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Symmetric Collinear Central Configurations for Four Masses

by

Sobia Shaheen

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First of all, I dedicate this research project to Allah Almighty, The most merciful and beneficent, creator and Sustainer of the earth

And

Dedicated to Prophet Muhammad (peace be upon him) whom, the world where we live and breathe owes its existence to his blessings

And

Dedicated to my parents and Siblings, who pray for me and always pave the way to success for me

And

Dedicated to my teachers, who are a persistent source of inspiration and encouragement for me

CERTIFICATE OF APPROVAL

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(MMT-181011)

THESIS EXAMINING COMMITTEE

S. No.	Examiner	Name	Organization
(a)	External Examiner	Dr. Azad A. Siddiqui	NUST, Islamabad
(b)	Internal Examiner	Dr. Mohammad Afzal	CUST, Islamabad
(c)	Supervisor	Dr. Abdul Rehman Kashif	CUST, Islamabad

Dr. Abdul Rehman Kashif
Thesis Supervisor
February, 2021

Dr. Muhammad Sagheer
Head
Dept. of Mathematics
February, 2021

Dr. M. Abdul Qadir
Dean
Faculty of Computing
February, 2021

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(Sobia Shaheen)

Registration No: MMT-181011

Abstract

We discuss collinear central configurations for four masses which are symmetric about the center of mass. We consider four masses which are placed at the line with two pairs of equal masses. We find the equation of motion of the fifth mass being negligible and not influence the motion of four masses. After evaluating the equation of motion of the fifth body, we calculate the positions of Lagrange points for different intervals and check the stability of Lagrange points finding eigenvalues analysis using Mathematica. In the end, we discuss the permissible regions of motion of m_5 for all the cases.

Contents

List of Figures

List of Tables

Abbreviations

2BP	Two-Body Problem
3BP	Three-Body Problem
4BP	Four-Body Problem
5BP	Five-Body Problem
CFBP	Collinear Four-Body Problem
CC's	Central Configuration
$\mathbf{M}_s$	Mass of the Sun
NBP	n-Body Problem
RCFBP	Restricted Collinear Five-Body Problem
R	Region
SI	System International

Symbols

symbol	name	unit
$\mathbf{G}$	Universal gravitational constant	$m^3kg^{-1}s^{-2}$
$\mathbf{F}$	Gravitational force	Newton
r	Distance	Meter
$\mathbf{P}$	Linear momentum	$kg\ m\ s^{-1}$
$\mathbf{L}$	Angular momentum	$kg\ m^2\ s^{-1}$
m_i	Point masses	kg
$\mathbb{R}$	Real number	
$\ni$	Such that	
$\forall$	For all	
$\in$	Belongs to	

Chapter 1

Introduction

Celestial mechanics is the branch of astronomy that deals with the motions of heavenly objects in outer space. Historically, celestial mechanics applies principles of physics (classical mechanics) to astronomical objects, such as stars and planets to produce ephemeris data. Modern analytic celestial mechanics started with Isaac Newton's Principia of 1687 [1]. The name "celestial mechanics" is more recent than that. Newton wrote that the field should be called "rational mechanics." The term "dynamics" came in a little later with Gottfried Leibniz [2], and over a century after Newton, Pierre-Simon Laplace [3] introduced the term "celestial mechanics." Prior to Kepler there was little connection between exact, quantitative prediction of planetary positions, using geometrical or arithmetical techniques, and contemporary discussions of the physical causes of the planetary motion. In a special case, the law for two point particles as they interact by gravitational force with each other is,

$$\mathbf{F} = G\frac{m_1 m_2}{r^3}\mathbf{r}, \tag{1.1}$$

where G is a universal gravitational constant and r is the distance of masses m_1 and m_2 from each other. In classical mechanics, the two-body problem (2BP) is to predict the motion of two massive objects which are abstractly viewed as point particles. The 2BP [4] is most common in the case of a gravity that occurs in astronomy to determine orbits of objects such as satellites, planets and stars. Newton solved 2BP by using his fundamental law of gravity. Newtonian mechanics is a mathematical model whose purpose is to explore the motions of the various objects in the universe. The basic concept of this model were first enunciated by sir Isaac Newton in a work entitled "***Philosophiae Naturalis Principia Mathematica***". This work, which was published in 1687. The problem has no significant solution if $n \geq 3$. Although we have a restricted 3-body problems (3BP) that provide us with a particular solution.

The 3BP [5] is the problem of taking the initial positions and velocities of three point masses and solving for their subsequent motion according to Newton's laws of motion and Newton's law of universal gravitation. The 3BP is a special case of the n-body problem. The 3BP is one of the oldest problems in classical dynamics that continues to throw up surprises. It has challenged scientists from Newtons time to the present. It arose in an attempt to understand the Sun's effect on the motion of the Moon around the Earth. NBP also known as many body problem [6].

Henri Poincare work on 3BP [7] has ended the classical era of research. The many body problem was first formulated precisely by Newton. In its form where the object involve point masses: "it may be stated as given at any time the position and velocities of three or more massive particles moving under their mutual gravitational forces, the mass also being known, calculated their positions and velocities at any other time".

The NBP [8] which predicts the individual motion of a system of celestial bodies that gravitationally attract with each other. The statement of the problem is what would be the orbit, if we are given n celestial objects interacting with each others under the gravitational forces. Mathematicians and astronomers continued to work on the NBP over the last four centuries. First of all, Kepler in his planetary motion laws [9] defining the elliptical trajectories of planets around the Sun. Most important works in science history in which Newton derived and formulated Kepler's law. Newton turned his attention to comparatively more difficult systems, after the justification of Kepler's laws.

The general problem of n-bodies, where n is greater than three, has been criticized vigorously with numerical techniques on powerful computers. Celestial mechanics in the solar system [10] is ultimately an n-body problem, but the special configuration and relative smallness of the perturbations have allowed quite accurate descriptions of motions (valid for limited time periods) with various approximations and procedures without any attempt to solve the complete problem of n-bodies. Examples are the restricted three-body problem to determine the effect of Jupiters perturbations of the asteroids and the use of successive approximations of series solutions to sequentially add the effects of smaller and smaller perturbations for the motion of the Moon. In the general n-body problem, all bodies have arbitrary masses, initial velocities, and positions; the bodies interact through Newtons law of gravitation [11], and one attempts to determine the subsequent motion of all the bodies. Many numerical solutions for the motion of quite large numbers of gravitating particles have been successfully completed where the precise motion of individual particles is usually less important than the statistical behaviour of the group [12].

The straight line solution of the n-body problem were first published by Moulton. He managed n masses on a straight line and solve the problem of the values of the masses at n arbitrary collinear points so that they remained collinear under suitable initial projections. These solutions are sometimes referred to as Moulton solutions [13]. In the $20th$ century, understanding the dynamics of globular cluster star systems became an important NBP [14].

1.1 Central Configuration

A Central Configuration(CC's) [15] is a special arrangement of point masses interacting by Newton's law of gravitation with the following property (the gravitational acceleration vector produced on each mass by all others should point toward the center of mass and proportional to the distance to the center of mass). Central configurations may be studied in Euclidean spaces of any dimension, although only dimensions one, two, and three are directly relevant for celestial mechanics. CC's [16] play an important role in the study of the Newtonian NBP [17]. For example, they lead to the only explicit solutions of the equations of motion, they govern the behaviour of solutions near collisions, and they influence the topology of the integral manifolds. Palmore [18] proposed many theorems in the study of points of equilibrium in the planar NBP [19].

Under Newton's law of universal gravitation, bodies placed at rest in a central configuration will maintain the configuration as they collapse to a collision at their center of mass. Systems of bodies in a two-dimensional central configuration can orbit stably around their center of mass, maintaining their relative positions, with circular orbits around the center of mass or in elliptical orbits with the center of mass at a focus of the ellipse. These are the only possible stable orbits in three-dimensional space in which the system of particles always remains similar to its initial configuration [20]. Two central configurations are considered to be equivalent if they are similar, that is, they can be transformed into each other by some combination of rotation, translation, and scaling. With this definition of equivalence, there is only one configuration of one or two points, and it is always central. Ouyang and Xie discussed several special analysis solutions for four-body problems, and showing that such solutions reduce the to Lagrange solutions [21] when two masses are similarly reduced. Zhiming and Yisui (1988) [22] described the finiteness of the central configurations for the general four-body problem [23]. Roy and Steves [24] addressed several theoretical approaches about the four-body problem.

We will address the existence of a continuous family of balancing solutions for the above mentioned four-body collinear problem [**?**], which involve two symmetrical configurations of two pairs of masses. Approximately 67% of our galaxy stars are known to be included in the multi-stellar system, that is why it is very important to understand the four-body problem and the five-body problem. CC is useful to understand the gravitational NBP [25].

1.2 Thesis Contribution

In this thesis we have reviewed [26] collinear four-body problem which involves two symmetrical arrangements for four masses. The masses are m_1, m_2, m_3 and

m_4 respectively as shown in figure (3.1), and the remaining fifth mass placed in the same plane under the influence of gravity of four masses, does not influence the movement of four masses. Firstly we studied the CC of the four masses. In the second part we discuss the motion of infinitesimal mass in the gravitational field of these masses, and also explore the possible position of equilibrium points of fifth mass and investigate their stability. Lastly we draw permissible regions and investigate the movement of the infinitesimal mass.

1.3 Dissertation Outlines

A brief overview of the contents of the thesis is provided below.

Chapter 1 introduction of the problem and aim of this research is briefly discussed.

In **Chapter 2** we have described the basic definitions and fundamental ideas of celestial mechanics. In the portion of this chapter, the two-body problem is briefly discussed.

In **Chapter 3** we have reviewed the research paper [26].

In **Chapter 4** we discuss and explain some key features regarding 5^{th} body. We also discussed the dynamics of 5^{th} body under the action of gravitational field of m_1 m_2 m_3 and m_4, all of them are collinear.

Chapter 5 provides the concluding remarks of the thesis.

References used in the thesis are mentioned in **Bibliography**.

Chapter 2

Preliminaries

This chapter includes the basic definitions and basic concepts that will help us better understanding of our objective research.

2.1 Basic Definitions

Definition 2.1.1. (Celestial Mechanics)
"Celestial mechanics is the branch of astronomy that deals with the motions of objects in outer space. Historically, celestial mechanics applies principles of physics (classical mechanics) to astronomical objects, such as stars and planets, to produce ephemeris data. Actually celestial mechanics is the science devoted to the study of the motion of the celestial bodies on the basis of the laws of gravitation. It was founded by Newton and it is the oldest of the chapters of Physical Astronomy."[27]

Definition 2.1.2. (Mechanics)
"Mechanics is a branch of physics concerned with motion or change in position of physical objects. It is sometimes further subdivided into:
1. **Kinematics**, which is concerned with the geometry of the motion,
2. **Dynamics**, which is concerned with the physical causes of the motion,
3. **Statics**, which is concerned with conditions."[28]

Definition 2.1.3. (Scalar)

"A scalar is an element of a field which is used to define a vector space. A quantity described by multiple scalars, such as having both direction and magnitude."[29]

Definition 2.1.4. (Vector)

"Other quantities of physics, such as displacement, velocity, momentum, force etc require for their specification a direction as well as magnitude. Such quantities are called **Vectors**."[29]

Definition 2.1.5. (Principle of conservation of momentum)

"The principle of conservation of momentum states that the total momentum of a system of objects remains constant provided no resultant external force acts on the system."[18]

Definition 2.1.6. (Angular velocity)

"Angular velocity, ω is the rate of change of angular displacement with respect to time."[18]

Definition 2.1.7. (Central force)

"Suppose that a force acting on a particle of mass m such that

(a) it is always directed from m toward or away from a fixed point O,

(b) its magnitude depends only on the distance r from O.

then we call the force a central force or central force field with O as the center of force. In symbols $\mathbf{F}$ is a central force if and only if

$$\mathbf{F} = f(r)\mathbf{r}_1 = f(r)\tfrac{\mathbf{r}}{r},$$

where $\mathbf{r}_1 = \frac{\mathbf{r}}{r}$ is a unit vector in the direction of $\mathbf{r}$. The central force is one of attraction towards O or repulsion from O according as *f(r)* < 0 or *f(r)* > 0 respectively."[18]

Definition 2.1.8. (Degree of freedom)

"The number of coordinates required to specify the position of a system of one or more particles is called number of degree of freedom of the system.
Example: A particle moving freely in space requires 3 coordinates, e.g. (x, y, z), to specify its position. Thus the number of degree of freedom is 3."[28]

Definition 2.1.9. (Center of mass)

"Let $r_1, r_2, ..., r_n$ be the position vector of a system of n particles of masses $m_1, m_2, ...m_n$ respectively. The center of mass or centroid of the system of particles is defined as that point having position vector

$$\hat{\mathbf{r}} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + ... + m_n\mathbf{r}_n}{m_1 + m_2 + ... + m_n} = \frac{1}{\mathbf{M}}\sum_{\nu=1}^{n} m_\nu \mathbf{r}_\nu,$$

where

$$\mathbf{M} = \sum_{\nu=1}^{n} m_\nu,$$

is the total mass of the system."[28]

Definition 2.1.10. (Center of gravity)

"If a system of particles is in a uniform gravitational field, the center of mass is sometimes called the center of gravity."[28]

Definition 2.1.11. (Torque)

"If a particle with a position vector $\mathbf{r}$ moves in a force field $\mathbf{F}$, we define $\boldsymbol{\tau}$ as torque or moment of the force as

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}.$$

The magnitude of $\boldsymbol{\tau}$ is

$$\tau = rF \sin\theta.$$

The magnitude of torque is a measure of the turning effect produced on the particle by the force."[28]

Definition 2.1.12. (Momentum)

"The linear momentum $\mathbf{p}$ of an object with mass m and velocity $\mathbf{v}$ is defined as:

$$\mathbf{p} = m\mathbf{v}.$$

Under certain circumstances the linear momentum of a system is conserved. The linear momentum of a particle is related to the net force acting on that object:

$$\mathbf{F} = m\mathbf{a} = m\frac{d\mathbf{v}}{dt} = \frac{d}{dt}(m\mathbf{v}) = \frac{d\mathbf{p}}{dt}.$$

The rate of change of linear momentum of a particle is equal to the net force acting on the object, and is pointed in the direction of the force. If the net force acting on an object is zero, its linear momentum is constant (conservation of linear momentum). The total linear momentum $\mathbf{p}$ of a system of particles is defined as the vector sum of the individual linear momentum.

$$\mathbf{p} = \sum_{1}^{n} \mathbf{p}_i.\text{"}[28]$$

Definition 2.1.13. (Point-like particle)

"A point-like particle is an idealization of particles mostly used in different fields of physics. Its defining features is the lacks of spatial extension:being zero-dimensional, it does not take up space. A point-like particle is an appropriate representation of an object whose structure, size and shape is irrelevant in a given context. e.g., from far away, a finite-size mass (object) will look like a point-like particle."[28]

Definition 2.1.14. (Angular momentum)

"Angular momentum for a point-like particle of mass m with linear momentum $\mathbf{p}$ about a point O, defined by the equation

$$\mathbf{L} = \mathbf{r} \times \mathbf{p},$$

where $\mathbf{r}$ is the vector from the point O to the particle. The torque about the point O acting on the particle is equal to the rate of change of the angular momentum about the point O of the particle i.e.,

$$\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt}.\text{"}[28]$$

Definition 2.1.15. (Inertial frame of reference)

"An inertial frame of reference is a frame of reference that is not undergoing acceleration. In an inertial frame of reference, a physical object with zero net force acting on it moves with a constant velocity (which might be zero) or equivalently, it is a frame of reference in which Newton's first law of motion holds."[30]

Definition 2.1.16. (Lagrange Points)

"Lagrange Points are positions in space where the gravitational forces of a two body system like the Sun and the Earth produce enhanced regions of attraction and repulsion. These can be used by spacecraft to reduce fuel consumption needed to remain in position. At Lagrange points, the gravitational pull of two large masses precisely equals the centripetal force required for a small object to move with them. Lagrange points are named in honor of Italian-French mathematician Josephy-Louis Lagrange."[31]

Definition 2.1.17. (Equilibrium solution)

"The **Equilibrium solution** can guide us through the behaviour of the equation that represents the problem without actually solving it. These solutions can be found only if we meet the sufficient condition of all rates equal to zero. If we have two variables then

$$\dot{x} = \dot{y} = \ddot{x} = \ddot{y} = ... = x^{(n)} = y^{(n)} = 0.$$

These solutions may be stable or unstable. The stable solutions regarding in celestial Mechanics helps us find parking spaces where if a satellite or any object placed, it will remain there for ever. These type of places are also found along the Jupiter's orbital path where bodies called Trojan are present. These equilibrium points with respect to Celestial Mechanics are also called Lagrange points named after a French mathematician and astronomer Joseph-Louis Lagrange. He was first to find these equilibrium points for the Sun-Earth system. He found that three of these five points were collinear.

Procedure for stability analysis and equilibrium points:

We need to follow the following steps to check the stability of equilibrium points.

1) Determine the equilibrium points, $\mathbf{x}^*$, solving $\Omega(\mathbf{x}^*) = \mathbf{0}$.
2) Construct the Jacobian matrix, $J(\mathbf{x}^*) = \frac{\partial \Omega}{\partial \mathbf{x}^*}$.
3) Compute eigenvalues of $\Omega(\mathbf{x}^*)$: $det|\Omega(\mathbf{x}^*) - \lambda\ I| = 0$.
4) Stability or instability of $\mathbf{x}^*$ based on the real parts of eigenvalues.
5) Point is stable, if all eigenvalues have real parts negative.
6) Unstable, If at least one eigenvalue has a positive real part.
7) Otherwise, there is no conclusion,

(i.e, require an investigation of higher order terms)."[17]

2.2 Permissible Regions of Motion in Plane of Motion

"The Jacobian constant of motion is one of the most important constants of the dynamical system, which represent the motion of the infinitesimal body. Because it can be used to sketch the zero velocity curves. Of course it can be used to explore the regions of permissible motion of the infinitesimal body."[32]

2.3 Kepler's Laws of Planetary Motion

"Kepler's three laws of planetary motion can be described as follows:

1. All planets are moving in an elliptical path with Sun at one focus.
2. The radius vector drawn from the Sun to a planet sweeps out equal areas in equal time intervals.
3. The cube of the semi major axis of the planetary orbits are proportional to the square of the planets periods of revolution. Mathematically, Kepler's third law can be written as:

$$T^2 = \left(\frac{4\pi^2}{GM_s}\right) r^3,$$

where T is the time period, r is the semi major axis, M_s is the mass of Sun and G is the universal gravitational constant."[33]

2.4 Newton's Laws of Motion [34]

"The following three laws of motion given by Newton are considered the axioms of mechanics:

1. **First law of motion**

   Every particle persists in a state of rest or of uniform motion in a straight line unless acted upon by a force.

2. **Second law of motion**

   If $\mathbf{F}$ is the external force acting on a particle of mass m which as a reaction is moving with velocity $\mathbf{v}$, then

   $$\mathbf{F} = \frac{d}{dt}(m\mathbf{v}) = \frac{d\mathbf{P}}{dt}.$$

   If m is independent of time this becomes

   $$\mathbf{F} = m\frac{d\mathbf{v}}{dt} = m\mathbf{a},$$

   where $\mathbf{a}$ is the acceleration of the particle.

3. **Third law of motion**

   For every action, there is an equal and opposite reaction."[33]

2.4.1 Newton's Universal Law of Gravitation

"Every particle of matter in the universe attracts every other particle of matter with a force which is directly proportional to the product of the masses and inversely proportional to the square of the distance between them. Hence, for any two particles separated by a distance r, the magnitude of the gravitational force $\mathbf{F}$ is:

$$\mathbf{F} = G\frac{m_1 m_2}{r^3}\mathbf{r}$$

where G is universal gravitational constant. Its numerical value in SI units is $6.67408 \times 10^{-11} m^3 kg^{-1} s^{-2}$."[35]

2.5 Two Body Problem

"The two-body problem, first studied and solved by Newton, states: Suppose that the positions and velocities of two massive bodies moving under their mutual gravitational force are given at any time t, then what should their position and velocities be for any other time t, if the masses are known? Example include a planet orbiting around a star (Earth-Sun, Moon-Earth), two stars orbiting around each other, satellite orbiting around orbit. The two-body problem is very important because of the following facts:

1. It is the only gravitational problem in celestial mechanics, apart from rather restricted solutions of three body problem, for which we have a complete and general solution.
2. A wide range of practical orbital motion problems can be treated as approximate two-body problems.
3. The two-body solution may be used to provide approximate orbital parameters and predictions or serve as a starting point for the generation of analytical solutions valid to higher orders of accuracy.

2.5.1 The Solution to the Two-Body Problem

The governing law for the two-body is Newton's universal gravitational law:

$$\mathbf{F} = G\frac{m_1 m_2}{r^3}\mathbf{r}, \tag{2.1}$$

for two masses m_1 and m_2 separated by a distance of $\mathbf{r}$, and G the universal gravitational constant. The aim here is to determine the path of the particles for any time t, if the initial positions and velocities are known. In figure (2.1), the force of attraction $\mathbf{F}_1$ is directed along r towards m, while the force $\mathbf{F}_2$ on M is in opposite direction. By Newton's third law,

$$\mathbf{F}_1 = -\mathbf{F}_2. \tag{2.2}$$

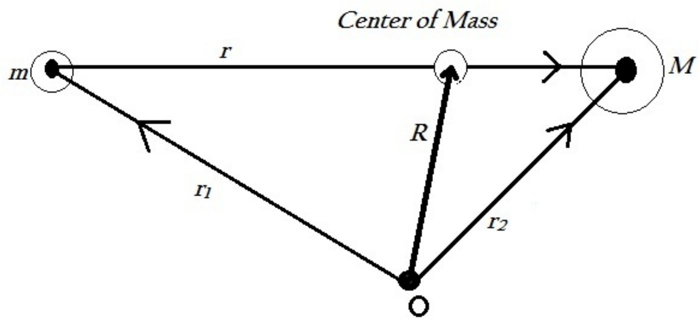

FIGURE 2.1: Center of mass of two-body system

From figure (2.1),

$$\mathbf{F}_1 = G\frac{mM}{r^3}\mathbf{r}. \tag{2.3}$$

Using Newton's second law of motion and by equations (2.1) and (2.2), the equation of motion of the particles under their mutual gravitational attractions are given by

$$m\ddot{\mathbf{r}}_1 = m\frac{d^2\mathbf{r}_1}{dt^2} = G\frac{mM}{r^3}\mathbf{r}, \tag{2.4}$$

$$M\ddot{\mathbf{r}}_2 = M\frac{d^2\mathbf{r}_2}{dt^2} = -G\frac{mM}{r^3}\mathbf{r}, \tag{2.5}$$

where $\mathbf{r}_1$ and $\mathbf{r}_2$ be the position vectors from the reference O as shown in Figure (2.1). Adding equation (2.4) and (2.5), we get:

$$m\ddot{\mathbf{r}}_1 + M\ddot{\mathbf{r}}_2 = \mathbf{0}, \tag{2.6}$$

integrating above equations yields:

$$m\dot{\mathbf{r}}_1 + M\dot{\mathbf{r}}_2 = \mathbf{c}_1, \tag{2.7}$$

that the total linear momentum of the system i.e. $m\mathbf{v}_m + M\mathbf{v}_M = \mathbf{c}_1$ is a constant. Again integrating equation (2.7) implies:

$$m\mathbf{r}_1 + M\mathbf{r}_2 = \mathbf{c}_1 t + \mathbf{c}_2, \tag{2.8}$$

where $\mathbf{c}_1$ and $\mathbf{c}_2$ are constant vectors.

Using the definition of center of mass in $2BP$, $\mathbf{R}$ is defined as:

$$\begin{aligned} (m+M)\mathbf{R} &= m\mathbf{r}_1 + M\mathbf{r}_2, \\ M_t\mathbf{R} &= m\mathbf{r}_1 + M\mathbf{r}_2, \end{aligned} \tag{2.9}$$

where $M_t = m + M$. Taking the derivative of equation (2.9) and comparing with equation (2.21), we get

$$M_t\dot{\mathbf{R}} = \mathbf{c}_1 \quad \Rightarrow \quad \dot{\mathbf{R}} = \frac{\mathbf{c}_1}{M_t} = constant$$

show that $\dot{\mathbf{R}} = \mathbf{v}_c$ (velocity of center of mass) is constant.

Subtracting the equations (2.4) and (2.5) gives:

$$\ddot{\mathbf{r}}_1 - \ddot{\mathbf{r}}_2 = \frac{GM}{r^3}r + \frac{Gm}{r^3}\mathbf{r}, \tag{2.10}$$

$$\ddot{\mathbf{r}}_1 - \ddot{\mathbf{r}}_2 = G(m+M)\frac{\mathbf{r}}{r^3}$$

$$\Rightarrow \ddot{\mathbf{r}} = \mu\frac{\mathbf{r}}{r^3}$$

$$\Rightarrow \ddot{\mathbf{r}} + \mu\frac{\mathbf{r}}{r^3} = \mathbf{0}, \tag{2.11}$$

where $\mu = G(m+M)$ is defined as reduced mass and $\mathbf{r}_1 - \mathbf{r}_2 = -\mathbf{r}$, see figure (2.1).

Taking the cross product of $\mathbf{r}$ with equation (2.11) we obtain:

$$\mathbf{r} \times \mu\ddot{\mathbf{r}} + \frac{\mu^2}{r^3}\mathbf{r} \times \mathbf{r} = \mathbf{0}$$
$$\Rightarrow \mathbf{r} \times \ddot{\mathbf{r}} = \mathbf{0}, \tag{2.12}$$

integrating above equation yields:

$$\mathbf{r} \times \dot{\mathbf{r}} = \mathbf{L}, \tag{2.13}$$

where $\mathbf{L}$ is a constant vector. We may write equation (2.12),

$$\Rightarrow \mathbf{r} \times \mu\ddot{\mathbf{r}} = \mathbf{0},$$
$$\Rightarrow \mathbf{r} \times \mathbf{F} = \mathbf{0}, \tag{2.14}$$

where $\mathbf{F} = \mu\ddot{\mathbf{r}} = \mu\mathbf{a}$ (μ is reduced mass i.e. constant).

From the definition of torque and angular momentum:

$$\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt} = \mathbf{r} \times \mathbf{F}. \tag{2.15}$$

Comparing equations (2.14) and (2.15), we get:

$$\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt} = \mathbf{r} \times \mathbf{F} = \mathbf{0},$$
$$\frac{d\mathbf{L}}{dt} = \mathbf{0}$$

$$\Rightarrow \mathbf{L} = \text{constant},$$

i.e. angular momentum of the system is constant.

Radial and transverse components of velocity and acceleration:

If polar coordinates r and θ are taken in this plane as in figure (2.2), the velocity components along and perpendicular to the radius vector joining m to M

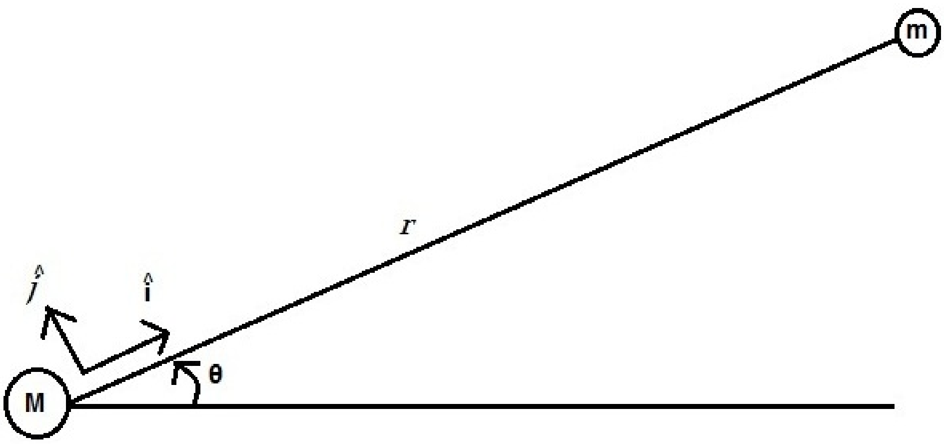

FIGURE 2.2: Radial and transverse components of velocity and acceleration

are $\dot{r}$ and $r\dot{\theta}$, then,

$$\dot{\mathbf{r}} = \frac{d\mathbf{r}}{dt} = \dot{r}\mathbf{i} + r\dot{\theta}\mathbf{j}, \tag{2.16}$$

where $\hat{\imath}$ and $\hat{\jmath}$ are unit vectors along and perpendicular to the radius vector. Hence, by equations (2.13) and (2.16),

$$\mathbf{r} \times (\dot{r}\hat{\imath} + r\dot{\theta}\hat{\jmath}) = r^2\dot{\theta}\hat{k} = L\hat{k}, \tag{2.17}$$

where $\hat{k}$ is a unit vector perpendicular to the plane of the orbit. We may then write

$$r^2\dot{\theta} = L, \tag{2.18}$$

where the constant L is seen to be twice the rate of description of area by the radius vector. This is the mathematical form of Kepler's second law.

Now taking the scalar product of $\dot{\mathbf{r}}$ with equation (2.11), we get:

$$\dot{\mathbf{r}}.\frac{d^2\mathbf{r}}{dt^2} + \mu\frac{\dot{\mathbf{r}}.\mathbf{r}}{r^3} = 0,$$

which may be integrated to give:

$$\frac{1}{2}\dot{\mathbf{r}}.\dot{\mathbf{r}} - \frac{mu}{r} = C, \tag{2.19}$$

$$\frac{1}{2}v^2 - \frac{\mu}{r} = C, \tag{2.20}$$

where C is a constant. This is the energy conservation form of the system. The quantity C is not the total energy; $\frac{1}{2}\mu^2$ is related to the kinetic energy and $\frac{-mu}{r}$ to the potential energy of the system i.e. total energy is conserved.
Recall that from celestial mechanics, components of acceleration vector along and perpendicular to the radius vector (see figure 2.2):

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\hat{\imath} + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\hat{\jmath},$$

using above equation in (2.11), we get

$$\ddot{r} - r\dot{\theta}^2 = -\frac{\mu}{r^2}, \tag{2.21}$$

$$\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0. \tag{2.22}$$

Integrating equation (2.21) gives the angular momentum integral:

$$r^2\dot{\theta} = L, \tag{2.23}$$

making the usual substitution of

$$u = \frac{1}{r}, \tag{2.24}$$

and eliminating the time between equation (2.20) and (2.22), implies:

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{L^2}. \tag{2.25}$$

The general solution of above equation is :

$$u = \frac{\mu}{L^2} + A\cos(\theta - \theta_0), \tag{2.26}$$

where A and θ_0 are two constants of integration. Substitute $u = \frac{1}{r}$ in above equation:

$$\frac{1}{r} = \frac{\mu}{L^2} + A\cos(\theta - \theta_0)$$

$$\Rightarrow r = \frac{\frac{L^2}{\mu}}{1 + \frac{L^2 A}{\mu}\cos(\theta - \theta_0)},$$

is the polar form of the equation of the conic and may be written as:

$$r = \frac{p}{1 + e\cos(\theta - \theta_0)},$$

where

$$p = \frac{L^2}{\mu},$$
$$e = \frac{AL^2}{\mu}.$$

Eccentricity e classifies the trajectory of one celestial body around another. Thus:
(i) If $0 < e < 1$ then the orbit is elliptical,
(ii) If $e = 1$ then the orbit is a parabolic,
(iii) If $e > 1$ then the orbit is a hyperbolic.
Hence the solution of the two-body problem is a conic, includes Kepler's first law as a special case." [36]

Chapter 3

Symmetric Collinear Central Configurations for Four Masses

3.1 Introduction

In this research work [26], we set up a collinear four-body problem (CFBP), with four positive masses. The masses are m_1, m_2, m_3 and m_4 respectively. There are two pairs of equal masses, that are symmetric about the center of mass moving in such a way that their configuration is always in a line. The geometric configuration can be seen in figure (3.1). The central configuration of four-bodies in a line is a geometric configuration of four-bodies in which the gravitational forces are balanced in such a way that the four bodies rotate together about their center of mass and maintain this configuration i.e., in a line all the time.

3.2 Characterization of the Collinear Configuration

Suppose that n point positive masses $(m_1, m_2, ..., m_n), m_i \in \mathbb{R}^+, i = 1, ...n$, and $\mathbf{r}_i \in \mathbb{R}^2, i = 1, ...n$ are n mass position vectors, and the Euclidean distance between any two masses are $\mathbf{r}_{ij}, i, j = 1, ...n$. The classical equation of motion for n positive masses has the form

$$m_i\ddot{\mathbf{r}}_i = \sum_{j=1 j\neq i}^{n} m_i m_j \frac{\mathbf{r}_j - \mathbf{r}_i}{|\mathbf{r}_j - \mathbf{r}_i|^3}, \qquad i = 1, ...n, \tag{3.1}$$

$\mathbf{r}_i$ is position vector of the ith body, m_i is the mass of the ith body and the units are chosen so that the gravitational constant $G = 1$. A central configuration is a particular configuration of the n bodies where the accelaration vector of each body is proportional to its position vector, and the constant of proportionality is the same for the n-bodies. Therefore, a central configuration is a configuration that satisfies the equation

$$-\omega^2(\mathbf{r}_i - \mathbf{c}) = \sum_{j=1 j\neq i}^{n} m_j \frac{\mathbf{r}_j - \mathbf{r}_i}{|\mathbf{r}_j - \mathbf{r}_i|^3}, \qquad i = 1, ...n, \tag{3.2}$$

where ω is a constant and $\omega \neq 0$, $\mathbf{c}$ is the center of mass.
For four bodies, we put $n = 4$ in equation (3.2), we will get central configuration for general four-body problem which are given below:

$$m_2 \frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|^3} + m_3 \frac{\mathbf{r}_3 - \mathbf{r}_1}{|\mathbf{r}_3 - \mathbf{r}_1|^3} + m_4 \frac{\mathbf{r}_4 - \mathbf{r}_1}{|\mathbf{r}_4 - \mathbf{r}_1|^3} = -\omega^2(\mathbf{r}_1 - \mathbf{c}), \tag{3.3}$$

$$m_1 \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3} + m_3 \frac{\mathbf{r}_3 - \mathbf{r}_2}{|\mathbf{r}_3 - \mathbf{r}_2|^3} + m_4 \frac{\mathbf{r}_4 - \mathbf{r}_2}{|\mathbf{r}_4 - \mathbf{r}_2|^3} = -\omega^2(\mathbf{r}_2 - \mathbf{c}), \tag{3.4}$$

$$m_1 \frac{\mathbf{r}_1 - \mathbf{r}_3}{|\mathbf{r}_1 - \mathbf{r}_3|^3} + m_2 \frac{\mathbf{r}_2 - \mathbf{r}_3}{|\mathbf{r}_2 - \mathbf{r}_3|^3} + m_4 \frac{\mathbf{r}_4 - \mathbf{r}_3}{|\mathbf{r}_4 - \mathbf{r}_3|^3} = -\omega^2(\mathbf{r}_3 - \mathbf{c}), \tag{3.5}$$

$$m_1 \frac{\mathbf{r}_1 - \mathbf{r}_4}{|\mathbf{r}_1 - \mathbf{r}_4|^3} + m_2 \frac{\mathbf{r}_2 - \mathbf{r}_4}{|\mathbf{r}_2 - \mathbf{r}_4|^3} + m_3 \frac{\mathbf{r}_3 - \mathbf{r}_4}{|\mathbf{r}_3 - \mathbf{r}_4|^3} = -\omega^2(\mathbf{r}_4 - \mathbf{c}). \tag{3.6}$$

Assuming $m_1 = m_4 = M$ and $m_2 = m_3 = m$. The masses m_1, m_2, m_3 and m_4 are fixed at $\mathbf{r}_1 = (-b, 0)$, $\mathbf{r}_2 = (-a, 0)$, $\mathbf{r}_3 = (a, 0)$ and $\mathbf{r}_4 = (b, 0)$ respectively (see figure (3.1)), with the above conditions the center of mass of four collinear masses

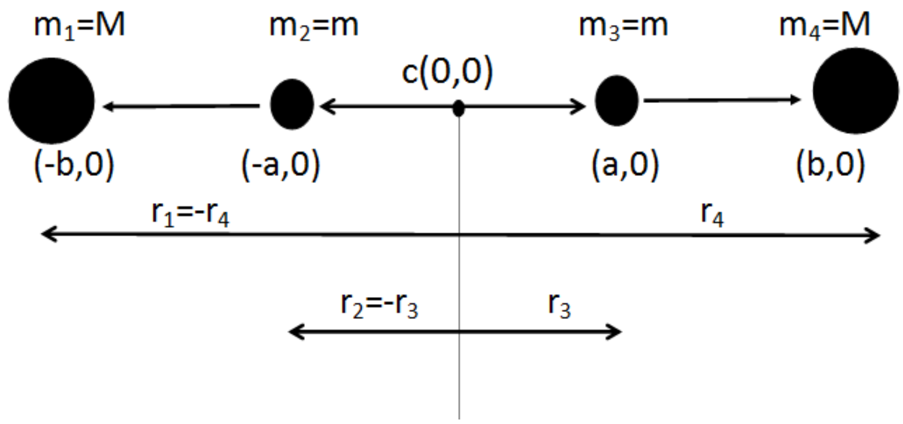

FIGURE 3.1: Symmetric collinear equilibrium configuration

will shift at the origin.

i.e.,

$$\mathbf{c} = (0,0)\,.$$

Using $m_1 = m_4 = M$ and $m_2 = m_3 = m$, the positions of $m_1 - m_4$ and the value of $c = (0,0)$ in equations (3.3) – (3.6) become

$$\frac{bM}{4|b|^3} + \frac{(-a+b)m}{|-a+b|^3} + \frac{(a+b)m}{|a+b|^3} - b\omega^2 \;= 0, \tag{3.7}$$

$$\frac{am}{4|a|^3} + \frac{(a-b)M}{|a-b|^3} + \frac{(a+b)M}{|a+b|^3} - a\omega^2 \;= 0, \tag{3.8}$$

$$\frac{am}{4|a|^3} + \frac{(-a-b)M}{|-a-b|^3} + \frac{(-a+b)M}{|-a+b|^3} + a\omega^2 \;= 0, \tag{3.9}$$

$$\frac{(-a-b)m}{4|-a-b|^3} + \frac{(a-b)m}{|a-b|^3} + \frac{bM}{4|b|^3} + b\omega^2 \;= 0. \tag{3.10}$$

Equation (3.7) is same as equation (3.10) and equation (3.8) is similar to equation (3.9), therefore we are left the following two equations as shown below:

$$\frac{M}{4b^2} + \frac{m}{(-a+b)^2} + \frac{m}{(a+b)^2} - b\omega^2 \ = 0, \tag{3.11}$$

$$\frac{m}{4a^2} + \frac{M}{(a-b)^2} + \frac{M}{(a+b)^2} - a\omega^2 \ = 0. \tag{3.12}$$

Here we assume $\omega = 1$, this will not effect the generality of the problem. Solving equations (3.11) and (3.12), we get

$$m = f_1(a,b)/f_2(a,b), \tag{3.13}$$

where

$$f_1(a,b) = 4a^2(a-b)^2(a+b)^2(a^5 - 2a^3b^2 - 8a^2b^3 + ab^4 - 8b^5),$$

$$f_2(a,b) = a^8 - 68a^6b^2 - 122a^4b^4 - 68a^2b^6 + b^8,$$

and

$$M = g_1(a,b)/g_2(a,b), \tag{3.14}$$

where

$$\begin{aligned} g_1(a,b) =& (4(-8a^9b^2 + a^8b^3 + 8a^7b^4 - 4a^6b^5 + 8a^5b^6 + 6a^4b^7 \\ & - 8a^3b^8 - 4a^2b^9 + b^{11})) \\ g_2(a,b) =& a^8 - 68a^6b^2 - 122a^4b^4 - 68a^2b^6 + b^8. \end{aligned}$$

Our next goal is to check the positivities of the masses m and M, which are defined in equations (3.13) and (3.14) i.e., we need to find the values of a and b for which our masses are positive. Because the masses are functions of distance parameters a and b, so we need to find the values of a and b, for which m and M are positive.

Here we take $b = 1$ (without loss of generality) and solving the masses expressions for a, we get the following interval for a, for which m and M are positive.

Case(i): $0.417221 < a < 1$

Case(ii): $1 < a < 2.39681$

Here we define the following proposition.

3.2.1 Proposition

We have two cases:

Case (I): If $b = 1$ (without loss of generality) then, $0.417221 < a < 1$ or $1 < a < 2.39681$

Case (II): If we choose $a = 1$ then same intervals as $b = 1$.

Chapter 4

Dynamics of 5^{th} Body

4.1 Introduction

In this chapter we discuss and explain some key features regarding 5^{th} body. The motion of m_5 will not effect the gravitational field of m_1, m_2, m_3 and m_4, because $m_5 << m_1, m_2, m_3, m_4$. We call this problem restricted collinear five-body problem (RC5BP). Equation of restricted collinear five-body problem of 5^{th} particles,

$$\ddot{\mathbf{r}}_5 = m_1 \frac{\mathbf{r}_1 - \mathbf{r}_5}{|\mathbf{r}_1 - \mathbf{r}_5|^3} + m_2 \frac{\mathbf{r}_2 - \mathbf{r}_5}{|\mathbf{r}_2 - \mathbf{r}_5|^3} + m_3 \frac{\mathbf{r}_3 - \mathbf{r}_5}{|\mathbf{r}_3 - \mathbf{r}_5|^3} + m_4 \frac{\mathbf{r}_4 - \mathbf{r}_5}{|\mathbf{r}_4 - \mathbf{r}_5|^3}. \tag{4.1}$$

We now introduce a coordinate system that is rotating about the center of mass with uniform angular speed ω. Let (x, y) be the coordinates of m_5 in this new rotating frame (non-inertial frame). We can convert equation (4.1) from fixed inertial frame to the rotating coordinates system with the following orthogonal system,

$$\mathbf{e}_1 = e^{iwt}, \quad \mathbf{e}_2 = ie^{iwt},$$

where ω is angular speed and "t" represents time. The position vector of m_5 in the rotating frame is,

$$\mathbf{r}_5 = x(t)\ \mathbf{e}_1 + y(t)\ \mathbf{e}_2, \tag{4.2}$$

choosing ω, (without loss of generality) and taking first and second derivatives of equation (4.2) yield,

$$\left.\begin{aligned} \dot{\mathbf{r}}_5 &= (\dot{x} - y)e^{it} + i(x + \dot{y})e^{it}, \\ \ddot{\mathbf{r}}_5 &= (\ddot{x} - 2\dot{y} - x)e^{it} + i(\ddot{y} + 2\dot{x} - y)e^{it}. \end{aligned}\right\} \tag{4.3}$$

Using equation (4.3) in equation (4.1), the planer equations of motion of m_5 in rotating frame in component form are,

$$\ddot{x} - 2\dot{y} - x = \left[M\left(\frac{x+b}{r_{51}^3} + \frac{x-b}{r_{54}^3}\right) + m\left(\frac{x+a}{r_{52}^3} + \frac{x-a}{r_{53}^3}\right)\right], \tag{4.4}$$

$$\ddot{y} + 2\dot{x} - y = \left[M\left(\frac{1}{r_{51}^3} + \frac{1}{r_{54}^3}\right)y + m\left(\frac{1}{r_{52}^3} + \frac{1}{r_{53}^3}\right)y\right], \tag{4.5}$$

where mutual distances are described as,

$$\left.\begin{aligned} r_{51} &= \sqrt{(x+b)^2 + y^2}, \\ r_{52} &= \sqrt{(x+a)^2 + y^2}, \\ r_{53} &= \sqrt{(x-a)^2 + y^2}, \\ r_{54} &= \sqrt{(x-b)^2 + y^2}. \end{aligned}\right\} \tag{4.6}$$

Multiply equation (4.4) by $\dot{x}$ and equation (4.5) by $\dot{y}$ to obtain

$$\ddot{x}\dot{x} - 2\dot{x}\dot{y} - x\dot{x} = \left[M\dot{x}\left(\frac{x+b}{r_{51}^3} + \frac{x-b}{r_{54}^3}\right) + m\dot{x}\left(\frac{x+a}{r_{52}^3} + \frac{x-a}{r_{53}^3}\right)\right], \tag{4.7}$$

$$\ddot{y}\dot{y} + 2\dot{x}\dot{y} - y\dot{y} = \left[M\dot{y} \left(\frac{1}{r_{51}^3} + \frac{1}{r_{54}^3} \right) y + m\dot{y} \left(\frac{1}{r_{52}^3} + \frac{1}{r_{53}^3} \right) y \right]. \tag{4.8}$$

Adding these equations, we get

$$\begin{aligned} \ddot{x}\dot{x} + \ddot{y}\dot{y} - (x\dot{x} + y\dot{y}) &= \frac{M}{r_{51}^3}(x\dot{x} + b\dot{x} + y\dot{y}) + \frac{M}{r_{54}^3}(x\dot{x} - b\dot{x} + y\dot{y}) \\ &+ \frac{m}{r_{52}^3}(x\dot{x} + a\dot{x} + y\dot{y}) + \frac{m}{r_{53}^3}(x\dot{x} - a\dot{x} + y\dot{y}). \end{aligned} \tag{4.9}$$

Note that

$$\ddot{x}\dot{x} + \ddot{y}\dot{y} = \frac{1}{2}\frac{d}{dt}\left(\dot{x}^2 + \dot{y}^2\right) = \frac{1}{2}\frac{dv^2}{dt}, \tag{4.10}$$

where v is the speed of the infinitesimal mass relative to the rotating frame. Similarly,

$$x\dot{x} + y\dot{y} = \frac{1}{2}\frac{d}{dt}(x^2 + y^2) \tag{4.11}$$

Using equations (4.10) and (4.11) in equation (4.9), we get the following equation,

$$\begin{aligned} \frac{1}{2}\frac{dv^2}{dt} - \frac{1}{2}\frac{d}{dt}\left(x^2 + y^2\right) &= \frac{M}{r_{51}^3}(x\dot{x} + b\dot{x} + y\dot{y}) + \frac{M}{r_{54}^3}(x\dot{x} - b\dot{x} + y\dot{y}) \\ &+ \frac{m}{r_{52}^3}(x\dot{x} + a\dot{x} + y\dot{y}) + \frac{m}{r_{53}^3}(x\dot{x} - a\dot{x} + y\dot{y}). \end{aligned} \tag{4.12}$$

Equation (4.6) gives

$$2r_{51}\frac{dr_{51}}{dt} = 2(x + b)\dot{x} + 2y\dot{y},$$

or

$$\frac{dr_{51}}{dt} = \frac{1}{r_{51}}(x\dot{x} + y\dot{y} + b\dot{x}). \tag{4.13}$$

Also we know that

$$\frac{d}{dt}\frac{1}{r_{51}} = -\frac{1}{r_{51}^2}\frac{dr_{51}}{dt}. \tag{4.14}$$

Using equation (4.13) in equation (4.14) we get

$$\frac{d}{dt}\frac{1}{r_{51}} = -\frac{1}{r_{51}^3}(x\dot{x} + y\dot{y} + b\dot{x}). \tag{4.15}$$

Similarly, we can obtain the following equations

$$\frac{d}{dt}\frac{1}{r_{52}} = -\frac{1}{r_{52}^3}(x\dot{x} + y\dot{y} + a\dot{x}), \tag{4.16}$$

$$\frac{d}{dt}\frac{1}{r_{53}} = -\frac{1}{r_{53}^3}(x\dot{x} + y\dot{y} - a\dot{x}), \tag{4.17}$$

$$\frac{d}{dt}\frac{1}{r_{54}} = -\frac{1}{r_{54}^3}(x\dot{x} + y\dot{y} - b\dot{x}). \tag{4.18}$$

Using equations (4.15) – (4.18) into equation (4.12) yields

$$\begin{aligned}\frac{1}{2}\frac{dv^2}{dt} - \frac{1}{2}\frac{d}{dt}\left(x^2 + y^2\right) = &-M\frac{d}{dt}\frac{1}{r_{51}} - M\frac{d}{dt}\frac{1}{r_{54}} \\ &-m\frac{d}{dt}\frac{1}{r_{52}} - m\frac{d}{dt}\frac{1}{r_{53}}.\end{aligned} \tag{4.19}$$

The above equation becomes

$$\frac{d}{dt}\left[\frac{1}{2}v^2 - \frac{1}{2}(x^2 + y^2) + \frac{M}{r_{51}} + \frac{M}{r_{54}} + \frac{m}{r_{52}} + \frac{m}{r_{53}}\right] = 0,$$

which means the bracketed expression is a constant

$$\frac{1}{2}v^2 - \frac{1}{2}(x^2 + y^2) + \frac{M}{r_{51}} + \frac{M}{r_{54}} + \frac{m}{r_{52}} + \frac{m}{r_{53}} = C. \tag{4.20}$$

The constant C is known as the Jacobi constant, (named after the German mathematician Carl Jacobi who discovered it in 1836). Jacobi's constant [37] may be interpreted as the total energy of the m_5 relative to the rotating frame. C is a constant of the motion of the m_5 in the collinear restricted five-body problem, here

- $\frac{1}{2}v^2$ is the Kinetic energy per unit mass relative to the rotating frame.
- $-\frac{1}{r_{51}}$, $-\frac{1}{r_{52}}$, $-\frac{1}{r_{53}}$ and $-\frac{1}{r_{54}}$ are gravitational potential energies of the masses m_1, m_2, m_3 and m_4.
- $-\frac{1}{2}(x^2 + y^2)$ interpreted as the potential energy of the centrifugal force of m_5

induced by the rotation of the reference frame. Rewriting equation (4.20) as

$$v^2 = (x^2+y^2) - 2M\left(\frac{1}{r_{51}}+\frac{1}{r_{54}}\right) - 2m\left(\frac{1}{r_{52}}+\frac{1}{r_{53}}\right) + 2C. \tag{4.21}$$

Since v^2 cannot be negative, it must be true that,

$$(x^2+y^2) - 2M\left(\frac{1}{r_{51}}+\frac{1}{r_{54}}\right) - 2m\left(\frac{1}{r_{52}}+\frac{1}{r_{53}}\right) + 2C \geq 0. \tag{4.22}$$

Trajectories of the m_5 in regions where this inequality is violated are not allowed. The boundaries between forbidden and allowed regions of motion are found by setting $v^2 = 0$, i.e.,

$$(x^2+y^2) - 2M\left(\frac{1}{r_{51}}+\frac{1}{r_{54}}\right) - 2m\left(\frac{1}{r_{52}}+\frac{1}{r_{53}}\right) + 2C = 0. \tag{4.23}$$

For a given value of the Jacobi constant the curves of zero velocity are determined by this equation. These boundaries cannot be crossed by a infinitesimal mass (spacecraft) moving within an allowed region.

4.2 Equilibrium Solutions

The equations (4.4) and (4.5) do not have an analytical solution of a closed form, we can use these equations to determine the location of the equilibrium points. These are the places in space where the infinitesimal mass m_5 would have zero velocity and acceleration, i.e., where m_5 appears at rest permanently relative to the masses m_1,m_2,m_3 and m_4 respectively. When located at an equilibrium point (also called libration point / Lagrange point), a body will apparently stay there. These solutions can be found only if we meet the sufficient condition of all rates equal to zero,

$$\dot{x} = \dot{y} = \ddot{x} = \ddot{y} = 0.$$

To find the zero's (x, y) or equilibrium points / Lagrange point, we need to solve these equations numerically or drawing contour plot using Mathematica. The classification of equilibrium points for (RC5BP) discussed in the following cases.

4.3 Case-I(1): When $a \in (0.417221, 1)$ and $b = 1$

i. When $a \in (0.417221\ ,\ 0.647221)$ there exist 7 equilibrium points.

ii. When $a \in (0.647221\ ,\ 0.937221)$ there exist 5 equilibrium points.

iii. When $a \in (0.937221\ ,\ 0.957221)$ there exist 3 equilibrium points.

iv. When $a \in (0.957221\ ,\ 0.999999)$ there exist 1 equilibrium points.

In addition, these intervals can be discuss in cases compared to other similar equilibrium points. The intersections of the non-linear algebraic equations $U_x = 0$, and $U_y = 0$ define the positions of the equilibrium points. The intersections of $U_x = 0$ (blue) and $U_y = 0$ (orange), respectively. The red dots represent the position of the masses and black dots represent the position of equilibrium points. See figures ((4.1) to (4.4)) below

4.3.1 Seven Equilibrium Point

We start our analysis with case-I(1) for equilibrium points where seven equilibrium points exist on x-axis and y-axis. It is numerically checked that the behaviour of the equilibrium points does not change throughout in (0.41722 , 0.63722), therefore we choose $a = 0.427221 \in (0.417221,\ 0.637221)$ and draw contour plot for the infinitesimal particle and check the position of equilibrium points.

The contour plot for $a = 0.427221$ the corresponding value of b, M and m are 1, 0.0116734 and 0.281741 respectively. Contour plot for these values shows that the L_1, L_2 , L_4 , L_6 and L_7 are collinear along x-axis, while L_3 and L_5 are collinear along

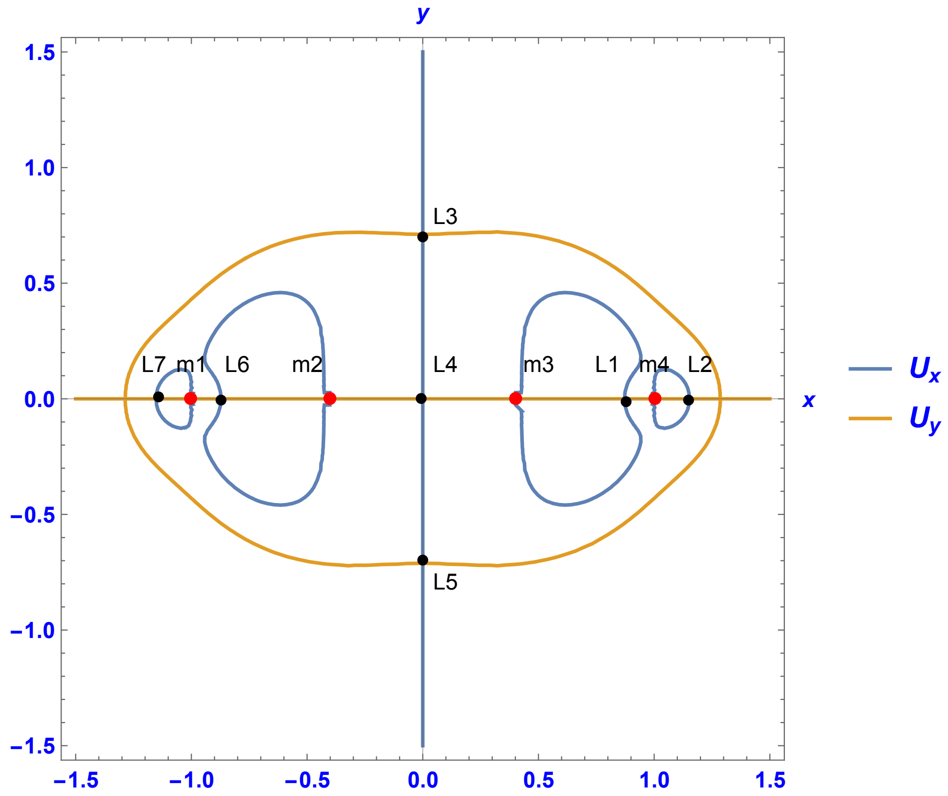

FIGURE 4.1: Positions of the masses (Red dots); equilibrium points of m_5 particle (Black dots). Here $a = 0.427221$ and the corresponding masses are $m = 0.281741$, $M = 0.0116734$.

the y-axis. In figure (4.1) the black dots denote the position of equilibrium points and the red dots represent the position of the masses $m_1 - m_4$. The equilibrium points are the points of the intersection of $U_x = 0$ (blue) and $U_y = 0$ (orange).

4.3.2 Five Equilibrium Point

The present analysis for equilibrium points where five equilibrium points exist on x-axis. It is numerically checked that the behaviour of the equilibrium points does not change throughout in (0.6372 , 0.9372), therefore we choose $a = 0.657221 \in (0.637221\ ,\ 0.937221)$ and draw contour plot for the infinitesimal particle and check the position of equilibrium points.

The contour plot for $a = 0.657221$ the corresponding value of b, M and m are 1, 0.668281 and 0.110794 respectively. Contour plot for these values shows that the L_1, L_2 , L_3 , L_4 and L_5 all are collinear along x-axis. In figure (4.2) the black dots

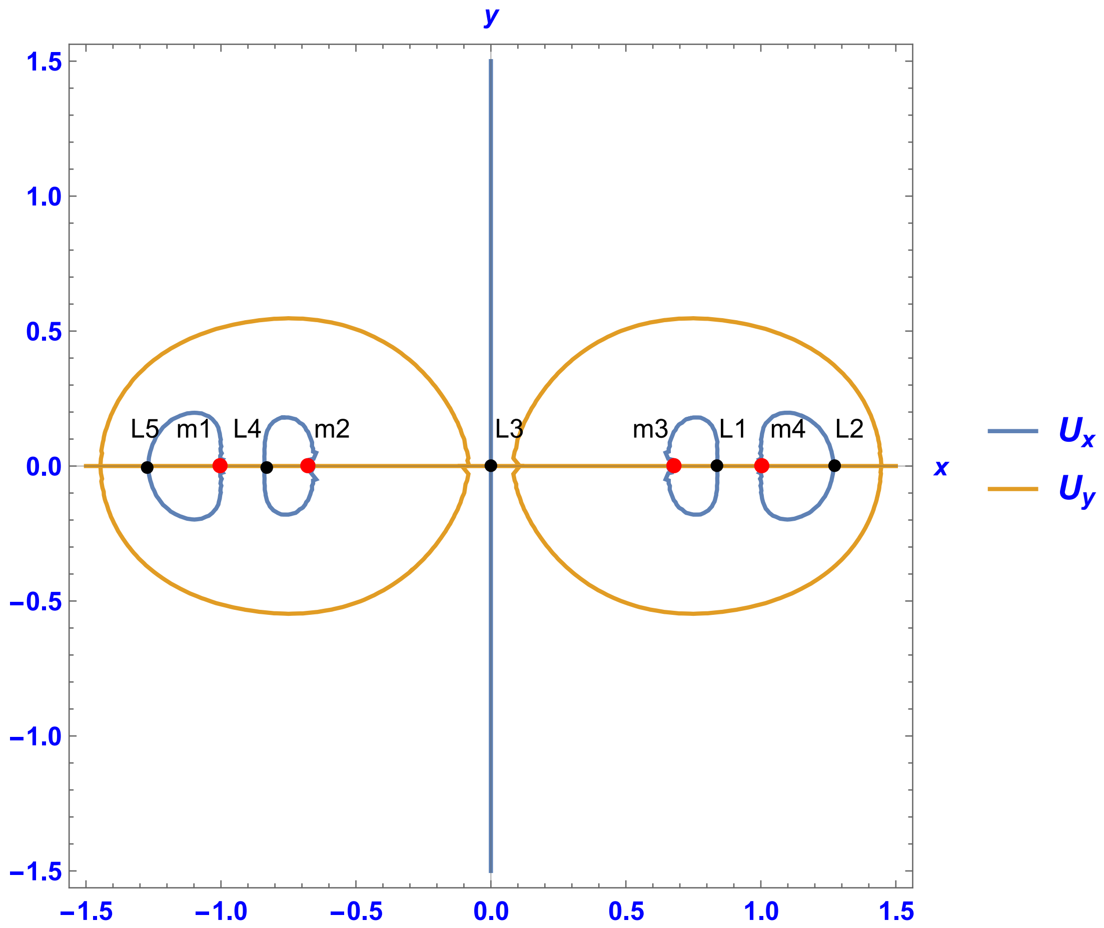

FIGURE 4.2: Positions of the masses (Red dots); equilibrium points of m_5 particle (Black dots). Here $a = 0.657221$ and the corresponding masses are $m = 0.110794$, $M = 0.668281$.

denote the position of equilibrium points and the red dots represent the position of the masses $m_1 - m_4$. The equilibrium points are the points of the intersection of $U_x = 0$ (blue) and $U_y = 0$ (orange).

4.3.3 Three Equilibrium Points

The present analysis for equilibrium points where three equilibrium points exist on x-axis. It is numerically checked that the behaviour of the equilibrium points does not change throughout in (0.9372 , 0.9572), therefore we choose $a = 0.947221 \in (0.937221 \ , \ 0.957221)$ and draw contour plot for the infinitesimal particle and check the position of equilibrium points.

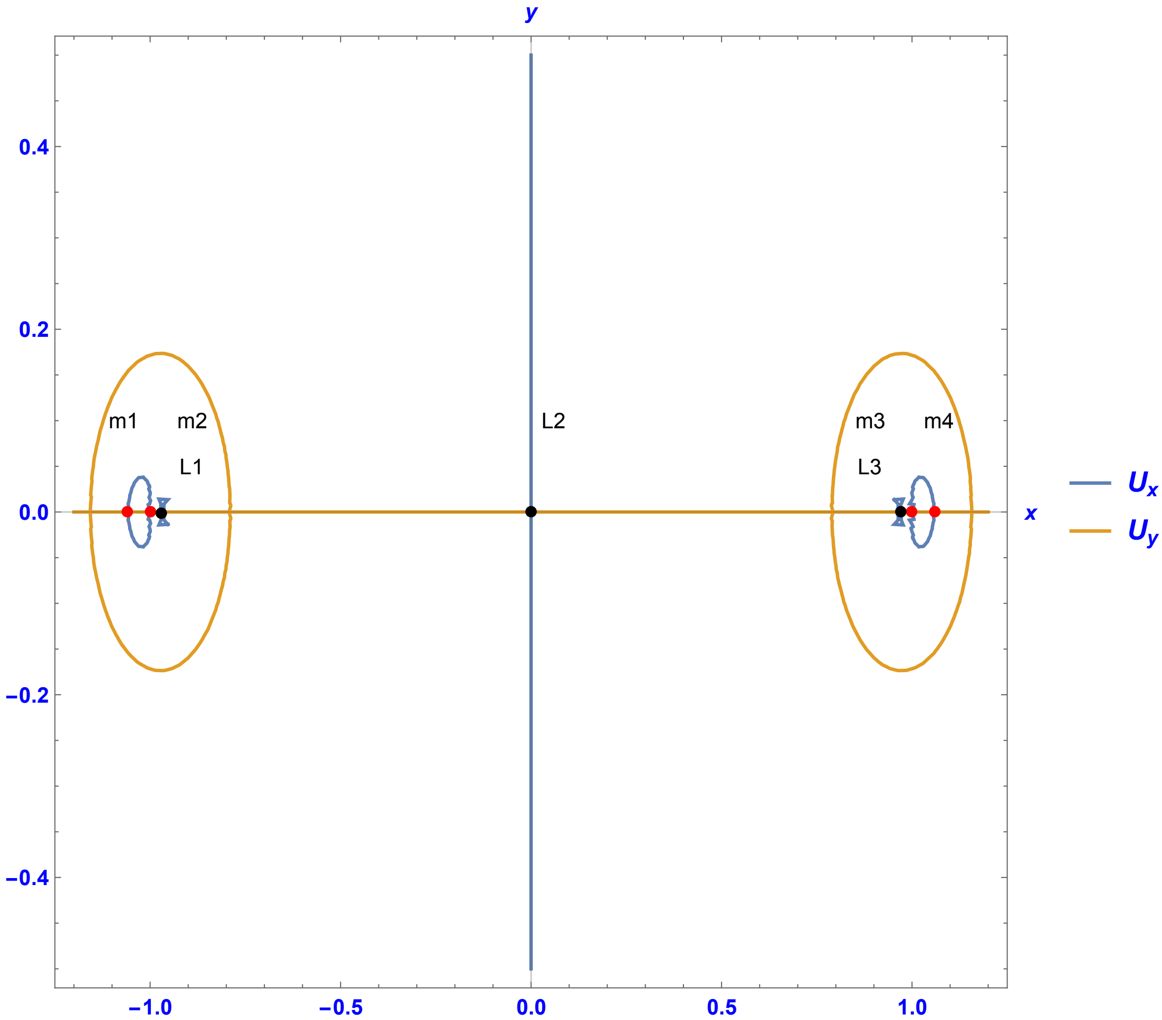

FIGURE 4.3: Positions of the masses (Red dots); equilibrium points of m_5 particle (Black dots). Here $a = 0.947221$ and the corresponding masses are $m = 0.00278174$, $M = 0.00263451$.

The contour plot for $a = 0.947221$ the corresponding value of b, M and m are 1, 0.00263451 and 0.00278174 respectively. Contour plot for these values shows that the L_1, L_2 and L_3 all are collinear along x-axis.

In figure (4.3) the black dots denote the position of equilibrium points and the red dots represent the position of the masses $(m_1 - m_4)$. The equilibrium points are the points of the intersection of $U_x = 0$ (blue) and $U_y = 0$ (orange), red dots represent the masses and black dots represent the equilibrium points.

4.3.4 One Equilibrium Point

We further continue our analysis for equilibrium points where one equilibrium point exist on x-axis. It is numerically checked that the behaviour of the equilibrium points does not change throughout in (0.957221 , 0.999999), therefore we choose $a = 0.967221 \in (0.95722\ ,\ 0.99999)$ and draw contour plot for the infinitesimal particle and check the position of equilibrium point.

The contour plot for $a = 0.967221$ the corresponding value of b, M and m are 1, 0.00103865 and 0.00107389 respectively. Contour plot for these values shows that the L_1 are collinear along x-axis. In figure (4.4) the black dots denote the position of equilibrium points and the red dots represent the position of the masses $m_1 - m_4$. The equilibrium points are the points of the intersection of $U_x = 0$ (blue) and $U_y = 0$ (orange).

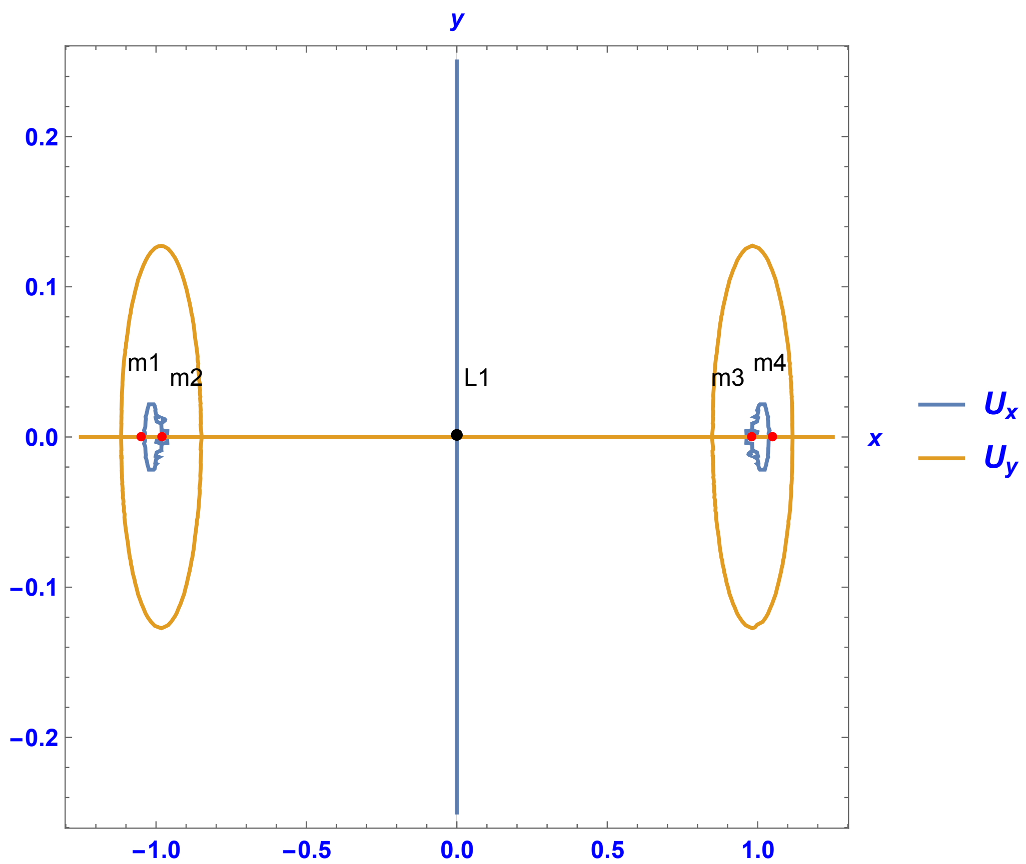

FIGURE 4.4: Positions of the masses (Red dots); equilibrium points of m_5 particle (Black dots). Here $a = 0.967221$ and the corresponding masses are $m = 0.00107389$, $M = 0.00103865$.

4.4 Case-I(2): When $a \in (1, 2.39)$ and $b = 1$

i. When $a \in (1.0\ ,\ 1.16)$ there exist 5 equilibrium points.

ii. When $a \in (1.16\ ,\ 1.29)$ there exist 5 equilibrium points.

iii. When $a \in (1.29\ ,\ 1.57)$ there exist 7 equilibrium points.

iv. When $a \in (1.57\ ,\ 2.39)$ there exist 7 equilibrium points.

4.4.1 Five Equilibrium Point

We start our analysis with case-I(2) for equilibrium points where five equilibrium points exist on x-axis. It is numerically checked that the behaviour of the equilibrium points does not change throughout in $(1.0\ ,\ 1.16)$, therefore we choose $a = 1.15 \in (1.0\ ,\ 1.16)$ and draw contour plot for the infinitesimal particle and check the position of equilibrium points.
The contour plot for $a = 1.15$ the corresponding value of b, M and m are 1, 0.0256555 and 0.0222474 respectively. Contour plot for these values shows that the L_1, L_2 , L_3 , L_4 and L_5 all are collinear along x-axis and the contour plot for this case is given below. In figure (4.5) the black dots denote the position of equilibrium points and the red dots represent the position of the masses m_1, m_2, m_3 and m_4. The equilibrium points are the points of the intersection of $U_x = 0$ (blue) and $U_y = 0$ (orange).

4.4.2 Five Equilibrium Point

We continue our analysis for equilibrium points where five equilibrium points exist on x-axis. It is numerically checked that the behaviour of the equilibrium points does not change throughout in $(1.16\ ,\ 1.29)$, therefore we choose $a = 1.28 \in (1.16\ ,\ 1.29)$ and draw contour plot for the infinitesimal particle and check the position of equilibrium points.

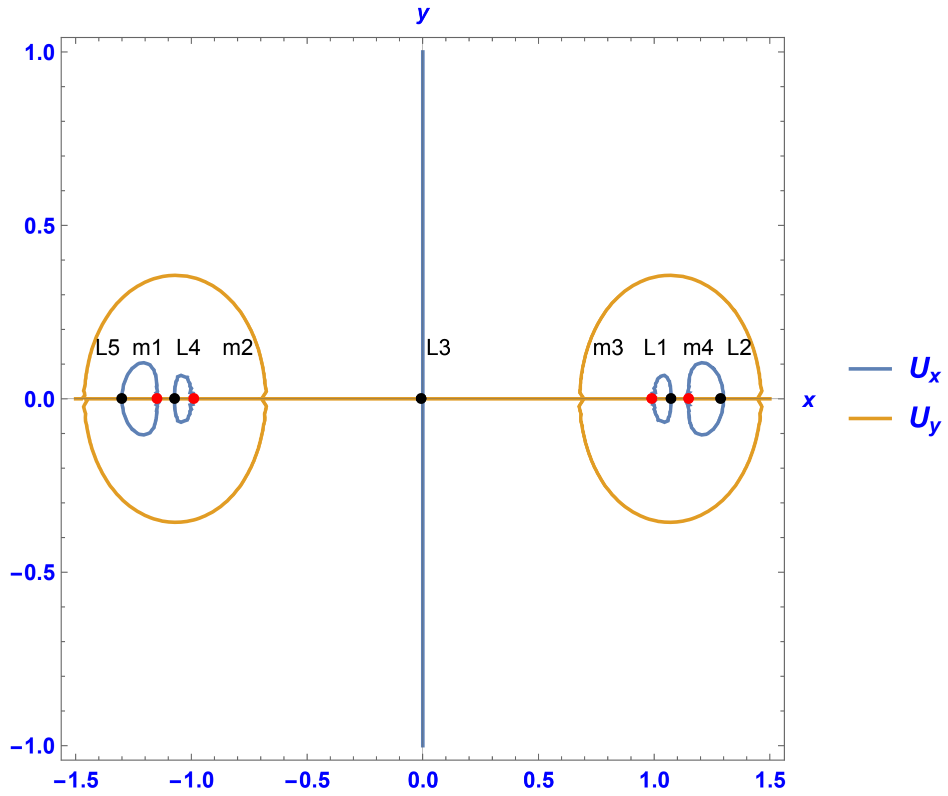

FIGURE 4.5: Positions of the masses (Red dots); equilibrium points of m_5 particle (Black dots). Here $a = 1.15$ and the corresponding masses are $m = 0.0222474$, $M = 0.0256555$.

The contour plot for $a = 1.28$ the corresponding value of b, M and m are 1, 0.0979731 and 0.0753434 respectively. Contour plot for these values shows that the L_1, L_2 , L_3 , L_4 and L_5 all are collinear along x-axis.

In figure (4.6) the black dots denote the position of equilibrium points and the red dots represent the position of the masses $m_1 - m_4$. The equilibrium points are the points of the intersection of $U_x = 0$ (blue) and $U_y = 0$ (orange).

4.4.3 Seven Equilibrium Point

The present analysis for equilibrium points where seven equilibrium points exist on x-axis and y-axis. It is numerically checked that the behaviour of the equilibrium points does not change throughout in (1.29 , 1.57), therefore we choose $a = 1.56 \in (1.29,\ 1.57)$ and draw contour plot for the infinitesimal particle and

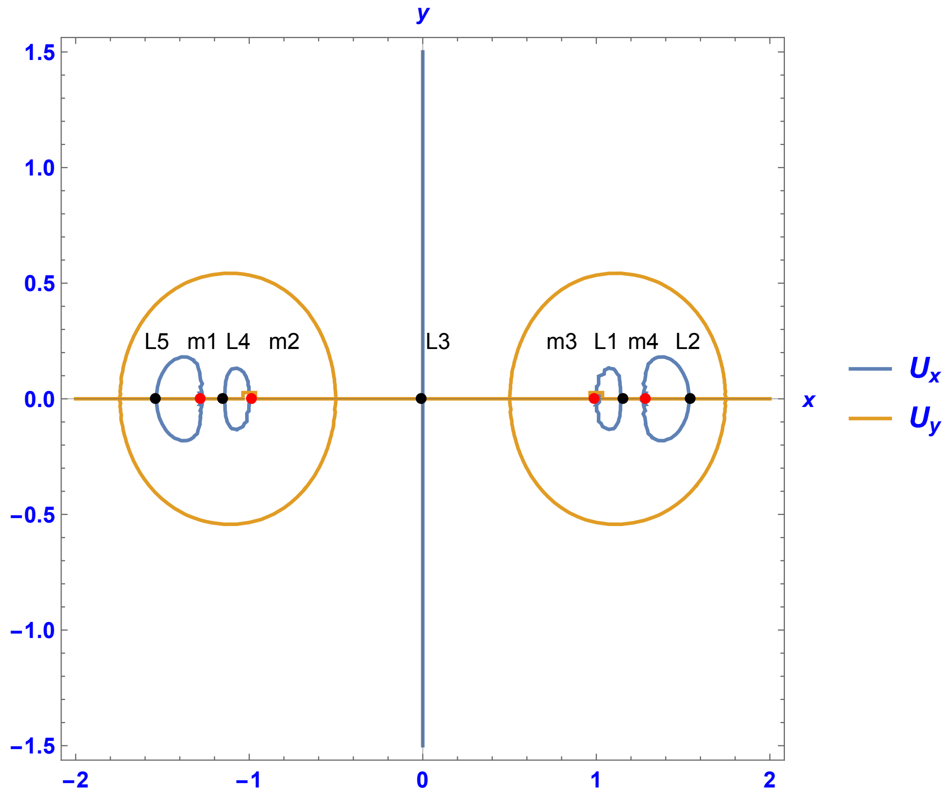

FIGURE 4.6: Positions of the masses (Red dots); equilibrium points of m_5 particle (Black dots). Here $a = 1.28$ and the corresponding masses are $m = 0.0753434$, $M = 0.0979731$.

check the position of equilibrium points.

The contour plot for $a = 1.56$ the corresponding value of b, M and m are 1, 0.458729 and 0.264957 respectively. Contour plot for these values shows that the L_1, L_2 , L_3 , L_6 and L_7 are collinear along x-axis, while L_4 and L_5 are collinear along the y-axis.
In figure (4.7) the black dots denote the position of equilibrium points and the red dots represent the position of the masses m_1, m_2, m_3 and m_4. The equilibrium points are the points of the intersection of $U_x = 0$ (blue) and $U_y = 0$ (orange).

4.4.4 Seven Equilibrium Point

We further continue our analysis for equilibrium points where seven equilibrium points exist on x-axis and y-axis. It is numerically checked that the behaviour of

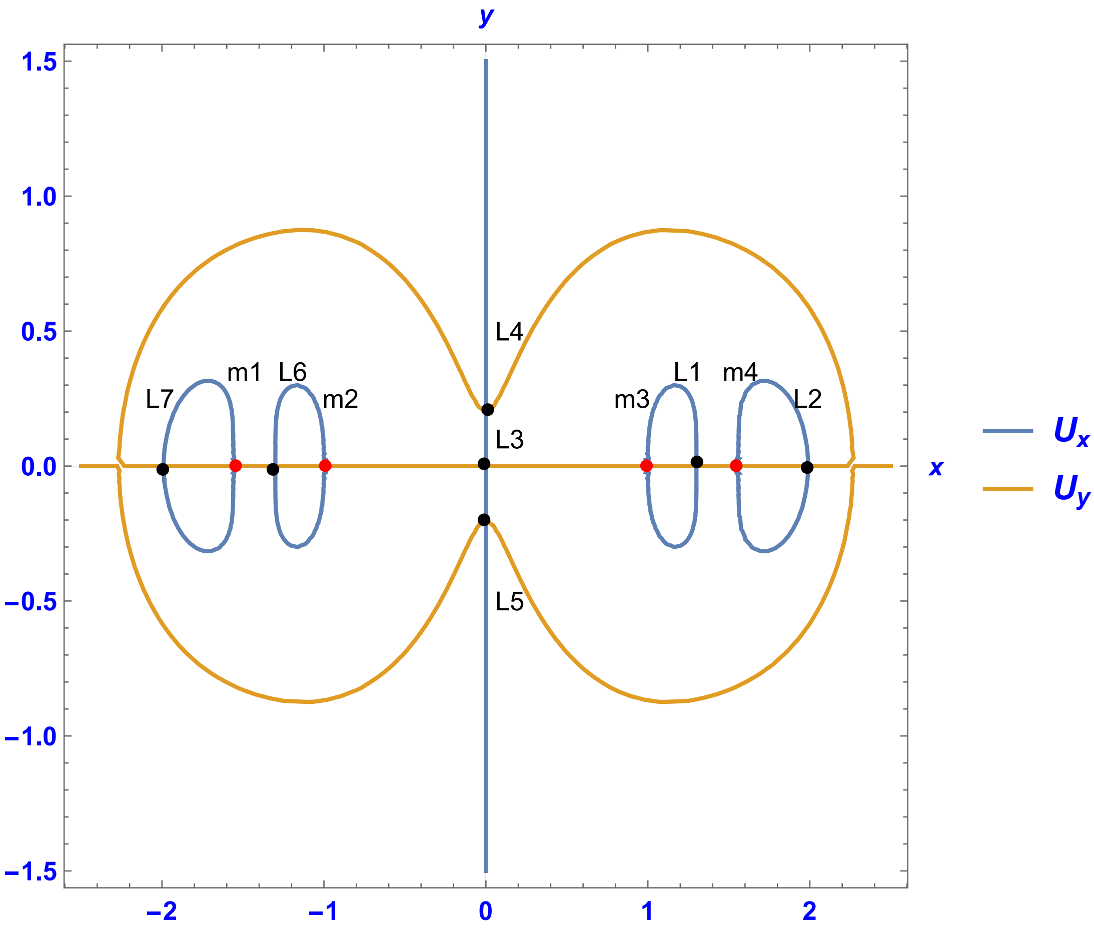

FIGURE 4.7: Positions of the masses (Red dots); equilibrium points of m_5 particle (Black dots). Here $a = 1.56$ and the corresponding masses are $m = 0.264957$, $M = 0.458729$.

the equilibrium points does not change throughout in $(1.57\ ,\ 2.39)$, therefore we choose $a = 2.0 \in (1.57,\ 2.39)$ and draw contour plot for the infinitesimal particle and check the position of equilibrium points.

The contour plot for $a = 2.0$ the corresponding value of b, M and m are 1, 1.771799 and 0.501345 respectively. Contour plot for these values shows that the L_1, L_2 , L_4 , L_6 and L_7 are collinear along x-axis, while L_3 and L_5 are collinear along the y-axis.

In figure (4.8) the black dots denote the position of equilibrium points and the red dots represent the position of the masses m_1, m_2, m_3 and m_4. The equilibrium points are the points of the intersection of $U_x = 0$ (blue) and $U_y = 0$ (orange), red dots represent the masses and black dots represent the equilibrium points.

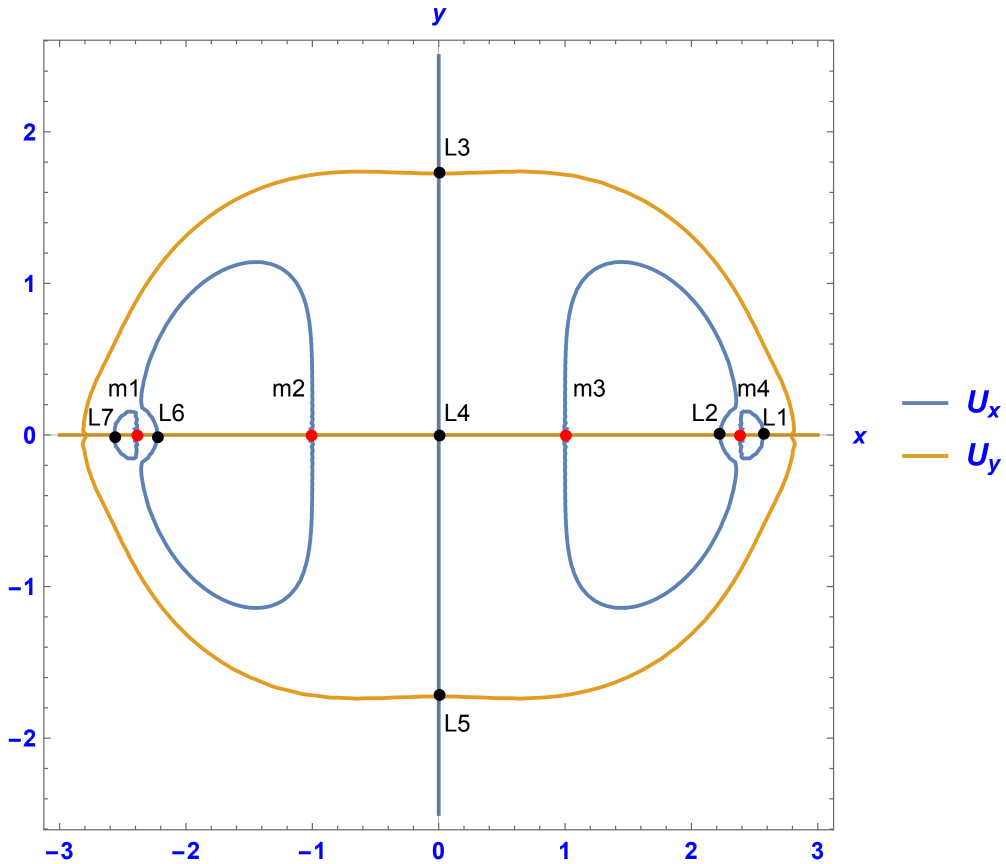

FIGURE 4.8: Positions of the masses (Red dots); equilibrium points of m_5 particle (Black dots). Here $a = 2.0$ and the corresponding masses are $m = 0.501345$, $M = 1.771799$.

4.5 Stability Analysis

This section is devoted to the mathematical analysis of the stability of Lagrange points in the system of a (RC5BP) for which we consider the slight displacement from the Lagrange points to the infinitesimal body as well as the small velocity. If the infinitesimal small body's motion departs from the point of the proximity, and it never returns, then this point of equilibrium is known as unstable. But if the body is oscillating about the point it seems to be stable.

We will now check whether the Lagrange points are either stable or unstable. To check the stability, with the support of the Jacobian matrix of individual values, followed by the procedure given on page 11.

We know that for case-I(1)(i) $a = 0.427221$ and corresponding coordinates of L_1

are $(0.8866, -0.01331)$, we get the following Jakobian matrix form as

$$A = \begin{pmatrix} 22.4148 & 2.468 \\ 2.468 & -9.54388 \end{pmatrix}$$

The eigenvalues of matrix A are: $(22.6043, -9.73334)$, eigenvalues of matrix A have not negative real part, therefore L_1 is unstable.

For the remaining values we will apply the same procedure and present the stability analysis of corresponding Lagrange points which are in the following table.

S.No	**Lagrange points**	**Eigenvalues**	**Stability**
1	L_1(0.8866, -0.01331)	(22.6043, -9.73334)	unstable
2	L_2(1.139, -0.004016)	(11.3928, -4.1958)	unstable
3	L_3(0.00158, 0.7025)	(2.20632, 0.820232)	unstable
4	L_4(-.006322, 0.0006322)	(15.5184, -6.25918)	unstable
5	L_5(0.00158, -.7152)	(2.18046, 0.806673)	unstable
6	L_6(-0.8834, -0.004016)	(21.8782, -9.43306)	unstable
7	L_7(-1.152, 0.00528)	(9.25988, -3.12918)	unstable

Table 4.1: Stability Analysis for $a = 0.427221$, $b = 1$,
and corresponding masses are $m = 0.281741$, $M = 0.0116734$.

Stability analysis for case-I(1)(ii) is shown in the following table:

S.No	**Lagrange points**	**Eigenvalues**	**Stability**
1	L_1(0.8384, -0.001516)	(70.0012, -33.4965)	unstable
2	L_2(0.9866, -0.001516)	(54506.2, -27251.6)	unstable
3	L_3(-0.00064, 0.002274)	(2.82841, 0.0858117)	unstable
4	L_4(-0.8371, -0.005371)	(69.877, -33.3881)	unstable
5	L_5(-0.9853, -0.005705)	(35018.8, -17507.3)	unstable

Table 4.2: Stability Analysis for $a = 0.657221$, $b = 1$,
and the corresponding masses are $m = 0.110794$, $M = 0.668281$.

Stability analysis for case-I(1)(iii) is shown in the following table:

S.No	Lagrange points	Eigenvalues	Stability
1	L_1(0.9781, 0.005725)	(598.64, -279.687)	unstable
2	L_2(0.002246, -0.0004922)	(1.02363, 0.988184)	unstable
3	L_3(-0.9771, -0.0004922)	(647.632, -322.163)	unstable

Table 4.3: Stability Analysis for $a = 0.947221$, $b = 1$,
and corresponding masses are $m = 0.00278174$, $M = 0.0.00263451$.

Stability analysis for case-I(1)(iv) is shown in the following table:

S.No	Lagrange points	Eigenvalues	Stability
1	L_1(0.9871, -0.001105)	(1223.96, -607.334)	unstable

Table 4.4: Stability Analysis for $a = 0.967221$, $b = 1$,
and corresponding masses are $m = 0.00107389$, $M = 0.00103865$.

Stability analysis for case-I(2)(i) is shown in the following table:

S.No	Lagrange points	Eigenvalues	Stability
1	L_1(1.299, 0.004726)	(16.356, -6.67769)	unstable
2	L_2(0.9869, -0.003493)	(20600.2, -10298.2)	unstable
3	L_3(-0.007602, 0.004726)	(1.16118, 0.919415)	unstable
4	L_4(-0.9856, 0.01294)	(7076.82, -3534.16)	unstable
5	L_5(-1.298, -0.01171)	(16.5351, -6.76555)	unstable

Table 4.5: Stability Analysis for $a = 1.15$, $b = 1$,
and the corresponding masses are $m = 0.0222474$, $M = 0.0256555$.

Stability analysis for case-I(2)(ii) is shown in the following table:

S.No	Lagrange points	Eigenvalues	Stability
1	L_1(1.527, 0.009316)	(12.3347, -4.66698)	unstable
2	L_2(1.139, 0.009316)	(126.095, -61.1407)	unstable
3	L_3(0.0006617, 0.000494)	(1.5356, 0.7322)	unstable
4	L_4(-1.146, -0.008328)	(125.289, -60.8129)	unstable
5	L_5(-1.534, 0.000494)	(11.501, -4.2551)	unstable

Table 4.6: Stability Analysis for $a = 1.28$, $b = 1$,
and the corresponding masses are $m = 0.075354$, $M = 0.0979731$.

Stability analysis for case-I(2)(iii) is shown in the following table:

S.No	Lagrange points	Eigenvalues	Stability
1	L_1(1.98, -0.004131)	(9.17278, -3.08637)	unstable
2	L_2(1.308,0.01152)	(65.2752, -31.0545)	unstable
3	L_3(-0.00665, 0.1993)	(2.89795, 0.103895)	unstable
4	L_4(-0.00665, 0.01152)	(3.11382, -0.0567179)	unstable
5	L_5(0.008997, -0.2075)	(2.88147, 0.116132)	unstable
6	L_6(-1.305, -0.01978)	(64.4198, -30.4685)	unstable
7	L_7(-1.994, -0.01978)	(8.44097, -2.72004)	unstable

Table 4.7: Stability Analysis for $a = 1.56$, $b = 1$,
and the corresponding masses are $m = 0.264957$, $M = 0.458729$.

Stability analysis for case-I(2)(iv) is shown in the following table:

S.No	Lagrange points	Eigenvalues	Stability
1	L_1(2.544, 0.01152)	(8.27679, -2.63826)	unstable
2	L_2(1.621,0.01152)	(34.3704, -15.079)	unstable
3	L_3(-0.00665, 1.201)	(1.70063, 1.30674)	unstable
4	L_4(-0.00665, 0.01152)	(8.33692, -2.66775)	unstable
5	L_5(0.008997, -1.715)	(1.70203, 1.29405)	unstable
6	L_6(-1.603, -0.01978)	(33.2415, -15.0797)	unstable
7	L_7(-2.541, -0.004131)	(8.39078, -2.69537)	unstable

Table 4.8: Stability Analysis for $a = 2.0$, $b = 1$,
and corresponding masses are $m = 0.501345$, $M = 1.771799$.

4.6 Permissible Regions of Motion

The Jacobian constant of motion in equation (4.22) is one of the most important constant of the dynamical system, which represent the movement of the infinitesimal body, because it can be used to sketch the zero velocity curves.Of course it can be used to explore the regions of permissible motion of the infinitesimal body. The motion of infinitesimal body depends on the value of the Jacobi constant. For choosing different values of Jacobi constant C in equation (4.22) gives us two different regions.

One is permissible region of motion for infinitesimal particle and the second region where motion of infinitesimal particle is not allowed [32].

Now we need to explore these possibilities in our geometry i.e., Four masses are placed m_1, m_2, m_3 *and* m_4 are on the x-axis and the infinitesimal mass m_5 is moving in the gravitational field of these four masses. We draw here regions for different value of Jacobi constant for equation (4.22) at Mathematica and we get two regions,

- Permissible (White region), where m_5 can freely move.

- Colored (Blue), where the motion of m_5 is not allowed.

Permissible regions for case-I(1)(i):

Choosing the value of $C = 1.0$ and using in equation (4.22) and then we get two regions in figure (4.10), we can clearly see that the blue region represent the excluded area, where the infinitesimal mass m_5 can not move or enter, and the white region that is permissible region represent where the infinitesimal mass can easily move or enter.

As soon as, we raise the value of C from (1 to 1.4), the allowable area is getting shorter. We can see this in figures from (4.9) to (4.12). In these statistics, we can see that the four masses are trapped, and the infinitesimal mass can not reach those masses.

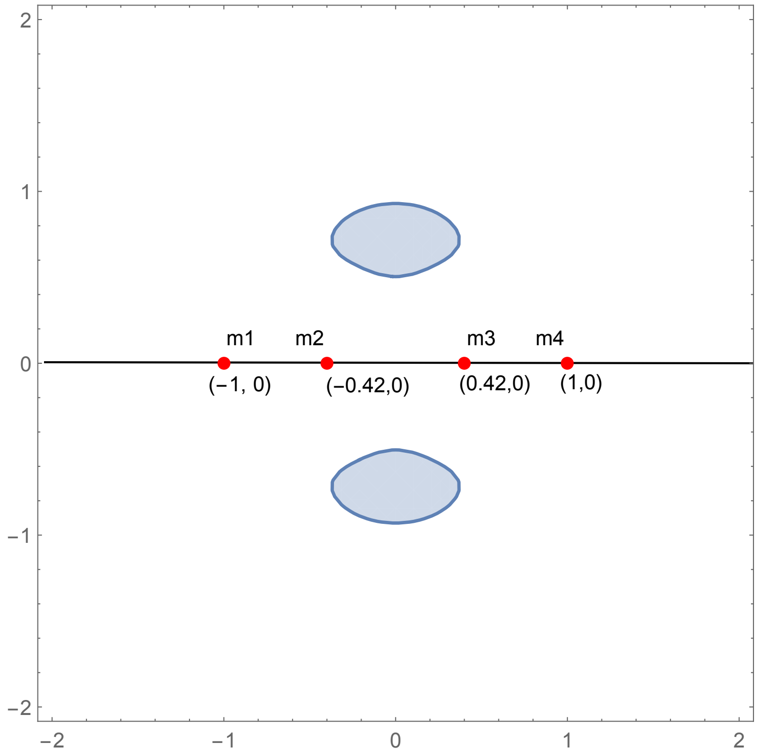

FIGURE 4.9: Permissible regions (white) of motion for $C = 1.0$

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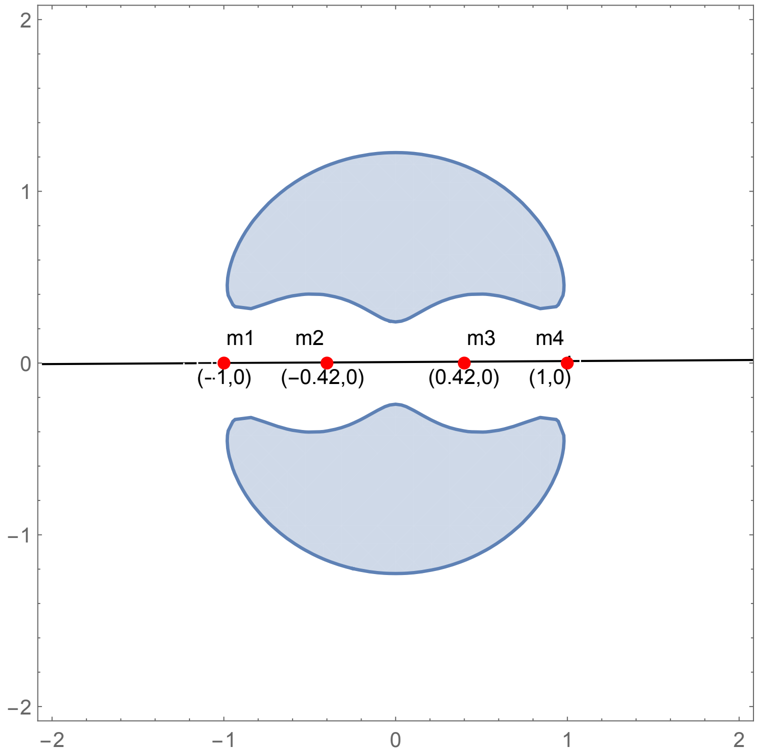

FIGURE 4.10: Permissible regions (white) of motion for $C = 1.2$

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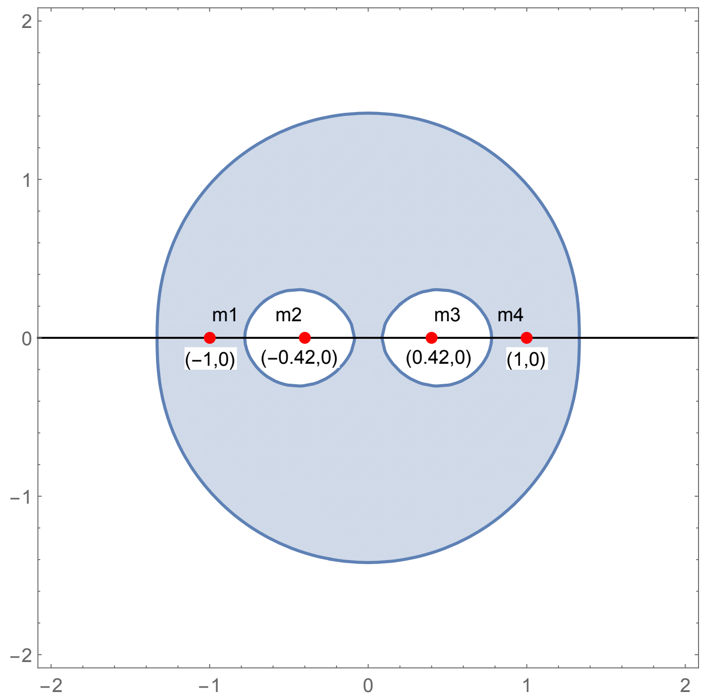

FIGURE 4.12: Permissible regions (white) of motion for $C = 1.4$

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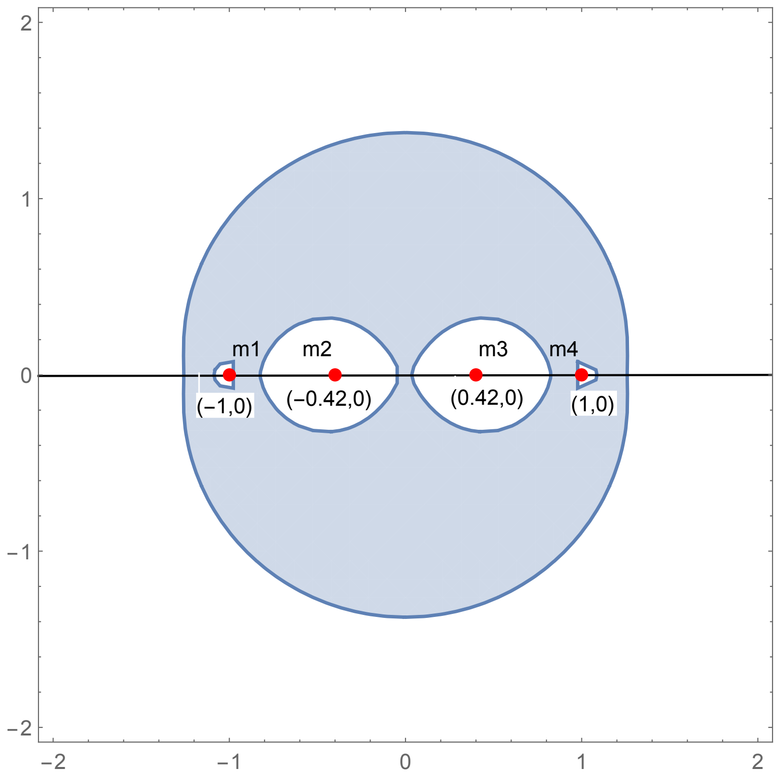

FIGURE 4.11: Permissible regions (white) of motion for $C = 1.35$

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Permissible regions for case-I(1)(ii):

When the value of C from (1.05 to1.35), for $a = 0.65$ the permissible area is getting shorter.

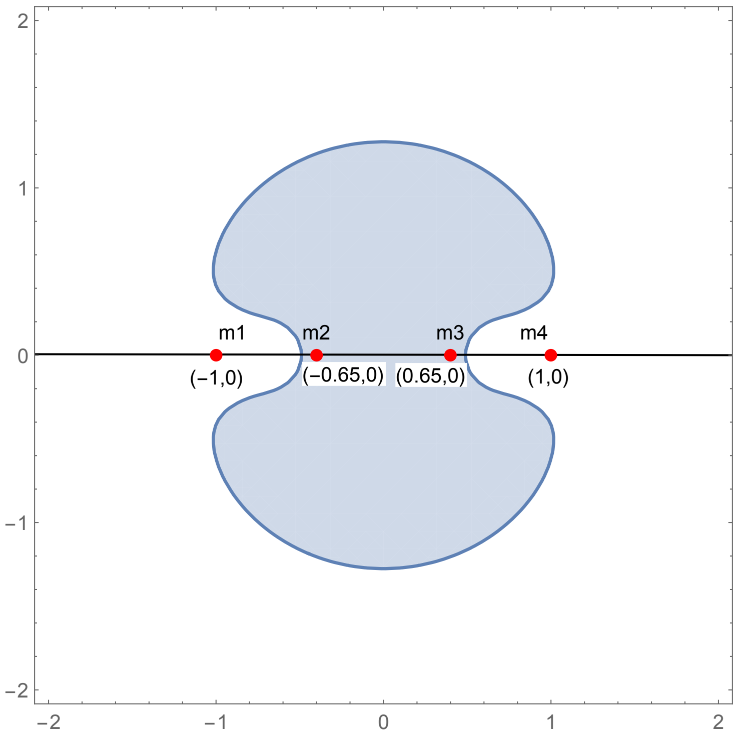

FIGURE 4.13: Permissible regions (white) of motion for $C = 1.05$

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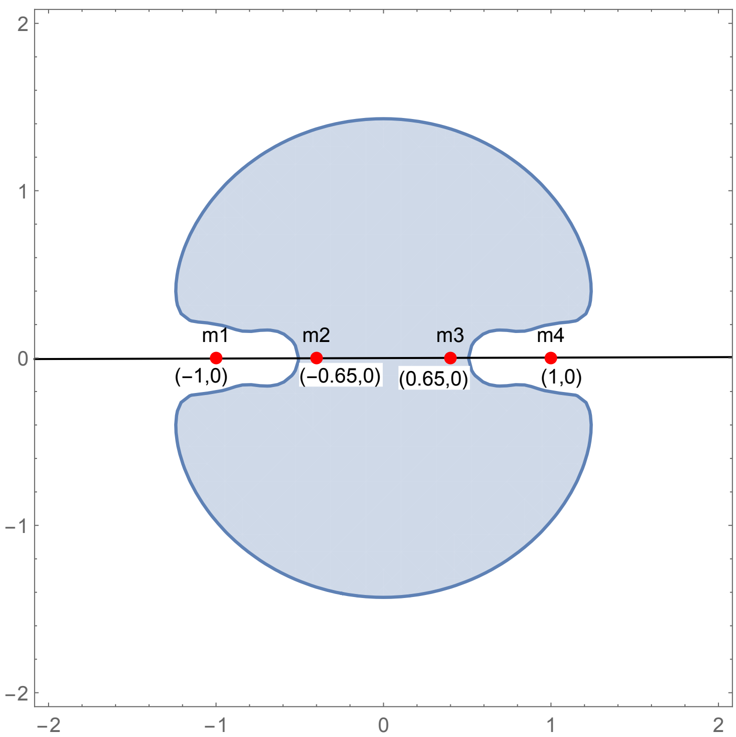

FIGURE 4.14: Permissible regions (white) of motion for $C = 1.25$

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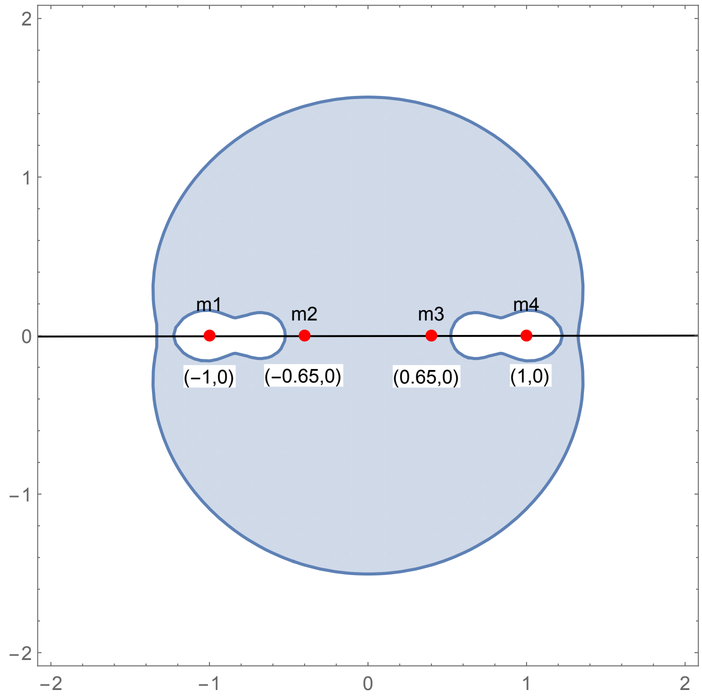

FIGURE 4.15: Permissible regions (white) of motion for $C = 1.35$

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Permissible regions for case-I(1)(iii):

When the value of C from (0.3 to 1.4), for $a = 0.94$ the permissible area is getting shorter.

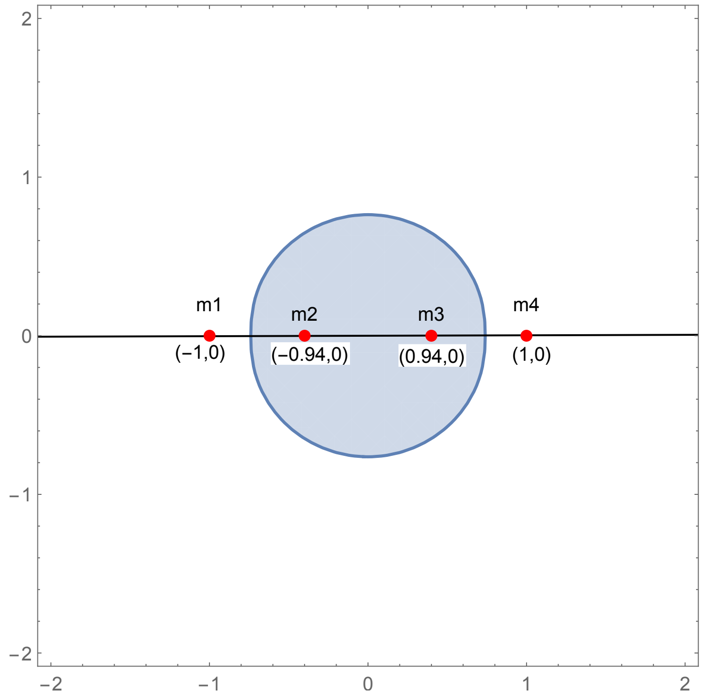

FIGURE 4.16: Permissible regions (white) of motion for $C = 0.3$

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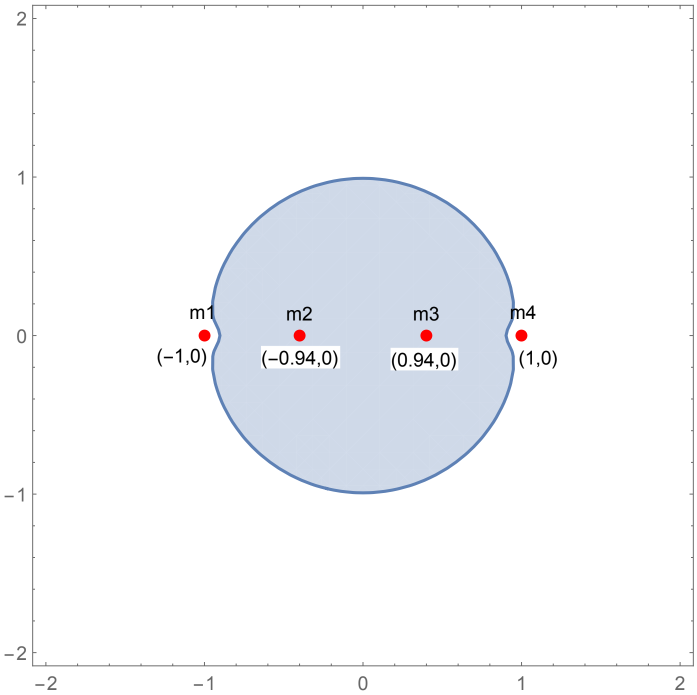

FIGURE 4.17: Permissible regions (white) of motion for $C = 0.5$
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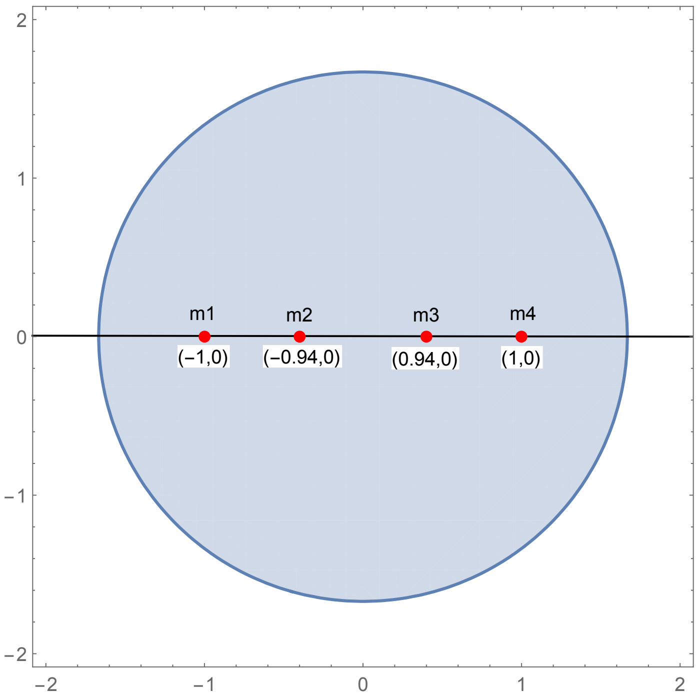

FIGURE 4.18: Permissible regions (white) of motion for $C = 1.4$
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Permissible regions for case-1(1)(iv):

When the value of C from (0.55 to 1.35), for $a = 0.96$ the permissible area is getting shorter.

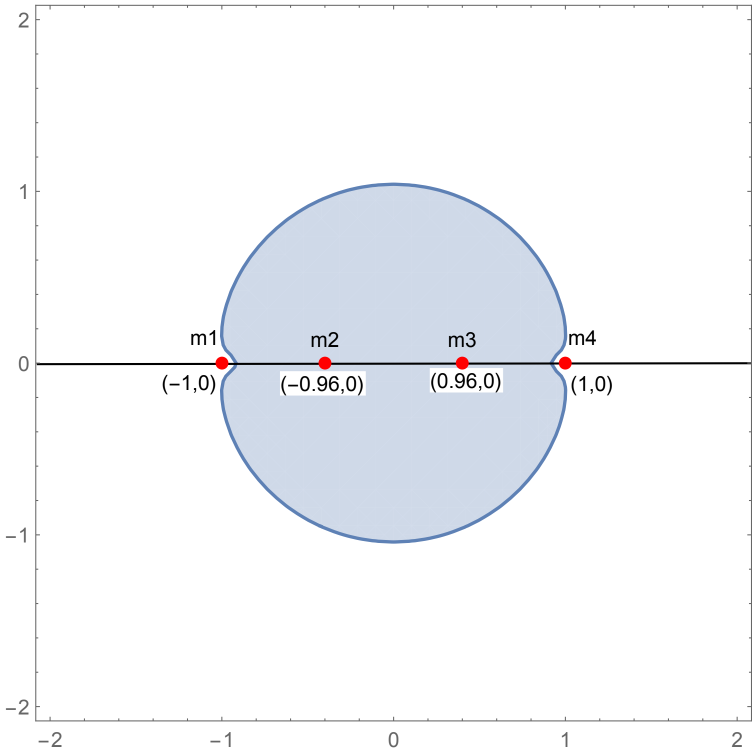

FIGURE 4.19: Permissible regions (white) of motion for $C = 0.55$
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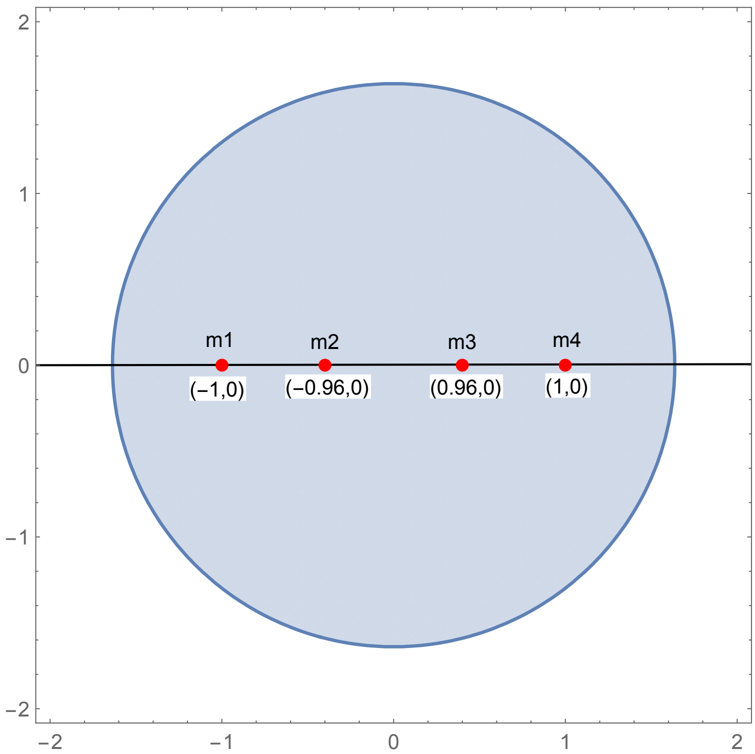

FIGURE 4.20: Permissible regions (white) of motion for $C = 1.35$
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Permissible regions for case-I(2)(i):

When the value of C from (0.4 to 1.5), for $a = 1.15$ the permissible area is getting shorter.

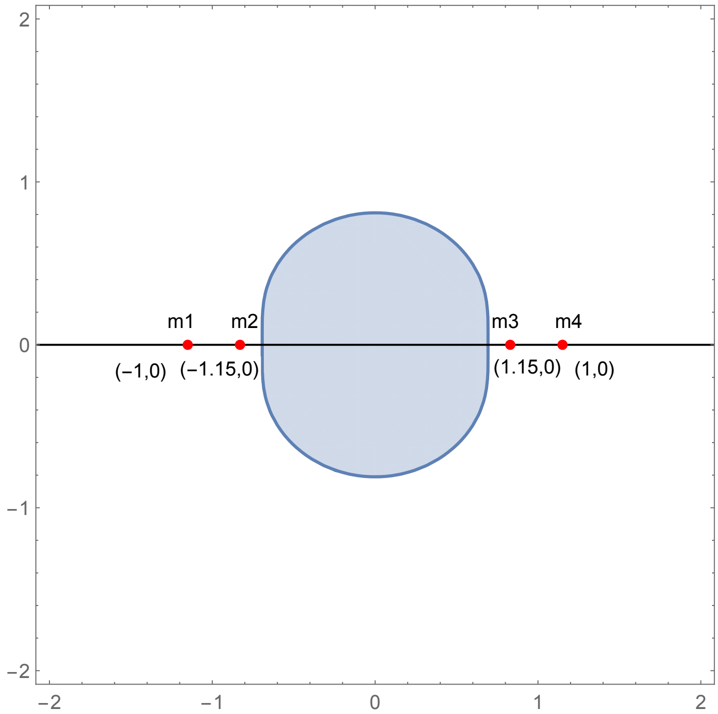

FIGURE 4.21: Permissible regions (white) of motion for $C = 0.4$

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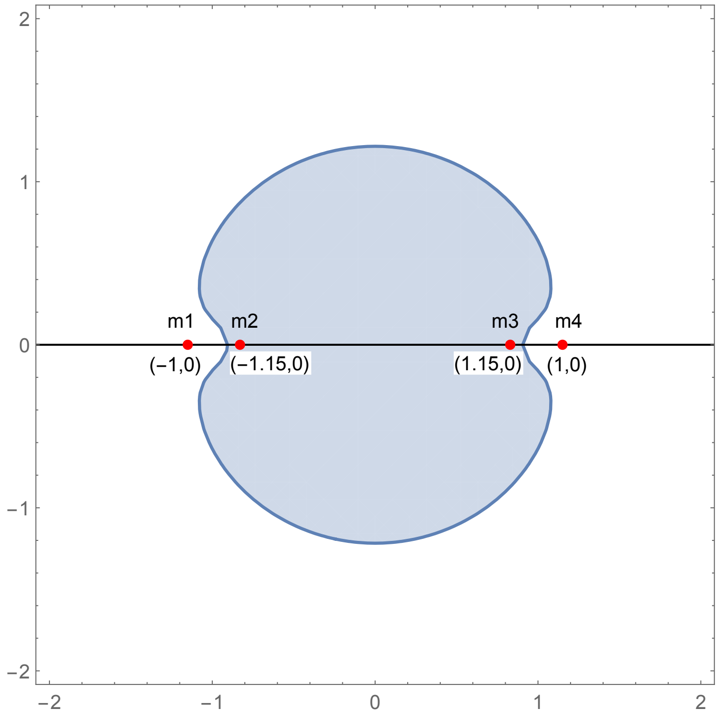

FIGURE 4.22: Permissible regions (white) of motion for $C = 0.8$

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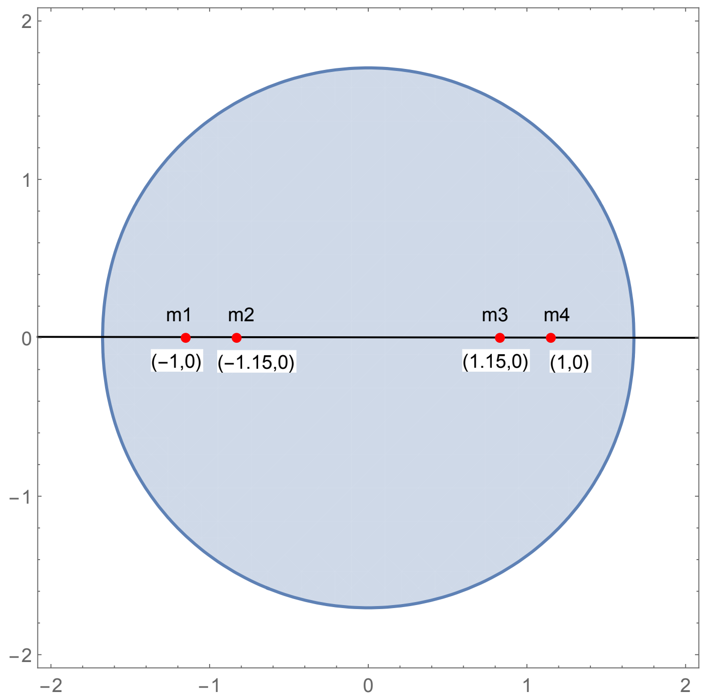

FIGURE 4.23: Permissible regions (white) of motion for $C = 1.5$

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Permissible regions for case-I(2)(ii):

When the value of C from (0.4 to 1.7), for $a = 1.28$ the permissible area is getting shorter.

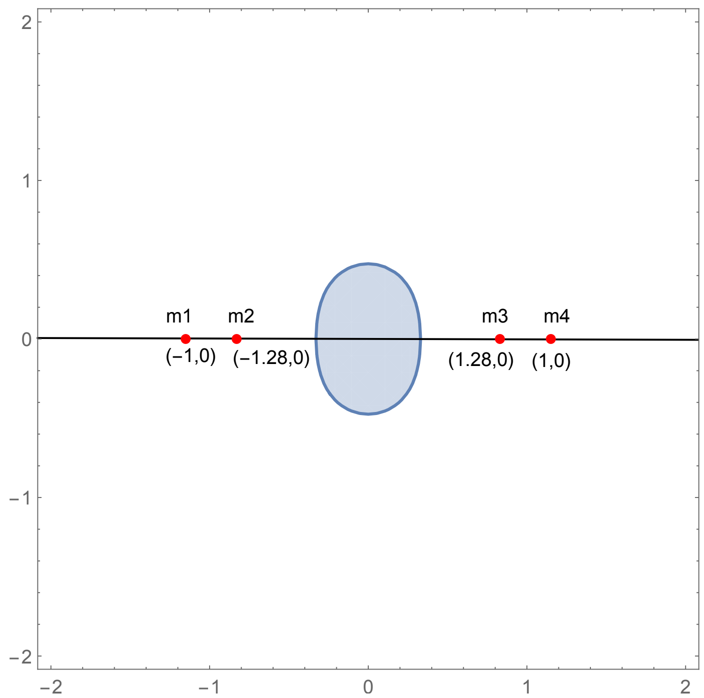

FIGURE 4.24: Permissible regions (white) of motion for $C = 0.4$

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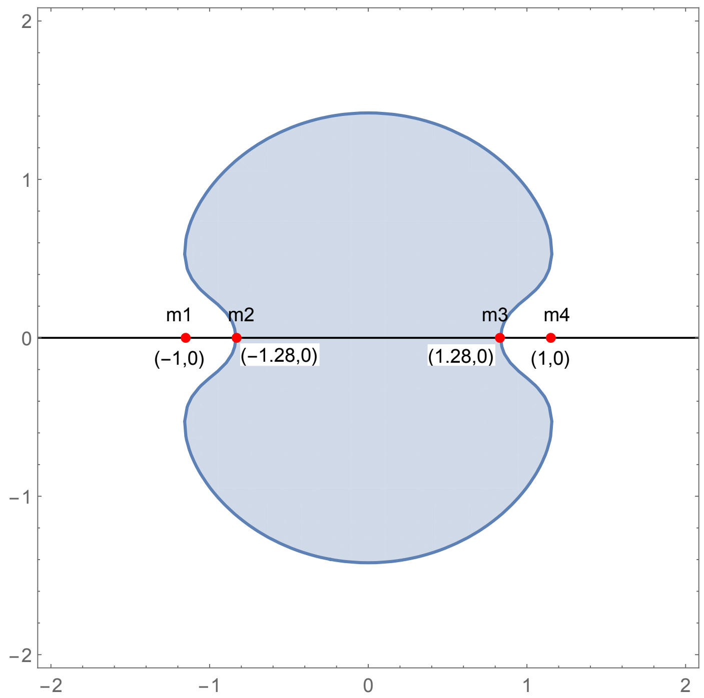

FIGURE 4.25: Permissible regions (white) of motion for $C = 1.2$
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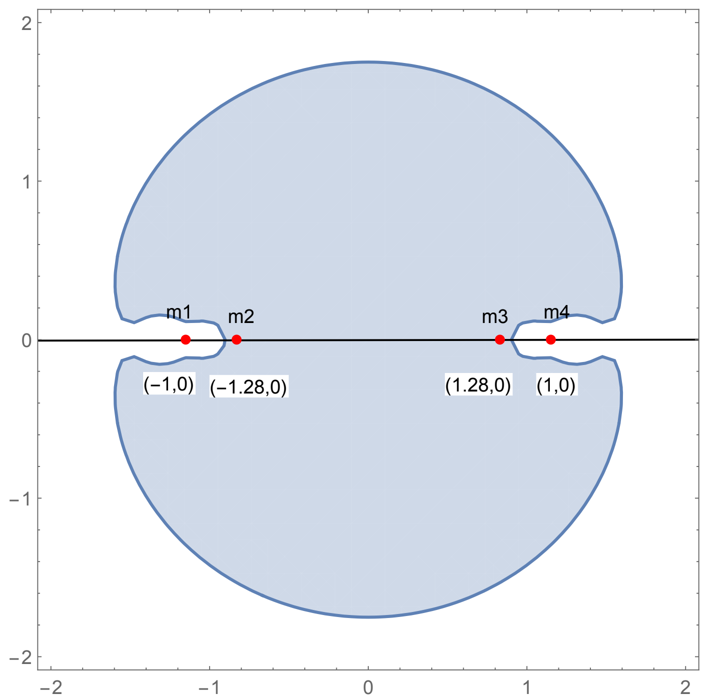

FIGURE 4.26: Permissible regions (white) of motion for $C = 1.7$
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Permissible regions for case-I(2)(iii):

When the value of C from (1.35 to 1.9), for $a = 1.56$ the permissible area is getting shorter.

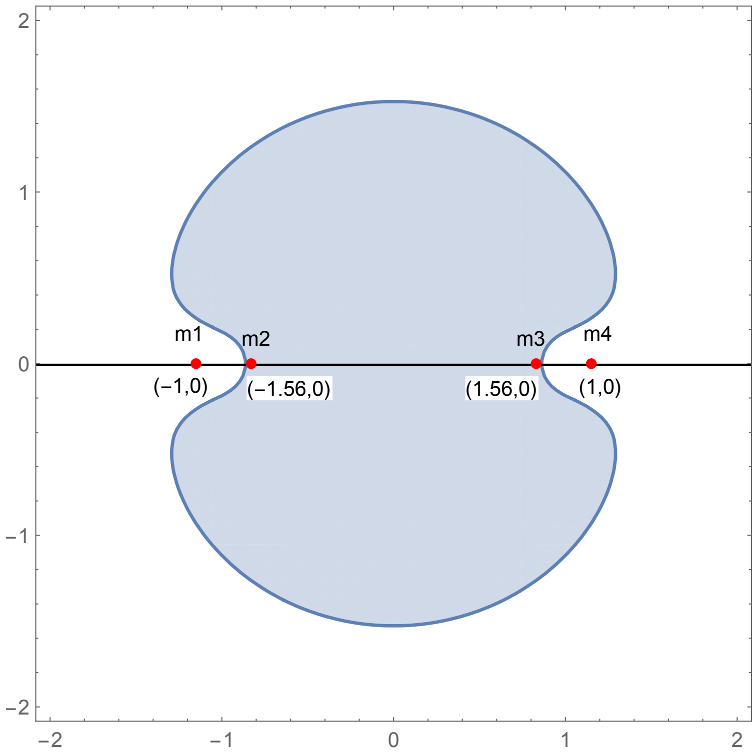

FIGURE 4.27: Permissible regions (white) of motion for $C = 1.35$

.

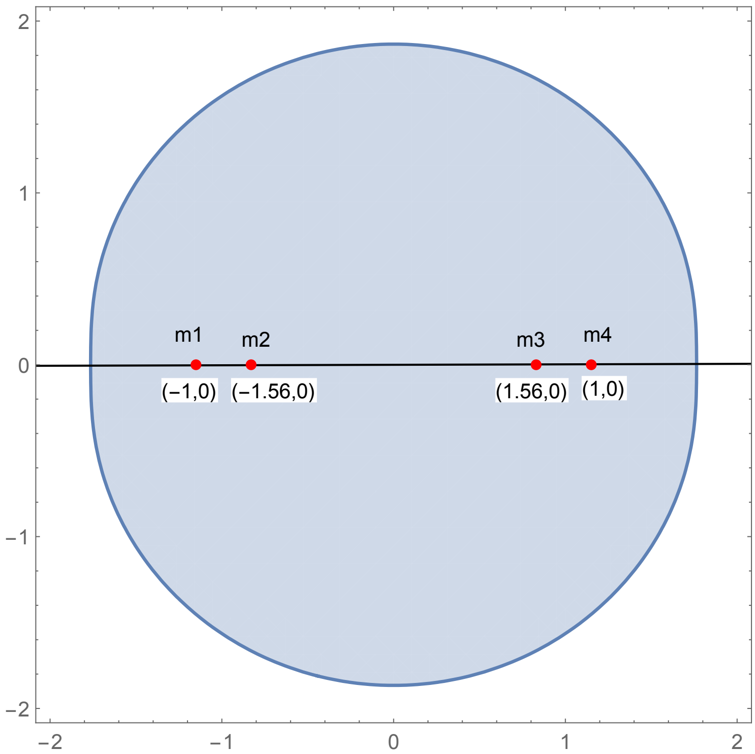

FIGURE 4.28: Permissible regions (white) of motion for $C = 1.9$

.

Permissible regions for case-I(2)(iv):

When the value of C from (5.5 to 5.9), for $a = 2.39$ the permissible area is getting shorter.

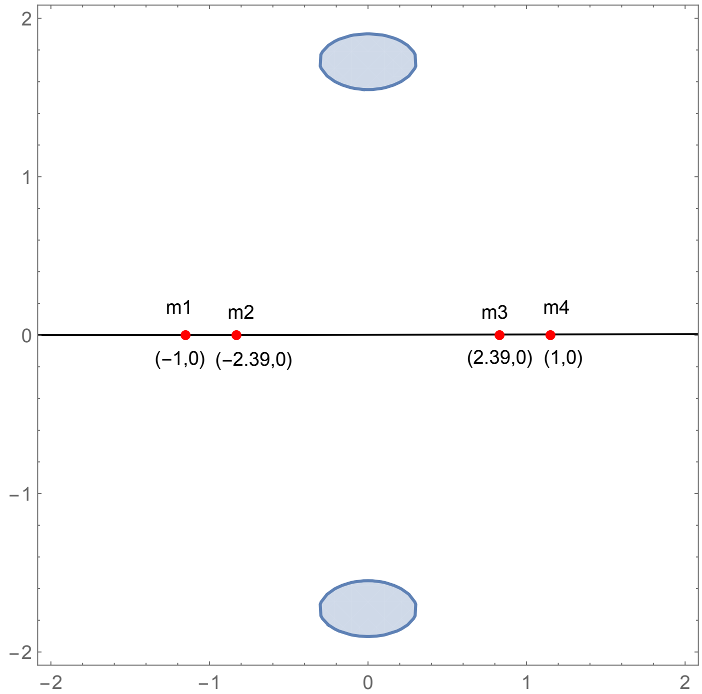

FIGURE 4.29: Permissible regions (white) of motion for $C = 5.5$

.

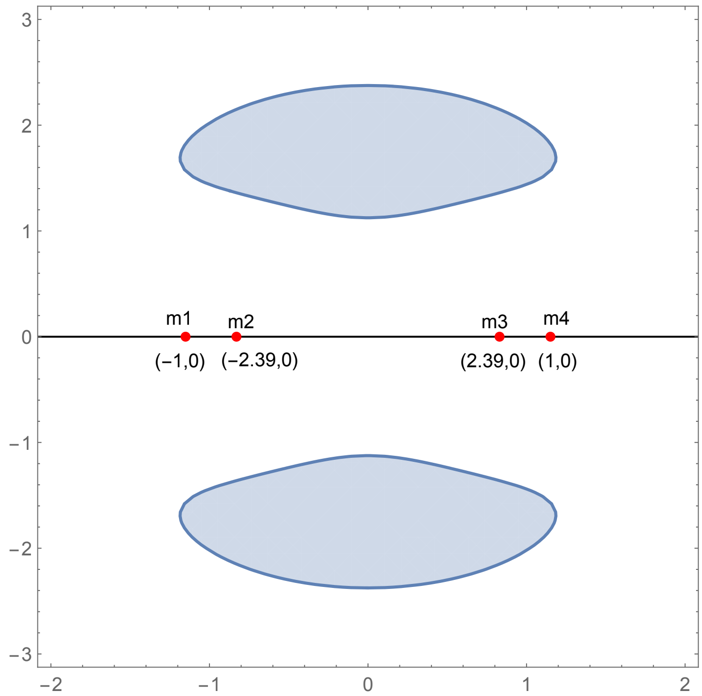

FIGURE 4.30: Permissible regions (white) of motion for $C = 5.9$

.

So we concluded that if the value of C is raised, then the allowable region is short and the infinitesimal mass can not approach the masses.

Chapter 5

Conclusion

In this thesis we discussed four-body collinear configurations with two pairs of equal masses which are symmetric about the center of mass. In this arrangement, we use the symmetry conditions to find the equation of motion. We also studied the motion of an infinitesimal mass m_5 under influence of the gravitational force of four masses m_1, m_2, m_3 and m_4 respectively.
There are two pairs of masses which are at the line, moving in such a way that their central configuration is always in a line. After finding the equation of motion of m_5 of negligible mass (not influence the motion of four masses), we calculated the position of the Lagrange points for different intervals, and examined the stability of Lagrange points for finding eigenvalues using Mathematica. We investigated different intervals and found different Lagrange points in these intervals. The parameters involved in the equation of motion of m_5 were influenced the positions of the Lagrange points. The numerical investigation of these values revealed that all the Lagrange points lying in the xy-plane are unstable.

Lastly, we discussed the Jacobian constant (energy of the infinitesimal mass in rotating frame). The motion of infinitesimal body depends on the value of the Jacobi constant. For choosing different values of Jacobi constant C gives us two different regions. One is permissible region of motion for infinitesimal particle and the second region where motion of infinitesimal particle is not allowed. We also explored all possible region of motion (permissible region) of m_5 by changing the value of C are shown in figures $(4.9 - 4.30)$ for different cases.

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