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Williamson Fluid Flow with Joule Heating, Thermal Radiation and Chemical Reaction

by

Adil Hussain

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degree of Master of Philosophy

in the
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This thesis is dedicated to my beloved, family specially to,

My Father Mr. Jehan Bahadar*,*

My Mother *and* ***My Brother Mr. Ilyas Ahmad***

A determined and aristocratic embodiment who educate me to belief in ***ALLAH****,*
without his constant money generation this THESIS would never have been completed and without constant compelling to make me study I would never have been admitted to this institution.

CERTIFICATE OF APPROVAL

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Abstract

The aim of this thesis is to analyze and investigate numerically the Williamson fluid model with MHD, Joul heating, concentration and chemical reaction of the flow of an electrically conducting nanofluid past a nonlinear stretching sheet through a porous medium. The governing nonlinear boundary value problem involving the partial differential equations is reduced to a system of nonlinear ordinary differential equations by using appropriate similarity transformations. The nonlinear boundary values problem is solved numerically by using well known shooting technique opted in the computational software Matlab. The influence of some important physical parameters such as viscosity, thermal conductivity, radiation and Williamson parameters on velocity profile, temperature distribution, concentration profile, Nusselt number, skin friction and Sherwood number are studied and presented in graphical and tabular forms. It is observed that by rising the values of the viscosity parameter, the skin friction and local Nusselt number are decreased. By rising the values of Williamson parameter, the Sherwood number also shows a decreasing behaviour.

Contents

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Abbreviations

IVPs	Initial Value Problems
MHD	Magnetohydrodynamics
ODEs	Ordinary Differential Equations
PDEs	Partial Differential Equations
RK	Runge-Kutta

Symbols

C	Concentration
Cf_x	Local Skin Friction
C_p	Specific Heat
C_∞	Ambient Concentration
δ	Local Williamson Fluid
Ec	Local Eckert Number
ε	Thermal Conductivity
f	Dimensionless velocity
k^*	Absorption Coeffficient
k	Thermal Conductivity
M	Magnetic Parameter
m	Stretching Parameter
Nu_x	Local Nusselt Number
Pr	Prandtl Number
q_r	Heat Flux
R	Radiation Parameter
Re_x	Local Reynolds Number
ρ_∞	Fluid Density
Sc	Schmidt Number
Sh_x	Sherwood Number
T	Temperature
T_∞	Constant Ambient Temperature
T_w	Temperature of the Sheet
u	x-component of fluid velocity
v	y-component of fluid velocity

Greek Symbol

α	Thermal Diffusivity
η	Similarity Variable
μ	Fluid Viscosity
ν	Kinematic Viscosity
ϕ	Dimensionless Concentration
ψ	Stream Function
ρ	Density
σ^*	Stefan-Boltzmann
θ	Dimensionless Temperature

Subscripts

∞	Ambient Condition
w	Condition on Surface

Chapter 1

Introduction

Various enormous efforts have been made in the flowing of fluid due to stretching sheet operation over the last two decades to provide valuable descriptions of industrial processes and standard manufacturing, such as food preservation processes, petroleum filtering operation, polymer manufacturing, crystal manufacturing and paper production [1]. As a result, many researchers have made efforts toward the novel findings that serve this field, due to and for the sake of the established importance of these topics [1]. The flow in the boundary layer was investigated, on a continuous solid surface with constant speed. Sakiadis [1], analyzed heat transfer in two dimensional flows past a flat past a flat moving sheet. Due to ambient fluid entrainment, this condition represents a separate class of boundary-layer problem, with a solution that differs significantly from that of boundary-layer flow over a semi-infinite flat plate [1]. The first contributions to this area were made by Sakiadis, who discussed the fluid flow due to a stretching surface [1].

The impact of heat transport phenomenon in fluid flows over a stretching sheet has significant consideration in scientific and technical developments. These achievements include high-temperature steel rolling, metal extrusion, metal working, paper manufacture, glass fibre production, crystal flowing, and continuous casting [1]. Crane [2], the problem of Blasius fluid flow due to a stretching sheet was investigated, which is important in the field of plastic film drawing. In 1970, he continued his work to stretching surface [2]. He analytically analyzed Nusselt number and skin friction. Chen and Char [3], investigate the mechanism of wall heat flux and its effect on a linearly stretching plate.

The heat transfer properties of a continuously micropolar boundary layer were investigated as fluid flowed over a linearly stretching, continuous sheet by Mohammadein and Gorla [4]. Shortly after, the Mohammadein, Gorla and Liu [4],[5], perceived the significance of hydromagnetic flow over a stretching sheet in general by considering heat and mass transfer. Cortell [6], the concept of an impermeable stretching sheet was used to present a steady fluid of an approach to the fluid behaviour based on flow and mass transfer. In a later study, Chen [7] attempted to complete his work on MHD non-Newtonian power-law fluid over a stretching sheet and some significant mechanisms. In the presence of heat generation/absorption and thermal radiation, the problem of a power-law fluid past a stretching surface with a magnetohydrodynamic has been addressed [7].

An increasing number of investigations on the exponentially stretching sheet have been performed recently by various researchers [8–10]. Many authors [11–16],take the problem of the nonlinearly stretching sheet. Because of its widespread applications in the chemical and petroleum industries, biological sciences, and geophysics, the analysis of non-Newtonian fluid has received a significant attention.

In particular, boundary layer flows of non-Newtonian fluids over a stretching surface are prevalent in several industrial processes, for example, drawing of plastic films, extrusion of a polymer sheet from a dye, oil recovery, food processing, paper production and numerous others [14]. The well-known Navier-Stokes equations are ineffective for describing the flow behaviour of non-Newtonian materials [14]. On the other hand, non-Newtonian materials have a variety of constitutive relations proposed in the literature due to their versatility. Such materials have been classified into three subcategories known as differential, integral and rate types [14]. Flow investigation using a variable-thickness by a stretching sheet has a wide range of technological applications. Although this concept is well-understood for flat sheets but there is little informations available for variable thickness sheets [14].

In this thesis, we apply the Williamson model, that was presented by Williamson in 1929 [17] . Because of the use of this form of fluid model, more study is being done in this area [18–23]. Extrusion of plastic or metal sheets, is the most valuable application in the study of heat transfer. Observation of cooling and heat transfer is very important in the extrusion process because of its effects on final product [23]. In the present work, an attempts to examine the Williamson flow of fluid due to non-linear a stretching sheet with viscous dissipation and thermal radiation [24].

1.1 Thesis Contributions

In this thesis, first the work of Ahmed M. Megahed [24] reviewed in detail and then extended by MHD, Joul Heating, concentration and chemical reaction. The non-dimensional PDEs are converted into the dimensionless system of ODEs by using the similarity transformation and boundary conditions. The influence of different parameters like Williamson parameter δ, radiation parameter R, thermal conductivity ε, viscosity parameter α, Eckert number Ec on the skin friction $\frac{1}{2}(Re_x)^{\frac{1}{2}}Cf_x$, local Nusselt number $(Re_x)^{-\frac{1}{2}}Nu_x$ and Sherwood number $Re_x^{-\frac{1}{2}}Sh_x$ which has been discussed in graphical and tabular forms.

1.2 Thesis Layout

A brief outline of the present thesis is given below.

Chapter 2 discuss few fundamental definitions and terminologies, which will be later used in theis thesis.

Chapter 3 shows the numerical investigation of the Williamson fluid flow with viscous dissipatin and thermal radiation. The results are obtained by using the shooting method.

Chapter 4 extends the model discussed in chapter 3 by including MHD, joule heating and concentration.

Chapter 5 contains the conclusion of the thesis.

References used in the thesis are specified in **Biblography**.

Chapter 2

Preliminaries

This chapter contains some basic definitions, termenologies and governing laws, widely used in the rest of the thesis. This chapter will help a reader who is interested in the research problems discussed in the next chapters.

2.1 Some Basic Definitions

Definition 2.1.1 (Fluid)
"A fluid is a substance that deforms continuously under the application of a shear stress no matter how small the shear stress may be."[25]

Definition 2.1.2 (Fluid Statics)
"The study of fluid at rest is called fluid statics."[26]

Definition 2.1.3 (Fluid Mechanics)
"Fluid mechanics is that branch of science which deals with the behaviour of the fluid (liquid or gass) at rest as well as in motion."[26]

Definition 2.1.4 (Fluid Dynamics)
"The study of fluids in motion if the pressure forces are also considered is called fluid dynamics."[26]

Definition 2.1.5 (Fluid Kinematics)

"The study of fluids in motion, where pressure forces are not considered, is called fluid kinematics."[26]

Definition 2.1.6 (Viscosity)

"Viscosity is defined as the property of a fluid which offers resistance to the movement of one layer of fluid over another adjacent layer of the fluid. Mathematiclly,

$$\mu = \frac{\tau}{\frac{\partial u}{\partial y}}$$

where μ is viscosity coefficient, τ is shear stress and $\frac{\partial u}{\partial y}$ represents the velocity gradient."[26]

Definition 2.1.7 (Kinematic Viscosity)

"It is defined as the ratio between the dynamic viscosity and density of fluid. It is denoted by Greek symbol ν called nu. Mathematically,

$$\nu = \frac{\mu}{\rho}.\text{"[26]}$$

Definition 2.1.8 (Thermal Diffusivity)

"The rate at which heat diffuses by conducting through a material depends on the thermal diffusivity and can be defined as:

$$\alpha = \frac{k}{\rho C_p},$$

where α is the thermal diffusivity, k is the thermal conductivity, ρ is the density and C_p is the specific heat at constant pressure."[27]

Definition 2.1.9 (Thermal Conductivity)

"The Fourier heat conduction law states that the heat flow is proportional to the temperature gradient. The coefficient of proportionality is a material parameter known as the thermal conductivity which may be a function of a number of variables."[28]

2.2 Types of Fluid

Definition 2.2.1 (Real Fluid)

"A fluid, which possesses viscosity, is known as a real fluid. In actual practice, all the fluids are real fluids. Some of its examples are petrol, air etc."[26]

Definition 2.2.2 (Ideal Fluid)

"A fluid, which is incomperssible and is having no viscosity, is known as an ideal fluid. Ideal fluid is only an imaginary fluid as all the fluids, which exist, have some viscosity."[26]

Definition 2.2.3 (Newtonian Fluid)

"A real fluid in which the shear stress is directly proportional to the rate of shear strain (or velocity gradient), is known as a Newtonian fluid. Some of its examples are water, air, alcohol, glycerol, thin motor oil etc."[26]

Definition 2.2.4 (Non-Newtonian Fluid)

"A real fluid in which the shear stress is not directly proportional to the rate of shear strain (or velocity gradient), is known as a Non-Newtonain fluid. Some of its examples blood, saliva, Soap solutions, cosmetics, and toothpaste etc.

$$\tau_{xy} \propto \left(\frac{du}{dy}\right)^m, \qquad m \neq 1$$

$$\tau_{xy} = \mu \left(\frac{du}{dy}\right)^m."[26]$$

2.3 Types of Flow

Definition 2.3.1 (Compressible Flow)

"Compressible flow is that type of flow in which the density of the fluid changes from point to point or in other words the density (ρ) is not constant for the fluid. Thus, mathematically, for compressible flow

$$\rho \neq c,$$

where c is constant."[26]

Definition 2.3.2 (Incompressible Flow)

"Incompressible flow is that type of flow in which the density is constant for the fluid. Liquids are generally incompressible while gases are compressible. Mathematically, for incompressible flow

$$\rho = c,$$

where c is constant."[26]

Definition 2.3.3 (Rotational Flow)

"Rotational flow is that type of flow in which the fluid particles while flowing along stream-lines, also rotate about their own axis."[26]

Definition 2.3.4 (Irrotational Flow)

"If the fluid particles while flowing along sream-lines, do not rotate about their own axis that type of flow is called irrotational flow."[26]

Definition 2.3.5 (External Flow)

"Flows over bodies immersed in an unbounded fluid are termed external flows."[25]

Definition 2.3.6 (Internal Flow)

"Flows completely bounded by solid surfaces are called internal or pipe or duct flows. The flow of water in a pipe is an example of internal flow."[25]

Definition 2.3.7 (Steady Flow)

"Steady flow is defined as that type of flow in which the fluid characteristics like velocity, pressure, density etc., at a point do not change with time. Thus for steady flow, mathematically, we have

$$\frac{\partial Q}{\partial t} = 0$$

where Q is any fluid property."[26]

Definition 2.3.8 (Unsteady Flow)

"Unsteady flow is that type of flow, in which the velocity, pressure or denstiy at a point changes with respect to time. Thus, mathematically, for unsteady flow,

$$\frac{\partial Q}{\partial t} \neq 0$$

where Q is any fluid property."[26]

2.4 Modes of Heat Transfer

Definiton 2.4.1 (Heat Transfer)

"Heat transfer is a branch of engineering that deals with the transfer of thermal energy from one point to another within a medium or from one medium to another due to the occurrence of a temperature difference."[28]

Definition 2.4.2 (Thermal Radiation)

"Thermal radiation is defined as radiant (electromagnetic) energy emitted by a medium and is due solely to the temperature of the medium."[28]

Definition 2.4.2 (Conduction)

"The transfer of heat within a medium due to a diffusion process is called conduction."[28]

Definition 2.4.2 (Convection)

"Convection heat transfer is usually defined as energy transport effected by the motion of a fluid. The convection heat transfer between two dissimilar media is governed by Newtons law of cooling. It states that the heat flow is proportional to the difference of the temperatures of the two media."[28]

2.5 Dimensionless Numbers

Definition 2.5.1 (Prandtle Number)

"This number expresses the ratio of the momentum diffusivity (viscosity) to the thermal diffusivity. Mathematically, it can be defined as

$$P_r = \frac{\mu C_p}{k} = \frac{\nu}{\alpha}$$

where μ represent the dynamic viscosity, C_p denotes the specific heat capacity, k represent thermal conductivity, ν represent kinematic viscosity and α represent thermal diffusivity. With small P_r numbers ($P_r < 1$), the molecules heat transfer by conduction predominates over that by convection. With ($P_r > 1$), it is opposite case."[29]

Definition 2.5.2 (Nusselt Number)

"A hot surface is cooled by a cold fluid stream. The heat from the hot surface, which is maintained at a constant temperature, is diffused through a boundary layer and converted away by the cold stream. Mathematically,

$$Nu_x = \frac{qL}{k}$$

where q stands for the convection heat transfer, L for the characteristic length and k stands for average thermal conductivity."[30]

Definition 2.5.3 (Eckert Number)

"It is a dimensionless number used in continum mechanics. It describes the relation between flows and the boundary layer enthalpy difference and it is used for characterized heat dissipation. Mathematically,

$$E_c = \frac{u^2}{C_p \nabla T}$$

where C_p denotes the specific heat capacity, u is flow velocity and ∇T is temperature difference. "[25]

Definition 2.5.4 (Skin Friction)

"It expresses the dynamic friction resistance originating in viscous fluid flow around a fixed wall. Mathematically,

$$C_{fx} = \frac{\tau_w}{\frac{1}{2}\rho w_\infty^2}$$

where τ_w denotes the shear stress on the wall, ρ is the fluid denstiy and w_∞ is the free fluid flow."[29]

Definition 2.5.5 (Sherwood Number)

"It is a non-dimensional quantity which shows the ratio of the mass transport by convection to the transfer of mass by diffusion. Mathematically,

$$Sh_x = \frac{kL}{D}$$

where L is characteristic length, D is the mass diffusivity and k is the mass transfer coefficient."[29]

Definition 2.5.6 (Reynolds Number)

"It is defined as the ratio of inertia force of a flowing fluid and the viscous force of the fluid. Mathematically,

$$Re_x = \frac{VL}{\nu}$$

where V denotes the free stream velocity, L is the characteristic length and ν stands for kinematic viscosity."[26]

2.6 Governing Laws

Definition 2.6.1 (Continuity Equation)

"The principle of conservation of mass states that the time rate of change of mass in a fixed volume is equal to the net rate of flow of mass across the surface. The mathematical statement of the principle results in the following equation, known as the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \mathrm{v}) = 0.\text{"}[28]$$

Definition 2.6.2 (Momentum Equation)

"The principle of conservation of linear momentum (or Newton's Second Law) states that the time rate of change of linear momentum of a given set of particles is equal to the vector sum of all the external forces acting on the particles of the set, provided Newton's third law of action and reaction governs the internal forces. Mathematically, it can be represented as

$$\frac{\partial}{\partial t}(\rho v) + \nabla.[(\rho v)v] = \nabla.\sigma + \rho g.\text{"}[28]$$

Definition 2.6.3 (Energy Equation)

"The law of conservation of energy (or the First Law of Thermodynamics) states that the time rate of change of the total energy is equal to the sum of the rate of work done by applied forces and change of heat content per unit time.

$$\frac{\partial \rho e}{\partial t} + \nabla.\rho v e = -\nabla.q + Q + \phi.\text{"}[28]$$

Chapter 3

Williamson Fluid Flow over a Nonlinearly Stretching Porous Sheet with Viscous Dissipation and Thermal Radiation

3.1 Introduction

In this chapter, the numerical solution of Williamson fluid over a nonlinearly stretching porous sheet in the presence of viscous dissipation and thermal radiation is addressed. The governing nonlinear partial differential equations are converted into a system of dimensionless ordinary differential equation by using some appropriate similiarity transformations. In order to solve the ODEs, the shooting method is applied. For the numerical computations, the computational software MATLAB is opted. The numerical solution of the system of ODEs is analysed for various values of the physical parameters of interest. Tables and graphs are used to show the numerical results. This chapter provides a detail review of [24].

3.2 Problem Formulation

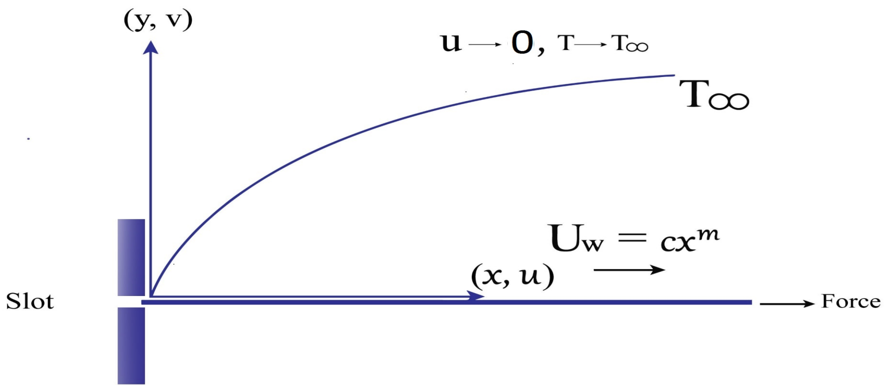

FIGURE 3.1: Fluid Flow over a nonlinearly stretching sheet.

Here a 2-D Williamson flowing of fluid over a nonlinearly stretchable sheet with thermal radiation and viscous dissipation has been considered. It is also assumed that the variable stretching velocity of the porous medium is $u_w = cx^m$. The governing equations are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{3.1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{1}{\rho_\infty}\frac{\partial}{\partial y}\left(\mu(T)\frac{\partial u}{\partial y} + \mu(T)\frac{\Gamma}{\sqrt{2}}\left(\frac{\partial u}{\partial y}\right)^2\right), \tag{3.2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{1}{\rho_\infty c_p}\frac{\partial}{\partial y}\left(K(T)\frac{\partial T}{\partial y}\right) + \frac{\mu(T)}{\rho_\infty c_p}\left(1 + \frac{\Gamma}{\sqrt{2}}\frac{\partial u}{\partial y}\right)\left(\frac{\partial u}{\partial y}\right)^2 - \frac{1}{\rho_\infty c_p}\frac{\partial q_r}{\partial y}. \tag{3.3}$$

The affilliated BCs are:

$$\left.\begin{aligned} &u = cx^m, \quad T_w(x) = T_\infty + Ax^r, \quad v = 0 \qquad at \;\; y = 0 \\ &T \to T_\infty, \quad u \to 0 \qquad as \;\; y \to \infty \end{aligned}\right\} \tag{3.4}$$

In this model, u and v are the velocity components, which are represented in xy-directions, respectively.

The fluid density is ρ_∞ and heat flux term is q_r while the c_p is the specific heat at a constant perssure.
The temperature T^4 can be extended by using the Taylor series about T_∞ as follows

$$T^4 = T_\infty^4 + 4T_\infty^3(T - T_\infty) + 6T_\infty^2(T - T_\infty)^2 + ...$$

Ignoring the highest order terms, we have

$$\begin{aligned}
T^4 &= T_\infty^4 + 4T_\infty^3(T - T_\infty),\\
T^4 &= T_\infty^4 + 4T_\infty^3 T - 4T_\infty^4,\\
T^4 &= 4T_\infty^3 T - 3T_\infty^3.
\end{aligned}$$

According to the Rosseland approximation we can use the q_r as a function of temperature as follows.

$$q_r = -\frac{4\sigma*}{3k*}\frac{\partial T^4}{\partial y}.$$

Here the Stefano-Boltzmann constant is σ^*, while the absorption coefficient is k^*. [23]
For the conversion of (3.1)-(3.3) into the system of ODEs, the following similirity transformations have been used

$$\left.\begin{aligned}
&\eta = \left(\frac{cx^{m-1}}{\nu_\infty}\right)^{\frac{1}{2}} y,\\
&\psi(x,y) = (cx^{m+1}\nu_\infty)^{\frac{1}{2}} f(\eta),\\
&\theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty}.
\end{aligned}\right\} \tag{3.5}$$

In the above formulae the kinematic viscosity is ν_∞, while the stream function isψ (x, y), which obeys the continuity equation (3.1) by using the following:

$$u = \frac{\partial \psi}{\partial y}, \ \ v = -\frac{\partial \psi}{\partial x}. \tag{3.6}$$

We also know that

$$\mu = \mu_\infty e^{-\alpha\theta}, \quad K = K_\infty(1 + \varepsilon\theta) \tag{3.7}$$

where μ_∞ is the viscostiy and K_∞ represents thermal conductivity. [31]

From (3.5), η andψ can be expressed as below:

- $\eta = \left(\dfrac{cx^{m-1}}{\nu_\infty}\right)^{\frac{1}{2}} y = \left(\dfrac{c}{\nu_\infty}\right)^{\frac{1}{2}} x^{\frac{m-1}{2}} y$

- $\psi = (cx^{m+1}\nu_\infty)^{\frac{1}{2}} f(\eta) = (c\nu_\infty)^{\frac{1}{2}} x^{\frac{m+1}{2}} f(\eta)$

The detailed process for converting (3.1)-(3.3) into the non-dimensional form has been described as follows

- $$\begin{aligned} u = \frac{\partial\psi}{\partial y} &= (c\nu_\infty)^{\frac{1}{2}} x^{\frac{m+1}{2}} f'(\eta) \left(\frac{c}{\nu_\infty}\right)^{\frac{1}{2}} x^{\frac{m-1}{2}} \\ &= cx^m f'(\eta). \end{aligned} \tag{3.8}$$

- $$\begin{aligned} v = -\frac{\partial\psi}{\partial x} &= -(c\nu_\infty)^{\frac{1}{2}} \left[x^{\frac{m+1}{2}} f'(\eta) \left(\frac{c}{\nu_\infty}\right)^{\frac{1}{2}} y \left(\frac{m-1}{2}\right) x^{\frac{m-3}{2}} + f(\eta)\left(\frac{m+1}{2}\right) x^{\frac{m-1}{2}}\right] \\ &= -(c\nu_\infty)^{\frac{1}{2}} \left(\frac{c}{\nu_\infty}\right)^{\frac{1}{2}} \left(\frac{m-1}{2}\right) x^{m-1} y f'(\eta) - (c\nu_\infty)^{\frac{1}{2}} \left(\frac{m+1}{2}\right) x^{\frac{m-1}{2}} f(\eta) \\ &= -c\left(\frac{m-1}{2}\right) x^{m-1} y f'(\eta) - (c\nu_\infty)^{\frac{1}{2}} \left(\frac{m+1}{2}\right) x^{\frac{m-1}{2}} f(\eta) \\ &= -c\left(\frac{m-1}{2}\right) x^{m-1} y f'(\eta) - (c\nu_\infty)^{\frac{1}{2}} \left(\frac{m+1}{2}\right) x^{\frac{m-1}{2}} f(\eta). \end{aligned} \tag{3.9}$$

- $$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial}{\partial x}\left(cx^m f'(\eta)\right), \\ &= c\left[x^m f''(\eta)\left(\frac{c}{\nu_\infty}\right)^{\frac{1}{2}} y \left(\frac{m-1}{2}\right) x^{\frac{m-3}{2}} + f'(\eta) m x^{m-1}\right] \\ &= cx^m \left(\frac{c}{\nu_\infty}\right)^{\frac{1}{2}} \left(\frac{m-1}{2}\right) x^{\frac{m-3}{2}} y f''(\eta) + cmx^{m-1} f'(\eta) \\ &= c^{\frac{3}{2}} \left(\frac{1}{(\nu_\infty)^{\frac{1}{2}}}\right) \left(\frac{m-1}{2}\right) x^{\frac{3m-3}{2}} y f''(\eta) + cmx^{m-1} f'(\eta). \end{aligned} \tag{3.10}$$

- $$\begin{aligned} \frac{\partial v}{\partial y} &= \frac{\partial}{\partial y}\left[-c\left(\frac{m-1}{2}\right) x^{m-1} y f'(\eta) - (c\nu_\infty)^{\frac{1}{2}} \left(\frac{m+1}{2}\right) x^{\frac{m-1}{2}} f(\eta)\right] \\ &= -c\left(\frac{m-1}{2}\right) x^{m-1}\left[y f''(\eta)\left(\frac{c}{\nu_\infty}\right)^{\frac{1}{2}} x^{\frac{m-1}{2}} + f'(\eta)\right] \\ &\quad - (c\nu_\infty)^{\frac{1}{2}} \left(\frac{m+1}{2}\right) x^{\frac{m-1}{2}} f'(\eta) \left(\frac{c}{\nu_\infty}\right)^{\frac{1}{2}} x^{\frac{m-1}{2}} \end{aligned}$$

$$= -c\left(\frac{m-1}{2}\right)x^{m-1}\left(\frac{c}{\nu_\infty}\right)^{\frac{1}{2}}x^{\frac{m-1}{2}}yf''(\eta) - c\left(\frac{m-1}{2}\right)x^{m-1}f'(\eta)$$
$$- cx^{m-1}\left(\frac{m+1}{2}\right)f'(\eta)$$
$$= -c^{\frac{3}{2}}\frac{1}{(\nu_\infty)^{\frac{1}{2}}}\left(\frac{m-1}{2}\right)x^{\frac{3m-3}{2}}yf''(\eta) - cx^{m-1}f'(\eta)\left(\frac{m-1}{2}+\frac{m+1}{2}\right)$$
$$= -c^{\frac{3}{2}}\left(\frac{1}{(\nu_\infty)^{\frac{1}{2}}}\right)\left(\frac{m-1}{2}\right)x^{\frac{3m-3}{2}}yf''(\eta) - cmx^{m-1}f'(\eta). \tag{3.11}$$

Adding (3.10) and (3.11) we get (3.1)

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = \left[c^{\frac{3}{2}}\left(\frac{1}{(\nu_\infty)^{\frac{1}{2}}}\right)\left(\frac{m-1}{2}\right)x^{\frac{3m-3}{2}}yf''(\eta) + cmx^{m-1}f'(\eta)\right],$$
$$+\left[-c^{\frac{3}{2}}\left(\frac{1}{(\nu_\infty)^{\frac{1}{2}}}\right)\left(\frac{m-1}{2}\right)x^{\frac{3m-3}{2}}yf''(\eta) - cx^{m-1}f'(\eta)\right].$$
$$\Rightarrow \quad \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = 0. \tag{3.12}$$

Hence the continuity equation is satisfied, identically. Now the detailed procedure for the conversion of (3.2) into the non-dimensional form is as follows,

- $$\frac{\partial u}{\partial y} = cx^m f''(\eta)\left(\frac{c}{\nu_\infty}\right)^{\frac{1}{2}}x^{\frac{m-1}{2}}$$
$$= c^{\frac{3}{2}}\left(\frac{1}{(\nu_\infty)^{\frac{1}{2}}}\right)x^{\frac{3m-1}{2}}f''(\eta).$$
- $$u\frac{\partial u}{\partial x} = cx^m f'(\eta)\left[c^{\frac{3}{2}}\left(\frac{1}{(\nu_\infty)^{\frac{1}{2}}}\right)\left(\frac{m-1}{2}\right)x^{\frac{3m-3}{2}}yf''(\eta) + cmx^{m-1}f'(\eta)\right]$$
$$= cx^m f'(\eta)c^{\frac{3}{2}}\left(\frac{1}{(\nu_\infty)^{\frac{1}{2}}}\right)\left(\frac{m-1}{2}\right)x^{\frac{3m-3}{2}}yf''(\eta) + cx^m f'(\eta)cmx^{m-1}f'(\eta)$$
$$= c^{\frac{5}{2}}\frac{1}{(\nu_\infty)^{\frac{1}{2}}}x^{\frac{5m-3}{2}}yf'(\eta)\left(\frac{m-1}{2}\right)f''(\eta) + c^2mx^{2m-1}f'^2(\eta). \tag{3.13}$$
- $$v\frac{\partial u}{\partial y} = \left[-c\left(\frac{m-1}{2}\right)x^{m-1}yf'(\eta)\right.$$
$$\left.-(c\nu_\infty)^{\frac{1}{2}}\left(\frac{m+1}{2}\right)x^{\frac{m-1}{2}}f(\eta)\right]c^{\frac{3}{2}}\left(\frac{1}{(\nu_\infty)^{\frac{1}{2}}}\right)x^{\frac{3m-1}{2}}f''(\eta)$$
$$= -c\left(\frac{m-1}{2}\right)x^{m-1}yf'(\eta)c^{\frac{3}{2}}\left(\frac{1}{(\nu_\infty)^{\frac{1}{2}}}\right)x^{\frac{3m-1}{2}}f''(\eta)$$
$$-(c\nu_\infty)^{\frac{1}{2}}\left(\frac{m+1}{2}\right)x^{\frac{m-1}{2}}f(\eta)c^{\frac{3}{2}}\left(\frac{1}{(\nu_\infty)^{\frac{1}{2}}}\right)x^{\frac{3m-1}{2}}f''(\eta)$$

$$
\begin{aligned}
&= -yf'(\eta)\frac{c^{\frac{5}{2}}}{(\nu_\infty)^{\frac{1}{2}}}\left(\frac{m-1}{2}\right)x^{\frac{5m-3}{2}}f''(\eta) - c^2 f(\eta)\left(\frac{m+1}{2}\right)x^{2m-1}f''(\eta)\\
&= -\frac{c^{\frac{5}{2}}}{(\nu_\infty)^{\frac{1}{2}}}x^{\frac{5m-3}{2}}yf'(\eta)\left(\frac{m-1}{2}\right)f''(\eta)\\
&\quad - c^2x^{2m-1}f(\eta)\left(\frac{m+1}{2}\right)f''(\eta). \qquad (3.14)
\end{aligned}
$$

- $\dfrac{\partial u}{\partial y} = c^{\frac{3}{2}}\left(\dfrac{1}{(\nu_\infty)^{\frac{1}{2}}}\right)x^{\frac{3m-1}{2}}f''(\eta).$

- $$\begin{aligned}\left(\frac{\partial u}{\partial y}\right)^2 &= \left(c^{\frac{3}{2}}\left(\frac{1}{(\nu_\infty)^{\frac{1}{2}}}\right)x^{\frac{3m-1}{2}}f''(\eta)\right)^2\\ &= \left(c^{\frac{3}{2}}\left(\frac{1}{(\nu_\infty)^{\frac{1}{2}}}\right)x^{\frac{3m-1}{2}}\right)^2 f''^2(\eta).\end{aligned}$$

- $\mu\dfrac{\partial u}{\partial y} = \mu_\infty e^{-\alpha\theta}c^{\frac{3}{2}}\left(\dfrac{1}{(\nu_\infty)^{\frac{1}{2}}}\right)x^{\frac{3m-1}{2}}f''(\eta).$

- $\mu\dfrac{\Gamma}{\sqrt{2}}\left(\dfrac{\partial u}{\partial y}\right)^2 = \mu_\infty e^{-\alpha\theta}\dfrac{\Gamma}{\sqrt{2}}\left(c^{\frac{3}{2}}\dfrac{1}{(\nu_\infty)^{\frac{1}{2}}}x^{\frac{3m-1}{2}}\right)^2 f''^2(\eta).$

- $\dfrac{\partial}{\partial y}\left[\mu(T)\dfrac{\partial u}{\partial y} + \mu(T)\dfrac{\Gamma}{\sqrt{2}}\left(\dfrac{\partial u}{\partial y}\right)^2\right]$

$$
\begin{aligned}
&= \frac{\partial}{\partial y}\Bigg[\mu_\infty e^{-\alpha\theta}c^{\frac{3}{2}}\left(\frac{1}{(\nu_\infty)^{\frac{1}{2}}}\right)x^{\frac{3m-1}{2}}f''(\eta)\\
&\quad + \mu_\infty e^{-\alpha\theta}\frac{\Gamma}{\sqrt{2}}\left(c^{\frac{3}{2}}\left(\frac{1}{(\nu_\infty)^{\frac{1}{2}}}\right)x^{\frac{3m-1}{2}}\right)^2 f''^2(\eta)\Bigg]\\
&= \mu_\infty c^{\frac{3}{2}}\left(\frac{1}{(\nu_\infty)^{\frac{1}{2}}}\right)x^{\frac{3m-1}{2}}\Bigg[e^{-\alpha\theta}f'''(\eta)\left(\frac{c}{\nu_\infty}\right)^{\frac{1}{2}}x^{\frac{m-1}{2}}\\
&\quad + f''(\eta)e^{-\alpha\theta}(-\alpha\theta')\left(\frac{c}{\nu_\infty}\right)^{\frac{1}{2}}x^{\frac{m-1}{2}}\Bigg]\\
&\quad + \mu_\infty\frac{\Gamma}{\sqrt{2}}\left(c^{\frac{3}{2}}\left(\frac{1}{(\nu_\infty)^{\frac{1}{2}}}\right)x^{\frac{3m-1}{2}}\right)^2\Bigg[2e^{-\alpha\theta}f''(\eta)f'''(\eta)\left(\frac{c}{\nu_\infty}\right)^{\frac{1}{2}}x^{\frac{m-1}{2}}\\
&\quad + e^{-\alpha\theta}(-\alpha\theta')\left(\frac{c}{\nu_\infty}\right)^{\frac{1}{2}}x^{\frac{m-1}{2}}f''^2(\eta)\Bigg]\\
&= \frac{\mu_\infty}{\nu_\infty}c^2x^{2m-1}e^{-\alpha\theta}f'''(\eta) - \frac{\mu_\infty}{\nu_\infty}c^2x^{2m-1}e^{-\alpha\theta}(\alpha\theta')f''(\eta)\\
&\quad + \frac{2\mu_\infty\Gamma}{\sqrt{2}}\left(\frac{c^{\frac{3}{2}}}{\nu_\infty^{\frac{1}{2}}}x^{\frac{3m-1}{2}}\right)\left(\frac{c^{\frac{3}{2}}}{\nu_\infty^{\frac{1}{2}}}x^{\frac{3m-1}{2}}\right)e^{-\alpha\theta}\left(\frac{c}{\nu_\infty}\right)^{\frac{1}{2}}x^{\frac{m-1}{2}}f''(\eta)f'''(\eta)\\
&\quad - \frac{\mu_\infty\Gamma}{\sqrt{2}}\left(\frac{c^{\frac{3}{2}}}{\nu_\infty^{\frac{1}{2}}}x^{\frac{3m-1}{2}}\right)\left(\frac{c^{\frac{3}{2}}}{\nu_\infty^{\frac{1}{2}}}x^{\frac{3m-1}{2}}\right)e^{-\alpha\theta}(\alpha\theta')\left(\frac{c}{\nu_\infty}\right)^{\frac{1}{2}}x^{\frac{m-1}{2}}f''^2(\eta)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\mu_\infty}{\nu_\infty} c^2 x^{2m-1} e^{-\alpha\theta} f'''(\eta) - \frac{\mu_\infty}{\nu_\infty} c^2 x^{2m-1} e^{-\alpha\theta} (\alpha\theta') f''(\eta) \\
&\quad + \mu_\infty \left(\frac{\sqrt{2} c^{\frac{3}{2}} x^{\frac{3m-1}{2}}}{\sqrt{\nu_\infty}} \right) \Gamma \frac{c^2}{\nu_\infty} x^{2m-1} e^{-\alpha\theta} f''(\eta) f'''(\eta) \\
&\quad - \frac{\mu_\infty \Gamma}{\sqrt{2}} \left(\frac{c^{\frac{3}{2}} x^{\frac{3m-1}{2}}}{\sqrt{\nu_\infty}} \right) \frac{c^2}{\nu_\infty} x^{2m-1} e^{-\alpha\theta} (\alpha\theta') f''^2(\eta) \\
&= \frac{\mu_\infty}{\nu_\infty} c^2 x^{2m-1} e^{-\alpha\theta} f'''(\eta) - \frac{\mu_\infty}{\nu_\infty} c^2 x^{2m-1} e^{-\alpha\theta} (\alpha\theta') f''(\eta) + \frac{\mu_\infty}{\nu_\infty} c^2 x^{2m-1} e^{-\alpha\theta} \delta f''(\eta) f'''(\eta) \\
&\quad - \frac{\mu_\infty}{\nu_\infty} \frac{\Gamma}{\sqrt{2}} \left(\frac{c^{\frac{3}{2}} x^{\frac{3m-1}{2}}}{\sqrt{\nu_\infty}} \right) c^2 x^{2m-1} e^{-\alpha\theta} (\alpha\theta') f''^2(\eta) \\
&= \frac{\mu_\infty}{\nu_\infty} c^2 x^{2m-1} e^{-\alpha\theta} f'''(\eta) - \frac{\mu_\infty}{\nu_\infty} c^2 x^{2m-1} e^{-\alpha\theta} (\alpha\theta') f''(\eta) + \frac{\mu_\infty}{\nu_\infty} c^2 x^{2m-1} e^{-\alpha\theta} \delta f''(\eta) f'''(\eta) \\
&\quad - \frac{1}{2} \frac{\mu_\infty}{\nu_\infty} \left(\frac{\sqrt{2} c^{\frac{3}{2}} x^{\frac{3m-1}{2}}}{\sqrt{\nu_\infty}} \right) \Gamma c^2 x^{2m-1} e^{-\alpha\theta} (\alpha\theta') f''^2(\eta) \\
&= \frac{\mu_\infty}{\nu_\infty} c^2 x^{2m-1} e^{-\alpha\theta} f'''(\eta) - \frac{\mu_\infty}{\nu_\infty} c^2 x^{2m-1} e^{-\alpha\theta} (\alpha\theta') f''(\eta) + \frac{\mu_\infty}{\nu_\infty} c^2 x^{2m-1} e^{-\alpha\theta} \delta f''(\eta) f'''(\eta) \\
&\quad - \frac{1}{2} \frac{\mu_\infty}{\nu_\infty} c^2 x^{2m-1} e^{-\alpha\theta} (\alpha\theta') \delta f''^2(\eta) \\
&= \frac{\mu_\infty}{\nu_\infty} e^{-\alpha\theta} c^2 x^{2m-1} \Big[-\frac{\delta}{2} \alpha\theta' f''^2(\eta) + -\alpha\theta' f''(\eta) + f'''(\eta) + \delta f''(\eta) f'''(\eta) \Big] \\
&= \rho_\infty e^{-\alpha\theta} c^2 x^{2m-1} \Big[(1 + \delta f''(\eta)) f'''(\eta) - \alpha\theta' f''(\eta)(1 + \frac{\delta}{2} f''(\eta)) \Big] \\
&= \rho_\infty e^{-\alpha\theta} c^2 x^{2m-1} \Big[(1 + \delta f''(\eta)) f'''(\eta) \\
&\quad - \alpha\theta' f''(\eta)(1 + \frac{\delta}{2} f'')(\eta) \Big] \\
&= \rho_\infty e^{-\alpha\theta} c^2 x^{2m-1} \Big[(1 + \delta f''(\eta)) f'''(\eta) \\
&\quad - \alpha\theta' f''(\eta)(1 + \frac{\delta}{2} f''(\eta)) \Big]
\end{aligned}
$$

where $\delta = (\frac{\sqrt{2} c^{\frac{3}{2}} x^{\frac{3m-1}{2}}}{\sqrt{\nu_\infty}})\Gamma.$

Adding (3.13) and (3.14), left side of (3.2) becomes

$$
\begin{aligned}
u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} &= \left[c^{\frac{5}{2}} \left(\frac{1}{(\nu_\infty)^{\frac{1}{2}}} \right) x^{\frac{5m-3}{2}} y f'(\eta) \left(\frac{m-1}{2} \right) f''(\eta) + c^2 m x^{2m-1} f'^2(\eta) \right] \\
&\quad + \Bigg[- \left(\frac{c^{\frac{5}{2}}}{(\nu_\infty)^{\frac{1}{2}}} \right) x^{\frac{5m-3}{2}} y f'(\eta) \left(\frac{m-1}{2} \right) f''(\eta) \\
&\qquad - c^2 x^{2m-1} f(\eta) \left(\frac{m+1}{2} \right) f''(\eta) \Bigg] \\
&= c^2 m x^{2m-1} f'^2(\eta) - c^2 x^{2m-1} f(\eta) \left(\frac{m+1}{2} \right) f''(\eta). \qquad (3.15)
\end{aligned}
$$

Now the right side of (3.2), becomes

$$
\begin{aligned}
&\frac{1}{\rho_\infty}\frac{\partial}{\partial y}\left(\mu(T)\frac{\partial u}{\partial y}+\mu(T)\frac{\Gamma}{\sqrt{2}}\left(\frac{\partial u}{\partial y}\right)^2\right)\\
&\quad = e^{-\alpha\theta}c^2x^{2m-1}\Big[(1+\delta f''(\eta))f'''(\eta)-\alpha\theta' f''(\eta)(1+\frac{\delta}{2}f''(\eta))\Big]. \qquad (3.16)
\end{aligned}
$$

Now comparing (3.15) and (3.16), we get

$$
\begin{aligned}
& f'^2(\eta)c^2mx^{2m-1}-c^2x^{2m-1}f(\eta)f''(\eta)\\
&\quad = e^{-\alpha\theta}c^2x^{2m-1}\Big[f'''(\eta)(1+\delta f''(\eta))-\alpha\left(1+\frac{\delta}{2}f''(\eta)\right)f''(\eta)\theta'(\eta)\Big].\\
\Rightarrow\quad & \left(mf'^2(\eta)-f(\eta)\left(\frac{m+1}{2}\right)f''(\eta)\right)c^2x^{2m-1}\\
&\quad = e^{-\alpha\theta}c^2x^{2m-1}\Big[-\alpha f''(\eta)\left(1+\frac{\delta}{2}f''(\eta)\right)\theta'(\eta)+(1+\delta f''(\eta))f'''(\eta)\Big]. \qquad (3.17)\\
\Rightarrow\quad & -f(\eta)\left(\frac{m+1}{2}\right)f''(\eta)+mf'^2(\eta)=e^{-\alpha\theta}\Big[f'''(\eta)(1+\delta f''(\eta))\\
&\qquad -\alpha\theta' f''(\eta)\left(1+\frac{\delta}{2}f''(\eta)\right)\Big]\\
\Rightarrow\quad & e^{-\alpha\theta}\left(f'''(\eta)(1+\delta f''(\eta))-\alpha\left(1+\frac{\delta}{2}f''(\eta)\right)\theta'(\eta)f''(\eta)\right)\\
&\qquad +f(\eta)\left(\frac{m+1}{2}\right)f''(\eta)-mf'^2(\eta)=0. \qquad (3.18)
\end{aligned}
$$

Next, to find the dimensionless form of (3.3), the procedure is as follows

- $$
\begin{aligned}
&\theta(\eta)=\frac{T-T_\infty}{T_w-T_\infty},\\
&\Rightarrow T-T_\infty=\theta(\eta)(T_w-T_\infty)\\
&\qquad\qquad =\theta(\eta)(T_\infty+Ax^r-T_\infty).\\
&\Rightarrow T=T_\infty+\theta(\eta)Ax^r.
\end{aligned}
$$

- $$
\begin{aligned}
\frac{\partial T}{\partial x}&=\frac{\partial}{\partial x}(T_\infty+\theta(\eta)Ax^r)\\
&=\frac{\partial}{\partial x}(T_\infty)+\frac{\partial}{\partial x}(A\theta(\eta)x^r),\\
&=A\frac{\partial}{\partial x}(\theta(\eta)x^r)=A\left[\theta(\eta)rx^{r-1}+x^r\theta'(\eta)\left(\frac{c}{\nu_\infty}\right)^{\frac{1}{2}}y\left(\frac{m-1}{2}\right)x^{\frac{m-3}{2}}\right]\\
&=Arx^{r-1}\theta(\eta)+A\left(\frac{c}{\nu_\infty}\right)^{\frac{1}{2}}\left(\frac{m-1}{2}\right)yx^{\frac{2r+m-3}{2}}\theta'(\eta). \qquad (3.19)
\end{aligned}
$$

- $$\frac{\partial T}{\partial y}=\frac{\partial}{\partial y}(T_\infty+\theta(\eta)Ax^r)$$

$$
\begin{aligned}
&= \frac{\partial}{\partial y}(T_\infty) + \frac{\partial}{\partial y}(Ax^r\theta(\eta)) \\
&= Ax^r \left(\frac{c}{\nu_\infty}\right)^{\frac{1}{2}} x^{\frac{m-1}{2}} \theta'(\eta) \\
&= A \left(\frac{c}{\nu_\infty}\right)^{\frac{1}{2}} x^{\frac{2r+m-1}{2}} \theta'(\eta).
\end{aligned} \tag{3.20}
$$

- $$\frac{\partial^2 T}{\partial y^2} = A \left(\frac{c}{\nu_\infty}\right)^{\frac{1}{2}} x^{\frac{2r+m-1}{2}} \theta''(\eta) \left(\frac{c}{\nu_\infty}\right)^{\frac{1}{2}} x^{\frac{m-1}{2}}$$
  $$= A \left(\frac{c}{\nu_\infty}\right) x^{r+m-1} \theta''(\eta).$$
- $$\left(\frac{\partial u}{\partial y}\right)^2 = \left(c^{\frac{3}{2}} \left(\frac{1}{(\nu_\infty)^{\frac{1}{2}}}\right) x^{\frac{3m-1}{2}} \right)^2 f''^2(\eta). \tag{3.21}$$
- $$
\begin{aligned}
q_r &= -\frac{4}{3}\frac{\delta^*}{k^*}\frac{\partial T^4}{\partial y} \\
&= -\frac{4}{3}\frac{\delta^*}{k^*}\frac{\partial}{\partial y}(4T_\infty^3 T - 3T_\infty^4) \\
&= -\frac{4}{3}\frac{\delta^*}{k^*}(4T_\infty^3 \frac{\partial T}{\partial y}) \\
&= -\frac{16}{3}\frac{\delta^*}{k^*} T_\infty^3 \frac{\partial T}{\partial y}.
\end{aligned}
$$
- $$
\begin{aligned}
\frac{\partial q_r}{\partial y} &= -\frac{16}{3}\frac{\delta^*}{k^*} T_\infty^3 \frac{\partial^2 T}{\partial y^2} \\
&= -\frac{16}{3}\frac{\delta^*}{k^*} T_\infty^3 \frac{Ac}{\nu_\infty} x^{r+m-1} \theta''(\eta).
\end{aligned} \tag{3.22}
$$

Using (3.19) and (3.20) in the left side of (3.3), we get

$$
\begin{aligned}
u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} &= cx^m f'(\eta) Arx^{r-1}\theta(\eta) + cx^m f'(\eta) A \left(\frac{c}{\nu_\infty}\right)^{\frac{1}{2}} \left(\frac{m-1}{2}\right) yx^{\frac{2r+m-3}{2}} \theta'(\eta) \\
&\quad + \left[-c\left(\frac{m-1}{2}\right) x^{m-1} y f'(\eta) - (c\nu_\infty)^{\frac{1}{2}} \left(\frac{m+1}{2}\right) x^{\frac{m-1}{2}} f(\eta) \right] \\
&\quad \left[A \left(\frac{c}{\nu_\infty}\right)^{\frac{1}{2}} x^{\frac{2r+m-1}{2}} \theta'(\eta) \right] \\
&= Acrx^{m+r-1} f'(\eta)\theta(\eta) + Ac^{\frac{3}{2}} \left(\frac{1}{(\nu_\infty)^{\frac{1}{2}}}\right) x^{\frac{3m+2r-3}{2}} \left(\frac{m-1}{2}\right) y f'(\eta)\theta'(\eta) \\
&\quad - Ac^{\frac{3}{2}} \left(\frac{1}{(\nu_\infty)^{\frac{1}{2}}}\right) x^{\frac{3m+2r-3}{2}} \left(\frac{m-1}{2}\right) y f'(\eta)\theta'(\eta) \\
&\quad - Ac\left(\frac{m+1}{2}\right) x^{r+m-1} f(\eta)\theta'(\eta) \\
&= Acrx^{m+r-1} f'(\eta)\theta(\eta) - Ac\left(\frac{m+1}{2}\right) x^{r+m-1} f(\eta)\theta'(\eta).
\end{aligned} \tag{3.23}
$$

Now the terms on the right side of (3.3) has been converted into the dimensionless form as flollows.

- $$\frac{1}{\rho_\infty c_p}\frac{\partial}{\partial y}\left(K(T)\frac{\partial T}{\partial y}\right) = \frac{1}{\rho_\infty c_p}\frac{\partial}{\partial y}\left((K_\infty + K_\infty\varepsilon\theta(\eta))\left(A\left(\frac{c}{\nu_\infty}\right)^{\frac{1}{2}} x^{\frac{2r+m-1}{2}}\theta'(\eta)\right)\right)$$

$$= \frac{1}{\rho_\infty c_p}\left[\frac{\partial}{\partial y}\left(K_\infty\left(Ax^{\frac{2r+m-1}{2}}\left(\frac{c}{\nu_\infty}\right)^{\frac{1}{2}}\theta'(\eta)\right)\right)\right]$$
$$+ \frac{\partial}{\partial y}\left[K_\infty\varepsilon\theta(\eta)\left(Ax^{\frac{2r+m-1}{2}}\left(\frac{c}{\nu_\infty}\right)^{\frac{1}{2}}\theta'(\eta)\right)\right]$$
$$= \frac{1}{\rho_\infty c_p}\left[K_\infty Ax^{\frac{2r+m-1}{2}}\left(\frac{c}{\nu_\infty}\right)^{\frac{1}{2}}\theta''(\eta)\left(\frac{c}{\nu_\infty}\right)^{\frac{1}{2}}x^{\frac{m-1}{2}}\right.$$
$$\left.+ K_\infty\varepsilon Ax^{\frac{2r+m-1}{2}}\left(\frac{c}{\nu_\infty}\right)^{\frac{1}{2}}\left(\theta(\eta)\theta''(\eta)\left(\frac{c}{\nu_\infty}\right)^{\frac{1}{2}}x^{\frac{m-1}{2}} + \theta'(\eta)\theta'(\eta)\left(\frac{c}{\nu_\infty}\right)^{\frac{1}{2}}x^{\frac{m-1}{2}}\right)\right]$$
$$= \frac{1}{\rho_\infty c_p}\left[K_\infty A\left(\frac{c}{\nu_\infty}\right)x^{r+m-1}\theta''(\eta) + K_\infty\varepsilon\theta(\eta)Ax^{r+m-1}\left(\frac{c}{\nu_\infty}\right)\theta(\eta)\theta''(\eta)\right.$$
$$\left.+ K_\infty\varepsilon Ax^{r+m-1}\left(\frac{c}{\nu_\infty}\right)\theta'^2(\eta)\right]$$
$$= \frac{K_\infty Ac}{\rho_\infty c_p\nu_\infty}x^{r+m-1}(\theta''(\eta) + \varepsilon\theta'^2(\eta) + \varepsilon\theta(\eta)\theta''(\eta))$$
$$= \frac{K_\infty Ac}{c_p\mu_\infty}x^{r+m-1}(\theta''(\eta) + \varepsilon\theta'^2(\eta) + \varepsilon\theta(\eta)\theta''(\eta))$$
$$= \frac{1}{p_r}Acx^{r+m-1}(\theta''(\eta) + \varepsilon\theta'^2(\eta) + \varepsilon\theta(\eta)\theta''(\eta)). \tag{3.24}$$

- $$\frac{\mu}{\rho_\infty c_p}\left(1 + \frac{\Gamma}{\sqrt{2}}\frac{\partial u}{\partial y}\right)\left(\frac{\partial u}{\partial y}\right)^2$$

$$= \frac{\mu}{\rho_\infty c_p}\left(1 + \frac{\Gamma}{\sqrt{2}}c^{\frac{3}{2}}\left(\frac{1}{(\nu_\infty)^{\frac{1}{2}}}\right)x^{\frac{3m-1}{2}}f''(\eta)\right)\left(c^{\frac{3}{2}}\left(\frac{1}{(\nu_\infty)^{\frac{1}{2}}}\right)x^{\frac{3m-1}{2}})\right)^2 f''^2(\eta)$$
$$= \frac{2}{2}\frac{\mu}{\rho_\infty c_p}\left(1 + \frac{\Gamma}{\sqrt{2}}c^{\frac{3}{2}}\left(\frac{1}{(\nu_\infty)^{\frac{1}{2}}}\right)x^{\frac{3m-1}{2}}f''(\eta)\right)\left(c^{\frac{3}{2}}\left(\frac{1}{(\nu_\infty)^{\frac{1}{2}}}\right)x^{\frac{3m-1}{2}}\right)^2 f''^2(\eta)$$
$$= \frac{\mu}{\rho_\infty c_p}\left(1 + \frac{\sqrt{2}}{2}\frac{c^{\frac{3}{2}}}{\sqrt{\nu_\infty}}x^{\frac{3m-1}{2}}\Gamma f''(\eta)\right)\left(\frac{c^3}{v_\infty}x^{3m-1}\right)f''^2(\eta)$$
$$= \frac{\mu}{\rho_\infty c_p}\left(1 + \frac{1}{2}\left(\frac{\sqrt{2}c^{\frac{3}{2}}x^{\frac{3m-1}{2}}}{\sqrt{\nu_\infty}}\right)\Gamma f''(\eta)\right)\left(\frac{c^3}{\nu_\infty}x^{3m-1}\right)f''^2(\eta)$$
$$= \frac{\mu}{\rho_\infty c_p}\left(1 + \frac{\delta}{2}f''(\eta)\right)\frac{c^3}{\nu_\infty}x^{3m-1}f''^2(\eta)$$
$$= \frac{\mu}{\rho_\infty c_p\nu_\infty}c^3x^{3m-1}\left(1 + \frac{\delta}{2}f''(\eta)\right)f''^2(\eta)$$
$$= \frac{\mu_\infty e^{-\alpha\theta}}{\rho_\infty c_p\nu_\infty}c^3x^{3m-1}\left(1 + \frac{\delta}{2}f''(\eta)\right)f''^2(\eta)$$

$$= \frac{\mu_\infty e^{-\alpha\theta}}{\mu_\infty c_p} c^3 x^{3m-1} \left(1 + \frac{\delta}{2} f''(\eta)\right) f''^2(\eta)$$
$$= \frac{e^{-\alpha\theta}}{c_p} c^3 x^{3m-1} \left(1 + \frac{\delta}{2} f''(\eta)\right) f''^2(\eta). \tag{3.25}$$

- $$\frac{1}{\rho_\infty c_p} \frac{\partial q_r}{\partial y} = \frac{1}{\rho_\infty c_p} \left(-\frac{16}{3} \frac{\delta^*}{k^*} T_\infty^3 \frac{Ac}{\nu_\infty} x^{r+m-1} \theta''(\eta)\right)$$
$$= -\frac{1}{\rho_\infty c_p \nu_\infty} \frac{16}{3} \frac{\delta^*}{k^*} T_\infty^3 Acx^{r+m-1} \theta''(\eta). \tag{3.26}$$

Using (3.24)-(3.26) the right side of (3.3), becomes

$$\begin{aligned}
&\frac{1}{\rho_\infty c_p} \frac{\partial}{\partial y}\left(k(T)\frac{\partial T}{\partial y}\right) + \frac{\mu(T)}{\rho_\infty c_p}\left(1 + \frac{\Gamma}{\sqrt{2}}\frac{\partial u}{\partial y}\right)\left(\frac{\partial u}{\partial y}\right)^2 - \frac{1}{\rho_\infty c_p}\frac{\partial q_r}{\partial y} \\
&= \frac{1}{P_r} Acx^{r+m-1}(\theta''(\eta) + \varepsilon\theta'^2(\eta) + \varepsilon\theta(\eta)\theta''(\eta)) \\
&\quad + \frac{e^{-\alpha\theta}}{c_p} c^3 x^{3m-1}\left(1 + \frac{\delta}{2} f''(\eta)\right) f''^2(\eta) \\
&\quad - \left(-\frac{1}{\rho_\infty c_p \nu_\infty}\frac{16}{3}\frac{\delta^*}{k^*} T_\infty^3 Acx^{r+m-1}\theta''(\eta)\right) \\
&= \frac{1}{P_r} Acx^{r+m-1}(\theta''(\eta) + \varepsilon\theta'^2(\eta) + \varepsilon\theta(\eta)\theta''(\eta)) \\
&\quad + \frac{e^{-\alpha\theta}}{c_p} c^3 x^{3m-1}\left(1 + \frac{\delta}{2} f''(\eta)\right) f''^2(\eta) + \frac{1}{\rho_\infty c_p \nu_\infty}\frac{16}{3}\frac{\delta^*}{k^*} T_\infty^3 Acx^{r+m-1}\theta''(\eta) \\
&= \frac{1}{P_r} Acx^{r+m-1}(\theta''(\eta) + \varepsilon\theta'^2(\eta) + \varepsilon\theta(\eta)\theta''(\eta)) \\
&\quad + \frac{e^{-\alpha\theta}}{c_p} c^3 x^{3m-1}\left(1 + \frac{\delta}{2} f''(\eta)\right) f''^2(\eta) \\
&\quad + \frac{1}{\rho_\infty c_p \nu_\infty}\frac{16}{3}\frac{\delta^*}{k^*} T_\infty^3 Acx^{r+m-1}\theta''(\eta) \\
&= \frac{1}{P_r} Acx^{r+m-1}(\theta''(\eta) + \varepsilon\theta'^2(\eta) + \varepsilon\theta(\eta)\theta''(\eta)) \\
&\quad + \frac{e^{-\alpha\theta}}{c_p} c^3 x^{3m-1}\left(1 + \frac{\delta}{2} f''(\eta)\right) f''^2(\eta) + \frac{K_\infty}{\mu_\infty c_p}\frac{16\delta^* T_\infty^3}{3K_\infty k^*} Acx^{r+m-1}\theta''(\eta) \\
&= \frac{1}{P_r} Acx^{r+m-1}(\theta''(\eta) + \varepsilon\theta'^2(\eta) + \varepsilon\theta(\eta)\theta''(\eta)) \\
&\quad + \frac{e^{-\alpha\theta}}{c_p} c^3 x^{3m-1}\left(1 + \frac{\delta}{2} f''(\eta)\right) f''^2(\eta) + \frac{1}{P_r} RAcx^{r+m-1}\theta''(\eta).
\end{aligned} \tag{3.27}$$

Now comparing (3.23) and (3.27), we get

$$Acrx^{m+r-1} f'(\eta)\theta(\eta) - Ac\left(\frac{m+1}{2}\right) x^{r+m-1} f(\eta)\theta'(\eta)$$
$$= \frac{1}{P_r} Acx^{r+m-1}(\theta''(\eta) + \varepsilon\theta'^2(\eta) + \varepsilon\theta(\eta)\theta''(\eta))$$

$$
\begin{aligned}
&+\frac{e^{-\alpha\theta}}{c_p}c^3x^{3m-1}f''^2(\eta)\left(1+\frac{\delta}{2}f''(\eta)\right)+\frac{1}{P_r}RAcx^{r+m-1}\theta''(\eta)\\
\Rightarrow\quad &-f(\eta)\left(\frac{m+1}{2}\right)\theta'(\eta)+rf'(\eta)\theta(\eta)=\frac{1}{P_r}R\theta''(\eta)\\
&+\frac{e^{-\alpha\theta}c^2}{Ac_p}x^{2m-r}\left(1+\frac{\delta}{2}f''(\eta)\right)f''^2(\eta)+\frac{1}{P_r}(\varepsilon\theta(\eta)\theta''(\eta)+\theta''(\eta)+\varepsilon\theta'^2(\eta))\\
\Rightarrow\quad &\frac{1}{P_r}(\varepsilon\theta(\eta)\theta''(\eta)+\theta''(\eta)+\varepsilon\theta'^2(\eta))+\frac{1}{P_r}R\theta''(\eta)-r\theta(\eta)f'(\eta)+f(\eta)\left(\frac{m+1}{2}\right)\theta'(\eta)\\
&+Ec\left(1+\frac{\delta}{2}f''(\eta)\right)f''^2(\eta)e^{-\alpha\theta}=0\\
\Rightarrow\quad &\frac{1}{P_r}(\varepsilon\theta'^2(\eta)+(1+\varepsilon\theta(\eta)+R)\theta''(\eta)+f(\eta)\left(\frac{m+1}{2}\right)\theta'(\eta)-rf'(\eta)\theta(\eta)\\
&+Ecf''^2(\eta)\left(1+\frac{\delta}{2}f''(\eta)\right)e^{-\alpha\theta}=0. \qquad (3.28)
\end{aligned}
$$

The BCs are converted into non-dimensionless form through the following procedure

- $u=cx^m,$ at $y=0.$

$$
\begin{aligned}
\Rightarrow\quad & cx^mf'(\eta)=cx^m,\\
\Rightarrow\quad & f'(\eta)=1, && \text{at}\quad \eta=0.\\
\Rightarrow\quad & f'(\eta)=1, && \text{at}\quad \eta=0.\\
\Rightarrow\quad & f'(0)=1.
\end{aligned}
$$

- $v=0,$ at $y=0.$

$$
\begin{aligned}
\Rightarrow\quad & -c\left(\frac{m-1}{2}\right)x^{m-1}yf'(\eta)-(c\nu_\infty)^{\frac{1}{2}}\left(\frac{m+1}{2}\right)x^{\frac{m-1}{2}}f(\eta)=0,\\
& && \text{at}\quad y=0.\\
\Rightarrow\quad & -(c\nu_\infty)^{\frac{1}{2}}\left(\frac{m+1}{2}\right)x^{\frac{m-1}{2}}f(0)=0, && \text{at}\quad \eta=0.\\
\Rightarrow\quad & f(0)=0.
\end{aligned}
$$

- $T=T_\infty+Ax^r,$ at $y=0.$

$$
\begin{aligned}
\Rightarrow\quad & T-T_\infty=Ax^r, && \text{at}\quad y=0.\\
\Rightarrow\quad & (T_w-T_\infty)\theta(\eta)=Ax^r, && \text{at}\quad \eta=0.\\
\Rightarrow\quad & Ax^r\theta(0)=Ax^r,\\
\Rightarrow\quad & \theta(0)=1.
\end{aligned}
$$

- $u\rightarrow 0,$ as $y\rightarrow\infty.$

$$
\Rightarrow\quad cx^mf'(\eta)\rightarrow 0, \qquad \text{as}\quad \eta\rightarrow\infty.
$$

$$\Rightarrow \quad f'(\eta) \to 0, \qquad \text{as} \quad \eta \to \infty.$$

- $$T \to T_\infty, \qquad \text{as} \quad y \to \infty.$$

$$\Rightarrow \quad T - T_\infty = (T_w - T_\infty)\theta(\eta), \qquad \text{as} \quad \eta \to \infty.$$

$$\Rightarrow \quad T = T_\infty + (T_w - T_\infty)\theta(\eta), \qquad \text{as} \quad \eta \to \infty.$$

$$\Rightarrow \quad T \to T_\infty, \qquad \text{as} \quad \eta \to \infty.$$

$$\Rightarrow \quad T_\infty + (T_w - T_\infty)\theta(\eta) \to T_\infty, \qquad \text{as} \quad \eta \to \infty.$$

$$\Rightarrow \quad \theta(\eta) \to 0, \qquad \text{as} \quad \eta \to \infty.$$

Hence the converted ODEs are in the following form

$$\begin{aligned} &e^{-\alpha\theta}\left((1+\delta f''(\eta))f'''(\eta) - \alpha f''(\eta)\theta'(\eta)\left(1+\frac{\delta}{2}f''(\eta)\right)\right) \\ &+ f(\eta)\left(\frac{m+1}{2}\right)f''(\eta) - m f'^2(\eta) = 0. \end{aligned} \tag{3.29}$$

$$\begin{aligned} &\frac{1}{P_r}(\varepsilon\theta'^2(\eta) + (1+R+\varepsilon\theta(\eta))\theta''(\eta)) + -r f'(\eta)\theta(\eta) + Ec\left(1+\frac{\delta}{2}f''(\eta)\right)f''^2(\eta)e^{-\alpha\theta} \\ &+ \left(\frac{m+1}{2}\right)\theta'(\eta)f(\eta) = 0. \end{aligned} \tag{3.30}$$

The associated BCs in the dimensionless form:

$$\left.\begin{aligned} &\theta(0) = 1, \quad f(0) = 0, \quad f'(0) = 1, \\ &\theta \to 0, \quad f' \to 0 \quad at \quad \eta \to \infty. \end{aligned}\right\} \tag{3.31}$$

Different parameters used in (3.29) and (3.30) are:

$$R = \frac{16\delta^* T_\infty^3}{3K_\infty k^*}, \quad \delta = \left(\frac{\sqrt{2}c^{\frac{3}{2}}x^{\frac{3m-1}{2}}}{\sqrt{\nu_\infty}}\right)\Gamma, \quad Pr = \frac{\mu_\infty c_p}{K_\infty}$$

$$Ec = \frac{U_w^2}{c_p(T_w - T_\infty)} = \frac{c^2 x^{2m-r}}{Ac_p}, \quad r = 2m = \frac{2}{3}.$$

The local skin friction is given as

$$Cf_x = \frac{(\tau_w)_{y=0}}{\rho_\infty U_w^2}. \tag{3.32}$$

To get the dimensionless form of Cf_x the following procedure is worked out

$$\tau_w = \mu\left(\frac{\partial u}{\partial y} + \frac{\Gamma}{\sqrt{2}}\left(\frac{\partial u}{\partial y}\right)^2\right) \tag{3.33}$$

$$\begin{aligned}
Cf_x &= \frac{\mu\left(\frac{\partial u}{\partial y} + \frac{\Gamma}{\sqrt{2}}\left(\frac{\partial u}{\partial y}\right)^2\right)_{y=0}}{\rho_\infty U_w^2} \\
&= \frac{\mu\frac{\partial u}{\partial y}\left(1 + \frac{\Gamma}{\sqrt{2}}\frac{\partial u}{\partial y}\right)_{y=0}}{\rho_\infty c^2 x^{2m}} \\
&= \frac{\mu_\infty e^{-\alpha\theta(0)}\left(\frac{c^{\frac{3}{2}}}{\nu_\infty^{\frac{1}{2}}}\right)x^{\frac{3m-1}{2}}f''(0)}{\rho_\infty c^2 x^{2m}}\left(1 + \frac{\Gamma}{\sqrt{2}}\left(\frac{c^{\frac{3}{2}}}{\nu_\infty^{\frac{1}{2}}}\right)x^{\frac{3m-1}{2}}f''(0)\right) \\
&= \frac{\rho_\infty}{\rho_\infty}\nu_\infty^{\frac{1}{2}}c^{\frac{-1}{2}}x^{\frac{-(m+1)}{2}}e^{-\alpha\theta(0)}f''(0)\left(1 + \frac{\Gamma}{\sqrt{2}}\left(\frac{c^{\frac{3}{2}}}{\nu_\infty^{\frac{1}{2}}}\right)x^{\frac{3m-1}{2}}f''(0)\right) \\
&= \frac{\nu_\infty^{\frac{1}{2}}}{c^{\frac{1}{2}}x^{\frac{(m+1)}{2}}}e^{-\alpha\theta(0)}f''(0)\left(\frac{-2}{2} - \frac{2\Gamma}{2\sqrt{2}}\left(\frac{c^{\frac{3}{2}}}{\nu_\infty^{\frac{1}{2}}}\right)x^{\frac{3m-1}{2}}f''(0)\right) \\
&= -\frac{1}{Re_x^{\frac{1}{2}}}e^{-\alpha\theta(0)}f''(0)\left(1 + \frac{\left(\frac{\sqrt{2}c^{\frac{3}{2}}x^{\frac{3m-1}{2}}}{\nu_\infty^{\frac{1}{2}}}\right)\Gamma}{2}f''(0)\right) \\
&= -Re_x^{\frac{-1}{2}}\left(1 + \frac{\delta}{2}f''(0)\right)e^{-\alpha\theta(0)}f''(0)
\end{aligned} \tag{3.34}$$

Reynolds number is defined as $Re = \frac{U_w x}{\nu_\infty}$.

Local Nusselt number is defined as

$$Nu_x = \frac{xq_w}{K_\infty(T_w - T_\infty)}. \tag{3.35}$$

To get the dimensionless form of Nu_x the following procedure is followed

$$q_w = -\left(K + \frac{16\sigma^* T_\infty^3}{3k^*}\right)\left(\frac{\partial T}{\partial y}\right)_{y=0} \tag{3.36}$$

$$\begin{aligned}
Nu_x &= -\frac{x\left(K + \frac{16\sigma^* T_\infty^3}{3k^*}\right)\left(\frac{\partial T}{\partial y}\right)}{K_\infty(T_w - T_\infty)} \\
&= -\frac{x\left(K + \frac{16\sigma^* T_\infty^3}{3k^*}\right)}{K_\infty(T_w - T_\infty)}\left(\frac{\partial T}{\partial y}\right)_{y=0} \\
&= -\frac{x\left(K + \frac{16\sigma^* T_\infty^3}{3k^*}\right)}{K_\infty(T_w - T_\infty)}(T_w - T_\infty)\left(\frac{c}{\nu_\infty}\right)^{\frac{1}{2}}x^{\frac{m-1}{2}}\theta'(0)
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{\left(K_\infty(1+\varepsilon\theta(0)) + \frac{16\sigma^* T_\infty^3}{3k^*}\right)}{K_\infty}\left(\frac{c}{\nu_\infty}\right)^{\frac{1}{2}} x^{\frac{m+1}{2}}\theta'(0)\\
&= -\frac{K_\infty}{K_\infty}\left(1+\varepsilon\theta(0) + \frac{16\sigma^* T_\infty^3}{3k^* K_\infty}\right)\left(\frac{c}{\nu_\infty}\right)^{\frac{1}{2}} x^{\frac{m+1}{2}}\theta'(0)\\
&= -Re_x^{\frac{1}{2}}\left(1+R+\varepsilon\theta(0)\right)\theta'(0). \qquad (3.37)
\end{aligned}
$$

3.3 Solution Methodology

In this section, to get the approximate solution of the ordinary differential equations (3.29)-(3.30) along with the boundary conditions (3.31), the shooting method has been used. To get the approximate results, the domain of the problem has been taken as [0, η_∞] instead of [0, ∞), where η_∞ is an appropriate finite positve real number. First of all, we need to convert these equations into a system of first order differential equations. Let us use the following notations.

$$
\begin{aligned}
&f = h_1,\ f' = h_1' = h_2,\ f'' = h_2' = h_3,\ f''' = h_3'\\
&\theta = h_4,\ \theta' = h_4' = h_5,\ \theta'' = h_5.
\end{aligned}
$$

Using the above notations, the system of equations (3.29)-(3.30) with the boundary conditions (3.31) is transformed into the following system of five first order differential equation

$$
\begin{aligned}
&h_1' = h_2, && h_1(0) = 0,\\
&h_2' = h_3, && h_2(0) = 1,\\
&h_3' = \frac{e^{\alpha\theta}}{1+dh_3}\left[\frac{\alpha}{e^{\alpha\theta}}h_5h_3\left(1+\frac{\delta}{2}h_3\right) - \left(\frac{m+1}{2}\right)h_1h_3 + mh_2^2\right], &&\\
& && h_3(0) = p,\\
&h_4' = h_5, && h_4(0) = 0,\\
&h_5' = \frac{Pr}{1+R+\varepsilon h_4}\left[rh_2h_4 - \frac{\varepsilon}{Pr}h_5^2 - \left(\frac{m+1}{2}\right)h_1h_5\right. &&\\
&\qquad \left. - \frac{Ec}{e^{\alpha h_4}}\left(1+\frac{\delta}{2}h_3\right)h_3^2\right], && h_5(0) = q.
\end{aligned}
$$

In the above IVP, the missing conditions p and q, are to be chosen to satisfy the following relation.

$$h_2(p,q) = 0, \tag{3.38}$$

$$h_4(p,q) = 0. \tag{3.39}$$

Here $h_2(p,q)$ and $h_4(p,q)$ are the values of h_2 and h_4 at $\eta = \eta_\infty$ for the chosen values of the missing conditions p and q. Newton's method with the following itrative scheme will be used to solve the above two equations with two variables.

$$\begin{bmatrix} p_{n+1} \\ q_{n+1} \end{bmatrix} = \begin{bmatrix} p_n \\ q_n \end{bmatrix} - \begin{bmatrix} \frac{\partial h_2}{\partial p} & \frac{\partial h_2}{\partial q} \\ \frac{\partial h_4}{\partial p} & \frac{\partial h_4}{\partial q} \end{bmatrix}^{-1}_{(p_n,q_n)} \begin{bmatrix} h_2 \\ h_4 \end{bmatrix}_{(p_n,q_n)} \tag{3.40}$$

To find the partial derivative of h_2 and h_4, the following notations have been introduce.

$$\frac{\partial h_1}{\partial p} = h_6, \quad \frac{\partial h_2}{\partial p} = h_7, \quad \frac{\partial h_3}{\partial p} = h_8, \quad \frac{\partial h_4}{\partial p} = h_9, \quad \frac{\partial h_5}{\partial p} = h_{10}.$$

$$\frac{\partial h_1}{\partial q} = h_{11}, \quad \frac{\partial h_2}{\partial q} = h_{12}, \quad \frac{\partial h_3}{\partial q} = h_{13}, \quad \frac{\partial h_4}{\partial q} = h_{14}, \quad \frac{\partial h_5}{\partial q} = h_{15}.$$

Using the above notations in (3.40), we get

$$\begin{bmatrix} p_{n+1} \\ q_{n+1} \end{bmatrix} = \begin{bmatrix} p_n \\ q_n \end{bmatrix} - \begin{bmatrix} h_7 & h_{12} \\ h_9 & h_{14} \end{bmatrix}^{-1}_{(p_n,q_n)} \begin{bmatrix} h_2 \\ h_4 \end{bmatrix}_{(p_n,q_n)} \tag{3.41}$$

The itrative proccess (3.41) will be continued untill the following criteria is met,

$$max\,\{|h_2(p,q)|, |h_4(p,q)|\} < \epsilon,$$

where ϵ has been taken as 10^{-10}.

Now differentiate the last system of five equations with respect to p and q, we get the following equations:

$$
\begin{aligned}
h_6' &= h_7, && h_6(0) = 0,\\
h_7' &= h_8, && h_7(0) = 0,\\
h_8' &= \frac{(1+\delta h_3)\alpha h_9 e^{\alpha h_4} - e^{\alpha h_4}(\delta h_8)}{(1+\delta h_3)^2}\left[\frac{\alpha}{e^{\alpha h_4}} h_5 h_3 \left(1+\frac{\delta}{2}h_3\right) - \left(\frac{m+1}{2}\right) h_1 h_3 + m h_2^2\right]\\
&\quad + \frac{e^{\alpha h_4}}{1+\delta h_3}\left[\alpha e^{\alpha h_4}(-\alpha h_9) h_5 h_3 \left(1+\frac{\delta}{2}h_3\right) + \frac{\alpha}{e^{\alpha h_4}} h_{10} h_3 \left(1+\frac{\delta}{2}h_3\right)\right.\\
&\quad + \frac{\alpha}{e^{\alpha h_4}} h_5 h_8 \left(1+\frac{\delta}{2}h_3\right) + \frac{\alpha}{e^{\alpha h_4}} h_5 h_3 \left(\frac{\delta}{2}h_8\right)\\
&\quad \left. - \left(\frac{m+1}{2}\right)(h_1 h_8 + h_6 h_3) + 2m h_2 h_7\right], && h_8(0) = 1,\\
h_9' &= h_{10}, && h_9(0) = 0,\\
h_{10}' &= -\frac{Pr\varepsilon}{(1+R\varepsilon h_4)^2} h_9 \left[r h_2 h_4 - \left(\frac{m+1}{2}\right) h_1 h_5 - \frac{Ec}{e^{\alpha h_4}\left(1+\frac{\delta}{2}h_3\right)} h_3^2\right]\\
&\quad + \frac{Pr}{1+R\varepsilon h_4}\left[r(h_2 h_9 + h_7 h_4) - \frac{2\varepsilon}{Pr} h_5 h_{10} - \left(\frac{m+1}{2}\right)(h_1 h_{10} + h_6 h_5)\right.\\
&\quad - Ec e^{\alpha h_4}(-\alpha h_9)\left(1+\frac{\delta}{2}h_3\right) h_3^2 - \frac{Ec}{e^{\alpha h_4}}\left(\frac{\delta}{2}h_8\right) h_3^2\\
&\quad \left. - \frac{2Ec}{e^{\alpha h_4}}\left(1+\frac{\delta}{2}h_3\right) h_3 h_8\right], && h_{10}(0) = 0,\\
h_{11}' &= h_{12}, && h_{11}(0) = 0,\\
h_{12}' &= h_{13}, && h_{12}(0) = 0,\\
h_{13}' &= \frac{(1+\delta h_3)\alpha h_{14} e^{\alpha h_4} - e^{\alpha h_4}(\delta h_{13})}{(1+\delta h_3)^2}\left[\frac{\alpha}{e^{\alpha h_4}} h_5 h_3 \left(1+\frac{\delta}{2}h_3\right)\right.\\
&\quad \left. - \left(\frac{m+1}{2}\right) h_1 h_3 + m h_2^2\right] + \frac{e^{\alpha h_4}}{1+\delta h_3}\left[\alpha e^{-\alpha h_4}(-\alpha h_{14}) h_5 h_3 \left(1+\frac{\delta}{2}h_3\right)\right.\\
&\quad + \frac{\alpha}{e^{\alpha h_4}} h_{15} h_3 \left(1+\frac{\delta}{2}h_3\right) + \frac{\alpha}{e^{\alpha h_4}} h_5 h_{13} \left(1+\frac{\delta}{2}h_3\right)\\
&\quad \left. + \frac{\alpha}{e^{\alpha h_4}} h_5 h_3 \left(\frac{\delta}{2}h_{13}\right) - \left(\frac{m+1}{2}\right)(h_1 h_{13} + h_{11} h_3) + 2m h_2 h_{12}\right],\\
& && h_{13}(0) = 0,\\
h_{14}' &= h_{15}, && h_{14}(0) = 0,\\
h_{15}' &= -\frac{Pr\varepsilon h_{14}}{(1+R\varepsilon h_4)^2}\left[r h_2 h_4 - \frac{\varepsilon h_5^2}{Pr} - \left(\frac{m+1}{2}\right) h_1 h_5 - \frac{Ec}{e^{\alpha h_4}}\left(1+\frac{\delta}{2}\right) h_3^2\right]\\
&\quad + \frac{Pr}{(1+R+\varepsilon h_4)}\left[r(h_2 h_{14} + h_{12} h_4) - \frac{2\varepsilon}{Pr} h_5 h_{15} - \left(\frac{m+1}{2}\right)(h_1 h_{15} + h_{11} h_5)\right.\\
&\quad \left. - Ec e^{-\alpha h_4}(-\alpha h_{14})\left(1+\frac{\delta}{2}h_3\right) h_3^2 - \frac{Ec}{e^{\alpha h_4}}\left(\frac{\delta}{2}h_{13}\right) h_3^2 - \frac{2Ec}{e^{\alpha h_4}}\left(1+\frac{\delta}{2}h_3\right) h_3 h_{13}\right],\\
& && h_{15}(0) = 1.
\end{aligned}
$$

3.4 Discussion on Tables and Graphs

In this section, the numerical solutions are addressed in detail for velocity and temperature profiles, using tables and graphs. For the verification of the code, the obtained results are compared with those of Gorla and Sidawi [32] in Table 3.1. For the observation of the effect of different physical parameters like Williamson fluid parameter δ, viscosity parameter α, radiation parameter R, thermal conductivity ε, Eckert number Ec on skin friction $\frac{1}{2}(Re_x)^{\frac{1}{2}}Cf_x$ and Nusselt number $(Re_x)^{-\frac{1}{2}}Nu_x$, the calculations are executed numericaly and presented in Table 3.2. For the rising values of α, the local skin friction and local Nusselt number decrease. By increasing the values of Williamson parameter δ, the local skin friction and local Nusselt number decrease. Similarly increaing the values of ϵ, the local skin friction decreases while the local Nusselt number is increased. By increasing the values of R and Ec, the local skin friction and local Nusselt number decrease. In this table, I_p and I_q are the intervals from which thee missing conditions p and q can be chosen.

Figure 3.2 shows the behaviour of velocity $f'(\eta)$ for various values of α. By increasing the values of α, the velocity $f'(\eta)$ is found to decrease. Figure 3.3 shows the representation of temperature $\theta(\eta)$ for different values of α. By increasing the value of α, the temperature $\theta(\eta)$ increases, which is an understandable behaviour. Figure 3.4 represents the velocity $f'(\eta)$ for different values of Williamson parameter δ. By improving the values of δ, the velocity distribution $f'(\eta)$ is found to decrease. In Figure 3.5, it is observed that by increasing the values of δ, the temperature $\theta(\eta)$ shows an increasing behaviour.

Figure 3.6 shows the behaviour of the temperature field $\theta(\eta)$ for the values of thermal conductivity ε. By rising the values of ε, the temperature $\theta(\eta)$ shows a natural increasing behaviour. Figure 3.7 shows the behaviour of the temperature $\theta(\eta)$ for the values of R. The temperature $\theta(\eta)$ increases by rising the values of R. Actually with an increase in the thermal radiation, the heat transfer increase because of which the temperature $\theta(\eta)$ icreases. Figure 3.8 represents the temperature $\theta(\eta)$ for various values of the Eckert number Ec, with the constant values of the rest of the parameters. By increasing the values of Ec, the temperature $\theta(\eta)$ is found to increase.

TABLE 3.1: Comparison of present Nusselt number $Nu_x Re_x^{-\frac{1}{2}}$ with that of Gorla and Sidawi [32] for various value of Pr when $\delta = \alpha = \varepsilon = R = Ec = 0$

Pr	**Gorla and Sidawi [32]**	**Present study**
20.0	3.35391	3.353902
7.00	1.89546	1.895453
2.00	0.91142	0.911358
0.20	0.16912	0.169117
0.07	0.06562	0.065531

TABLE 3.2: Values of $(Re_x)^{\frac{1}{2}} Cf_x$ and $(Re_x)^{-\frac{1}{2}} Nu_x$ for values of $\alpha, \delta, \varepsilon, R, and\ Ec$ with $m = \frac{1}{3}, r = \frac{2}{3}, Pr = 2.0$

α	δ	ε	R	Ec	$(Re_x)^{\frac{1}{2}} Cf_x$	$(Re_x)^{\frac{-1}{2}} Nu_x$	I_p	I_q
0.5	0.2	0.2	0.2	0.2	0.546574	1.27389	[-0.9,-0.6]	[-0.9,-0.5]
0.0					0.659036	1.33147	[-0.7,-0.1]	[-1.3,-0.1]
1.0					0.439825	1.20470	[-0.9,-0.8]	[-0.9,-0.8]
	0.2				0.546574	1.27389	[-0.9,-0.6]	[-0.9,-0.7]
	0.0				0.567824	1.28575	[-0.9,-0.8]	[-0.9,-0.8]
	0.5				0.506501	1.24862	[-0.9,-0.8]	[-1.0,-0.1]
		0.2			0.546574	1.27389	[-0.9,-0.6]	[-0.9,-0.8]
		0.0			0.549230	1.21855	[-0.9,-0.6]	[-0.9,-0.7]
		0.5			0.543218	1.34923	[-0.9,-0.7]	[-0.7,-0.1]
			0.0		0.549638	1.19305	[-0.9,-0.6]	[-0.9,-0.5]
			0.5		0.542856	1.37819	[-0.9,-0.7]	[-0.9,-0.6]
			1.0		0.538195	1.51929	[-0.9,-0.5]	[-0.9,-0.4]
			0.2	0.0	0.548241	1.36146	[-0.9,-0.7]	[-0.9,-0.6]
				0.2	0.546574	1.27389	[-0.7,-0.6]	[-0.9,-0.5]
				0.5	0.544103	1.14333	[-0.9,-0.7]	[-0.8,-0.1]

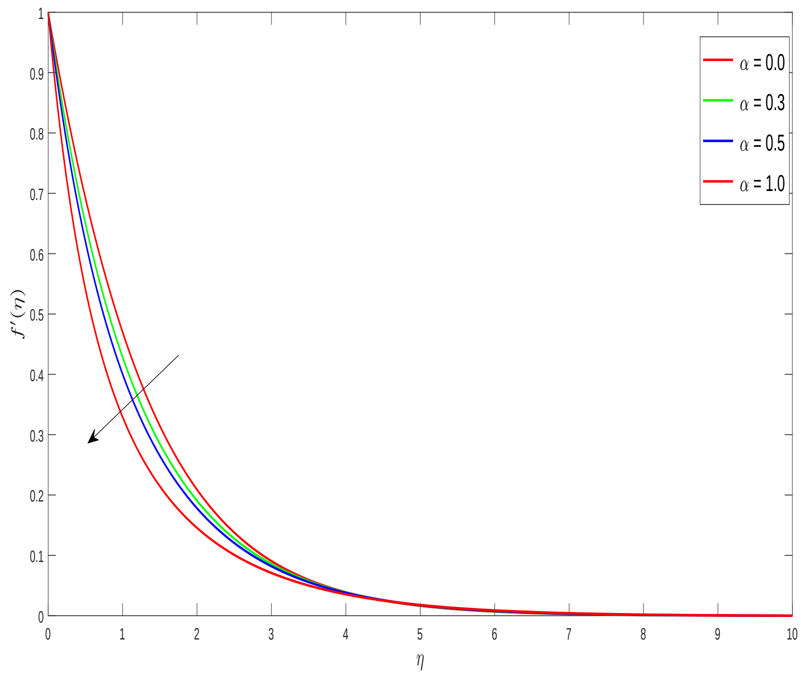

FIGURE 3.2: Velocity $f'(\eta)$ for the values of α with $\delta = \varepsilon = R = Ec = 0.2$, $m = \frac{1}{3},\ Pr = 2.0$

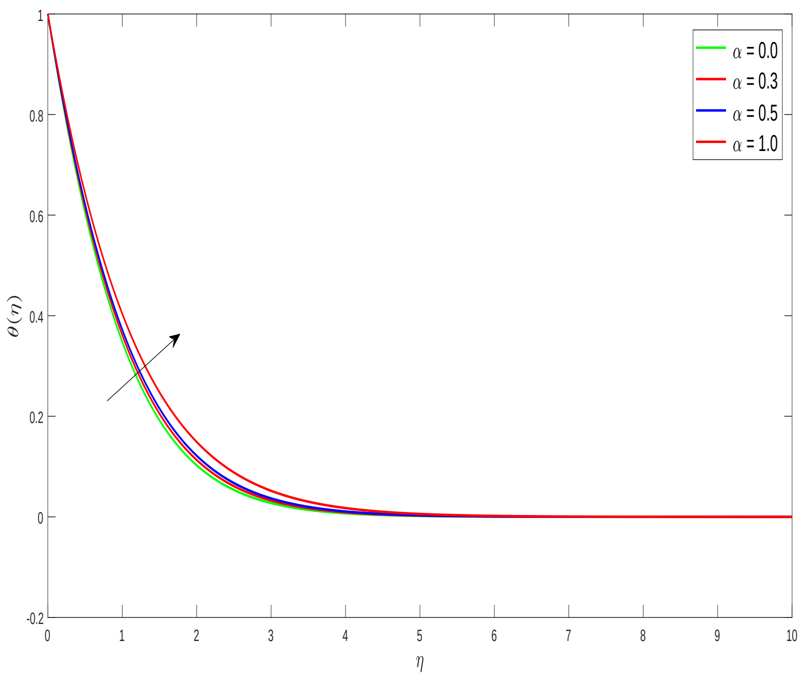

FIGURE 3.3: Temperature $\theta(\eta)$ for the values of α with $\delta = \varepsilon = Ec = R = 0.2,\ Pr = 2.0,\ m = \frac{1}{3}$

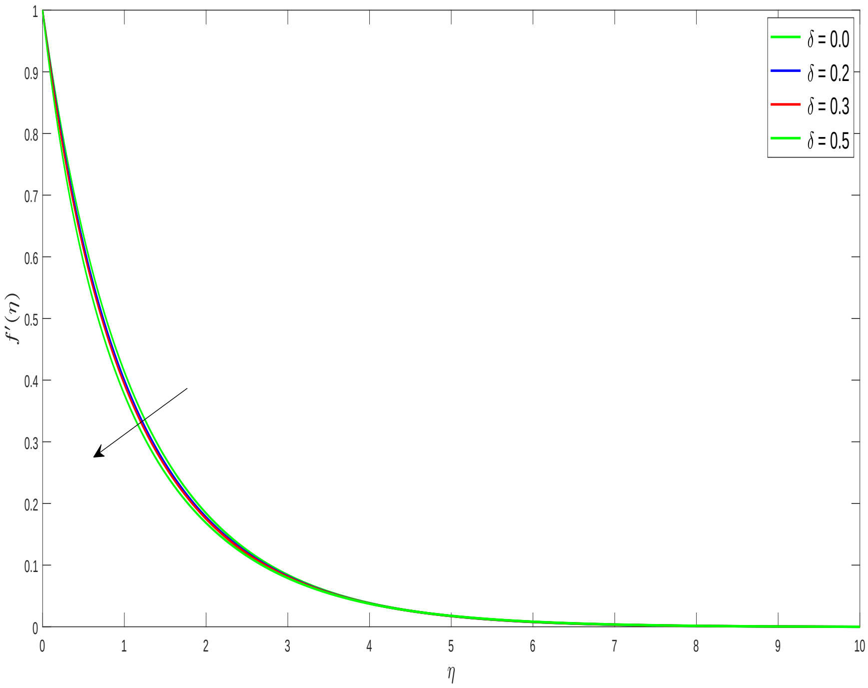

FIGURE 3.4: Velocity $f'(\eta)$ for the of values δ with $\varepsilon = Ec = R = 0.2$, $Pr = 2.0$, $\alpha = 0.5,\ m = \frac{1}{3}$

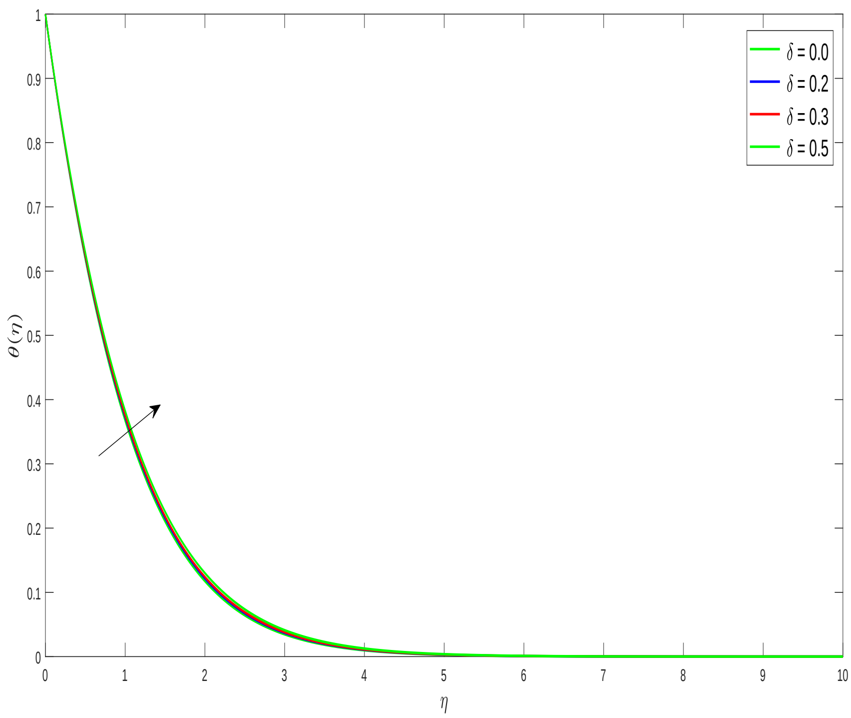

FIGURE 3.5: Temperature $\theta(\eta)$ for various values δ with $\varepsilon = Ec = R = 0.2$, $\alpha = 0.5,\ Pr = 2.0,\ m = \frac{1}{3}$

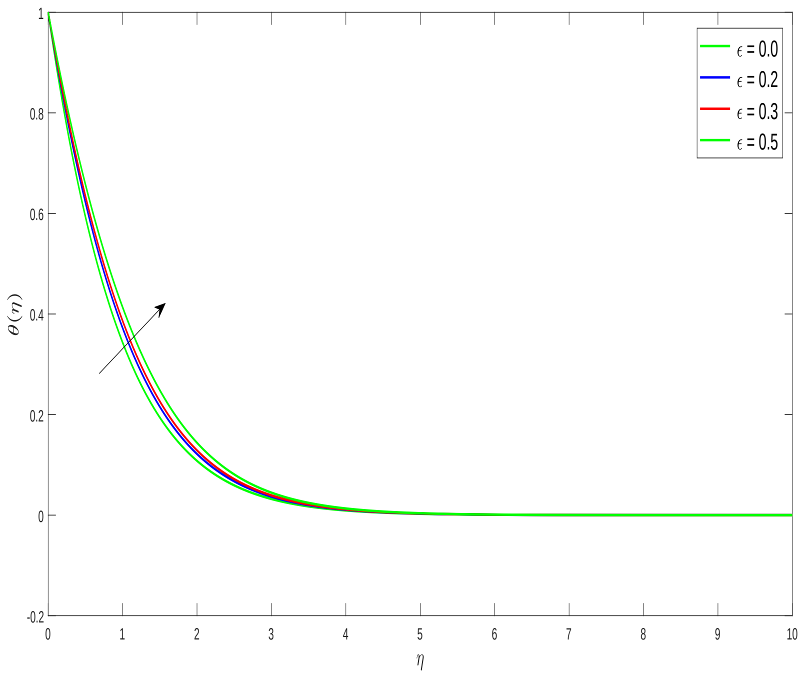

FIGURE 3.6: Temperature $\theta(\eta)$ for various values of ε with $\delta = Ec = R = 0.2$, $Pr = 2.0,\ m = \frac{1}{3}$

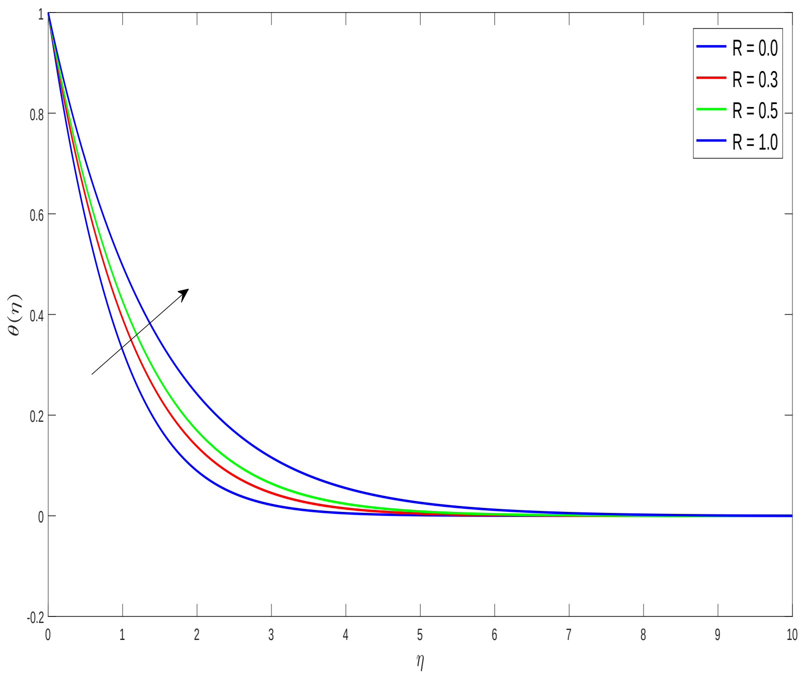

FIGURE 3.7: Temperature $\theta(\eta)$ for various values of R with $\delta = \varepsilon = Ec = 0.2$, $Pr = 2.0,\ m = \frac{1}{3}$

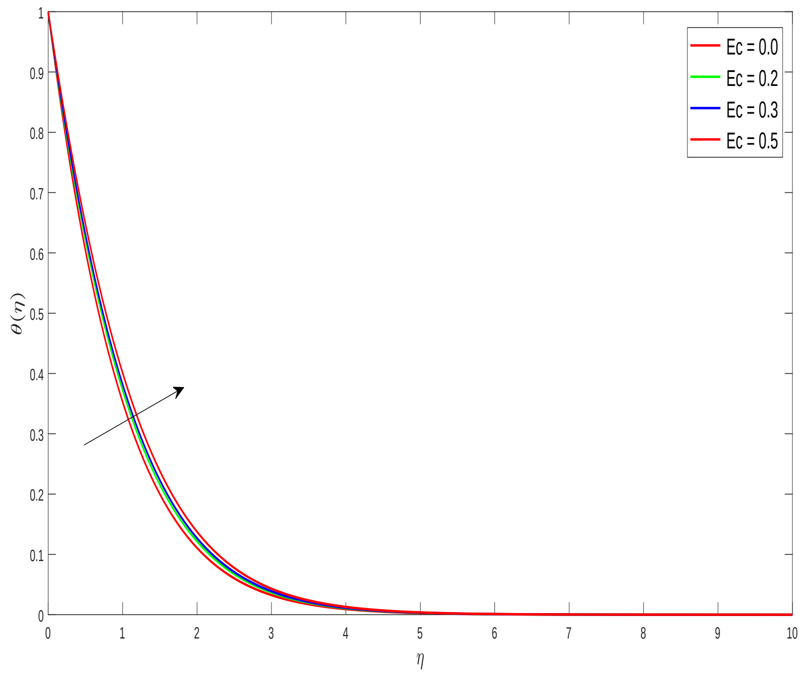

FIGURE 3.8: Temperature $\theta(\eta)$ for various values of Ec with $\delta = \varepsilon = Ec = 0.2$, $Pr = 2.0,\ m = \frac{1}{3}$

Chapter 4

Williamson fluid flow with MHD, Joule Heating, Concentration and Chemical Reaction

4.1 Introduction

This chapter discuses an extension of the mathematical model discussed in Chapter 3. The MHD term is in the momentum equation while the Joule heating is added to the energy equation. Furthermore, the concentration equation is also incorporated along with the chemical reaction. The governing nonlinear partial differential equations are converted into a system of dimensionless ordinary differential equations by using some appropriate similiarity transformations. In order to solve the ODEs, the shooting method is applied. For the numerical computations, the computational software MATLAB is opted. At the end of this chapter, the numerical results are discused for various parameters affecting the velocity, temperature and concentration profiles.

4.2 Mathematical Modeling

It has been aimed to analyse a 2-D Williamson fluid flowing over a nonlinearly stretching sheet. In the model, u and v are the velocity components in the xy-direction, respectively. In addition, the concentration of fluid is also examined with the assistance of the

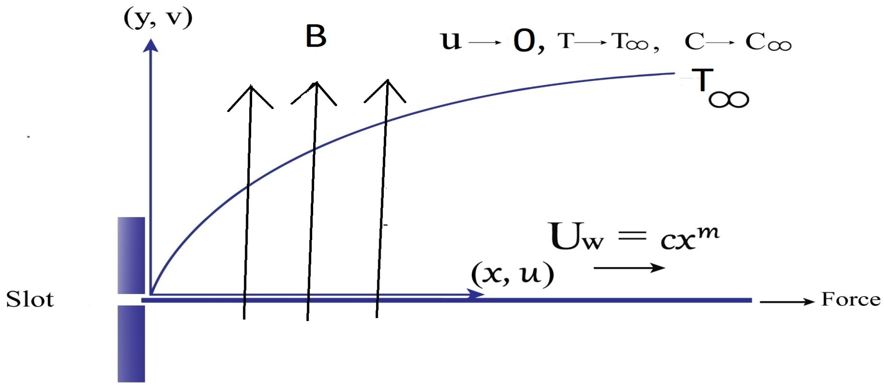

FIGURE 4.1: Flow over a nonlinearly stretching sheet is seen in this diagram.

concentration equation under the effect of chemical reaction. The system of equtions is as follows

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{4.1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{1}{\rho_\infty}\frac{\partial}{\partial y}\left(\mu(T)\frac{\partial u}{\partial y} + \mu(T)\frac{\Gamma}{\sqrt{2}}\left(\frac{\partial u}{\partial y}\right)^2\right) - \frac{\sigma B^2(x)}{\rho}u, \tag{4.2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{1}{\rho_\infty c_p}\frac{\partial}{\partial y}\left(k(T)\frac{\partial T}{\partial y}\right) + \frac{\mu(T)}{\rho_\infty c_p}\left(1 + \frac{\Gamma}{\sqrt{2}}\frac{\partial u}{\partial y}\right)\left(\frac{\partial u}{\partial y}\right)^2 - \frac{1}{\rho_\infty c_p}\frac{\partial q_r}{\partial y}$$
$$+ \frac{\sigma B^2(x)}{\rho c_p}u^2, \tag{4.3}$$

$$u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} = D\frac{\partial^2 C}{\partial y^2} - k_1(x)(C - C_\infty). \tag{4.4}$$

The associated BCs are:

$$\left.\begin{aligned} &u = cx^m, \quad T_w(x) = T_\infty + Ax^r, \quad v = 0 \qquad at \;\; y = 0 \\ &T \to T_\infty, \quad u \to 0, \quad C \to C_\infty \qquad as \;\; y \to \infty. \end{aligned}\right\} \tag{4.5}$$

For the conversion of (4.1)-(4.3) into the system of ODEs, the following similirity transformations have been used

$$\left.\begin{aligned} \eta &= \left(\frac{cx^{m-1}}{\nu_\infty}\right)^{\frac{1}{2}} y, \qquad \psi(x,y) = (cx^{m-1}\nu_\infty)^{\frac{1}{2}} f(\eta), \\ \theta(\eta) &= \frac{T-T_\infty}{T_w-T_\infty}, \qquad \phi(\eta) = \frac{C-C_\infty}{C_w-C_\infty}. \end{aligned}\right\} \tag{4.6}$$

The identical satisfaction of (4.1) is not different from that of the continuity equation in Chapter 3. To convert (4.2) into the dimensionless form, the following procedure has been incorporated

$$\psi = (cx^{m+1}\nu_\infty)^{\frac{1}{2}} f(\eta) = (c\nu_\infty)^{\frac{1}{2}} x^{\frac{m+1}{2}} f(\eta)$$

- $$u = \frac{\partial \psi}{\partial y} = (c\nu_\infty)^{\frac{1}{2}} x^{\frac{m+1}{2}} f'(\eta) \left(\frac{c}{\nu_\infty}\right)^{\frac{1}{2}} x^{\frac{m-1}{2}} = cx^m f'(\eta). \tag{4.7}$$
- $$\begin{aligned} v = -\frac{\partial \psi}{\partial x} &= -c\left(\frac{m-1}{2}\right) x^{m-1} y f'(\eta) - (c\nu_\infty)^{\frac{1}{2}} \left(\frac{m+1}{2}\right) x^{\frac{m-1}{2}} f(\eta) \\ &= -c\left(\frac{m-1}{2}\right) x^{m-1} y f'(\eta) - (c\nu_\infty)^{\frac{1}{2}} \left(\frac{m+1}{2}\right) x^{\frac{m-1}{2}} f(\eta). \end{aligned} \tag{4.8}$$

To get the left side of (4.2), we proceed as follows,

$$\begin{aligned} u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} = &\left[c^{\frac{5}{2}}\left(\frac{1}{(\nu_\infty)^{\frac{1}{2}}}\right)\left(\frac{m-1}{2}\right) x^{\frac{5m-3}{2}} y f'(\eta) f''(\eta) + c^2 m x^{2m-1} f'^2(\eta)\right] \\ &+ \left[-\frac{c^{\frac{5}{2}}}{(\nu_\infty)^{\frac{1}{2}}} x^{\frac{5m-3}{2}} y f'(\eta) \left(\frac{m-1}{2}\right) f''(\eta) \right. \\ &\left. - c^2 x^{2m-1} f(\eta) \left(\frac{m+1}{2}\right) f''(\eta)\right] \\ &= c^2 m x^{2m-1} f'^2(\eta) - c^2 x^{2m-1} f(\eta) \left(\frac{m+1}{2}\right) f''(\eta). \end{aligned} \tag{4.9}$$

Next, to get the right side of (4.2), we convert (4.2) into the dimensionless form, we need only the following conversion. The rest of the conversions are already included in the last chapter,

$$\begin{aligned} \frac{\sigma B^2}{\rho} u &= \frac{\sigma B^2}{\rho} cx^m f'(\eta) \\ &= \frac{\sigma (B_0 x^{\frac{m-1}{2}})^2}{\rho} cx^m f'(\eta) \end{aligned}$$

$$
\begin{aligned}
&= \frac{\sigma B_0^2 x^{m-1}}{\rho} c x^m f'(\eta) \\
&= \frac{\sigma B_0^2 x^{2m-1}}{\rho} c f'(\eta).
\end{aligned} \tag{4.10}
$$

As a result, the right side of (4.2) gets the following form:

$$
\begin{aligned}
&\frac{1}{\rho_\infty}\frac{\partial}{\partial y}\left(\mu(T)\frac{\partial u}{\partial y} + \mu(T)\frac{\Gamma}{\sqrt{2}}\left(\frac{\partial u}{\partial y}\right)^2\right) - \frac{\sigma B^2}{\rho}u \\
&= \frac{1}{\rho_\infty}\frac{\mu_\infty}{\nu_\infty} e^{-\alpha\theta} c^2 x^{2m-1}\Big[(1+\delta f''(\eta))f'''(\eta) \\
&\quad - \alpha\theta' f''(\eta)(1+\frac{\delta}{2}f'')(\eta)\Big] - \frac{\sigma B_0^2 x^{2m-1} c}{\rho} f'(\eta) \\
&= \frac{1}{\rho_\infty}\rho_\infty e^{-\alpha\theta} c^2 x^{2m-1}\Big[(1+\delta f''(\eta))f'''(\eta) \\
&\quad - \alpha\theta' f''(\eta)(1+\frac{\delta}{2}f''(\eta))\Big] - \frac{\sigma B_0^2 x^{2m-1} c}{\rho} f'(\eta) \\
&= e^{-\alpha\theta} c^2 x^{2m-1}\Big[(1+\delta f''(\eta))f'''(\eta) - \alpha\theta' f''(\eta)(1+\frac{\delta}{2}f''(\eta))\Big] \\
&\quad - \frac{\sigma B_0^2 x^{2m-1} c}{\rho} f'(\eta).
\end{aligned} \tag{4.11}
$$

Comparing (4.9) and (4.11), we get

$$
\begin{aligned}
& c^2 m x^{2m-1} f'^2(\eta) - c^2\frac{m+1}{2} x^{2m-1} f(\eta) f''(\eta) \\
&\quad = e^{-\alpha\theta} c^2 x^{2m-1}\Big[(1+\delta f''(\eta))f'''(\eta) - \alpha\theta' f''(\eta)(1+\frac{\delta}{2}f''(\eta))\Big] \\
&\qquad - \frac{\sigma B_0^2 x^{2m-1} c}{\rho} f'(\eta). \\
\Rightarrow\quad & c^2 x^{2m-1}\left(-f(\eta)\left(\frac{m+1}{2}\right) f''(\eta)\right) + m f'^2(\eta) \\
&\quad = e^{-\alpha\theta} c^2 x^{2m-1}\Big[-\alpha\left(1+\frac{\delta}{2}f''(\eta)\right)\theta' f''(\eta) + f'''(\eta)\left(1+\delta f''(\eta)\right)\Big] - \frac{\sigma B_0^2 x^{2m-1} c}{\rho} f'. \\
\Rightarrow\quad & m f'^2(\eta) - \left(\frac{m+1}{2}\right) f(\eta) f''(\eta) = e^{-\alpha\theta}\Big[f'''(\eta)\left(1+\delta f''(\eta)\right) \\
&\qquad - \alpha f''(\eta)\left(1+\frac{\delta}{2}f''\right)\theta'(\eta)\Big] - \frac{\sigma B_0^2}{\rho c} f'(\eta). \\
\Rightarrow\quad & e^{-\alpha\theta}\left((1+\delta f''(\eta))f'''(\eta) - f''(\eta)\alpha f''(\eta)\left(1+\frac{\delta}{2}\theta'(\eta)\right)\right) - m f'^2(\eta) - M^2 f'(\eta) \\
&\quad + f(\eta)\left(\frac{m+1}{2}\right) f''(\eta) = 0.
\end{aligned} \tag{4.12}
$$

Next, the left side of (4.3) takes the following dimensionless forms

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=Acrx^{m+r-1}f'(\eta)\theta(\eta)-Ac\left(\frac{m+1}{2}\right)x^{r+m-1}f(\eta)\theta'(\eta). \tag{4.13}$$

To convert the right side of (4.3), we need only the following conversion.

$$\begin{aligned}\frac{\sigma B^2(x)}{\rho c_p}u^2&=\frac{\sigma B_0^2}{\rho c_p}\left(cx^m f'(\eta)\right)^2\\&=\frac{\sigma B^2}{\rho c_p}c^2x^{2m}f'^2(\eta)\\&=\frac{\sigma B_0^2x^{m-1}}{\rho c_p}x^{2m}c^2f'^2(\eta)\\&=\frac{\sigma B_0^2}{\rho c_p}c^2x^{3m-1}f'^2(\eta).\end{aligned}\tag{4.14}$$

The conversion of the rest of the terms into the dimensionless form is already discussed in the previous chapter. Thus the right side of (4.3) becomes

$$\begin{aligned}&\frac{1}{\rho_\infty c_p}\frac{\partial}{\partial y}\left(k(T)\frac{\partial T}{\partial y}\right)+\frac{\mu(T)}{\rho_\infty c_p}\left(1+\frac{\Gamma}{\sqrt{2}}\frac{\partial u}{\partial y}\right)\left(\frac{\partial u}{\partial y}\right)^2-\frac{1}{\rho_\infty c_p}\frac{\partial q_r}{\partial y}+\frac{\sigma B^2(x)}{\rho c_p}u^2\\&=\frac{1}{P_r}Acx^{r+m-1}\left(\varepsilon\theta(\eta)\theta''(\eta)+\theta''(\eta)+\varepsilon\theta'^2(\eta)\right)\\&\quad+\frac{e^{-\alpha\theta}}{c_p}c^3x^{3m-1}\left(1+\frac{\delta}{2}f''(\eta)\right)f''^2(\eta)+\frac{1}{P_r}RAcx^{r+m-1}\theta''(\eta)\\&\quad+\frac{\sigma B_0^2}{\rho c_p}c^2x^{3m-1}f'^2(\eta).\end{aligned}\tag{4.15}$$

Now comparing (4.13) and (4.15), we get

$$\begin{aligned}&Acrx^{m+r-1}f'(\eta)\theta(\eta)-Ac\theta'(\eta)\left(\frac{m+1}{2}\right)x^{r+m-1}f(\eta)\\&=\frac{1}{P_r}Acx^{r+m-1}\left(\varepsilon\theta''(\eta)\theta(\eta)+\theta''(\eta)+\varepsilon\theta'^2(\eta)\right)+\frac{e^{-\alpha\theta}}{c_p}c^3x^{3m-1}\left(1+\frac{\delta}{2}f''(\eta)\right)f''^2(\eta)\\&\quad+\frac{1}{P_r}RAcx^{r+m-1}\theta''(\eta)+\frac{\sigma B_0^2}{\rho c_p}c^2x^{3m-1}f'(\eta).\\&\Rightarrow\ rf'(\eta)\theta(\eta)-f(\eta)\left(\frac{m+1}{2}\right)\theta'(\eta)\\&\quad=\frac{1}{P_r}\left(\varepsilon\theta''(\eta)\theta(\eta)+\theta''(\eta)+\varepsilon\theta'^2(\eta)\right)+\frac{e^{-\alpha\theta}c^2}{Ac_p}x^{2m-r}\left(1+\frac{\delta}{2}f''(\eta)\right)f''^2(\eta)\\&\quad+\frac{1}{P_r}R\theta''(\eta)+\frac{\sigma B_0^2}{\rho c_p}\frac{c^2x^{2m-r}}{Ac}f'^2(\eta).\end{aligned}$$

$$\Rightarrow \frac{1}{P_r}\left(\varepsilon\theta''(\eta)\theta(\eta)+\theta''(\eta)+\varepsilon\theta'^2(\eta)\right)-r\theta(\eta)f'(\eta)+\frac{1}{P_r}R\theta''(\eta)+f(\eta)\left(\frac{m+1}{2}\right)\theta'(\eta)$$
$$+M^2Ecf'^2(\eta)+Ecf''^2(\eta)\left(1+\frac{\delta}{2}f''(\eta)\right)e^{-\alpha\theta}=0,$$
$$\Rightarrow \frac{1}{P_r}\left(\varepsilon\theta'^2(\eta)+(1+\varepsilon\theta(\eta))\theta''(\eta)+R\right)+f(\eta)\left(\frac{m+1}{2}\right)\theta'(\eta)-f'(\eta)r\theta(\eta)$$
$$+Ec\left(1+\frac{\delta}{2}f''(\eta)\right)f''^2(\eta)e^{-\alpha\theta}+M^2Ecf'^2(\eta)=0. \tag{4.16}$$

To get the dimensionless form of (4.4), we proceed as follows

$$C=C_\infty+\phi(\eta)(C_w-C_\infty).$$

- $$\frac{\partial C}{\partial x}=(C_w-C_\infty)\phi'(\eta)\left(\frac{c}{\nu_\infty}\right)^{\frac{1}{2}}y\left(\frac{m-1}{2}\right)x^{\frac{m-1}{2}-1}$$
$$=(C_w-C_\infty)\left(\frac{c}{\nu_\infty}\right)^{\frac{1}{2}}\left(\frac{m-1}{2}\right)x^{\frac{m-3}{2}}y\phi'(\eta).$$
- $$u\frac{\partial C}{\partial y}=\left(cx^mf'(\eta)\right)\left[(C_w-C_\infty)\left(\frac{c}{\nu_\infty}\right)^{\frac{1}{2}}\left(\frac{m-1}{2}\right)x^{\frac{m-3}{2}}y\phi'(\eta)\right]$$
$$=\frac{c^{\frac{3}{2}}}{(\nu_\infty)^{\frac{1}{2}}}(C_w-C_\infty)\left(\frac{m-1}{2}\right)x^{\frac{3m-3}{2}}yf'(\eta)\phi'(\eta). \tag{4.17}$$
- $$\frac{\partial C}{\partial y}=(C_w-C_\infty)\phi'(\eta)\left(\frac{c}{\nu_\infty}\right)^{\frac{1}{2}}x^{\frac{m-1}{2}}.$$
- $$v\frac{\partial C}{\partial y}=\left(-c\left(\frac{m-1}{2}\right)x^{m-1}yf'(\eta)-(c\nu_\infty)^{\frac{1}{2}}x^{\frac{m-1}{2}}\left(\frac{m+1}{2}\right)f(\eta)\right)$$
$$\left((C_w-C_\infty)\phi'(\eta)\left(\frac{c}{\nu_\infty}\right)^{\frac{1}{2}}x^{\frac{m-1}{2}}\right)$$
$$=-\frac{(c)^{\frac{3}{2}}}{(\nu_\infty)^{\frac{1}{2}}}\left(\frac{m-1}{2}\right)(C_w-C_\infty)x^{\frac{3m-3}{2}}yf'(\eta)\phi'(\eta)$$
$$-c(C_w-C_\infty)x^{m-1}\left(\frac{m+1}{2}\right)f(\eta)\phi'(\eta). \tag{4.18}$$

Adding (4.17) and (4.18), we get the left side of (4.4), i.e.

$$u\frac{\partial C}{\partial y}+v\frac{\partial C}{\partial y}=-c(C_w-C_\infty)x^{m-1}\left(\frac{m+1}{2}\right)f(\eta)\phi'(\eta). \tag{4.19}$$

To get the right side of (4.4), the following proccess will be helpful

- $$\frac{\partial C}{\partial y}=(C_w-C_\infty)\phi'(\eta)\left(\frac{c}{\nu_\infty}\right)^{\frac{1}{2}}x^{\frac{m-1}{2}}$$

- $$\frac{\partial^2 C}{\partial y^2} = (C_w - C_\infty)\left(\frac{c}{\nu_\infty}\right)^{\frac{1}{2}} x^{\frac{m-1}{2}} \phi''(\eta)\left(\frac{c}{\nu_\infty}\right)^{\frac{1}{2}} x^{\frac{m-1}{2}}$$
$$= (C_w - C_\infty)\left(\frac{c}{\nu_\infty}\right) x^{m-1}\phi''(\eta)$$
- $$D\frac{\partial^2 C}{\partial y^2} = D(C_w - C_\infty)\left(\frac{c}{\nu_\infty}\right) x^{m-1}\phi''(\eta) \tag{4.20}$$
- $$k_1(x)(C - C_\infty) = -k_1(x)\left(C_\infty + \phi(\eta)(C_w - C_\infty) - C_\infty\right)$$
$$= -k_1(x)\phi(\eta)(C_w - C_\infty). \tag{4.21}$$

Adding (4.20) and (4.21), we get the right side of (4.4) which is

$$D\frac{\partial^2 C}{\partial y^2} - k_1(x(C - C_\infty) = \left(D(C_w - C_\infty)\left(\frac{c}{\nu_\infty}\right) x^{m-1}\phi''(\eta)\right)$$
$$- \left(k_1(x)\phi(\eta)(C_w - C_\infty)\right). \tag{4.22}$$

Comparing (4.19) and (4.22), we get the following dimensionless form of (4.4)

$$- c(C_w - C_\infty)x^{m-1}\left(\frac{m+1}{2}\right) f(\eta)\phi'(\eta) = \left(D(C_w - C_\infty)\left(\frac{c}{\nu_\infty}\right) x^{m-1}\phi''(\eta)\right)$$
$$- \left(k_1(x)\phi(\eta)(C_w - C_\infty)\right).$$
$$\Rightarrow \; D(C_w - C_\infty)\left(\frac{c}{\nu_\infty}\right) x^{m-1}\phi''(\eta) - k_1(x)\phi(\eta)(C_w - C_\infty)$$
$$+ c(C_w - C_\infty)x^{m-1}\left(\frac{m+1}{2}\right) f(\eta)\phi'(\eta) = 0,$$
$$\Rightarrow \; \phi''(\eta) - \frac{\nu_\infty}{D}\frac{k_1}{(cx^{m-1})}\phi(\eta) + \frac{\nu_\infty}{D}\left(\frac{m+1}{2}\right) f(\eta)\phi'(\eta) = 0,$$
$$\Rightarrow \; \phi''(\eta) - ScK_r\phi(\eta) + Sc\left(\frac{m+1}{2}\right) f(\eta)\phi'(\eta) = 0,$$
$$\Rightarrow \; \phi''(\eta) - ScK_r\phi(\eta) + Sc\left(\frac{m+1}{2}\right) f(\eta)\phi'(\eta) = 0. \tag{4.23}$$

Finally, the dimensionless form of the governing mathematical model is as follows

$$e^{-\alpha\theta}\left((1 + \delta f''(\eta))f'''(\eta) - \alpha\theta(\eta)' f''(\eta)\left(1 + \frac{\delta}{2}f''(\eta)\right)\right) + \left(\frac{m+1}{2}\right) f(\eta)f''(\eta)$$
$$- mf'^2(\eta) - M^2 f'(\eta) = 0, \tag{4.24}$$
$$\frac{1}{P_r}\left(\varepsilon\theta'^2(\eta) + (1 + R + \varepsilon\theta(\eta))\theta''(\eta)\right) + \left(\frac{m+1}{2}\right) f(\eta)\theta'(\eta) - rf'(\eta)\theta(\eta)$$
$$+ Ec\left(1 + \frac{\delta}{2}f''(\eta)\right) f''^2(\eta)e^{-\alpha\theta} + M^2 Ecf'^2(\eta) = 0, \tag{4.25}$$

$$\phi'' + Sc\left(\frac{m+1}{2}\right)f(\eta)\phi'(\eta) - ScK_r\phi(\eta) = 0. \tag{4.26}$$

Now, the procedure for the conversion of the boundary conditions into the dimensionless form, has been presented below

- $u = cx^m,$ at $y = 0.$

$\Rightarrow \quad u = cx^m f'(\eta),$ at $\eta = 0.$

$\Rightarrow \quad cx^m = cx^m f'(\eta),$ at $\eta = 0.$

$\Rightarrow \quad f'(\eta) = 1,$ at $\eta = 0.$

$\Rightarrow \quad f'(0) = 1.$

- $v = 0,$ at $y = 0.$

$\Rightarrow \quad -c\left(\frac{m-1}{2}\right)x^{m-1}yf'(\eta) - (c\nu_\infty)^{\frac{1}{2}}\left(\frac{m+1}{2}\right)x^{\frac{m-1}{2}}f(\eta) = 0,$

at $\eta = 0.$

$\Rightarrow \quad -(c\nu_\infty)^{\frac{1}{2}}\left(\frac{m+1}{2}\right)x^{\frac{m-1}{2}}f(0) = 0,$ at $\eta = 0.$

$\Rightarrow \quad f(0) = 0.$

- $T = T_\infty + Ax^r,$ at $y = 0.$

$\Rightarrow \quad T - T_\infty = Ax^r,$ at $y = 0.$

$\Rightarrow \quad (T_w - T_\infty)\theta(\eta) = Ax^r,$ at $\eta = 0.$

$\Rightarrow \quad Ax^r\theta(0) = Ax^r,$ at $\eta = 0.$

$\Rightarrow \quad \theta(0) = 1.$

- $C = C_w,$ at $y = 0.$

$\Rightarrow \quad C_\infty + \phi(\eta)(C_w - C_\infty) = C_w,$ at $\eta = 0.$

$\Rightarrow \quad \phi(\eta)(C_w - C_\infty) = C_w - C_\infty,$ at $\eta = 0.$

$\Rightarrow \quad \phi(\eta) = 1,$ at $\eta = 0.$

$\Rightarrow \quad \phi(0) = 1.$

- $u \to 0,$ as $y \to \infty.$

$\Rightarrow \quad cx^m f'(\eta) \to 0,$ as $\eta \to \infty.$

$\Rightarrow \quad f'(\eta) \to 0,$ as $\eta \to \infty.$

- $T \to T_\infty,$ as $y \to \infty.$

$$\Rightarrow \; T_\infty + (T_w - T_\infty)\theta(\eta) \to T_\infty, \qquad \text{as} \quad \eta \to \infty.$$

$$\Rightarrow \; \theta(\eta) \to 0, \qquad \text{as} \quad \eta \to \infty.$$

- $$C \to C_\infty, \qquad \text{as} \quad y \to \infty.$$

$$\Rightarrow \; C_\infty + \phi(\eta)(C_w - C_\infty) \to C_\infty, \qquad \text{as} \quad \eta \to \infty.$$

$$\Rightarrow \; \phi(\eta)(C_w - C_\infty) \to 0, \qquad \text{as} \quad \eta \to \infty.$$

$$\Rightarrow \; \phi(\eta) \to 0, \qquad \text{as} \quad \eta \to \infty.$$

Hence, the associated BCs in the dimensionless form, are:

$$\left.\begin{array}{l} f(0) = 0, \; f'(0) = 1, \; \theta(0) = 1, \; \phi(0) = 1 \\ f' \to 0, \quad \theta \to 0, \quad \phi \to 0 \quad at \quad \eta \to \infty. \end{array}\right\} \tag{4.27}$$

Different parameters used in equations (4.24)-(4.26), have the following formulae

$$R = \frac{16\delta^* T_\infty^3}{3k_\infty k^*}, \quad \delta = \frac{\sqrt{2} c^{\frac{3}{2}} x^{\frac{3m-1}{2}}}{\sqrt{\nu_\infty}} \Gamma, \quad Pr = \frac{\mu_\infty c_p}{k_\infty}, \quad M = -\frac{\sigma B_0^2}{\rho c}$$

$$Ec = \frac{U_w^2}{c_p(T_w - T_\infty)} = \left(\frac{c^2 x^{2m-r}}{Acp}, \quad r\right) = 2m = \frac{2}{3}, \quad Sc = \frac{\nu_\infty}{D}, \quad K_r = \frac{2k_1}{(n+1)ax^{n-1}}.$$

The local Sherwood number is defined as,

$$Sh_x = \frac{xq_m}{Dm(C_w - C_\infty)}. \tag{4.28}$$

To get the dimensionless form of Sh_x, the following procedure will be helpful

$$q_m = -D_m \left(\frac{\partial C}{\partial y}\right)_{y=0}. \tag{4.29}$$

$$\begin{aligned} Sh_x &= -\frac{xD_m}{D_m(C_w - C_\infty)} \left(\frac{\partial C}{\partial y}\right)_{y=0} \\ &= -\frac{x}{C_w - C_\infty}(C_w - C_\infty)\left(\frac{a}{\nu_\infty}\right)^{\frac{1}{2}} x^{\frac{m-1}{2}} \phi'(0) \\ &= -x\left(\frac{a}{\nu_\infty}\right)^{\frac{1}{2}} x^{\frac{m-1}{2}} \phi'(0) \\ &= -x^{\frac{m+1}{2}} \left(\frac{a}{\nu_\infty}\right)^{\frac{1}{2}} \phi'(0) \\ &= -\left(\frac{ax^m x}{\nu_\infty}\right)^{\frac{1}{2}} \phi'(0) \end{aligned}$$

$$= -\left(Re_x\right)^{\frac{1}{2}} \phi'(0)$$
$$\Rightarrow \quad \left(Re_x\right)^{\frac{-1}{2}} Sh_x = -\phi'(0). \tag{4.30}$$

4.3 Solution Methodology

In this section, to get the approximate solution of the ordinary differential equations (4.24)-(4.26) along with the boundary condition (4.27), the shooting method has been used. First of all, we need to convert these equations into a system of first order differential equations. Let us use the following notations

$$f = h_1, \quad f' = h_1' = h_2, \quad f'' = h_2' = h_3, \quad f''' = h_3',$$
$$\theta = h_4, \quad \theta' = h_4' = h_5, \quad \theta'' = h_5,$$
$$\phi = t_1, \quad \phi' = t_1' = t_2, \quad \phi'' = t_2'.$$

Now, the system of ODEs (4.24)-(4.25) with the boundary conditions (4.27) is transformed into the following system of first order differential equations,

$$h_1' = h_2, \qquad h_1(0) = 0,$$
$$h_2' = h_3, \qquad h_2(0) = 1,$$
$$h_3' = \frac{e^{\alpha\theta}}{1 + dy_3}\left[\frac{\alpha}{e^{\alpha\theta}} h_5 h_3 \left(1 + \frac{\delta}{2} h_3\right) - \left(\frac{m+1}{2}\right) h_1 h_3 + m h_2^2 + M^2 h_2\right],$$
$$h_3(0) = l,$$
$$h_4' = h_5, \qquad h_4(0) = 0,$$
$$h_5' = \frac{Pr}{1 + R + \varepsilon h_4}\left[r h_2 h_4 - \frac{\varepsilon}{Pr} h_5^2 - \left(\frac{m+1}{2}\right) h_1 h_5 \right.$$
$$\left. - \frac{Ec}{e^{\alpha h_4}}\left(1 + \frac{\delta}{2} h_3\right) h_3^2 - M^2 Ec h_2^2\right], \qquad h_5(0) = m.$$

In the above IVP, the missing conditions l and m are to be chosen to satisfy the following relations,

$$\left(h_2(l, m)\right)_{\eta=\eta_\infty} = 0, \qquad \left(h_4(l, m)\right)_{\eta=\eta_\infty} = 0.$$

Here $h_2(l, m)$ and $h_4(l, m)$ are the values of h_2 and h_4 at $\eta = \eta_\infty$ for the chosen values of the missing conditions l and m.

Newton's method with the following itrative scheme will be used to solve for the above two equations with two variables.

$$\begin{bmatrix} l_{n+1} \\ m_{n+1} \end{bmatrix} = \begin{bmatrix} l_n \\ m_n \end{bmatrix} - \begin{bmatrix} \frac{\partial h_2}{\partial u} & \frac{\partial h_2}{\partial v} \\ \frac{\partial h_4}{\partial u} & \frac{\partial h_4}{\partial v} \end{bmatrix}^{-1}_{(l_n,m_n)} \begin{bmatrix} h_2 \\ h_4 \end{bmatrix}_{(l_n,m_n)}. \tag{4.31}$$

To find the partial derivative of h_2 and h_4, the following notations have been intoduced

$$\frac{\partial h_1}{\partial l} = h_6, \quad \frac{\partial h_2}{\partial l} = h_7, \quad \frac{\partial h_3}{\partial l} = h_8, \quad \frac{\partial h_4}{\partial l} = h_9, \quad \frac{\partial h_5}{\partial l} = h_{10},$$

$$\frac{\partial h_1}{\partial m} = h_{11}, \quad \frac{\partial h_2}{\partial m} = h_{12}, \quad \frac{\partial h_3}{\partial m} = h_{13}, \quad \frac{\partial h_4}{\partial m} = h_{14}, \quad \frac{\partial h_5}{\partial m} = h_{15}.$$

Using the above notations in (4.32), we get

$$\begin{bmatrix} l_{n+1} \\ m_{n+1} \end{bmatrix} = \begin{bmatrix} l_n \\ m_n \end{bmatrix} - \begin{bmatrix} h_7 & h_{12} \\ h_9 & h_{14} \end{bmatrix}^{-1}_{(l_n,m_n)} \begin{bmatrix} h_2 \\ h_4 \end{bmatrix}_{(l_n,m_n)} \tag{4.32}$$

Now differentiate the system of first order ODEs with respect to l and m, we get the following ODEs,

$$\begin{aligned}
h_6' &= h_7, && h_6(0) = 0,\\
h_7' &= h_8, && h_7(0) = 0,\\
h_8' &= \frac{(1+\delta h_3)\alpha h_9 e^{\alpha h_4} - e^{\alpha h_4}(\delta h_8)}{(1+\delta h_3)^2}\left[\frac{\alpha}{e^{\alpha h_4}} h_5 h_3 \left(1+\frac{\delta}{2}h_3\right) - \left(\frac{m+1}{2}\right) h_1 h_3 + m y_2^2\right]\\
&\quad + \frac{e^{\alpha h_4}}{1+\delta h_3}\Bigg[\alpha e^{\alpha h_4}(-\alpha h_9) h_5 h_3 \left(1+\frac{\delta}{2}h_3\right) + \frac{\alpha}{e^{\alpha h_4}} h_{10} h_3 \left(1+\frac{\delta}{2}h_3\right)\\
&\quad + \frac{\alpha}{e^{\alpha h_4}} h_5 h_8 \left(1+\frac{\delta}{2}h_3\right) + \frac{\alpha}{e^{\alpha h_4}} h_5 h_3 \left(\frac{\delta}{2}h_8\right) - \left(\frac{m+1}{2}\right)(h_1 h_8 + h_6 h_3)\\
&\quad + 2m h_2 h_7 + M^2 h_7\Bigg],\\
& && h_8(0) = 1,\\
h_9' &= h_{10}, && h_9(0) = 0,\\
h_{10}' &= -\frac{Pr\varepsilon}{(1+R\varepsilon h_4)^2} h_9 \left[r h_2 h_4 - \left(\frac{m+1}{2}\right) h_1 h_5 - \frac{Ec}{e^{\alpha h_4}\left(1+\frac{\delta}{2}h_3\right)} h_3^2\right]
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{Pr}{1+R\varepsilon h_4}\Bigg[r(h_2h_9+h_7h_4) - \frac{2\varepsilon}{Pr}h_5h_{10} - \left(\frac{m+1}{2}\right)(h_1h_{10}+h_6h_5) \\
&- Ece^{\alpha h_4}(-\alpha h_9)\left(1+\frac{\delta}{2}h_3\right)h_3^2 - \frac{Ec}{e^{\alpha h_4}}\left(\frac{\delta}{2}h_8\right)h_3^2 - \frac{2Ec}{e^{\alpha h_4}}\left(1+\frac{\delta}{2}h_3\right)h_3h_8 \\
&- 2M_2Ech_2h_7\Bigg],
\end{aligned}
$$

$$h_{10}(0) = 0,$$

$$h'_{11} = h_{12}, \qquad h_{11}(0) = 0,$$

$$h'_{12} = h_{13}, \qquad h_{12}(0) = 0.$$

$$
\begin{aligned}
h'_{13} = &\frac{(1+\delta h_3)\alpha h_{14}e^{\alpha h_4} - e^{\alpha h_4}(\delta h_{13})}{(1+\delta h_3)^2}\left[\frac{\alpha}{e^{\alpha h_4}}h_5h_3\left(1+\frac{\delta}{2}h_3\right) - \left(\frac{m+1}{2}\right)h_1h_3 + mh_2^2\right] \\
&+ \frac{e^{\alpha h_4}}{(1+\delta h_3)}\Bigg[\alpha e^{-\alpha h_4}(-\alpha h_{14})h_5h_3\left(1+\frac{\delta}{2}h_3\right) + \frac{\alpha}{e^{\alpha h_4}}h_{15}h_3\left(1+\frac{\delta}{2}h_3\right) \\
&+ \frac{\alpha}{e^{\alpha h_4}}h_5h_{13}\left(1+\frac{\delta}{2}h_3\right) + \frac{\alpha}{e^{\alpha h_4}}h_5h_3\left(\frac{\delta}{2}h_{13}\right) \\
&- \left(\frac{m+1}{2}\right)(h_1h_{13}+h_{11}h_3) + 2mh_2h_{12} + M^2h_{12}\Bigg],
\end{aligned}
$$

$$h_{13}(0) = 0,$$

$$h'_{14} = h_{15}, \qquad h_{14}(0) = 0,$$

$$
\begin{aligned}
h'_{15} = &-\frac{Pr\varepsilon h_{14}}{(1+R\varepsilon h_4)^2}\left[rh_2h_4 - \frac{\varepsilon h_5^2}{Pr} - \left(\frac{m+1}{2}\right)h_1h_5 - \frac{Ec}{e^{\alpha h_4}}\left(1+\frac{\delta}{2}\right)h_3^2\right] \\
&+ \frac{Pr}{(1+R+\varepsilon h_4)}\Bigg[r(h_2h_{14}+h_{12}h_4) - \frac{2\varepsilon}{Pr}h_5h_{15} - \left(\frac{m+1}{2}\right)(h_1h_{15}+h_{11}h_5) \\
&- Ece^{-\alpha h_4}(-\alpha h_{14})\left(1+\frac{\delta}{2}h_3\right)h_3^2 - \frac{Ec}{e^{\alpha h_4}}\left(\frac{\delta}{2}h_{13}\right)h_3^2 - \frac{2Ec}{e^{\alpha h_4}}\left(1+\frac{\delta}{2}h_3\right)h_3h_{13} \\
&- 2M^2h_2h_{12}\Bigg],
\end{aligned}
$$

$$h_{15}(0) = 1.$$

The stopping criteria for the Newton's method is set as:

$$max\,\{|h_2(\eta_\infty)|, |h_4(\eta_\infty)|\} < \epsilon.$$

Using the notations, the system (4.26) with the boundary conditions (4.27) is converted into the following first order differential equations

$$
\begin{aligned}
b_1' &= b_2, & b_1(0) &= 1,\\
b_2' &= Sc\ k_r\ b_1 - Sc\left(\frac{m+1}{2}\right) c_1\ b_2, & b_2(0) &= s,\\
b_3' &= b_4, & b_3(0) &= 0,\\
b_4' &= Sc\ k_r\ b_3 - Sc\left(\frac{m+1}{2}\right) c_1\ b_4, & b_4(0) &= 1.
\end{aligned}
$$

The missing condition s is updated by the Newtons method and process will be continued until the following criteria is met:

$$(b_1(\eta_\infty))_{s=s_n} < \epsilon.$$

where ϵ is an arbitrarily small positive number. From now onward ϵ has been taken 10^{-10}.

In the above IVP, the missing condition s is to be chosen to satisfy the following relation.

$$b_1(\eta_\infty)_s = 0.$$

Here $b_1(\eta_\infty)_s$ are the value of b_1 at $\eta = \eta_\infty$ for the chosen value of the missing conditon s. Newton's method with the following iterative scheme will be used to solve the above equation.

$$s_{n+1} = s_n - \frac{(b_1(\eta_\infty))_{s=s_n}}{\left(\frac{\partial b_1(\eta_\infty)}{\partial s}\right)_{s=s_n}}. \tag{4.33}$$

To find the partial derivative of b_1 the following notation has been introduced

$$\frac{\partial b_1}{\partial s} = t_3,$$

Using the above notation in (4.33), we get

$$s_{n+1} = s_n - \frac{(b_1(\eta_\infty))_{s=s_n}}{(b_3(\eta_\infty))_{s=s_n}}.$$

4.4 Results and Discussions

In this section, the numerical solutions are addressed in detail to examine the impact of different parameters on velocity $f'(\eta)$, concentration $\phi(\eta)$ and temperature $\theta(\eta)$. The impact of different factors like Williamson parameter δ, viscosity parameter α, radiation parameter R, thermal conductivity parameter ε, local Eckert number Ec on the skin friction $\frac{1}{2}(Re_x)^{\frac{1}{2}}Cf_x$, local Nusselt number $(Re_x)^{-\frac{1}{2}}Nu_x$ and Sherwood number $Re_x^{-\frac{1}{2}}Sh_x$ has been shown through the tables and graphs.

Table 4.1 discusses the effect of different parameters including the magnetic parameter M, Schmidt number Sc and chemical reaction Kr on local skin friction $\frac{1}{2}(Re_x)^{\frac{1}{2}}Cf_x$ has been shown in this table. By rising the value α, the local skin friction $\frac{1}{2}(Re_x)^{\frac{1}{2}}Cf_x$ decreases. Silmilarly, by rising the values of Willaimson parameter δ, radiation parameter R, thermal conductivity parameter ε, and Eckert number Ec, the local skin friction $\frac{1}{2}(Re_x)^{\frac{1}{2}}Cf_x$ shows a decreasing behaviour. Furthermore, by improving the values of Schmidt numebr Sc, the local skin friction $\frac{1}{2}(Re_x)^{\frac{1}{2}}Cf_x$, is found to be decreased. In this table, I_l and I_m are the intervals from which the missing conditions l and m can be chosen.

Table 4.2 discusses the effect of Williamson parameter δ, viscosity parameter α, thermal conductivity parameter ε, radiation parameter R, Eckert number Ec, magnetic parameter M, chemical reaction parameter Kr and Schmidt number Sc, on local Nusselt number $(Re_x)^{\frac{-1}{2}}Nu_x$. By improving the value α, the local Nusselt number $(Re_x)^{\frac{-1}{2}}Nu_x$ is found to decrease. By rising the values of Willaimson parameter δ, thermal conductivity parameter ε, radiation parameter R and Eckert number Ec, the local Nusselt number $(Re_x)^{\frac{-1}{2}}Nu_x$ shows a decreasing behaviour. Furthermore, by improving the values of Schmidt numebr Sc, the local Nusselt number $(Re_x)^{\frac{-1}{2}}Nu_x$ decreases. Simialrly, in this table , I_l and I_m are the intervals from which the missing conditions l and m can be chosen.

Table 4.3 discusses the effect of Williamson parameter δ, viscosity parameter α, thermal conductivity parameter ε, radiation parameter R, Eckert number Ec, magnetic parameter M, chemical reaction parameter Kr and Schmidt number Sc, on local Sherwood number $(Re_x)^{-\frac{1}{2}}Sh_x$. By rising the value of the viscosity parameter α, the local Sherwood number $(Re_x)^{-\frac{1}{2}}Sh_x$ is found to decrease. By improving the values of Willaimson parameter δ, the local Sherwood number $(Re_x)^{-\frac{1}{2}}Sh_x$ shows a decreasing behaviour. In this table, I_l, I_m and I_s are the intervals from which the missing conditions l, m and s can be chosen.

Figure 4.2 shows the detailed behaviour of the velocity $f'(\eta)$ for the values of viscoity parameter α, with the constant values of the rest of the parameters. By increasing the value of the viscoity parameter α, the velocity $f'(\eta)$ is found to decrease. Figure 4.3 shows the representation of the temperature profile $\theta(\eta)$ for the values of viscosity parameter α, and the rest of the parameters are considered as a constant.
Figure 4.4 represents the velocity $f'(\eta)$ for different values of Williamson parameter δ. By increasing the values of Williamson parameter δ, the velocity profile $f'(\eta)$ is found to decrease.

Figure 4.5 shows the behaviour of the temperature $\theta(\eta)$ for different values of Williamson parameter δ. By improving the values of δ, the temperature profile $\theta(\eta)$ is found to increase, which is an understandable behaviour. Figure 4.6 shows the behaviour of the temperature field $\theta(\eta)$ for the values of the thermal conductivity ε. By rising the values of ε, the temperature $\theta(\eta)$ shows a natural increasing behaviour.
Figure 4.7 shows the behaviour of the temperature $\theta(\eta)$ for different values of the radiation parameter R, with the values of the rest of the parameters as constants. The temperature $\theta(\eta)$ increases by rising the value of the radiation parameter R. Actually, with an increase in the thermal radiation, the heat transfer increases because of which the temperature $\theta(\eta)$ is increased.

Figure 4.8 represents the temperature $\theta(\eta)$ for various values of Eckert number Ec. By increasing the values of Ec, the temperature $\theta(\eta)$ is found to increase. Figure 4.9 shows the behaviour of the concentration profile $\phi(\eta)$ for different values of magnetic parameter M. By rising the values of the magnetic parameter M, the concentration $\phi(\eta)$ shows a decreasing behaviour. Figure 4.10 represents the velocity $f'(\eta)$ for the value of magnetic parameter M. By scaling the values of the magnetic parameter M, the velocity profile $f'(\eta)$ is found to decrease. Figure 4.11 shows the behaviour of the concentration profile $\phi(\eta)$ for various values of chemical reaction parameter Kr. By rising the values of the chemical reaction parameter Kr, the concentratation $\phi(\eta)$ shows an increasing behaviour. Figure 4.12 represents the concentration $\phi(\eta)$ for different vlaues of Schmidt number Sc, with the constant values of the rest of the parameters. By rising the values of Schmidt number Sc, the concentration $\phi(\eta)$ is found to decrease. Figure 4.13 represents the behaviour of the temperture $\theta(\eta)$ for various values of Schmidt number Sc.

By improving the values of Schmidt number Sc, the temperature $\theta(\eta)$ shows an increasing behaviour.

Figure 4.14 represents the behaviour of the concentration $\phi(\eta)$ for various values of the radiation parameter R, with the rest of the parameters set to constant levels. The concentration $\phi(\eta)$ displays an increasing behaviour as the values of the radiation parameter R are improved.

TABLE 4.1: Values of $(Re_x)^{\frac{1}{2}}Cf_x$, for $\alpha,\ \varepsilon,\ \delta,\ R,\ Kr,\ Sc,\ Ec,\ M$ with $m=\frac{1}{3}$ and $Pr=2.0$

α	δ	ε	R	Ec	M	Kr	Sc	I_l	I_m	$(Re_x)^{\frac{1}{2}}Cf_x$
0.5	0.2	0.2	0.2	0.2	0.2	1	0.2	[-0.9,-0.6]	[-0.9,-0.5]	0.567727
0.0								[-0.7,-0.1]	[-1.3,-0.1]	0.685700
1.0					0.3			[-0.9,-0.8]	[-0.9,-0.8]	0.474920
	0.2	0.0			0.2			[-0.9,-0.8]	[-1.0,-0.1]	0.570457
	0.5	0.2				1	0.6	[-0.9,-0.6]	[-0.9,-0.7]	0.523903
	0.0					2	0.2	[-0.9,-0.8]	[-0.9,-0.8]	0.590572
		0.5				3		[-0.9,-0.6]	[-0.9,-0.7]	0.564283
			0.0			1	0.7	[-0.9,-0.6]	[-0.9,-0.8]	0.570876
			0.2	0.0	0.2		0.2	[-0.9,-0.7]	[-0.9,-0.6]	0.569618
				0.5		1		[-0.9,-0.7]	[-0.8,-0.1]	0.564927
			0.5	0.2	0.4	5		[-0.9,-0.8]	[-0.7,-0.5]	0.621859
			1.0		0.2	1	2.6	[-0.9,-0.7]	[-0.9,-0.5]	0.559121

TABLE 4.2: Values of $(Re_x)^{-\frac{1}{2}}Nu_x$, for $\alpha,\ \delta,\ \varepsilon,\ R,\ Ec,\ M,\ Kr,\ Sc$ with $m = \frac{1}{3}$, and $Pr = 2.0$

α	δ	ε	R	Ec	M	Kr	Sc	I_l	I_m	$(Re_x)^{-\frac{1}{2}}Nu_x$
0.5	0.2	0.2	0.2	0.2	0.2	1	0.2	[-0.9,-0.6]	[-0.9,-0.5]	1.25018
0.0								[-0.7,-0.1]	[-1.3,-0.1]	1.30979
1.0					0.3			[-0.9,-0.8]	[-0.9,-0.8]	1.14659
	0.2	0.0			0.2	1		[-0.9,-0.8]	[-1.0,-0.1]	1.19610
	0.5	0.2					0.6	[-0.9,-0.6]	[-0.9,-0.7]	1.22244
	0.0					2	0.2	[-0.9,-0.8]	[-0.9,-0.8]	1.26289
		0.5				3		[-0.9,-0.6]	[-0.9,-0.8]	1.32364
			0.0			1	0.7	[-0.9,-0.6]	[-0.9,-0.8]	1.17152
			0.2	0.0	0.2		0.2	[-0.9,-0.7]	[-0.9,-0.6]	1.34822
				0.5				[-0.9,-0.7]	[-0.8,-0.1]	1.10408
			0.5	0.2	0.4	5		[-0.9,-0.7]	[-0.7,-0.5]	1.27654
			1.0		0.2	1	2.6	[-0.9,-0.7]	[-0.8,-0.1]	1.10408

TABLE 4.3: Values of $(Re_x)^{-\frac{1}{2}}Sh_x$ for $\alpha,\ \delta,\ Ec,\ M,\ Kr,\ Sc,\ \varepsilon,\ R$ with $m = \frac{1}{3}$, and $Pr = 2.0$

α	δ	ε	R	Ec	M	Kr	Sc	I_l	I_m	Is	$(Re_x)^{-\frac{1}{2}}Sh_x$
0.5	0.2	0.2	0.2	0.2	0.2	1	0.2	[-0.9,-0.6]	[-0.9,-0.5]	[-0.9,-0.3]	0.48578
0.0								[-0.7,-0.1]	[-1.3,-0.1]	[-0.9,-0.1]	0.48864
1.0					0.3			[-0.9,-0.8]	[-0.9,-0.8]	[-0.9,-0.4]	0.48120
	0.2	0.0			0.2			[-0.9,-0.8]	[-1.0,-0.1]	[-0.9,-0.1]	0.48588
	0.5	0.2					0.6	[-0.9,-0.6]	[-0.9,-0.7]	[-0.8,-0.3]	0.40552
	0.0					2	0.2	[-0.9,-0.8]	[-0.9,-0.8]	[-0.9,-0.1]	0.66459
		0.5				3		[-0.9,-0.6]	[-0.9,-0.8]	[-0.9,-0.4]	0.80232
			0.0			1	0.7	[-0.9,-0.8]	[-0.7,-0.5]	[-0.9,-0.4]	0.93203
			0.2	0.0	0.2		0.2	[-0.9,-0.7]	[-0.9,-0.6]	[-0.9,-0.2]	0.48585
				0.5			0.2	[-0.9,-0.7]	[-0.8,-0.1]	[-0.9,-0.2]	0.48568
			0.5	0.2	0.4	5	0.2	[-0.9,-0.8]	[-0.7,-0.5]	[-0.9,-0.2]	1.02252
			1.0		0.2	1	2.6	[-0.9,-0.7]	[-0.7,-0.5]	[-0.9,-0.4]	1.83634

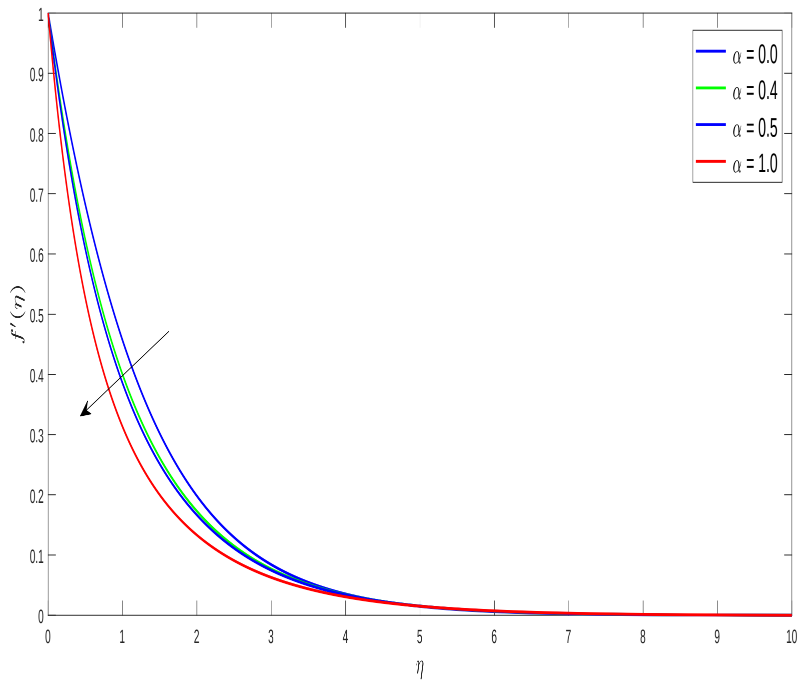

FIGURE 4.2: Impact of various values of α on velocity $f'(\eta)$ with $\delta = \varepsilon = R = M = Ec = Sc = 0.2,\ Pr = 2.0,\ Kr = 1,\ m = \frac{1}{3}$

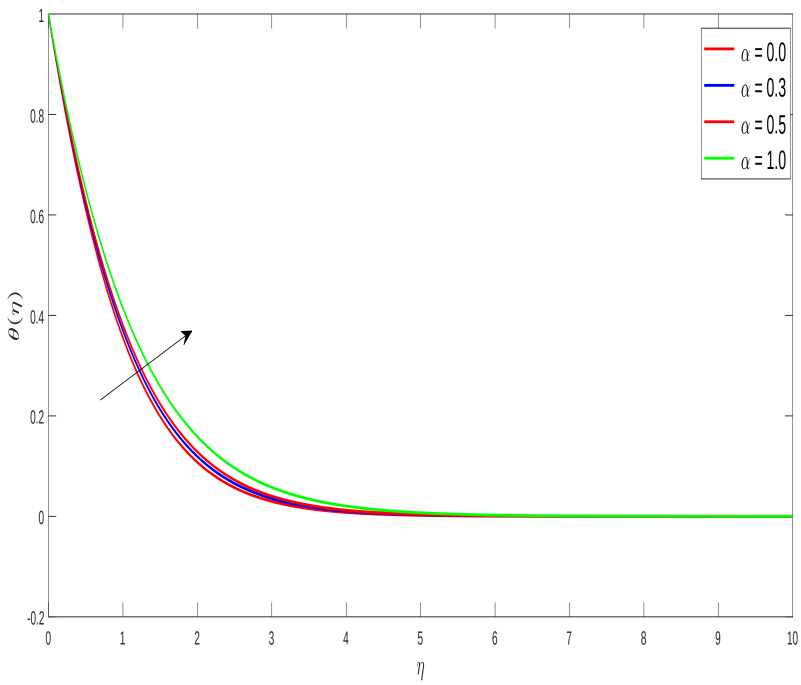

FIGURE 4.3: Impact of various values of α on temperature $\theta(\eta)$ with $\delta = \varepsilon = R = M = Ec = Sc = 0.2,\ Pr = 2.0,\ Kr = 1,\ m = \frac{1}{3}$

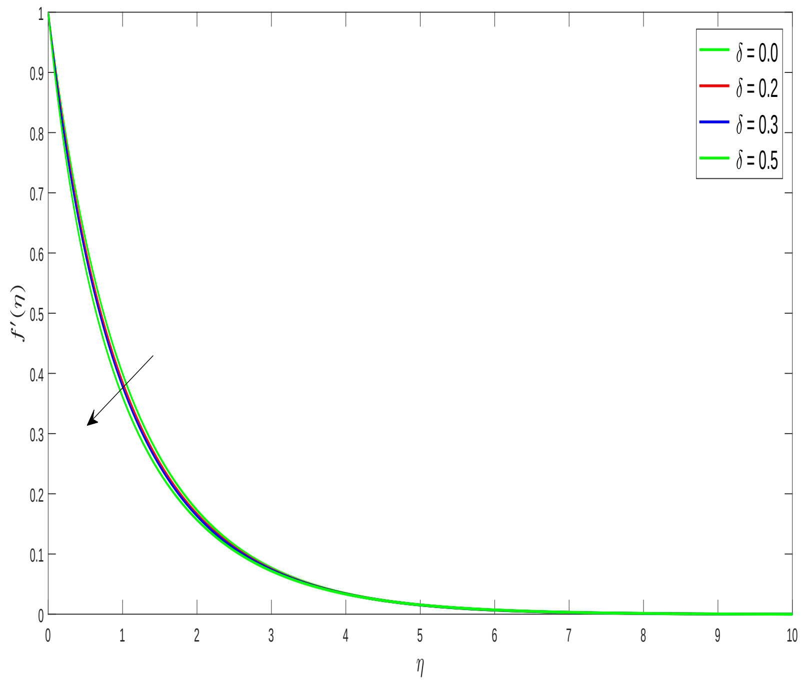

FIGURE 4.4: Impact of various values of δ on velocity $f'(\eta)$ with $\varepsilon = Ec = M = Sc = R = 0.2,\ \alpha = 0.5,\ Pr = 2.0,\ Kr = 1,\ m = \frac{1}{3}$

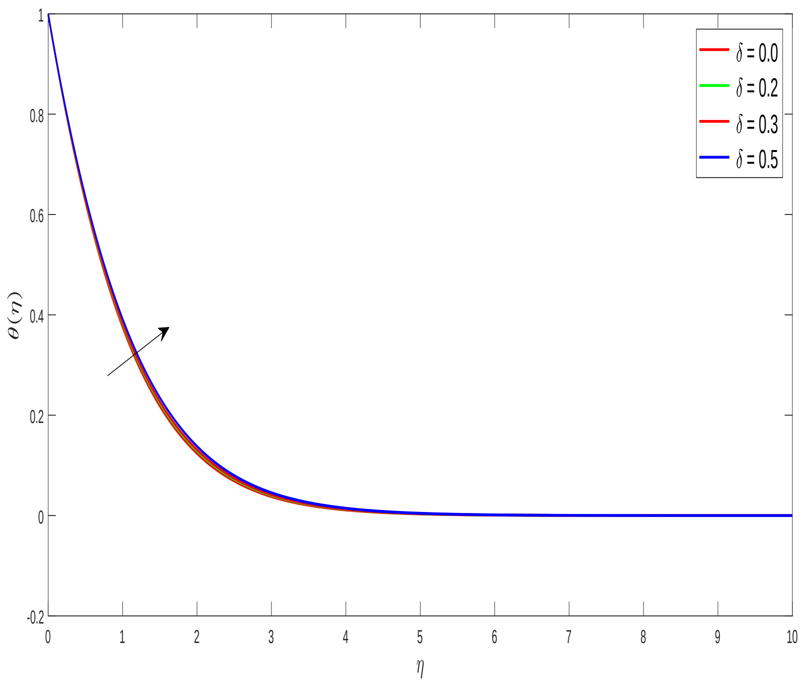

FIGURE 4.5: Impact of various values of δ on temperature $\theta(\eta)$ with $\varepsilon = Ec = M = Sc = R = 0.2,\ \alpha = 0.5,\ Pr = 2.0,\ Kr = 1,\ m = \frac{1}{3}$

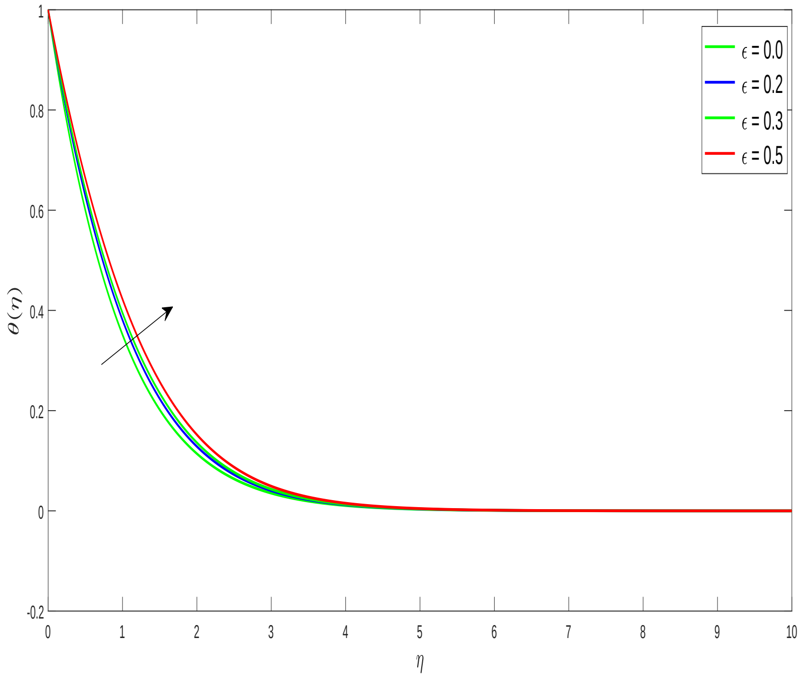

FIGURE 4.6: Impact of various values of ε on temperature $\theta(\eta)$ with $\delta = Ec = M = Sc = R = 0.2,\ \alpha = 0.5,\ Pr = 2.0,\ Kr = 1,\ m = \frac{1}{3}$

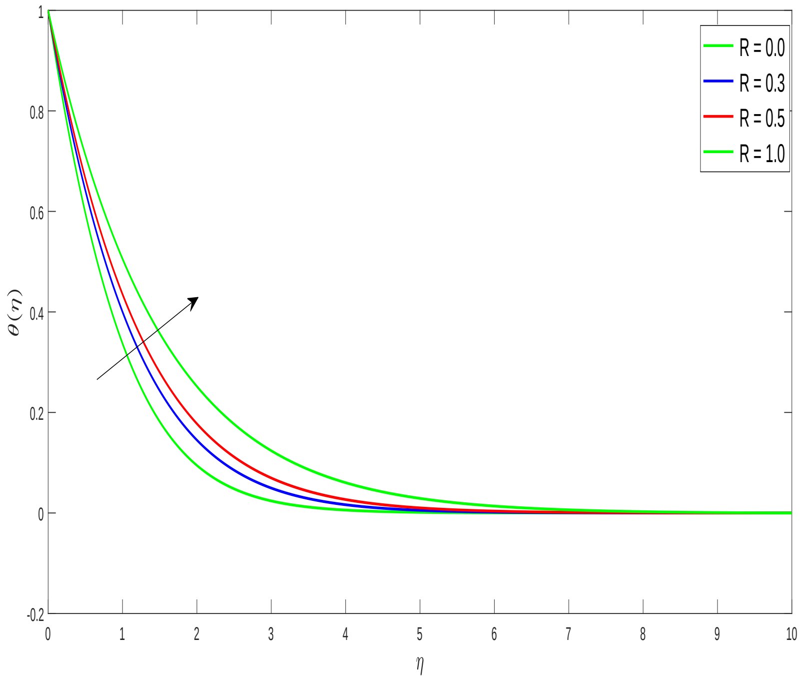

FIGURE 4.7: Impact of various values R on temperature $\theta(\eta)$ with $\delta = \varepsilon = Ec = M = Sc = R = 0.2,\ \alpha = 0.5,\ Pr = 2.0,\ Kr = 1,\ m = \frac{1}{3}$

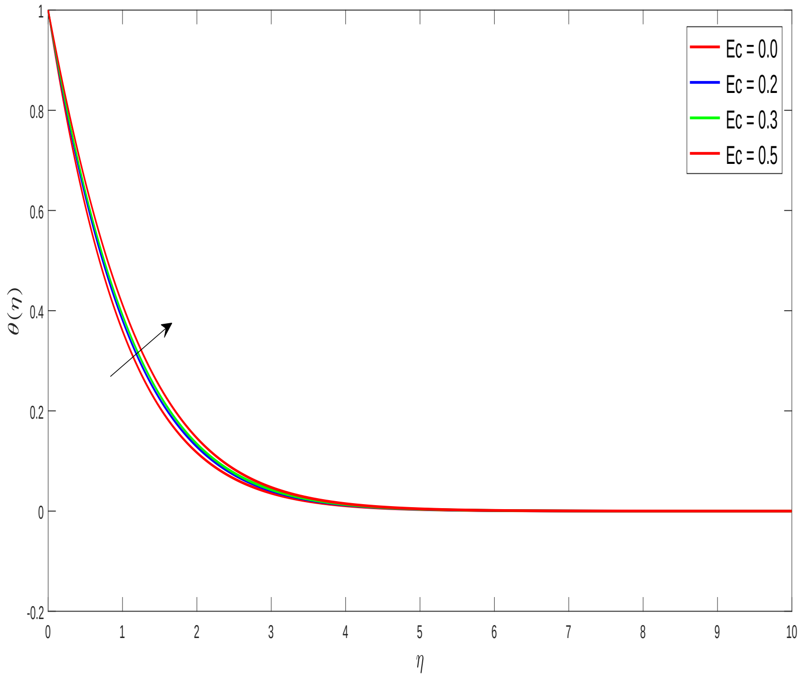

FIGURE 4.8: Impact of various values of Ec on temperature $\theta(\eta)$ with $\delta = \varepsilon = M = Sc = R = 0.2$, $\alpha = 0.5$, $Pr = 2.0$, $Kr = 1$, $m = \frac{1}{3}$

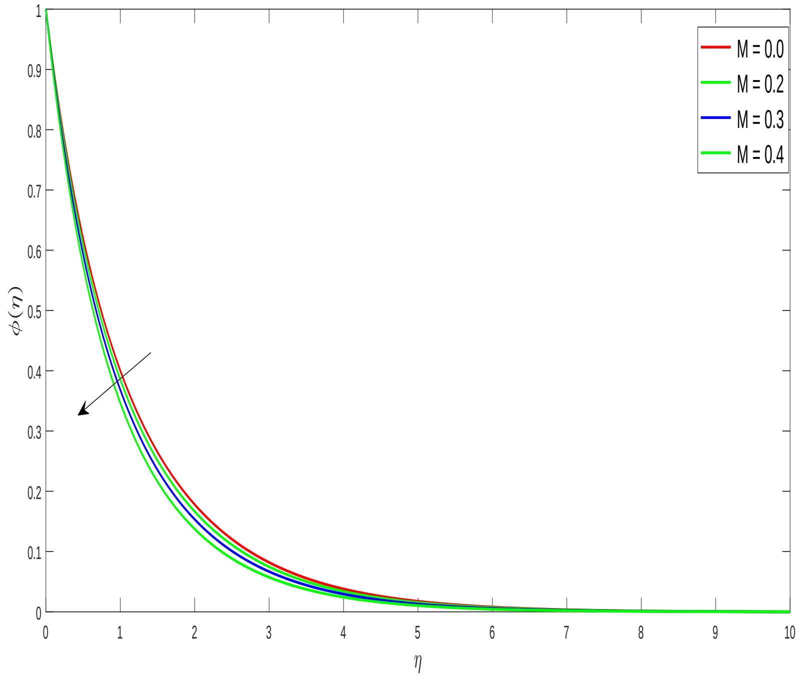

FIGURE 4.9: Impact of various values of M on concentration $\phi(\eta)$ with $\delta = \varepsilon = Ec = Sc = R = 0.2$, $\alpha = 0.5$, $Pr = 2.0$, $Kr = 1$, $m = \frac{1}{3}$

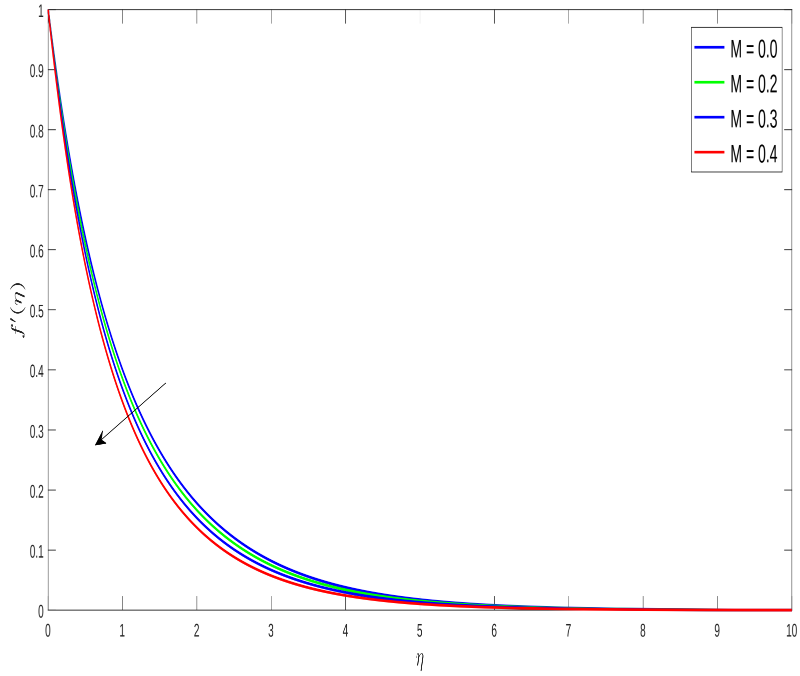

FIGURE 4.10: Impact of various values of M on velocity $f'(\eta)$ with $\delta = \varepsilon = Ec$ $= Sc = R = 0.2,\ \alpha = 0.5,\ \ Pr = 2.0,\ \ Kr = 1,\ \ m = \frac{1}{3}$

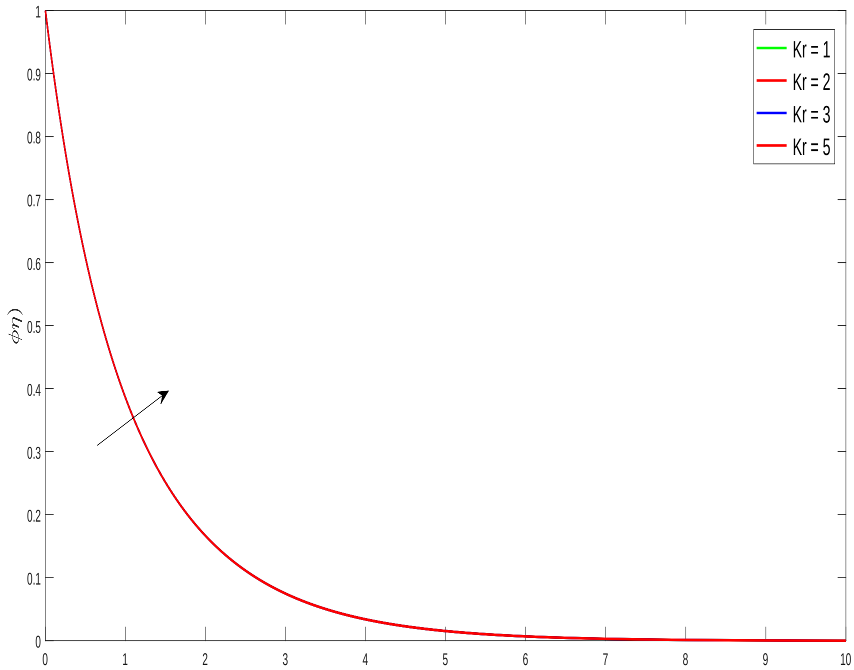

FIGURE 4.11: Impact of various values of Kr on concentration $\phi(\eta)$ with $\delta = \varepsilon = Ec$ $= Sc = R = M = 0.2,\ \alpha = 0.5,\ \ Pr = 2.0,\ \ m = \frac{1}{3}$

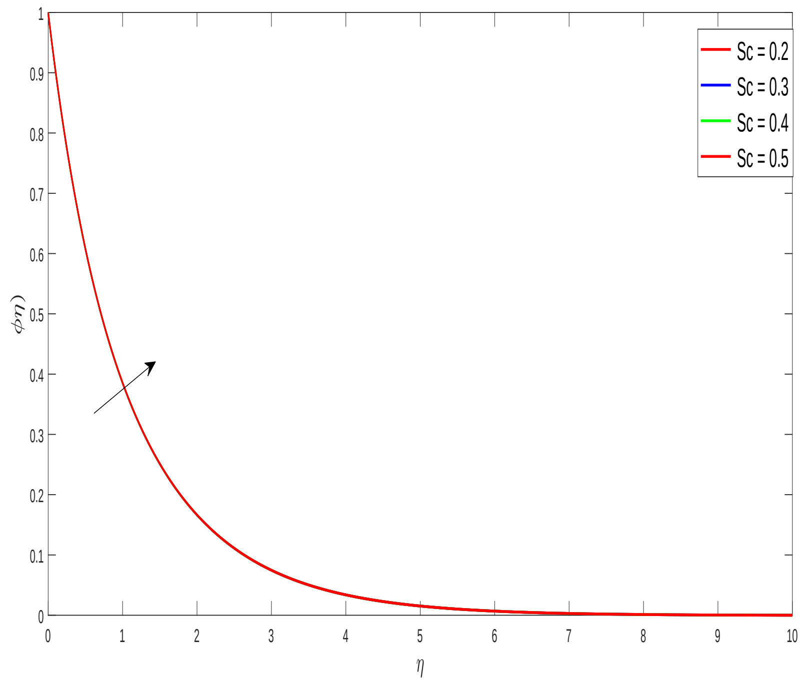

FIGURE 4.12: Impact of various values ofSc on concentration $\phi(\eta)$ with $\delta = \varepsilon = Ec$ $= R = M = 0.2,\ \alpha = 0.5,\ Pr = 2.0,\ m = \frac{1}{3}$

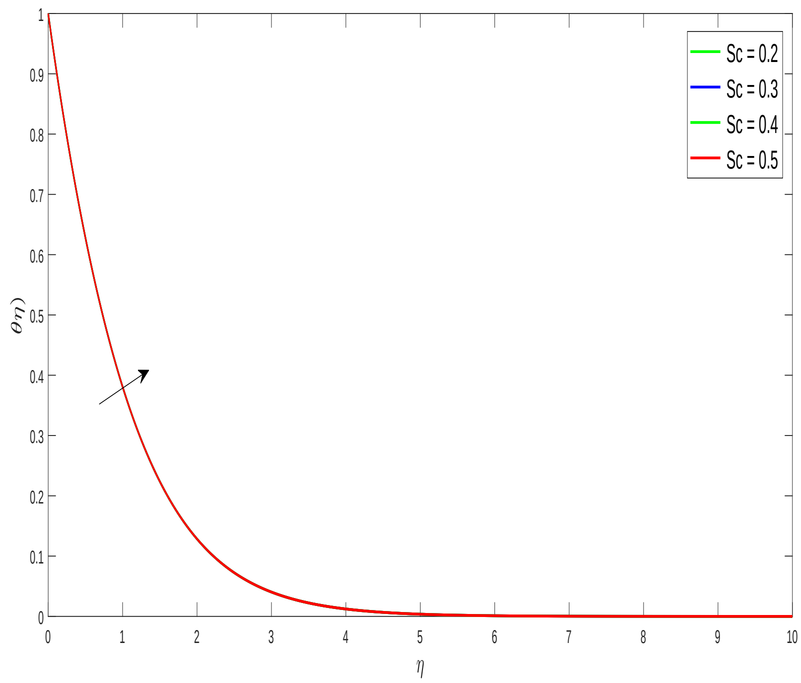

FIGURE 4.13: Impact of various values of Sc on Temperature $\theta(\eta)$ with $\delta = \varepsilon = Ec$ $= R = M = 0.2,\ \alpha = 0.5,\ Pr = 2.0,\ m = \frac{1}{3}$

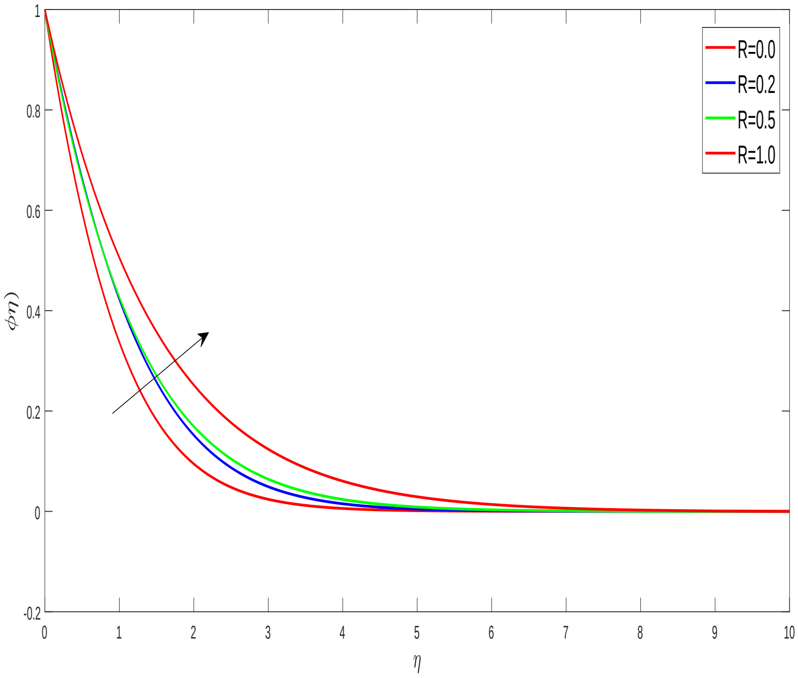

FIGURE 4.14: Impact of various values of R on Concentratrion $\phi(\eta)$ with $\delta = \varepsilon = Ec = R = M = 0.2$, $\alpha = 0.5$, $Pr = 2.0$, $m = \frac{1}{3}$

Chapter 5

Conclusion

In this thesis, the work of Ahmed M. Megahed [24] is reviewed and extended by MHD, Joul heating and chemical reaction. First of all, momentum, energy and concentration equations are changed into the ODEs by using some similarity transformations. By using the shooting method, numerical solution is found for the transformed ODEs. Using different values of the governing physical parameters, the results are presented in the form of tables and graphs for velocity, temperature and concentration profiles. The achievements of the current research can be summarized as below:

- Expanding the values of viscosity parameter α, the velocity profile decreases while the temperature profile shows an opposite behaviour.
- For improving the values of the radiation parameter R, the temperature $\theta(\eta)$ is increased.
- Due to the increasing the values of ε, the temperature profile increases.
- Expanding the values of Williamson parameter δ, the velocity profile is decreased.
- The velocity profile is decreased due to expanding the values of the chemical reaction paramater K_r.
- By rising the values of Eckert number Ec, the temperature distribution $\theta(\eta)$ is also increased.
- By expanding the values of the magnetic parameter M, the concentration profile is observed to increase.

- Due to the ascending values of Schimdt number Sc, the numerical values of Sherwood number are increased.

- Due to the rising values of the chemical reaction parameter Kr, the values of Nusselt number are observed to rise while Sherwood number is decreased.

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