CAPITAL UNIVERSITY OF SCIENCE AND TECHNOLOGY, ISLAMABAD

Boundary Layer Flow through Porous Medium in the Presence of Slip Effects, Joule Heating and Inclined Magnetic Field

by

Muhammad Bilal Javed

A thesis submitted in partial fulfillment for the degree of Master of Philosophy

in the

Faculty of Computing
Department of Mathematics

2021

I dedicating this heartfelt effort to my beloved family who supported me to conduct this research study and the respected teachers for helping and guiding me to make a final output.

CERTIFICATE OF APPROVAL

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Acknowledgement

All the appreciations are for **ALLAH** who is ever most lasting and sustainer of this universe. Everything belongs to him whatever is in the havens and whatever on the earth. Countless respect and endurance for **Prophet Muhammad (Peace be upon him)** the fortune of knowledge, who took the humanity out of ignorance and shows the right path.

I would first like to thank my Supervisor **Dr. Dur-e-Shehwar Sagheer**, Assistant Professor, Capital University of Science and Technology, Islamabad who helped me to solve and analyze the problem. The doors towards the supervisor were always open whenever I ran into a trouble spot or had a question about my research and writing. She consistently steered me in the right direction whenever she thought I needed it. It was a great honor for me to work and study under her care and guidance. Also special thanks to **Dr. Muhammad Sagheer**, head of department who also helped several times. I would also be thankful to my respected teachers **Dr. Shafqat Hussain**, **Dr. Muhammad Afzal**, **Dr. Abdul Rehman Kashif**, **Dr. Rashid Ali**, and **Dr. Samina Batul** for their valuable guidance and help throughout this MPhil session.

My heartiest gratitude to my Parents and family who put their efforts in making me a good human being. I also feel grateful to my dearest friends **Sajid Ali Satti**, **Qasim Mehmood**, **Haseeb Janjua**, **Faisal Khan Niazi**, **Farhan Ilyas**, **Bilal Ali**, **Usman Khan** and **Haris Abbasi,** who never let me down and always fortified me throughout the hard period of my research work.

May Almighty Allah shower his countless blessings on my Supervisor **Dr. Dur-e-Shehwar Sagheer**, and all those who assisted me in any way during completion of my thesis.

(Muhammad Bilal javed)

Abstract

This dissertation aims to examine the Darcy–Brinkman flow over a stretching sheet in the presence of porous dissipation. The governing equations of the described flow are nonlinear PDEs. Appropriate similarity transformation are applied to transform the PDEs into a set of ordinary differential equations. The resulting system of nonlinear equations is solved numerically under the velocity and thermal slip conditions, by shooting technique. The numerical results obtained by shooting method are validated by in built MATLAB routine bvp4c. An excellent agreement was observed. The variation of parameters was studied for different flow quantities of interest and results are presented in the form of tables and graphs.

Contents

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Abbreviations

BC	Boundary Condition
BVP	Boundary Value Problem
IVP	Initial Value Problem
MHD	Magnetohydrodynamics
ODEs	Ordinary Differential Equations
PDEs	Partial Differential Equations
RK	Runge-Kutta

Symbols

C_p	Specific heat
E_c	Eckert number
$g^{'}(\xi)$	Dimensionless velocity profile
K^*	Absorption coefficient
M	Magnetic field term
N_{Rx}	Nusselt number
P_m	Porosity parameter
P_r	Prandtl number
S	Suction/injection parameter
S_{fx}	skin friction coefficient
t	angle of magnetic field
T	Nanofluid temperature
T_∞	Ambient temperature
T_s	Temperature of sheet
u	Velocity component along X -axis
U_s	Velocity
γ	Brink man parameter
β	Velocity slip parameter
δ	Thermal slip parameter
ν	Kinematic viscosity
μ_e	Permeability
ϵ	Species diffusivity parameter
ρ	Density

μ	Permeability
κ	Thermal conductivity
α	Constant
ξ	Dimensionless variable
$\theta(\xi)$	Dimensionless temperature profile

Chapter 1

Introduction

Fluid dynamics is a branch of applied science that studies how liquids and gases move. Fluid dynamics is studied by scientists from various fields. This science has received a great deal of scientific attention, propelling it to the forefront of technological advancement. Fluid dynamics is a subject that is particularly amenable to cross-fertilization with other sciences and engineering disciplines since fluids have the potential to carry matter and transmit force.

The porous medium containing continuous solid phase with pores/vacant spaces to enable a fluid to pass through or around them. Porosity is the fraction of vacant space to the total volume. Varieties of naturally occurring and artificial porous media are available including rocks like stand stone, dolomite, pumice, limestone, beach sand, cloth sponge, gall bladder with stones, foamed plastic, lathes packed with pebbles, rye bread, surface layer that is endothermic, drug permeation through skin, catalyst pellets and human lung. Investigators paid attention towards applications of porous medium in industrial and engineering fields[1]. These applications are filtration as well as purification techniques, water seepage in river beds, drying of spongy materials in textile production, pollutants movement into soil and aquifers, chemically saturation of porous materials, migration of humidity via engineering framework and transport of heat and mass via packed bed reactors columns[1]. Darcy law states that "Flow is linearly dependent on the pressure gradient and the gravitational force".

This law is usually considered as the macroscopic equation of motion for Newtonian fluids in porous media at low Reynolds numbers and when the medium is densely packed(lower permeability). The porous medium will contain considerable voids when the pore distribution is sparse and the pores are big, creating viscous shear in addition to Darcy's resistance. In that situation, both the standard viscous resistance term (Brinkman term) and the Darcy resistance term (Darcy–Brinkman is the name of this model) should be considered [1–7].

A stretching flow is an elastic flat sheet that is stretched in its own plane with a velocity that varies with distance from a fixed point. Various manufacturing, industrial, and engineering operations manufacture the sheeting material. When melt material is forced through an extrusion die to make polymer sheets, it cools and solidifies away before reaching the cooling phase of the die. Fiber spinning, glass blowing, hot rolling, continuous casting, thin film flow and many other operations use boundary layer flow generated by stretching sheets [8–14].

A number of researchers have studied boundary layer flow over a stretching sheet in the presence of a Darcy porous medium [15–21].

Waqar and Pop [21] Performed a Darcy–Brinkman flow over a stretching sheet. When a fluid is forced to move as a result of sheet stretching, it accumulates velocity and kinetic energy, which is then converted into heat energy. The phrase for viscous dissipation in the energy equation has been changed in the presence of a porous media, and this process is known as porous dissipation. In the consequences of viscous dissipation were overlooked by the authors. Furthermore, even while simulating Darcy flow, the authors ignored the porous dissipation factors. To the best of our knowledge, no one has investigated the Darcy–Brinkman flow across a stretching sheet in the presence of frictional heating and porous dissipation. Slip condition has a wide range of commercial and practical uses, particularly in micro- and nanochannels. Slip conditions, which greatly influence fluid motion at the fluid–solid interface, are necessary to examine heat transfer flows more precisely.

The phenomenon of slip condition has many industrial and practical applications, especially in microchannels or nanochannels. To study heat transfer flows more

accurately, slip conditions are required, which strongly influence fluid motion at the fluid solid interface. Zhang et al. [22] investigated the heat transfer performance in microchannel under the slip flow regime and constant heat flux boundary condition by considering into account the effects of velocity slip and temperature jump. Hooman and Ejlali [23] showed that the combined effects of temperature jump and velocity slip on forced convection in both parallel plate and circular microchannels for fully developed gas– liquid slip flows. Hussanan et al. [24] studied the Newtonian heating problem with additional effects of velocity slip and free convection on heat transfer flow over a vertical plate.
Liu and Guo [25] used second-order slip condition while studying analytical solution of fractional Maxwell flow under magnetic field. Jing et al. [26] investigated the hydraulic resistance and heat transfer rate in elliptical microchannel with the velocity slip for different length ratios. Andersson [27] obtained the analytical solution for the slip flow over a stretching sheet. Turkyilmazoglu [28] performed the heat and mass transfer analysis of MHD flow over a stretching sheet in presence of velocity and thermal slip effects. Yazdi et al. [29] studied the effects of viscous dissipation on MHD flow over a porous stretching sheet in the presence of slip and convective boundary conditions. Hsiao [30] examined the MHD stagnation point flow of nanofluid towards a stretching sheet with slip boundary conditions.

1.1 Thesis Contribution

The key focus of the current study is to perform the analysis for the Flow of the Boundary Layer Flow through Porous Medium in the Presence of Slip Effects, Joule Heating and Inclined Magnetic Field under the conditions of velocity, thermal slip and porous dissipation. In this work, we convert a system of PDEs into non-linear ODEs using similarity transformations. Shooting technique with fourth order RK method is used to obtained the numerical results. To verify validity of our results comparative review was made between the shooting method and the built-in bvp4c function [31] in MATLAB. In case of special cases, contrast among available results was also analyzed. The effects of a variety of relative parameter on velocity profile $g'(\xi)$ and temperature profile $\theta(\xi)$ are examined.

1.2 Layout of Thesis

A brief overview of the contents of the thesis is provided below.

Chapter 2: includes some basic definitions and terminologies, which are useful to understand the concepts discussed later on.

Chapter 3: consists of a elaborated review of boundary layer flow through Porous medium with slip condition.

Chapter 4: gives the details of the generalization of the work presented in chapter 3, under the influence of Joule heating, field of magnetic attraction, viscous dissipation, and radiation of heat.

Chapter 5: contains the conclusion of the whole research.

All the references used in this study are provided in the **Bibliography**.

Chapter 2

Basic Definitions and Governing Equations

We will discuss some basic concepts, terminologies, basic rules, and dimensionless numbers in this chapter, which will be useful in carrying out the work in the next chapters. This section contains few essentials definitions and laws of the fluids, which will be used in the upcoming discussions.

2.1 Basic Terminologies

This section contains, some basic terminologies and definitions from fluid dynamics which are needed for our main work.

Definition 2.1.1. (Fluid)
"Fluid is a substance which exists in three primary phases solid, gas and liquid. (At very high temperatures, it also exist as plasmas). A substance in the liquid or gas phase is referred to as a fluid. Distinction between a solid and a fluid is made on the basis of the substances ability to resist an applied shear (or tangential) stress that to change its shape. A solid can resist an applied shear stress by deforming, whereas a fluid deforms continuously under the influence of shear stress

, no matter how small. In solid stress is proportional to strain, but in fluids, stress is proportional to strain rate."[32]

Definition 2.1.2. (Fluid Mechanics)
"Fluid mechanics is that branch of science which deals with the behavior of the fluid (liquids or gases) at rest as well as in motion."[33]

Definition 2.1.3. (Fluid Statics)
"The study of fluid at rest is called fluid statics."[33]

Definition 2.1.4. (Fluid Dynamics)
"The study of fluid if the pressure forces are also considered for the fluids in motion is called fluid dynamics."[34]

Definition 2.1.5. (Viscosity)
"Viscosity is defined as the property of a fluid which offers resistance to the movement of one layer of fluid over another adjacent layer of the fluid. Mathematically,

$$\mu = \frac{\tau}{\frac{\partial u}{\partial y}},$$

where μ is viscosity coefficient, τ is shear stress and $\frac{\partial u}{\partial y}$ represents the velocity gradient."[33]

Definition 2.1.6. (Density)
"The density of a substance is its mass per unit volume. The symbol most often used for density is ρ although the Latin letter D can also be used. Mathematically,

$$\rho = \frac{m}{V}.\text{"[34]}$$

Definition 2.1.7. (Kinematic Viscosity)
"It is defined as the ratio between the dynamic viscosity and density of fluid. It is denoted by symbol ν. Mathematically,

$$\nu = \frac{\mu}{\rho}.\text{"[33]}$$

Definition 2.1.8. (Magnethydromechanics)

"MHD is concerned with the mutual interaction of fluid flow and magnetic fields. The fluids in question must be electrically conducting and non-magnetic, which limits us to liquid metals, hot ionized gases (plasma's) and strong electrolytes."[33]

Definition 2.1.9. (Nano Fluids)

""A nano fluid is a fluid containing nano meter-sized particles, called nano particles. These fluids are engineering colloidal suspensions of nano particles in a base fluid. The nano particles used in nano fluids are typically made of metals, oxides, carbides, or carbon nano tubes." Common base fluids include water, ethylene glycerol and oil."[33]

Definition 2.1.10. (Pressure)

"Pressure is defined as the physical force exerted on an object. The force applied is perpendicular to the surface of objects per unit area. Mathematically,

$$P = \frac{F}{A}.\text{''}[34]$$

Definition 2.1.11. (Boundary Condition)

"Boundary conditions (b.c) are constraints necessary for the solution of a boundary value problem. A boundary value problem is a differential equation (or system of differential equations) to be solved in a domain on whose boundary a set of conditions is known. It is opposed to the initial value problem, in which only the conditions on one extreme of the interval are known.

Boundary value problems are extremely important as they model a vast amount of phenomena and applications, from solid mechanics to heat transfer, from fluid mechanics to acoustic diffusion. They arise naturally in every problem based on a differential equation to be solved in space, while initial value problems usually refer to problems to be solved in time." [34]

2.2 Classification of Fluids

In this section, types of fluids are discussed which further help in understanding nature of fluid motion.

Definition 2.2.1. (Ideal Fluid)
"A fluid which is in compressible and has no viscosity, is known as an ideal fluid. Ideal fluid is only an imaginary fluid as all the fluids, which exist, have some viscosity."[35]

Definition 2.2.2. (Real Fluid)
"A fluid which possesses viscosity, is known as a real fluid. In actual practice, all the fluids are real fluids."[35]

Definition 2.2.3. (Newton law of Viscosity)
"The relationship between the shear stress and shear rate of a fluid subjected to a mechanical stress. The ratio of shear stress to shear rate is a constant, for a given temperature and pressure, and is defined as the viscosity or coefficient of Newton Law of viscosity. Mathematically,

$$\tau_{yx} = \mu \frac{du}{dy}.\text{''}[35]$$

Definition 2.2.4. (Newtonian Fluid)
"A real fluid in which the shear stress is directly proportional to the rate of shear strain (or velocity gradient), is known as a Newtonian fluid. Mathematically, it can be written as:

$$\tau_{xy} \propto \left(\frac{du}{dy}\right)$$

,

$$\tau_{xy} = \mu \left(\frac{du}{dy}\right).$$

where

- μ= Dynamic viscosity, τ_{xy}= Shear stress exerted by the fluid, and $\dfrac{du}{dy}$= Velocity gradient perpendicular to the direction of the shear."

  water and alcohol etc, are the common examples of Newtonian fluid.[35]

Definition 2.2.5. (Non-Newtonian Fluid)
"A real fluid in which the shear stress is not directly proportional to the rate of shear strain (or velocity gradient), is known as a non-Newtonian fluid. Mathematically, it can be expressed as

$$\tau_{xy} \propto \kappa \left(\frac{du}{dy}\right)^n,$$

$$\tau_{xy} = \kappa \left(\frac{du}{dy}\right)^n,$$

where

- κ= Flow consistency coefficient, $\dfrac{du}{dy}$=Shear rate, and n= Flow behaviour index."

  Paints, blood and biological fluids etc, are good examples of Non-Newtonian fluids.[35]

2.3 Types of Fluid Flow

Fluid flow is studied in fluid Mechanics and deals with fluid dynamics. This section gives the following six types of fluid flow.

Definition 2.3.1. (Laminar and Turbulent Flow)
"Laminar flow is defined as that type of flow in which the fluid particles move along well-defined paths or stream line and all the stream-lines are straight and parallel.
Turbulent flow is that type of flow in which the fluids particles move in a zig-zag way."
For Examples:

(i) Oil transport and Blood flow in arteries are the examples of Laminar flow,

(i) Lava flow, atmosphere and ocean currents are the examples of Turbulent flow.[35]

Definition 2.3.2. (Compressible and Incompressible Flows)
"Compressible flow is that type of flow in which the density of the fluid changes from point to point or in other words the density ρ is not constant for the fluid. Mathematically,

$$\rho \neq k,$$

In-compressible flow is that type of flow in which the density is constant for the fluid. Liquids are generally in-compressible while gases are compressible. Mathematically,

$$\rho = k,$$

where k is a constant."
For examples:

- Natural gas is the example of Compressible flow and water is the example of Incompressible flow.[35]

Definition 2.3.3. (Steady and Unsteady Flows)
"If the flow characteristics such as depth of flow, velocity of flow, rate of flow at any point in open channel flow do not change with respect to time, the flow is said to be steady flow.
Mathematically,

$$\frac{\partial Q}{\partial t} = 0,$$

if at any point in open channel flow, the velocity of flow, depth of flow or rate of flow changes with respect to time, the flow is said to be unsteady. Mathematically,

$$\frac{\partial Q}{\partial t} \neq 0,$$

where, Q is any property of flow."
For examples:

(*a*) Pipes, nozzles, diff-users, and pumps(steady flow),

(*b*) Tidal effects, passage of a flood wave(unsteady flow).[35]

Definition 2.3.4. (Uniform and Non-uniform Flows)
"If the flow velocity is assumed to have the same speed and direction at every point within the fluid, it is said to be uniform.
If at a given instant, the velocity is not the same at every point, the flow is non-uniform."
For examples:

(*i*) Flow of water in a pipe of constant diameter at constant velocity is the example of uniform flow,

(*ii*) Tapering pipe at either decreasing or increasing flow rate is the example of non-uniform flow.[35]

Definition 2.3.5. (One,Two and Three dimensional Flows)
"One, two or three dimensional flow refers to the number of space coordinate required to describe a flow. It appears that any physical flow is generally three-dimensional, but these are difficult to calculate and call for as much simplification as possible. This is achieved by ignoring changes to flow in any of the directions, thus reducing the complexity. It may be possible to reduce a three-dimensional problem to a two-dimensional one, even a one-dimensional one at times."[35]

Definition 2.3.6. (Rotational and Irrotational Flow)
"Flow is Rotational when the particles of fluids are all rotating about their own axis in addition to their other movements.
Irrotational flow is when the individual particles are not rotating around their axis."[35]

Definition 2.3.7. (Viscous Flows)
"When two fluid layers move relative to each other, a friction force develops between them and the slower layer tries to slow down the faster layer. This internal

resistance to flow is quantified by the fluid property viscosity, which is a measure of internal stickiness of the fluid. Viscosity is caused by cohesive forces between the molecules in liquids and by molecular collisions in gases. There is no fluid with zero viscosity, and thus all fluid flows involve viscous effects to some degree. Flows in which the frictional effects are significant are called viscous flows."[35]

2.4 Properties of Fluid

This section contains some important properties of fluid. These properties are necessary for our goals to be achieved in this dissertation.

Definition 2.4.1. (Shear Stress)
"In shear stress a force is tending to cause deformation in a material or fluid. The direction of force in this case is always parallel to the material. Shear stress can be represented as η by following relation[35]

$$\eta = \frac{F}{A}.\text{''}$$

Definition 2.4.2. (Normal Stress)
"Normal stress is the component of stress in which force acts perpendicular to the unit surface area."[35]

Definition 2.4.3. (Thermal Conductivity)
"Thermal conductivity κ is the property of a material related to its ability to transfer heat.
Mathematically , it is given by

$$\kappa = \frac{q^* \nabla L}{S \nabla T},$$

where q^* is the heat passing through a surface area S and the effect of a temperature difference ∇T over a distance is ∇L. Here L, S and ∇T are all assumed to be of unit measurement. The SI unit of thermal conductivity is $\dfrac{W}{m.b}$ and its dimension is $\left[MLT^{-1}\theta^{-1}\right]$." [35]

Definition 2.4.4. (Ratio of thermal conductivity and specific heat)
"It is the ratio of the thermal conductivity of fluid or material to the specific heat capacity of fluid or material.
Mathematical formulation is:

$$\alpha = \frac{\kappa}{\rho C_p},$$

where

- κ = Thermal conductivity of material, ρ = Density and C_p = Specific heat capacity." [35]

2.5 Properties of Heat Transfer in Fluid

Heat transfer is the branch of thermal engineering that deals with the generation, use, conversion and exchange of thermal energy between physical systems. Following are the modes of heat transfer.

Definition 2.5.1. (Conduction)
"The flow of heat transfer through liquid or solid with rapid vibration between neighboring molecules and atoms is called conduction. In other words motion of free electrons from one atom to another is known as conduction.
Mathematically, it can be written as

$$q = -\kappa A\left(\frac{\nabla T}{\nabla n}\right),$$

where κ denotes the constant of the thermal conductivity, A is the area and ∇T denotes gradient of temperature respectively."
For example:

(i) Picking up a hot cup of tea,

(ii) After a car is turned on, the engine becomes hot,

(iii) A radiator is a good example of conduction.[35]

Definition 2.5.2. (Convection)
"It is a mechanism in which heat transfer occurs due to the motion of molecules within the fluid such as air and water. A mathematical expression for convection phenomena is

$$q = hA\left(T_f - T_\infty\right),$$

where q, h, A, T_f and T_∞ denote the the rate of convection, heat transfer coefficient, the area, the temperature of the surface and the temperature away from the surface respectively."
For example:
If meat is still frozen when it's time to start cooking, it will defrost more quickly when placed under running water than if it is immersed in water. The reason is the convection, or movement of the water and its heat circulation, will transfer heat more quickly into the frozen meat than if the meat sits immersed in water and has to absorb heat energy through conduction. [35]

Convection is further categorized as free or natural, forced and mixed. An overview is as written below:

Definition 2.5.3. (Free Convection)
"It is the process, in which heat transfer is caused by the temperature differences. It effects the density of the fluids and the fluid motion is not developed by an external source. It occurs only in the presence of gravitational force and also known as free convection."
For example:
Natural convection can create a noticeable difference in temperature within a home. Often this becomes places where certain parts of the house are warmer and certain parts are cooler.[36]

Definition 2.5.4. (Forced Convection)

"It is the type of convection in which some external source is used too induced a force on the fluid's system for the transportation of heat. External source may be a pump, fan or a suction device."

For example:

(a) The sweat that our body produces is for effective heat transfer. So when the fan is off, the air around us absorbs the water vapor until its saturated. After that it stops and we start feeling more hot. So when we switch on the fan the air around us starts moving, so the air never gets saturated completely and hence the sweat keeps evaporating by absorbing our body heat and we feel cooler.

(b) Forced convection creates a more uniform and therefore comfortable temperature throughout the entire home. This reduces cold spots in the house, reducing the need to crank the thermostat to a higher temperature, or putting on sweaters.[36]

Definition 2.5.5. (Mixed Convection)

"When both natural and forced convection affect the heat transfer process at the same time, then this mechanism is called mixed convection."

For example:

A fan blowing upward on a hot plate. Since heat naturally rises, the air being forced upward over the plate adds to the heat transfer.[36]

Definition 2.5.6. (Radiation)

"Radiation is the energy transfer due to the release of photons or electromagnetic waves from a surface volume. Radiation does not require any medium to transfer heat. The energy produced by radiation is transformed by electromagnetic waves. Mathematical formulation for this phenomenon is:

$$\Bbbk = E\sigma A \left[\Delta T\right]^4 ,$$

where

(a) E is the emissivity of the scheme

(b) σ is the constant of Stephan-Boltzmann $\left(5.670 * 10^{-8}\frac{W}{m^2k^4}\right)$

(c) ΔT is the variation of the temperature,

(d) A is the area,

(e) $\Bbbk$ is the amount of heat transferred."

For examples:

(i) Electromagnetic radiation, such as radio waves, microwaves, infrared, visible light, ultraviolet, x-rays, and gamma radiation (γ).

(ii) Particle radiation, such as alpha radiation (α), beta radiation (β), proton radiation and neutron radiation (particles of non-zero rest energy).

(iii) Acoustic radiation, such as ultrasound, sound, and seismic waves (dependent on a physical transmission medium).[35]

Definition 2.5.7. (Thermal Radiation)
"The process by which heat is transferred from a body by virtue of its temperature, without the aid of any intervening medium, is called thermal radiation. Sometimes radiant energy is taken to be transported by electromagnetic waves while at other times it is supposed to be transported by particle like photons. Radiation is found to travel at the speed of light in vacuum. The term electromagnetic radiation encompasses many types of radiation such as:

(i) Short wave radiation like gamma rays, x-rays and microwave.

(*ii*) Long wave radiation like radio wave and thermal radiation. The cause for the emission of each type of radiation is different. Thermal radiation is emitted by a medium due to its temperature." [35]

Definition 2.5.8. (Thermal Diffusivity)
"The rate at which heat diffuses by conducting through a material depends on the thermal diffusivity and can be defined as:

$$\alpha = \frac{\kappa}{\rho C_p},$$

where α is the thermal diffusion, κ is the thermal conductivity, ρ is the density and C_p is the specific heat at constant pressure."[37]

Definition 2.5.9. (Mass Transfer)
"Mass transfer is the flow of molecules from one body to another when these bodies are in contact or within a system consisting of two components when the distribution of materials is not uniform."
For example:
When copper plate is placed on steel plate, some molecules from either side will diffuse into the other side. When salt is placed in a glass and water poured over it, after sufficient time the salt molecules will diffuse into water body. Usually mass transfer takes place from a location where the particular component is proportionately low. [37]

Definition 2.5.10. (Boundary layer)
"The fundamental concept of the boundary layer was suggested by L.P randtl (1904). A boundary layer is a thin layer of viscous fluid close to the solid surface of a wall in contact with a moving stream in which (within its thickness) the flow velocity varies from zero at the wall (where the flow sticks to the wall because of its viscosity) up to the start of free stream at the boundary.
In spite of its relative thinness, the boundary layer is very important for initiating processes of dynamic interaction between the flow and the body. The boundary layer determines the aerodynamic drag and lift of the flying vehicle, or the energy loss for fluid flow in channels ." [37]

Definition 2.5.11. (Joule Heating)

"It is the procedure in which heat is generated by passing an electric current through a conductor. It is also known as oh-mic heating or resistive heating" [37]

Definition 2.5.12. (Porous Medium)

"A material containing the pores in it is called porous material or a porous medium. Pores are usually filled with fluid, i.e., liquid or gases. A porous medium is often considered by its porosity." [37]

Definition 2.5.13. (Magnetic Field)

"A region around a magnetic material or a moving electric charge within which the force of magnetism acts." [37]

2.6 Dimensionless Numbers

Definition 2.6.1. (Prandtl Number)

"The prandtl number is the connecting link between the velocity field and the temperature field. The prandtl number is dimensionless Mathematically,

$$P_r = \frac{\nu}{\alpha} = \frac{\frac{\mu}{\rho}}{\frac{\kappa}{\rho C_p}} = \frac{\mu C_p}{\kappa},$$

where μ represents the dynamic viscosity, C_p the specific heat and κ stands for thermal conductivity.

This number expresses the ratio of the momentum diffusivity (viscosity) to the thermal diffusivity. It characterizes the physical properties of a fluid with convective and diffusive heat transfers. It describes, for example, the phenomena connected with the energy transfer in a boundary layer. It expresses the degree of similarity between velocity and diffusive thermal fields or, alternatively, between hydrodynamic and thermal boundary layers." [38]

Definition 2.6.2. (Nusselt Number)

"The hot surface is cooled by a cold fluid stream. The heat from the hot surface, which is maintained at a constant temperature, is diffused through a boundary layer and convected away by the cold stream. Mathematically,

$$N_u = \frac{qL}{\kappa}$$

where q stands for the convection heat transfer, L for the characteristic length and κ stands for thermal conductivity." [36]

Definition 2.6.3. (Reynolds Number)
"It is a dimensionless number which is used to clarify the different flow behaviors like turbulent or laminar flow. It helps to measure the ratio between inertial force and the viscous force.
Mathematically expressed as

$$R_e = \frac{\frac{\rho U^2}{L}}{\frac{\mu U}{L^2}} \Rightarrow R_e = \frac{LU}{\nu},$$

where U denotes the velocity of the fluid with respect to object, L the characteristics length. At low Reynolds number, laminar flow arises where the viscous forces are dominant. At high Reynolds number, Turbulent flow arises where the inertial forces are dominant." [34]

Definition 2.6.4. (Skin Friction Coefficient)
"The steady flow of an in-compressible gas or liquid in a long pipe of internal. The mean velocity is denoted by U. The skin friction coefficient can be defined as

$$C_f = \frac{2\tau_w}{\rho U^2},$$

where

- τ_w denotes the wall shear stress, ρ is the density, and U is the free stream velocity." [39]

Definition 2.6.5. (Porosity Parameter)

"Porosity is an important parameter for the characterization of materials. It corresponds to the volume of interstice that can contain fluid, related to the total volume of the material.The description of the pore system can be refined by considering the pore size distribution." [34]

Definition 2.6.6. (Brinkmann Parameter)

"It is the ratio between heat produced by viscous dissipation and heat transported by molecular conduction. i.e., the ratio of viscous heat generation to external heating. The higher its value, the slower the conduction of heat produced by viscous dissipation and hence the larger the temperature rise.

$$\mathrm{Br} = \frac{\mu u^2}{\kappa(T_w - T_0)} = \mathrm{Pr}\,\mathrm{Ec},$$

where,

(*i*) μ is the dynamic viscosity,

(*ii*) u is the flow velocity,

(*iii*) κ is the thermal conductivity,

(*iv*) T_0 is the bulk fluid temperature,

(*v*) T_w is the wall temperature,

(*vi*) P_r is the prandtl number,

(*vii*) E_c is the Eckert number." [34]

Definition 2.6.7. (Suction Parameter)

"Suction is the force that a partial vacuum exerts upon a solid, liquid, or a gas. When the pressure in one part of a system is reduced relative to another, the fluid in the higher pressure region will exert a force relative to the region of lowered pressure." [34]

Definition 2.6.8. (Eckert Number)

"The Eckert number (E_c) is a dimensionless number used in continuum mechanics. It expresses the relationship between a flow's kinetic energy and the boundary layer enthalpy difference, and is used to characterize heat transfer dissipation. It is defined as

$$\mathrm{Ec} = \frac{u^2}{C_p \Delta T} = \frac{\text{Advective Transport}}{\text{Heat Dissipation Potential}},$$

where,

(*i*) μ is the local flow velocity of the continuum,

(*ii*) C_p is the constant-pressure local specific heat of the continuum,

(*iii*) ΔT is the difference between wall temperature and local temperature." [34]

Definition 2.6.9. (Velocity Slip Parameter)

"Velocity slip parameter γ on the radial velocity profiles. It indicates that the slip parameter has a significant effect on radial velocity distributions; there is a peak for the radial velocity profiles (maximum) that decreases rapidly and moves to the disk as the slip parameter γ increases." [34]

2.7 Generalized Governing Laws for Fluid Motion

In this section some basic laws are discussed which are necessary for the further discussion. In later part of this section generalized equations such as continuity

equation, momentum equation and energy equation are presented.
Several conservation laws such as the laws of conservation of mass, conservation of energy and conservation of momentum are of great use for the research community. Historically, the conservation laws were first applied to a fixed quantity of matter called a closed system or just a system, and then extended to regions in space called control volumes. The conservation relations are also called balance equations since any conserved quantity must balance during a process.

2.8 Continuity Equation

In the conservation of mass of fluid entering and leaving the control volume, the resulting mass balance is called the equation of continuity. This equation reflects the fact that mass is conserved.
For any fluid, conservation of mass is expressed by scalar equation:

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho V) = 0. \tag{2.1}$$

For the steady flow (2.1) can be written as:

$$\nabla.(\rho V) = 0. \tag{2.2}$$

For incompressible flow (2.2) becomes:

$$\nabla.V = 0. \tag{2.3}$$

For incompressible and irrotational flow, the (2.3) is transformed in terms of velocity potential ϕ, which is given by:

$$\nabla^2 \phi = 0. \tag{2.4}$$

(2.4) is known as Laplace equation." [35]

2.9 Conservation of Momentum

The product of the mass and the velocity of a body is called the linear momentum or just the momentum of the body, and the momentum of a rigid body of mass m moving with a velocity V is mV. Newtons second law states that the acceleration of a body is proportional to the net force acting on it and is inversely proportional to its mass, and that the rate of change of the momentum of a body is equal to the net force acting on the body. Therefore, the momentum of a system remains constant when the net force acting on it is zero, and thus the momentum of such systems is conserved. This is known as the conservation of momentum principle. Mathematically, it can be written as:

$$\frac{\partial}{\partial t}(\rho u) + \nabla.[(\rho u)u] = \nabla.T + \rho g,"$$

where

- u = velocity, ρ = Density, g = acceleration, T= Time. [35]

2.10 Law of Conservation of Energy

The law of conservation of energy states that the time rate of change of the total energy is equal to the sum of the rate of work done by the applied forces and change of heat content per unit time.

$$\frac{\partial \rho}{\partial t} + \nabla.\rho u = -\nabla.q + Q + \phi,$$

where ϕ is a dissipation function." [35]

2.11 Solution Methodlogy

Shooting method used to solved the higher order nonlinear ordinary differential equation. To implement this technique, first convert the higher order ODEs to the system of first order ODEs. In the shooting method, first we assume the missing initial conditions and differential equation are then integrated numerically through Runge-Kutta method as an initial value problem. The accuracy of the assumed missing initial condition is then checked by comparing the calculated values of the dependent variables at the terminal point with their given value there. If the boundary conditions are not fulfilled upto the required accuracy, with the new set of initial conditions, which are modified by Newton,s method. The method is repeated again until the required accuracy is achieved. [35]

Chapter 3

Boundary Layer Flow through Porous Medium with Slip Conditions

3.1 Introduction

In this Chapter the detailed review of the article [1] "Boundary Layer Flow through Darcy Brinkman Porous Medium in the Presence of Slip Effects and Porous Dissipation" is presented. The goal of this research is to look at the Darcy Brinkman flow across a permeable stretched sheet in the presence of viscous and porous dissipation at different speeds and temperatures. The conversion of non-linear PDE's describing the proposed flow problem to a set of ODEs has been carried out by employing appropriate similarity transformation. Shooting method is incorporated for the solution of the proposed flow equations. The impact of flow parameters on the non-dimensional temperature and velocity profiles has been demonstrated by the aid of tables and graph. The limiting case of the present study affirms that the obtained numerical result reflect a very good agreement with those from open literature.

3.2 Mathematical Modeling

Consider the flow across a porous media with a permeable stretching surface. In Cartesian dimensions, the $x - axis$ and $y - axis$ are perpendicular to the sheet, which is being stretched with velocity $U_s = \alpha x$.
The temperature of the sheet and the ambient temperature are $T_s = T_\infty + cx^2$, and $T_s > T_\infty$. The governing equations for the boundary layer assumption in the presence of viscous dissipation are,
Continuity Equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{3.1}$$

Momentum Equation:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\epsilon^2 \mu_e}{\rho}\left(\frac{\partial^2 u}{\partial^2 y}\right) - \frac{\mu \epsilon^2}{\rho K^*}u. \tag{3.2}$$

Energy Equation:

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{k}{\rho C_p}\left(\frac{\partial^2 T}{\partial^2 y}\right) + \frac{\epsilon^2}{\rho C_p}\left[\mu_e\left(\frac{\partial^2 u}{\partial^2 y}\right) + \frac{\mu u^2}{K^*}\right]. \tag{3.3}$$

along with boundary conditions

$$\left.\begin{array}{llllll} u = \alpha x + \beta_1\left(\frac{\partial u}{\partial y}\right), & v = -V_0, & T = T_s + \delta_1\left(\frac{\partial T}{\partial y}\right) & \text{at} & y = 0, \\ u \to 0, & T \to T_\infty & \text{at} & & y \to \infty & . \end{array}\right\} \tag{3.4}$$

Here u and v components of velocity along the x and y directions, respectively.
Introducing the similarity transformation

$\xi = \sqrt{\frac{\alpha}{v}}y,$ $\qquad u = \alpha x g'(\xi),$

$v = -\sqrt{\alpha v}g(\xi),$ $\qquad \theta(\xi) = \frac{T - T_\infty}{T_s - T_\infty}.$

The detailed procedure for the conversion of (3.1)-(3.3)into ordinary differential equations has been described in the upcoming discussion.

$\frac{\partial u}{\partial x} = \alpha g'(\xi).$

$\frac{\partial v}{\partial y} = -\alpha g^{'}(\xi).$

Using the above derivation Continuity equation is trivially satisfied:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \alpha g^{'}(\xi) - \alpha g^{'}(\xi) = 0. \tag{3.5}$$

To exhibit the procedure for the conversion of momentum equation (3.2) the dimensionless form following derivatives are evaluated:

- $u = \alpha x g^{'}(\xi)$
  $u\frac{\partial u}{\partial x} = \alpha^2 x (g^{'})^2(\xi).$

- $v = -\sqrt{\alpha v} g(\xi)$
  $v\frac{\partial u}{\partial y} = -\alpha^2 x g g^{''}(\xi).$

- $\frac{\partial u}{\partial y} = \alpha x g^{''}(\xi)\frac{\partial \xi}{\partial y}$
  $\frac{\partial u}{\partial y} = \alpha x g^{''}(\xi)\sqrt{\frac{\alpha}{v}}.$

- $\frac{\partial^2 u}{\partial y^2} = \alpha x g^{'''}(\xi)\sqrt{\frac{\alpha}{v}}\sqrt{\frac{\alpha}{v}}$
  $\frac{\partial^2 u}{\partial y^2} = \frac{\alpha^2 x}{v} g^{'''}(\xi).$

Hence the dimensionless form of (3.2) becomes

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\epsilon^2 \mu_e}{\rho}\left(\frac{\partial^2 u}{\partial^2 y}\right) - \frac{\mu \epsilon^2 u}{\rho K^*}$$

$$\Rightarrow \quad \alpha^2 x g^{'} - \alpha^2 x g g^{''} = \frac{\epsilon^2 \mu_e}{\rho}\left(\frac{\alpha^2 x}{v} g^{''} - \frac{\mu \epsilon^2 \alpha x}{\rho K^*} g^{'}\right)$$

$$\Rightarrow \quad \alpha^2 x (g^{'2} - g g^{''}) = \alpha^2 x \left(\frac{\epsilon^2 \mu_e}{\frac{\mu}{v} v} g^{''} - \frac{\mu \epsilon^2}{\rho \alpha K^*} g^{'}\right)$$

$$\Rightarrow \quad \gamma g^{'''} - (g^{'})^2 + g g^{''} - P_m g^{'} = 0. \tag{3.6}$$

The conversion of energy equation (3.3) into dimensionless form is clarified by carrying out the following calculations:

- $T = (T_s - T_\infty)\theta + T_\infty,$
  $T_s = T_\infty + cx^2,$
  $T = (T_s + cx^2 - T_\infty)\theta + T_\infty,$
  $T = cx^2\theta + T_\infty.$

- $\frac{\partial T}{\partial x} = \frac{\partial}{\partial x}(cx^2\theta) + T_\infty$
  $\frac{\partial T}{\partial x} = 2cx\theta.$

- $\frac{\partial T}{\partial y} = \frac{\partial}{\partial y}(cx^2\theta + T_\infty)$
  $= cx^2\theta' \frac{\partial \xi}{\partial y}$
  $= cx^2\theta' \sqrt{\frac{\alpha}{\upsilon}}.$

- $\frac{\partial^2 T}{\partial y^2} = cx^2\theta'' \sqrt{\frac{\alpha}{\upsilon}}\sqrt{\frac{\alpha}{\upsilon}}$
  $= cx^2\theta'' \frac{\alpha}{\upsilon}.$

- $\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(\alpha x g'(\xi))$
  $= \alpha x g' \sqrt{\frac{\alpha}{\upsilon}}.$

- $u\frac{\partial T}{\partial x} = \alpha 2cx\theta x g'(\xi)$
  $= 2\alpha(x^2)cg'\theta.$

- $v\frac{\partial T}{\partial y} = -\sqrt{\alpha\upsilon} g cx^2\theta' \sqrt{\frac{\alpha}{\upsilon}}$
  $= -\alpha cx^2\theta' g.$

Incorporating all the above derivatives the energy equation (3.3) takes the form:

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{k}{\rho C_p}\left(\frac{\partial^2 T}{\partial^2 y}\right)+\frac{\epsilon^2}{\rho C_p}\left[\mu_e\left(\frac{\partial u}{\partial y}\right)^2+\frac{\mu u^2}{K*}\right]$$

$$\begin{aligned}2c\alpha x^2g^{'}\theta-\alpha cx^2g\theta^{'}&=\frac{k}{\rho C_p}(cx^2\theta^{''}\frac{\alpha}{\upsilon})+\frac{\epsilon^2}{\rho C_p}\left[\mu_e\left(\alpha xg^{''}\sqrt{\frac{\alpha}{\upsilon}}\right)^2+\frac{\mu\left(\alpha xg^{'}\right)^2}{K*}\right]\\
&=\alpha cx^2\left(\frac{k}{\frac{\mu}{\upsilon}}C_p\upsilon\right)\theta^{''}+\frac{\epsilon^2}{\rho C_p}\left[\mu_e\alpha^2x^2\frac{\alpha}{\upsilon}(g^{''})^2+\frac{\mu\left(\alpha xg^{'}\right)^2}{K^*}\right]\\
&=\alpha cx^2\frac{1}{P_r}\theta^{''}+\frac{\alpha^3x^2}{C_p}\left[\frac{\epsilon^{2\mu_e}}{\rho\upsilon}\left(g^{''}\right)^2+\frac{\epsilon^2\mu\left(\alpha xg^{'}\right)^2}{\rho K^*}\right]\\
&=\alpha cx^2\left(\frac{1}{P_r}\right)\theta^{''}+\frac{\alpha^2}{cC_p}\left(\frac{\epsilon^2\mu_e}{\frac{\mu}{\upsilon}}\upsilon(g^{''})^2+\frac{\epsilon^2\mu(g^{'})^2}{\rho\alpha K*}\right)\\
&=\alpha cx^2\left(\frac{1}{P_r}\theta^{''}\right)+E_c\left[\gamma(g^{''})^2+P_m(g^{'})^2\right]\end{aligned}$$

$$\Rightarrow\quad \alpha cx^2(2g^{'}\theta-g\theta^{'})=\alpha cx^2\left(\frac{1}{P_r}\theta^{''}\right)+E_c\left[\gamma(g^{''})^2+P_m(g^{'})^2\right]$$

$$\Rightarrow\quad \frac{1}{P_r}\theta^{''}-2g^{'}\theta-g\theta^{'}+E_c\left[\gamma(g^{''})^2+P_m(g^{'})^2\right]=0.$$

$$\Rightarrow\quad \frac{1}{P_r}\theta^{''}-2g^{'}\theta-g\theta^{'}+E_c\left[\gamma(g^{''})^2+P_m(g^{'})^2\right]=0. \tag{3.7}$$

3.3 Boundary Conditions

The procedure for the conversion of boundary conditions into dimensionless form is described below:

- $u=\alpha x+\beta_1(\frac{\partial u}{\partial y})$ at $y=0$

  $\xi=0$ at $y=0$

  $\frac{\partial u}{\partial y}=\alpha xg^{''}(\xi)\sqrt{\frac{\alpha}{\upsilon}}$

  $\beta_1=\frac{\beta}{\sqrt{\frac{\alpha}{\upsilon}}}$

  substituting the value of β_1 and $\frac{\partial u}{\partial y}$ in above equation yields:

$$u = \alpha x + \frac{\beta}{\sqrt{\frac{\alpha}{\upsilon}}}\alpha x\sqrt{\frac{\alpha}{\upsilon}}g^{''}(\xi)$$
$$u = \alpha x + \alpha x\beta g^{''}(\xi)$$
$$\Rightarrow \quad \alpha x g^{'}(\xi) = \alpha x(1 + \beta g^{''}(\xi))$$
$$\Rightarrow \quad g^{'}(0) = 1 + \beta g^{''}(0) \quad \text{at} \quad \xi = 0.$$

- $$T = T_s + \delta_1(\frac{\partial T}{\partial y}) \quad \text{at} \quad y = 0$$
  $$\xi = 0 \qquad \text{at} \qquad y = 0$$
  $$\frac{\partial T}{\partial y} = cx^2\theta^{'}\sqrt{\frac{\alpha}{\upsilon}}$$
  $$\delta_1 = \frac{\delta}{\sqrt{\frac{\alpha}{\upsilon}}}$$
  substitute the values of δ_1 and $\frac{\partial T}{\partial y}$ in above equation then
  $$cx^2\theta(\xi) + T_\infty = cx^2 + T_\infty + \frac{\delta}{\sqrt{\frac{\alpha}{\upsilon}}}cx^2\theta^{'}\sqrt{\frac{\alpha}{\upsilon}}$$
  $$\Rightarrow \quad cx^2\theta(\xi) = cx^2 + cx^2\delta\theta^{'}(\xi)$$
  $$\Rightarrow \quad cx^2\theta(\xi) = cx^2(1 + \delta\theta^{'}(\xi))$$
  $$\Rightarrow \quad \theta(0) = 1 + \delta\theta^{'}(0) \quad \text{at} \quad \xi = 0.$$

- $$\upsilon = -V_0 \quad \text{at} \quad y = 0$$
  $$-\sqrt{\alpha\upsilon g(\xi)} = -V_0$$
  $$g(0) = -\frac{V_0}{-\sqrt{\alpha\upsilon}} \quad \text{at} \quad \xi = 0.$$

Now considering the right boundary i.e,

$$\xi \longrightarrow \infty \quad \text{at} \quad y \to \infty$$
$$u \longrightarrow 0$$

- $$u = \alpha x g^{'}(\xi)$$
  $$0 = \alpha x g^{'}(\infty)$$

$$\Rightarrow \quad g^{'}(\xi) \longrightarrow 0 \quad \text{at} \quad (\xi) \to \infty.$$

- $$T \to T_\infty$$
  $$T = T_\infty + cx^2\theta(\xi)$$
  $$T_\infty = T_\infty + cx^2\theta(\xi)$$

$\Rightarrow \quad cx^2\theta(\xi) = 0$

$\Rightarrow \quad \theta(\infty) \longrightarrow 0 \quad \text{at} \quad (\xi) \rightarrow \infty.$

The ultimate dimensionless form of the governing model is:

$$\gamma g^{'''} - (g^{'})^2 + gg^{'} - P_m g^{'} = 0. \tag{3.8}$$

$$\frac{1}{P_r}\theta^{''} + g\theta^{'} - 2g^{'}\theta + E_c\left[\gamma(g^{''})^2 + P_m g^{'}\right] = 0. \tag{3.9}$$

along with the boundary conditions:

$$\left.\begin{array}{lll} g(0) = S, & g^{'}(0) = 1 + \beta g^{''}(0), & g^{'}(\infty) = 0, \\ \theta(0) = 1 + \delta\theta^{'}(0), & \theta(\infty) = 0. & \end{array}\right\} \tag{3.10}$$

3.4 Dimensionless Constant

Skin friction

- $$S_{fx} = \frac{\mu}{\rho U_s^2}\left(\frac{\partial T}{\partial y}\right) \quad \text{at} \quad y = 0$$
$$= \frac{\mu}{\rho\alpha x^2}\alpha x\sqrt{\frac{\alpha}{\upsilon}}g^{''}(0)$$
$$= \frac{\mu}{\rho\alpha x}\sqrt{\frac{\alpha}{\upsilon}}g{''}(0)$$
$$= \frac{\mu}{\frac{\mu}{\upsilon}}\alpha x\sqrt{\frac{\alpha}{\upsilon}}g^{''}(0)$$
$$= \frac{\sqrt{\upsilon}\sqrt{\upsilon}}{\sqrt{\alpha}\sqrt{\alpha}}x\sqrt{\frac{\alpha}{\upsilon}}g^{''}(0)$$
$$\frac{\sqrt{\alpha}}{\sqrt{\upsilon}}x = g^{''}(0)$$

Finally,

$$\Rightarrow \quad S_{fx}R_e x^{\frac{1}{2}} = g^{''}(0). \tag{3.11}$$

Nusselt number

- $$\begin{aligned} N_{Rx} &= -\frac{x}{T_s - T_\infty}\left(\frac{\partial T}{\partial y}\right) \quad \text{at} \quad y=0 \\ &= -\frac{x}{T_\infty + cx^2 - T_\infty}\left(\frac{\partial T}{\partial y}\right)_{y=0} \\ &= -\frac{x}{cx^2}\left(cx^2\theta'\sqrt{\frac{\alpha}{\upsilon}}\right) \\ &= -x\theta'\sqrt{\frac{\alpha}{\upsilon}} \end{aligned}$$

$$\frac{1}{x\sqrt{\frac{\alpha}{\upsilon}}} = -\theta'(0)$$

Hence,

$$\Rightarrow \quad N_{Rx} R_e x^{\frac{-1}{2}} = \theta'(0). \tag{3.12}$$

Following table gives some dimensionless physical parameters:

TABLE 3.1: Dimensionless Physical Parameters

Physical Parameters with No Dimensions	Notations	Definitions
Brinkmann parameter	γ	$\varepsilon^2 \frac{\mu_e}{\mu}$
Porosity parameter	P_m	$\frac{\mu\varepsilon^2}{\rho\alpha K^*}$
Suction/injection parameter	S	$\frac{V_0}{\sqrt{\alpha\nu}}$
Prandtl number	P_r	$\frac{\mu C_p}{\kappa}$
Eckert number	E_c	$\frac{\alpha^2}{cC_p}$
Velocity slip parameter	β	$\beta_1\sqrt{\frac{\alpha}{\nu}}$
Thermal slip parameter	δ	$\delta_1\sqrt{\frac{\alpha}{\nu}}$

3.5 Solution Methodology

To obtain the numerical solution of equations (3.8) and (3.9) subject to boundary conditions, shooting method is used. One can easily observe that (3.8) is independent of g so it can be solved seperately first solution of this equation is pluged into the second equation later. Let us use the following notations:

$$g = y_1, \qquad g' = y_1' = y_2, \qquad g'' = y_2' = y_3, \qquad g''' = y_3'.$$

Following system of ordinary differential equations is obtained by using above notations for (3.8) :

$$\left.\begin{aligned} y_1' &= y_2, & y_1(0) &= s, \\ y_2' &= y_3, & y_2(0) &= 1 + \beta y_3(0), \\ y_3' &= \frac{1}{\gamma}\left[y_2^2 - y_1 y_3 + P_m y_2\right], & y_3(0) &= t. \end{aligned}\right\} \tag{3.13}$$

In order to achieve approximate numerical results, (3.13) is solved by RK-4 method. The domain of our problem is considered to be bounded i.e. $[0, \eta_\infty]$, where η_∞ is a positive number for which the variation in the fluid property is negligible after $\eta = \eta_\infty$. Assumed missing initial condition t is such that:

$$y_2(\eta_\infty, t) = 0.$$

Newton method is used to solve above non linear algebraic equation and solution will provide the missing initial condition t. Further we start the iteration by $t = t^{(0)}$

$$t^{(i+1)} = t^{(i)} - \left[\left(\frac{\partial y_2}{\partial t}\right)^{-1}(y_2(\eta, t))\right]_{t=t^{(0)}}$$

To incorporate Newton,s method, we further use the following notations:

$$\frac{\partial y_1}{\partial t} = y_4, \quad \frac{\partial y_2}{\partial t} = y_5, \quad \frac{\partial y_3}{\partial t} = y_6. \tag{3.14}$$

As a result of these new notations, the Newton,s iterative scheme gets the form:

$$t^{(i+1)} = t^{(i)} - \left[\left(\frac{\partial y_5}{\partial t}\right)^{-1}(y_2(\eta, t))\right]_{t=t^{(i)}},$$

where i is the number of iterations $(i = 0, 1, 2, 3...)$. Now differentiating the system of three first order ordinary differential equations (3.16) with respect to t, we get three more equations. Following system of initial value problems is obtained:

$$
\begin{aligned}
y_1' &= y_2, & y_1(0) &= 0,\\
y_2' &= y_3, & y_2(0) &= 1 + \beta y_3(0),\\
y_3' &= \frac{1}{\gamma}\left[y_2^2 - y_1 y_3 + P_m y_2\right], & y_3(0) &= t,\\
y_4' &= y_5, & y_4(0) &= 0,\\
y_5' &= y_6, & y_5(0) &= \beta,\\
y_6' &= \frac{1}{\gamma}\left[2y_2 y_5 - y_1 y_6 - y_3 y_4 + P_m y_5\right], & y_6(0) &= 1.
\end{aligned}
$$

The required stopping criteria for shooting method is set as follows

$$|y_2(\eta, t)| < \epsilon,$$

where ϵ is finitely small positive number up to 10^{-8}.

Now considering equation (3.9) which is coupled in g and θ. We will use shooting technique once again to solve this equation. This time we will incorporate the solution of g in (3.9) appropriately to get solution for θ. For this purpose the following notation have been taken in account.

$$
\begin{aligned}
\theta &= y_1, & \theta' &= y_1' = y_2, & \theta'' &= y_2',\\
g &= d, & g' &= d_1, & g'' &= d_2.
\end{aligned}
$$

As a result the following system of ODEs is obtained:

$$
\left.
\begin{aligned}
y_1' &= y_2, & y_1(0) &= 1 + G y_2(0),\\
y_2' &= P_r\left[2d_1 y_1 - d y_2 - E_c(\gamma d_2^2 + P_m d_1^2)\right], & y_2(0) &= r.
\end{aligned}
\right\} \quad (3.15)
$$

In order to achieve approximate numerical results, (3.15) is solved by RK-4 method. The domain of our problem is considered to be bounded i.e. $[0, \eta_\infty]$, where η_∞ is a positive number and for which the variation in the solution is negligible after $\eta = \eta_\infty$. Assumed missing initial condition r is such that:

$$y_2(\eta_\infty, r) = 0.$$

Newton,s method is used to refine the value of missing initial condition t. We start the iteration by $r = r^{(0)}$ in the following formula:

$$r^{(i+1)} = r^{(i)} - \left[\left(\frac{\partial y_2}{\partial r} \right)^{-1} (y_2(\eta, r)) \right]_{r=r^{(0)}},$$

To incorporate Newton method, we further use the following notations:

$$\frac{\partial y_1}{\partial r} = y_3, \quad \frac{\partial y_2}{\partial r} = y_4. \tag{3.16}$$

As a result of these new notations, the Newton,s iterative scheme gets the form:

$$r^{(i+1)} = r^{(i)} - \left[\left(\frac{\partial y_4}{\partial r} \right)^{-1} (y_2(\eta, r)) \right]_{r=r^{(i)}},$$

where i is the number of iterations $(i = 0, 1, 2, 3...)$. Now differentiating the system of three first order ODEs (3.16) with respect to t, we get another system of ODEs, of first order. Hence following system of IVPs is obtained:

$$\begin{aligned}
y_1^{'} &= y_2, & y_1(0) &= 1 + Gy_2(0),\\
y_2^{'} &= P_r \left[2d_1 y_1 - dy_2 - E_c(\gamma d_2^2 + P_m d_1^2) \right], & y_2(0) &= r,\\
y_3^{'} &= y_4, & y_3(0) &= G,\\
y_4^{'} &= P_r \left[2d_1 y_3 - dy_4 \right], & y_4(0) &= 1.
\end{aligned}$$

The required stopping criteria for shooting method is set as follows

$$|y_2(\eta, r)| < \epsilon,$$

where ϵ is finitely small positive number up to 10^{-8}.

3.6 Validation of Codes

For validation of the numerical code Tables 3.2 and 3.3 have been presented and the result compared with the results obtained by in built MATLAB routine bvp4c. [1]

TABLE 3.2: Skin friction coefficient $S_{fx}Re_x^{\frac{1}{2}} = -g''(0)$ for slip case $\beta = 1.0$. By using shooting method.

Physical Parameters			$S_{fx}Re_x^{\frac{1}{2}} = -g''(0)$	
γ	S	P_m	bvp4c Salman, et al. [1]	Present value
1.0	1.0	0.5	0.610511	0.610503
2.0	1.0	0.5	0.500008	0.500008
3.0	1.0	0.5	0.439566	0.439510
0.5	0.0	0.3	0.550438	0.550422
0.5	1.0	0.3	0.712228	0.712119
0.5	2.0	0.5	0.808872	0.808832
2.0	0.5	0.0	0.406493	0.406483
2.0	0.5	0.4	0.452006	0.452000
2.0	0.5	0.8	0.485908	0.485889

3.7 Results and Discussion

This section is dedicated for elaboration of effects of some important physical parameters on velocity profile $g'(\xi)$ and temperature profile $\theta(\xi)$. In the present survey, the shooting method has been opted for reproducing the solution of $g'(\xi)$ and $\theta(\xi)$. The results are presented in different tables and graphs.

FIGURE 3.1 depicts the effect of Brinkman viscosity ratio parameter γ on velocity profile $g'(\xi)$. It shows that velocity increases on increasing the values of Brinkman viscosity ratio parameter. This also makes the sense because the

TABLE 3.3: Local Nusselt number $N_{Rx}Re_x^{\frac{-1}{2}} = -\theta'(0)$ when $\beta = 1.0$ and $S = 0.5$. By shooting method

Physical Parameters					$N_{Rx}Re_x^{\frac{-1}{2}} = -\theta'(0)$	
P_r	E_c	δ	P_m	γ	shooting method	bvp4c
0.7	0.5	1.0	0.4	2.0	0.456141	0.0506995
1.2	0.5	1.0	0.4	2.0	0.538161	0.5000024
6.8	0.5	1.0	0.4	2.0	0.738928	0.06060773
3.0	0.0	1.0	0.4	2.0	0.738024	0.04100318
3.0	0.6	1.0	0.4	2.0	0.642319	0.06060773
3.0	1.2	1.0	0.4	2.0	0.546560	0.06060773
3.0	1.0	0.0	0.4	2.0	2.208602	0.06060773
3.0	1.0	0.6	0.4	2.0	0.820808	0.06060773
3.0	1.0	1.2	0.4	2.0	0.504071	0.06060773
3.0	1.0	1.0	0.0	2.0	0.640207	0.16773413
3.0	1.0	1.0	0.5	2.0	0.566207	0.08500874
3.0	1.0	1.0	1.0	2.0	0.517044	0.03473116
3.0	1.0	1.0	0.4	2.0	0.619665	0.04087228
3.0	1.0	1.0	0.4	2.0	0.578480	0.06060773
3.0	1.0	1.0	0.4	2.0	0.546450	0.10072825

brinkman viscosity ratio number appears with the velocity gradient term in the momentum equation, consequently large values of brinkman viscosity parameter increases the velocity.

FIGURE 3.2 portrays the effects of porosity parameter P_m on velocity profile. It was noticed the velocity and momentum boundary layer decreases by increasing porosity parameter.

FIGURE 3.3 **and** 3.4 we can see that, by increasing the suction parameter S and slip parameter β , the velocity of fluid decreases and momentum boundary layer becomes thinner.

FIGURE 3.5 demonstrate how the Brinkman viscosity number affects the temperature profile $\theta(\xi)$. The rising function of γ is the thermal boundary layer.

FIGURE 3.6 **and** (3.7) increase in the suction velocity S and porosity parameter P_m decreases the fluid temperature with in the boundary layer.

FIGURE 3.8 describe how the Prandtl number affects the temperature field $\theta(\xi)$. The Prandtl number Pr will increase the temperature inside the boundary layer drops as the temperature outside drops. This is because increasing the prandtl number one lowers the fluid's heat conductivity. As a result of the stretching, the heat transfer rate lowers, and the boundary layer shrinks.

FIGURE 3.9 the Eckert number Ec has an impact on the temperature profile. As expected, the thermal boundary layer grows since the Eckert number Ec rises, as the Eckert number increases fluid friction between neighbouring layers, converting kinetic energy to heat energy.

FIGURE 3.10 illustrates that increase in the thermal slip parameter δ, reduces the temperature and thermal boundary layer.

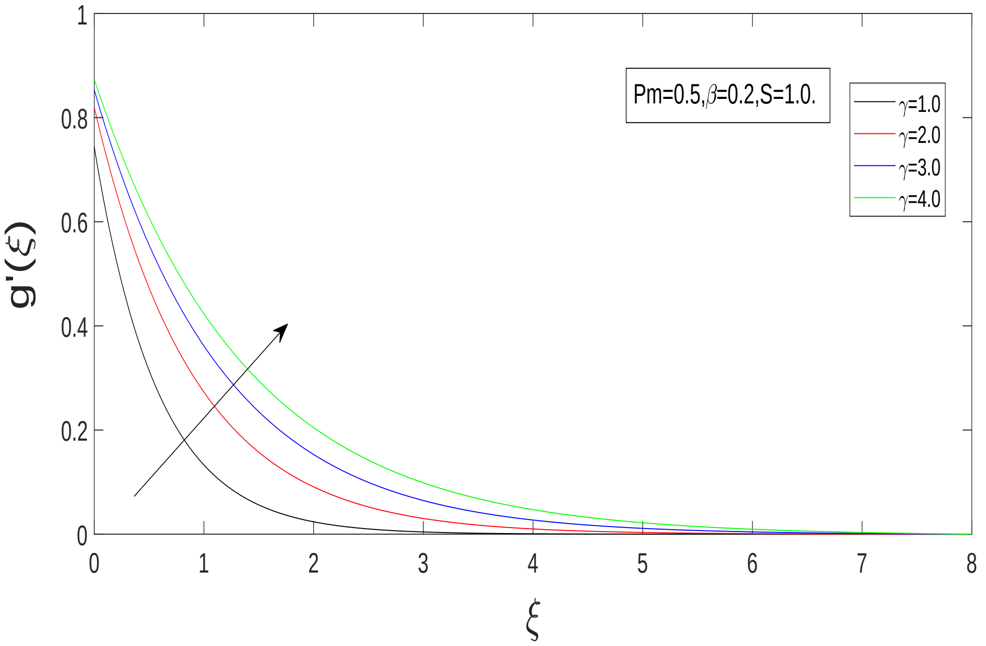

FIGURE 3.1: Variation of velocity as a function of viscosity ratio γ

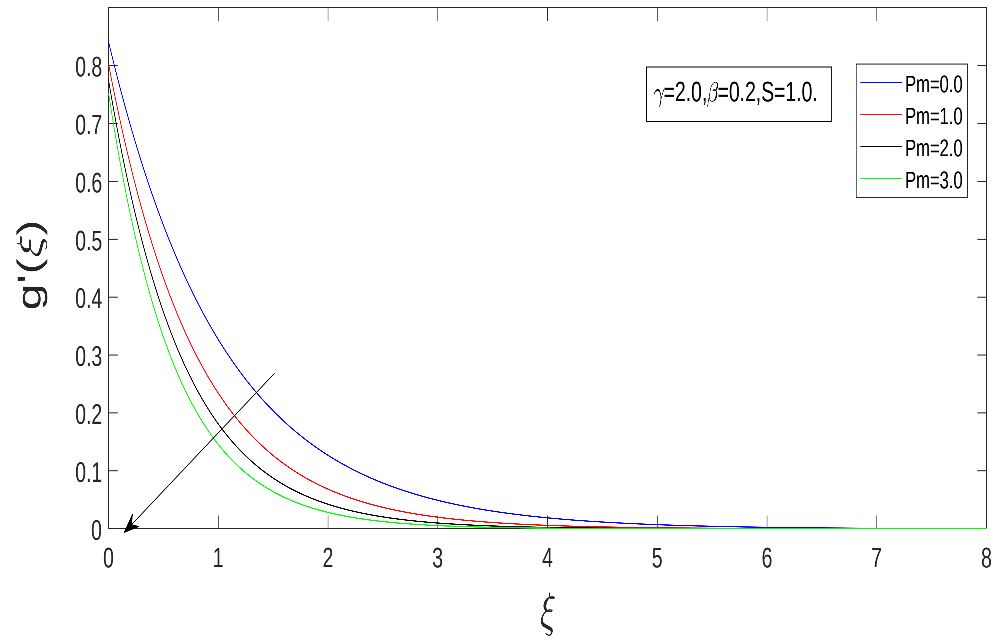

FIGURE 3.2: Variation of velocity with porosity parameter P_m

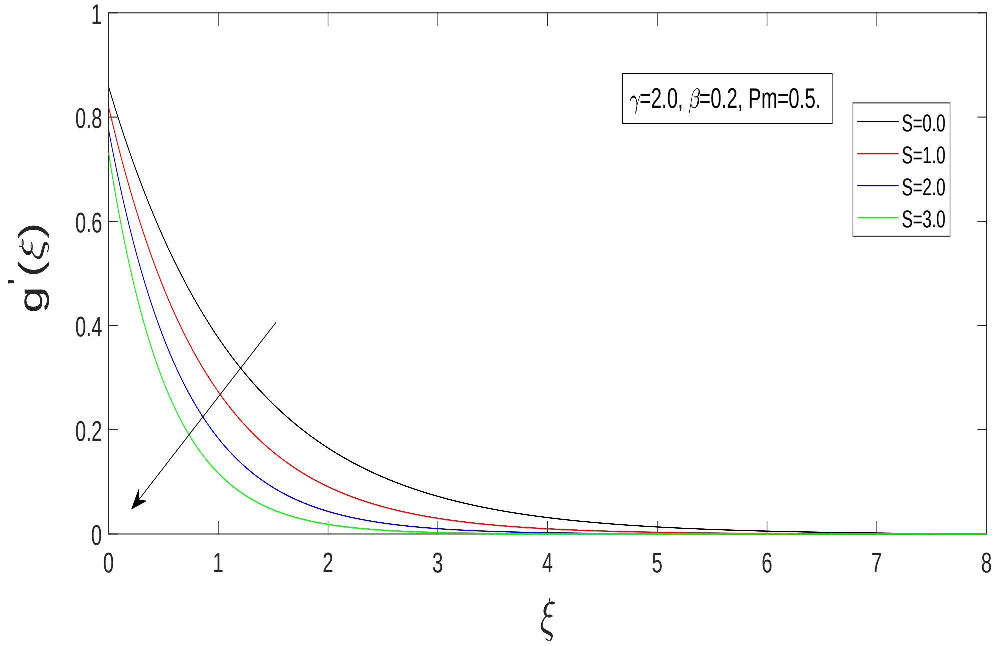

FIGURE 3.3: Variation of velocity with suction parameter S

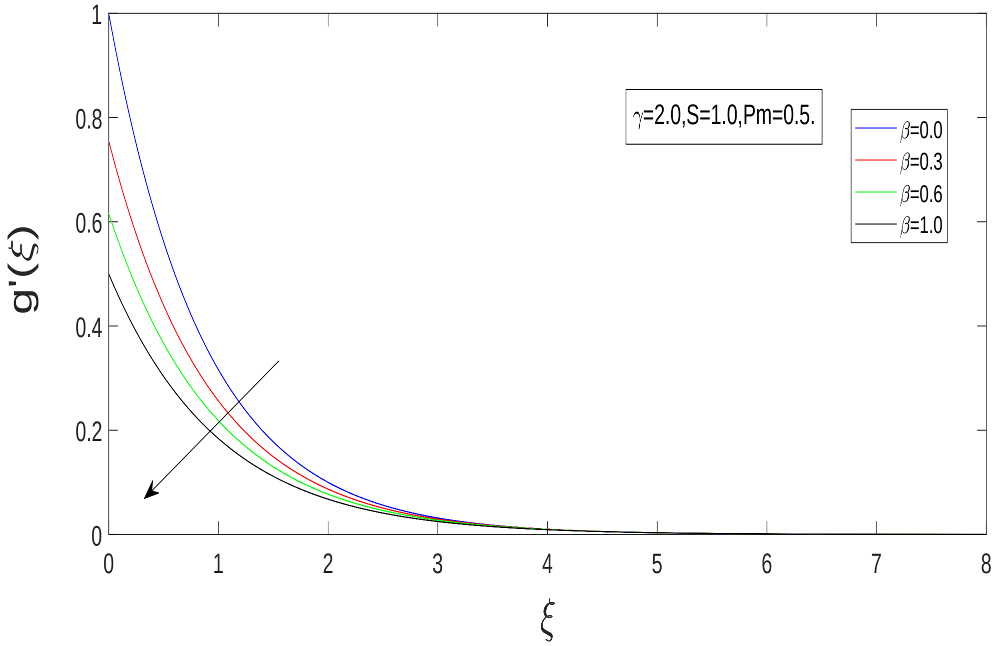

FIGURE 3.4: Variation in velocity as a function of the velocity slip parameter β

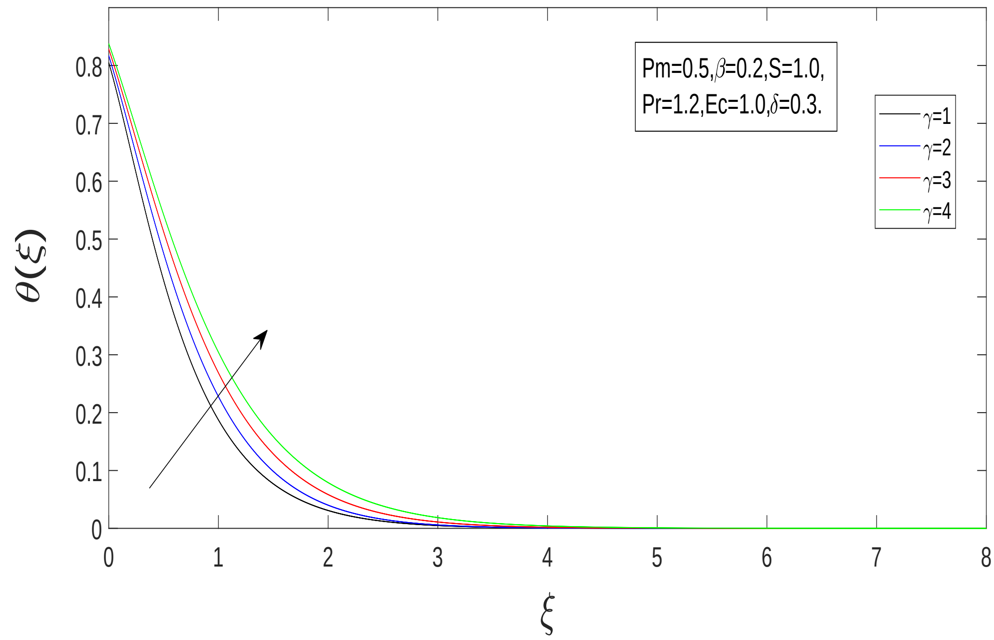

FIGURE 3.5: Temperature variation using the brinkman parameter γ

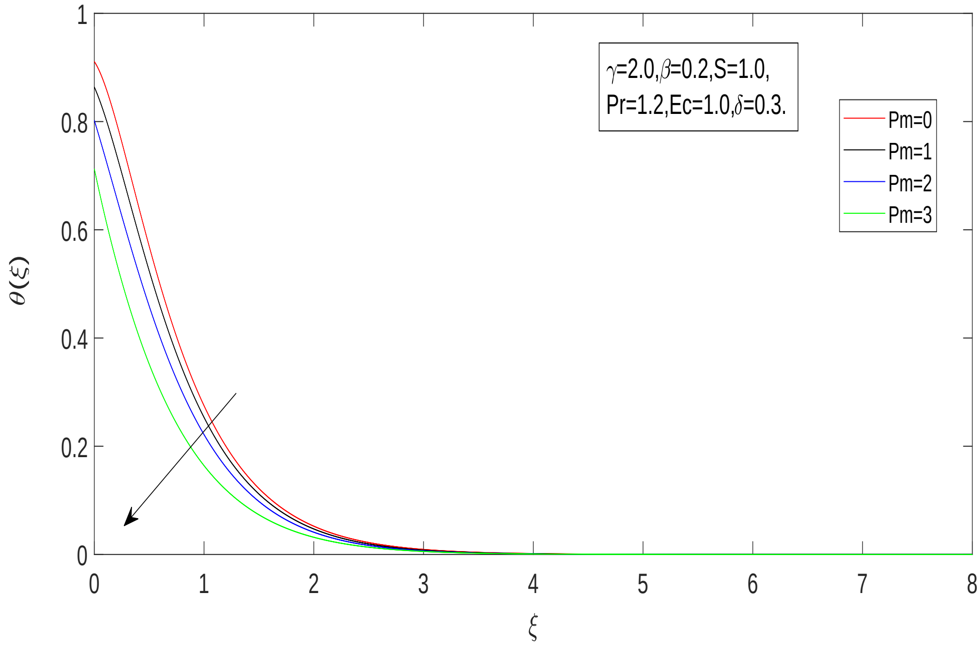

FIGURE 3.6: Variation of temperature with porosity parameter P_m

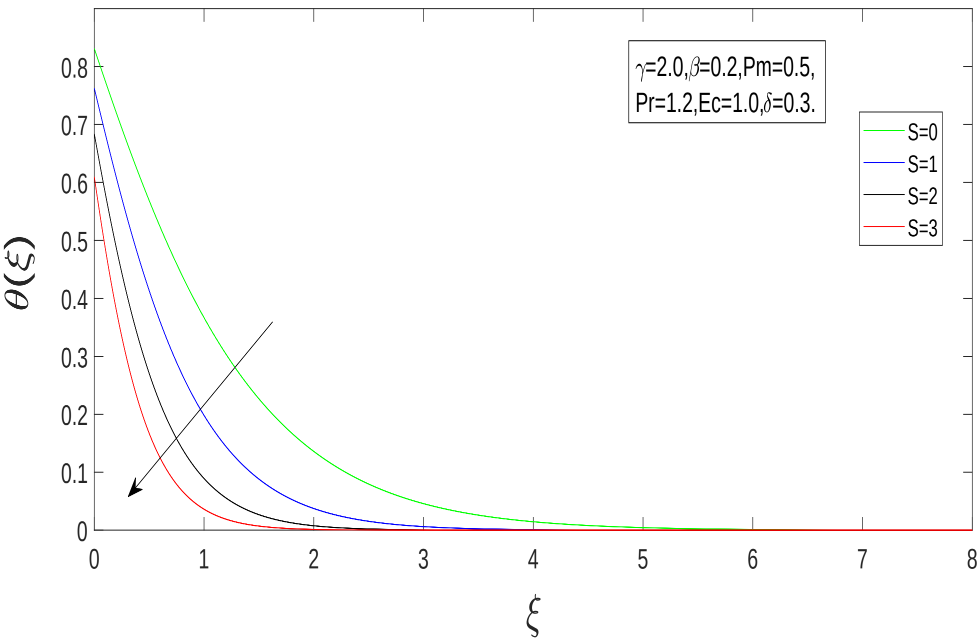

FIGURE 3.7: Temperature fluctuation as a function of suction S

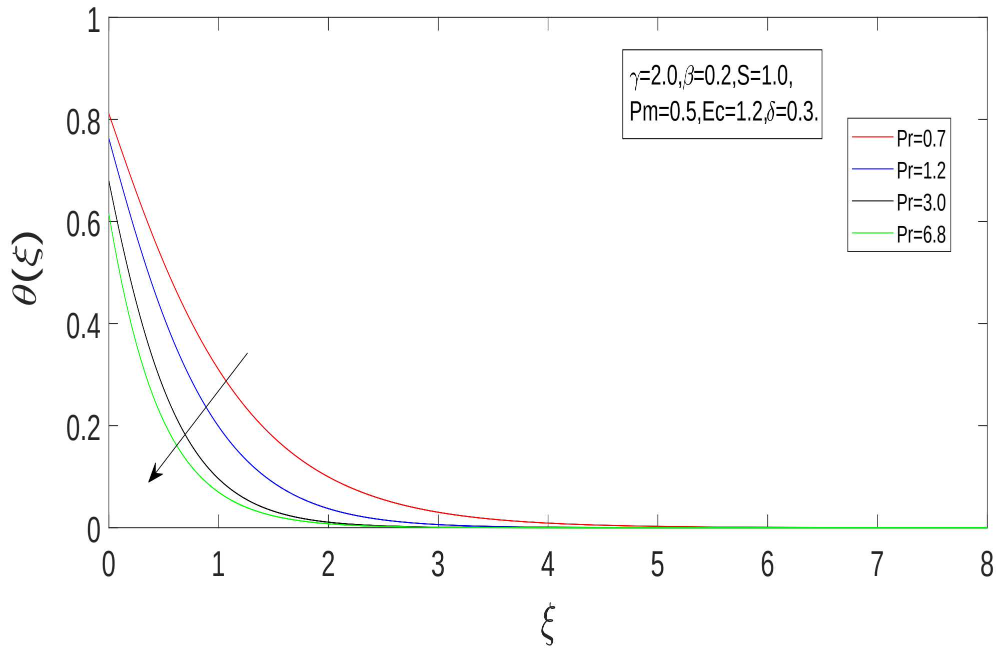

FIGURE 3.8: Variation of temprature with Prandtl parameter P_r

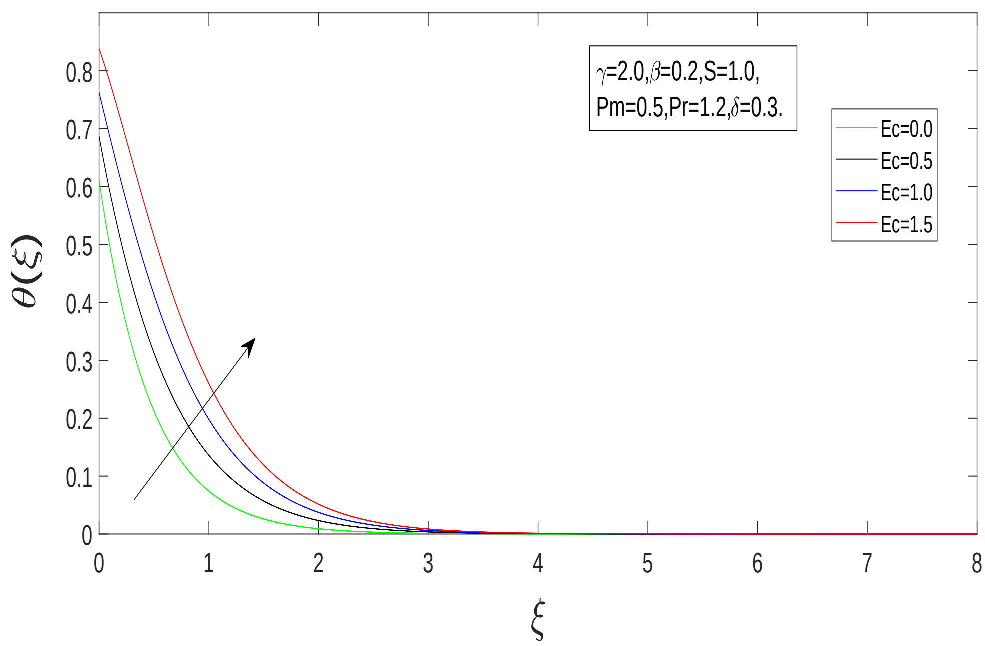

FIGURE 3.9: Variation of temprature with Eckert number E_c

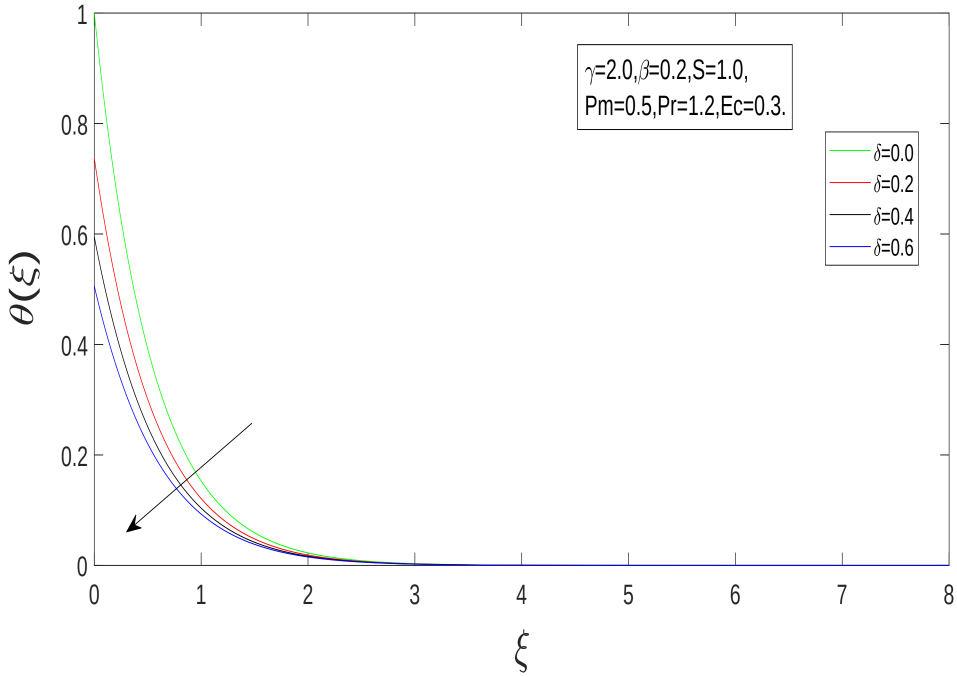

FIGURE 3.10: Variation of temprature with Eckert number δ

Chapter 4

Boundary Layer Flow through Darcy-Brinkman Porous Medium in the Presence of Slip Effects Under the Effect of Joule Heating and Inclined Magnetic Field

4.1 Introduction

This chapter contains the extension of the model given by Kausar, et al. [1] by considering boundary layer flow through a Darcy-Brinkman porous medium with convective boundary conditions and joule heating in the presence of a magnetic field. Furthermore, by using the similarity transformations, the nonlinear PD Es are transformed into a system of ODEs. The numerical solution of ODEs is obtained by applying numerical technique known as shooting method. At the end of this chapter, the final results are discussed for significant parameters that have impact on the $g^{'}(\xi)$ and $\theta(\xi)$ which are shown in tables and graphs.

4.2 Mathematical Modeling

Consider the flow across a porous media with a permeable stretching surface. In Cartesian dimensions, the $x-axis$ and $y-axis$ are perpendicular to the sheet, which is being stretched with velocity $U_s = \alpha x$.
The temperature of the sheet and the ambient temperature are $T_s = T_\infty + cx^2$, and $T_s > T_\infty$. The governing equations for the boundary layer assumption in the presence of viscous dissipation are,
Continuity Equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{4.1}$$

Momentum Equation:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\epsilon^2 \mu_e}{\rho}\left(\frac{\partial^2 u}{\partial y^2}\right) - \frac{\mu\epsilon^2}{\rho K^*}u + \frac{\sigma}{\rho}uB_0^2 \sin^2 t. \tag{4.2}$$

Energy Equation:

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{\kappa}{\rho C_p}\left(\frac{\partial^2 T}{\partial y^2}\right) + \frac{\epsilon^2}{\rho C_p}\left[\mu_e\left(\frac{\partial u}{\partial y}\right)^2 + \frac{\mu u^2}{K^*} + \frac{\sigma}{\rho C_p}B_0^2 u^2 \sin^2 t\right]. \tag{4.3}$$

along with boundary conditions

$$\left.\begin{array}{llll} u = \alpha x + \beta_1\left(\frac{\partial u}{\partial y}\right), & v = -V_0, & T = T_s + \delta_1\left(\frac{\partial T}{\partial y}\right) & \text{at} \quad y = 0, \\ u \to 0, & T \to T_\infty & \text{at} & y \to \infty \quad . \end{array}\right\} \tag{4.4}$$

Here inclined magnetic field term added to momentum equation and joule heating added to energy equation.

- $\frac{\sigma}{\rho}B_0 \sin^2 tu = \frac{\sigma}{\rho}\frac{\alpha\rho M}{\sigma}\sin^2 t\alpha x g'(\xi)$

- $\frac{\sigma}{\rho C_p}B_0^2 u^2 \sin^2 t = \frac{\sigma}{\rho C_p}\frac{\alpha\rho M(\alpha x g')^2}{\sigma}\sin^2 t$

The components of velocity along the x and y directions, respectively, are u and υ in the above equation. The similarity transformations are introduced.

$$\xi = \sqrt{\frac{\alpha}{\upsilon}}y, \qquad u = \alpha x g'(\xi),$$

$$v = -\sqrt{\alpha\upsilon}g(\xi), \qquad \theta(\xi) = \frac{T - T_\infty}{T_s - T_\infty}.$$

The detailed procedure for the verification of the continuity equation (4.1) has been discussed in Chapter 3. The conversion of momentum equation (4.2) is mostly same as shown in Chapter 3.

Let us consider magnetic field term:

- $\frac{\sigma}{\rho}B_0 \sin^2 tu = \frac{\sigma}{\rho}\frac{\alpha\rho M}{\sigma}\sin^2 t\alpha x g'(\xi).$

Plugin the above term in (4.2)

$$u\frac{\partial u}{\partial x} + \upsilon\frac{\partial u}{\partial y} = \frac{\epsilon^2\mu_e}{\rho}\left(\frac{\partial^2 u}{\partial y^2}\right) - \frac{\mu\epsilon^2}{\rho K^*}u + \frac{\sigma}{\rho}B_0^2 \sin^2 tu$$

$$\Rightarrow \quad \alpha^2 x(g')^2(\xi) - \alpha^2 x g g''(\xi) = \frac{\epsilon^2\mu_e}{\rho}\left(\frac{\alpha^2 x}{\upsilon}g'''(\xi)\right) - \frac{\mu\epsilon^2}{\rho\alpha K^*}\alpha^2 x g'(\xi)$$

$$\Rightarrow \quad + \frac{\sigma}{\rho}\frac{\alpha\rho M}{\sigma}(\sin^2 t)\alpha x g'(\xi)$$

$$\Rightarrow \quad \alpha^2 x\left((g')^2 - gg''\right) = \alpha^2 x\left(\frac{\epsilon^2\mu_e}{\frac{\mu}{\upsilon}\upsilon}g''' - \frac{\mu\epsilon^2}{\rho\alpha K^*}g' + M(\sin^2 t)g'\right)$$

$$\Rightarrow \quad (g')^2 - gg'' = \gamma g'' - P_m g' + M(\sin^2 t)g'$$

$$\Rightarrow \quad \gamma g''' - (g')^2 + gg'' - P_m g' + M(\sin^2 t)g' = 0. \tag{4.5}$$

Let us consider the term due to joule heating :

- $\frac{\sigma}{\rho C_p}B_0^2 u^2 \sin^2 t = \frac{\sigma}{\rho C_p}\frac{\alpha\rho M(\alpha x g')^2}{\sigma}\sin^2 t.$

To convert the energy equation (4.3) into dimensionless form:

$$u\frac{\partial u}{\partial x} + \upsilon\frac{\partial T}{\partial y} = \frac{\kappa}{\rho C_p}\left(\frac{\partial^2 T}{\partial y^2}\right) + \frac{\epsilon^2}{\rho C_p}\left[\mu_e(\frac{\partial u}{\partial y})^2 + \frac{\mu u^2}{K^*}\right] + \frac{\sigma B_0^2 u^2 \sin^2 t}{\rho C_p}$$

$$\Rightarrow \quad 2c\alpha x^2 g'\theta - \alpha c x^2 g\theta' = \alpha c x^2\frac{\kappa}{\mu C_p}\theta'' + \frac{\epsilon^2}{\rho C_p}\left[\mu_e\frac{\alpha^3 x^2}{\upsilon}(g'')^2 + \frac{\mu\alpha^2 x^2}{K}^*(g')^2\right]$$

$$\Rightarrow \quad +\frac{\sigma}{\rho C_p}\frac{\alpha\rho M\alpha x(g')^2}{\sigma}\sin^2 t$$

$$\Rightarrow \quad 2\alpha cx^2 g'\theta - \alpha cx^2 g\theta' = \alpha cx^2 \tfrac{\kappa}{\mu C_p}\theta'' + \tfrac{\alpha^3 x^2}{C_p}\left[\frac{\epsilon^2\mu_e}{\rho\upsilon}(g'')^2 + \frac{\epsilon^2\mu}{\rho\alpha K^*}(g')^2\right]$$

$$\Rightarrow \quad +\frac{\alpha^3 x^2 M}{C_p}\sin^2 t(g')^2$$

$$\Rightarrow \quad \alpha cx^2(2g'\theta - g\theta') = \alpha cx^2\left[\frac{1}{P_r}\theta'' + \frac{1}{c}\left(\frac{\epsilon^2\mu_e}{\frac{\mu}{\upsilon}\upsilon}\frac{\alpha^2}{C_p}g''(^2\right) + \frac{\epsilon^2\mu}{\rho\alpha K^*}\frac{\alpha^2}{C_p}(g')^2\right]$$

$$\Rightarrow \quad +\frac{\alpha^2}{C_p}M\sin^2 t(g')^2$$

$$\Rightarrow \quad 2g'\theta - g\theta' = \frac{1}{P_r}\theta'' + \frac{\alpha^2}{cC_p}\gamma(g'')^2 + \frac{\alpha^2}{cC_p}P_m(g')^2 + \frac{\alpha^2}{cC_p}M\sin^2 t(g')^2$$

$$\Rightarrow \quad \frac{1}{P_r}\theta'' - 2g'\theta + g\theta' + E_c\left[\gamma(g'')^2 + P_m(g')^2 + M\sin^2 t(g')^2\right] = 0. \tag{4.6}$$

The detailed procedure for the conversion of boundary conditions into dimensionless form is similar to that discussed in Chapter 3.

The ultimate dimensionless form of the governing model is:

$$\gamma g''' - (g')^2 + gg'' - P_m g' + \sin^2 tMg' = 0. \tag{4.7}$$

$$\frac{1}{P_r}\theta'' - 2g'\theta + g\theta' + E_c\gamma(g'')^2 + E_cP_m(g')^2 + E_cM\sin^2 t(g')^2 = 0. \tag{4.8}$$

The transformed BCs are stated below:

$$\left.\begin{aligned} &g(0) = S, && g'(0) = 1 + \beta g''(0), \quad g'(\infty) = 0,\\ &\theta(0) = 1 + \delta\theta'(0), && \theta(\infty) = 0. \end{aligned}\right\} \tag{4.9}$$

The dimensionless form of Skin-friction and Nusselt Number is evaluated as follows:

- $$\begin{aligned} S_{fx} &= \frac{\mu}{\rho U_s^2}\left(\frac{\partial T}{\partial y}\right) \quad \text{at} \quad y = 0\\ &= \frac{\mu}{\rho\alpha x^2}\alpha x\sqrt{\frac{\alpha}{\upsilon}}g''(0)\\ &= \frac{\mu}{\rho\alpha x}\sqrt{\frac{\alpha}{\upsilon}}g_{''}(0) \end{aligned}$$

$$= \frac{\mu}{\frac{\mu}{\upsilon}} \alpha x \sqrt{\frac{\alpha}{\upsilon}} g''(0)$$

$$= \frac{\sqrt{\upsilon}\sqrt{\upsilon}}{\sqrt{\alpha}\sqrt{\alpha}} x \sqrt{\frac{\alpha}{\upsilon}} g''(0)$$

$$\Rightarrow \frac{\sqrt{\alpha}}{\sqrt{\upsilon}} x = g''(0)$$

$$\Rightarrow \quad S_{fx} R_e x^{\frac{1}{2}} = g''(0). \tag{4.10}$$

- $$N_{Rx} = -\frac{x}{T_s - T_\infty} \left(\frac{\partial T}{\partial y}\right) \quad \text{at} \quad y = 0$$

$$= -\frac{x}{T_\infty + cx^2 - T_\infty} \left(\frac{\partial T}{\partial y}\right)_{y=0}$$

$$= -\frac{x}{cx^2} \left(cx^2 \theta' \sqrt{\frac{\alpha}{\upsilon}}\right)$$

$$= -x\theta' \sqrt{\frac{\alpha}{\upsilon}}$$

$$\Rightarrow \frac{1}{x\sqrt{\frac{\alpha}{\upsilon}}} = -\theta'(0)$$

$$\Rightarrow \quad N_{Rx} R_e x^{\frac{-1}{2}} = \theta'(0). \tag{4.11}$$

4.3 Solution Methodology

Shooting method is used for the system of ordinary differential equations(4.7) and (4.8) subjected to boundary condition (4.9). Following notation are being considered for further work:

$$g = y_1, \qquad g' = y_1' = y_2, \qquad g'' = y_2' = y_3, \qquad g''' = y_3'.$$

Following system of ODEs is obtained by using above notations in equation (4.7) :

$$\left.\begin{aligned} y_1' &= y_2, & y_1(0) &= s, \\ y_2' &= y_3, & y_2(0) &= 1 + \beta y_3(0), \\ y_3' &= \frac{1}{\gamma}\left[y_2^2 - y_1 y_3 + P_m y_2 - sin^2 t M y_2\right], & y_3(0) &= p. \end{aligned}\right\} \tag{4.12}$$

In order to achieve approximate numerical results, (4.12) is solved by RK-4 method. The domain of our problem is considered to be bounded i.e. $[0, \eta_\infty]$, where η_∞ is a positive number and for which the variation in the solution is negligible after $\eta = \eta_\infty$. Assumed missing initial condition p such that:

$$y_2(\eta_\infty, p) = 0,$$

Newton,s method is used to refine the value of missing initial condition p further we start the iteration by $p = p^{(0)}$

$$p^{(i+1)} = p^{(i)} - \left[\left(\frac{\partial y_2}{\partial p} \right)^{-1} (y_2(\eta, p)) \right]_{p=p^{(0)}}$$

To incorporate Newton,s method, we further use the following notations:

$$\frac{\partial y_1}{\partial p} = y_4, \quad \frac{\partial y_2}{\partial p} = y_5, \quad \frac{\partial y_3}{\partial p} = y_6. \tag{4.13}$$

As a result of these new notations, the Newton,s iterative scheme gets the form:

$$p^{(i+1)} = p^{(i)} - \left[\left(\frac{\partial y_5}{\partial p} \right)^{-1} (y_2(\eta, p)) \right]_{p=p^{(i)}},$$

where i is the number of iterations $(i = 0, 1, 2, 3...)$. Now differentiating the system of three first order ODE's (4.12) with respect to t, we get another system of ODE's, of first order. Hence following system of IV P's is obtained:

$$\begin{aligned}
y_1^{'} &= y_2, & y_1(0) &= s, \\
y_2^{'} &= y_3, & y_2(0) &= 1 + \beta p, \\
y_3^{'} &= \frac{1}{\gamma}\left[y_2^2 - y_1 y_3 + P_m y_2 + sin^2 t M y_2\right], & y_3(0) &= p, \\
y_4^{'} &= y_5, & y_4(0) &= 0, \\
y_5^{'} &= y_6, & y_5(0) &= \beta, \\
y_6^{'} &= \frac{1}{\gamma}\left[2y_2 y_5 - y_1 y_6 - y_3 y_4 + P_m y_5 + sin^2 t M y_5\right], & y_6(0) &= 1.
\end{aligned}$$

The required stopping criteria for shooting method is set as follows

$$|y_2(\eta, p)| < \epsilon,$$

where ϵ is finitely small positive number up to 10^{-8}.
For the numerical solution of (4.8), again we use RK-4 method through the following notation have been taken in an account.

$$\theta = y_1, \qquad \theta' = y_1' = y_2, \qquad \theta'' = y_2',$$
$$g = d, \qquad g' = d_1, \qquad g'' = d_2.$$

Following system of ODEs is obtained by using above notations in equation (4.8) :

$$\left.\begin{aligned} y_1' &= y_2, & y_1(0) &= 1 + Gy_2(0), \\ y_2' &= P_r\left[2d_1y_1 - dy_2 - E_c(\gamma d_2^2 + P_m d_1^2) + E_c M sin^2 t d_1^2\right], & y_2(0) &= q. \end{aligned}\right\} \tag{4.14}$$

In order to achieve approximate numerical results, (3.15) is solved by RK-4 method. The domain of our problem is considered to be bounded i.e. $[0, \eta_\infty]$, where η_∞ is a positive number and for which the variation in the solution is negligible after $\eta = \eta_\infty$.
Assumed missing initial condition q such that:

$$y_2(\eta_\infty, q) = 0,$$

Newton,s method is used to refine the value of missing initial condition q further we start the iteration by $q = q^{(0)}$

$$q^{(i+1)} = q^{(i)} - \left[\left(\frac{\partial y_2}{\partial r}\right)^{-1}(y_2(\eta, q))\right]_{q=q^{(0)}}$$

Following derivatives are necessary to incorporate Newton method:

$$\frac{\partial y_1}{\partial q} = y_3, \quad \frac{\partial y_2}{\partial q} = y_4. \tag{4.15}$$

As a result of these new notations, the Newton iterative scheme gets the form:

$$q^{(i+1)} = q^i - \left[\left(\frac{\partial y_4}{\partial r} \right)^{-1} (y_2(\eta, q)) \right]_{q=q^i},$$

where i is the number of iterations ($i = 0, 1, 2, 3...$). Now differentiating the system of two first order ODE's (3.16) with respect to q two more equation of first order. Hence following system of IVP's is obtained

$$\begin{aligned}
y_1^{'} &= y_2, & y_1(0) &= 1 + Gq, \\
y_2^{'} &= P_r \left[2d_1 y_1 - d y_2 - E_c \left(\gamma d_2^2 + P_m d_1^2 \right) + E_c M sin^2 t d_1^2 \right], & y_2(0) &= q, \\
y_3^{'} &= y_4, & y_3(0) &= G, \\
y_4^{'} &= P_r [2d_1 y_3 - d y_4], & y_4(0) &= 1.
\end{aligned}$$

The required stopping criteria for shooting method is set as follows

$$|y_2(\eta, q)| < \epsilon,$$

where ϵ is finitely small positive number up to 10^{-8}.

4.4 Validation of Codes

For validation of the numerical code Tables 4.1 and 4.2 have been presented and the result compared with the results obtained by in built MATLAB routine bvp4c.

4.5 Results and Discussion

The mathematical outcomes of the equations are discussed in this unit by using tables and graphs. The impact of various parameters such as Brinkman

TABLE 4.1: Skin friction coefficient $S_{fx}Re_x^{\frac{1}{2}} = -g''(0)$ for slip case $\beta = 1.0$. Comparison between shooting method and bvp4c

Physical Parameters			$S_{fx}Re_x^{\frac{1}{2}} = -g''(0)$	
γ	S	P_m	shooting method	bvp4c
1.0	1.0	0.5	0.610442	0.6105112
2.0	1.0	0.5	0.500002	0.5000024
3.0	1.0	0.5	0.439312	0.4395283
0.5	0.0	0.3	0.550411	0.5504380
0.5	1.0	0.3	0.712201	0.7122794
0.5	2.0	0.5	0.808832	0.8088720
2.0	0.5	0.0	0.406400	0.4068926
2.0	0.5	0.4	0.452003	0.4519865
2.0	0.5	0.8	0.485901	0.4859061

TABLE 4.2: Local Nusselt number $N_{Rx}Re_x^{\frac{-1}{2}} = -\theta'(0)$ when $\beta = 1.0$ and $S = 0.5$. Comparison between shooting method and bvp4c

Physical Parameters					$N_{Rx}Re_x^{\frac{-1}{2}} = -\theta'(0)$	
P_r	E_c	δ	P_m	γ	shooting method	bvp4c
0.7	0.5	1.0	0.4	2.0	0.456111	0.062191
1.2	0.5	1.0	0.4	2.0	0.538103	0.062191
6.8	0.5	1.0	0.4	2.0	0.738872	0.042228
3.0	0.0	1.0	0.4	2.0	0.738005	0.062191
3.0	0.6	1.0	0.4	2.0	0.642305	0.062191
3.0	1.2	1.0	0.4	2.0	0.546516	0.062191
3.0	1.0	0.0	0.4	2.0	2.208547	0.062191
3.0	1.0	0.6	0.4	2.0	0.820800	0.062191
3.0	1.0	1.2	0.4	2.0	0.504005	0.062191
3.0	1.0	1.0	0.0	2.0	0.640112	0.197152
3.0	1.0	1.0	0.5	2.0	0.566145	0.044660
3.0	1.0	1.0	1.0	2.0	0.517002	0.035473
3.0	1.0	1.0	0.4	2.0	0.619600	0.076326
3.0	1.0	1.0	0.4	2.0	0.578435	0.062191
3.0	1.0	1.0	0.4	2.0	0.546419	0.102838

viscosity ratio parameter, porosity parameter , suction parameter, slip parameter, Prandtl number and Eckert number are observed graphically. This section is dedicated for elaboration of effects of some important physical parameters on velocity profile $g'(\xi)$ and temperature profile $\theta(\xi)$. Here shooting method has been opted for reproducing the solution of $g'(\xi)$ and $\theta(\xi)$.

These physical parameters have a direct impact on $g'(\xi)$ and $\theta(\xi)$. The heat and mass transfer rate for fixed values of Pm, Pr, Ec, S, β, δ and γ are analyzed numerically as shown in figures.

FIGURE 4.1 shows how the Brinkman viscosity ratio parameter affects the velocity profile. By increasing the value of the Brinkman viscosity ratio number, the velocity profile lowers. Because the Brinkman viscosity ratio number appears in the momentum equation alongside the velocity gradient part, higher values of the Brinkman viscosity parameter reduce velocity.

FIGURE 4.2 illustrates the impact of porosity parameter on the velocity distribution. It can be seen that the velocity profile increases by enlarging the porosity parameter.

FIGURE 4.3 and 4.4 reflect the influence of the suction parameter S and the slip parameter β on the velocity profiles. The velocity of the fluid decreases, and the momentum boundary layer thin.

FIGURE 4.5 clarifies the influence of Brinkman viscosity number γ on the temperature profile. Thermal boundary layer is an increasing function of γ.

FIGURE 4.6 and 4.7 depicts that with an increase in suction velocity S and porosity parameter increase within the boundary layer, Pm lowers the fluid temperature.

FIGURE 4.8 The temperature profile is shown with the impacts of the Prandtl number. The temperature inside the boundary layer decreases as the Prandtl number Pr rises. Because the heat transfer rate from the stretching sheet reduces as the fluid's Prandtl number (thermal conductivity) increases, the thermal boundary layer shrinks.

FIGURE 4.9 show the effect of the Eckert number Ec on temperature profile; as the Eckert number rises, fluid friction between adjoining layers rises, causing kinetic energy to be converted to heat energy.

FIGURE 4.10 shows that the effects of thermal slip parameter. It is noted that increasing the thermal slip parameter reduces the temperature and thermal boundary layer.

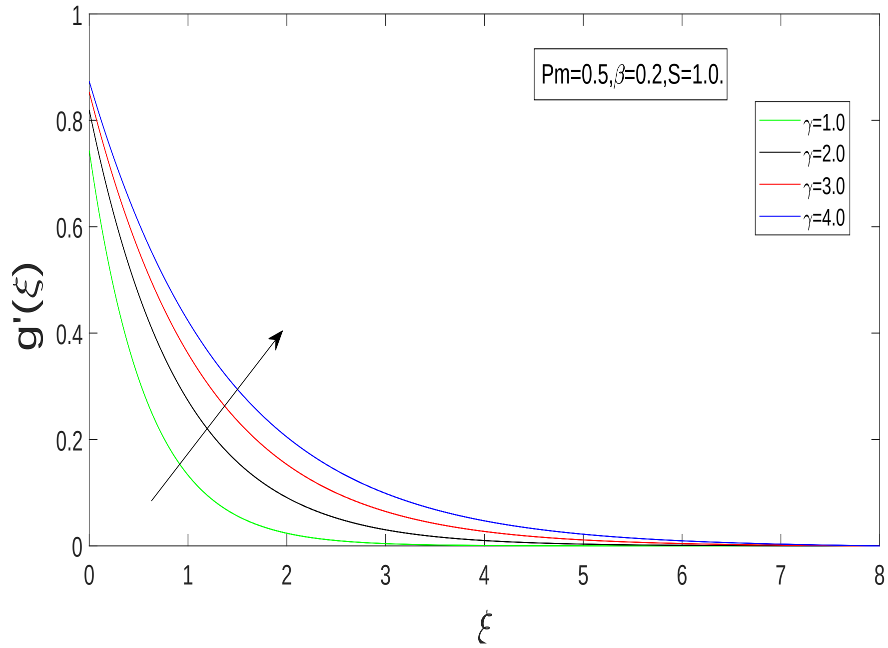

FIGURE 4.1: Variation of velocity as a function of viscosity ratio γ.

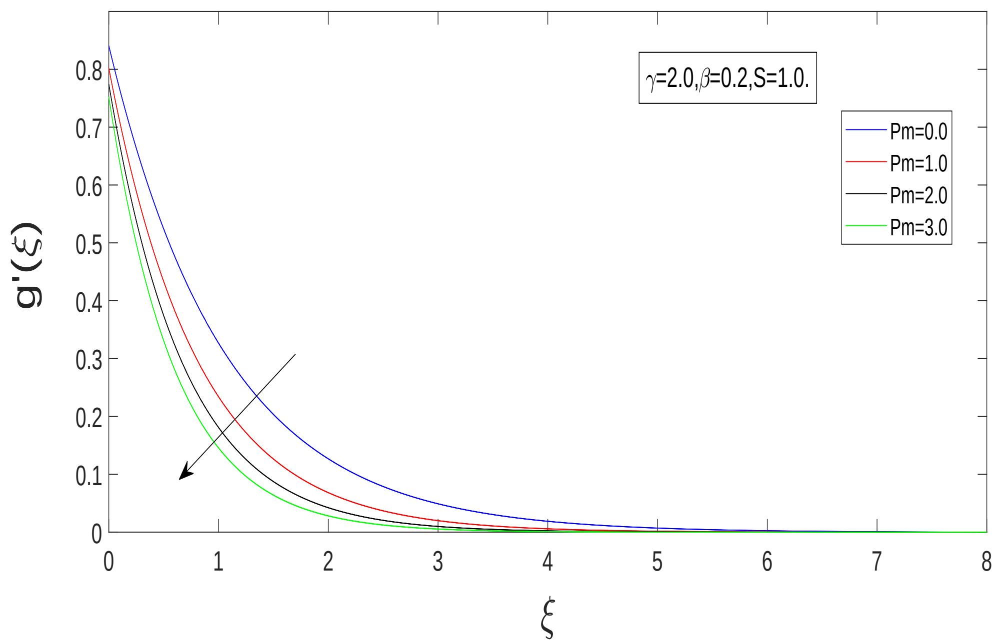

FIGURE 4.2: Variation of velocity with viscosity ratio parameter P_m.

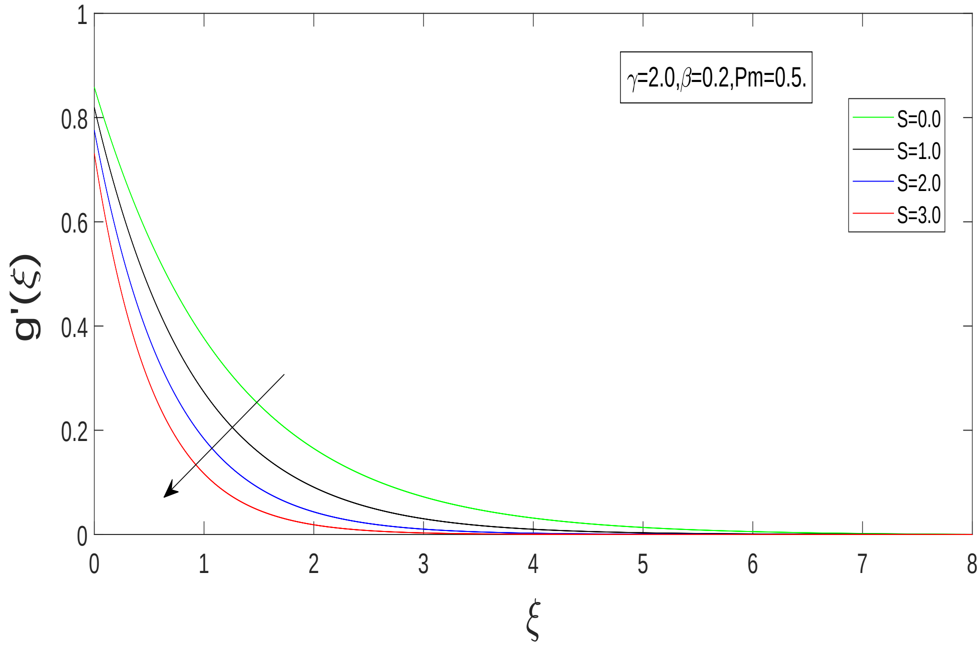

FIGURE 4.3: Variation in velocity as a function of the suction parameter S.

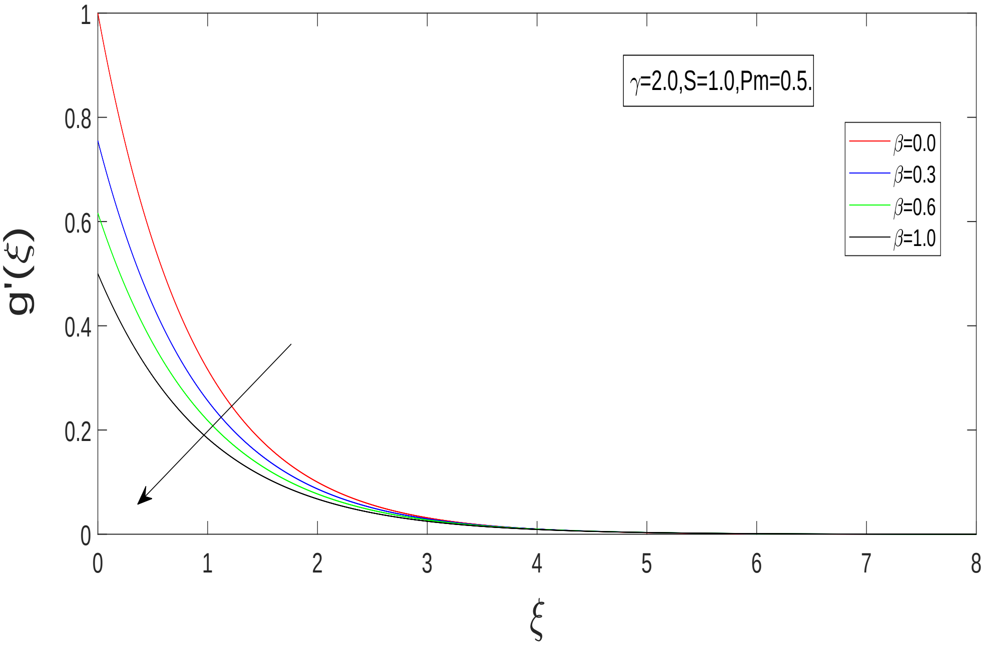

FIGURE 4.4: Variation of velocity with the β velocity slip parameter.

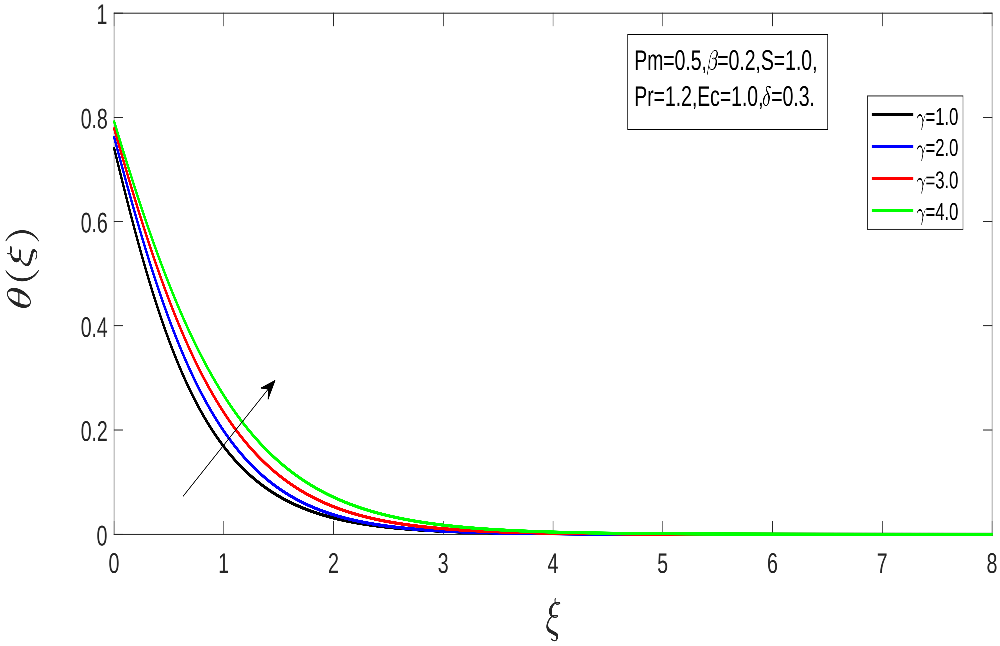

FIGURE 4.5: Variation of temperature with Brinkman parameter γ.

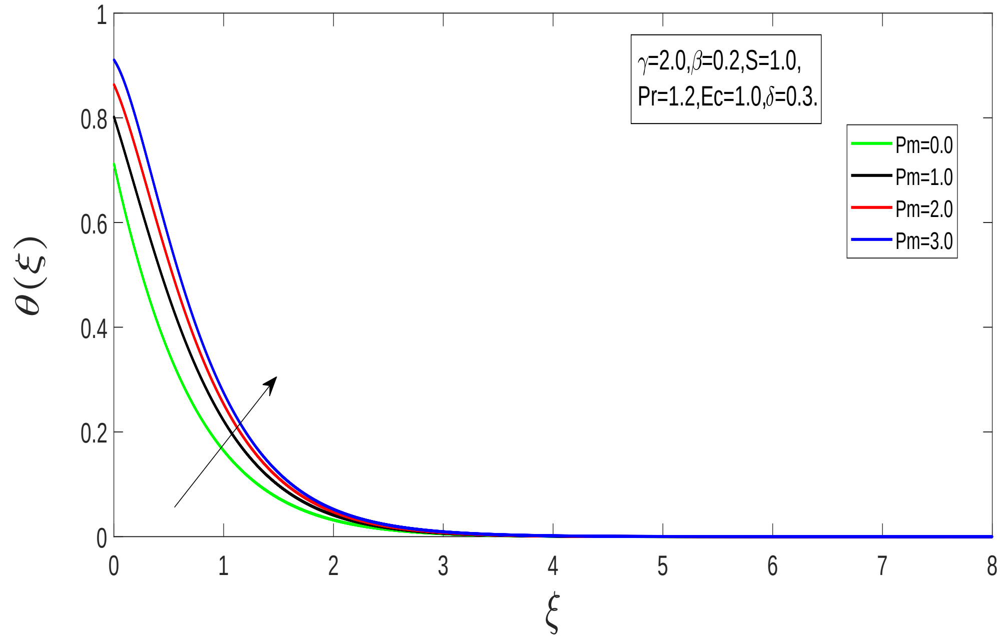

FIGURE 4.6: Variation of temperature with porosity parameter P_m.

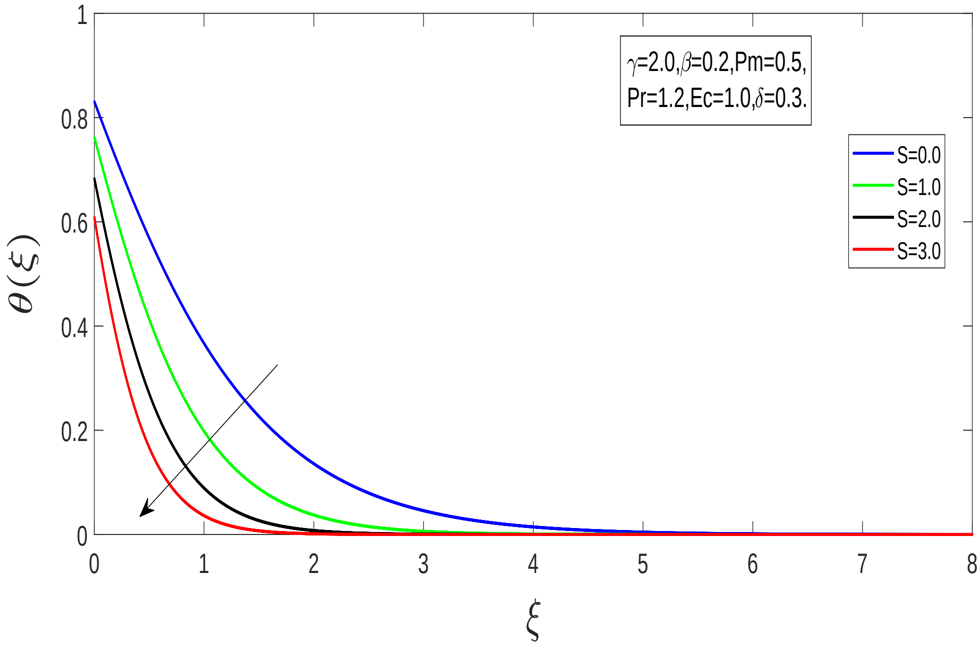

FIGURE 4.7: Variation of temprature with suction parameter S.

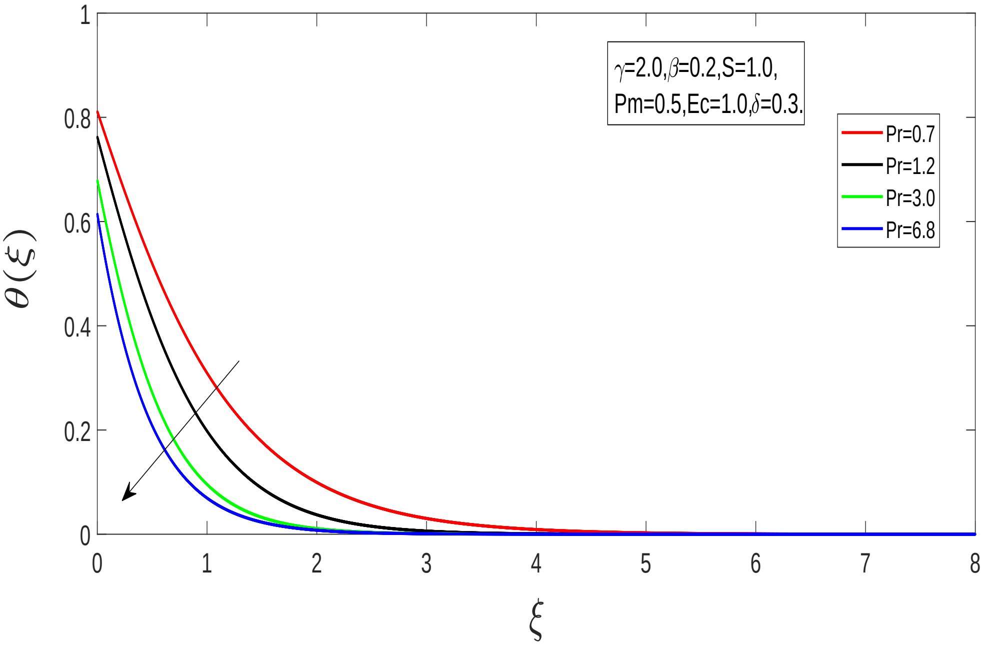

FIGURE 4.8: Variation of temprature with suction parameter P_r.

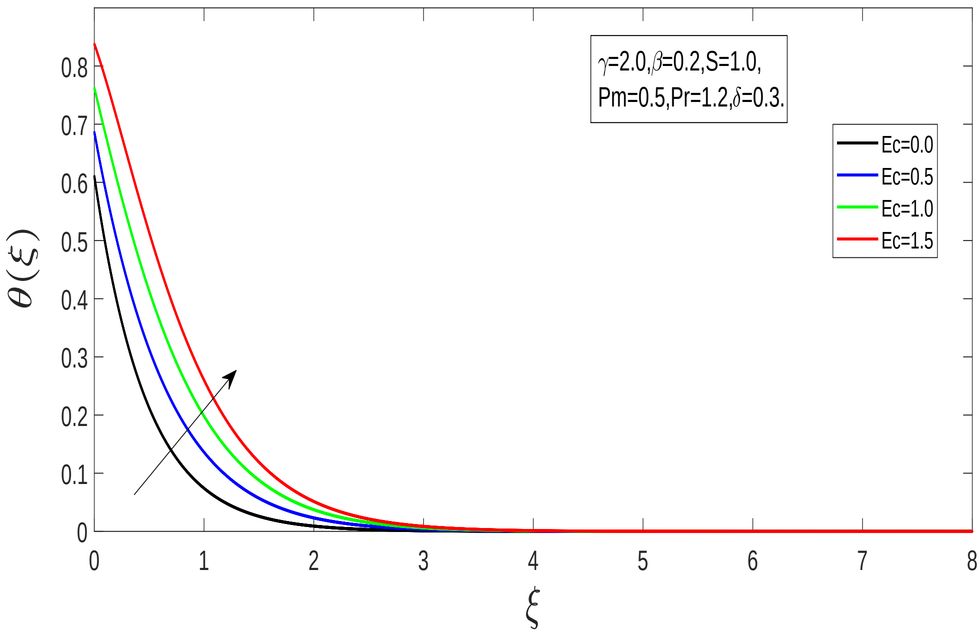

FIGURE 4.9: Variation of temprature with Eckert number E_c.

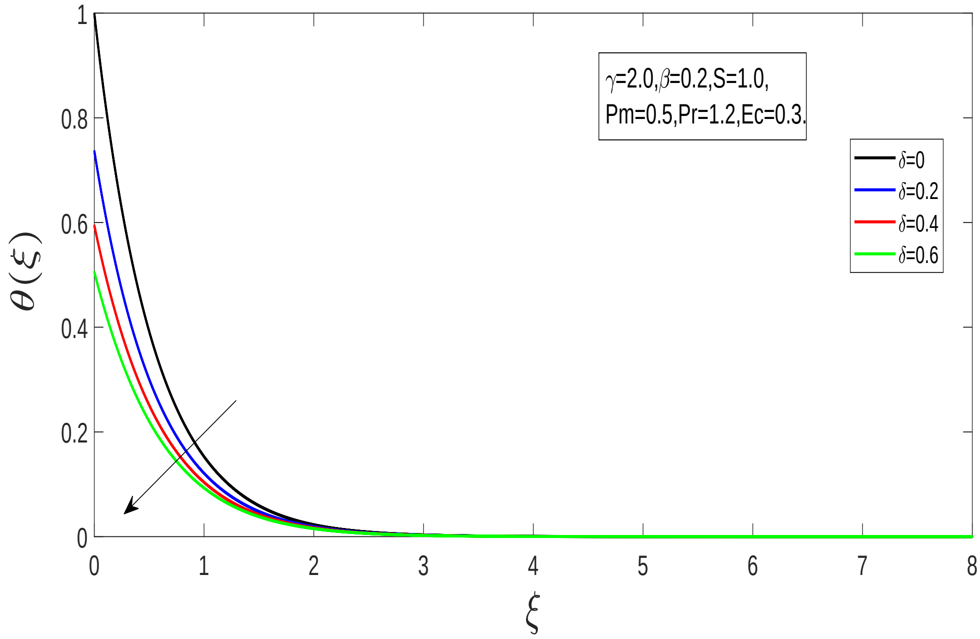

FIGURE 4.10: Variation of temprature with thermal slip parameter γ.

Chapter 5

Conclusion

This research provides the elaborated review of boundary layer flow through Darcy Brinkman porous medium in the presence of slip effects and porous dissipation. Numerical results are explained with help of graphs and tables. Further the model prepared by [1] is extended by adding the effects of joule heating and inclined magnetic field. The obtained mathematical model contains nonlinear PDEs of continuity equation, momentum equation and energy equation. These PDEs are converted into a system of nonlinear ODEs by applying the similarity transformation. The numerical results of ODEs are obtained by shooting technique. The dimensionless velocity behavior and temperature distribution have been analyzed for various parameters. The numerical results are explained through different figures and tables.

5.1 Concluding Remarks

After achieving the numerical results follows signification observation are recorded:

- Both velocity and temperature are increases with the increase of viscosity ratio γ.

- Both velocity and temperature decreases with the increase of porosity parameter P_m.
- Velocity and temperature are decreases by increase the suction parameter S.

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