CAPITAL UNIVERSITY OF SCIENCE AND TECHNOLOGY, ISLAMABAD

Natural Convection of Casson Fluid in Inclined Porous Cavity under Effect of Magnetic Field and Viscous Dissipation

by

Muhammad Zeeshan

A thesis submitted in partial fulfillment for the degree of Master of Philosophy

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This thesis is dedicated to my lovely parents. Today what I have become, it's because of their endless efforts and prayers.

CERTIFICATE OF APPROVAL

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(Muhammad Zeeshan)

Abstract

The present numerical study investigates the natural convection of Casson fluid in a porous, partially heated square enclosure taking into account the effects of horizontal magnetic field, cavity inclination and viscous dissipation. The bottom wall of the cavity has been partially heated, whereas the left and right walls are taken as cold at constant temperature. The top wall and the remaining portions of the bottom wall are adiabatic. The Galerkin weighted residual based finite element method is used to discretize the governing partial differential equations which are then solved iteratively. Results are presented in the form of streamlines, isotherms and two dimensional plots. The governing parameters are Darcy number ($Da = 10^{-4}$), Rayleigh number ($Ra = 10^5$), Prandtl number ($Pr = 7$), Casson parameter ($\gamma = 0.1 - 10$), Hartmann number ($Ha = 0 - 100$), cavity inclination ($\phi = 0^\circ - 90^\circ$) and Eckert number ($Ec = 10^{-6} - 10^{-4}$).

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Abbreviations

FEM	Finite element methods
FVM	Finite volume methods
GFEM	Galerkin finite element method
PDEs	Partial differential equations

Symbols

B_0	Magnetic Field Parameter
c_p	Specific Heat, (J/(kg.K))
Da	Darcy Number
Ec	Eckert Number
g	Acceleration due to Gravity (m/s^2)
Ha	Hartmann Number
k	Thermal Conductivity (W/(m.K)
K	Permeability of the Porous Medium (m^2)
L	Length (m)
Nu	Nusselt Number
Nu_{avg}	Average Nusselt Number
p	Pressure, (Pa)
P	Dimensionless Pressure
Pr	Prandtl Number
Ra	Rayleigh Number
T	Temperature, (K)
u, v	Velocity Components (ms^{-1})
U, V	Dimensionless Velocities
x, y	Cartesian Coordinates
X, Y	Dimensionless Cartesian Coordinates
Greek symbols	
α	Thermal Diffusivity, (m^2 /s)
β	Coefficient of Thermal Expansion of fluid, (K^{-1})

γ	Casson Parameter
ϕ	Cavity Inclination
σ	Specific Heat Ratio
ρ	Density of the Fluid, (kg/m^3)
μ	Dynamic Viscosity, (m^2 /s)
ν	Kinematic Viscosity, (m^2 /s)
ϵ	Length of Heated Zone

Subscripts

c	Cold Wall
h	Hot Wall

Chapter 1

Introduction

The analysis of natural convective flow has been receiving a great observation in recent years based on various applications in many fields of engineering. Heat transportation in porous media has great importance in science as well as in engineering applications. For example, in bio-mechanics, chemical engineering, food technology, mechanical engineering etc. [1]. Many other applications of the heat transmission phenomenon can be found in heatings and cooling system of buildings, textile, foods, vehicles, chemical industries etc. [2]. Many researchers have studied the flow behaviour of non-Newtonian fluids because many things, for instance, ketchup, mud, lubricants, blood with a little shear amount and certain care products are considered as non-Newtonian. Non-Newtonian fluids can be described by some useful models, like micropolar fluids [3], Viscoplastic fluids [4], Casson fluids [4] and power law fluids [4].

Du and Bilgen [5] examined the natural convection with uniform heat generation for large values of the Rayleigh number. A vertical two-dimensional porous cavity is considered for the investigation. The left and right walls of the cavity are retained at fixed temperature. The left wall is considered as cooled and the right wall is kept hot. It has found that the transmission of heat in the porous cavity cannot be done through the right wall. Due to the temperature difference between vertical walls, heat transmission can be observed from the right wall towards the left wall through the porous medium. Baytas and Pop [6] presented the numerical analysis for steady-state free convection flow in an inclined porous enclosure. The inclined walls of the enclosure were taken with temperature difference and keeping the horizontal walls as adiabatic. For $Ra = 10^4$, the major process for heat transmission is convection due to boundary layer flow. Saeid [7] studied the natural convection, heat transfer in a two-dimensional permeable enclosure. Horizontal walls assumed to have temperature difference and the steep sides are regarded as insulated. The cavity's heated portion is supposed to have a spatial sinusoidal temperature. The study showed that the length of the heated segment decreases the amount of heat transferred per unit of area. Basak et al. [8] analyzed the natural convection in the square cavity for uniform as well as for the non-uniform heated lower wall. A penalty finite element approach was used to get the findings for velocity profile, temperature profile, and for heat transfer rate. For $Da = 10^{-5}$, $Pr = 0.71$ and $Ra = 10^6$, a thermal boundary layer is estimated to 75%, formed inside the cavity for uniform heating, approximately 60% within the cavity for non uniform heating. Lamsaadi et al. [9] studied the movement of non-Newtonian fluid and heat transfer rate in a container that is vertical, by employing the power law model. The upper and lower walls of the rectangular container are taken as insulated and the left, as well as the right side wall of the cavity, are assumed to have constant heat flux. In the case of a particular power law index, increasing Ra enhances the heat transfer through convection, while reducing the power law index, for a fixed value of Ra, has a comparable effect. However, for shear thinning behaviour, the convection is more sensitive to changes in Ra. Basak et al. [10] examined energy flows as a result of natural convection with a heated bottom wall, cooled from each of the side walls and insulated upper wall of enclosure. It is monitored that at the middle regime of the bottom heated wall, non-uniform heating consequences in a greater heat transfer rate.

Aleshkova and Sheremet [11] investigated the mathematical simulation of transient conjugate natural convection in the porous enclosure. The porous enclosure is taken with finite thickness heat conduction walls in the presence of heat source, under the influence of governing dimensionless parameters. Dimensionless parameters considered for the mathematical simulation of the problem were Rayleigh number and Darcy number with ranges $10^4 \leq Ra \leq 10^6$ and $10^{-5} \leq Da \leq 10^{-4}$.

A porous two-dimensional cavity and a heat source with constant temperature for the complete process at the bottom wall are used. An insignificant variation of heat transfer rate is observed at an increasing value of Ra with $Da = 10^{-5}$. Lam and Prakash [12] studied the thermal and entropy generation properties correlated with the natural convection in a porous enclosure having high-temperature heat sources on horizontal walls. The Galerkin-based FEM was employed for this objective. An increasing behaviour is observed for the average Nusselt number with raising porosity, Darcy number and Rayleigh number. Computational analysis of natural convection and nanofluid flow in an enclosure with top wavy wall studied by Sheremet et al. [13]. The bottom left most corner is considered to have the heated portion of the cavity. For the considered range of Rayleigh number ($Ra = 10^4 - 10^5$), a development of convective flow near the horizontal walls is observed. Convective flow inside the cavity is increased as the heat source length of the corner heater is increased along the horizontal axis. Ghalambaz et al. [14] examined the variation of local and average Nusselt numbers with the influence of viscous dissipation. In a square enclosure filled with nanofluid, the ups and downs in the heat transfer near the hot and cooled walls are observed under the variation of Eckert number and Rayleigh number. A significant effect for the fluid flow and transfer of heat is observed near the top and bottom left most corner of the cavity. Typical results are presented for the values $Ec = 0.001$ and $Ra = 100$. It is also observed that the Nusselt number near the hot and cold wall is not equal. Kefayati [15] investigated the fluctuation of the average Nusselt number for natural convection under the influence of the Rayleigh number and Darcy number. Results are depicted for parameter values of $Ra = 10^5$ and $Da = (10^{-3} - 10^{-1})$. It has found that the increasing the Rayleigh number and Darcy number, the average Nusselt number shows increasing behaviour in the cavity. Abdelraheem et al. [16] studied numerically the flow behaviour of non-Newtonian fluid using collocated FVM. The physical model considered for the problem was an open, porous, inclined cavity with inclination $(0^\circ - 180^\circ)$. By assuming the sides which are perpendicular to the heated wall as adiabatic. The results evaluated for the governing parameters including inclination angle $(0^\circ - 180^\circ)$, Rayleigh number $(10^3 - 10^9)$ and Darcy number $(10^{-2} - 10^{-4})$ show that the velocities, local and average heat transfer rate decreases with variations of inclination. An increase of Rayleigh number enhances the streamlines strength and greater convection heat transfer rate. Casson fluid is classified as non-Newtonian fluid and may be described as, it is such kind of non-Newtonian fluids which is a shear thinning fluid. It behaves like a fluid with less viscosity or with greater viscosity depending on the shear rate. The yield stress in Casson fluid is significant. It acts like a solid when shear stress is smaller than yield stress due to infinite viscosity. Casson fluid behaves like a fluid and move when shear stress is greater than yield stress due to zero viscosity at infinite shear rate [17]. Tomato sauce, jelly, soup, honey, human blood and juices are examples of Casson fluid [18]. There are numerous utilization for Casson fluid in medicine and pharmacy. For example, human blood, proteins, hemoglobin, white blood cells, fibrinogen, red blood cells. For the particular description of non-Newtonian fluids such as, printing ink, soaps, sugar solution, muds, condensed milk, glues, melts, paints, tomato paste, shampoos, a constitutive governing relation is required. For this purpose, Casson model purposed for such kind of fluids which are viscous with excessive viscosity [19]. Natural convection of Casson fluid in a two dimensional square container was firstly presented by Pop and Sheremet [19]. The cavity's horizontal walls are supposed to be insulated. The cavity's left vertical side is heated from the left and the other vertical side of the cavity is cooled. By considering parameters like Rayleigh number $Ra = 10^5$, Casson dimensionless parameter $\gamma = (0.1 - 5.0)$, Prandtl number $Pr = 0.1, 0.7, 7.0$, Eckert number $Ec = (0 - 1.0)$, it has been found that when the Eckert number is increased, the temperature rises slightly and the heat transfer rate decreases. Natural convection in a trapezoidal container was presented by Hamid et al. [2]. The container was filled with Casson fluid for numerous heated lengths. For various ranges of Rayleigh numbers ($10^4 \leq Ra \leq 10^{5.5}$) and heated segments of various lengths, it is observed that if the Rayleigh number is bigger, the streamlines have taken up the major portion of the cavity. Aghighi et al. [20] explored the natural convection of viscoplastic fluids with the Casson model in a square enclosure and heated from side walls. Galerkin weighted residual FEM was used for the analysis. The impact of natural convection on heat transfer and flow characteristics were analyzed for the range of Rayleigh number $(10^3 - 10^6)$.

Analysis depicted that the rise in Ra values causes an increase in the velocity's magnitude as well as due to stronger convection effect with an increase in temperature variations. Sivasankaran et al. [21] investigated the Casson fluid flowing within a square porous enclosure, in the presence of thermal radiation. The thermal stratification is found when the Casson parameter is set to a higher value in the presence of radiation and it results in significant energy transport throughout the cavity. Aneja et al. [22] analyzed the movement of Casson fluid in a square enclosure with the aid of using partially heating a portion of the lower wall and taking the cavity's left and right walls as cooled simultaneously. It has been referred that under the influence of the Darcy number, Rayleigh number and Casson fluid parameter, the rate of heat transmission seems to be rising.

1.1 Thesis Contribution

The objective of this study is to review the work of Aneja et al. [22] and extend the analysis for the natural convection with Casson fluid under the effect of cavity inclination, horizontal magnetic field and viscous dissipation in a permeable square enclosure. For this purpose, the corresponding mathematical model is established and then Galerkin finite element method (GFEM) is utilized for the solution of governing partial differential equations (PDEs) including momentum equation, continuity equation and energy equation. The primary goal of this study is to examine the variation of heat transmission rate of Casson fluid in the presence of applied horizontal magnetic field. Viscouss dissipation effect is also under observation for the same problem. Viscous dissipation is the irreversible process by which the heat is internally generated due to the work done by a fluid on neighbouring layers subject to the shear forces. Governing parameters, Darcy number Da, Rayleigh number, Prandtl number Pr, Eckert number and Casson fluid parameter are considered for the analysis. All the numerical results obtained for these parameters are expressed in terms of streamlines, isotherms and 2D plots.

1.2 Thesis Classification

This study is further classified into following chapters.

Chapter 2 gives the fundamental definitions, terminologies, examples and information about parameters involved in the problem that will help in understanding problem deeply. It provides the detail about dimensionless parameters which are used to convert the problem in dimensionless form. Further, it describes the finite element method which is used to solve the partial differential equations by converting it into algebraic equations.
Chapter 3 illustrates the complete formulation (mathematical and graphical) of the problem and validation of results. It gives the review of considered problem by representing the results in the form of streamlines, isotherms and 2D plots.

Chapter 4 presents the modification of the problem by considering some additional conditions on the problem which is reviewed in **Chapter 3**. Additional conditions includes cavity inclination, horizontal magnetic field and viscous dissipation inside the cavity. It also expresses the effect of corresponding parameters related to additional conditions on considered problem.

Chapter 5 shows the major findings and conclusions about the flow of Casson fluid, heat transfer rate and the internal heat generation in enclosure.

Bibliography shows references of all the articles and books which are helpful in this thesis.

Chapter 2

Basic Definitions and Concepts

In this chapter, we can find basic definitions, classifications of fluid, types of flows, heat transfer phenomenon, governing laws, governing equations and dimensionless parameters. At the end of this chapter, the major steps of numerical methodology which are used to solve the proposed problem, are discussed to understand the solution procedure of problem.

2.1 Basic Definitions

Definition 2.1. (Fluid)
"A fluid is a substance that deforms continuously under the application of shear (tangential) stress no matter how small it"[23]. A fluid covers liquids and gases.

Definition 2.2. (Stress and Strain)
"The force of resistance per unit area, offered by a body against deformation is known as stress. Mathematically stress is written as $\sigma = \frac{P}{A}$
where σ shows stress, P represents external force and A is for cross-sectional area"[24].
Strain can be described as "When a body is subjected to some external force, there is some change of dimension of the body. The ratio of change of dimension of the body to the original dimension is known as strain. Strain is dimensionless"[24].

Definition 2.3. (Shear Stress and Shear Strain)

"The stress induced in a body, when subjected to two equal and opposite forces which are acting tangentially across the resisting section, as a result of which the body tends to shear off across the section, is known as shear stress.
The corresponding strain is known as shear strain"[24].

Definition 2.4. (Viscous Stress Tensor)

"The viscous stresses, which originate from the friction between the fluid and the surface of an element, are described by the stress tensor τ. In Cartesian coordinates the general form is given by

$$\tau = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix}$$

The notation τ_{ij} means by convention that the particular stress component of fects a plane perpendicular to the $i-$axis, in the direction of the $j-$axis. The components τ_{xx}, τ_{yy}, and τ_{zz} represent the normal stresses, the other components of τ stand for the shear stresses, respectively"[25].

Definition 2.5. (Fluid Mechanics)

"Fluid mechanics is the branch of science which deals with the behaviour of the fluids (liquids or gases) at rest as well as in motion"[26].
Fluid mechanics is widely used in design of modern engineering systems, aircraft, refrigerator, compressor, pumps, automobile, recent models of car, transportation of water, wind turbines, bio medical devices, breathing machines, cooling of electronic components and many natural phenomena like weather patterns, rain cycle, heating and ventilation system etc.

Definition 2.6. (Fluid Dynamics)

"Fluid dynamics is the study of fluid motion with the forces causing flow".[26]

Definition 2.7. (Fluid Kinematics)

"Fluid kinematics is defined as that branch of science which deals with motion of particles without considering the forces causing the motion"[26].

Definition 2.8. (Fluid Statics)

Fluid statics is concerned with the behaviour of a fluid at rest or nearly so. The key problem in fluid statics is to determine the pressure distribution in the fluid.

Definition 2.9. (Viscosity) or (Dynamic Viscosity)

"Viscosity is defined as the property of a fluid which offers resistance to the movement of one layer of fluid over another adjacent layer of the fluid"[26].

Definition 2.10. (Kinematic Viscosity) "It is defined as the ratio between the dynamic viscosity and density of fluid. It is denoted by Greek symbol ν called (nu). Thus, mathematically"[26],

$$\nu = \frac{\text{Viscosity}}{\text{Density}} = \frac{\mu}{\rho}.$$

2.2 Newton's Law of Viscosity

"It states that the shear stress (τ) on a fluid element layer is directly proportional to the rate of shear strain. The constant of proportionality is called the co-efficient of viscosity. Mathematically,

$$\tau = \mu \frac{du}{dy},$$

where τ is the shear stress, μ is the coefficient of viscosity, u is the velocity, y is the coordinate measured in a direction perpendicular to the direction of flow velocity u"[26].

2.3 Types of Fluid

The basic types of fluid are as follows:

Definition 2.11. (Ideal and Real Fluids)

"An imaginary fluid which is incompressible and has zero viscosity is called an ideal fluid. Practically, all the fluids have some viscosity and are called real fluids"[27].

Definition 2.12. (Newtonian Fluids)

"Fluids which obey the Newton's law of viscosity are known as Newtonian fluids. These are real fluids in which there is a linear relation between the magnitude of shear stress and the resulting velocity gradient. Some of the Newtonian fluids are air, water, glycerine, kerosene and molten metals"[27].

Definition 2.13. (Non-Newtonian Fluids)

"Fluids which do not obey the Newton's law of viscosity are known as non-Newtonian fluids. These are real fluids in which there is a non-linear relation between the magnitude of shear stress and velocity gradient. Some of the non-Newtonian fluids are slurries, polymer solutions, suspensions, grouts, human blood, thick lubricating oil, toothpaste and gels"[27].

2.4 Classification of Fluids Flow

The flow of fluid can be categorized as:

Definition 2.14. (Steady and Unsteady Flow)

"Steady flow is defined as that type of flow in which the fluid characteristics like velocity, pressure, density, etc., at a point do not change with time".

"Unsteady flow is that type of flow, in which the velocity, pressure or density at a point changes with respect to time"[26].

Definition 2.15. (Uniform and Non-uniform Flows)

"Uniform flow is defined as that type of flow in which the velocity at any given time does not change with respect to space".

"Non-uniform flow is that type of flow in which the velocity at any given time changes with respect to space"[26].

Definition 2.16. (Laminar and Turbulent Flows.)

"Laminar flow is defined as that type of flow in which the fluid particles move along well defined paths or streamline and all the streamlines are straight and

parallel. This type of flow is also called stream-line flow or viscous flow".
"Turbulent flow is that type of flow in which the fluid particles move in a zig-zag way. Due to the movement of fluid particles in a zig-zag way, the eddies formation takes place which responsible for high energy loss"[26].

Definition 2.17. (Rotational and Irrotational Flows)
"Rotational flow is that type of flow in which the fluid particles while flowing along streamlines, also rotate about their own axis".
"If the fluid particles while flowing along streamlines, do not rotate about their own axis then that type of flow is called irrotational flow"[26].

Definition 2.18. (Compressible and Incompressible Flow)
"Compressible flow is that type of flow in which the density of the fluid changes from point to point or in other words ρ is not constant for the fluid"[26].

"Incompressible flow is that type of flow in which the density is constant for the fluid flow. Liquids are generally incompressible while gases are compressible"[26].

2.5 Streamlines, Stream Function and its Properties

Definition 2.19. (Streamline) "A streamline may be defined as an imaginary line drawn through a flowing fluid in such a way that the tangent to it at any point gives the direction of the velocity of flow at that point. Thus, streamlines indicate the direction of motion of particles at each point. Since the streamlines are tangent to the velocity vector at every point in the flow field, there can be no flow across a streamline. The differential equation for streamlines in two-dimensional flow field is as follows:

$$\frac{dx}{u} = \frac{dy}{v}$$

Here, dx and dy be the components of the displacement along x and y directions, respectively and u and v be the components of the velocity $\mathbf{V}$ along x and y directions, respectively"[27].

Definition 2.20. (Stream Function)

"Stream function is defined as the scalar function of space and time, such that its partial derivative with respect to any direction gives the velocity component at right angles to that direction. It is denoted by Greek letter $\psi(Psi)$ and defined only for two dimensional flow"[26].

Stream function characterizes the flow of fluid for rotational and irrotational flow. The flow of fluid will be possibly irrotational if the stream function satisfies the Laplace equation.

2.6 Porous Material

Definition 2.21. (Porous Material)

"A solid containing holes or voids, either connected or non-connected, dispersed within it in either a regular or random manner known as porous material provided that holes occur relatively frequently within the solid"[28].

"Pores are either interconnected or non-interconnected. A fluid can flow through a porous material only if at least some of the pores are interconnected"[28].
Some natural porous materials are beach sand, limestone, sandstone, wood, loaf of bread and human lung etc.

Definition 2.22. (Porosity)

"The porosity of a porous material is the fraction of the bulk volume of the material occupied by voids. The symbol usually employed for this parameter is ϕ. Thus

$$\phi = \frac{V_p}{V_B} = \frac{\text{Volume of pores}}{\text{Bulk volume}},$$

which is a dimensionless quantity.
Since that portion of the bulk volume not occupied by pores is occupied by the solid grains or matrix of the material, it follows that" [28]

$$1 - \phi = \frac{V_s}{V_B} = \frac{\text{Volume of solids}}{\text{Bulk volume}} .$$

Definition 2.23. (Permeability)

"Permeability is the property of a porous material which characterizes the ease with which a fluid may be made to flow through the material by an applied pressure gradient. Permeability is the fluid conductivity of the porous material".

"If horizontal linear flow of an incompressible fluid is established through a sample of porous material of length L in the direction of flow, and cross sectional area A, then the permeability K of the material is defined as

$$K = \frac{q\mu}{A(\Delta P/L)}$$

Here q is the fluid flow rate in volume per unit time, μ is the viscosity of the fluid and ΔP is the applied pressure difference across the length of the specimen"[28].

2.7 Heat Transfer Phenomenon

Definition 2.24. (Heat)

"Heat is defined as an energy that flows due to a difference in temperature. It flows in a direction from higher to lower temperature"[29].

Definition 2.25. (Heat Transfer)

"The science that deals with the mechanism and the rate of energy transfer due to a difference in temperature is known as heat transfer"[29].

2.7.1 Modes of Heat Transfer

"The process of heat transfer takes place by three distinct modes: conduction, convection and radiation"[29].

Definition 2.26. (Conduction)

"The mechanism of heat transfer due to a temperature gradient in a stationary medium is called conduction. The medium may be a solid or a fluid".

"A popular example of conduction heat is that when one end of a metallic spoon is dipped into a cup of hot tea, the other end becomes gradually hot"[29].

2.7.1.1 Fourier's Law

"The law which describes the rate of heat transfer in conduction is known as Fourier's law. According to this law, the rate of heat transfer per unit area normal to the direction of heat flow is directly proportional to the temperature gradient along that direction.
Mathematically,

$$q_x = -k\frac{dT}{dx}$$

where
$T(x)$= temperature distribution in a medium through which heat is being conducted.
$\frac{dT}{dx}$= Temperature gradient in $x-$direction, q_x is the heat flow rate per unit area normal to the heat flow direction. The minus sign indicates that heat flows in the direction of decreasing temperature and k is the thermal conductivity or the proportionality constant, a transport property of the medium through which heat is conducted"[29].

Definition 2.27. (Convection)
"The mode by which heat is transferred between a solid surface and the adjacent fluid in motion when there is a temperature difference between the two is known as convection heat transfer. The temperature of the fluid stream refers either to its bulk or free stream temperature"[29].
"It is experienced that an increase in fluid motion cools the surface faster which means that the rate of heat transfer increases with an increase in the fluid motion. This implies that the heat is being convected away from the surface by the stream of fluid and hence the mode of heat transfer is known as convective heat transfer"[29].

The convection is of two types:

Definition 2.28. (Forced Convection)
"In forced convection, the fluid is forced to flow over a solid surface by external means such as fan, pump or atmospheric wind"[29].

Definition 2.29. (Natural Convection)
"When the fluid motion is caused by the buoyancy forces that are induced by density differences due to the variation in temperature in the fluid, the convection is called natural or free convection"[29].

There are many systems and processes like heat transfer in buildings, solar collectors, electronic cooling and heat exchangers, which are based on convective heat transfer.

Definition 2.30. (Radiation)
"The exchange of radiant energy (energy in the form of electromagnetic waves emitted by any substance at a finite temperature) in all directions and at all wavelengths from a very low one to a very high one, between two bodies at different temperatures is defined as heat transfer between the bodies by radiation"[29].

"Radiation heat transmission does not always necessitate the presence of a medium, and that it is most efficient in a vacuum"[29].

2.8 Magnetohydrodynamics (MHD)

"Magnetohydrodynamics is the branch of continuum mechanics which deals with the motion of an electrically conducting fluid in the presence of a magnetic field" [30]. Examples of such fluid are liquid metals, plasmas, electrolytes and salt water.

Some applications of magnetohydrodynamics can be found in Sunspots, electromagnetic pump and the flow meter. Magnetohydrodynamics is applied in fission and fusion, astrophysics and geophysics, metallurgy, etc. In geophysics and astrophysics, to study the solar structures and stellar, inter- stellar matter, radio propagation through the ionosphere, magnetohydrodynamics is applied.

2.9 Governing Equations

2.9.1 Continuity Equation

"The law of conservation of mass states that mass can neither be created nor be destroyed. For a control volume, it can be stated as:
Rate at which mass enters the region = Rate at which mass leaves the region + Rate of accumulation of mass in the region" [29].
Mathematically, for compressible fluid flow:

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \mathbf{V}) = 0 \tag{2.1}$$

where ρ is the fluid density, t is for time and $\mathbf{V}$ shows the fluid velocity vector. which is known as the continuity equation of a compressible flow.
In case of a steady flow, $\frac{\partial \rho}{\partial t} = 0$
(2.1) $\implies \nabla.(\rho \mathbf{V}) = 0$
In case of incompressible flow, $\rho =$constant.
The continuity equation for incompressible flow becomes,

$$\nabla.\mathbf{V} = 0$$

2.9.2 Momentum Equation

"The principle of conservation of momentum is, in effect, an application of Newton's second law of motion to an element of the fluid. It is stated that the rate at which the momentum of the fluid mass is changing is equal to the net external force acting on the mass"[31]. The two sources of forces are:

1. **Body Forces**
   "Body forces act directly on the volumetric mass of the fluid element. These forces 'act at a distance'; examples are gravitational, electric and magnetic forces"[32].

Body forces can be specified as force per unit mass.

$$\text{Body force} = \frac{\text{Force}}{\text{Mass}}.$$

2. **Surface Forces**

   "Surface forces act directly on the surface of the fluid element. They are due to only two sources: (i) the pressure distribution acting on the surface, imposed by the outside fluid surrounding the fluid element, and (ii) the shear and normal stress distributions acting on the surface, also imposed by the outside fluid 'tugging' or 'pushing' on the surface by means of friction"[32]. Surface forces are specified as force per unit area.

   $$\text{Surface force} = \frac{\text{Force}}{\text{Area}}.$$

   For example, Pressure, shear force.

Conservation of momentum can be expressed mathematically by relation

$$\frac{\partial(\rho\mathbf{V})}{\partial t} + \nabla.(\rho\mathbf{V})\mathbf{V} = \rho\mathbf{f} + \nabla.\mathbf{T} \tag{2.2}$$

Equation (2.2) is the momentum equation for both compressible and incompressible fluid.

In case of incompressible fluid,

(2.2) $\Longrightarrow$

$$\rho\left[\frac{\partial\mathbf{V}}{\partial t} + (\mathbf{V}.\nabla)\mathbf{V}\right] = \rho\mathbf{f} + \nabla.\mathbf{T} \tag{2.3}$$

where $\mathbf{f}$ shows the body forces and $\mathbf{T}$ represents resultant of surface forces, $\mathbf{V}$ indicates the velocity of fluid and ρ is the desity of fluid.

2.9.3 Navier Stokes Equations

Consider the constitutive relation for Newtonian fluid as:

$$\mathbf{T} = -pI + \mu(\nabla\mathbf{V} + (\nabla\mathbf{V})^t) \tag{2.4}$$

where identity tensor is represented by I, p represents pressure, viscosity or dynamic viscosity is represted as μ, T shows resultant of surface forces and t indicates the transpose.

Let $\mathbf{V}$ be the velocity vector having components (u, v, w). Then

$$\nabla\mathbf{V}+(\nabla\mathbf{V})^t = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{bmatrix} + \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix} = \begin{bmatrix} 2\frac{\partial u}{\partial x} & \frac{\partial v}{\partial x}+\frac{\partial u}{\partial y} & \frac{\partial w}{\partial x}+\frac{\partial u}{\partial z} \\ \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x} & 2\frac{\partial v}{\partial y} & \frac{\partial w}{\partial y}+\frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial z}+\frac{\partial w}{\partial x} & \frac{\partial v}{\partial z}+\frac{\partial w}{\partial y} & 2\frac{\partial w}{\partial z} \end{bmatrix} \tag{2.5}$$

using the relation,

$$\tau = \mu(\nabla\mathbf{V} + (\nabla\mathbf{V})^{\mathbf{t}}) \tag{2.6}$$

x-component of τ can be expressed in components form as:

$$\tau_{xx} = 2\mu\frac{\partial u}{\partial x}; \tau_{xy} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right); \tau_{xz} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right). \tag{2.7}$$

For only x-component, we have from (2.4)

$$(\nabla.\mathbf{T})_x = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z}$$

using values from (2.7) and simplify

$$(\nabla.\mathbf{T})_x = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + \mu\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$$

$$(\nabla.\mathbf{T})_x = -\frac{\partial p}{\partial x} + \mu\Delta u + \mu\frac{\partial}{\partial x}(\nabla.\mathbf{V}).$$

In case of incompressible fluid, $\nabla.\mathbf{V} = 0$, so

$$(\nabla.\mathbf{T})_x = -\frac{\partial p}{\partial x} + \mu\Delta u.$$

Similarly for y and z component, we have

$$(\nabla.\mathbf{T})_y = -\frac{\partial p}{\partial y} + \mu\Delta v.$$

$$(\nabla.\mathbf{T})_z = -\frac{\partial p}{\partial z} + \mu\Delta w.$$

From (2.3) , x, y and z-component of momentum equation can be expressed in the form:

$$\rho\left[\frac{\partial u}{\partial t}+(\mathbf{V}.\nabla)u\right]=\rho f_x-\frac{\partial p}{\partial x}+\mu\Delta u, \tag{2.8}$$

$$\rho\left[\frac{\partial v}{\partial t}+(\mathbf{V}.\nabla)v\right]=\rho f_y-\frac{\partial p}{\partial y}+\mu\Delta v, \tag{2.9}$$

$$\rho\left[\frac{\partial w}{\partial t}+(\mathbf{V}.\nabla)w\right]=\rho f_z-\frac{\partial p}{\partial z}+\mu\Delta w. \tag{2.10}$$

Equations (2.8), (2.9), (2.10) are referred as Navier-Stokes equations.

2.9.4 Energy Equation

2.9.4.1 First Law of Thermodynamics

"The first law of thermodynamics that will be applied to an element of the fluid states that the rate of change of the total energy (intrinsic plus kinetic) of the fluid as it flows is equal to the sum of the rate at which work is being done on the fluid by external forces and the rate at which heat is being added by conduction[31]".

As the law of conservation of energy based on first law of thermodynamics, so by using first law of thermodynamics,

Total energy= E = sum of internal forces + kinetic energy.

$$E=i+\frac{V^2}{2}$$
$$E=i+\frac{1}{2}(u^2+v^2+w^2)$$

Therefore

$$\rho\frac{DE}{Dt}=\text{div}(k\text{grad}T)-\nabla.(p\mathbf{V})+\frac{\partial u\tau_{xx}}{\partial x}+\frac{\partial u\tau_{yx}}{\partial y}+\frac{\partial u\tau_{zx}}{\partial z}+\frac{\partial v\tau_{xy}}{\partial x}+\frac{\partial v\tau_{yy}}{\partial y}+\frac{\partial v\tau_{zy}}{\partial z}+\frac{\partial w\tau_{xz}}{\partial x}+\frac{\partial w\tau_{yz}}{\partial y}+\frac{\partial w\tau_{zz}}{\partial z}+S_E$$

where S_E represent the source term.

2.10 Dimensionless Numbers

Dimensionless numbers are used in fluid mechanics to make the problems easier by neglecting certain effects in flow field. The terminology dimensionless is infact involve in fluid mechanics problems that may differ in scales. A dimensionless quantity is the basically ratio of two quantities which are dimensionally equal. It can also be express in the ratio of product of quantities in numerator and denominator.

There are many dimensionless parameters in mechanics which are used to show the stability condition of the numerical solution. In case of free convection or natural convection, following dimensionless parameters may be used:

Definition 2.31. (Prandtl Number)

It is a dimensionless quantity used in fluid mechanics can be expressed as, "the ratio of the momentum diffusivity (viscosity) to the thermal diffusivity. It characterizes the physical properties of a fluid with convective and diffusive heat transfers"[33].

$$Pr = \frac{\text{Momentum diffusitivity}}{\text{Thermal diffusivity}} = \frac{\nu}{\alpha}$$

where ν = momentum diffusivity or kinematic viscosity, α= thermal diffusivity.

It is important to keep in mind that:

For small values of Prandtl number, $Pr << 1$, heat transfer by conduction dominated over convection. When Prandtl number is, $Pr >> 1$, heat transfer by convection dominated over conduction. For the case when Prandtl number is about 1 or $Pr = 1$, indicates that both convection and conduction are at about same rate.

Definition 2.32. (Rayleigh Number)

"It characterizes the free convection heat transfer along a heat-exchanging surface. It expresses the buoyancy to diffusion ratio or, alternatively, the free convection thermal instability in fluids"[33].

$$Ra = \frac{g\beta\Delta T L^3}{\nu\alpha}$$

where g is gravitational acceleration, L is characteristic length, δT represent temperature difference, β shows volume thermal expansion coefficient, ν represent kinematic viscosity, α indicate thermal diffusivity.

Definition 2.33. (Nusselt Number)

"It is a dimensionless number expresses the ratio of convective heat transfer to conductive heat transfer across boundary.

$$Nu = \frac{\text{Convective heat transfer}}{\text{Conductive heat transfer}} = \frac{h}{\frac{k}{L}} = \frac{hL}{k}$$

where h is for convective heat transfer coefficient, k shows thermal conductivity, L is characteristic length".

Definition 2.34. (Darcy Number)

"It is a dimensionless number expresses the ratio of the permeability of the medium to its cross-sectional area usually the diameter squared. It is commonly used in heat transfer through porous media.

$$Da = \frac{K}{L^2}$$

where K is for permeability of the medium and L is characteristic length".

Definition 2.35. (Hartmann Number)

"It expresses the ratio of the induced electrodynamic (magnetic) force to the hydrodynamic force of the viscosity. It characterizes the magnetic field influence on the flow of viscous, electrically conducting fluid"[33]. Mathematically, Hartmann number can be written as:

$$Ha = B_0 L \sqrt{\frac{\sigma}{\rho\nu}}, \tag{2.11}$$

where B_0, L, σ, ρ and ν are magnetic field strength, characteristic length, electrical conductivity, fluid density and kinematic viscosity respectively".

Definition 2.36. (Eckert Number)

"The Eckert number relates the kinetic energy to the enthalpy of a fluid"[34]. It is given by

$$Ec = \frac{\cup^2}{c_p \Delta T}$$

where $\cup$, c_p and ΔT represent buoyancy velocity scale, specific heat capacity of fluid and temperature difference.

"The Eckert number is used to characterize the influence of self-heating of a fluid as a consequence of dissipation effects. At high flow velocities, the temperature profile in a fluidic system is not just dominated by the temperature gradients that are present in the system, but also be effects of dissipation due to internal friction of the fluid. This will result in self-heating and thus in a change of the temperature profile. The Eckert number allows judging if the effects of self-heating due to dissipation can be neglected ($Ec << 1$) or not"[34].

2.11 Finite Element Methods

For the approximate solution of many engineering problems, a numerical technique finite element method is used. There are many complex problems in science and industries where exact solutions are difficult to obtain. We need to obtain the approximated solution for such type of problems. The finite element method (FEM) approximate the solution of complete region or domain by considering the sub-domains or elements. In this way a piecewise approximation is obtained for the whole domain.

"The solution of a continuum problem by FEM is approximated by the following step-by-step process:

1. **Discretize the Continuum** Divide the solution region into non-overlapping elements or sub-regions. The finite element discretization allows a variety of element shapes, for example, triangles, quadrilaterals. Each element is formed by the connection of a certain number of nodes. The number of nodes employed to form an element depends on the type of element (or interpolation function).

2. **Select Interpolation or Shape Functions** The next step is to choose the type of interpolation function that represents the variation of the field variable over an element. The number of nodes form an element; the nature and number of unknowns at each node decide the variation of a field variable within the element.

3. **Form Element Equations (Formulation)** To determine the matrix equations that express the properties of the individual elements by forming an element Left Hand Side (LHS) matrix and load vector.

4. **Assemble the Element Equations to Obtain a System of Simultaneous Equations** To find the properties of the overall system, we must assemble all the individual element equations, that is, to combine the matrix equations of each element in an appropriate way such that the resulting matrix represents the behaviour of the entire solution region of the problem. The boundary conditions must be incorporated after the assemblage of the individual element contributions.

5. **Solve the System of Equations** The resulting set of algebraic equations, now solved to obtain the nodal values of the field variable, for example, temperature.

   The finite element method involves the discretization of both the domain and the governing equations. In this process, the variables are represented in a piece-wise manner over the domain"[35].

2.11.1 Elements and Shape Functions

Since in the numerical technique FEM both the governing equations and the problem domain is discretized. So, the approximation of solution is obtained by dividing the whole solution region into many small regions. These small regions are called elements. This approximation is obtained by using suitable known function. Such type of functions are known as shape functions, or basis functions, or interpolating functions. By interpolating the nodal values within each element, interpolating functions used to calculate the value of field variable. As the polynomials can be differentiated, or integrated easily and also by increasing the degree of polynomial the accuracy can be upgraded. Therefore, polynomial functions are used mostly as shape functions.

2.11.2 Formulation of FEM Model

There are three methods for formulation of FEM model, Direct method, Variational method and weighted Residual approach.

2.11.2.1 Weighted Residual Method

Method of weighted residual is useful method to find approximate solutions if the physical problem is described as a differential equation of the form

$$\mathcal{L}u = f \tag{2.12}$$

where $f(x)$ is known function, $u(x)$ represent dependent variable and it is considered as unknown function, $\mathcal{L}$ shows differential operator for spatial derivative of u. Using weighted residual method, approximate solution or trial solution $\tilde{u}(x)$ which satisfy boundary conditions is assumed. Since $\tilde{u}$ is an approximate solution, so it will not satisfies the differential equation (2.12).

$$R(\tilde{u}(x)) = \mathcal{L}\tilde{u} - f \neq 0 \tag{2.13}$$

where

$$\tilde{u}(x) = \sum_{i=1}^{N} u_i \phi_i(x) \tag{2.14}$$

and

$$\tilde{u(x)} = u_1\phi_1(x) + u_2\phi_2(x) + u_3\phi_3(x) + ... + u_N\phi_N(x).$$

The functions ϕ_i are used as basis functions. As the function space ϕ has finite dimensions, in general the expression (2.14) cannot satisfy the differential equation

(2.12) in the domain for each point. This implies that the approximate solution or trial solution $\tilde{u}$ cannot be same like u (exact solution. By letting N grow, the approximate solution becomes very close to the exact solution.
From Residual (2.13), by finding a way to make this residual small or approximately zero, the approximated solution of BVP can be evaluated. In finite element method (FEM), the approximate solution can be obtained by making suitable number of weighted integrals of residual over the domain Ω, be zero.

$$\int_{\Omega} \omega R d\Omega = 0 \tag{2.15}$$

where $w = \{\omega_i; i = 1, 2, ..., N\}$ is the suitable collection of weighting functions, which shows that obtained approximated soulution is for N being finite.

$$\omega(x) = \sum_{i=1}^{N} \omega_i \psi_i(x) \tag{2.16}$$

where $\psi_i(x)$ are known functions and ω_i are parameters.
By using (2.16) in (2.15), we get

$$[A]\mathbf{u} = \mathbf{F}$$

as a system of algebraic equations. By solving above system for N-unknowns, $u_i's$ provided that a suitable weight function ω is selected.

2.11.2.2 Selection of Weight Function using Galerkin Method

Weight functions are choosen to be the trial functions themselves in a Galerkin weighted residual method, *i.e.*

$$\omega_i = \frac{\partial \tilde{u}}{\partial u_i} = \phi_i, (i = 1, 2, 3, ..., N)$$

The unrevealed coefficients in the estimated outcomes are evaluated by placing the integral over Ω (domain) of the weighted residual to zero.
In standard FEM the Galerkin method is often used to derive the element equation.

Chapter 3

Natural Convection of Casson Fluid in Partially Heated Porous Cavity

In this chapter, the features of natural convection of Casson fluid in permeable enclosure, reviewed numerically. Considered problem is governed by dimensionless coupled partial differential equations. Finite element method in conjunction with Galerkin weighted residual method is used for solution of problem. The impact of governing parameters is examined through streamlines, isotherms and graphs.

3.1 Problem Description

Consider a square porous cavity with dimension L filled with incompressible Casson fluid. A constant temperature T_c along side walls is considered during whole investigation. The upper wall and the section of bottom wall assumed to be adiabatic as shown in Figure 3.1. The bottom wall is partially heated and it is varying to study the effect of heated area on fluid flow and heat transfer characteristics. The Casson fluid is considered to satisfy the Boussinesq approximation.

The symbolic diagram of the problem under investigation is shown in Figure 3.1.

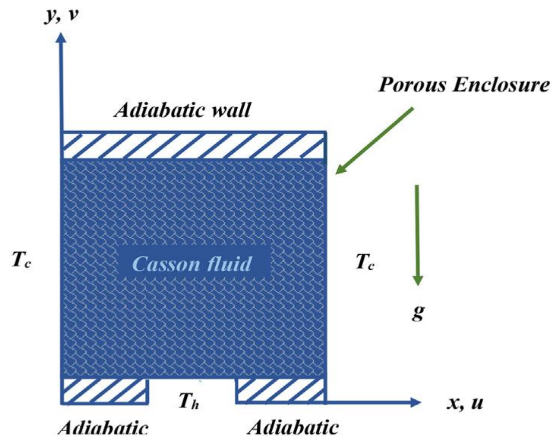

FIGURE 3.1: Physical description of the problem

3.2 Governing Equations

Under assumptions mentioned as above, the governing system for Casson fluid flow in porous cavity which describes the flow with corresponding boundary conditions are:

Momentum Equation:

Momentum equation for u-velocity

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(1 + \frac{1}{\gamma}\right)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - \frac{\nu}{K}u \tag{3.1}$$

Momentum equation for v-velocity

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(1 + \frac{1}{\gamma}\right)\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) - \frac{\nu}{K}v - g\beta(T - T_c) \tag{3.2}$$

Continuity Equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{3.3}$$

Energy Equation:

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) \tag{3.4}$$

Boundary Conditions:

The imposed boundary conditions are expressed as:

- On the cavity's left wall

$$\text{At} \quad 0 < y < L \quad \text{and} \quad x = 0; \quad u = v = 0, \quad T = T_c$$

- On the cavity's right wall

$$\text{At} \quad 0 < y < L \quad \text{and} \quad x = L; \quad u = v = 0, \quad T = T_c$$

- On the cavity's bottom portions

$$\text{At} \quad \frac{(1-\epsilon)\,L}{2} \leq x \leq \frac{(1+\epsilon)\,L}{2} \quad \text{and} \quad y = 0; \quad u = v = 0, T = T_h$$

$$\text{At} \quad 0 < x < \frac{(1-\epsilon)\,L}{2} \quad , \quad \frac{(1+\epsilon)\,L}{2} < x < L \quad \text{and} \quad y = 0; u = v = 0, \frac{\partial T}{\partial y} = 0$$

- On the cavity's top wall

$$\text{At} \quad 0 < x < L, \quad y = L; u = v = 0, \quad \frac{\partial T}{\partial y} = 0$$

In all the above dimensional governing equations and boundary conditions, u and v represent velocities along $x-$axis and $y-$axis respectively. Further, ρ is density of fluid, ν represents kinematic viscosity, T shows fluid temperature, volume expansion coefficient is represented by β, γ shows Casson fluid parameter, K is permeability of the porous medium, g is for gravitational acceleration and α is the thermal diffusivity.

3.2.1 Dimensionless Governing Equations

By using the transformation given in [22], dimensionless form of governing equations (3.1), (3.2), (3.3) and (3.4) can be converted as:

$$U = \frac{uL}{\alpha}, V = \frac{vL}{\alpha}, X = \frac{x}{L}, Y = \frac{y}{L}, \theta = \frac{T - T_c}{T_h - T_c}, \gamma = \mu_B \sqrt{2\pi_c} \frac{\partial p}{\partial y}$$

Momentum equation:

Momentum equation for u-velocity

$$\frac{\alpha U}{L}\frac{\partial\left(\frac{\alpha U}{L}\right)}{\partial\left(LX\right)}+\frac{\alpha V}{L}\frac{\partial\left(\frac{\alpha U}{L}\right)}{\partial\left(LY\right)}=-\frac{1}{\rho}\frac{\partial\left(\frac{\rho\alpha^2P}{L^2}\right)}{\partial\left(LX\right)}+\nu\left(1+\frac{1}{\gamma}\right)\left(\frac{\partial^2\left(\frac{\alpha U}{L}\right)}{\partial\left(LX\right)^2}+\frac{\partial^2\left(\frac{\alpha U}{L}\right)}{\partial\left(LY\right)^2}\right)-\frac{\nu}{K}\left(\frac{\alpha U}{L}\right)$$

$$\frac{\alpha^2}{L^3}\left[U\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial Y}\right]=\frac{\alpha^2}{L^3}\left(-\frac{\partial P}{\partial X}\right)+\nu\left(1+\frac{1}{\gamma}\right)\frac{\alpha}{L^3}\left(\frac{\partial^2 U}{\partial X^2}+\frac{\partial^2 U}{\partial Y^2}\right)-\frac{\alpha}{L}\frac{\nu}{K}U$$

$$\frac{\alpha^2}{L^3}\left[U\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial Y}\right]=\frac{\alpha^2}{L^3}\left[-\frac{\partial P}{\partial X}+\frac{\nu}{\alpha}\left(1+\frac{1}{\gamma}\right)\left(\frac{\partial^2 U}{\partial X^2}+\frac{\partial^2 U}{\partial Y^2}\right)-\frac{L^2}{\alpha}\frac{\nu}{K}U\right]$$

$$U\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial Y}=-\frac{\partial P}{\partial X}+\frac{\nu}{\alpha}\left(1+\frac{1}{\gamma}\right)\left(\frac{\partial^2 U}{\partial X^2}+\frac{\partial^2 U}{\partial Y^2}\right)-\frac{\frac{\nu}{\alpha}}{\frac{K}{L^2}}U$$

$$U\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial Y}=-\frac{\partial P}{\partial X}+Pr\left(1+\frac{1}{\gamma}\right)\left(\frac{\partial^2 U}{\partial X^2}+\frac{\partial^2 U}{\partial Y^2}\right)-\frac{Pr}{Da}U \tag{3.5}$$

Momentum equation for v-velocity

$$\frac{\alpha U}{L}\frac{\partial\left(\frac{\alpha V}{L}\right)}{\partial\left(LX\right)}+\frac{\alpha V}{L}\frac{\partial\left(\frac{\alpha V}{L}\right)}{\partial\left(LY\right)}=-\frac{1}{\rho}\frac{\partial\left(\frac{\rho\alpha^2P}{L^2}\right)}{\partial\left(LY\right)}+\nu\left(1+\frac{1}{\gamma}\right)\left(\frac{\partial^2\left(\frac{\alpha V}{L}\right)}{\partial\left(LX\right)^2}+\frac{\partial^2\left(\frac{\alpha V}{L}\right)}{\partial\left(LY\right)^2}\right)-\frac{\nu}{K}\left(\frac{\alpha V}{L}\right)-g\beta\left(T_h-T_c\right)\theta$$

$$\frac{\alpha^2}{L^3}\left[U\frac{\partial V}{\partial X}+V\frac{\partial V}{\partial Y}\right]=\frac{\alpha^2}{L^3}\left[-\frac{\partial P}{\partial Y}\right]+\frac{\alpha}{L^3}\nu\left(1+\frac{1}{\gamma}\right)\left(\frac{\partial^2 V}{\partial X^2}+\frac{\partial^2 V}{\partial Y^2}\right)-\frac{\alpha\nu}{LK}V-g\beta\left(T_h-T_c\right)\theta$$

$$U\frac{\partial V}{\partial X}+V\frac{\partial V}{\partial Y}=-\frac{\partial P}{\partial Y}+\frac{\nu}{\alpha}\left(1+\frac{1}{\gamma}\right)\left(\frac{\partial^2 V}{\partial X^2}+\frac{\partial^2 V}{\partial Y^2}\right)-\frac{L^2\nu}{\alpha K}V-\frac{L^3}{\alpha^2}g\beta\left(T_h-T_c\right)\theta$$

$$U\frac{\partial V}{\partial X}+V\frac{\partial V}{\partial Y}=-\frac{\partial P}{\partial Y}+Pr\left(1+\frac{1}{\gamma}\right)\left(\frac{\partial^2 V}{\partial X^2}+\frac{\partial^2 V}{\partial Y^2}\right)$$
$$-\frac{Pr}{Da}V-\frac{g\beta\left(T_h-T_c\right)L^3}{\nu^2}\frac{\nu^2}{\alpha^2}\theta$$

$$U\frac{\partial V}{\partial X}+V\frac{\partial V}{\partial Y}=-\frac{\partial P}{\partial Y}+Pr\left(1+\frac{1}{\gamma}\right)\left(\frac{\partial^2 V}{\partial X^2}+\frac{\partial^2 V}{\partial Y^2}\right)-\frac{Pr}{Da}V-RaPr\theta \quad (3.6)$$

Continuity Equation:

$$\frac{\partial\left(\frac{\alpha U}{L}\right)}{\partial\left(LX\right)}+\frac{\partial\left(\frac{\alpha V}{L}\right)}{\partial\left(LY\right)}=0$$

$$\frac{\alpha}{L^2}\left(\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}\right)=0$$

$$\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}=0 \quad (3.7)$$

Energy Equation:

$$\frac{\alpha}{L}U\frac{\partial\left(\Delta T\theta+T_c\right)}{\partial\left(LX\right)}+\frac{\alpha}{L}V\frac{\partial\left(\Delta T\theta+T_c\right)}{\partial\left(LY\right)}=\alpha\left(\frac{\partial^2\left(\Delta T\theta+T_c\right)}{\partial\left(LX\right)^2}+\frac{\partial^2\left(\Delta T\theta+T_c\right)}{\partial\left(LY\right)^2}\right)$$

$$\frac{\alpha}{L^2}\Delta T\left[U\frac{\partial\theta}{\partial X}+V\frac{\partial\theta}{\partial Y}\right]=\frac{\alpha}{L^2}\Delta T\left[\frac{\partial^2\theta}{\partial X^2}+\frac{\partial^2\theta}{\partial Y^2}\right]$$

$$U\frac{\partial\theta}{\partial X}+V\frac{\partial\theta}{\partial Y}=\frac{\partial^2\theta}{\partial X^2}+\frac{\partial^2\theta}{\partial Y^2} \quad (3.8)$$

Dimensionless parameters used in governing equations (3.5) and (3.6) are as follows:

$$Pr=\frac{\nu}{\alpha},\quad Da=\frac{K}{L^2},\quad Ra=\frac{g\beta\left(T_h-T_c\right)L^3Pr}{\nu^2}$$

Dimensionless boundary conditions can be written as:

- On the cavity's left wall

$$\text{At}\quad 0<Y<1\quad\text{and}\quad X=0;U=V=0,\theta=0$$

- On the cavity's right wall

$$\text{At} \quad 0 < Y < 1 \quad \text{and} \quad X = 1; \quad U = V = 0, \theta = 0$$

- On the cavity's bottom portions

$$\text{At} \qquad \frac{1-\epsilon}{2} \leq X \leq \frac{1+\epsilon}{2} \quad \text{and} \quad Y = 0; \quad U = V = 0, \theta = 1$$

$$\text{At} \quad 0 < X < \frac{1-\epsilon}{2} \;,\; \frac{1+\epsilon}{2} < X < 1 \;\text{and}\; Y = 0; \quad U = V = 0, \frac{\partial \theta}{\partial Y} = 0$$

- On the cavity's top wall

$$\text{At} \quad 0 < X < 1 \quad \text{and}\; Y = 1; \quad U = V = 0, \frac{\partial \theta}{\partial Y} = 0$$

For the heated region of the bottom wall, the average Nusselt number is given by

$$Nu_{avg} = \int_{\frac{1-\epsilon}{2}}^{\frac{1+\epsilon}{2}} Nu(X) dX,$$

where $Nu(X) = -\left(\frac{\partial \theta}{\partial Y}\right)_{Y=0}$ is the local Nusselt number.

3.3 Weak Formulation

In order to solve the non-dimensional governing equations (3.5), (3.6), (3.7), (3.8) together with boundary conditions, we have used finite element method in conjunction with Galerkin weighted residual method. First, we integrated the system together with weight function to get integral form or weak form of the system which is valid for entire solution domain. Finally, we used the set of approximated trial functions valid only over a part of the domain to get approximated solution.
Strong form of governing equations:

$$U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} = -\frac{\partial P}{\partial X} + Pr\left(1 + \frac{1}{\gamma}\right)\left(\frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Y^2}\right) - \frac{Pr}{Da}U, \tag{3.9}$$

$$U\frac{\partial V}{\partial X} + V\frac{\partial V}{\partial Y} = -\frac{\partial P}{\partial Y} + Pr\left(1 + \frac{1}{\gamma}\right)\left(\frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial Y^2}\right) - \frac{Pr}{Da}V - RaPr\theta, \quad (3.10)$$

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0, \quad (3.11)$$

$$U\frac{\partial \theta}{\partial X} + V\frac{\partial \theta}{\partial Y} = \frac{\partial^2 \theta}{\partial X^2} + \frac{\partial^2 \theta}{\partial Y^2}. \quad (3.12)$$

Here U, V, θ and P are solution spaces defined on continuously varying infinite dimensional space Ω, the matter of fact is that the achievement of the solution in such a large space is not possible. The main objective is to find some suitable spaces to get functions with finite properties or parameters.

Let $W = (H^1(\Omega), H^1(\Omega), H^1(\Omega))$ be the test space for velocity components and temperature and $Q = L_2(\Omega)$ is the test space for pressure. Let w and q be the respective test functions such that $w \in W$ and $q \in Q$. For the variational formulation the components of momentum and energy equations are multiplied by test function $w \in W$ while the continuity equation is multiplied by test function $q \in Q$. The weak formulation of the strong form of governing PDEs from (3.9) to (3.12) can be noted as:

$$\int_\Omega \left(U\frac{\partial U}{\partial X}\right) w + \left(V\frac{\partial U}{\partial Y}\right) w \, d\Omega = -\int_\Omega \frac{\partial P}{\partial X} w \, d\Omega + Pr\left(1 + \frac{1}{\gamma}\right)\int_\Omega \left(\frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Y^2}\right) w \, d\Omega - \frac{Pr}{Da}\int_\Omega U w \, d\Omega$$

$$\int_\Omega \left(U\frac{\partial V}{\partial X} + V\frac{\partial V}{\partial Y}\right) w \, d\Omega = -\int_\Omega \frac{\partial P}{\partial Y} w \, d\Omega + Pr\left(1 + \frac{1}{\gamma}\right)\int_\Omega \left(\frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial Y^2}\right) w \, d\Omega - \frac{Pr}{Da}\int_\Omega V w \, d\Omega - RaPr\int_\Omega \theta w \, d\Omega$$

$$\int_\Omega \left(\frac{\partial U}{\partial X}\right) q + \left(\frac{\partial V}{\partial Y}\right) q \, d\Omega = 0$$

$$\int_\Omega \left(U\frac{\partial \theta}{\partial X}\right) w + \left(V\frac{\partial \theta}{\partial Y}\right) w \, d\Omega = \int_\Omega \left(\frac{\partial^2 \theta}{\partial X^2} + \frac{\partial^2 \theta}{\partial Y^2}\right) w d\Omega$$

for all $(w,\ q) \in W \times Q$.

Consider

$$\int_\Omega \left(U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y}\right) w \, d\Omega = \\ -\int_\Omega \frac{\partial P}{\partial X} w \, d\Omega + Pr\left(1+\frac{1}{\gamma}\right)\int_\Omega \left(\frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Y^2}\right) w \, d\Omega - \frac{Pr}{Da}\int_\Omega U w \, d\Omega$$

Using Green's theorem for Laplacian term as

$$\int_\Omega \psi \Delta \phi \, d\Omega = -\int_\Omega \nabla\phi\nabla\psi \, d\Omega + \int_\Omega \psi \left(\nabla\phi.\mathbf{n}\right) \, d\Gamma$$

$$\int_\Omega \left(U\frac{\partial U}{\partial X}\right) w + \left(V\frac{\partial U}{\partial Y}\right) w \, d\Omega = \\ -\int_\Omega \frac{\partial P}{\partial X} w \, d\Omega + Pr\left(1+\frac{1}{\gamma}\right)\left[-\int_\Omega \frac{\partial U}{\partial X}\frac{\partial w}{\partial X} \, d\Omega + \oint_\Gamma w\frac{\partial U}{\partial X} n_x \, d\Gamma\right] + \\ Pr\left(1+\frac{1}{\gamma}\right)\left[-\int_\Omega \frac{\partial U}{\partial Y}\frac{\partial w}{\partial Y} \, d\Omega + \oint_\Gamma w\frac{\partial U}{\partial Y} n_y \, d\Gamma\right] - \frac{Pr}{Da}\int_\Omega U w \, d\Omega$$

$$\int_\Omega \left(U\frac{\partial U}{\partial X}\right) w + \left(V\frac{\partial U}{\partial Y}\right) w \, d\Omega = \\ -\int_\Omega \frac{\partial P}{\partial X} w \, d\Omega + Pr\left(1+\frac{1}{\gamma}\right)\left[-\int_\Omega \frac{\partial U}{\partial X}\frac{\partial w}{\partial X} \, d\Omega + \oint_\Gamma w\frac{\partial U}{\partial X} n_x \, d\Gamma\right] + \\ Pr\left(1+\frac{1}{\gamma}\right)\left[-\int_\Omega \frac{\partial U}{\partial Y}\frac{\partial w}{\partial Y} \, d\Omega + \oint_\Gamma w\frac{\partial U}{\partial Y} n_y \, d\Gamma\right] - \frac{Pr}{Da}\int_\Omega U w \, d\Omega$$

$$\int_\Omega \left(U\frac{\partial U}{\partial X}\right) w + \left(V\frac{\partial U}{\partial Y}\right) w \, d\Omega = \\ -\int_\Omega \frac{\partial P}{\partial X} w \, d\Omega - Pr\left(1+\frac{1}{\gamma}\right)\int_\Omega \frac{\partial U}{\partial X}\frac{\partial w}{\partial X} + \frac{\partial U}{\partial Y}\frac{\partial w}{\partial Y} \, d\Omega - \\ \frac{Pr}{Da}\int_\Omega U w \, d\Omega \quad (3.13)$$

which is the weak form of $x-$component of momentum equation.

Similarly, we obtain the weak form for $y-$component of momentum equation as follows:

$$\int_\Omega \left(U\frac{\partial V}{\partial X}\right) w + \left(V\frac{\partial V}{\partial Y}\right) w\, d\Omega = \\ -\int_\Omega \frac{\partial P}{\partial Y} w\, d\Omega + Pr\left(1+\frac{1}{\gamma}\right)\int_\Omega \left(\frac{\partial^2 V}{\partial X^2}+\frac{\partial^2 V}{\partial Y^2}\right) w\, d\Omega - \frac{Pr}{Da}\int_\Omega V w\, d\Omega - \\ RaPr\int_\Omega \theta w\, d\Omega$$

$$\int_\Omega \left(U\frac{\partial V}{\partial X}\right) w + \left(V\frac{\partial V}{\partial Y}\right) w\, d\Omega = -\int_\Omega \frac{\partial P}{\partial Y} w\, d\Omega + \\ Pr\left(1+\frac{1}{\gamma}\right)\left[-\int_\Omega \frac{\partial V}{\partial X}\frac{\partial w}{\partial X}\, d\Omega + \oint_\Gamma w\frac{\partial V}{\partial X} n_x\, d\Gamma - \int_\Omega \frac{\partial V}{\partial Y}\frac{\partial w}{\partial Y}\, d\Omega + \oint_\Gamma w\frac{\partial V}{\partial Y} n_y\, d\Gamma\right] \\ -\frac{Pr}{Da}\int_\Omega V w\, d\Omega - RaPr\int_\Omega \theta w\, d\Omega$$

$$\int_\Omega \left(U\frac{\partial V}{\partial X} + V\frac{\partial V}{\partial Y}\right) w\, d\Omega = -\int_\Omega \frac{\partial P}{\partial Y} w\, d\Omega + Pr\left(1+\frac{1}{\gamma}\right) \\ \left[-\int_\Omega \frac{\partial V}{\partial X}\frac{\partial w}{\partial X}\, d\Omega + \oint_\Gamma w\frac{\partial V}{\partial X} n_x\, d\Gamma - \int_\Omega \frac{\partial V}{\partial Y}\frac{\partial w}{\partial Y}\, d\Omega + \oint_\Gamma w\frac{\partial V}{\partial Y} n_y\, d\Gamma\right] - \\ \frac{Pr}{Da}\int_\Omega U w\, d\Omega - RaPr\int_\Omega \theta w\, d\Omega$$

$$\int_\Omega \left(U\frac{\partial V}{\partial X} + V\frac{\partial V}{\partial Y}\right) w\, d\Omega = -\int_\Omega \frac{\partial P}{\partial Y} w\, d\Omega - \\ Pr\left(1+\frac{1}{\gamma}\right)\int_\Omega \left(\frac{\partial V}{\partial X}\frac{\partial w}{\partial X} + \frac{\partial V}{\partial Y}\frac{\partial w}{\partial Y}\right) d\Omega - \frac{Pr}{Da}\int_\Omega V w\, d\Omega - RaPr\int_\Omega \theta w\, d\Omega \tag{3.14}$$

which is weak form of $y-$component of momentum equation.

Now for continuity equation, we get

$$\int_\Omega \left(\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y}\right) q\, d\Omega = 0 \tag{3.15}$$

The energy equation in the same way becomes:

$$\int_\Omega \left(U\frac{\partial \theta}{\partial X} + V\frac{\partial \theta}{\partial Y}\right) w\, d\Omega = \int_\Omega \left(\frac{\partial^2 \theta}{\partial X^2} + \frac{\partial^2 \theta}{\partial Y^2}\right) w d\Omega$$

$$\int_{\Omega}\left(U\frac{\partial\theta}{\partial X}\right)w+\left(V\frac{\partial\theta}{\partial Y}\right)w\,d\Omega=\int_{\Omega}w\frac{\partial}{\partial X}\left(\frac{\partial\theta}{\partial X}\right)\,d\Omega+\int_{\Omega}w\frac{\partial}{\partial Y}\left(\frac{\partial\theta}{\partial Y}\right)\,d\Omega$$

Using Green's theorem

$$\int_{\Omega}\left(U\frac{\partial\theta}{\partial X}\right)w+\left(V\frac{\partial\theta}{\partial Y}\right)w\,d\Omega=-\int_{\Omega}\frac{\partial\theta}{\partial X}\frac{\partial w}{\partial X}\,d\Omega+\oint_{\Gamma}w\frac{\partial\theta}{\partial X}n_x\,d\Gamma-\int_{\Omega}\frac{\partial\theta}{\partial Y}\frac{\partial w}{\partial Y}\,d\Omega+\oint_{\Gamma}w\frac{\partial\theta}{\partial Y}n_y\,d\Gamma \quad (3.16)$$

Next we approximate the infinite test and solution space, using Galerkin discretization scheme to meet required accuracy. Consider the approximated solution Components as $U_h, V_h, P_h, \theta_h \approx U, V, P, \theta$, where $(W_h, Q_h) \approx (W, Q)$.
Now to find $(U_h, V_h, P_h, \theta_h)$

$$\int_{\Omega}\left(U_h\frac{\partial U_h}{\partial X}+V_h\frac{\partial U_h}{\partial Y}\right)w_h\,d\Omega=-\int_{\Omega}\frac{\partial P_h}{\partial X}w_h\,d\Omega+Pr\left(1+\frac{1}{\gamma}\right)\int_{\Omega}\left(\frac{\partial U_h}{\partial X}\frac{\partial w_h}{\partial X}+\frac{\partial U_h}{\partial Y}\frac{\partial w_h}{\partial Y}\right)\,d\Omega-\frac{Pr}{Da}\int_{\Omega}U_hw_h\,d\Omega \quad (3.17)$$

$$\int_{\Omega}\left(U_h\frac{\partial V_h}{\partial X}+V_h\frac{\partial V_h}{\partial Y}\right)w_h\,d\Omega=-\int_{\Omega}\frac{\partial P_h}{\partial Y}w_h\,d\Omega+Pr\left(1+\frac{1}{\gamma}\right)\int_{\Omega}\left(\frac{\partial V_h}{\partial X}\frac{\partial w_h}{\partial X}+\frac{\partial V_h}{\partial Y}\frac{\partial w_h}{\partial Y}\right)\,d\Omega-\frac{Pr}{Da}\int_{\Omega}V_hw_h\,d\Omega-RaPr\int_{\Omega}\theta_hw_h\,d\Omega \quad (3.18)$$

$$\int_{\Omega}\left(\frac{\partial U_h}{\partial X}+\frac{\partial V_h}{\partial Y}\right)q_h\,d\Omega=0 \quad (3.19)$$

$$\int_{\Omega}\left(U_h\frac{\partial\theta_h}{\partial X}+V_h\frac{\partial\theta_h}{\partial Y}\right)w_h\,d\Omega=-\int_{\Omega}\left(\frac{\partial\theta_h}{\partial X}\frac{\partial w_h}{\partial X}+\frac{\partial\theta_h}{\partial Y}\frac{\partial w_h}{\partial Y}\right)d\Omega \quad (3.20)$$

FEM approximation is achieved by using the approximate trial solution functions and trial test functions. These functions are the linear combination of nodal unknowns and shape functions which are linearly independent. Given below are the trial solution functions:

$$\mathbf{U_h}=\sum_{j=1}^{m}u_j\phi_j, \mathbf{V_h}=\sum_{j=1}^{m}v_j\phi_j, \mathbf{\theta_j}=\sum_{j=1}^{m}\theta_j\phi_j, \mathbf{P_h}=\sum_{j=1}^{l}p_j\psi_j.$$

Also test approximated functions can be defined for test spaces as

$w_h = \sum_{i=1}^{m} w_i\phi_i$ and $q_h = \sum_{i=1}^{l} q_i\psi_i$, where ϕ_j and ψ_j are shape functions.
By using these approximation in Eqs. (3.17), (3.18), (3.19), (3.20) weak formulation can be expressed as:

(3.17) $\implies$

$$\int_\Omega \left[\left(\sum u_j\phi_j\right)\frac{\partial}{\partial X}\left(\sum u_j\phi_j\right) + \sum v_j\phi_j\frac{\partial}{\partial Y}\left(\sum u_j\phi_j\right)\right]\phi_i\, d\Omega = -\int_\Omega \frac{\partial}{\partial X}\left(\sum p_j\psi_j\right)\phi_i\, d\Omega - Pr\left(1+\frac{1}{\gamma}\right)\int_\Omega\left[\frac{\partial}{\partial X}\left(\sum u_j\phi_j\right)\frac{\partial\phi_i}{\partial X} + \frac{\partial}{\partial Y}\left(\sum u_j\phi_j\right)\frac{\partial\phi_i}{\partial Y}\right]d\Omega - \frac{Pr}{Da}\int_\Omega\left(\sum u_j\phi_j\right)\phi_i\, d\Omega$$

$$\int_\Omega \left[\left(\sum \phi_j u_j\right)\frac{\partial\phi_j}{\partial X} + \left(\sum v_j\phi_j\right)\frac{\partial\phi_j}{\partial Y}\right]\phi_i d\Omega\{u_j\} + \sum\int_\Omega \frac{\partial\psi_j}{\partial X}\phi_i\, d\Omega\{p_j\} + Pr\left(1+\frac{1}{\gamma}\right)\sum\int_\Omega\left(\frac{\partial\phi_j}{\partial X}\frac{\partial\phi_i}{\partial X} + \frac{\partial\phi_j}{\partial Y}\frac{\partial\phi_i}{\partial Y}\right)d\Omega\{u_j\} - \frac{Pr}{Da}\sum\int_\Omega \phi_j\phi_i\, d\Omega\{u_j\} = 0 \tag{3.21}$$

(3.18) $\implies$

$$\int_\Omega \left[\left(\sum u_j\phi_j\right)\frac{\partial}{\partial X}\left(\sum v_j\phi_j\right) + \sum v_j\phi_j\frac{\partial}{\partial Y}\left(\sum v_j\phi_j\right)\right]\phi_i\, d\Omega = -\int_\Omega \frac{\partial}{\partial Y}\left(\sum p_j\psi_j\right)\phi_i\, d\Omega - Pr\left(1+\frac{1}{\gamma}\right)\int_\Omega\left[\frac{\partial}{\partial X}\left(\sum v_j\phi_j\right)\frac{\partial\phi_i}{\partial X} + \frac{\partial}{\partial Y}\left(\sum v_j\phi_j\right)\frac{\partial\phi_i}{\partial Y}\right]d\Omega - \frac{Pr}{Da}\int_\Omega\left(\sum v_j\phi_j\right)\phi_i\, d\Omega - RaPr\int_\Omega\left(\sum \theta_j\phi_j\right)\phi_i\, d\Omega$$

$$\int_\Omega \left[\left(\sum \phi_j u_j\right)\frac{\partial\phi_j}{\partial X} + \left(\sum v_j\phi_j\right)\frac{\partial\phi_j}{\partial Y}\right]\phi_i d\Omega\{v_j\} + \sum\int_\Omega \frac{\partial\psi_j}{\partial Y}\phi_i\, d\Omega\{p_j\} + Pr\left(1+\frac{1}{\gamma}\right)\sum\int_\Omega\left(\frac{\partial\phi_j}{\partial X}\frac{\partial\phi_i}{\partial X} + \frac{\partial\phi_j}{\partial Y}\frac{\partial\phi_i}{\partial Y}\right)d\Omega\{v_j\} + \frac{Pr}{Da}\sum\int_\Omega \phi_j\phi_i\, d\Omega\{v_j\} + RaPr\sum\int_\Omega \phi_i\phi_j\, d\Omega\{\theta_j\} = 0 \tag{3.22}$$

(3.19) $\implies$

$$\int_\Omega \frac{\partial}{\partial X}\left(\sum u_j\phi_j\right) + \frac{\partial}{\partial Y}\left(\sum v_j\phi_j\right)\psi_i\, d\Omega = 0$$

$$\sum \int_{\Omega} \frac{\partial \phi_j}{\partial X} \psi_i \, d\Omega \{u_j\} + \sum \int_{\Omega} \frac{\partial \phi_j}{\partial Y} \psi_i \, d\Omega \{v_j\} = 0 \tag{3.23}$$

(3.20) $\implies$

$$\int_{\Omega} \left[\sum u_j \phi_j \frac{\partial}{\partial X} \left(\sum \theta_j \phi_j \right) + \sum v_j \phi_j \frac{\partial}{\partial Y} \left(\sum \theta_j \phi_j \right) \right] \phi_i \, d\Omega = \int_{\Omega} \left[\frac{\partial}{\partial X} \left(\sum \theta_j \phi_j \right) \frac{\partial \phi_i}{\partial X} + \frac{\partial}{\partial Y} \left(\sum \theta_j \phi_j \right) \frac{\partial \phi_i}{\partial Y} \right] d\Omega$$

$$\int_{\Omega} \left[\left(\sum u_j \phi_j \right) \frac{\partial \phi_j}{\partial X} + \left(\sum v_j \phi_j \frac{\partial \phi_j}{\partial Y} \right) \right] \phi_i \, d\Omega \{\theta_j\} + \sum \int_{\Omega} \left(\frac{\partial \phi_j}{\partial X} \frac{\partial \phi_i}{\partial X} + \frac{\partial \phi_j}{\partial Y} \frac{\partial \phi_i}{\partial Y} \right) d\Omega \{\theta_j\} = 0 \tag{3.24}$$

The discrete system of non-linear algebraic equations in matrix form can be written as:

$$\begin{bmatrix} A_{11} & A_{12} & B_1 & A_{14} \\ A_{21} & A_{22} & B_2 & A_{24} \\ B_1^t & B_2^t & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix} \begin{bmatrix} \mathbf{U_h} \\ \mathbf{V_h} \\ \mathbf{P_h} \\ \theta_{\mathbf{h}} \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} \tag{3.25}$$

Here A, X, F are called block stiffness matrix, block solution vector and block load vector. The entries in block stiffness matrix for the corresponding local element are represented in terms of ϕ and ψ. Local elemental equations from eqs. (3.21), (3.22), (3.23), and (3.24) which are written in block stiffness matrix are given below as:

$$A_{11}^{ij} = \int_{\Omega} \left[\left(\sum \phi_j u_j \right) \frac{\partial \phi_j}{\partial X} + \left(\sum v_j \phi_j \right) \frac{\partial \phi_j}{\partial Y} \right] \phi_i d\Omega - Pr \left(1 + \frac{1}{\gamma} \right) \sum \int_{\Omega} \left(\frac{\partial \phi_j}{\partial X} \frac{\partial \phi_i}{\partial X} + \frac{\partial \phi_j}{\partial Y} \frac{\partial \phi_i}{\partial Y} \right) d\Omega + \frac{Pr}{Da} \sum \int_{\Omega} \phi_j \phi_i \, d\Omega$$

$$A_{12}^{ij} = 0, \quad A_{14}^{ij} = 0, \quad A_{21}^{ij} = 0,$$

$$A_{22}^{ij} = \int_{\Omega} \left[\left(\sum \phi_j u_j \right) \frac{\partial \phi_j}{\partial X} + \left(\sum v_j \phi_j \right) \frac{\partial \phi_j}{\partial Y} \right] \phi_i d\Omega -$$
$$Pr \left(1 + \frac{1}{\gamma} \right) \sum \int_{\Omega} \left(\frac{\partial \phi_j}{\partial X} \frac{\partial \phi_i}{\partial X} + \frac{\partial \phi_j}{\partial Y} \frac{\partial \phi_i}{\partial Y} \right) d\Omega + \frac{Pr}{Da} \sum \int_{\Omega} \phi_j \phi_i \, d\Omega$$

$$A_{24}^{ij} = RaPr \sum \int_{\Omega} \phi_i \phi_j \, d\Omega$$

$$A_{33}^{ij} = 0$$
$$A_{34}^{ij} = 0$$
$$A_{41}^{ij} = 0$$
$$A_{42}^{ij} = 0$$
$$A_{43}^{ij} = 0$$

$$A_{44}^{ij} = \int_{\Omega} \left[\left(\sum u_j \phi_j \right) \frac{\partial \phi_j}{\partial X} + \left(\sum v_j \phi_j \frac{\partial \phi_j}{\partial Y} \right) \right] \phi_i \, d\Omega +$$
$$\sum \int_{\Omega} \left(\frac{\partial \phi_j}{\partial X} \frac{\partial \phi_i}{\partial X} + \frac{\partial \phi_j}{\partial Y} \frac{\partial \phi_i}{\partial Y} \right) d\Omega$$

The entries A_{13}, A_{23} and A_{31}, A_{32} are the pressure matrices with their respective transposes can be written as

$$B_1^{ij} = \sum \int_{\Omega} \frac{\partial \psi_j}{\partial X} \phi_i \, d\Omega \qquad B_2^{ij} = \sum \int_{\Omega} \frac{\partial \psi_j}{\partial Y} \phi_i \, d\Omega$$

$$\left(B_1^{ij} \right)^t = \sum \int_{\Omega} \frac{\partial \phi_j}{\partial X} \psi_i \, d\Omega \qquad \left(B_2^{ij} \right)^t = \sum \int_{\Omega} \frac{\partial \phi_j}{\partial Y} \psi_i \, d\Omega$$

3.4 Results and Discussion

Analysis of natural convection of Casson fluid is depicted for the numerical results in terms of streamlines, isotherms, and 2D plots. Results are presented for fixed value $\gamma = 0.1$ and $\gamma = 5.0$ with varying values of Rayleigh number (Ra) and Darcy number (Da) in the range ($10^4 \leq Ra \leq 10^6$) and ($10^{-5} \leq Da \leq 10^{-2}$). The effect of Da and Ra on average Nusselt number in the aforementioned range are presented in the form of $2D$ plots.

The effect of Da and Ra on streamlines within the considered range with fixed value of Casson parameter $\gamma = 0.1$ is presented in Figure 3.2. It can be observe that in the cavity, two counter-rotating symmetric vortices developed along the vertical mid-plane. The streamlines show that by increasing values of $Da = 10^{-5}, 10^{-4}, 10^{-3}, 10^{-2}$, the streamlines move downward to the bottom heated portion and the center of both counter rotating vortices moved toward the left and right cooled wall near heated portion of cavity. In this way the circulation of fluid increases near the bottom heated portion very close to cooled walls of the cavity. This phenomenon can be clearly seen for $Ra = 10^6$ and at high value of $Da = 10^{-2}$. As increasing the Casson parameter γ, decreasing effect of the effective viscosity can be notice for Casson fluid. So, for $\gamma = 5.0$ and taking the same range of Ra and Da, the above phenomenon can be observed more clearly in terms of streamlines in Figure 3.4. The heat transfer effect in the cavity for considered range of Ra and Da is presented in terms of isotherms in Figure 3.3. At the different values of Darcy number (Da), $\gamma = 0.1, \epsilon = 0.2$, isotherms are moving toward the heated portion of cavity. This phenomenon indicates that the heat transfer rate within the cavity increases close to the heated portion of cavity. As a result heat transfer rate near the adiabatic wall gradually decreasing for the higher values of Darcy number $Da = 10^6$. By increasing Casson parameter from $\gamma = 0.1$ to $\gamma = 5.0$, the effective viscosity of Casson fluid decreases. As a consequence, cavity's average heat transfer rate (Nu_{avg}) increases. This phenomenon can be clearly observed in Figure 3.5.

The variation of average Nusselt number (Nu_{avg}) due to the effect of Ra and Da are presented in Figure 3.6 and Figure 3.7 for different values of Casson parameter $\gamma = 0.1, 1.0$ and $\gamma = 5.0$. The impact of Da on average Nusselt number for different values of Ra is expressed in Figure 3.6. Results show that for small value of $Da = 10^{-5} - 10^{-4}$, average Nusselt number increases but heat transfer rate is too small for small values $Ra = 10^4$ and $Ra = 10^5$. For $Da = 10^{-4} - 10^{-2}$, average Nusselt number show increasing behaviour for both Ra and Casson parameter γ.

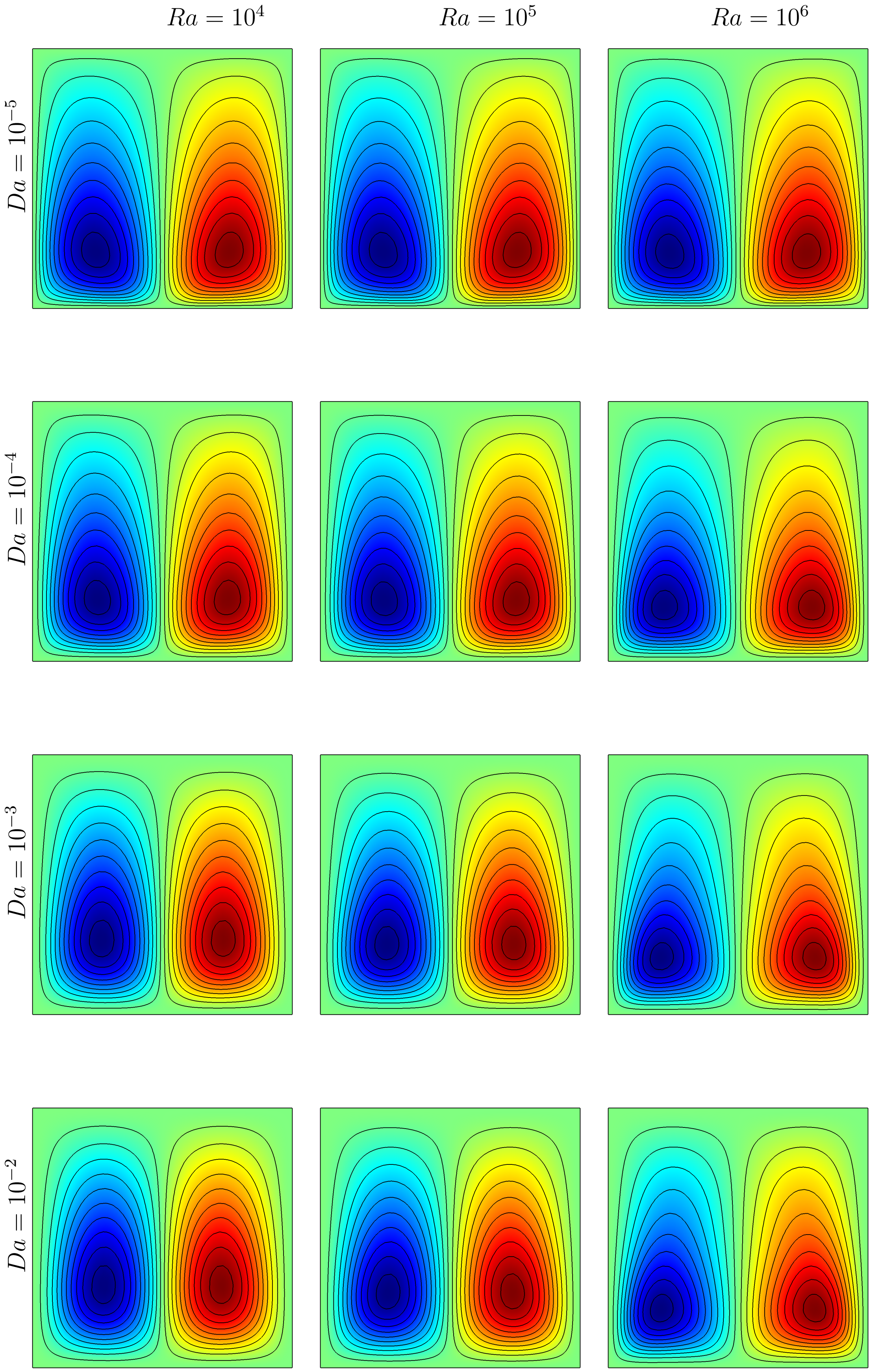

FIGURE 3.2: Influence of Da and Ra on streamlines for fixed value of $\gamma = 0.1$, $\epsilon = 0.2$, $Pr = 7$.

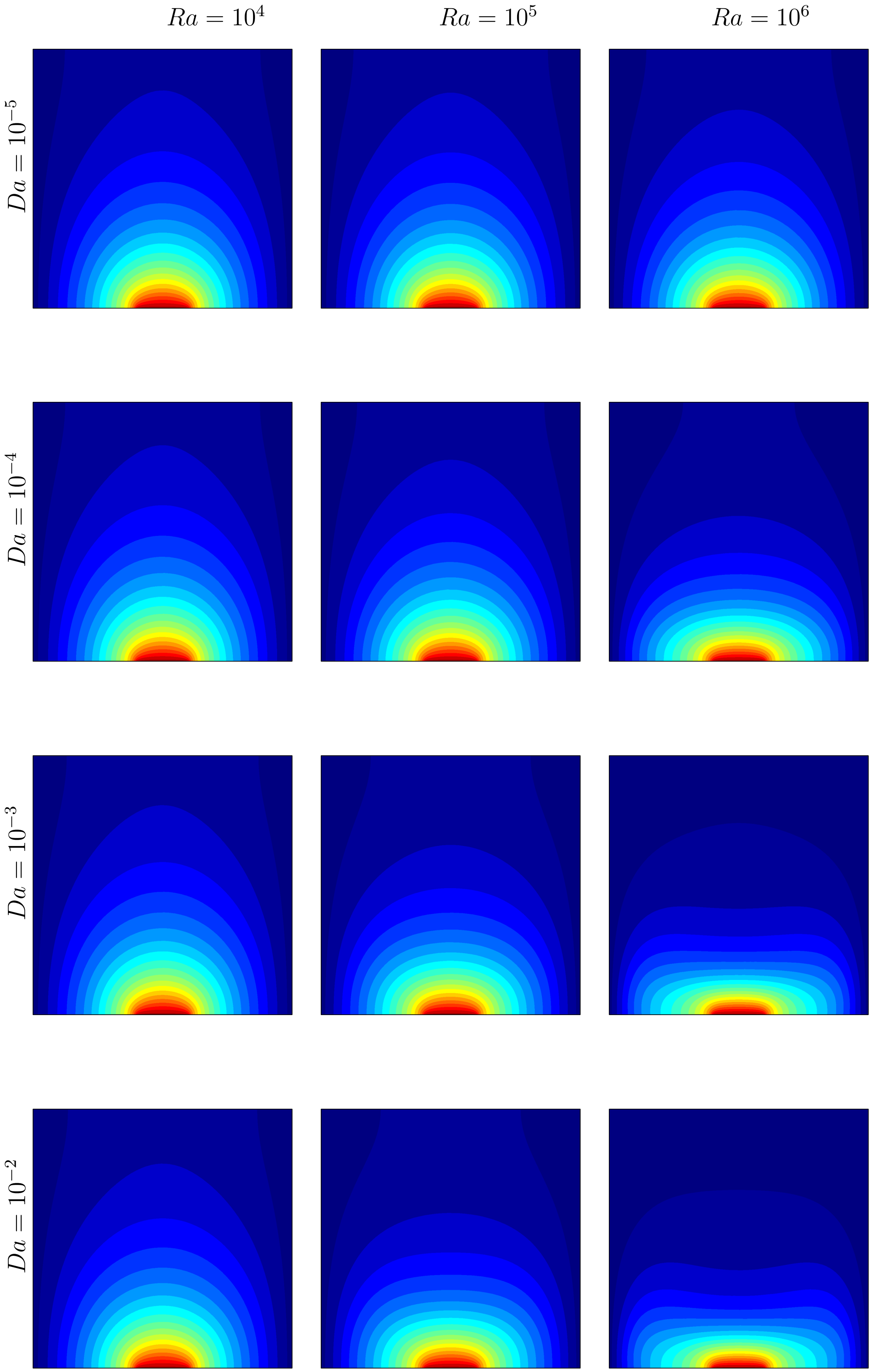

FIGURE 3.3: Influence of Da and Ra on isotherms for fixed value of $\gamma = 0.1$, $\epsilon = 0.2$, $Pr = 7$.

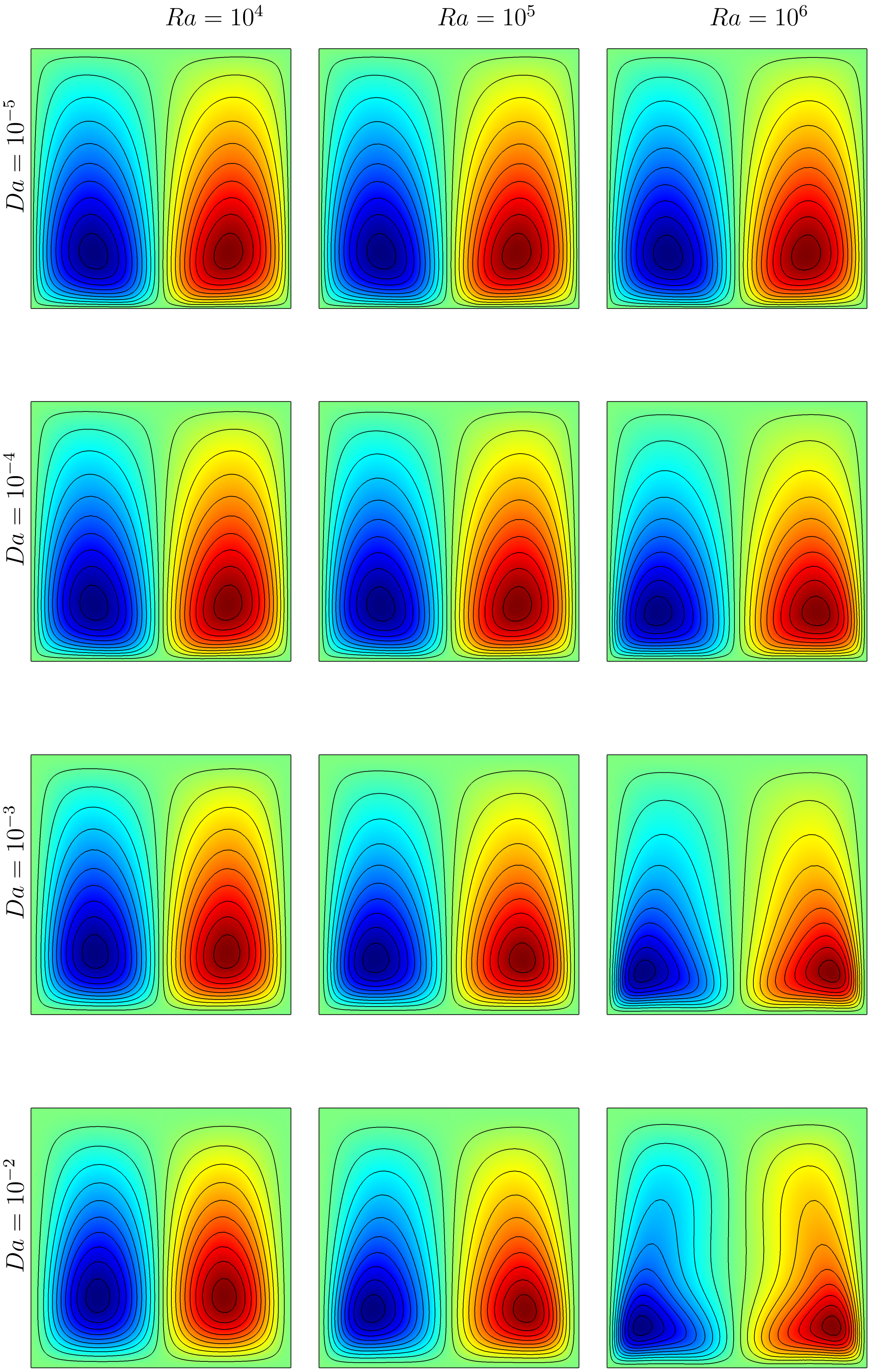

FIGURE 3.4: Influence of Da and Ra on streamlines for fixed value of $\gamma = 5.0$, $\epsilon = 0.2$, $Pr = 7$.

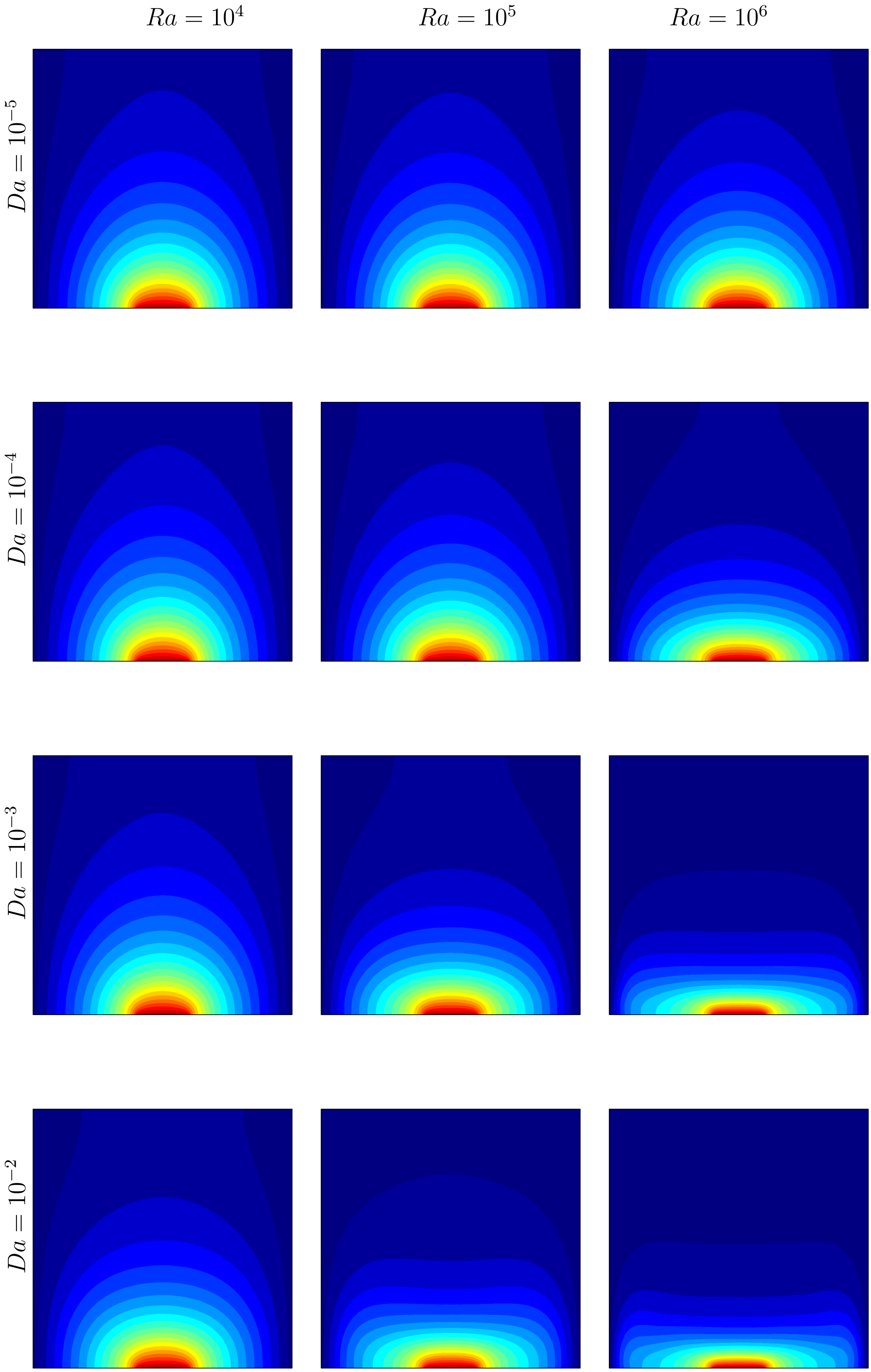

FIGURE 3.5: Influence of Da and Ra on isotherms for fixed value of $\gamma = 5.0$, $\epsilon = 0.2$, $Pr = 7$.

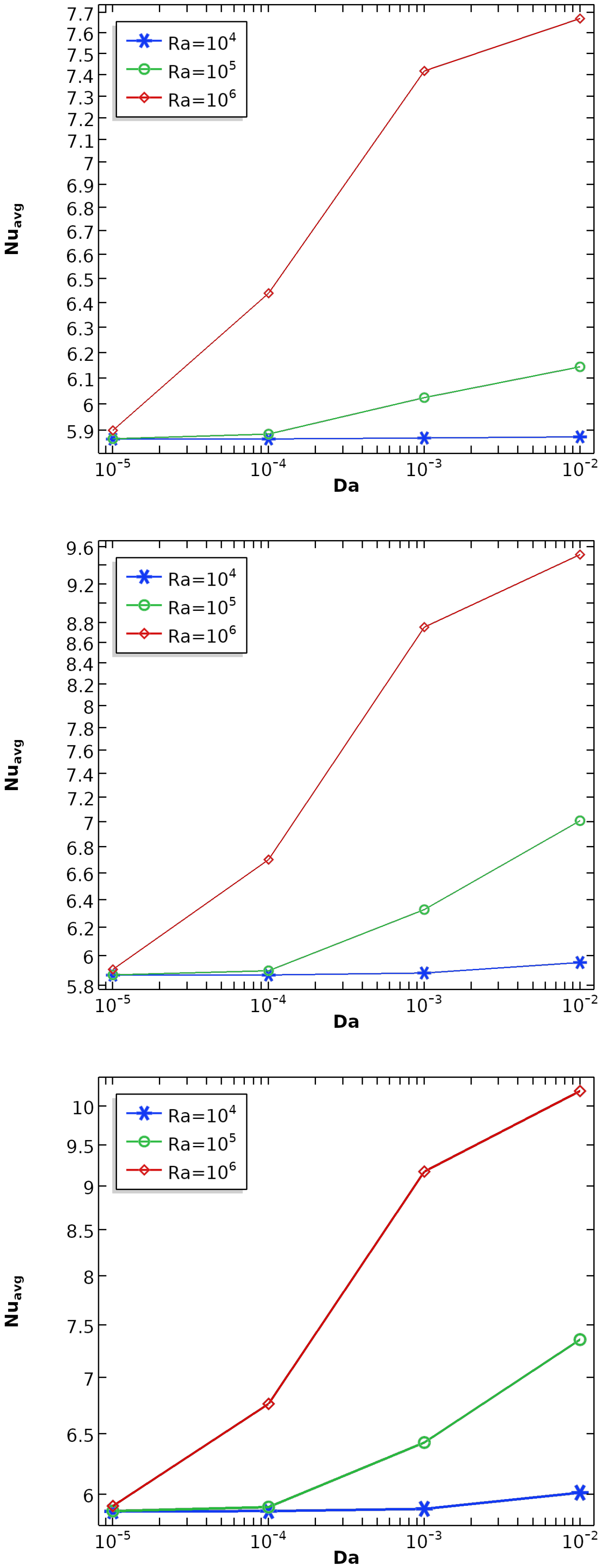

FIGURE 3.6: Influence of Da on average Nusselt number for different value of $\gamma = 0.1, 1.0, 5.0$ (top to bottom) $\epsilon = 0.2$, $Pr = 7$.

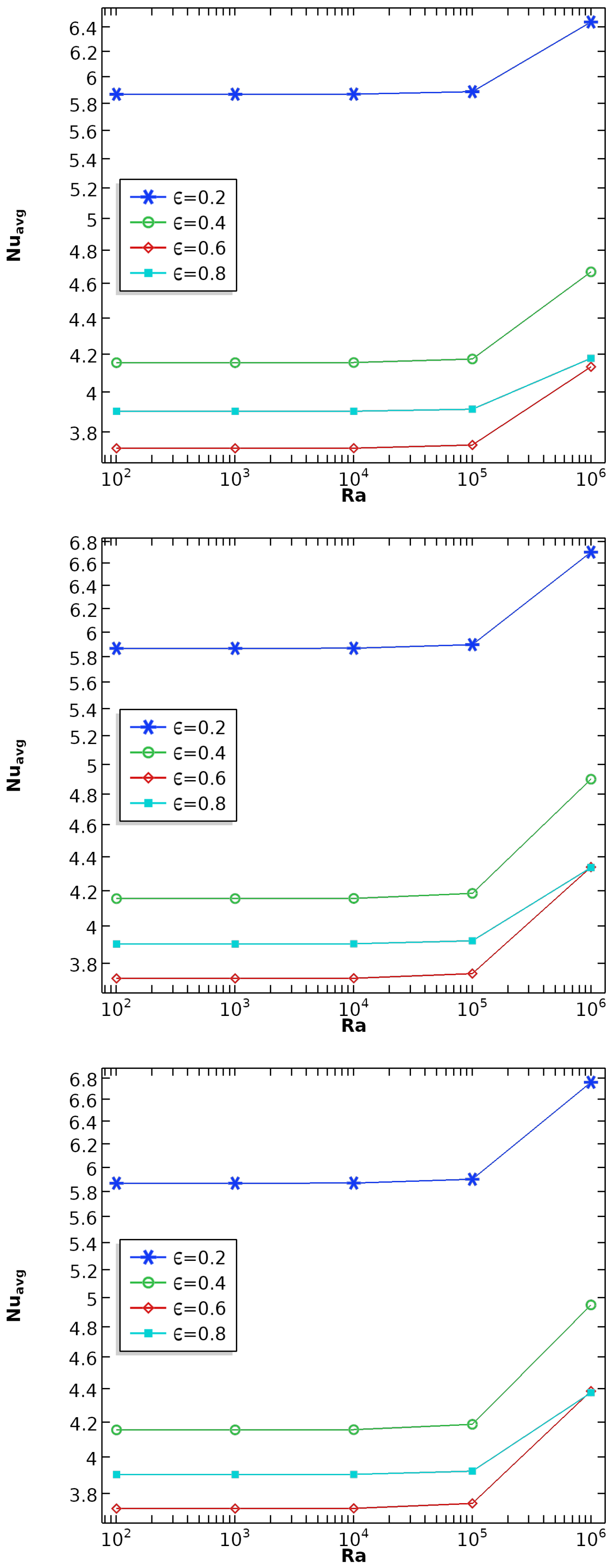

FIGURE 3.7: Influence of Ra on average Nusselt number at different value of ϵ, for $\gamma = 0.1, 1.0, 5.0$ (top to bottom), $Da = 10^{-4}$ and $Pr = 7$.

The effect of Ra on average Nusselt number for varies values of $\epsilon = 0.2, 0.4, 0.6$ and $\epsilon = 0.8$ is exhibited in Figure 3.7. Two dimensional plots show that for varies values of ϵ, small values of Ra ($10^2 \leq Ra \leq 10^5$) does not effect significantly the rate of heat transfer. For each value $\epsilon = 0.2, 0.4, 0.6, 0.8$, ($Nu_{avg}$) increases with increasing values of Ra which can be observed in the range $Ra = (10^5 - 10^6)$.

It is observed in literature [2], [36], [37], [38] that the two bullous appreared in cavities to show the strong and weak streamlines are positioned as strong bullou is along cavity's left side and weak is along the cavity's right side. Review of [22] shows that the negative sign of buoyancy term in governing momentum eq. (3.2) alter the position of two bullous. Numerical findings are presented as streamlines and isotherms accordingly.

Chapter 4

Natural Convection of Casson Fluid in Inclined Porous Cavity under Effect of Magnetic Field and Viscous Dissipation

Natural convection in cavities filled with non-Newtonian fluid investigated by many researchers. Based on literature analysis, it is observed that the natural convection in permeable cavities saturated with non-Newtonian fluids specially Casson fluid has been given more attention in past few years. Casson fluid also have many applications in textile printing, electronics manufacturing, printing, food processing and biological modelling. [39, 40]. Investigation of convective flow of Casson fluid in a partially heated container is the main objective of the study, under the effect of magnetic field, cavity inclination and viscous dissipation effect on heat transfer. These observations are basically apply and investigated with results presented in [22].

4.1 Problem Description

The geometry of the present work is a square cavity with dimension L, filled with Casson fluid and porous medium. The heat transfer characteristics and movement of fluid are under observation with the horizontal magnetic field, cavity inclination and viscous dissipation effect. Figure 4.1 depicts the problem's geometry. A constant temperature $T = T_h$ is considered for the section of the partially heated bottom wall and the rest of the portion of bottom wall including top wall is considered as adiabatic walls. Horizontal magnetic field is represented by B_0 and ϕ shows the cavity inclination.

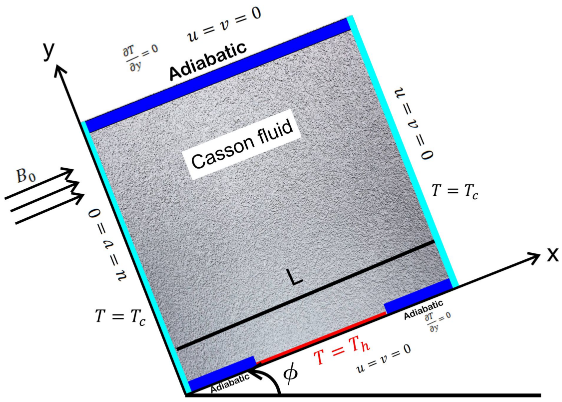

FIGURE 4.1: Schematic cavity under magnetic field and cavity inclination

4.1.1 Governing Equations in Dimensional Form

Under all the aforementioned assumptions, governing equations including momentum equation, continuity equation and energy equation with boundary conditions are given as:

Momentum Equation:

Momentum equation for u-velocity

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(1+\frac{1}{\gamma}\right)\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) - \frac{\nu}{K}u + g\beta\sin\phi\,(T-T_c) \quad (4.1)$$

Momentum equation for v-velocity

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(1+\frac{1}{\gamma}\right)\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right) - \frac{\nu}{K}v + g\beta\cos\phi\,(T-T_c) - \frac{\sigma B_0^2}{\rho}v \quad (4.2)$$

Continuity Equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (4.3)$$

Energy Equation:

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)$$
$$+\frac{\sigma B_0^2}{\rho c_p}v^2+\frac{\nu}{c_p}\left(1+\frac{1}{\gamma}\right)\left[2\left\{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right\}+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2\right] \quad (4.4)$$

where u, v represent horizontal and vertical velocities of fluid flow, ρ is the fluid density, Casson fluid parameter is represented by γ, ν is kinematic viscosity, g shows gravitational constant, ϕ is the cavity inclination, p shows pressure, B_0 is the magnetic field strength, σ is for electrical conductivity, k represents thermal conductivity, c_p shows specific heat, thermal diffusivity is represented by α and β is for expansion coefficient.

Boundary Conditions:

Boundary conditions in the dimensional form will be taken as same as mentioned in **chapter 3**.

4.1.2 Dimensionless Form of Governing Equation

To convert dimensional governing equations to dimensionless form of governing equations, following dimensionless parameters have been used [36],[41].

$$U=\frac{uL}{\alpha},V=\frac{vL}{\alpha},X=\frac{x}{L},Y=\frac{y}{L},\theta=\frac{T-T_c}{T_h-T_c},\gamma=\frac{\mu_B\sqrt{2\pi_c}}{\frac{\partial p}{\partial y}},P=\frac{pL^2}{\rho\alpha^2}$$
$$Pr=\frac{\nu}{\alpha},Da=\frac{K}{L^2},Ra=\frac{g\beta\left(T_h-T_c\right)L^3Pr}{\nu^2},Ha=B_0L\sqrt{\frac{\sigma}{\rho\nu}},Ec=\frac{\left(\frac{\alpha}{L}\right)^2}{c_p\Delta T}$$

Momentum equation for U-velocity

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(1+\frac{1}{\gamma}\right)\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)-\frac{\nu}{K}u+g\beta\sin\phi\left(T-T_c\right)$$

$$\frac{\alpha U}{L}\frac{\partial\left(\frac{\alpha U}{L}\right)}{\partial\left(LX\right)}+\frac{\alpha V}{L}\frac{\partial\left(\frac{\alpha U}{L}\right)}{\partial\left(LY\right)}=-\frac{1}{\rho}\frac{\partial\left(\frac{\rho\alpha^2P}{L^2}\right)}{\partial\left(LX\right)}+\nu\left(1+\frac{1}{\gamma}\right)\left(\frac{\partial^2\left(\frac{\alpha U}{L}\right)}{\partial\left(LX\right)^2}+\frac{\partial^2\left(\frac{\alpha U}{L}\right)}{\partial\left(LY\right)^2}\right)$$
$$-\frac{\nu}{K}\left(\frac{\alpha U}{L}\right)+g\beta\left(\Delta T\theta+T_c-T_c\right)\sin\phi$$

$$\left[\frac{\alpha^2}{L^3}\left(U\frac{\partial U}{\partial X}\right)+\frac{\alpha^2}{L^3}\left(V\frac{\partial U}{\partial Y}\right)\right]=\frac{\alpha^2}{L^3}\left(-\frac{\partial P}{\partial X}\right)+\nu\left(1+\frac{1}{\gamma}\right)\frac{\alpha}{L^3}\left(\frac{\partial^2 U}{\partial X^2}+\frac{\partial^2 U}{\partial Y^2}\right)$$
$$-\frac{\alpha}{L}\frac{\nu}{K}U+g\beta\Delta T\theta\sin\phi$$

$$\frac{\alpha^2}{L^3}\left[U\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial Y}\right]=\frac{\alpha^2}{L^3}\left[-\frac{\partial P}{\partial X}+\frac{\nu}{\alpha}\left(1+\frac{1}{\gamma}\right)\left(\frac{\partial^2 U}{\partial X^2}+\frac{\partial^2 U}{\partial Y^2}\right)\right]$$
$$+\frac{\alpha^2}{L^3}\left[-\frac{L^2}{\alpha}\frac{\nu}{K}U+\frac{g\beta\Delta T L^3}{\alpha^2}\theta\sin\phi\right]$$

$$U\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial Y}=-\frac{\partial P}{\partial X}+\frac{\nu}{\alpha}\left(1+\frac{1}{\gamma}\right)\left(\frac{\partial^2 U}{\partial X^2}+\frac{\partial^2 U}{\partial Y^2}\right)$$
$$-\frac{\frac{\nu}{\alpha}}{\frac{K}{L^2}}U+\frac{g\beta\Delta T L^3}{\nu^2}\frac{\nu^2}{\alpha^2}\theta\sin\phi$$

$$U\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial Y}=-\frac{\partial P}{\partial X}+Pr\left(1+\frac{1}{\gamma}\right)\left(\frac{\partial^2 U}{\partial X^2}+\frac{\partial^2 U}{\partial Y^2}\right)-\frac{Pr}{Da}U+RaPr\theta\sin\phi$$

Momentum equation for V-velocity

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(1+\frac{1}{\gamma}\right)\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)-\frac{\nu}{K}v-\frac{\sigma B_0^2}{\rho}v+g\beta\cos\phi\,(T-T_c)$$

$$\frac{\alpha U}{L}\frac{\partial\left(\frac{\alpha V}{L}\right)}{\partial(LX)}+\frac{\alpha V}{L}\frac{\partial\left(\frac{\alpha V}{L}\right)}{\partial(LY)}=-\frac{1}{\rho}\frac{\partial\left(\frac{\rho\alpha^2 P}{L^2}\right)}{\partial(LY)}+\nu\left(1+\frac{1}{\gamma}\right)\left(\frac{\partial^2\left(\frac{\alpha V}{L}\right)}{\partial(LX)^2}+\frac{\partial^2\left(\frac{\alpha V}{L}\right)}{\partial(LY)^2}\right)$$
$$-\frac{\nu}{K}\left(\frac{\alpha V}{L}\right)-\frac{\sigma B_0^2}{\rho}\frac{\alpha}{L}V+g\beta\,(T_h-T_c)\,\theta\cos\phi$$

$$\frac{\alpha^2}{L^3}\left[U\frac{\partial V}{\partial X}+V\frac{\partial V}{\partial Y}\right]=\frac{\alpha^2}{L^3}\left[-\frac{\partial P}{\partial Y}\right]+\frac{\alpha}{L^3}\nu\left(1+\frac{1}{\gamma}\right)\left(\frac{\partial^2 V}{\partial X^2}+\frac{\partial^2 V}{\partial Y^2}\right)$$
$$-\frac{\alpha\nu}{Lk}V-\frac{\sigma B_0^2}{\rho}\frac{\alpha}{L}V+g\beta\,(T_h-T_c)\,\theta\cos\phi$$

$$\frac{\alpha^2}{L^3}\left[U\frac{\partial V}{\partial X}+V\frac{\partial V}{\partial Y}\right]=\frac{\alpha^2}{L^3}\left[-\frac{\partial P}{\partial Y}+\frac{\nu}{\alpha}\left(1+\frac{1}{\gamma}\right)\left(\frac{\partial^2 V}{\partial X^2}+\frac{\partial^2 V}{\partial Y^2}\right)\right]$$
$$+\frac{\alpha^2}{L^3}\left[-\frac{L^2}{\alpha}\frac{\nu}{K}V-\frac{\sigma B_0^2}{\rho}\frac{L^2}{\alpha}V+\frac{g\beta\cos\phi\Delta T L^3}{\alpha^2}\theta\right]$$

$$U\frac{\partial V}{\partial X}+V\frac{\partial V}{\partial Y}=-\frac{\partial P}{\partial Y}+\frac{\nu}{\alpha}\left(1+\frac{1}{\gamma}\right)\left(\frac{\partial^2 V}{\partial X^2}+\frac{\partial^2 V}{\partial Y^2}\right)$$
$$-\frac{\frac{\nu}{\alpha}}{\frac{K}{L^2}}V-\frac{\alpha\sigma B_0^2}{\rho}\frac{L^2}{\alpha^2}V+\frac{g\beta\cos\phi\Delta T L^3}{\nu^2}\frac{\nu^2}{\alpha^2}\theta$$

$$U\frac{\partial V}{\partial X}+V\frac{\partial V}{\partial Y}=-\frac{\partial P}{\partial Y}+Pr\left(1+\frac{1}{\gamma}\right)\left(\frac{\partial^2 V}{\partial X^2}+\frac{\partial^2 V}{\partial Y^2}\right)-\frac{Pr}{Da}V$$
$$-Ha^2PrV+RaPr\theta\cos\phi$$

Continuity Equation:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

$$\frac{\partial\left(\frac{\alpha U}{L}\right)}{\partial(LX)}+\frac{\partial\left(\frac{\alpha V}{L}\right)}{\partial(LY)}=0$$

$$\left[\frac{\alpha}{L^2}\left(\frac{\partial U}{\partial X}\right)+\frac{\alpha}{L^2}\left(\frac{\partial V}{\partial Y}\right)\right]=0$$

$$\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}=0$$

Energy equation:

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)$$
$$+\frac{\sigma B_0^2}{\rho c_p}v^2+\frac{\nu}{c_p}\left(1+\frac{1}{\gamma}\right)\left[2\left(\frac{\partial u}{\partial x}\right)^2+2\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2\right]$$

$$\frac{\alpha}{L}U\frac{\partial(\Delta T\theta)}{\partial(LX)}+\frac{\alpha}{L}V\frac{\partial(\Delta T\theta)}{\partial(LY)}=\alpha\left(\frac{\partial^2(\Delta T\theta)}{\partial(LX)^2}+\frac{\partial^2(\Delta T\theta)}{\partial(LY)^2}\right)+\frac{\sigma B_0^2}{\rho c_p}\left(\frac{\alpha}{L}V\right)^2$$
$$+\frac{\nu}{c_p}\left(1+\frac{1}{\gamma}\right)\left[2\left\{\left(\frac{\partial\left(\frac{\alpha}{L}U\right)}{\partial(LX)}\right)^2+\left(\frac{\partial\left(\frac{\alpha}{L}V\right)}{\partial(LY)}\right)^2\right\}+\left(\frac{\partial\left(\frac{\alpha}{L}U\right)}{\partial(LY)}+\frac{\partial\left(\frac{\alpha}{L}V\right)}{\partial(LX)}\right)^2\right]$$

$$\frac{\alpha}{L^2}\Delta T\left[U\frac{\partial\theta}{\partial X}+V\frac{\partial\theta}{\partial Y}\right]=\frac{\alpha}{L^2}\Delta T\left(\frac{\partial^2\theta}{\partial X^2}+\frac{\partial^2\theta}{\partial Y^2}\right)+\frac{\sigma B_0^2}{\rho c_p}\frac{\alpha^2}{L^2}V^2$$
$$+\frac{\nu}{c_p}\left(1+\frac{1}{\gamma}\right)\left[\frac{\alpha^2}{L^4}2\left\{\left(\frac{\partial U}{\partial X}\right)^2+\left(\frac{\partial V}{\partial Y}\right)^2\right\}+\frac{\alpha^2}{L^4}\left(\frac{\partial U}{\partial Y}+\frac{\partial V}{\partial X}\right)^2\right]$$

$$U\frac{\partial\theta}{\partial X}+V\frac{\partial\theta}{\partial Y}=\frac{L^2}{\alpha\Delta T}\frac{\alpha\Delta T}{L^2}\left[\left(\frac{\partial^2\theta}{\partial X^2}+\frac{\partial^2\theta}{\partial Y^2}\right)+\frac{L^2}{\alpha\Delta T}\frac{\alpha^2}{L^2}\frac{\sigma B_0^2}{\rho c_p}V^2\right]$$
$$+\frac{L^2}{\alpha\Delta T}\frac{\nu}{c_p}\frac{\alpha^2}{L^4}\left(1+\frac{1}{\gamma}\right)\left[2\left\{\left(\frac{\partial U}{\partial X}\right)^2+\left(\frac{\partial V}{\partial Y}\right)^2\right\}+\left(\frac{\partial U}{\partial Y}+\frac{\partial V}{\partial X}\right)^2\right]$$

$$U\frac{\partial\theta}{\partial X}+V\frac{\partial\theta}{\partial Y}=\frac{\partial^2\theta}{\partial X^2}+\frac{\partial^2\theta}{\partial Y^2}+\alpha\frac{\sigma B_0^2}{\rho c_p\Delta T}V^2$$
$$+\frac{\alpha\nu}{L^2c_p\Delta T}\left(1+\frac{1}{\gamma}\right)\left[2\left\{\left(\frac{\partial U}{\partial X}\right)^2+\left(\frac{\partial V}{\partial Y}\right)^2\right\}+\left(\frac{\partial U}{\partial Y}+\frac{\partial V}{\partial X}\right)^2\right]$$

$$U\frac{\partial\theta}{\partial X}+V\frac{\partial\theta}{\partial Y}=\frac{\partial^2\theta}{\partial X^2}+\frac{\partial^2\theta}{\partial Y^2}+Ha^2EcPrV^2$$
$$+EcPr\left(1+\frac{1}{\gamma}\right)\left[2\left\{\left(\frac{\partial U}{\partial X}\right)^2+\left(\frac{\partial V}{\partial Y}\right)^2\right\}+\left(\frac{\partial U}{\partial Y}+\frac{\partial V}{\partial X}\right)^2\right]$$

Hence dimensionless system of governing equations (4.1), (4.2), (4.3) and (4.4) with boundary conditions can be noted as:

Momentum equation for U-velocity

$$U\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial Y}=-\frac{\partial P}{\partial X}+Pr\left(1+\frac{1}{\gamma}\right)\left(\frac{\partial^2U}{\partial X^2}+\frac{\partial^2U}{\partial Y^2}\right)-\frac{Pr}{Da}U+RaPr\theta\sin\phi \quad (4.5)$$

Momentum equation for V-velocity

$$U\frac{\partial V}{\partial X}+V\frac{\partial V}{\partial Y}=-\frac{\partial P}{\partial Y}+Pr\left(1+\frac{1}{\gamma}\right)\left(\frac{\partial^2V}{\partial X^2}+\frac{\partial^2V}{\partial Y^2}\right)-\frac{Pr}{Da}V$$
$$-Ha^2PrV+RaPr\theta\cos\phi \quad (4.6)$$

Continuity Equation:

$$\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}=0 \quad (4.7)$$

Energy Equation:

$$U\frac{\partial\theta}{\partial X}+V\frac{\partial\theta}{\partial Y}=\frac{\partial^2\theta}{\partial X^2}+\frac{\partial^2\theta}{\partial Y^2}+Ha^2EcPrV^2$$
$$+EcPr\left(1+\frac{1}{\gamma}\right)\left[2\left\{\left(\frac{\partial U}{\partial X}\right)^2+\left(\frac{\partial V}{\partial Y}\right)^2\right\}+\left(\frac{\partial U}{\partial Y}+\frac{\partial V}{\partial X}\right)^2\right] \quad (4.8)$$

Boundary Conditions:

- For the cavity's left wall

$$\text{At}\quad 0<Y<1\quad\text{and}\quad X=0; U=V=0,\ \theta=0$$

- For the cavity's Right wall

$$\text{At}\quad 0<Y<1\quad\text{and}\quad X=1;\ U=V=0,\ \theta=0$$

- For the cavity's bottom portions

$$\text{At}\quad \frac{1-\epsilon}{2}\le X\le\frac{1+\epsilon}{2}\quad\text{and}\quad Y=0;\ U=V=0,\ \theta=1$$

$$\text{At}\quad 0<X<\frac{1-\epsilon}{2},\ \frac{1+\epsilon}{2}<X<1\ \text{and}\ Y=0; U=V=0,\ \frac{\partial\theta}{\partial Y}=0$$

- For the cavity's top wall

$$\text{At}\quad 0<X<1\quad\text{and}\quad Y=1;\ U=V=0,\ \frac{\partial\theta}{\partial Y}=0$$

4.2 Weak Formulation

This section will present the weak formulation for all governing equations with the help of weighted residual method and $w\in W$ and $q\in Q$ as weight function. Momentum equation and energy equation will be multiplied by weight function w and the continuity equation will be multiplied by q to get the weak form.

Weak formulation for x-momentum equation (4.5), represented as:

$$\int_\Omega \left(U\frac{\partial U}{\partial X}\right) w + \left(V\frac{\partial U}{\partial Y}\right) w \, d\Omega =$$
$$-\int_\Omega \frac{\partial P}{\partial X} w \, d\Omega + Pr\left(1+\frac{1}{\gamma}\right)\int_\Omega \left(\frac{\partial^2 U}{\partial X^2}+\frac{\partial^2 U}{\partial Y^2}\right) w \, d\Omega - \frac{Pr}{Da}\int_\Omega U w \, d\Omega$$
$$+RaPr\int_\Omega \theta \sin\phi w d\Omega$$

$$\int_\Omega \left(U\frac{\partial U}{\partial X}\right) w + \left(V\frac{\partial U}{\partial Y}\right) w \, d\Omega =$$
$$-\int_\Omega \frac{\partial P}{\partial X} w \, d\Omega + Pr\left(1+\frac{1}{\gamma}\right)\int_\Omega \left(\frac{\partial^2 U}{\partial X^2}+\frac{\partial^2 U}{\partial Y^2}\right) w \, d\Omega - \frac{Pr}{Da}\int_\Omega U w \, d\Omega$$
$$+RaPr\sin\phi\int_\Omega \theta w d\Omega$$

Using Green's theorem for Laplacian term as

$$\int_\Omega \psi\Delta\phi \, d\Omega = -\int_\Omega \nabla\phi\nabla\psi \, d\Omega + \int_\Omega \psi(\nabla\phi.\hat{n}) \, d\Gamma$$

$$\int_\Omega \left(U\frac{\partial U}{\partial X}\right) w + \left(V\frac{\partial U}{\partial Y}\right) w \, d\Omega =$$
$$-\int_\Omega \frac{\partial P}{\partial X} w \, d\Omega + Pr\left(1+\frac{1}{\gamma}\right)\left[-\int_\Omega \frac{\partial U}{\partial X}\frac{\partial w}{\partial X} \, d\Omega + \oint_\Gamma w\frac{\partial U}{\partial X} n_x \, d\Gamma\right] +$$
$$Pr\left(1+\frac{1}{\gamma}\right)\left[-\int_\Omega \frac{\partial U}{\partial Y}\frac{\partial w}{\partial Y} \, d\Omega + \oint_\Gamma w\frac{\partial U}{\partial Y} n_y \, d\Gamma\right] - \frac{Pr}{Da}\int_\Omega U w \, d\Omega$$
$$+RaPr\sin\phi\int_\Omega \theta w d\Omega$$

$$\int_\Omega \left(U\frac{\partial U}{\partial X}\right) w + \left(V\frac{\partial U}{\partial Y}\right) w \, d\Omega =$$
$$-\int_\Omega \frac{\partial P}{\partial X} w \, d\Omega + Pr\left(1+\frac{1}{\gamma}\right)\left[-\int_\Omega \frac{\partial U}{\partial X}\frac{\partial w}{\partial X} \, d\Omega + \oint_\Gamma w\frac{\partial U}{\partial X} n_x \, d\Gamma\right] +$$
$$Pr\left(1+\frac{1}{\gamma}\right)\left[-\int_\Omega \frac{\partial U}{\partial Y}\frac{\partial w}{\partial Y} \, d\Omega + \oint_\Gamma w\frac{\partial U}{\partial Y} n_y \, d\Gamma\right] - \frac{Pr}{Da}\int_\Omega U w \, d\Omega +$$
$$RaPr\sin\phi\int_\Omega \theta w d\Omega$$

$$\int_\Omega \left(U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y}\right) w \, d\Omega =$$
$$-\int_\Omega \frac{\partial P}{\partial X} w \, d\Omega - Pr\left(1+\frac{1}{\gamma}\right)\int_\Omega \frac{\partial U}{\partial X}\frac{\partial w}{\partial X} + \frac{\partial U}{\partial Y}\frac{\partial w}{\partial Y} \, d\Omega -$$
$$\frac{Pr}{Da}\int_\Omega U w \, d\Omega + RaPr\sin\phi\int_\Omega \theta w d\Omega \quad (4.9)$$

Weak formulation of y-momentum equation (4.6), presented as:

$$\int_\Omega \left(U\frac{\partial V}{\partial X}\right) w + \left(V\frac{\partial V}{\partial Y}\right) w\, d\Omega =$$
$$-\int_\Omega \frac{\partial P}{\partial Y} w\, d\Omega + Pr\left(1+\frac{1}{\gamma}\right)\int_\Omega \left(\frac{\partial^2 V}{\partial X^2}+\frac{\partial^2 V}{\partial Y^2}\right) w\, d\Omega - \frac{Pr}{Da}\int_\Omega V w\, d\Omega$$
$$-Ha^2 Pr\int_\Omega V w d\Omega + Ra Pr\int_\Omega \theta\cos\phi w d\Omega$$

$$\int_\Omega \left(U\frac{\partial V}{\partial X} + V\frac{\partial V}{\partial Y}\right) w\, d\Omega =$$
$$-\int_\Omega \frac{\partial P}{\partial Y} w\, d\Omega + Pr\left(1+\frac{1}{\gamma}\right)\int_\Omega \left(\frac{\partial^2 V}{\partial X^2}+\frac{\partial^2 V}{\partial Y^2}\right) w\, d\Omega - \frac{Pr}{Da}\int_\Omega V w\, d\Omega$$
$$-Ha^2 Pr\int_\Omega V w d\Omega + Ra Pr\cos\phi\int_\Omega \theta w d\Omega$$

Using Green's theorem for Laplacian term as

$$\int_\Omega \psi\Delta\phi\, d\Omega = -\int_\Omega \nabla\phi\nabla\psi\, d\Omega + \int_\Omega \psi\left(\nabla\phi.\hat{\mathbf{n}}\right)\, d\Gamma$$

$$\int_\Omega \left(U\frac{\partial V}{\partial X} + V\frac{\partial V}{\partial Y}\right) w\, d\Omega =$$
$$-\int_\Omega \frac{\partial P}{\partial Y} w\, d\Omega + Pr\left(1+\frac{1}{\gamma}\right)\left[-\int_\Omega \frac{\partial V}{\partial X}\frac{\partial w}{\partial X}\, d\Omega + \oint_\Gamma w\frac{\partial V}{\partial X} n_x\, d\Gamma\right] +$$
$$Pr\left(1+\frac{1}{\gamma}\right)\left[-\int_\Omega \frac{\partial V}{\partial Y}\frac{\partial w}{\partial Y}\, d\Omega + \oint_\Gamma w\frac{\partial V}{\partial Y} n_y\, d\Gamma\right] - \frac{Pr}{Da}\int_\Omega V w\, d\Omega$$
$$-Ha^2 Pr\int_\Omega V w d\Omega + Ra Pr\cos\phi\int_\Omega \theta w d\Omega$$

$$\int_\Omega \left(U\frac{\partial V}{\partial X} + V\frac{\partial V}{\partial Y}\right) w\, d\Omega =$$
$$-\int_\Omega \frac{\partial P}{\partial Y} w\, d\Omega + Pr\left(1+\frac{1}{\gamma}\right)\left[-\int_\Omega \frac{\partial V}{\partial X}\frac{\partial w}{\partial X}\, d\Omega + \oint_\Gamma w\frac{\partial V}{\partial X} n_x\, d\Gamma\right]$$
$$+Pr\left(1+\frac{1}{\gamma}\right)\left[-\int_\Omega \frac{\partial V}{\partial Y}\frac{\partial w}{\partial Y}\, d\Omega + \oint_\Gamma w\frac{\partial V}{\partial Y} n_y\, d\Gamma\right] - \frac{Pr}{Da}\int_\Omega V w\, d\Omega$$
$$-Ha^2 Pr\int_\Omega V w d\Omega + Ra Pr\cos\phi\int_\Omega \theta w d\Omega$$

$$\int_\Omega \left(U\frac{\partial V}{\partial X} + V\frac{\partial V}{\partial Y}\right) w\, d\Omega =$$
$$-\int_\Omega \frac{\partial P}{\partial Y} w\, d\Omega - Pr\left(1+\frac{1}{\gamma}\right)\int_\Omega \frac{\partial V}{\partial X}\frac{\partial w}{\partial X} + \frac{\partial V}{\partial Y}\frac{\partial w}{\partial Y}\, d\Omega -$$
$$\frac{Pr}{Da}\int_\Omega V w\, d\Omega - Ha^2 Pr\int_\Omega V w d\Omega + Ra Pr\cos\phi\int_\Omega \theta w d\Omega \quad (4.10)$$

Continuity equation weak formulation:

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0$$

$$\int_{\Omega} \left(\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} \right) q \, d\Omega \;=\; 0 \tag{4.11}$$

Similarly for energy Eq. (4.8), weak formulation calculated as:

$$U\frac{\partial \theta}{\partial X} + V\frac{\partial \theta}{\partial Y} = \frac{\partial^2 \theta}{\partial X^2} + \frac{\partial^2 \theta}{\partial Y^2} + Ha^2 EcPrV^2$$
$$+EcPr\left(1+\frac{1}{\gamma}\right)\left[2\left(\frac{\partial U}{\partial X}\right)^2 + 2\left(\frac{\partial V}{\partial Y}\right)^2 + \left(\frac{\partial U}{\partial Y} + \frac{\partial V}{\partial X}\right)^2\right]$$

Using Green's theorem

$$\int_{\Omega} \left(U\frac{\partial \theta}{\partial X} + V\frac{\partial \theta}{\partial Y} \right) w \, d\Omega = -\int_{\Omega} \frac{\partial \theta}{\partial X}\frac{\partial w}{\partial X} \, d\Omega + \oint_{\Gamma} w \frac{\partial \theta}{\partial X} n_x \, d\Gamma - \int_{\Omega} \frac{\partial \theta}{\partial Y}\frac{\partial w}{\partial Y} \, d\Omega$$
$$+\oint_{\Gamma} w \frac{\partial \theta}{\partial Y} n_y \, d\Gamma + Ha^2 EcPr \int_{\Omega} V^2 w \, d\Omega$$
$$+EcPr\left(1+\frac{1}{\gamma}\right)\int_{\Omega}\left[2\left(\frac{\partial U}{\partial X}\right)^2 + 2\left(\frac{\partial V}{\partial Y}\right)^2 + \left(\frac{\partial U}{\partial Y} + \frac{\partial V}{\partial X}\right)^2\right] w \, d\Omega$$

$$\int_{\Omega} \left(U\frac{\partial \theta}{\partial X} \right) w + \left(V\frac{\partial \theta}{\partial Y} \right) w \, d\Omega =$$
$$-\int_{\Omega} \frac{\partial \theta}{\partial X}\frac{\partial w}{\partial X} \, d\Omega - \int_{\Omega} \frac{\partial \theta}{\partial Y}\frac{\partial w}{\partial Y} \, d\Omega + Ha^2 EcPr \int_{\Omega} V^2 w \, d\Omega$$
$$+EcPr\left(1+\frac{1}{\gamma}\right)\int_{\Omega}\left[2\left(\frac{\partial U}{\partial X}\right)^2 + 2\left(\frac{\partial V}{\partial Y}\right)^2 + \left(\frac{\partial U}{\partial Y} + \frac{\partial V}{\partial X}\right)^2\right] w \, d\Omega$$

$$\int_{\Omega} \left(U\frac{\partial \theta}{\partial X} + V\frac{\partial \theta}{\partial Y} \right) w \, d\Omega = -\int_{\Omega} \frac{\partial \theta}{\partial X}\frac{\partial w}{\partial X} + \frac{\partial \theta}{\partial Y}\frac{\partial w}{\partial Y} \, d\Omega + Ha^2 EcPr \int_{\Omega} V^2 w \, d\Omega$$
$$+EcPr\left(1+\frac{1}{\gamma}\right)\left[2\int_{\Omega}\left(\frac{\partial U}{\partial X}\frac{\partial U}{\partial X}\right) w \, d\Omega + 2\int_{\Omega}\left(\frac{\partial V}{\partial Y}\frac{\partial V}{\partial Y}\right) w \, d\Omega\right]$$
$$+EcPr\left(1+\frac{1}{\gamma}\right)\int_{\Omega}\left(\frac{\partial U}{\partial Y} + \frac{\partial V}{\partial X}\right)\left(\frac{\partial U}{\partial Y} + \frac{\partial V}{\partial X}\right) w \, d\Omega$$

$$\int_{\Omega}\left(U\frac{\partial\theta}{\partial X}+V\frac{\partial\theta}{\partial Y}\right)w\,d\Omega=-\int_{\Omega}\frac{\partial\theta}{\partial X}\frac{\partial w}{\partial X}+\frac{\partial\theta}{\partial Y}\frac{\partial w}{\partial Y}\,d\Omega+Ha^2EcPr\int_{\Omega}V^2w\,d\Omega$$
$$+EcPr\left(1+\frac{1}{\gamma}\right)\left[2\int_{\Omega}\left(\frac{\partial U}{\partial X}\right)^2w\,d\Omega+2\int_{\Omega}\left(\frac{\partial V}{\partial Y}\right)^2w\,d\Omega\right]$$
$$+EcPr\left(1+\frac{1}{\gamma}\right)\int_{\Omega}\left(\frac{\partial U}{\partial Y}+\frac{\partial V}{\partial X}\right)^2w\,d\Omega \quad (4.12)$$

Finally weak formulation for governing equations are represented by (4.9), (4.10), (4.11) and (4.12).

Now to approximate the infinite test and solution space, using Galerkin discretization scheme to meet required accuracy. Consider the approximated solution Components as $U_h, V_h, P_h, \theta_h \approx U, V, P, \theta$.

Now to find $(U_h, V_h, P_h, \theta_h)$

$$\int_{\Omega}\left(U_h\frac{\partial U_h}{\partial X}+V_h\frac{\partial U_h}{\partial Y}\right)w_h\,d\Omega=$$
$$-\int_{\Omega}\frac{\partial P_h}{\partial X}w_h\,d\Omega-Pr\left(1+\frac{1}{\gamma}\right)\int_{\Omega}\frac{\partial U_h}{\partial X}\frac{\partial w_h}{\partial X}+\frac{\partial U_h}{\partial Y}\frac{\partial w_h}{\partial Y}\,d\Omega-$$
$$\frac{Pr}{Da}\int_{\Omega}U_hw_h\,d\Omega+RaPr\sin\phi\int_{\Omega}\theta_hw_hd\Omega \quad (4.13)$$

$$\int_{\Omega}\left(U_h\frac{\partial V_h}{\partial X}+V_h\frac{\partial V_h}{\partial Y}\right)w_h\,d\Omega=$$
$$-\int_{\Omega}\frac{\partial P_h}{\partial Y}w_h\,d\Omega-Pr\left(1+\frac{1}{\gamma}\right)\int_{\Omega}\frac{\partial V_h}{\partial X}\frac{\partial w_h}{\partial X}+\frac{\partial V_h}{\partial Y}\frac{\partial w_h}{\partial Y}\,d\Omega-$$
$$\frac{Pr}{Da}\int_{\Omega}V_hw_h\,d\Omega-Ha^2Pr\int_{\Omega}V_hw_hd\Omega+RaPr\cos\phi\int_{\Omega}\theta_hw_hd\Omega \quad (4.14)$$

$$\int_{\Omega}\left(\frac{\partial U_h}{\partial X}+\frac{\partial V_h}{\partial Y}\right)q_h\,d\Omega=0 \quad (4.15)$$

$$\int_{\Omega}\left(U_h\frac{\partial\theta_h}{\partial X}+V_h\frac{\partial\theta_h}{\partial Y}\right)w_h\,d\Omega=$$
$$-\int_{\Omega}\left(\frac{\partial\theta_h}{\partial X}\frac{\partial w_h}{\partial X}+\frac{\partial\theta_h}{\partial Y}\frac{\partial w_h}{\partial Y}\right)d\Omega+Ha^2EcPr\int_{\Omega}V_h^2w_h\,d\Omega$$
$$+EcPr\left(1+\frac{1}{\gamma}\right)\left[2\int_{\Omega}\left(\frac{\partial U_h}{\partial X}\right)^2w_h\,d\Omega+2\int_{\Omega}\left(\frac{\partial V_h}{\partial Y}\right)^2w_h\,d\Omega\right]$$
$$+EcPr\left(1+\frac{1}{\gamma}\right)\int_{\Omega}\left(\frac{\partial U_h}{\partial Y}+\frac{\partial V_h}{\partial X}\right)^2w_h\,d\Omega \quad (4.16)$$

for $(w_h, q_h) \in W \times Q$.

FEM approximation is achieved by using the approximate trial solution functions and trial test functions. These functions are the linear combination of nodal unknowns and shape functions which are linearly independent. Given below are the trial solution functions: $U_h = \sum_{j=1}^{m} u_j\phi_j$, $V_h = \sum_{j=1}^{m} v_j\phi_j$, $\theta_h = \sum_{j=1}^{m} \theta_j\phi_j$, $P_h = \sum_{j=1}^{l} p_j\psi_j$. Also trial approximated functions can be defined for test spaces as $w_h = \sum_{i=1}^{m} w_i\phi_i$ and $q_h = \sum_{i=1}^{l} q_i\psi_i$, where ϕ_j and ψ_j are shape functions.

By using these approximation in Equations (4.13), (4.14), (4.15), (4.16) we get the weak formulation as

(4.13) $\Longrightarrow$

$$\begin{aligned}
\int_\Omega \left[\left(\sum u_j\phi_j\right)\frac{\partial}{\partial X}\left(\sum u_j\phi_j\right) + \sum v_j\phi_j\frac{\partial}{\partial Y}\left(\sum u_j\phi_j\right)\right]\phi_i\, d\Omega = \\
-\int_\Omega \frac{\partial}{\partial X}\left(\sum p_j\psi_j\right)\phi_i\, d\Omega + Pr\left(1+\frac{1}{\gamma}\right)\int_\Omega\left[\frac{\partial}{\partial X}\left(\sum u_j\phi_j\right)\frac{\partial\phi_i}{\partial X}\right. \\
\left. + \frac{\partial}{\partial Y}\left(\sum u_j\phi_j\right)\frac{\partial\phi_i}{\partial Y}\right] d\Omega - \frac{Pr}{Da}\int_\Omega\left(\sum u_j\phi_j\right)\phi_i\, d\Omega \\
+ RaPr\sin\phi\int_\Omega \sum \theta_j\phi_j\phi_i d\Omega
\end{aligned}$$

$$\begin{aligned}
\int_\Omega \left[\left(\sum \phi_j u_j\right)\frac{\partial\phi_j}{\partial X} + \left(\sum v_j\phi_j\right)\frac{\partial\phi_j}{\partial Y}\right]\phi_i d\Omega\{u_j\} + \sum\int_\Omega \frac{\partial\psi_j}{\partial X}\phi_i\, d\Omega\{p_j\} \\
+ Pr\left(1+\frac{1}{\gamma}\right)\sum\int_\Omega\left(\frac{\partial\phi_j}{\partial X}\frac{\partial\phi_i}{\partial X} + \frac{\partial\phi_j}{\partial Y}\frac{\partial\phi_i}{\partial Y}\right) d\Omega\{u_j\} + \frac{Pr}{Da}\sum\int_\Omega \phi_j\phi_i\, d\Omega\{u_j\} \\
- RaPr\sin\phi\int_\Omega\sum\phi_j\phi_i d\Omega\{\theta_j\} = 0 \quad (4.17)
\end{aligned}$$

(4.14) $\Longrightarrow$

$$\begin{aligned}
\int_\Omega \left[\left(\sum u_j\phi_j\right)\frac{\partial}{\partial X}\left(\sum v_j\phi_j\right) + \sum v_j\phi_j\frac{\partial}{\partial Y}\left(\sum v_j\phi_j\right)\right]\phi_i\, d\Omega = \\
-\int_\Omega \frac{\partial}{\partial Y}\left(\sum p_j\phi_j\right)\phi_i\, d\Omega + Pr\left(1+\frac{1}{\gamma}\right)\int_\Omega\left[\frac{\partial}{\partial X}\left(\sum v_j\phi_j\right)\frac{\partial\phi_i}{\partial X}\right. \\
\left. + \frac{\partial}{\partial Y}\left(\sum v_j\phi_j\right)\frac{\partial\phi_i}{\partial Y}\right] d\Omega - \frac{Pr}{Da}\int_\Omega\left(\sum v_j\phi_j\right)\phi_i\, d\Omega \\
- Ha^2 Pr\int_\Omega\sum v_j\phi_j\phi_i d\Omega + RaPr\cos\phi\int_\Omega\sum\theta_j\phi_j\phi_i d\Omega
\end{aligned}$$

$$\int_\Omega \left[\left(\sum u_j\phi_j\right)\frac{\partial\phi_j}{\partial X}+\left(\sum v_j\phi_j\right)\frac{\partial\phi_j}{\partial Y}\right]\phi_i d\Omega\{v_j\}+\sum\int_\Omega \frac{\partial\phi_j}{\partial Y}\phi_i\, d\Omega\{p_j\}$$
$$+Pr\left(1+\frac{1}{\gamma}\right)\sum\int_\Omega\left(\frac{\partial\phi_j}{\partial X}\frac{\partial\phi_i}{\partial X}+\frac{\partial\phi_j}{\partial Y}\frac{\partial\phi_i}{\partial Y}\right)d\Omega\{v_j\}+\frac{Pr}{Da}\sum\int_\Omega \phi_j\phi_i\, d\Omega\{v_j\}$$
$$+Ha^2 Pr\sum\int_\Omega \phi_j\phi_i d\Omega\{v_j\}-Ra Pr\sin\phi\sum\int_\Omega \phi_j\phi_i d\Omega\{\theta_j\}=0 \quad (4.18)$$

(4.15) $\Longrightarrow$

$$\int_\Omega \frac{\partial}{\partial X}\left(\sum u_j\phi_j\right)+\frac{\partial}{\partial Y}\left(\sum V_j\phi_j\right)\psi_i\, d\Omega=0$$

$$\sum\int_\Omega \frac{\partial\phi_j}{\partial X}\psi_i\, d\Omega\{u_j\}+\sum\int_\Omega \frac{\partial\phi_j}{\partial Y}\psi_i\, d\Omega\{v_j\} = 0 \quad (4.19)$$

(4.16) $\Longrightarrow$

$$\int_\Omega\left[\sum u_j\phi_j\frac{\partial}{\partial X}\left(\sum\theta_j\phi_j\right)+\sum v_j\phi_j\frac{\partial}{\partial Y}\left(\sum\theta_j\phi_j\right)\right]\phi_i\, d\Omega=$$
$$-\int_\Omega\left[\frac{\partial}{\partial X}\left(\sum\theta_j\phi_j\right)\frac{\partial\phi_i}{\partial X}+\frac{\partial}{\partial Y}\left(\sum\theta_j\phi_j\right)\frac{\partial\phi_i}{\partial Y}\right]d\Omega$$
$$+Ha^2 Ec Pr\int_\Omega\left(\sum v_j\phi_j\right)^2\phi_i\, d\Omega$$
$$+EcPr\left(1+\frac{1}{\gamma}\right)\left[2\int_\Omega\left(\frac{\partial\sum u_j\phi_j}{\partial X}\right)^2\phi_i\, d\Omega+2\int_\Omega\left(\frac{\partial\sum v_j\phi_j}{\partial Y}\right)^2\phi_i\, d\Omega\right]$$
$$+EcPr\left(1+\frac{1}{\gamma}\right)\int_\Omega\left(\frac{\partial\sum u_j\phi_j}{\partial Y}+\frac{\partial\sum v_j\phi_j}{\partial X}\right)^2\phi_i\, d\Omega \quad (4.20)$$

The discrete, non-linear system of algebraic equations in matrix form can be written as

$$\underbrace{\begin{bmatrix} A_{11} & A_{12} & B_1 & A_{14}\\ A_{21} & A_{22} & B_2 & A_{24}\\ B_1^t & B_2^t & A_{33} & A_{34}\\ A_{41} & A_{42} & A_{43} & A_{44}\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}\mathbf{U_h}\\ \mathbf{V_h}\\ \mathbf{P_h}\\ \theta_\mathbf{h}\end{bmatrix}}_{X}=\underbrace{\begin{bmatrix}F_1\\ F_2\\ F_3\\ F_4\end{bmatrix}}_{F} \quad (4.21)$$

Here A, X, F are called block stiffness matrix, block solution vector and block load vector. The entries in block stiffness matrix for the corresponding local elements are specified in the form of ϕ and ψ.

4.3 Results and Discussion

The results of the horizontal magnetic field, viscous dissipation in an inclined, porous cavity that has been partially heated, filled with Casson fluid are presented as in isotherms, streamlines profile and $2D$ plots. The controlling parameters for the analysis are the Hartmann number $0 \leq Ha \leq 100$, the cavity inclination $0^\circ \leq \phi \leq 90^\circ$, the Eckert number $10^{-6} \leq Ec \leq 10^{-4}$, the Casson parameter $0.1 \leq \gamma \leq 10$ and the fixed value of $\epsilon = 0.2$, $Pr = 7$, $Da = 10^{-4}$, $Ra = 10^5$.
Figure 4.2 shows the streamlines against the increasing value of Hartman number and inclination angle. The number of vortices grows as the magnetic field or Ha grows, whereas the strength of streamlines reduces as the Lorentz force increases. Figure 4.2 shows that for the considered range of Hartmann number, magnetic field has less significant effect on streamlines. Two cells of streamlines converges to their own center. It can also observe that by increasing cavity inclination, streamlines become stronger and cover most part of the cavity. The two symmetric bullous of streamlines convert into one strong bullou and move near the center of the cavity.
Figure 4.3 demonstrates the heat transmission in container under the effect of Hartmann number and cavity inclination in terms of isotherm. At $\phi = 0^\circ$ increasing Ha, the velocity of fluid is reduced due to which the dimensionless temperature is also reduced near the top adiabetic wall and near the both cooled wall. Like streamlines less significant changes can be observed for considered values $Ha = 0, 50, 100$. By increasing cavity inclination, the temperature is found to be gradually decreases near adiabatic wall . The effect of viscous dissipation and Casson fluid parameter on temperature in the cavity is presented in figure 4.4 in terms of isotherms. Figure 4.4 shows the increasing behaviour in the range $10^{-6} - 10^{-4}$ for each value of Casson fluid parameter γ in the range $(0.1 - 10)$. Temperature distribution in cavity move from bottom heated portion to the top adiabatic wall. For small value of $Ec = 10^{-6}$ and $Ec = 10^{-5}$, heat transfer in the

cavity increases with an increase of Casson parameter. For large value $Ec = 10^{-4}$, a comparable behaviour can be examine due to decrease of temperature in the cavity for increase in γ.

The effect of cavity inclination ($0° \leq \phi \leq 90°$), Casson fluid parameter γ ($0.1 \leq \gamma \leq 10$), Hartmann number $0 \leq Ha \leq 100$ and Eckert number $10^{-6} \leq Ec \leq 10^{-4}$ on average Nusselt number Nu_{avg} represented in figure 4.5, 4.6, 4.7 and 4.8. Figure 4.5 expresses that the average Nusselt number is a function that decreases for both cavity inclination and Hartmann number. For different values of Hartmann nummber $Ha = 0, 25, 50, 75, 100$, the average Nusselt number is at its maximum at $\phi = 0°$ and at its lowest at $\phi = 90°$. It can be observed that in the range $20° \leq \phi \leq 80°$, Nu_{avg} significantly changes.

Figure 4.6 demonstrates that for small values $Ec = 10^{-6}$ and $Ec = 10^{-5}$, Nu_{avg} is at highest and increses with increasing values of γ. At higher value $Ec = 10^{-4}$, a massive drop of average Nusselt number noticed at $\gamma = 0.1$. For higher values of γ, velocity , thermal boundary layer thickness and effective viscosity decreases. As an outcome, the cavity's overall heat transfer rate improves. Figure 4.7 reveals that the relation between average Nu and γ is a direct relation. When $Ha = 0$ (no magnetic field), heat transfer rate increasing with an increase of γ. When horizontal magnetic field Ha is applied the thickness of the boundary layer rises as Ha increases and the velocity magnitude in the cavity decreases. As a result the average Nusselt number decreases.

Figure 4.8 provides the influence of cavity inclination for varies values of Casson parameter γ on average Nusselt number. Heat transfer rate in the cavity reduces due to the cavity inclination. For small value of $\gamma = 0.1$, the Casson fluid have vital effect of viscosity. So by increasing inclination of cavity the heat transfer rate in the cavity decreases and at lowest for $\phi = 90°$. For higher value of Casson parameter, fluid's effective viscosity decreases, the flow of fluid increases. As a result, the effect of cavity inclination on Nu_{avg} is less as compared to $\gamma = 0.1$.

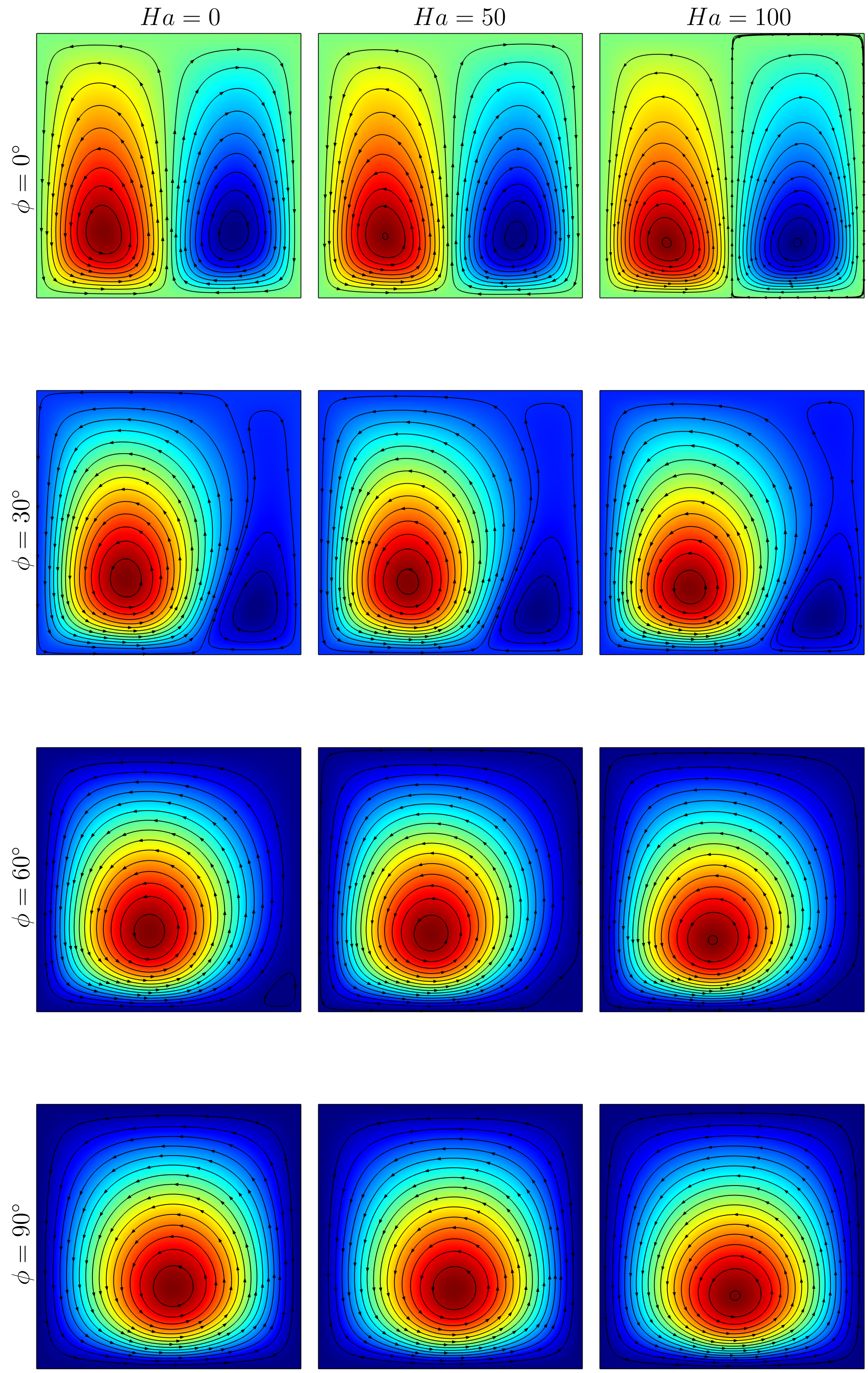

FIGURE 4.2: Influence of Ha and ϕ on streamlines for fixed value of $\gamma = 0.1$, $\epsilon = 0.2$, $Pr = 7, Da = 10^{-4}, Ra = 10^5$.

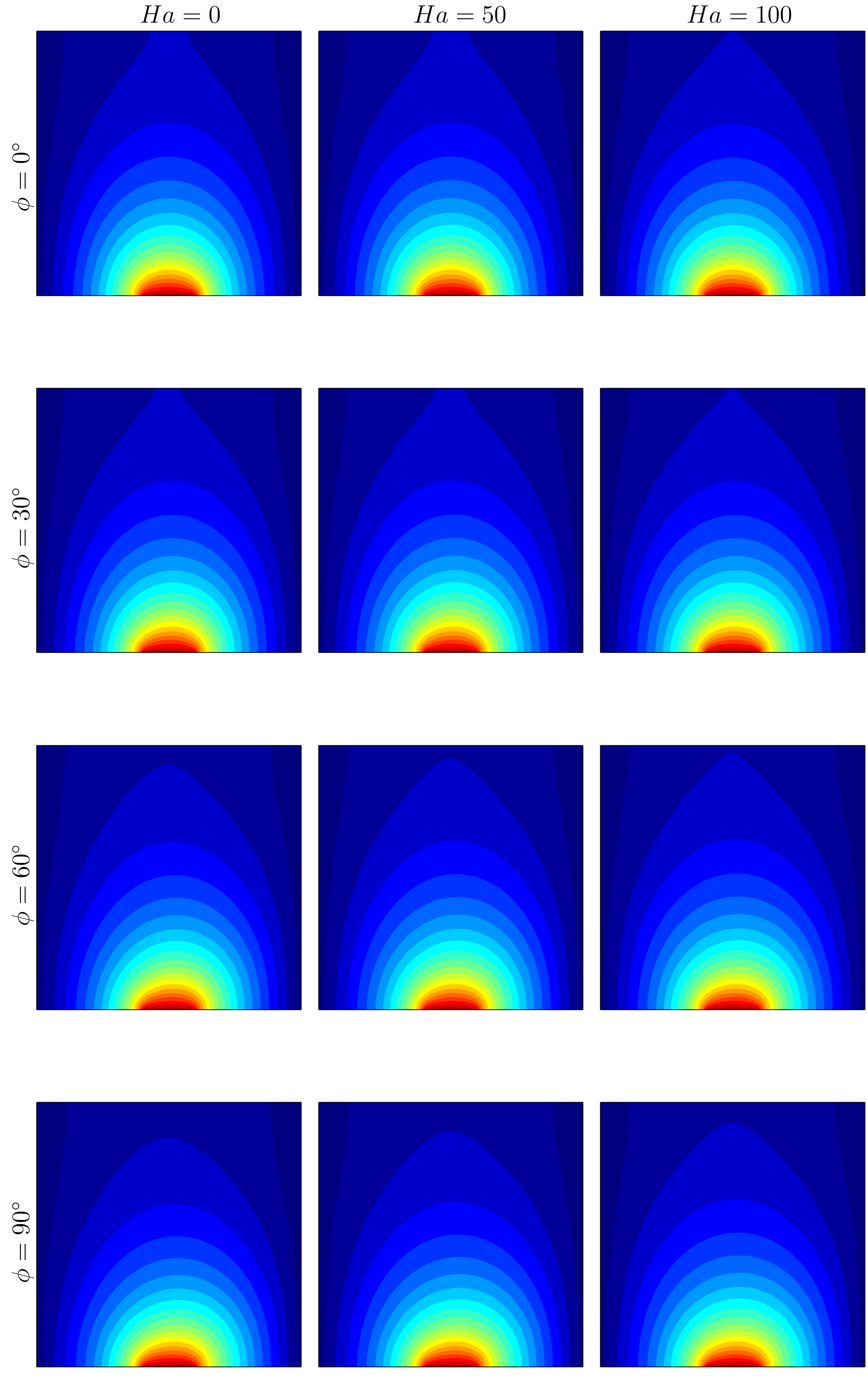

FIGURE 4.3: Influence of Ha and ϕ on isotherms for fixed value of $Pr = 7$, $\gamma = 0.1$, $\epsilon = 0.2$, $Da = 10^{-4}$, $Ra = 10^5$.

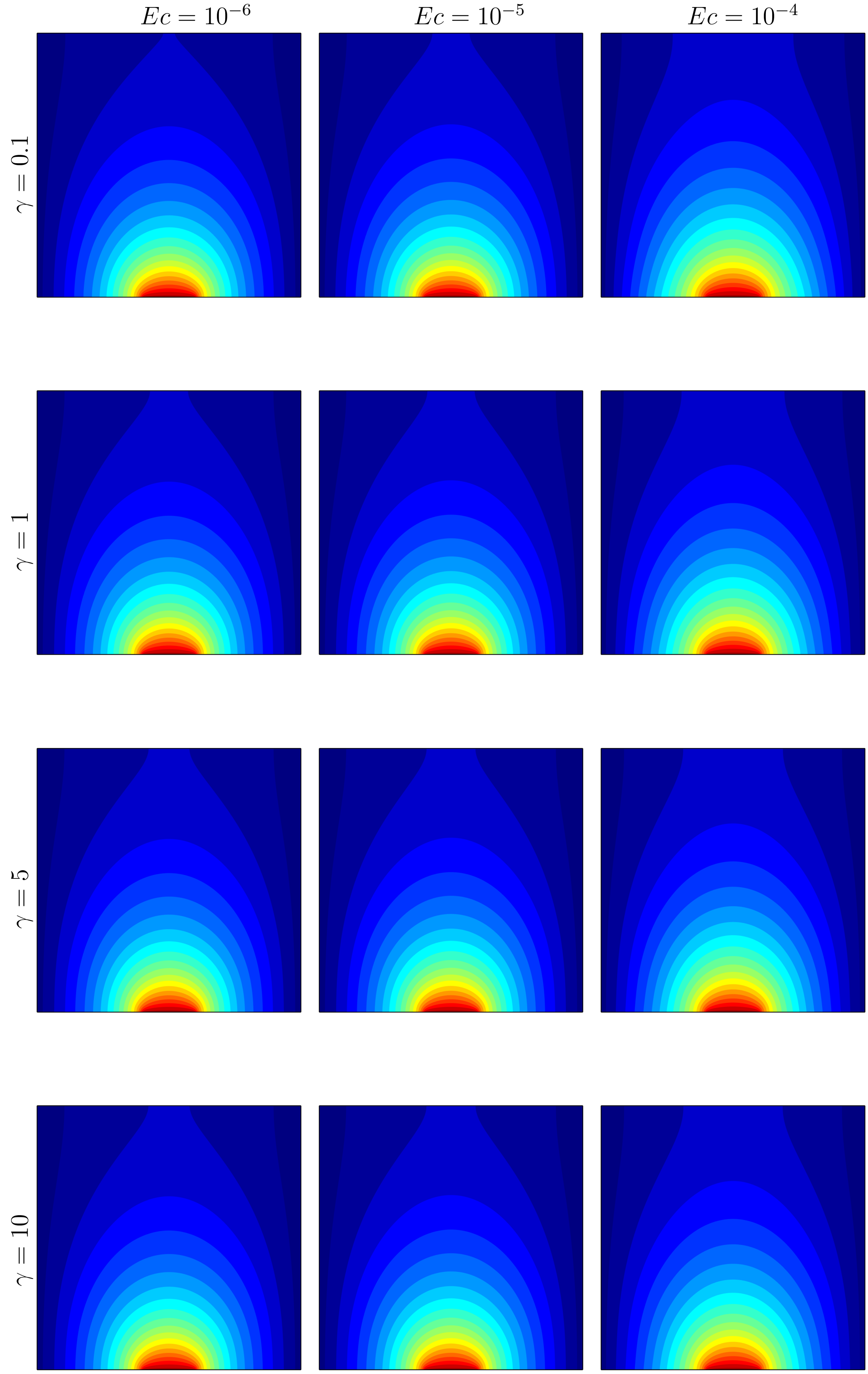

FIGURE 4.4: Influence of Ec and γ on isotherms for fixed value of $\epsilon = 0.2$, $Pr = 7, Da = 10^{-4}, Ra = 10^5$
.

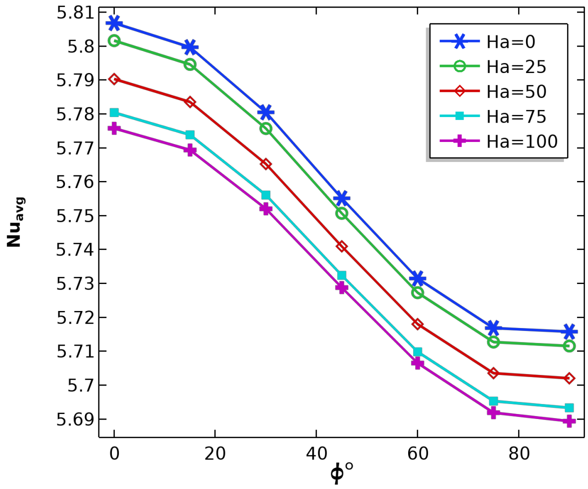

FIGURE 4.5: Influence of Ha and ϕ on average Nusselt number for $\gamma = 0.1$, $\epsilon = 0.2$, $Pr = 7, Da = 10^{-4}, Ra = 10^5$.

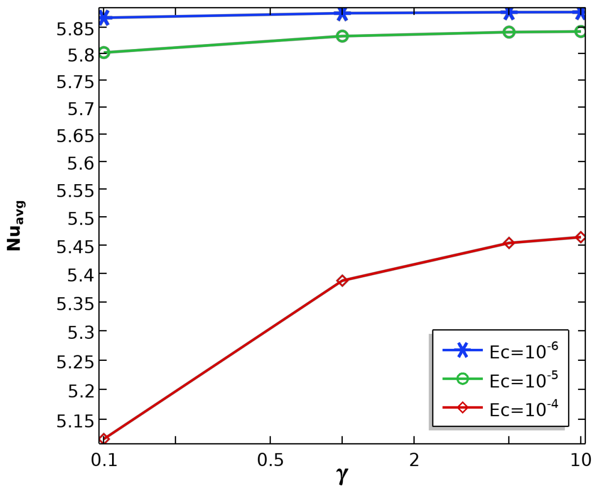

FIGURE 4.6: Influence of γ and Ec on average Nusselt number for fixed value of $\epsilon = 0.2$, $Pr = 7, Da = 10^{-4}, Ra = 10^5$.

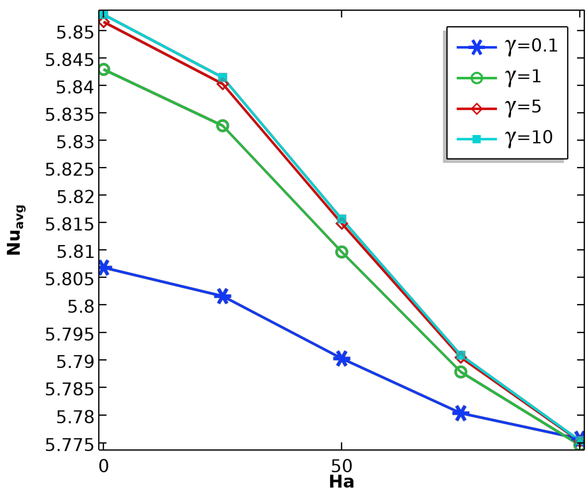

FIGURE 4.7: Influence of γ and Ha on average Nusselt number for fixed value of $\epsilon = 0.2$, $Pr = 7, Da = 10^{-4}, Ra = 10^5$.

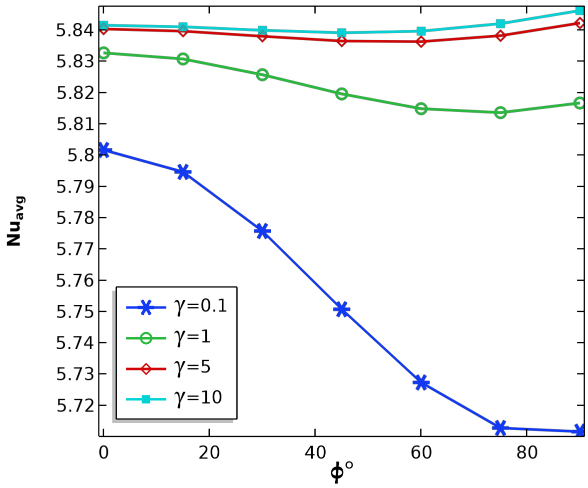

FIGURE 4.8: Influence of γ and ϕ on average Nusselt number for fixed value of $\epsilon = 0.2,\ Pr = 7, Da = 10^{-4}, Ra = 10^5$.

Chapter 5

Conclusions

In order to investigate the natural convection of Casson fluid, an inclined square cavity with partially heated from bottom is considered. The left as well as right wall of the cavity kept cool. Upper wall and the section of the bottom warm wall is supposed to be adiabetic. The governing equations for the problem are first transformed into dimensionless by using relevant transformation and later on solved numerically by GFEM. The effect of significant dimensionless parameters such as Raleigh number, Hartmann number, Eckert number and Darcy number with the cavity inclination on the heat transfer and the flow of has been substantially noticed. The numerical results are analysed for the flow behaviour and temperature of Casson fluid in the form of streamlines, isotherms and 2D graphs for different values of relative dimensionless parameters. The average Nusselt number (Nu_{avg}) in the cavity for different values of cavity inclination ϕ, Eckert number Ec, Casson parameter γ and for magnetic field parameter Ha plotted to interpret heat transfer rate.

In current investigations, the work of Aneja et al. [22] is extended by considering cavity inclination ϕ and horizontal magnetic field B_0. In addition, internal heat generation due to fluid flow is under observation by adding viscous dissipation effect in energy equation. The average Nusselt number in the cavity is observed using plots for separate values of cavity inclination ϕ, Casson parameter γ, viscous dissipation parameter Ec and for magnetic field Ha.

The major findings from the analysis are listed below as:

1. Increasing the magnetic field's effect, the fluid flow velocities decreases as a result the temperature inside cavity decreases or moves to the heated portion.
2. Increasing the cavity inclination, the flow of Casson fluid cover most part of the cavity.
3. For small value of Casson fluid parameter $\gamma = 0.1$, higher value of Eckert number increases the internal heat generation of fluid.
4. Low average Nusselt number for higher values of cavity inclination in the range $(75^\circ \leq \phi < 90^\circ)$ at Hartmann number $Ha = 100$.
5. For small Casson parameter $\gamma = 0.1$, cavity inclination decreases the average Nusselt number.
6. For small value of Casson parameter $\gamma = 0.1$, both Harmann number and cavity inclination decreases the average Nusselt number by increasing ϕ and Ha.
7. Increase in viscouss dissipation effect of fluid, decreases the average Nusselt number(Nu_{avg}).
8. For large value of Eckert number Ec, Casson parameter significantly effect on average Nusselt number(Nu_{avg}). For small value of Casson parameter $\gamma = 0.1$, heat transfer rate is low for higher value of Eckert number $Ec = 10^{-4}$ but increases in Casson parameter, average Nusselt number also increases.
9. For low value of Eckert number $10^{-5} \leq Ec \leq 10^{-4}$, (Nu_{avg}) increases with an increase of γ. For large value $Ec = 10^{-4}$, a massive drop of heat transfer rate is observed in cavity for $\gamma = 0.1$ but gradually increases with an increase of γ.

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