CAPITAL UNIVERSITY OF SCIENCE AND TECHNOLOGY, ISLAMABAD

Six Masses in a Symmetrical Restricted Collinear Central Configuration

by

Muhammad Awais Yaqoob Mughal

A thesis submitted in partial fulfillment for the degree of Master of Philosophy

in the

Faculty of Computing
Department of Mathematics

2021

I dedicate this sincere effort to my beloved parents, wife, son Ibrahim Mughal and my elegant teachers whose devotions and contributions to my life are really worthless and whose deep consideration on part of my academic career, made me consolidated and inspired me as I am up to this grade now.

CERTIFICATE OF APPROVAL

Six Masses in a Symmetrical Restricted Collinear Central Configuration

by

Muhammad Awais Yaqoob Mughal

(MMT173014)

THESIS EXAMINING COMMITTEE

S. No.	Examiner	Name	Organization
(a)	External Examiner	Dr. Jamil Ahmed	NUTECH, Islamabad
(b)	Internal Examiner	Dr. Muhammad Sagheer	CUST, Islamabad
(c)	Supervisor	Dr. Abdul Rehman Kashif	CUST, Islamabad

Dr. Abdul Rehman Kashif
Thesis Supervisor
November, 2021

Dr. Muhammad Sagheer
Head
Dept. of Mathematics
November, 2021

Dr. M. Abdul Qadir
Dean
Faculty of Computing
November, 2021

Author's Declaration

I, **Muhammad Awais Yaqoob Mughal** hereby state that my M.Phil thesis titled "**Six Masses in a Symmetrical Restricted Collinear Central Configuration**" is my own work and has not been submitted previously by me for taking any degree from Capital University of Science and Technology, Islamabad or anywhere else in the country/abroad.

At any time if my statement is found to be incorrect even after my graduation, the University has the right to withdraw my M.Phil Degree.

(Muhammad Awais Yaqoob Mughal)
Registration No: MMT173014

Plagiarism Undertaking

I solemnly declare that research work presented in this thesis titled "**Six Masses in a Symmetrical Restricted Collinear Central Configuration**" is solely my research work with no significant contribution from any other person. Small contribution/help wherever taken has been duly acknowledged and that complete thesis has been written by me.

I understand the zero tolerance policy of the HEC and Capital University of Science and Technology towards plagiarism. Therefore, I as an author of the above titled thesis declare that no portion of my thesis has been plagiarized and any material used as reference is properly referred/cited.

I undertake that if I am found guilty of any formal plagiarism in the above titled thesis even after award of M.Phil Degree, the University reserves the right to withdraw/revoke my M.Phil degree and that HEC and the University have the right to publish my name on the HEC/University website on which names of students are placed who submitted plagiarized work.

(Muhammad Awais Yaqoob Mughal)

Registration No: MMT173014

Acknowledgement

First of all, I would like to pay my cordial gratitude to the Almighty **ALLAH** who created us as a human being with the great boon of intellect. I would like to pay my humble gratitude to the **ALLAH** Almighty, for blessing us with the **Holy Prophet Mohammad** (sallallahu alaihy waaalehi wassalam) for whom the whole universe is being created. He (sallallahu alaihy waaalehi wassalam) removed evil from the society and brought us out of darkness.

I would like to express my special gratitude to my kind supervisor **Dr. Abdul Rehman Kashif** for his constant motivation. He was always there whenever I found any problem. I really thankful to his efforts and guidance throughout my thesis and proud to be a student of such an intelligent supervisor. May **ALLAH** bless him with all kind of happiness and success in his life and may all his wishes come true.

My heartiest and sincere salutations to my parents, who put their unmatchable efforts in making me a good human being. My deepest gratitude to my brothers who are the real pillars of my life. They always encouraged me and showed their everlasting love, care and support throughout my life. The love from my brothers is priceless.

I would also like to thanks to my research fellows Mohsin Ali for his fruitful help and kind support in my research period.

May **ALLAH** Almighty shower His choicest blessings and prosperity on all those who helped me in any way during the completion of my thesis.

(Muhammad Awais Yaqoob Mughal)

Abstract

The first part of this thesis is devoted to a collinear central configurations for five masses m_1, m_2, m_3, m_4 and m_5. The five big masses m_1, m_2, m_3, m_4 and m_5 are placed along a line with two pairs of equal masses and one mass is at the origin. In the second part a test particle m_6 of negligible mass (i.e $m_6 << m_1, m_2, m_3, m_4, m_5$) is considered in the gravitational field of collinear five big masses. With this condition i.e $m_6 << m_1, m_2, m_3, m_4, m_5$, it is called restricted six body problem. The equation of motion of m_6 is obtain under the gravitational field of five collinear masses using the universal law of Newtonian gravity. After evaluating the equation of motion of m_6 the equilibrium points and their linear stability is investigated. The permissible region of motion of m_6 is also studied using different values of Jacobian constant.

Contents

Author's Declaration iv

Plagiarism Undertaking v

Acknowledgement vi

Abstract vii

List of Figures x

List of Tables xii

Abbreviations xiii

Symbols xiv

1 Introduction **1**
- 1.1 Central Configuration . . . 3
- 1.2 Thesis Contribution . . . 4
- 1.3 Dissertation Outlines . . . 4

2 Preliminaries **5**
- 2.1 Basic Definitions . . . 5
- 2.2 Kepler's Laws of Planetary Motion . . . 10
- 2.3 Newton's Laws of Motion . . . 11
  - 2.3.1 Newton's Universal Law of Gravitation . . . 11
- 2.4 Two Body Problem . . . 12
  - 2.4.1 The Solution to the Two-Body Problem . . . 12
  - 2.4.2 Transverse and Radial Components of Velocity and Acceleration: . . . 15

3 Six Masses in a Symmetrical Restricted Collinear Central Configuration **19**
- 3.1 Introduction . . . 19
- 3.2 Problem Formulation . . . 19

3.2.1 Central Configuration for Collinear Five Masses 22

4 Introduction **24**
4.1 Dynamics of 6^{th} Particle 24
4.2 Equilibrium Solutions 28
4.3 Case 1: When $a \in (0.417221\ ,\ 0.494666)$ 29
4.4 Case 2: When $a \in (1.75768\ ,\ 2.0)$ 32
4.5 Stability Analysis of Equilibrium Points 36
4.6 Permitted Areas of Motion for the Test Particle: 39
4.6.1 Permitted areas when a=0.417222 and b=1.0 40
4.6.2 Permissible areas when a=0.427221 and b=1.0 41
4.6.3 Permissible areas when a=0.447221 and b=1.0 43
4.6.4 Permissible areas when a=0.487221 and b=1.0 45
4.6.5 Permitted areas when a=1.75778 and b=1.0 47
4.6.6 Permitted areas when a=1.84768 and b=1.0 49
4.6.7 Permitted areas when a=1.97768 and b=1.0 51
4.6.8 Permitted areas when a=1.98768 and b=1.0 53

5 Conclusions **56**

Bibliography **57**

List of Figures

2.1 Center of mass of two body system 13
2.2 Radial and transverse velocity and acceleration components. 16

3.1 Symmetric collinear equilibrium configuration for five masses. . . . 20

4.1 Positions of the equilibrium points (Red dots), and the corresponding positions of the masses (Black dots). Here $a = 0.417222$, $b = 1$, $m = 0.2905$ and $M = 0$. 29
4.2 Positions of the equilibrium points (Red dots), and the corresponding positions of the masses (Black dots). Here $a = 0.427221$, $b = 1$, $m = 0.2767$ and $M = 0.0164$. 30
4.3 Positions of the equilibrium points (Red dots), and the corresponding positions of the masses (Black dots). Here $a = 0.447221$, $b = 1$, $m = 0.2454$ and $M = 0.0637$. 31
4.4 Positions of the equilibrium points (Red dots), and the corresponding positions of the masses (Black dots). Here $a = 0.487221$, $b = 1$, $m = 0.1074$ and $M = 0.4341$. 32
4.5 Positions of the equilibrium points (Red dots), and the corresponding positions of the masses (Black dots). Here $a = 1.75778$, $b = 1$, $m = 0$ and $M = 0.8$. 33
4.6 Positions of the equilibrium points (Red dots), and the corresponding positions of the masses (Black dots). Here $a = 1.84768$, $b = 1$, $m = 0.2103$ and $M = 1.0134$. 34
4.7 Positions of the equilibrium points (Red dots), and the corresponding positions of the masses (Black dots). Here $a = 1.97768$, $b = 1$, $m = 0.7551$ and $M = 1.3633$. 35
4.8 Positions of the equilibrium points (Red dots), and the corresponding positions of the masses (Black dots). Here when $a = 1.98768$, $b = 1$, $m = 0.8115$ and $M = 1.3928$. 35
4.9 Permitted areas for motion of C = 1.0 40
4.10 Permitted areas for motion of C = 1.2 40
4.11 Permitted areas for motion of C = 1.3 41
4.12 Permitted areas for motion of C = 1.6 41
4.13 Permitted areas for motion of C = 1.07 42
4.14 Permitted areas for motion of C = 1.19 42
4.15 Permitted areas for motion of C = 1.32 43
4.16 Permitted areas for motion of C = 1.37 43

4.17 Permitted areas for motion of C = 1.04 44
4.18 Permitted areas for motion of C = 1.24 44
4.19 Permitted areas for motion of C = 1.37 45
4.20 Permitted areas for motion of C = 1.49 45
4.21 Permitted areas for motion of C = 2.1 46
4.22 Permitted areas for motion of C = 2.5 46
4.23 Permitted areas for motion of C = 2.6 47
4.24 Permitted areas for motion of C = 3.1 47
4.25 Permitted areas for motion of C = 3.0 48
4.26 Permitted areas for motion of C = 3.3 48
4.27 Permitted areas for motion of C = 3.9 49
4.28 Permitted areas for motion of C = 4.4 49
4.29 Permitted areas for motion of C = 3.3 50
4.30 Permitted areas for motion of C = 4.7 50
4.31 Permitted areas for motion of C = 5.0 51
4.32 Permitted areas for motion of C = 5.8 51
4.33 Permitted areas for motion of C = 4.5 52
4.34 Permitted areas for motion of C = 6.5 52
4.35 Permitted areas for motion of C = 7.4 53
4.36 Permitted areas for motion of C = 8.4 53
4.37 Permitted areas for motion of C = 6.9 54
4.38 Permitted areas for motion of C = 7.9 54
4.39 Permitted areas for motion of C = 8.3 55
4.40 Permitted areas for motion of C = 9.4 55

List of Tables

4.1 The stability analysis for: $a = 0.427221$, $M = 0.0164$, $m = 0.2767$. . 36
4.2 The stability analysis for: $a = 0.447221$, $M = 0.0637$, $m = 0.2454$. . 37
4.3 The stability analysis for: $a = 0.487221$, $M = 0.4341$, $m = 0.1074$. . 37
4.4 The stability analysis for: $a = 1.84768$, $M = 1.0134$, $m = 0.2103$. . . 38
4.5 The stability analysis for: $a = 1.97768$, $M = 1.3638$, $m = 0.7551$. . . 38
4.6 The stability analysis for: $a = 1.98768$, $M = 1.3928$, $m = 0.8115$. . . 39

Abbreviations

CCs	Central Configurations
D	Denominator
$\mathbf{M}_s$	Mass of the Sun
N	Numerator
R	Region
R3BP	Restricted Three-Body Problem
RC6BP	Restricted Collinear Six-Body Problem
SI	System International
2BP	Two-Body Problem
3BP	Three-Body Problem
4BP	Four-Body Problem
5BP	Five-Body Problem

Symbols

symbol	name	unit
$\mathbf{L}$	Angular momentum	$kg\ m^2\ s^{-1}$
$\mathbf{F}$	Gravitational force	Newton
$\mathbf{P}$	Linear momentum	$kg\ m\ s^{-1}$
$\in$	Belongs to	
r	Distance	Meter
$\forall$	For all	
m_i	Point masses	kg
$\mathbb{R}$	Real number	
$\ni$	Such that	
G	Universal gravitational constant	$m^3 kg^{-1} s^{-2}$

Chapter 1

Introduction

Celestial mechanics, in the broadest sense is the application of classical mechanics to the motion of celestial bodies acted on by any of several types of forces [1]. By far the most important force experienced by these bodies, and much of the time the only important force is that of their mutual gravitational attraction. The term celestial mechanics is sometimes assumed to refer only to the analysis developed for the motion of point mass particles moving under their mutual gravitational attractions with emphasis on the general orbital motions of solar system bodies. The term astrodynamics is often used to refer to the celestial mechanics of artificial satellites motion. Dynamic astronomy is a much broader term, which in addition to celestial mechanics and astrodynamics is usually interpreted to include all aspects of celestial body motion (e.g. rotation, tidal evolution, mass and mass distribution determinations for stars and galaxies, fluid motions in nebula and so forth) [2]. The two-body problem is to predict the motion of two massive objects which are abstractly viewed as point particles. The problem assumes that the two objects interact only with one another the only force affecting each object arises from the other one and all other objects are ignored [3]. The most prominent case of the classical two-body problem is the gravitational case arising in astronomy for predicting the orbits (or escapes from orbit) of objects such as satellites, planets and stars. During the past century, celestial mechanics had principally devoted to the study of the three-body problem. Due to the difficulty in handling additional parameters in the four-body problems very little analytical work has been carried out for more than three bodies. The general problem of n bodies where n is greater than three has been tackled vigorously with numerical techniques on powerful computers. Celestial mechanics in the solar system is ultimately an n-body problem but the special configurations and relative smallness of the perturbations have allowed quite accurate descriptions of motions (valid for limited time periods) with various approximations and procedures without any attempt to solve the complete problem of n bodies. A two-point-particle model of such a system nearly always describes its behavior well enough to provide useful insights and predictions [4]. In the $17th$ century from (1609 and 1619), Kepler initially defined the planet's elliptical orbits around the sun using the law of motion. ***"Philosophiae Naturalis Principia Mathematica",*** into the history of science, one of the most important works in which Isaac Newton derived the law of Kepler. The force of gravity between two point particles is

$$\mathbf{F} = G\frac{m_1 m_2}{d^3}\mathbf{d}, \tag{1.1}$$

where G is universal gravitational constant and d is the distance between the masses m_1 and m_2. After that, he concentrated on comparatively complex systems, but despite several challenges, he was unable to make a breakthrough in a three-body problem (3BP) in his life [5]. In the history, specific 3BP was first studied with the Inclusion of the moon with earth and the sun. This particular 3BP is called "the main problem of the lunar theory," which has been studied extensively with a variety of methods beginning with Newton [6]. In the recent days, a 3BP is any problem in classical mechanics or quantum mechanics that models the motion of three particle. There is no closed form solution for general 3BP like two body problem. Because the general 3BP is chaotic in nature for most of the initial condition, therefore numerical techniques are required to solve the general 3BP. Poincare have proposed restricted 3BP [7], where two massive bodies moves in circle about their center of mass and attract (but not attracted by) the third particle of infinitesimal mass. Lagrange [8] first solved the restricted three body problem in which he obtained five equilibrium points (stable or unstable). The stable points are important for sending space probes at these points (for example

Sotto and WAMP are parked of L_1).

1.1 Central Configuration

The idea of dynamics represented by complete masses collision or the rotating equilibrium, we are led to the idea of a CC. In CC, *"the acceleration of the ith mass must be proportional to its position (relative to the center of mass of the system)"*; thus, $\ddot{r}_i = \lambda r_i \; \forall \; i = 1, 2, 3, ...n$. CC is common and basic concept in the study of NBP. Consequently, for years the question of few bodies in CC and general has fascinated considerable attention (see for example Albouy and Llibre [9] and Shoaib and Faye [10]). Moulton first published linear solutions to the NBP [11]. Palmore [12] proposed many theorems in the study of points of equilibrium in the planar NBP. Papadakis and Kanavos [13] studied the restricted photo gravitational 5BP, they investigated the movement of a mass less object on a sphere. Kulesza et al. [14] have more recently examined a restricted rhomboidal 5BP. The masses are arranged in the same plane as the 5th point is mass less and the other masses on the vertices of the rhombus. Ollongren [15] studied a restricted 5BP with three bodies of equal mass m of the equilateral triangle placed on the vertices; rotating in circular orbits in triangular plane under the mutual gravitational attraction around its gravitational center.Under the gravitational attraction of other bodies a 5th body with negligible mass as opposed to m moves in the plane. Other notable studies are Kalvouridis [16] and Markellos et al. [17] on the restricted 5BP. Another restrictive approach used to investigate 5BP is some sort of symmetries added. For example, on a particular case of the 5BP, Roberts [18] addressed relative equilibria. He investigated a CC which consists of five bodies, four bodies are situated at the vertices of the rhombus and the 5^{th} body is in the middle. Mioc and Blaga [19] explain the similar problem but in the post Newtonian field of Manev.

The CC of the 5BP were addressed by Albouy and Llibre [9]. They studied on a sphere with a larger 5th mass at its center they considered four equal masses. More recent studies on the symmetrically restricted 5BP include Shoaib et al. [20, 21]. Lee and Santoprete [22] also studies on the symmetrically restricted 5BP. Similarly Gidea and Llibre [23], and Marchesin and Vidal [24] discussed on the

symmetrically restricted 5BP. As yet, in the non-collinear general four and 5BP, the basic interest has been on the same question: Is there a fixed arrangement of bodies and unique CC for a given set of masses?
Ouyang and Xie [25] investigated about a four-body collinear problem and Mello and Fernandes [26] discussed a rhomboidal 4BP and 5BP.

1.2 Thesis Contribution

The main objective of this thesis is to study the motion of infinitesimal mass m_6 and its equilibrium points (stable or unstable) under the gravitational field of five big mass m_1, m_2, m_3, m_4 and m_5. The region of permissible region of motion of m_6 are also studied using the Jacobian constant c.

1.3 Dissertation Outlines

This thesis is divided into five chapters.

In **Chapter-1** the research goals are briefly discussed.

Chapter-2 covers numerous fundamental concepts related to celestial mechanics, Kepler's laws of planetary motion and Newton's laws of motion.

In **Chapter-3** central configurations of 5 collinear masses are discussed.

In **Chapter-4** equation of motion of m_6 is obtained under gravitational field of five masses and a qualitative analysis has been done of this equation.

Chapter-5 provides the concluding remarks of the thesis.

Chapter 2

Preliminaries

This chapter is composed of the fundamental meanings, fundamental principles and laws which will help us to fully understand our research work.

2.1 Basic Definitions

Definition 2.1.1. (Motion)
"Motion is the phenomenon in which an object changes its position over time. Motion is mathematically described in terms of displacement, distance, velocity, acceleration, speed, and time." [27]

Definition 2.1.2. (Mechanics)
"Mechanics is a branch of physics concerned with motion or change in position of physical objects." [28]

Definition 2.1.3. (Scalar)
"A scalar is a quantity having magnitude but no direction. Such as mass, length, time, temperature and any real numbers." [29]

Definition 2.1.4. (Vector)
"A **vector** is a quantity having both magnitude and direction. Such as displacement, velocity, force and acceleration." [29]

Definition 2.1.5. (Field)

"A field is a physical quantity, represented by a number or tensor, that has a value for each point in space and time." [28]

Definition 2.1.6. (Scalar Field)

"If at every point in a region, a scalar function has a defined value, the region is called a scalar field. i.e.,

$$f : \mathbb{R}^3 \to \mathbb{R}." \text{ [28]}$$

Definition 2.1.7. (Vector Field)

"If at every point in a region, a vector function has a defined value, the region is called a vector field.

$$f : \mathbb{R}^3 \to \mathbb{R}^3." \text{ [28]}$$

Definition 2.1.8. (Conservative Vector Field)

"A vector field $\mathbf{V}$ is conservative if and only if there exists a continuously differentiable scalar field f such that $\mathbf{V} = -\nabla f$ or equivalently if and only if

$$\nabla \times \mathbf{V} = \texttt{Curl}\,\mathbf{V} = \mathbf{0}." \text{ [28]}$$

Definition 2.1.9. (Uniform Force Field)

"A force field which has constant magnitude and direction is called a uniform or constant force field. If the direction of the field is taken as negative z direction and magnitude is constant $F_0 > 0$, then the force field is given by

$$\mathbf{F} = -F_0\hat{\mathbf{k}}." \text{ [28]}$$

Definition 2.1.10. (Central Force)

"Suppose that a force acting on a particle of mass m such that

(a) it is always directed from m towards or away from a fixed point O,

(b) its magnitude depends only on the distance r from O.

Then we call the force a central force or central force field with O as the center of force. In symbols $\mathbf{F}$ is a central force if and only if

$$\mathbf{F} = f(r)\mathbf{r}_1 = f(r)\tfrac{\mathbf{r}}{r},$$

where $\mathbf{r}_1 = \frac{\mathbf{r}}{r}$ is a unit vector in the direction of $\mathbf{r}$. The central force is one of attraction towards O or repulsion from O according as $f(r) < 0$ or $f(r) > 0$ respectively." [28]

Definition 2.1.11. (Degree of Freedom)

"The number of coordinates required to specify the position of a system of one or more particles is called number of degree of freedom of the system." [28]

Definition 2.1.12. (Center of Mass)

"Let $\mathbf{r}_1$, $\mathbf{r}_2$,..., $\mathbf{r}_n$ be the position vector of a system of n particles of masses $m_1, m_2, ...m_n$ respectively. The center of mass or centroid of the system of particles is defined as the point having the position vector

$$\hat{\mathbf{r}} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + ... + m_n\mathbf{r}_n}{m_1 + m_2 + ... + m_n} = \frac{1}{\mathbf{M}}\sum_{\nu=1}^{n} m_\nu \mathbf{r}_\nu,$$

where

$$\mathbf{M} = \sum_{\nu=1}^{n} m_\nu,$$

is the total mass of the system." [28]

Definition 2.1.13. (Center of Gravity)

"The gravitational force on an extended body is the vector sum of the gravitational forces acting on the individual elements (the atoms) of the body. Instead of considering all those individual elements, we can say that The gravitational force F_g on a body effectively acts at a single point, called the center of gravity (cog) of the body." [30]

Definition 2.1.14. (Torque)

"A quantity called torque τ as the product of the two factors and write it as

$$\boldsymbol{\tau} = r \times \mathbf{F}.$$

The magnitude of $\boldsymbol{\tau}$ is

$$\boldsymbol{\tau} = rF\sin\theta,$$

where r is he perpendicular distance between the rotation axis at 0 and an extended line running through the vector F, and θ is the angle between the position and force vectors." [30]

Definition 2.1.15. (Momentum)

"The linear momentum $\mathbf{p}$ of an object with mass m and velocity $\mathbf{v}$ is defined as:

$$\mathbf{p} = m\mathbf{v}.$$

Under certain circumstances the linear momentum of a system is conserved. The linear momentum of a particle is related to the net force acting on that object:

$$\mathbf{F} = m\mathbf{a} = m\frac{d\mathbf{v}}{dt} = \frac{d}{dt}(m\mathbf{v}) = \frac{d\mathbf{p}}{dt}.$$

The rate of change of linear momentum of a particle is equal to the net force acting on the object, and is pointed in the direction of the force. If the net force acting on an object is zero, its linear momentum is constant (conservation of linear momentum). The total linear momentum $\mathbf{p}$ of a system of particles is defined as the vector sum of the individual linear momentum.

$$\mathbf{p} = \sum_{1}^{n} \mathbf{p}_i.\text{"}$$ [28]

Definition 2.1.16. (Angular Momentum)

"Angular momentum $\mathbf{L}$ of a particle of mass m and linear momentum $\mathbf{p}$ is a vector quantity defined as:

$$\mathbf{L} = \mathbf{r} \times \mathbf{p},$$

Where $\mathbf{r}$ is a position vector of a particle relative to an origin O, that is in an inertial frame. Where magnitude of $\mathbf{L}$ is given by

$$\mathbf{L} = m\mathbf{vr}\sin\phi.\text{"}$$ [30]

Definition 2.1.17. (Angular Velocity)
Angular velocity $\boldsymbol{\omega}$ is a vector quantity and is described as the rate of change of angular displacement which specifies the angular speed or rotational speed of an object and the axis about which the object is rotating. The amount of change of angular displacement of the particle at a given period of time is called angular velocity.

Definition 2.1.18. (Principle of Conservation of Momentum)
"If the net external force acting on a particle is zero, the angular momentum will remain unchanged. This is called the principle of conservation of momentum." [28]

Definition 2.1.19. (Inertial Frame of Reference)
"If no forces act on an object, any reference frame with respect to which the acceleration of the object remains zero is an inertial reference frame." [30]

Definition 2.1.20. (Lagrange Points)
"Let us search for possible equilibrium points of the mass m_3 in the rotating refere nce frame. Such points are termed Lagrange points. Hence, in the rotating frame, the mass m_3 would remain at rest if placed at one of the Lagrange points. It is, thus, clear that these points are fixed in the rotating frame. The Lagrange points satisfy $\dot{\mathbf{r}} = \ddot{\mathbf{r}} = \mathbf{0}$ in the rotating frame." [31]

Definition 2.1.21. (Celestial Mechanics)
Celestial Mechanics is defined as the science of studying the motion of celestial bodies. Basically it is that branch of astronomy which deals with the motion of heavenly bodies in space.

Definition 2.1.22. (Equilibrium Solution)
The **Equilibrium solution** can lead us through the behavior of the equation that describes the problem without really solving it. These solutions are only possible if we satisfy the necessary condition of all rates being equal to zero. If we have two variables then

$$x' = y' = x'' = y'' = ... = x^{(n)} = y^{(n)} = 0.$$

These solutions may be stable or unstable. The stable solutions in celestial mechanics assist us in locating parking spaces where a satellite or other object may be put and remain there indefinitely. These type of places are also found along the Jupiter's orbital path where bodies called trojan are present. These equilibrium points with respect to Celestial Mechanics are also called Lagrange points named after a French mathematician and astronomer Joseph-Louis Lagrange. He was first to find these equilibrium points for the Sun-Earth system. He found that three of these five points were collinear.

Procedure for stability analysis and equilibrium points:

We need to follow the following steps to check the stability of equilibrium points.

1) Determine the equilibrium points, $\mathbf{u}^*$, solving $\phi(\mathbf{u}^*) = \mathbf{0}$.

2) Construct the Jacobian matrix, $J(\mathbf{u}^*) = \frac{\partial \phi}{\partial \mathbf{u}^*}$.

3) Compute eigenvalues of $\phi(\mathbf{u}^*)$: $det|\phi(\mathbf{u}^*) - \beta\ I| = 0$.

4) Stability or instability of $\mathbf{u}^*$ based on the real parts of eigenvalues.

5) Point is stable, if all eigenvalues have real parts negative.

6) Unstable, If at least one eigenvalue has a positive real part.

7) Otherwise, there is no conclusion, (i.e, require an investigation of higher order terms).

2.2 Kepler's Laws of Planetary Motion

"Kepler's three laws of planetary motion can be described as follows:

1. All planets move in elliptical orbits with the sun at one focus.

2. A line joining any planet to the sun sweeps out equal areas in equal times.

3. The square of the period of any planet is proportional to the cube of the semi-major axis of its orbit. Mathematically, Kepler's third law can be written as:

$$T^2 = Cr^3$$

where T is the time period, r is the semi major axis and C is the constant." [32]

2.3 Newton's Laws of Motion

The following three laws of motion given by Newton are considered the axioms of mechanics:

1. **First law of motion**

   "If no force acts on a body, the body's velocity cannot change; that is, the body cannot accelerate. In other words, if the body is at rest, it stays at rest. If it is moving, it continues to move with the same velocity (same magnitude and same direction)."

2. **Second law of motion**

   "The net force on a body is equal to the product of the body's mass and its acceleration. In equation form,

   $$\mathbf{F}_{net} = m\mathbf{a}.\text{''}$$

3. **Third law of motion**

   "When two bodies interact, the forces on the bodies from each other are always equal in magnitude and opposite in direction." [30]

2.3.1 Newton's Universal Law of Gravitation

"Newton proposed a force law that we call Newton's law of gravitation: Every particle attracts any other particle with a gravitational force of magnitude.

$$\mathbf{F} = G\frac{m_1 m_2}{d^3}\mathbf{d}$$

Here m_1 and m_2 are the masses of the particles, d is the distance between them, and G is the gravitational constant. Its numerical value in SI units is $6.67408 \times 10^{-11} m^3 kg^{-1} s^{-2}$." [30]

2.4 Two Body Problem

"An isolated dynamical system consisting of two freely moving point objects exerting forces on one another is conventionally termed a two-body problem. Suppose that the first object is of mass m_1 and is located at position vector $\mathbf{r}_1$. Likewise, the second object is of mass m_2 and is located at position vector $\mathbf{r}_2$. Let the first object exert a force $\mathbf{f}_{21}$ on the second. The equations of motion of our two objects are thus

$$m_1 \frac{d^2\mathbf{r}_1}{dt^2} = -\mathbf{f}$$

$$m_2 \frac{d^2\mathbf{r}_2}{dt^2} = \mathbf{f}.\text{"}[31]$$

2.4.1 The Solution to the Two-Body Problem

Newton's universal gravitational law is the governing law for the two bodies:

$$\mathbf{F} = G\frac{m_1 m_2}{d^3}\mathbf{d}, \tag{2.1}$$

for two masses, m_1 and m_2 are separated by a distance $\mathbf{d}$, and the universal gravitational constant is G.

The force of attraction $\mathbf{F}_{12}$ on mass m_1 is directed along the vector $\mathbf{d}$ towards the mass m_2, while the force $\mathbf{F}_{21}$ on m_1 is in the opposite direction.

By Newton's third law,

$$\mathbf{F}_{12} = -\mathbf{F}_{21}. \tag{2.2}$$

From Figure 2.1,

$$\mathbf{F}_{12} = G\frac{m_1 m_2}{d^3}\mathbf{d}. \tag{2.3}$$

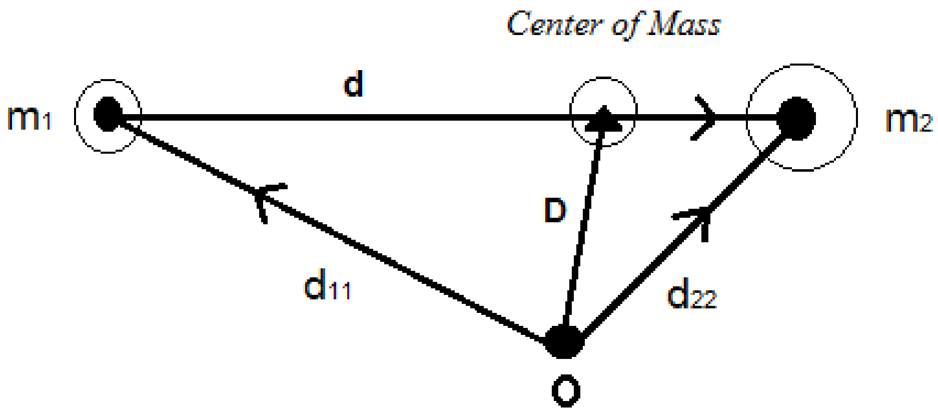

FIGURE 2.1: Center of mass of two body system

Now let vectors $\mathbf{d}_{11}$ and $\mathbf{d}_{22}$ be directed from some fixed reference point O to the particles of mass m_1 and mass m_2 respectively. Using the Newton's 2nd law of motion and equations (2.2), (2.3), the equations of motion of the particles under their mutual gravitational attractions are then given by the two equations

$$m_1\mathbf{d}''_{11} = G\frac{m_1m_2}{d^3}\mathbf{d}, \tag{2.4}$$

$$m_2\mathbf{d}''_{22} = -G\frac{m_1m_2}{d^3}\mathbf{d}, \tag{2.5}$$

Adding equations (2.4) and (2.5) gives

$$m_1\mathbf{d}''_{11} + m_2\mathbf{d}''_{22} = \mathbf{0}. \tag{2.6}$$

giving two integrals

$$m_1\mathbf{d}'_{11} + m_2\mathbf{d}'_{22} = \mathbf{k}_1. \tag{2.7}$$

and

$$m_1\mathbf{d}_{11} + m_2\mathbf{d}_{22} = \mathbf{k}_1 t + \mathbf{k}_2, \tag{2.8}$$

where $\mathbf{k}_1$ and $\mathbf{k}_2$ are constant vectors. But if $\mathbf{D}$ is the position vector of G (the centre of mass of the two masses m_1 and m_2), $\mathbf{D}$ is defined as

$$(m_1 + m_2)\mathbf{D} = m_1\mathbf{d}_{11} + m_2\mathbf{d}_{22},$$

$$m_t\mathbf{D} = m_1\mathbf{d}_{11} + m_2\mathbf{d}_{22}, \tag{2.9}$$

where $m_t = m_1 + m_2$. Differentiate the equation (2.9) and compare it with equation (2.7).

$$m_t\mathbf{D}' = \mathbf{k}_1 \quad \Rightarrow \quad m_t\mathbf{D} = \mathbf{k}_1 t + \mathbf{k}_2.$$

These relations show that the centre of mass of the system moves with constant velocity. Equations (2.4) and (2.5) may be written as

$$\mathbf{d}_{11}'' = G\frac{m_2}{d^3}\mathbf{d}, \tag{2.10}$$

$$\mathbf{d}_{22}'' = -G\frac{m_1}{d^3}\mathbf{d}. \tag{2.11}$$

Subtracting equation (2.10) from equation (2.11) gives

$$\mathbf{d}_{11}'' - \mathbf{d}_{22}'' = \frac{Gm_2}{d^3}\mathbf{d} + \frac{Gm_1}{d^3}\mathbf{d}, \tag{2.12}$$

$$\begin{aligned} \mathbf{d}_{11}'' - \mathbf{d}_{22}'' &= G(m_1 + m_2)\frac{\mathbf{d}}{d^3} \\ \Rightarrow \mathbf{d}'' &= \beta\frac{\mathbf{d}}{d^3} \\ \Rightarrow \mathbf{d}'' - \beta\frac{\mathbf{d}}{d^3} &= \mathbf{0}, \end{aligned} \tag{2.13}$$

where $\beta = G(m_1 + m_2)$ is described as a reduction in mass and $\mathbf{d}_{11} - \mathbf{d}_{22} = -\mathbf{d}$, as seen in Figure 2.1.

Taking the vector product of $\mathbf{d}$ with equation (2.13) we obtain

$$\begin{aligned} \mathbf{d} \times \beta\mathbf{d}'' + \frac{\beta^2}{d^3}\mathbf{d} \times \mathbf{d} &= \mathbf{0}, \\ \Rightarrow \mathbf{d} \times \beta\mathbf{d}'' &= \mathbf{0}. \end{aligned} \tag{2.14}$$

Integrating, we have

$$\mathbf{d} \times \mathbf{d}' = \mathbf{H}, \tag{2.15}$$

where $\mathbf{H}$ is a constant vector. The equation (2.14) should be written as,

$$\begin{aligned} \mathbf{d} \times \beta\mathbf{d}'' &= \mathbf{0}, \\ \Rightarrow \mathbf{d} \times \mathbf{F} &= \mathbf{0}, \end{aligned} \tag{2.16}$$

where $\mathbf{F} = \beta \mathbf{d}''$.

The description of angular momentum and torque is taken from Chapter 2:

$$\boldsymbol{\tau} = \mathbf{d} \times \mathbf{F} = \frac{d\mathbf{H}}{dt}. \tag{2.17}$$

When equations (2.16) and (2.17) are compared, we get:

$$\boldsymbol{\tau} = \mathbf{d} \times \mathbf{F} = \frac{d\mathbf{H}}{dt} = \mathbf{0},$$
$$\frac{d\mathbf{H}}{dt} = \mathbf{0},$$

$$\Rightarrow \mathbf{H} = constant.$$

This means that the angular momentum is constant.

From Chapter 2, we know the definition that angular momentum is constant or conserved, if external torque of an object is equal to zero, and we know that external torque of an object is equal to zero, angular momentum is a vector quantity that requires both a magnitude and a direction to be fully described.

2.4.2 Transverse and Radial Components of Velocity and Acceleration:

If polar coordinates d and θ are taken in this plane as in Figure 2.2, the velocity components along and perpendicular to the radius vector joining m_1 to m_2 are d' and $d\theta'$, then

$$\mathbf{d}' = d'\hat{\imath} + d\theta'\hat{\jmath}, \tag{2.18}$$

where $\hat{\imath}$ and $\hat{\jmath}$ are unit vectors along and perpendicular to the radius vector. Thus, by means of equations (2.15) and (2.18),

$$\mathbf{d} \times (d'\hat{\imath} + d\theta'\hat{\jmath}) = d^2\theta'\hat{k} = H\hat{k}, \tag{2.19}$$

where $\hat{k}$ is a unit vector perpendicular to the plane of the orbit. We may then write

$$d^2\theta' = H, \tag{2.20}$$

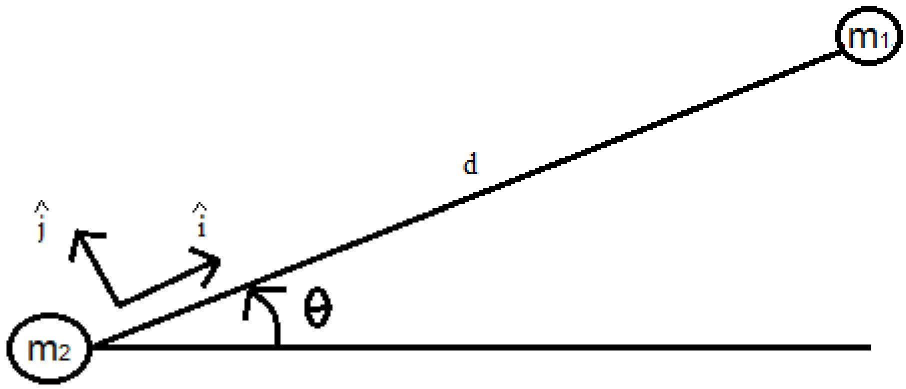

FIGURE 2.2: Radial and transverse velocity and acceleration components.

where the constant H is seen to be twice the rate of description of area by the radius vector. This is the mathematical form of Kepler's second law. If the scalar product of $\mathbf{d}'$ with equation (2.13) is now taken, we obtain

$$\mathbf{d}'.\frac{d^2\mathbf{d}}{dt^2} + \beta\frac{\mathbf{d}'.\mathbf{d}}{d^3} = 0.$$

After integrating, we have get

$$\frac{1}{2}\mathbf{d}'.\mathbf{d}' - \frac{m_1 v}{d} = C,$$

or

$$\frac{1}{2}u^2 - \frac{\beta}{d} = C, \tag{2.21}$$

where C is a constant of integration. This is a sort of energy system preservation. The C quantity does not include absolute energy, $\frac{1}{2}\beta^2$ is associated with KE, and $\frac{-\beta}{d}$ is associated with PE of the system's, i.e., the system's total energy is constant. Recall that from celestial mechanics, components of acceleration vector along and perpendicular to the radius vector (see Figure 2.2):

$$\mathbf{a} = (d'' - d\theta'^2)\hat{i} + \frac{1}{d}\frac{d}{dt}(d^2\theta')\hat{j}.$$

Using above equation in (2.13), we get

$$d'' - d\theta'^2 = -\frac{\beta}{d^2}, \tag{2.22}$$

$$\frac{1}{d}\frac{d}{dt}(d^2\theta') = 0. \tag{2.23}$$

Integrating equation (2.24) gives the angular momentum integral:

$$d^2\theta' = H, \tag{2.24}$$

making the usual substitution of

$$v = \frac{1}{d}, \tag{2.25}$$

the absence of time between equations (2.23) and (2.24), therefore, means that:

$$\frac{d^2v}{d\theta^2} + v = \frac{\beta}{H^2}. \tag{2.26}$$

The general solution of above equation is:

$$v = \frac{\beta}{H^2} + B\cos(\theta - \theta_0), \tag{2.27}$$

where B and θ_0 are two constants of integration. Substitute $v = \frac{1}{d}$ in above equation:

$$\frac{1}{d} = \frac{\beta}{H^2} + B\cos(\theta - \theta_0),$$

or

$$d = \frac{\frac{H^2}{\beta}}{1 + \frac{H^2B}{\beta}\cos(\theta - \theta_0)},$$

is the conic equation's polar form, can be expressed as:

$$d = \frac{q}{1 + e\cos(\theta - \theta_0)},$$

where,

$$q = \frac{H^2}{\beta},$$

$$e = \frac{BH^2}{\beta}.$$

Eccentricity e describes the path of one heavenly body around another. Thus:

(i) The orbital motion is elliptical if $0 < e < 1$,

(ii) The orbital motion is parabolic if $e = 1$,

(iii) The orbital motion is hyperbolic if $e > 1$.

Hence the solution of the two-body problem is a conic, includes Kepler's first law as a special case.

Chapter 3

Six Masses in a Symmetrical Restricted Collinear Central Configuration

3.1 Introduction

In this chapter the CC's of five masses which are placed in a symmetrical collinear configurations is studied. The pair of larger mass and one large mass are placed at the corners and collinear arrangement respectively i.e., ($m_1 = m_2 = m_3 = M$. The pair of smaller masses m_4 and m_5 are set in the middles of collinear arrangement. The CC's equation are obtained for these masses. After solving CC's equation for M and m in terms of a and b, a numerical investigation has been done for positivity of M and m.

3.2 Problem Formulation

For the n-body problem the classical motion of the equation has the form

$$m_j \mathbf{q}_j'' = \sum_{k=0 \ k\neq j}^{n} m_j m_k \frac{\mathbf{q}_k - \mathbf{q}_j}{|\mathbf{q}_k - \mathbf{q}_j|^3}, \qquad j = 1, ..., n, \tag{3.1}$$

here we choose gravitational units, so that $G = 1$. For the central configurations, the acceleration vector is proportional to position vector i.e.,

$$\mathbf{q}_j'' = -\boldsymbol{\omega}^2(\mathbf{q}_j - \mathbf{c}), \tag{3.2}$$

where $\boldsymbol{\omega}$ is angular speed and $\mathbf{c}$ is the center of mass of five collinear masses m_1, m_2, m_3, m_4 and m_5. Under these conditions equations (3.1) take the following form,

$$-\boldsymbol{\omega}^2\ \ (\mathbf{q}_j - \mathbf{c}) = \sum_{k=0\ \ k\neq i}^{n} m_k \frac{\mathbf{q}_k - \mathbf{q}_j}{|\mathbf{q}_k - \mathbf{q}_j|^3}, \qquad j = 1, ..., n, \tag{3.3}$$

substituting n=5 in the equation (3.3), we set up the following CC equations for the general five body problem as,

$$m_2\frac{\mathbf{q}_2 - \mathbf{q}_1}{|\mathbf{q}_2 - \mathbf{q}_1|^3} + m_3\frac{\mathbf{q}_3 - \mathbf{q}_1}{|\mathbf{q}_3 - \mathbf{q}_1|^3} + m_4\frac{\mathbf{q}_4 - \mathbf{q}_1}{|\mathbf{q}_4 - \mathbf{q}_1|^3} + m_5\frac{\mathbf{q}_5 - \mathbf{q}_1}{|\mathbf{q}_5 - \mathbf{q}_1|^3} = -\boldsymbol{\omega}^2(\mathbf{q}_1 - \mathbf{c}), \tag{3.4}$$

$$m_1\frac{\mathbf{q}_1 - \mathbf{q}_2}{|\mathbf{q}_1 - \mathbf{q}_2|^3} + m_3\frac{\mathbf{q}_3 - \mathbf{q}_2}{|\mathbf{q}_3 - \mathbf{q}_2|^3} + m_4\frac{\mathbf{q}_4 - \mathbf{q}_2}{|\mathbf{q}_4 - \mathbf{q}_2|^3} + m_5\frac{\mathbf{q}_5 - \mathbf{q}_2}{|\mathbf{q}_5 - \mathbf{q}_2|^3} = -\boldsymbol{\omega}^2(\mathbf{q}_2 - \mathbf{c}), \tag{3.5}$$

$$m_1\frac{\mathbf{q}_1 - \mathbf{q}_3}{|\mathbf{q}_1 - \mathbf{q}_3|^3} + m_2\frac{\mathbf{q}_2 - \mathbf{q}_3}{|\mathbf{q}_2 - \mathbf{q}_3|^3} + m_4\frac{\mathbf{q}_4 - \mathbf{q}_3}{|\mathbf{q}_4 - \mathbf{q}_3|^3} + m_5\frac{\mathbf{q}_5 - \mathbf{q}_3}{|\mathbf{q}_5 - \mathbf{q}_3|^3} = -\boldsymbol{\omega}^2(\mathbf{q}_3 - \mathbf{c}), \tag{3.6}$$

$$m_1\frac{\mathbf{q}_1 - \mathbf{q}_4}{|\mathbf{q}_1 - \mathbf{q}_4|^3} + m_2\frac{\mathbf{q}_2 - \mathbf{q}_4}{|\mathbf{q}_2 - \mathbf{q}_4|^3} + m_3\frac{\mathbf{q}_3 - \mathbf{q}_4}{|\mathbf{q}_3 - \mathbf{q}_4|^3} + m_5\frac{\mathbf{q}_5 - \mathbf{q}_4}{|\mathbf{q}_5 - \mathbf{q}_4|^3} = -\boldsymbol{\omega}^2(\mathbf{q}_4 - \mathbf{c}), \tag{3.7}$$

$$m_1\frac{\mathbf{q}_1 - \mathbf{q}_5}{|\mathbf{q}_1 - \mathbf{q}_5|^3} + m_2\frac{\mathbf{q}_2 - \mathbf{q}_5}{|\mathbf{q}_2 - \mathbf{q}_5|^3} + m_3\frac{\mathbf{q}_3 - \mathbf{q}_5}{|\mathbf{q}_3 - \mathbf{q}_5|^3} + m_4\frac{\mathbf{q}_4 - \mathbf{q}_5}{|\mathbf{q}_4 - \mathbf{q}_5|^3} = -\boldsymbol{\omega}^2(\mathbf{q}_5 - \mathbf{c}). \tag{3.8}$$

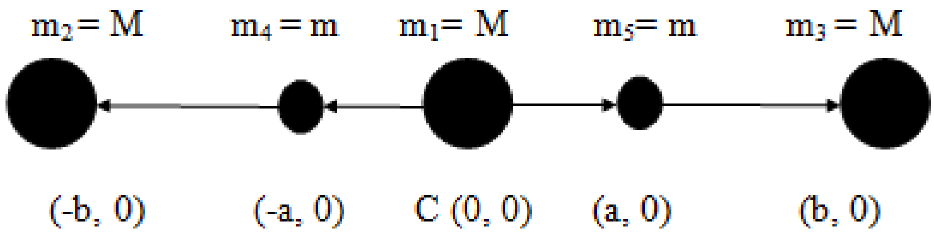

FIGURE 3.1: Symmetric collinear equilibrium configuration for five masses.

Now taking the five positive masses into consideration (the masses are m_1, m_2, m_3, m_4 and m_5) which are fixed at $\mathbf{q}_1 = (0, 0)$, $\mathbf{q}_2 = (-b, 0)$, $\mathbf{q}_3 = (b, 0)$, $\mathbf{q}_4 = (-a, 0)$

and $\mathbf{q}_5 = (a, 0)$ respectively. The following assumptions are made for the masses i.e.,

$$m_1 = m_2 = m_3 = M \quad \textit{and} \quad m_4 = m_5 = m. \tag{3.9}$$

The centre of mass for five-bodies can be written as,

$$\mathbf{c} = \frac{m_1\mathbf{q}_1 + m_2\mathbf{q}_2 + m_3\mathbf{q}_3 + m_4\mathbf{q}_4 + m_5\mathbf{q}_5}{m_1 + m_2 + m_3 + m_4 + m_5}. \tag{3.10}$$

After using the values of $\mathbf{q}_1 - \mathbf{q}_5$ and $m_1 = m_2 = m_3 = M$, $m_4 = m_5 = m$, the equation (3.10) will become,

$$\mathbf{c} = (0, 0)\,. \tag{3.11}$$

Taking $\boldsymbol{\omega} = 1$ and using equations (3.9) and (3.11) in equations (3.4)−(3.8), the equation (3.4) is identically satisfied and remaining four equations reduce to the following equations:

$$\frac{m}{(a+b)^2} + \frac{5M}{4b^2} + \frac{m}{(a-b)^2} - b = 0, \tag{3.12}$$

$$\frac{m}{(a+b)^2} - \frac{5M}{4b^2} + \frac{m}{(a-b)^2} + b = 0, \tag{3.13}$$

$$\frac{M}{(a+b)^2} + \frac{M}{(a-b)^2} + \frac{M}{a^2} + \frac{m}{4a^2} - a = 0, \tag{3.14}$$

$$\frac{M}{(a+b)^2} + \frac{M}{(a-b)^2} - \frac{M}{a^2} - \frac{m}{4a^2} + a = 0. \tag{3.15}$$

It can be easily checked that equations (3.12) and (3.14) are similar to (3.13) and (3.15) respectively, therefore we are left with the following two equations,

$$\frac{m}{(a+b)^2} - \frac{5M}{4b^2} + \frac{m}{(a-b)^2} + b = 0, \tag{3.16}$$

$$\frac{M}{(a+b)^2} + \frac{M}{(a-b)^2} + \frac{M}{a^2} + \frac{m}{4a^2} - a = 0. \tag{3.17}$$

Solving equations (3.16) and equation (3.17) for m and M, we obtain

$$m = \frac{h_1(a,b)}{h_2(a,b)}, \tag{3.18}$$

$$M = \frac{h_3(a,b)}{h_4(a,b)}, \tag{3.19}$$

where

$$h_1(a,b) = -\left(\frac{b}{(a+b)^2} + \frac{5a}{4b^2} + \frac{b}{a^2} + \frac{b}{(a-b)^2}\right), \tag{3.20}$$

$$h_2(a,b) = -\frac{5}{16a^2b^2} - \left(\frac{1}{(a-b)^2} + \frac{1}{(a+b)^2}\right)\left(\frac{1}{a^2} + \frac{1}{(a-b)^2} + \frac{1}{(a+b)^2}\right), \tag{3.21}$$

$$\begin{aligned} h_3(a,b) =& (4(4a^3(a-b)^4b^2(a+b)^2 + 4a^3(a-b)^2b^2(a+b)^4 \\ &+ b(a-b)^4b^2(a+b)^4)), \end{aligned} \tag{3.22}$$

and

$$\begin{aligned} h_4(a,b) =& (16a^2(a-b)^4b^2 + 32a^2(a-b)^2b^2(a+b)^2 \\ &+ 16(a-b)^4b^2(a+b)^2 + 5(a-b)^4(a+b)^4 \\ &+ 16a^2b^2(a+b)^4 + 16(a-b)^2b^2(a+b)^4). \end{aligned} \tag{3.23}$$

Our next goal is to verify positivity of the masses of m and M, as described in equations (3.18) and (3.19) i.e. to find the values of a and b such that m and M are positive. Because the masses are functions of distance parameters a and b, so we need to find the values of a and b, for which m and M are positive. Here we take $b = 1$ (without loss of generality) and solving the masses expressions for a, we get the following interval for a, for which m and M are positive.
(i) $0.417221 < a < 0.494666$ and (ii) $1.75768 < a < 2$.

3.2.1 Central Configuration for Collinear Five Masses

From the above analysis there exist following two cases for a for which m and M are positive.

Case-I: $0.417221 < a < 0.494666$, $b = 1$,

Case-II: $1.75768 < a < 2$, $b = 1$.

For above values of a and b central configuration for collinear five masses will always exist i.e., the masses will always move in a line. In the next chapter the dynamics of infinitesimal mass will be discussed.

Chapter 4

Introduction

In this chapter, we discuss the dynamics of 6^{th} particle m_6 moving in the plane according to the gravitational field which is formed by the attraction of 5 masses $(m_1, m_2, m_3, m_4, m_5)$ moving in a straight line according to their configuration as discussed in the previous chapter. The motion of m_6 will not effect the gravitational field of m_1, m_2, m_3, m_4 and m_5, because $m_6 << m_1, m_2, m_3, m_4$ and m_5. This problem is called restricted collinear six-body problem (RC6BP).

4.1 Dynamics of 6^{th} Particle

Equation of motion of six body m_6 is,

$$\begin{aligned}\mathbf{q}_6'' = &\frac{\mathbf{q}_1 - \mathbf{q}_6}{|\mathbf{q}_1 - \mathbf{q}_6|^3}\, m_1 + \frac{\mathbf{q}_2 - \mathbf{q}_6}{|\mathbf{q}_2 - \mathbf{q}_6|^3}\, m_2 + \frac{\mathbf{q}_3 - \mathbf{q}_6}{|\mathbf{q}_3 - \mathbf{q}_6|^3}\, m_3 \\ &+ \frac{\mathbf{q}_4 - \mathbf{q}_6}{|\mathbf{q}_4 - \mathbf{q}_6|^3}\, m_4 + \frac{\mathbf{q}_5 - \mathbf{q}_6}{|\mathbf{q}_5 - \mathbf{q}_6|^3}\, m_5. \end{aligned} \tag{4.1}$$

Now we are introducing a system of coordinates revolving around the center of mass at a uniform angular velocity of $\boldsymbol{\omega} = 1$. Let (x_6, y_6) be the coordinates of m_6 in this new rotating frame (non-inertial frame). Converting equation (4.1) from fixed inertial frame to the rotating coordinates system with the following orthogonal system,

$$\mathbf{e}_1 = e^{it}, \quad \mathbf{e}_2 = ie^{it},$$

where t is the time. In this rotating frame, the position vector of m_6 is,

$$\mathbf{q}_6 = x_6(t)\ \mathbf{e}_1 + y_6(t)\ \mathbf{e}_2. \tag{4.2}$$

Taking 1^{st} derivative and then 2^{nd} derivative of equation (4.2) yield,

$$\left.\begin{aligned} \mathbf{q}_6' &= (x_6' - y_6)e^{it} + i(x_6 + y_6')e^{it}, \\ \mathbf{q}_6'' &= (x_6'' - 2y_6' - x_6)e^{it} + i(y_6'' + 2x_6' - y_6)e^{it}. \end{aligned}\right\} \tag{4.3}$$

Using (4.3) in (4.1), the planer equations of motion of m_6 in rotating frame in component form are,

$$x_6'' - 2y_6' - x_6 = -\left[M\left(\frac{x_6}{q_{61}^3} + \frac{x_6+b}{q_{62}^3} + \frac{x_6-b}{q_{63}^3}\right) + m\left(\frac{x_6+a}{q_{64}^3} + \frac{x_6-a}{q_{65}^3}\right)\right], \tag{4.4}$$

$$y_6'' + 2x_6' - y_6 = -\left[M\left(\frac{y_6}{q_{61}^3} + \frac{y_6}{q_{62}^3} + \frac{y_6}{q_{63}^3}\right) + m\left(\frac{y_6}{q_{64}^3} + \frac{y_6}{q_{65}^3}\right)\right], \tag{4.5}$$

where corresponding distances are described as,

$$\left.\begin{aligned} q_{61} &= \sqrt{x_6^2 + y_6^2}, \\ q_{62} &= \sqrt{(x_6+b)^2 + y_6^2}, \\ q_{63} &= \sqrt{(x_6-b)^2 + y_6^2}, \\ q_{64} &= \sqrt{(x_6+a)^2 + y_6^2}, \\ q_{65} &= \sqrt{(x_6-a)^2 + y_6^2}. \end{aligned}\right\} \tag{4.6}$$

Multiplying the equation (4.4) with x_6' and the equation (4.5) with y_6' to obtain,

$$\begin{aligned} x_6''x_6' - 2y_6'x_6' - x_6x_6' = &- Mx_6'\left(\frac{x_6}{q_{61}^3} + \frac{x_6+b}{q_{62}^3} + \frac{x_6-b}{q_{63}^3}\right) \\ &+ mx_6'\left(\frac{x_6+a}{q_{64}^3} + \frac{x_6-a}{q_{65}^3}\right), \end{aligned} \tag{4.7}$$

$$\begin{aligned} y_6''y_6' - 2x_6'y_6' - x_6y_6' = &- My_6'\left(\frac{x_6}{q_{61}^3} + \frac{x_6+b}{q_{62}^3} + \frac{x_6-b}{q_{63}^3}\right) \\ &+ my_6'\left(\frac{x_6+a}{q_{64}^3} + \frac{x_6-a}{q_{65}^3}\right). \end{aligned} \tag{4.8}$$

Adding (4.7) and (4.8) and after some simplification we get the following equation,

$$\begin{aligned} x_6''x_6' + y_6''y_6' - (x_6x_6' + y_6y_6') = &- \frac{M}{q_{61}^3} \ (x_6x_6' + y_6y_6') \\ &- \frac{M}{q_{62}^3} \ (x_6x_6' + bx_6' + y_6y_6') \\ &- \frac{M}{q_{63}^3} \ (x_6x_6' - bx_6' + y_6y_6') \\ &- \frac{m}{q_{64}^3} \ (x_6x_6' + ax_6' + y_6y_6') \\ &- \frac{m}{q_{65}^3} \ (x_6x_6' - ax_6' + y_6y_6')\,. \end{aligned} \tag{4.9}$$

We can easily write the expression $x_6''x_6' + y_6''y_6'$ as

$$x_6''x_6' + y_6''y_6' = \frac{1}{2}\frac{d}{dt}\ \left(x_6'^2 + y_6'^2\right) = \frac{1}{2}\frac{dv^2}{dt}, \tag{4.10}$$

where v is the speed of the infinitesimal mass relative to the rotating frame. Likewise,

$$x_6x_6' + y_6y_6' = \frac{1}{2}\frac{d}{dt}\left(x_6^2 + y_6^2\right). \tag{4.11}$$

From equation (4.6) we obtain,

$$q_{61}^2 = x_6^2 + y_6^2. \tag{4.12}$$

Taking derivative of eqution (4.12)

$$2q_{61}\frac{d}{dt}q_{61} = 2x_6x_6' + 2y_6y_6',$$

or

$$\frac{d}{dt}q_{61} = \frac{1}{q_{61}}(x_6x_6' + y_6y_6'). \tag{4.13}$$

It follows that

$$\frac{d}{dt}\left(\frac{1}{q_{61}}\right) = -\frac{1}{q_{61}^2}\frac{d}{dt}q_{61}. \tag{4.14}$$

Using (4.13) in (4.14), we obtain

$$\frac{d}{dt}\left(\frac{1}{q_{61}}\right) = -\frac{1}{q_{61}^3}(x_6x_6' + y_6y_6'). \tag{4.15}$$

Similarly, we obtain the following relations

$$\frac{d}{dt}\left(\frac{1}{q_{62}}\right) = -\frac{1}{q_{62}^3}(x_6x_6' + bx_6' + y_6y_6'), \tag{4.16}$$

$$\frac{d}{dt}\left(\frac{1}{q_{63}}\right) = -\frac{1}{q_{63}^3}(x_6x_6' - bx_6 + y_6y_6'), \tag{4.17}$$

$$\frac{d}{dt}\left(\frac{1}{q_{64}}\right) = -\frac{1}{q_{64}^3}(x_6x_6' + ax_6 + y_6y_6'), \tag{4.18}$$

$$\frac{d}{dt}\left(\frac{1}{q_{65}}\right) = -\frac{1}{q_{65}^3}(x_6x_6' - ax_6 + y_6y_6'). \tag{4.19}$$

Using equation (4.10) and (4.11) and equation from (4.15) $-$ (4.19) in equation (4.9)

$$\begin{aligned}\frac{1}{2}\frac{dv^2}{dt} - \frac{1}{2}\frac{d}{dt}(x_6^2 + y_6^2) =& M\frac{d}{dt}\left(\frac{1}{q_{61}}\right) + M\frac{d}{dt}\left(\frac{1}{q_{62}}\right) + M\frac{d}{dt}\left(\frac{1}{q_{63}}\right) + \\ & m\frac{d}{dt}\left(\frac{1}{q_{64}}\right) + m\frac{d}{dt}\left(\frac{1}{q_{65}}\right),\end{aligned} \tag{4.20}$$

or

$$\begin{aligned}&\frac{d}{dt}\left[\frac{1}{2}v^2 - \frac{1}{2}(x_6^2 + y_6^2) - M\left(\frac{1}{q_{61}}\right) - M\left(\frac{1}{q_{62}}\right)\right. \\ &\left. - M\left(\frac{1}{q_{63}}\right) - m\left(\frac{1}{q_{64}}\right) - m\left(\frac{1}{q_{65}}\right)\right] = 0.\end{aligned} \tag{4.21}$$

The bracketed expression in equation (4.21) is a constant i.e.,

$$\frac{1}{2}v^2 - \frac{1}{2}(x_6^2 + y_6^2) - M\left(\frac{1}{q_{61}} + \frac{1}{q_{62}} + \frac{1}{q_{63}}\right) - m\left(\frac{1}{q_{64}} + \frac{1}{q_{65}}\right) = C. \tag{4.22}$$

Here constant C is known as the Jacobian constant (named after Carl Jacobi, a German mathematician who discovered it 1836). Jacobian constant may be interpreted as the total energy of the m_6 relative to the rotating frame. C is a constant of the motion of the m_6 in the restricted collinear six-body problem, here

- $-\frac{1}{2}v^2$ is the kinetic energy, corresponding to the rotating frame per unit mass.

- $-\frac{1}{q_{61}}$,$-\frac{1}{q_{62}}$,$-\frac{1}{q_{63}}$,$-\frac{1}{q_{64}}$, and $-\frac{1}{q_{65}}$ are potential gravitational energies along the horizontal axis of the masses m_1, m_2, m_3, m_4, m_5.
- $-\frac{1}{2}(x_6^2 + y_6^2)$ interpreted as the potential energy of the centrifugal force of m_6 induced by the rotation of the reference frame.

Rewritting the equation (4.22) as,

$$v^2 = (x_6^2 + y_6^2) + 2M\left(\frac{1}{q_{61}} + \frac{1}{q_{62}} + \frac{1}{q_{63}}\right) + 2m\left(\frac{1}{q_{64}} + \frac{1}{q_{65}}\right) + 2C. \tag{4.23}$$

Since v^2 cannot be negative, it must be true that,

$$(x_6^2 + y_6^2) + 2M\left(\frac{1}{q_{61}} + \frac{1}{q_{62}} + \frac{1}{q_{63}}\right) + 2m\left(\frac{1}{q_{64}} + \frac{1}{q_{65}}\right) + 2C \geq 0. \tag{4.24}$$

In regions where this inequality is violated, the trajectories of m_6 are not allowed. The boundary between restricted and allowable region can be obtained by putting $v^2 = 0$, i.e.,

$$(x_6^2 + y_6^2) + 2M\left(\frac{1}{q_{61}} + \frac{1}{q_{62}} + \frac{1}{q_{63}}\right) + 2m\left(\frac{1}{q_{64}} + \frac{1}{q_{65}}\right) + 2C = 0. \tag{4.25}$$

For a given value of the Jacobian constant the curves of zero velocity are determined by this equation. These boundaries cannot be crossed by a infinitesimal mass (spacecraft) moving within an allowed region.

4.2 Equilibrium Solutions

The equations (4.4) and (4.5) do not have an analytic solution. To analyze the positions of the equilibrium points we will use these equations. These are the positions of m_6 in space where there will be zero velocity and acceleration with the infinitesimal mass m_6, i.e. compared with the masses of m_1, m_2, m_3, m_4 and m_5, where m_6 always appears at rest. Such solutions will only be found if all rates in equations (4.4) and (4.5) are equal to zero i.e.,$x_6' = y_6' = x_6'' = y_6'' = 0$. So finally the equation (4.4) and (4.5) becomes.

$$M\left(\frac{x_6}{q_{61}^3}+\frac{x_6+b}{q_{62}^3}+\frac{x_6-b}{q_{63}^3}\right)+m\left(\frac{x_6+a}{q_{64}^3}+\frac{x_6-a}{q_{65}^3}\right)=0, \tag{4.26}$$

$$M\left(\frac{y_6}{q_{61}^3}+\frac{y_6}{q_{62}^3}+\frac{y_6}{q_{63}^3}\right)+m\left(\frac{y_6}{q_{64}^3}+\frac{y_6}{q_{65}^3}\right)=0. \tag{4.27}$$

Because equations (4.26) and (4.27) are non-linear algebraic equations, therefore we neet to solve these equations numerically by mathematica [33] under the conditions on a and b i.e., (Case-I and Case II) given in the previous chapter, section 3.2.1. First we will equilibrium points for Case-I(section 3.2.1).

4.3 Case 1: When $a \in (0.417221\ ,\ 0.494666)$

We start our analysis by taking the value of $a = 0.417222$ (left hand side of interval). In this case only two equilibrium points exist (see Figure 4.1).

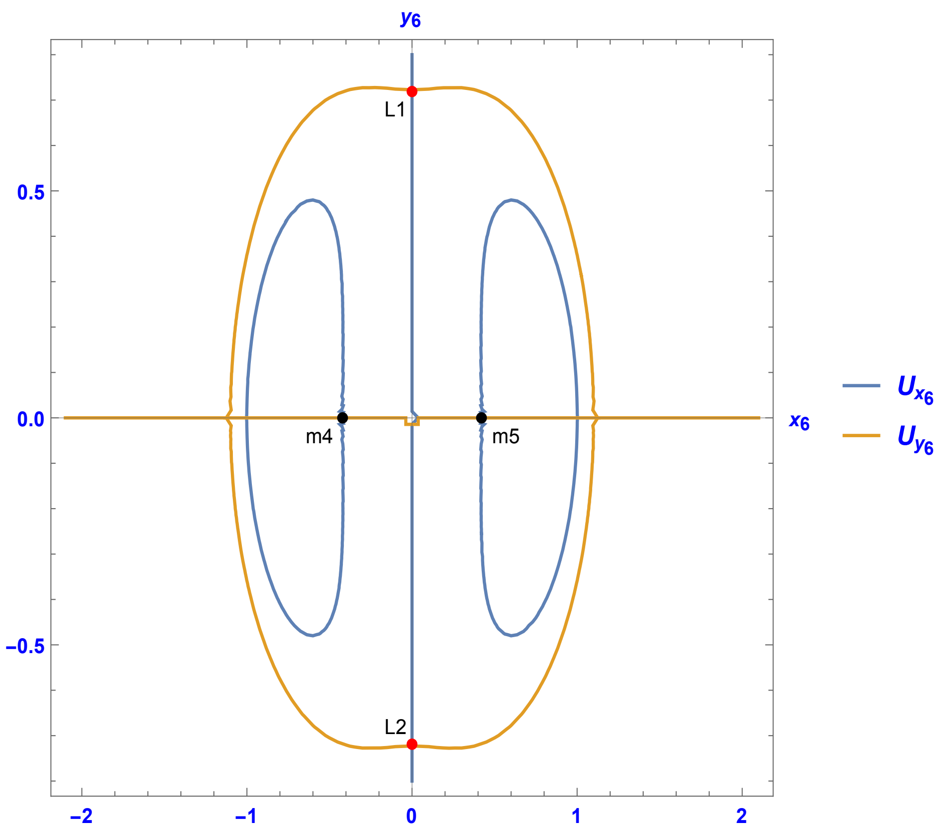

FIGURE 4.1: Positions of the equilibrium points (Red dots), and the corresponding positions of the masses (Black dots). Here $a = 0.417222$, $b = 1$, $m = 0.2905$ and $M = 0$.

The corresponding value of masses in this case are $m = 0.2905$ and $M = 0$. It can be easily concluded here that we have only three masses left in this case and the rest of two masses become zero. We call this case as degenerate restricted collinear six body problem. The intersections of the non-linear equations $U_{x_6} = 0$ and $U_{y_6} = 0$ describe the position of the equilibrium points. The intersections of $U_{x_6} = 0$ (blue) and $U_{y_6} = 0$ (orange), respectively. Primary masses are represented by black dots, whereas equilibrium points are represented by red dots.

Eight Equilibrium Points.

A critical analysis of equilibrium points has been done for $a \in (0.417221, 0.49666)$ by varying the value of a and we found that there always exist eight equilibrium points for any value of a in this interval. So choosing any value of $a \in (0.417221, 0.49666)$ will not effect the number of equilibrium points in this case. Because the behavior of the number of equilibrium points does not change for $a \in (0.417221, 0.49666)$, so we have chosen three different values of a. First taking $a = 0.427221$ and solving equations (4.26) and (4.27) numerically then we get eight equilibrium points. See Figure 4.2 and Table 4.1.

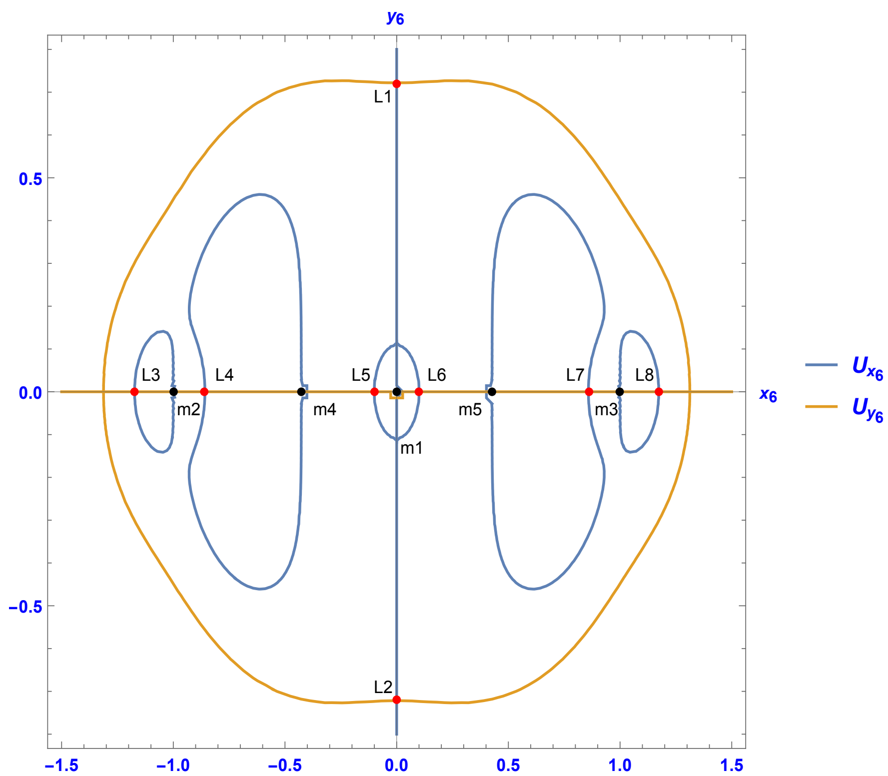

FIGURE 4.2: Positions of the equilibrium points (Red dots), and the corresponding positions of the masses (Black dots). Here $a = 0.427221$, $b = 1$, $m = 0.2767$ and $M = 0.0164$.

In Figure 4.2 the red dots show position of equilibrium points of m_6 and black dots represent the positions of the masses m_1–m_5. One can easily see that two equilibrium points are along y-axis and the remaining six are lying along x-axis. The value of the corresponding masses m and M are 0.2767 and 0.01648. Now taking $a = 0.447221$ and $a = 0.487221$ and solving equations (4.26) and (4.27), we can see the position of equilibrium points are varying along x-axis and y-axis respectively. The number of equilibrium points remains the same. See Figures 4.3 and 4.4.

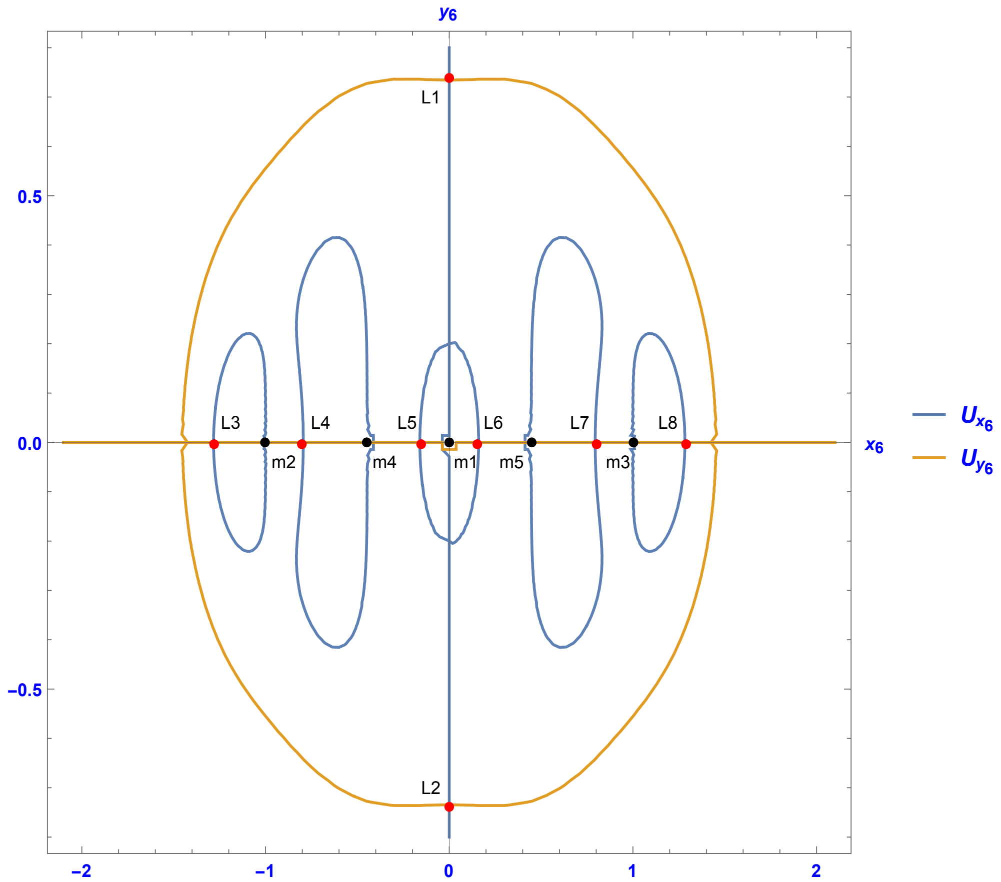

FIGURE 4.3: Positions of the equilibrium points (Red dots), and the corresponding positions of the masses (Black dots). Here $a = 0.447221$, $b = 1$, $m = 0.2454$ and $M = 0.0637$.

The value of masses in these cases are $m = 0.2454$ and $M = 0.637$ and $m = 0.1074$ and $M = 0.4341$. The numerical value of equilibrium points can be seen in Tables 4.2 and 4.3.

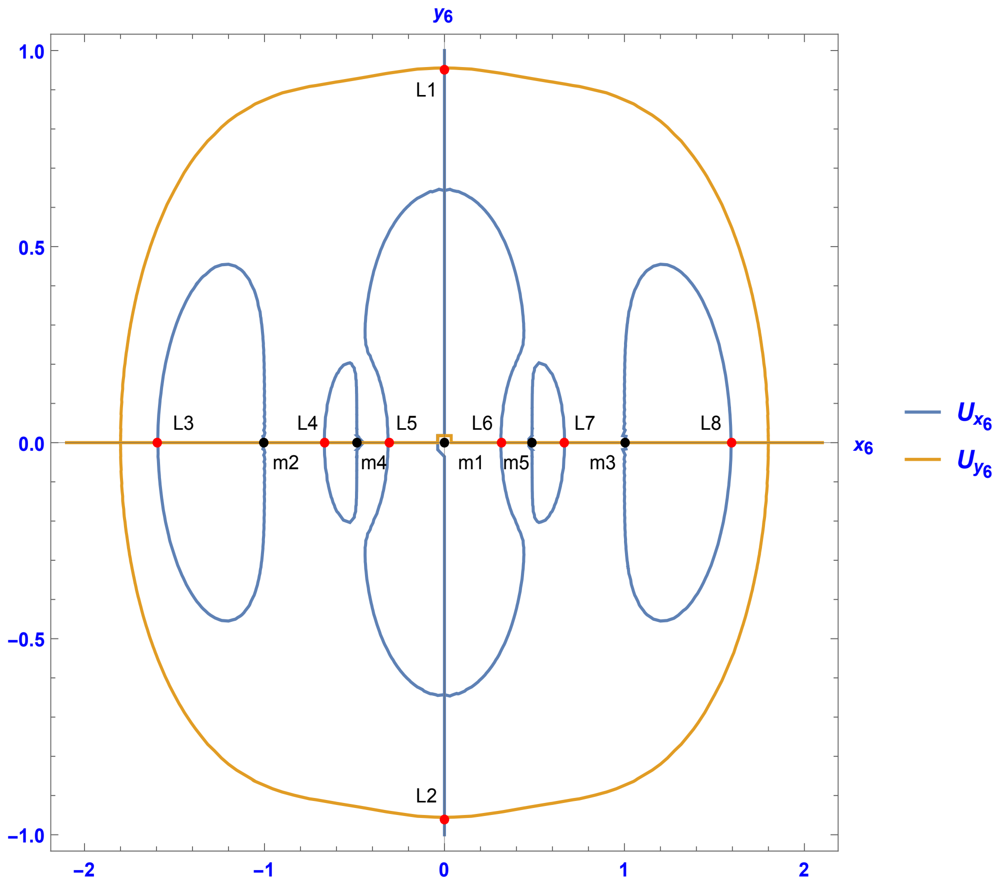

FIGURE 4.4: Positions of the equilibrium points (Red dots), and the corresponding positions of the masses (Black dots). Here $a = 0.487221$, $b = 1$, $m = 0.1074$ and $M = 0.4341$.

4.4 Case 2: When $a \in (1.75768\ ,\ 2.0)$

We start our analysis by taking the value of $a = 1.75778$ (left hand side of interval). In this case only four equilibrium points exist (see Figure 4.5). The corresponding value of masses in this case are $m = 0$ and $M = 0.8$. It can be easily concluded here that we have only three masses left in this case and the rest of three masses become zero. We call this case as degenerate restricted collinear six body problem. The intersections of the non-linear equations $U_{x_6} = 0$ and $U_{y_6} = 0$ describe the position of the equilibrium points. The intersections of $U_{x_6} = 0$ (blue) and $U_{y_6} = 0$ (orange), respectively. Primary masses are represented by black dots, whereas equilibrium points are represented by red dots.

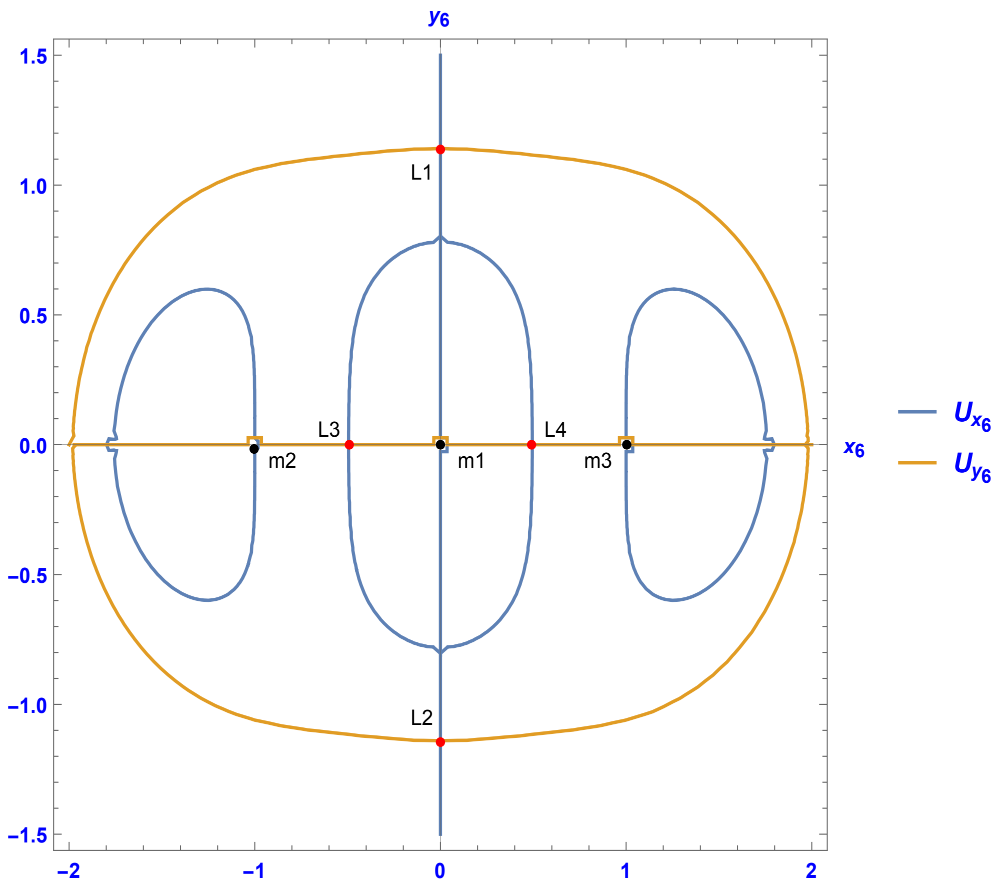

FIGURE 4.5: Positions of the equilibrium points (Red dots), and the corresponding positions of the masses (Black dots). Here $a = 1.75778$, $b = 1$, $m = 0$ and $M = 0.8$.

Eight Equilibrium Points.

A critical analysis of equilibrium points has been done for $a \in (1.75768, 2.0)$ by varying the value of a and we found that there always exist eight equilibrium points for any value of a in this interval. So choosing any value of $a \in (1.75768, 2.0)$ will not effect the number of equilibrium points in this case. Because the behavior of the number of equilibrium points does not change for $a \in (1.75768, 2.0)$, so we have chosen three different values of a. First taking $a = 1.84768$ and solving equations (4.26) and (4.27) numerically then we get eight equilibrium points. See Figure 4.6 and Table 4.4. In Figure 4.6 the red dots show position of equilibrium points of m_6 and black dots represent the positions of the masses m_1–m_5. One can easily see that two equilibrium points are along y-axis and the remaining six are lying along x-axis. The value of the corresponding masses m and M are 0.2103 and 1.0134. Now taking $a = 1.97768$ and $a = 1.98768$ and solving equations (4.26) and (4.27), we can see the position of equilibrium points are varying along x-axis and y-axis respectively. The number of equilibrium points remains the same.

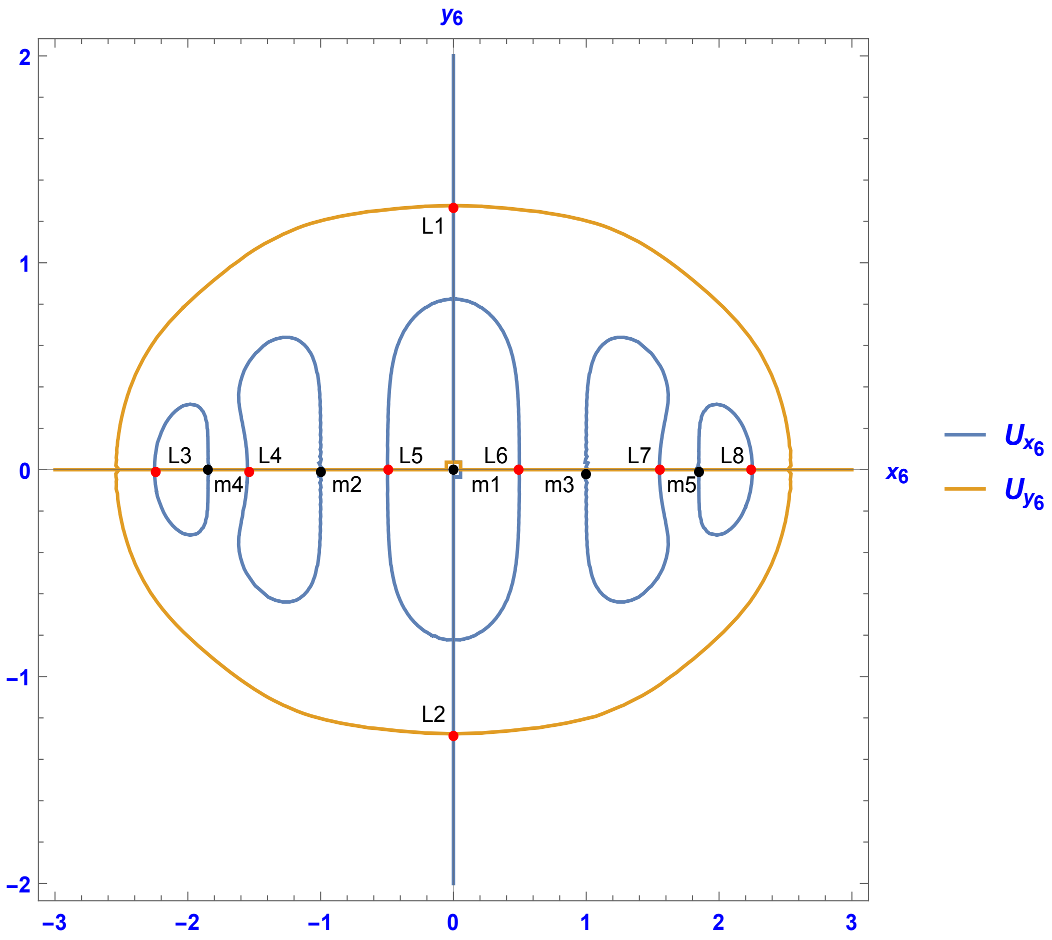

FIGURE 4.6: Positions of the equilibrium points (Red dots), and the corresponding positions of the masses (Black dots). Here $a = 1.84768$, $b = 1$, $m = 0.2103$ and $M = 1.0134$.

See Figures 4.7 and 4.8. The value of masses in these cases are $m = 0.7551$ and $M = 1.3633$ and $m = 0.8115$ and $M = 1.39928$. The numerical value of equilibrium points can be seen in Tables 4.5 and 4.6.

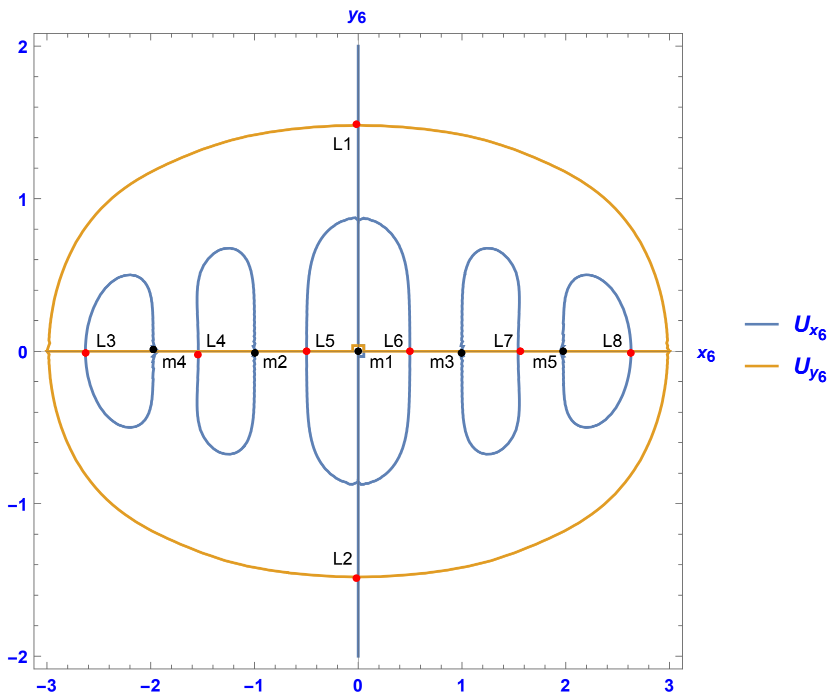

FIGURE 4.7: Positions of the equilibrium points (Red dots), and the corresponding positions of the masses (Black dots). Here $a = 1.97768$, $b = 1$, $m = 0.7551$ and $M = 1.3633$.

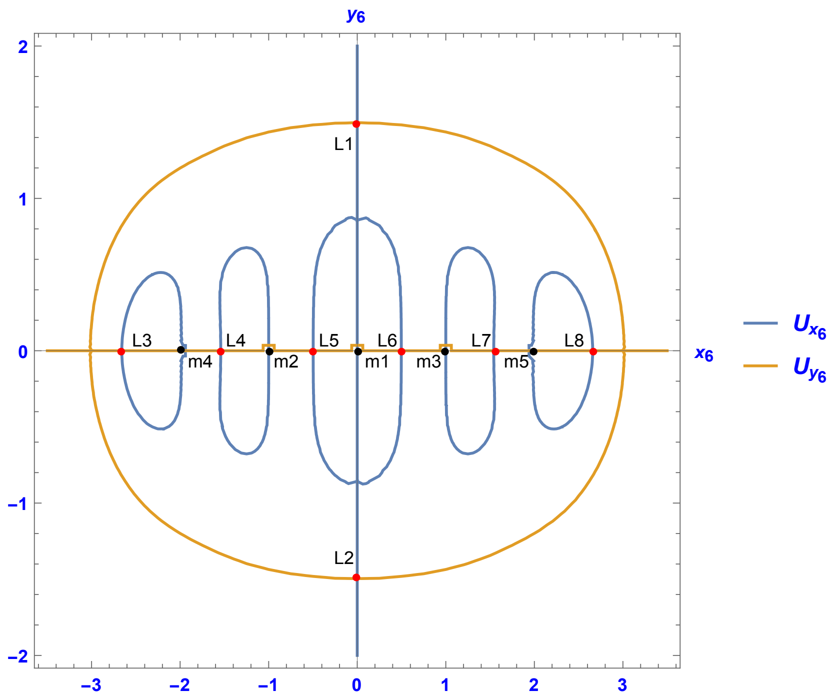

FIGURE 4.8: Positions of the equilibrium points (Red dots), and the corresponding positions of the masses (Black dots). Here when $a = 1.98768$, $b = 1$, $m = 0.8115$ and $M = 1.3928$.

4.5 Stability Analysis of Equilibrium Points

This section deals with the analysis of the stability of equilibrium points in RC6BP. Here, we check if the points of equilibrium are either stable or unstable. For checking stability, we perform an individual eigenvalue analysis for each equilibrium points discussed in last section.

Eigenvalues of Case-1

Choosing $a = 0.427221$ from $(0.417221\ ,\ 0.494666)$ and the corresponding value of $M = 0.0164$, and $m = 0.2767$ respectively. The corresponding equilibrium points is $L_1(0, 0.72)$, (see Figure 4.2). For the stability analysis of L_1, we will follow the procedure given in chapter 2. The Jacobian matrix form for $L_1(0, 0.72)$ is.

$$A = \begin{pmatrix} 0.766654 & 0 \\ 0 & 2.23833 \end{pmatrix}.$$

The eigenvalues of matrix A are: $(2.23833, 0.766654)$. Likewise, all eigenvalues of equilibrium points L_i, where $i = 2, ..., 8$ in case-I are given below in Table 4.1. The same method also applied to each Lagrange points and the remaining eigen-values for L_2-L_8 are shown in Table 4.1.

Sr.No	Equilibrium points	Eigenvalues	Stability
1	L_2(0.001156, -0.72)	(2.23834, 0.766647)	unstable
2	L_3(-1.17, 0.0005508)	(9.21971, -3.10985)	unstable
3	L_4(-0.86, -0.00202)	(20.1515, -8.57453)	unstable
4	L_5(-0.093, -0.003489)	(60.6798, -28.8202)	unstable
5	L_6(0.095, -0.00202)	(58.4588, -27.7231)	unstable
6	L_7(0.860, -0.00202)	(20.1515, -8.57453)	unstable
7	L_8(1.18, -0.003489)	(8.10273, -2.55117)	unstable

TABLE 4.1: The stability analysis for: $a = 0.427221$, $M = 0.0164$, $m = 0.2767$.

The stability analysis for the case-I for $a = 0.447221$ and $a = 0.487221$ are shown in the following Tables 4.2 and 4.3.

Sr.NO	**Equilibrium points**	**Eigenvalues**	**Stability**
1	L_1(0.00420, 0.74)	(2.2278, 0.754944)	unstable
2	L_2(0.004725, -0.74)	(2.22781, 0.754932)	unstable
3	L_3(-1.28, -0.0005508)	(7.82068, -2.41034)	unstable
4	L_4(-0.80, -0.00202)	(28.6246, -12.811)	unstable
5	L_5(-0.150, -0.003489)	(59.9836, -28.4802)	unstable
6	L_6(0.150, -0.00202)	(60.022, -28.5071)	unstable
7	L_7(0.80, -0.00202)	(28.6246, -12.811)	unstable
8	L_8(1.29, -0.003489)	(7.20644, -2.10318)	unstable

Table 4.2: The stability analysis for: $a = 0.447221$, $M = 0.0637$, $m = 0.2454$.

Sr.No	**Equilibrium points**	**Eigenvalues**	**Stability**
1	L_1(0.004725, 0.9517)	(2.38995, 0.619583)	unstable
2	L_2(-0.004725, -0.9607)	(2.36083, 0.626358)	unstable
3	L_3(-1.592, -0.00225)	(5.6331, -1.31655)	unstable
4	L_4(-0.6662, -0.00225)	(65.0828, -31.0374)	unstable
5	L_5(-0.3071, -0.00225)	(71.1689, -34.0795)	unstable
6	L_6(0.3165, 0.00225)	(75.0762, -36.0329)	unstable
7	L_7(0.6662, 0.00225)	(65.0828, -31.0374)	unstable
8	L_8(1.592, -0.00225)	(5.6331, -1.31655)	unstable

Table 4.3: The stability analysis for: $a = 0.487221$, $M = 0.4341$, $m = 0.1074$.

Eigenvalues of Case-2

Choosing $a = 1.84768$ from (1.75768 , 2.0) and the corresponding value of, $M = 1.0134$, and $m = 0.2103$ respectively. The corresponding equilibrium points is $L_1(-0.006749, 1.268)$, (see Figure 4.5). All eigenvalues of remaining equilibrium points L_i, where $i = 2, ..., 8$ in case-II are shown in Table 4.4.

Sr,NO	**Equilibrium points**	**Eigenvalues**	**Stability**
1	L_2(0.006749, -1.284)	(2.3645, 0.621417)	unstable
2	L_3(-2.248, -0.01215)	(8.83549, -2.91737)	unstable
3	L_4(-1.546, -0.01215)	(29.3734, -13.1664)	unstable
4	L_5(-0.4927, -0.00405)	(34.2778, -15.6373)	unstable
5	L_6(0.4927, -0.00405)	(34.2778, -15.6373)	unstable
6	L_7(1.559, -0.00405)	(30.7504, -13.8728)	unstable
7	L_8(2.248, -0.00405)	(8.84434, -2.92213)	unstable

TABLE 4.4: The stability analysis for: $a = 1.84768$, $M = 1.0134$, $m = 0.2103$.

The stability analysis of $a = 1.97768$ and $a = 1.98768$ are shown in the following Tables 4.5 and 4.6.

Sr.No	**Equilibrium points**	**Eigenvalues**	**Stability**
1	L_1(-0.01324, 1.483)	(2.35066, 0.644164)	unstable
2	L_2(-0.01324, -1.483)	(2.35066, 0.644164)	unstable
3	L_3(-2.622, -0.008828)	(7.50806, -2.25399)	unstable
4	L_4(-1.549, -0.01766)	(37.4441, -17.1862)	unstable
5	L_5(-0.4927, -0.00405)	(46.0733, -21.5345)	unstable
6	L_6(0.4927, -0.00405)	(46.0733, -21.5345)	unstable
7	L_7(1.559, -0.00405)	(38.1039, -17.55)	unstable
8	L_8(2.635, -0.008828)	(7.1617, -2.08081)	unstable

TABLE 4.5: The stability analysis for: $a = 1.97768$, $M = 1.3638$, $m = 0.7551$.

Sr.No	Equilibrium points	Eigenvalues	Stability
1	L_1(0, 1.504)	(2.33926, 0.647765)	unstable
2	L_2(0, -1.504)	(2.35066, 0.644164)	unstable
3	L_3(-2.655, -0.00405)	(7.29814, -2.14906)	unstable
4	L_4(-1.548, -0.00405)	(37.9709, -17.4836)	unstable
5	L_5(-0.4927, -0.00405)	(47.0475, -22.0215)	unstable
6	L_6(0.4927, -0.00405)	(47.0475, -22.0215)	unstable
7	L_7(1.548, -0.00405)	(37.9709, -17.4836)	unstable
8	L_8(2.655, -0.008828)	(7.29814, -2.14906)	unstable

TABLE 4.6: The stability analysis for: $a = 1.98768$, $M = 1.3928$, $m = 0.8115$.

4.6 Permitted Areas of Motion for the Test Particle:

In the equation (4.24) the Jacobian constant of motion is one of the most important constants of dynamical system, which represent the motion of the infinitesimal body. Because it can be used to sketch the zero velocity curves, which can be used to explore the regions of permissible motion of the infinitesimal body. The value of the Jacobian constant effects the motion of an infinitesimal body. The first is an area where infinitesimal particle motion is permitted, while the second is a region where infinitesimal particle motion is prohibited.

In our geometry, we must now investigate these possibilities i.e., on the x-axis, five masses are placed: m_1, m_2, m_3, m_4, and m_5, with the infinitesimal mass m_6 moving in the gravitational field of $m_1 - m_5$. In Mathematica, we draw regions for different values of the Jacobian constant for equation (4.24), and we obtain two regions, which are following:

i. Permissible region of motion (white area), where m_6 can freely move.

ii. Shaded area (blue), where the motion of m_6 is not allowed.

One can easily see the Figures 4.9−4.12 by increasing the value of the Jacobian constant C from $C = 1$ to $C = 1.6$, the white region of motion of m_6 is reducing and for $C = 1.6$ the masses m_4 and m_5 are completely trapped. So for this value of C the m_6 can not reach around m_4 and m_5.

4.6.1 Permitted areas when a=0.417222 and b=1.0

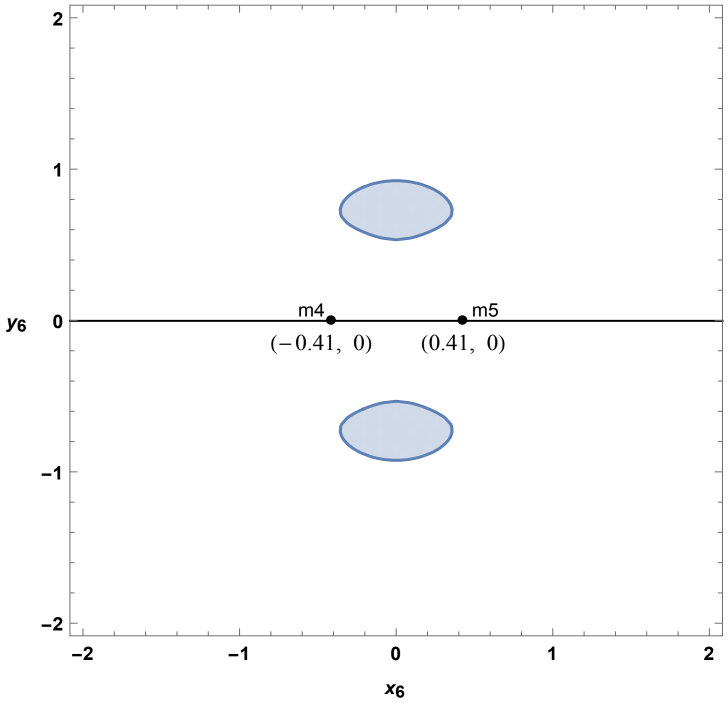

FIGURE 4.9: Permitted areas for motion of C = 1.0

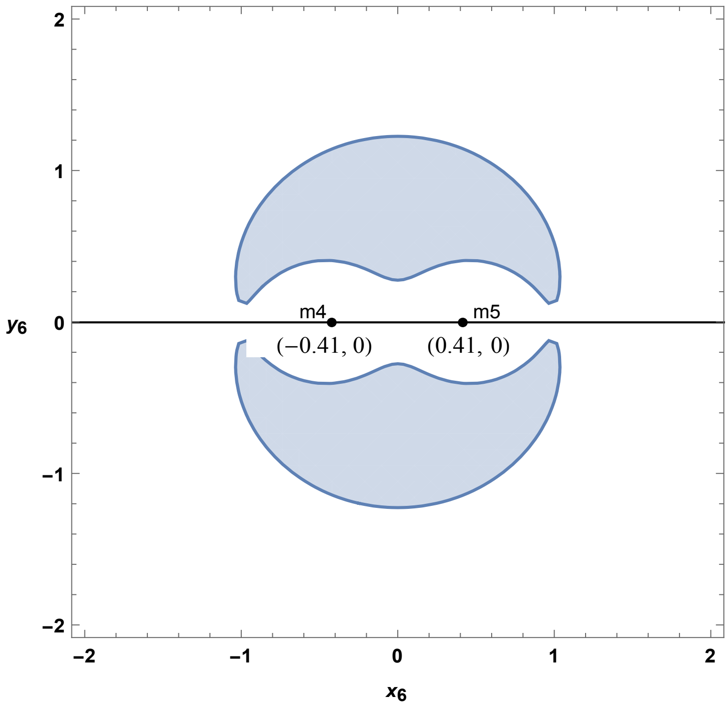

FIGURE 4.10: Permitted areas for motion of C = 1.2

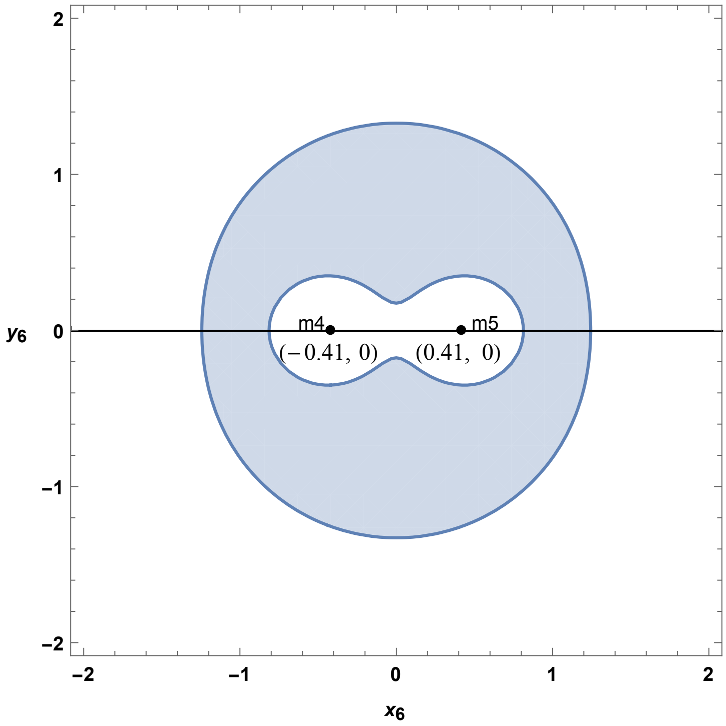

FIGURE 4.11: Permitted areas for motion of C = 1.3

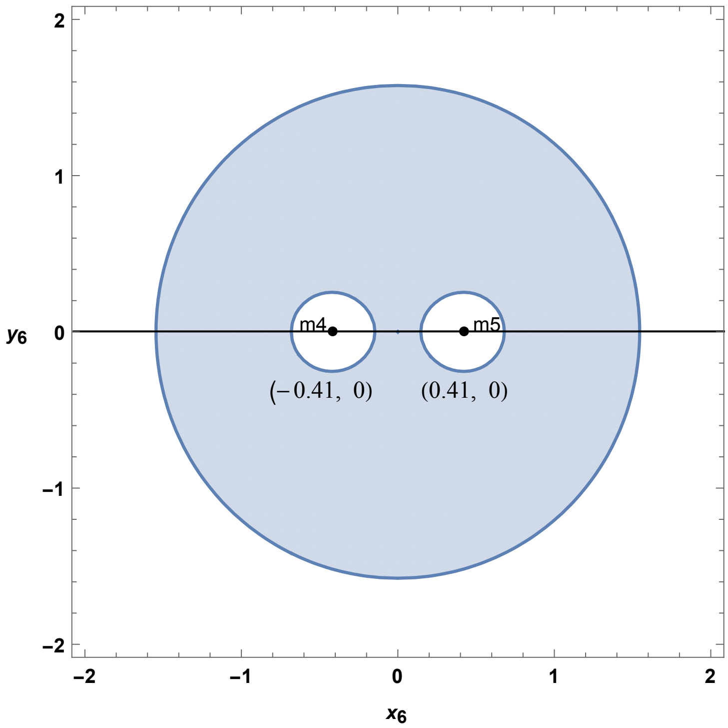

FIGURE 4.12: Permitted areas for motion of C = 1.6

4.6.2 Permissible areas when a=0.427221 and b=1.0

When the value of C from (1.07 to 1.37), the permissible area is getting shorter.

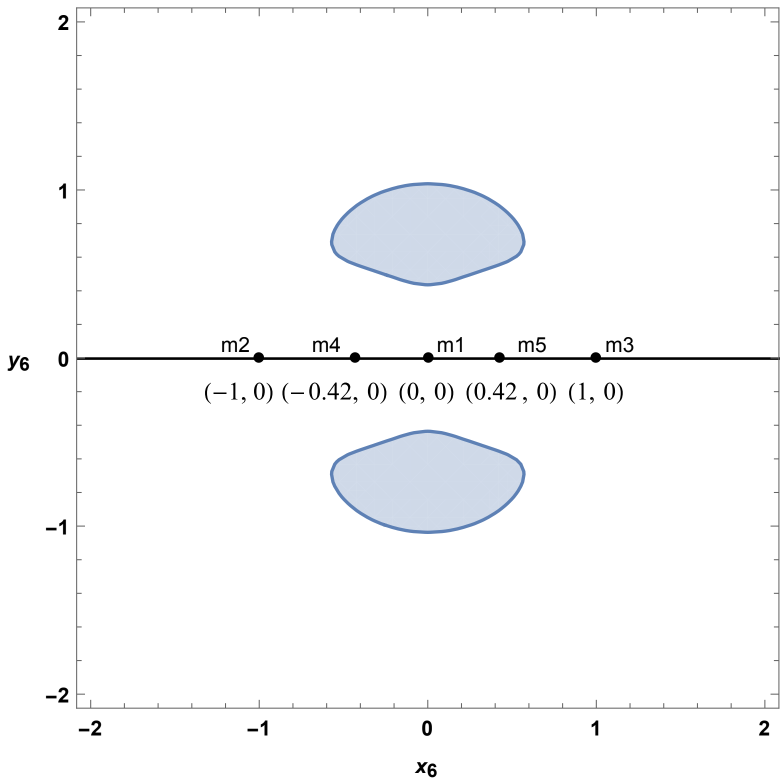

FIGURE 4.13: Permitted areas for motion of C = 1.07

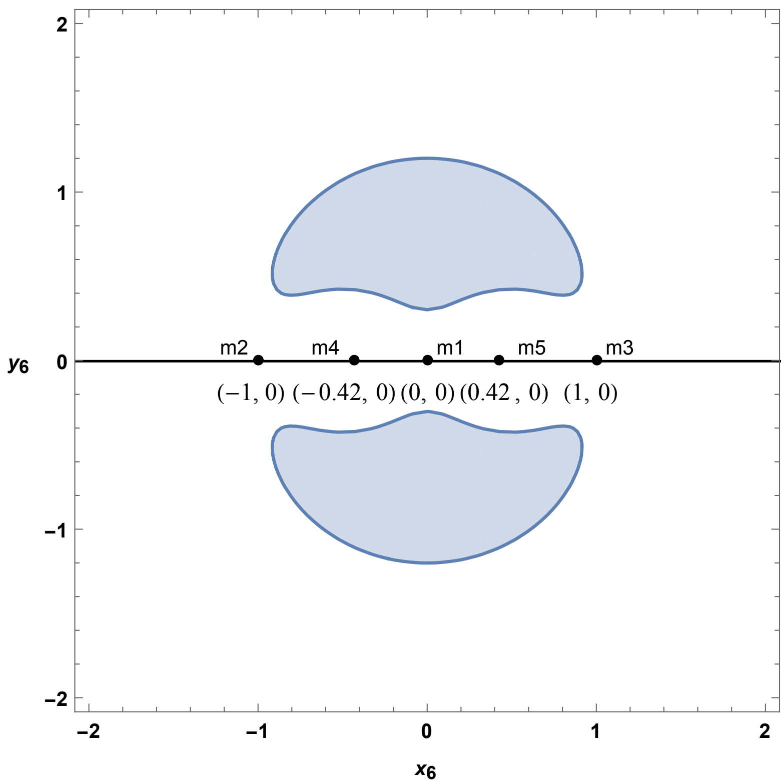

FIGURE 4.14: Permitted areas for motion of C = 1.19

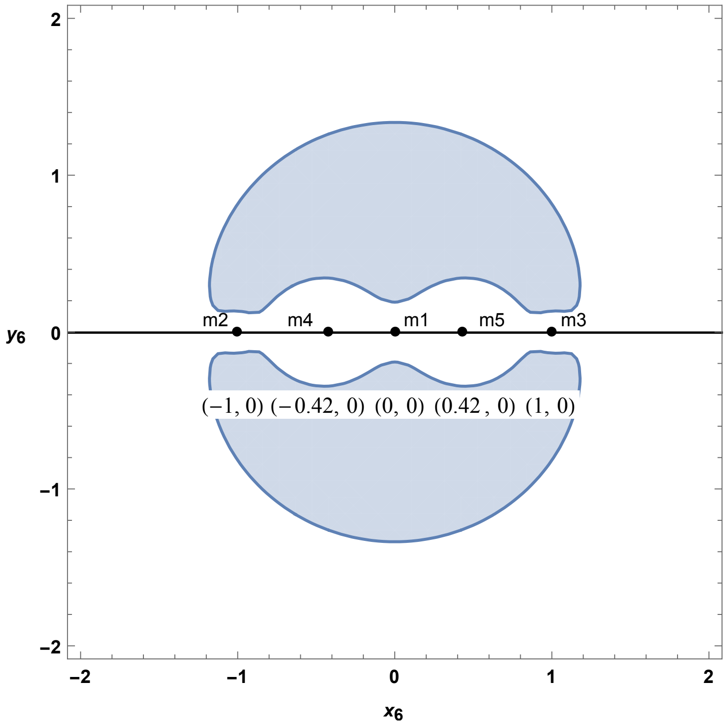

FIGURE 4.15: Permitted areas for motion of C = 1.32

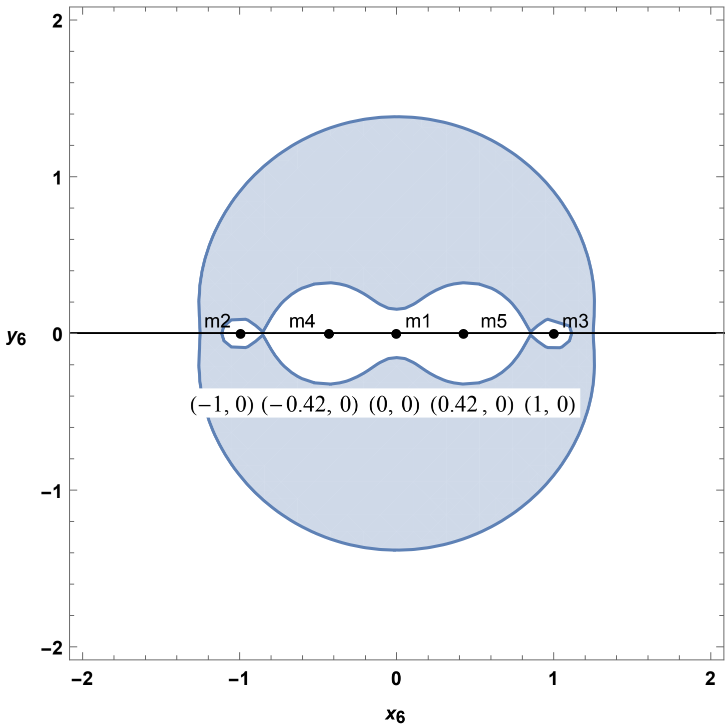

FIGURE 4.16: Permitted areas for motion of C = 1.37

4.6.3 Permissible areas when a=0.447221 and b=1.0

When the value of C from (1.04 to 1.49), the permissible area is getting shorter.

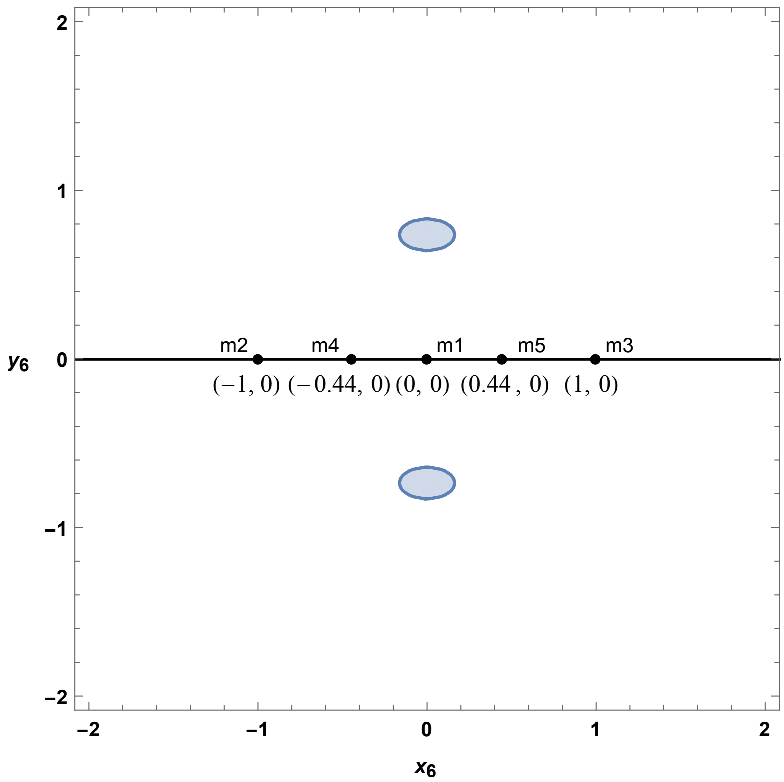

FIGURE 4.17: Permitted areas for motion of C = 1.04

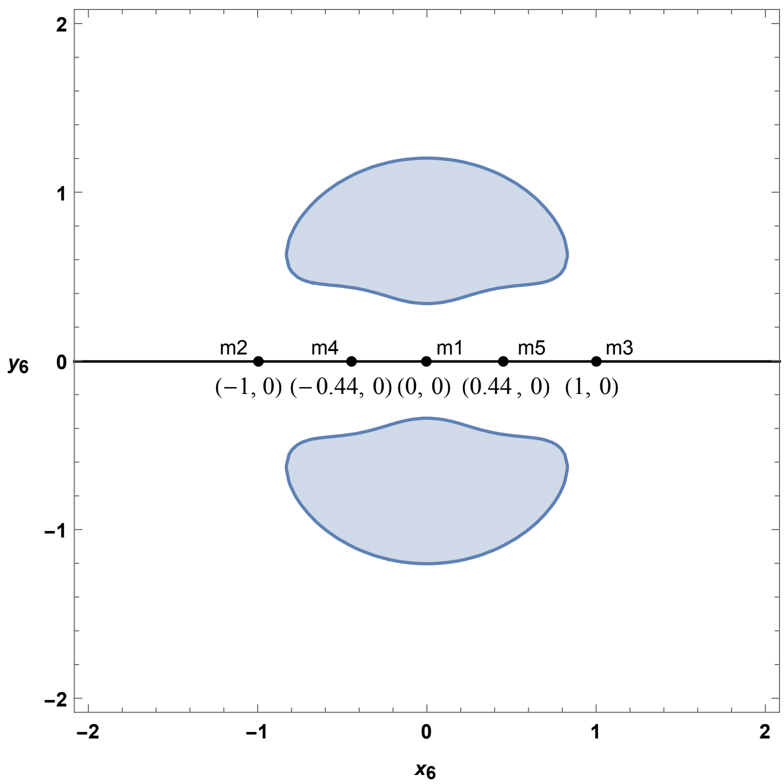

FIGURE 4.18: Permitted areas for motion of C = 1.24

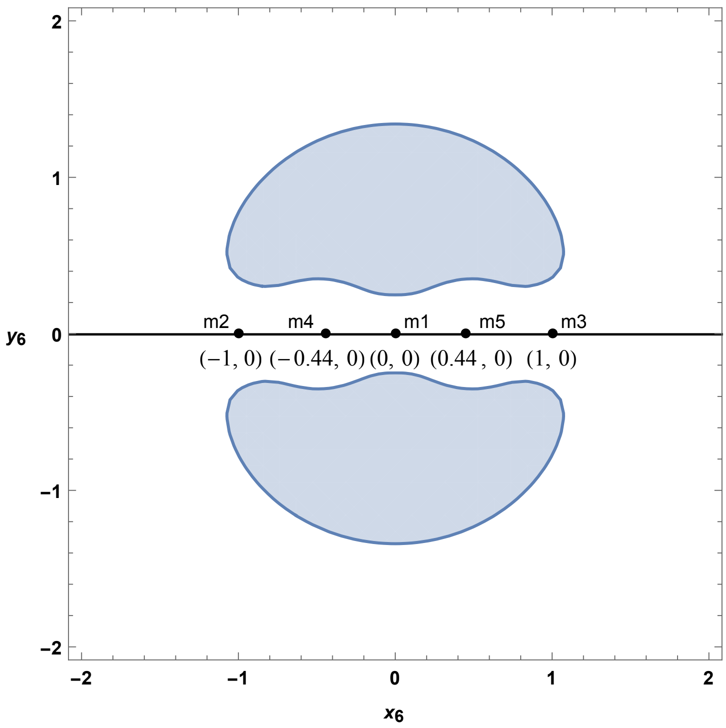

FIGURE 4.19: Permitted areas for motion of C = 1.37

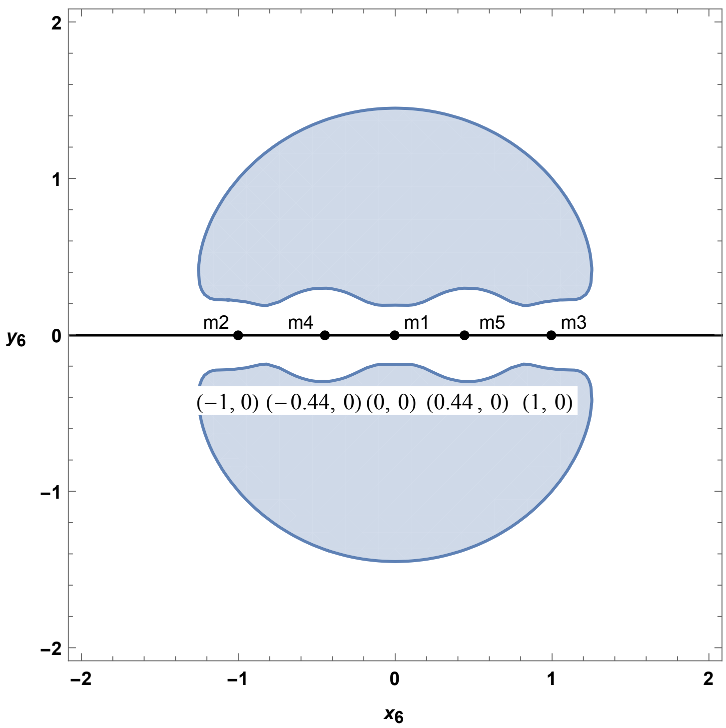

FIGURE 4.20: Permitted areas for motion of C = 1.49

4.6.4 Permissible areas when a=0.487221 and b=1.0

When the value of C from (2.1 to 3.1), the permissible area is getting shorter.

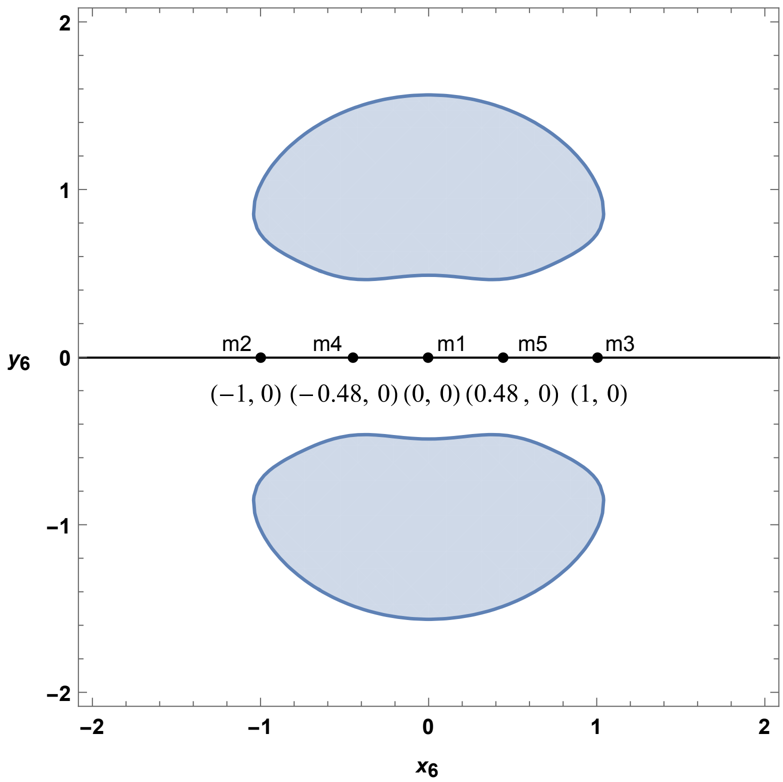

FIGURE 4.21: Permitted areas for motion of C = 2.1

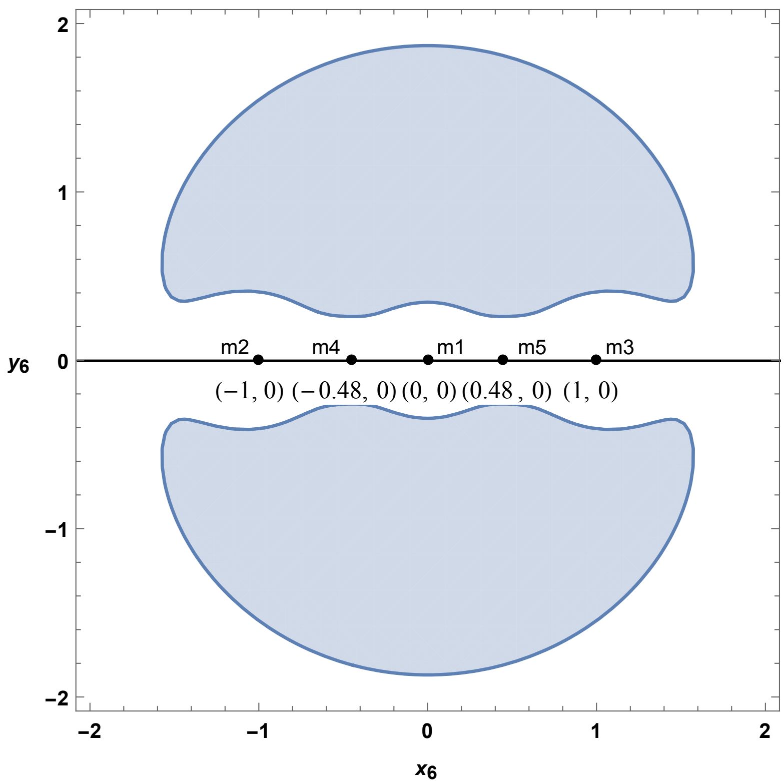

FIGURE 4.22: Permitted areas for motion of C = 2.5

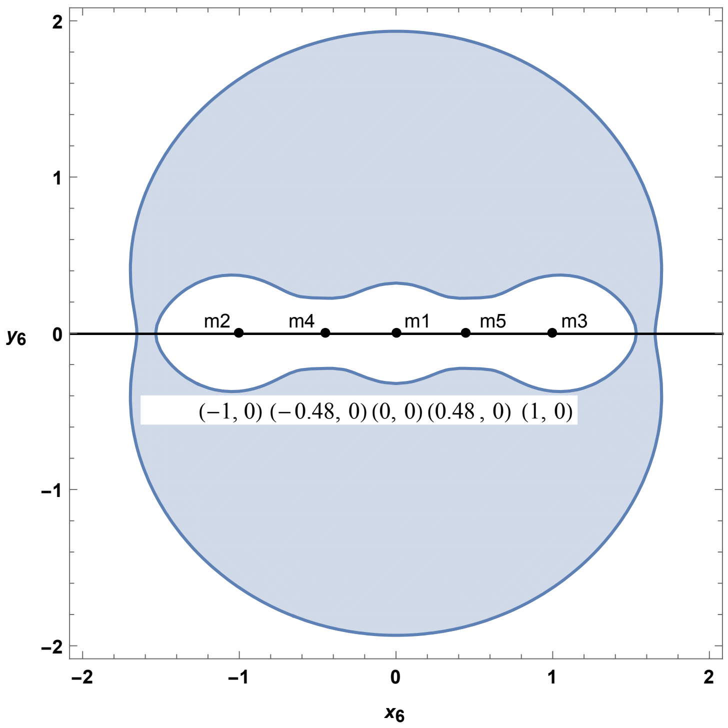

FIGURE 4.23: Permitted areas for motion of C = 2.6

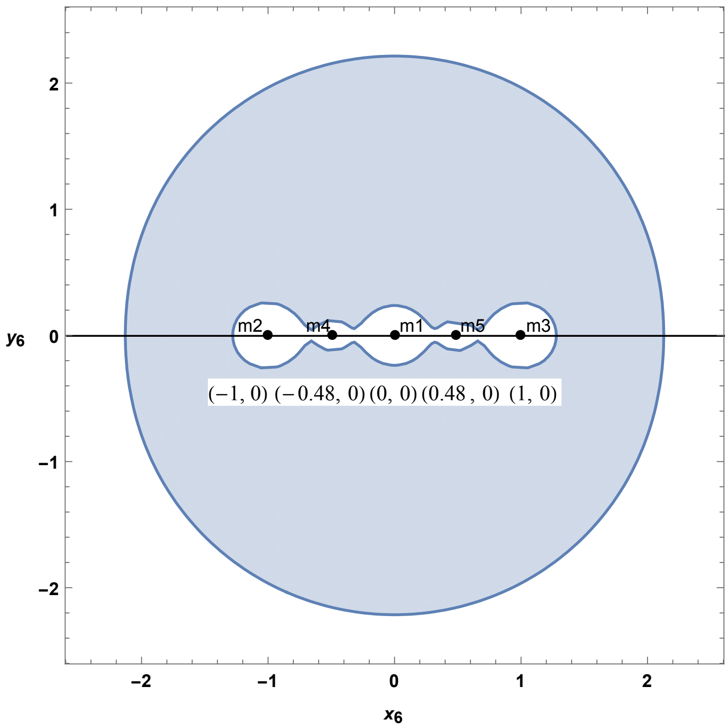

FIGURE 4.24: Permitted areas for motion of C = 3.1

4.6.5 Permitted areas when a=1.75778 and b=1.0

When the value of C from (3.0 to 4.4), the permissible area is getting shorter.

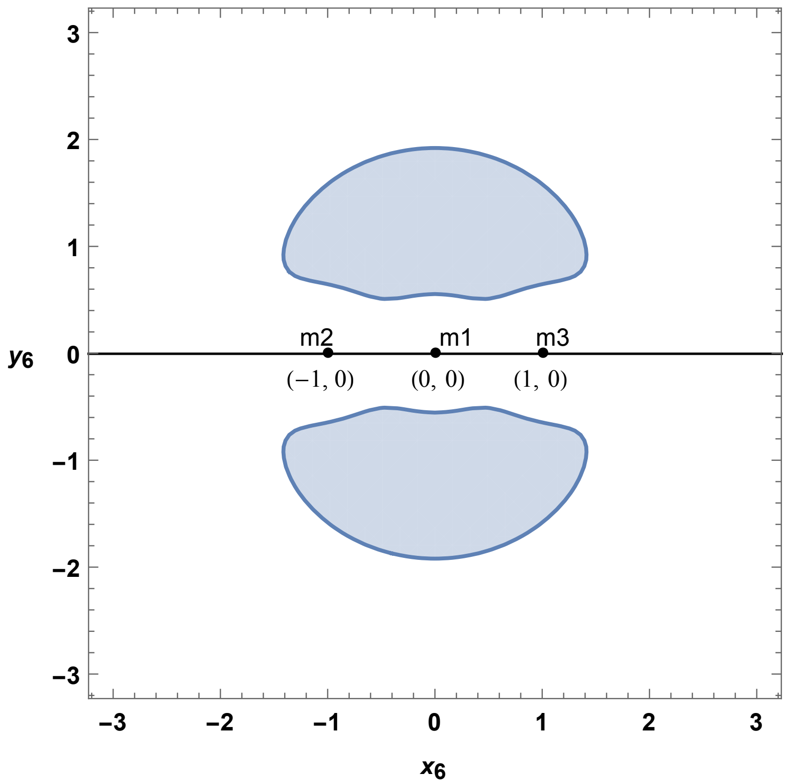

FIGURE 4.25: Permitted areas for motion of C = 3.0

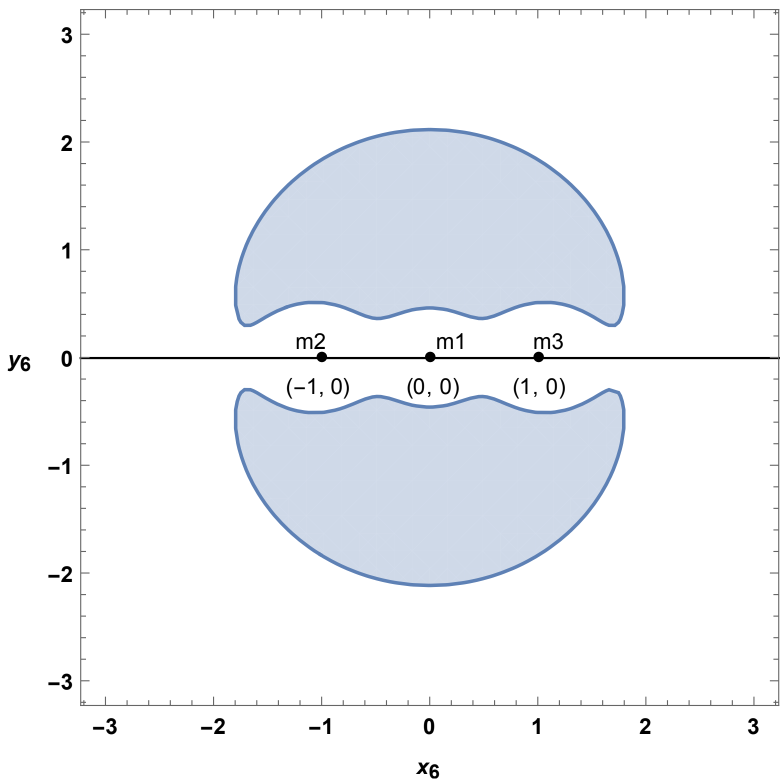

FIGURE 4.26: Permitted areas for motion of C = 3.3

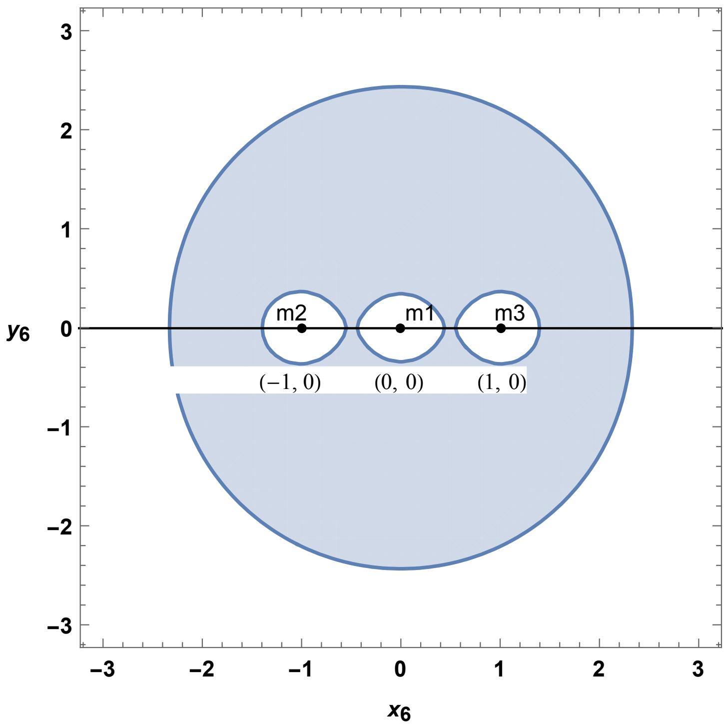

FIGURE 4.27: Permitted areas for motion of C = 3.9

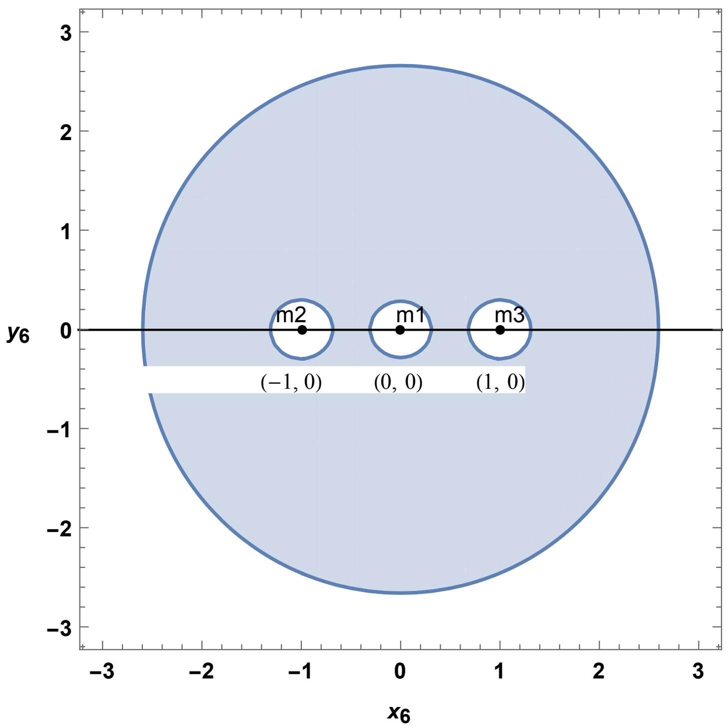

FIGURE 4.28: Permitted areas for motion of C = 4.4

4.6.6 Permitted areas when a=1.84768 and b=1.0

When the value of C from (3.3 to 5.8), the permissible area is getting shorter.

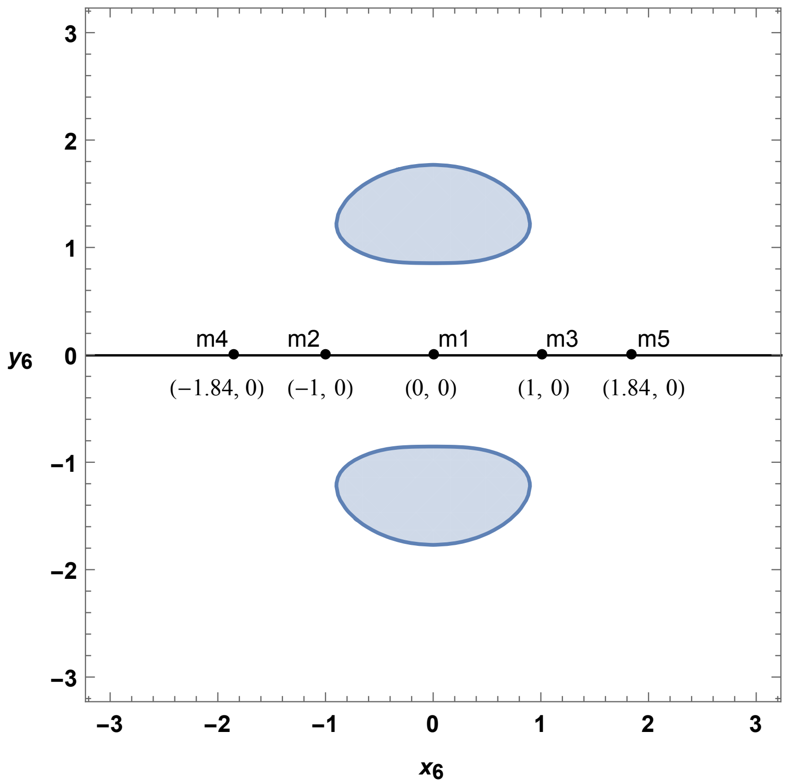

FIGURE 4.29: Permitted areas for motion of C = 3.3

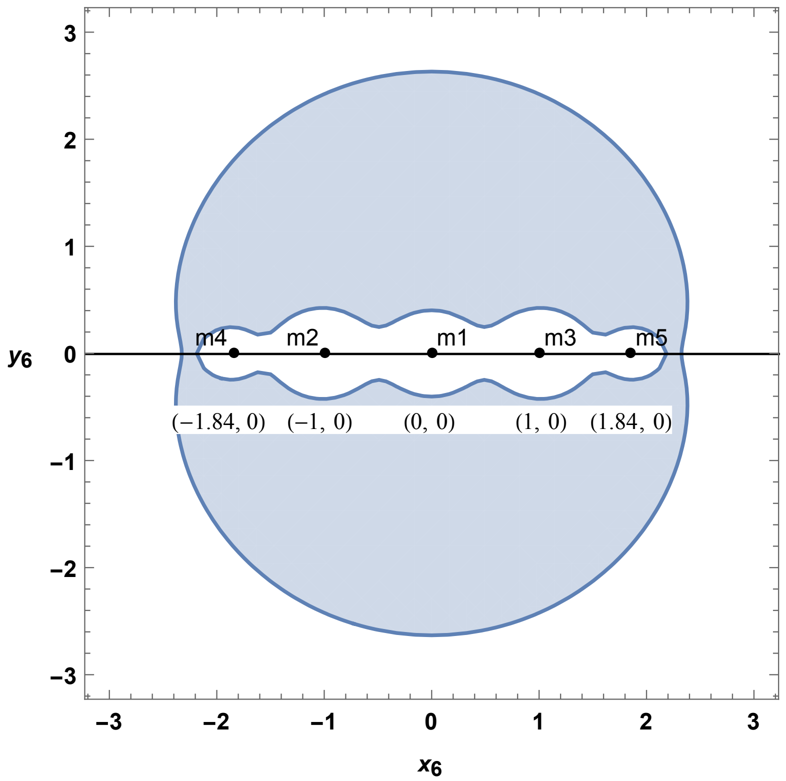

FIGURE 4.30: Permitted areas for motion of C = 4.7

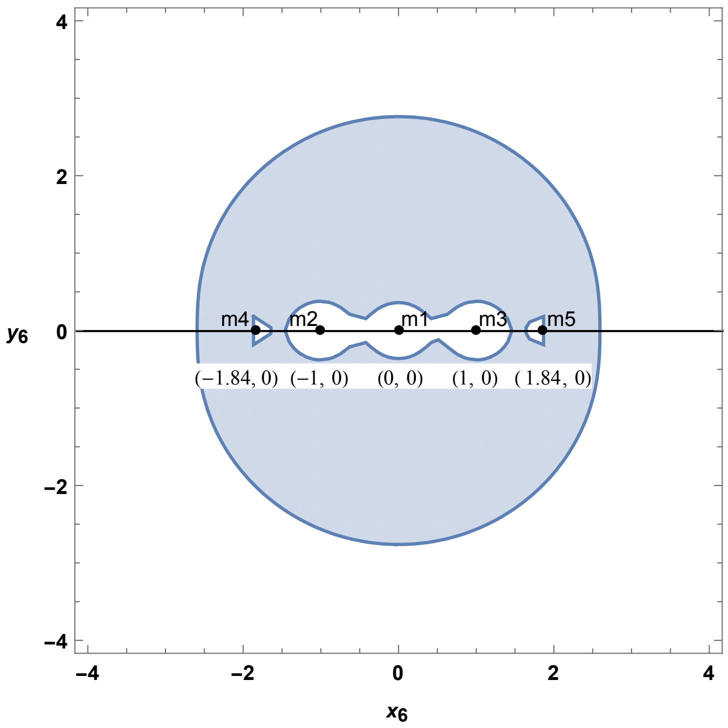

FIGURE 4.31: Permitted areas for motion of C = 5.0

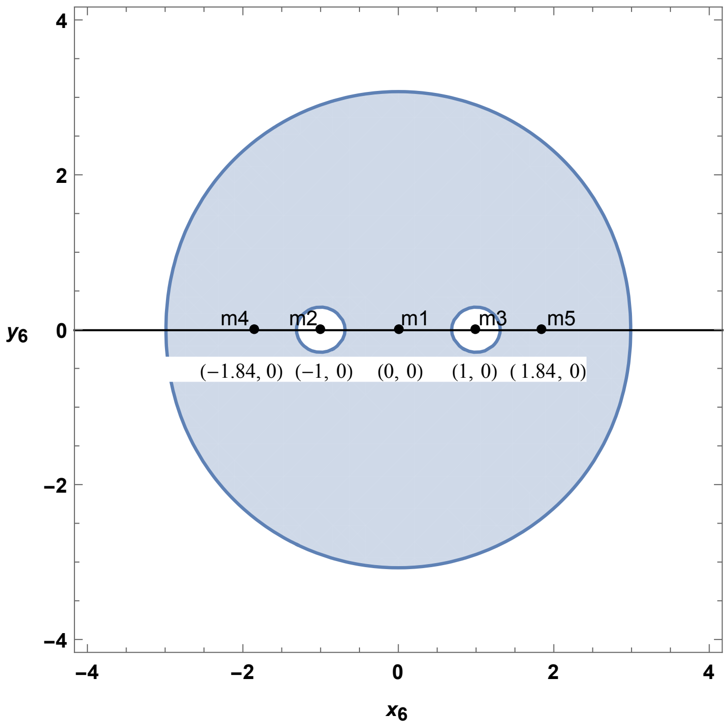

FIGURE 4.32: Permitted areas for motion of C = 5.8

4.6.7 Permitted areas when a=1.97768 and b=1.0

When the value of C from (4.5 to 8.4), the permissible area is getting shorter.

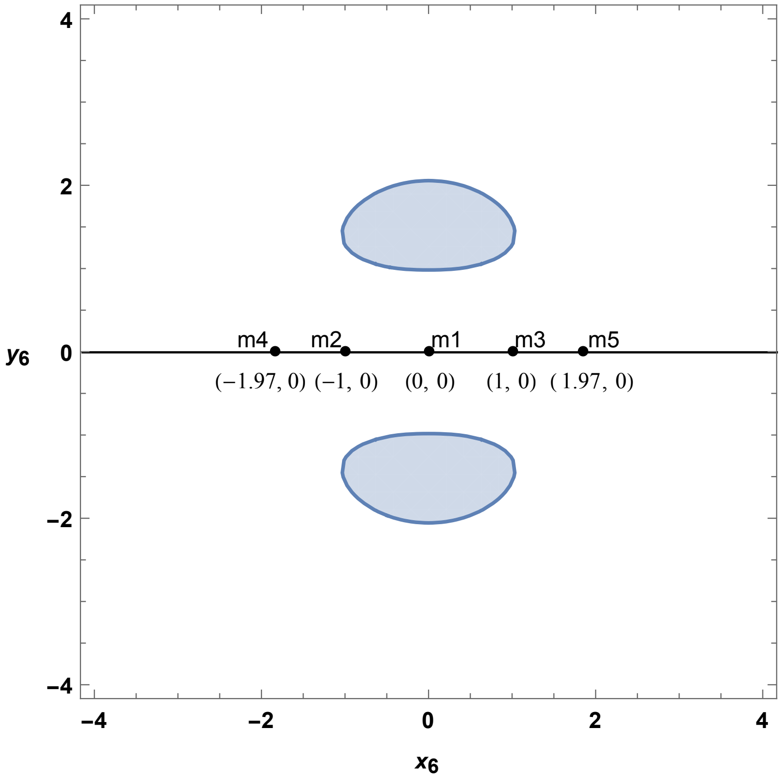

FIGURE 4.33: Permitted areas for motion of C = 4.5

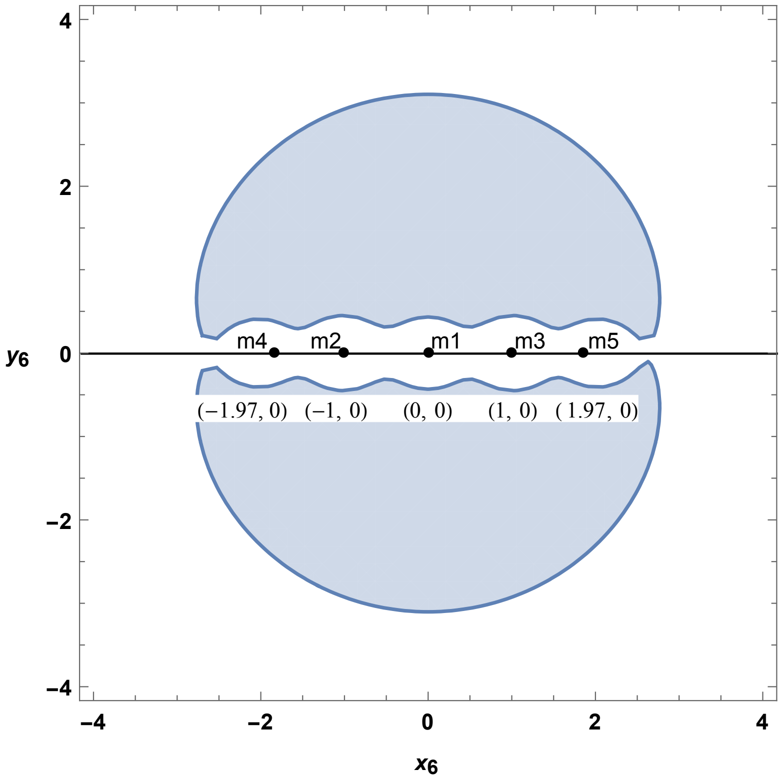

FIGURE 4.34: Permitted areas for motion of C = 6.5

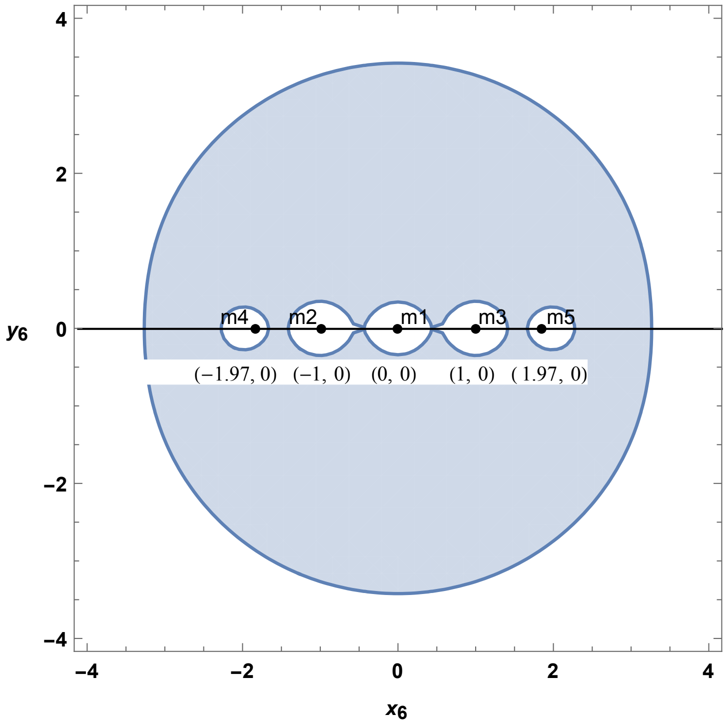

FIGURE 4.35: Permitted areas for motion of C = 7.4

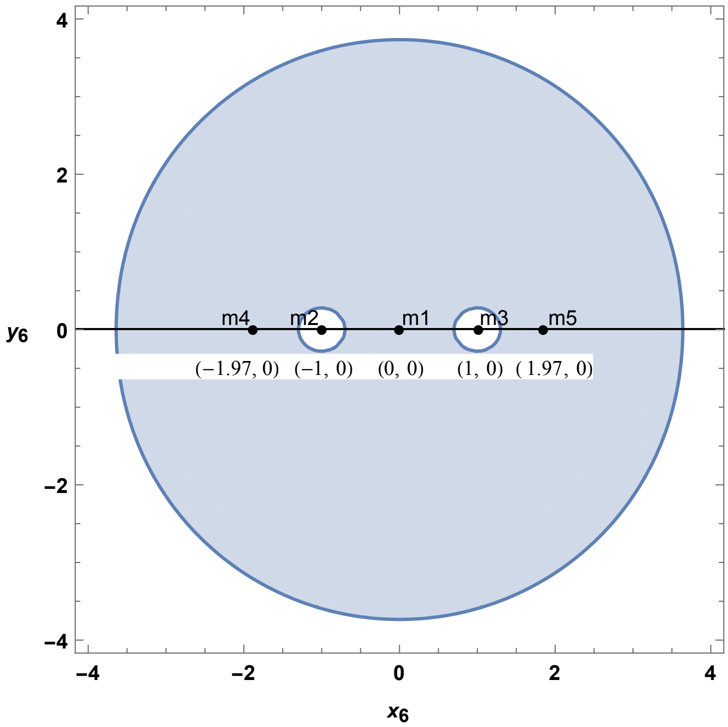

FIGURE 4.36: Permitted areas for motion of C = 8.4

4.6.8 Permitted areas when a=1.98768 and b=1.0

When the value of C from (6.9 to 9.4), the permissible area is getting shorter.

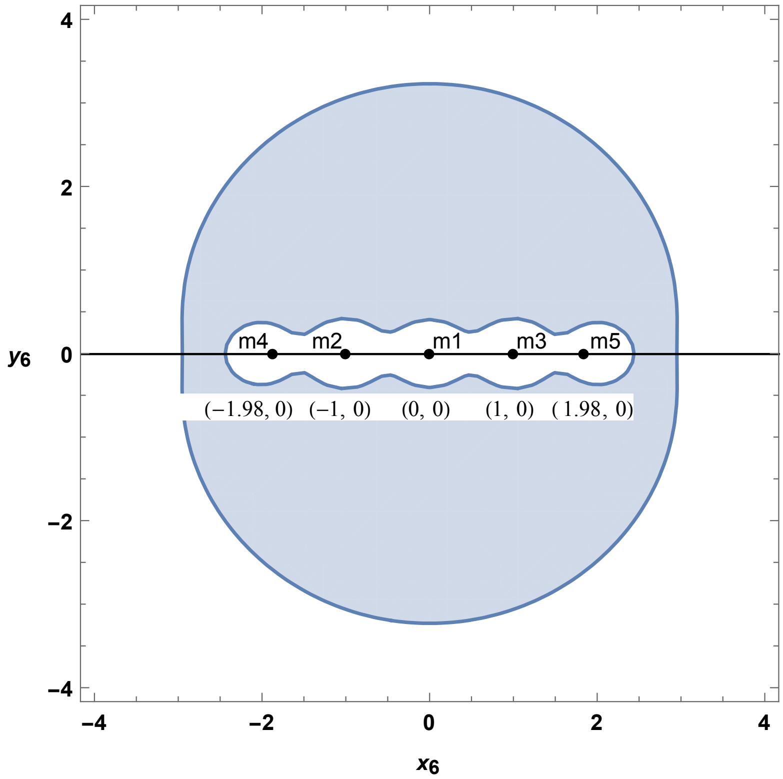

FIGURE 4.37: Permitted areas for motion of C = 6.9

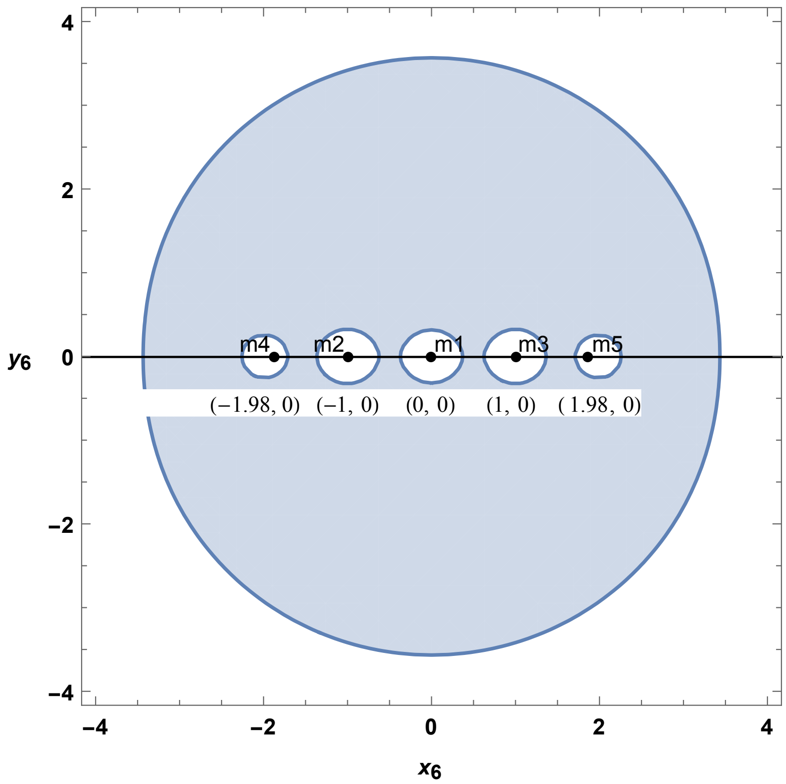

FIGURE 4.38: Permitted areas for motion of C = 7.9

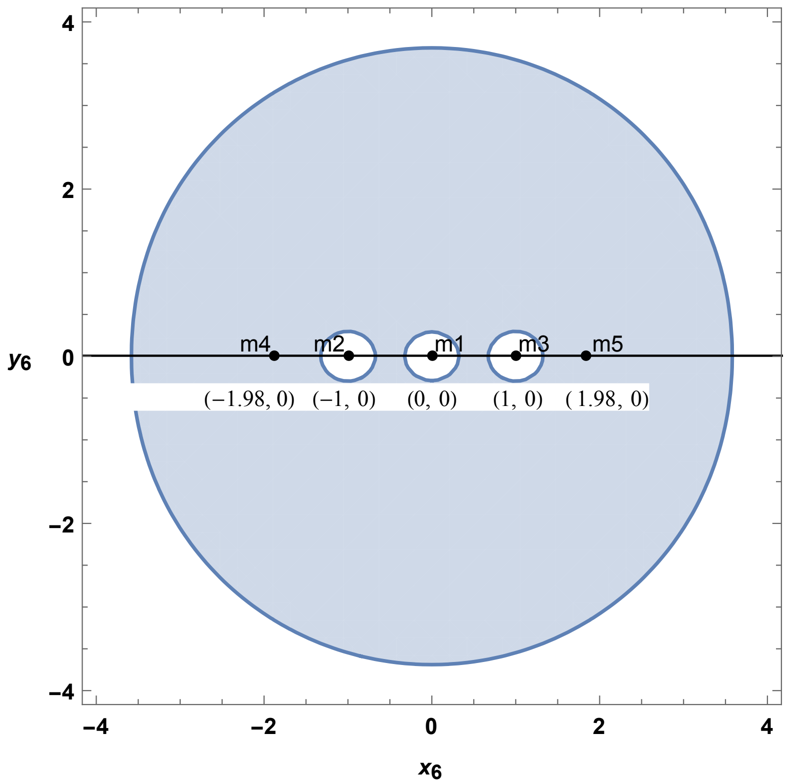

FIGURE 4.39: Permitted areas for motion of C = 8.3

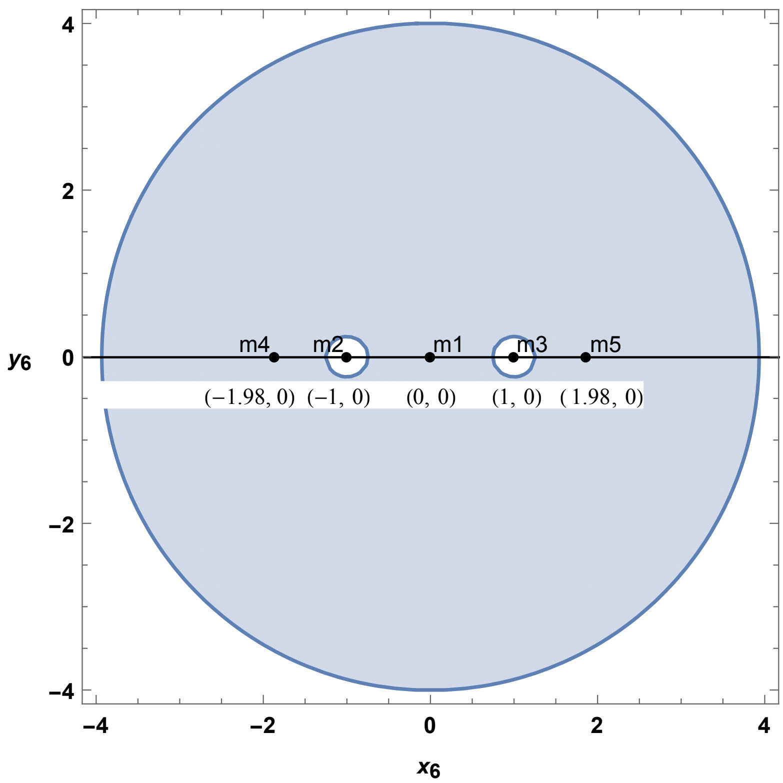

FIGURE 4.40: Permitted areas for motion of C = 9.4

Chapter 5

Conclusions

In the first part of this thesis the CC's of symmetric collinear five masses $m_1 - m_5$ have been discussed. The pair of smaller masses are placed in the middles and pair of larger masses are kept at the origin and the corners along the horizontal axis. Central configurations equations have been obtained and solved for the distance parameter a keeping $b = 1$ fixed. There exist two cases for CC' the value of a i.e., $a \in (0.417221, 0.49666)$ and $1.75768 < a < 2$ for which $m_1 - m_5$ are positive. In the second part of this research the dynamics of test particle m_6 is studied (especially the equilibrium points) under the influence of the gravitational field of $m_1 - m_5$. There always exist eight equilibrium points in both cases of CC's (i.e., for $a \in (0.417221, 0.494666)$ and for $1.75768 < a < 2$). It has been showed that the all equilibrium points are unstable using the local linear stability analysis. Lastly, we discussed the Jacobian constant (energy of the infinitesimal mass in rotating frame). The motion of infinitesimal body depends on the value of the Jacobian constant. For choosing different values of Jacobian constant C gives us two different regions. One is permitted region of motion for infinitesimal particle and the second region where motion of infinitesimal particle is not allowed. We also explored all possible region of motion (permissible region) of m_6 by changing the value of C (see Figures 4.9−4.40) for different cases.

Bibliography

[1] F. R. Moulton, *An introduction to celestial mechanics.* Courier Corporation, 1970.

[2] D. Brouwer and G. M. Clemence, *Methods of Celestial Mechanics.* Elsevier, 2013.

[3] L. G. Taff, "Celestial mechanics: a computational guide for the practitioner," *A Wiley-Interscience Publication*, 1985.

[4] H. Poincaré, *New Methods of Celestial Mechanics*, vol. 13. Springer Science & Business Media, 1992.

[5] G. Duboshin, "Celestial mechanics. basic problems and methods.,"

[6] C. Simo, "Relative equilibrium solutions in the four body problem," *Celestial Mechanics*, vol. 18, no. 2, pp. 165–184, 1978.

[7] J. Barrow-Green, *Poincaré and the three body problem*, vol. 2. American Mathematical Soc., 1997.

[8] C. Zhang, F. Topputo, F. Bernelli-Zazzera, and Y.-S. Zhao, "Low-thrust minimum-fuel optimization in the circular restricted three-body problem," *Journal of Guidance, Control, and Dynamics*, vol. 38, no. 8, pp. 1501–1510, 2015.

[9] A. Albouy and J. Llibre, "Spatial central configurations for the 1+ 4 body problem," *cmds*, vol. 292, p. 1, 2002.

[10] M. Shoaib and I. Faye, "Collinear equilibrium solutions of four-body problem," *Journal of Astrophysics and Astronomy*, vol. 32, no. 3, pp. 411–423, 2011.

[11] F. R. Moulton, "The straight line solutions of the problem of n bodies," *The Annals of Mathematics*, vol. 12, no. 1, pp. 1–17, 1910.

[12] J. I. Palmore, "Collinear relative equilibria of the planar n-body problem," *Celestial mechanics*, vol. 28, no. 1, pp. 17–24, 1982.

[13] K. Papadakis and S. Kanavos, "Numerical exploration of the photogravitational restricted five-body problem," *Astrophysics and Space Science*, vol. 310, no. 1-2, pp. 119–130, 2007.

[14] M. Kulesza, M. Marchesin, and C. Vidal, "Restricted rhomboidal five-body problem," *Journal of Physics A: Mathematical and Theoretical*, vol. 44, no. 48, p. 485204, 2011.

[15] A. Ollongren, "On a particular restricted five-body problem an analysis with computer algebra," *Journal of symbolic computation*, vol. 6, no. 1, pp. 117–126, 1988.

[16] T. Kalvouridis, "A planar case of the n+ 1 body problem: the'ring'problem," *Astrophysics and space science*, vol. 260, no. 3, pp. 309–325, 1998.

[17] V. Markellos, K. Papadakis, and E. Perdios, "The plane restricted five-body problem," in *Joint European and National Astronomical Meeting*, p. 15, 1997.

[18] G. E. Roberts, "A continuum of relative equilibria in the five-body problem," *Physica D: Nonlinear Phenomena*, vol. 127, no. 3-4, pp. 141–145, 1999.

[19] V. Mioc and C. Blaga, "A class of relative equilibria in the manev five-body problem," *Hvar Observatory Bulletin*, vol. 23, pp. 41–48, 1999.

[20] M. Shoaib, B. Steves, and A. Széll, "Stability analysis of quintuple stellar and planetary systems using a symmetric five-body model," *New Astronomy*, vol. 13, no. 8, pp. 639–645, 2008.

[21] M. Shoaib, A. Sivasankaran, and A. Kashif, "Central configurations in the collinear 5-body problem," *Turkish Journal of Mathematics*, vol. 38, no. 3, pp. 576–585, 2014.

[22] T.-L. Lee and M. Santoprete, "Central configurations of the five-body problem with equal masses," *Celestial Mechanics and Dynamical Astronomy*, vol. 104, no. 4, pp. 369–381, 2009.

[23] M. Gidea and J. Llibre, "Symmetric planar central configurations of five bodies: Euler plus two," *Celestial Mechanics and Dynamical Astronomy*, vol. 106, no. 1, pp. 89–107, 2010.

[24] M. Marchesin and C. Vidal, "Spatial restricted rhomboidal five-body problem and horizontal stability of its periodic solutions," *Celestial Mechanics and Dynamical Astronomy*, vol. 115, no. 3, pp. 261–279, 2013.

[25] T. Ouyang and Z. Xie, "Collinear central configuration in four-body problem," *Celestial Mechanics and Dynamical Astronomy*, vol. 93, no. 1-4, pp. 147–166, 2005.

[26] L. F. Mello and A. C. Fernandes, "New spatial central configurations in the 5-body problem," *Anais da Academia Brasileira de Ciências*, vol. 83, no. 3, pp. 763–774, 2011.

[27] L. Wåhlin, "The deadbeat universe," 1997.

[28] M. R Spiegel, "Theory and problems of theoretical mechanics (schaums outline)," 2021.

[29] M. R. Spiegel, S. Lipschutz, and J. Liu, *Schaum's Outlines: Mathematical Handbook of Formulas and Tables*, vol. 2. McGraw-Hill New York, 2009.

[30] D. Halliday, R. Resnick, and J. Walker, *Fundamentals of physics*. John Wiley & Sons, 2013.

[31] R. Fitzpatrick, *An introduction to celestial mechanics*. Cambridge University Press, 2012.

[32] P. A. Tipler and G. Mosca, *Physics for scientists and engineers*. Macmillan, 2007.

[33] S. Wagon, *Mathematica in action: problem solving through visualization and computation*. Springer, 2010.