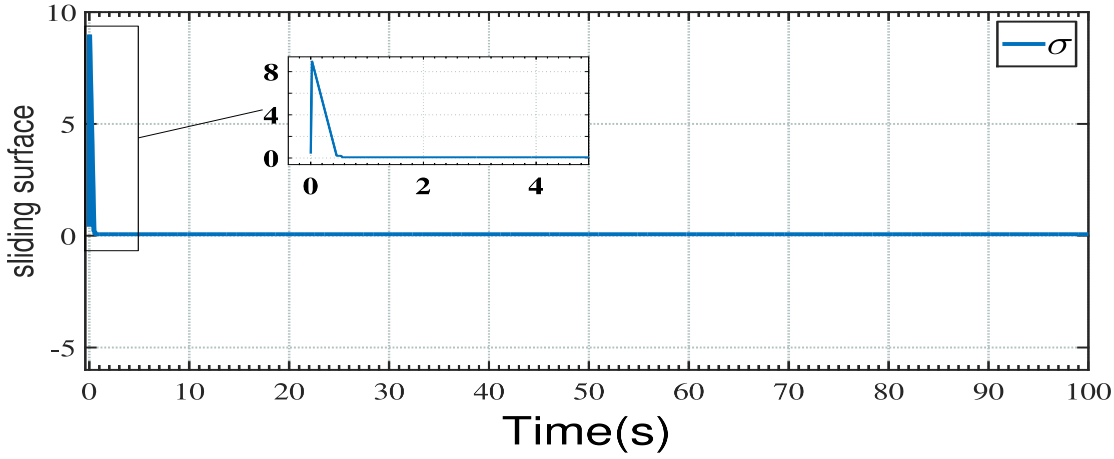


(e)

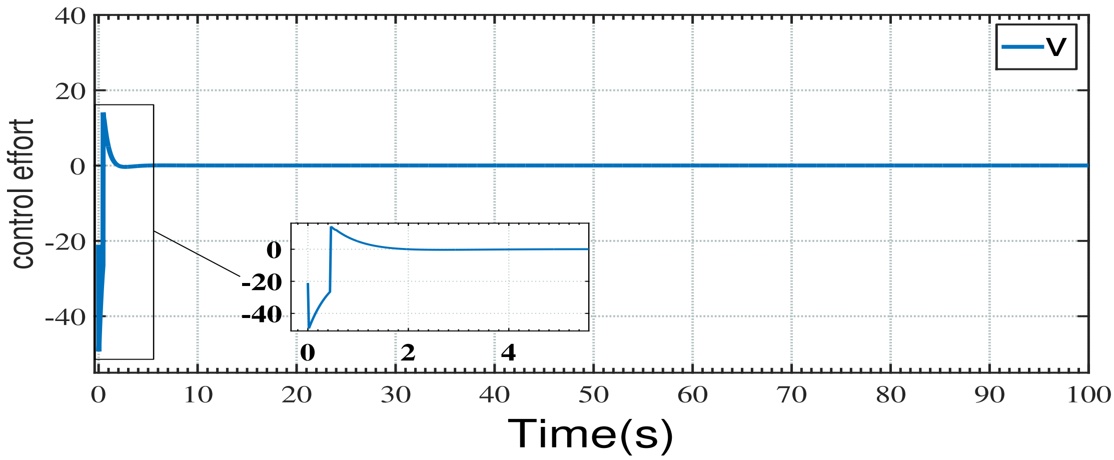


(f)

FIGURE 4.6: *Synchronization of 4D Hyperchaotic Financial System, (a) Synchronization of interest rate corresponding to initial condition* $[x_1(0)\ , y_1(0) = (3, -2)]$, *(b) Synchronization of investment demand corresponding to initial condition* $[x_2(0), y_2(0) = (1, 3)]$, *(c) Synchronization of price index corresponding to initial condition* $[x_3(0), y_3(0) = (2, -1)]$,*(d) Synchronization of average profit margins corresponding to initial condition* $[x_4(0), y_4(0) = (-3, -4)]$, *(e) and (f) Time history of the errors* e_1, e_2, e_3 *and* e_4



(a)

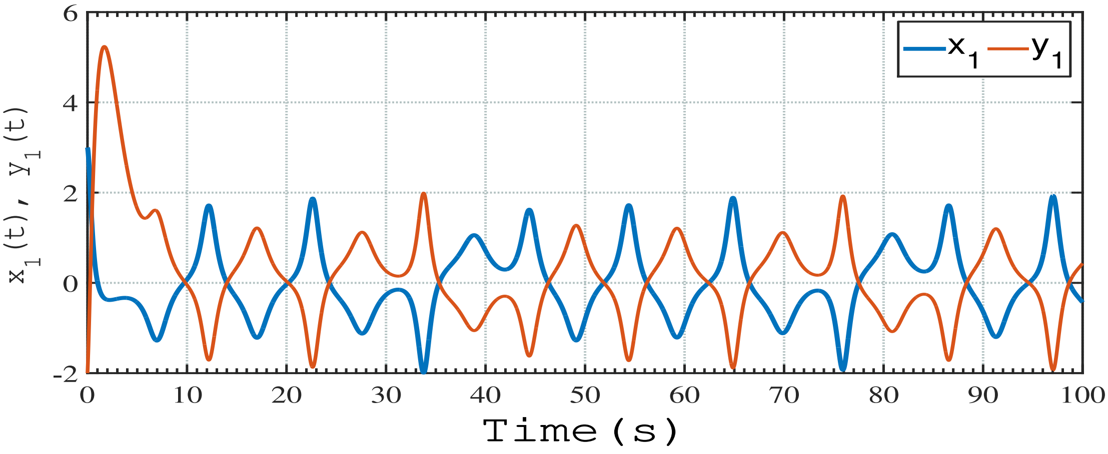


(b)

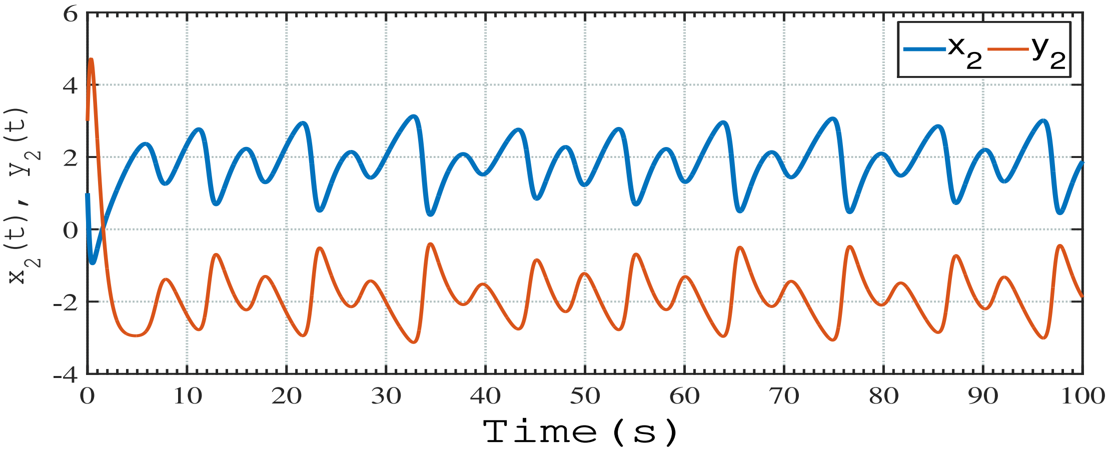
FIGURE 4.7: *Synchronization of 4D Hyperchaotic Financial System, (a) Sliding manifold σ (b) Control effort v*

4.3.6 Anti-Synchronization of 4D Hyperchaotic Financial System

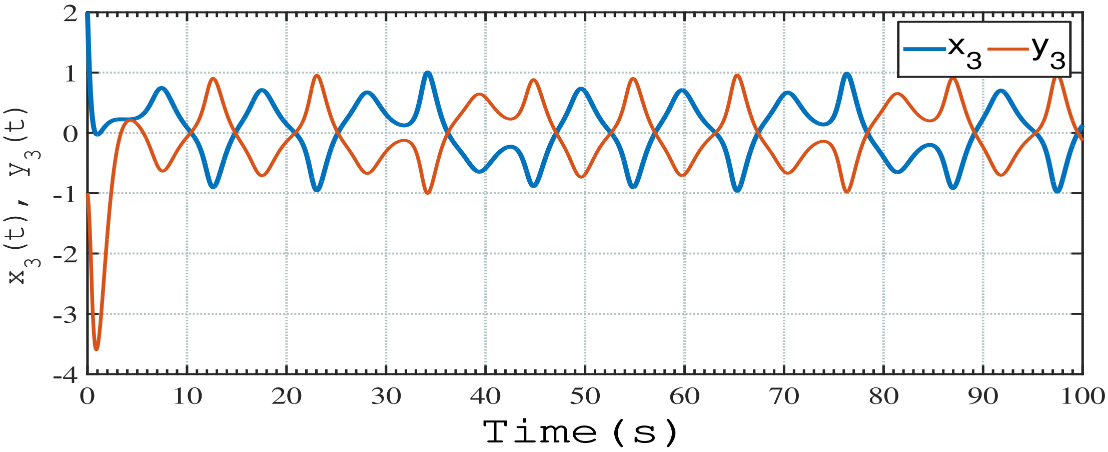
For Anti-synchronization set q= -1 in eq (4.18):



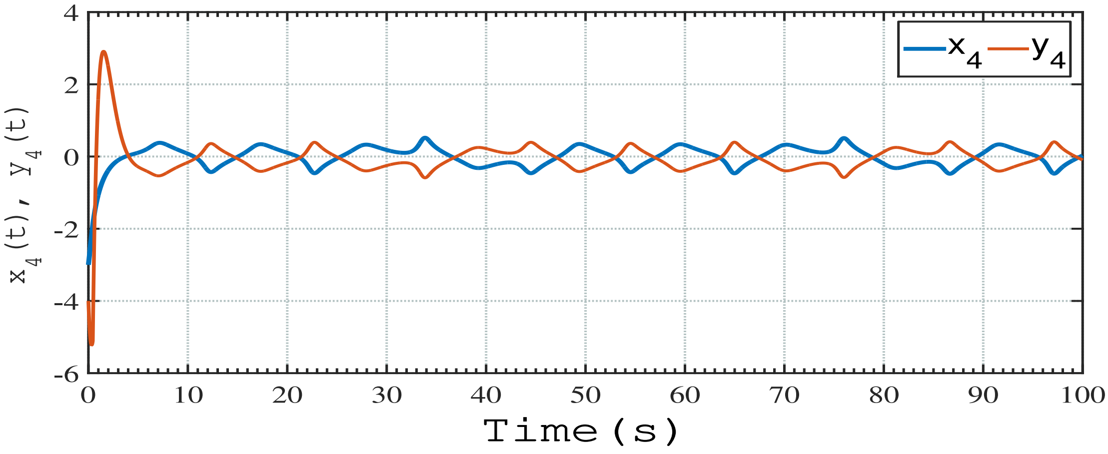
(a)



(b)

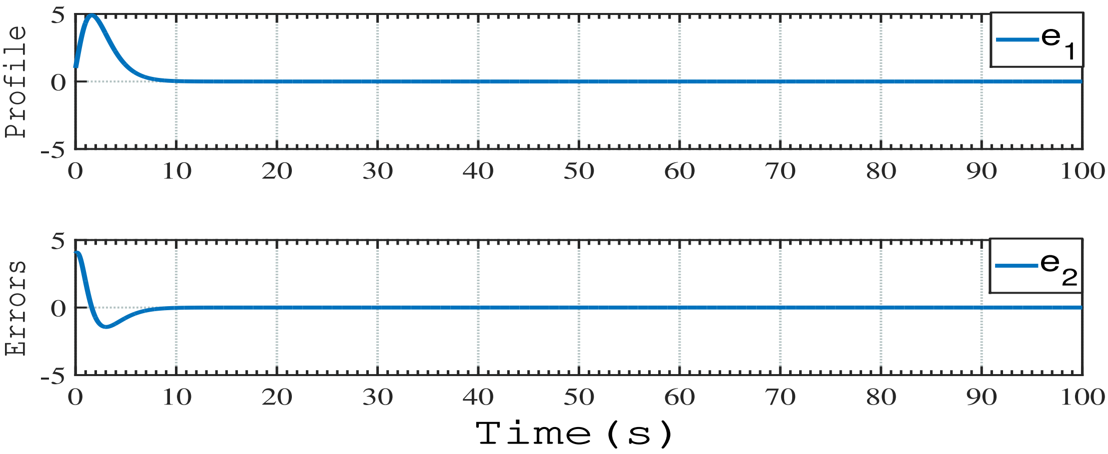


(c)



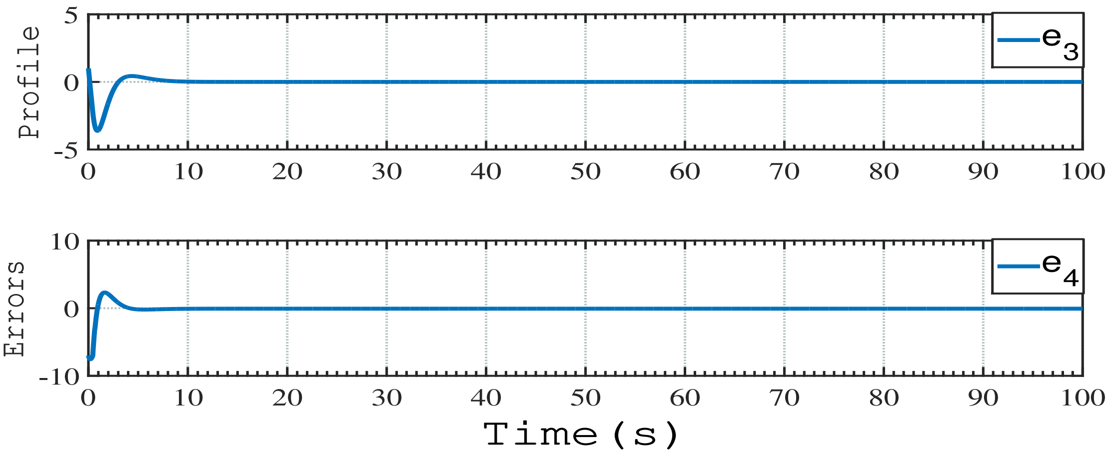
4
2
0
-2
-4
-6
0 10 20 30 40 50 60 70 80 90 100
x4(t), y4(t)
Time(s)
x4
y4

(d)



Profile
Errors
5
0
-5
0 10 20 30 40 50 60 70 80 90 100
e1
e2
Time(s)

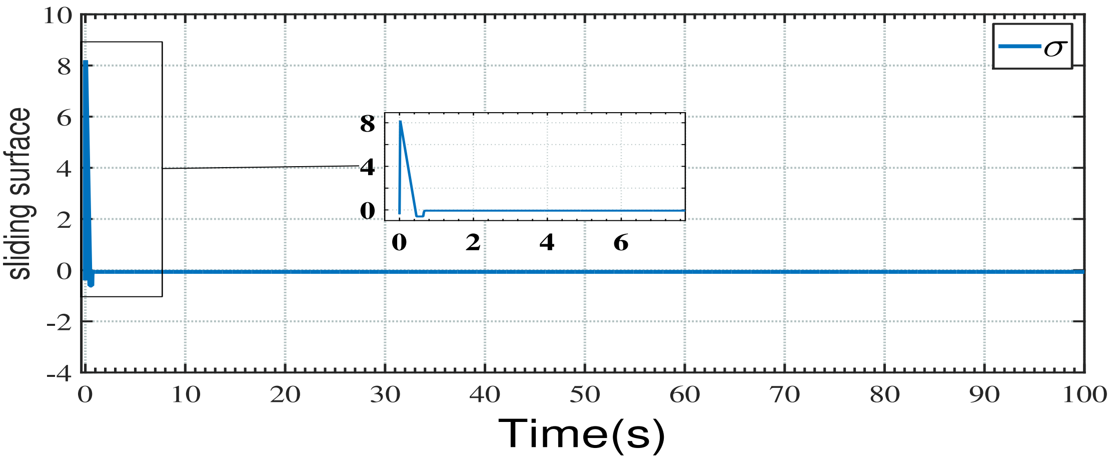
(e)



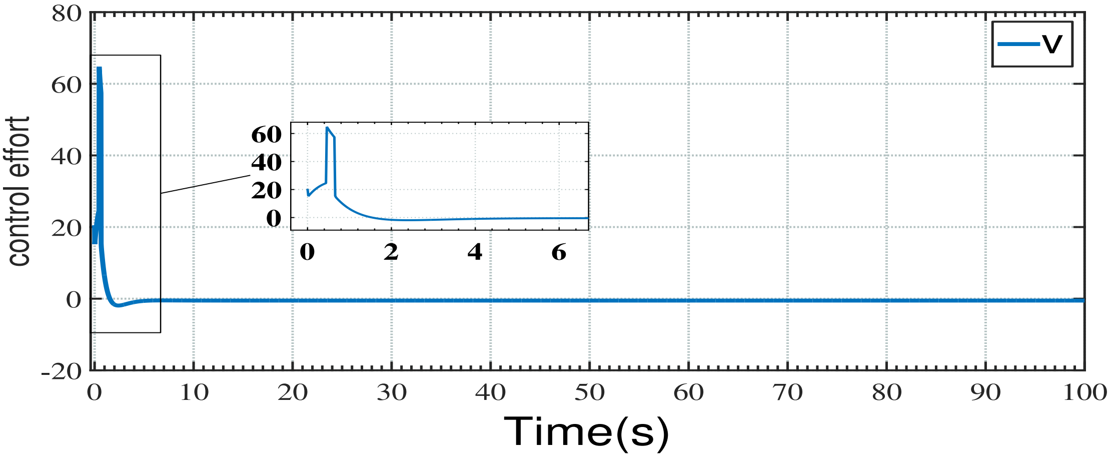
Profile
Errors
5
0
-5
10
-10
0 10 20 30 40 50 60 70 80 90 100
e3
e4
Time(s)

(f)

FIGURE 4.8: *Anti-Synchronization of 4D Hyperchaotic Financial System, (a) Anti-Synchronization of interest rate corresponding to initial condition* $[x_1(0), y_1(0) = (3, -2)]$*, (b) Anti-Synchronization of investment demand corresponding to initial condition* $[x_2(0), y_2(0) = (1, 3)]$*, (c) Anti-Synchronization of price index corresponding to initial condition* $[x_3(0), y_3(0) = (2, -1)]$*,(d) Anti-Synchronization of average profit margins corresponding to initial condition* $[x_4(0), y_4(0) = (-3, -4)]$*, (e) and (f) Time history of the errors* e_1*,* e_2*,* e_3 *and* e_4



(a)



(b)

FIGURE 4.9: *Anti-Synchronization of 4D Hyperchaotic Financial System, (a) Sliding manifold* σ *(b) Control effort* v

4.4 Adaptive Integral Sliding Mode Control

In this section AISMC is dispensed for synchronization and anti-synchronization of financial chaotic systems. In this method the parameters are expected unknown and are estimated using ADISMC.

4.4.1 3 Dimensional System

Let $\hat{a}, \hat{b}, \hat{c}$ be estimate value of a, b, c and let $\tilde{a} = a - \hat{a}, \tilde{b} = b - \hat{b}, \tilde{c} = c - \hat{c}$ be errors.
Hence system 4.1 and 4.2 with external perturbations are shown below:

$$\begin{aligned} \dot{x_1} &= x_3 + x_2x_1 - \hat{a}x_1 - \tilde{a}x_1 \\ \dot{x_2} &= 1 - \hat{b}x_2 - \tilde{b}x_2 - x_1^2 \\ \dot{x_3} &= -x_1 - \hat{c}x_3 - \tilde{c}x_3 \end{aligned} \tag{4.13}$$

$$\begin{aligned} \dot{y_1} &= y_3 + y_2y_1 - \hat{a}y_1 - \tilde{a}y_1 + h_1v_1 + u_1 \\ \dot{y_2} &= 1 - \hat{b}y_2 - \tilde{b}y_2 - y_1^2 + h_2v_2 + u_2 \\ \dot{y_3} &= -y_1 - \hat{c}y_3 - \tilde{c}y_3 + h_3v_3 + u_3 \end{aligned} \tag{4.14}$$

External disturbances are given below:

$$\begin{aligned} \dot{v_1} &= 2y_1y_2 - 0.4v_1 \\ \dot{v_2} &= -2y_1y_3 - 0.8v_2 \\ \dot{v_3} &= -1.2y_1y_2 - 0.5v_3 \end{aligned} \tag{4.15}$$

The error signals are given below:

$$\begin{aligned} e_1 &= y_1 - qx_1 \\ e_2 &= y_2 - qx_2 \\ e_3 &= y_3 - qx_3 \end{aligned} \tag{4.16}$$

For synchronization select $q = 1$ and for anti-synchronization select $q = -1$.
By taking derivative of error signals dynamics become:

$$\dot{e_1} = \dot{y_1} - q\dot{x_1} = (y_3 + y_2y_1 - \hat{a}y_1 - \tilde{a}y_1 + h_1v_1) + u_1 - q(x_3 + x_2x_1 - \hat{a}x_1 - \tilde{a}x_1)$$
$$\dot{e_2} = \dot{y_2} - q\dot{x_2} = (1 - \hat{b}y_2 - \tilde{b}y_2 - y_1^2 + h_2v_2) + u_2 - q(1 - \hat{b}x_2 - \tilde{b}x_2 - x_1^2)$$
$$\dot{e_3} = \dot{y_3} - q\dot{x_3} = (-y_1 - \hat{c}y_3 - \tilde{c}y_3 + h_3v_3) + u_3 - q(-x_1 - \hat{c}x_3 - \tilde{c}x_3) \tag{4.17}$$

By choosing

$$u_1 = -(y_3 + y_2y_1 - \hat{a}y_1 + h_1v_1) + q(x_3 + x_2x_1 - \hat{a}x_1) + e_2$$
$$u_2 = -(1 - \hat{b}y_2 - y_1^2 + h_2v_2) + q(1 - \hat{b}x_2 - x_1^2) + e_3 \tag{4.18}$$
$$u_3 = -(-y_1 - \hat{c}y_3 + h_3v_3) + q(-x_1 - \hat{c}x_3) + v$$

In 4.18 v represents the new input, the system 4.17 can be written as:

$$\dot{e_1} = -\tilde{a}y_1 + q(\tilde{a}x_1) + e_2$$
$$\dot{e_2} = -\tilde{b}y_2 + q(\tilde{b}x_2) + e_3 \tag{4.19}$$
$$\dot{e_3} = -\tilde{c}y_3 + q(\tilde{c}x_3) + v$$

By using AISMC, choose the nominal system for 4.19 as:

$$\dot{e_1} = e_2$$
$$\dot{e_2} = e_3 \tag{4.20}$$
$$\dot{e_3} = v_0$$

Defining the Hurwitz sliding surface for nominal system 4.20 as:

$\sigma_0 = (1 + \frac{d}{dt})^2 e_1$

$\sigma_0 = e_1 + 2e_2 + e_3$

By taking derivative:

$\dot{\sigma_0} = \dot{e_1} + 2\dot{e_2} + \dot{e_3}$

Putting values:

$\dot{\sigma_0} = e_2 + 2e_3 + v_0$

By choosing

$v_0 = -e_2 - 2e_3 - k\sigma_0,\ k > 0$ we have $\dot{\sigma_0} = -k\sigma_0$.

Hence the error system 4.17 is asymptotically stable.

Now choose the integral sliding surface for the system 4.16 as under:

$\sigma = \sigma_0 + z$

$\sigma = e_1 + 2e_2 + e_3 + z$

Where z is some integral term discussed later.

To circumvent the reaching phase, choose $z(0)$ in such a way $\sigma(0) = 0$.

Take $v = v_0 + v_s$ where, v_0 is nominal input and v_s is discontinuous term computed later.
By taking derivative:
$\dot{\sigma} = \dot{e_1} + 2\dot{e_2} + \dot{e_3} + \dot{z}$
Putting values:
$\dot{\sigma} = (-\tilde{a}y_1 + q(\tilde{a}x_1) + e_2) + 2(-\tilde{b}y_2 + q(\tilde{b}x_2) + e_3) + (-\tilde{c}y_3 + q(\tilde{c}x_3) + v) + \dot{z}$
By choosing Lyapunov function:
$V = \frac{1}{2}\sigma^2 + \frac{1}{2}(\tilde{a}^2 + \tilde{b}^2 + \tilde{c}^2)$
Sketch the adaptive laws for $\tilde{a}, \hat{a}, \tilde{b}, \hat{b}, \tilde{c}, \hat{c}$ and compute v_s just like that $\dot{V} < 0$.
Appraised a Lyapunov function: $V = \frac{1}{2}\sigma^2 + \frac{1}{2}(\tilde{a}^2 + \tilde{b}^2 + \tilde{c}^2)$.
Afterwards $\dot{V} < 0$ if the adaptive laws for $\tilde{a}, \hat{a}, \tilde{b}, \hat{b}, \tilde{c}, \hat{c}$ and the value of v_s are chosen as:
$\dot{z} = -e_2 - 2e_3 - k\sigma_0, k > 0 - v_0,\ v_s = -k\sigma - ksign(\sigma_0)$

$$\begin{aligned}
\dot{\tilde{a}} &= \sigma e_1 - k\tilde{a} \\
\dot{\tilde{b}} &= 2\sigma e_2 - k\tilde{b} \\
\dot{\tilde{c}} &= \sigma e_3 - k\tilde{c} \\
\dot{\hat{a}} &= -\sigma e_1 + k\tilde{a} \\
\dot{\hat{b}} &= -2\sigma e_2 + k\tilde{b} \\
\dot{\hat{c}} &= -\sigma e_3 + k\tilde{c}
\end{aligned} \tag{4.21}$$

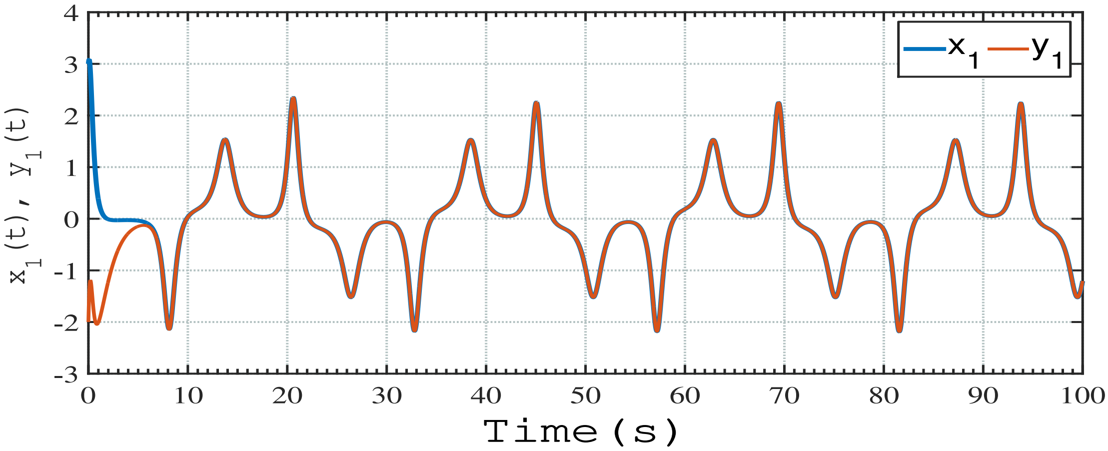
Proof:
Since
$\dot{V} = \sigma\dot{\sigma} + \tilde{a}\dot{\tilde{a}} + \tilde{b}\dot{\tilde{b}} + \tilde{c}\dot{\tilde{c}}$
$= \sigma(-\tilde{a}e_1 - 2\tilde{b}e_2 - \tilde{c}e_3 - ksign(\sigma)) + \tilde{a}\dot{\tilde{a}} + \tilde{b}\dot{\tilde{b}} + \tilde{c}\dot{\tilde{c}}$
$= \tilde{a}(-\sigma e_1 + \dot{\tilde{a}}) + \tilde{b}(-2\sigma e_2 + \dot{\tilde{b}}) + \tilde{c}(-\sigma e_3 + \dot{\tilde{c}}) - k\sigma^2$
By putting

$$\begin{aligned}
\dot{\tilde{a}} &= \sigma e_1 - k\tilde{a} \\
\dot{\tilde{b}} &= 2\sigma e_2 - k\tilde{b} \\
\dot{\tilde{c}} &= \sigma e_3 - k\tilde{c} \\
\dot{\hat{a}} &= -\sigma e_1 + k\tilde{a} \\
\dot{\hat{b}} &= -2\sigma e_2 + k\tilde{b} \\
\dot{\hat{c}} &= -\sigma e_3 + k\tilde{c}
\end{aligned} \tag{4.22}$$

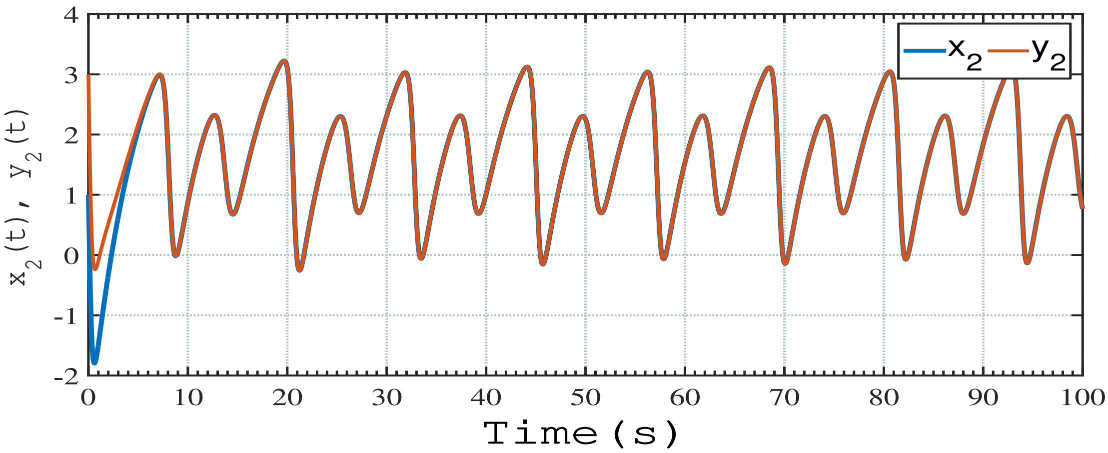
We have
$\dot{V} = -k\sigma^2 - k_1\tilde{a}^2 - k_2\tilde{b}^2 - k_3\tilde{c}^2.$
For this we can say that $\sigma, \tilde{a}, \tilde{b}, \tilde{c} \rightarrow 0$. Since $\sigma \rightarrow 0$, therefore $e = (e_1, e_2, e_3) \rightarrow 0$.
In simulations, the initial conditions are selected as: $x(0) = [3, 1, 2]^T$, $y(0) = [-2, 3, -1]^T$.
The true value of the unknown parameters are chosen as: $a = 0.9, b = 0.2, c = 1.5$.

4.4.2 Synchronization of 3D Financial Chaotic System

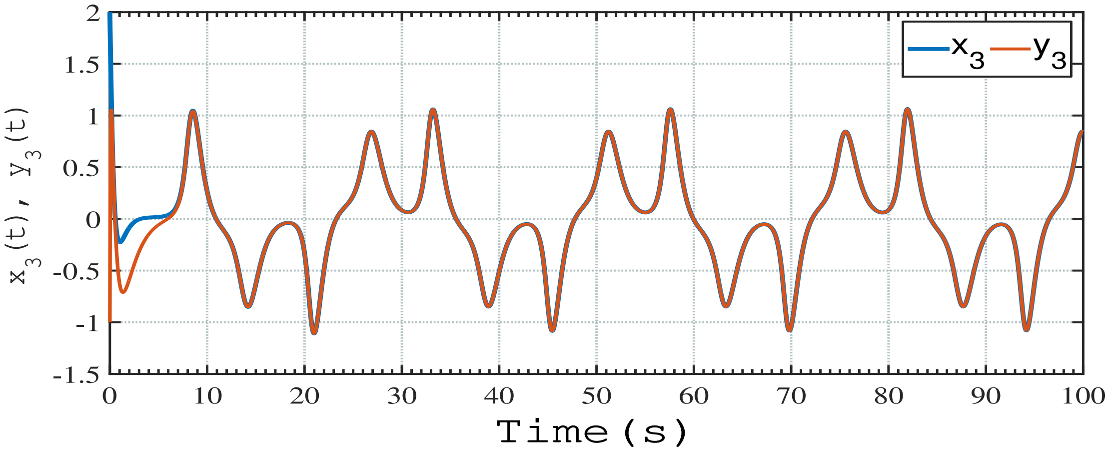
For Synchronization set q=1 in eq (4.16):



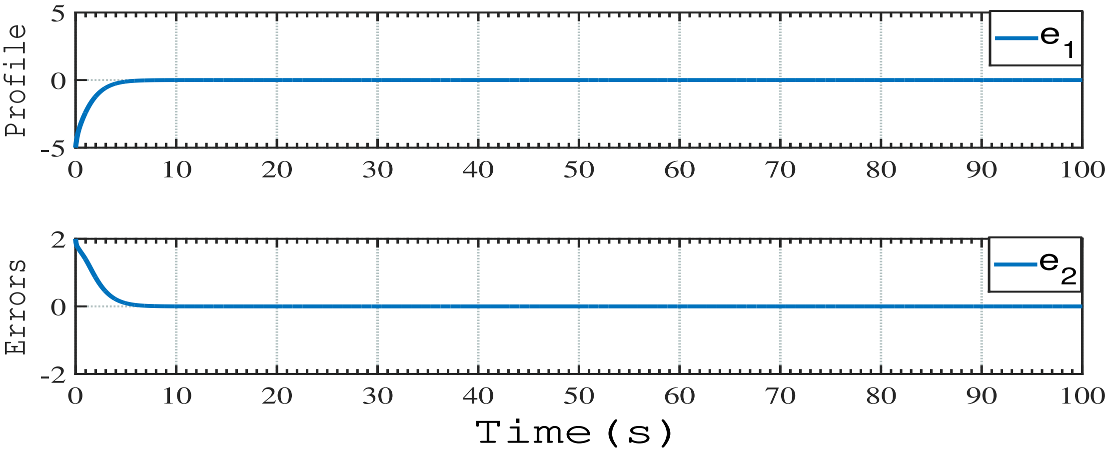
(a)



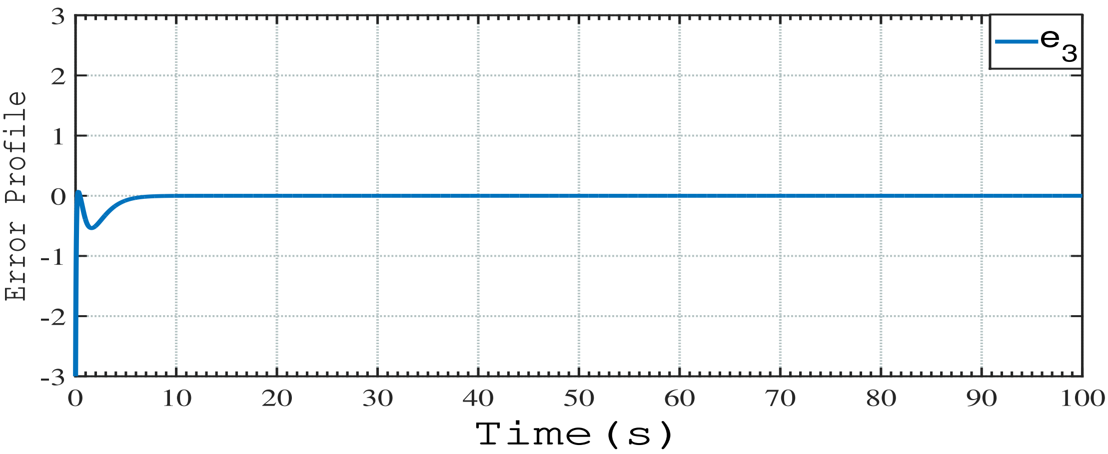
(b)


x3(t), y3(t)
Time(s)
x3
y3
2
1.5
1
0.5
0
-0.5
-1
-1.5
0 10 20 30 40 50 60 70 80 90 100

(c)

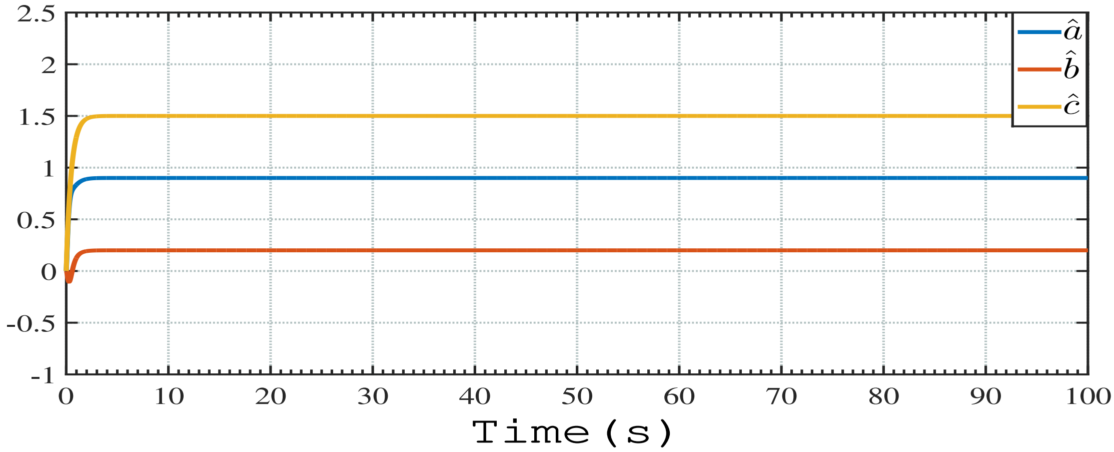

Profile
Errors
e1
e2
5
0
-5
2
0
-2
0 10 20 30 40 50 60 70 80 90 100
Time(s)

(d)

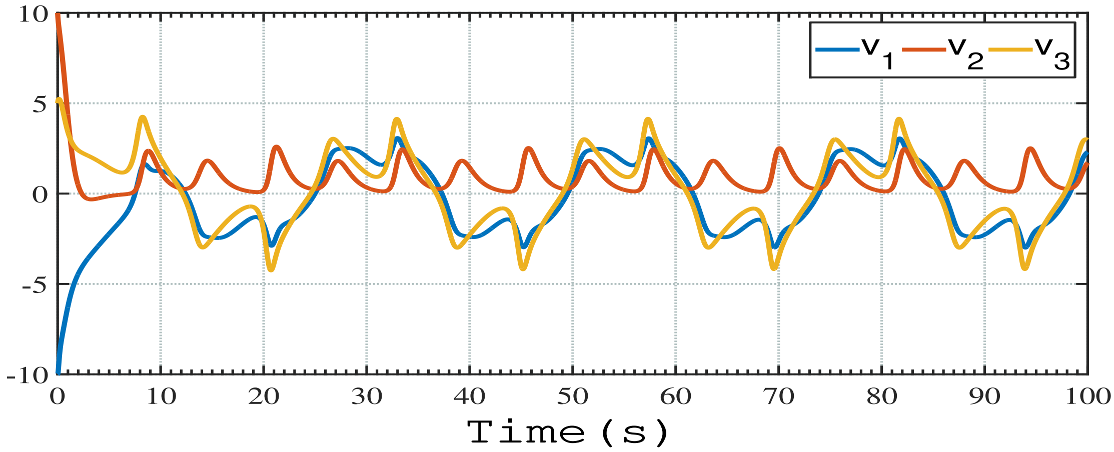

Error Profile
e3
3
2
1
0
-1
-2
-3
0 10 20 30 40 50 60 70 80 90 100
Time(s)

(e)

FIGURE 4.10: *Synchronization of 3D Financial Chaotic System with adaptation of parameters, (a) Synchronization of interest rate corresponding to initial condition $[x_1(0), y_1(0) = (3, -2)]$, (b) Synchronization of investment demand corresponding to initial condition $[x_2(0), y_2(0) = (1, 3)]$, (c) Synchronization of price index corresponding to initial condition $[x_3(0), y_3(0) = (2, -1)]$, (d) and (e) Time history of the errors e_1, e_2 and e_3*

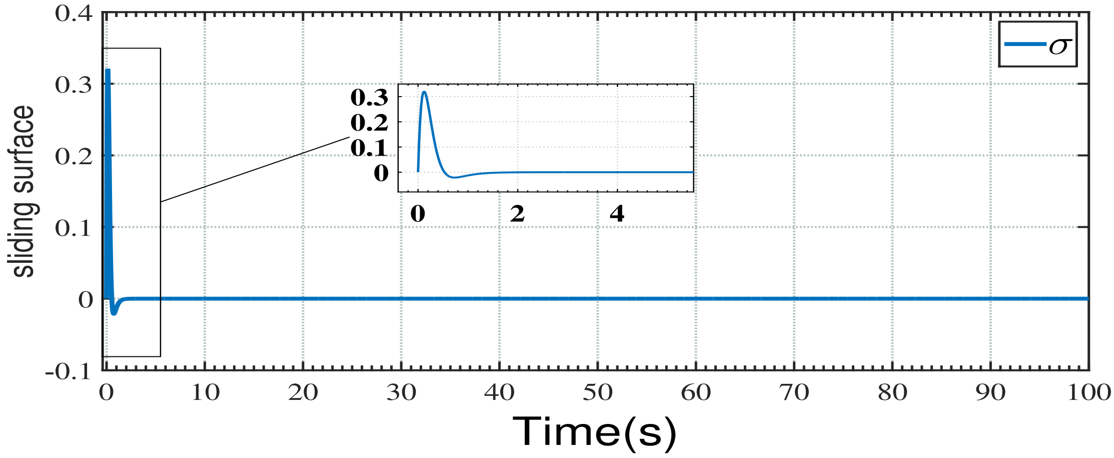


(a)

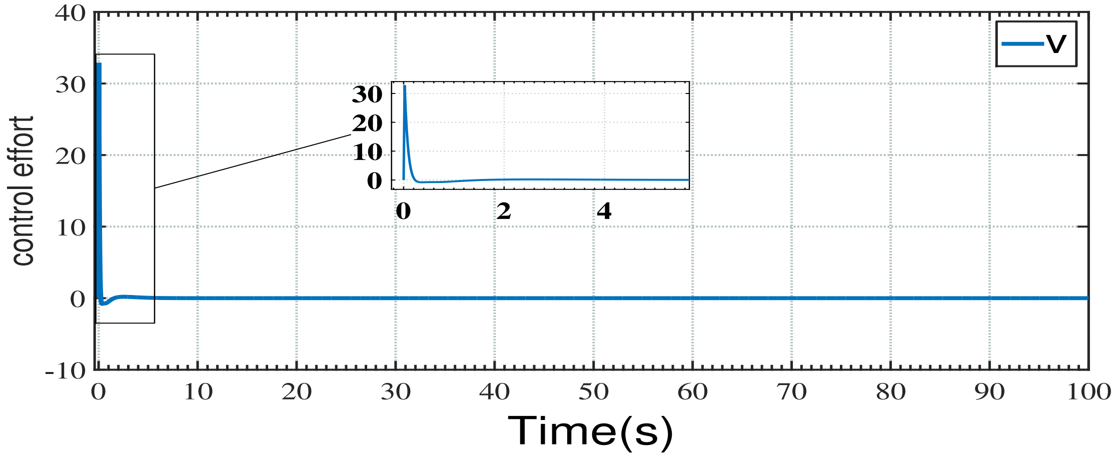


(b)

FIGURE 4.11: *Synchronization of 3D Financial Chaotic System, (a) $\hat{a}$, $\hat{b}$, $\hat{c}$ represents the adaptation of unknown parameters, (b) v_1, v_2, v_3 represents the time varying disturbances.*



(a)

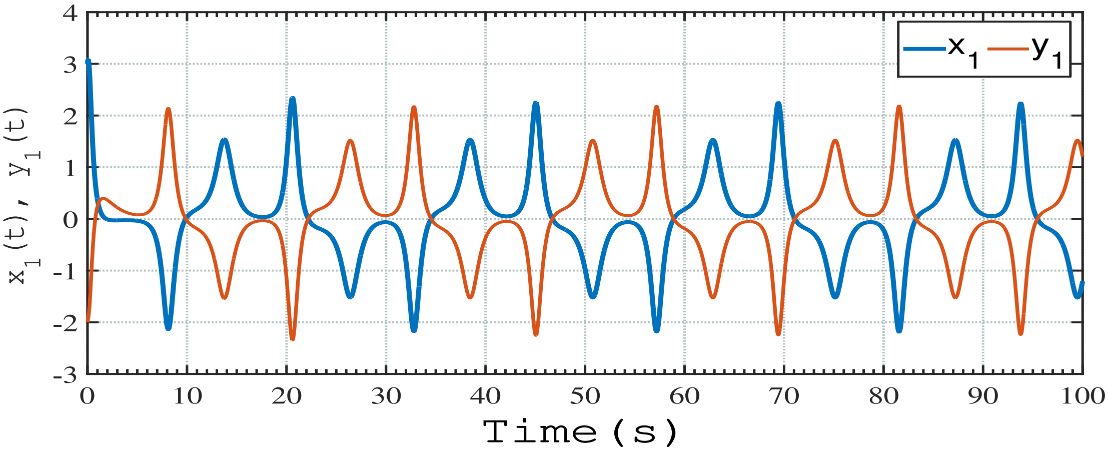


(b)

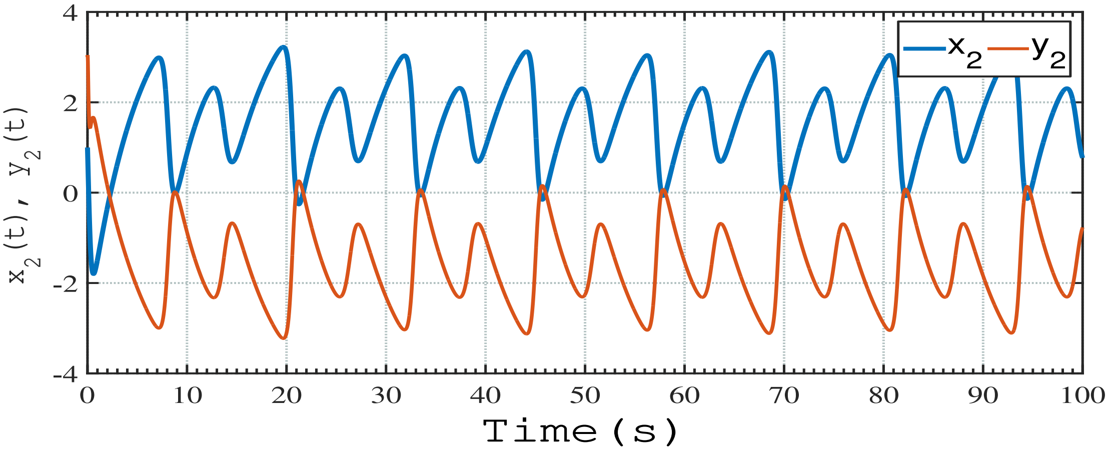
FIGURE 4.12: *Synchronization of 3D Financial Chaotic System with adaptation of parameters, (a) Sliding manifold σ (b) Control effort v*

4.4.3 Anti-Synchronization of 3D Financial Chaotic System

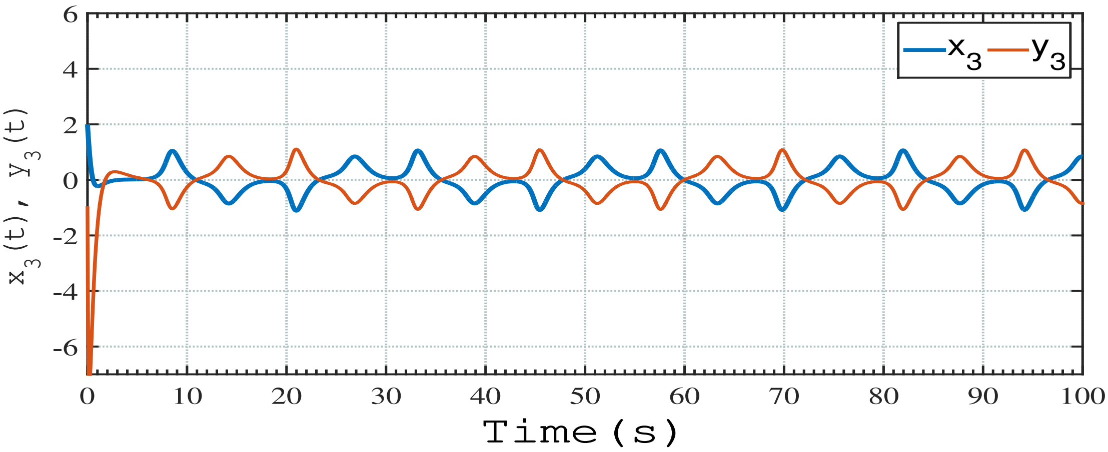
For Anti-synchronization set q=-1 in eq (4.16):


x1 y1
4
3
2
1
0
-1
-2
-3
0 10 20 30 40 50 60 70 80 90 100
Time(s)
x1(t), y1(t)

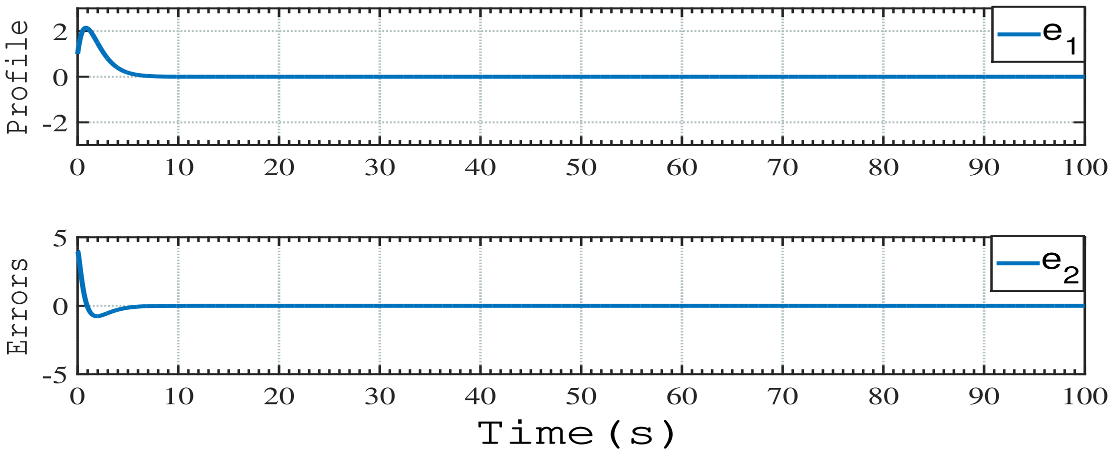
(a)


x2 y2
4
2
0
-2
-4
0 10 20 30 40 50 60 70 80 90 100
Time(s)
x2(t), y2(t)

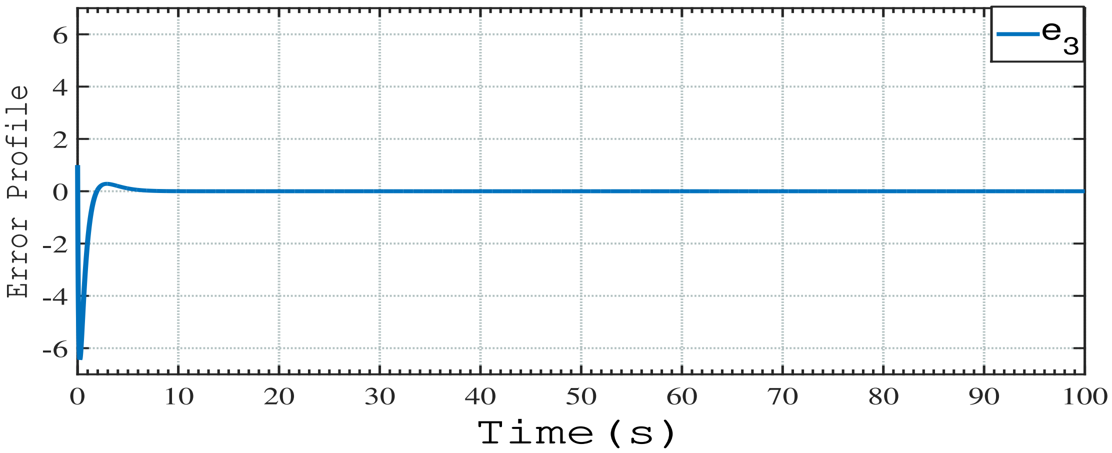
(b)


x3 y3
6
4
2
0
-2
-4
-6
0 10 20 30 40 50 60 70 80 90 100
Time(s)
x3(t), y3(t)

(c)

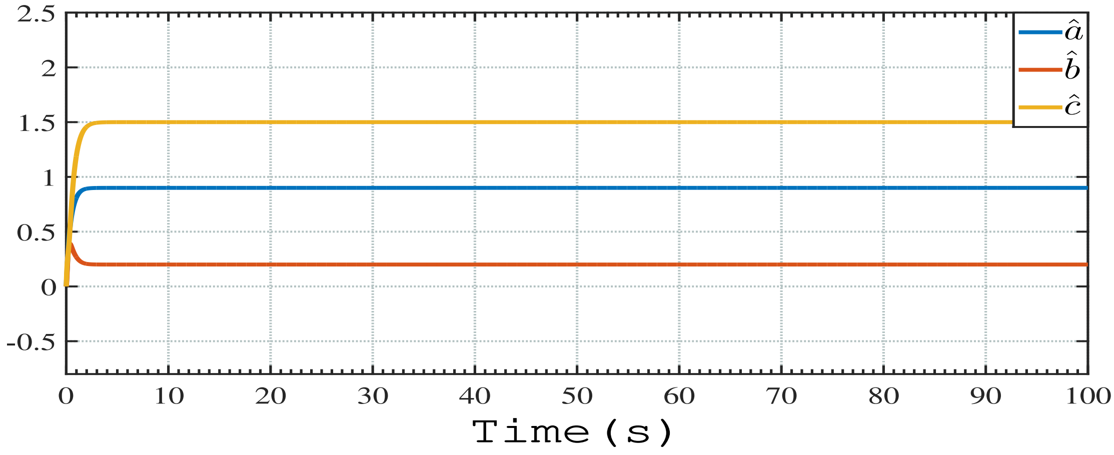


(d)

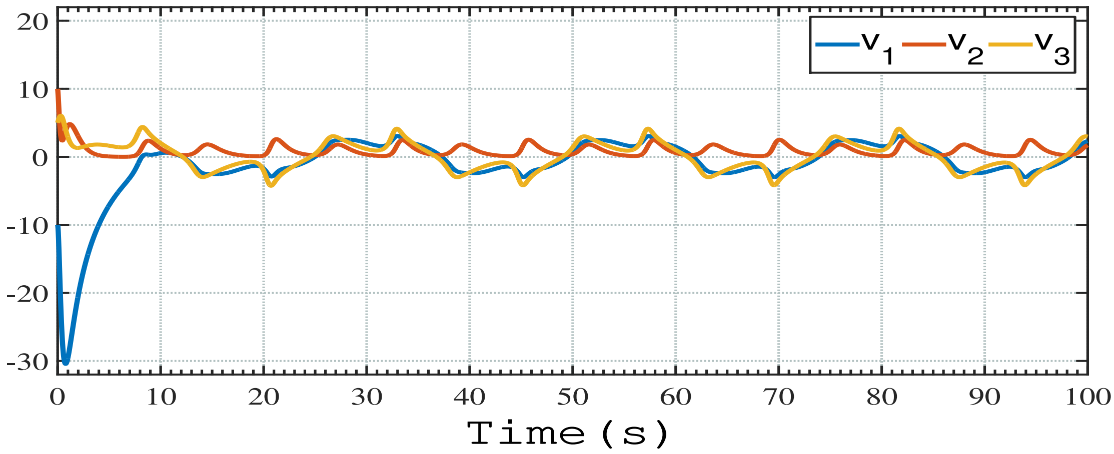


(e)

FIGURE 4.13: *Anti-Synchronization of 3D Financial Chaotic System with adaptation of parameters, (a) Anti-Synchronization of interest rate corresponding to initial condition* $[x_1(0), y_1(0) = (3, -2)]$, *(b) Anti-Synchronization of investment demand corresponding to initial condition* $[x_2(0), y_2(0) = (1, 3)]$, *(c) Anti-Synchronization of price index corresponding to initial condition* $[x_3(0), y_3(0) = (2, -1)]$, *(d) and (e) Time history of the errors* e_1, e_2 *and* e_3

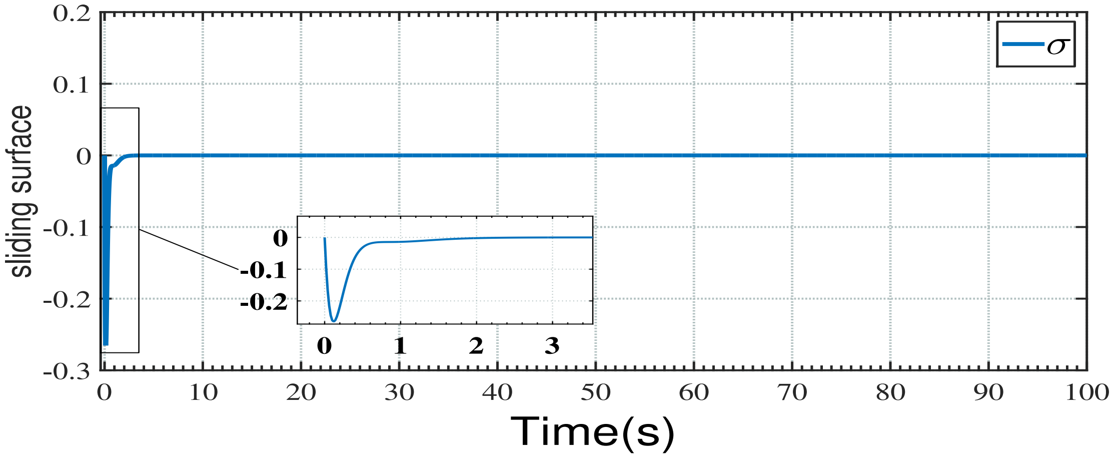


(a)

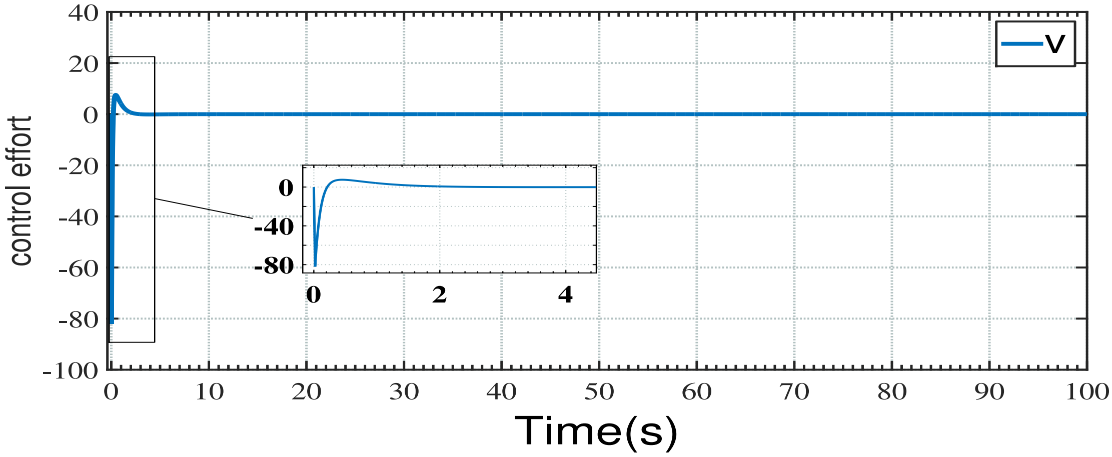


(b)

FIGURE 4.14: *Anti-Synchronization of 3D Financial Chaotic System, (a) $\hat{a}$, $\hat{b}$, $\hat{c}$ represents the adaptation of unknown parameters, (b) v_1, v_2, v_3 represents the time varying disturbances.*



(a)



(b)

FIGURE 4.15: *Anti-Synchronization of 3D Financial Chaotic System with adaptation of parameters, (a) Sliding manifold σ (b) Control effort v*

4.4.4 4 Dimensional System

Let $\hat{a}, \hat{b}, \hat{c}, \hat{d}, \hat{k}$ be estimate value of a, b, c, d, k and let $\tilde{a} = a - \hat{a}, \tilde{b} = b - \hat{b}$, $\tilde{c} = c - \hat{c}, \tilde{d} = d - \hat{d}, \tilde{k} = k - \hat{k}$ be errors.

Thus system 4.3 and 4.4 with external perturbations are mentioned below:

$$\begin{aligned} \dot{x_1} &= x_3 + x_2x_1 - \hat{a}x_1 - \tilde{a}x_1 + x_4 \\ \dot{x_2} &= 1 - \hat{b}x_2 - \tilde{b}x_2 - x_1^2 \\ \dot{x_3} &= -x_1 - \hat{c}x_3 - \tilde{c}x_3 \\ \dot{x_4} &= -\hat{d}x_1x_2 - \tilde{d}x_1x_2 - \hat{k}x_4 - \tilde{k}x_4 \end{aligned} \tag{4.23}$$

$$\begin{aligned} \dot{y_1} &= y_3 + y_2y_1 - \hat{a}y_1 - \tilde{a}y_1 + y_4 + h_1v_1 + u_1 \\ \dot{y_2} &= 1 - \hat{b}y_2 - \tilde{b}y_2 - y_1^2 + h_2v_2 + u_2 \\ \dot{y_3} &= -y_1 - \hat{c}y_3 - \tilde{c}y_3 + h_3v_3 + u_3 \\ \dot{y_4} &= -\hat{d}y_1y_2 - \tilde{d}y_1y_2 - \hat{k}y_4 - \tilde{k}y_4 + h_4v_4 + u_4 \end{aligned} \tag{4.24}$$

External disturbances are given below:

$$\begin{aligned} \dot{v_1} &= 2y_2y_3 - 0.4v_1 \\ \dot{v_2} &= -2y_1y_4 - 0.8v_2 \\ \dot{v_3} &= -1.2y_1y_2 - 0.5v_3 \\ \dot{v_4} &= y_2y_3 - 0.5v_4 \end{aligned} \tag{4.25}$$

By defining the error signals:

$$\begin{aligned} e_1 &= y_1 - qx_1 \\ e_2 &= y_2 - qx_2 \\ e_3 &= y_3 - qx_3 \\ e_4 &= y_4 - qx_4 \end{aligned} \tag{4.26}$$

For synchronization select $q = 1$ and select $q = -1$ for anti-synchronization. By taking derivative of error signals we get error dynamics:

$$\begin{aligned} \dot{e_1} &= \dot{y_1} - q\dot{x_1} \\ \dot{e_1} &= (y_3 + y_2y_1 - \hat{a}y_1 - \tilde{a}y_1 + y_4 + h_1v_1) + u_1 - q(x_3 + x_2x_1 - \hat{a}x_1 - \tilde{a}x_1 + x_4) \\ \dot{e_2} &= \dot{y_2} - q\dot{x_2} \\ \dot{e_2} &= (1 - \hat{b}y_2 - \tilde{b}y_2 - y_1^2 + h_2v_2) + u_2 - q(1 - \hat{b}x_2 - \tilde{b}x_2 - x_1^2) \\ \dot{e_3} &= \dot{y_3} - q\dot{x_3} \\ \dot{e_3} &= (-y_1 - \hat{c}y_3 - \tilde{c}y_3 + h_3v_3) + u_3 - q(-x_1 - \hat{c}x_3 - \tilde{c}x_3) \\ \dot{e_4} &= \dot{y_4} - q\dot{x_4} \\ \dot{e_4} &= (-\hat{d}y_1y_2 - \tilde{d}y_1y_2 - \hat{k}y_4 - \tilde{k}y_4 + h_4v_4) + u_4 - q(-\hat{d}x_1x_2 - \tilde{d}x_1x_2 - \hat{k}x_4 - \tilde{k}x_4) \end{aligned} \tag{4.27}$$

By choosing

$$\begin{aligned}
u_1 &= -(y_3 + y_2y_1 - \hat{a}y_1 + y_4 + h_1v_1) + q(x_3 + x_2x_1 - \hat{a}x_1 + x_4) + e_1 \\
u_2 &= -(1 - \hat{b}y_2 - y_1^2 + h_2v_2) + q(1 - \hat{b}x_2 - x_1^2) + e_2 \\
u_3 &= -(-y_1 - \hat{c}y_3 + h_3v_3) + q(-x_1 - \hat{c}x_3) + e_3 \\
u_4 &= -(-\hat{d}y_1y_2 - \hat{k}y_4 + h_4v_4) + q(-\hat{d}x_1x_2 - \hat{k}x_4) + v
\end{aligned} \tag{4.28}$$

Where v is the new input, the system 4.27 can be written as:

$$\begin{aligned}
\dot{e_1} &= -\tilde{a}y_1 + q(\tilde{a}x_1) + e_2 \\
\dot{e_2} &= -\tilde{b}y_2 + q(\tilde{b}x_2) + e_3 \\
\dot{e_3} &= -\tilde{c}y_3 + q(\tilde{c}x_3) + e_4 \\
\dot{e_4} &= -\tilde{d}y_1y_2 - \tilde{l}y_4 + q(\tilde{d}x_1x_2) + q(\tilde{l}x_4) + v
\end{aligned} \tag{4.29}$$

By using AISMC, choose the nominal system for 4.29 as:

$$\begin{aligned}
\dot{e_1} &= e_2 \\
\dot{e_2} &= e_3 \\
\dot{e_3} &= e_4 \\
\dot{e_4} &= v_0
\end{aligned} \tag{4.30}$$

Defining the Hurwitz sliding surface for nominal system 4.30 as:
$\sigma_0 = (1 + \frac{d}{dt})^3 e_1$
$\sigma_0 = e_1 + 3e_2 + 3e_3 + e_4$
By taking derivative:
$\dot{\sigma}_0 = \dot{e_1} + 3\dot{e_2} + 3\dot{e_3} + \dot{e_4}$
$\dot{\sigma}_0 = e_2 + 3e_3 + 3e_4 + v_0$
If we choose $v_0 = -e_2 - 3e_2 - 3e_4 - k\sigma_0,\ k > 0$
By putting v_0 we get $\dot{\sigma}_0 = -k\sigma_0$.
So we can say that error system 4.30 is asymptotically stable.
Now choose integral sliding surface for the system 4.29 as:
$\sigma = \sigma_0 + z$
$\sigma = e_1 + 3e_2 + 3e_3 + e_4 + z$
Where, z is some integral term discussed later. To circumvent the reaching phase, choose $z(0)$ such that $\sigma(0) = 0$.
Choose $v = v_0 + v_s$ where, v_0 is nominal input and v_s is discontinuous term computed later.

By taking derivative:

$\dot{\sigma} = \dot{e_1} + 3\dot{e_2} + 3\dot{e_3} + \dot{e_4} + \dot{z}$

$\dot{\sigma} = (-\tilde{a}y_1 + q(\tilde{a}x_1) + e_2) + 3(-\tilde{b}y_2 + q(\tilde{b}x_2) + e_3) + 3(-\tilde{c}y_3 + q(\tilde{c}x_3) + e_4) + (-\tilde{d}y_1y_2 - \tilde{l}y_4 + q(\tilde{d}x_1x_2) + q(\tilde{l}x_4) + v) + \dot{z}$

By choosing Lyapunov function:

$V = \frac{1}{2}\sigma^2 + \frac{1}{2}(\tilde{a}^2 + \tilde{b}^2 + \tilde{c}^2 + \tilde{d}^2 + \tilde{k}^2)$

Design the adaptive laws for $\tilde{a}, \hat{a}, \tilde{b}, \hat{b}, \tilde{c}, \hat{c}, \tilde{d}, \hat{d}, \tilde{k}, \hat{k}$ and compute v_s such that $\dot{V} < 0$.

Consider a Lyapunov function:

$V = \frac{1}{2}\sigma^2 + \frac{1}{2}(\tilde{a}^2 + \tilde{b}^2 + \tilde{c}^2 + \tilde{d}^2 + \tilde{k}^2)$. Afterward $\dot{V} < 0$ if the adaptive laws for $\tilde{a}, \hat{a}, \tilde{b}, \hat{b}, \tilde{c}, \hat{c}, \tilde{d}, \hat{d}, \tilde{k}, \hat{k}$ and the value of v_s are chosen as:

$\dot{z} = -e_2 - 3e_3 - 3e_4 - k\sigma_0, k > 0 - v_0$, $v_s = -k\sigma - ksign(\sigma_0)$

$$\begin{aligned}
\dot{\tilde{a}} &= \sigma e_1 - k_1\tilde{a} \\
\dot{\tilde{b}} &= 3\sigma e_2 - k_2\tilde{b} \\
\dot{\tilde{c}} &= 3\sigma e_3 - k_3\tilde{c} \\
\dot{\tilde{d}} &= \sigma x_1x_2 - \sigma y_1y_2 - k_4\tilde{d} \\
\dot{\tilde{k}} &= \sigma e_4 - k_5\tilde{k} \\
\dot{\hat{a}} &= -\sigma e_1 + k_1\tilde{a} \\
\dot{\hat{b}} &= -3\sigma e_2 + k_2\tilde{b} \\
\dot{\hat{c}} &= -3\sigma e_3 + k_3\tilde{c} \\
\dot{\hat{d}} &= -\sigma x_1x_2 + \sigma y_1y_2 + k_4\tilde{d} \\
\dot{\hat{k}} &= -\sigma e_4 + k_5\tilde{k}
\end{aligned} \tag{4.31}$$

Proof:

Since

$\dot{V} = \sigma\dot{\sigma} + \tilde{a}\dot{\tilde{a}} + \tilde{b}\dot{\tilde{b}} + \tilde{c}\dot{\tilde{c}} + \tilde{d}\dot{\tilde{d}} + \tilde{k}\dot{\tilde{k}}$

By putting values:

$= \sigma(-\tilde{a}e_1 - 3\tilde{b}e_2 - 3\tilde{c}e_3 - \tilde{d}x_1x_2 - \tilde{k}e_4 - \tilde{d}y_1y_2 - ksign(\sigma)) + \tilde{a}\dot{\tilde{a}} + \tilde{b}\dot{\tilde{b}} + \tilde{c}\dot{\tilde{c}} + \tilde{d}\dot{\tilde{d}} + \tilde{k}\dot{\tilde{k}}$

$= \tilde{a}(-\sigma e_1 + \dot{\tilde{a}}) + \tilde{b}(-3\sigma e_2 + \dot{\tilde{b}}) + \tilde{c}(-3\sigma e_3 + \dot{\tilde{c}}) + \tilde{d}(-\sigma x_1x_2 + \sigma y_1y_2 + \dot{\tilde{d}}) + \tilde{k}(-\sigma e_4 + \dot{\tilde{l}}) - k\sigma^2$

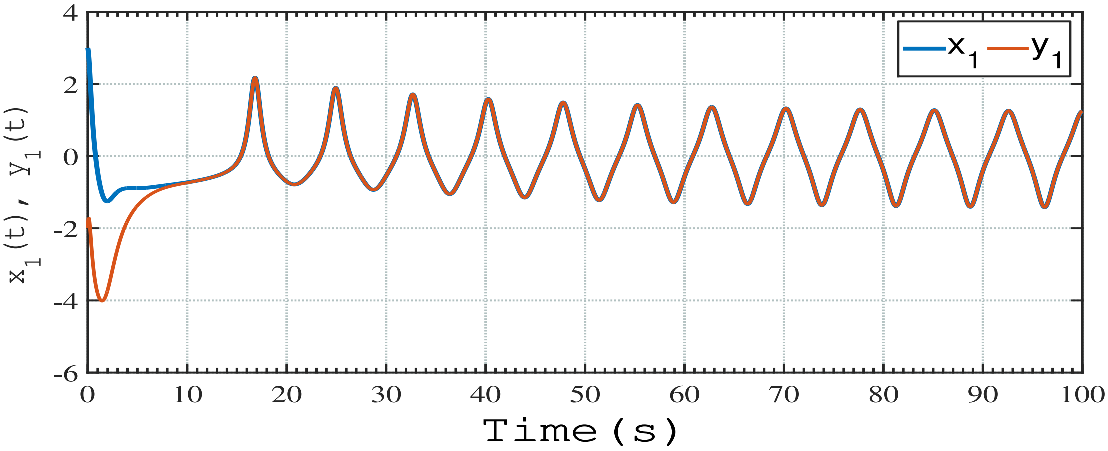
By putting

$$\begin{aligned}
\dot{\tilde{a}} &= \sigma e_1 - k_1\tilde{a} \\
\dot{\tilde{b}} &= 3\sigma e_2 - k_2\tilde{b} \\
\dot{\tilde{c}} &= 3\sigma e_3 - k_3\tilde{c} \\
\dot{\tilde{d}} &= \sigma x_1 x_2 - \sigma y_1 y_2 - k_4\tilde{d} \\
\dot{\tilde{k}} &= \sigma e_4 - k_5\tilde{k} \\
\dot{\hat{a}} &= -\sigma e_1 + k_1\tilde{a} \\
\dot{\hat{b}} &= -3\sigma e_2 + k_2\tilde{b} \\
\dot{\hat{c}} &= -3\sigma e_3 + k_3\tilde{c} \\
\dot{\hat{d}} &= -\sigma x_1 x_2 + \sigma y_1 y_2 + k_4\tilde{d} \\
\dot{\hat{k}} &= -\sigma e_4 + k_5\tilde{k}
\end{aligned} \tag{4.32}$$

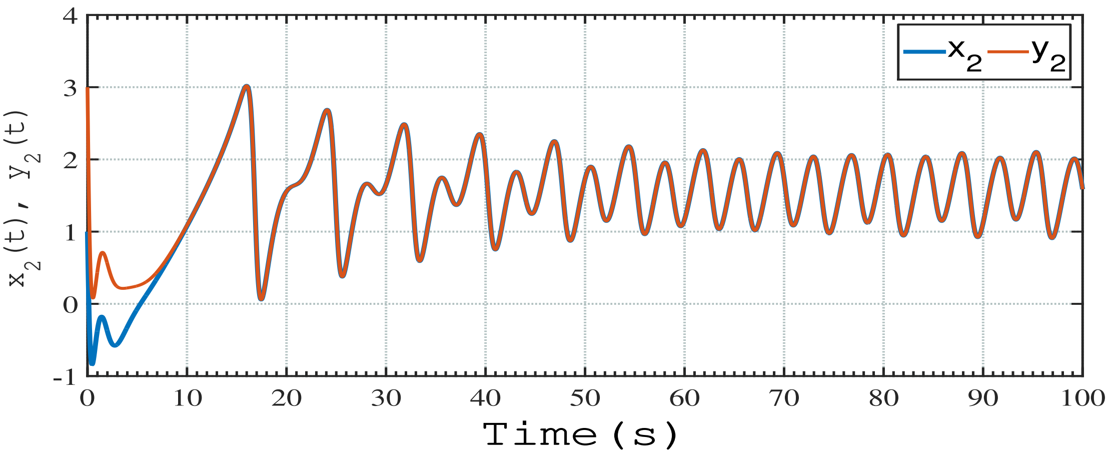
We have
$\dot{V} = -k\sigma^2 - k_1\tilde{a}^2 - k_2\tilde{b}^2 - k_3\tilde{c}^2 - k_3\tilde{d}^2 - k_3\tilde{k}^2$.
For we terminate that $\sigma, \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}, \tilde{k} \to 0$. Since $\sigma \to 0$
therefore $e = (e_1, e_2, e_3, e_4) \to 0$.
In simulations, the initial conditions are taken as: $x(0) = [3, 1, 2, -3]^T$, $y(0) = [-2, 3, -1, -4]^T$.
The true value of the unknown parameters are chosen as: $a = 0.9,\ b = 0.2,\ c = 1.5,\ d = 0.2,\ k = 1$.

4.4.5 Synchronization of 4D HyperChaotic Financial System

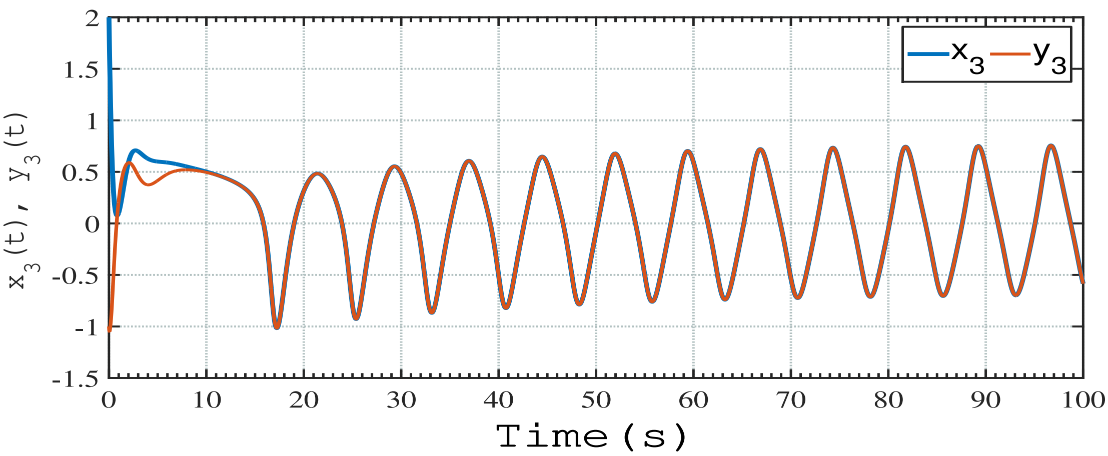
For synchronization put $q = 1$


4
2
0
-2
-4
-6
0
10
20
30
40
50
60
70
80
90
100
Time(s)
x1(t), y1(t)
x1
y1

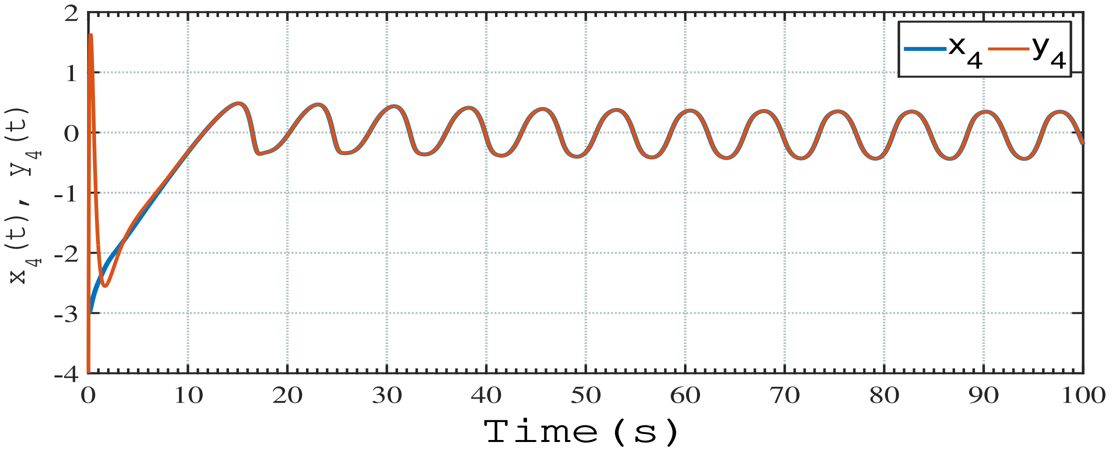
(a)


4
3
2
1
0
-1
0
10
20
30
40
50
60
70
80
90
100
Time(s)
x2(t), y2(t)
x2
y2

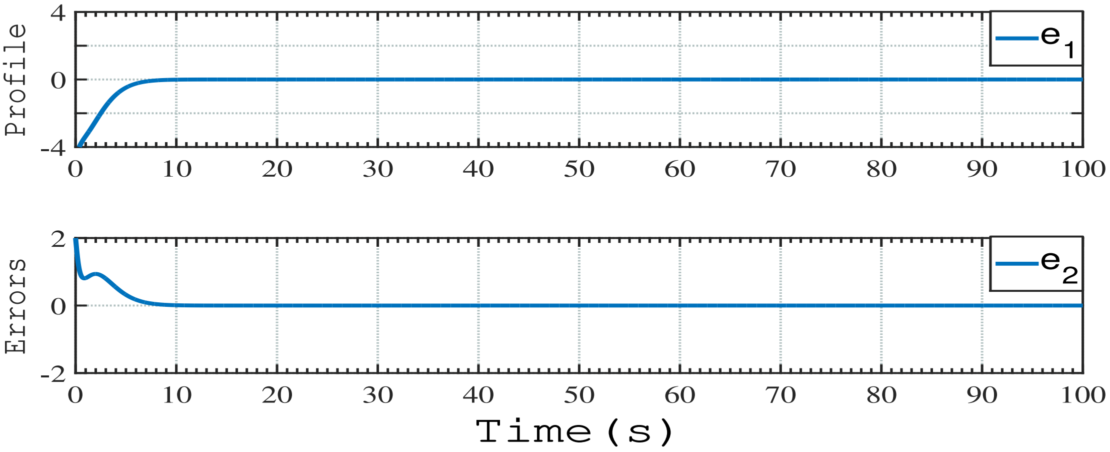
(b)


2
1.5
1
0.5
0
-0.5
-1
-1.5
0
10
20
30
40
50
60
70
80
90
100
Time(s)
x3(t), y3(t)
x3
y3

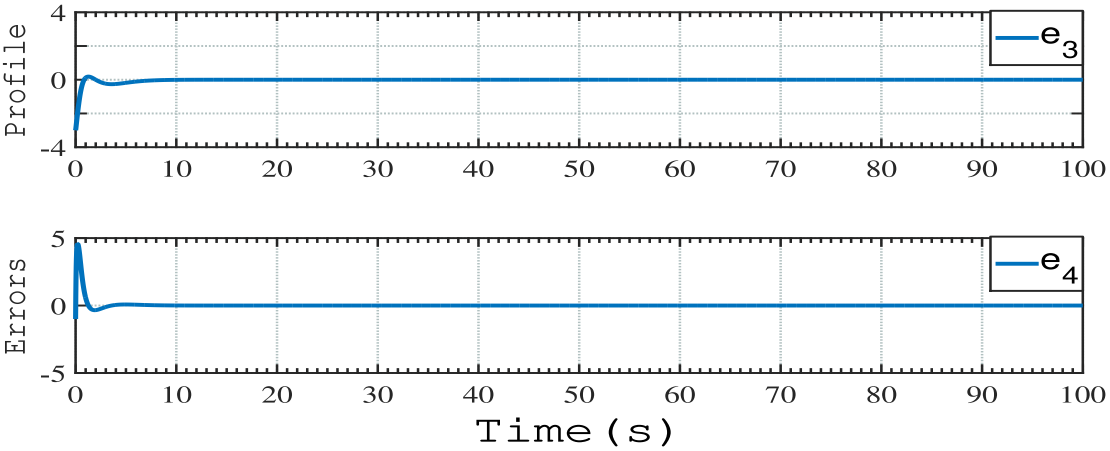
(c)


x4
y4
x4(t), y4(t)
Time(s)
2
1
0
-1
-2
-3
-4
0
10
20
30
40
50
60
70
80
90
100

(d)

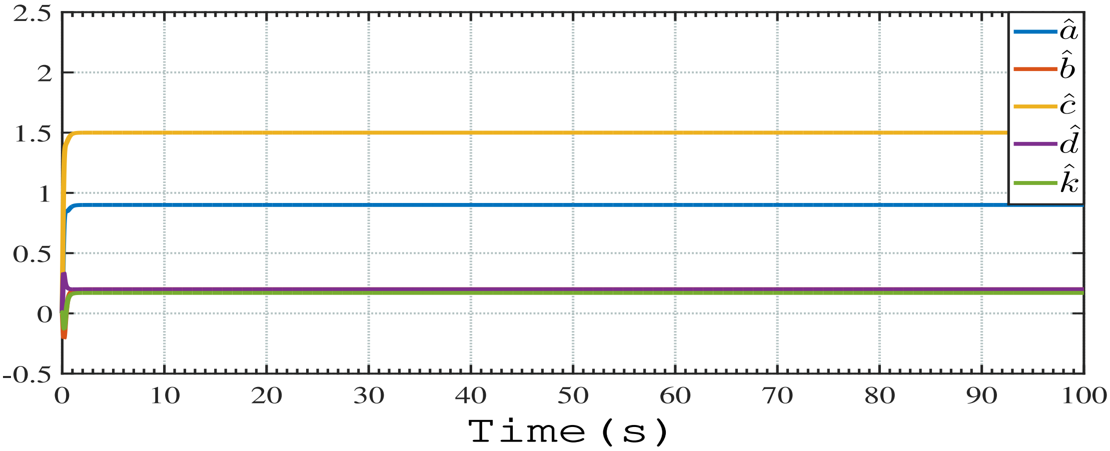

e1
e2
Profile
Errors
Time(s)
4
0
-4
2
0
-2
0
10
20
30
40
50
60
70
80
90
100

(e)

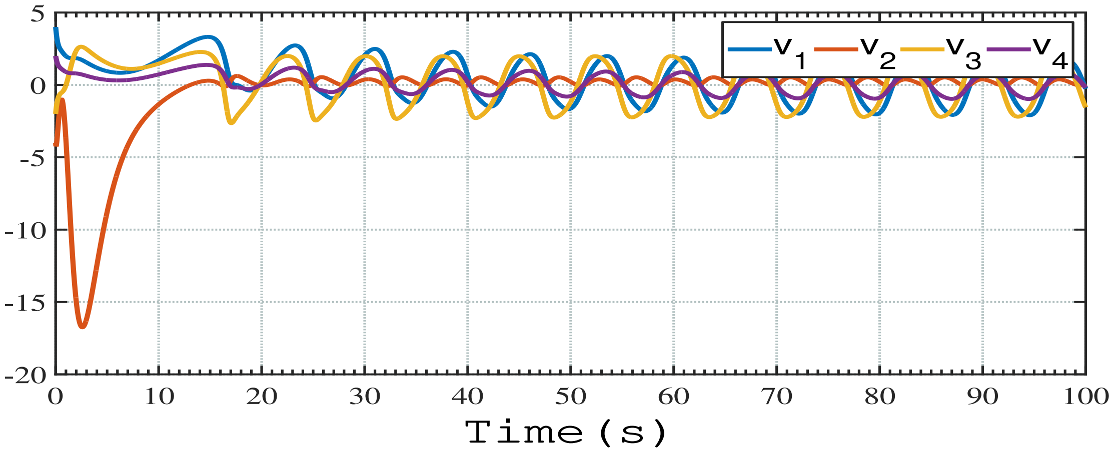

e3
e4
Profile
Errors
Time(s)
4
0
-4
5
0
-5
0
10
20
30
40
50
60
70
80
90
100

(f)

FIGURE 4.16: *Synchronization of 4D Hyperchaotic Financial System with adaptation of parameters, (a) Synchronization of interest rate corresponding to initial condition* $[x_1(0), y_1(0) = (3, -2)]$, *(b) Synchronization of investment demand corresponding to initial condition* $[x_2(0), y_2(0) = (1, 3)]$, *(c) Synchronization of price index corresponding to initial condition* $[x_3(0), y_3(0) = (2, -1)]$,*(d) Synchronization of average profit margins corresponding to initial condition* $[x_4(0), y_4(0) = (-3, -4)]$, *(e) and (f) Time history of the errors* e_1, e_2, e_3 *and* e_4

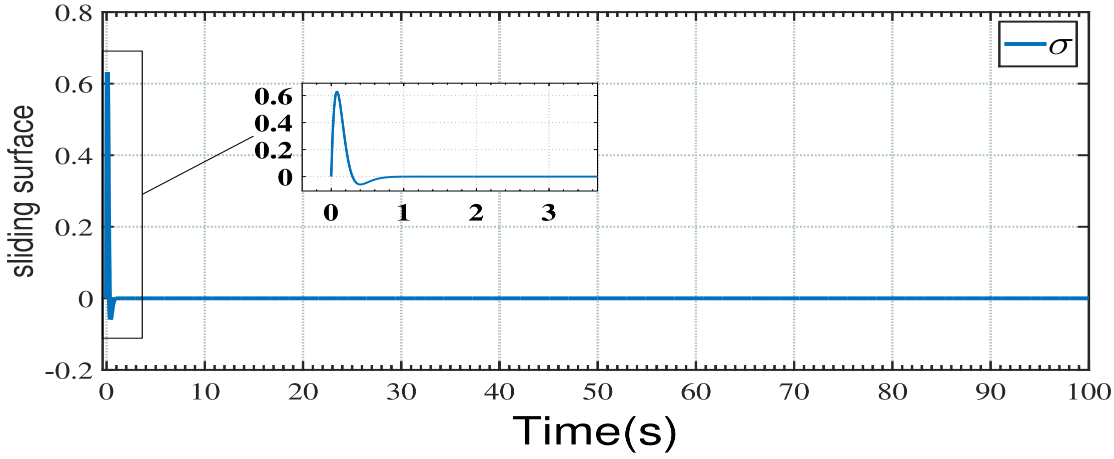


(a)

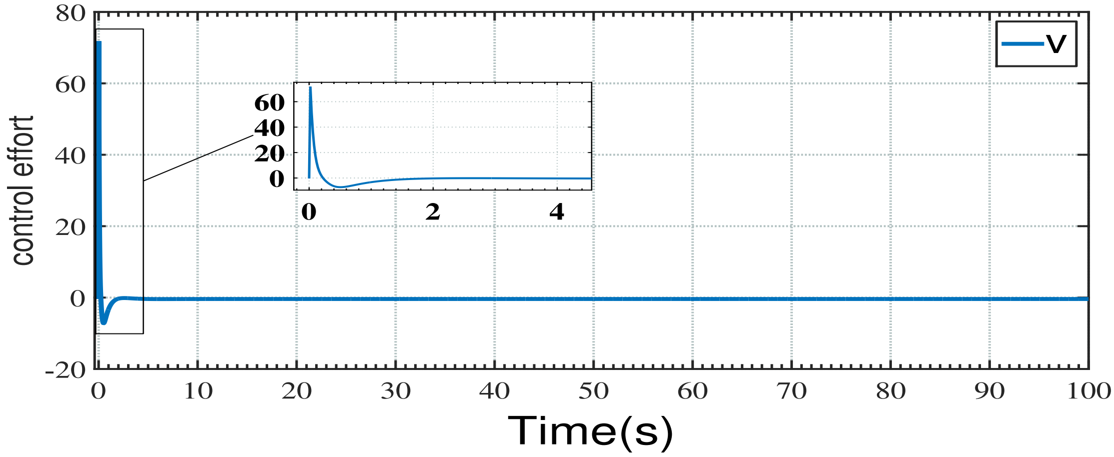


(b)

FIGURE 4.17: *Synchronization of 4D Hyperchaotic Financial System, (a)* $\hat{a}$, $\hat{b}$, $\hat{c}$, $\hat{d}$, $\hat{k}$ *represents the adaptation of unknown parameters, (b)* v_1, v_2, v_3, v_4 *represents the time varying disturbances.*



(a)

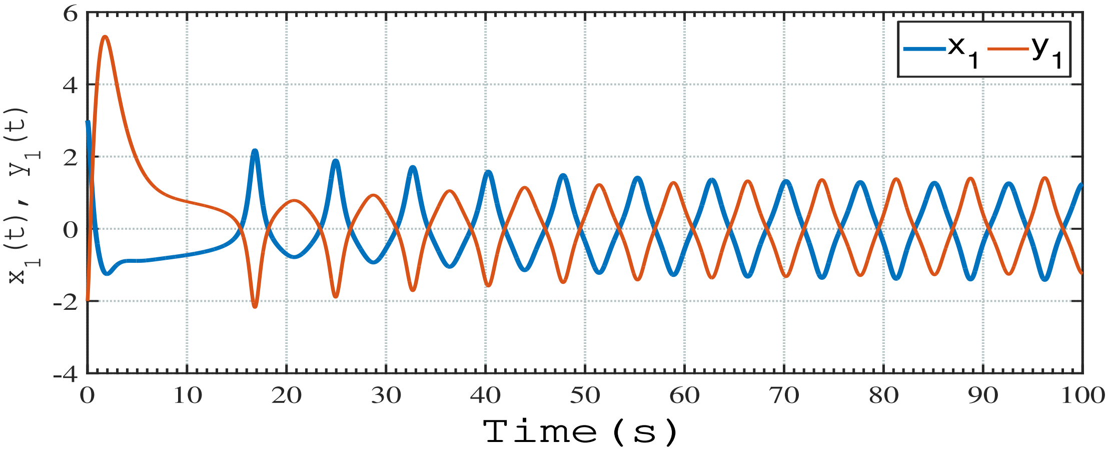


(b)

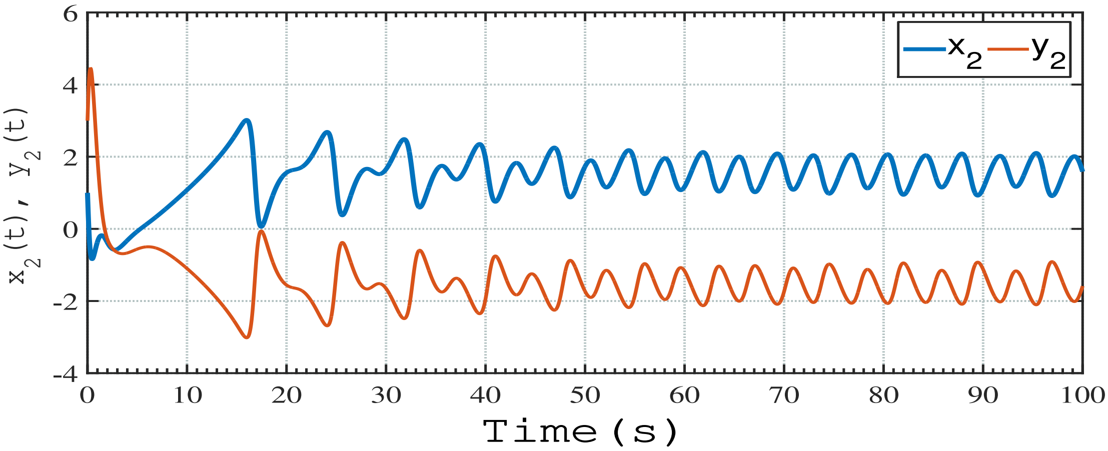
FIGURE 4.18: *Synchronization of 4D Hyperchaotic Financial System with adaptation of parameters, (a) Sliding manifold σ (b) Control effort v*

4.4.6 Anti-Synchronization of 4D HyperChaotic Financial System

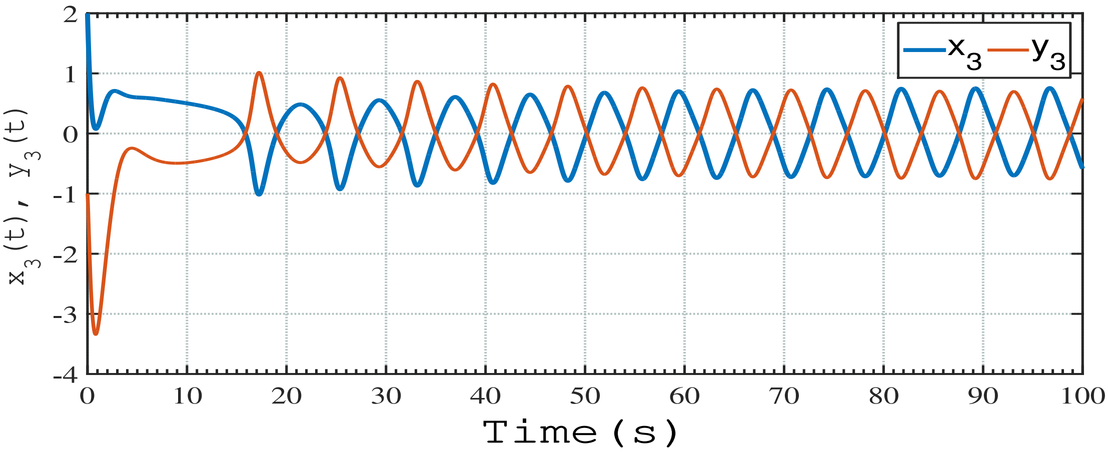
For Anti-synchronization put $q = -1$


6
4
2
0
-2
-4
0
10
20
30
40
50
60
70
80
90
100
Time(s)
x1(t), y1(t)
x1
y1

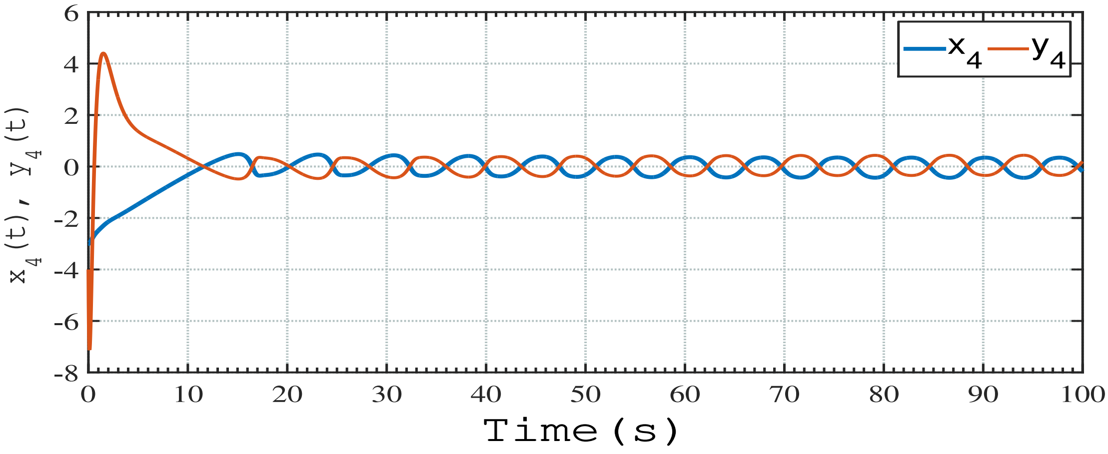
(a)


6
4
2
0
-2
-4
0
10
20
30
40
50
60
70
80
90
100
Time(s)
x2(t), y2(t)
x2
y2

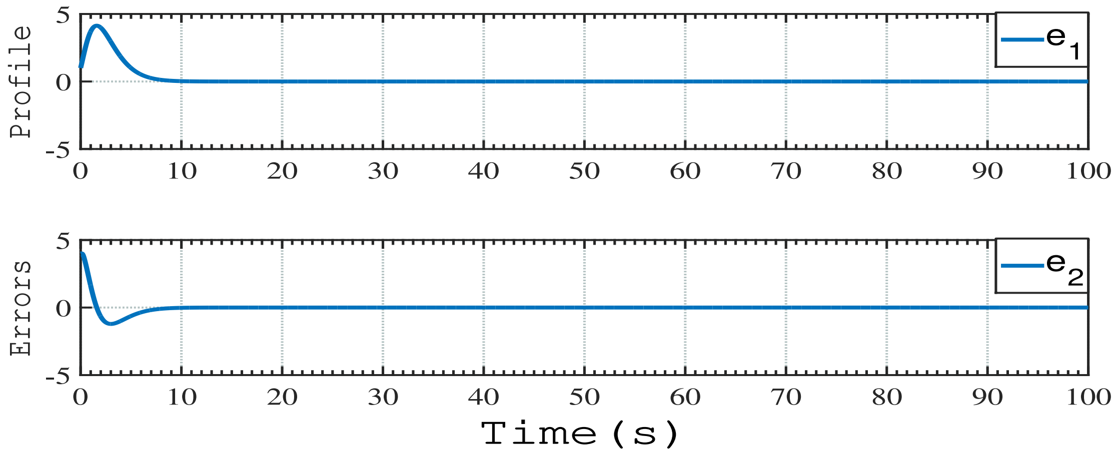
(b)


2
1
0
-1
-2
-3
-4
0
10
20
30
40
50
60
70
80
90
100
Time(s)
x3(t), y3(t)
x3
y3

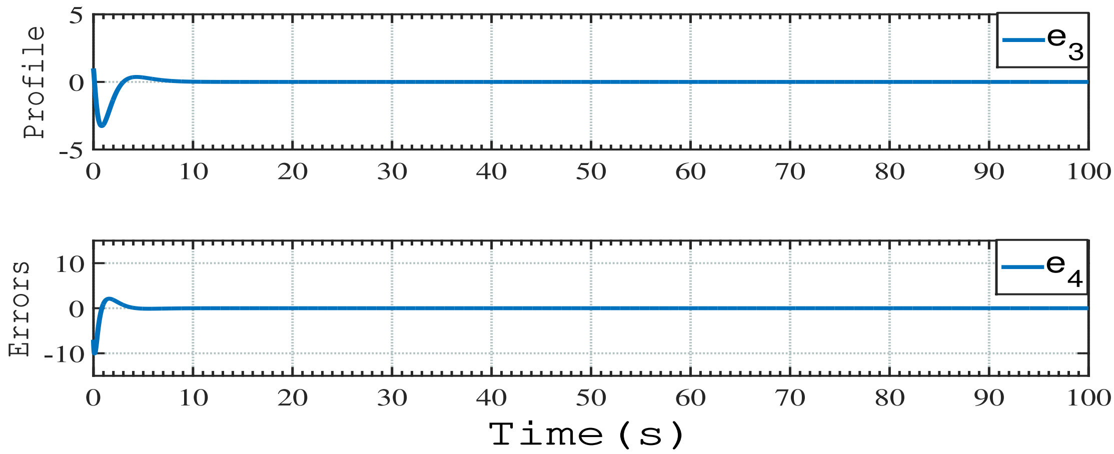
(c)


6
4
2
0
-2
-4
-6
-8
x4(t), y4(t)
0
10
20
30
40
50
60
70
80
90
100
Time(s)
x4
y4

(d)

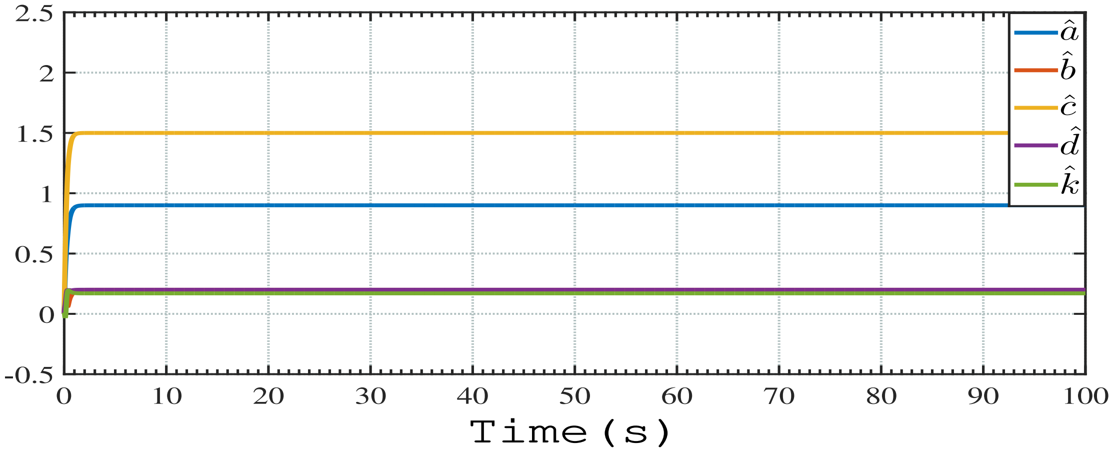

5
0
-5
Profile
Errors
0
10
20
30
40
50
60
70
80
90
100
Time(s)
e1
e2

(e)

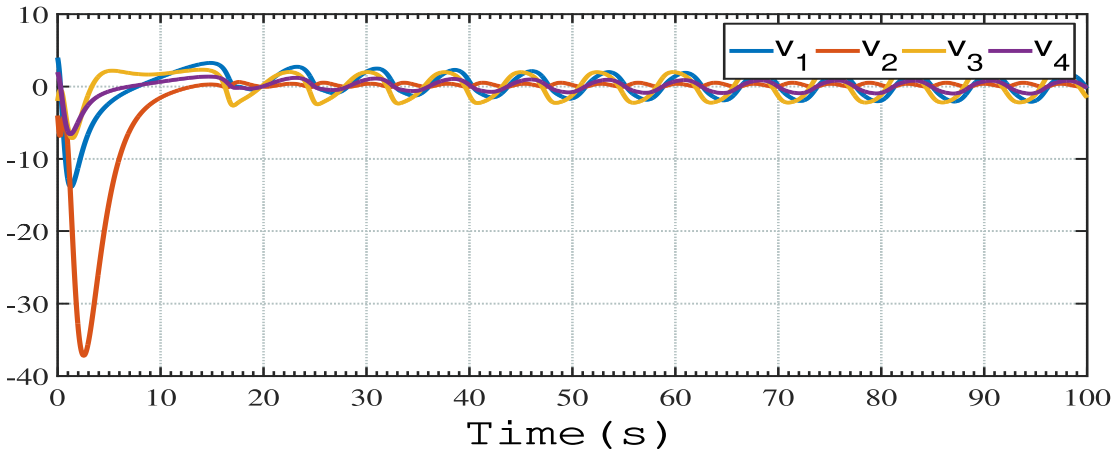

5
0
-5
10
0
-10
Profile
Errors
0
10
20
30
40
50
60
70
80
90
100
Time(s)
e3
e4

(f)

Figure 4.19: *Anti-Synchronization of 4D Hyperchaotic Financial System with adaptation of parameters, (a) Anti-Synchronization of interest rate corresponding to initial condition* $[x_1(0), y_1(0) = (3, -2)]$*, (b) Anti- Synchronization of investment demand corresponding to initial condition* $[x_2(0), y_2(0) = (1, 3)]$*, (c) Anti-Synchronization of price index corresponding to initial condition* $[x_3(0), y_3(0) = (2, -1))]$*,(d) Anti- Synchronization of average profit margins corresponding to initial condition* $[x_4(0), y_4(0) = (-3, -4)]$*, (e) and (f) Time history of the errors* e_1*,* e_2*,* e_3 *and* e_4

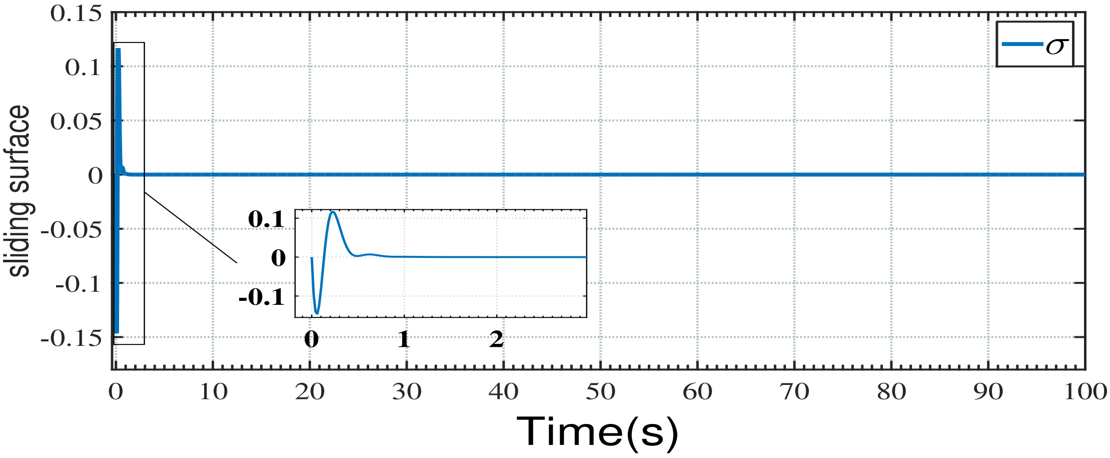


(a)

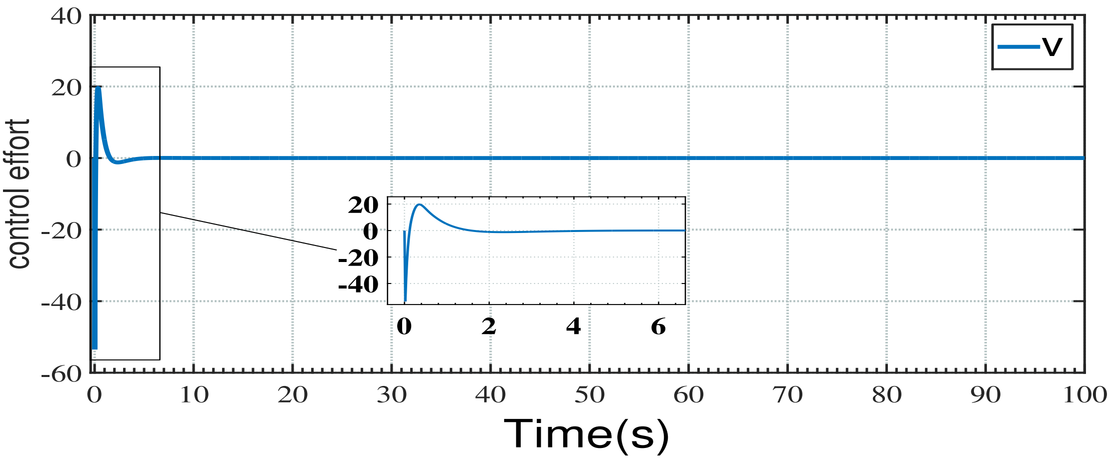


(b)

Figure 4.20: *Anti-Synchronization of 4D Hyperchaotic Financial System, (a)* $\hat{a}$, $\hat{b}$, $\hat{c}$, $\hat{d}$, $\hat{k}$ *represents the adaptation of unknown parameters, (b)* v_1, v_2, v_3, v_4 *represents the time varying disturbances.*



(a)



(b)

FIGURE 4.21: *Anti-Synchronization of 4D Hyperchaotic Financial System with adaptation od parameters, (a) Sliding manifold σ (b) Control effort v*

Chapter 5

Conclusion and Future work

5.1 Introduction

In this particular research work complete synchronization of financial chaotic systems is carried out by using first order SMC and AISMC, this chapter is aimed to explain outcomes and conclusion of this research thesis.

5.2 Performance Analysis

The performance of proposed work is summarized in Table 5.1, based on different features in simulated results. After analyzing, we conclude that the adaptive integral sliding mode control (AISMC) carries substantial marks in case of robustness.

TABLE 5.1: Comparative analysis of FOSMC and AISMC.

Attributes	**First Order SMC**	**Adaptive Integral SMC**
Synchronization	Yes	Yes
Anti-Synchronization	Yes	Yes
Robustness	No	Yes
Computational Complexity	No	Yes
Chattering	Yes	Yes

5.3 Conclusion

These studies work is definitely the synchronization and anti-synchronization scheme between two financial chaotic systems. Two cases are believed first is systems with known parameters, and 2nd is systems with unknown parameters. In first case the synchronization and anti-synchronization are accomplished by utilizing first order SMC, whilst in second case the AISMC is applied. To use the AISMC, the error strategy is converted into a certain structure including nominal part and some unknown terms. The unknown terms are calculated adaptively. Than the error strategy is stabilized utilizing integral sliding mode control. The stabilizing controller towards error strategy is established featuring its the nominal control and some compensator control. The controller and also adapted law are derived so then derivative with the Lyapunov function set off rigidly negative. Numerical simulations are demonstrated to endorse the proposed schemes introduced during this work.

5.4 Future Research Directions

After the completion of this research work, some future directions are recommend.

- Apply proposed control algorithms to other financial chaotic systems.
- Implement the proposed control strategies to the practical financial chaotic systems.
- Compare the simulated results with practical results of financial chaotic systems.

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