CAPITAL UNIVERSITY OF SCIENCE AND TECHNOLOGY, ISLAMABAD

Common Fixed Point Theorems for $\mathcal{T}_{gF}$-Contractions in Extended b-metric-like Spaces

by

Nisar Ahmed Kiani

A thesis submitted in partial fulfillment for the degree of Master of Philosophy

in the

Faculty of Computing
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I dedicate my dissertation work to

my beloved ***family*** *specially*

my parents*,*

my wife*,*

and

my loving sisters*.*

A special feeling of gratitude is for

my brothers*,*

May God bless them with happiness,

and marvelous health.

CERTIFICATE OF APPROVAL

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Abstract

Motivated by the idea of b-metric-like spaces and extended b-metric spaces, the idea of extended b-metric-like spaces is introduced in this dissertation. The idea of $\mathcal{T}_{gF}$-contraction is introduced by Yu et al. recently. Authors presented some common fixed point results on such mappings. Meanwhile extended b-metric spaces are introduced by Kamran et al. with certain fixed point results. Combining the both ideas a theorem on common fixed point is proved on extended b-metric-like space. These results generalize many already existing results. An example is also provided to validate the result.

Contents

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Abbreviations

BCP	Banach Contraction Principle
$\mathcal{T}_{gF}$**-Contraction**	$(T, g)_F - Contraction$

Symbols

d	Distance function
$\lim$	Limit
$\liminf$	Limit infimum
$\mathbb{N}$	Set of natural numbers
$\mathbb{R}$	Set of real numbers
(X, d)	Metric space
(X, d_b)	b-metric-like space
(X, d_β)	b-metric space
(X, d_θ)	Extended-b-metric space
(X, ρ)	Partial metric space
(X, σ)	Metric-like space
$(X, d_{b\mu})$	Extended b-metric-like space
$\mathbb{Z}$	Set of integers
$\in$	Belongs to
$\Rightarrow$	Implies that
$\forall$	For all
∞	Infinity

Chapter 1

Introduction

1.1 Background

Mathematics has an important role in scientific knowledge that is why it is called mother of all the other sciences. Mathematics has a lot of applications for humans in every field of life. Mathematics is divided into many branches and each branch has its own significance. One of the important branches of mathematics is known as functional analysis. In the early decades of twentieth century, functional analysis is originated from classical analysis. Mainly, vector space and different operators are focused in functional analysis. It is also related to topology, abstract linear algebra and modern geometry. It is originated from approximation theory, calculus of variations, ordinary and partial linear differential equations and linear integral equations has great impact on the development of modern ideas. At its earliest stage, it was used to solve differential equations and has many wide applications for non-linear problems. Recently, functional analytic methods are very useful in different areas of mathematics.

In functional analysis, fixed point theory is a valuable and dominant theory. Fixed point theory provides sufficient conditions for the existence of solution of different problems. The concept of fixed point theory has a lot of applications in different fields of science, such as in the area of numerical analysis, polynomial interpolation, error estimation, optimization theory, mathematical economics, variational inequalities, approximation theory and finite difference methods.

Poincare [1] was the first mathematician who studied the field of fixed point theory in 1886 and substantiate various fixed point results. Later on Brouwer [2] considered the equation $T(\eta) = \eta$ and established the solution of this equation by proving a fixed point theorem in 1910. He also worked to prove fixed point results for the shapes like square and a sphere. In 1922, a notable mathematician Stephan Banach [3] demonstrated a significant fixed point result in the field of functional analysis acknowledged as Banach contraction principle. This result is declared to be the most fundamental in the field of fixed point theory. The two remarkable applications come from this principle. The first one is that it guarantees the existence and uniqueness of fixed point of a contraction mapping. The second and the very emotive one is that it developed an approach to determine the fixed point of a contractive mapping. This principle occupies a significant part in the field of functional analysis. Afterwards, Banach contraction principle has been extending in various directions. Different mathematicians used different approaches to extend this principle, by either replacing the contraction condition or taking the different spaces [4–7].

Nadler [8] also extended the Banach contraction principle from single valued to multivalued contraction mappings. On the other hand few authors used different spaces like pseudo metric space [9], metric like space [10], partially ordered space [11]. The b-metric space is one of the interesting generalization of the metric space which was initiated by Bakhtin [12] and Czerwik [13]. They established the idea of b-metric space and then used the same idea to set up some fixed point theorems for generalizing the Banach contraction principle.

Huang [14] introduce cone metric spaces and prove some fixed point theorems of contractive mappings on cone metric spaces. Many fixed point theorems are generalized on cone metric spaces [15–17]. An interesting generalization of metric space is established by Ma et al. [18] known as C^* valued metric space. Many researcher extended a number of fixed point results in this metric [19–21].

In 2013, on the basis of the concepts of b-metric space, partial mretric space and metric like space, Alghamdi et al. [22] introduced b-metric-like spaces. By providing some supportive results, authors proved fixed point results in expansive

mappings. They also worked on the b-metric like spaces which are partially ordered and proved fixed point theorems. In 2014, Zhu et al. [23] introduced the notion of qausi b-metric-like spaces. He also gives the criteria for the convergence and completeness, and proved some results showing fixed points in qausi b-metric-like space. While in 2015, Chen et al. [24] also worked on b-metric like space and he generalizes many related results.
In 2018 Yu et al. [25] introduce a new concept of $\mathcal{T}_{gF}$-contraction in b-metric-like spaces and investigate some fixed point theorems about such contraction. Concurrently Kamran et al. [26] introduce the concept of extended b-metric space and establish some fixed point theorems for self-mappings defined on such spaces. Many researcher worked on this new notation and extended already existing results in literature [27–29]. In this dissertation, the main focus is to work on b-metric like space its examples, completeness, convergence and common fixed point theorems for $\mathcal{T}_{gF}$-contractions in b-metric-like spaces. The detailed review of article "Common fixed point theorems for $\mathcal{T}_{gF}$-contractions in b-metric-like spaces" Yu et al. [25] is presented. By using concept of b-metric like spaces, extended b-metric spaces and $\mathcal{T}_{gF}$-contractions in b-metric-like spaces the definitions of extended b-metric like spaces have been introduced in this thesis. The concept of $\mathcal{T}_{gF}$-contractions in extended b-metric-like spaces is established. A result regarding common fixed point theorems about such contractions in extended b-metric-like spaces is provided with an example. This result generalize the result of Yu et al. [25].

1.2 Thesis Layout

Following are the details of work, which have been done this thesis.

1. **Chapter 2**:
   This chapter consists of brief literature review of metric fixed point theory. Focus is on basic notations, definitions and results regarding metric spaces. This chapter includes seven sections. First section contains the definitions and examples of metric spaces. Second sections include some mappings on metric space. Section third to sixth include the definitions and examples of

different spaces. The last section contain Banach contraction principle and its generalizations on different spaces.

2. **Chapter 3**:

   This chapter contains the detailed review of article "Common fixed point theorems for $\mathcal{T}_{gF}$-contractions in b-metric-like spaces" by Yu et al. [25].

3. **Chapter 4**:

   In this chapter motivated by the idea of extended b-metric spaces the definition of extended b-metric like space was introduced. $\mathcal{T}_{gF}$-contraction in extended b-metric-like space is also introduced. Some fixed point theorems for $\mathcal{T}_{gF}$-contraction on extended b-metric-like space is presented which generalize many already existing results.

4. **Chapter 5**:

   The conclusion is given in this chapter.

Chapter 2

Preliminaries

In this chapter we will recall some initiatory definitions and examples from the evaluation of extended b-metric-like spaces. The main intent of this chapter is to present the elementary results, definitions and examples that will be used in the subsequent chapters.

2.1 Metric Space

Functional analysis is an important branch of mathematical analysis which is originated from classical analysis. Its development started about more than a century but now a days functional analytic methods are used in various fields of applied mathematics and other sciences.

In abstract approach one usually start from a set of elements satisfying certain axioms. The theory then construct of logical consequences which results from axioms and derived as therefore once or for all. The idea of using abstract spaces in a systematic manners goes back to M. Frechet (1906) and is justified by its enormous use in different fields. In this chapter we consider metric spaces, which are fundamental in functional analysis because they have a similar role to real line $\mathbb{R}$ in calculus . In fact it generalizes $\mathbb{R}$ and provide a basis for uniform treatment of important problems in various branches of analysis.

The concept of metric space and related ideas are discussed in the upcoming section along with suitable examples.

Definition 2.1.1. Metric Space

"A metric space is a pair (X, d), where X is a non-empty set and d is a metric on X, i.e., a function defined on $X \times X$ such that for all $x, y, z \in X$ we have:

$(M1)$ d is real-valued, finite and nonnegative;

$(M2)$ $d(x, y) = 0$ if and only if $x = y$;

$(M3)$ $d(x, y) = d(y, x)$;

$(M4)$ $d(x, y) \leq d(x, z) + d(z, y)$;

The pair (X, d) is called metric space on X."[30]

Example 2.1.1.

Let $X = C[a, b]$ be set of all real valued continuous function on interval $[a, b]$. A mapping $d : X \times X \rightarrow \mathbb{R}$ given by

$$d(\alpha, \beta) = \max_{t \in [a,b]} \quad |\alpha(t), \beta(t)| \qquad \forall \;\; \alpha, \beta \;\in X,$$

then d is metric on X.

Example 2.1.2.

Let X be set of all bounded sequences of complex numbers; i.e every element of X is a complex sequence

$$x = (\alpha_1, \alpha_2, \alpha_3, ...) \qquad \text{briefly} \quad x = (\alpha_j)$$

such that for all $j = 1, 2, 3....$ we have $|\alpha_j| \leq c_x$ where c_x is a real number. we define a metric $d : X \times X \rightarrow \mathbb{R}$ by

$$d(x, y) = \sup_{j \in \mathbb{N}} \quad |\alpha_j - \beta_j|$$

where $y = (\beta_j) \in X$ and sup denotes the supremum (the least upper bound) with $\mathbb{N} = \{1, 2, 3, ...\}$. The metric space thus obtained is generally denoted by l^∞.

Example 2.1.3.

Consider a set Y which is non empty, it can be made a metric space (Y, d_0), where $d_0 : Y \times Y \rightarrow \mathbb{R}$ is a function

$$d_0(\eta_1, \eta_2) = \begin{cases} 0 & \eta_1 = \eta_2 \\ 1 & \eta_1 \neq \eta_2 \end{cases}$$

(M1), (M2)and (M3) are satisfied obviously.

(M4.) In order to prove the rectangular inequality

i. If $\eta_1 = \eta_2 = \eta_3$, then $d_0(\eta_1, \eta_2) = 0, d_0(\eta_1, \eta_3) = 0, d_0(\eta_2, \eta_3) = 0$

$$\Rightarrow d_0(\eta_1, \eta_3) = d_0(\eta_1, \eta_2) + d_0(\eta_2, \eta_3). \tag{2.1}$$

ii. If $\eta_1 \neq \eta_2 = \eta_3$, then $d_0(\eta_1, \eta_2) = 1, d_0(\eta_1, \eta_3) = 1, d_0(\eta_2, \eta_3) = 0$

$$\Rightarrow d_0(\eta_1, \eta_3) = d_0(\eta_1, \eta_2) + d_0(\eta_2, \eta_3). \tag{2.2}$$

iii. If $\eta_1 = \eta_2 \neq \eta_3$, then $d_0(\eta_1, \eta_2) = 0, d_0(\eta_1, \eta_3) = 1, d_0(\eta_2, \eta_3) = 1$

$$\Rightarrow d_0(\eta_1, \eta_3) = d_0(\eta_1, \eta_2) + d_0(\eta_2, \eta_3). \tag{2.3}$$

iv. If $\eta_1 \neq \eta_2 \neq \eta_3$, then $d_0(\eta_1, \eta_2) = 1, d_0(\eta_1, \eta_3) = 1, d_0(\eta_2, \eta_3) = 1$

$$\Rightarrow d_0(\eta_1, \eta_3) \leq d_0(\eta_1, \eta_2) + d_0(\eta_2, \eta_3). \tag{2.4}$$

From Equations (2.1), (2.2), (2.3) and (2.4), we conclude that

$$\Rightarrow d_0(\eta_1, \eta_3) \leq d_0(\eta_1, \eta_2) + d_0(\eta_2, \eta_3). \quad \forall \quad \eta_1, \eta_2, \eta_3 \in Y.$$

Hence d_0 is a metric on Y. It is called discrete metric and has special properties.

In fact, for each positive integer m, $d_m : Y \times Y \to \mathbb{R}$ is a function

$$d_m(\eta_1, \eta_2) = \begin{cases} 0 & \eta_1 = \eta_2 \\ m & \eta_1 \neq \eta_2, m \in \mathbb{Z}^+ \end{cases}$$

is a *discrete metric* on Y. Also we can call it *generalized discrete metric.*

Definition 2.1.2. Open and Closed Ball

"let (X, d) be a metric space the set

$$S(x_\circ, r) = \{x \in X : d(x_\circ, x) < r\}, \text{ where } r > 0$$

is called an open ball of radius r and centre $x_\circ$.

The set

$$S(x_\circ, r) = \{x \in X : d(x_\circ, x) \leq r\}, \text{ where } r > 0$$

is called closed ball of radius r and centre $x_\circ$."[31]

Definition 2.1.3. Open and Closed Set

"A subset M of a metric space X is set to be open if it contains a ball about each of its point. A subset K of x is said to be closed if its complement is open, that is, $K^C = X - K$ is open."[30]

Example 2.1.4.

The closed interval [1,2] of real numbers $\mathbb{R}$ is a closed set.

Example 2.1.5.

Consider (X, d) be a metric space the set

$$S(x_\circ, r') = \{x \in X : d(x_\circ, x) < r'\}, \quad \text{where} \quad r' > 0$$

is an open set.

$$S(x_\circ, r_\prime) = \{x \in X : d(x_\circ, x) \leq r'\}, \quad \text{where} \quad r' > 0$$

is a closed set.

Example 2.1.6.

Let (X, d) be a metric space then each singleton set $\{u\}$ is a closed subset of M. Hence every finite set is closed.

Example 2.1.7.

Consider $X = \mathbb{R}^2$, define $d : X \times X \to \mathbb{R}$ by

$$d(x, y) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

then the set

$$S' = \{(x, y) \in X : x^2 + y^2 < 1\}$$

is and open set.

Definition 2.1.4. Neighbourhood

"Suppose that (X, d) is a metric space. We call a set U a neighbourhood of $x \in X$ if there exists an open set $V \subseteq U$ with $x \in V$."[32]

Definition 2.1.5. Convergence of a Sequence

"Suppose (x_n), $n \in \mathbb{N}$ is a sequence in a metric space (X, d). We say x_0 is a limit of (x_n) if for every neighbourhood U of x_0 there exists $n_0 \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq n_0$. We write

$$x_0 = \lim_{n\to\infty} x_n \text{ or } x_n \to x \text{ as } n \to \infty.$$

If the sequence has a limit we say it is convergent, otherwise we say it is divergent sequence."[32]

Definition 2.1.6. Cauchy Sequence

"A sequence (x_n) in a metric space (X, d) is said to be Cauchy if for every $\epsilon > 0$ there is an $N = N(\epsilon)$ such that

$$d(x_m, x_n) < \epsilon, \qquad \text{for every } m, n > N.$$ "[30]

Definition 2.1.7. Completeness

"The space (X, d) is said to be complete if every Cauchy sequence in X converges in X."[30]

Example 2.1.8.

Consider $X = \mathbb{R}$, and consider a sequence $\{\alpha_n\} = \dfrac{1}{n}$, with

$$d(\alpha, \beta) = |\alpha - \beta|,$$

then $\{\alpha_n\}$ is convergent and $\lim_{n\to\infty} d(\alpha_n, 0) = 0$. Now consider $\epsilon > 0$, choose $N > \dfrac{2}{\epsilon}$, then for any $n, m > N$,

$$\begin{aligned}\left|\frac{1}{n} - \frac{1}{m}\right| &\leq \frac{1}{n} + \frac{1}{m} \\ &< \frac{1}{N} + \frac{1}{N} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon.\end{aligned}$$

Therefore $\{\alpha_n\}$ is a Cauchy sequence.

Example 2.1.9.

Consider $X = \mathbb{R}$, and consider a sequence $\{\alpha_n\} = \frac{n^2-1}{n^2}$, with

$$d(\alpha, \beta) = |\alpha - \beta|.$$

Consider $\epsilon > 0$, choose $N > \sqrt{\frac{2}{\epsilon}}$, then for any $n, m > N$,

$$\begin{aligned}\left|\frac{n^2-1}{n^2} - \frac{m^2-1}{m^2}\right| &\leq \frac{1}{n^2} + \frac{1}{m^2} \\ &< \frac{1}{N^2} + \frac{1}{N^2} \\ &= \frac{2}{N^2} \\ &< \epsilon.\end{aligned}$$

Therefore $\{\alpha_n\}$ is a Cauchy sequence.

Example 2.1.10.

(i) The closed interval $[0, 1]$ in $\mathbb{R}$ is a complete metric space with usual metric on $\mathbb{R}$.

(ii) Every finite dimensional metric space is complete.

(iii) Closed subspace of a complete space is complete.

Remark 2.1.1.

Every convergent sequence is Cauchy sequence but converse is not true.

Definition 2.1.8. Topology

"A topology on a set X in a family $\mathbb{F}$ of subset of X which satisfies the following axioms:

(1) ϕ and X are in $\mathbb{F}$.

(2) The union of any sub collection of $\mathbb{F}$ is a member of $\mathbb{F}$.

(3) The intersection of any finite sub collection of $\mathbb{F}$ is a member of $\mathbb{F}$.

Together the pair $(X, \mathbb{F})$ is called a topological space."[33]

Example 2.1.11.

let $X = \{0, 1\}$ then if we let $\mathbb{F} = \{\phi, \{0\}, \{1\}, X, \}$ then $(X, \mathbb{F})$ is a topological space. This is true because (1) can be verified by inspection, (2) and (3) required that certain subset of X are elements of $\mathbb{F}$, but if we can choose $\mathbb{F}$ to be all subset of $\mathbb{F}$, which make (2) and (3) hold.

Definition 2.1.9. The Metric Topology

"The metric topology on a metric space M is the topology obtained by taking as open sets the collection of all sets $\mathbb{F}$ in M which have the property $S \in \mathbb{F}$ provided each point $x \in S$ is the center of some open ball $U(x, r)$ (for $r > 0$), which also lies in S."[33]

Example 2.1.12.

Consider $X = \mathbb{R}$ with metric $d = |x - y|$, we can generate collection of open sets as

$$\tau = \{U \subseteq \mathbb{R} : \forall\ x \in U, \exists\ (x - \epsilon, x + \epsilon) \subset U\},$$

then τ satisfy all the conditions of topology. so, τ is called metric topology.

2.2 Some Mapping on Metric Space

This section addresses some important mappings on metric spaces. These mappings play a fundamental role in the field of fixed point theory.

Definition 2.2.1. Continuous Mapping

" Let (X, d_X) and (Y, d_Y) be metric spaces and $A \subseteq X$. A function $f : A \to Y$ is said to be continuous at $a \in A$, if for every $\epsilon > 0$, there exists some $\delta > 0$ such that

$$d_Y(f(x), f(a)) < \epsilon \quad \text{whenever} \quad x \in A \quad \text{and} \quad d_X(x, a) < \delta. \tag{2.5}$$

If f is continuous at every point of A, then it is said to be continuous on A."[31]

Remark 2.2.1.

(i) If one positive number δ satisfies this condition (2.5), then every positive number $\delta_1 < \delta$ also satisfies it. This is obvious because whenever $x \in A$ and $d_x(x, a) < \delta_1$, it is also true that $x \in A$ and $d_x(x, a) < \delta_1$. Therefore, such a number δ is far from being unique.

(ii) If a is a limit point of A and $\{x_n\}$ is a sequence of points of A such that $x_n \to a$, it follows from the continuity of f at a that $f(x_n) \to f(a)$.

Example 2.2.1.

Consider $X = \mathbb{R}$ and a mapping $T : X \to X$ defined on a usual metric space (X, d) as follows:

$$T(x) \;=\; x^5 \quad \text{where} \quad x \;\in\; X,$$

then T is a continuous mapping.

Example 2.2.2.

Consider $X = \mathbb{R}$ and (X, d) be a metric space and $I : X \to X$ be an identity function, then I is continuous on $\mathbb{R}$.

Example 2.2.3.

Consider (U, d_0) be a discrete metric space. Then any map $T : U \to V$ is continuous. For every $\epsilon > 0$ we choose $\delta = 1$. Then

$$B(x, \delta) = B(x, 1) = \{x\}$$

for all $x \in U$ and the condition $T(x) \in B(T(x), \epsilon)$ is obviously satisfied.

Definition 2.2.2. Lipschitzian Mapping

"Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be Lipschitzian if there exist a constant $\alpha \geq 0$ with

$$d\left(T(x), T(y)\right) \leq \alpha d\left(x, y\right) \text{ for all } x, y \in X.$$

Notice that a Lipschitzian map is necessarily continuous. The smallest α for which above inequality holds is said to be Lipschitz constant for T and is denoted by L." [34]

Example 2.2.4.

Consider set of real numbers $\mathbb{R}$ with usual metric $d(l_1, l_2) = |l_1 - l_2|$, $\forall\, l_1, l_2 \in \mathbb{R}$. A mapping $T : \mathbb{R} \to \mathbb{R}$ define by $T(l) = 2l$, then

$$\begin{aligned} d(T(l_1), T(l_2)) &= d(2l_1, 2l_2) \\ &= |2l_1 - 2l_2| \\ &= 2|l_1 - l_2| \\ &= 2d(l_1, l_2). \end{aligned}$$

So, T is a Lipschitzian mapping and its Lipschitz constant is 2.

Definition 2.2.3. Contraction Mapping

"Let $X = (X, d)$ be a metric space. A mapping $T : X \to X$ is called a contraction on X if there exist a positive real number $\alpha < 1$ such that for all $x, y \in X$

$$d\left(T(x), T(y)\right) \leq \alpha \; d\left(x, y\right)$$

Geometrically this means that any point x and y has images that are closer together than those points x and y; more precisely, the ratio $\dfrac{d\left(T(x), T(y)\right)}{d\left(x, y\right)}$ does not exceed a constant α which is strictly less than 1." [30]

Example 2.2.5.

Consider $X = [0, 1]$ with usual metric. A mapping $T : X \to X$ define by

$$T(x) = \frac{1}{2 + x}.$$

Then

$$\begin{aligned} d\left(T(x), T(y)\right) &= d\left(\frac{1}{2+x}, \frac{1}{2+y}\right) \\ &= \left|\frac{1}{2+x} - \frac{1}{2+y}\right| \end{aligned}$$

$$\begin{aligned}
&= \left| \frac{x-y}{(2+x)(2+y)} \right| \\
&= \frac{1}{|(2+x)(2+y)|} \left| x-y \right| \\
&= \frac{1}{(2+x)(2+y)} d(x,y) \\
&\leq \frac{1}{4} d(x,y)
\end{aligned}$$

$$d\left(T\left(x\right), T\left(y\right)\right) \leq \ \frac{1}{4} d(x,y),$$

hence, T is contraction mapping with contraction constant $\alpha = \frac{1}{4}$.

Example 2.2.6.

Consider (X, d) be a metric space and

$$d(\phi, \psi) = \mid \phi - \psi \mid,$$

then define a mapping $T : X \to X$ by

$$T(\phi) = \frac{\phi}{5} + 3$$

$$\begin{aligned}
d(T\phi, T\psi) &= \left| \left(\frac{\phi}{5} + 3 \right) - \left(\frac{\psi}{5} + 3 \right) \right| \\
&= \left| \frac{\phi}{5} - \frac{\psi}{5} \right| \\
&= \frac{1}{5} \left| \phi - \psi \right|
\end{aligned}$$

$\Rightarrow \alpha = \dfrac{1}{5} < 1,$

then T is a contraction with $\alpha = \dfrac{1}{5} < 1.$

Definition 2.2.4. Contractive Mapping

"Consider (X, d) be a metric space and F be a self map on X then, F is called a contractive mapping if, for all $\alpha, \beta \in X$

$$d(F(\alpha), F(\beta)) < d(\alpha, \beta)$$

where $\alpha \neq \beta$."[34]

Example 2.2.7.

Let $X = [1, \infty)$ with usual metric d. A mapping $T : X \to X$ define by $T(u) = \dfrac{1}{u}$, then

$$\begin{aligned} d(T(u), T(v)) &= d\left(\frac{1}{u}, \frac{1}{v}\right) \\ &= \left|\frac{1}{u} - \frac{1}{v}\right| \\ &= \left|\frac{v-u}{uv}\right| \\ &= \left|\frac{u-v}{uv}\right| \\ &= \left|\frac{1}{uv}\right| |u-v| \\ &< |u-v| \\ &= d(u,v) \end{aligned}$$

$$\Rightarrow d(T(u), T(v)) < d(u,v) \quad \forall \quad u, v \in X,$$

which implies T is contractive mapping.

Example 2.2.8.

Consider $X = \mathbb{R}$ and (X, d) be usual metric space. Let T be a self-mapping on X defined by

$$T(\alpha) = \alpha + \frac{1}{\alpha} \quad , \quad \forall \ \alpha \in X, \tag{2.6}$$

then T is contractive but not a contraction.

Definition 2.2.5. Non-Expansive Mapping

"Let $T : X \to X$ be a mapping on metric space (X, d) into itself. We call T a non-expansive if,

$$d(T(\alpha), T(\beta)) \leq d(\alpha, \beta)$$

for all $\alpha, \beta \in X$."[35]

Example 2.2.9.

Let $X = \mathbb{R}$ with usual metric. A mapping $T : X \to X$ define by $T(w) = w$. Then

$$\begin{aligned} d(T(w_1), T(w_2)) &= d(w_1, w_2) \\ &= |w_1 - w_2| \\ &= d(w_1, w_2) \end{aligned}$$

$$\Rightarrow d(T(w_1), T(w_2)) = d(w_1, w_2) \quad \forall \quad w_1, w_2 \in X,$$

which implies T is non-expansive mapping.

Remark 2.2.2.

Contraction ⇒ Contractive ⇒ Non-expansive ⇒ Lipschitzian.

In past years many generalizations of metric space are introduced and discussed. All these ideas intrigued many mathematicians to generalize various fixed point theorems. Some very important generalizations of metric spaces will appears in upcoming sections.

2.3 b-metric Space

The notion of b-metric space was firstly presented by Bakhtin [12] in 1989. Also in 1993, Czerwik [13] gave its formal definition. Another mathematician Bourbaki [36] also worked on this idea. This section includes the definition and examples of the said space.

Definition 2.3.1. b-metric Space

"Let X be a nonempty set and $d_\beta : X \times X \to [0, \infty)$ be a function satisfying the following conditions:

$d_\beta 1$: $d_\beta(x, y) = 0$ if and only if $x = y$.

$d_\beta 2$: $d_\beta(x, y) = d_\beta(y, x)$ for all $x, y \in X$.

$d_\beta 3$: $d_\beta(x, y) \leq s[d_\beta(x, z) + d_\beta(z, y)]$ for all $x, y, z \in X$, where $s \geq 1$.

The function d_β is called a d_β-metric and the space (X, d_β) is called a d_β-metric space, in short, bMS."[13]

Remark 2.3.1.

(i) For $s = 1$ the above definition reduce to the definition of metric space.

(ii) In general b-metric is not a continuous function.

Example 2.3.1.

Let $Y = [0, 1]$ and $d_\beta : Y \times Y \to [0, \infty)$ be defined by

$$d_\beta(y_1, y_2) = (y_1 - y_2)^2 \text{ for all } y_1, y_2 \in Y$$

then (X, d_β) is a $b-$metric space with $s = 2$.

Example 2.3.2.

Let (X, d) be a metric space. Then for a real number $m > 1$. we define a function $d_\beta : X \times X \to \mathbb{R}^+$ by

$$d_\beta(\alpha, \beta) = (d(\alpha, \beta))^m,$$

this gives d_β as a b-metric space with its coefficient $\kappa = 2^{m-1}$.

For proof we will use the inequality

$$\left(\frac{\alpha + \beta}{2}\right)^m \leq \frac{\alpha^m + \beta^m}{2}$$

$$\frac{(\alpha + \beta)^m}{2^m} \leq \frac{\alpha^m + \beta^m}{2}$$

$$(\alpha + \beta)^m \leq 2^{m-1}\left(\alpha^m + \beta^m\right).$$

$d_\beta 1,\ d_\beta 2$ are trivially satisfied, to prove $d_\beta 3$ we proceed as:

since for every $\alpha, \beta, \gamma \in X$ we get

$$\begin{aligned} d_\beta(\alpha, \gamma) &= (d(\alpha, \gamma))^m \\ &\leq [d_\beta(\alpha, \beta) + d_\beta(\beta, \gamma)]^m \\ &\leq 2^{m-1}[d(\alpha, \beta)^m + d(\beta, \gamma)^m] \\ &\leq 2^{m-1}[d_\beta(\alpha, \beta) + d_\beta(\beta, \gamma)]. \end{aligned}$$

Hence d_β is a b-metric space with coefficient 2^{m-1}.

Definition 2.3.2. Convergent, Cauchy Sequence and Completeness
"Let (Y, d_β) be a b-metric space. A sequence $\{\alpha_n\}$ in Y is said to be:

(i) Cauchy if and only if

$$\lim_{m,n\to\infty} d_\beta(\alpha_m, \alpha_n) = 0 \ \text{ as } \ m, n \to \infty.$$

(ii) Convergent if and only if there exist $\alpha \in Y$ such that

$$d_\beta(\alpha_n, \alpha) = 0 \ \text{ as } \ n \to \infty,$$

and we write

$$\lim_{n\to\infty} \alpha_n = \alpha.$$

(iii). The b-metric space (Y, d_β) is complete if every Cauchy sequence is convergent in d_β." [26]

2.4 Extended b-metric Space

This section is dedicated to the notion of extended b-metric space. Kamran et al. [26] introduced a new type of generalized b-metric space and termed it as extended b-metric space.

Definition 2.4.1. Extended b-metric Space
"Let X be a nonempty set and $\theta : X \times X \to [1, \infty)$. A function $d_\theta : X \times X \to [0, \infty)$ is called an extended b-metric if, for all $x, y, z \in X$, it satisfies

$(d_\theta 1)$: $d_\theta(x, y) = 0$ iff $x = y$;

$(d_\theta 2)$: $d_\theta(x, y) = d_\theta(y, x)$;

$(d_\theta 3)$: $d_\theta(x, y) \leq \theta(x, y)\ [d_\theta(x, z) + d_\theta(z, y)]$.

The pair (X, d_θ) is called an extended b-metric space, in short extended-bMS." [26]

Remark 2.4.1.

(i) If $\theta(x,y) = s$ for $s \geq 1$, then we obtain the definition of b-metric space.

(ii) Further if $\theta(x,y) = 1$, then we obtain the definition of metric space.

Example 2.4.1.

Let $Y = [0,1]$. Define $\theta : Y \times Y \to [1,\infty)$ as,

$$\theta(y_1,y_2) = \frac{y_1+y_2+1}{y_1+y_2}.$$

Also introduce $d_\theta : Y \times Y \to [0,\infty)$ as

$$d_\theta(y_1,y_2) = \begin{cases} \dfrac{1}{y_1 y_2} & \forall \quad y_1,y_2 \in (0,1], \quad y_1 \neq y_2, \\ 0 & \forall \quad y_1,y_2 \in [0,1], \quad y_1 = y_2 \end{cases}$$

with

$$d_\theta(y_1,0) = d_\theta(0,y_1) = \frac{1}{y_1} \ \ \forall \ \ y_1 \in (0,1],$$

then (Y,d_θ) is an extended b-metric space.

Definition 2.4.2. Convergent, Cauchy Sequence and Completeness

"Let (X,d_θ) be an extended b-metric space.

(i) A sequence $\{x_n\}$ in X is said to converge to $x \in X$, if for every $\epsilon > 0$ there exists $N = N(\epsilon) \in N$ such that

$$d_\theta(x_n,x) < \epsilon$$

for all $n \geq N$. In this case, we write $\lim_{n\to\infty} x_n = x$.

(ii) A sequence $\{x_n\}$ in X is said to be Cauchy, if for every $\epsilon > 0$ there exists $N = N(\epsilon) \in N$ such that

$$d_\theta(x_m,x_n) < \epsilon$$

for all $m,n \geq N$.

An extended b-metric space (X,d_θ) is complete if every Cauchy sequence in X is convergent. Note that, in general a b-metric is not a continuous functional and thus so is an extended b-metric."[26]

2.5 Partial Metric Space

This section provides another generalization of metric space known as Partial metric space. In 1980, the idea of Partial metric space is presented by Steve Matthews [37]. Matthews was working in the field of computer science. For his studies, he had to encounter the self distances which are non-zero. Matthews gave a new idea of metric space in which the self distances are non-zero. His work was first published in 1994. This section includes the definition and examples of partial metric space.

Definition 2.5.1. Partial Metric Space

"A partial metric on a set X is a function $\rho : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$:

(ρ_1) $x = y \;\Leftrightarrow\; \rho(x,x) = \rho(x,y) = \rho(y,y)$;

(ρ_2) $\rho(x,x) \leq \rho(x,y)$;

(ρ_3) $\rho(x,y) = \rho(y,x)$;

(ρ_4) $\rho(x,z) \leq \rho(x,y) + \rho(y,z) - \rho(y,y)$.

The space (X, ρ) is a partial metric space."[38]

Example 2.5.1.

Consider $X = \mathbb{R}^+$ define $\rho : X \times X \to \mathbb{R}^+$ by

$$\rho(\alpha, \beta) = \max(\alpha, \beta) \;\forall\; \alpha, \beta \in \mathbb{R}^+,$$

then $(\mathbb{R}^+, \rho)$ is a partial metric space.

Example 2.5.2.

Let Y denotes the set of all intervals $[\alpha, \beta]$, for any real numbers $\alpha \leq \beta$. Let $\rho : Y \times Y \to [0, \infty)$ be function such that

$$\rho\;\left([\alpha_1, \beta_1], [\alpha_2, \beta_2]\right) = \max(\beta_1, \beta_2) - \min(\alpha_1, \alpha_2),$$

then (Y, ρ) is partial metric space.

Definition 2.5.2. Convergent, Cauchy Sequence and Completeness

"Let (Y, ρ) be a partial metric space.

(i). A sequence $\{\alpha_n\}$ in a partial metric space (Y, ρ) converges to a point $\alpha \in Y$ if and only if

$$\rho(\alpha, \alpha) = \lim_{n\to\infty} \rho(\alpha, \alpha_n).$$

(ii). A sequence $\{\alpha_n\}$ in a partial metric space (Y, ρ) is called a Cauchy sequence

$$\lim_{m,n\to\infty} \rho(\alpha_m, \alpha_n),$$

exist.

(iii). A partial metric space (Y, ρ) is called complete if and only if every Cauchy sequence $\{\alpha_n\}$ in Y converges to a point $\alpha \in Y$ such that

$$\rho(\alpha, \alpha) = \lim_{m,n\to\infty} \rho(\alpha_m, \alpha_n)."[39]$$

2.6 Metric Like Space

The generalized form of partial metric space is metric like space. In 2012, the idea of metric like space was presented by Amini-Harandi [10]. This section includes the definitions and examples of metric like space.

Definition 2.6.1. Metric Like Space

"A mapping $\sigma : X \times X \to \mathbb{R}^+$, where X is a nonempty set, is said to be metric-like on X if for any $x, y, z \in X$, the following three conditions hold true:

(σ_1) $\sigma(x, y) = 0 \Rightarrow x = y$;

(σ_2) $\sigma(x, y) = \sigma(y, x)$;

(σ_3) $\sigma(x, z) \leq \sigma(x, y) + \sigma(y, z)$.

The pair (X, σ) is called a metric-like space. A metric-like on X satisfies all of the conditions of a metric except that $\sigma(x, x)$ may be positive for $x \in X$."[10]

Remark 2.6.1.

Every partial metric space is a metric-like space but not conversely in general.

Example 2.6.1.

Let $Y = \{0, 1\}$, and let

$$\sigma(\alpha,\beta)=\begin{cases}2, & \text{if } \alpha=\beta=0,\\ 1, & \text{otherwise,}\end{cases}$$

then (Y,σ) is a metric-like space, but since $\sigma(0,0)\nleq\sigma(0,1)$, hence (Y,σ) is not a partial metric space.

Example 2.6.2.

Consider the set $X=[0,\infty)$, and $\sigma : X\times X\to\mathbb{R}$ by

$$\sigma(a_1,a_2)=\max\{a_1,a_2\},$$

we claim that σ is a metric-like space as:

(σ_1)

$$\sigma(a_1,a_2)=\max\{a_1,a_2\}=0$$

$$\Rightarrow a_1=a_2=0$$

If the maximum is 0, then the other values of this function should must be less than 0, which is not possible due to the given domain $X=[0,\infty)$. So, the other values will also be 0.

(σ_2)

$$\sigma(a_1,a_2)=\max\{a_1,a_2\}=\max\{a_2,a_1\}=\sigma(a_2,a_1)$$

(σ_3)

$$\begin{aligned}\sigma(a_1,a_3)&=\max\{a_1,a_3\}\\&\leq\max\{a_1,a_2,a_3\}\\&\leq\max\{a_1,a_2\}+\max\{a_2,a_3\}\end{aligned}$$

So,

$$\sigma(a_1,a_3)\leq\sigma(a_1,a_2)+\sigma(a_2,a_3)$$

Definition 2.6.2. Convergent, Cauchy Sequence and Completeness

"A sequence $\{\alpha_n\}$ in a metric-like space (Y,σ) converges to a point $\alpha\in Y$ if and only if

$$\lim_{n\to+\infty} \sigma(\alpha_n, \alpha) = \sigma(\alpha, \alpha).$$

A sequence $\{\alpha_n\}$ of elements of Y is called σ-Cauchy if the limit $\lim_{m,n\to+\infty} \sigma(\alpha_m, \alpha_n)$ exists and is finite.

The metric-like space (Y, σ) is called complete if for each σ-Cauchy sequence $\{\alpha_n\}$, there is some $\alpha \in Y$ such that

$$\lim_{n\to+\infty} \sigma(\alpha_n, \alpha) = \sigma(\alpha, \alpha) = \lim_{m,n\to+\infty} \sigma(\alpha_m, \alpha_n).\text{” [10]}$$

2.7 Banach Contraction Principle (BCP) and its Generalizations

Stefan Banach proved Banach contraction principle (BCP) in 1922. BCP is known to be one of the fundamental outcomes of fixed point theory. The Banach contraction principle (BCP) provides us with a unique fixed point. Fixed point is a useful tool in mathematics which can be used to prove the existence of solution of a differential equation, integral equation and eigenvalue equation. Fixed point theorems play an important role in both pure and applied mathematics. Present section is providing the definition and examples of fixed point and some classical fixed point results.

Definition 2.7.1. Fixed Point

“A fixed point of a mapping $T : X \to X$ of a set X into itself is an $x \in X$ which is mapped onto itself, that is,

$$Tx = x,$$

the image coincides with x.

Geometrically for a real valued function the fixed point of a mapping $y = f(x)$ are the points of intersection of graph of $y = f(x)$ and line $y = x$. For example the following graph shows the points of intersection of $y = x^3$ and $y = x$.

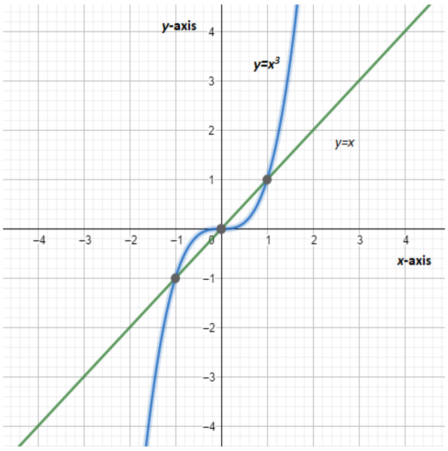

FIGURE 2.1: Three fixed points

The above graph represents a function having three fixed points."[30]

Example 2.7.1.

Consider $S = \mathbb{R}$ and $S : \mathbb{R} \to \mathbb{R}$ be a mapping defined as

$$S(s) = \frac{s}{4} + 3.$$

S has a unique fixed point $s = 4$.

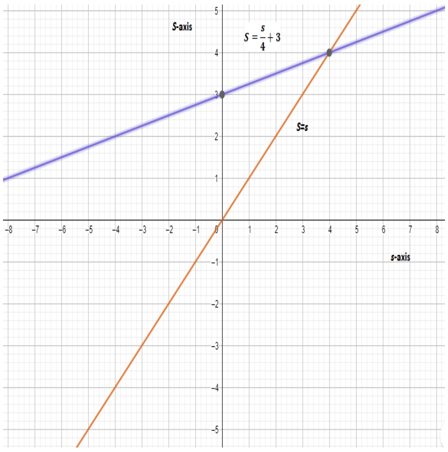

FIGURE 2.2: One fixed point

Example 2.7.2.

Consider $S = \mathbb{R}$ and $S : \mathbb{R} \to \mathbb{R}$ be a mapping defined as

$$S(s) = s + 3,$$

S has no fixed point.

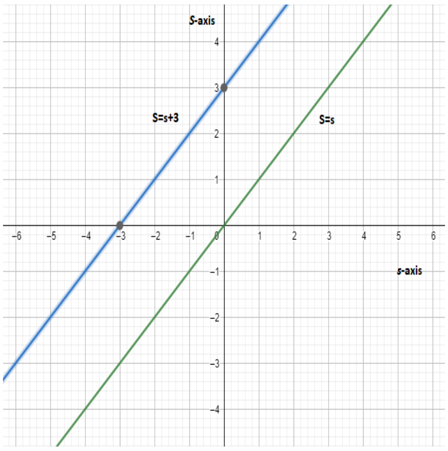

FIGURE 2.3: No fixed point

Definition 2.7.2 (Weakly Compatible, Coincidence, Point of Coincidence and Common Fixed Point).
"Let Y be a nonempty set, g and h be self-mappings on y and

$$B(g,h) = \{y \in Y : g(y) = h(y)\}.$$

The pair g and h are called weakly compatible if

$$g(hy) = h(gy), \text{ for all } y \in B(g,h).$$

Furthermore $u = g(y) = h(y)$ for some $y \in Y$, then y is called coincidence of g and h, and u is called point of coincidence of g and h. If $y = u$, then u is called the common fixed point of g and h." [40]

Example 2.7.3.
Let $Y = \mathbb{R}$ define $g, h : Y \to Y$ by

$$g(y) = 2y + 1 \qquad , \qquad h(y) = 3y + 1$$

for $y = 0 \in Y \Rightarrow g(y) = h(y) = 1 = u \Rightarrow y = 0$ is coincidence of g and h, and $u = 1$ is point of coincidence of g and h.

Example 2.7.4.

Let $Y = [0,1]$ define $g, h : Y \to Y$ by

$$g(y) = \frac{y^2}{16} \qquad , \qquad h(y) = \frac{y}{4}$$

for $y = 0 \in Y \Rightarrow g(y) = h(y) = 0 = u$, also $g(hy) = h(gy) = 0$

g, h are weakly compatible, and $u = 0$ is point of coincidence of g and h.

As $y = u = 0 \Rightarrow 0$ is common fixed point of g and h.

Theorem 2.7.3 Banach Contraction Principle

"Consider a metric space $X = (X, d)$, where $X \neq \emptyset$. Suppose that X is a complete and let $T : X \to X$ be a contraction on X. Then T has precisely one fixed point."[30]

Example 2.7.5.

Consider the metric space $(\mathbb{R}, d)$ where d is the usual metric, define as

$$d(\alpha, \beta) = |\alpha - \beta|.$$

The function $f : \mathbb{R} \to \mathbb{R}$ is define as

$$f(\alpha) = \frac{\alpha}{a} + b$$

is a contraction if $a > 1$. In this specific case we can find a fixed point, since a fixed point means that $f(\alpha) = \alpha$, we want $\alpha = \dfrac{\alpha}{a} + b$. Solving for α gives us $\alpha = \dfrac{ab}{a-1}$

Example 2.7.6.

Consider a mapping $g : \left(0, \dfrac{1}{4}\right) \to \left(0, \dfrac{1}{4}\right)$ where $g(\alpha) = \alpha^2$ is a contraction with respect to the usual metric and has no fixed point. Infact,

$$g(\alpha) = \alpha \Rightarrow \alpha^2 = \alpha \Rightarrow \alpha^2 - \alpha = 0,$$

$$\Rightarrow \alpha(\alpha - 1) = 0 \Rightarrow \alpha = 0, 1.$$

But both $0, 1 \notin \left(0, \frac{1}{4}\right)$.

Definition 2.7.4. Compact Metric Space

"A metric space X is said to be compact if every sequence in X has a convergent subsequence. A subset M of X is said to be compact if M is compact subspace of X, that is, if every sequence in M has a convergent subsequence whose limit is an element of M."[30]

Edelstein [41] established the following fixed point result, popularly named as Edelstein theorem.

Theorem 2.7.5

"Let (X, d) be a compact metric space, and let T be a mapping on X. Assume

$$d(Tx, Ty) < d(x, y) \text{ for all } x, y \in X$$

with $x \neq y$. Then T has a unique fixed point."[41]

In 1992 Matthews [37] established the following fixed point result on partial metric space.

Theorem 2.7.6 (The Partial Metric Contraction Mapping Theorem)

"For each complete partial metric $\rho : X^2 \to \mathbb{R}$, and for each function $f : X \to X$ such that

$$\exists\, 0 \leq c < 1 \text{ for all } x, y \in X,$$

$$\rho(f(x), f(y)) \leq c.\rho(x, y)$$

then, there exist a unique $a \in X$, such that

$$a = f(a), \text{ and } \rho(a, a) = 0.\text{"} \ [37]$$

Czerwik [13] established the following fixed point result on b-metric space in 1993.

Theorem 2.7.7 (Extension of BCP on b-metric Space)

"Let (X, d_β) be a complete b-metric space with constant $s \geq 1$ and suppose that $T : X \to X$ satisfies

$$d_\beta(Tx, Ty) \leq \phi(d_\beta(x, y)),$$

for all $x, y \in X$, where $\phi : [0, \infty) \to [0, \infty)$ is increasing and

$$\lim_{n\to\infty} \phi^n(t) = 0,$$

for each $t \geq 0$, then T has a unique fixed point $x^* \in X$ and

$$\lim_{n\to\infty} T^n(x) = x^*,$$

for each $x \in X$." [13]

In 2017 Kamran et. al [26] established the following fixed point result on extended b-metric Space.

Theorem 2.7.8 (Extension of BCP on Extended b-metric Space)
"Let (X, d_θ) be a complete extended b-metric space, such that d_θ is a continuous functional. Let $T : X \to X$ satisfy:

$$d_\theta(Tx, Ty) \leq k d_\theta(x, y) \ \forall\ x, y \in X,$$

where $k \in [0, 1)$ be such that for each $x_0 \in X$,

$$\lim_{n,m\to\infty} \theta(x_n, x_m) < \frac{1}{k},$$

here $x_n = T^n x_0, n = 1, 2, ...$ Then T has precisely one fixed point ξ. Moreover for each $y \in X$, $T^n y \to \xi$." [26]

Harandi [10] established the following fixed point result on b-metric space in 2012.

Theorem 2.7.9 (Extension of BCP on Metric Like Space)
"Let (X, σ) be a complete metric-like space, and let $T : X \to X$ be a map such that

$$\sigma(Tx, Ty) \leq \psi(M(x, y)),$$

for all $x, y \in X$, where

$$M(x, y) = \max\{\sigma(x, y), \sigma(x, Tx), \sigma(y, Ty), \sigma(x, Ty), \sigma(y, Tx), \sigma(x, x), \sigma(y, y)\},$$

where $\psi : [0,\infty) \to [0,\infty)$ is a nondecreasing function satisfying

$$\psi(t) < t \;\forall\; t > 0, \;\; \lim_{s\to t^+} \psi(s) < t, \;\; \forall\; t > 0 \text{ and } \lim_{t\to\infty}(t - \psi(t)) = \infty.$$

Then T has a fixed point."[10]

Example 2.7.7.

Let $Y = \{0, 1, 2\}$. Define $\sigma : Y \times Y \to \mathbb{R}^+$ as follows:

$$\sigma(0,0) = 0, \;\; \sigma(1,1) = 3, \;\; \sigma(2,2) = 1, \;\; \sigma(0,1) = \sigma(1,0) = 7,$$

$$\sigma(0,2) = \sigma(2,0) = 3, \;\; \sigma(1,2) = \sigma(0,2) = 4,$$

then (Y, σ) is a complete metric-like space. Note that σ is not a partial metric on Y because

$$\sigma(0,1) \neq \sigma(0,2) + \sigma(2,1) - \sigma(2,2),$$

define the map $T : Y \to Y$ by

$$T0 = 0, \;\; T1 = 2, \;\text{ and } T2 = 0,$$

then

$$\sigma(T_x, T_y) \leq \frac{3}{4}\sigma(x,y) \leq \frac{3}{4}\mathcal{M}(x,y)$$

for each $x, y \in Y$, hence all the required hypotheses of Theorem 2.7.9 are satisfied. Therefore T has a unique fixed point.

Chapter 3

Common Fixed Point Theorems on b-metric-like Spaces

3.1 Introduction

In this chapter, we present the concept of $\mathcal{T}_{gF}$-contraction and investigate some fixed point theorems for such contraction in b-metric-like spaces. Moreover, an example is given to support one of our results.

3.2 b-metric-like Space

This section is dedicated to the notion of b-metric-like Space.

Definition 3.2.1. ***b*-metric-like Space**

A b-metric-like on a nonempty set X is a function $d_b : X \times X \to [0, \infty)$ such that, for all $\alpha, \beta, \gamma \in X$ and a constant $b \geq 1$, the following three conditions hold true:

(b1): if $d_b(\alpha, \beta) = 0$ then $\alpha = \beta$;

(b2): $d_b(\alpha, \beta) = d_b(\beta, \alpha)$;

(b3): $d_b(\alpha, \gamma) \leq b[d_b(\alpha, \beta) + d_b(\beta, \gamma)]$.

The pair (X, d_b) is then called a b-metric-like space with coefficient b. [22]

Remark 3.2.1.

Each b-metric-like d_b on X generalizes a topology τ_b on X whose base is the family of open b-balls

$$B_b(x,\epsilon) = \{z \in X : |d_b(x,z) - d_b(x,x)| < \epsilon\},$$

for all $x \in X$ and $\epsilon > 0$. [22]

Example 3.2.1.

We take $Y = [0,\infty)$. Consider $d_b : Y^2 \to [0,\infty)$, define a function by

$$d_b(\alpha_1,\alpha_2) = (\alpha_1 + \alpha_2)^2,$$

then it is a b-metric-like space and its constant is 2.

(b1)and (b2) are obvious.

(b3):

$$\begin{aligned} d_b(\alpha_1,\alpha_2) &= (\alpha_1+\alpha_2)^2 \\ &\leq (\alpha_1+\alpha_3+\alpha_3+\alpha_2)^2 \\ &= (\alpha_1+\alpha_3)^2 + (\alpha_3+\alpha_2)^2 + 2(\alpha_1+\alpha_3)(\alpha_3+\alpha_2) \\ &\leq 2[(\alpha_1+\alpha_3)^2 + (\alpha_3+\alpha_2)^2] \\ &= 2[d_b(\alpha_1,\alpha_3) + d_b(\alpha_3,\alpha_2)] \\ \Rightarrow \quad d_b(\alpha_1,\alpha_2) &\leq 2[d_b(\alpha_1,\alpha_3) + d_b(\alpha_3,\alpha_2)]. \end{aligned}$$

Hence, the given function is a b-metric-like space.

Since self distances is non-zero hence d_b is not a b-metric space.

Example 3.2.2.

Let $M = [0,\infty)$. Consider $d_b : M^2 \to [0,\infty)$, define as

$$d_b(\alpha,\beta) = (\max\{\alpha,\beta\})^2,$$

then (M,d_b) is a b-metric-like space and its constant is 2.

(b1)and (b2) are obvious.

(b3):

$$\begin{aligned} d_b(\alpha,\gamma) &= (\max\{\alpha,\gamma\})^2 \\ &\le (\max\{\alpha,\beta,\gamma\})^2 \\ &\le (\max\{\alpha,\beta\} + (\max\{\beta,\gamma\})^2 \\ &\le 2[(\max\{\alpha,\beta\})^2 + (\max\{\beta,\gamma\})^2] \\ \Rightarrow d_b(\alpha,\gamma) &\le 2[d_b(\alpha,\beta) + d_b(\beta,\gamma)]. \end{aligned}$$

Hence, the given function is a b-metric-ike space.

Definition 3.2.2 (Cauchy, Convergence and Completeness).
A sequence $\{y_m\}$ in a b-metric-like space (Y, d_b) converges to a point $y \in Y$ if and only if

$$d_b(y,y) = \lim_{m\longrightarrow\infty} d_b(y, y_m).$$

A sequence $\{y_n\}$ in a b-metric-like space (Y, d_b) is called a Cauchy sequence if

$$\lim_{m,n\longrightarrow\infty} d_b(y_n, y_m)$$

exists.
A b-metric-like space is called complete if every Cauchy sequence $\{y_n\}$ in Y converges with respect to τ_b to a point $y \in Y$ such that

$$\begin{aligned} \lim_{m\longrightarrow\infty} d_b(y, y_m) &= d_b(y,y) \\ &= \lim_{m,n\longrightarrow\infty} d_b(y_n, y_m). \end{aligned}$$ [22]

3.3 Fixed Point Results for $\mathcal{T}_{gF}$-Contraction of Geraghty Type in b-metric-like Spaces

In this section, we introduce the concept of $\mathcal{T}_{gF}$-contraction and investigate common fixed point theorems for such contraction in b-metric-like spaces. Moreover, an example is given to support one of our results.

Definition 3.3.1.

Let C be class of all functions $\gamma : [0, \infty) \to [0, 1)$ that satisfy the condition:

$$\lim_{m \longrightarrow \infty} \gamma(t_m) = 1 \text{ implies } \lim_{m \longrightarrow \infty} (t_m) = 0.$$

Theorem 3.3.2

Let (Y, d) be a complete metric space and $S : Y \to Y$ be a mapping. If S satisfies

$$d(S(u), S(v)) \leq \gamma(d(u, v))d(u, v) \quad \text{for any} \quad u, v \in Y,$$

where $\gamma \in C$, then S has a unique fixed point.[42]

Theorem 3.3.3

Let (Y, σ) be a complete metric-like space and $S : Y \to Y$ be a mapping. If there exists $\gamma \in C$ such that

$$\sigma(S(u), S(v)) \leq \gamma(S(u, v))S(u, v) \quad \text{for all} \quad u, v \in Y,$$

where

$$S(u, v) = \sigma(u, v)|\sigma(u, S_u) - \sigma(v, S_v)|,$$

then T has a unique fixed point.[43]

Lemma 3.3.1.

Let g and h be weakly compatible self maps of a set Y. If g and h have a unique point of coincidence $u = g(y) = h(y)$, then u is the unique common fixed point of g and h. [40]

Proof.

Since $u = g(y) = h(y)$ and g and h are weakly compatible, we have $g(u) = g(hy) = h(gy) = h(u)$: i.e., $g(u) = h(u)$ is a point of coincidence of g and h. But u is the only point of coincidence of g and h, so $u = g(u) = h(u)$. Moreover if $v = g(v) = h(v)$, then v is a point of coincidence of g and h, and therefore $v = u$ by uniqueness. Thus u is a unique common fixed point of g and h. □

Definition 3.3.4. $\mathcal{T}_{gF}$-Contraction

Let (Y, d_b) be a b-metric-like space with coefficient $b \geq 1$ and $T, g : Y \to Y$ be two mappings. We say that the pair $\mathcal{T}_{gF}$ is a $\mathcal{T}_{gF}$-contraction of Geraghty type if there exists $\gamma \in C$ such that

$$d_b(\mathcal{T}y, \mathcal{T}z) \leq \gamma(Fg(y,z))Fg(y,z) \text{ for all } y, z \in Y, \tag{3.1}$$

where

$$Fg(y,z) = \frac{1}{b^2}\left[d_b(gy, gz) + |d_b(gy, \mathcal{T}y) - d_b(gz, \mathcal{T}z)|\right]. \text{ [25]}$$

Lemma 3.3.2.

Let (Y, d_b) be a b-metric-like space, $\mathcal{T}$ and g be self-mappings on Y such that $(\mathcal{T}, g)$ is a $\mathcal{T}_{gF}$-contraction of Geraghty type. If $w \in Y$ is a point of coincidence of $\mathcal{T}$ and g, then $d_b(w, w) = 0$. [25]

Proof.

Suppose that $w \in Y$ is a point of coincidencevof $\mathcal{T}$ and g, then there exists $v \in Y$ such that $\mathcal{T}v = gv = w$.

Assume $d_b(w, w) > 0$, we get

$$d_b(w, w) = d_b(\mathcal{T}v, \mathcal{T}v) \leq \gamma(Fg(v,v))Fg(v,v),$$

since

$$Fg(v,v) = \frac{1}{b^2}[d_b(g_v, g_v) + |d_b(gv, \mathcal{T}v) - d_b(gv, \mathcal{T}v)|] = \frac{1}{b^2}d_b(w, w)$$

then we have

$$d_b(w, w) < \frac{1}{b^2}d_b(w, w),$$

which is a contradiction, hence $d_b(w, w) = 0$. □

Theorem 3.3.5

Consider (Y, d_b) be a b-metric-like space with coefficient $b \geq 1$, $\mathcal{T}, g : Y \to Y$ be two mappings with $\mathcal{T}Y \subseteq gY$ and gY is complete. If the pair $(\mathcal{T}, g)$ is a $\mathcal{T}_{gF}$-contraction of Geraghty type, then $\mathcal{T}$ and g have a unique point of coincidence.

In addition, if $\mathcal{T}$ and g are weakly compatible, then $\mathcal{T}$ and g have a unique common fixed point. [25]

Proof.
For an arbitrary $y_0 \in Y$, since $\mathcal{T}Y \subseteq gY$, we can construct a sequence $\{x_m\}$ by

$$x_m = gy_m = \mathcal{T}y_{m-1} \tag{3.2}$$

for all $m \in \mathbb{Z}^+$. Now, we prove that $\mathcal{T}$ and g have a point of coincidence. If there exists some $m_0 \in \mathbb{Z}^+$ such that $d_b(x_{m_0}, x_{m_0+1}) = 0$, then $x_{m_0} = x_{m_0+1}$, which implies $gy_{m_0} = \mathcal{T}y_{m_0}$, thus, x_{m_0} is a coincidence point of $\mathcal{T}$ and g, so $v_0 = gy_{m_0} = \mathcal{T}y_{m_0}$ is a point of coincidence of $\mathcal{T}$ and g. We assume that $d_b(x_m, x_{m+1}) > 0$ for all $m \in mathbbZ^+$. From (3.1), we have

$$\begin{aligned} d_b(x_m, x_{m+1}) &= d_b(\mathcal{T}y_{m-1}, \mathcal{T}y_m) \\ &\leq \gamma(F_g(y_{m-1}, y_m))F_g(y_{m-1}, y_m) \end{aligned} \tag{3.3}$$

where

$$F_g(y_{m-1}, y_m) = \frac{1}{b^2}[d_b(gy_{m-1}, gy_m) + |d_b(gy_m, \mathcal{T}y_{m-1}) - d_b(gy_m, \mathcal{T}y_m)|]$$

$$= \frac{1}{b^2}[d_b(x_{m-1}, x_m) + |d_b(x_{m-1}, x_m) - d_b(x_m, x_{m+1})|]$$

Assume that there exists $m_0 \in z^+$ such that

$$d_b(x_{m_0-1}, x_{m_0}) \leq d_b(x_{m_0}, x_{m_0})$$

By (3.3), we get

$$\begin{aligned} d_b(x_{m_0}, x_{m_0}) &= d_b(\mathcal{T}y_{m_0-1}, \mathcal{T}y_{m_0}) \\ &\leq \gamma(Fg(y_{m_0-1}, y_{m_0}))Fg(y_{m_0-1}, y_{m_0}) \\ &< Fg(y_{m_0-1}, y_{m_0}) \\ &= \frac{1}{b^2}[d_b(gy_{m_0-1}, gy_{m_0}) + |d_b(gy_{m_0-1}, \mathcal{T}y_{m_0-1}) - d_b(gy_{m_0}, \mathcal{T}y_{m_0})|] \end{aligned}$$

$$= \frac{1}{b^2}[d_b(x_{m_0-1}, x_{m_0}) + |d_b(x_{m_0-1}, x_{m_0}) - d_b(x_{m_0}, x_{m_0+1})|]$$
$$= \frac{1}{b^2} d_b(x_{m_0}, x_{m_0+1}))$$
$$\leq d_b(x_{m_0}, x_{m_0+1}),$$

which is a contradiction. Thus, we obtain

$$d_b(x_{m_0-1}, x_{m_0}) > d_b(x_{m_0}, x_{m_0+1}) \tag{3.4}$$

for all $m \in \mathbb{Z}^+$. Therefore, there exists $c \geq 0$ such that

$$\lim_{m\to\infty} d_b(x_{m_0-1}, x_{m_0}) = c \tag{3.5}$$

(3.3) and (3.4) yield that

$$\begin{aligned} d_b(x_m, x_{m+1}) &= d_b(\mathcal{T} y_{m-1}, v y_m) \\ &\leq \gamma(Fg(y_{m-1}, y_m)) Fg(y_{m-1}, y_m) \\ &= \gamma\left[\frac{1}{b^2}(2d_b(x_{m-1}, x_m) d_b(x_m, x_m + 1))\right] . \frac{1}{b_2}(2d_b(x_{m-1}, x_m) - d_b(x_m, x_{m+1})) \\ &\leq \gamma\left[\frac{1}{b^2}(2d_b(x_{m-1}, x_m) d_b(x_m, x_m + 1))\right] .(2d_b(x_{m-1}, x_m) - d_b(x_m, x_{m+1}) \\ &< 2d_b(x_{m-1}, x_m) - d_b(x_m, x_{m+1}) \end{aligned} \tag{3.6}$$

Taking $m \to \infty$ in (3.6), we get

$$\lim_{m\to\infty} \gamma\left[\frac{2d_b(x_{m-1}, x_m) - d_b(x_m, x_{m+1})}{b^2}\right] = 1$$

hence,

$$\lim_{m\to\infty} \frac{2d_b(x_{m-1}, x_m) - d_b(x_m, x_{m+1})}{b^2} = 0.$$

On the other hand,

$$\lim_{m\to\infty} \frac{2d_b(x_{m-1}, x_m) - d_b(x_m, x_{m+1})}{b^2} = \frac{c^2}{b}$$

therefore $c = 0$. Hence,

$$\lim_{m\to\infty} d_b(x_{m-1}, x_m) = 0. \tag{3.7}$$

Now we prove that

$$\lim_{n,m\to\infty} d_b(x_n, x_m) = 0. \tag{3.8}$$

If (3.8) does not hold, then there exists $\epsilon > 0$, for which we can find two subsequences $\{x_{n(j)}\}$ and $\{y_{m(j)}\}$ of $\{y_m\}$, where $n(j)$ is the smallest index for which $n(j) > m(j) > j$ with

$$d_b(x_n(j), x_m(j)) \geq \epsilon, \qquad d_b(x_{n(j)-1}, x_m(j)) < \epsilon. \tag{3.9}$$

Applying (3.1) and (3.9), we have

$$\begin{aligned}\epsilon &\leq d_b(x_{n(j)}, x_{m(j)})\\ &= d_b(\mathcal{T}y_{n(j)-1}, \mathcal{T}y_{m(j)-1})\\ &\leq \gamma(F_g(y_{n(j)-1}, y_{m(j)-1})F_g(y_{n(j)-1}, y_{m(j)-1})\\ &< F_g(y_{n(j)-1}, y_{m(j)-1}),\end{aligned} \tag{3.10}$$

where

$$\begin{aligned}F_g(y_{n(j)-1}, y_{m(j)-1}) &= \frac{1}{b^2}\left[d_b(gy_{n(j)-1}, gy_{m(j)-1}) + |d_b(gy_{m(j)-1}, \mathcal{T}y_{m(j)-1}) - d_b(gy_{m(j)-1}, \mathcal{T}y_{m(j)-1})|\right]\\ &= \frac{1}{b^2}\left[d_b(x_{n(j)-1}, x_{m(j)-1}) + |d_b(x_{n(j)-1}, x_{n(j)})d_b(x_{m(j)-1}, x_{m(j)})|\right].\end{aligned} \tag{3.11}$$

Next we discuss two cases.

Case 1: Case of $b > 1$. Applying (3.7), (3.10), and (3.11), we obtain

$$\epsilon \leq \liminf_{m\to\infty} \frac{1}{b^2} d_b(x_{n(j)-1}, x_{m(j)-1}). \tag{3.12}$$

Moreover, from (3.9), we have

$$\begin{aligned}d_b(x_{n(j)-1}, x_{m(j)-1}) &\leq b d_b(x_{n(j)-1}, x_{m(j)}) + b d_b(x_{m(j)}, x_{m(j)-1})\\ &< b\epsilon + b d_b(x_{m(j)}, x_{m(j)-1}).\end{aligned}$$

Taking $\lim_{m\to\infty}$ in the above inequalities, we have

$$\liminf_{m\to\infty} d_b(x_{n(j)-1}, x_{m(j)-1}) \leq b\epsilon \tag{3.13}$$

(3.12) and (3.13) imply $\epsilon \leq \frac{\epsilon}{b}$, which is a contradiction.
Case 2: Case of $b = 1$. From (3.9), we have

$$\begin{aligned}\epsilon &\leq d_b(x_{n(j)}, x_{m(j)})\\ &\leq d_b(x_{n(j)}, x_{n(j)-1}) + d_b(x_{n(j)-1}, x_{m(j)-1}) + d_b(x_{m(j)-1}, x_{n(j)})\\ &\leq d_b(x_{n(j)}, x_{n(j)-1}) + d_b(x_{n(j)-1}, x_{m(j))} + 2d_b(x_{m(j)-1}, x_{m(j)})\\ &< d_b(x_{n(k)}, x_{n(k)-1}) + \epsilon + 2d_b(x_{m(j)-1}, x_{m(j)}).\end{aligned} \tag{3.14}$$

By (3.7), taking $\lim_{m\to\infty}$ in (3.14), we have

$$\lim_{m\to\infty} d_b(x_{n(j)-1}, x_{m(j)-1}) = \epsilon \tag{3.15}$$

Since $b = 1$, by (3.10) and (3.11), we have

$$\begin{aligned}\epsilon &\leq \gamma(Fg(y_{n(k)-1}, y_{m(k)-1}))Fg(y_{n(k)-1}, y_{m(k)-1})\\ &< d_b(x_{n(k)-1}, x_{m(k)-1}) + |d_b(x_{n(k)-1}, x_{n(k)})d_b(x_{m(k)-1}, x_{m(k)})|.\end{aligned} \tag{3.16}$$

(3.7), (3.15) and (3.16) yield

$$\lim_{m\to\infty} \gamma(Fg(y_{n(k)-1}, y_{m(k)-1}))Fg(y_{n(k)-1}, y_{m(k)-1}) = \epsilon \tag{3.17}$$

From (3.11) and (3.15), and taking $b = 1$ into account, we get

$$\lim_{m\to\infty} Fg(y_{n(k)-1}, y_{m(k)-1}) = \epsilon,$$

which together with (3.17) implies

$$\lim_{m\to\infty} \gamma Fg(y_{n(k)-1}, y_{m(k)-1}) = \epsilon,$$

thus

$$\lim_{m\to\infty} Fg(y_{n(k)-1}, y_{m(k)-1}) = 1,$$

which is a contradiction with

$$\lim_{m\to\infty} Fg(y_{n(k)-1}, y_{m(k)-1}) = \epsilon,$$

From the above discussions, we get that (3.8) holds. Therefore, the sequence $\{x_m\} = \{gy_m\}$ is a Cauchy sequence in gY. Since gY is complete, then there exist $w, u \in Y$ such that $w = gu$, and the following equalities hold:

$$\lim_{m,n\to\infty} d_b(x_m, w) = d_b(w, w) = \lim_{m,n\to\infty} d_b(x_m, x_n) = \lim_{m,n\to\infty} d_b(x_m, gu) = 0. \quad (3.18)$$

By (3.1), we have

$$d_b(x_m, \mathcal{T}u) = d_b(\mathcal{T}y_{m-1}, \mathcal{T}u) \leq \gamma(Fg(y_{m-1}, u))Fg(y_{m-1}, u) < Fg(y_{m-1}, u), \quad (3.19)$$

where

$$\begin{aligned} Fg(y_{m-1}, u) &= \frac{1}{b^2}[d_b(gy_{m-1}, gu) + |d_b(gy_{m-1}, \mathcal{T}y_{m-1)} - d_b(gu, \mathcal{T}u)|] \\ &= \frac{1}{b^2}[d_b(x_{m-1}, w) + |d_b(x_{m-1}, x_m) - d_b(w, \mathcal{T}u)|]. \end{aligned} \quad (3.20)$$

Next, we prove $b(\mathcal{T}u, w) = 0$ in two cases:
Case 1. $b > 1$. Suppose $d_b(\mathcal{T}u, w) > 0$. Letting $\lim_{m\to\infty}$ in (3.19), applying (3.20), we obtain

$$\liminf_{m\to\infty} d_b(x_m, \mathcal{T}u) \leq \frac{1}{b^2} d_b(w, \mathcal{T}u). \quad (3.21)$$

By the triangle inequality, we get $d_b(w, \mathcal{T}u) \leq bd_b(x_m, w) + bd_b(x_m, \mathcal{T}u)$, which yields

$$d_b(w, \mathcal{T}u) \leq b \liminf_{m\to\infty} d_b(y_m, \mathcal{T}u). \quad (3.22)$$

Applying (3.22), we have

$$\liminf_{m\to\infty} d_b(x_m, \mathcal{T}u) \geq \frac{1}{b} d_b(w, \mathcal{T}u) > 0.$$

From (3.21) and (3.22), we get

$$d_b(w, \mathcal{T}u) \leq \frac{1}{b} d_b(w, \mathcal{T}u) < d_b(w, \mathcal{T}u),$$

this is a contradiction, therefore $d_b(\mathcal{T}u, w) = 0$.

Case 2. $b = 1$. Taking $\lim\limits_{m\to\infty}$ in (3.20), and taking $b = 1$ into account, we obtain

$$\lim_{m\to\infty} Fg(y_{m-1}, u) = d_b(w, \mathcal{T}u). \tag{3.23}$$

On the other hand, from (3.1), we have

$$\begin{aligned} d_b(w, \mathcal{T}u) &\leq d_b(w, x_m) + d_b(x_m, \mathcal{T}u) \\ &= d_b(w, x_m) + d_b(\mathcal{T}y_{m-1}, \mathcal{T}u) \end{aligned}$$

$$\begin{aligned} &\leq d_b(w, x_m) + \gamma(Fg(y_{m1}, u))Fg(y_{m-1}, u) \qquad (3.24) \\ &< d_b(w, x_m) + Fg(y_{m-1}, u). \end{aligned}$$

Letting $\lim\limits_{m\to\infty}$ in (3.24), by (3.23), we get $\lim\limits_{m\to\infty} \gamma(Fg(y_{m-1}, u) = 1$, hence $\lim\limits_{m\to\infty} \gamma(Fg(y_{m-1}, u) = 0$, by (3.23), we get $d_b(\mathcal{T}u, w) = 0$. The above two cases mean $d_b(\mathcal{T}u, w) = 0$, which implies $\mathcal{T}u = w$, thus $\mathcal{T}u = w = gu$. Therefore, $\mathcal{T}$ and g have a coincidence point u, and w is a point of coincidence of $\mathcal{T}$ and g. By Lemma 3.3.1, we get $d_b(w, w) = 0$. Suppose that w_1 is also a point of coincidence of $\mathcal{T}$ and g, then we can find $u_1 \in Y$ such that $\mathcal{T}u_1 = w_1 = gu_1$ and $d_b(w_1, w_1) = 0$. Now, we prove $d_b(w, w_1) = 0$ by contradiction. Suppose $d_b(w, w_1) > 0$, applying (3.1), we have

$$d_b(w, w_1) = d_b(\mathcal{T}u, \mathcal{T}u_1) \leq \gamma(Fg(u, u_1))Fg(u, u_1) < Fg(u, u_1), \tag{3.25}$$

where

$$\begin{aligned} Fg(u, u_1) &= \frac{1}{b^2}[d_b(gu, gu_1) + |d_b(gu, \mathcal{T}u) - b(gu_1, \mathcal{T}u_1)|] \\ &= \frac{1}{b^2}[d_b(w, w_1) + |d_b(w, w) - b(w_1, w_1)|] \qquad (3.26) \\ &= \frac{1}{b^2} d_b(w, w_1) \end{aligned}$$

From (3.25) and (3.26), we obtain

$$d_b(w, w_1) < \frac{1}{b^2} d_b(w, w_1),$$

which is a contradiction, thus $d_b(w, w_1) = 0$, which implies $w = w_1$, therefore $\mathcal{T}$ and g have a unique point of coincidence. Moreover, $\mathcal{T}$ and g are weakly compatible, then we have $\mathcal{T}w = gw$. Let $\mathcal{T}w = gw = v$. From the uniqueness of the point of coincidence, we have $\mathcal{T}w = gw = v = w$, that is, $\mathcal{T}w = gw = w$. Therefore, $\mathcal{T}$ and g have a unique common fixed point. □

Letting $g = Iy$ (identity mapping) in Theorem 3.3.5, we can get the following.

Corollary 1.
Let (Y, d_b) be a complete b-metric-like space with coefficient $b \geq 1$, and $\mathcal{T} : Y \to Y$ be a mapping. If there exists $\gamma \in C$ such that

$$d_b(\mathcal{T}y, \mathcal{T}z) \leq \gamma(F(y, z))F(y, z),$$

for any $y, z \in Y$, where

$$F(y, z) = \frac{1}{b^2}[d_b(y, z) + |d_b(y, \mathcal{T}y) - d_b(z, \mathcal{T}z)|],$$

then $\mathcal{T}$ has a unique fixed point. [25]

Taking $b = 1$ in Corollary 1, we have the following.

Corollary 2.
Let (Y, σ) be a complete metric-like space and $\mathcal{T} : Y \to Y$ be a mapping. If there exists $\gamma \in C$ such that

$$\sigma(\mathcal{T}y, \mathcal{T}z) \leq \gamma(F(y, z))F(y, z) \quad \text{for any} \quad y, z \in Y,$$

where

$$F(y, z) = \sigma(y, z) + |\sigma(y, \mathcal{T}y)\sigma(z, \mathcal{T}z)|,$$

then $\mathcal{T}$ has a unique fixed point. [25]

Taking $b = 1$ in Theorem 3.3.5, we have the following.

Corollary 3.

Let (Y, d_b) be a b-metric-like space and $\mathcal{T}, g : Y \times Y \to Y$ be two mappings with $\mathcal{T}Y \subseteq gY$ and gY is complete. Suppose that there exists $\gamma \in C$ such that

$$d_b(\mathcal{T}y, \mathcal{T}z) \leq \gamma(Fg(y,z))Fg(y,z),$$

where

$$Fg(y,z) = d_b(gy, gz) + |d_b(gy, \mathcal{T}y) - d_b(gz, \mathcal{T}z)|,$$

then $\mathcal{T}$ and g have a unique point of coincidence. In addition, if $\mathcal{T}$ and g are weakly compatible, then $\mathcal{T}$ and g have a unique common fixed point. [25]

Now, we use an example to illustrate the validity of our main result.

Example 3.3.1.

Let $Y = \{0, 1, 2\}$. Define $d_b : Y \times Y \to \mathbb{R}$ by $d_b(0,0) = 0,\ d_b(1,1) = 3,\ d_b(2,2) = 1,\ d_b(0,1) = d_b(1,0) = 8,\ d_b(0,2) = d_b(2,0) = 1,\ d_b(1,2) = d_b(2,1) = 4$. It is easy to prove that (X, d_b) is a complete b-metric-like space with coefficient $b = \frac{8}{5}$.
Consider $\mathcal{T} : Y \to Y$ as $\mathcal{T}0 = 0,\ \mathcal{T}1 = 2,\ \mathcal{T}2 = 0$. Take

$$\gamma(s) = \begin{cases} \dfrac{1}{1 + \dfrac{s}{100}}, & s > 0, \\ \dfrac{1}{3}, & s = 0. \end{cases}$$

By the following cases, we prove

$$d_b(\mathcal{T}y, \mathcal{T}z) \leq \gamma(F(y,z))F(y,z) \text{ for any } y, z \ \in Y,$$

where

$$F(y,z) = \frac{1}{b^2}[d_b(y,z) + |d_b(y, \mathcal{T}y) - b(z, \mathcal{T}z)|].$$

Case 1: $(y,z) = (0,0), (y,z) = (2,2), (y,z) = (0,2)$. Since

$$d_b(\mathcal{T}0, \mathcal{T}0) = d_b(0,0) = 0, d_b(\mathcal{T}2, \mathcal{T}2) = d_b(0,0) = 0, d_b(\mathcal{T}0, \mathcal{T}2) = d_b(0,0) = 0,$$

then

$$d_b(\mathcal{T}y, \mathcal{T}z) \leq \gamma(F(y,z))F(y,z),$$

holds for $(y,z) = (0,0),\ (y,z) = (2,2),\ (y,z) = (0,2)$.

Case 2: $(y,z) = (0,1)$. We get

$$d_b(\mathcal{T}0, \mathcal{T}1) = d_b(0,2) = 1,$$

and

$$F(0,1) = \frac{25}{64}\left[d_b(0,1) + |d_b(0,\mathcal{T}0) - d_b(1,\mathcal{T}1)|\right] = \frac{300}{64},$$

hence

$$d_b(\mathcal{T}0, \mathcal{T}1) = 1 < \gamma(F(0,1))F(0,1) = \left(\frac{1}{1 + \left(\frac{1}{100}\right)\left(\frac{300}{64}\right)}\right)\left(\frac{300}{64}\right) = \frac{300}{94}.$$

Case 3: $(y,z) = (1,1)$. We get

$$d_b(\mathcal{T}1, \mathcal{T}1) = d_b(2,2) = 1$$

and

$$F(1,1) = \frac{25}{64}[d_b(1,1) + |d_b(1,\mathcal{T}1) - d_b(1,\mathcal{T}1)|] = \frac{75}{64},$$

hence

$$d_b(\mathcal{T}1, \mathcal{T}1) = 1 < \gamma(F(1,1))F(1,1) = \left(\frac{1}{1 + \left(\frac{1}{100}\right)\left(\frac{75}{64}\right)}\right)\left(\frac{75}{64}\right) = \frac{750}{715}.$$

Case 4: $(y,z) = (1,2)$. We get

$$d_b(\mathcal{T}1, \mathcal{T}2) = d_b(2,0) = 1,$$

and

$$F(1,2) = \frac{25}{64}[d_b(1,2) + |d_b(1,\mathcal{T}1) - d_b(2,\mathcal{T}2)|] = \frac{175}{64},$$

hence

$$d_b(\mathcal{T}1, \mathcal{T}2) = 1 < \gamma(F(0,1))F(0,1) = \left(\frac{1}{1 + \left(\frac{1}{100}\right)\left(\frac{175}{64}\right)}\right)\left(\frac{175}{64}\right) = \frac{1750}{815}.$$

From the above discussions, we know that

$$d_b(\mathcal{T}y, \mathcal{T}z) \leq \gamma(F(y,z))F(y,z) \text{ for any } y, z \in Y,$$

where

$$F(y,z) = \frac{1}{b^2}[d_b(y,z) + |d_b(y,\mathcal{T}y) - b(z,\mathcal{T}z)|].$$

By Corollary 1, we obtain that $\mathcal{T}$ has a unique fixed point, 0 is the unique fixed point of $\mathcal{T}$. [25]

Chapter 4

Common Fixed Point Theorems on Extended b-metric-like Spaces

This chapter is the extention of the results presented in [25]. In the start of this chapter we introduced extended b-metric-like spaces and some other definitions which will be used in the main result.

4.1 Extended b-metric-like Space

This section comprises of a very important generalization of b-metric-like space known as extended b-metric-like space.

Definition 4.1.1. Extended b-metric-like Space

Consider a set Y which is non-empty and $\mu : Y \times Y \to [1, \infty)$. A mapping $d_{b\mu} : Y \times Y \to [0, \infty)$ is said to be an extended b-metric like if for all $y_1, y_2, y_3 \in Y$, the following conditions are satisfied

$(d_{b\mu}1) : d_{b\mu}(y_1, y_2) = 0 \Rightarrow y_1 = y_2;$

$(d_{b\mu}2) : d_{b\mu}(y_1, y_2) = d_{b\mu}(y_2, y_1);$

$(d_{b\mu}3) : d_{b\mu}(y_1, y_3) \leq \mu(y_1, y_3)[d_{b\mu}(y_1, y_2) + d_{b\mu}(y_2, y_3)],$

then $(Y, d_{b\mu})$ is known as extended b-metric like space.

Remark 4.1.1.

(1) It is worth to mention that b-metric-like space is a special case of extended b-metric space when $\mu(y_1, y_2) = b$ with $b \geq 1$.

(2) The metric-like space is a special case of extended b-metric-like space when $\mu(y_1, y_2) = b$ with $b = 1$.

Example 4.1.1.

Let $Y = \{1, 2, 3, ...\}$ and $d_{b\mu} : Y \times Y \to [0, \infty)$ defined as

$$d_{b\mu}(y_1, y_2) = (y_1 - y_2)^2.$$

Consider a function $\mu : Y \times Y \to [1, \infty)$ defined as

$$\mu(y_1, y_2) = \frac{y_1 + y_2 + 2}{y_1 + y_2},$$

then $(Y, d_{b\mu})$ is an extended b-metric like space.

$(d_{b\mu}1)$ and $(d_{b\mu}2)$ are obvious.

$(d_{b\mu}3)$: To prove

$$d_{b\mu}(y_1, y_3) \leq \mu(y_1, y_3)[d_{b\mu}(y_1, y_2) + d_{b\mu}(y_2, y_3)],$$

we proceed as follows:

$$\begin{aligned} d_{b\mu}(y_1, y_3) &= (y_1 - y_3)^2 \\ &\leq \left[(y_1 - y_2)^2 + (y_2 - y_3)^2\right] \\ &\leq 2\left[(y_1 - y_2)^2 + (y_2 - y_3)^2\right] \\ &\leq \frac{y_1 + y_3 + 2}{y_1 + y_3}\left[(y_1 - y_2)^2 + (y_2 - y_3)^2\right] \\ &= \mu(y_1, y_3)\left[d_{b\mu}(y_1, y_2) + d_{b\mu}(y_2, y_3)\right], \end{aligned}$$

hence proved that it is an extended b-metric-like space.

Example 4.1.2.

Consider a set $Y = [0, \infty)$ and $d_{b\mu} : Y \times Y \to [0, \infty)$ and is defined as

$$d_{b\mu}(y_1, y_2) = \{\max(y_1, y_2)\}^2.$$

Consider a function $\mu : Y \times Y \to [1, \infty)$ defined as

$$\mu(y_1, y_2) = 2y_1 + y_2 + 2,$$

then $(\mu, d_{b\mu})$ is an extended b-metric like space.

$(d_{b\mu}1)$ and $(d_{b\mu}2)$ are obvious.

$(d_{b\mu}3)$: To prove $d_{b\mu}(y_1, y_3) \leq \mu(y_1, y_3)\{d_{b\mu}(y_1, y_2) + d_{b\mu}(y_2, y_3)\}$, we proceed as follows:

$$\begin{aligned}
\{\max(y_1, y_3)\}^2 &\leq \{\max(y_1 + y_2), (y_2 + y_3)\}^2 \\
&\leq \{\max(y_1, y_2) + (y_2, y_3)\}^2 \\
&\leq \{\max(y_1, y_2) + \max(y_2, y_3\}^2 \\
&\leq 2\left[\{\max(y_1, y_2)\}^2 + \{\max(y_2, y_3)\}^2\right] \\
&\leq (2y_1 + y_3 + 2)\left[\{\max(y_1, y_2)\}^2 + \max\{(y_2, y_3)\}^2\right] \\
\Rightarrow d_{b\mu}(y_1, y_3) &\leq \mu(y_1, y_3)\{d_{b\mu}(y_1, y_2) + d_{b\mu}(y_2, y_3)\},
\end{aligned}$$

hence proved that it is an extended b-metric-like space.
Some necessary definition and concepts are given in upcoming discussion. These concepts will help in proving the main result.

Definition 4.1.2.
Consider an extended b-metric like space $(Y, d_{b\mu})$. It induces a topology $\tau_{d_{b\mu}}$ on Y based on the family of open $d_{b\mu}$-balls

$$\mathbb{B}_{d_{b\mu}}(y, \epsilon) = \{z \in Y : |d_{b\mu}(y, z) - d_{b\mu}(y, y)| < \epsilon\},$$

for all $\epsilon > 0$ and $y \in Y$.

Definition 4.1.3.
Assume that $(Y, d_{b\mu})$ is an extended b-metric-like space and $\mu : Y \times Y \to [1, \infty)$. Consider a sequence $\{y_m\}$ in Y and $y \in Y$, then

(1) $\{y_m\}$ is said to converge to a point $y \in Y$, iff we have

$$\lim_{m\to\infty} d_{b\mu}(y, y_m) = d_{b\mu}(y, y).$$

(2) $\{y_m\}$ is a Cauchy sequence if and only if

$$\lim_{m,n\to\infty} d_{b\mu}(y_m, y_n),$$

exists.

(3) An extended b-metric like space $(Y, d_{b\mu})$ is called complete iff each sequence $\{y_m\}$ in Y which is Cauchy in Y is convergent to $y \in Y$ that is

$$\lim_{m,n\to\infty} d_{b\mu}(y_m, y_n) = d_{b\mu}(y, y) = \lim_{m\to\infty} d_{b\mu}(y_m, y).$$

Definition 4.1.4. $\mathcal{T}_{gF}$-Contraction

Let $(Y, d_{b\mu})$ be an extended b-metric-like space with coefficient $\mu : Y \times Y \to [1, \infty)$ and $\mathcal{T}, g : Y \to Y$ be two mappings. We say that the pair $(\mathcal{T}, g)$ is a $\mathcal{T}_{gF}$-contraction of Geraghty type if there exists $\gamma : [0, \infty) \to [0, 1)$, which satisfy the condition

$$\lim_{m\to\infty} \gamma(t_m) = 1 \Rightarrow \lim_{m\to\infty} (t_m) = 0,$$

such that

$$d_{b\mu}(\mathcal{T}y, \mathcal{T}) \leq \gamma(Fg(y, z))Fg(y, z) \tag{4.1}$$

for all $y, z \in Y$, where

$$Fg(y, z) = \frac{1}{\mu(y, z)^2}[d_{b\mu}(gy, gz) + |d_{b\mu}(gy, \mathcal{T}y) - d_{b\mu}(gz, \mathcal{T}z)|].$$

Lemma 4.1.1.

Let (Y, d_b) be an extended b-metric-like space, $\mathcal{T}$ and g be self-mappings on Y such that the pair $(\mathcal{T}, g)$ is a $\mathcal{T}_{gF}$-contraction of Geraghty type. If $w \in Y$ is a point of coincidence of $\mathcal{T}$ and g, then $d_{b\mu}(w, w) = 0$.

Proof.

Suppose that $w \in Y$ is a point of coincidence of $\mathcal{T}$ and g, then there exists $v \in Y$ such that $\mathcal{T}v = gv = w$.

Assume $d_{b\mu}(w, w) > 0$, we get

$$d_{b\mu}(w, w) = d_{b\mu}(\mathcal{T}v, \mathcal{T}v) \leq \gamma(F_g(v, v))F_g(v, v)$$

since

$$\begin{aligned} F_g(v, v) &= \frac{1}{\mu(y, z)^2}\left[d_{b\mu}(gv, gv) + |d_{b\mu}(gv, \mathcal{T}v) - d_{b\mu}(gv, \mathcal{T}v)|\right] \\ &= \frac{1}{\mu(y, z)^2} d_{b\mu}(w, w), \end{aligned}$$

then we have

$$d_{b\mu}(w, w) < \frac{1}{\mu(y, z)^2} d_b(w, w),$$

which is a contradiction, hence $d_{b\mu}(w, w) = 0$. □

4.2 Main Result

Theorem 4.2.1

Let $(Y, d_{b\mu})$ be an extended b-metric-like space with coefficient $\mu : Y \times Y \to [1, \infty)$, and $\mathcal{T}, g : Y \to Y$ be two mappings with $\mathcal{T}Y \subseteq gY$ and gY is complete. Then $\mathcal{T}$ and g have a unique point of coincidence if:

(i) If the pair $(\mathcal{T}, g)$ is a $\mathcal{T}_{gF}$-contraction of Geraghty type.

(ii) For $J \in (0, 1)$ and for an arbitrary $y_0 \in Y$, $\lim_{m,n\to\infty} \mu(y_m, y_n) < \frac{1}{J}$ with $x_m = gy_m = \mathcal{T}y_{m-1}$.

In addition, if $\mathcal{T}$ and g are weakly compatible, then $\mathcal{T}$ and g have a unique common fixed point.

Proof.

Consider an arbitrary $y_0 \in Y$, since $\mathcal{T}Y \subseteq gY$, we can construct a sequence $\{x_m\}$ by

$$x_m = gy_m = \mathcal{T}y_{m-1} \tag{4.2}$$

for all $m \in \mathbb{Z}^+$. Now, we prove that $\mathcal{T}$ and g have a point of coincidence. If there exists some $m_0 \in \mathbb{Z}^+$ such that

$$d_{b\mu}(x_{m_0}, x_{m_0+1}) = 0, \text{ then } x_{m_0} = x_{m_0+1},$$

which implies $gy_{m_0} = \mathcal{T}y_{m_0}$, thus, x_{m_0} is a coincidence point of $\mathcal{T}$ and g, so $v_0 = gy_{m_0} = \mathcal{T}y_{m_0}$ is a point of coincidence of $\mathcal{T}$ and g. We assume that

$$d_{b\mu}(x_m, x_{m+1}) > 0 \text{ for all } m \in \mathbb{Z}^+.$$

From (4.1), we have

$$\begin{aligned} d_{b\mu}(x_m, x_{m+1}) &= d_{b\mu}(\mathcal{T}y_{m-1}, \mathcal{T}y_m) \\ &\leq \gamma(F_g(y_{m-1}, y_m))F_g(y_{m-1}, y_m) \end{aligned} \tag{4.3}$$

where

$$\begin{aligned} F_g(y_{m-1}, y_m) &= \frac{1}{\{\mu(y_{m-1}, y_m)\}^2} \\ &[d_{b\mu}(gy_{m-1}, gy_m) + |d_{b\mu}(gy_m, \mathcal{T}y_{m-1}) - d_{b\mu}(gy_m, \mathcal{T}y_m)|] \\ &= \frac{1}{\{\mu(y_{m-1}, y_m)\}^2}[d_{b\mu}(x_{m-1}, x_m) + |d_{b\mu}(x_{m-1}, x_m) - d_{b\mu}(x_m, x_{m+1})|] \end{aligned}$$

Assume that there exists $m_0 \in \mathbb{Z}^+$ such that

$$d_{b\mu}(x_{m_0-1}, x_{m_0}) \leq d_{b\mu}(x_{m_0}, x_{m_0})$$

By (4.3), we get

$$\begin{aligned} d_{b\mu}(x_{m_0}, x_{m_0+1}) &= d_{b\mu}(\mathcal{T}y_{m_0-1}, \mathcal{T}y_{m_0}) \\ &\leq \gamma(Fg(y_{m_0-1}, y_{m_0}))Fg(y_{m_0-1}, y_{m_0}) \\ &< Fg(y_{m_0-1}, y_{m_0}) \\ &= \frac{1}{\{\mu(y_{m_0-1}, y_{m_0})\}^2} \\ &[d_{b\mu}(gy_{m_0-1}, gy_{m_0}) + |d_{b\mu}(gy_{m_0-1}, \mathcal{T}y_{m_0-1}) - d_{b\mu}(gy_{m_0}, \mathcal{T}y_{m_0})|] \end{aligned}$$

$$= \frac{1}{\{\mu(y_{m_0-1}, y_{m_0})\}^2}$$
$$[d_{b\mu}(x_{m_0-1}, x_{m_0}) + |d_{b\mu}(x_{m_0-1}, x_{m_0}) - d_{b\mu}(x_{m_0}, x_{m_0+1})|]$$
$$= \frac{1}{\{\mu(y_{m_0-1}, y_{m_0})\}^2}$$
$$[d_{b\mu}(x_{m_0-1}, x_{m_0}) - d_{b\mu}(x_{m_0-1}, x_{m_0}) + d_{b\mu}(x_{m_0}, x_{m_0+1})]$$
$$= \frac{1}{\{\mu(y_{m_{0-1}}, y_{m_0})\}^2} d_{b\mu}(x_{m_0}, x_{m_0+1}))$$
$$\leq d_{b\mu}(x_{m_0}, x_{m_0+1})$$

which is a contradiction. Thus, we obtain

$$d_{b\mu}(x_{m_0-1}, x_{m_0}) > d_{b\mu}(x_{m_0}, x_{m_0+1}) \text{ for all } m \in \mathbb{Z}^+. \tag{4.4}$$

Therefore, there exists $c \geq 0$ such that

$$\lim_{m\to\infty} d_{b\mu}(x_{m_0-1}, x_{m_0}) = c \tag{4.5}$$

(4.3) and (4.4) yield that

$$\begin{aligned}
d_{b\mu}(x_m, x_{m+1}) &= d_{b\mu}(\mathcal{T} y_{m-1}, \mathcal{T} y_m) \\
&\leq \gamma(Fg(y_{m-1}, y_m))Fg(y_{m-1}, y_m) \\
&= \gamma\left[\frac{1}{\{\mu(y_{m-1}, y_m)\}^2}(2d_{b\mu}(x_{m-1}, x_m)d_{b\mu}(x_m, x_{m+1}))\right]. \\
&\frac{1}{\{\mu(y_{m-1}, y_m)\}_2}\left(2d_{b\mu}(x_{m-1}, x_m) - d_{b\mu}(x_m, x_{m+1})\right) \qquad (4.6) \\
&\leq \gamma\left[\frac{1}{\{\mu(y_{m-1}, y_m)\}^2}(2d_{b\mu}(x_{m-1}, x_m) - d_{b\mu}(x_m, x_{m+1}))\right]. \\
&(2d_{b\mu}(x_{m-1}, x_m) - d_{b\mu}(x_m, x_{m+1})) \\
&< 2d_{b\mu}(x_{m-1}, x_m) - d_{b\mu}(x_m, x_{m+1})
\end{aligned}$$

Taking $\lim_{m\to\infty}$ in (4.6), we get

$$\lim_{m\to\infty} \gamma\left[\frac{2d_{b\mu}(x_{m-1}, x_m) - d_{b\mu}(x_m, x_{m+1})}{\{\mu(y_{m-1}, y_m)\}^2}\right] = 1$$

hence,

$$\lim_{m\to\infty} \frac{2d_{b\mu}(x_{m-1}, x_m) - d_{b\mu}(x_m, x_{m+1})}{\{\mu(y_{m-1}, y_m)\}^2} = 0.$$

On the other hand,

$$\lim_{m\to\infty} \frac{2d_{b\mu}(x_{m-1}, x_m) - d_{b\mu}(x_m, x_{m+1})}{\{\mu(y_{m-1}, y_m\}^2} = \frac{c}{\lim\limits_{m\to\infty} \{\mu(y_{m-1}, y_m\}^2}$$

therefore $c = 0$. Hence,

$$\lim_{m\to\infty} d_{b\mu}(x_{m-1}, x_m) = 0. \tag{4.7}$$

Now we prove that

$$\lim_{n,m\to\infty} d_{b\mu}(x_n, x_m) = 0. \tag{4.8}$$

If (4.8) does not hold, then there exists $\epsilon > 0$, for which we can find two subsequences $\{x_{n(j)}\}$ and $\{y_{m(j)}\}$ of $\{y_m\}$, where $n(j)$ is the smallest index for which $n(j) > m(j) > j$ with

$$d_{b\mu}(x_n(j), x_m(j)) \geq \epsilon, \qquad d_{b\mu}(x_{n(j)-1}, x_m(j)) < \epsilon. \tag{4.9}$$

Applying (4.1) and (4.9), we have

$$\begin{aligned}
\epsilon &\leq d_{b\mu}(x_{n(j)}, x_{m(j)}) \\
&= d_{b\mu}(\mathcal{T}y_{n(j)-1}, \mathcal{T}y_{m(j)-1}) \\
&\leq \gamma(F_g(y_{n(j)-1}, y_{m(j)-1})F_g(y_{n(j)-1}, y_{m(j)-1}) \\
&< F_g(y_{n(j)-1}, y_{m(j)-1}),
\end{aligned} \tag{4.10}$$

where

$$\begin{aligned}
F_g(y_{n(j)-1}, y_{m(j)-1}) &= \frac{1}{\{\mu(y_{n(j)-1}, y_{m(j)-1})\}^2}\cdot \\
&\quad \left[d_{b\mu}(gy_{n(j)-1}, gy_{m(j)-1}) + \left|d_{b\mu}(gy_{m(j)-1}, \mathcal{T}y_{m(j)-1}) - d_{b\mu}(gy_{m(j)-1}, \mathcal{T}y_{m(j)-1})\right|\right] \\
&= \frac{1}{\{\mu(y_{n(j)-1}, y_{m(j)-1})\}^2}\cdot \\
&\quad \left[d_{b\mu}(x_{n(j)-1}, x_{m(j)-1}) + \left|d_{b\mu}(x_{n(j)-1}, x_{n(j)}) - d_{b\mu}(x_{m(j)-1}, x_{m(j)})\right|\right].
\end{aligned} \tag{4.11}$$

Next we discuss two cases.

Case 1: Case of $\mu(y_{n(j)-1}, y_{m(j)-1}) > 1$. Applying (4.7), (4.10), and (4.11), we obtain

$$\epsilon \leq \liminf_{m\to\infty} \frac{1}{\{\mu(y_{n(j)-1}, y_{m(j)-1})\}^2} d_{b\mu}(x_{n(j)-1}, x_{m(j)-1}). \tag{4.12}$$

Moreover, from (4.9), we have

$$\begin{aligned} d_{b\mu}(x_{n(j)-1}, x_{m(j)-1}) \leq\ & \mu(y_{n(j)-1}, y_{m(j)-1}) d_{b\mu}(x_{n(j)-1}, x_{m(j)}) + \\ & \mu(y_{n(j)-1}, y_{m(j)-1}) d_{b\mu}(x_{m(j)}, x_{m(j)-1}) \\ & < \mu(y_{n(j)-1}, y_{m(j)-1})\epsilon + \\ & \mu(y_{n(j)-1}, y_{m(j)-1}) d_{b\mu}(x_{m(j)}, x_{m(j)-1}). \end{aligned}$$

Taking $\liminf_{m\to\infty}$ in the above inequalities, we have

$$\liminf_{m\to\infty} d_{b\mu}(x_{n(j)-1}, x_{m(j)-1}) \geq \frac{\epsilon}{J} \tag{4.13}$$

(4.12) and (4.13) imply $\epsilon \leq J\epsilon$, which is a contradiction.

Case 2: Case of $\mu(y_{n(j)-1}, y_{m(j)-1}) = 1$. From (4.9), we have

$$\begin{aligned} \epsilon &\leq d_{b\mu}(x_{n(j)}, x_{m(j)}) \\ &\leq d_{b\mu}(x_{n(j)}, x_{n(j)-1}) + d_{b\mu}(x_{n(j)-1}, x_{m(j)-1}) + d_{b\mu}(x_{m(j)-1}, x_{n(j)}) \\ &\leq d_{b\mu}(x_{n(j)}, x_{n(j)-1}) + d_{b\mu}(x_{n(j)-1}, x_{m(j))} + 2d_{b\mu}(x_{m(j)-1}, x_{m(j)}) \\ &< d_{b\mu}(x_{n(k)}, x_{n(k)-1}) + \epsilon + 2d_{b\mu}(x_{m(j)-1}, x_{m(j)}). \end{aligned} \tag{4.14}$$

By (4.7), taking $\liminf_{m\to\infty}$ in (4.14), we have

$$\liminf_{m\to\infty} d_{b\mu}(x_{n(j)-1}, x_{m(j)-1}) = \epsilon \tag{4.15}$$

Since $\mu(y_{n(j)-1}, y_{m(j)-1}) = 1$, by (4.10) and (4.11), we have

$$\begin{aligned} \epsilon &\leq \gamma(Fg(y_{n(k)-1}, y_{m(k)-1}))Fg(y_{n(k)-1}, y_{m(k)-1}) \\ &< d_{b\mu}(x_{n(k)-1}, x_{m(k)-1}) + |d_{b\mu}(x_{n(k)-1}, x_{n(k)}) - d_{b\mu}(x_{m(k)-1}, x_{m(k)})|. \end{aligned} \tag{4.16}$$

(4.7), (4.15) and (4.16) yield

$$\liminf_{m\to\infty} \gamma(Fg(y_{n(k)-1}, y_{m(k)-1}))Fg(y_{n(k)-1}, y_{m(k)-1}) = \epsilon \tag{4.17}$$

From (4.11) and (4.15), and taking $\mu(y_{n(j)-1}, y_{m(j)-1} = 1$ into account, we get

$$\liminf_{m\to\infty} Fg(y_{n(k)-1}, y_{m(k)-1}) = \epsilon,$$

which together with (4.17) implies

$$\liminf_{m\to\infty} \gamma\{Fg(y_{n(k)-1}, y_{m(k)-1})\} = \epsilon,$$

thus

$$\liminf_{m\to\infty} Fg(y_{n(k)-1}, y_{m(k)-1}) = 1,$$

which is a contradiction with

$$\liminf_{m\to\infty} Fg(y_{n(k)-1}, y_{m(k)-1}) = \epsilon,$$

From the above discussions, we get that equation (4.8) holds. Therefore, the sequence $\{x_m\} = \{gy_m\}$ is a Cauchy sequence in gY. Since gY is complete, then there exist $w, u \in Y$ such that $w = gu$, and the following equalities hold:

$$\lim_{m,n\to\infty} d_{b\mu}(x_m, w) = d_{b\mu}(w, w) = \lim_{m,n\to\infty} d_{b\mu}(x_m, x_n) = \lim_{m,n\to\infty} d_b(x_m, gu) = 0. \tag{4.18}$$

By (4.1), we have

$$d_{b\mu}(x_m, \mathcal{T}u) = d_{b\mu}(\mathcal{T}y_{m-1}, Tu) \leq \gamma(Fg(y_{m-1}, u))Fg(y_{m-1}, u) < Fg(y_{m-1}, u), \tag{4.19}$$

where

$$\begin{aligned} Fg(y_{m-1}, u) &= \frac{1}{\{\mu(y_{m-1}, u)\}^2}[d_{b\mu}(gy_{m-1}, gu) + |d_{b\mu}(gy_{m-1}, \mathcal{T}y_{m-1)} - d_{b\mu}(gu, \mathcal{T}u)|] \\ &= \frac{1}{\{\mu(y_{m-1}, u)\}^2}[d_{b\mu}(x_{m-1}, w) + |d_{b\mu}(x_{m-1}, x_m) - d_{b\mu}(w, \mathcal{T}u)|]. \end{aligned} \tag{4.20}$$

Next, we prove $d_{b\mu}(\mathcal{T}u, w) = 0$ in two cases:

Case 1. $\mu(y_{m-1}, u) > 1$. Suppose $d_{b\mu}(\mathcal{T}u, w) > 0$. Letting $\lim_{m\to\infty}$ in (4.19), applying (4.20), we obtain

$$\liminf_{m\to\infty} d_{b\mu}(x_m, \mathcal{T}u) \le J^2 d_{b\mu}(w, Tu). \tag{4.21}$$

By the triangle inequality, we get

$$d_{b\mu}(w, \mathcal{T}u) \le \mu(y_{m-1}, u) d_{b\mu}(x_m, w) + \mu(y_{m-1}, u) d_{b\mu}(x_m, \mathcal{T}u),$$

which yields

$$\begin{aligned} d_{b\mu}(w, \mathcal{T}u) &\le \liminf_{m\to\infty} \mu(y_{m-1}, u) d_{b\mu}(y_m, \mathcal{T}u) \\ &< \frac{1}{J} \liminf_{m\to\infty} d_{b\mu}(y_m, \mathcal{T}u) \end{aligned} \tag{4.22}$$

Applying (4.22), we have

$$\liminf_{m\to\infty} d_{b\mu}(x_m, \mathcal{T}u) \ge J d_{b\mu}(w, \mathcal{T}u) > 0.$$

From (4.21) and (4.22), we get

$$J^2 d_{b\mu}(w, \mathcal{T}u) \ge J d_{b\mu}(w, \mathcal{T}u),$$

$$\Rightarrow J d_{b\mu}(w, \mathcal{T}u) \ge d_{b\mu}(w, \mathcal{T}u),$$

this is a contradiction, therefore $d_{b\mu}(\mathcal{T}u, w) = 0$.

Case 2. $\mu(y_{m-1}, u) = 1$. Taking $m \to \infty$ in (4.20), and taking $\mu(y_{m-1}, u) = 1$ into account, we obtain

$$\lim_{m\to\infty} Fg(y_{m-1}, u) = d_{b\mu}(w, \mathcal{T}u). \tag{4.23}$$

On the other hand, from (4.1), we have

$$\begin{aligned} d_{b\mu}(w, \mathcal{T}u) &\le d_{b\mu}(w, x_m) + d_{b\mu}(x_m, \mathcal{T}u) \\ &= d_{b\mu}(w, x_m) + d_{b\mu}(\mathcal{T}y_{m-1}, \mathcal{T}u) \\ &\le d_{b\mu}(w, x_m) + \gamma(Fg(y_{m1}, u))Fg(y_{m-1}, u) \\ &< d_{b\mu}(w, x_m) + Fg(y_{m-1}, u). \end{aligned} \tag{4.24}$$

Letting $\lim\limits_{m\to\infty}$ in (4.24), by (4.23), we get

$$\lim_{m\to\infty}\gamma(Fg(y_{m-1},u)=1.$$

Hence,

$$\lim_{m\to\infty}Fg(y_{m-1},u)=0,$$

by (4.23), we get

$$d_{b\mu}(\mathcal{T}u,w)=0.$$

The above two cases mean $d_{b\mu}(\mathcal{T}u,w)=0$, which implies $\mathcal{T}u=w$, thus $\mathcal{T}u=w=gu$. Therefore, $\mathcal{T}$ and g have a coincidence point u, and w is a point of coincidence of T and g.

By Lemma 4.1.1, we get $d_{b\mu}(w,w)=0$. Suppose that w_1 is also a point of coincidence of $\mathcal{T}$ and g, then we can find $u_1\in Y$ such that $\mathcal{T}u_1=w_1=gu_1$ and $d_{b\mu}(w_1,w_1)=0$. Now, we prove $d_{b\mu}(w,w_1)=0$ by contradiction. Suppose $d_{b\mu}(w,w_1)>0$, applying (3.1), we have

$$d_{b\mu}(w,w_1)=d_{b\mu}(\mathcal{T}u,\mathcal{T}u_1)\le\gamma(Fg(u,u_1))Fg(u,u_1)<Fg(u,u_1),\tag{4.25}$$

where

$$\begin{aligned}Fg(u,u_1)&=\frac{1}{\{\mu(w,w_1)\}^2}\left[d_{b\mu}(gu,gu_1)+|d_{b\mu}(gu,\mathcal{T}u)-d_{b\mu}(gu_1,\mathcal{T}u_1)|\right]\\&=\frac{1}{\{\mu(w,w_1)\}^2}\left[d_{b\mu}(w,w_1)+|d_{b\mu}(w,w)-d_{b\mu}(w_1,w_1)|\right]\\&=\frac{1}{\{\mu(w,w_1)\}^2}d_b(w,w_1)\end{aligned}\tag{4.26}$$

From (4.25) and (4.26), we obtain

$$d_{b\mu}(w,w_1)<\frac{1}{\{\mu(w,w_1)\}^2}d_{b\mu}(w,w_1),$$

which is a contradiction, thus

$$d_{b\mu}(w,w_1)=0,\quad\Rightarrow\quad w=w_1,$$

therefore $\mathcal{T}$ and g have a unique point of coincidence. Moreover, $\mathcal{T}$ and g are weakly compatible, then we have $\mathcal{T}w=gw$. Let $\mathcal{T}w=gw=v$. From the uniqueness of the point of coincidence, we have $\mathcal{T}w = gw = v = w$, that is, $\mathcal{T}w = gw = w$. Therefore, $\mathcal{T}$ and g have a unique common fixed point. □

Letting $g = I_y$ (identity mapping) in Theorem 4.2.1, we can get the following.

Corollary 4.

Let $(Y, d_{b\mu})$ be a complete extended b-metric-like space and $\mu : Y\times Y \to [1,\infty)$, and $\mathcal{T} : Y \to Y$ be a mapping. If there exists $\gamma \in C$ such that

$$d_{b\mu}(\mathcal{T}y, \mathcal{T}z) \leq \gamma(F(y,z))F(y,z) \quad \text{for any} \quad y, z \in Y,$$

where

$$F(y,z) = \frac{1}{\{\mu(y,z)\}^2}[d_{b\mu}(y,z) + |d_{b\mu}(y,\mathcal{T}y) - d_{b\mu}(z,\mathcal{T}z)|],$$

then $\mathcal{T}$ has a unique fixed point.

If we take $\mu(y,z) = b$ in corollary 4, we have the following.

Corollary 5.

Let (Y, d_b) be a complete b-metric-like space and $b \geq 1$, and $\mathcal{T} : Y \to Y$ be a mapping. If there exists $\gamma \in C$ such that

$$d_b(\mathcal{T}y, \mathcal{T}z) \leq \gamma(F(y,z))F(y,z) \quad \text{for any} \quad y, z \in Y,$$

where

$$F(y,z) = \frac{1}{b^2}[d_b(y,z) + |d_b(y,\mathcal{T}y) - d_b(z,\mathcal{T}z)|],$$

then $\mathcal{T}$ has a unique fixed point.

Taking $b = 1$ in Corollary 5, we have the following.

Corollary 6.

Let (Y, σ) be a complete metric-like space and $T : Y \to Y$ be a mapping. If there exists $\gamma \in C$ such that

$$\sigma(\mathcal{T}y, \mathcal{T}z) \leq \gamma(F(y,z))F(y,z) \quad \text{for any} \quad y, z \in Y,$$

where

$$F(y,z) = \sigma(y,z) + |\sigma(y,\mathcal{T}y)\sigma(z,\mathcal{T}z)|,$$

then $\mathcal{T}$ has a unique fixed point.

If we take $\mu(y,z) = b$ in Theorem 4.2.1, we have the following.

Corollary 7.

Let (Y, d_b) be a b-metric-like space with coefficient $b \geq 1$, and $\mathcal{T}, g : Y \to Y$ be two mappings with $\mathcal{T}Y \subseteq gY$ and gY is complete. If the pair $(\mathcal{T}, g)$ is a $\mathcal{T}_{gF}$-contraction of Geraghty type, then $\mathcal{T}$ and g have a unique point of coincidence. In addition, if $\mathcal{T}$ and g are weakly compatible, then $\mathcal{T}$ and g have a unique common fixed point.

Taking $b = 1$ in corollary 7, we have the following.

Corollary 8.

Let (Y, σ) be a metric-like space and $\mathcal{T}, g : Y \times Y \to Y$ be two mappings with $\mathcal{T}Y \subseteq gY$ and gY is complete. Suppose that there exists $\gamma \in C$ such that

$$\sigma(\mathcal{T}y, \mathcal{T}z) \leq \gamma(Fg(y,z))Fg(y,z),$$

where

$$Fg(y,z) = \sigma(gy, gz) + |\sigma(gy, \mathcal{T}y) - \sigma(gz, \mathcal{T}z)|,$$

then $\mathcal{T}$ and g have a unique point of coincidence. In addition, if $\mathcal{T}$ and g are weakly compatible, then $\mathcal{T}$ and g have a unique common fixed point.

Example 4.2.1.

Let $Y = \{0,1,2\}$. Define $d_{b\mu} : Y \times Y \to R$ by $d_{b\mu}(0,0) = 0$, $d_{b\mu}(1,1) = 3$, $d_{b\mu}(2,2) = 1$, $d_{b\mu}(0,1) = d_{b\mu}(1,0) = 8$, $d_{b\mu}(0,2) = d_{b\mu}(2,0) = 1$, $d_{b\mu}(1,2) = d_{b\mu}(2,1) = 4$, be a complete extended b metric like space with $\theta : Y \times Y \to [1,\infty)$ by

$$\theta(y,z) = \frac{y+z+1}{y+z}.$$

Consider $\mathcal{T} : Y \to Y$ as

$$\mathcal{T}0 = 0, \ \mathcal{T}1 = 2, \ \mathcal{T}2 = 0.$$

Take

$$\gamma(s) = \begin{cases} \dfrac{1}{1+\dfrac{s}{100}}, & s > 0, \\ \dfrac{1}{3}, & s = 0. \end{cases}$$

By the following cases, we prove

$$d_{b\mu}(\mathcal{T}y, \mathcal{T}z) \leq \gamma(F(y,z))F(y,z) \ \text{ for any } \ y, z \in Y,$$

where

$$F(y,z) = \frac{1}{\{\mu(y,z)\}^2}[d_{b\mu}(y,z) + |d_{b\mu}(y,\mathcal{T}y) - d_{b\mu}(z,\mathcal{T}z)|].$$

(case 1): $(x,y) = (0,0)$

$$\begin{aligned} d_{b\mu}(\mathcal{T}x, \mathcal{T}y) &= d_{b\mu}(\mathcal{T}0, \mathcal{T}1) \\ &= d_{b\mu}(0,0) \\ &= 0 \end{aligned}$$

$$F(x,y) = F(0,0) = 0$$

$$\gamma(F(0,0)) = \frac{1}{3}$$

$$\begin{aligned} d_{b\mu}(\mathcal{T}0, \mathcal{T}1) &= 0 \\ &= \gamma(F(0,0))F(0,0) \\ &= 0 \end{aligned}$$

(case 2): $(x,y) = (1,1)$

$$\begin{aligned} d_{b\mu}(\mathcal{T}x, \mathcal{T}y) &= d_{b\mu}(\mathcal{T}1, \mathcal{T}1) \\ &= d_{b\mu}(2,2) \\ &= 1 \end{aligned}$$

$$F(x,y) = F(1,1) = 4$$

$$\gamma(F(1,1)) = \frac{25}{26}$$

$$\begin{aligned} d_{b\mu}(\mathcal{T}1, \mathcal{T}1) &= 1 \\ &< \gamma(F(1,1))F(1,1) \\ &= \frac{50}{13} \end{aligned}$$

(case 3): $(x, y) = (2, 2)$

$$\begin{aligned} d_{b\mu}(\mathcal{T}x, \mathcal{T}y) &= d_{b\mu}(\mathcal{T}2, \mathcal{T}2) \\ &= d_{b\mu}(0, 0) \\ &= 0 \end{aligned}$$

$$F(x, y) = F(2, 2) = \frac{16}{25}$$

$$\gamma(F(2, 2)) = \frac{625}{629}$$

$$\begin{aligned} d_{b\mu}(\mathcal{T}2, \mathcal{T}2) &= 0 \\ &< \gamma(F(2,2))F(2,2) \\ &= \frac{400}{629} \end{aligned}$$

(case 4): $(x, y) = (0, 1)$

$$\begin{aligned} d_{b\mu}(\mathcal{T}x, \mathcal{T}y) &= d_{b\mu}(\mathcal{T}0, \mathcal{T}1) \\ &= d_{b\mu}(0, 2) \\ &= 1 \end{aligned}$$

$$F(x, y) = F(0, 1) = 3$$

$$\gamma(F(0, 1)) = \frac{100}{103}$$

$$\begin{aligned} d_{b\mu}(\mathcal{T}0, \mathcal{T}1) &= 1 \\ &< \gamma(F(0,1))F(0,1) \\ &= \frac{300}{103} \end{aligned}$$

(case 5): $(x, y) = (0, 2)$

$$\begin{aligned} d_{b\mu}(\mathcal{T}x, \mathcal{T}y) &= d_{b\mu}(\mathcal{T}0, \mathcal{T}2) \\ &= d_{b\mu}(0, 0) \\ &= 0 \end{aligned}$$

$$F(x, y) = F(0, 2) = \frac{8}{9}$$

$$\gamma(F(2, 2)) = \frac{225}{227}$$

$$\begin{aligned} d_{b\mu}(\mathcal{T}0, \mathcal{T}2) &= 0 \\ &< \gamma(F(0, 2))F(0, 2) \\ &= \frac{200}{227} \end{aligned}$$

(case 6): $(x, y) = (1, 2)$

$$\begin{aligned} d_{b\mu}(\mathcal{T}x, \mathcal{T}y) &= d_{b\mu}(\mathcal{T}1, \mathcal{T}2) \\ &= d_{b\mu}(2, 0) \\ &= 1 \end{aligned}$$

$$F(x, y) = F(2, 2) = \frac{63}{16}$$

$$\gamma(F(2, 2)) = \frac{1600}{1663}$$

$$\begin{aligned} d_{b\mu}(\mathcal{T}1, \mathcal{T}2) &= 1 \\ &< \gamma(F(1, 2))F(1, 2) \\ &= \frac{6300}{1663} \end{aligned}$$

From the above discussions, we know that

$$d_{b\mu}(\mathcal{T}y, \mathcal{T}z) \leq \gamma(F(y, z))F(y, z) \quad \text{for any} \quad y, z \in Y,$$

where

$$F(y,z) = \frac{1}{\{\mu(y,z)\}^2}[d_{b\mu}(y,z) + |d_{b\mu}(y,\mathcal{T}y) - d_{b\mu}(z,\mathcal{T}z)|].$$

By Corollary 4, we obtain that $\mathcal{T}$ has a unique fixed point, 0 is the unique fixed point of $\mathcal{T}$.

Chapter 5

Conclusion and Future Work

The dissertation comes to its end in the following manners:

- The dissertation is started with brief introduction, pointing out the history and work done by many mathematicians related to the article.
- As supportive material, some abstract spaces like metric space, partial metric space, b-metric space and metric-like space, convergence, completeness and Cauchy criteria are elaborated with proper examples.
- A section is mentioned for brief discussion on fixed point theory. This helps to understand the existence and uniqueness of the fixed point in main results.
- Different mappings are also elaborated for better understanding, that are used in the main results.
- The idea of common fixed point in the sense of metric spaces, b-metric spaces and metric-like spaces under specific contraction mappings is demonstrated. The work of Yu et al. [25] "Common fixed point theorems for $\mathcal{T}_{gF}$-contraction in b-metric-like spaces" is investigated with detailed description.
- One result in the setting of extended b-metric-like spaces is established. These results are the extensions of the results presented by Yu et al. [25].
- In future,

i. The application of given result can be provided.

ii. Using the idea of extended b-metric-like space, one can establish further results.

iii. The idea of new-extended b-metric-like space can be incorporated.

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