CAPITAL UNIVERSITY OF SCIENCE AND TECHNOLOGY, ISLAMABAD

Symmetric Collinear Central Configurations for Two Pairs of Equal Masses

by

Hina Warasat

A thesis submitted in partial fulfillment for the degree of Master of Philosophy

in the

Faculty of Computing
Department of Mathematics

2021

First of all, I dedicate this research project to Allah Almighty, The most merciful and beneficent, creator and Sustainer of the earth

And

Dedicated to Prophet Muhammad (peace be upon him) whom, the world where we live and breathe owes its existence to his blessings

And

Dedicated to my parents and Siblings, who pray for me and always pave the way to success for me

And

Dedicated to my teachers, who are a persistent source of inspiration and encouragement for me

CERTIFICATE OF APPROVAL

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Acknowledgement

In the name of Allah, the Most Gracious and the Most Merciful Alhamdulillah, all praises to Creator and Sustainer of the universe for the strengths and His blessing in completing this thesis. This study is nothing, but an effort to understand and articulate the principles of one of the several hundred thousand phenomena, with a tool, the brain, a precious gift from Almighty.

I would like to express my sincerest appreciation to my enthusiastic supervisor, **Dr. Abdul Rehman Kashif** for his supervision, assistance, and immense knowledge. I am sincerely thankful to him for his constant support, motivation, and patience. His invaluable help of constructive comments and suggestions throughout the thesis work have contributed to the success of this research. It has been an amazing experience and I thank him wholeheartedly, not only for his tremendous support, but also for introducing me to the world of research.

I am truly thankful to my teachers at Capital University of Science and Technology, **Dr. Muhammad Sagheer**, **Dr. Rashid Ali**, **Dr. Shafqat Hussain**, **Dr. M. Afzal**, **Dr. Samina Rashid** and **Dr. Dur e Shehwar**. They are excellent teachers and I learnt alot from them throughout the course of my M.Phil study.

My deepest gratitude goes to my beloved parents and siblings for tolerating my mood swings and being patient with me. A special thanks to my father for believing in me, for giving me the determination to overcome many trying moments to pursue my dreams. Heartily thanks to my mother for her love and moral support.

Finally, I am grateful to the management staff of Capital University of Science and Technology, Islamabad for providing a friendly environment for studies.

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Abstract

In this thesis, we set up a collinear four body problem with positive four masses m_1, m_2, m_3 and m_4 respectively. There are two pairs of equal masses, that are symmetric about the center of mass moving in such a way that their configuration is always in a line. The pair of larger masses lie in the middle and the pair of smaller masses lie at the corners each. We find the equations of motion of four massive bodies m_1, m_2, m_3 and m_4. By using the condition of positivity on masses, we obtain the constraints on the distance parameter a and b which are positive. So our central configuration exists for this arrangement of masses. i.e., they will be collinear all the time. We assume that the 5^{th} mass in the same plane of masses is very small and it does not disturb the movement of the four massess m_1, m_2, m_3 and m_4 respectively. We investigate equilibrium points for m_5 and look for the stability of these points. Lastly, we discuss the jacobian constant (i.e., energy of the infinitesimal mass in rotating frame) and region of possible motion of m_5 in central configuration region.

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Abbreviations

2BP	Two-Body Problem
3BP	Three-Body Problem
4BP	Four-Body Problem
5BP	Five-Body Problem
CC	Central Configuration
D	Denominator
$\mathbf{M}_s$	Mass of the Sun
NBP	N-Body Problem
N	Numerator
R3BP	Restricted Three-Body Problem
RC5BP	Restricted Collinear Five-Body Problem
R	Region
SI	System International

Symbols

symbol	Name	Unit
$\mathbf{G}$	Universal gravitational constant	$m^3kg^{-1}s^{-2}$
$\mathbf{F}$	Gravitational force	Newton
r	Distance	Meter
$\mathbf{P}$	Linear momentum	$kg\ m\ s^{-1}$
$\mathbf{L}$	Angular momentum	$kg\ m^2\ s^{-1}$
m_i	Point masses	kg
$\mathbb{R}$	Real number	
$\ni$	Such that	
$\forall$	For all	
$\in$	Belongs to	

Chapter 1

Introduction

Celestial mechanics is the branch of astronomy that deals with the motions of objects in outer space. Historically, celestial mechanics applies principles of physics (classical mechanics) to astronomical objects, such as stars and planets, to produce ephemeris data. Actually celestial mechanics is the science devoted to the study of the motion of the celestial bodies on the basis of the laws of gravitation. It was founded by Newton and it is the oldest of the chapters of Physical Astronomy. Celestial mechanics in the solar system is ultimately an n-body problem, but the special configurations [1] and relative smallness of the perturbations have allowed quite accurate descriptions of motions (valid for limited time periods) with various approximations and procedures without any attempt to solve the complete problem of n bodies. All bodies have arbitrary masses, initial velocities and positions in the general n-body problem, and the bodies interact with the Newton law of gravity and attempt to establish the subsequent motion of all bodies. Many numerical solutions have been successfully developed for the motion of quite a large number of gravitating particles where accurate movement is usually less important than the statistical behaviour of the group. In classical mechanics, the two-body problem is to predict the motion of two massive objects which are abstractly viewed as point particles. The problem assumes that the two objects interact only with one another; the only force influencing each object is the gravitational force arises from the other one, and all other objects are ignored.

The three-body problem is the problem [2] of taking the initial positions and velocities (or momenta) of three point masses and solving for their subsequent motion according to Newton's laws of motion and Newton's law of universal gravitation. Alexis Clairaut succeeded in providing an approximation for the 3BP after twenty years of Isaac Newton's death. In 1767, Euler found collinear motions, in which three bodies of any masses move proportionately along a fixed straight line. The Euler's three-body problem is the special case in which two of the bodies are fixed in space (this should not be confused with the circular restricted three-body problem, in which the two massive bodies describe a circular orbit and are only fixed in a synodic reference frame).

In the 19th century, many famous astronomers and mathematicians such as Carl Gustav Jacob Jacobi, Lagrange and Euler [3] worked on NBP. The general solution to the problem remained unsolved until 1991, when Qiudong Wang, a professor at the University of Arizona published "The global solution of n-body Problem" [4]. At the same time Leonhard Euler also works on the 3BP. In classical mechanics, the two-body problem is to predict the motion of two massive objects which are abstractly viewed as point particles. The problem assumes that the two objects interact only with one another; the only force affecting each object is the gravitational force arises from the other one, and all other objects are ignored. Inspired by the circular restricted three-body problem, the four-body problem can be greatly simplified by considering a smaller body to have a small mass compared to the other three massive bodies, which in turn are approximated to describe circular orbits. This is known as the bicircular restricted four-body problem (also known as bicircular model) and it can be traced back to 1960 in a NASA report written by Su-Shu Huang [5]. This formulation has been highly relevant in the astrodynamics, mainly to model spacecraft trajectories in the Earth-Moon system with the addition of the gravitational attraction of the Sun. The former formulation of the bicircular restricted four-body problem can be problematic when modelling other systems that are the Earth-Moon-Sun, so the formulation was generalized by Negri and Prado to expand the application range and improve the accuracy without loss of simplicity.

1.1 Central Configuration

A central configuration (CC) is a special arrangement of point masses interacting by Newton's laws of gravitation with the following property "*the acceleration of the ith mass must be proportional to its position (relative to the center of mass of the system)*"; thus, $\ddot{\mathbf{r}}_i = \lambda \mathbf{r}_i \; \forall \; i = 1, 2, 3, ..., n$. CC is common and basic concept in the study of NBP. The first non-trivial example of CC were discussed by Euler [6] who studied three bodies in a line. When two masses are equal, one can get a CC by putting an arbitrary mass at their mid point. For three unequal masses it is not obvious that CC exists. But Euler showed that there is exactly one equivalance class collinear CC for each possible ordering of masses in a line . Lagrange found [7] CC for an equilateral triangle, not only for the unequal mass, but also for any three different masses. Moreover it is the only non-collinear CC for 3BP. Moulton [8] first published linear solutions to the NBP. On a straight line Moulton put n masses so that they remained collinear and then solved the mass value problem at n arbitrary collinear points. Palmore [9] proposed many theorems in the study of points of equilibrium in the planar NBP [10].

The two most popular methods of restriction are to ignore one body mass and introduce symmetries of some kind. Papadakis and Kanavos [11] studied the restricted photo gravitational 5BP [12]. For any four equal distant particles under the gravitational attraction, they investigate the movement of a massless object on a sphere. Ouyang and Xie [13] discussed several special analysis solutions for four body problems, showing that such solutions reduce the Copenhagen problem to Lagrange solutions when two masses are similarly reduced. Zhiming and Yisui [14] investigated the finiteness of the central configurations for the general 4-body problem. It was shown that there are 12 central configurations for each of the masses for the collinear four-body problem. Roy and Steve [15] addressed several special theoretical approaches about the four-body problems. We will address the existence of a continuous family of balancing solutions for the above mentioned four-body collinear problems, which involve two symmetrical configurations of two pairs of masses.

1.1.1 Restricted Few-Body Problem

In restricted three body problem (R3BP), a body of negligible mass moves under the gravitational field of two massses bodies [16]. In R3BP there exist five equilibrium points three collinear (unstable) and two non-collinear (stable) point. The restricted three body problem is a very well known problem in celesial mechanics. It has great theoretical, practical and historical background. It has been greatly studied by many mathematician, astronomer and physist in different aspect in last five decades [17]. A systematic analysis of periodic orbits was done in the problem of the two-dimensional, elliptic, R3PB [18]. The position and stability of the five points of equilibrium in the planar, circular R3BP is investigated when a variety of studies of drag forces act on the third body [19]. For equal masses, Yan *et al.* studied the existence and linear stability of periodic orbits [20]. In the (R3BP), the presence of transversal ejection-collision orbits discussed [21]. Conley *et.al.*, discussed new long periodic solutions in plane, of the R3PB [22]. Simmons and Bakker gave analysis (linear stability) of a rhomboidal 4BP and show that collisions (isolated binary) can be regularized at origin [23]. Prokopenya discussed the stability of the equilibrium solutions in the elliptic restricted many-body problem [24]. Planar central configurations of the four-body problem with three equal masses discussed in [25]. Santos discussed each equilibrium solution must be defined by the primaries along a diagonal [26].

1.2 Thesis Contribution

We set up a collinear four body problem with positive masses m_1, m_2, m_3 and m_4 respectively. There are two pairs of equal masses, that are symmetric about the center of mass moving in such a way that their configuration is always in a line. There is small mass m_5 moving in the same plane under the influence of gravity of four masses. We analyze the possible positions of equilibrium points/Lagrange points and their stability in gravitational field of masses. Lastly, we discuss the jacobian constant for infinitesimal mass in the presence of big masses.

1.3 Thesis Outline

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This thesis is further composed of four chapters:

Chapter 2 demonstrates some important definitions, concept and laws that are helpful in understandingthe present work.

Chapter 3 interprets a review analysis of the study performed by [27].

Chapter 4 discuss the dyanamic of 5th body under the gravitational field of m_1 to m_4 which are colinear.

Chapter 5 summarizes the whole study and includes the conclusion arising from the entire discussion.

References used in the thesis are mentioned in **Biblography**.

Chapter 2

Preliminaries

We need to recall the basic definitions, concepts, terminologies and laws from existing literature related to our research work.

2.1 Definitions

Definition 2.1.1. Mechanics

"Mechanics is the branch of physics dealing with the study of motion. It is sometimes further subdivided into:

1. **Kinematics**, "Kinematics is the branch of mechanics concerned with the motion of objects without reference to the forces which cause the motion."

2. **Dynamics**, "Dynamics is the branch of mechanics concerned with the motion of bodies under the action of forces."

3. **Statics** "Statics is the branch of mechanics concerned with the conditions under which no motion is apparent." [28]

Definition 2.1.2. Velocity

"The velocity of an object is the rate of change of its position with respect to a frame of reference, and is a function of time.”[29]

Definition 2.1.3. Acceleration

“Acceleration is a vector quantity that is defined as the rate at which an object changes its velocity. ”[29]

Definition 2.1.4. Force

“Force is an external agent capable of changing the state of rest or motion of a particular body. It has a magnitude and a direction.”[29]

Definition 2.1.5. Central Force

“Suppose that a force acting on a particle of mass m is such that it is always directed from m towards or away from a fixed point O and its magnitude depends only on the distance r from O. Then we call the force a central force or central force field with the O as the center of the force field. Mathematically, F is central force if and only if,

$$\mathbf{F} = f(r)\mathbf{r}_1 = f(r)\frac{\mathbf{r}}{r},$$

where $\mathbf{r}_1 = \frac{\mathbf{r}}{r}$ is a unit vector in the direction of $\mathbf{r}$. The central force is one of attraction towards O or repulsion from O according as $f(r) < 0$ or $f(r) > 0$ respectively.”[29]

Definition 2.1.6. Degree of Freedom

“The number of coordinates required to specify the position of a system of one or more particles is called number of degree of freedom of the system. Example: A particle moving freely in space requires 3 coordinates, e.g. $(x; y; z)$, to specify its position. Thus the number of degree of freedom is 3.”[28]

Definition 2.1.7. Scalar

“Scalars are quantities that are fully described by a magnitude (or numerical value) alone e.g speed, pressure, mass, density, energy etc."[28]

Definition 2.1.8. Vector

"Vectors are quantities that are fully described by both a magnitude and a direction. e.g., velocity, momentum, force etc."[28]

Definition 2.1.8. Field

"A field is a physical quantity associated with every point of space time. The physical quantity may be either in vector form, scalar form or tensor form."[28]

Definition 2.1.9. Scalar Field

"If at every point in a region, a scalar function has a defined value, the region is called a scalar field. i.e.,

$$f : \mathbb{R}^3 \longrightarrow \mathbb{R},$$

e.g., Temperature and pressure fields around the earth."[28]

Definition 2.1.10. Vector Field

"If at every point in a region, a vector function has a defined value, the region is called a vector field.

$$f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3,$$

e.g., Tangent vector around a smooth curve."[28]

Definition 2.1.11. Conservative Vector Field

"A vector field $\mathbf{V}$ is conservative if and only if there exists a contentiously differentiable scalar field f such that $\mathbf{V}$ = -∇f or equivalently if and only if,

$$\nabla \times \mathbf{V} = curl\mathbf{V} = \mathbf{0}."$$

Definition 2.1.12. Tensor Field

"If at every point in a region, a tensor function has a defined value, the region is called a tensor field. e.g. Riemann curvature tensor, Stress-energy-momentum tensor, Electromagnetic tensor."[28]

Definition 2.1.13. Uniform Force Field

"A force field which has constant magnitude and direction is called a uniform or constant force field. If the direction of the field is taken as negative z direction and magnitude is constant $F_0 > 0$, then the force field is given by:

$$\mathbf{F} = -F_0\hat{\mathbf{k}}.\text{''}[28]$$

Definition 2.1.14. Center of Mass

"Let $r_1, r_2, ..., r_n$ be the position vector of a system of n particles of masses $m_1, m_2, ..., m_n$ respectively. The center of mass or centroid of the system of particles is defined as that point having position vector,

$$\hat{\mathbf{r}} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + ... + m_n\mathbf{r}_n}{m_1 + m_2.... + m_n} = \frac{1}{M}\sum_{i=1}^{n} m_i\mathbf{r}_i,$$

where

$$M = \sum_{i=1}^{n} m_i,$$

is the total mass of the system."[28]

Definition 2.1.15. Center of Gravity

"If a system of particles is in a uniform gravitational field, the center of mass is sometimes called the center of gravity."[28]

Definition 2.1.16. Torque

"If a particle with a position vector r moves in a force field $\mathbf{F}$, we define τ as torque or moment of the force as:

$$\tau = \mathbf{r} \times \mathbf{F}.$$

The magnitude of τ is,

$$\tau = rF\sin\theta.$$

The magnitude of torque is a measure of the turning effect produced on the particle by the force."[28]

Definition 2.1.17. Momentum

"The linear momentum p of an object with mass m and velocity v is defined as:

$$\mathbf{P} = m\mathbf{v},$$

under certain circumstances the linear momentum of a system is conserved. The linear momentum of a particle is related to the net force acting on that object:

$$\mathbf{F} = m\mathbf{a} = m\frac{d\mathbf{v}}{dt} = \frac{d}{dt}(m\mathbf{v}) = \frac{d\mathbf{P}}{dt}.$$

The rate of change of linear momentum of a particle is equal to the net force acting on the object, and is pointed in the direction of the force. If the net force acting on an object is zero, its linear momentum is constant (conservation of linear momentum). The total linear momentum p of a system of particles is defined as the vector sum of the individual linear momentum:

$$\mathbf{P} = \sum_{i=1}^{n} \mathbf{P}_i."$$

Definition 2.1.18. Point-like Particle

"A point-like particle is an idealization of particles mostly used in different fields of physics. Its defining features is the lacks of spatial extension:being zero-dimensional, it does not take up space. A point-like particle is an appropriate representation of an object whose structure, size and shape is irrelevant in a given context. e.g., from far away, a finite-size mass (object) will look like a point-like particle."[28]

Definition 2.1.19. Angular Momentum

"The cross product of position vector of a rotating body about axis of rotation and linear momentum is called angular momentum. Mathematically

$$\mathbf{L} = \mathbf{r} \times \mathbf{P},$$

where r is the vector from the point O to the particle. The torque about the point O acting on the particle is equal to the rate of change of the angular momentum about the point O of the particle i.e.,

$$\tau = \frac{d\mathbf{L}}{dt}."$$

Definition 2.1.20. Holonomic and Non-holonomic Constraints

"The limitations on the motion are often called constraints. If the constraint condition can be expressed as an equation:

$$\phi(\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_n, t) = \mathbf{0},$$

connecting the position vector of the particles and the time, then the constraints are called holonomic. If it cannot be so expressed it is called non-holonomic."[28]

Definition 2.1.21. Inertial Frame of Reference

"A frame of reference that remains at rest or moves with constant velocity with respect to other frames of reference is called inertial frame of reference. Actually, an unaccelerated frame of reference is an inertial frame of reference.

In this frame of reference a body does not acted upon by external forces. Newton's laws of motion are valid in all inertial frames of reference. All inertial frames of reference are equivalent."[28]

2.1.26. Non-Inertial Frame of Reference

"A non-inertial reference frame is a frame of reference that is undergoing acceleration with respect to an inertial frame. While the laws of motion are the same in all inertial frames, in non-inertial frames, they vary from frame to frame depending on the acceleration."[28]

Definition 2.1.22. Lagrange Points

"A point in space where a small body with negligible mass under the gravitational influence of two large bodies will remain at rest relative to the larger ones. These points are locations in an orbital arrangement of two large bodies where a third smaller body, affected solely by gravity, is capable of maintaining a stable position relative to the two larger bodies. A lagrange point is also known as a equilibrium point and Liberation point named after a French mathematician and atronomer Joseph-Louis Lagrange. He was first to find these equlibrium points for the earth, sun, and moon system. He found five points out of these three are collinear."[30]

Definition 2.1.23. Equilibrium Solution

"The Equilibrium solution can guide us through the behaviour of the equation that represents the problem without actually solving it. These solutions can be found only if we meet the sufficient condition of all rates equal to zero. If we have two variables then

$$\dot{x} = \dot{y} = \ddot{x} = \ddot{y} = ... = x^{(n)} = y^{(n)} = 0.$$

These solutions may be stable or unstable. The stable solutions regarding in celestial Mechanics helps us find parking spaces where if a satellite or any object placed, it will remain there for ever. These type of places are also found along the Jupiter's orbital path where bodies called trojan are present. These equilibrium points with respect to Celestial Mechanics are also called Lagrange points named after a French mathematician and astronomer Joseph-Louis Lagrange. He was first to find these equilibrium points for the Sun-Earth system. He found that three of these five points were collinear.

2.1.23.1. Procedure for Stability Analysis and Equilibrium Points:

We need to follow the following steps to check the stability of equilibrium points.

1) Determine the equilibrium points, $\mathbf{x}^*$, solving $\Omega(\mathbf{x}^*) = \mathbf{0}$.
2) Construct the Jacobian matrix, $J(\mathbf{x}^*) = \frac{\partial \Omega}{\partial \mathbf{x}^*}$.
3) Compute eigenvalues of $\Omega(\mathbf{x}^*)$: $det|\Omega(\mathbf{x}^*) - \lambda\ I| = 0$.
4) Stability or instability of $\mathbf{x}^*$ based on the real parts of eigenvalues.
5) Point is stable, if all eigenvalues have real parts negative.
6) Unstable, If at least one eigenvalue has a positive real part.
7) Otherwise, there is no conclusion, (i.e, require an investigation of higher order terms)."[30]

Definition 2.1.24. Galilean Transformation

"In physics, a Galilean transformation is used to transform between the coordinates of two reference frames which differ only by constant relative motion within the constructs of Newtonian physics. Without the translations in space and time the group is the homogeneous Galilean group.
Galilean transformations, also called Newtonian transformations, set of equations in classical physics that relate the space and time coordinates of two systems moving at a constant velocity relative to each other.[30]"

2.2 Newton's Laws of Motion

"The following three laws of motion given by Newton are considered the axioms of mechanics:

1. **First Law of Motion**

   Every body continues in its state of rest, or uniform motion in a straight line, unless compelled to change that state by forces impressed on it.

2. **Second Law of Motion**

   If $\mathbf{F}$ is the external force acting on a particle of mass m which as a reaction is moving with velocity $\mathbf{v}$, then

$$\mathbf{F} = \frac{d}{dt}(m\mathbf{v}) = \frac{d\mathbf{P}}{dt}.$$

   If m is independent of time this becomes

$$\mathbf{F} = m\frac{d}{dt}(\mathbf{v}) = m\mathbf{a},$$

   where $\mathbf{a}$ is the acceleration of the particle.

3. **Third Law of Motion**

   To every action there is always opposed an equal reaction; or, the mutual actions of two bodies on each other are always equal, and directed to contrary parts."[30]

2.3.1. Newton's Universal Law of Gravitation

"Every particle of matter in the universe attracts every other particle of matter with a force which is directly proportional to the product of the masses and inversely proportional to the square of the distance between them. Hence, for any Preliminaries 12 two particles separated by a distance r, the magnitude of the gravitational force $\mathbf{F}$ is:

$$\mathbf{F} = G\frac{m_1 m_2}{r^2}\mathbf{r},$$

where G is universal gravitational constant. Its numerical value in SI units is $6.67408 \times 10^{-11} m^3 kg^{-1} s^{-2}$."[30]

2.3 Kepler's Laws of Planetary Motion

"Kepler's three laws of planetary motion can be described as follows:

1. The planetary orbits are all ellipses that are confocal with the Sun (i.e., the Sun lies at one of the foci of each ellipse.

2. The radius vector connecting each planet to the Sun sweeps out equal areas in equal time intervals.

3. The square of the orbital period of each planet is proportional to the cube of its orbital major radii. Mathematically, Kepler's third law can be written as:

$$T^2 = \left(\frac{4\pi^2}{GM_s}\right)r^3,$$

   where T is the time period, r is the semi major axis, M_s is the mass of sun and G is the universal gravitational constant." [30]

2.4 Two Body Problem

The two-body problem(2BP) first studied and solved by Newton, states: "Suppose that the positions and velocities of two massive bodies moving under their mutual gravitational force are given at any time t, then what should their position and velocities be for any other time t, if the masses are known? Example include a planet orbiting around a star (Earth-Sun, Moon-Earth), two stars orbiting around each other, satellite orbiting around orbit."[31]

2.5 Three Body Problem

"In the three-body problem, three bodies move in space under their mutual gravitational interactions as described by Newton's theory of gravity. Solutions of this problem require that future and past motions of the bodies be uniquely determined based solely on their present positions and velocities. In general, the motions of the bodies take place in three dimensions (3D), and there are no restrictions on their masses nor on the initial conditions. Thus, we refer to this as the general three-body problem."[31]

2.6 The Equations of Motion in the N-Body Problem

"The 2BP deals with much of the essential work in astrodynamics, but we also have to model the real world by including other bodies. The next logical step, then, is to generate 3BP formulas. The n-body problem is a further generalization of three body problems. In general, it takes a fixed number of integration constants to solve general differential equations of movements in the n-body problem. Consider a basic gravity question in which over time we have constant acceleration, $a(t) = a_0$. We get the velocity, $v(t) = a_0 t + v_0$, if we integrate this equation. Again integrating provides, $r(t) = r_0 + v_0 t + \frac{1}{2} a_0 t^2$. In order to complete the solution, the initial conditions must be known. This example is a straight-forward analytical approach using the initial values, or a function of integration time and constants, called movement integrals. Unfortunately, this isn't always the easy scenario. If initial conditions alone do not provide a solution, the order of differential equations, often called the degrees of freedom of the dynamic system, can be lowered by integrals of motion. Ideally, if the number of integrals equals the order of differential equations, we can reduce it to order zero. These integrals are constant functions of the initial conditions, as well as the position and velocity of at any time, hence the term constants of the motion.

For the n-body problem, a system of $3n$ second order differential equations, we need $6n$ integrals of motion for a complete solution. Conservation of linear momentum provides six, conservation of energy one, and conservation of total angular momentum three, for a total of ten. There are no laws analogous to Kepler's first two laws to obtain additional constants, thus we are left with a system of order $6n - 10$ for $n \geq 3$.

These equations for n bodies $n \geq 3$, deny all attempts at closed-form solutions. H. Brun, in 1887, showed that there were no other algebraic integrals. Although Poincareé later generalized Brun's work, we still have only the ten known integrals. They give us insight into the motions within the three body and n-body problems. Conservation of total linear momentum assumes no external forces are on the system.

First, here we set up the equations of motions of n massive particles of masses $m_i (i = 1, 2, 3, ..., n)$" their mutual radius vectors are given by r_{ij} where,

$$\mathbf{r}_{ij} = \mathbf{r}_j - \mathbf{r}_i. \tag{2.1}$$

"From Newton's laws of motion and the law of gravitation,

$$m_i \ddot{\mathbf{r}}_i = G \sum_{j=1, j \neq i}^{n} \frac{m_i m_j}{r_{ij}^3} \mathbf{r}_{ij}, \text{"} \tag{2.2}$$

here we note that $\mathbf{r}_{ij}$ implies that the vector between m_i and m_j is directed for m_i to m_j, thus

$$\mathbf{r}_{ij} = -\mathbf{r}_{ji}. \tag{2.3}$$

The set of equations are the required equation of motion for n-body problem G being the constant of gravitation."[31]

Chapter 3

Symmetric Collinear Equilibrium Configurations for Two Pairs of Equal Masses

In this research work, we set up a collinear four body problem [27] which involves two symmetrical arrangements of two pairs of masses. The masses are m_1, m_2, m_3 and m_4 respectively. The pair of larger masses lie in the middle and the pair of smaller masses lie at the corners each. Consider the positive masses moving in such away that their configuration is always a straight line. The geometric configuration can be seen in Figure 3.1.

3.1 Problem Formulation

Suppose that the masses, m_1, m_2, m_3 and m_4 which are fixed at $(-b, 0)$,$(-a, 0)$, $(a, 0)$ and $(b, 0)$, respectively. Consider the positive masses $m_1 = m_4 = m$ and $m_2 = m_3 = M$ moving in such a way that their configuration is always a straight line. We assume that the 5^{th} mass in the same plane of masses is very small and it does not disturb the movement of the four massess m_1, m_2, m_3 and m_4 respectively.

Our problem involves two sections: First, the characterization of the central configuration of symmetric collinear equilibrium configurations and secondly, discuss the motion of the 5^{th} body of mass m_5, finding equilibrium points for m_5 and look for the stability of these points, (which is discussed in the next chapter in detail).

3.2 Characterization of the Collinear Configuration

The classical equation of motion for the n-body problem has the form

$$m_i\ddot{\mathbf{r}}_i = \sum_{j=0 j\neq i}^{n} m_i m_j \frac{(r_j - \mathbf{r}_i)}{|\mathbf{r}_j - \mathbf{r}_i|^3}, \qquad i = 1, ...n, \tag{3.1}$$

where the units are chosen so that universal gravitational constant G is equal to one. A central configuration is a particular configuration of the n bodies where the acceleration vector of each body is proportional to its position vector, and the constant of proportionality is the same for the n bodies. Therefore, a central configuration is a configuration that satisfies the equation

$$-\omega^2(\mathbf{r}_i - \mathbf{c}) = \sum_{j=0 j\neq i}^{n} m_i m_j \frac{(r_j - \mathbf{r}_i)}{|\mathbf{r}_j - \mathbf{r}_i|^3}, \qquad i = 1, ...n, \tag{3.2}$$

where ω is a angular speed and c is the center of mass.
For four bodies, we put $n = 4$ in equation (3.2), we will get central configuration (CC) for general four-body problem (4BP) equations are given below: For four bodies, we put $n = 4$ in equation (3.2), we will get central configuration (CC) for general four-body problem (4BP) equations are given below:

$$\frac{m_2(r_2 - \mathbf{r}_1)}{|\mathbf{r}_2 - \mathbf{r}_1|^3} + \frac{m_3(r_3 - \mathbf{r}_1)}{|\mathbf{r}_3 - \mathbf{r}_1|^3} + \frac{m_4(r_4 - \mathbf{r}_1)}{|\mathbf{r}_4 - \mathbf{r}_1|^3} = -\omega^2(\mathbf{r}_1 - \mathbf{c}), \tag{3.3}$$

$$\frac{m_1(r_1 - \mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|^3} + \frac{m_3(r_3 - \mathbf{r}_2)}{|\mathbf{r}_3 - \mathbf{r}_2|^3} + \frac{m_4(r_4 - \mathbf{r}_2)}{|\mathbf{r}_4 - \mathbf{r}_2|^3} = -\omega^2(\mathbf{r}_2 - \mathbf{c}), \tag{3.4}$$

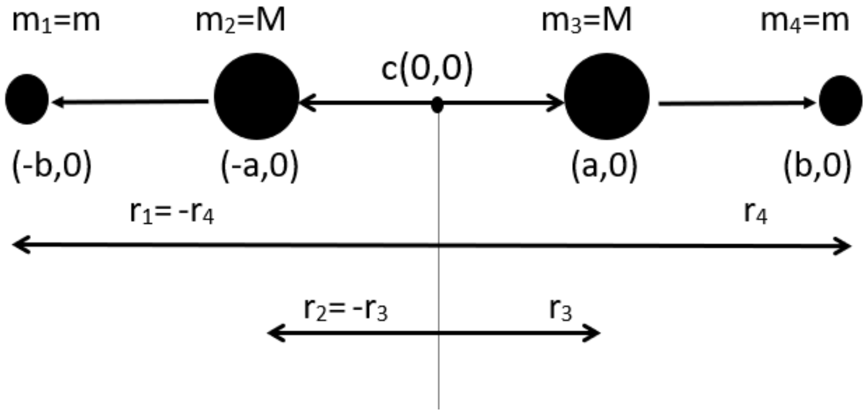

FIGURE 3.1: Symmetric collinear equilibrium configuration

$$\frac{m_1(r_1 - \mathbf{r}_3)}{|\mathbf{r}_1 - \mathbf{r}_3|^3} + \frac{m_2(r_2 - \mathbf{r}_3)}{|\mathbf{r}_2 - \mathbf{r}_3|^3} + \frac{m_4(r_4 - \mathbf{r}_3)}{|\mathbf{r}_4 - \mathbf{r}_3|^3} = -\omega^2(\mathbf{r}_3 - \mathbf{c}), \tag{3.5}$$

$$\frac{m_1(r_1 - \mathbf{r}_4)}{|\mathbf{r}_1 - \mathbf{r}_4|^3} + \frac{m_2(r_2 - \mathbf{r}_4)}{|\mathbf{r}_2 - \mathbf{r}_4|^3} + \frac{m_3(r_3 - \mathbf{r}_4)}{|\mathbf{r}_3 - \mathbf{r}_4|^3} = -\omega^2(\mathbf{r}_4 - \mathbf{c}). \tag{3.6}$$

Suppose that the masses, m_1, m_2, m_3 and m_4 which are fixed at $\mathbf{r}_1 = (-b, 0)$, $\mathbf{r}_2 = (-a, 0)$, $\mathbf{r}_3 = (a, 0)$, and $\mathbf{r}_4 = (b, 0)$, respectively Figure 3.1, with the above conditions the center of mass of four collinear masses will shift at the origin.
i.e.,

$$\mathbf{c} = (0, 0) \tag{3.7}$$

Using $m_1 = m_4 = m$ $m_2 = m_3 = M$ and equation (3.8) in equations (3.4) – (3.7) become

$$\frac{bm}{4|b|^3} + \frac{(-a+b)M}{|-a+b|^3} + \frac{(a+b)M}{|a+b|^3} - b\omega^2 \ = 0, \tag{3.8}$$

$$\frac{aM}{4|a|^3} + \frac{(a-b)m}{|a-b|^3} + \frac{(a+b)m}{|a+b|^3} - a\omega^2 \ = 0, \tag{3.9}$$

$$-\frac{aM}{4|a|^3} + \frac{(-a-b)m}{|-a-b|^3} + \frac{(-a+b)m}{|-a+b|^3} + a\omega^2 \ = 0, \tag{3.10}$$

$$\frac{(-a-b)M}{|-a-b|^3} + \frac{(a-b)M}{|a-b|^3} + \frac{bm}{4|b|^3} + b\omega^2 \ = 0. \tag{3.11}$$

Due to symmetry, equation (3.9) is same as equation (3.12) and equation (3.10) is same to equation (3.11). Therefore, we are left the following two equations:

$$\frac{m}{4b^2} + \frac{(-a+b)M}{(-a+b)^2} + \frac{M}{(a+b)^2} - b\omega^2 \ = 0, \tag{3.12}$$

$$\frac{M}{4a^2} + \frac{m}{(a-b)^2} + \frac{m}{(a+b)^2} - a\omega^2 \ = 0. \tag{3.13}$$

Solving Eq. (3.13) and (3.14) for m and M and taking ω=1 (without loss of generality), we get

$$m = f_1(a,b)/f_2(a,b), \tag{3.14}$$

where

$$f_1(a,b) = 4(a-b)^2b^2(a+b)^2(-4a^4+4a^5+5a^3b+8a^4b+a^2b^2+4a^3b^2-ab^3-b^4),$$

and

$$\begin{aligned} f_2(a,b) =& a^7+a^6b-35a^5b^2+32a^6b^2+29a^4b^3+64a^5b^3-29a^3b^4 \\ &+64a^4b^4+35a^2b^5+64a^3b^5-ab^6+32a^2b^6-b^7. \end{aligned}$$

$$M = g_1(a,b)/g_2(a,b), \tag{3.15}$$

where

$$g_1(a,b) = 4(a^{10}+a^9b-3a^8b^2-11a^7b^3-5a^6b^4+a^4b^6+7a^3b^7+8a^2b^8),$$

and

$$\begin{aligned} g_2(a,b) =& a^7+a^6b-35a^5b^2+32a^6b^2+29a^4b^3+64a^5b^3-29a^3b^4+64a^4b^4 \\ &+35a^2b^5+64a^3b^5-ab^6+32a^2b^6-b^7. \end{aligned}$$

Our next goal is to check the positivities of the masses m and M, which are defined in equation (3.15) and (3.16) i.e., we need to find the values of a and b for which our masses are positive. Because the masses are functions of distance parameters a and b, so we need to find the values of a and b, for which m and M are positive. Here we take $b = 1$ (without loss of generality) and solving the masses expressions for a, we get the following interval for a, for which m and M are positive.

i. $0.473223 < a < 1$

ii. $a > 2.39681$

when we take $a = 1$ (without loss of generality) and solving the masses expressions for b, we get following interval for b.

i. $0 < b < 0.417221$

ii. $1 < b < 2.67857$

for which m and M are positive. Here we define the following proposition.

3.2.1 Proposition

We have two cases:

Case(I): If $b = 1$ (without loss of generality) then $0.473223 < a < 1$ or $a > 2.39681$.

Case(II): When $a = 1$(without loss of generality) then $0 < b < 0.417221$ or $1 < b < 2.67857$.

Chapter 4

Dynamics of 5^{th} Body

4.1 Introduction

In this section we discuss the dynamic of 5^{th} particle in the plane moving according to the gravitational field formed by the attraction of four masses moving in a straight line as shown in the previous section. The motion of m_5 will not effect the gravitational field of m_1,m_2,m_3 and m_4, because $m_5 << m_1, m_2, m_3$ and m_4. This problem called symmetric collinear equilibrium configurations for two pairs of equal masses. Equation of motion of 5^{th} particles,

$$\ddot{\mathbf{r}}_5 = m_1 \frac{\mathbf{r}_1 - \mathbf{r}_5}{|\mathbf{r}_1 - \mathbf{r}_5|^3} + m_2 \frac{\mathbf{r}_2 - \mathbf{r}_5}{|\mathbf{r}_2 - \mathbf{r}_5|^3} + m_3 \frac{\mathbf{r}_3 - \mathbf{r}_5}{|\mathbf{r}_3 - \mathbf{r}_5|^3} + m_4 \frac{\mathbf{r}_4 - \mathbf{r}_5}{|\mathbf{r}_4 - \mathbf{r}_5|^3}. \tag{4.1}$$

Now introduce a coordinate system that is rotating about the center of mass with uniform angular speed ω. Let (x, y) be the coordinates of m_5 in this new rotating frame (non-inertial frame). We can convert equation (4.1) from fixed inertial frame to the rotating coordinates system with the following orthogonal transformation,

$$\mathbf{e}_1 = e^{iwt}, \quad \mathbf{e}_2 = ie^{iwt},$$

where "t" represents time. The position vector of m_5 in the rotating frame is,

$$\mathbf{r}_5 = x(t)\ \mathbf{e}_1 + y(t)\ \mathbf{e}_2, \tag{4.2}$$

choosing $\omega = 1$, (without loss of generality) and taking first and second derivatives of equation (4.2) yield,

$$\left.\begin{aligned} \dot{\mathbf{r}}_5 =& (\dot{x} - y)e^{it} + i(x + \dot{y})e^{it}, \\ \ddot{\mathbf{r}}_5 =& (\ddot{x} - 2\dot{y} - x)e^{it} + i(\ddot{y} + 2\dot{x} - y)e^{it}. \end{aligned}\right\} \tag{4.3}$$

Using equation (4.3) in equation (4.1), the planner equations of motion of m_5 in rotating frame in component form are,

$$\ddot{x} - 2\dot{y} = x - \left[m\left(\frac{x+b}{r_{51}^3} + \frac{x-b}{r_{54}^3}\right) + M\left(\frac{x+a}{r_{52}^3} + \frac{x-a}{r_{53}^3}\right)\right], \tag{4.4}$$

$$\ddot{y} + 2\dot{x} = y - \left[m\left(\frac{1}{r_{51}^3} + \frac{1}{r_{54}^3}\right)y + M\left(\frac{1}{r_{52}^3} + \frac{1}{r_{53}^3}\right)y\right], \tag{4.5}$$

where mutual distances are described as,

$$\left.\begin{aligned} r_{51} &= \sqrt{(x+b)^2 + y^2}, \\ r_{52} &= \sqrt{(x+a)^2 + y^2}, \\ r_{53} &= \sqrt{(x-a)^2 + y^2}, \\ r_{54} &= \sqrt{(x-b)^2 + y^2}. \end{aligned}\right\} \tag{4.6}$$

Multiply Equation (4.4) by $\dot{x}$ and Equation (4.5) by $\dot{y}$ to obtain

$$\ddot{x}\dot{x} - 2\dot{x}\dot{y} - x\dot{x} = -\left[m\dot{x}\left(\frac{x+b}{r_{51}^3} + \frac{x-b}{r_{54}^3}\right) + M\dot{x}\left(\frac{x+a}{r_{52}^3} + \frac{x-a}{r_{53}^3}\right)\right], \tag{4.7}$$

$$\ddot{y}\dot{y} - 2\dot{x}\dot{y} - y\dot{y} = -\left[m\dot{y}\left(\frac{1}{r_{51}^3} + \frac{1}{r_{54}^3}\right)y + M\dot{y}\left(\frac{1}{r_{52}^3} + \frac{1}{r_{53}^3}\right)y\right]. \tag{4.8}$$

Adding the left and right sides of these equations to get

$$\begin{aligned}\ddot{x}\dot{x}+\ddot{y}\dot{y}+\ddot{z}\dot{z}-(x\dot{x}+y\dot{y}) &= -\frac{m}{r_{51}^{3}}\left(\dot{x}x+b\dot{x}+y\dot{y}\right)-\frac{m}{r_{54}^{3}}\left(\dot{x}x-b\dot{x}+y\dot{y}\right)\\ &\quad -\frac{M}{r_{51}^{3}}\left(\dot{x}x+a\dot{x}+y\dot{y}\right)-\frac{M}{r_{51}^{3}}\left(\dot{x}x-a\dot{x}+y\dot{y}\right), \end{aligned} \tag{4.9}$$

Note that

$$\ddot{x}\dot{x}+\ddot{y}\dot{y}+\ddot{z}\dot{z}=\frac{1}{2}\frac{d}{dt}\left(\dot{x}^{2}+\dot{y}^{2}+\dot{z}^{2}\right)=\frac{1}{2}\frac{dv^{2}}{dt}, \tag{4.10}$$

where v is the speed of the secondary mass relative to the rotating frame. Similarly,

$$x\dot{x}+y\dot{y}=\frac{1}{2}\frac{d}{dt}(x^{2}+y^{2}), \tag{4.11}$$

From equation (4.6) gives

$$r_{51}^{2}=(x+b)^{2}+y^{2},$$

Therefore

$$2r_{51}\frac{dr_{1}}{dt}=2(x+b)\dot{x}+2y\dot{y},$$

or

$$\frac{dr_{1}}{dt}=\frac{1}{r_{51}}(x\dot{x}+x\dot{x}+y\dot{y}).$$

It follows that

$$\frac{d}{dt}\frac{1}{r_{51}}=-\frac{1}{r_{51}^{2}}\frac{dr_{1}}{dt}=-\frac{1}{r_{51}^{3}}(x\dot{x}+y\dot{y}+b\dot{x}). \tag{4.12}$$

Similary, we can obtain the following expressions

$$\frac{d}{dt}\frac{1}{r_{52}}=-\frac{1}{r_{52}^{3}}(x\dot{x}+y\dot{y}+a\dot{x}), \tag{4.13}$$

$$\frac{d}{dt}\frac{1}{r_{53}}=-\frac{1}{r_{53}^{3}}(x\dot{x}+y\dot{y}-a\dot{x}), \tag{4.14}$$

$$\frac{d}{dt}\frac{1}{r_{54}} = -\frac{1}{r_{54}^3}(x\dot{x} + y\dot{y} - b\dot{x}). \tag{4.15}$$

Using equations (4.10)-(4.15) into equation (4.9) yields

$$\frac{1}{2}\frac{dv^2}{dt} - \frac{1}{2}\frac{d}{dt}(x^2+y^2) = m\frac{d}{dt}\frac{1}{r_{51}} + m\frac{d}{dt}\frac{1}{r_{54}} + M\frac{d}{dt}\frac{1}{r_{52}} + M\frac{d}{dt}\frac{1}{r_{53}},$$

Alternatively, upon rearranging terms the above equation becomes

$$\frac{d}{dt}\left[\frac{1}{2}v^2 - \frac{1}{2}(x^2+y^2) - m\left(\frac{1}{r_{51}} + \frac{1}{r_{54}}\right) - M\left(\frac{1}{r_{52}} + \frac{1}{r_{53}}\right)\right] = 0,$$

which means the bracketed expression is a constant

$$\frac{1}{2}v^2 - \frac{1}{2}(x^2+y^2) - m\left(\frac{1}{r_{51}} + \frac{1}{r_{54}}\right) - M\left(\frac{1}{r_{52}} + \frac{1}{r_{53}}\right) = C. \tag{4.16}$$

The constant C is known as the jacobi constant,(named after the German mathematician Carl Jacobi who discovered it in 1836). Jacobi's constant [32] may be interpreted as the total energy of the m_5 relative to the rotating frame. C is a constant of the motion of the m_5 in the collinear restricted five body problem,here

- $\frac{1}{2}v^2$ is the kinetic energy per unit mass relative to the rotating frame.

- $\frac{m}{r_{51}}$,-$\frac{M}{r_{52}}$,-$\frac{M}{r_{53}}$, and -$\frac{m}{r_{54}}$ are gravitational potential energies of the masses.

- $\frac{1}{2}(x^2+y^2)$ interpreted as the potential energy of the centrifugal force of m_5 induced by the rotation of the refrence frame.

rewriting equation

$$v^2 = (x^2+y^2) + 2m\left(\frac{1}{r_{51}} + \frac{1}{r_{54}}\right) + 2M\left(\frac{1}{r_{52}} + \frac{1}{r_{53}}\right) + 2C. \tag{4.17}$$

Since v^2 cannot be negative,it must be true that,

$$(x^2 + y^2) + 2m\left(\frac{1}{r_{51}} + \frac{1}{r_{54}}\right) + 2M\left(\frac{1}{r_{52}} + \frac{1}{r_{53}}\right) + 2C \geq 0. \tag{4.18}$$

Trajectories of the secondary body in regions where this inequality is violated are not allowed. The boundaries between forbidden and allowed regions of motion are found by setting $v^2 = 0$, i.e.,

$$(x^2 + y^2) + 2m\left(\frac{1}{r_{51}} + \frac{1}{r_{54}}\right) + 2M\left(\frac{1}{r_{52}} + \frac{1}{r_{53}}\right) + 2C = 0. \tag{4.19}$$

For a given value of the Jacobi constant the curves of zero velocity are determined by this equation. These boundaries cannot be crossed by a infinitesimal mass (spacecraft) moving within an allowed region.

4.2 Equilibrium Solutions

The equations (4.4) and (4.5), do not have an analytical solution or a closed form solution. we need to analyze the qualitative analysis of equations (4.4) and (4.5) to determine the location of the equilibrium points. These are the places in space where the infinitesimal mass m_5 would have zero velocity and acceleration,i.e, where m_5 appears at rest permanently relative to the primaries m_1,m_2,m_3 and m_4 respectively. when located at an equilibrium point (also called libration point / Lagrange point), a body will apparently stay there. These solutions can be found only if we meet the sufficient condition of all rates equal to zero,

$$\dot{x} = \dot{y} = \ddot{x} = \ddot{y} = 0.$$

To find the equilibrium points / Lagrange point, we need to solve these equations numerically or drawing contour plot using Mathematica. The classification of equilibrium points/ Lagrange points for symmetric collinear equilibrium configurations for two pairs of equal masses are,

4.3 Case-I(1): When $a \in (0.47, 1)$ and $b = 1$

i. When $a \in (0.5\ ,\ 0.76)$ there exist 7 equilibrium points.

ii. When $a \in (0.76\ ,\ 0.79)$ there exist 5 equilibrium points.

iii. When $a \in (0.79\ ,\ 0.88)$ there exist 3 equilibrium points.

iv. When $a \in (0.88.\ ,\ 0.99)$ there exist 0 equilibrium points.

In addition, these intervals can be discussed in cases compared to other similar equilibrium points. The intersections of the non-linear algebraic equations $U_x = 0$, and $U_y = 0$ define the positions of the equilibrium points.

The intersections of $U_x = 0$ (blue) and $U_y = 0$ (orange), respectively. The red dots represent the position of the masses and black dots represent the position of equilibrium points.

4.3.1 Seven Equilibrium Points

We start our analysis with the first case for equilibrium points where seven equilibrium points exist on the x-axis and y-axis. It is numerically checked that the behaviour of equilibrium points do not change throughout in $(0.5, 0.76)$, therefore we choose $a = 0.6 \in (0.5\ ,\ 0.76)$ and draw contour plots for the infinitesimal particle and check the position of equilibrium points.

4.3.1.1 Contour-Plot

For $a = 0.6$ the corresponding value of b , M and m are 1 , 0.268 and 0.054 respectively. Contour plot for these values shows that the equilibrium points L_1, L_2,L_3,L_4 and L_5 are collinear along the x-axis L_6 and L_7 are collinear along the y-axis.

In Figure 4.1 the black dot denote the position of equilibrium points and red dots represent the position of masses m_1 to m_4. The equilibrium points are the points of the intersections of $U_x = 0$ (blue) and $U_y = 0$ (orange).

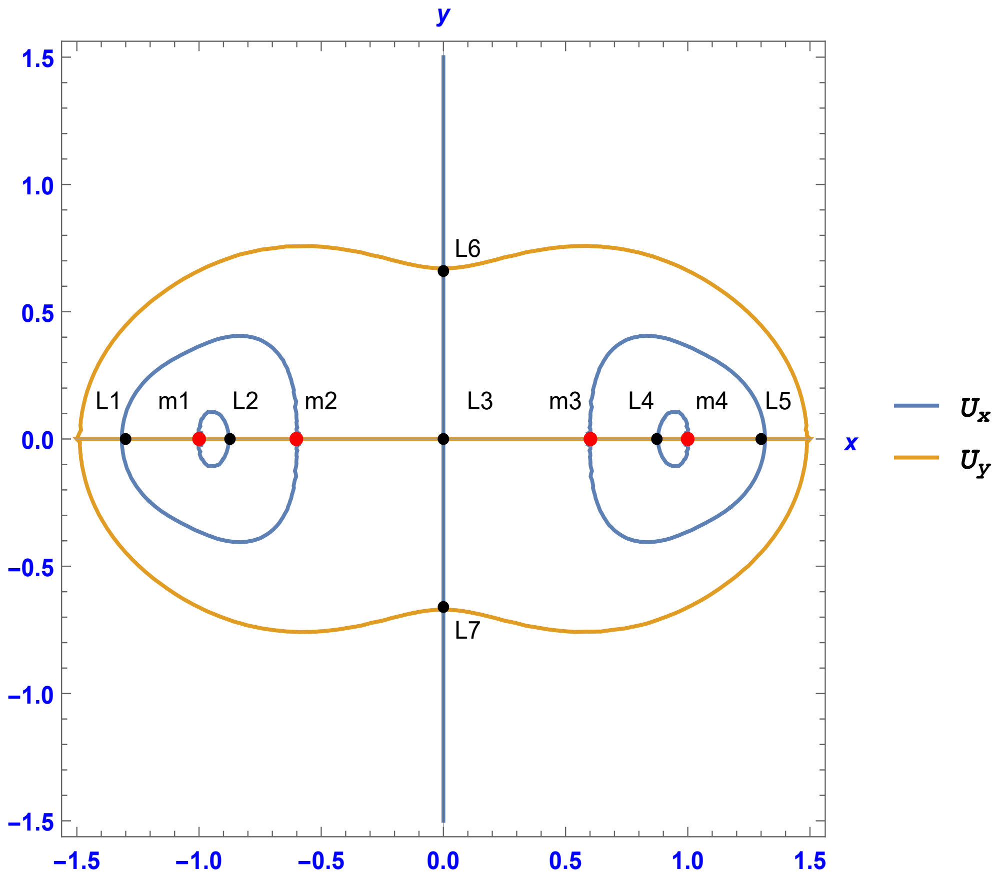

FIGURE 4.1: Positions of the masses (Red dots); equilibrium points of m_5 (Black dots). Here $a = 0.6$ and the corresponding masses are $m = 0.0359$, $M = 0.2123$.

4.3.2 Five Equilibrium Points

We continue our analysis with the second case for equilibrium points where five equilibrium points exist on the x-axis and y-axis. It is numerically checked that the behaviour of equilibrium points do not change throughout in $(0.76, 0.79)$, therefore we choose $a = 0.78 \in (0.76\ ,\ 0.79)$ and draw contour plots for the infinitesimal particle and check the position of equilibrium points.

4.3.2.1 Contour-Plot

For $a = 0.78$ the corresponding value of b , M and m are 1 , 0.2123 and 0.0359 respectively. Contour plot for these values shows that the equilibrium points L_1, L_2,L_3,L_4 and L_5 are collinear along the x-axis. In Figure 4.2 the black dot denote the position of equilibrium points and red dots represent the position of masses

m_1 to m_4. The equilibrium points are the points of the intersections of $U_x = 0$ (blue) and $U_y = 0$ (orange).

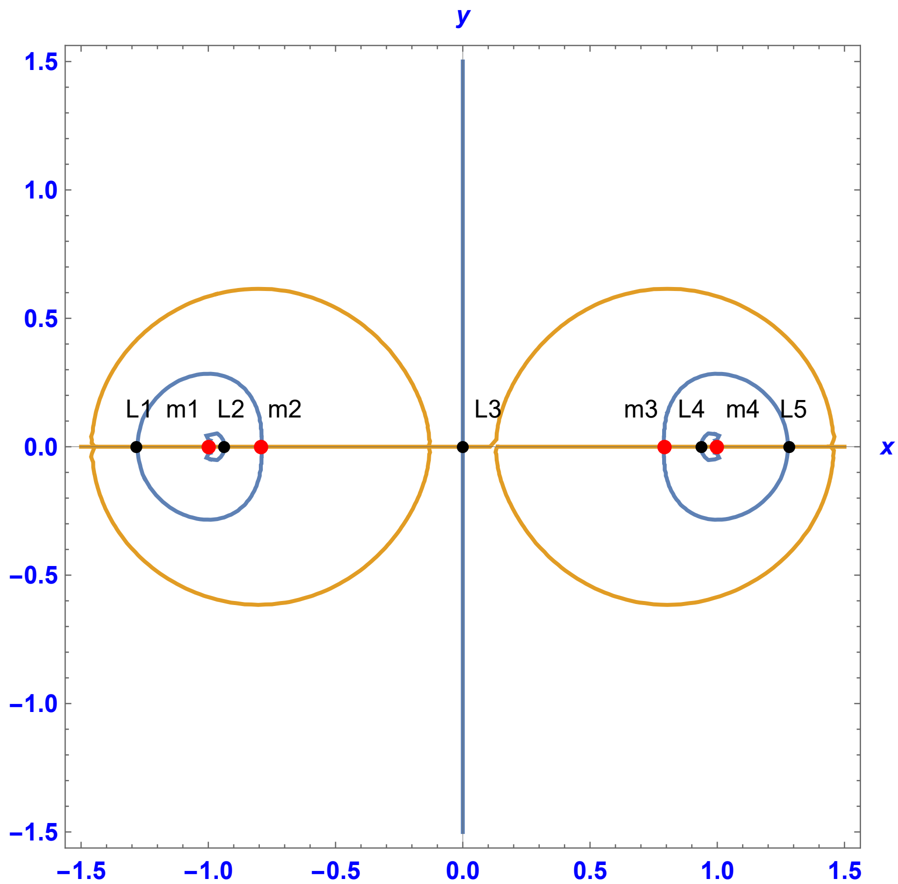

FIGURE 4.2: Positions of the masses (Red dots); equilibrium points of m_5 (Black dots). Here $a = 0.78$ and the corresponding masses are $m = 0.0359$, $M = 0.2123$.

4.3.3 Three Equilibrium Points

We present our analysis with the third case for equilibrium points where three equilibrium points exist on the x-axis and y-axis. It is numerically checked that the behaviour of equilibrium points do not change throughout in $(0.79, 0.88)$, therefore we choose $a = 0.85 \in (0.79\ ,\ 0.88)$ and draw contour plots for the infinitesimal particle and check the position of equilibrium points.

4.3.3.1 Contour-Plot

For $a = 0.85$ the corresponding value of b , M and m are 1 , 0.1175 and 0.012 respectively. Contour plot for these values shows that the equilibrium points L_1, L_2 and L_3 are collinear along the x-axis. In Figure 4.3 the black dot denote the position of equilibrium points and red dots represent the position of masses m_1 to m_4. The equilibrium points are the points of the intersections of $U_x = 0$ (blue) and $U_y = 0$ (orange).

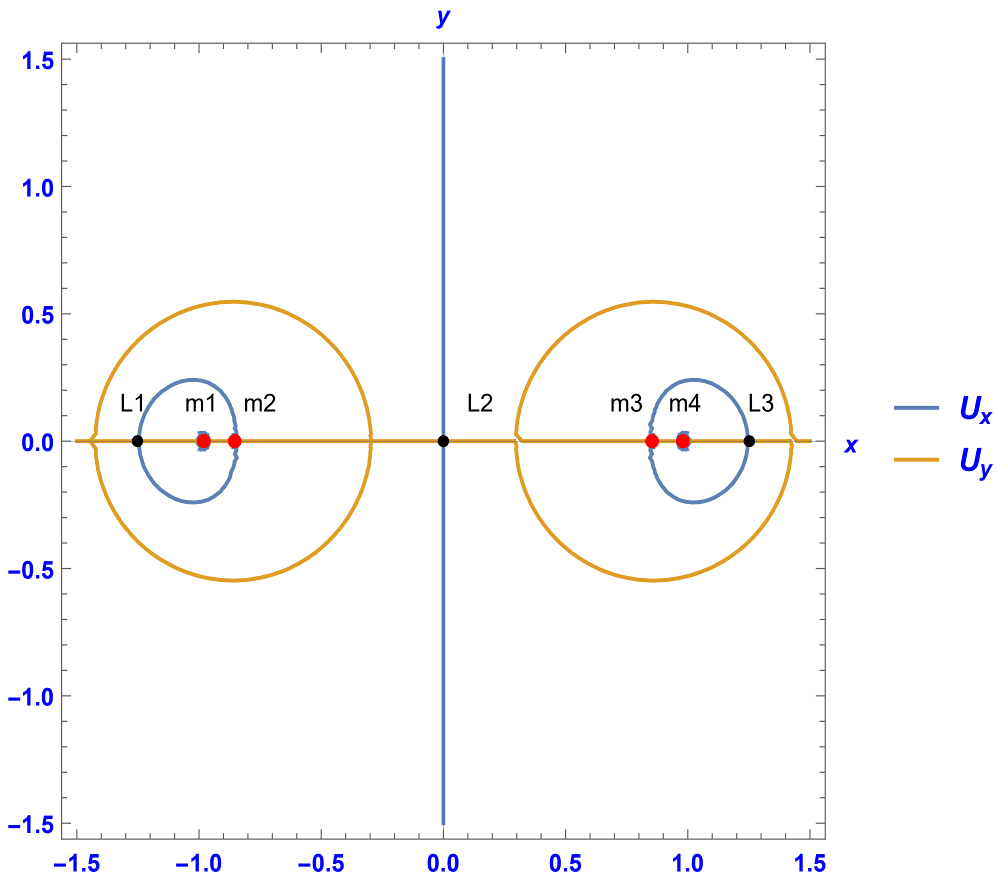

FIGURE 4.3: Positions of the masses (Red dots); equilibrium points of m_5 (Black dots). Here $a = 0.85$ and the corresponding masses are $m = 0.012$, $M = 0.1175$.

4.3.4 Zero Equilibrium Points

We further continue our analysis with the foruth case for equilibrium points where no equilibrium point exist on the x-axis and y-axis. It is numerically checked that the behaviour of equilibrium points do not change throughout in $(0.88, 0.99)$,therefore we choose $a = 0.95 \in (0.88\ ,\ 0.99)$ and draw contour plots for the infinitesimal particle and check the position of equilibrium points.

4.3.4.1 Contour-Plot

For $a = 0.95$ the corresponding value of b , M and m are 1 , 0.0781 and 0.0057 respectively. Contour plot for these values shows that no equilibrium point exist.

In Figure 4.4 no equilibrium point exist. The equilibrium points are the points of the intersections of $U_x = 0$ (blue) and $U_y = 0$ (orange).

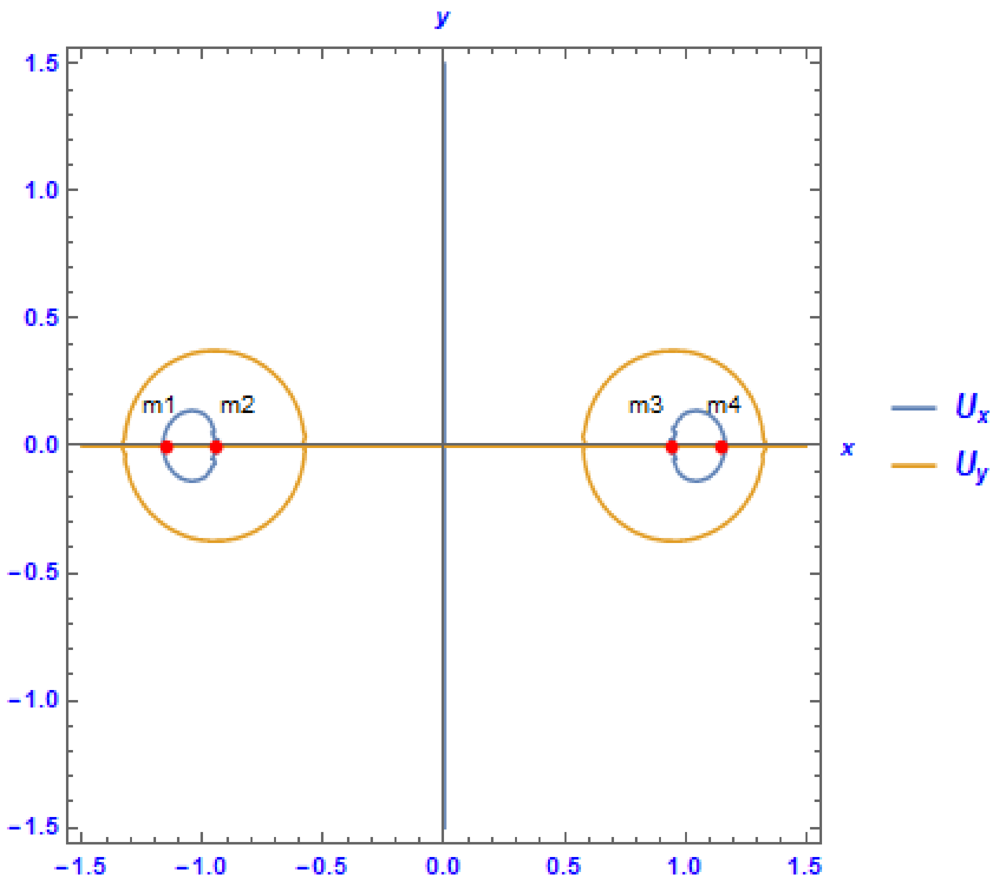

FIGURE 4.4: Positions of the masses (Red dots); equilibrium points of m_5 (Black dots). Here $a = 0.95$ and the corresponding masses are $m = 0.0057$, $M = 0.0781$.

4.4 Case-I(2): When $a > 2.3$ and $b = 1$

i. When $a \in (2.45\ ,\ 3.25)$ there exist 7 equilibrium points.

ii. When $a \in (3.25\ ,\ 4.15)$ there exist 7 equilibrium points.

iii. When $a \in (4.15\ ,\ 5.00)$ there exist 7 equilibrium points.

4.4.1 Seven Equilibrium Points

We start our analysis with the first case for equilibrium points where seven equilibrium points exist on the x-axis and y-axis.It is numerically checked that the behaviour of the equilibrium points do not change throughout in $(2.45, 3.25)$ therefore we choose $a = 2.95 \in (2.45\ ,\ 3.25)$ and draw contour plots for the infinitesimal particle and check the position of equilibrium points.

4.4.1.1 Contour-Plot

For $a = 2.95$ the corresponding value of b , M and m are 1 , 2.665 and 8.785 respectively. Contour plot for these values shows that the equilibrium points L_1, L_2,L_3,L_4 and L_5 are collinear along the x-axis L_6 and L_7 are collinear along the y-axis. In Figure 4.5 the black dot denote the position of equilibrium points and red dots represent the position of masses m_1 to m_4. The equilibrium points are the points of the intersections of $U_x = 0$ (blue) and $U_y = 0$ (orange).

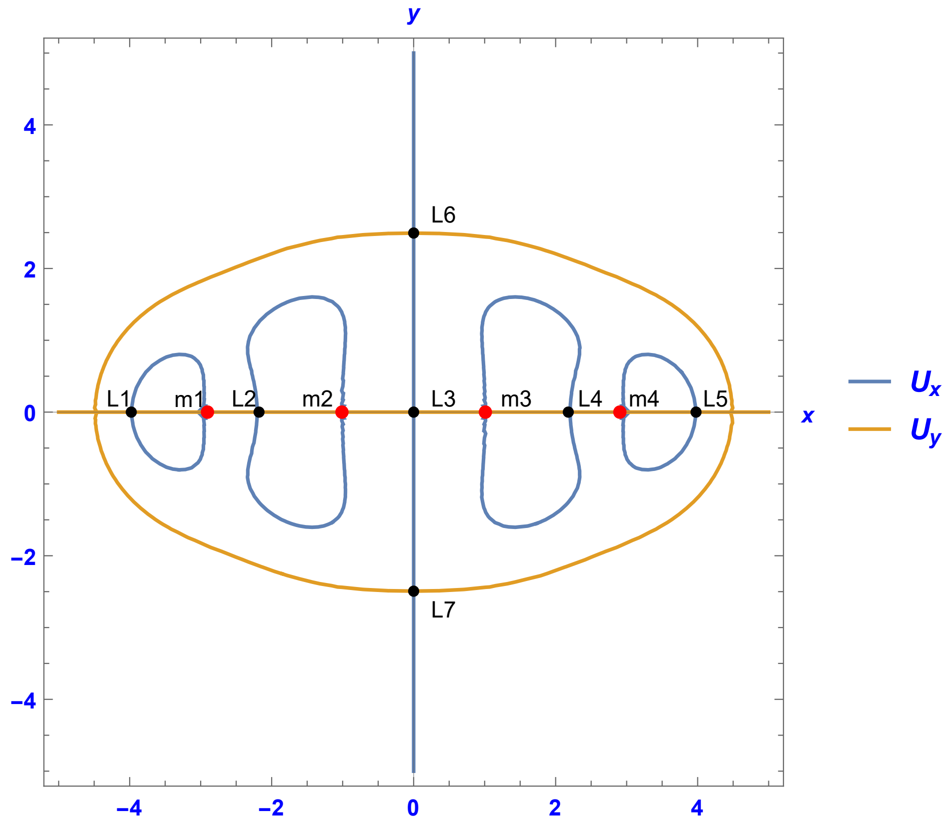

FIGURE 4.5: Positions of the masses (Red dots); equilibrium points of m_5 (Black dots). Here $a = 2.95$ and the corresponding masses are $m = 8.785$, $M = 2.665$.

4.4.2 Seven Equilibrium Points

We continue our analysis with the second case for equilibrium points where seven equilibrium points exist on the x-axis and y-axis. It is numerically checked that the behaviour of equilibrium points do not change throughout in $(3.25, 4.15)$, therefore we choose $a = 3.75 \in (3.25\ ,\ 4.15)$ and draw contour plots for the infinitesimal particle and check the position of equilibrium points.

4.4.2.1 Contour-Plot

For $a = 3.75$ the corresponding value of b , M and m are 1 , 12.511 and 19.980 respectively. Contour plot for these values shows that the equilibrium points L_1, L_2,L_3,L_4 and L_5 are collinear along the x-axis L_6 and L_7 are collinear along the y-axis. In Figure 4.6 the black dot denote the position of equilibrium points and red dots represent the position of masses m_1 to m_4. The equilibrium points are the points of the intersections of $U_x = 0$ (blue) and $U_y = 0$ (orange).

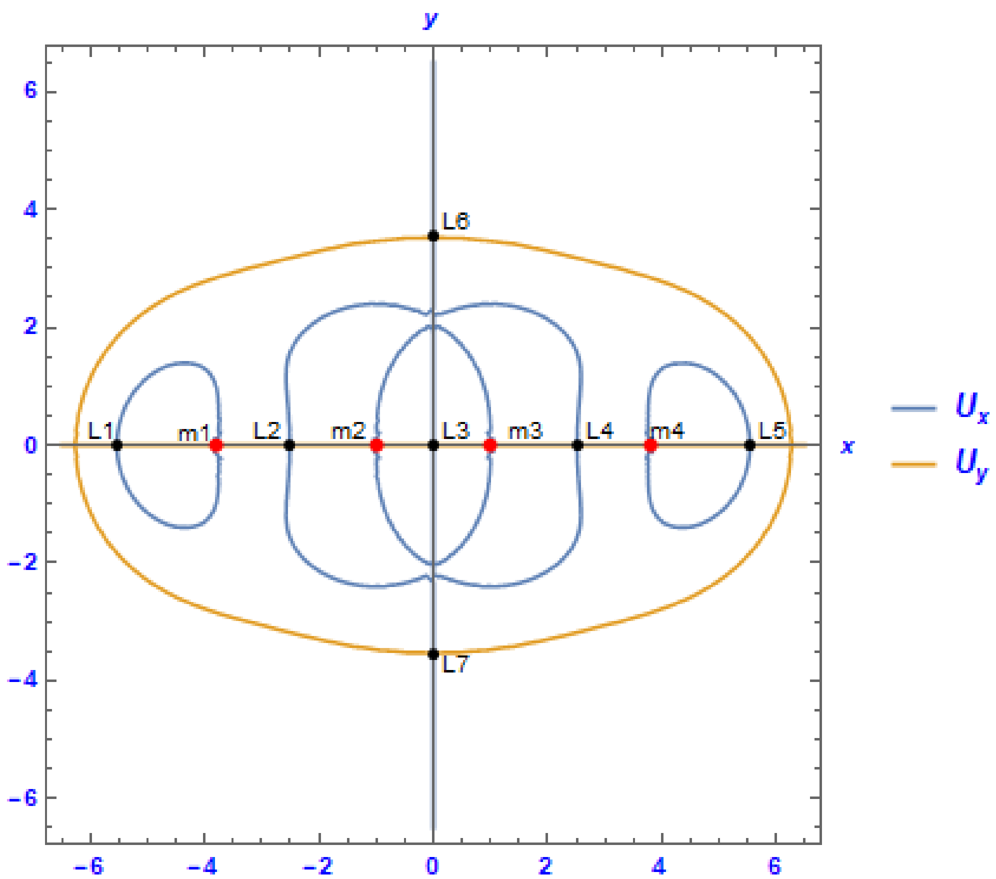

FIGURE 4.6: Positions of the masses (Red dots); equilibrium points of m_5 (Black dots). Here $a = 3.75$ and the corresponding masses are $m = 19.980$, $M = 12.511$.

4.4.3 Seven Equilibrium Points

We present our analysis with the third case for equilibrium points where seven equilibrium points exist on the x-axis and y-axis. It is numerically checked that the behaviour of equilibrium points do not change throughout in $(4.15, 5)$, therefore we choose $a = 4.55 \in (4.15\ ,\ 5)$ and draw contour plots for the infinitesimal particle and check the position of equilibrium points.

4.4.3.1 Contour-Plot

For $a = 4.55$ the corresponding value of b , M and m are 1 , 33.208 and 37.106 respectively. Contour plot for these values shows that the equilibrium points L_1, L_2,L_3,L_4 and L_5 are collinear along the x-axis L_6 and L_7 are collinear along the y-axis. In Figure 4.7 the black dot denote the position of equilibrium points and red dots represent the position of masses m_1 to m_4. The equilibrium points are the points of the intersections of $U_x = 0$ (blue) and $U_y = 0$ (orange).

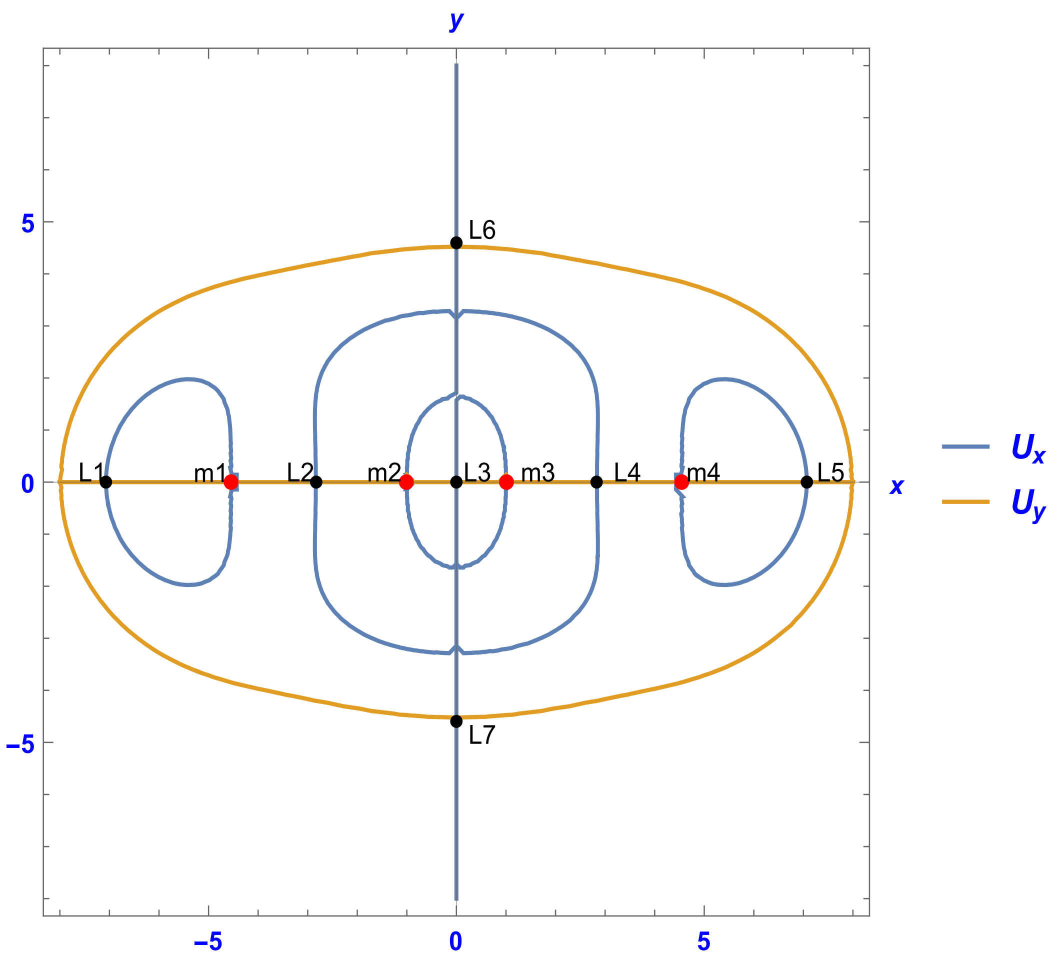

FIGURE 4.7: Positions of the masses (Red dots); equilibrium points of m_5 (Black dots). Here $a = 0.6$ and the corresponding masses are $m = 37.106$, $M = 33.206$.

4.5 Stability Analysis

This section is devoted to the mathematical analysis of the stability of equilibrium points in the system of a restricted collinear 5-body problem for which we considered the slight displacement from the equilibrium points to the infinitesimal body as well as the small velocity. If the infinitesimal small body's motion departs from the point of proximity, and it never returns, then this point of equilibrium is unkown as unstable. But if the body is oscillating about the point it seems to be stable. We will now check whether the equilibrium points are either stable or unstable. To check stability, with the support of the Jacobian matrix of individual values, followed by the procedure given on page 12.

For case-I(1)(i) $a = 0.6$ and corresponding coordinates of equilibrium point L_1 are (-1.246,0), we get following Jacobian matrix form as

$$A = \begin{pmatrix} 172.414 & 0 \\ 0 & -84.7071 \end{pmatrix}.$$

The eigenvalues of matrix A are: $(172.414, -84.7071)$, because eigenvalue have not negative real part therefore L_1 is unstable.

For the remaining values we will apply the same procedure and present the stability analysis of corresponding Lagrange point which are given in the following table.

S.No	**Lagrange points**	**Eigenvalues**	**Stability**
1	L_1(-1.246, 0)	(9.79358, -3.39679)	Unstable
2	L_2(-0.886, 0)	(88.1497, -42.5749)	Unstable
3	L_3(0, 0.001)	(7.53753, -2.26876)	Unstable
4	L_4(0.886, 0)	(104.198, -50.5749)	Unstable
5	L_5 (1.246, 0)	(9.85628, -3.425675)	Unstable
6	L_6(0, 0.787)	(1.63959, -1.111759)	Unstable
7	L_7(0, -0.787)	(1.63959, -1.11759)	Unstable

Table 4.1: Stability analysis for: $a = 0.6$, $b = 1$, $m = 0.054$, $M = 0.268$.

Stability analysis for case-I(1)(ii) is shown in the following table:

S.No	Lagrange points	Eigenvalues	Stability
1	L_1(-1.284, 0)	(7.07425, -2.03712)	Unstable
2	L_2(-0.9119, 0)	(253.29, -125.145)	Unstable
3	L_3(0, 0.001)	(3.00234, -0.00117089)	Unstable
4	L_4(0.916,0)	(257.026, -127.013)	Unstable
5	L_5((1.288,0)	(7.07425, -2.03712)	Unstable

TABLE 4.2: Stability analysis for: $a = 0.781$, $b = 1$, $m = 0.0359$, $M = 0.2123$.

Stability analysis for case-I(1)(iii) is shown in the following table:

S.No	Lagrange points	Eigenvalues	Stability
1	L_1(1.256,0)	(6.81993, -1.90996)	Unstable
2	L_2(0, 0)	(1.72751, 0.636245)	Unstable
3	L_3(-1.255,0)	(6.8719, -1.93595)	Unstable

TABLE 4.3: Stability Analysis for : $a = 0.85$, $b = 1$, $m = 0.012$, $M = 0.1175$.

Stability analysis for case-I(1)(iv) is shown in the following table::

S.No	Lagrange points	Eigenvalues	Stability
1	L_1(-2.826, 0)	(179.355, -88.1774)	Unstable
2	L_2(-2.168, 0)	(25.1715, -11.0857)	Unstable
3	L_3(0, 0.001)	(23.8406, -10.4203)	Unstable
4	L_4(2.031, 0)	(19.3065, -8.15327)	Unstable
5	L_5(2.841, 0)	(145.794, -71.397)	Unstable
6	L_6(0, 1.753)	(2.73861, 0.689486)	Unstable
7	L_7(0, -0.1738)	(21.8952, -8.97028)	Unstable

TABLE 4.4: Stability analysis for : $a = 2.95$, $b = 1$, $m = 8.785$, $M = 2.665$.

Stability analysis for case-I(2)(i) is shown in the following table:

S.No	Lagrange points	Eigenvalues	Stability
1	L_1(-3.129, 0)	(31.3896, -14.1948)	Unstable
2	L_2(-2.117,0)	(22.0595, -9.52974)	Unstable
3	L_3(0, 0.001)	(26.63, -11.815)	Unstable
4	L_4(2.031,0)	(20.6812, -8.84058)	Unstable
5	L_5(3.096,0)	(39.2961, -18.1481)	Unstable
6	L_6(0,1.955)	(2.65444, 0.606435)	Unstable
7	L_7(0,-1.945)	(2.66811, 0.607784)	Unstable

TABLE 4.5: Stability analysis for: $a = 3.75$, $b = 1$, $m = 19.9880$, $M = 12.511$.

Stability analysis for case-I(2)(ii) is shown in the following table:

S.No	Lagrange points	Eigenvalues	Stability
1	L_1(-2.168, 0)	(24.2522, -10.6261)	Unstable
2	L_2(-4.141, 0)	(1.51779, 0.741104)	Unstable
3	L_3(0, 0.001)	(19.9936, -8.49682)	Unstable
4	L_4(4.105, 0)	(1.53974, 0.730129)	Unstable
5	L_5(2.168, 0)	(24.2522, -10.6261)	Unstable
6	L_6(0, 2.562)	(1.7362, 0.731988)	Unstable
7	L_7(0, -2.547)	(1.74549, 0.729817)	Unstable

TABLE 4.6: Stability analysis for : $a = 4.15$, $b = 1$, $m = 37.106$, $M = 33.208$.

4.6 Permissible Regions of Motion in Plane of Motion

The Jacobian constant of motion (4.18) is one of the most important constant of the dynamical system, which represent the movement of the infinitesimal body

[33]. The motion of infinitesimal body depends on the value of the Jacobi constant. For choosing different values of Jacobi constant C in equation (4.18) gives us two different regions. One is permissible region of motion for infinitesimal particle and the second region where motion of infinitesimal particle is not allowed. Now we need to explore these possibilities in our geometry i.e., Four masses are placed m_1, m_2, m_3 and m_4 are on the x-axis and the infinitesimal mass m_5 is moving in the gravitational field of these four masses. We draw here regions for different value of Jacobi constant (4.18) at Mathematica and get two regions,

- permissible region (white), where m_5 can freely move.
- Colored region (Blue), where the motion of m_5 is not allowed.

Case-I(1): When $a \in (0.47, 1)$ **and** $b = 1$

Choosing the value of $C = 1.1$ and using in equation (4.18) and then drawing the graph shown in Figure 4.8, get two region so clearly see that the blue region represent the excluded area, where the infinitesimal mass m_5 cannot move or enter, and the white region represent where the infinitesimal mass can easily move or enter. As soon as, increase the value of C, the allowable area is getting shorter. We can see Figures from 4.8-4.11. In these statistics, we can see that the four masses are trapped, and the infinitesimal mass can not approach those masses.

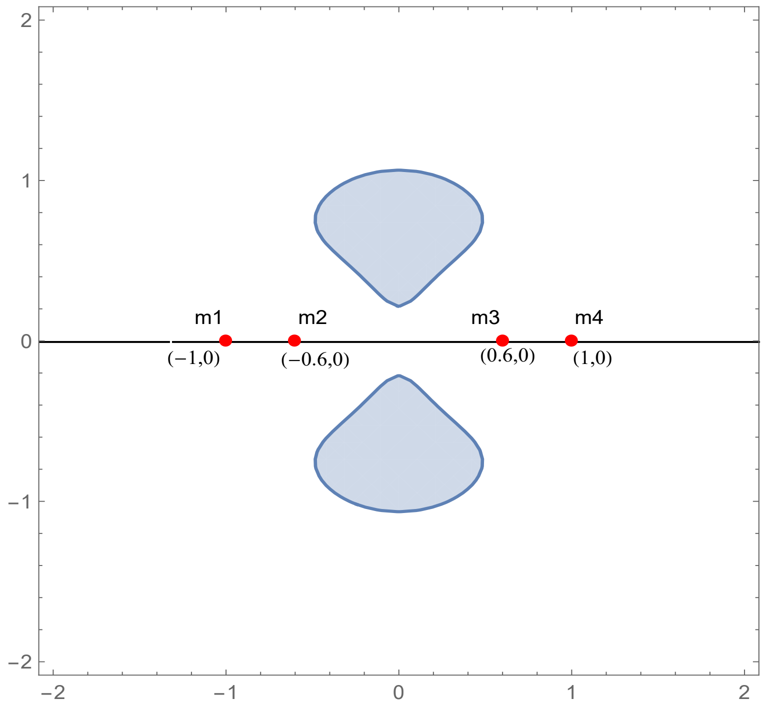

FIGURE 4.8: Permissible regions (white) of motion for $C = 1.2$.

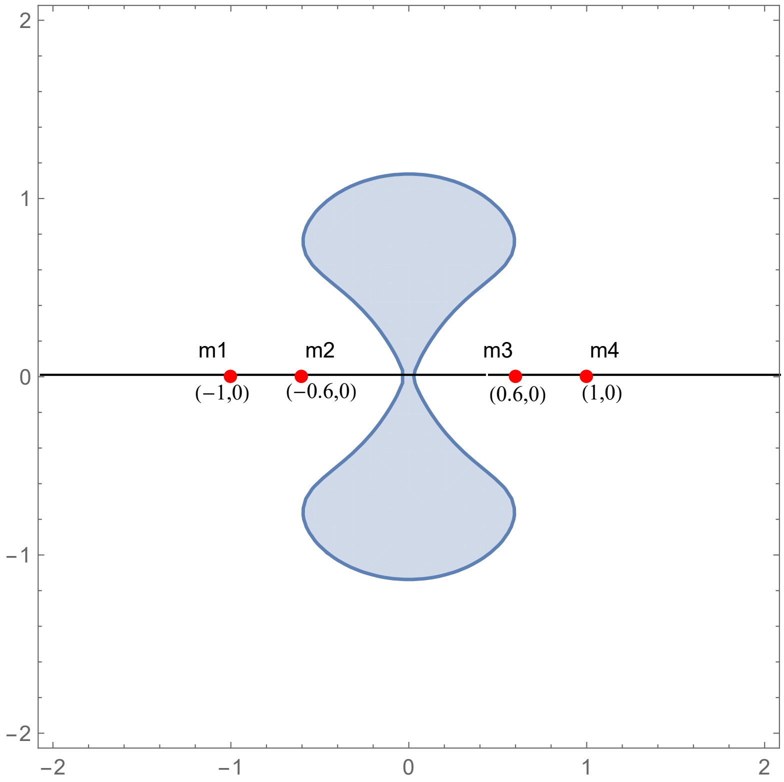

FIGURE 4.9: Permissible regions (white) of motion for $C = 1.3$.

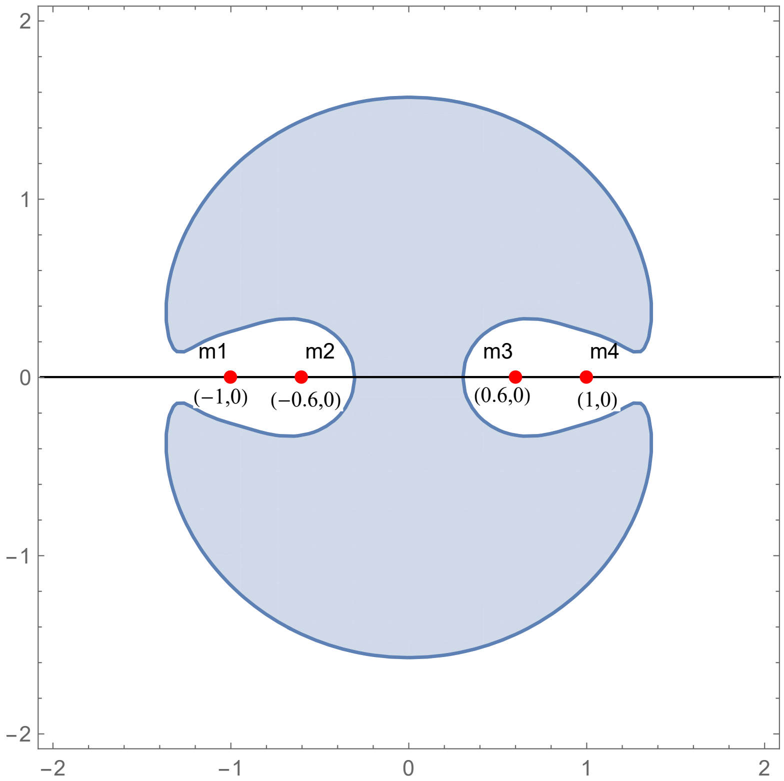

FIGURE 4.10: Permissible regions (white) of motion for $C = 1.6$.

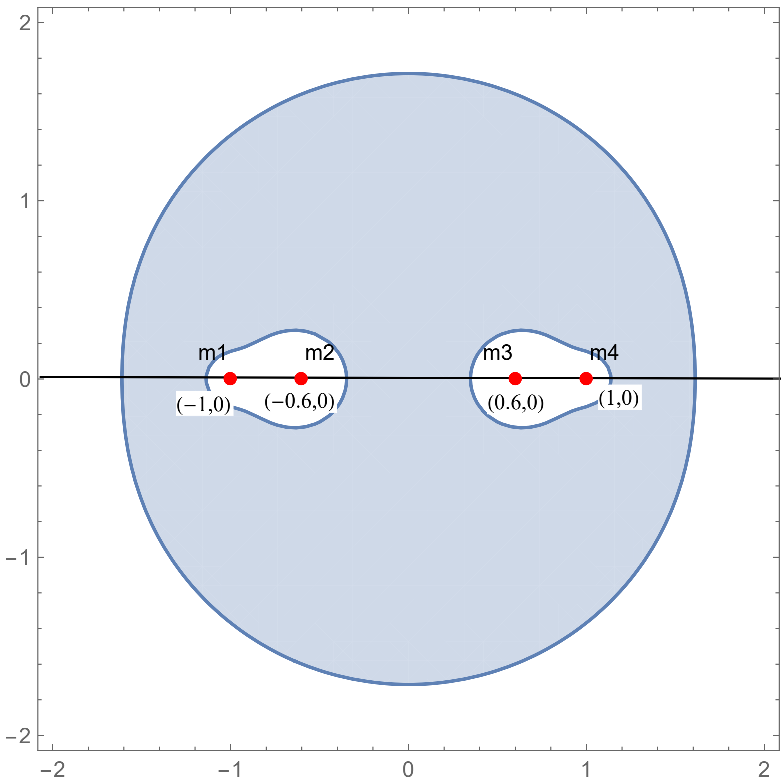

FIGURE 4.11: Permissible regions (white) of motion for $C = 1.9$.

Case-I(2): When $a > 2.3$ and $b = 1$

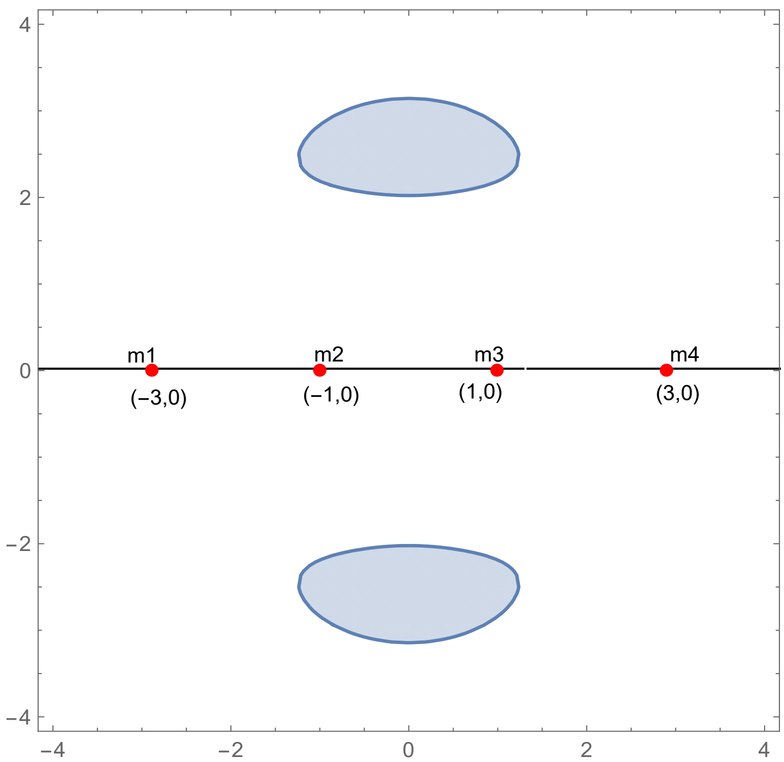

FIGURE 4.12: Permissible regions (white) of motion for $C = 12$.

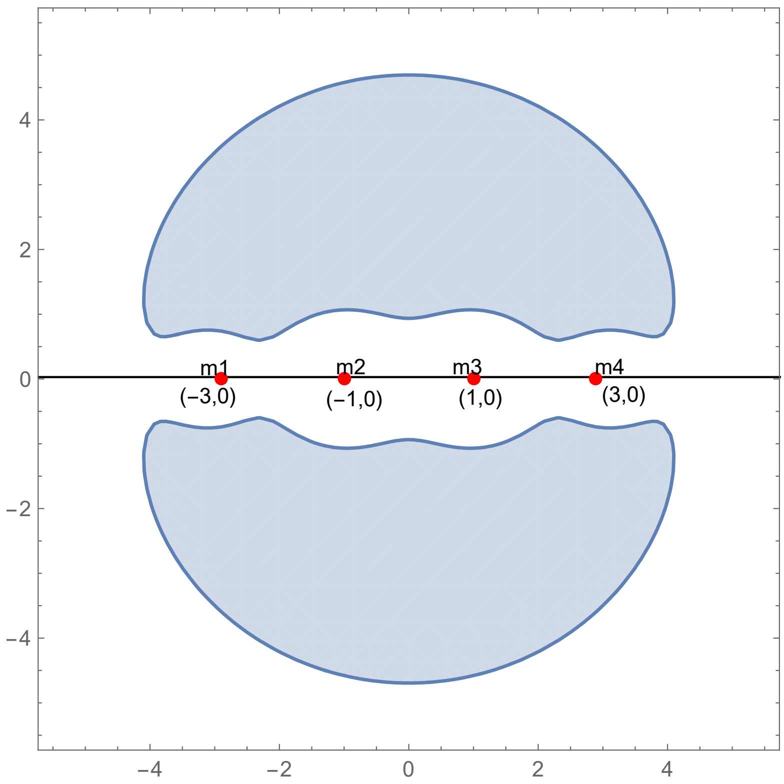

FIGURE 4.13: Permissible regions (white) of motion for $C = 16$.

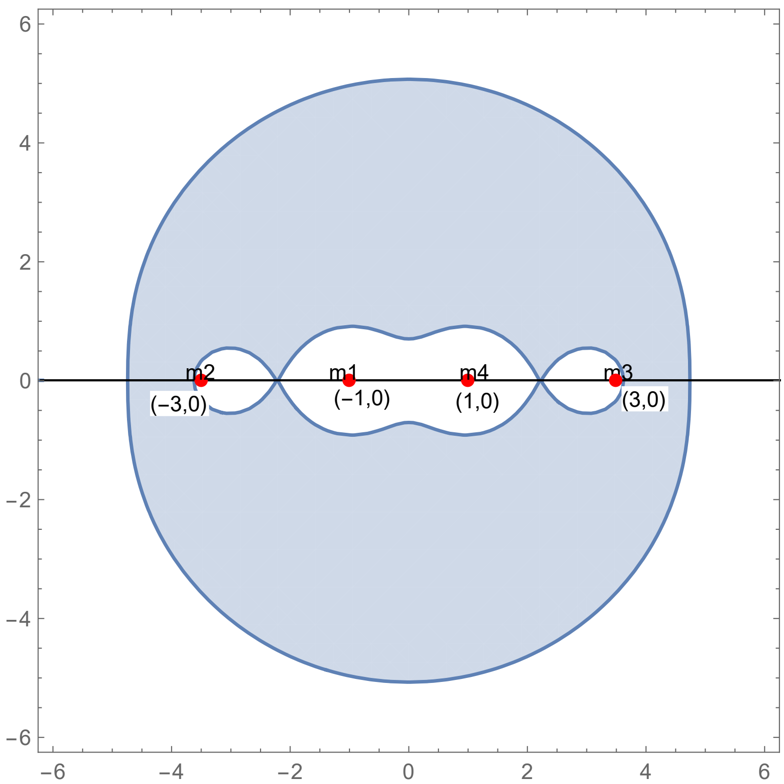

FIGURE 4.14: Permissible regions (white) of motion when $C = 17.5$.

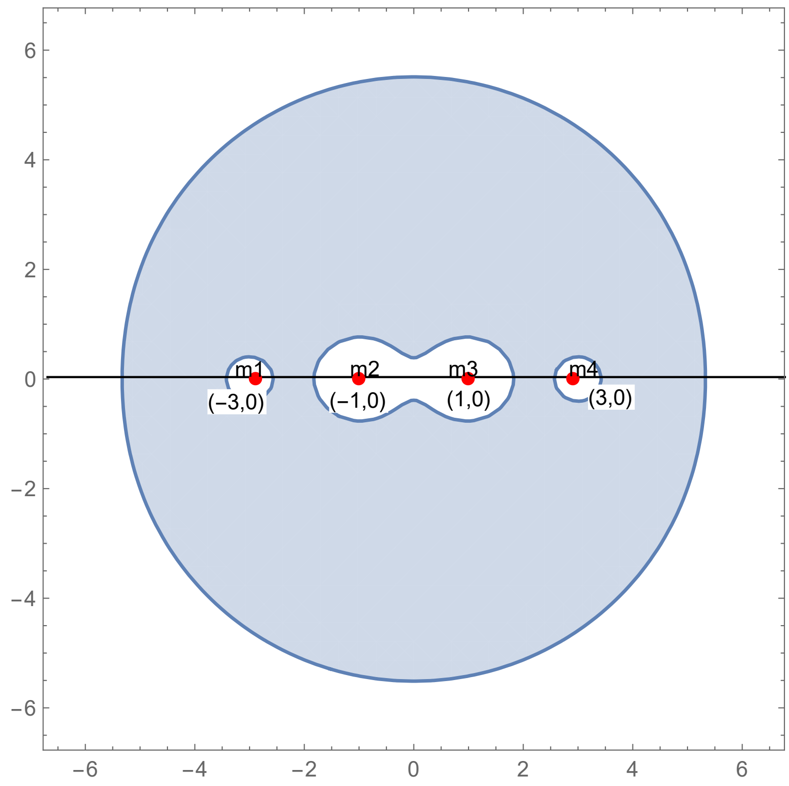

FIGURE 4.15: Permissible regions (white) of motion for $C = 19.5$.

4.7 Case-II(1): When $b \in (0, 0.41)$ and $a = 1$

i. When $b \in (0.11\ ,\ 0.27)$ there exist 7 equilibrium points.

ii. When $b \in (0.27\ ,\ 0.39)$ there exist 7 equilibrium points.

4.7.1 Seven Equilibrium Points

We start our analysis with the first case for equilibrium points where seven equilibrium points exist on the x-axis and y-axis. It is numerically checked that the behaviour of the equilibrium points do not change throughout in $(0.11, 0.27)$, therefore we choose $b = 0.25 \in (0.11\ ,\ 0.27)$ and draw contour plots for the infinitesimal particle and check the position of equilibrium points.

4.7.1.1 Contour-Plot

For $b = 0.25$ the corresponding value of a , M and m are 1 , 1.268 and 0.282

respectively. Contour plot for these values shows that the equilibrium points L_1, L_2,L_3,L_4 and L_5 are collinear along the x-axis L_6 and L_7 are collinear along the y-axis. In Figure 4.16 the black dot denote the position of equilibrium points and red dots represent the position of masses m_1 to m_4. The equilibrium points are the points of the intersections of $U_x = 0$ (blue) and $U_y = 0$ (orange).

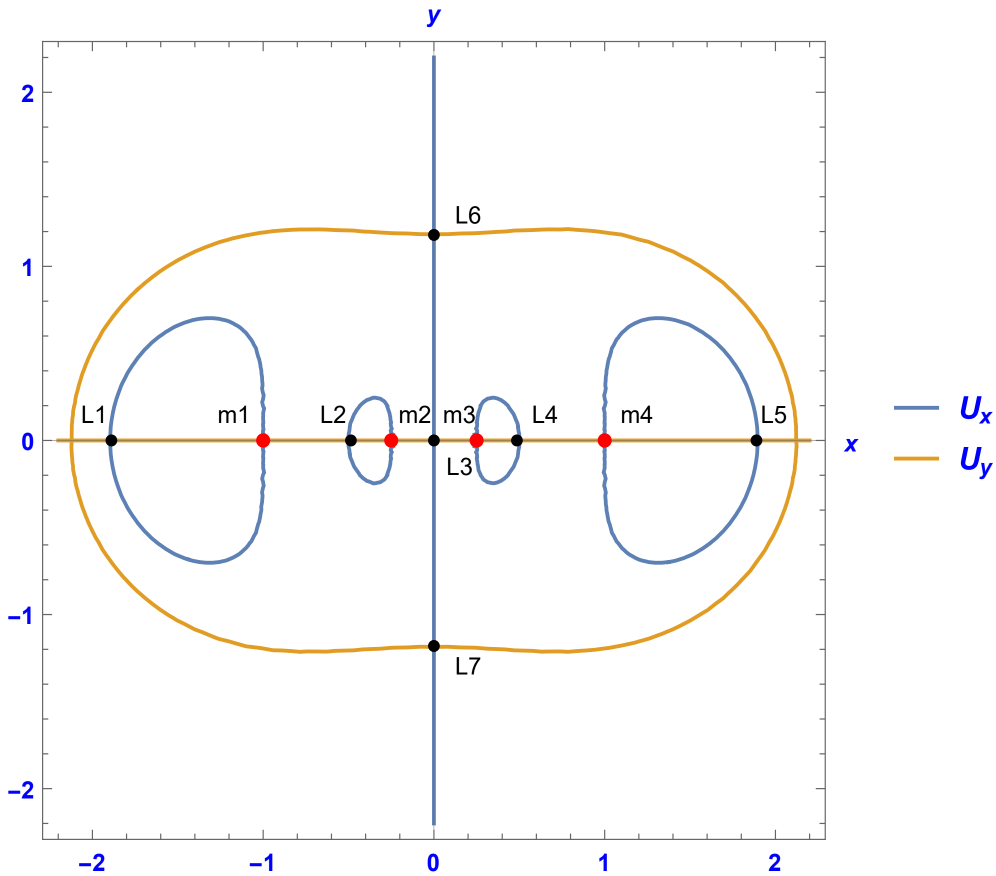

FIGURE 4.16: Positions of the masses (Red dots); equilibrium points of m_5 (Black dots). Here $b = 0.25$ and the corresponding masses are $m = 0.282$, $M = 1.268$.

4.7.2 Seven Equilibrium Points

We continue our analysis with the second case for equilibrium points where seven equilibrium points exist on the x-axis and y-axis. It is numerically checked that the behaviour of equilibrium points do not change throughout in $(0.27, 0.39)$,therefore we choose $b = 0.37 \in (0.27\ ,\ 0.39)$ and draw contour plots for the infinitesimal particle and check the position of equilibrium points.

4.7.2.1 Contour-Plot

For $b = 0.37$ the corresponding value of a , M and m are 1 , 0.1895 and 0.3120 respectively. Contour plot for these values shows that the equilibrium points L_1, L_2,L_3,L_4 and L_5 are collinear along the x-axis L_6 and L_7 are collinear along the y-axis. In Figure 4.17 the black dot denote the position of equilibrium points and red dots represent the position of masses m_1 to m_4. The equilibrium points are the points of the intersections of $U_x = 0$ (blue) and $U_y = 0$ (orange).

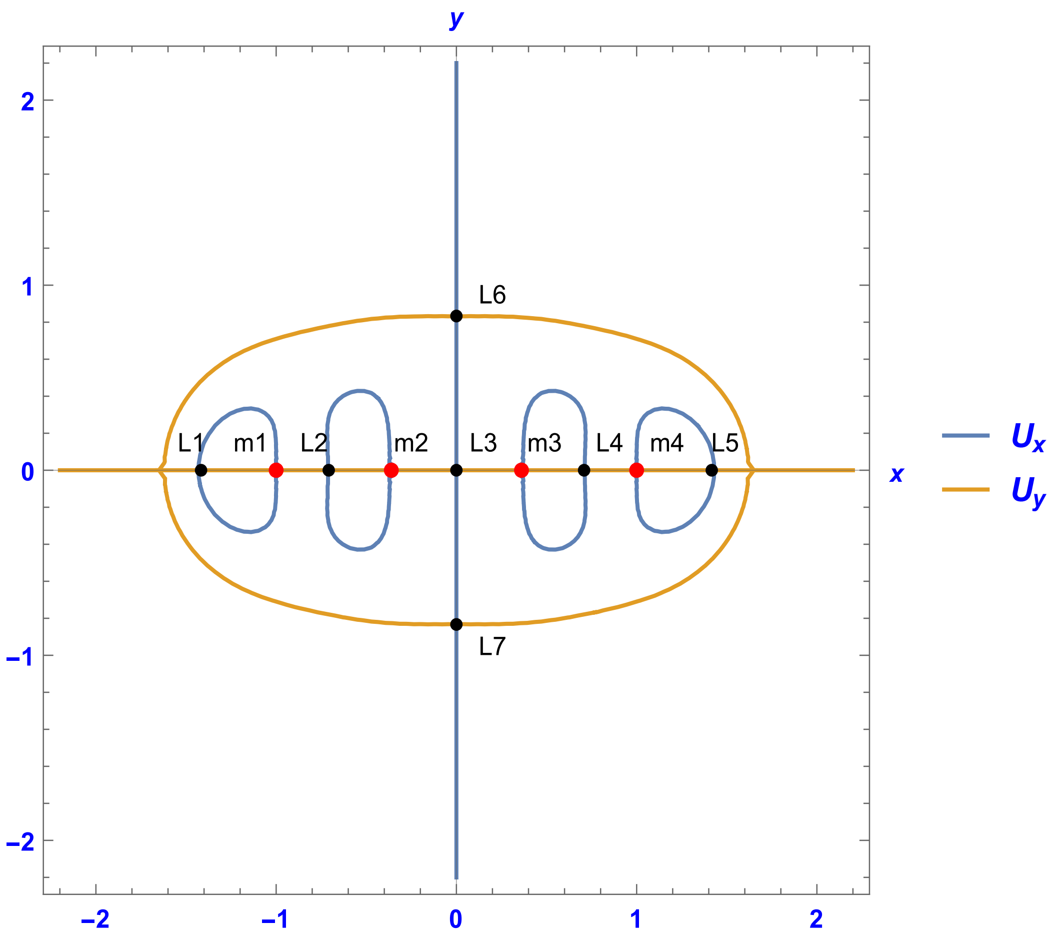

FIGURE 4.17: Positions of the masses (Red dots); equilibrium points of m_5 (Black dots). Here $b = 0.37$ and the corresponding masses are $m = 0.3120$, $M = 0.1895$.

4.8 Case-II(2): When $b \in (1\ ,\ 2.6)$ and $a = 1$

i. When $b \in (1.5\ ,\ 2.1)$ there exist 7 equilibrium points.

ii. When $b \in (2.1\ ,\ 2.5)$ there exist 5 equilibrium points.

4.8.1 Seven Equilibrium Points

We start our analysis with the first case for equilibrium points where equilibrium points are exist on the x-axis and y-axis. It is numerically checked that the behaviour of equilibrium points do not change throughout in $(1.5, 2.1)$, therefore we choose $b = 1.7 \in (1.5\ ,\ 2.1)$ and draw contour plots for the infinitesimal particle and check the position of equilibrium points.

4.8.1.1 Contour-Plot

For $b = 1.7$ the corresponding value of a , M and m are 1 , 1.067 and 0.33366 respectively. Contour plot for these values shows that the equilibrium points L_1, L_2,L_3,L_4 and L_5 are collinear along the x-axis L_6 and L_7 are collinear along the y-axis. In Figure 4.18 the black dot denote the position of equilibrium points and red dots represent the position of masses m_1 to m_4. The equilibrium points are the points of the intersections of $U_x = 0$ (blue) and $U_y = 0$ (orange).

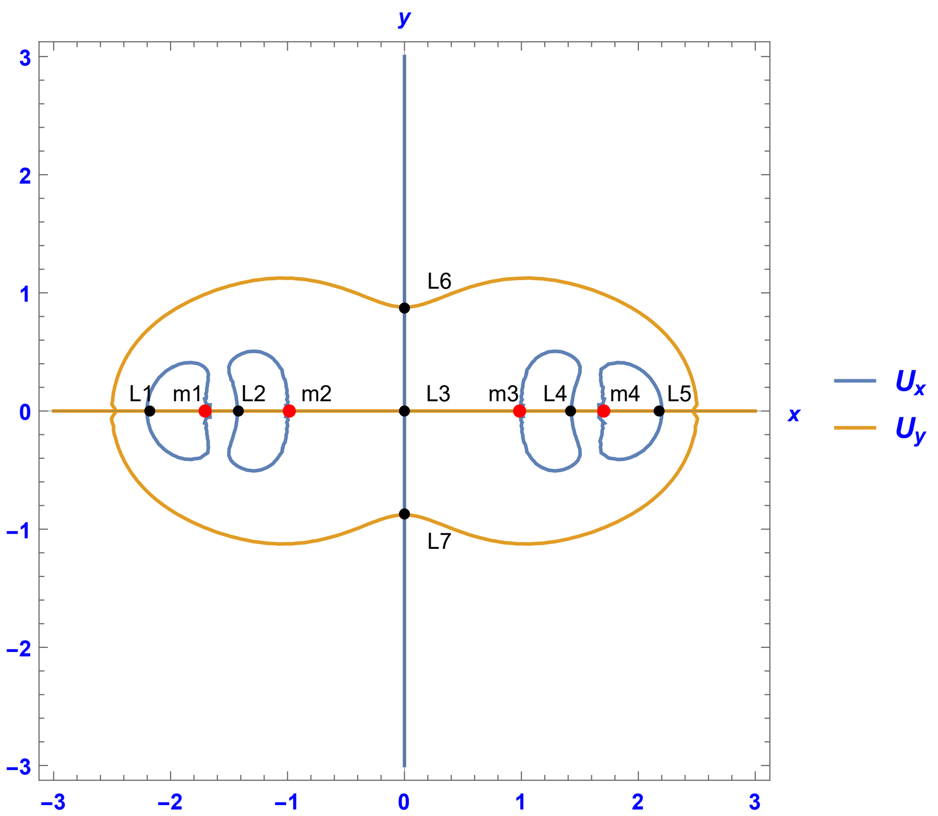

FIGURE 4.18: Positions of the masses (Red dots); equilibrium points of m_5 (Black dots). Here $b = 1.7$ and the corresponding masses are $m = 0.3366$, $M = 1.067$.

4.8.2 Five Equilibrium Points

We continue our analysis with the second case for equilibrium points where five equilibrium points exist on the x-axis and y-axis. It is numerically checked that the behaviour of equilibrium points do not change throughout in $(2.1, 2.5)$,therefore we choose $b = 2.3 \in (2.1\ ,\ 2.5)$ and draw contour plots for the infinitesimal particle and check the position of equilibrium points.

4.8.2.1 Contour-Plot

For $b = 2.3$ the corresponding value of a , M and m are 1 , 2.643 and 0.4959 respectively. Contour plot for these values shows that the equilibrium points L_1, L_2,L_3,L_4 and L_5 are collinear along the x-axis L_6 and L_7 are collinear along the y-axis. In Figure 4.19 the black dot denote the position of equilibrium points and red dots represent the position of masses m_1 to m_4. The equilibrium points are the points of the intersections of $U_x = 0$ (blue) and $U_y = 0$ (orange).

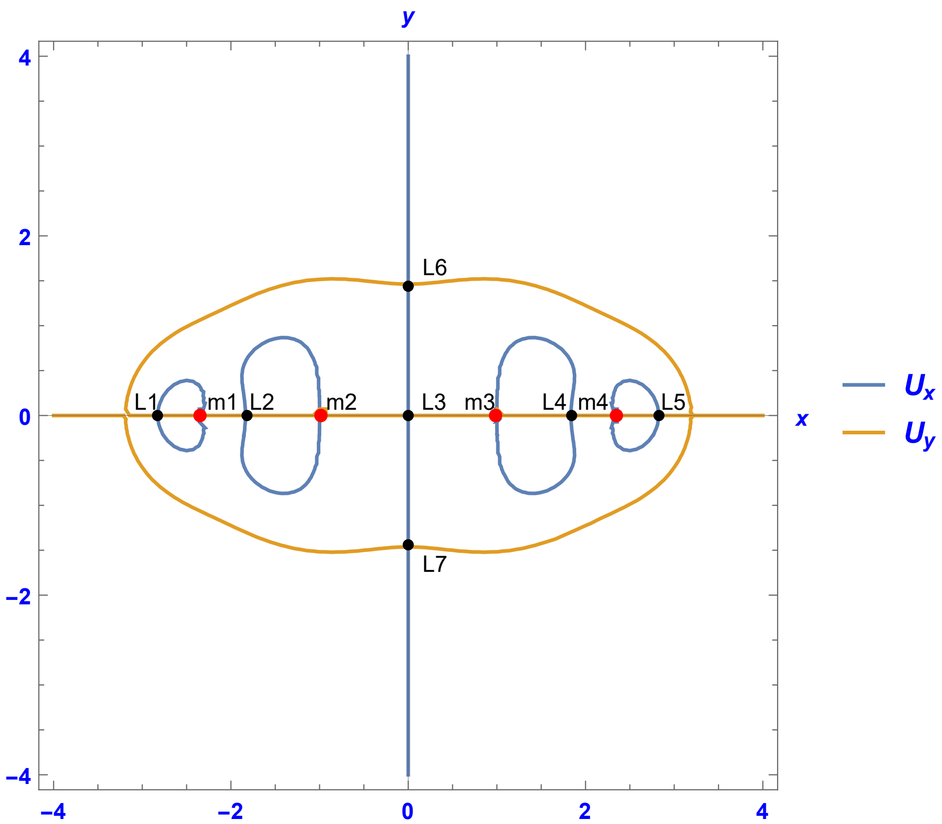

FIGURE 4.19: Positions of the masses (Red dots); equilibrium points of m_5 (Black dots). Here $b = 2.3$ and the corresponding masses are $m = 0.4959$, $M = 2.643$.

4.9 Stability analysis

Stability analysis of equilibrium points for the interval $(0.11\ ,\ 0.27)$**:**

Choosing $b = 0.25$ from $(0.11\ ,\ 0.27)$ and the corresponding value of $a, M = 1.268$, $m = 0.282$, and $L_1(-1.878, 0)$, the Jacobian matrix form is

$$A = \begin{pmatrix} 5.04499 & 0 \\ 0 & -1.02249 \end{pmatrix}.$$

The eigenvalues of matrix A are: $(5.04499, -1.02249)$, because eigenvalues are not negative real part therefore L_1 is unstable.

For the remaining values we will apply the same procedure and present the stability analysis of corresponding Lagrange point which are given below.

S.No	**Lagrange points**	**Eigenvalues**	**Stability**
1	L_1(-1.878, 0)	(5.04499, -1.02249)	Unstable
2	L_2(-0.4811, 0)	(67.1511, -32.0756)	Unstable
3	L_3(0, 0.001)	(7.53753, -2.26876)	Unstable
4	L_4(0.4811, 0)	(63.0676, -30.0338)	Unstable
5	L_5 (1.878, 0)	(4.92815, -0.964073)	Unstable
6	L_6(0, 1.179)	(2.11512, 0.894266)	Unstable
7	L_7(0, -1.179)	(2.18459, 0.922405)	Unstable

TABLE 4.7: Stability analysis for: $a = 1$, $b = 0.25$, $m = 0.054$, $M = 0.268$.

Stability analysis for case-II(1)(ii) is shown in the following table:

S.No	Lagrange points	Eigenvalues	Stability
1	L_1(-1.361, 0)	(20.2438, -8.62189)	Unstable
2	L_2(-0.763, 0)	(68.3844, -32.6922)	Unstable
3	L_3(0, 0.001)	(38.2621, -17.631)	Unstable
4	L_4(0.763, 0)	(74.7611, -35.8805)	Unstable
5	L_5 (1.361, 0)	(21.8877, -9.44387)	Unstable
6	L_6(0, 0.787)	(2.66576, 0.794342)	Unstable
7	L_7(0, -0.787)	(2.65556, 0.793036)	Unstable

TABLE 4.8: Stability analysis for: $a = 1$, $b = 0.37$, $m = 0.312091$ and $M = 0.189591$.

Stability analysis for case-II(2)(i) is shown in the following table:

S.No	Lagrange points	Eigenvalues	Stability
1	L_1(-2.031, 0)	(7.32383, -2.16191)	Unstable
2	L_2(-1.372, 0)	(113.872, -55.4361)	Unstable
3	L_3(0, 0.001)	(4.34528, -0.672642)	Unstable
4	L_4(1.372, 0)	(91.6803, -44.3401)	Unstable
5	L_5 (2.031, 0)	(6.66774, -1.83387)	Unstable
6	L_6(0, 0.7353)	(1.93914, 0.979198)	Unstable
7	L_7(0, -0.7353)	(1.90027, 1.00126)	Unstable

TABLE 4.9: Stability analysis for: $a = 1$, $b = 1.7$, $m = 0.33366$, $M = 1.067$.

Stability analysis for case-II(2)(ii) is shown in the following table:

S.No	Lagrange points	Eigenvalues	Stability
1	L_1(-2.827, 0)	(20.5951, -8.79754)	Unstable
2	L_2(-1.937, 0)	(12.952, -4.97599)	Unstable
3	L_3(0, 0.001)	(14.3772, -5.6886)	Unstable
4	L_4(1.937, 0)	(13.2555, -5.12774)	Unstable
5	L_5 (2.827, 0)	(13.8319, -5.41597)	Unstable
6	L_6(0, 1.491)	(2.22278, 0.952095)	Unstable
7	L_7(0,-1.491)	(2.24102, 0.978238)	Unstable

TABLE 4.10: Stability analysis: $a = 1$, $b = 2.3$, $m = 0.4959$ and $M = 2.643$.

4.10 Permissible Regions of Motion in Plane of Motion

Case-II(1): When $b \in (0, 0.41)$ **and** $a = 1$

The Jacobian constant of motion is one of the most important constant of the dynamical system, which represent the movement of the infinitesimal body. Now we need to explore these possibilities in our geometry i.e., Four masses are placed m_1, m_2, m_3 and m_4 are on the x-axis and the infinitesimal mass m_5 is moving in the gravitational field of these four masses. Choosing the value of $C = 3.5$ and using in equation (4.18) and then drawing the graph shown in figure (4.20), we get two region so we can clearly see that the blue region represent the excluded area, where the infinitesimal mass m_5 cannot move or enter, and the white region represent where the infinitesimal mass can easily move or enter.

As soon as, increase the value of C, the allowable area is getting shorter. We can see Figures from 4.20-4.23. In these statistics, we can see that the four masses are trapped, and the infinitesimal mass can not approach those masses.

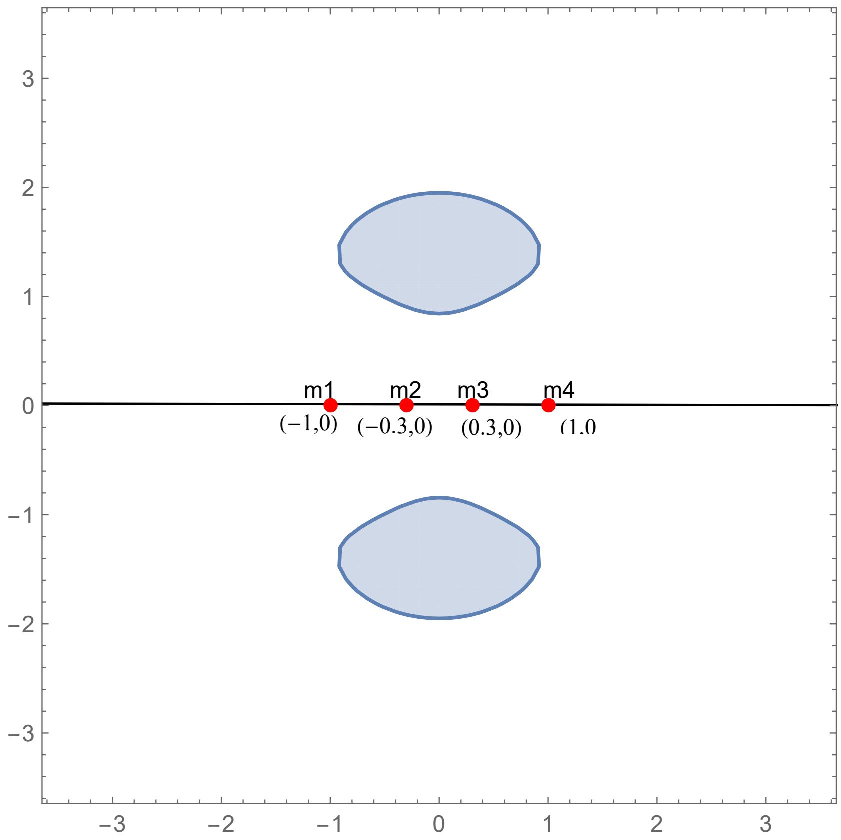

FIGURE 4.20: Permissible regions (white) of motion for $C = 3.5$.

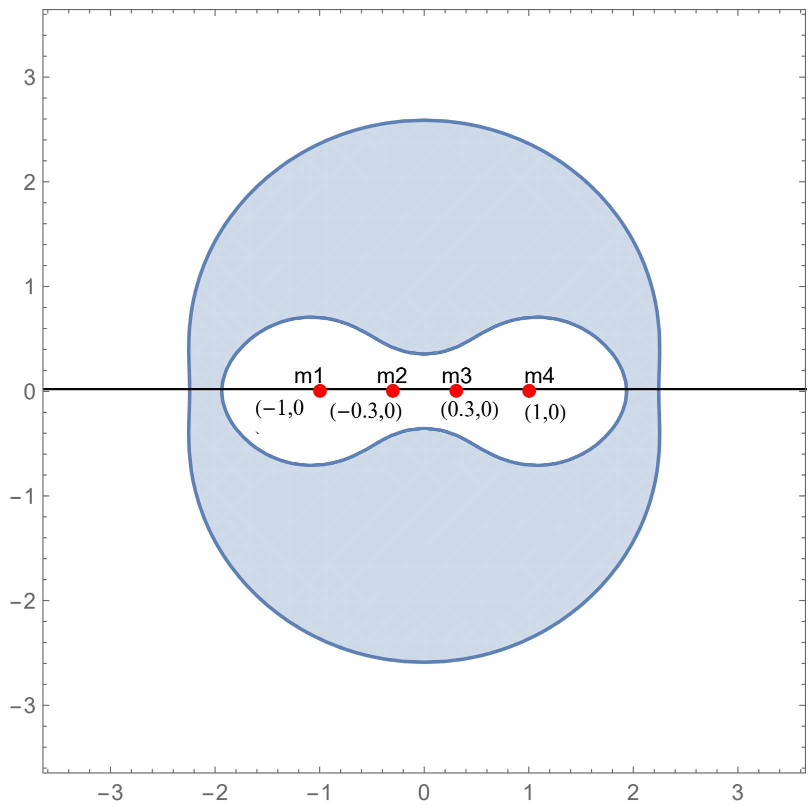

FIGURE 4.21: Permissible regions (white) of motion for $C = 5$.

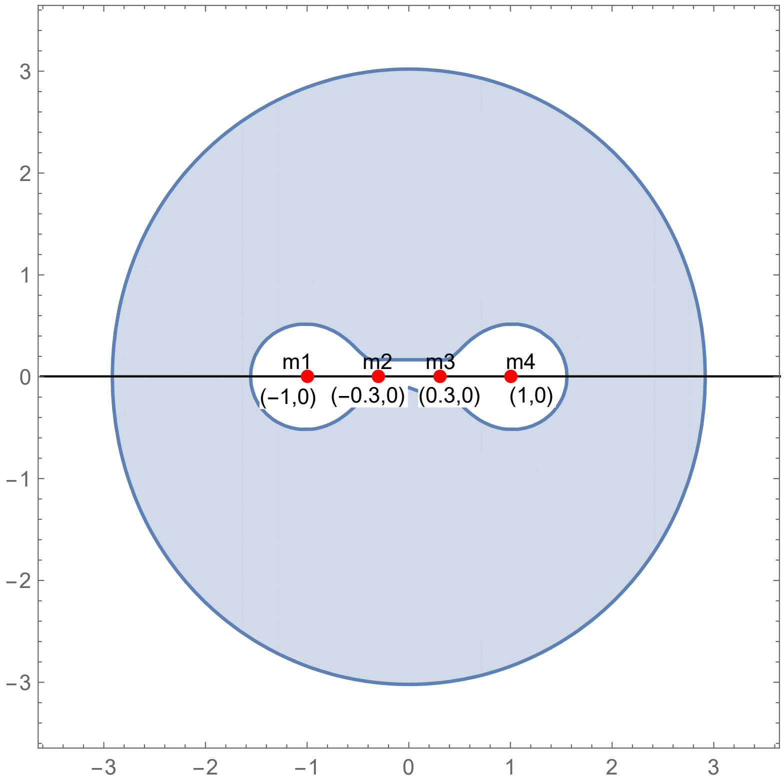

FIGURE 4.22: Permissible regions (white) of motion for $C = 6$.

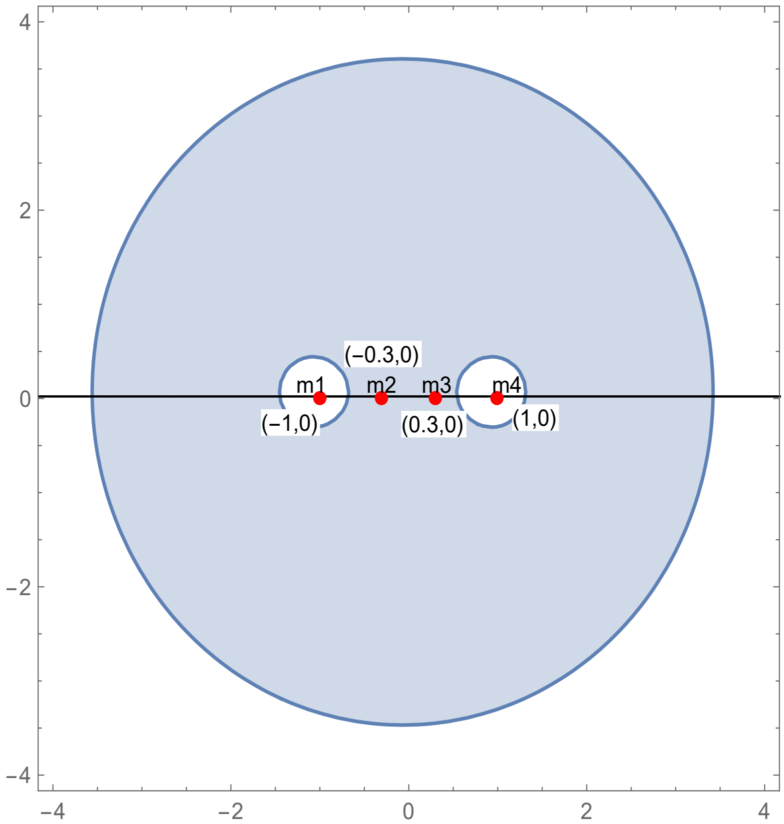

FIGURE 4.23: Permissible regions (white) of motion for $C = 7.5$.

Case-II(2): When $b \in (1, 2.6)$ and $a = 1$

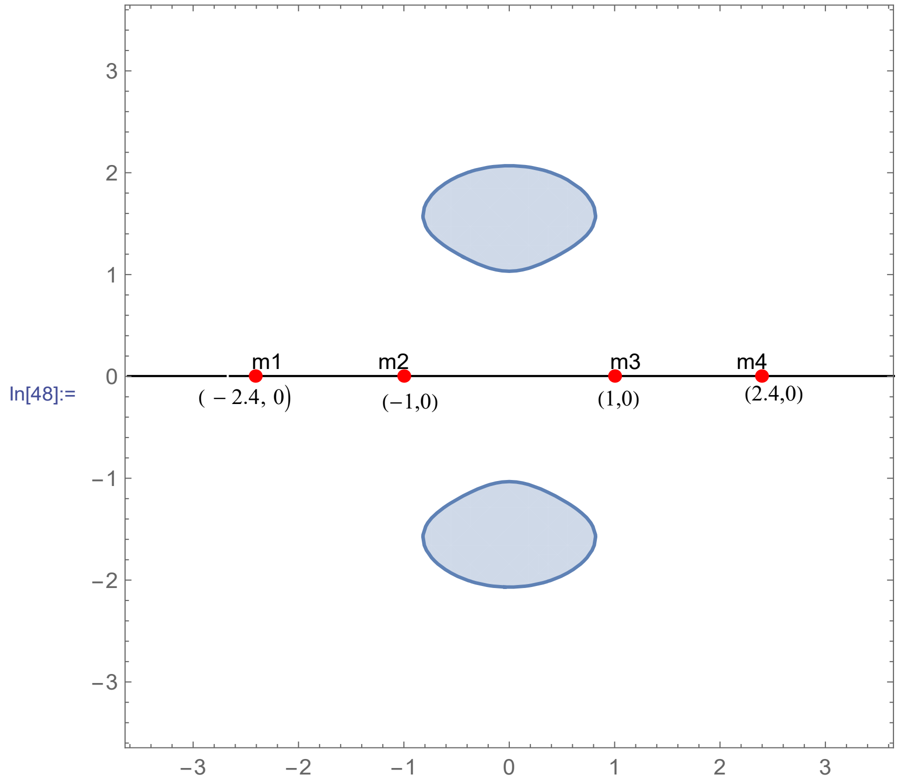

FIGURE 4.24: Permissible regions (white) of motion for $C = 5.5$.

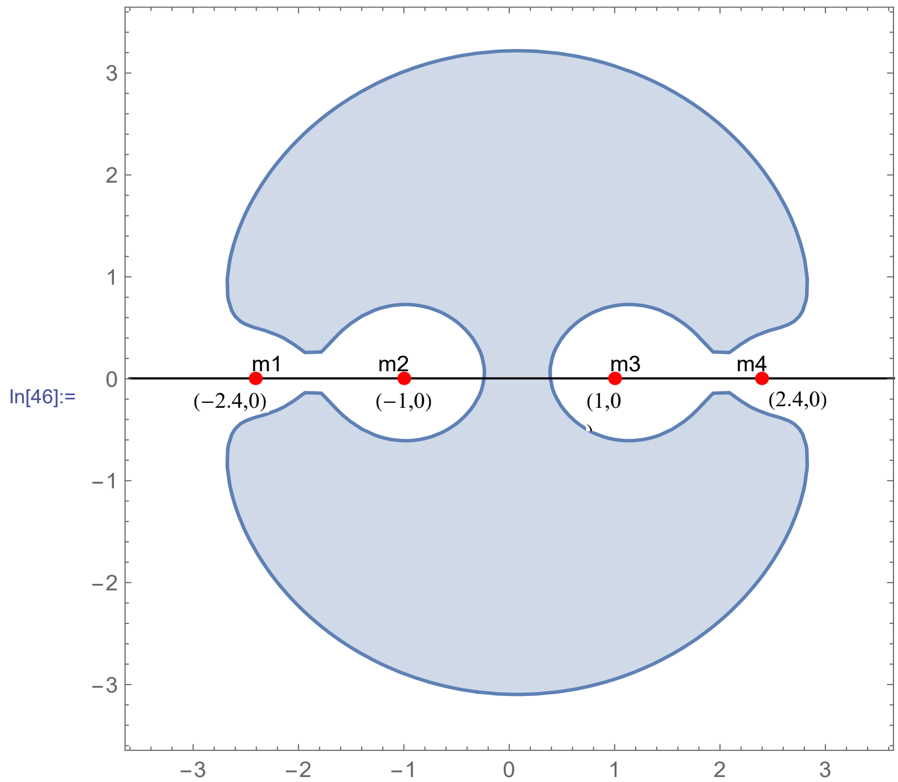

FIGURE 4.25: Permissible regions (white) of motion for $C = 6.5$.

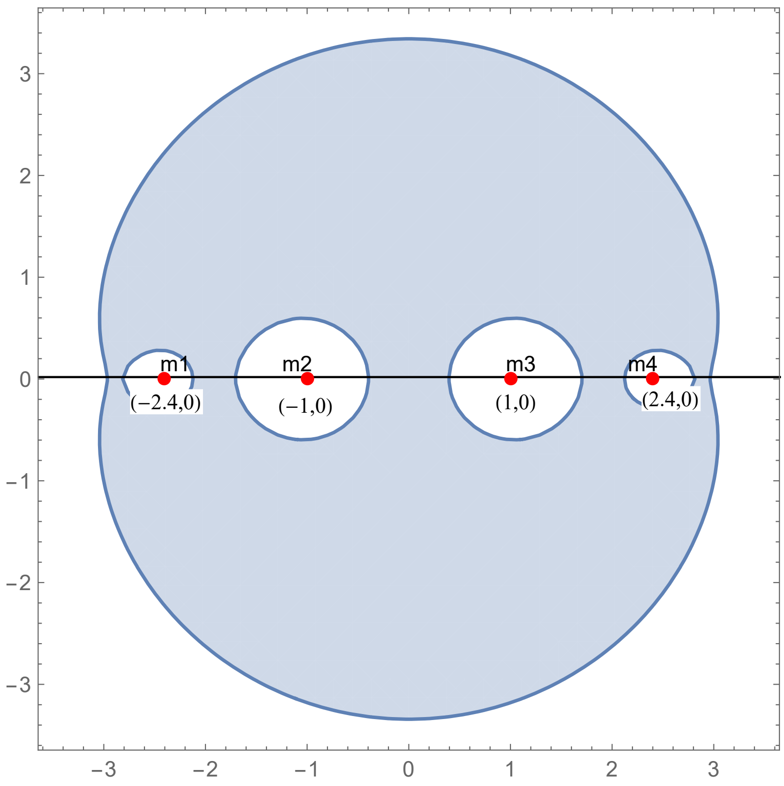

FIGURE 4.26: Permissible regions (white) of motion for $C = 7.5$.

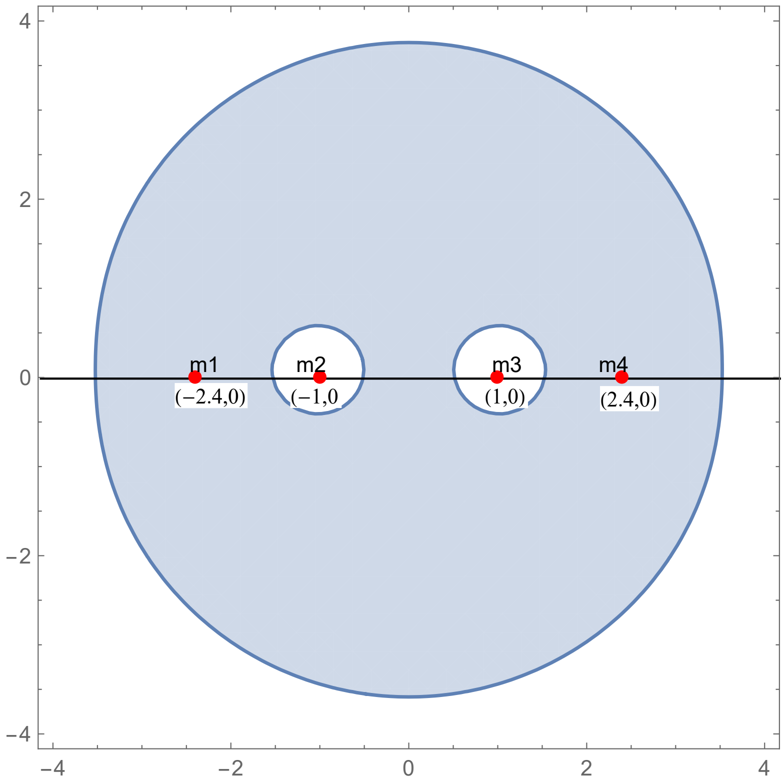

FIGURE 4.27: Permissible regions (white) of motion for $C = 8.5$.

Chapter 5

Conclusion

In this thesis we discussed the motion of an infinitesimal mass m_5 in the xy-plane under influence of the gravitational force of four masses m_1, m_2, m_3 and m_4 respectively. There are two pairs of equal masses, that are symmetric about the center of mass moving in such a way that their configuration is always in a line. The pair of larger masses lie in the middle and the pair of smaller masses lie at the corners each. The central configurations of four-bodies in a line is a geometric configurations of four-bodies in which the gravitational forces are balanced in such a way that the four bodies rotate together about their center of mass and maintain this configuration i.e. in a line all the time. With this arrangement of masses, we used the central configuration condition to find the equations of motion of four massive bodies m_1, m_2, m_3 and m_4. Solving equations for M and m (using the condition of positivity on masses) we get the constraints on the distance parameter a and b which are positive. In the second part of our problem discussed the motion of infinitesimal particle under the gravitational field of four massive bodies. Supposed that the infinitesimal mass m_5 is such that it does not affect the gravitational field of four big masses. We investigated all possible situation of the positions of equilibrium points/Lagrange points of m_5 according to the central configuration criterion for the four massive bodies. We also investigated the linear stability of these equilibrium points. Lastly, discuss the jacobian constant for infinitesimal mass in the presence of big masses. The Jacobian constant of motion is one of the most important constant of the dynamical system, which represent the movement of the infinitesimal body. The movement of the infinitesimal body depends on the Jacobi constant's value. The Jacobi constant C in the equation (4.18) gives us two different regions for choosing different values. One is a permissible region of motion for infinitesimal particle and the second is a region where infinitesimal particle are not allowed to move. By changing the value of C for the case(I) and case(II), also analyzed all permissible region of $m5$. All permissible regions (white region) of motion of m_5 are shown in figures for different cases.

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