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Effect of Thermal Radiation on MHD Nanofluid Flow over a Non-linear Stretching Sheet through Porous Medium

by

Naeem Muzaffar

A thesis submitted in partial fulfillment for the
degree of Master of Philosophy

in the

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DEDICATED

to

my father

(late) mother

wife

and

children

CERTIFICATE OF APPROVAL

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Abstract

In this analysis, the boundary layer viscous flow of radiactive MHD nanofluids flow and heat transfer over a non-linearly-stretching sheet through porous medium is presented. Velocity and thermal slip conditions are considered instead of no slip conditions at the boundary. A similarity transformation set is used to transform the governing partial differential equations into non-linear ODEs. The reduced equations are solved numerically using the Shooting method. The influence of the governing parameters on the dimensionless velocity, temperature, nanoparticle concentration as well as the skin friction coefficient, Nusselt number and local Sherwod number are analyzed. It is found that as the velocity slip parameter increases, the velocity profile is decreased and the skin friction and heat transfer decreased while the mass transfer is increased. Increasing the thermal slip parameter causes decrease in the heat and mass transfer rates. The results are presented in both graphical and tabular forms.

Contents

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Abbreviations

BVPs	Boundary Value Problems
IVPs	Initial Value Problems
MHD	Magnetohydrodynamics
ODEs	Ordinary Differential Equations
PDEs	Partial Differential Equations
RK	Runge-Kutta

Symbols

$B(x)$	Magnetic field strength (unit: $A \cdot m^{-1}$)
c	Constant parameter
C	Nanoparticle concentration(unit: $mol \cdot m^{-3}$)
C_{fx}	Skin-friction coefficient (Pascal)
C_∞	Ambient concentration (unit: $mol \cdot m^{-3}$)
C_w	Nanoparticle concentration (unit:$mol \cdot m^{-3}$)
C_p	Specific heat capacity (unit:$J \cdot kg^{-1} \cdot K$)
D_T	Brownian diffusion coefficient
D_b	Thermophoresis diffusion coefficient
D_1	Thermal slip parameter
E_c	Eckert number
f	Dimensionless stream function
L_e	Lewis number
M	Magnetic parameter
N_1	Velocity slip factor
N_x	Nusselt number
N_b	Brownian motion parameter
N_t	thermophoresis parameter
n	Non-linear stretching parameter
P_r	Prandtl number
Sh_x	Sherwood number
T	Fluid temperature (unit: K)
(u, v)	x and y components of fluid velocity (unit: m/s)
u_w	Fluid velocity of the wall along the x-axis (unit: m/s)

(x, y)	Cartesian coordinates along the sheet (unit: m)

Greek Symbols

σ	Electrical conductivity (unit: $S \cdot m^{-1}$)
ψ	Stream function
η	Dimensionless similarity variable
μ	Dynamic viscosity of the base fluid (unit: kg/(m·s))
ν	Kinematic viscosity (unit: $m_2 \cdot s^{-1}$)
ρ_f	Density of fluid (unit: $kg \cdot m^{-3}$)
$(\rho c)_f$	Heat capacity of the base fluid (unit: $kg/m \cdot s^2$)
$(\rho c)_\nu$	Heat capacity of the nanoparticle (unit: $kg/m \cdot s^2$)
θ	Dimensionless temperature (unit: K)
p	Pressure (unit: N/m^2)
ϕ	Nanoparticle volume fraction
ϕ_w	Nanoparticle volume fraction at the stretching surface
ϕ_∞	Ambient nanoparticle volume fraction
q_w	Surface heat flux (unit: $\mathrm{W/m^2}$)
q_m	Surface mass flux
λ	Velocity slip parameter
δ	Thermal slip parameter
T_w	Temperature at the surface (unit: K)
T_∞	Fluid temperature far away from the surface (unit:K)
Re_x	Reynolds number

Subscripts

f	Fluid
w	Condition on the sheet
∞	Ambient condition

Chapter 1

Introduction

In many engineering processes, viscous fluid flow is an important factor including crystal growing, electronic chips or cooling of metallic sheets, paper and glass fiber production etc. [1]. Hence, the cooling process results in the end product of desired characteristics [1]. Sakiadis [2] introduced axi-symmetric analysis and 2D boundary layer fluid flow while Kumaran and Ramanaiah [3] analysed flow over a quadratic stretched surface. Sajid et al. [4] examined the viscous flow due to curved stretching surface. The Shooting method with RK4 was used to obtain the similarity solution of the problem. The physical quantities of interest such as Skin friction coefficient and fluid velocity are obtained and discussed under the influencce of nondimensional curvature. It is observed that boundary layer thickness increased due to nondimensional curvature while skin friction coefficient decreased. Sanni et al. [5] observed the said problem with non-linear power law velocity over the curved stretching sheet. Sandeep et al. [6] analysed the complication related to stagnation point, mass and heat flow behaviour in the presence of magnetic field over a stretching surface. The slip flow of the magnetic properties-convective boundary layer through a non-isothermal, continuously moving non-linear radiating plate immersed in Darcian porous materials using the numerical fourth-fifth order Runge-Kutta Fehlberg was worked by Uddin et al. [7]. The fluid containing nanoparticles is called Nanofluid. These fluids are concoted colloidal solution in a base fluid. Their composition utilize metals like copper or silver, carbides (SiC), oxides (CuO, Al_2O_3), Nitrides (AIN) or non metals (Carbon nano tubes). The base fluid can be water or toluene. The combination of base fluid particle is chosen according to the desired application of nanofluid. Nanofluids have many engineering and biomedical utilizations. Stability, sufficient viscosity, better wetting and dispersion are the benefits of nanofluids. The range of nanoparticles is $1 - 100$ nm in diameter. Experimentations have shown that only five percent nanoparticle volume fraction is required for effective heat transfer [8].

Fakour et al. [9] examined transfer of heat and flow of nanofluid in a permeable channel with magnetic field. Hamad et al. [10] reported similar solutions in terms of energy transfer and viscous fluid over non-linearity sheet stretching using RK4 technique. Das [11] analysed the boundary layer flow with the partial slip over a non-linear stretched surface at specific temperature. Kumar et al. [12] studied the problem related to flow of electrically charged fluid and transfer of heat under the action of magnetic field and heat source over a stretching surface. Uddin et al. [13] presented the $2D$ magnetohydrodynamic boundary layer flow of a charged Newtonian nanofluid over a stretching surface in a quiescent fluid. Uddin et al. [14] presented the impact of heat and mass transfer, based on 2D laminar mixed convective boundary layer nanofluid flow. Water based transfer of heat of steady viscous fluid in existence of charged particles over a stretched surface is examined by Rashidi et al. [15]. The unsteady MHD squeezing flow through two parallel (boundary layer flow) discs is investigated by Azimi and Riaz [16]. In this research similarity transformation was used to transform the PDEs into ODEs, and finally ODEs were solved using Shooting method with RK4 scheme.

Boundary slip results are widely applicable for cleaning synthetic heart valves. By using homotopy analysis technique, Mustafa et al. [17] investigated the slip impact of nanofluid in a network with wall slip, on flowing movements. In recent times, Malvandi et al. [18] simulated the joint impact of viscous flow and thermal slip on turbulent boundary layer flow from a nanofluid over a stretching surface. Khan et al. [19] analyzed the role of boundary layer and slip velocity of Copper-water and Copper-kerosene nanofluid on 2D and axi-symmetric stretching flow. The study of MHD flow and heat transfer over a stretched surface was given by

Turkyilmazoglu [20], taken into account second order slip. Turkyilmazoglu [21] analyzed the magnetohydrodynamics slip flow of an ionized, non-Newtonian liquids over a stretched surface. Turkyilmazoglu [22] investigated the relationship between specific nanofluids and ordinary flow properties by incorporating a scaling method that significantly simplifies the assessment of flow and process conditions including skin friction and temperature profile. Turkyilmazoglu [23] examined the thermal energy liquid film principle of conventional Nusselt in case where material from some widely applied nanoparticles are introduced to the liquid phase. The mixture solution that is nanofluid, is numerically examined either when the nanoparticles are randomly scattered throughout the condensate control volume that would be the most common template in the research, or when the particle concentration through the film can differ from the outer wall of the condensate film throughout the case of the modified nanoparticles of Buongiorno. Rashidi et al. [24] investigated the free convective heat and mass transfer in a linear $2D$ magnetohydrodynamics stretching vertical surface in a pervious material, by using the homotopy analysis approach.

Thermal radiation is the phenomena of release of energy from a heated surface in the form of electromagnetic radiations. Hayat and Qasim [25] worked on the MHD flow with heat transmission over a stretched surface utilizing Joule heating and thermodiffusion and concluded that R_d and θ are in inverse relationship to each other. Sheikholeslami et al. [26] demonstrated the influence of R_d on magnetohydrodynamics flow between two horizontal rotating plates and obtained a conclusion that the Nusselt number have direct impact on radiation parameter and inverse impact on other active parameters while the thickness of concentration boundary layer and radiation parameter are in inverse relationship to each other.

A porous medium is any solid containing sufficient open spaces in between for fluid flow. Porous media has numerous applications in many areas of applied science including mechanics, engineering, biophysics and material science etc. [27]. Saffman [28] worked on the dispersion of dynamically neutral material quantity in a fluid flow through porous medium. Jugjai et al. [29] analyzed that porous medium burner is more advantegous than ordinary open burner due to low emission of pollutants and enhanced evaporation of droplet spray. Khanafer and Vafai [30] demonstrated that combination of nanofluids and porous medium has wide range of potential to increase heat transfer in thermal systems.

A thorough search on the impact of thermal radiation and porosity factor on MHD nanofluid flow over a non-linear stretching sheet in the existence of a magnetic field, yeilded only a few information. This thesis prolongate the study of of Ramya et al. [8]. The flow over peripheral layer and transfer of heat of a nanofluid that shows wall slip effects are taken into the account and the PDEs are changed into ODEs and solved numerically using Shooting method with RK4.

1.1 Thesis Contribution

The main purpose of this thesis is to demonstrate the impact of thermal radiation and porosity factor on MHD nanofluid flow over a non-linear stretched surface. The governing PDEs are transformed into set of non-linear ODEs using suitable similarity transformations. Moreover, Shooting method is employed to obtain numerical results of obtained ODEs. The numerically obtained results are computed by using MATLAB. The impact of various physical parameters have been discussed in table and graphs.

1.2 Contents of Thesis

This thesis is classified into following four chapters:

Chapter 2 consists on basic definitions, terminologies and governing PDEs which are useful for upcoming chapters.

Chapter 3 based on review work of Ramya et al. [8]. The set of governing non-linear PDEs is converted into non-linear ODEs by using set of adequate similarity transformations and then solved by Shooting technique with RK4. The results obtained by ODEs are discussed through table and graphs.

Chapter 4 extends the work of Ramya et al. [8] through porous medium. The non-linear ODEs are solved using Shooting technique with RK4. Numerical outcomes for different physical parameters are discussed through table and graphs.

Chapter 5 presents the conclusion of thesis.

Chapter 2

Basic Definitions and Governing Equations

A few basic definitions, terminologies and governing laws will be presented in this unit, which will be useful in continuing the work for the next units.

2.1 Basic Definitions

Definition 2.1. (**Fluid**)
"A fluid is a substance that deforms continuously under the application of a shear (tangential) stress no matter how small the shear stress may be." [31].

Definition 2.2. (**Fluid Mechanics**)
"Fluid mechanics is that branch of science which deals with the behaviour of the fluid (liquids or gasses) at rest as well as in motion" [32].

Definition 2.3. (**Fluid Statics**)
"The study of fluid at rest is called fluid statics" [32].

Definition 2.4. (**Fluid Dynamics**)
"The study of fluid if the pressure forces are also considered for the fluid in motion, that branch of science is called fluid dynamics" [32].

Definition 2.5. **(Viscosity)**

"Viscosity is defined as the property of a fluid which offers resistance to the movement of one layer of fluid over another adjacent layer of the fluid. Mathematically,

$$\mu = \frac{\tau}{\frac{du}{dy}},$$

where μ is the viscosity coefficient τ is the shear stress and $\frac{du}{dy}$ represents the velocity gradient" [32].

Definition 2.6. **(Kinematic Viscosity)**

"It is defined as the ratio between the dynamic viscosity and density of fluid. It is denoted by symbol ν called **nu**. Mathematically,

$$\nu = \frac{\mu}{\rho},$$

where ν is the kinematic viscosity and ρ denote density respectively" [32].

2.2 Types of Fluid

"The fluid may be classified into following five types.

- Ideal fluid
- Real fluid
- Newtonian fluid
- Non-Newtonian fluid, and
- Ideal plastic fluid" [32].

Definition 2.7. **(Ideal Fluid)**

"A fluid, which is incompressible and has no viscosity, is known as an ideal fluid. Ideal fluid is only an imaginary fluid as all the fluids, which exist, have some viscosity" [32].

Definition 2.8. **(Real Fluid)**
"A fluid, which possesses viscosity, is known as a real fluid. In actual practice, all the fluids are real fluids" [32].

Definition 2.9. **(Newtonian Fluid)**
"A real fluid, in which the shear stress is directly proportional to the rate of shear strain (or velocity gradient), is known as a Newtonian fluid" [32].

Definition 2.10. **(Non-Newtonian Fluid)**
"A real fluid in which the shear stress is not directly proportional to the rate of shear strain (or velocity gradient), is known as a non-Newtonian fluid" [32].

Definition 2.11. **(Ideal Plastic Fluid)**
"A fluid, in which shear stress is more than the yield value and shear stress is proportional to the rate of shear strain(or velocity gradient), is known as ideal plastic fluid" [32].

2.3 Heat Transfer Mechanism and related Properties

Definition 2.12. **(Heat Transfer)**
"Heat transfer is a branch of engineering that deals with the transfer of thermal energy from one point to another within a medium or from one medium to another due to the occurrence of a temperature difference" [33].

Definition 2.13. **(Conduction)**
"The transfer of heat within a medium due to a diffusion process is called conduction" [33].

Definition 2.14. **(Convection)**
"Convection heat transfer is usually defined as energy transport effected by the motion of a fluid. The convection heat transfer between two dissimilar media is governed by Newtons law of cooling" [33].

Definition 2.15. (Thermal Radiation)
"Thermal radiation is defined as radiant (electromagnetic) energy emitted by a medium and is solely to the temperature of the medium. Radiant energy exchange between surfaces or between a region and its surroundings is described by the Stefan-Boltzmann law, which states that the radiant energy transmitted is proportional to the difference of the fourth power of the temperatures of the surfaces. The proportionality parameter is known as the Stefan-Boltzmann constant" [33].

Definition 2.16. (Thermal Conductivity)
"The Fourier heat conduction law states that the heat flow is proportional to the temperature gradient. The coefficient of proportionality is a material parameter known as the thermal conductivity which may be a function of a number of variables." [33].

2.4 Types of Fluid Flow

" The fluid flow is classified as:

1. Steady and unsteady flows;
2. Uniform and non-uniform flows;
3. Laminar and turbulent flows;
4. Compressible and incompressible flows;
5. Rotational and irrotational flows
6. One, two and three-dimensional flows" [32].

Definition 2.17. (Steady Flow)
"Steady flow is defined as that type of flow in which the fluid characteristics like velocity, pressure, density, etc., at a point do not change with time. Thus for steady flow, mathematically, we have:

$$\left(\frac{\partial \mathbf{V}}{\partial t}\right)_{x_0\ y_0\ z_0} = \left(\frac{\partial p}{\partial t}\right)_{x_0\ y_0\ z_0} = \left(\frac{\partial \rho}{\partial t}\right)_{x_0\ y_0\ z_0} = 0\text{"}\ [32].$$

Definition 2.18. (Unsteady Flow)

"Unsteady flow is defined as that type of flow in which the fluid characteristics like velocity, pressure, density, etc., at a point changes with respect to time. Thus for unsteady flow, mathematically, we have:

$$\left(\frac{\partial \mathbf{V}}{\partial t}\right)_{x_0 \ y_0 \ z_0} = \left(\frac{\partial p}{\partial t}\right)_{x_0 \ y_0 \ z_0} \neq 0\text{" [32].}$$

Definition 2.19. (Uniform Flow)

"Uniform flow is defined as that type of flow in which velocity at any given time does not change with respect to space (i.e., length of direction of the flow). Mathematically, for uniform flow

$$\left(\frac{\partial \mathbf{V}}{\partial s}\right)_{t=constant} = 0,$$

where $\partial \mathbf{V}$ = change of velocity and ∂ s =length of flow in the direction S" [32].

Definition 2.20. (Non-uniform Flow)

"Non-uniform flow is that type of flow in which velocity at any given time changes with respect to space. Thus, mathematically, for non-uniform flow

$$\left(\frac{\partial \mathbf{V}}{\partial s}\right)_{t=constant} \neq 0,$$

where $\mathbf{V}$ is the velocity and s is the displacement" [32].

Definition 2.21. (Laminar Flow)

"Laminar flow is defined as that type of flow in which fluid particles move along well-defined paths or stream line and all the stream-lines are straight and parallel. Thus the particles move in laminas or layers gliding smoothly over the adjacent layer. This type of flow is also called stream-line floew or viscous flow" [32].

Definition 2.22. (Turbulent Flow)

"Turbulent flow is that type of flow in which the fluid particles move in a zig-zag way. Due to movement of fluid particles in zig-zag way the eddies formation takes place which are responsible for high energy loss." [32].

Definition 2.23. (Compressible Flow)

"Compressible flow is that type of flow in which the density of fluid changes from point to point or in other words the density (ρ) is not constant for the fluid. Thus, mathematically, for compressible flow

$$\rho(x, y, z, t) \neq c,$$

where 'c' is a constant" [32].

Definition 2.24. (Incompressible Flow)

"Incompressible flow is that type of flow in which the density is constant for the fluid flow. Liquids are generally incompressible while gasses are compressible. Mathematically for compressible flow:

$$\rho(x, y, z, t) = c,$$

where 'c' is constant" [32].

Definition 2.25. (Rotational Flow)

"Rotational flow is that type of flow in which the fluid particles while flowing along stream-lines, also rotate about their own axis" [32].

Definition 2.26. (Irrotational Flow)

"An Irrotational flow is that type of flow in which the fluid particles while flowing along stream-lines, do not rotate about their own axis" [32].

Definition 2.27. (One-dimensional Flow)

"One dimensional flow is that type of flow in which the flow parameter such as velocity is a function of time and one space co-ordinate only. The variation of velocities in other two mutually perpendicular directions is assumed negligible. Hence, mathematically, for one dimension flow

$$u = f(x), v = 0 \text{ and } w = 0,$$

where u, v and w are velocities components in x, y and z directions respectively" [32].

Definition 2.28. (Two-dimensional Flow)
"Two-dimensional flow is that type of flow in which velocity is a function of time and two rectangular space co-ordinate say x and y. For a steady two-dimensional the velocity is a function of two space co-ordinate only. The variation of the velocity in the third direction is negligible. Thus mathematically for two-dimensional flow

$$u = f_1(x, y), v = f_2(x, y) \text{ and } w = 0\text{''} \ [32].$$

Definition 2.29. (Three-dimensional Flow)
"Two-dimensional flow is that type of flow in which velocity is a function of time and three mutually perpendicular directions. But for a steady three-dimensional flow the fluid parameters are function of three space co-ordinates(x, y and z) only. Thus mathematically for three-dimensional flow

$$u = f_1(x, y, z), v = f_2(x, y, z) \text{ and } w = f_3(x, y, z)\text{''} \ [32].$$

2.5 Fundamental Equation of Flow

2.5.1 Continuity Equation

"The principle of conservation of mass can be stated as the time rate of change of mass in a fixed volume is equal to the net rate of flow of mass across the surface. The mathematical statement of the principle results in the following equation, known as the continuity (of mass) equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0, \tag{2.1}$$

where ρ is the density (kg/m^3) of the medium, $\mathbf{V}$ the velocity vector (m/s), and ∇ is the nabla or del operator. The continuity equation in (2.1) is in conservation (or divergence) form since it can be derived directly from an integral statement of mass conservation. By introducing the material derivative or Eulerian derivative operator D/Dt

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla, \tag{2.2}$$

the continuity equation (2.1) can be expressed in the alternate, non-conservation (or advective) form

$$\frac{\partial \rho}{\partial t} + \mathbf{V} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{V} = \frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{V} \tag{2.3}$$

For steady-state conditions the continuity equation becomes

$$\nabla \cdot (\rho \mathbf{V}) = 0 \tag{2.4}$$

When the density changes following a fluid particle are negligible, the continuum is termed incompressible and we have $\frac{D\rho}{Dt} = 0$. The continuity equation (2.3) then becomes

$$\nabla \cdot \mathbf{V} = 0, \tag{2.5}$$

which is often referred to as the incompressibility condition or incompressibility constraint" [33].

2.5.2 Momentum Equation

"The principle of conservation of linear momentum (or Newton's Second Law of motion) states that the time rate of change of linear momentum of a given set of particles is equal to the vector sum of all the external forces acting on the particles of the set, provided Newton's Third Law of action and reaction governs the internal forces. Newton's Second Law can be written as

$$\frac{\partial}{\partial t}(\rho \mathbf{V}) + \nabla \cdot (\rho \mathbf{V} \otimes \mathbf{V}) = \nabla \cdot \sigma + \rho \mathbf{f}, \tag{2.6}$$

where $\otimes$ is the tensor (or dyadic) product of two vectors, σ is the Cauchy stress tensor (N/m^2) and `f` is the body force vector, measured per unit mass and normally taken to be the gravity vector. Equation (2.6)describes the motion of a continuous medium, and in fluid mechanics they are also known as the Navier equations. The form of the momentum equation shown in (2.6) is the conservation (divergence) form that is most often utilized for compressible flows. This equation may be simplified to a form more commonly used with incompressible flows. Expanding the first two derivatives and collecting terms

$$\rho \left(\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \nabla \cdot \mathbf{V} \right) + \mathbf{V} \left(\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{V} \right) = \nabla \cdot \sigma + \rho \mathbf{f} \tag{2.7}$$

The second term in parentheses is the continuity equation (2.1) and neglecting this term allows (2.7) to reduce to the non-conservation (advective) form

$$\rho \left(\frac{D\mathbf{V}}{Dt} \right) = \nabla \cdot \sigma + \rho \mathbf{f} \tag{2.8}$$

where the material derivative (2.2) has been employed.
The principle of conservation of angular momentum can be stated as the time rate of change of the total moment of momentum of a given set of particles is equal to the vector sum of the moments of the external forces acting on the system. In the absence of distributed couples, the principle leads to the symmetry of the stress tensor:

$$\sigma = (\sigma)^T, \tag{2.9}$$

where the superscript T denotes the transpose of the enclosed quantity" [33].

2.5.3 Law of Conservation of Energy

"The law of conservation of energy (or the First Law of Thermodynamics) states that the time rate of change of the total energy is equal to the sum of the rate of work done by applied forces and the change of heat content per unit time. In the general case, the First Law of Thermodynamics can be expressed in conservation form as

$$\frac{\partial \rho e^t}{\partial t} + \nabla \cdot \rho \mathbf{v} e^t = -\nabla \cdot \mathbf{q} + \nabla \cdot (\sigma \cdot \mathbf{v}) + Q + \rho \mathbf{f} \cdot \mathbf{v}, \tag{2.10}$$

where $e^t = e + 1/2\mathbf{v} \cdot \mathbf{v}$ is the total energy (J/m^3), e is the internal energy, $\mathbf{q}$ is the heat flux vector (W/m^2) and Q is the internal heat generation (W/m^3)" [33].

2.6 Dimensionless Parameters

Definition 2.30. Reynolds Number(Re)

"It is defined as the ratio of inertia force of a flowing fluid and the viscous force of the fluid. The expression for the Reynold's number is defined as

$$\begin{aligned}\text{Inertia force}(F_i) &= \text{Mass} \times \text{ Acceleration of flowing fluid} \\ &= \rho \times \text{ Volume} \times \frac{\text{Velocity}}{\text{Time}} = \rho \times \frac{\text{Volume}}{\text{Time}} \times \text{ Velocity} \\ &= \rho \times AV \times V \ \ (\because \text{Volume per sec} = \text{Area} \times \text{Velocity} = A \times V) \\ &= \rho AV^2\end{aligned}$$

$$\begin{aligned}\text{viscous force}(F_v) &= \text{Shear stress} \times \text{Area} \quad \left(\because \tau = \mu \frac{du}{dy} \therefore \text{Force} = \tau \times \text{Area}\right) \\ &= \tau \times A \\ &= \left(\mu \frac{du}{dy}\right) \times A = \mu \cdot \frac{V}{L} \times A \quad \left(\because \frac{du}{dy} = \frac{V}{L}\right)\end{aligned}$$

By definition Reynold's number,

$$\begin{aligned}Re &= \frac{F_i}{F_v} = \frac{\rho AV^2}{\mu \cdot \frac{V}{L} \times A} = \frac{\rho VL}{\mu} \\ &= \frac{V \times L}{\mu/\rho} = \frac{V \times L}{\nu} \quad \left(\because \frac{\mu}{\rho} = \nu = \text{Kinematic Viscosity}\right)\end{aligned}$$ " [32].

Definition 2.31. Prandtl Number(P_r)

"The Prandtl number is a dimensionless quantity that puts the viscosity of a fluid in correlation with the thermal conductivity. It therefore assesses the relation between momentum transport and thermal transport capacity of a fluid. It is defined as

$$P_r = \frac{\nu}{\alpha},$$

$$P_r = \frac{\mu C_p}{\rho k} = \frac{\eta}{\rho \alpha} = \frac{\eta C_p}{\lambda} = \frac{\text{momentum transport}}{\text{heat transport}} \tag{2.11}$$

where we have used the thermal diffusivity α which is defined as

$$\alpha = \frac{\lambda}{\rho C_p}$$

The Prandtl number is an example of a dimensionless number that is an intrinsic property of a fluid. Fluids with small Prandtl numbers are free-flowing liquids with high thermal conductivity and are therefore a good choice for heat conducting liquids" [34].

Definition 2.32. Nusselt Number(Nu)

"Nusselt number is an important parameter that can contribute to a better rate of heat exchange. It is basically a function of Reynolds and Prandtl number. The correlation is provided as

$$Nu = C(Re)^m(Pr)^n = h * D/(k) \tag{2.12}$$

This is the so-called Dittus Boelter-type correlation.

Where Nu = Nusselt number; Re = Reynolds number; Pr = Prandtl number; h = heat transfer coefficient (W/m^2k); and D = inner diameter of the tube (m)" [35].

Definition 2.33. Force Coefficient(C_f)

$$\text{"}C_f = \frac{\tau_w}{\frac{1}{2}\varrho w^2} = \frac{F}{\frac{1}{2}\varrho w^2 A} = 2N_e \tag{2.13}$$

$\tau_w(Pa)$ – stress component of circumfluenced body; $\varrho(kgm^{-3})$ – fluid density; $w(ms^{-1})$ – flow velocity; $F(N)$ – force;$A(m^2)$ – drag area; Ne (–) – Newton number.

It is important mainly in aerodynamics and expresses the resistance-to-inertia forces ratio. As a vector, the force F has a drag component F_D and that of uplift F_L" [36].

Definition 2.34. Eckert Number(E_c)

"The Eckert number relates the kinetic energy to the enthalpy of a fluid. It is given by

$$E_c = \frac{{v_{ref}}^2}{C_p T_{char}} = \frac{\eta}{C_p \lambda} = \frac{\text{kinetic energy}}{\text{enthalpy}} \tag{2.14}$$

The Eckert number is used to characterize the influence of self-heating of a fluid as a consequence of dissipation effects. At high flow velocities, the temperature profile in a fluidic system is not just dominated by the temperature gradients that are present in the system, but also be effects of dissipation due to internal friction of the fluid. This will result in self-heating and thus in a change of the temperature profile. The Eckert number allows judging if the effects of self-heating due to dissipation can be neglected ($Ec \ll 1$) or not" [34].

Definition 2.35. Sherwood Number(Sh)

"The Sherwood is defined as the ratio of convective to diffusive mass transfer. It is a dimensionless number and the mass transfer equivalent of Nu" [37].

Definition 2.36. Lewis Number(L_e)

"The Lewis number puts in correlation the mass diffusion and the thermal conductivity of a fluid. Similar to the Prandtl number, which correlates momentum transport and thermal transport properties of a fluid, and similar to the Schmidt number, which correlates momentum transport and mass transport of a fluid, the Lewis number correlates mass transport to thermal transport properties of the fluid. It is defined as

$$L_e = \frac{D}{\alpha} = \frac{\text{mass transport}}{\text{heat transport}} \tag{2.15}$$

Similar to the Prandtl and the Schmidt numbers, the Lewis number is a material constant" [34].

2.7 Solution Methodology

Consider the second order two point boundary value problem (BVP):

$$u'' = f(x, u, u'), \tag{2.16}$$

subjected to the boundary conditions:

$$u(0) = 0, \quad u(\alpha) = \xi,$$

where ξ is some known constant. In order to apply the shooting method for (BVP)(2.16), we first convert the equation (2.16) into a system of two first order ODEs. Using the notation, $u = u_1, \quad u = u_1' = u_2, \quad u'' = u_1'' = u_2'$, we have

$$u_1' = u_2, \tag{2.17}$$

$$u_2' = f(x, u_1, u_2). \tag{2.18}$$

The associated boundary conditions reduced as:

$$u_1(0) = 0, \quad u_1(\alpha) = \xi,$$

by considering $u_2(0) = \eta$, the first order system of Eqs. (2.17) and (2.18) together with $u_1(0) = 0, u_2(0) = \eta$ is an initial value problem (IVP) and can be solved by using the Runge-Kutta method of order fourth(RK4). Then we get both u_1 and u_2 computed at the decided nodes. If $u_1(\alpha)$ is sufficiently close to ξ, then this u_1 is an approximate solution, If not we have to choose another value of ξ and the process is repeated again. Newton method is used to refine the initial guess. This process is continued until a satisfactory accuracy is achieved. Its main advantage is its efficiency and fastness. If the solution is extremely sensitive to the assumed initial condition, then parallel shooting method is applied (see Na [38] for details).

Chapter 3

Transfer of Heat and MHD Viscous Nanofluid Flow over a Non-linear Deformed Surface

This chapter comprises the transfer of heat and viscous boundary layer flow of nanofluids in the existence of magnetic field over a non-linear stretching surface. For the conversion of the governing PDES into non-linear ODEs, the similarity transformation package is being used. Simplified equations are solved by applying Shooting technique with RK4. It examines the effect of boundary conditions on the nondimensional temperature, velocity, concentration of nanoparticle, Nusselt and Sherwood number. It has noticed that the velocity profile is reduced by increasing the values of velocity slip parameter λ. Results of converted ODEs are given in the form of table and graphs. The present study is the review of Ramya et al. [8].

3.1 Problem Formulation

Consider 2D incompressible and a steady viscous flow of an electrically conducting fluid over a non-linear deformed surface in Figure 3.1. The surface is expanded with a fixed origin velocity of $u_w = ax^n$ considering T_w and C_w as constant.

The nanoparticle fraction and ambient temperature have constant values C_∞ and T_∞ respectively. The flow is determined by consedring 2D governing equations consisting continuity, momentum and energy transfer.

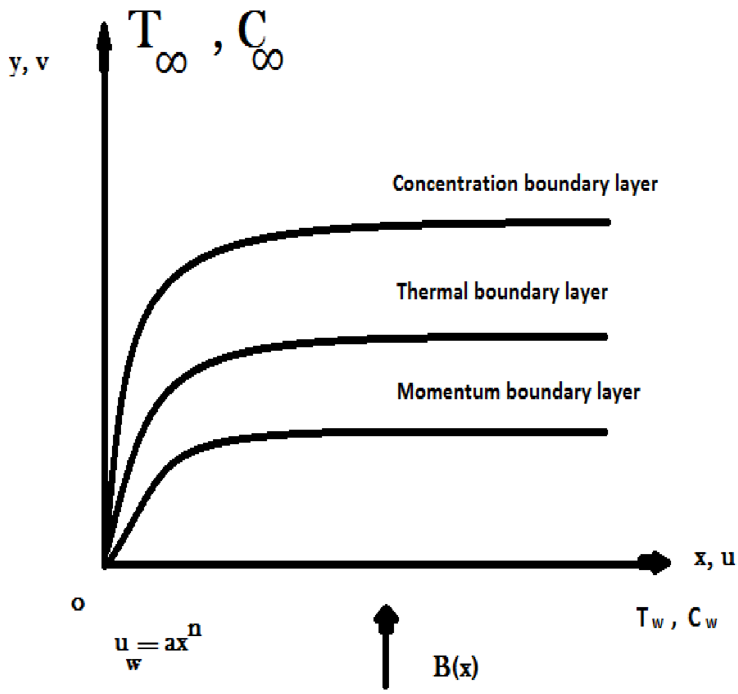

FIGURE 3.1: Geometry of the physical model

3.1.1 The Governing Equations

The set of governing partial differential equations are given below [8]

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{3.1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} - \frac{\sigma B^2}{\rho}u, \tag{3.2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} + \frac{\nu}{C_p}\left(\frac{\partial u}{\partial y}\right)^2 + \tau\left[D_B\frac{\partial T}{\partial y}\frac{\partial C}{\partial y} + \frac{D_T}{T_\infty}\left(\frac{\partial T}{\partial y}\right)^2\right], \tag{3.3}$$

$$u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} = D_B\frac{\partial^2 C}{\partial y^2} + \frac{D_T}{T_\infty}\left(\frac{\partial^2 T}{\partial y^2}\right). \tag{3.4}$$

In equations (3.1) to (3.4) T denotes the fluid temperature, C is the nanoparticle concentration, where ν represent the kinematic viscosity, ρ_f represent the

fluid density, specific heat capacity at constant pressure is represented by C_p, the thermophoresis diffusion coefficient is represented by D_w, also D_T is the Brownian diffusion coefficient. T_∞ is the fluid of temperature far away from the stretching sheet, thermal diffusivity is $\alpha = \frac{k}{\rho C_p}\,(m^2/s)$, and ratio between effective heat capacity of the fluid is $\tau = \dfrac{(\rho c)_p}{(\rho c)_f}$, $B(x) = B_0 x^{n-1/2}$ is the magnetic field.
The associated boundary conditions for the above system have been taken as:

$$\begin{cases} u = u_w + N\nu\left(\frac{\partial u}{\partial y}\right), v = 0, T = T_w + D\left(\frac{\partial T}{\partial y}\right), C = C_w, \text{ at } y = 0. \\ u \to 0, v \to 0, T = T_\infty, C = C_\infty, \text{ at } y \to \infty. \end{cases} \tag{3.5}$$

Here $u_w = ax^n$ is stretching velocity, temperature at the sheet is $T_w = T_\infty + bx^{2n}$, c, b are constant. $D = D_1 x^{-\left(\dfrac{n+1}{2}\right)}$ is the thermal slip factor which changes with x, where D_1 is the starting value, and $N = N_1 x^{-\left(\dfrac{n+1}{2}\right)}$ is the thermal slip factor that varies with x, whereas D_1 is the initial value, and also no slip case is recovered when $N = D = 0$.
For the conversion of (3.5) to (3.8), into the dimensionless form, the following similarity transformation has been applied:

$$\begin{cases} \eta = y\sqrt{a(n+1)/2\nu}\, x^{\frac{n-1}{2}}, \\ u = ax^n f'(\eta), \\ v = -\sqrt{\frac{(n+1)a\nu}{2}}\, x^{n-1/2}\left[f + \frac{n-1}{n+1}\left(y\sqrt{\frac{a(n+1)}{2\nu}}\, x^{\frac{n-1}{2}}\right) f'(\eta)\right], \\ T_w = T_\infty + bx^{2n}\theta, \Phi = (C - C_\infty)/(C_w - C_\infty). \end{cases} \tag{3.6}$$

On using, the set of above transformations into Eq. (3.1), the continuity equation satisfied.

$$\begin{aligned} \because \qquad u &= ax^n f', \\ \frac{\partial u}{\partial x} &= a\frac{\partial}{\partial x}(x^n f'), \\ \frac{\partial u}{\partial x} &= anx^{n-1} f' + ax^n f''\left(\frac{\partial \eta}{\partial x}\right), \\ \because \qquad \frac{\partial \eta}{\partial x} &= y\sqrt{\frac{a(n+1)}{2\nu}}\left(\frac{n-1}{2}\right) x^{\frac{n-3}{2}} \end{aligned}$$

$$\frac{\partial u}{\partial x} = anx^{n-1}f' + ay\left(\frac{n-1}{2}\right)\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{3n-3}{2}}f''. \tag{3.7}$$

Differentiating v w.r.t. y i.e.

$$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\left(-\sqrt{\frac{(n+1)\,a\nu}{2}}x^{n-1/2}\left[f(\eta) + \frac{n-1}{n+1}\eta f'(\eta)\right]\right),$$

$$\frac{\partial v}{\partial y} = -\sqrt{\frac{(n+1)\,a\nu}{2}}x^{n-1/2}\left[f'(\eta)\frac{\partial \eta}{\partial y} + \frac{n-1}{n+1}\eta f''(\eta)\frac{\partial \eta}{\partial y} + \frac{\partial \eta}{\partial y}f'(\eta)\right],$$

$$\because \qquad \frac{\partial \eta}{\partial y} = \sqrt{\frac{a(n+1)}{2\nu}}x^{(n-1)/2}$$

$$\frac{\partial v}{\partial y} = -anx^{\frac{n-1}{2}}f'(\eta) - ay\sqrt{\frac{a(n+1)}{2\nu}}\left(\frac{n-1}{2}\right)x^{\frac{3n-3}{2}}f''(\eta). \tag{3.8}$$

On using Eqs. (3.7) and (3.8) into Eq. (3.1), the continuity equation satisfied. For dimensionless form of momentum equation, differentiating u w.r.t. y:

$$\frac{\partial u}{\partial y} = a\frac{\partial}{\partial y}\left(x^n f'(\eta)\right),$$

$$\frac{\partial u}{\partial y} = ax^n f''\frac{\partial \eta}{\partial y},$$

$$\because \qquad \frac{\partial \eta}{\partial y} = \sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{n-1}{2}}$$

$$\frac{\partial u}{\partial y} = a\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{3n-1}{2}}f''(\eta). \tag{3.9}$$

Again differentiating:

$$\frac{\partial^2 u}{\partial y^2} = \frac{a^2(n+1)}{2\nu}x^{2n-1}f'''(\eta). \tag{3.10}$$

On using Eqs. (3.6), (3.7), (3.9) and (3.10), into Eq. (3.2).

$$\left(ax^n f'(\eta)\right)\left(anx^{n-1}f' + ay\left(\frac{n-1}{2}f''\right)\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{3n-3}{2}}\right)$$

$$+\left(-\sqrt{\frac{(n+1)\,a\nu}{2}}x^{n-1/2}\left[f(\eta) + \frac{n-1}{n+1}\eta f'(\eta)\right]\right)\left(a\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{3n-1}{2}}f''(\eta)\right)$$

$$= \nu\left(\frac{a^2(n+1)}{2\nu}x^{2n-1}f'''(\eta)\right) - \frac{\sigma B^2}{\rho}\left(ax^n f'(\eta)\right),$$

by simplifying above equation and using $B = B_0 x^{n-1/2}$

$$\frac{a^2(n+1)x^{2n-1}}{2}f''' + \frac{1}{2}a^2(n+1)x^{2n-1}ff'' - \frac{1}{2}a^2y(n-1)x^{5n-3}\sqrt{\frac{a(n+1)}{2\nu}}f'f''$$
$$+\frac{a^2y(n-1)x^{5n-3}}{2}\sqrt{\frac{a(n+1)}{2\nu}}f'f'' - a^2nx^{2n-1}f'^2 - \frac{\sigma B_0^2\left(x^{n-1/2}\right)^2 ax^n}{\rho} = 0,$$

the third and fourth terms on the L.H.S. of above equation will be cancelled to each other and rest equation will be multiplied by $\frac{2}{(n+1)a^2x^{2n-1}}$ then its finally becomes:

$$f''' + ff'' - \left(\frac{2n}{n+1}\right)f'^2 - Mf' = 0, \tag{3.11}$$

where $M = \frac{2\sigma B_0^2}{a\rho(n+1)}$ and $B = B_0 x^{n-1/2}$.

To make dimensionless form of energy equation:

Differentiating T w.r.t. x from (3.6)

$$\frac{\partial T}{\partial x} = \frac{\partial}{\partial x}\left(T_\infty + bx^{2n}\theta(\eta)\right),$$
$$\frac{\partial T}{\partial x} = 0 + b\left(\theta 2nx^{2n-1} + x^{2n}\theta'(\eta)\frac{\partial \eta}{\partial x}\right),$$
$$\because \quad \frac{\partial \eta}{\partial x} = y\sqrt{\frac{a(n+1)}{2\nu}}\left(\frac{n-1}{2}\right)x^{\frac{n-3}{2}}$$
$$\frac{\partial T}{\partial x} = 2bn\theta x^{2n-1} + b\left(\frac{n-1}{2}\right)\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{5n-3}{2}}\theta'. \tag{3.12}$$

Differentiating T:

$$\frac{\partial T}{\partial y} = \frac{\partial}{\partial y}\left(T_\infty + bx^{2n}\theta(\eta)\right),$$
$$\frac{\partial T}{\partial y} = 0 + bx^{2n}\theta'(\eta)\frac{\partial \eta}{\partial y},$$
$$\because \quad \frac{\partial \eta}{\partial y} = \sqrt{a(n+1)/2\nu}\,x^{(n-1)/2}$$
$$\frac{\partial T}{\partial y} = b\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{5n-1}{2}}\theta'. \tag{3.13}$$

Differentiating again:

$$\frac{\partial^2 T}{\partial y^2} = ab\frac{(n+1)}{2\nu}x^{3n-1}\theta'' \tag{3.14}$$

Differentiating C from Eq. (3.6), w.r.t. y:

$$\frac{\partial C}{\partial y} = \frac{\partial}{\partial y}\left(\left(C_w - C_\infty\right)\Phi + C_\infty\right),$$
$$\frac{\partial C}{\partial y} = \left(C_w - C_\infty\right)\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{n-1}{2}}\Phi'(\eta). \tag{3.15}$$

On using Eqs. (3.6), (3.9), (3.12), (3.13) and (3.14), into Eq. (3.3):

$$\left(ax^n f'(\eta)\right)\left(2bn\theta x^{2n-1} + b\left(\frac{n-1}{2}\right)\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{5n-3}{2}}\theta'\right)$$
$$+\left(-\sqrt{\frac{(n+1)a\nu}{2}}x^{n-1/2}\left[f(\eta) + \frac{n-1}{n+1}\eta f'(\eta)\right]\right)\left(b\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{5n-1}{2}}\theta'\right)$$
$$= \alpha\left(ab\frac{(n+1)}{2\nu}x^{3n-1}\theta''\right) + \frac{\nu}{C_p}\left(a\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{3n-1}{2}}f''(\eta)\right)^2$$
$$+\tau\left(D_B\left(b\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{5n-1}{2}}\theta'\right)\right)\left(\left(C_w - C_\infty\right)\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{n-1}{2}}\Phi'(\eta)\right)$$
$$+\tau\frac{D_T}{T_\infty}\left(b\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{5n-1}{2}}\theta'\right)^2,$$

by simplifying:

$$2abnx^{3n-1}\theta f' + aby\frac{n-1}{2}\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{7n-3}{2}}f'\theta' - \sqrt{\frac{a\nu(n+1)x^{n-1}}{2}}fbx^{5n-1/2}$$
$$\sqrt{\frac{a(n+1)}{2\nu}}\theta' - \sqrt{\frac{a\nu(n+1)}{2}}x^{n-1/2}\left(\frac{n-1}{n+1}\right)y\sqrt{\frac{a(n+1)}{2\nu}}x^{n-1/2}bf'x^{5n-1/2}$$
$$\sqrt{\frac{a(n+1)}{2\nu}}\theta' = \alpha ab\frac{n+1}{2\nu}x^{3n-1}\theta''$$
$$+a^3\frac{\nu}{C_p}\frac{n+1}{2\nu}x^{3n-1}f''^2 + \tau D_B ab\left(C_w - C_\infty\right)\frac{n+1}{2\nu}x^{3n-1}\Phi'\theta'$$
$$+\tau ab^2\frac{D_t}{T_\infty}\frac{n+1}{2\nu}x^{5n-1}\theta'^2,$$

again simplifying

$$2abnx^{3n-1}\theta f' + aby\frac{n-1}{2}\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{7n-3}{2}}f'\theta' - ab\frac{n+1}{2}x^{3n-1}f\theta'$$
$$-aby\frac{n+1}{2}\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{7n-3}{2}}f'\theta' = \alpha ab\frac{n+1}{2\nu}x^{3n-1}\theta'' + a^3\frac{\nu}{C_p}\frac{n+1}{2\nu}x^{3n-1}f''^2$$
$$+\tau D_B ab\left(C_w - C_\infty\right)\frac{n+1}{2\nu}x^{3n-1}\Phi'\theta' + \tau ab^2\frac{D_T}{T_\infty}\frac{n+1}{2\nu}x^{5n-1}\theta'^2,$$

second and fourth term on the L.H.S. of above equation will be cancelled to each other, and multiplying by $\dfrac{2}{ab(n+1)x^{3n-1}}$ the above equation:

$$\frac{4n}{n+1}\theta f' - f\theta' = \frac{\alpha}{\nu}\theta'' + \frac{a^2\nu}{bC_p\nu}f''^2 + \tau D_B\frac{(C_w - C_\infty)}{\nu}\Phi'\theta' + \tau b\frac{D_T}{T_\infty}\frac{x^{2n}}{\nu}\theta'^2,$$

using $bx^{2n} = T_w - T_\infty$ and $P_r = \dfrac{\nu}{\alpha}$, above equation gets the following form:

$$\frac{1}{P_r}\theta'' + f\theta' - \frac{4n}{n+1}\theta f' + \frac{a^2}{bC_p}f''^2 + \frac{\tau D_B(C_w - C_\infty)}{\nu}\Phi'\theta' + \frac{\tau D_T(T_w - T_\infty)}{T_\infty\nu}\theta'^2 = 0,$$

finally its becomes:

$$\frac{1}{P_r}\theta'' + f\theta' - \frac{4n}{n+1}\theta f' + N_b\Phi'\theta' + N_t\theta'^2 + E_c f''^2 = 0, \tag{3.16}$$

where $N_t = \dfrac{(\rho c)_f D_T(T_w - T_\infty)}{(\rho c)_f T_\infty \alpha}$, $N_b = \dfrac{(\rho c)_p D_B(C_w - C_\infty)}{(\rho c)_f \nu}$, and $E_c = \dfrac{u_w^2}{C_p(T_w - T_\infty)}$.
The dimensionless form of nanoparticles concentration equation Eq.(3.4):
Differentiating C from Eq. (3.6):

$$\frac{\partial C}{\partial x} = \frac{\partial}{\partial x}\left((C_w - C_\infty)\Phi + C_\infty\right),$$
$$\frac{\partial C}{\partial x} = (C_w - C_\infty)\sqrt{\frac{a(n+1)}{2\nu}}y\left(\frac{n-1}{2}\right)x^{\frac{n-3}{2}}\Phi'(\eta). \tag{3.17}$$

Differentiating Eq. (3.15) w.r.t. y again:

$$\frac{\partial^2 C}{\partial y^2} = (C_w - C_\infty)\frac{a(n+1)}{2\nu}x^{n-1}\Phi'' \tag{3.18}$$

On using Eqs. (3.6), (3.14), (3.15), (3.17), and (3.18) into Eq. (3.4).

$$\begin{aligned}
&(ax^n f'(\eta))\left((C_w - C_\infty)\sqrt{\frac{a(n+1)}{2\nu}}y\left(\frac{n-1}{2}\right)x^{\frac{n-3}{2}}\Phi'(\eta)\right)\\
&+\left(-\sqrt{\frac{(n+1)a\nu}{2}}x^{n-1/2}\left[f(\eta) + \frac{n-1}{n+1}\left(y\sqrt{\frac{a(n+1)}{2\nu}}x^{(n-1)/2}\right)f'(\eta)\right]\right)\\
&\left((C_w - C_\infty)\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{n-1}{2}}\Phi'(\eta)\right)\\
&= D_B\left((C_w - C_\infty)\frac{a(n+1)}{2\nu}x^{n-1}\Phi''\right) + \frac{D_T}{T_\infty}\left(ab\frac{(n+1)}{2\nu}x^{3n-1}\theta''\right),
\end{aligned}$$

after simplification it becomes:

$$
\begin{aligned}
&(ax^n f'(\eta))\left((C_w - C_\infty)\sqrt{\frac{a(n+1)}{2\nu}}y\left(\frac{n-1}{2}\right)x^{\frac{n-3}{2}}\Phi'(\eta)\right) \\
&-\sqrt{\frac{(n+1)\,a\nu}{2}}x^{n-1/2}\left((C_w - C_\infty)\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{n-1}{2}}f(\eta)\Phi'\right) - \\
&\sqrt{\frac{(n+1)\,a\nu}{2}}x^{n-1/2}\frac{n-1}{n+1}\left(y\sqrt{\frac{a(n+1)}{2\nu}}x^{(n-1)/2}\right)f'(\eta)\cdot \\
&\left((C_w - C_\infty)\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{n-1}{2}}\Phi'(\eta)\right) \\
&= D_B\left((C_w - C_\infty)\frac{a(n+1)}{2\nu}x^{n-1}\Phi''\right) + \frac{D_T}{T_\infty}\left(ab\frac{(n+1)}{2\nu}x^{3n-1}\theta''\right),
\end{aligned}
$$

taking similar terms common from first and third terms on the L.H.S. of above Eq.

$$
\begin{aligned}
&ax^{3n-3/2}y\frac{C_w - C_\infty}{2}\sqrt{\frac{a(n+1)}{2\nu}}\left[(n-1)-(n-1)\right]f'\Phi' \\
&= D_B\left((C_w - C_\infty)\frac{a(n+1)}{2\nu}x^{n-1}\Phi''\right) + \frac{D_T}{T_\infty}\left(ab\frac{(n+1)}{2\nu}x^{3n-1}\theta''\right) \\
&+\sqrt{\frac{(n+1)\,a\nu}{2}}x^{n-1/2}\frac{n-1}{n+1}\left(y\sqrt{\frac{a(n+1)}{2\nu}}x^{(n-1)/2}\right)f'(\eta) \\
&\left((C_w - C_\infty)\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{n-1}{2}}\Phi'(\eta)\right),
\end{aligned}
$$

rearranging the above equation after multiplying by $\dfrac{2\nu}{D_B(C_w - C_\infty)x^{n-1}}$ the reciprocal of the term involving Φ''

$$
\Phi'' + \frac{D_T}{T_\infty}\left(\frac{bx^{2n}}{D_B(C_w - C_\infty)}\right)\theta'' + \frac{\nu}{D_B}f\Phi' = 0,
$$

$\because bx^{2n} = T_w - T_\infty$, $L_e = \dfrac{\nu}{D_B}$, $N_t = \dfrac{\tau D_T(T_w - T_\infty)}{\nu T_\infty}$, $N_b = \dfrac{\tau D_B(C_w - C_\infty)}{\nu}$,
therefore above equation finally becomes:

$$
\Phi'' + \frac{N_t}{N_b}\theta'' + Lef\Phi' = 0. \tag{3.19}
$$

The mathematical procedure for the conversion of the dimensional boundary conditions Eq. (3.5), into the dimensionless form, is explained as:

$$\because \qquad u = u_w + N\nu\left(\frac{\partial u}{\partial y}\right), \text{at } y = 0$$

$$\Rightarrow x^n f'(\eta) = ax^n + N\nu\left(a\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{3n-1}{2}}f''(\eta)\right), \text{at } \eta = 0.$$

$$\because \qquad N = N_1 x^{-(n-1/2)}, \text{ and } \lambda = N_1\sqrt{\frac{a\nu(n+1)}{2}}$$

$$\therefore \qquad ax^n f'(0) = ax^n + \lambda a x^n f''(0)$$

$$\Rightarrow f' = 1 + \lambda f'', \ \eta = 0.$$

$$\because \qquad v = 0 \text{ at } y = 0,$$

$$\Rightarrow f = 0 \text{ at } \eta = 0, \text{ by using Eq. (3.5)}$$

$$T = T_w + D\left(\frac{\partial T}{\partial y}\right) \text{ at } y = 0$$

$$\because \qquad T = T_\infty + bx^{2n}\theta(\eta), T_w = T_\infty + bx^{2n}, D = D_1 x^{-(n-1/2)}$$

$$\Rightarrow T_\infty + bx^{2n}\theta(\eta) = T_\infty + bx^{2n} + D_1 x^{-(n-1/2)}\left(b\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{5n-1}{2}}\theta'(\eta)\right)$$

$$\because \qquad \delta = D_1\sqrt{\frac{a(n+1)}{2\nu}}$$

$$\therefore \qquad bx^{2n}\theta(\eta) = bx^{2n} + \delta b x^{2n}\theta'(\eta)$$

$$\Rightarrow \theta(0) = 1 + \delta\theta'(0), \ at \ \eta = 0$$

As

$$u \to 0, v \to 0, T = T_\infty, C = C_\infty, at \ y \to \infty, \text{ then from. (3.5)}$$

$$f' \to 0, \theta \to 0, \Phi \to 0, at \ \eta \to \infty$$

The final governing ODEs are as follows:

$$f''' + ff'' - \left(\frac{2n}{n+1}\right)f'^2 - Mf' = 0, \tag{3.20}$$

$$\frac{1}{P_r}\theta'' + f\theta' - \frac{4n}{n+1}\theta f' + N_b\Phi'\theta' + N_t\theta'^2 + E_c f''^2 = 0, \tag{3.21}$$

$$\Phi'' + \frac{N_t}{N_b}\theta'' + L_e f\Phi' = 0. \tag{3.22}$$

The associated boundary conditions (3.5) get the following form:

$$\begin{cases} f'(0) = 1 + \lambda f''(0), f = 0, \theta(0) = 1 + \delta\theta'(0), \Phi = 0, \ at \ \eta = 0, \\ f' \to 0, \theta \to 0, \Phi \to 0, at \ \eta \to \infty, \end{cases} \tag{3.23}$$

following parameters were used in the above equation:

$$\begin{cases} L_e = \dfrac{\nu}{D_B}, P_r = \dfrac{\nu}{\alpha}, N_b = \dfrac{\tau D_B(C_w - C_\infty)}{\nu}, \\ N_t = \dfrac{\tau D_T(T_w - T_\infty)}{\nu T_\infty}, M = \dfrac{2\sigma B_0^2}{a\rho(n+1)}, \\ \delta = D_1\sqrt{\dfrac{a(n+1)}{2\nu}}, \lambda = N_1\sqrt{\dfrac{a\nu(n+1)}{2}}, \\ E_c = \dfrac{u_w^2}{C_p(T_w - T_\infty)}. \end{cases}$$

3.1.2 Physical Quantities of Interest

For the dimensionless form of Skin friction coefficient C_{fx} following steps are required:

$$C_{fx} = \frac{\mu}{\rho u_w^2}\left[\frac{\partial u}{\partial y}\right]_{y=0}, \tag{3.24}$$

$$\Rightarrow C_{fx} = \frac{\nu}{u_w^2}\left(a\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{3n-1}{2}}f''(\eta)\right) \quad \because \nu = \frac{\mu}{\rho}, \text{using (3.9)}$$

$$C_{fx} = a\frac{\sqrt{\nu}}{a^2x^{2n}}\sqrt{\frac{a(n+1)}{2}}x^{\frac{3n-1}{2}}f''(\eta),$$

$$C_{fx} = \sqrt{\frac{\nu}{ax^nx}}\sqrt{\frac{n+1}{2}}f'',$$

$\because \quad Re_x = u_w x/\nu$ is local Reynolds number,

$$\therefore \quad Re_x C_{fx} = \sqrt{\frac{n+1}{2}}f'' \tag{3.25}$$

The dimensionlessn form of Nusselt number Nu_x is given as:

$$Nu_x = \frac{xq_w}{k(T_w - T_\infty)}, \tag{3.26}$$

$\because \quad q_w = -k\left[\dfrac{\partial T}{\partial y}\right]_{y=0}$ is heat flux at the surface

$$\therefore \qquad Nu_x = \frac{-xk}{k(T_w - T_\infty)}\left(b\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{5n-1}{2}}\theta'\right)$$

$$Nu_x = -\frac{x}{bx^{2n}}\left(b\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{5n-1}{2}}\theta'\right), \quad \because \quad T_w - T_\infty = bx^{2n}$$

$$Nu_x = -\sqrt{\frac{ax^n x}{\nu}}\sqrt{\frac{n+1}{2}}\theta', \quad \because \; Re_x = u_w x/\nu$$

$$(Re_x)^{-1/2}Nu_x = -\sqrt{\frac{n+1}{2}}\theta'. \tag{3.27}$$

Following calculations are required for the dimensionless form of Sherwood Number Sh_x:

$$Sh_x = \frac{xq_m}{D_B(C_w - C_\infty)}, \tag{3.28}$$

$$\because \qquad q_m = -D_b\left[\frac{\partial C}{\partial y}\right]_{y=0} \text{ is mass flux at the surface}$$

$$\therefore \qquad Sh_x = \frac{-xD_B}{D_B(C_w - C_\infty)}\left((C_w - C_\infty)\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{n-1}{2}}\Phi'(\eta)\right),$$

$$Sh_x = -\sqrt{\frac{a}{\nu}}\sqrt{\frac{n+1}{2}}x^{\frac{n}{2}+\frac{1}{2}}\Phi',$$

$$Sh_x = -Re_x^{\frac{1}{2}}\sqrt{\frac{n+1}{2}}\Phi', \quad \because \; Re_x = u_w x/\nu$$

$$Re_x^{-\frac{1}{2}}Sh_x = -\sqrt{\frac{n+1}{2}}\Phi'. \tag{3.29}$$

3.2 Numerical Technique

In this thesis Shooting method has been used to solve the transformed system of ODEs (3.20) to (3.22) by assuming the missing initial conditions subject to the boundary condition (3.23). The system of BVP (3.17)-(3.19) is converted into IVP for applying shooting method. By solving Eq. (3.20) independently the results obtained as f, f', f'' and f''', then these results will be used in couple Eqs. (3.21) and (3.22). For this purpose the following notations has been used.

$$f = f_1, f' = f_1' = f_2,$$

$$f'' = f_1'' = f_2' = f_3,$$
$$f''' = f_1''' = f_2'' = f_3'.$$

The resulting IVP takes the following form:

$$f_1' = f_2, \qquad f_1(0) = 0, \tag{3.30}$$
$$f_2' = f_3, \qquad f_2(0) = 1 + \lambda\xi, \tag{3.31}$$
$$f_3' = -f_1 f_3 + \frac{2n}{n+1} f_2^2 + (M + P) f_2, \qquad f_3(0) = \xi, \tag{3.32}$$

where ξ is the missing initial condition. The IVP has been solved by using RK4 method. Since the unbounded domain can not be used for the numerical computations, so the domain of the IVP has been taken as $[0, \eta_\infty]$ instead of $[0, \infty)$, where η_∞ is an appropriate positive real number with choosen initial guess ξ such that:

$$f_2\left(\eta_\infty, \xi\right) = 0.$$

To solve the previous equation, Newton's method has been used with following iterative procedure

$$\xi^{n+1} = \xi^n - \frac{f_2(\eta_\infty, \xi^n)}{\left(\frac{\partial f_2(\eta_\infty, \xi^n)}{\partial \xi}\right)}.$$

In order to obtain the derivatives w.r.t. ξ, following notations will be used

$$\frac{\partial f_1}{\partial \xi} = f_4, \quad \frac{\partial f_2}{\partial \xi} = f_5, \quad \frac{\partial f_3}{\partial \xi} = f_6.$$

Hence the Newton's iterative scheme gets the following form

$$\xi^{n+1} = \xi^n - \frac{f_2(\eta_\infty, \xi^n)}{f_5(\eta_\infty, \xi^n)}.$$

By differentiating Eqs. (3.30), (3.31) and (3.32) w.r.t. ξ three more equations will be appeared. Consequently, IVP takes the following form:

$$f_1' = f_2, \quad f_1(0) = 0,$$

$$
\begin{aligned}
f_2' &= f_3, & f_2(0) &= 1 + \lambda\xi,\\
f_3' &= -f_1 f_3 + \frac{2n}{n+1} f_2^2 + (M) f_2, & f_3(0) &= \xi,\\
f_4' &= f_5, & f_4(0) &= 0,\\
f_5' &= f_6, & f_5(0) &= \lambda,\\
f_6' &= -(f_1 f_6 + f_3 f_4) + \frac{4n}{n+1} f_2 f_5 + (M) f_5, & f_6(0) &= 1,
\end{aligned}
$$

The Newton's iterative process is repeated untill the following condition is met.

$$
|f_2(\eta_\infty, \xi)| < \epsilon,
$$

here ϵ is taken as 10^{-6}. Similarly by solving Eqs. (3.25) and (3.26) along with the boundary conditions (3.27), where the missing initial conditions $\theta'(0)$ and $\Phi'(0)$ are denoted by ψ and χ respectively. The notations used for this purpose are given as follows:

$$
\begin{aligned}
\theta &= Y_1,\\
\theta' &= Y_1' = Y_2,\\
\theta'' &= Y_1'' = Y_2',\\
\frac{\partial Y_1}{\partial \psi} &= Y_5, \quad \frac{\partial Y_2}{\partial \psi} = Y_6, \quad \frac{\partial Y_1}{\partial \chi} = Y_9, \quad \frac{\partial Y_2}{\partial \chi} = Y_{10},\\
\Phi &= Y_3,\\
\Phi' &= Y_3' = Y_4,\\
\Phi'' &= Y_3'' = Y_4',\\
\frac{\partial Y_3}{\partial \psi} &= Y_7, \quad \frac{\partial Y_4}{\partial \psi} = Y_8, \quad \frac{\partial Y_3}{\partial \chi} = Y_{11}, \quad \frac{\partial Y_4}{\partial \chi} = Y_{12}.
\end{aligned}
$$

By using these notations, we get the following first order ODEs

$$
\begin{aligned}
Y_1' &= Y_2, & Y_1(0) &= 1 + \delta\psi,\\
Y_2' &= P_r \left[-f_1 Y_2 + \frac{4n}{n+1} f_2 Y_1 - N_b Y_2 Y_4 - N_t Y_2^2 - E_c f_3^2 \right], & Y_2(0) &= \psi,\\
Y_3' &= Y_4, & Y_3(0) &= 1,
\end{aligned}
$$

$$
\begin{aligned}
Y_4' &= -L_e f_1 Y_4 - \left(\frac{N_t}{N_b}\right) P_r \\
&\quad \left[-f_1 Y_2 + \frac{4n}{n+1} f_2 Y_1 - N_b Y_2 Y_4 - N_t Y_2^2 - E_c f_3^2\right], & Y_4(0) &= \chi, \\
Y_5' &= Y_6, & Y_5(0) &= \delta, \\
Y_6' &= P_r \\
&\quad \left[-f_1 Y_6 + \frac{4n}{n+1} f_2 Y_5 - N_b (Y_6 Y_4 + Y_2 Y_8) - 2N_t Y_2 Y_6\right], & Y_6(0) &= 1, \\
Y_7' &= Y_8, & Y_7(0) &= 0, \\
Y_8' &= -L_e f_1 Y_8 - \left(\frac{N_t}{N_b}\right) P_r \\
&\quad \left[-f_1 Y_6 + \frac{4n}{n+1} f_2 Y_5 - N_b (Y_4 Y_6 + Y_2 Y_8) - 2N_t Y_2 Y_6\right], & Y_8(0) &= 0, \\
Y_9' &= Y_{10}, & Y_9(0) &= 0, \\
Y_{10}' &= P_r \\
&\quad \left[-f_1 Y_{10} + \frac{4n}{n+1} f_2 Y_9 - N_b (Y_4 Y_{10} + Y_2 Y_{12}) - 2N_t Y_2 Y_{10}\right], & Y_{10}(0) &= 0, \\
Y_{11}' &= Y_{12}, & Y_{11}(0) &= 0, \\
Y_{12}' &= -L_e f_1 Y_{12} - \left(\frac{N_t}{N_b}\right) P_r \\
&\quad \left[-f_1 Y_{10} + \frac{4n}{n+1} f_2 Y_9 - N_b (Y_{10} Y_4 + Y_2 Y_{12}) - 2N_t Y_2 Y_{10}\right], & Y_{12}(0) &= 1.
\end{aligned}
$$

The domain of the above problem has been taken as $[0, \eta_\infty]$ instead of $[0, \infty)]$, (where η_∞ is a finite positive number for which the variations in the solution are negligible after $\eta = \eta_\infty$) because the numerical calculations can not be performed on an unbounded domain. The ideal missing conditions ψ and χ are assumed to satisfy the following relations.

$$Y_1(\eta_\infty, \psi, \chi) = 0, \tag{3.33}$$

$$Y_3(\eta_\infty, \psi, \chi) = 0. \tag{3.34}$$

The above system of equations will be solved by the Newton's method governed by the following formulation.

$$\begin{bmatrix} \psi^{(n+1)} \\ \chi^{(n+1)} \end{bmatrix} = \begin{bmatrix} \psi^{n} \\ \chi^{n} \end{bmatrix} - \begin{bmatrix} Y_5 & Y_9 \\ Y_7 & Y_{11} \end{bmatrix}^{-1} \begin{bmatrix} Y_1^n \\ Y_3^n \end{bmatrix}_{(\eta_\infty,\psi,\chi)}$$

The Newton's iterative process is repeated up till the following condition is met.

$$max\{|f_2(\eta_\infty)|, \quad |Y_1(\eta_\infty)|, \quad |Y_3(\eta_\infty)|\} \leq \epsilon,$$

where ϵ is a small positive number. For the computational purpose, ϵ has been given the value $\epsilon = 10^{-8}$ whereas η_∞ is set as 5.

3.3 Graphical Results with Explanation

A computational attempt was performed for multiple values of the velocity slip parameter λ, thermal slip parameter δ, magnetic parameter(M), non-linear stretching parameter n, Prandtl number P_r, Eckert number E_c and Brownian motion parameter N_b has been performed. The parametric study results are portrayed in Figures 3.2 - 3.11. For the clarification of the correctness of the numerical model implemented, the comparability of current outputs relating to the values of $[-\theta'(0)]$, mass flow rate for $M = 0, \lambda = 0, \delta = 0$ shall be rendered with the accessible Ramya et al. [8]. The measurement for coefficient of skin friction, rate of heat transfer and rate of mass transfer is shown in Table 3.1. Numerical solutions will be obtained in this thesis, using the Shooting process.

Figure 3.2 shows the impact on dimensionalless velocity for different values of n and M. It has been noticed that there is an indirect relation of velocity f' with n and M. The magnetic field, that controverses the position of the magnetic field introduced, creates a retarding body force as per the Lorentz force. Enhanced values of M rises the retarding body force and consequently velocity decreases. As a result of increase magnetic field there is a reduction in boundary layer thickness.

Figure 3.3 demonstrates the direct relationship of M and n on dimensionless temperature. Figure 3.4 depicts the relationship between concentration profile and M. It has been observed that there is a direct impact of M on concentration profile.

TABLE 3.1: Table for $-f''(0)$, $(-\theta'(0))$ and $(-\Phi'(0))$ for different values of M, λ and δ
when $P_r = 5, N_b = N_t = 0.3, L_e = 2, E_c = 0.1$

			Ramya et al. [8]			Present study		
M	λ	δ	$-f''(0)$	$-\theta'(0)$	$-\Phi'(0)$	$-f''(0)$	$-\theta'(0)$	$-\Phi'(0)$
0.1	0.2	0.2	0.8787	1.9429	0.7594	0.86855	1.9424	0.7593
0.3			0.9548	1.8862	0.7416	0.9595	1.8861	0.7409
0.5			1.0252	1.8302	0.7211	1.10434	1.8579	0.7225
1.0	0.3	0.1	1.0705	1.9407	0.9199	1.06268	1.9412	0.9185
	0.4		0.9768	1.9255	0.9465	0.95665	1.9252	0.9430
	0.5		0.8974	1.9041	0.9632	0.87120	1.9044	0.9675
	0.1	0.3	1.3195	1.5029	0.4478	1.38040	1.5921	0.44090
		0.4	1.3195	1.3521	0.3017	1.38040	1.4574	0.32088
		0.5	1.3195	1.2199	0.2081	1.38040	1.3409	0.2082

Heat is evolved due to resistive Lorentz force that enters the fluid flow. Thus for a stiffened magnetic field thermal and nanoparticle concentration boundary layer thickness is thicker. Figure 3.5 displays the impact of λ on velocity profile. It has been observed that there is an indirect relation between f' and velocity slip parameter λ. This is due to slip condition that there is a difference between the stretched sheet velocity and velocity of the fluid. Figures 3.6 and 3.7 show the influence of variable velocity slip parameter λ on $\theta(\eta)$ and $\Phi(\eta)$. It is analysed that θ and Φ has a direct relationship with λ in the presence of magnetic field and thermal jump. Figures 3.8 and 3.9 displays the impact of δ on temperature profile θ and volume fraction of the nano particles.

It has been noticed that there is an indirect relationship of thermal slip δ with temperature and concentration profiles. The thermal boundary layer thickness decreases as the value of the thermal slip parameter increases even though a little quantity of heat is transported up to the fluid from stretched sheet.

Figure 3.10 demonstrates the influence on dimensionless temperature of the Eckert number (E_c). It has been observed that the temperature increases with increasing E_c values and also increases the thickness of the thermal boundary layer. It is because at stretching sheet surface the rate of heat transfer is decreased. Figure 3.11 shows the results of parameter Brownian motion Concentration N_b.

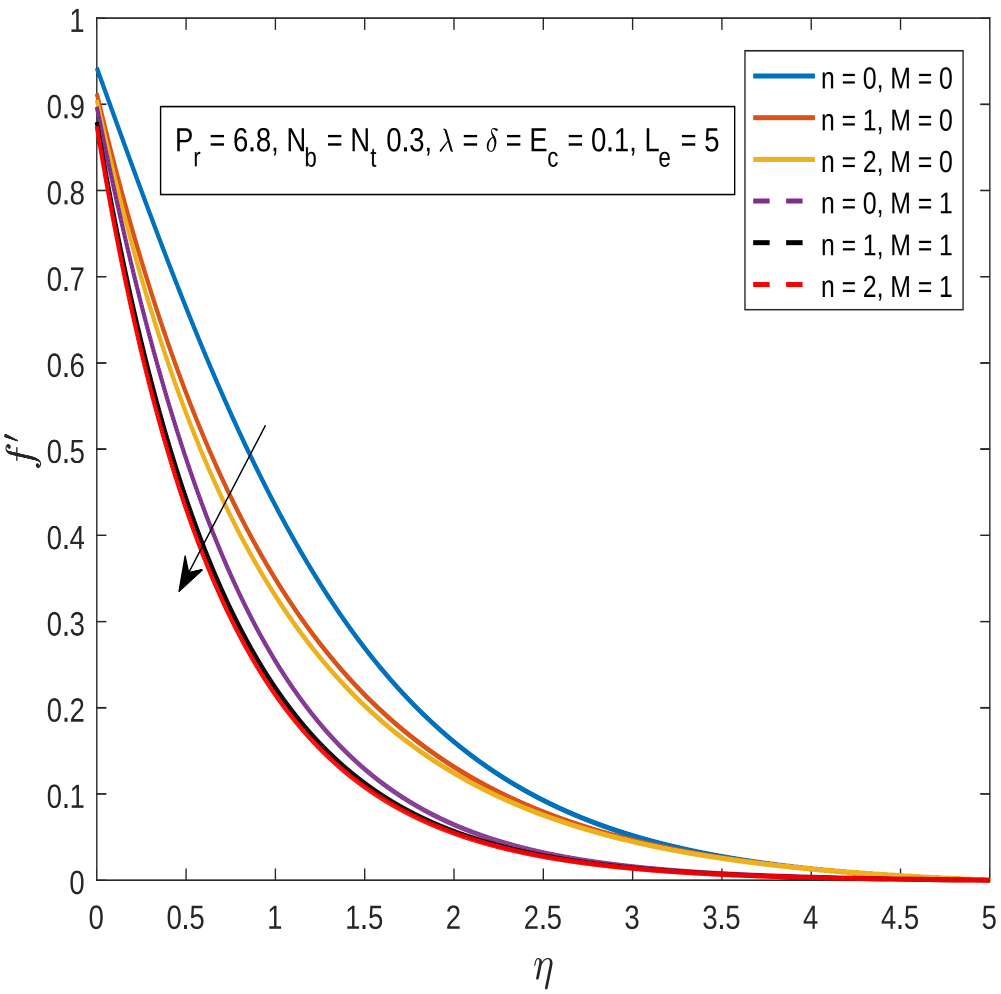

FIGURE 3.2: impact of f' for saveral values of M and n.

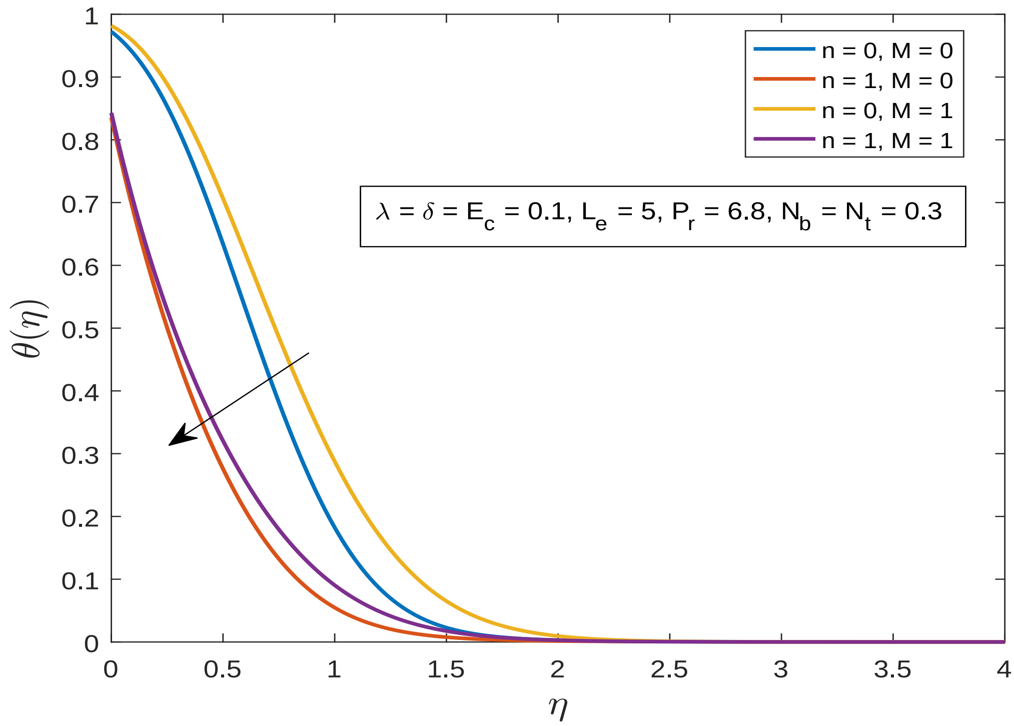

FIGURE 3.3: Impact of $\theta(\eta)$ for saveral values of M and n.

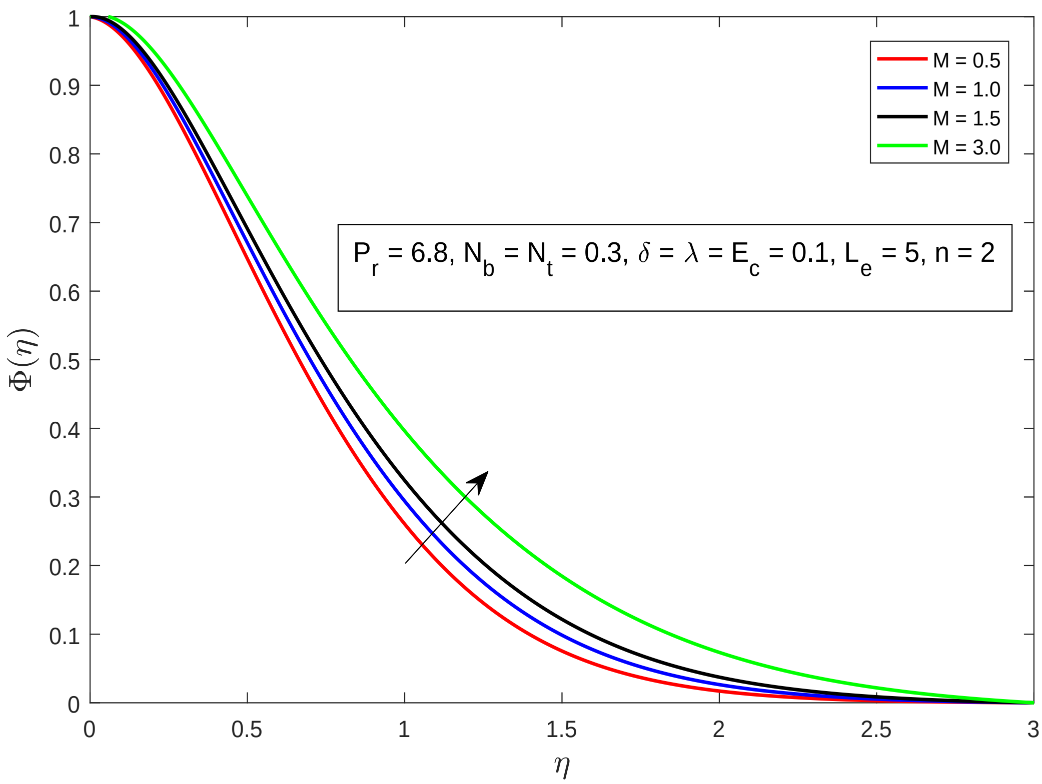

FIGURE 3.4: Effect of M on $\Phi(\eta)$.

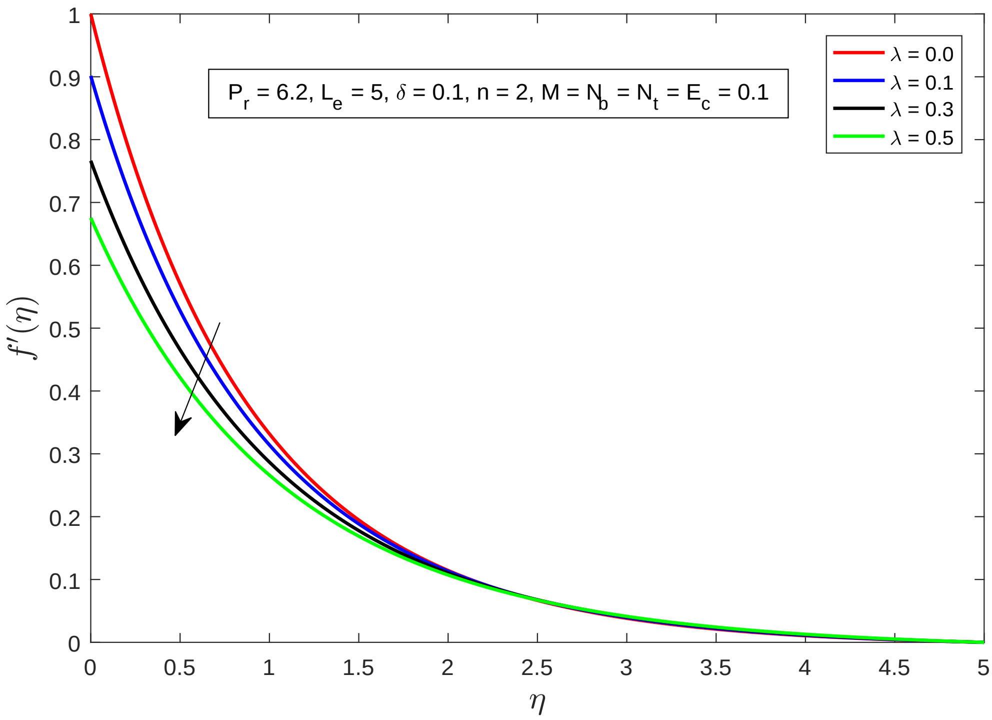

FIGURE 3.5: Effect of λ on $f'(\eta)$.

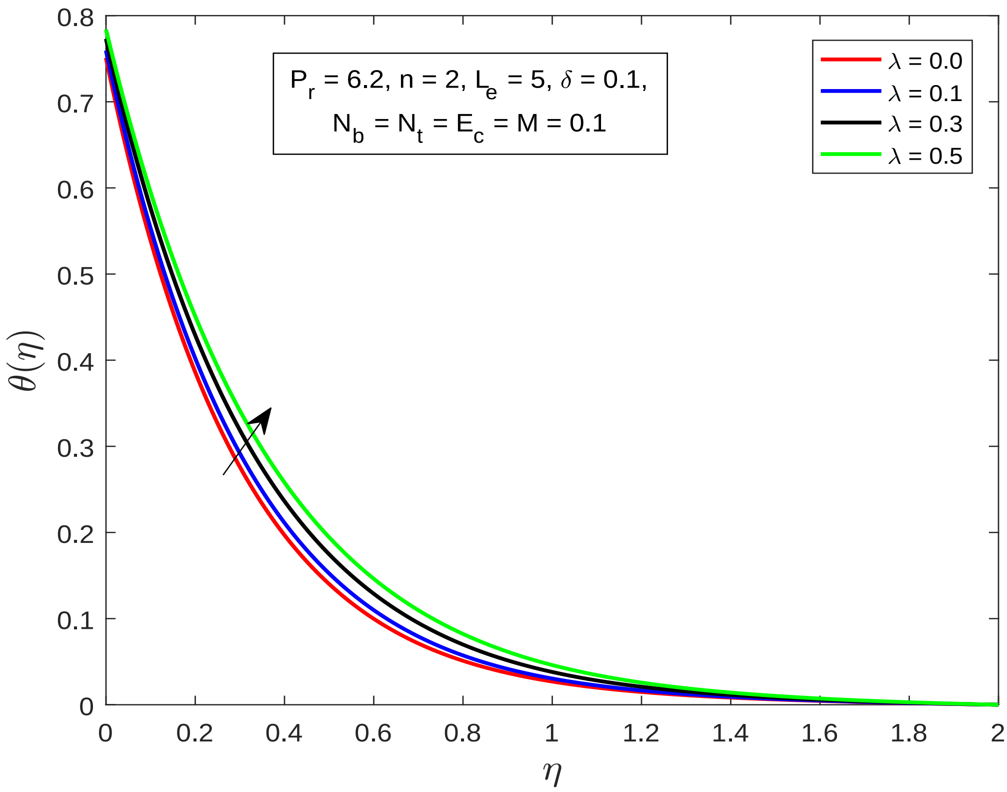

FIGURE 3.6: Effect of λ on $\theta(\eta)$.

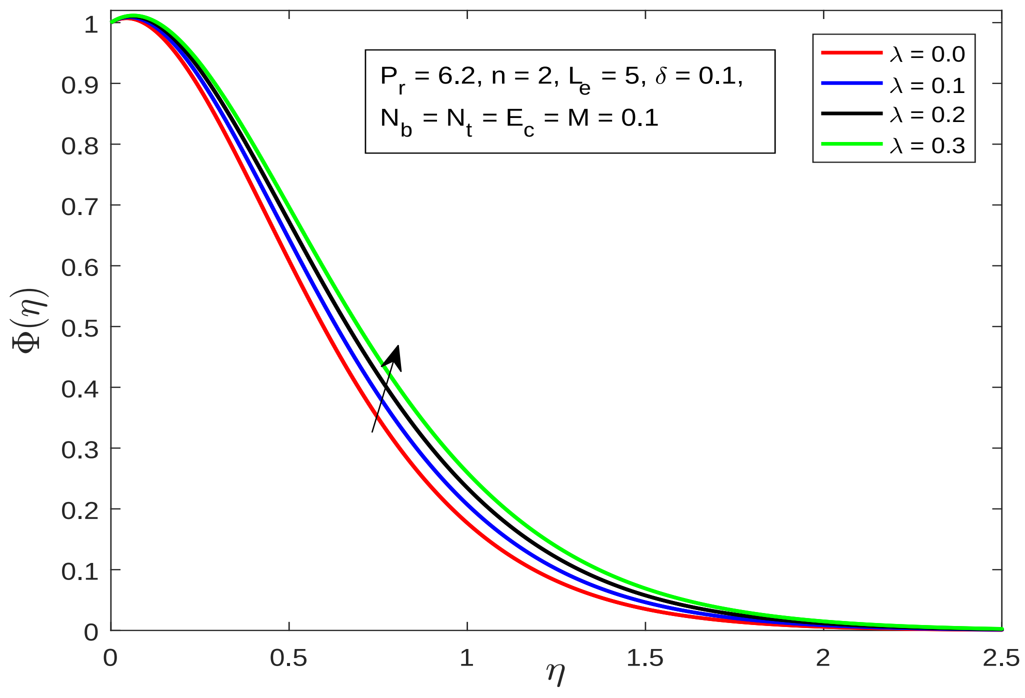

FIGURE 3.7: Influence of λ on $\Phi(\eta)$.

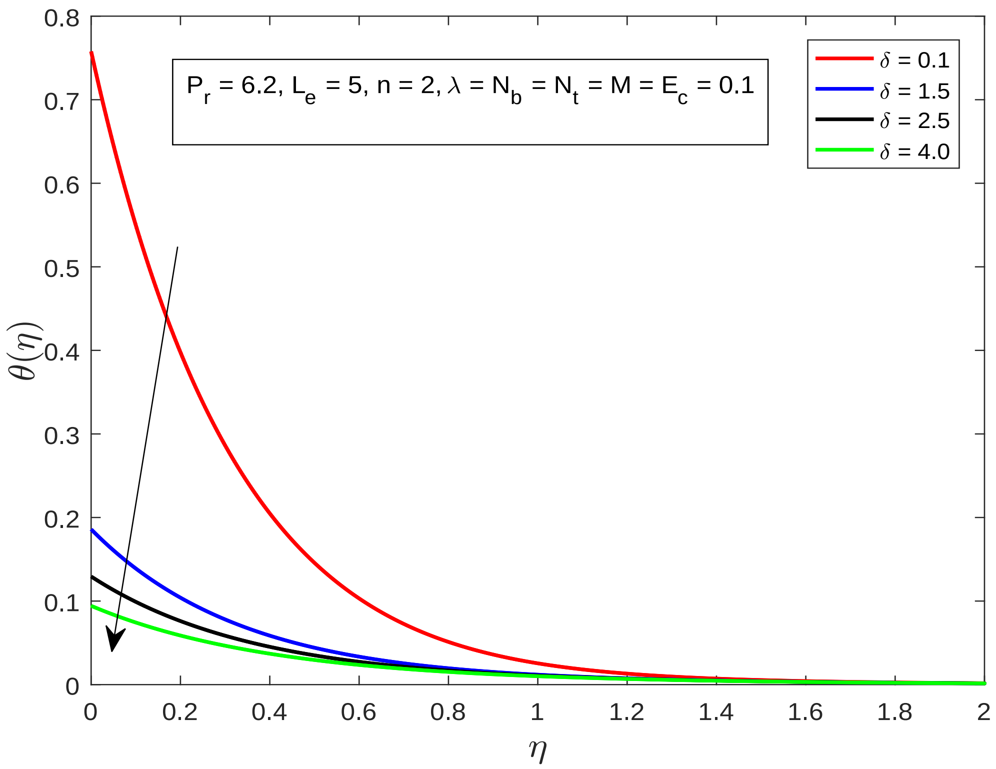

FIGURE 3.8: Impact of δ on $\theta(\eta)$.

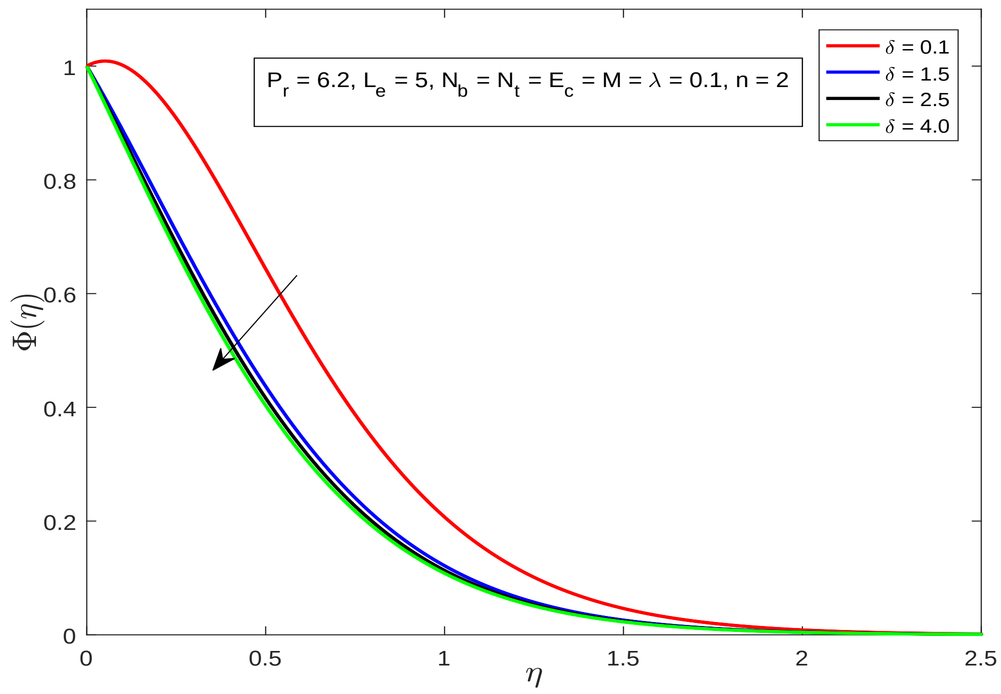

FIGURE 3.9: Impact of δ on $\Phi(\eta)$.

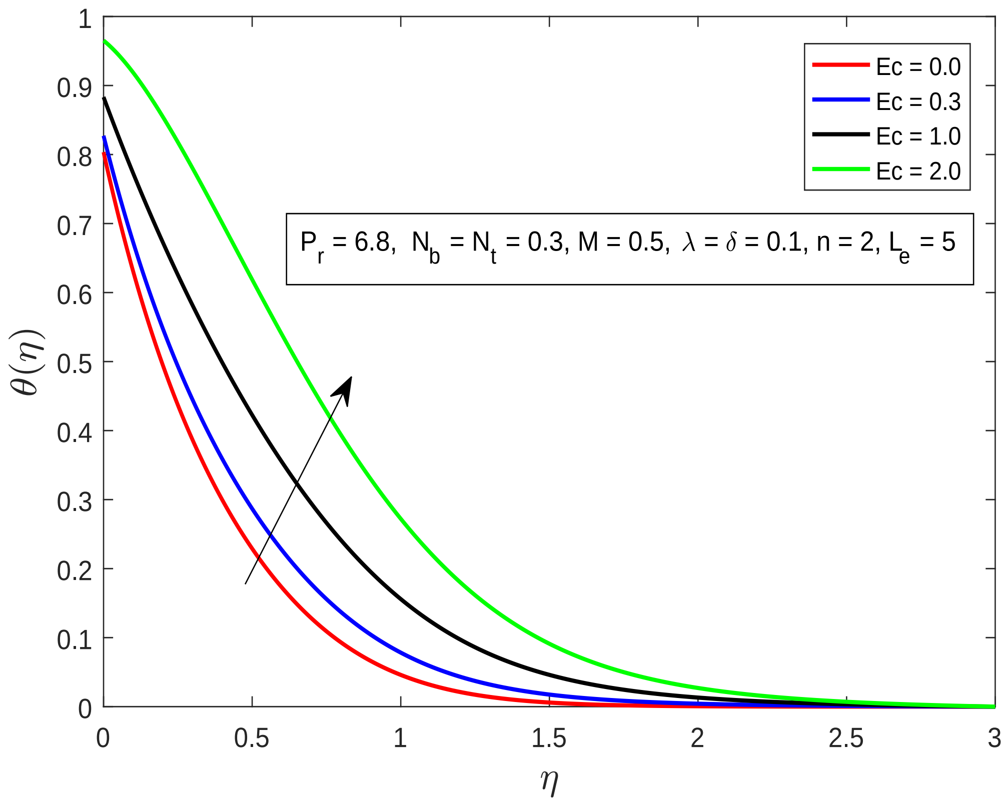

FIGURE 3.10: Influence of E_c on $\theta(\eta)$.

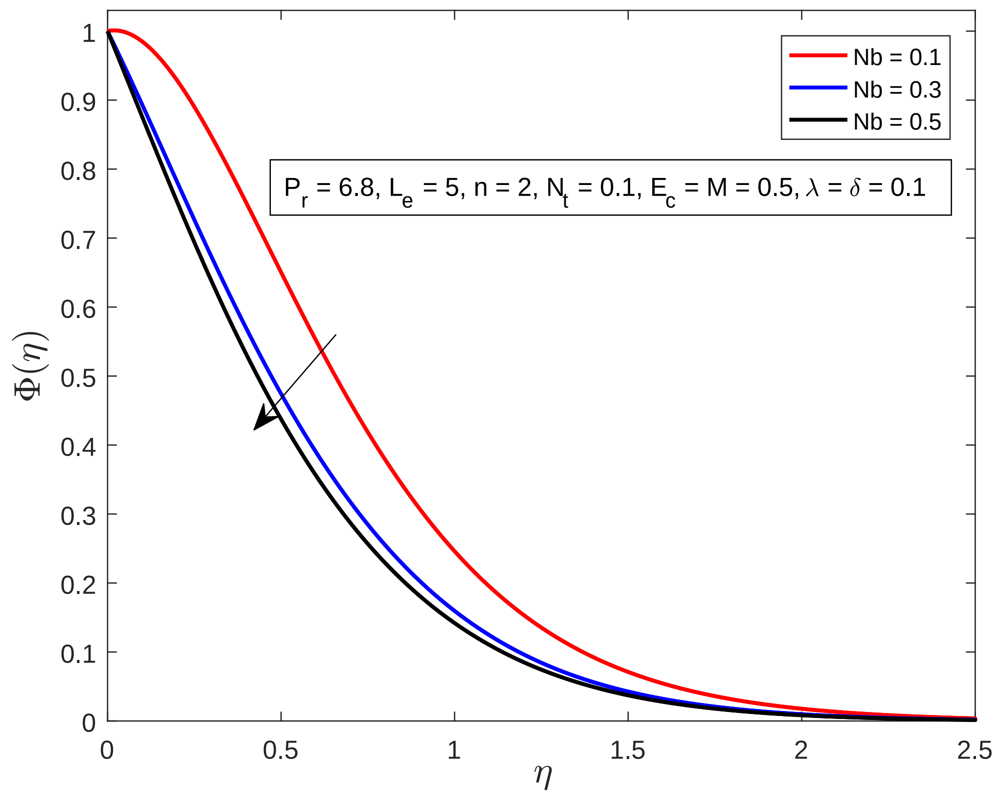

FIGURE 3.11: Impact of N_b on $\Phi(\eta)$.

Chapter 4

MHD Radiative Nanofluid Flow over a Non-linear Stretching Sheet Through a Porous Medium

This chapter extends the work of Ramya et al. [8] presented in Chapter 3. The effect of heat radiation flux and porosity factor on the magnetohydrodynamics flow of viscous nanofluid over a non-linear stretched surface in the presence of magnetic field is analysed. The governing non-linear PDEs and their associated boundary conditions are transformed into system of ODEs through adequate transformation. The obtained system of ODEs are solved by using Shooting method with RK4. In the last section of this chapter, the results obtained from transformed ODEs are explained with the help of graphs and table.

4.1 Mathematical Modeling

Consider a 2D incompressible and steady viscous flow of an electrically charged fluid through a non-linear stretched sheet through a porous medium in the presence of thermal radiation in Figure 3.1. The flow occupied the space $y \geq 0$. Furthermore , the direction of flow is along x-axis and y axis is perpendicular to it. A surface is expanded with a velocity of $u_w = ax^n$ and T_w(wall temperature), C_w(nanoparticle fraction), C_∞ (nanoparticle fraction) and T_∞ (ambient temperature) are considered constant.

The governing PDEs are as follows:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{4.1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} - \frac{\sigma B^2}{\rho}u - \frac{\nu}{K}u, \tag{4.2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} + \frac{\nu}{C_p}\left(\frac{\partial u}{\partial y}\right)^2 + \tau\left[D_B\frac{\partial T}{\partial y}\frac{\partial C}{\partial y} + \frac{D_T}{T_\infty}\left(\frac{\partial T}{\partial y}\right)^2\right] - \frac{1}{\rho C_p}\frac{\partial q_r}{\partial y}, \tag{4.3}$$

$$u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} = D_B\frac{\partial^2 C}{\partial y^2} + \frac{D_T}{T_\infty}\left(\frac{\partial^2 T}{\partial y^2}\right). \tag{4.4}$$

In the above equations (4.1) to (4.4), T represents fluid temperature, nanoparticle concentration is represented by C, ν denotes kinematic viscosity, permeability is represented by K, thermal radiation is represented by q_r, ρ is fluid density, where at uniform pressure specific heat capacity is represented by C_p, D_B is the thermophoresis diffusion coefficient, Brownian diffusion coefficient is represented by D_T T_∞ is the fluid of temperature at infinity from the stretched surface.

The symbol $\alpha = \frac{k}{(\rho c)_f}$ (m^2/s) is the thermal diffusivity, $\tau = \frac{(\rho c)_p}{(\rho c)_f}$ is the ratio between effective heat capacity of the fluid and $B(x) = B_0 x^{n-1/2}$ is a variable magnetic field. The dimensional form of the boundary conditions is as follows:

$$\begin{cases} u = u_w + N\nu\left(\frac{\partial u}{\partial y}\right), v = 0, T = T_w + D\left(\frac{\partial T}{\partial y}\right), C = C_w, & \text{at } y = 0. \\ u \to 0, v \to 0, T = T_\infty, C = C_\infty, & \text{at } y \to \infty. \end{cases} \tag{4.5}$$

The radiative heat flux is given by:

$$q_r = -\frac{4\sigma^\star}{3k^\star}\frac{\partial T^4}{\partial y}, \tag{4.6}$$

where σ^* is the Stefan-Boltzmann constant and k^* is the coefficient of mean absorption.

For small difference in temperature, T^4 can be obtained by using Taylor series as:

$$T^4 = 4TT_\infty^3 - 3T_\infty^4. \tag{4.7}$$

On using (4.7) into (4.6)

$$q_r = -\frac{16\sigma^\star T_\infty^3}{3k^\star}\frac{\partial T}{\partial y}, \tag{4.8}$$

and differentiating w.r.t. y,

$$\frac{\partial q_r}{\partial y} = -\frac{16\sigma^\star T_\infty^3}{3k^\star}\frac{\partial^2 T}{\partial y^2}. \tag{4.9}$$

Now, (4.3) got the following form:

$$\begin{aligned} u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} &= \alpha\frac{\partial^2 T}{\partial y^2} + \frac{\nu}{C_p}\left(\frac{\partial u}{\partial y}\right)^2 \\ &+ \tau\left[D_B\frac{\partial T}{\partial y}\frac{\partial C}{\partial y} + \frac{D_T}{T_\infty}\left(\frac{\partial T}{\partial y}\right)^2\right] - \frac{1}{\rho C_p}\frac{16\sigma^\star T_\infty^3}{3k^\star}\frac{\partial^2 T}{\partial y^2}. \end{aligned} \tag{4.10}$$

For the conversion of (4.1) to (4.4) into the dimensionless form, the following similarity transformation has been applied [8].

$$\begin{cases} \eta = y\sqrt{a\left(n+1\right)/2\nu}x^{(n-1)/2}, \\ u = ax^n f'\left(\eta\right), \\ v = -\sqrt{\frac{(n+1)a\nu_f}{2}}x^{n-1/2}\left[f\left(\eta\right) + \frac{n-1}{n+1}\left(y\sqrt{\frac{a(n+1)}{2\nu}}x^{(n-1)/2}\right)f'\left(\eta\right)\right], \\ T = T_\infty + bx^{2n}\theta\left(\eta\right), \Phi\left(\eta\right) = \left(C - C_\infty\right)/\left(C_w - C_\infty\right). \end{cases} \tag{4.11}$$

The detailed procedure for the conversion of continuity Eq. (4.1) and concentration Eq. (4.4) has been discussed in Chapter 3.

Here, calculations for the dimensionless form of momentum Eq. (4.2) are given as follows:

$$\begin{aligned} \because \quad & u = ax^n f'\left(\eta\right), \\ & \frac{\partial u}{\partial x} = a\frac{\partial}{\partial x}\left(x^n f'\left(\eta\right)\right), \\ & \frac{\partial u}{\partial x} = anx^{n-1}f'\left(\eta\right) + ax^n f''\left(\eta\right)\left(\frac{\partial \eta}{\partial x}\right), \because \quad \frac{\partial \eta}{\partial x} = y\sqrt{\frac{a(n+1)}{2\nu}}\left(\frac{n-1}{2}\right)x^{\frac{n-3}{2}} \\ & \frac{\partial u}{\partial x} = anx^{n-1}f' + ay\left(\frac{n-1}{2}\right)\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{3n-3}{2}}f''. \end{aligned} \tag{4.12}$$

Differentiating u w.r.t. y:

$$\begin{aligned}
\frac{\partial u}{\partial y} &= a\frac{\partial}{\partial y}\left(x^n f'(\eta)\right),\\
\frac{\partial u}{\partial y} &= ax^n f''(\eta)\frac{\partial \eta}{\partial y},\\
\because \qquad \frac{\partial \eta}{\partial y} &= \sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{n-1}{2}}\\
\frac{\partial u}{\partial y} &= a\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{3n-1}{2}}f''(\eta). \qquad (4.13)
\end{aligned}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{a^2(n+1)}{2\nu}x^{2n-1}f'''(\eta). \qquad (4.14)$$

On using Eqs. (4.11), (4.12), (4.13), and (4.14), into Eq. (4.2), we have

$$\begin{aligned}
&(ax^n f'(\eta))\left(anx^{n-1}f' + ay\left(\frac{n-1}{2}f''\right)\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{3n-3}{2}}\right)\\
&+\left(-\sqrt{\frac{(n+1)\,a\nu}{2}}x^{n-1/2}\left[f(\eta)+\frac{n-1}{n+1}\eta f'(\eta)\right]\right)\left(a\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{3n-1}{2}}f''(\eta)\right)\\
&=\nu\left(\frac{a^2(n+1)}{2\nu}x^{2n-1}f'''(\eta)\right)-\frac{\sigma B^2}{\rho_f}(ax^n f'(\eta))-\frac{\nu}{K}(ax^n f').
\end{aligned}$$

By simplifying and using $B = B_0x^{n-1/2}$

$$\begin{aligned}
&\frac{a^2(n+1)x^{2n-1}}{2}f''' + \frac{a^2(n+1)x^{2n-1}}{2}ff'' - \frac{a^2y(n-1)x^{5n-3}}{2}\sqrt{a(n+1)/2\nu}f'f''\\
&+\frac{a^2y(n-1)x^{5n-3}}{2}\sqrt{a(n+1)/2\nu}f'f'' - a^2nx^{2n-1}f'^2 - \frac{\sigma B_0^2\left(x^{n-1/2}\right)^2 ax^n}{\rho_f}f'\\
&-\frac{\nu}{K}(ax^n f') = 0.
\end{aligned}$$

The $3rd$ and $4th$ terms on the L.H.S. of above equation will be cancelled to each other and rest equation will be multiplied by $\frac{2}{(n+1)a^2x^{2n-1}}$, then the above equation finally becomes:

$$f''' + ff'' - \left(\frac{2n}{n+1}\right)f'^2 - Mf' - Pf' = 0, \qquad (4.15)$$

where $M = \frac{2\sigma B_0^2}{a\rho_f(n+1)}$, $B = B_0x^{n-1/2}$ and $P = \frac{2\nu}{K(n+1)ax^{n-1}}$. The procedure for the conversion of energy equation (4.11) is given as:

Differentiating T w.r.t. x:

$$\frac{\partial T}{\partial x} = \frac{\partial}{\partial x}\left(T_\infty + bx^{2n}\theta\left(\eta\right)\right),$$
$$\frac{\partial T}{\partial x} = 0 + b\left(\theta 2nx^{2n-1} + x^{2n}\theta'\left(\eta\right)\frac{\partial \eta}{\partial x}\right),$$
$$\because \qquad \frac{\partial \eta}{\partial x} = y\sqrt{\frac{a(n+1)}{2\nu}}\left(\frac{n-1}{2}\right)x^{\frac{n-3}{2}}$$

$$\frac{\partial T}{\partial x} = 2bn\theta x^{2n-1} + b\left(\frac{n-1}{2}\right)\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{5n-3}{2}}\theta'. \tag{4.16}$$

Differentiating T w.r.t y:

$$\frac{\partial T}{\partial y} = \frac{\partial}{\partial y}\left(T_\infty + bx^{2n}\theta\left(\eta\right)\right),$$
$$\frac{\partial T}{\partial y} = 0 + bx^{2n}\theta'\left(\eta\right)\frac{\partial \eta}{\partial y},$$
$$\because \qquad \frac{\partial \eta}{\partial y} = \sqrt{a\left(n+1\right)/2\nu}x^{(n-1)/2}$$
$$\frac{\partial T}{\partial y} = b\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{5n-1}{2}}\theta'. \tag{4.17}$$

Again differentiating above equation:

$$\frac{\partial^2 T}{\partial y^2} = ab\frac{(n+1)}{2\nu}x^{3n-1}\theta''. \tag{4.18}$$

Differentiating C from Eq. (4.15), w.r.t. y:

$$C = \left(C_w - C_\infty\right)\Phi\left(\eta\right) + C_\infty,$$

$$\frac{\partial C}{\partial y}b = \frac{\partial}{\partial y}\left(\left(C_w - C_\infty\right)\Phi\left(\eta\right) + C_\infty\right),$$

$$\frac{\partial C}{\partial y} = \left(C_w - C_\infty\right)\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{n-1}{2}}\Phi'\left(\eta\right). \tag{4.19}$$

On using Eqs. (4.11), (4.16), (4.17), (4.18), and (4.19), into Eq. (4.10):

$$
\begin{aligned}
&(ax^n f'(\eta))\left(2bn\theta x^{2n-1} + b\left(\frac{n-1}{2}\right)\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{5n-3}{2}}\theta'\right) \\
&+\left(-\sqrt{\frac{(n+1)\,a\nu}{2}}x^{n-1/2}\left[f(\eta)+\frac{n-1}{n+1}\eta f'(\eta)\right]\right)\left(b\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{5n-1}{2}}\theta'\right) \\
&=\alpha\left(ab\frac{(n+1)}{2\nu}x^{3n-1}\theta''\right)+\frac{\nu}{C_p}\left(a\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{3n-1}{2}}f''(\eta)\right)^2 \\
&+\tau\left(D_B\left(b\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{5n-1}{2}}\theta'\right)\right)\left((C_w-C_\infty)\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{n-1}{2}}\Phi'(\eta)\right) \\
&+\tau\frac{D_T}{T_\infty}\left(b\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{5n-1}{2}}\theta'\right)^2-\frac{1}{\rho C_p}\frac{16\sigma^\star T_\infty^3}{3k^\star}ab\frac{(n+1)}{2\nu}x^{3n-1}\theta'',
\end{aligned}
$$

and simplifying above equation

$$
\begin{aligned}
&2abnx^{3n-1}\theta f' + aby\frac{n-1}{2}\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{7n-3}{2}}f'\theta' \\
&-\sqrt{\frac{a\nu(n+1)}{2}}x^{n-1/2}fbx^{5n-1/2}\sqrt{\frac{a(n+1)}{2\nu}}\theta' \\
&-\sqrt{\frac{a\nu(n+1)}{2}}x^{n-1/2}\left(\frac{n-1}{n+1}\right)y\sqrt{\frac{a(n+1)}{2\nu}}x^{n-1/2}bf'x^{5n-1/2}\sqrt{\frac{a(n+1)}{2\nu}}\theta' \\
&=\alpha ab\frac{n+1}{2\nu}x^{3n-1}\theta''+a^3\frac{\nu}{C_p}\frac{n+1}{2\nu}x^{3n-1}f''^2+\tau D_B ab\,(C_w-C_\infty)\,\frac{n+1}{2\nu}x^{3n-1}\Phi'\theta' \\
&+\tau ab^2\frac{D_T}{T_\infty}\frac{n+1}{2\nu}x^{5n-1}\theta'^2-\frac{1}{\rho C_p}\frac{16\sigma^\star T_\infty^3}{3k^\star}ab\frac{(n+1)}{2\nu}x^{3n-1}\theta''.
\end{aligned}
$$

Again simplifying:

$$
\begin{aligned}
&2abnx^{3n-1}\theta f' + aby\frac{n-1}{2}\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{7n-3}{2}}f'\theta' \\
&-ab\frac{n+1}{2}x^{3n-1}f\theta' - aby\frac{n+1}{2}\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{7n-3}{2}}f'\theta' \\
&=\alpha ab\frac{n+1}{2\nu}x^{3n-1}\theta''+a^3\frac{\nu}{C_p}\frac{n+1}{2\nu}x^{3n-1}f''^2 \\
&+\tau D_B ab\,(C_w-C_\infty)\,\frac{n+1}{2\nu}x^{3n-1}\Phi'\theta' \\
&+\tau ab^2\frac{D_T}{T_\infty}\frac{n+1}{2\nu}x^{5n-1}\theta'^2-\frac{1}{\rho C_p}\frac{16\sigma^\star T_\infty^3}{3k^\star}ab\frac{(n+1)}{2\nu}x^{3n-1}\theta''.
\end{aligned}
$$

Second and fourth terms on the R.H.S. of previous equation will be cancelled to each other and rest of the equation will be multiplied by $\dfrac{2\nu}{ab\alpha(n+1)x^{3n-1}}$.

$$\frac{4n}{n+1}\theta f'-f\theta' = \frac{\alpha}{\nu}\theta''+\frac{a^2\nu}{bC_p\nu}f''^2+\tau D_B\frac{(C_w-C_\infty)}{\nu}\Phi'\theta'+\tau b\frac{D_T}{T_\infty}\frac{x^{2n}}{\nu}\theta'^2-\frac{1}{\rho C_p\alpha}\frac{16\sigma^\star T_\infty^3}{3k^\star}\theta''.$$

In above equation we will use $bx^{2n} = T_w - T_\infty$, $\rho C_p\alpha = k, \alpha = \dfrac{K}{\rho C_p}(m^2/s)$, $P_r = \dfrac{\nu}{\alpha}$, and $R_d = \dfrac{4\sigma^\star T_\infty^3}{3kk^\star}$.

Hence the transformed form of energy equation is:

$$\frac{1}{P_r}\theta''+\frac{4R_d}{3}\theta''+f\theta'-\frac{4n}{n+1}\theta f'+\frac{a^2}{bC_p}f''^2+\frac{\tau D_B(C_w-C_\infty)}{\nu}\Phi'\theta'+\frac{\tau D_T(T_w-T_\infty)}{T_\infty\nu}\theta'^2 = 0,$$

and finally it becomes:

$$\left(\frac{1}{P_r}+\frac{4R_d}{3}\right)\theta''+f\theta'-\frac{4n}{n+1}\theta f'+N_b\Phi'\theta'+N_t\theta'^2+E_cf''^2 = 0, \tag{4.20}$$

where $N_b = \dfrac{(\rho c)_p D_B(C_w-C_\infty)}{(\rho c)_f\nu}$, $N_t = \dfrac{(\rho c)_f D_T(T_w-T_\infty)}{(\rho c)_f T_\infty\alpha}$ and $E_c = \dfrac{u_w^2}{C_p(T_w-T_\infty)}$.

The final dimensionless form of the governing model is

$$f'''+ff''-\left(\frac{2n}{n+1}\right)f'^2-(M+P)f' = 0, \tag{4.21}$$

$$\left(\frac{1}{P_r}+\frac{4R_d}{3}\right)\theta''+f\theta'-\frac{4n}{n+1}\theta f'+N_b\Phi'\theta'+N_t\theta'^2+E_cf''^2 = 0, \tag{4.22}$$

$$\Phi''+\frac{N_t}{N_b}\theta''+L_ef\Phi' = 0. \tag{4.23}$$

The associated boundary conditions are as follows:

$$\begin{cases} f'(0) = 1+\lambda f''(0), f = 0, \\ \theta(0) = 1+\delta\theta'(0), \Phi = 0, \;\; at\ \eta = 0. \\ f'\to 0, \theta\to 0, \Phi\to 0, at\ \eta\to\infty, \end{cases} \tag{4.24}$$

where the detailed procedure for the conversion of dimensional boundary conditions (4.5) into nondimensional form (4.24) has been discussed in Chapter 3.

Different parameters used in the previous equations are given as:

$$\begin{cases} L_e = \frac{\nu}{D_B},\ P_r = \frac{\nu}{\alpha}, N_b = \frac{\tau D_B(C_w - C_\infty)}{\nu}, \\ N_t = \frac{\tau D_T(T_w - T_\infty)}{\nu T_\infty}, M = \frac{2\sigma B_0^2}{a\rho(n+1)}, \\ \delta = D_1\sqrt{\frac{a(n+1)}{2\nu}}, \lambda = N_1\sqrt{\frac{a\nu(n+1)}{2}}, \\ E_c = \frac{u_w^2}{C_p(T_w - T_\infty)}. \end{cases}$$

4.1.1 Physical Quantities of Interest

Calculation for the nondimensionl form of Skin friction coefficient C_{fx} is given as.

$$C_{fx} = \frac{\mu}{\rho u_w^2}\left[\frac{\partial u}{\partial y}\right]_{y=0}, \tag{4.25}$$

$$C_{fx} = \frac{\nu}{u_w^2}\left(a\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{3n-1}{2}}f''(\eta)\right) \quad \because \nu = \frac{\mu}{\rho}, \text{using (4.17)}$$

$$C_{fx} = a\frac{\sqrt{\nu}}{a^2x^{2n}}\sqrt{\frac{a(n+1)}{2}}x^{\frac{3n-1}{2}}f''(\eta),$$

$$C_{fx} = \sqrt{\frac{\nu}{ax^nx}}\sqrt{\frac{n+1}{2}}f'',$$

$$\because \quad Re_x = u_wx/\nu \text{ is local Reynolds number,}$$

$$\therefore \quad Re_xC_{fx} = \sqrt{\frac{n+1}{2}}f'' \tag{4.26}$$

To obtained the dimensionless form of Nusselt number Nu_x the following steps are required.

$$Nu_x = \frac{xq_w}{k(T_w - T_\infty)}, \tag{4.27}$$

$$\because \quad q_w = -\left[\left(k + \frac{16\sigma^\star T_\infty^3}{3k^\star}\right)\frac{\partial T}{\partial y}\right]_{y=0} \text{ is heat radiative flux.}$$

$$Nu_x = -\frac{x}{k(T_w - T_\infty)}\left(k + \frac{16\sigma^\star T_\infty^3}{3k^\star}\right)\left(b\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{5n-1}{2}}\theta'(0)\right),$$

$$Nu_x = -\frac{x}{bx^{2n}}\left(1 + \frac{16\sigma^\star T_\infty^3}{3kk^\star}\right)\left(b\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{5n-1}{2}}\theta'(0)\right), \quad \because \quad T_w - T_\infty = bx^{2n},$$

$$Nu_x = -\left(1 + \frac{4R_d}{3}\right)\sqrt{\frac{ax^nx}{\nu}}\sqrt{\frac{n+1}{2}}\theta'(0), \quad \because R_d = \frac{4\sigma^\star T_\infty^3}{kk^\star}$$

$$(Re_x)^{-1/2} Nu_x = -\left(1+\frac{4Rd}{3}\right)\sqrt{\frac{n+1}{2}}\theta'(0). \quad \because \quad Re_x = u_w x/\nu \tag{4.28}$$

For the nondimensional form of Sherwood number Sh_x the following calculations are required.

$$Sh_x = \frac{xq_m}{D_B(C_w - C_\infty)}, \tag{4.29}$$

$$\because \quad q_m = -D_b\left[\frac{\partial C}{\partial y}\right]_{y=0} \quad \text{is mass flux at the surface}$$

$$Sh_x = \frac{-xD_B}{D_B(C_w - C_\infty)}\left((C_w - C_\infty)\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{n-1}{2}}\Phi'(\eta)\right),$$

$$Sh_x = -\sqrt{\frac{a}{\nu}}\sqrt{\frac{n+1}{2}}x^{\frac{n}{2}+\frac{1}{2}}\Phi',$$

$$Sh_x = -Re_x^{\frac{1}{2}}\sqrt{\frac{n+1}{2}}\Phi', \quad \because \ Re_x = u_w x/\nu$$

$$Re_x^{-\frac{1}{2}} Sh_x = -\sqrt{\frac{n+1}{2}}\Phi'. \tag{4.30}$$

4.2 Numerical Technique

In this thesis Shooting method has been used to solve the transformed system of ODEs (4.21) to (4.23) by assuming the missing initial conditions subject to the boundary condition (4.24). The system of BVP (4.21)-(4.23) is first converted into IVP for the application of shooting method. By solving Eq. (4.21) independently we obtained results as f, f', f'' and f''', then we will use these results in couple Eqs. (4.22) and (4.23). For this purpose the following notations has been used.

$$f = f_1, \qquad f' = f_1' = f_2,$$

$$f'' = f_1'' = f_2' = f_3, \qquad f''' = f_1''' = f_2'' = f_3'.$$

The resulting IVP takes the following form:

$$f_1' = f_2, \qquad f_1(0) = 0, \tag{4.31}$$

$$f_2' = f_3, \qquad f_2(0) = 1 + \lambda\xi, \tag{4.32}$$

$$f_3' = -f_1 f_3 + \frac{2n}{n+1}f_2^2 + (M+P)f_2, \qquad f_3(0) = \xi, \tag{4.33}$$

where ξ is the missing initial condition. The IVP has been solved by using RK4 method. Since the unbounded domain can not be used for the numerical computations, so the domain of the IVP has been taken as $[0, \eta_\infty]$ instead of $[0, \infty)$, where η_∞ is an appropriate positive real number with choosen initial guess ξ such that:

$$f_2\left(\eta_\infty, \xi\right) = 0.$$

To solve the previous equation, Newton's method has been used with following iterative procedure

$$\xi^{n+1} = \xi^n - \frac{f_2(\eta_\infty, \xi^n)}{\left(\frac{\partial f_2(\eta_\infty, \xi^n)}{\partial \xi}\right)}.$$

In order to obtain the derivatives w.r.t. ξ, following notations will be used

$$\frac{\partial f_1}{\partial \xi} = f_4, \quad \frac{\partial f_2}{\partial \xi} = f_5, \quad \frac{\partial f_3}{\partial \xi} = f_6.$$

Hence the Newton's iterative scheme gets the following form

$$\xi^{n+1} = \xi^n - \frac{f_2(\eta_\infty, \xi^n)}{f_5(\eta_\infty, \xi^n)}.$$

By differentiating Eqs. (4.31), (4.32) and (4.33) w.r.t. ξ three more equations will be appeared. As a result of these six ODEs, IVP takes the following form:

$$\begin{aligned}
&f_1' = f_2, && f_1(0) = 0,\\
&f_2' = f_3, && f_2(0) = 1 + \lambda\xi,\\
&f_3' = -f_1 f_3 + \frac{2n}{n+1} f_2^2 + (M+P) f_2, && f_3(0) = \xi,\\
&f_4' = f_5, && f_4(0) = 0,\\
&f_5' = f_6, && f_5(0) = \lambda,\\
&f_6' = -(f_1 f_6 + f_3 f_4) + \frac{4n}{n+1} f_2 f_5 + (M+P) f_5, && f_6(0) = 1,
\end{aligned}$$

The Newton's iterative process is repeated untill the following condition is met.

$$|f_2(\eta_\infty, \xi)| < \epsilon,$$

here ϵ is taken as 10^{-6}. Similarly by solving Eqs. (4.22) and (4.23) along with the boundary conditions (4.24), where the missing initial conditions $\theta'(0)$ and $\Phi'(0)$ are denoted by ψ and χ respectively.

The notations used for this purpose are given as follows:

$$
\begin{aligned}
\theta &= Y_1, \\
\theta' &= Y_1' = Y_2, \\
\theta'' &= Y_1'' = Y_2', \\
\frac{\partial Y_1}{\partial \psi} &= Y_5, \quad \frac{\partial Y_2}{\partial \psi} = Y_6, \quad \frac{\partial Y_1}{\partial \chi} = Y_9, \quad \frac{\partial Y_2}{\partial \chi} = Y_{10}, \\
\Phi &= Y_3, \\
\Phi' &= Y_3' = Y_4, \\
\Phi'' &= Y_3'' = Y_4', \\
\frac{\partial Y_3}{\partial \psi} &= Y_7, \quad \frac{\partial Y_4}{\partial \psi} = Y_8, \quad \frac{\partial Y_3}{\partial \chi} = Y_{11}, \quad \frac{\partial Y_4}{\partial \chi} = Y_{12}.
\end{aligned}
$$

By using these notations, we get the following first order ODEs:

$$
\begin{aligned}
&Y_1' = Y_2, && Y_1(0) = 1 + \delta\psi, \\
&Y_2' = \left(\frac{3P_r}{3 + 4R_dP_r}\right) \\
&\qquad \left[-f_1Y_2 + \frac{4n}{n+1}f_2Y_1 - N_bY_2Y_4 - N_tY_2^2 - E_cf_3^2\right], && Y_2(0) = \psi, \\
&Y_3' = Y_4, && Y_3(0) = 1, \\
&Y_4' = -L_ef_1Y_4 - \left(\frac{N_t}{N_b}\right)P_r \\
&\qquad \left[-f_1Y_2 + \frac{4n}{n+1}f_2Y_1 - N_bY_2Y_4 - N_tY_2^2 - E_cf_3^2\right], && Y_4(0) = \chi, \\
&Y_5' = Y_6, && Y_5(0) = \delta, \\
&Y_6' = \left(\frac{3P_r}{3 + 4R_dP_r}\right) \\
&\qquad \left[-f_1Y_6 + \frac{4n}{n+1}f_2Y_5 - N_b(Y_6Y_4 + Y_2Y_8) - 2N_tY_2Y_6\right], && Y_6(0) = 1, \\
&Y_7' = Y_8, && Y_7(0) = 0,
\end{aligned}
$$

$$Y_8' = -L_e f_1 Y_8 - \left(\frac{N_t}{N_b}\right) P_r \left[-f_1 Y_6 + \frac{4n}{n+1} f_2 Y_5 - N_b(Y_4 Y_6 + Y_2 Y_8) - 2N_t Y_2 Y_6\right], \qquad Y_8(0) = 0,$$

$$Y_9' = Y_{10}, \qquad Y_9(0) = 0,$$

$$Y_{10}' = \left(\frac{3P_r}{3 + 4R_d P_r}\right) \left[-f_1 Y_{10} + \frac{4n}{n+1} f_2 Y_9 - N_b(Y_4 Y_{10} + Y_2 Y_{12}) - 2N_t Y_2 Y_{10}\right], \qquad Y_{10}(0) = 0,$$

$$Y_{11}' = Y_{12}, \qquad Y_{11}(0) = 0,$$

$$Y_{12}' = -L_e f_1 Y_{12} - \left(\frac{N_t}{N_b}\right) P_r \left[-f_1 Y_{10} + \frac{4n}{n+1} f_2 Y_9 - N_b(Y_{10} Y_4 + Y_2 Y_{12}) - 2N_t Y_2 Y_{10}\right], \qquad Y_{12}(0) = 1.$$

The domain of the above problem is $[0, \eta_\infty]$ instead of $[0, \infty)$, where η_∞ is a finite positive number for which the variations in the solution are negligible after $\eta = \eta_\infty$ where ψ and χ are missing conditions and are assumed to satisfy the following relations.

$$Y_1(\eta_\infty, \psi, \chi) = 0, \tag{4.34}$$

$$Y_3(\eta_\infty, \psi, \chi) = 0. \tag{4.35}$$

The above system of equations will be solved by the Newton's method governed by the following formulation.

$$\begin{bmatrix} \psi^{(n+1)} \\ \chi^{(n+1)} \end{bmatrix} = \begin{bmatrix} \psi^n \\ \chi^n \end{bmatrix} - \begin{bmatrix} Y_5 & Y_9 \\ Y_7 & Y_{11} \end{bmatrix}^{-1} \begin{bmatrix} Y_1^n \\ Y_3^n \end{bmatrix}_{(\eta_\infty, \psi, \chi)}$$

The Newton's iterative process is repeated up till the following condition is met.

$$max\{|f_2(\eta_\infty)|, \quad |Y_1(\eta_\infty)|, \quad |Y_3(\eta_\infty)|\} \leq \epsilon,$$

where ϵ is a small positive number. For the computational purpose, ϵ has been given the value $\epsilon = 10^{-8}$ whereas η_∞ is set as 5.

4.3 Graphical Results

The main objective of graphical results is to explain the impact of different parameters for the $f'(\eta)$, $\theta(\eta)$ and $\Phi(\eta)$. Numerical results of the Skin friction coefficient $[-f''(0)]$, Nusselt number $[-\theta'(0)]$ and Sherwood number $[-\Phi'(0)]$ for the different values of some fixed parameters (magnetic number M, velocity slip variable λ, thermal slip parameter δ, heat radiative flux R, and porosity factor P) are shown in Table 4.1. The values of M and skin friction coefficient are directly related while by accelerating the values of λ, Skin friction coefficient decreased. Shooting method was employed to obtain numerical results. The parameters such as porosity(P), magnetic number M, Brownian factor N_b, Prandtl P_r, Thermophoretic number N_t, non-linear stretching parameter n, Lewis number L_e, velocity slip variable λ and Eckert number E_c observed graphically in Figures 4.1 to 4.16.

Figure 4.1 shows the impact of porosity factor(P) on f'. It has been observed that f' is decreasing by increasing the values of a porosity factor P. The increased porosity results in the increased resistance applied by surface to the fluid motion which in return reduces the fluid velocity. The impact of porosity factor P on temperature θ has been shown in Figure 4.2. The increased temperature θ is due to increased resistance offered by porosity P. Effect of porosity P on concentration has been shown in Figure 4.3. For increased porosity P nanoparticles's far movement has occurred due to intense gradiant augments close to the surface.
The effect of R_d on θ has been shown in Figure 4.4. The enhanced thermal radiation parameter R_d increases the temperature distribution. The exit of heat energy from the fluid surface due to incremental values of R_d causes an increase in temperature θ of nanofluids and hence cools the system. The increased values of R_d generate maximum energy to the system which in results elevate the temperature of the fluid and hence enhances the intensity of heat radiation flux. It is clearly observed that the responsibility of enhanced temperature is due to weakened mean absorption coefficient($k*$). Enhanced temperature is also effected by presence of magnetic field. Hence fine method of cooling can be achieved at minimum value of Rd. Figure 4.5 illustrated the effect of M on the velocity f'.

If M is equal to zero, hydrodynamic movement occurs and when M is greater than zero, the MHD flow occurs. By increasing M graph of f' decreases. This is because the Lorentz force in MHD flow is occurred due to existence of magnetic parameter M. Increased temperature and thermal boundary layer is due to direct relationship between Lorentz force and magnetic parameter. By increasing M, the retarding force also increases which in result decreases the velocity profile f'. And by increasing M, temperature and concentration profile also increases because of a resistive force called Lorentz force that crosses the fluid motion and hence heat is transformed. This phenomena has been presented in Figures 4.6 and 4.7.

TABLE 4.1: Table for $-f''(0)$, $-\theta'(0)$ and $-\Phi'(0)$ for different values of M, λ and δ when $P_r = 5, N_b = N_t = 0.3, L_e = 2$ and $E_c = 0.1$.

M	λ	δ	R	P	$-f''(0)$	$-\theta'(0)$	$-\Phi'(0)$
0.1	0.2	0.2	0.5	0.2	0.93309	1.80229	0.62928
0.3					0.99154	1.76143	0.62928
0.4					1.01883	1.74164	0.6156
0.5					1.04498	1.7222	0.6292
1.0	0.3	0.1			1.0316	1.73372	0.5244
	0.4				0.9293	1.9228	0.9439
	0.5		0.1	0.1	0.8466	1.92281	0.8914
	0.1	0.3			1.3342	1.42129	0.9130
		0.4			1.3342	1.26126	0.7180
		0.5			1.3342	1.31618	0.5629

Figures 4.8, 4.9 and 4.10 present the inverse relationship of n with f', θ and Φ

respectively. The inverse relationship between Lewis number L_e and Φ with η, has been shown in Figure 4.11. Lewis number L_e and concentration profile are inversely related.

Figure 4.12 presents the relationship of Pr with θ. It has been analysed that incremental values of P_r causes a reduction in thermal diffusivity which in return causes a decrease in temperature profile. Figure 4.13 shows the relationship between velocity profile f' and velocity slip parameter λ. It has been observed that by increasing the values λ velocity profile decreases. In case of slip condition, stretching sheet velocity is different from flow velocity close to the sheet. Figure 4.14 illustrated the impact of E_c on temperature profile θ. It has been noticed that there is direct relationship between E_c and $\theta(\eta)$. Collection of energy in the fluid region is due to enhanced E_c. Consequently frictional heating is produced due to squandering, created in response to viscosity and elastic deformation. The indirect relationship between thermal slip parameter δ and concentration parameter Φ has been shown in Figure 4.15. It is clear from the graph that increasing the value of thermal slip parameter a reduction in the concentration parameter was observed even for a small value. The effect of N_b and N_t on θ has been shown in Figure 4.16. It is noticed that increased temperature profile is due to enhanced N_b and N_t. The reason is that the thermophoretic force is generated due to temperature gradient and it generates a rapid flow far from the stretched surface.

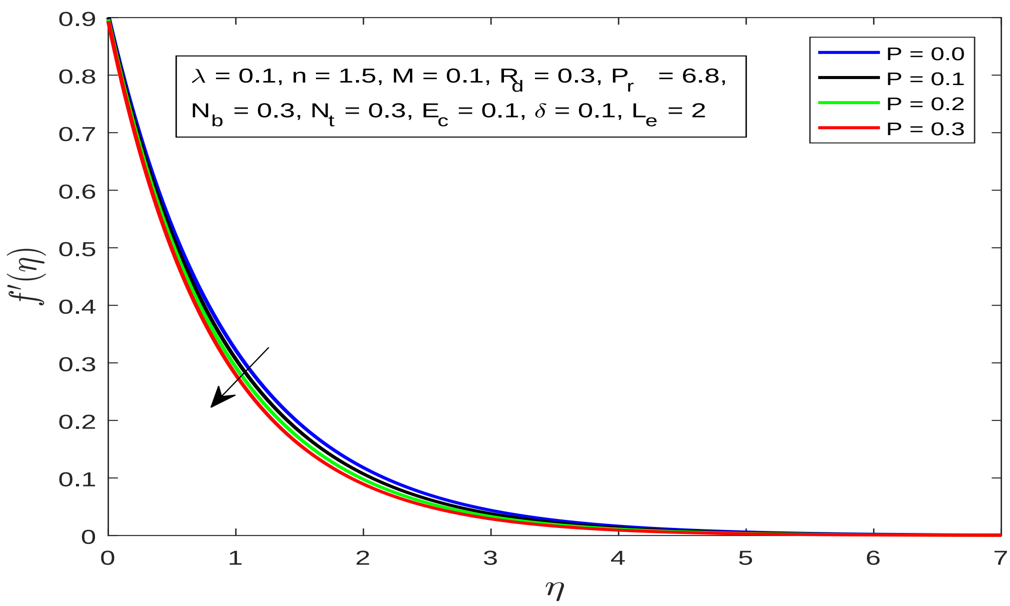

FIGURE 4.1: impact of P on velocity profile.

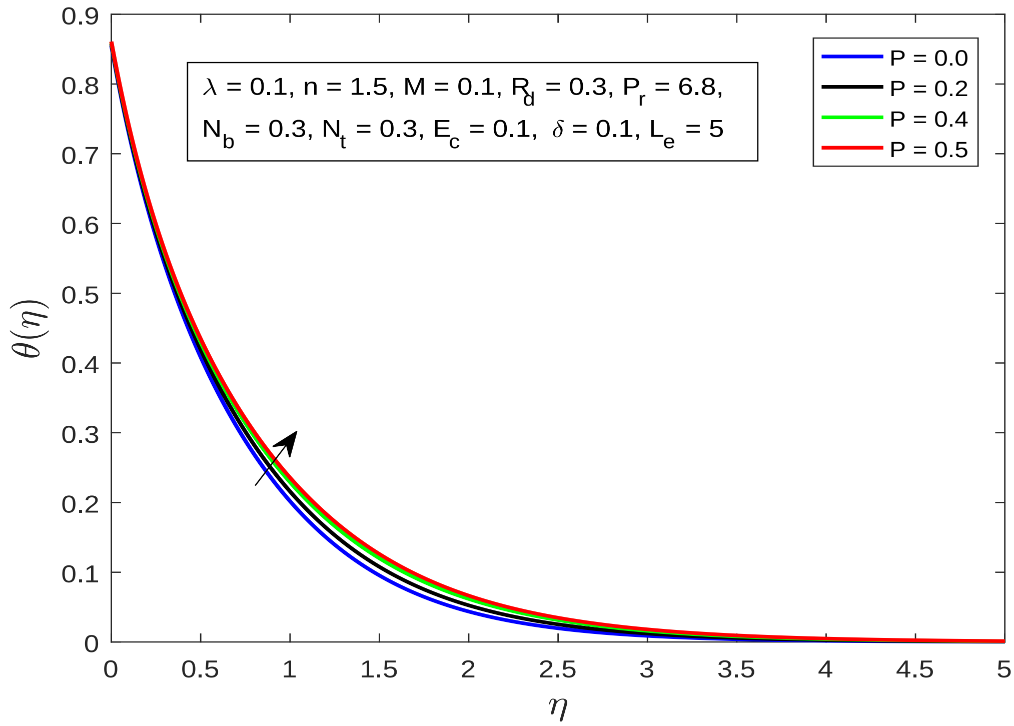

FIGURE 4.2: Effect of porosity P on temperature profile.

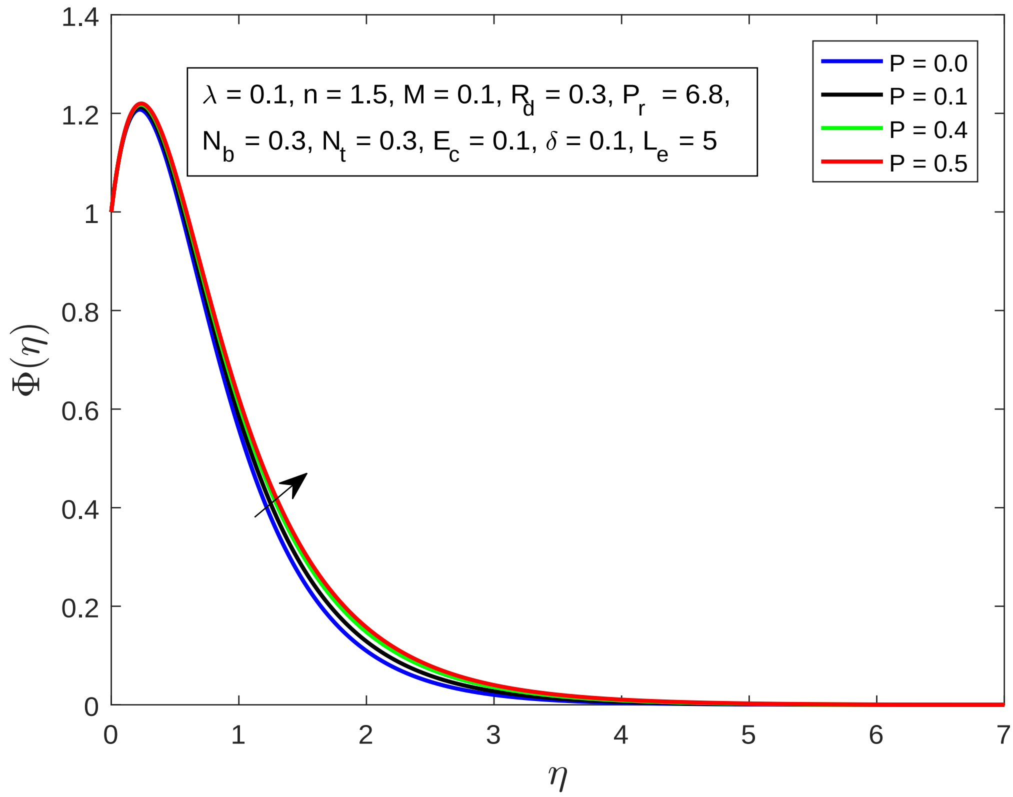

FIGURE 4.3: Effect of porosity P on Φ.

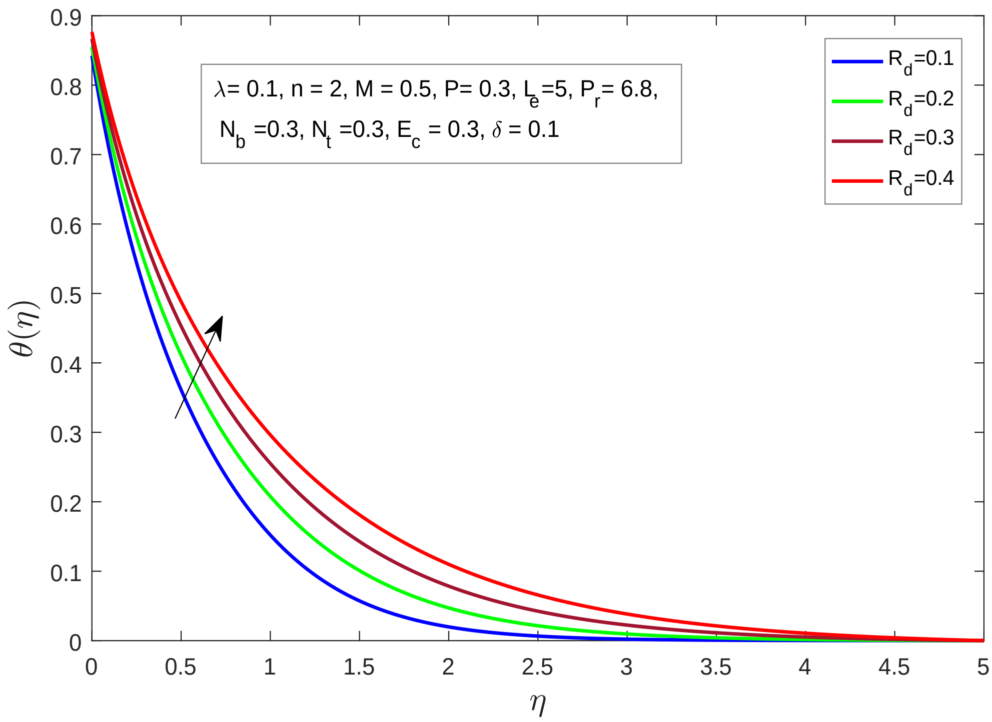

FIGURE 4.4: Effect of R_d on θ.

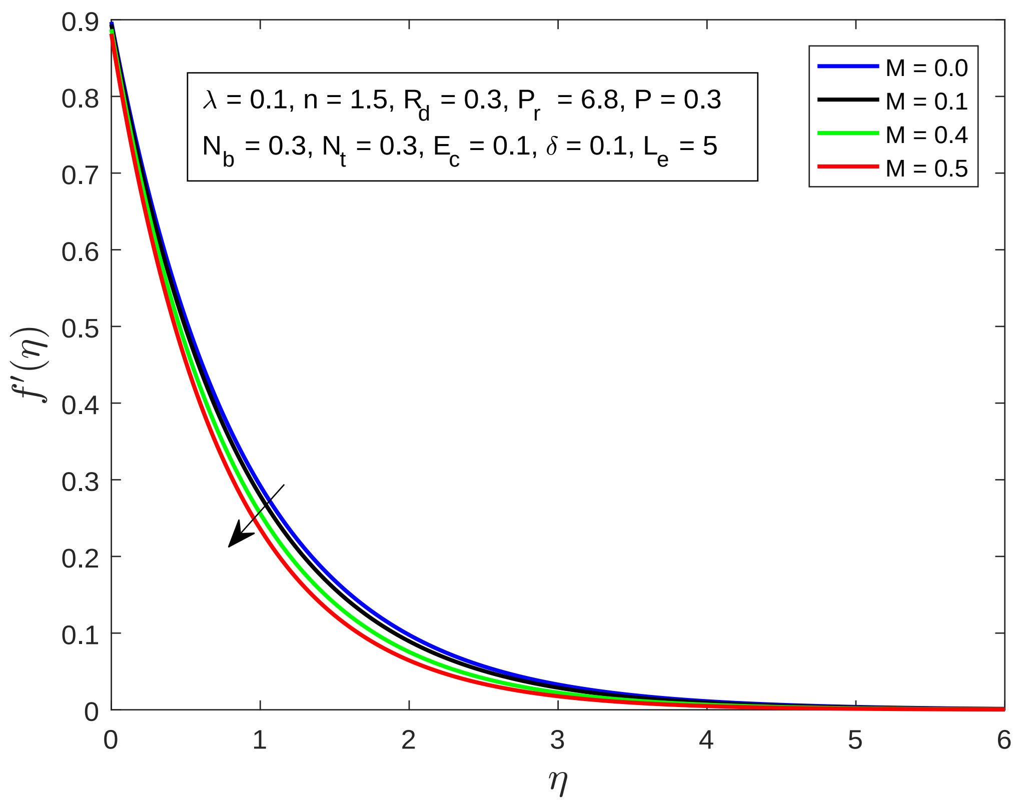

FIGURE 4.5: Effect of M on f'.

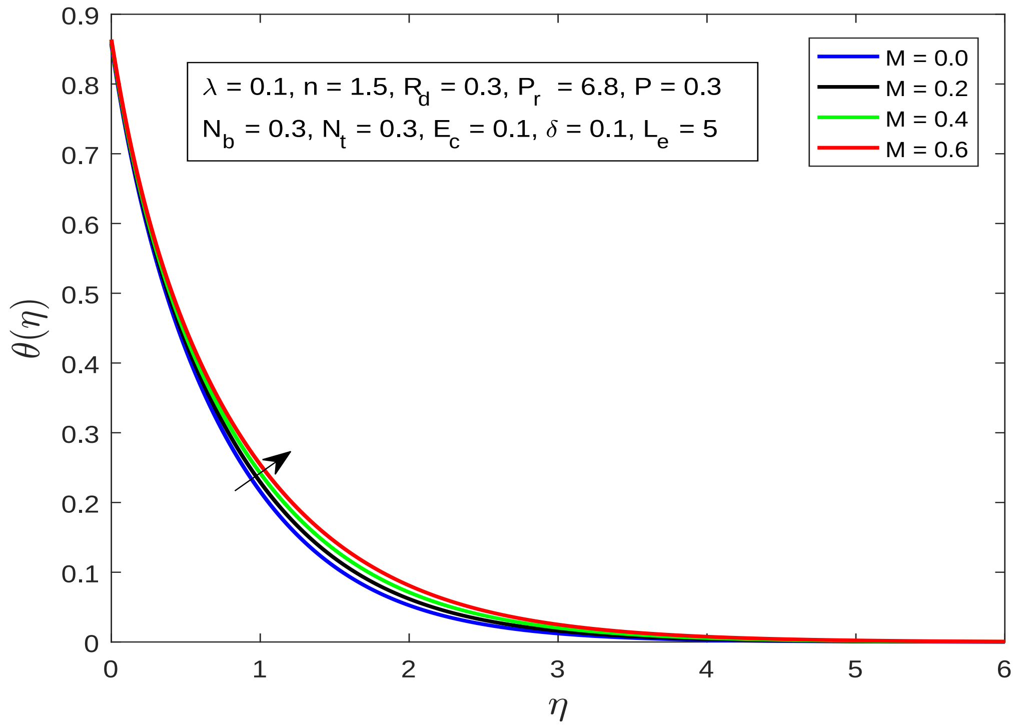

FIGURE 4.6: Effect of M on θ.

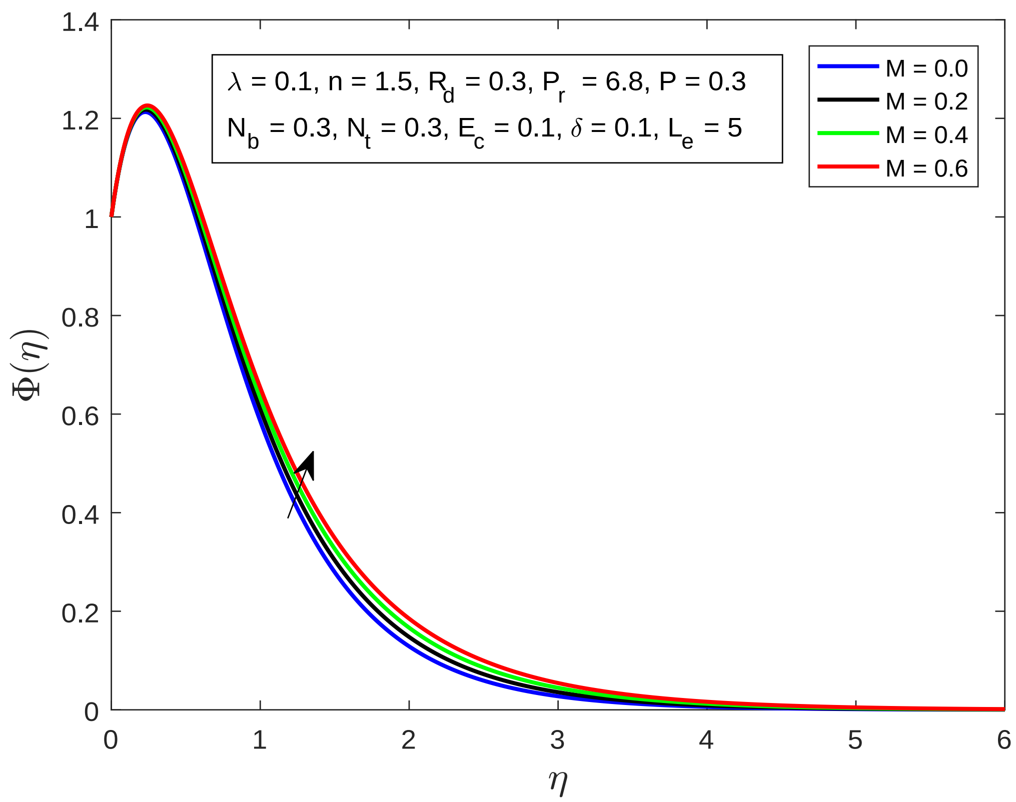

FIGURE 4.7: Effect of M on $\Phi(\eta)$.

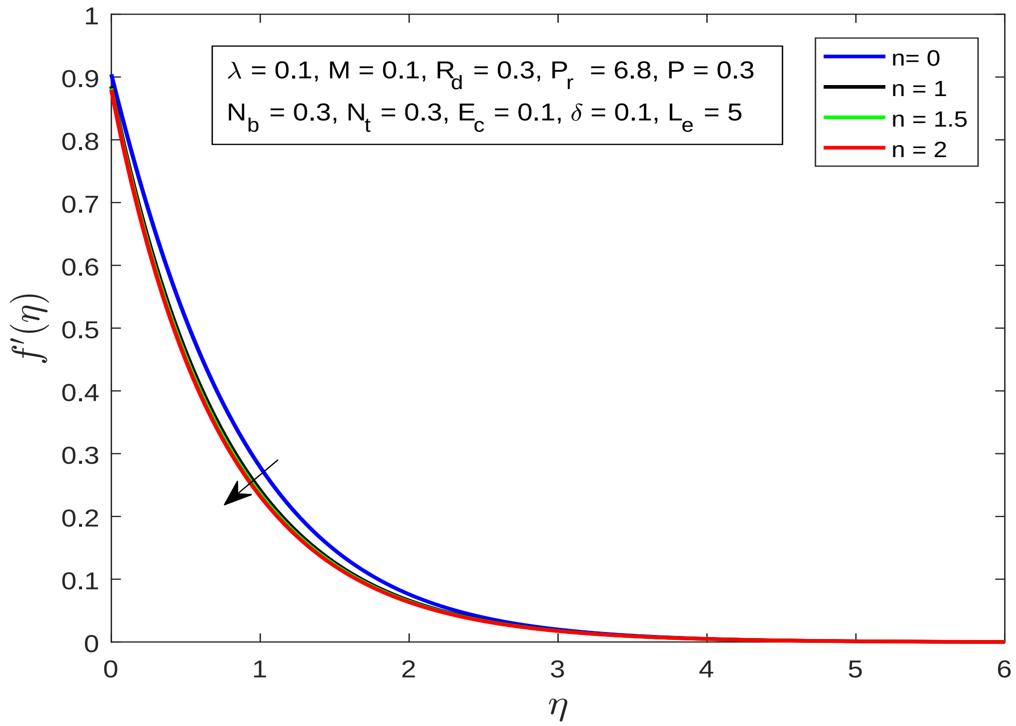

FIGURE 4.8: Effect of n on f'.

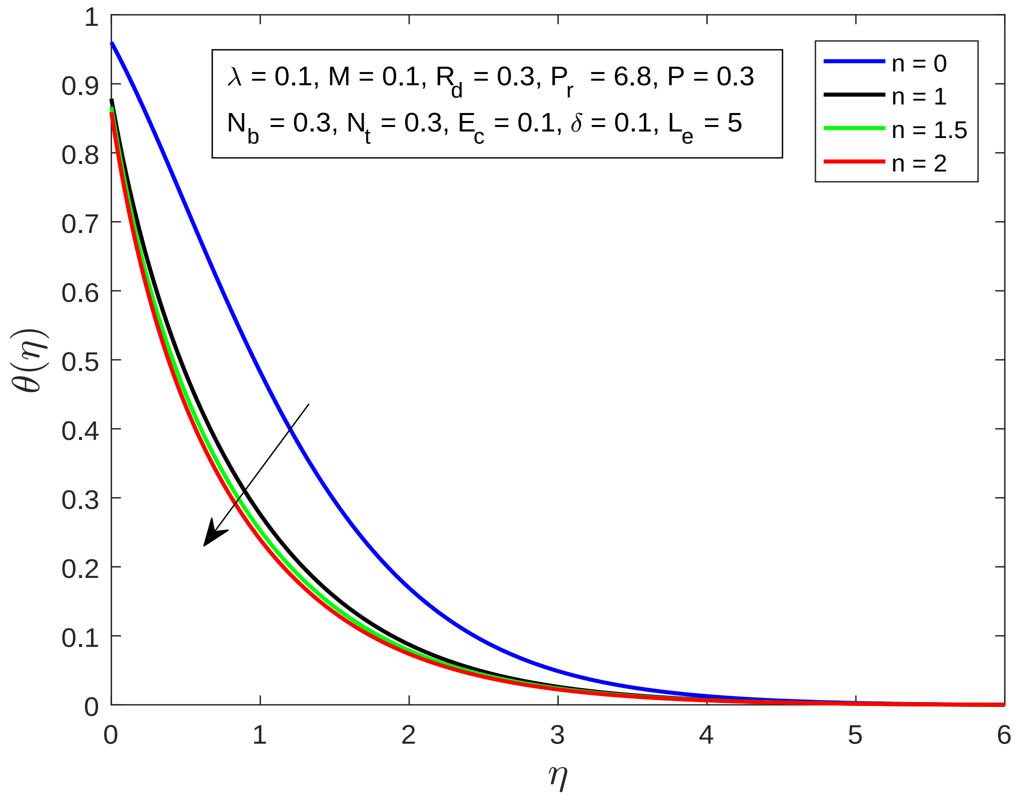

FIGURE 4.9: Effect of n on θ.

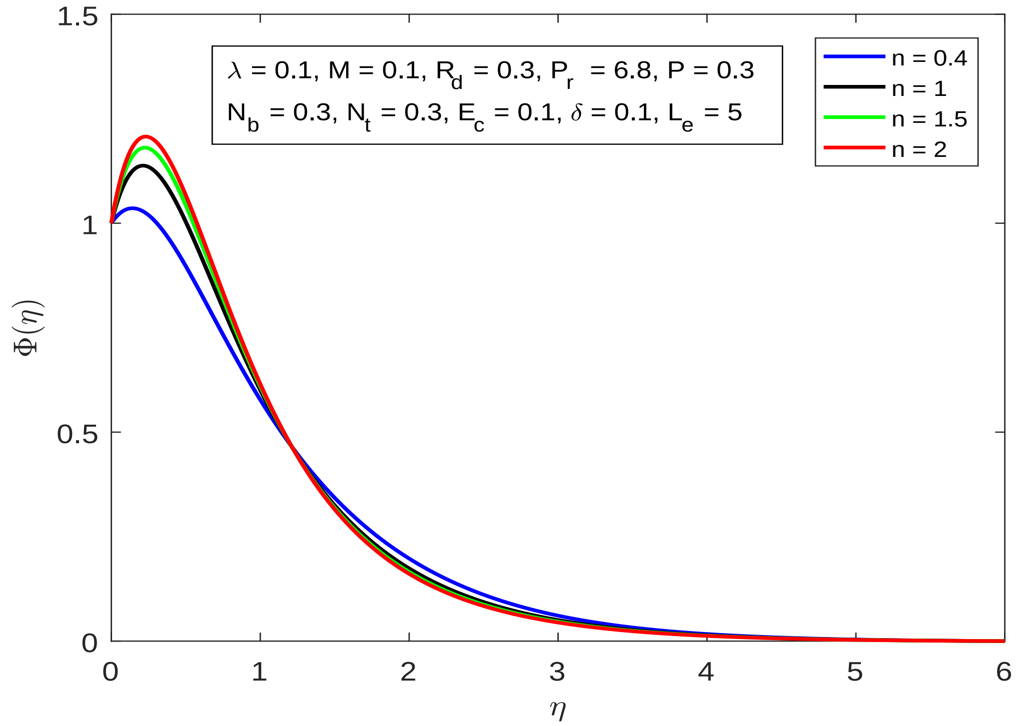

FIGURE 4.10: Effect of n on Φ.

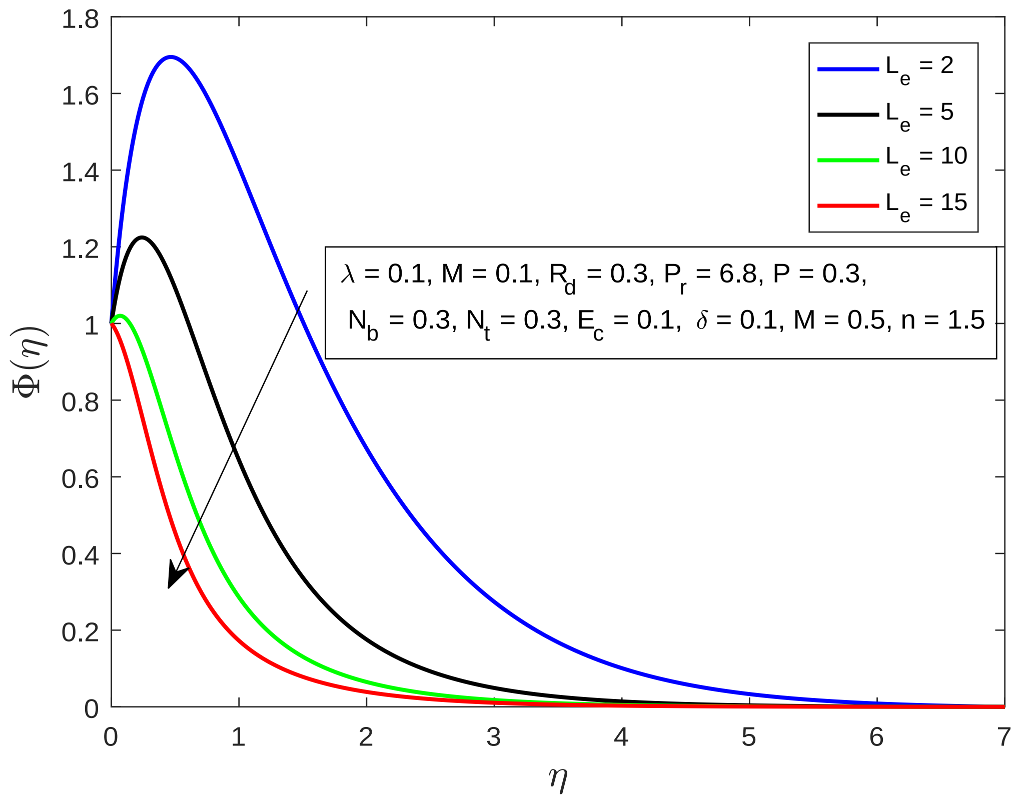

FIGURE 4.11: Effect of L_e on $\Phi(\eta)$.

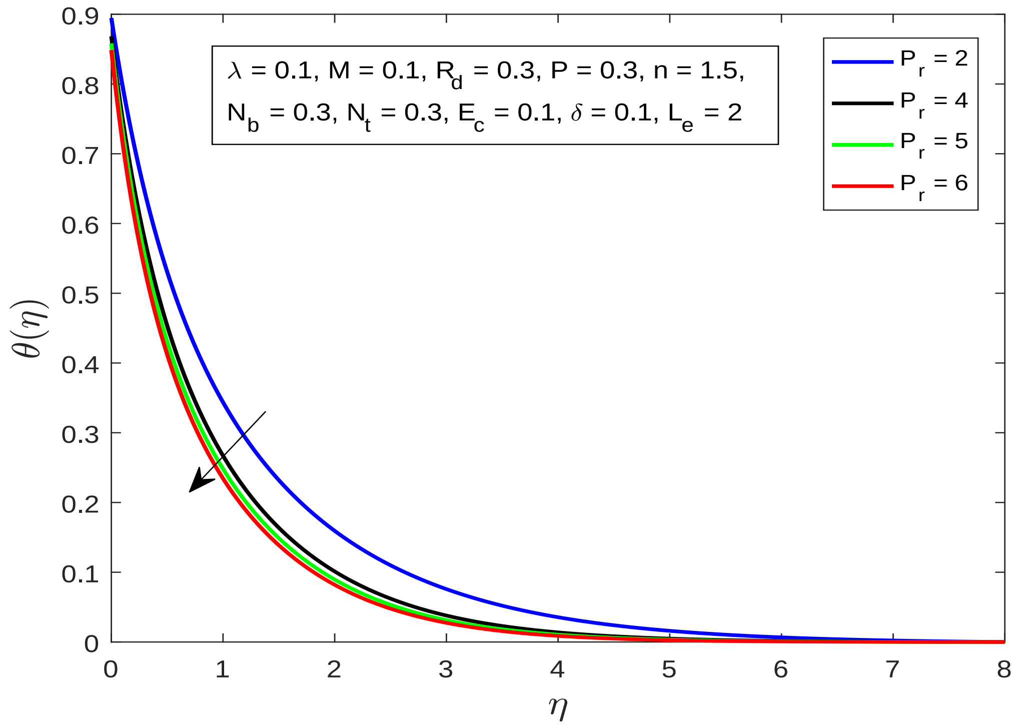

FIGURE 4.12: Effect of P_r on $\theta(\eta)$.

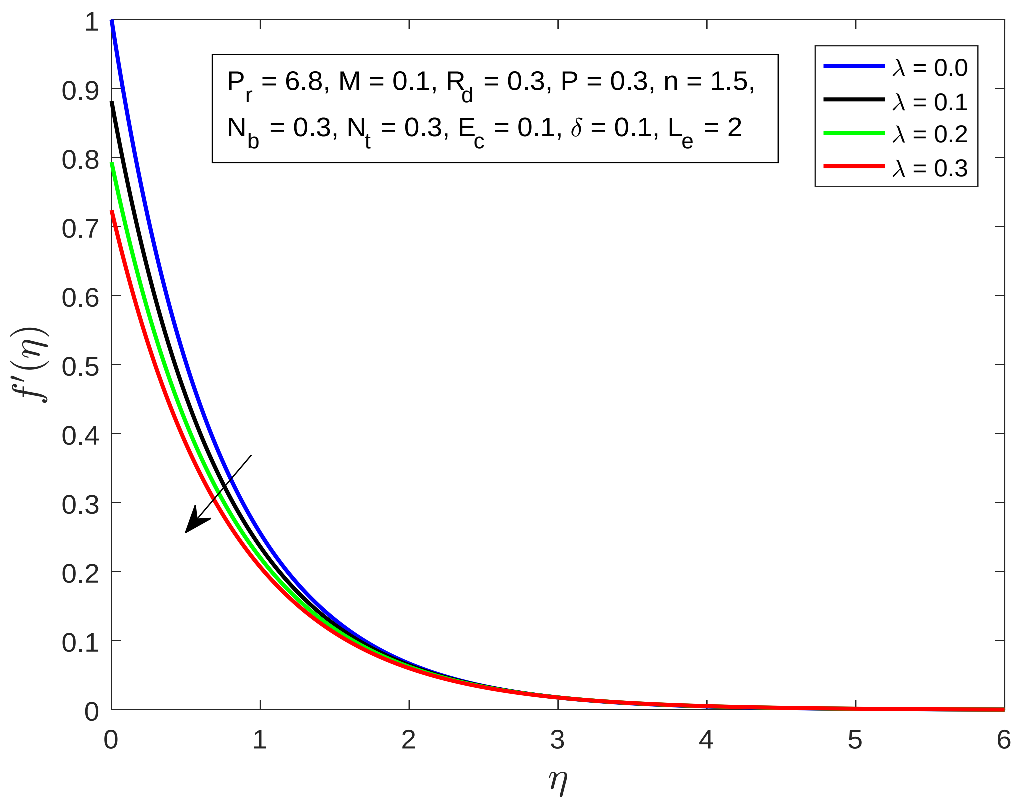

FIGURE 4.13: Effect of λ on f'.

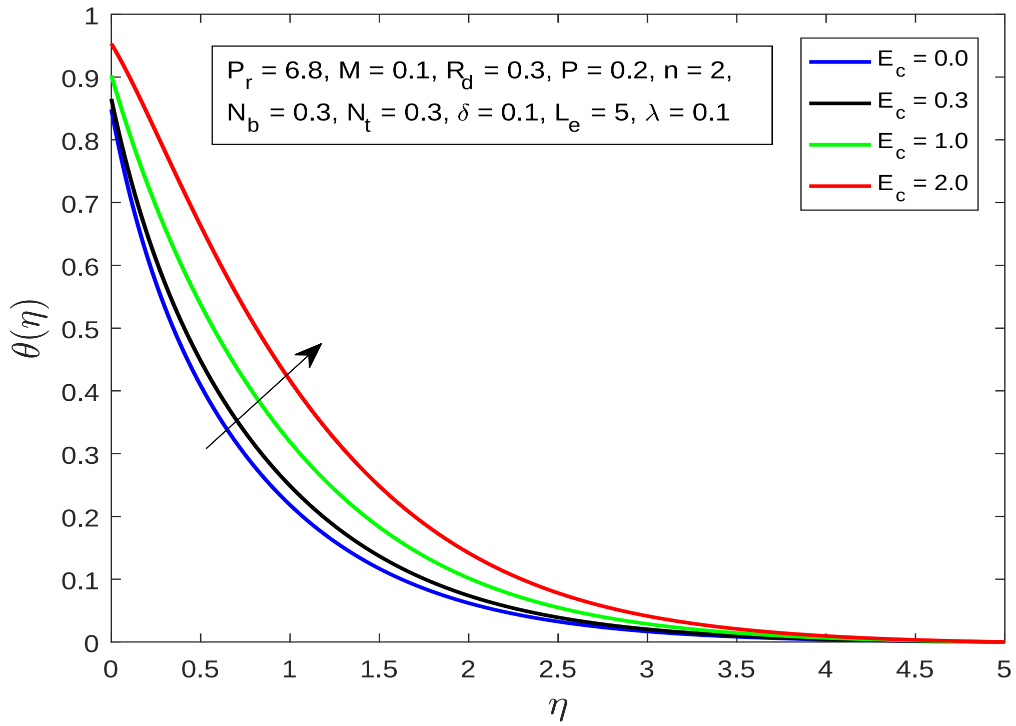

FIGURE 4.14: Effect of E_c on θ.

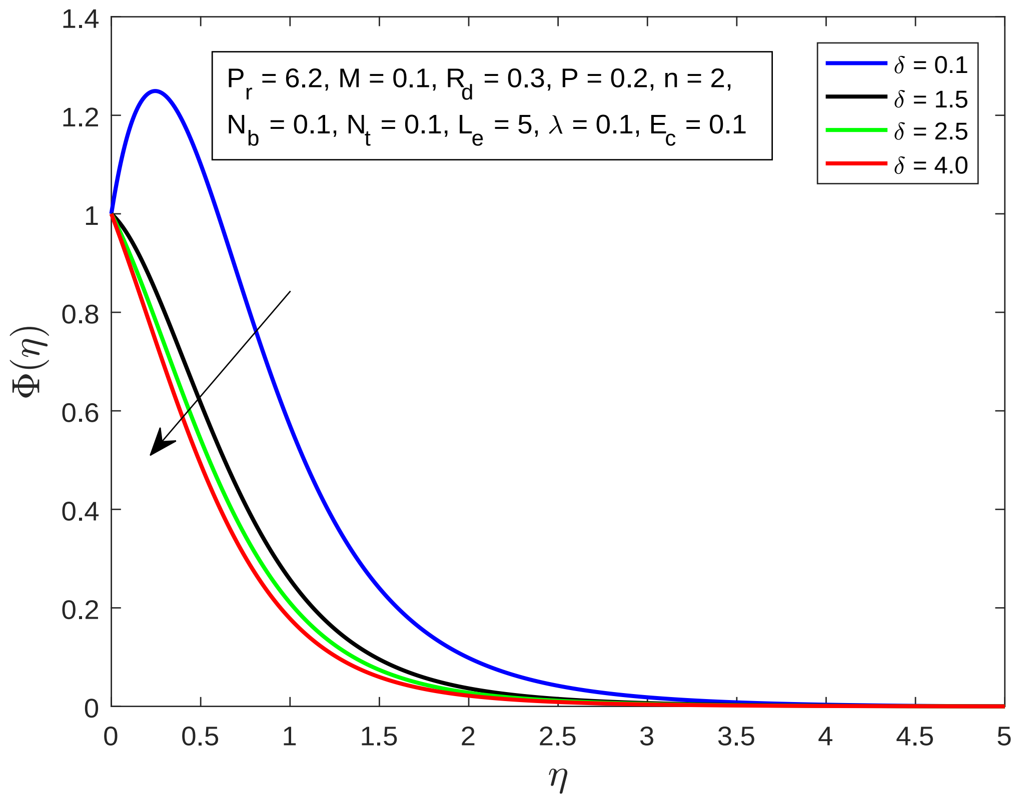

FIGURE 4.15: Effect of δ on $\Phi(\eta)$.

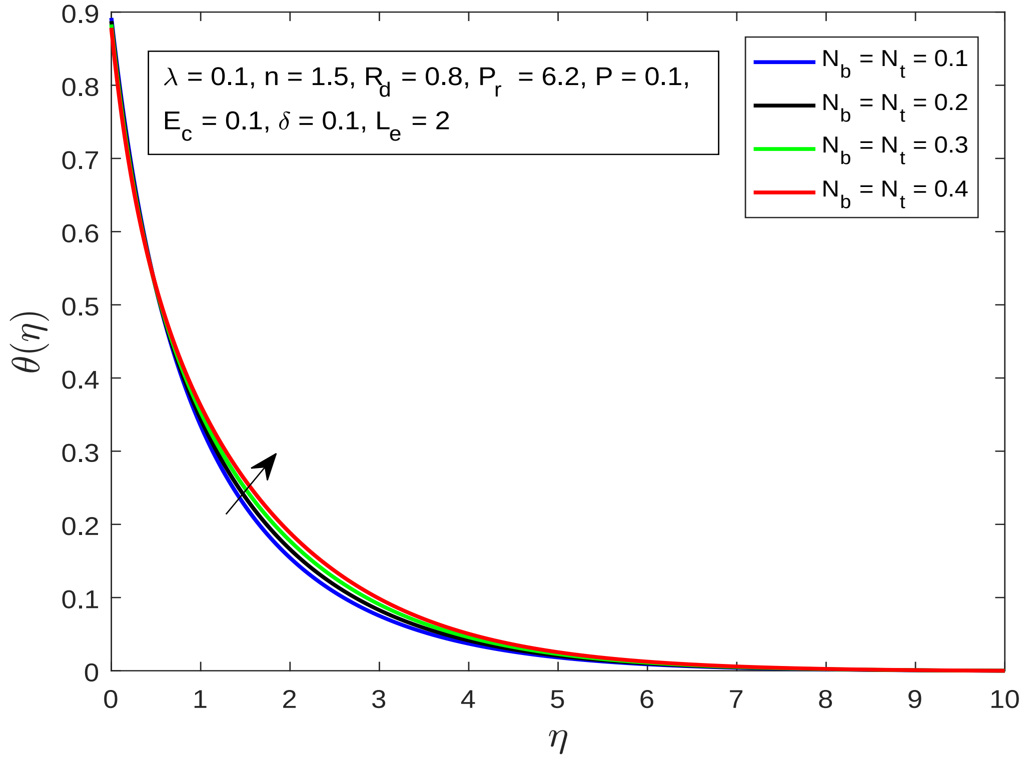

FIGURE 4.16: Effect of N_b and N_t on $\theta(\eta)$.

Chapter 5

Conclusion

This thesis reviewed the work of Ramya et al. [8] and extended the impact of velocity and heat radiation flux on MHD nanofluid flow over a non-linear stretching sheet through a porous medium in the presence of magnetic field. At first, the non-linear governing equations are converted into ODEs by using suitable transformations. Numerical results are obtained from transformed ordinary differential equations by using Shooting technique with RK4. Results are shown in the form of graphs and tables for different values of governing physical parameters i.e., Porosity factor P, heat radiative flux R_d, Prandtl number P_r, Eckert number E_c, Brownian motion parameter N_b, Thermophoresis parameter N_t, magnetic parameter M and Lewis number L_e on velocity, temperature and concentration profiles. It is concluded from the present work that:

1. Enhanced value of Brownian motion parameter N_b causes an increment in temperature profile θ. On the other hand, N_b causes reduction in concentration profile Φ.

2. By increasing the value Eckert number E_c, temperature profile θ increases.

3. Velocity slip parameter λ and velocity profile f' are in inverse relationship to each other.

4. Enhanced values of velocity slip parameter λ cause an increment in the temperature profile.

5. It is noticed that velocity slip parameter λ has a direct impact on concentration profile Φ.

6. By increasing the value of Lewis number L_e, concentration profile decreases.

7. There is direct relationship of thermophoresis parameter N_t with temperature profile θ and concentration profile Φ. It is because temperature gradient generates thermophoretic N_t so the fluid gets heatup and flows from the stretching sheet.

8. The velocity profile f' decreases while increasing the porosity factor P.

9. The values of porosity factor and temperature profile are directly related.

10. There is also direct relationship between porosity P and concentration profile.

11. By increasing the values of heat radiative flux R_d, temperature profile rises.

12. Concentration profile Φ decreases while increasing the Lewis number L_e .

13. Enhanced Prandtl number P_r causes reduction in temperature profile.

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