CAPITAL UNIVERSITY OF SCIENCE AND TECHNOLOGY, ISLAMABAD

Effect of Heat Generation and Chemical Reaction on MHD Fluid Flow over a Nonlinear Stretching Sheet

by

Sabica Zahid

A thesis submitted in partial fulfillment for the degree of Master of Philosophy

in the

Faculty of Computing
Department of Mathematics

2021

*I dedicate my dissertation work to my **family** and dignified **teachers**. A special feeling of gratitude to my loving parents who have supported me in my studies.*

CERTIFICATE OF APPROVAL

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Acknowledgement

In the name of **ALLAH**, who is the most merciful and beneficent, created the universe and blessed the mankind with intelligence and wisdom to explore its secret. Countless respect and love for **Prophet Muhammad (Peace Be Upon Him)**, the fortune of knowledge, who took the humanity out of ignorance and showed the right path.

I would like to express my deepest gratitude to my supervisor, **Dr. Muhammad Sagheer**, the Head of the Department of Mathematics, for the valuable guidance, encouragement and motivation. His guidance helped me in all the time of research and writing of this thesis. May Allah bless you sir with his blessings.

The acknowledge will surely incomplete if I don't express my deep indebtedness and cordial thanks **Mahzad Ahmed**, for his valuable suggestion and unending cooperation during my thesis. I am also thankful to my best friend **Shakria Azhar**, who helped me during this work directly or indirectly.

I would like to pay great tribute to my **parents**, for their prayers, moral support, encouragment and appreciation.

(Sabica Zahid)

Abstract

This thesis numerically investigates the influence of aligned magnetic field, heat generation and chemical reaction of the flow of an electrically conducting nanofluid past a nonlinear stretching sheet through a porous medium. The partial differential equtions governing the flow problems are converted to ordinary differential equtions by using suitable similarity tramsformations. The transformed equtions are then solved numerically with the help of shooting method. The influence of physical parameters such as nonlinear stretching sheet parameter n, magnetic field parameter M, heat generation parameter Q, Eckert number Ec, Prandtl number Pr, thermophoresis parameter Nt, Brownain motion parameter Nb, Lewis number Le and chemical reaction parameter γ_2 on the velocity profile, temperature distribution, concentration profile, skin friction coefficient, Nusselt number and Sherwood number are studied and presented in graphical and tabular forms. The temperature distribution is also influenced by the presence of Brownain motion parameter Nb, thermal radiation parameter and heat generation parameter.

Contents

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Abbreviations

IVPs	Initial value problems
MHD	Magnetohydrodynamics
ODEs	Ordineary differential equations
PDEs	Partial differential equations
RK	Runge-Kutta

Symbols

μ	Viscosity
ρ	Density
ν	Kinematic viscosity
τ	Stress tensor
k	Thermal conductivity
α	Thermal diffisuitivity
σ	Electrical conductivity
u	x-component of fluid velocity
v	y-component of fluid velocity
B_0	Magnetic field constant
k_0	Permeability constant
a	Stretching constant
T_w	Temperature of the wall
T_∞	Ambient temperature of the nanofluid
T	Temperature
q_r	Radiative heat flux
q	Heat generation constant
q_w	Heat flux
q_m	Mass flux
σ^*	Stefan Boltzmann constant
k^*	Absorption coefficient
ψ	Stream function
ζ	Similarity variable

C_f	Skin friction coefficient
Nu	Nusselt number
Nu_x	Local Nusselt number
Sh	Sherwood number
Sh_x	Local Sherwood number
Re	Reynolds number
Re_x	Local Reynolds number
R	Thermal radiation parameter
n	Stretching parameter
M	Magnetic parameter
G	Permeability parameter
Ec	Eckert number
Pr	Prandtl number
Q	Heat generation parameter
Nb	Brownain motion parameter
Nt	Thermophoresis parameter
γ_1	Chemical reaction parameter
L_e	Lewis number
f	Dimensionless velocity
θ	Dimensionless temperature
ϕ	Dimensionless concentration
C_∞	Ambient concentration
C	Concentration
C_w	Nanoparticles concentration at the stretching surface

Chapter 1

Introduction

The boundary layer problems associated to a stretched surface have drawn a lot of interest during the last few decades, owing to the large number of applications in engineering and industrial manufacturing processes. The boundary layer, defined as the layer of fluid in the region of a bounded area where viscosity effects are prominent, is a valuable concept in physics and fluid mechanics. Furthermore, it is an area in the flow where the fluid deforms with a relative velocity. Each primary fluid has a set of fundamental features that plays a vital role in its dynamics. The stretching and cooling rates are both important in the manufacturing process for the end product's effects. Crane [1] was the first to present the 2D flow of an incompressible liquids within the boundary layer along the stretching surface .

Magnetohydrodynamics (MHD) is a mix of three words: magneto refers to a magnetic field, hydro refers to water, and dynamics refers to motion. The study of the magnetic properties of an electrically conducting fluid, on the other hand, is known as hydromagnetic flow. Plasmas, liquid metals, salt water, and electrolytes are examples of magneto fluids. A lot of scholars discussed at flow models that included hydromagnetic phenomena. Pavlov [2], also studied at the MHD flow of viscous fluids along a stretching sheet. Sarpakaya [3] studied the flow of specific types of fluids in a magnetic field. Alfven [4] established the existence of electromagnetic-hydrodynamic waves. There are a lot of engineering applications of heat transfer in porous media like geothermal energy recovery, thermal energy storage, crude oil extraction and flow fittering media [5]. Nanofluids have nanometer-sized solid particles. They have unique physical and chemical properties. They possess better heat transfer ability and are used in nuclear reactor, transportation and electronics [6]. Buoyancy induced flow in a porous medium which is opposite to horizontal surface is investigated by Cgeng and Chang [7]. Natural convection-heat transfer from a horizontal plate in porous mediumis investegated by Nield and Bejan [8].

In order to improve the thermal conductivity of the base fluid, nanofluids are used. They are used in biomedical and engineering applications. Thermal conductivity of convectional heat transfer is because of suspension of solid particles.it raised the heat transfer coefficient. Solid metal has a higher heat conductivity than liquids. Stability, spreading and dispersion properties of nanofluid surface [9]. The thermal conductiviy of solid metal is higher than of base fluids.

Nanofluid increases the thermal conductivity of the base fluids. They are also used in cooling and other industrial process. Thermal conductivity of convectional heat transfer by suspension of solid psrticles is a recent advancement in this field [10]. Nanoparticles in nanofluids are made up of various metal oxides, carbides, nitrides or nonmetals. Experimental studies have demonstrated that nanofluids only require up to 5% volume fraction of nanoparticles to provide effective heat transfer increases [11].

Nanofluids are also used in fuel cell, nuclear reactors and transportation and many more [12]. Khan and Pop [13] reported that using Buongiorrios model [14] the flow of nanofluid over a strtching sheets. It accelerates the coefficient of heat trnsfer. Using finite element and finite difference approaches to solve a nonlinear stretching sheet Rana and Bhargara [15] conducted similar research. Makinde and Aziz [16] discussed the effect of convective surface boundary condition on the boundary layer flow over a stretching sheet. Khan Mustafa et al [17] discussed into the boundary layer flow for an exponential stretching sheet.

Khan and Shehzad [18] investigated the influence of thermophoresis and Brownain movement on nanofluid and heat transfer rate through a dynamic oscillation sheet.

Many authors [19–24] have contributed to the study of electrically conducting nanofluids in disciplines such as plasma studies, MHD pumps, MHD generators, and bearings. Other important considerations are the effects of viscous discipation, thermal radiation or heat generation on nanofluid boundary layer flow, as well as the features of heat transfer rate immeresed in porous medium. Ahmad et al. [25] studied how MHD viscous flow behaved in a porous material across an exponentially stretched surface with radiative effect. Williamson fluid film flow and heat transfer in the presence of thermal radiation through a porous material across a linear stretching sheet were investigated Shah et al. [26]. In their investigation, they discovered that raising the porosity parameter reduces the flow of thin films and the Lorentz force has an impact on the flow of liquid films. MHD boundary laye flow of nanofluid in porous medium was studied by Zeeshan et al. [27]. Pal and Mandal [28] showed the effect of thermal radiation and heat generation on convective nanofluid flow through a stagnation point in a porous medium. Rama and Chandra [29] used a hybrid technique to numerically investigate the effects of viscous dissipation on MHD boundary layer nanofluid flow through a nonlinear stretching sheet in a porous medium in their paper Haroun et al. [30] developed the spectral relaxation technique to investigate the effects of the chemical reaction, viscous dissipation and radiation on MHD nanofluid flow in a porous medium, and discovered that as the porosity parameter increases, the velocity field decreases while the temperature distribution increases. On the same topic, Geng et al. [31] investigated MHD nanofluid flow and rate of heat transfer between porous medium and stretching sheet.

Malvandi et al. [32] demonstrated a stagnation point nanofluid flow over a nonlinear stretching sheet using suction. They demonstrate that as the suction parameter is increased, the heat transfer rate increase, depending on the heat blowing parameter. Khan and Shehzad [33] investigated the influence of thermophoresis and Brownain motion on third-grade nnofluid and rate of heat transmission through an oscillation dynamic sheet.

The porous medium has many applications in biochemical catalytic vessels, energy diligences, transport development in human lungs and kidneys, thermal isolation, strategy of dense medium heat exchangers and geothermal processes, etc. Further, building material, mineral, leakage of water in stream beds and timber are few specimens of naturally obtainable porous medium.Even for the flow of non-Newtonian liquids, the established Darcy's law is usually applied. Chamkha and Rashad [34] discussed the effect of chemical reactions on MHD flow when heat is generated or absorbed by a uniform vertical permeable surface. The impact of chemical reactions with radiation on heat and mass transfer along the MHD flow was explained by Das [35].

1.1 Thesis Contributions

The current study focuses on numerical solution of MHD nanofluid flow with an angled magnetic field, heatt generation and chemical reaction. By using similarity transformations, the proposed nonlinear PDEs are trasnsformed into the systerm of ODEs . The shooting method is also to get the numerical results of nonlinear ODEs. The numerically results are computed by using MATLAB. The impact of significant parameters on velocity distribution, temperature distribution and concentration distribution, skin friction coefficient, local Nusselt number and local Sherwood number have been discussed in graphs and tables.

1.2 Layout of Thesis

A brief overview of the contents of the thesis is provided below.

Chapter 2 includes some basic definitions and terminologies, which are useful to understand the concepts discussed later on.

Chapter 3 provides an analytical investigation of MHD nanofluid flow in a porous medium caused by a nonlinear stretching sheet. The numerical results of the governing flow equations are derived by the shooting method.

Chapter 4 extends the model flow discussed in Chapter 3 by including the impacts of inclined magnetic field, heat generation and chemical reaction.

Chapter 5 provides the concluding remarks of the thesis.

References used in the thesis are mentioned in **Biblography**.

Chapter 2

Preliminaries

This chapter provides some basic definitions and regulating laws that will be useful in the upcomming chapters.

2.1 Some Basic Terminologies

Definition 2.1.1 (Fluid)
"A fluid is a substance that deforms continuously under the application of a shear (tangential) stress no matter how small the shear stress may be." [36]

Definition 2.1.2 (Fluid Mechanics)
"Fluid mechanics is that branch of science which deals with the behavior of the fluid (liquids or gases) at rest as well as in motion." [37]

Definition 2.1.3 (Fluid Dynamics)
"The study of fluid if the pressure forces are also considered for the fluids in motion, that branch of science is called fluid dynamics." [37]

Definition 2.1.4 (Fluid Statics)
"The study of fluid at rest is called fluid statics." [37]

Definition 2.1.5 (Viscosity)

"Viscosity is defined as the property of a fluid which offers resistance to the movement of one layer of fluid over another adjacent layer of the fluid. Mathematically,

$$\mu = \frac{\tau}{\frac{\partial u}{\partial y}},$$

where μ is viscosity coefficient, τ is shear stress and $\frac{\partial u}{\partial y}$ represents the velocity gradient." [37]

Definition 2.1.6 (Kinematic Viscosity)

"It is defined as the ratio between the dynamic viscosity and density of fluid. It is denoted by symbol ν called **nu**. Mathematically,

$$\nu = \frac{\mu}{\rho}.\text{" [37]}$$

Definition 2.1.7 (Thermal Conductivity)

"The Fourier heat conduction law states that the heat flow is proportional to the temperature gradient. The coefficient of proportionality is a material parameter known as the thermal conductivity which may be a function of a number of variables." [38]

Definition 2.1.8 (Thermal Diffusivity)

"The rate at which heat diffuses by conducting through a material depends on the thermal diffusivity and can be defined as,

$$\alpha = \frac{k}{\rho C_p},$$

where α is the thermal diffusivity, k is the thermal conductivity, ρ is the density and C_p is the specifc heat at constant pressure." [39]

2.2 Types of Fluid

Definition 2.2.1 (Ideal Fluid)

"A fluid, which is incompressible and has no viscosity, is known as an ideal fluid. Ideal fluid is only an imaginary fluid as all the fluids, which exist, have some viscosity." [37]

Definition 2.2.2 (Real Fluid)

"A fluid, which possesses viscosity, is known as a real fluid. In actual practice, all the fluids are real fluids." [37]

Definition 2.2.3 (Newtonian Fluid)

"A real fluid, in which the shear stress is directly proportional to the rate of shear strain (or velocity gradient), is known as a Newtonian fluid." [37]

Definition 2.2.4 (Non-Newtonian Fluid)

"A real fluid in which the shear stress is not directly proportional to the rate of shear strain (or velocity gradient), is known as a non-Newtonian fluid.

$$\tau_{xy} \propto \left(\frac{du}{dy}\right)^m, \qquad m \neq 1$$
$$\tau_{xy} = \mu\left(\frac{du}{dy}\right)^m."\ [37]$$

Definition 2.2.5 (Magnetohydrodynamics)

"Magnetohydrodynamics (MHD) is concerned with the mutual interaction of fluid flow and magnetic fields. The fluids in question must be electrically conducting and non-magnetic, which limits us to liquid metals, hot ionised gases (plasmas) and strong electrolytes." [40]

2.3 Types of Flow

Definition 2.3.1 (Rotational Flow)

"Rotational flow is that type of flow in which the fluid particles while flowing along stream-lines, also rotate about their own axis." [37]

Definition 2.3.2 (Irrotational Flow)

"Irrotational flow is that type of flow in which the fluid particles while flowing along stream-lines, do not rotate about their own axis then this type of flow is called irrotational flow." [37]

Definition 2.3.3 (Compressible Flow)

"Compressible flow is that type of flow in which the density of the fluid changes from point to point or in other words the density (ρ) is not constant for the fluid, Mathematically,

$$\rho \neq k,$$

where k is constant." [37]

Definition 2.3.4 (Incompressible Flow)

"Incompressible flow is that type of flow in which the density is constant for the fluid. Liquids are generally incompressible while gases are compressible, Mathematically,

$$\rho = k,$$

where k is constant." [37]

Definition 2.3.5 (Steady Flow)

"If the flow characteristics such as depth of flow, velocity of flow, rate of flow at any point in open channel flow do not change with respect to time, the flow is said to be steady flow. Mathematically,

$$\frac{\partial Q}{\partial t} = 0,$$

where Q is any fluid property." [37]

Definition 2.3.6 (Unsteady Flow)
"If at any point in open channel flow, the velocity of flow, depth of flow or rate of flow changes with respect to time, the flow is said to be unsteady. Mathematically,

$$\frac{\partial Q}{\partial t} \neq 0,$$

where Q is any fluid property." [37]

Definition 2.3.7 (Internal Flow)
"Flows completely bounded by solid surfaces are called internal or duct flows." [36]

Definition 2.3.8 (External Flow)
"Flows over bodies immersed in an unbounded fluid are said to be external flows." [36]

2.4 Modes of Heat Transfer

Definition 2.4.1 (Heat Transfer)
"Heat transfer is a branch of engineering that deals with the transfer of thermal energy from one point to another within a medium or from one medium to another due to the occurrence of a temperature difference." [38]
Definition 2.4.2 (Conduction)

"The transfer of heat within a medium due to a diffusion process is called conduction." [38]

Definition 2.4.3 (Convection)
"Convection heat transfer is usually defined as the energy transport affected by the motion of a fluid. The convection heat transfer between two dissimilar media is governed by Newtons law of cooling." [38]

Definition 2.4.4 (Thermal Radiation)
"Thermal radiation is defined as radiant (electromagnetic) energy emitted by a medium and is solely due to the temperature of the medium." [38]

2.5 Dimensionless Numbers

Definition 2.5.1 (Eckert Number)
"It is a dimensionless number used in continuum mechanics. It describes the relation between flows and the boundary layer enthalpy difference and it is used for characterized heat dissipation. Mathematically,

$$Ec = \frac{u^2}{C_p \nabla T}$$

where C_p denotes the specific heat." [36]

Definition 2.5.2 (Prandtl Number)
"It is the ratio between the momentum diffusivity ν and the thermal diffusivity α. Mathematically, it can be defined as

$$Pr = \frac{\nu}{\alpha} = \frac{\frac{\mu}{\rho}}{\frac{k}{C_p \rho}} = \frac{\mu C_p}{k}$$

where μ represents the dynamic viscosity, Cp denotes the specific heat and k stands for the thermal conductivity. The relative thickness of thermal and momentum boundary layer is controlled by Prandtl number. For small Pr, heat distributed rapidly corresponds to the momentum.” [36]

Definition 2.5.3 (Skin Friction Coefficient)

“The skin friction coefficient is typically defined as

$$C_f = \frac{2\tau_w}{\rho w_\infty^2}$$

where τ_w denotes the local wall shear stress and ρ is the density and w_∞ is the free stream velocity (usually taken outside the boundary layer or at the inlet). It expresses the dynamic friction resistance originating is viscous fluid flow around a fixed wall” [41]

Definition 2.5.4 (Nusselt Number)

“It is the relationship between the convective to the conductive heat transfer through the boundary layer of the surface. It is a dimensionless number which was first introduced by the German mathematician Nusselt. Mathematically, it is defined as

$$Nu = \frac{hL}{k}$$

where h stands for the convection heat transfer, L for the characteristic length and k stands for thermal conductivity.” [42]

Definition 2.5.5 (Sherwood Number)

“It is a nondimensional quantity which shows the ratio of the mass transport byconvection to the transfer of mass by diffusion.

Mathematically,

$$Sh = \frac{kL}{D}$$

where L is the characteristics length, D is the mass diffusivity and k is the mass transfer" coeffcient." [43]

Definition 2.5.6 (Reynolds Number)
"It is defined as the ratio of inertia force of a flowing fluid and the viscous force of the fluid. Mathematically,

$$Re = \frac{UL}{\nu},$$

where U denotes the free stream velocity, L is the characteristic length and ν stands for the kinematic viscosity." [37]

2.6 Governing Laws

2.6.1. Law of Consevation of Mass

"The principle of conservation of mass can be stated as the time rate of change of mass in fixed volume is equal to the net rate of flow of mass across the surface. Mathematically, it can be written as

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \mathbf{u}) = 0.\text{"} \ [38]$$

2.6.2. Law of Conservation of Momentum

"The momentum equation states that the time rate of change of linear momentum of a given set of particles is equal to the vector sum of all the external forces acting on the particles of the set, provided Newtons third law of action and reaction governs the internal forces. Mathematically, it can be written as:

$$\frac{\partial}{\partial t}(\rho \mathbf{u}) + \nabla.[(\rho \mathbf{u})\mathbf{u}] = \nabla.\mathbf{T} + \rho g.\text{"} \ [38]$$

2.6.3. Law of Consevation of Energy

"The law of conservation of energy states that the time rate of change of the total energy is equal to the sum of the rate of work done by the applied forces and change of heat content per unit time.

$$\frac{\partial \rho}{\partial t} + \nabla.\rho\mathbf{u} = -\nabla.\mathbf{q} + Q + \phi,$$

where ϕ is the dissipation function." [38]

2.7 Shooting Method

It is a numerical method for solving boundary value problems that are expressed as ordinary differential equations. The system of first order equation is created by converting the ordinary differential equation(s) into the system of first order equation(s). The condition(s) that are missing have been gussed. An appropriate methodology, such as the Runge-Kutta method of order four, is used to solve the resulting IVP. If the solution does not meet the requirements, the missing conditions are refined by some suitable numerical techniques e.g, Newton's method. To elaborate the shooting method a couple of boundary value problems have been considered in the upcomming discussion.

2.7.1 A Second Order Boundary Value Problem

Consider the nonlinear boundary value problem that follows.

$$\left.\begin{aligned} & f''(x) = f(x)f'(x) + 2f^2(x) \\ & f(0) = 0, \qquad f(G) = J. \end{aligned}\right\} \tag{2.1}$$

To reduce the order of the above boundary value problem, introduce the following notations.

$$f = Y_1 \qquad f' = Y_1' = Y_2 \qquad f'' = Y_2'. \tag{2.2}$$

As a result, (2.1) is transformed into the first order ODEs system shown below.

$$Y_1' = Y_2, \qquad\qquad Y_1(0) = 0, \tag{2.3}$$

$$Y_2' = Y_1Y_2 + 2Y_1^2, \qquad\qquad Y_2(0) = w, \tag{2.4}$$

where w is the missing initial condition.

The above IVP will be numerically solved by the RK-4 method. The missing condition w is to be chosen such that

$$(Y_1, w)_{x=G} = J. \tag{2.5}$$

For convenience, now onward $(Y_1, w)_{x=G}$ will be denoted by $Y_1(w)$.

Let us further denote $Y_1(w) - J$ by $H(w)$, so that

$$H(w) = 0. \tag{2.6}$$

The above equation can be solved by using Newton's method with the following iterative formula.

$$w^{(n+1)} = w^{(n)} - \left(\frac{H(w)}{\frac{\partial H}{\partial w}}\right)_{w=w^{(n)}}$$

or

$$w^{(n+1)} = w^{(n)} - \left(\frac{Y_1(w) - J}{\frac{\partial Y_1}{\partial w}}\right)_{w=w^{(n)}}. \tag{2.7}$$

To find $\frac{\partial Y_1}{\partial w}$, introduce the following notations.

$$\frac{\partial Y_1}{\partial w} = Y_3, \quad \frac{\partial Y_2}{\partial w} = Y_4. \tag{2.8}$$

As a result of these new notations, the Newton's iterative scheme, will then get the form.

$$w^{(n+1)} = w^{(n)} - \frac{Y_1(w^{(n)}) - J}{Y_3(w^{(n)})}. \tag{2.9}$$

Now differentiating the system of two first order ODEs (2.3)-(2.4) with respect to w, we get another system of ODEs, as follows.

$$Y_3' = Y_4, \qquad Y_3(0) = 0. \tag{2.10}$$

$$Y_4' = Y_3Y_2 + Y_1Y_4 + 4Y_1Y_3, \qquad Y_4(0) = 1. \tag{2.11}$$

Writing all the four ODEs (2.3) and (2.4), (2.10) and (2.11) together, we have the following initial value problem.

$$\left.\begin{aligned} Y_1' &= Y_2, & Y_1(0) &= 0. \\ Y_2' &= Y_1Y_2 + 2Y_1^2, & Y_2(0) &= w. \\ Y_3' &= Y_4, & Y_3(0) &= 0. \\ Y_4' &= Y_3Y_2 + Y_1Y_4 + 4Y_1Y_3, & Y_4(0) &= 1. \end{aligned}\right\} \tag{2.12}$$

The above system together will be solved numerically by the Runge-Kutta method of order four.

The stopping criteria for the Newton's technique is set as,

$$\mid Y_1(w) - J \mid < \epsilon,$$

where $\epsilon > 0$ is an arbitrarily small positive number.

The algorithmic form of the shooting method for this problem and similarly, all second order two point boundary value problems in the form of ordinary differential equations have been explained in the following sectopns.

Step I:

Convert the provided second order differential equation to a system of four first order ODEs, similar to the ones in (2.12).

Step II:

Choose a missing condition $w = w^{(0)}$ in (2.12) and solve this system by (say) RK-4 mathod.

Step III:

Choose a sufficiently small number $\epsilon > 0$.

Step IV:

If $|(Y_1, w^{(0)} - J)| < \epsilon$, then stop

If $|(Y_1, w^{(0)} - J)| > \epsilon$, then compute the next gusess $w^{(1)}$ by using (2.9) with $n = 0$ and go back to Step-II and take $w = w^{(1)}$ in it.

Continue repeating Step-II and then Step-IV until in Step-IV, we get

$$|(Y_1, w^{(k)} - J)| < \epsilon$$

for some $k \ \epsilon$ 0,1,2,3,......w, where w is the whole number.

Chapter 3

MHD Nanofluids Flow and Heat Transfer Induced by a Nonlinear Stretching Sheet in the Porous Medium

3.1 Introduction

The numerical analysis of MHD nanofluid flow across a nonlinear stretching sheet saturated in aporous medium in the presence of a magnetic field and heat radiation is discussed in this chapter. Using the similarity transformations, the governing nonlinear PDEs are transformed into a system of dimensionless ODEs.

In order to solve the ODEs, the shooting technique is implemented in MATLAB. The numerical solution for various parameters for the dimensionless velocity profile, temperature distribution, and concentration distribution is described at the end of this chapter.

Investigation of obtained numerical results are given through the tables and graphs.

3.2 Mathematical Modeling

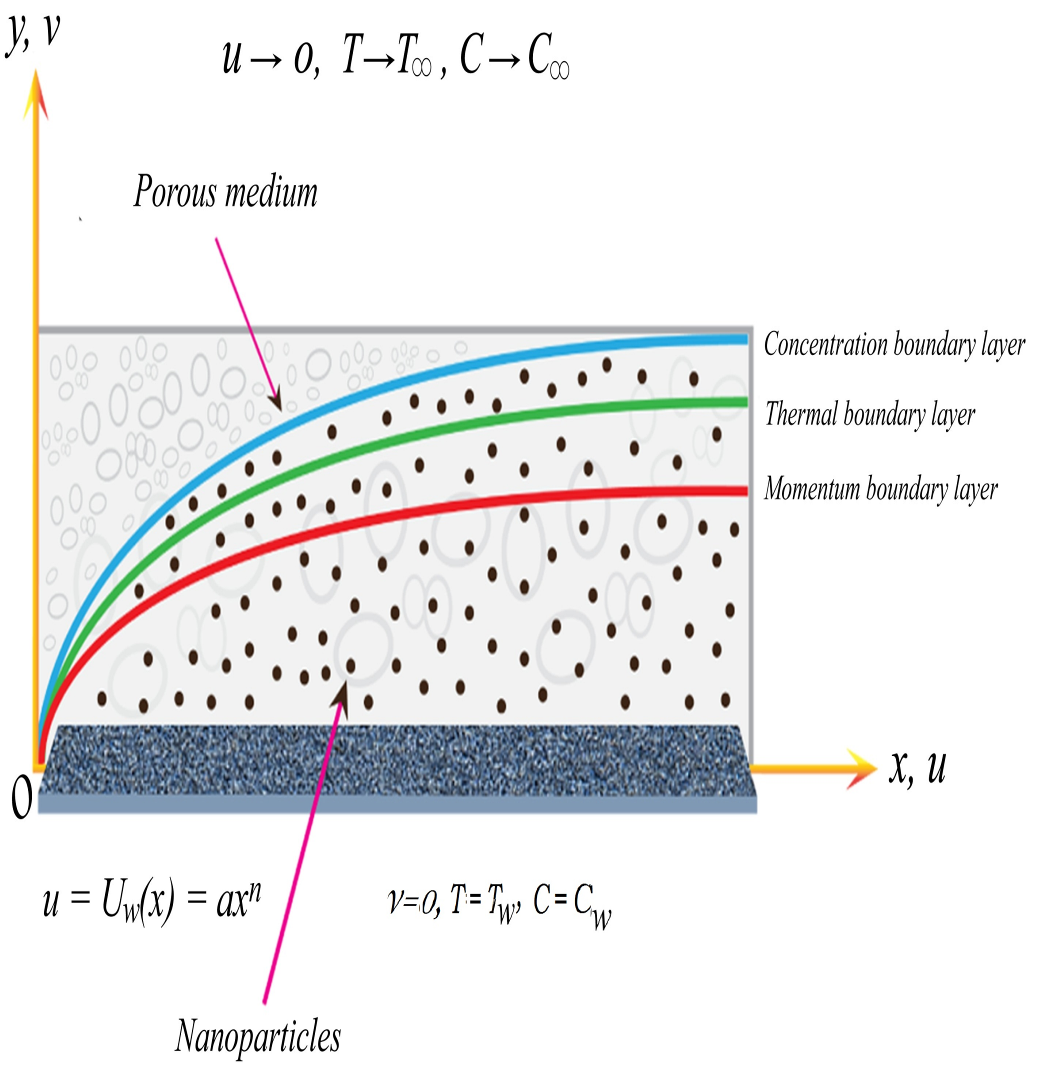

FIGURE 3.1: Systematic representation of physical model.

A $2D$ flow of MHD nanofluid past a nonlinear stretching sheet with $y = 0$ has been investigated. The flow is considered along y-axis with $y > 0$. It is assumed that the variable stretching velocity and the variable magnetic field of the nanofluid flow are $U_w(x){=}ax^n$ and $B(x){=}B_0x^{(n-1)/2}$.

At the stretching surface, the wall temperature T_w and the nanoparticle fraction C_w have been considered to be constant. The ambient temperature and nanoparticle fraction are indicated by T_∞ and C_∞, respectively, as y approaches infinity.

The set of equations describing the flow are as follows.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{3.1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} - \frac{\sigma B^2(x)u}{\rho_f} - \frac{\nu}{k}u, \tag{3.2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} + \tau\left(D_B\frac{\partial C}{\partial y}\frac{\partial T}{\partial y} + \frac{D_T}{T_\infty}\left(\frac{\partial T}{\partial y}\right)^2\right) \tag{3.3}$$

$$- \frac{1}{(\rho c_p)_f}\frac{\partial q_r}{\partial y} + \frac{\nu}{c_p}\left(\frac{\partial T}{\partial y}\right)^2,$$

$$u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} = D_B\frac{\partial^2 C}{\partial y^2} + \left(\frac{D_T}{T_\infty}\right)\frac{\partial^2 T}{\partial y^2}. \tag{3.4}$$

The associated BCs have been taken as.

$$\left.\begin{array}{lllll} y = 0: & U_w = ax^n, & v = 0, & T = T_w, & C = C_w, \\ y \to \infty: & u \to 0, & v \to 0, & T \to T_\infty, & C \to C_\infty \end{array}\right\} \tag{3.5}$$

In the above model, x axis is along the sheet, the direction perpendicular to the sheet is y, u and v are the horizontal and vertical velocities. The radiative heat flux is given by

$$q_r = -\frac{4\sigma^*}{3k^*}\frac{\partial T^4}{\partial y},$$

The Stefan-Boltzman constant is $\sigma*$, and the absorption coefficient is k^*. If the temperature difference is minimal, the temperature T^4 can be increased using the Taylor series to roughly T_∞, as shown below.

$$T^4 = T_\infty^4 + 4T_\infty^3(T - T_\infty) + 6T_\infty^2(T - T_\infty)^2 + ...$$

$$T^4 = T_\infty^4 + 4T_\infty^3(T - T_\infty),$$

$$T^4 = T_\infty^4 + 4T_\infty^3 T - 4T_\infty^4,$$

$$T^4 = -3T_\infty^4 + 4T_\infty^3 T.$$

$$T^4 = 4T_\infty^3 T - 3T_\infty^4.$$

For the conversion of the mathematical model (3.1)-(3.4) into asystem of ODEs, the following similarity transformations have been obtained

$$\left.\begin{aligned} &\zeta = y\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{n-1}{2}}, \qquad u = ax^n f'(\zeta) \\ &v = -x^{\frac{n-1}{2}}\sqrt{\frac{\nu a(n+1)}{2}}\left(f(\zeta) + \left(\frac{n-1}{n+1}\right)\zeta f'(\zeta)\right) \\ &\theta(\zeta) = \frac{T - T_\infty}{T_w - T_\infty}, \qquad \phi(\zeta) = \frac{C - C_\infty}{C_w - C_\infty}. \end{aligned}\right\} \tag{3.6}$$

The detailed procedure for the conversion of (3.1)-(3.4) into the dimensionless form has been presented below,

$$\begin{aligned} \zeta &= y\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{n-1}{2}}, \\ \frac{\partial\zeta}{\partial x} &= y\sqrt{\frac{a(n+1)}{2\nu}}\left(\frac{n-1}{2}\right)x^{\frac{n-3}{2}}, \\ \frac{\partial u}{\partial x} &= \frac{\partial}{\partial x}\left(af'(\zeta)x^n\right), \\ &= a\frac{\partial}{\partial x}\left(f'(\zeta)x^n\right), \\ &= a\left(nx^{n-1}f'(\zeta) + x^n f''(\zeta)\frac{\partial\zeta}{\partial x}\right), \\ &= a\left(nx^{n-1}f'(\zeta) + x^n f''(\zeta)y\sqrt{\frac{a(n+1)}{2\nu}}\left(\frac{n-1}{2}\right)x^{\frac{n-3}{2}}\right), \\ &= a\left(nx^{n-1}f'(\zeta) + x^{n-1}f''\zeta\left(\frac{n-1}{2}\right)\right), \\ &= ax^{n-1}\left(nf'(\zeta) + \zeta f''(\zeta)\left(\frac{n-1}{2}\right)\right). \end{aligned} \tag{3.7}$$

$$\begin{aligned} v &= -x^{\frac{n-1}{2}}\sqrt{\frac{\nu a(n+1)}{2}}\left(f(\zeta) + \left(\frac{n-1}{n+1}\right)\zeta f'(\zeta)\right). \\ \frac{\partial v}{\partial y} &= \frac{\partial}{\partial y}\left[-x^{\frac{n-1}{2}}\sqrt{\frac{(n+1)\nu a}{2}}\left(f(\zeta) + \left(\frac{n-1}{n+1}\right)\zeta f'(\zeta)\right)\right] \\ &= -x^{\frac{n-1}{2}}\sqrt{\frac{(n+1)\nu a}{2}}\left[f'(\zeta)\frac{\partial\zeta}{\partial y} + \left(\frac{n-1}{n+1}\right)\zeta f''(\zeta)\frac{\partial\zeta}{\partial y} + \left(\frac{n-1}{n+1}\right)f'(\zeta)\frac{\partial\zeta}{\partial y}\right] \\ &= -x^{\frac{n-1}{2}}\sqrt{\frac{(n+1)\nu a}{2}}\left[f'(\zeta) + \left(\frac{n-1}{n+1}\right)\zeta f''(\zeta)\right]\sqrt{\frac{(n+1)a}{2\nu_f}}x^{\frac{n-1}{2}} \\ &\quad - x^{\frac{n-1}{2}}\sqrt{\frac{(n+1)\nu a}{2}}\left(\left(\frac{n-1}{n+1}\right)f'(\zeta)\right)\sqrt{\frac{(n+1)a}{2\nu}}x^{\frac{n-1}{2}} \\ &= -\frac{a}{2}x^{n-1}(n+1)\left(f'(\zeta) + \left(\frac{n-1}{n+1}\right)\zeta f''(\zeta) + \left(\frac{n-1}{n+1}\right)f'(\zeta)\right) \end{aligned}$$

$$
\begin{aligned}
&= -\frac{a}{2}x^{n-1}\left(f'(\zeta)(n+1) + (n-1)\zeta f''(\zeta) + (n-1)f'(\zeta)\right) \\
&= -\frac{a}{2}x^{n-1}f'(\zeta)(n+1+n-1) - \frac{a}{2}x^{n-1}(n-1)\zeta f''(\zeta) \\
&= -\frac{a}{2}x^{n-1}2nf'(\zeta) - \frac{a}{2}x^{n-1}(n-1)\zeta f''(\zeta) \\
&= -ax^{n-1}nf'(\zeta) - ax^{n-1}\left(\frac{n-1}{2}\right)\zeta f''(\zeta). \qquad (3.8)
\end{aligned}
$$

Equation (3.1) is easily satisfied by using (3.7) and (3.8) as follows

$$
\begin{aligned}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= ax^{n-1}nf'(\zeta) + ax^{n-1}\left(\frac{n-1}{2}\right)\zeta f''(\zeta) - ax^{n-1}nf'(\zeta) \\
&\quad - ax^{n-1}\left(\frac{n-1}{2}\right)\zeta f''(\zeta), \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0. \qquad (3.9)
\end{aligned}
$$

Now, for the momentum equation (3.2) the following derivatives are required,

$$
\begin{aligned}
\frac{\partial u}{\partial y} &= \frac{\partial}{\partial y}(ax^n f'(\zeta)) \\
&= a\frac{\partial}{\partial y}(x^n f'(\zeta)) \\
&= ax^n f''(\zeta)\frac{\partial \zeta}{\partial y} \\
&= ax^n f''(\zeta)\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{n-1}{2}}. \qquad (3.10)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 u}{\partial y^2} &= ax^n f'''(\zeta)\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{n-1}{2}}\frac{\partial \zeta}{\partial y}, \\
&= af'''(\zeta)\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{n-1}{2}}x^n\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{n-1}{2}} \\
&= a^2x^{2n-1}f'''(\zeta)\left(\frac{n+1}{2\nu}\right). \qquad (3.11)
\end{aligned}
$$

$$
\begin{aligned}
u\frac{\partial u}{\partial x} &= ax^n f'(\zeta)\left(ax^{n-1}nf'(\zeta) + ax^{n-1}\left(\frac{n-1}{2}\right)\zeta f''(\zeta)\right) \\
&= a^2x^{2n-1}nf'^2(\zeta) + a^2x^{2n-1}\left(\frac{n-1}{2}\right)\zeta f'(\zeta)f''(\zeta). \qquad (3.12)
\end{aligned}
$$

$$
\begin{aligned}
v\frac{\partial u}{\partial y} &= -\sqrt{\frac{a\nu(n+1)}{2}}x^{\frac{n-1}{2}}\left(\zeta f'(\zeta)\left(\frac{n-1}{n+1}\right) + f(\zeta)\right)\left(ax^n f''(\zeta)\sqrt{\frac{a(n+1)}{2}}x^{\frac{n-1}{2}}\right) \\
&= -\sqrt{\frac{a\nu_f(n+1)}{2}}x^{\frac{n-1}{2}}f'(\zeta)\zeta\frac{n-1}{n+1}ax^n f''(\zeta)\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{n-1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&-\sqrt{\frac{a\nu(n+1)}{2}}x^{\frac{n-1}{2}}f(\zeta)ax^{n}f''(\zeta)\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{n-1}{2}}\\
&=-\frac{a^2(n+1)}{2}x^{2n-1}f'(\zeta)f''(\zeta)\zeta\left(\frac{n-1}{n+1}\right)\\
&-\frac{a^2(n+1)}{2}x^{2n-1}f(\zeta)f''(\zeta). \qquad (3.13)
\end{aligned}
$$

Using (3.12) and (3.13), the left side of (3.2) becomes

$$
\begin{aligned}
u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} &= a^2x^{2n-1}nf'^2(\zeta)+a^2x^{2n-1}\left(\frac{n-1}{2}\right)\zeta f'(\zeta)f''(\zeta)\\
&\quad-\frac{a^2(n+1)}{2\nu}x^{2n-1}f(\zeta)f''(\zeta)\Big)\\
&=a^2x^{2n-1}nf'^2(\zeta)+a^2x^{2n-1}\left(\frac{n-1}{2}\right)\zeta f'(\zeta)f''(\zeta)\\
&\quad-\frac{a^2(n+1)}{2}x^{2n-1}\xi f'(\zeta)f''(\zeta)\left(\frac{n-1}{n+1}\right)-\frac{a^2(n+1)}{2}x^{2n-1}f(\zeta)f''(\zeta)\\
&=\left(a^2x^{2n-1}nf'^2(\zeta)-\frac{a^2(n+1)}{2}x^{2n-1}f(\zeta)f''(\zeta)\right),\\
&=a^2x^{2n-1}\left(nf'^2(\zeta)-\left(\frac{n+1}{2}\right)f(\zeta)f''(\zeta)\right). \qquad (3.14)
\end{aligned}
$$

Using (3.11), in the right side of (3.2) becomes

$$
\begin{aligned}
\nu\left(\frac{\partial^2 u}{\partial y^2}\right)-\frac{\sigma B^2(x)}{\rho_f}u-\frac{\nu}{k}u &= \nu\left(a^2x^{2n-1}f'''(\zeta)\left(\frac{n+1}{2\nu}\right)\right)\\
&\quad-\frac{\sigma B^2(x)a}{\rho_f}x^{n-1}f'(\zeta)-\frac{\nu}{k}ax^{n-1}f'(\zeta)\\
&=a^2x^{2n-1}f'''(\zeta)\left(\frac{n+1}{2}\right)-\frac{\sigma B_0^2a}{\rho_f}x^{2n-1}f'(\zeta)-\frac{\nu}{k}a^2x^{2n-1}f'(\zeta). \qquad (3.15)
\end{aligned}
$$

Comparing (3.14) and (3.15), the dimensionless form of (3.2) can be written as

$$
\begin{aligned}
&u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\left(\frac{\partial^2 u}{\partial y^2}\right)-\frac{\sigma B^2(x)}{\rho_f}u-\frac{\nu}{k}u,\\
&a^2x^{2n-1}\left(nf'^2(\zeta)-\left(\frac{n+1}{2}\right)f(\zeta)f''(\zeta)\right)=a^2x^{2n-1}f'''(\zeta)\left(\frac{n+1}{2\nu}\right)\\
&-\frac{\sigma B_0^2a}{\rho}x^{2n-1}f'(\zeta)-\frac{\nu}{k}a^2x^{2n-1}f'(\zeta),\\
&\left(\frac{2n}{n+1}\right)f'^2(\zeta)-f(\eta)f''(\zeta)=f'''(\zeta)-Mf'-Gf'(\zeta).
\end{aligned}
$$

$$f'''(\zeta) = \left(\frac{2n}{n+1}\right) f'^2(\zeta) - f(\zeta)f''(\zeta) - (M+G)\,f'(\zeta)$$
$$.f'''(\zeta) - \left(\frac{2n}{n+1}\right) f'^2(\zeta) + f(\zeta)f''(\zeta) - (M+G)\,f'(\zeta) = 0 \tag{3.16}$$

Now, for the conversion of energy equation (3.3) the following derivatives are required,

$$T = \theta(\zeta)(T_w - T_\infty) + T_\infty.$$
$$\frac{\partial T}{\partial x} = (T_w - T_\infty)\theta'(\zeta)\frac{\partial \zeta}{\partial x},$$
$$= (T_w - T_\infty)y\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{n-3}{2}}\left(\frac{n-1}{2}\right)\theta'(\zeta) \tag{3.17}$$
$$\frac{\partial T}{\partial y} = (T_w - T_\infty)\theta'(\zeta)\frac{\partial \zeta}{\partial y}$$
$$= (T_w - T_\infty)\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{n-1}{2}}\theta'(\zeta). \tag{3.18}$$
$$\frac{\partial^2 T}{\partial y^2} = (T_w - T_\infty)\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{n-1}{2}}\theta''(\zeta)\frac{\partial \zeta}{\partial y},$$
$$\frac{\partial^2 T}{\partial y^2} = (T_w - T_\infty)\left(\frac{a(n+1)}{2\nu}\right)x^{n-1}\theta''(\zeta). \tag{3.19}$$
$$\left(\frac{\partial u}{\partial y}\right)^2 = \left(ax^{\frac{3n-1}{2}}\sqrt{\frac{a(n+1)}{2\nu}}f''(\zeta)\right)^2$$
$$\left(\frac{\partial u}{\partial y}\right)^2 = a^2x^{3n-1}\frac{a(n+1)}{2\nu}f''^2(\zeta). \tag{3.20}$$
$$\left(\frac{\partial T}{\partial y}\right)^2 = x^{n-1}\frac{(n+1)a}{2\nu}(T_w - T_\infty)^2\theta'^2(\zeta). \tag{3.21}$$
$$\tau\left(D_B\left(\frac{\partial T}{\partial y}\frac{\partial C}{\partial y}\right) + \frac{D_T}{T_\infty}\left(\frac{\partial T}{\partial y}\right)^2\right) = \tau\left(\frac{D_T}{T_\infty}x^{n-1}\frac{(n+1)a}{2\nu_f}(T_w - T_\infty)^2\theta'^2(\zeta)\right)$$
$$+\tau\left(D_B\left[x^{\frac{n-1}{2}}\sqrt{\frac{(n+1)a}{2\nu}}(T_w - T_\infty)\theta'(\zeta)\right]\left[x^{\frac{n-1}{2}}\sqrt{\frac{(n+1)a}{2\nu}}(C_w - C_\infty)\phi'(\zeta)\right]\right)$$
$$= \tau D_B x^{n-1}\left(\frac{(n+1)a}{2\nu}\right)(T_w - T_\infty)(C_w - C_\infty)\theta'(\zeta)\phi'(\zeta)$$
$$+\tau\frac{D_T}{T_\infty}x^{n-1}\left(\frac{(n+1)a}{2\nu}\right)(T_w - T_\infty)^2\theta'^2(\zeta)$$
$$= \frac{\tau D_B(C_w - C_\infty)}{\nu}x^{n-1}\left(\frac{(n+1)a}{2}\right)(T_w - T_\infty)\theta'(\eta)\phi'(\zeta)$$
$$+\frac{\tau D_T(T_w - T_\infty)}{T_\infty\nu}x^{n-1}\left(\frac{(n+1)a}{2}\right)(T_w - T_\infty)\theta'^2(\zeta), \tag{3.22}$$

$$
\begin{aligned}
\frac{\partial q_r}{\partial y} &= -\frac{16\sigma^*}{3k^*}T_\infty^3\frac{\partial^2 T}{\partial y^2},\\
\frac{\partial q_r}{\partial y} &= -\frac{16\sigma^*}{3k^*}T_\infty^3 x^{n-1}\frac{a(n+1)}{2\nu}(T_w-T_\infty)\theta''(\zeta). \qquad (3.23)\\
(T-T_\infty) &= ((T_w-T_\infty)\theta(\zeta)+T_\infty)-T_\infty\\
(T-T_\infty) &= (T_w-T_\infty)\theta(\zeta). \qquad (3.24)
\end{aligned}
$$

Using (3.17) and (3.18) in the left side of (3.3), we get

$$
\begin{aligned}
u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y} &= ax^n f'(\zeta)\left[(T_w-T_\infty)\left(\frac{n-1}{2x}\right)\zeta\theta'(\zeta)\right]+\left[-x^{\frac{n-1}{2}}\sqrt{\frac{a(n+1)}{2\nu}}\right.\\
&\left.\left[\left(\frac{n-1}{n+1}\right)\zeta f'(\zeta)+f(\zeta)\right]\right]\left[(T_w-T_\infty)\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{n-1}{2}}\theta'(\zeta)\right],\\
&= ax^{n-1}(T_w-T_\infty)\left(\frac{n-1}{2}\right)\zeta f'(\zeta)\theta'(\zeta)\\
&\quad -\left(\frac{(n+1)a}{2}\right)x^{n-1}(T_w-T_\infty)\left(\frac{n-1}{n+1}\right)\zeta\theta'(\zeta)f'(\zeta)\\
&\quad -\left(\frac{a(n+1)}{2}\right)x^{n-1}(T_w-T_\infty)f(\zeta)\theta'(\zeta),\\
&= ax^{n-1}\left(\frac{n-1}{2}\right)(T_w-T_\infty)\zeta f'(\zeta)\theta'(\zeta)\\
&\quad -ax^{n-1}\left(\frac{n-1}{2}\right)(T_w-T_\infty)\zeta f'(\zeta)\theta'(\zeta)\\
&\quad -ax^{n-1}\left(\frac{n+1}{2}\right)(T_w-T_\infty)f(\zeta)\theta'(\zeta)\\
u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y} &= -ax^{n-1}\left(\frac{n+1}{2}\right)(T_w-T_\infty)f(\zeta)\theta'(\zeta). \qquad (3.25)
\end{aligned}
$$

Using (3.19)-(3.23) in the right side of (3.3), we get

$$
\begin{aligned}
&\alpha\frac{\partial^2 T}{\partial y^2}+\tau\left(D_B\left(\frac{\partial T}{\partial y}\frac{\partial C}{\partial y}\right)+\frac{D_T}{T_\infty}\left(\frac{\partial T}{\partial y}\right)^2\right)+\frac{\nu}{Cp}\left(\frac{\partial u}{\partial y}\right)^2-\frac{1}{(\rho Cp)_f}\frac{\partial q_r}{\partial y}\\
&=\alpha\left(x^{n-1}\left(\frac{a(n+1)}{2\nu}\right)(T_w-T_\infty)\theta''(\zeta)\right)\\
&\quad+\frac{\tau D_B(C_w-C_\infty)}{\nu}x^{n-1}\left(\frac{(n+1)a}{2}\right)(T_w-T_\infty)\theta'(\zeta)\phi'(\zeta)\\
&\quad+\frac{\tau D_T(T_w-T_\infty)}{T_\infty\nu}x^{n-1}\left(\frac{(n+1)a}{2}\right)(T_w-T_\infty)\theta'^2(\zeta),
\end{aligned}
$$

$$+\frac{\nu}{Cp}\left(a^2x^{3n-1}\left(\frac{a(n+1)}{2\nu}\right)f''^2(\zeta)\right)$$
$$-\frac{1}{(\rho Cp)_f}\left(\frac{16\sigma^* T_\infty^3}{3k^*}(T_w-T_\infty)x^{n-1}\left(\frac{a(n+1)}{2\nu}\right)\theta''(\zeta)\right) \tag{3.26}$$

With the help of (3.25) and (3.26), the dimensionless form of (3.3) shown is obtained, as below.

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2}+\tau\left(D_B\left(\frac{\partial T}{\partial y}\frac{\partial C}{\partial y}\right)+\frac{D_T}{T_\infty}\left(\frac{\partial T}{\partial y}\right)^2\right)$$
$$+\frac{\nu}{Cp}\left(\frac{\partial u}{\partial y}\right)^2-\frac{1}{(\rho Cp)_{nf}}\frac{\partial q_r}{\partial y}.$$
$$\Rightarrow -ax^{n-1}\left[\left(\frac{n+1}{2}\right)(T_w-T_\infty)f(\zeta)\theta'(\zeta)\right]=\alpha\left[x^{n-1}\left(\frac{a(n+1)}{2\nu}\right)(T_w-T_\infty)\theta''(\zeta)\right]$$
$$+\frac{\tau D_B(C_w-C_\infty)}{\nu}x^{n-1}\left(\frac{(n+1)a}{2}\right)(T_w-T_\infty)\theta'(\zeta)\phi'(\zeta)+\frac{\tau D_T(T_w-T_\infty)}{T_\infty\nu},$$
$$x^{n-1}\left(\frac{(n+1)a}{2}\right)(T_w-T_\infty)\theta'^2(\zeta)+\frac{\nu}{Cp}\left(a^2x^{3n-1}\left(\frac{a(n+1)}{2\nu}\right)f''^2(\zeta)\right)$$
$$-\frac{1}{(\rho Cp)_f}\left(\frac{16\sigma^* T_\infty^3}{3k^*}(T_w-T_\infty)x^{n-1}\left(\frac{a(n+1)}{2\nu}\right)\theta''(\zeta)\right),$$
$$\Rightarrow -f(\zeta)\theta'(\zeta)=\frac{\alpha}{\nu}\theta''(\zeta)+\frac{(\rho Cp)_p}{(\rho Cp)_f}\frac{D_B(C_w-C_\infty)}{\nu}\theta'(\zeta)\phi'(\zeta)$$
$$+\frac{(\rho Cp)_p}{(\rho Cp)_f}\frac{D_T(T_w-T_\infty)}{T_\infty\nu}\theta'^2+\frac{\nu}{Cp(T_w-T_\infty)}a^2x^{2n}f''^2+\frac{16\sigma^* T_\infty^3}{(\rho Cp)_f\nu 3k^*}\theta''(\zeta)$$
$$-f\theta'=\frac{1}{\mathrm{Pr}}\theta''(\zeta)+Nb\theta'(\zeta)\phi'(\zeta)+Nt\theta'^2(\zeta)+Ecf''^2+\frac{4}{3Pr}R\theta''^2$$
$$\frac{1}{\mathrm{Pr}}\theta''(\zeta)+\frac{4}{3Pr}R\theta''^2+f\theta'+Nb\theta'(\zeta)\phi'(\zeta)+Nt\theta'^2(\zeta)+Ecf''^2=0$$
$$\left(1+\frac{4}{3}R\right)\theta''+Pr\left(f\theta'+Nb\theta'\phi'+Nt\theta'^2+Ecf''^2\right)=0. \tag{3.27}$$

Now, we include below the procedure for the conversion of equation (4.4) into the dimensionless form.

$$\phi(\zeta)=\frac{C-C_\infty}{C_w-C_\infty},$$
$$C=(C_w-C_\infty)\phi(\zeta)+C_\infty.$$
$$\frac{\partial C}{\partial x}=(C_w-C_\infty)\phi'(\zeta)\frac{\partial\zeta}{\partial x},$$

$$\frac{\partial C}{\partial x} = \left(\frac{n-1}{2}\right) x^{\frac{n-3}{2}} y \sqrt{\frac{a(n+1)}{2\nu}} (C_w - C_\infty)\phi'(\zeta). \tag{3.28}$$

$$\frac{\partial C}{\partial y} = (C_w - C_\infty)\phi'(\zeta)\frac{\partial \zeta}{\partial y},$$

$$\frac{\partial C}{\partial y} = x^{\frac{n-1}{2}} \sqrt{\frac{a(n+1)}{2\nu}} (C_w - C_\infty)\phi'(\zeta). \tag{3.29}$$

$$\frac{\partial^2 C}{\partial y^2} = x^{\frac{n-1}{2}} \sqrt{\frac{a(n+1)}{2\nu}} (C_w - C_\infty)\phi''(\zeta)\frac{\partial \zeta}{\partial y},$$

$$\frac{\partial^2 C}{\partial y^2} = x^{\frac{n-1}{2}} \sqrt{\frac{a(n+1)}{2\nu}} (C_w - C_\infty)\phi''(\zeta) \left(x^{\frac{n-1}{2}} \sqrt{\frac{a(n+1)}{2\nu}} \right),$$

$$\frac{\partial^2 C}{\partial y^2} = x^{n-1} \left(\sqrt{\frac{a(n+1)}{2\nu}} \right)^2 (C_w - C_\infty)\phi''(\zeta),$$

$$\frac{\partial^2 C}{\partial y^2} = x^{n-1} \frac{a(n+1)}{2\nu} (C_w - C_\infty)\phi''(\zeta). \tag{3.30}$$

$$\frac{\partial^2 T}{\partial y^2} = x^{n-1} \frac{a(n+1)}{2\nu} (T_w - T_\infty)\theta''(\zeta). \tag{3.31}$$

Using (3.28) and (3.29) in left hand side of (3.4)

$$\begin{aligned}
u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} &= ax^n f'(\zeta)\left(\left(\frac{n-1}{2}\right) x^{\frac{n-3}{2}} y \sqrt{\frac{a(n+1)}{2\nu}} (C_w - C_\infty)\phi'(\zeta) \right) \\
&+ \left(\frac{n-1}{2} x^{n-1} y a f'(\zeta) - x^{\frac{n-1}{2}} \frac{n+1}{2} \sqrt{\frac{2\nu a}{n+1}} f(\zeta) \right) x^{\frac{n-1}{2}} \sqrt{\frac{a(n+1)}{2\nu}} (C_w - C_\infty)\phi'(\zeta), \\
u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} &= a x^{\frac{3n-3}{2}} y \left(\frac{n-1}{2}\right) \sqrt{\frac{a(n+1)}{2\nu}} (C_w - C_\infty) f'(\zeta)\phi'(\zeta) \\
&\quad - x^{\frac{3n-3}{2}} y \left(\frac{n-1}{2}\right) \sqrt{\frac{a(n+1)}{2\nu}} (C_w - C_\infty) f'(\zeta)\phi'(\zeta) \\
&\quad - a x^{n-1} \left(\frac{n+1}{2}\right) (C_w - C_\infty) f(\zeta)\phi'(\zeta) \\
u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} &= -a x^{n-1} \left(\frac{n+1}{2}\right) (C_w - C_\infty) f(\zeta)\phi'(\zeta). \qquad (3.32)
\end{aligned}$$

Using (3.30) and (3.31) in right hand side of (3.4), the following is obtained

$$\begin{aligned}
D_B \frac{\partial^2 C}{\partial y^2} + \frac{D_T}{T_\infty}\frac{\partial^2 T}{\partial y^2} &= D_B x^{n-1} \left(\frac{a(n+1)}{2\nu} \right) (C_w - C_\infty)\phi''(\zeta) \\
&\quad + \frac{D_T}{T_\infty} x^{n-1} \left(\frac{a(n+1)}{2\nu} \right) (T_w - T_\infty)\theta''(\zeta) \qquad (3.33)
\end{aligned}$$

Comparing (3.32) and (3.33)

$$
\begin{aligned}
&-ax^{n-1}\left(\frac{n+1}{2}\right)(C_w-C_\infty)f(\zeta)\phi'(\zeta)\\
&=D_Bx^{n-1}\left[\frac{a(n+1)}{2\nu}\right](C_w-C_\infty)\phi''(\zeta)+\frac{D_T}{T_\infty}x^{n-1}\left[\frac{a(n+1)}{2\nu}\right](T_w-T_\infty)\theta''(\zeta).
\end{aligned}
$$

Dividing both side $D_Bx^{n-1}a\left(\frac{n+1}{2\nu}\right)(C_w-C_\infty)$

$$
\begin{aligned}
&-\frac{\nu}{D_B}f(\zeta)\phi'(\zeta)=\phi''(\zeta)+\frac{D_T(T_w-T_\infty)}{T_\infty D_B(C_w-C_\infty)}\theta''(\zeta)\\
\Rightarrow\quad &\phi''(\zeta)+Lef(\zeta)\phi'(\zeta)+\frac{D_T\tau(T_w-T_\infty)\nu}{T_\infty\nu D_B\tau(C_w-C_\infty)}\theta''(\zeta)=0.\\
\Rightarrow\quad &\phi''(\zeta)+Lef(\zeta)\phi'(\zeta)+\frac{Nt}{Nb}\theta''(\zeta)=0. \qquad (3.34)
\end{aligned}
$$

The corresponding BCs are transformed into the non-dimensional form through the following procedure,

$$
\begin{aligned}
&u=u_w(x)=ax^n, && at\quad y=0.\\
\Rightarrow\quad &u=af'(\zeta)x^n && at\quad \zeta=0\\
\Rightarrow\quad &af'(\zeta)=ax^n && at\quad \zeta=0\\
\Rightarrow\quad &f'(\zeta)=1, && at\quad \zeta=0.\\
\Rightarrow\quad &f'(0)=1.\\
&v=0 && at\quad y=0.\\
\Rightarrow\quad &-x^{\frac{n-1}{2}}\sqrt{\frac{2\nu a}{n+1}}\left(\frac{n+1}{2}\right)f(\zeta)-ax^{n-1}y\left(\frac{n-1}{2}\right)f'(\zeta)=0,\\
& && at\quad \zeta=0.\\
\Rightarrow\quad &-x^{\frac{n-1}{2}}\sqrt{\frac{a\nu(n+1)}{2}}f(0)=0, && at\quad \zeta=0.\\
\Rightarrow\quad &f(0)=0.\\
&T=T_w, && at\quad y=0.\\
\Rightarrow\quad &\theta(\zeta)(T_w-T_\infty)+T_\infty=T_w,\\
\Rightarrow\quad &\theta(\zeta)(T_w-T_\infty)=(T_w-T_\infty),
\end{aligned}
$$

$$\Rightarrow \quad \theta(\zeta) = 1, \qquad at \quad \zeta = 0.$$

$$\Rightarrow \quad \theta(0) = 1.$$

$$C = C_w, \qquad at \quad y = 0.$$

$$\Rightarrow \quad \phi(\zeta)(C_w - C_\infty) + C_\infty = C_w,$$

$$\Rightarrow \quad \phi(\zeta)(C_w - C_\infty) = (C_w - C_\infty),$$

$$\Rightarrow \quad \phi(\zeta) = 1, \qquad at \quad \zeta = 0.$$

$$\Rightarrow \quad \phi(0) = 1.$$

$$u \rightarrow 0, \qquad as \quad y \rightarrow \infty.$$

$$\Rightarrow \quad af'(\zeta)x^n \rightarrow 0,$$

$$\Rightarrow \quad ax^n f'(\zeta) \rightarrow 0,$$

$$\Rightarrow \quad f'(\zeta) \rightarrow 0, \qquad as \quad \zeta \rightarrow \infty.$$

$$T \rightarrow T_\infty, \qquad as \quad y \rightarrow \infty.$$

$$\Rightarrow \quad \theta(\zeta)(T_w - T_\infty) + T_\infty \rightarrow T_\infty,$$

$$\Rightarrow \quad \theta(\zeta)(T_w - T_\infty) \rightarrow 0, \qquad as \quad \zeta \rightarrow \infty.$$

$$\Rightarrow \quad \theta(\zeta) \rightarrow 0, \qquad as \quad \zeta \rightarrow \infty.$$

$$\Rightarrow \quad \theta(\infty) \rightarrow 0.$$

$$C \rightarrow C_\infty, \qquad as \quad y \rightarrow \infty.$$

$$\Rightarrow \quad \phi(\zeta)(C_w - C_\infty) + C_\infty \rightarrow C_\infty,$$

$$\Rightarrow \quad \phi(\zeta)(C_w - C_\infty) \rightarrow 0, \qquad as \quad \zeta \rightarrow \infty.$$

$$\Rightarrow \quad \phi(\zeta) \rightarrow 0, \qquad as \quad \zeta \rightarrow \infty.$$

The final dimensionless form of the governing model, is

$$f'''(\zeta) - \left(\frac{2n}{n+1}\right) f'^2(\zeta) + f(\zeta)f''(\zeta) - (M+G)\, f'(\zeta) = 0 \tag{3.35}$$

$$\left(1 + \frac{4}{3}R\right)\theta''(\zeta) + Pr\left(Nb\theta'(\zeta)\phi'(\zeta) + Nt\theta'^2(\zeta) + Ecf''^2(\zeta)\right) = 0. \tag{3.36}$$

$$\phi''(\zeta) + Lef(\zeta)\phi'(\zeta) + \frac{Nt}{Nb}\theta''(\zeta) = 0. \tag{3.37}$$

The associated BCs (3.5) in the dimensionless form are,

$$\left.\begin{array}{llll} f(0)=0, & f'(0)=1, & \theta(0)=1, & \phi(0)=1. \\ f'\to 0, & \theta\to 0, & \phi\to 0 & as\ \ \zeta\to\infty. \end{array}\right\} \tag{3.38}$$

Different dimensionless parameters used in equations (3.35) and (3.37) are formulated as follows

$$\begin{array}{llll} M=\dfrac{2\sigma B_0^2}{\rho_f a(n+1)}, & G=\dfrac{\nu}{k}, & R=\dfrac{4\sigma^* T_\infty^3}{kk^*}, & Nb=\dfrac{\tau D_B(C_w-C_\infty)}{\nu} \\ Pr=\dfrac{\nu}{\alpha}, & Ec=\dfrac{U_w^2}{c_p(T_w-T_\infty)}, & Le=\dfrac{\nu}{D_B}, & Nt=\dfrac{\tau D_T(T_w-T_\infty)}{T_\infty \nu}. \end{array}$$

The skin friction coefficient, is given as follows.

$$C_f=\frac{\mu}{\rho U_w^2(x)}\left(\frac{\partial u}{\partial y}\right)_{y=0}. \tag{3.39}$$

To achive the dimensionless form of C_f the following steps will be helpful, Since

$$\begin{aligned} C_f &= \frac{\mu}{\rho U_w(x)^2}\left(\frac{\partial u}{\partial y}\right)_{y=0}, \\ &= \frac{\mu}{\rho_f a^2 x^{2n}}\left(ax^n f''(\zeta) x^{\frac{n-1}{2}}\sqrt{\frac{(n+1)a}{2\nu}}\right), \\ C_f &= \frac{\mu}{\rho_f a^2 x^{2n}(1-\phi)^{2.5}}\left(ax^n f''(\zeta) x^{\frac{n-1}{2}}\sqrt{\frac{(n+1)a}{2\nu}}\right). \\ \Rightarrow \qquad &= \frac{\nu\rho a^{\frac{3}{2}} x^{\frac{3n-1}{2}}}{\rho_f a^2 x^{2n}(1-\phi)^{2.5}}\sqrt{\frac{(n+1)}{2\nu}} f''(\zeta) \\ &= \frac{\nu}{a^{\frac{1}{2}} x^{\frac{n+1}{2}}(1-\phi)^{2.5}}\left(\frac{n+1}{2}\right)^{\frac{1}{2}} f''(\zeta) \\ &= \frac{1}{Re_x^{\frac{1}{2}}(1-\phi)^{2.5}}\left(\frac{n+1}{2}\right)^{\frac{1}{2}} f''(\zeta). \end{aligned}$$

$$\Rightarrow \quad Re_x^{\frac{1}{2}} C_f = \frac{1}{(1-\phi)^{2.5}}\left(\frac{n+1}{2}\right)^{\frac{1}{2}} f''(\zeta), \tag{3.40}$$

where Re denotes the Reynolds number defined as $Re = \frac{xu_x(x)}{\nu}$.

Local Nusselt number is defined as follow.

$$Nu_x = \frac{xq_w}{k(T_w - T_\infty)}. \tag{3.41}$$

To achive the dimensionless form of Nu_x, the following steps will be helpful

Since

$$
\begin{aligned}
q_w &= -\left(\frac{\partial T}{\partial y}\right)_{y=0}, \qquad (3.42)\\
Nu_x &= -\frac{x}{k(T_w - T_\infty)}\left(\frac{\partial T}{\partial y}\right)_{y=0}.\\
Nu_x &= -\frac{x}{k(T_w - T_\infty)}\left(\frac{(n+1)a}{2\nu}\right)^{\frac{1}{2}} x^{\frac{n-1}{2}}(T_w - T_\infty)\theta'(\zeta).\\
&= -\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{n+1}{2}}\theta'(\zeta),\\
&= -\sqrt{\frac{n+1}{2}}\theta'(\zeta)\sqrt{\frac{ax^{n+1}}{\nu}},\\
&= -\sqrt{\frac{n+1}{2}}\theta'(\zeta)\sqrt{\frac{ax^{n+1}}{\nu}},\\
&= -\sqrt{\frac{n+1}{2}}\theta'(\zeta)Re_x^{\frac{1}{2}},\\
Re_x^{\frac{-1}{2}} Nu_x &= -\left(\frac{n+1}{2}\right)^{\frac{1}{2}}\theta'(\zeta). \qquad (3.43)
\end{aligned}
$$

The local Sherwood number are defined as

$$Sh_x = \frac{xq_m}{D_B(C_w - C_\infty)}. \tag{3.44}$$

To achive the dimensionless form of Sh_x, the following step will be helpful,

Since

$$
\begin{aligned}
q_m &= -D_B\left(\frac{\partial C}{\partial y}\right)_{y=0}, \qquad (3.45)\\
Sh_x &= -\frac{xD_B}{D_B(C_w - C_\infty)}\left(\frac{\partial C}{\partial y}\right)_{y=0}.
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{x}{(C_w - C_\infty)} x^{\frac{n-1}{2}} \sqrt{\frac{a(n+1)}{2\nu}} (C_w - C_\infty)\phi'(\zeta), \\
&= -x^{\frac{n+1}{2}} \sqrt{\frac{a(n+1)}{2\nu}} \phi'(\zeta), \\
&= -\sqrt{\frac{ax^{n+1}}{\nu}} \left(\frac{n+1}{2}\right)^{\frac{1}{2}} \phi'(\zeta), \\
&= -Re_x^{\frac{1}{2}} \left(\frac{n+1}{2}\right)^{\frac{1}{2}} \phi'(\zeta), \\
\frac{Sh_x}{Re_x^{\frac{1}{2}}} &= -\left(\frac{n+1}{2}\right)^{\frac{1}{2}} \phi'(\zeta), \\
Re_x^{-\frac{1}{2}} Sh_x &= -\left(\frac{n+1}{2}\right)^{\frac{1}{2}} \phi'(\zeta), \qquad (3.46)
\end{aligned}
$$

3.3 Numerical Method for Solution

The shooting method has been used to solve the ordinary differential equation (3.35). To get the numerical solution, the unbounded domain $[0, \infty[$ has been replace by the bounded domain $[0, \epsilon_\infty[$, where the real number $\epsilon_\infty > 0$ is chosen with the objective that the variations in the solution for $\epsilon > \epsilon_\infty$ are ignorable resulting in an asymptotic convergence of the solution. The following notations have been considered.

$$
f = Z_1, \qquad f' = Z_1' = Z_2, \qquad f'' = Z_1'' = Z_2' = Z_3, \qquad f''' = Z_3'.
$$

As a result the momentum equation is converted into the following system of first order ODEs.

$$
\begin{aligned}
Z_1' &= Z_2, && Z_1(0) = 0. \\
Z_2' &= Z_3, && Z_2(0) = 1. \\
Z_3' &= \left(\frac{2n}{n+1}\right) Z_2^2 - Z_1 Z_3 + (M+G)Z_2, && Z_3(0) = s.
\end{aligned}
$$

The above IVP will be numerically solved by the RK-4 method. The missing condition s is to be chosen in such a way that

$$Z_2(s) = 0.$$

It is important to mention that $Z_2(s)$ the value of $Z_2(s)$ at $\epsilon = \epsilon_\infty$ for the chosen missing condition s. Newton's method will be used to find s. This method has the following iterative scheme.

$$s^{(n+1)} = s^{(n)} - \frac{Z_2(s^{(n)})}{\left(\frac{\partial}{\partial s}(Z_2(s))\right)_{s=s^{(n)}}}.$$

We further introduce the following notations,

$$\frac{\partial Z_1}{\partial s} = Z_4, \qquad \frac{\partial Z_2}{\partial s} = Z_5, \qquad \frac{\partial Z_3}{\partial s} = Z_6.$$

As a result of these new notations, the Newton's iterative scheme gets the form

$$s^{(n+1)} = s^{(n)} - \frac{Z_2(s^{(n)})}{Z_5(s^{(n)})}.$$

Now differentiating the last system of three first order ODEs with respect to s, we get another system of ODEs, as follows

$$\begin{aligned}
Z_4' &= Z_5, & Z_4(0) &= 0.\\
Z_5' &= Z_6, & Z_5(0) &= 0.\\
Z_3' &= \left(\frac{4n}{n+1}\right) Z_2Z_5 - Z_1Z_6 - Z_4Z_3 + (M+G)Z_5, & Z_6(0) &= 1.
\end{aligned}$$

The stopping criteria for the Newton's technique is set as

$$\mid Z_2(s) \mid < \epsilon,$$

where $\epsilon > 0$ is an arbitrarily small positive number. From now onward ϵ has been taken as 10^{-10}.

The equation (3.36) and (3.37) will also be numerically solved by using the shooting method with by assuming f as a known function. For this, we utilize the following notions

$$\theta = Y_1, \quad \theta' = Y_1' = Y_2, \quad \theta'' = Y_2'.$$
$$\phi = Y_3, \quad \phi' = Y_3' = Y_4, \quad \phi'' = Y_4'.$$
$$A_1 = \left(1 + \frac{4}{3}R\right), \quad C_1 = f, \quad C_2 = f''.$$

As a result, the energy and concentration equation is transformed into the first-order ODE system shown below. As a result, the energy and concentration equation is transformed into the first order ODE system shown below,

$$\begin{aligned}
&Y_1' = Y_2, && Y_1(0) = 1.\\
&Y_2' = -\frac{Pr}{A_1}\left(C_1Y_2 + NbY_2Y_4 + NtY_2^2 + EcC_2^2\right), && Y_2(0) = p.\\
&Y_3' = Y_4, && Y_3(0) = 1.\\
&Y_4' = -LeC_1Y_4 - \frac{Nb}{Nt}\left(\frac{-Pr}{A_1}C_1Y_2 + NbY_2Y_4 + NtY_2^2 + EcC_2^2\right), && Y_4(0) = q.
\end{aligned}$$

In order to solve the aforementioned initial value problem, the RK-4 approach has been used. The missing conditions for the given system of equations should be chosen in such a way that

$$Y_1(p, q) = 0, \qquad Y_3(p, q) = 0.$$

Here $Y_1(p, q)$ and $Y_3(p, q)$ are the values of Y_1 and Y_3 at $\epsilon = \epsilon_\infty$ for some choice of the missing conditions p and q. To solve the above algebraic equations, we apply the Newton's method which has the following scheme,

$$\begin{bmatrix} p^{(n+1)} \\ q^{(n+1)} \end{bmatrix} = \begin{bmatrix} p^{(n)} \\ q^{(n)} \end{bmatrix} - \left(\begin{bmatrix} \frac{\partial Y_1}{\partial p} & \frac{\partial Y_1}{\partial q} \\ \frac{\partial Y_3}{\partial p} & \frac{\partial Y_3}{\partial q} \end{bmatrix}^{-1} \begin{bmatrix} Y_1 \\ Y_3 \end{bmatrix} \right)_{(p^{(n)},q^{(n)})}$$

Now, introduce the following notations.

$$\frac{\partial Y_1}{\partial p} = Y_5, \qquad \frac{\partial Y_2}{\partial p} = Y_6, \qquad \frac{\partial Y_3}{\partial p} = Y_7, \qquad \frac{\partial Y_4}{\partial p} = Y_8.$$
$$\frac{\partial Y_1}{\partial q} = Y_9, \qquad \frac{\partial Y_2}{\partial q} = Y_{10}, \qquad \frac{\partial Y_3}{\partial q} = Y_{11}, \qquad \frac{\partial Y_4}{\partial q} = Y_{12}.$$

As a result of these new notations, the Newton's iterative scheme gets the form:

$$\begin{bmatrix} p^{(n+1)} \\ q^{(n+1)} \end{bmatrix} = \begin{bmatrix} p^{(n)} \\ q^{(n)} \end{bmatrix} - \left(\begin{bmatrix} Y_5 & Y_9 \\ Y_7 & Y_{11} \end{bmatrix}^{-1} \begin{bmatrix} Y_1 \\ Y_3 \end{bmatrix} \right)_{(p^{(n)},q^{(n)})}$$

Now differentiating the last system of four first order ODEs first with respect to p and q then with respect to get another system of ODEs, as follows,

$$Y_5' = Y_6, \qquad Y_5(0) = 0.$$
$$Y_6' = \frac{-Pr}{A_1}\Bigg[C_1Y_6 + Nb(Y_6Y_4 + Y_2Y_8) + 2NtY_2Y_6\Bigg], \qquad Y_6(0) = 1.$$
$$.Y_7' = Y_8, \qquad Y_7(0) = 0.$$
$$Y_8' = -LeC_1Y_8 - \frac{Nt}{Nb}\left[\frac{-Pr}{A_1}\Big(C_1Y_6 + Nb\Big(Y_6Y_4 + Y_2Y_8\Big) + 2NtY_2Y_6\Big)\right],$$
$$Y_8(0) = 0.$$
$$Y_9' = Y_{10}, \qquad Y_9(0) = 0.$$
$$Y_{10}' = \frac{-Pr}{A_1}\Bigg[C_1Y_{10} + Nb(Y_{10}Y_4 + Y_2Y_{12}) + 2NtY_2Y_{10}\Bigg], \qquad Y_{10}(0) = 0.$$
$$Y_{11}' = Y_{12}, \qquad Y_{11}(0) = 0.$$
$$Y_{12}' = -LeC_1Y_{12} - \frac{Nt}{Nb}\left[\frac{-Pr}{A_1}\Bigg[C_1Y_{10} + Nb(Y_{10}Y_4 + Y_2Y_{12}) + 2NtY_2Y_{10}\Bigg]\right],$$
$$Y_{12}(0) = 1.$$

The stopping criteria for the Newton's method is set as.

$$max\{|Y_1(p,q)|, |Y_3(p,q)|\} < \epsilon.$$

3.4 Representation of Graphs and Tables

The impact of dimensionless parameters on the skin friction coefficient $(Re_x)^{\frac{1}{2}}C_f$, local Nusselt number $(Re_x)^{-\frac{1}{2}}Nu_x$ and local Sherwood number $(Re_x)^{-\frac{1}{2}}Sh_x$. respectively, has been throughly discussed in the graphs and tables. Table 3.1 explains the impact of nonlinear stretching parameter n, magnetic parameter M, thermal radiation parameter R, Eckert number Ec and Lewis number Le on $(Re_x)^{\frac{1}{2}}C_f$, $(Re_x)^{\frac{-1}{2}}Nu_x$ and $(Re_x)^{\frac{-1}{2}}Sh_x$. For the rising values of M, the skin fraction coefficient $(Re_x)^{\frac{1}{2}}C_f$ decreases. By increasing the values of Ec, the numerical values of the local Nusselt number is decreased and local Sherwood number is increased.

Figure 3.2 displays the impact of M on the velocity distribution. By rising the values of M, the velocity distribution shows the decreasing behavior due to the presence of the Lorentz force. Figure 3.3 describes the impact of M on the temperature distribution. The temperature distribution expands by rising the values of M. Figure 3.4 describes the impact of M, on the concentration distribution. Rising the values of M, the concentration distribution is increased.

Figure 3.5 shows the impact of the thermal radiation R on $\theta(\zeta)$. In this graph, it can be observed that on rising values of R, the temperature profile $\theta(\zeta)$ also increases. So, with on increase in thermal the radiation R, the rate of heat transfer reduces, and the temperature profile $\theta(\zeta)$ rises.

Figures 3.6 and 3.7 show the impact of permeability parameter G. The velocity profile $f'(\zeta)$ declines as G grows, while the temperature profile $\theta(\zeta)$ increases.The effect of G on the concentration profile is depicted in figure 3.8. The concentration distribution $\phi(\zeta)$ is enlarged by expanding the values of G. Figure 3.9 shows the influence of Prandtl number Pr on $\theta(\zeta)$. By the rising values of Pr, the temperature profile $\theta(\zeta)$ is decreased.

Figure 3.10 shows the relationship between Lewis numbers Le and the dimensional concentration distribution $\phi(\zeta)$. Concentration profile is observed to decrease for the rising values of Le and thus we get a small molecular diffusivity and thermal boundary layer. As can be observed in Figure 3.11, increasing Eckert number Ec result in an increases in the temperature profile.

Figure 3.12 and Figure 3.13 indicate the impact of Nb on the dimensionless temperature and concentration distributions. The behavior of the temperature distribution is increasing that for the and concentration profile is decreasing due to the accelerating values of Nb.

TABLE 3.1: Results of $(Re_x)^{\frac{1}{2}}C_f$, $-(Re_x)^{-\frac{1}{2}}Nu_x$ and $-(Re_x)^{-\frac{1}{2}}Sh_x$ some fixed parameters
$Pr = 6.2,\ G = 0.1,\ Nt = 0.1,\ Nb = 0.1$

M	n	R	Ec	Le	$(Re_x)^{\frac{1}{2}}C_f$	$-(Re_x)^{\frac{-1}{2}}Nu_x$	$-(Re_x)^{\frac{-1}{2}}Sh_x$
0.0	2.0	1.0	0.1	5.0	-1.406633	0.869857	1.552562
0.1					-1.458652	0.858868	1.544917
0.2					-1.508998	0.840327	1.537633
0.1	0.0				-0.540147	0.552600	0.922921
	1.0				-1.097633	0.716511	1.270009
	3.0				-1.746965	0.974691	1.778412
	2.0	0.0				1.052747	1.487514
		0.4				0.972294	1.500372
		0.8				0.891317	1.529852
		1.0	0.0			0.986436	1.439643
			0.2			0.722987	1.650481
			0.4			0.458127	1.862488
			0.1	6.0		0.844627	1.790302
				7.0		0.836676	2.012233
				8.0		0.830314	2.216101

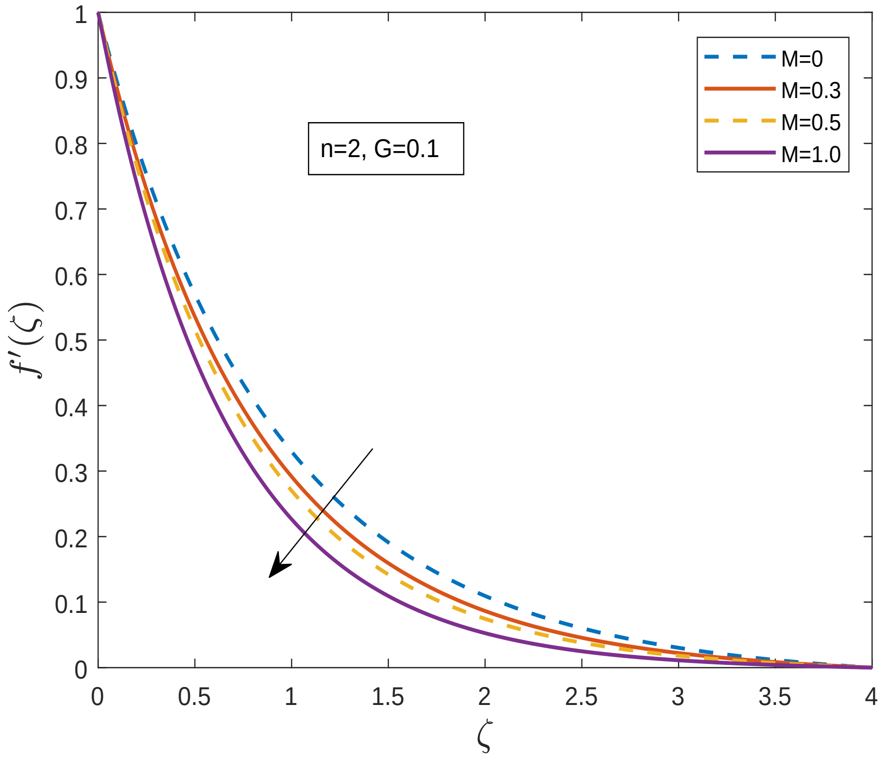

FIGURE 3.2: Impact of M on the velocity profile.

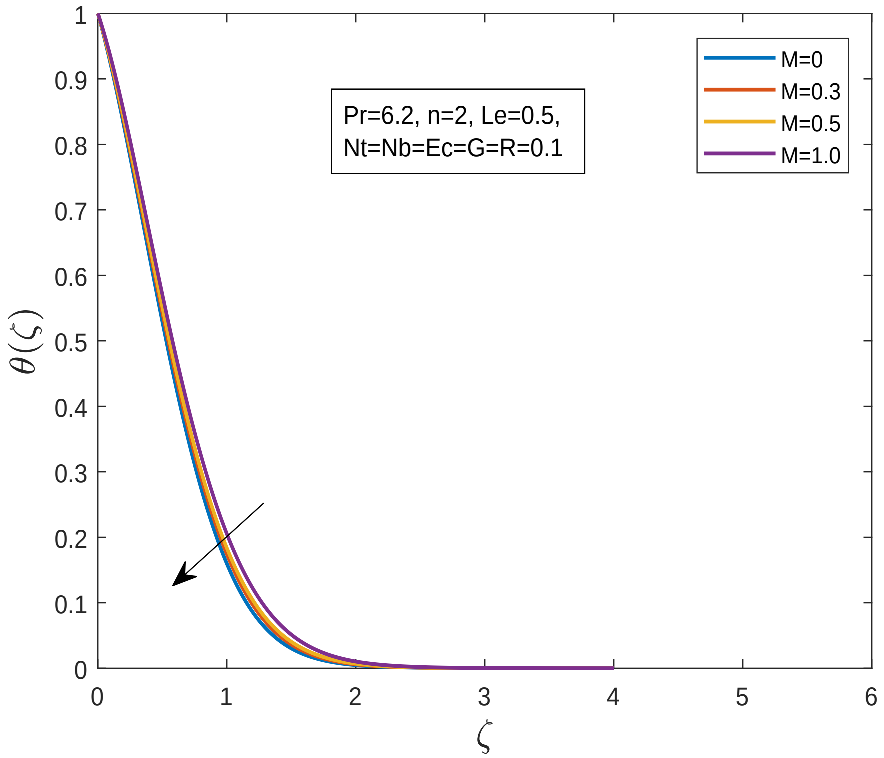

FIGURE 3.3: Impact of M on the temperature profile.

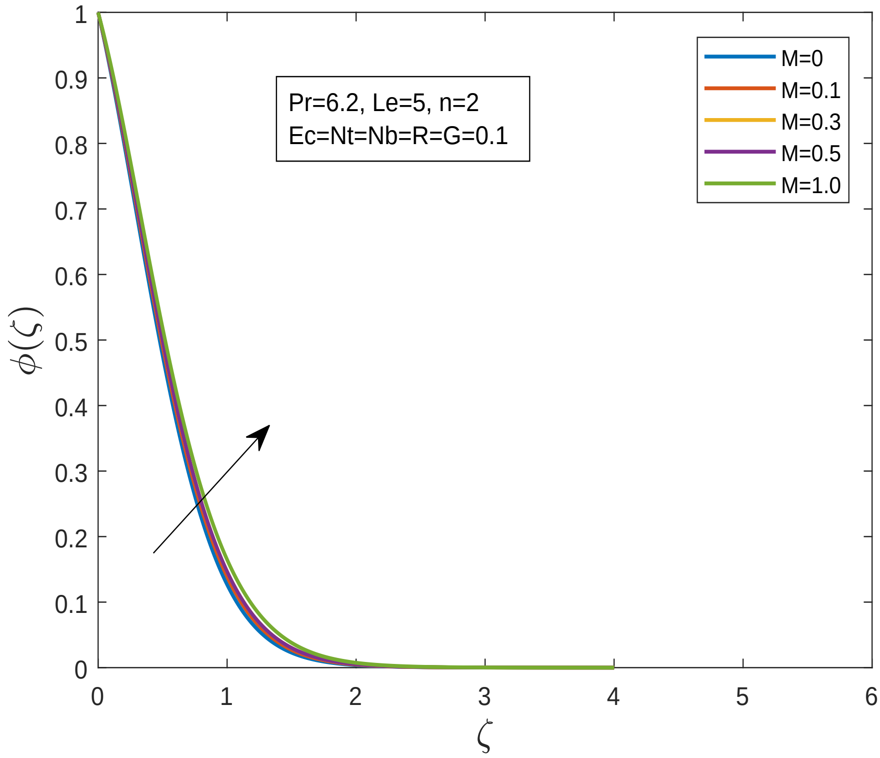

FIGURE 3.4: Impact of M on the concentration profile.

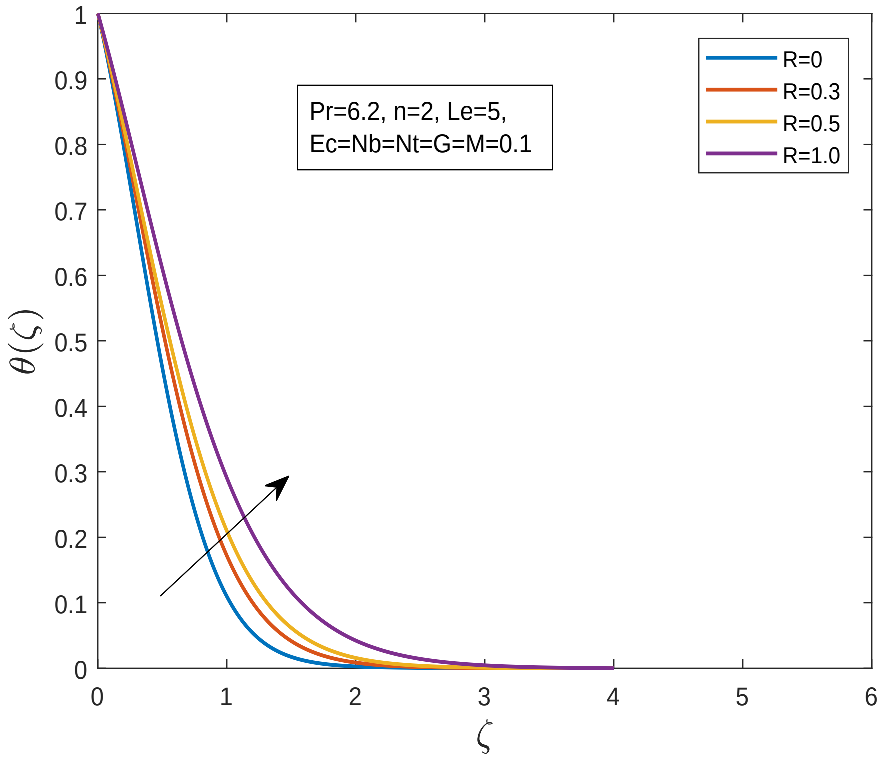

FIGURE 3.5: Impact of R on the temperature profile.

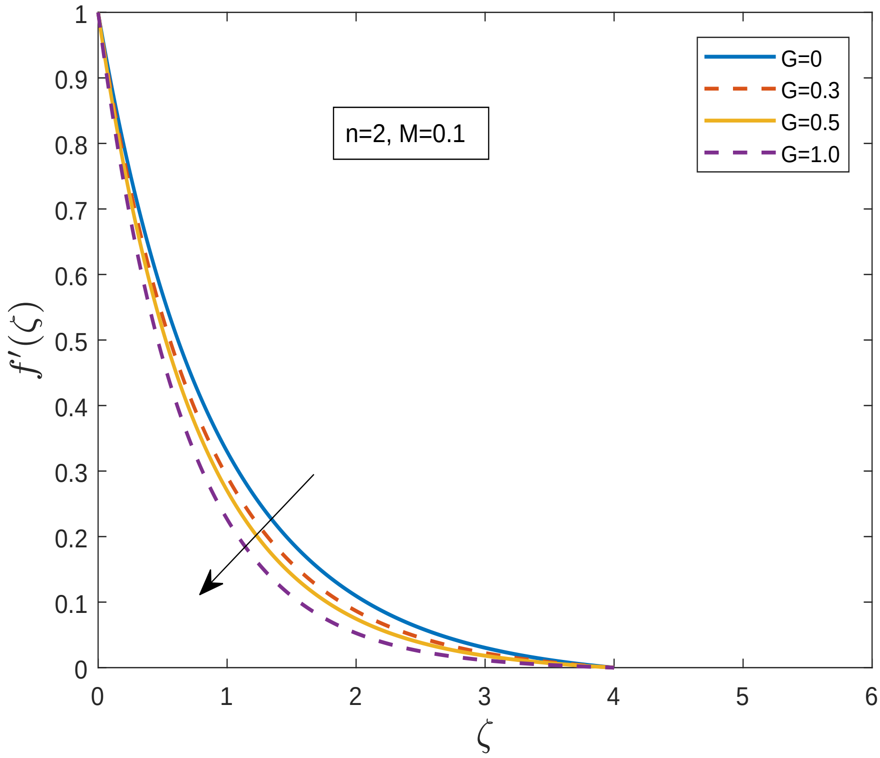

FIGURE 3.6: Impact of G on the velocity profile.

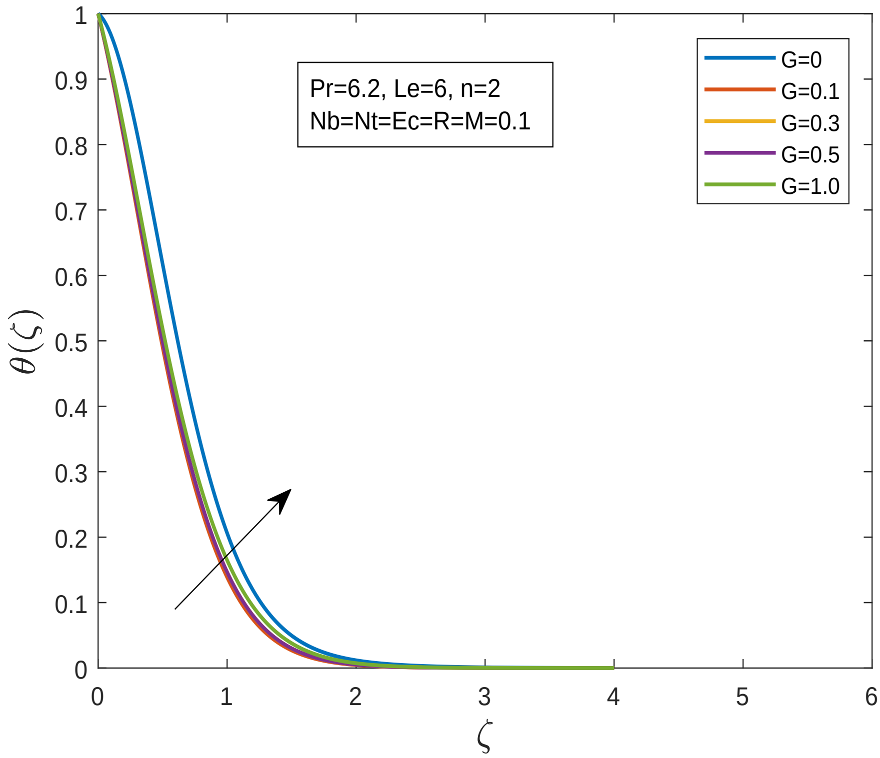

FIGURE 3.7: Impact of G on the temperature profile.

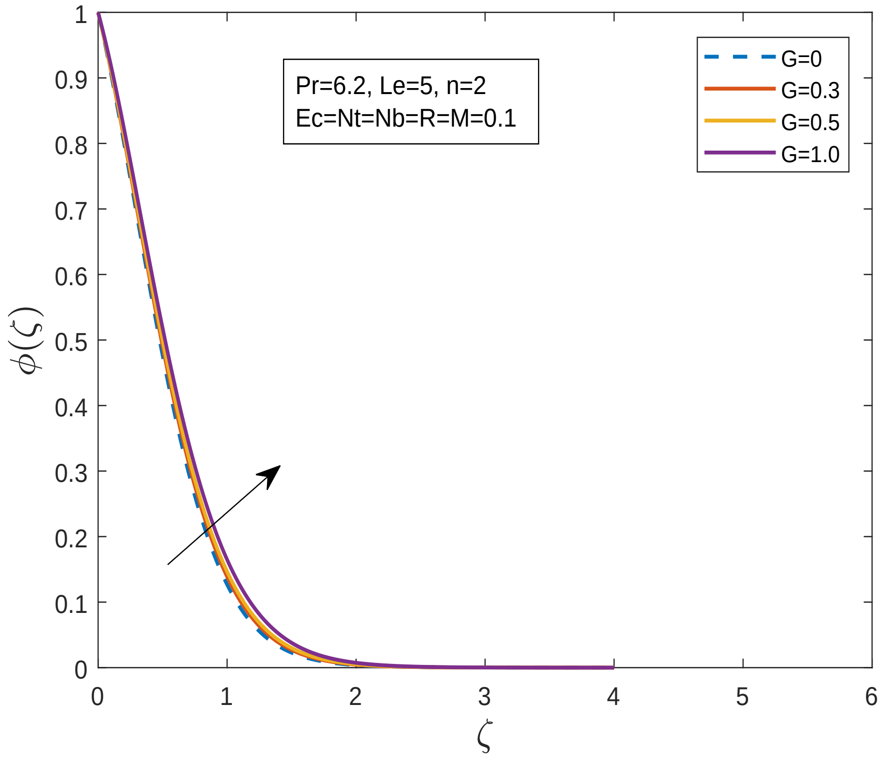

FIGURE 3.8: Impact of G on the concentration profile.

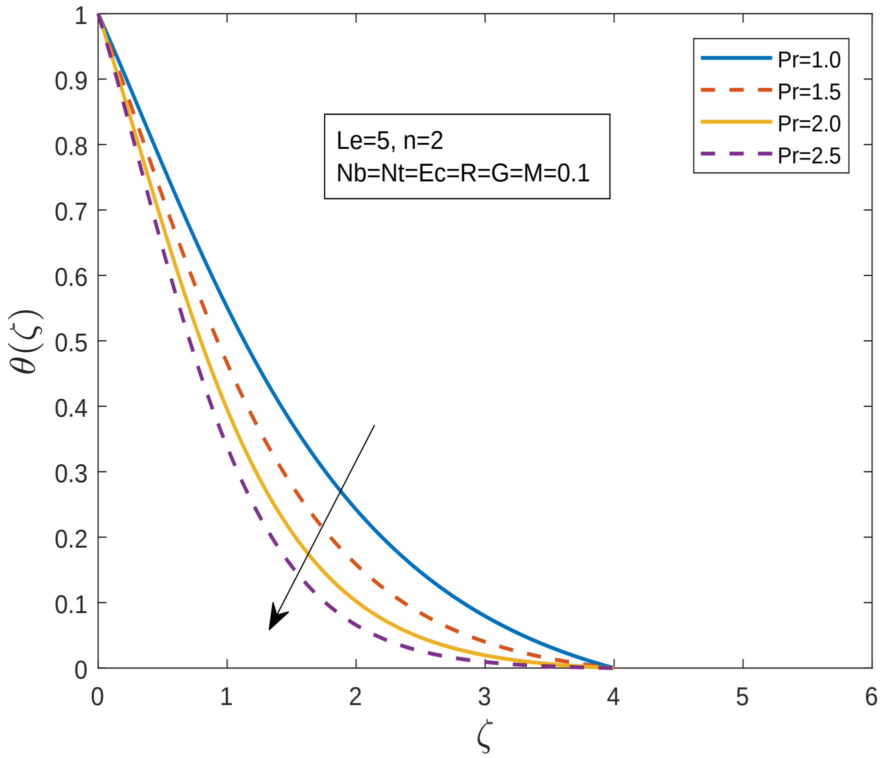

FIGURE 3.9: Impact of Pr on the temperature profile.

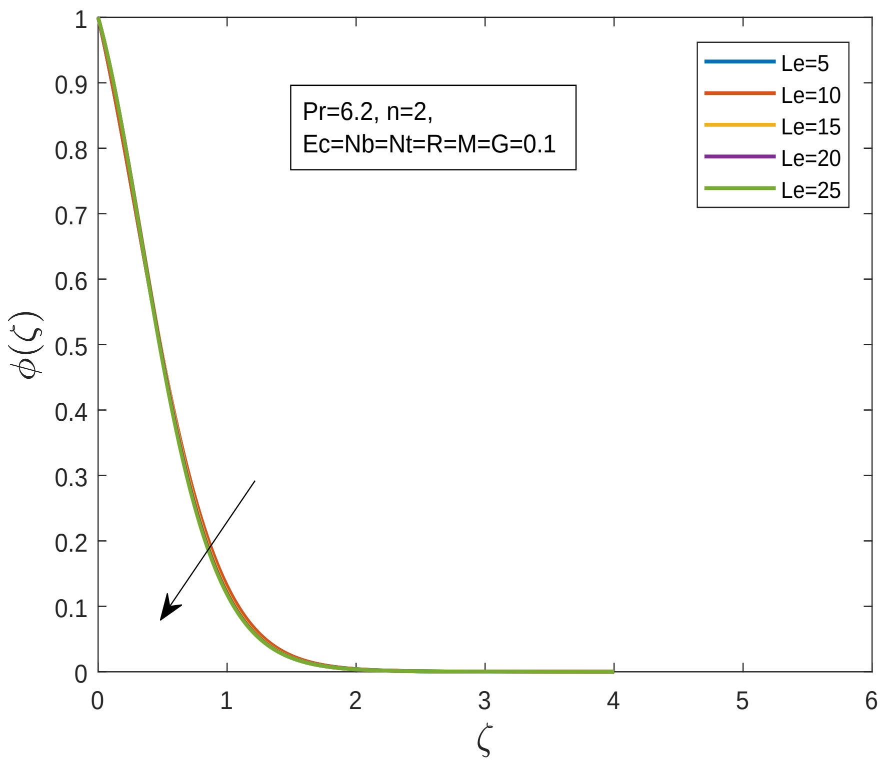

FIGURE 3.10: Impact of Le on the concentration profile.

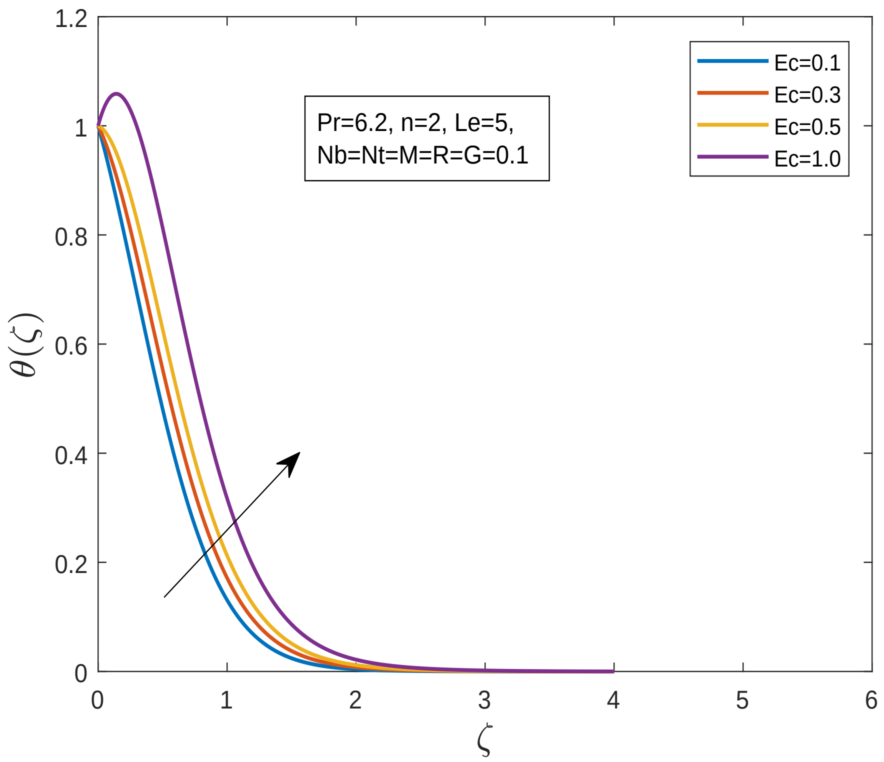

FIGURE 3.11: Impact of Ec on the temperature profile.

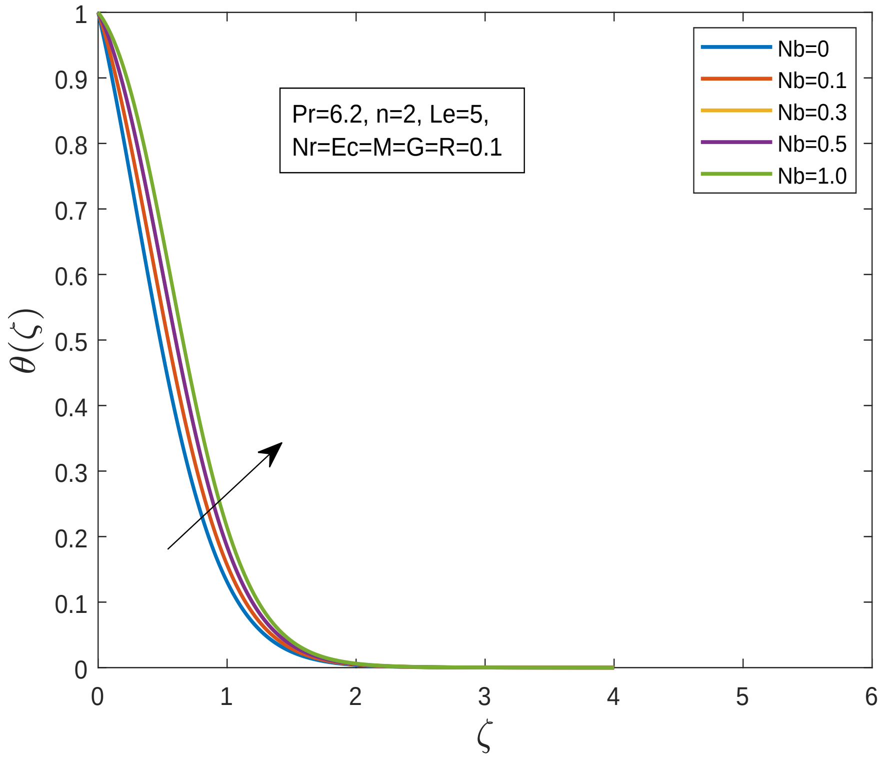

FIGURE 3.12: Impact of Nb on the temperature profile.

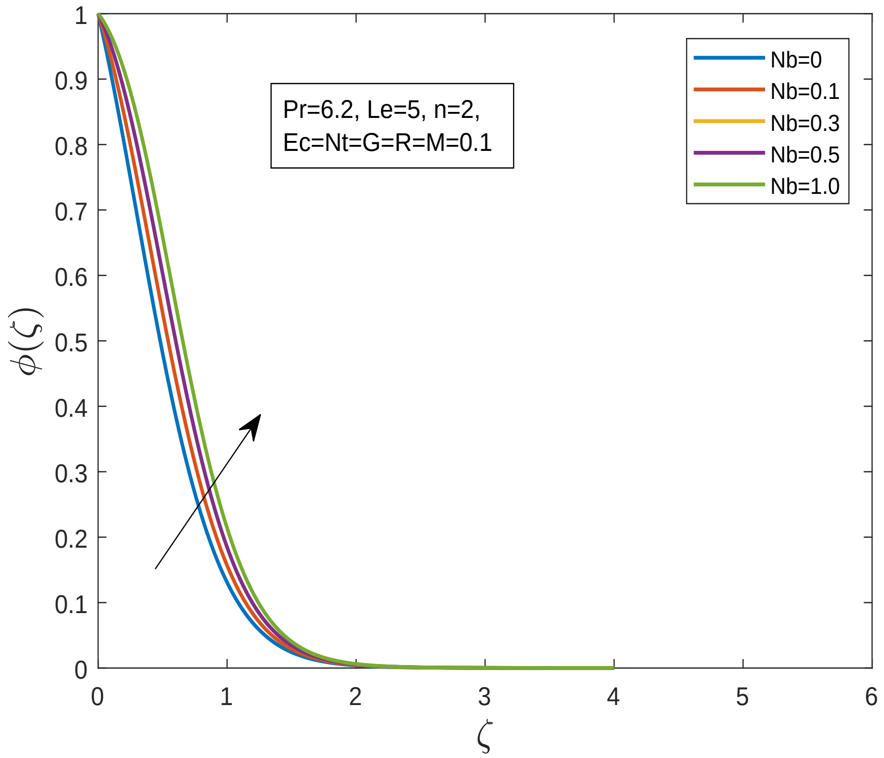

FIGURE 3.13: Impact of Nb on the concentration profile.

Chapter 4

Effect of Heat Generation and Chemical Reaction on MHD Fluid Flow over a Nonlinear Stretching Sheet

4.1 Introduction

This chapter contains the extension of the model [44] by considering aligned magnetic field in momentum equation. The heat generation are also included in the temperature equation. Furthermore chemical reaction is also taken into concentration equation. The governing nonlinear PDEs are converted into a system of dimensionless ODEs by utilizing the similarity transformations.

The numerical solution of ODEs is obtaind by applying numerical method known as shooting method. At the end of this chapter, the final results are discussed for significant parameters affecting $f'(\zeta)$, $\theta(\zeta)$ and $\phi(\zeta)$ which are shown in tables and graphs.

4.2 Mathematical Modeling

It is aimed to analyse the $2D$, MHD flow of nanofluid past a nonlinear stretching sheet and porous medium. $y > 0$ was occupied by the flow. The horizontal axis is used to apply a magnetic field of intensity B. In addition, x-axis is aligned with the flow direction, whereas the y- axis is perpendicular to it. Thermal radiation, viscous dissipation, and heat creation are all considered in the energy transport study. Moreover, the concentration equation under the effect of chemical reaction. By considering the above assumptions, the governing PDEs are.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{4.1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\left(\frac{\partial^2 u}{\partial y^2}\right) - \frac{\nu}{k}u - \frac{\sigma B^2(x)}{\rho_f}\sin^2(\gamma)u, \tag{4.2}$$

$$\begin{aligned} u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} &= \alpha\left(\frac{\partial^2 T}{\partial y^2}\right) + \frac{\nu}{C_p}\left(\frac{\partial u}{\partial y}\right)^2 - \frac{1}{(\rho C_p)_f}\left(\frac{\partial q_r}{\partial y}\right) \\ &+ \frac{q}{(\rho C_p)_f}(T - T_\infty) + \tau\left(D_B\frac{\partial C}{\partial y}\frac{\partial T}{\partial x} + \frac{D_T}{T_\infty}\left(\frac{\partial T}{\partial y}\right)^2\right), \end{aligned} \tag{4.3}$$

$$u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} = D_B\frac{\partial^2 C}{\partial y^2} + \left(\frac{D_T}{T_\infty}\right)\frac{\partial^2 T}{\partial y^2} - K_r(C - C_\infty). \tag{4.4}$$

The associated BCs have been taken as.

$$\left.\begin{aligned} &u = U_w(x) = ax^n, \quad v = 0, \quad T = T_w, \quad C = C_w \quad at \quad y = 0. \\ &u \to 0, \quad T \to T_\infty, \quad C \to C_\infty \quad as \quad y \to \infty. \end{aligned}\right\} \tag{4.5}$$

Following similarity transformation has been used to convert PDEs (4.1)-(4.4) into system of ODEs.

$$\left.\begin{aligned} &\zeta = y\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{n-1}{2}}, \qquad u = ax^n f'(\zeta) \\ &v = -x^{\frac{n-1}{2}}\sqrt{\frac{\nu a(n+1)}{2}}\left(f(\zeta) + \left(\frac{n-1}{n+1}\right)\zeta f'(\zeta)\right) \\ &\theta(\zeta) = \frac{T - T_\infty}{T_w - T_\infty}, \qquad \phi(\zeta) = \frac{C - C_\infty}{C_w - C_\infty}. \end{aligned}\right\} \tag{4.6}$$

where ζ denotes the similarity variable, f, θ, and ϕ are the dimensionless velocity, temperature and concentration.
The detailed procedure for the conversion of (4.1) has been discussed in chapter 3.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{4.7}$$

Now, I include the below procedure for the conversion of (4.2) into the dimensionless form.

$$u = ax^n f'(\xi). \tag{4.8}$$

$$v = -x^{n-1}ay\left(\frac{n-1}{2}\right)f'(\zeta) - x^{\frac{n-1}{2}}\left(\frac{n+1}{2}\right)\sqrt{\frac{2\nu a}{n+1}}f(\zeta). \tag{4.9}$$

The complete procedure for the conversion of (4.2) discussed in chapter 3.

$$f''' + ff'' - \left(\frac{2n}{n+1}\right)f'^2 - \left(Msin^2(\gamma) + G\right)f' \tag{4.10}$$

Now, we include below the procedure for the conversion of equation (4.3) into the dimensionless form. The (4.11)-(4.17) we have already derived in chapter 3.

$$\frac{\partial u}{\partial x} = ax^{\frac{3n-3}{2}}y\left(\frac{n-1}{2}\right)\sqrt{\frac{(n+1)a}{2\nu}}f''(\zeta) + nax^{n-1}f'(\zeta). \tag{4.11}$$

$$\frac{\partial u}{\partial y} = ax^{\frac{3n-1}{2}}\sqrt{\frac{(n+1)a}{2\nu}}f''(\zeta). \tag{4.12}$$

$$\frac{\partial T}{\partial x} = x^{\frac{n-3}{2}}y\left(\frac{n-1}{2}\right)\sqrt{\frac{(n+1)a}{2\nu}}(T_w - T_\infty)\theta'(\zeta). \tag{4.13}$$

$$\frac{\partial T}{\partial y} = x^{\frac{n-1}{2}}\sqrt{\frac{(n+1)a}{2\nu}}(T_w - T_\infty)\theta'(\zeta). \tag{4.14}$$

$$\frac{\partial^2 T}{\partial y^2} = x^{n-1}\frac{a(n+1)}{2\nu}(T_w - T_\infty)\theta''(\zeta). \tag{4.15}$$

$$\frac{\partial C}{\partial y} = x^{\frac{n-1}{2}}\sqrt{\frac{(n+1)a}{2\nu}}(C_w - C_\infty)\phi'(\zeta). \tag{4.16}$$

$$\left(\frac{\partial T}{\partial y}\right)^2 = x^{n-1}\frac{(n+1)a}{2\nu}(T_w - T_\infty)^2\theta'^2(\zeta). \tag{4.17}$$

$$\tau\left(D_B\left(\frac{\partial T}{\partial y}\frac{\partial C}{\partial y}\right)+\frac{D_T}{T_\infty}\left(\frac{\partial T}{\partial y}\right)^2\right)=\tau\left(\frac{D_T}{T_\infty}x^{n-1}\frac{(n+1)a}{2\nu}(T_w-T_\infty)^2\theta'^2(\zeta)\right)$$

$$+\tau\left(D_B\left[x^{\frac{n-1}{2}}\sqrt{\frac{(n+1)a}{2\nu}}(T_w-T_\infty)\theta'(\zeta)\right]\left[x^{\frac{n-1}{2}}\sqrt{\frac{(n+1)a}{2\nu}}(C_w-C_\infty)\phi'(\zeta)\right]\right),$$

$$=\tau D_B x^{n-1}\left(\frac{(n+1)a}{2\nu}\right)(T_w-T_\infty)(C_w-C_\infty)\theta'(\zeta)\phi'(\zeta)$$

$$+\tau\frac{D_T}{T_\infty}x^{n-1}\left(\frac{(n+1)a}{2\nu}\right)(T_w-T_\infty)^2\theta'^2(\zeta),$$

$$=\frac{\tau D_B(C_w-C_\infty)}{\nu}x^{n-1}\left(\frac{(n+1)a}{2}\right)(T_w-T_\infty)\theta'(\zeta)\phi'(\zeta)$$

$$+\frac{\tau D_T(T_w-T_\infty)}{T_\infty\nu}x^{n-1}\left(\frac{(n+1)a}{2}\right)(T_w-T_\infty)\theta'^2(\zeta),$$

$$=ax^{n-1}\left(\frac{n+1}{2}\right)Nb(T_w-T_\infty)\theta'(\zeta)\phi'(\zeta)$$

$$+ax^{n-1}\left(\frac{n+1}{2}\right)Nt(T_w-T_\infty)\theta'^2(\zeta),$$

$$=ax^{n-1}\left(\frac{n+1}{2}\right)(T_w-T_\infty)\left(Nb\theta'(\zeta)\phi'(\zeta)+Nt\theta'^2(\zeta)\right). \tag{4.18}$$

Left hand side of (4.3)

$$=-ax^{n-1}\left(\frac{n+1}{2}\right)(T_w-T_\infty)f(\zeta)\theta'(\zeta) \tag{4.19}$$

Right hand side of (4.3)

$$=\alpha a x^{n-1}\left(\frac{n+1}{2\nu}\right)(T_w-T_\infty)\theta''(\zeta)+\frac{\nu}{c_p}a^3x^{3n-1}\left(\frac{n+1}{2\nu}\right)f''^2(\zeta)$$

$$-\frac{1}{(\rho Cp)_f}\frac{16\sigma^*T_\infty^3}{3k^*}ax^{n-1}(T_w-T_\infty)\left(\frac{n+1}{2\nu}\right)\theta''(\zeta)+\frac{q}{(\rho Cp)_f}(T_w-T_\infty)\theta(\zeta)$$

$$+ax^{n-1}\left(\frac{n+1}{2}\right)(T_w-T_\infty)\left(Nb\theta'(\zeta)\phi'(\zeta)+Nt\theta'^2(\zeta)\right). \tag{4.20}$$

Comparing (4.19) and (4.20)

$$-ax^{n-1}\left[\left(\frac{n+1}{2}\right)(T_w-T_\infty)f(\zeta)\theta'(\zeta)\right]=\alpha\left[x^{n-1}\left(\frac{a(n+1)}{2\nu}\right)(T_w-T_\infty)\theta''(\zeta)\right]$$

$$\frac{\tau D_B(C_w - C_\infty)}{\nu} x^{n-1} \left(\frac{(n+1)a}{2}\right) (T_w - T_\infty)\theta'(\zeta)\phi'(\zeta) + \frac{\tau D_T(T_w - T_\infty)}{T_\infty \nu},$$

$$x^{n-1} \left(\frac{(n+1)a}{2}\right) (T_w - T_\infty)\theta'^2(\zeta) + \frac{\nu}{Cp} \left(a^2 x^{3n-1} \left(\frac{a(n+1)}{2\nu}\right) f''^2(\zeta)\right)$$

$$- \frac{1}{(\rho Cp)_f} \left(\frac{16\sigma^* T_\infty^3}{3k^*}(T_w - T_\infty)x^{n-1} \left(\frac{a(n+1)}{2\nu}\right) \theta''(\zeta)\right) + \frac{q}{(\rho Cp)_f}(T_w - T_\infty)\theta(\zeta),$$

$$- f(\zeta)\theta'(\zeta) = \frac{(\rho Cp)_p}{(\rho Cp)_f} \frac{D_B(C_w - C_\infty)}{\nu}\theta'(\zeta)\phi'(\zeta) + \frac{(\rho Cp)_p}{(\rho Cp)_f} \frac{D_T(T_w - T_\infty)}{T_\infty \nu}\theta'^2$$

$$+ \frac{\nu}{Cp(T_w - T_\infty)} a^2 x^{2n} f''^2 + \frac{16\sigma^* T_\infty^3}{(\rho Cp)_f \nu 3k^*}\theta''(\zeta) + \frac{\alpha}{\nu}\theta''(\zeta) + \frac{q}{(\rho Cp)_f} \frac{2}{a(n+1)x^{n-1}},$$

$$- f(\zeta)\theta'(\zeta) = \frac{1}{Pr}\theta''(\zeta) + Nb\theta'(\zeta)\phi'(\zeta) + Nt\theta'^2(\zeta) + Ecf''^2 + \frac{4}{3Pr}R\theta'' + \frac{2}{n+1}Q\theta$$

$$- f(\zeta)\theta'(\zeta) = \frac{1}{Pr} \left(1 + \frac{4}{3}R\right) \theta''(\zeta) + Nb\theta'(\zeta)\phi'(\zeta) + Nt\theta'^2(\zeta)$$

$$+ Ecf''^2(\zeta) + \left(\frac{2}{n+1}\right) Q\theta(\zeta),$$

$$\frac{1}{Pr} \left(1 + \frac{4}{3}R\right) \theta''(\zeta) + Nb\theta'(\zeta)\phi'(\zeta) + Nt\theta'^2(\zeta) + f(\zeta)\theta'(\zeta)$$

$$+ Ecf''^2(\zeta) + \left(\frac{2}{n+1}\right) Q\theta(\zeta) = 0,$$

$$\left(1 + \frac{4}{3}R\right) \theta''(\zeta) + Pr\Bigg(f(\zeta)\theta'(\zeta) + Nb\theta'(\zeta)\phi'(\zeta) + Nt\theta'^2(\zeta)$$

$$+ Ecf''^2(\zeta) + \left(\frac{2}{n+1}\right) Q\theta(\zeta)\Bigg) = 0. \tag{4.21}$$

Now, we include below the procedure for the conversion of equation (4.4) into the dimensionless form.

$$\phi(\eta) = \frac{C - C_\infty}{C_w - C_\infty},$$

$$C = (C_w - C_\infty)\phi(\zeta) + C_\infty.$$

$$\frac{\partial C}{\partial x} = (C_w - C_\infty)\phi'(\zeta)\frac{\partial \zeta}{\partial x},$$

$$\frac{\partial C}{\partial x} = \left(\frac{n-1}{2}\right) x^{\frac{n-3}{2}} y\sqrt{\frac{a(n+1)}{2\nu}}(C_w - C_\infty)\phi'(\zeta). \tag{4.22}$$

$$\frac{\partial C}{\partial y} = (C_w - C_\infty)\phi'(\zeta)\frac{\partial \zeta}{\partial y},$$

$$\frac{\partial C}{\partial y} = x^{\frac{n-1}{2}}\sqrt{\frac{a(n+1)}{2\nu}}(C_w - C_\infty)\phi'(\zeta). \tag{4.23}$$

$$\frac{\partial^2 C}{\partial y^2} = x^{\frac{n-1}{2}}\sqrt{\frac{a(n+1)}{2\nu}}(C_w - C_\infty)\phi''(\zeta)\frac{\partial \zeta}{\partial y},$$
$$\frac{\partial^2 C}{\partial y^2} = x^{\frac{n-1}{2}}\sqrt{\frac{a(n+1)}{2\nu}}(C_w - C_\infty)\phi''(\zeta)\left(x^{\frac{n-1}{2}}\sqrt{\frac{a(n+1)}{2\nu}}\right),$$
$$\frac{\partial^2 C}{\partial y^2} = x^{n-1}\left(\sqrt{\frac{a(n+1)}{2\nu}}\right)^2(C_w - C_\infty)\phi''(\zeta),$$
$$\frac{\partial^2 C}{\partial y^2} = x^{n-1}\frac{a(n+1)}{2\nu}(C_w - C_\infty)\phi''(\zeta). \tag{4.24}$$
$$\frac{\partial^2 T}{\partial y^2} = x^{n-1}\frac{a(n+1)}{2\nu}(T_w - T_\infty)\theta''(\zeta). \tag{4.25}$$

Using (4.22) and (4.23) in left hand side of (4.4)

$$u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} = ax^n f'(\zeta)\left(\left(\frac{n-1}{2}\right)x^{\frac{n-3}{2}}y\sqrt{\frac{a(n+1)}{2\nu}}(C_w - C_\infty)\phi'(\zeta)\right)$$
$$+\left(\frac{n-1}{2}x^{n-1}yaf'(\zeta) - x^{\frac{n-1}{2}}\frac{n+1}{2}\sqrt{\frac{2\nu a}{n+1}}f(\zeta)\right)x^{\frac{n-1}{2}}\sqrt{\frac{a(n+1)}{2\nu}}(C_w - C_\infty)\phi'(\zeta),$$
$$u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} = ax^{\frac{3n-3}{2}}y\left(\frac{n-1}{2}\right)\sqrt{\frac{a(n+1)}{2\nu}}(C_w - C_\infty)f'(\zeta)\phi'(\zeta)$$
$$- x^{\frac{3n-3}{2}}y\left(\frac{n-1}{2}\right)\sqrt{\frac{a(n+1)}{2\nu}}(C_w - C_\infty)f'(\zeta)\phi'(\zeta)$$
$$- ax^{n-1}\left(\frac{n+1}{2}\right)(C_w - C_\infty)f(\zeta)\phi'(\zeta),$$
$$u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} = -ax^{n-1}\left(\frac{n+1}{2}\right)(C_w - C_\infty)f(\zeta)\phi'(\zeta). \tag{4.26}$$

Using (4.24) and (4.25) in right hand side of (4.4)

$$D_B\frac{\partial^2 C}{\partial y^2} + \frac{D_T}{T_\infty}\frac{\partial^2 T}{\partial y^2} - K_r(C - C_\infty) = D_B x^{n-1}\left(\frac{a(n+1)}{2\nu}\right)(C_w - C_\infty)\phi''(\zeta)$$
$$+ \frac{D_T}{T_\infty}x^{n-1}\left(\frac{a(n+1)}{2\nu}\right)(T_w - T_\infty)\theta''(\zeta) - K_r(C_w - C_\infty)\phi(\zeta). \tag{4.27}$$

Comparing (4.26) and (4.27)

$$- ax^{n-1}\left(\frac{n+1}{2}\right)(C_w - C_\infty)f(\zeta)\phi'(\zeta) = D_B x^{n-1}\left(\frac{a(n+1)}{2\nu}\right)(C_w - C_\infty)\phi''(\zeta)$$

$$+\frac{D_T}{T_\infty}x^{n-1}\left(\frac{a(n+1)}{2\nu}\right)(T_w-T_\infty)\theta''(\zeta)-K_r(C_w-C_\infty)\phi(\zeta).$$

Dividing both side $D_Bx^{n-1}a\left(\frac{n+1}{2\nu}\right)(C_w-C_\infty)$

$$\begin{aligned}
&-\frac{\nu}{D_B}f(\zeta)\phi'(\zeta)=\phi''(\zeta)+\frac{D_T(T_w-T_\infty)}{T_\infty D_B(C_w-C_\infty)}\theta''(\zeta)-\frac{K_r2\nu}{D_B(n+1)ax^{n-1}}\phi(\zeta),\\
&h''(\zeta)+Lef(\zeta)\phi'(\zeta)+\frac{D_T\tau(T_w-T_\infty)\nu}{T_\infty\nu D_B\tau(C_w-C_\infty)}\theta''(\zeta)-\frac{\nu}{D_B}\frac{2K_r}{(n+1)ax^{n-1}}\phi(\zeta)=0,\\
&\phi''(\zeta)+Lef(\zeta)\phi'(\zeta)+\frac{Nt}{Nb}\theta''(\zeta)-\gamma_2Le\phi(\zeta)=0. \qquad (4.28)
\end{aligned}$$

Now discussing the procedure for conversion of boundary conditions into dimensionless form.

$$\begin{aligned}
&u=U_w(x)=ax^n, && at \quad y=0.\\
&u=af'(\zeta)x^n,\\
\Rightarrow\ &af'(\zeta)x^n=ax^n,\\
\Rightarrow\ &ax^nf'(\zeta)=ax^n,\\
\Rightarrow\ &f'(\zeta)=1, && at \quad \zeta=0.\\
\Rightarrow\ &f'(0)=1.\\
&v=0, && at \quad y=0.\\
\Rightarrow\ &-x^{\frac{n-1}{2}}\sqrt{\frac{(n+1)\nu a}{2}}\left(f(\zeta)+\zeta f'(\eta)\left(\frac{n-1}{n+1}\right)\right)=0, && at \quad \zeta=0.\\
\Rightarrow\ &-x^{\frac{n-1}{2}}\sqrt{\frac{(n+1)\nu a}{2}}f(\zeta)=0, && at \quad \zeta=0.\\
\Rightarrow\ &f(\zeta)=0,\\
\Rightarrow\ &f(0)=0.\\
&T=T_w, && at \quad y=0.\\
\Rightarrow\ &\theta(\zeta)(T_w-T_\infty)+T_\infty=T_w,\\
\Rightarrow\ &\theta(\zeta)(T_w-T_\infty)=(T_w-T_\infty),\\
\Rightarrow\ &\theta(\zeta)=1, && at \quad \zeta=0.\\
\Rightarrow\ &\theta(0)=1.
\end{aligned}$$

$$
\begin{aligned}
& C = C_w, && at \quad y = 0. \\
\Rightarrow \quad & \phi(\zeta)(C_w - C_\infty) + C_\infty = C_w, && \\
\Rightarrow \quad & \phi(\zeta)(C_w - C_\infty) = (C_w - C_\infty), && \\
\Rightarrow \quad & \phi(\zeta) = 1, && at \quad \zeta = 0. \\
\Rightarrow \quad & \phi(0) = 1. && \\
& u \to (0), && as \quad y \to \infty. \\
\Rightarrow \quad & af'(\zeta)x^n \to (0), && \\
\Rightarrow \quad & f'(\zeta) \to (0), && as \quad \zeta \to \infty. \\
\Rightarrow \quad & f'(\zeta) \to 0. && \\
& T \to T_\infty, && as \quad y \to \infty. \\
\Rightarrow \quad & \theta(\zeta)(T_w - T_\infty) + T_\infty \to T_\infty, && \\
\Rightarrow \quad & \theta(\zeta)(T_w - T_\infty) \to 0, && \\
\Rightarrow \quad & \theta(\zeta) \to 0, && as \quad \zeta \to \infty. \\
\Rightarrow \quad & \theta(\infty) \to 0. && \\
& C \to C_\infty, && as \quad y \to \infty. \\
\Rightarrow \quad & \phi(\zeta)(C_w - C_\infty) + C_\infty \to C_\infty, && \\
\Rightarrow \quad & \phi(\zeta)(C_w - C_\infty) \to 0, && \\
\Rightarrow \quad & \phi(\zeta) \to 0, && as \quad \zeta \to \infty. \\
\Rightarrow \quad & \phi(\infty) \to 0. &&
\end{aligned}
$$

The final dimensionless form of the governing model, is

$$f'''(\zeta) + f(\zeta)f''(\zeta)) - \left(\frac{2n}{n+1}\right) f'^2(\zeta) - \left(M \sin^2(\gamma) + G\right) f'(\zeta) = 0. \tag{4.29}$$

$$\left(1 + \frac{4}{3}R\right) \theta''(\zeta) + Pr\Bigg(f(\zeta)\theta'(\zeta) + Nb\theta'(\zeta)\phi'(\zeta) + Nt\theta'^2(\zeta)$$
$$+ Ecf''^2(\zeta) + \left(\frac{n+1}{2}\right)Q\theta(\zeta)\Bigg) = 0. \tag{4.30}$$

$$\phi''(\zeta) + Lef(\xi)\phi'(\zeta) + \frac{Nt}{Nb}\theta''(\zeta) - \gamma_1 Le\phi(\zeta) = 0. \tag{4.31}$$

The associated BCs (4.5) in the dimensionless form are,

$$\left.\begin{aligned} &f(0)=0, \quad f'(0)=1, \quad \theta(0)=1, \quad \phi(0)=1 \\ &f'(\infty)\to 0, \quad \theta(\infty)\to 0, \quad \phi(\infty)\to 0. \end{aligned}\right\} \tag{4.32}$$

Different parameters used in equations (4.29)-(4.31) are formulated as follows.

$$\begin{aligned} &M=\frac{\sigma B_0^2}{\rho_f a x^{-1}}, \quad K=\frac{\nu}{a k_0}, \quad G=\frac{v}{k}, \quad \gamma_1=\frac{2K_r}{(n+1)ax^{n-1}}, \\ &Pr=\frac{\nu}{\alpha}, \quad Ec=\frac{U_w^2}{(c_p)_f(T_w-T_\infty)}, \quad Q=\frac{qx}{(\rho c_p)_f U_w}, \quad Le=\frac{\nu}{D_B}, \\ &Nb=\frac{\tau D_B(C_w-C_\infty)}{\nu}, \quad Nt=\frac{\tau D_T(T_w-T_\infty)}{T_\infty \nu}, \quad R=\frac{4\sigma^* T_\infty^3}{kk^*}. \end{aligned}$$

4.3 Solution Methodology

In order to solve the system of ODEs (4.29) the shooting method has been used. The following notations have been cosidered.

$$f=Y_1, \quad f'=Y_1'=Y_2, \quad f''=Y_1''=Y_2'=Y_3, \quad f'''=Y_3'.$$

By using the notations, the equation (4.29) is converted into first order ODEs.

$$\begin{aligned} &Y_1'=Y_2, && Y_1(0)=0. \\ &Y_2'=Y_3, && Y_2(0)=0. \\ &Y_3'=\left(\frac{2n}{n+1}\right)Y_2^2-Y_1Y_3+(Msin^2\gamma+G)Y_2, && Y_3(0)=s. \end{aligned}$$

The above initial value problem will be numerically solved by RK-4. The missing condition 's' assumed to satisfy the following relation.

$$Y_2(\zeta_\infty)_s=0.$$

To solve the above algebaric equations we use the Newton's method which has the following iterative scheme.

$$s^{n+1} = s^n - \frac{(Y_2(\zeta_\infty))_{s=s^n}}{\left(\frac{\partial Y_2(\zeta_\infty)}{\partial s}\right)_{s=s^n}}.$$

We further introduce the following notations,

$$\frac{\partial Y_1}{\partial s} = Y_4, \qquad \frac{\partial Y_2}{\partial s} = Y_5, \qquad \frac{\partial Y_3}{\partial s} = Y_6.$$

As a result of these new notations, the Newton's iterative scheme.

$$s^{n+1} = s^n - \frac{(Y_2(\zeta_\infty))_{s=s^n}}{(Y_5(\zeta_\infty))_{s=s^n}}.$$

Now differentiating system of three first order ODEs with respect to s, we get three more ODEs.

$$\begin{aligned}
&Y_4' = Y_5, && Y_4(0) = 0.\\
&Y_5' = Y_6, && Y_5(0) = 0.\\
&Y_6' = \left(\frac{4n}{n+1}\right) Y_2Y_5 - Y_1Y_6 - Y_4Y_3 + (Msin^2\gamma + G)Y_5, && Y_6(0) = 1.
\end{aligned}$$

The missing condition s is updated by the Newton's method and process will be continued until the following criteria is met.

$$\mid (Y_2(\zeta_\infty))_{s=s^n} \mid < \epsilon.$$

where ϵ is an arbitrarily small positive number. From now onward ϵ has been taken as 10^{-10}.

Also, for equations (4.30) and (4.31), the following notation have been used.

$$\theta = Z_1, \qquad \theta' = Z_1' = Z_2, \qquad \theta'' = Z_2'.$$

$$\phi = Z_3, \qquad \phi' = Z_3' = Z_4, \qquad \phi'' = Z_4'.$$

$$A_1 = \left(1 + \frac{4}{3}R\right), \qquad A_2 = \left(\frac{2}{n+1}\right)$$

The system of equations (4.30) and (4.31), can be written in the form of the following first order coupled ODEs.

$$
\begin{aligned}
Z_1' &= Z_2, && Z_1(0) = 1.\\
Z_2' &= -\frac{Pr}{A_1}\left(C_1 Z_2 + NbZ_2Z_4 + NtZ_2^2 + EcC_2^2 + A_2QZ_1\right), && Z_2(0) = l.\\
Z_3' &= Z_4, && Z_3(0) = 1.\\
Z_4' &= -LeC_1Z_4 + \gamma_2 LeZ_3 + \frac{Nb}{Nt}\Bigg(\frac{Pr}{A_1}\Bigg(C_1Z_2 + NbZ_2Z_4\\
&\qquad + NtZ_2^2 + EcC_2^2 + A_2QZ_1\Bigg)\Bigg), && Z_4(0) = m.
\end{aligned}
$$

In order to solve the above mentioned initial value problem, the RK-4 approach has been used. The missing condition for the above system of equation should be chosen in such a way that.

$$
(Z_1(l,m))_{\zeta=\zeta_\infty} = 0, \qquad (Z_3(l,m))_{\zeta=\zeta_\infty} = 0.
$$

To solve the above algebaric equations, we apply the Newton's method which has the following scheme.

$$
\begin{bmatrix} l^{n+1} \\ m^{n+1} \end{bmatrix} = \begin{bmatrix} l^{n} \\ m^{n} \end{bmatrix} - \begin{bmatrix} \frac{\partial Z_1}{\partial l} & \frac{\partial Z_1}{\partial m} \\ \frac{\partial Z_3}{\partial l} & \frac{\partial Z_3}{\partial m} \end{bmatrix}^{-1} \begin{bmatrix} Z_1 \\ Z_3 \end{bmatrix}
$$

Now, introduce the following notations,

$$
\begin{aligned}
&\frac{\partial Z_1}{\partial l} = Z_5, && \frac{\partial Z_2}{\partial l} = Z_6, && \frac{\partial Z_3}{\partial l} = Z_7, && \frac{\partial Z_4}{\partial l} = Z_8.\\
&\frac{\partial Z_1}{\partial m} = Z_9, && \frac{\partial Z_2}{\partial m} = Z_{10}, && \frac{\partial Z_3}{\partial m} = Z_{11}, && \frac{\partial Z_4}{\partial m} = Z_{12}.
\end{aligned}
$$

As the result of these new notations, the Newton's iterative scheme gets the form.

$$
\begin{bmatrix} l^{n+1} \\ m^{n+1} \end{bmatrix} = \begin{bmatrix} l^{n} \\ m^{n} \end{bmatrix} - \begin{bmatrix} Z_5 & Z_9 \\ Z_7 & Z_{11} \end{bmatrix}^{-1} \begin{bmatrix} Z_1 \\ Z_3 \end{bmatrix}
$$

Now differentiating the system of four first order ODEs with respect to l, and m we get another system of ODEs, as follows.

$$Z_5' = Z_6, \qquad Z_5(0) = 0.$$

$$Z_6' = \frac{-Pr}{A_1}\Big(C_1 Z_6 + Nb(Z_6 Z_4 + Z_2 Z_8 + 2NtZ_2 Z_6 + A_2 Q Z_5\Big), \qquad Z_6(0) = 1.$$

$$Z_7' = Z_8, \qquad Z_7(0) = 0.$$

$$Z_8' = -LeC_1 Z_8 + \gamma_2 LeZ_7 - \frac{Nt}{Nb}\left(\frac{-Pr}{A_1}\Big(C_1 Z_6 + Nb(Z_6 Z_4 + Z_2 Z_8) + 2NtZ_2 Z_6 + A_2 Q Z_5\Big)\right), \qquad Z_8(0) = 0.$$

$$Z_9' = Z_{10}, \qquad Z_9(0) = 0.$$

$$Z_{10}' = \frac{-Pr}{A_1}\Big(C_1 Z_{10} + Nb(Z_{10} Z_4 + Z_2 Z_{12}) + 2NtZ_2 Z_{10} + A_2 Q Z_9\Big), \qquad Z_{10}(0) = 0.$$

$$Z_{11}' = Z_{12}, \qquad Z_{11}(0) = 0.$$

$$Z_{12}' = -LeC_1 Z_{12} + \gamma_2 LeZ_{11} - \frac{Nt}{Nb}\left(\frac{-Pr}{A_1}\Big(C_1 Z_{10} + Nb(Z_{10} Z_4 + Z_2 Z_{12}) + 2NtZ_2 Z_{10} + A_2 Q Z_9\Big)\right), \qquad Z_{12}(0) = 1.$$

The stopping criteria for the Newton's method is set as.

$$max\{|Z_1(\zeta_\infty)|, |Z_3(\zeta_\infty)|\} < \epsilon.$$

4.4 Representation of Graphs and Tables

The principle object is about to examine the impact of different parameters against the velocity , temperature and concentration distribution. The impact of different factors like nonlinear stretching parameter n, magnetic parameter M, thermal radiation R and Lewis number Le is observed graphically.

Numerical outcomes of the skin friction coefficient, local Nusselt number and local Sherwood number for the distinct values of some fixed parameters are shown in Tables 4.1-4.2.

Figure 4.1 displays the impact of M, on the velocity distribution. The rising values of M, shows decreasing behavior of velocity profile.Because M denotes the ratio of Lorentz forces to viscous forces, an increase in M causes the Lorentz force to become dominant, lowering the velocity of the fluid. Figure 4.1 describes the impact of M on $\theta(\zeta)$. The temperature distribution expands by enhancing the values of M. Figure 4.3 describes the impact of M, on the concentration distribution. Rising the values of M, the concentration distribution $h(\zeta)$ is increased due to the presence of Lorentz force.
Figure 4.4 shows the impact of thermal radiation R on the temperature distribution $\theta(\zeta)$. By enhancing the values of R, the temperature distribution $\theta(\zeta)$ is increased.

Figures 4.5 and 4.6 show the impact of permeability parameter G. For the rising values of G, the velocity profile $f'(\zeta)$ decreases and temperature profile $\theta(\zeta)$ increases. Figure 4.7 shows the impact of G on the concentration profile. By expanding the values of G, the concentration distribution $\phi(\zeta)$ is increased.
The impact of the Prandtl number Pr on temperature distributions is shown in Figure Figure 4.8. Because Pr is directly proportional to the viscous diffusion rate and inversely proportional to the thermal diffusivity, when Pr rises, the thermal diffusion rate decreases, and the fluid's temperature lowers substantially. There has also been a drop in the thickness of the thermal boundary layer.
Figure 4.9 shows the relationship between Lewis numbers Le and the dimensional concentration distribution. Concentration profile decreasing for the rising values of Le and thus we have get a small molecular diffusivity and thermal boundary layer.

The impact of the Eckert number Ec on $\theta(\zeta)$ is shown in Figure 4.10. The ratio of kinetic energy and enthalpy change of flow is defined by the Eckert number. The increase in $\theta(\zeta)$ is clearly demonstrated by increasing the values of Ec due to the decrease in heat transfer rate.
Figure 4.11 and Figure 4.12 indicate the impact of Nb on the dimensionless temperature and concentration distribution. The behavior of temperature distribution is increased and concentration profile is decreased due to the accelerating values

of Nb.

Figure 4.13 depicts the effect of heat generation Q on $\theta(\zeta)$. It is noticed that as Q grows, more heat is generated, resulting in an increase in $\theta(\zeta)$ and thermal boundary layer thickness.

Figure ref4.14 shows that increasing the value of the chemical reaction parameter γ_1 decreases the concentration profile.

TABLE 4.1: Results of $(Re_x)^{\frac{1}{2}} C_f$ for fixed parameter $\gamma = \pi/3$

n	M	G	$(Re_x)^{\frac{1}{2}} C_f$
2.0	0.0	0.1	-1.148511
2.0	0.1	0.1	-1.190984
2.0	0.2	0.1	-1.230917
2.0	0.3	0.1	-1.271937
2.0	0.1	0.0	-1.148518
2.0	0.1	0.1	-1.909850
2.0	0.1	0.2	-1.233099
2.0	0.1	0.3	-1.232092
3.0	0.5	0.3	-1.348219
3.0	0.6	0.3	-1.384820
3.0	0.3	0.5	-1.420492
3.0	0.3	0.6	-1.455298
3.0	0.3	0.7	-1.489297
0.1	2.0	2.0	2.612924
0.1	2.0	2.0	-2.774333
4.0	0.0	0.1	-1.248511

TABLE 4.2: Results of $-(Re_x)^{\frac{-1}{2}}Nu_x$ and $-(Re_x)^{\frac{-1}{2}}Sh_x$ some fixed parameters $\gamma = \pi/3,\ n = 2.0,\ Ec = 0.1,\ Q = 0.1,\ Nt = Nb = 0.1$

M	R	Pr	γ_1	Le	G	$-(Re_x)^{\frac{-1}{2}}Nu_x$	$-(Re_x)^{\frac{-1}{2}}Sh_x$
0.1	0.1	6.2	0.5	5.0	0.1	0.306217	1.921066
0.5	0.1	6.2	0.5	5.0	0.1	0.260764	1.879936
0.8	0.1	6.2	0.5	5.0	0.1	0.227314	1.852385
1.0	0.1	6.2	0.5	5.0	0.1	0.216324	0.227314
0.1	0.2	6.2	0.5	5.0	0.1	0.343682	1.910956
0.1	0.5	6.2	0.5	5.0	0.1	0.41531 5	1.889407
0.1	1.0	6.2	0.5	5.0	0.1	0.465659	1.870425
0.1	0.1	6.2	0.5	1.0	0.1	0.789389	0.590562
0.1	0.1	6.2	0.5	1.5	0.1	0.648659	0.827584
0.1	0.1	6.2	0.5	2.0	0.1	0.548951	1.036722
0.1	0.1	6.2	0.5	5.0	0.3	0.282851	1.899805
0.1	0.1	6.2	0.5	5.0	0.5	0.261016	1.879936
0.1	0.1	6.2	0.5	5.0	0.7	0.238115	1.861293
0.1	0.1	6.2	0.5	5.0	1.0	0.206133	1.861393
0.1	0.1	6.2	0.5	5.0	0.1	0.240460	1.835321
0.1	0.1	6.2	1.0	5.0	0.1	0.885558	1.45352
0.1	0.1	6.2	1.5	5.0	0.1	0.885558	1.453525
0.1	0.1	6.2	2.0	5.0	0.1	0.885558	1.453525
0.1	0.1	6.2	2.5	5.0	0.1	0.885558	1.453525
0.1	0.1	6.2	0.3	5.0	0.1	0.885556	1.453525
0.1	0.1	7.0	0.5	6.0	0.1	0.896640	1.459844
0.1	0.1	7.3	0.5	5.0	0.1	0.899149	1.463170
0.1	0.1	7.5	0.5	5.0	0.1	0.900381	1.465642
0.1	0.1	7.6	0.5	5.0	0.1	0.902055	1.472645

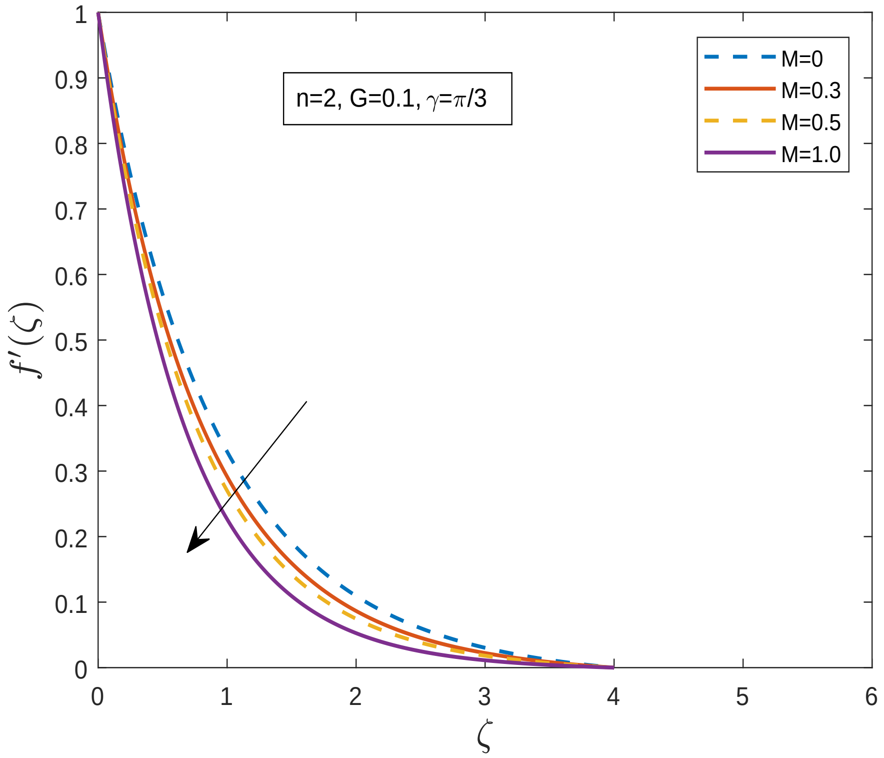

FIGURE 4.1: Impact of M on the velocity profile.

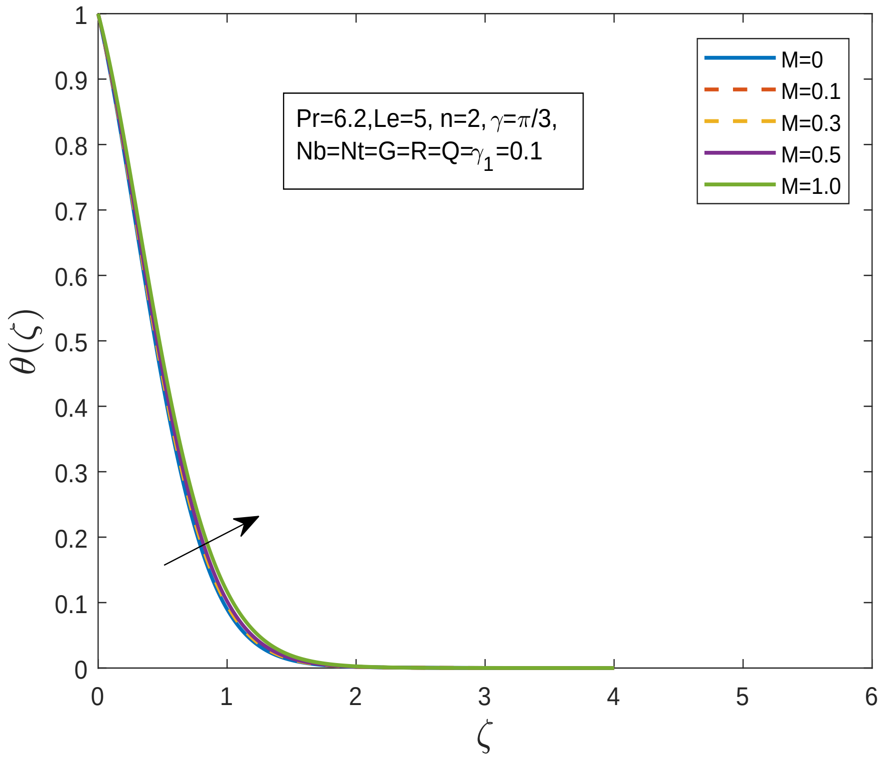

FIGURE 4.2: Impact of M on the temperature profile.

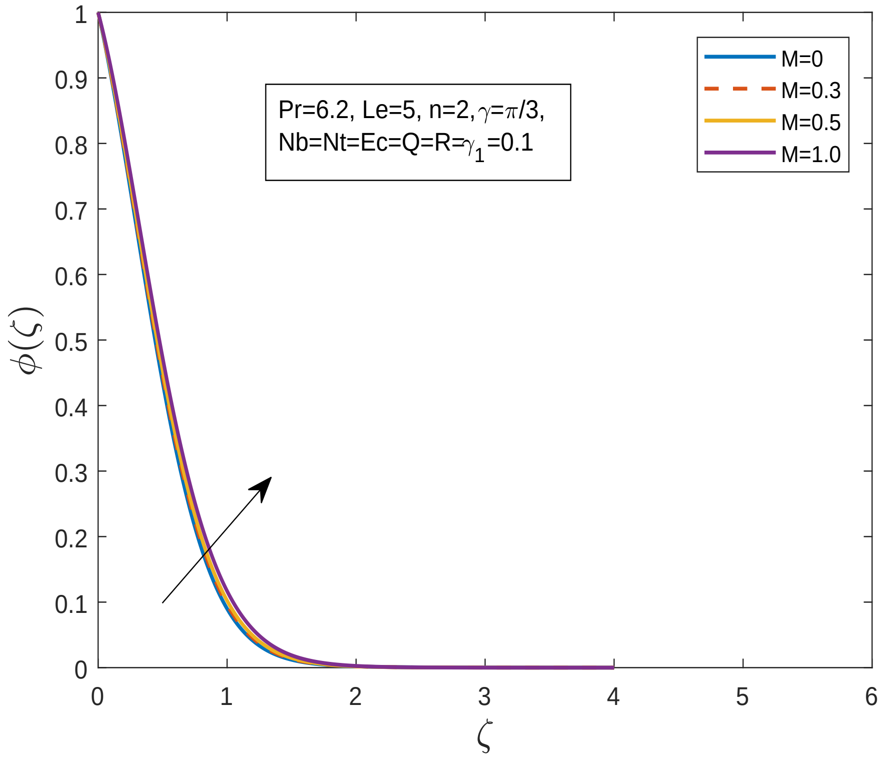

FIGURE 4.3: Impact of M on the concentration profile.

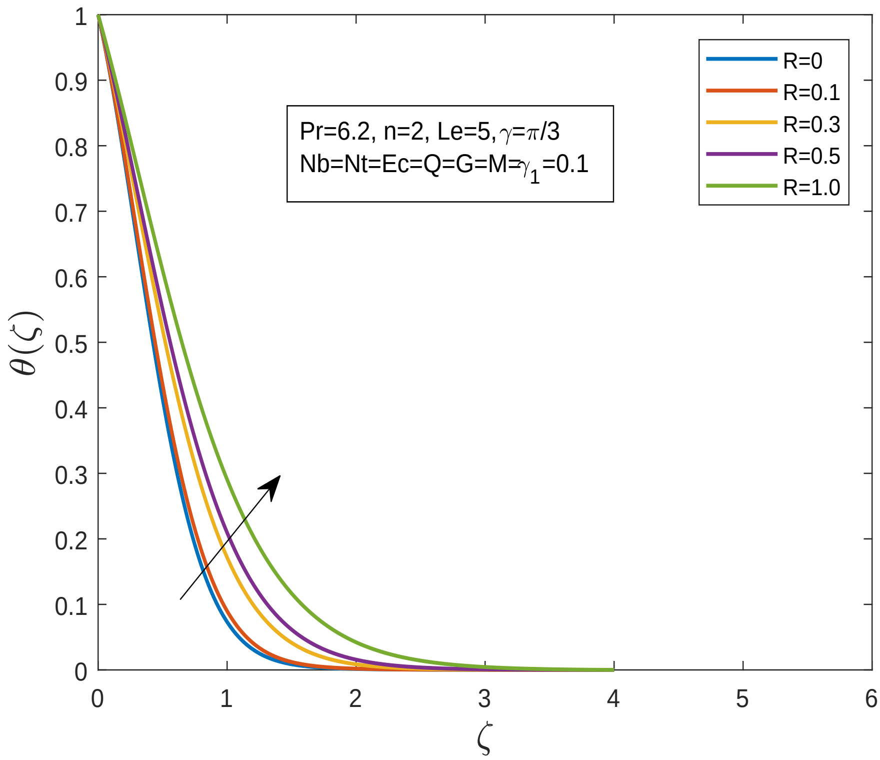

FIGURE 4.4: Impact of R on the temperature profile.

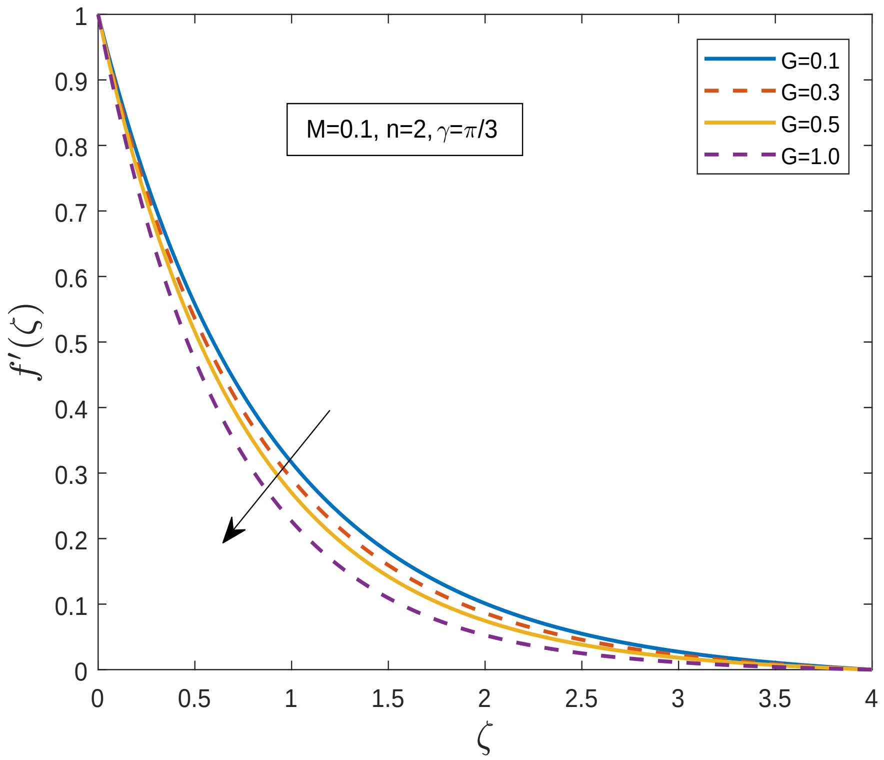

Figure 4.5: Impact of G on the velocity profile.

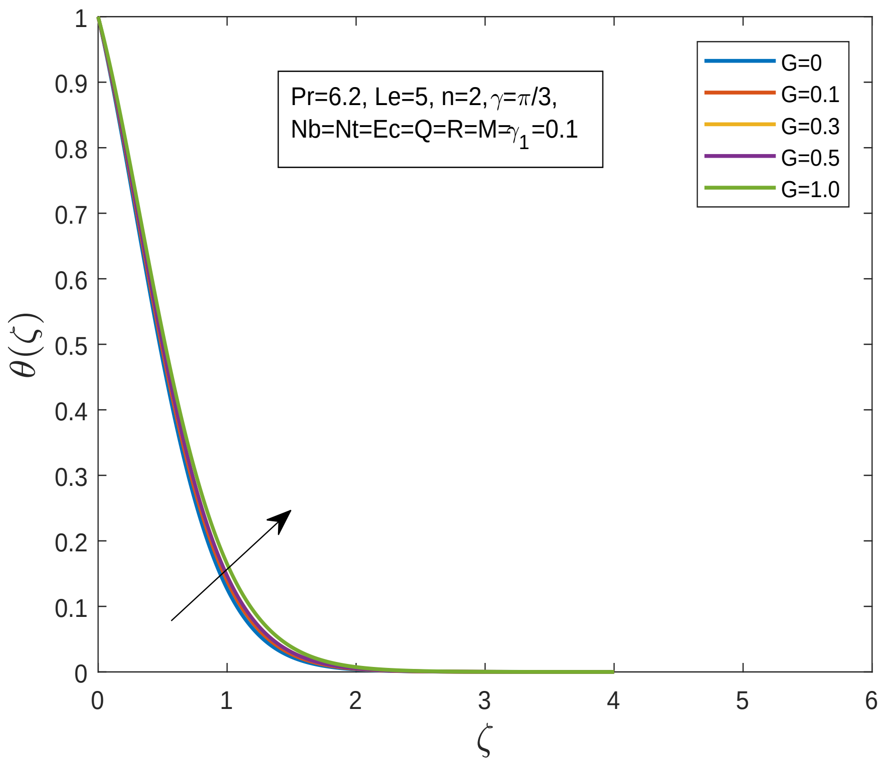

Figure 4.6: Impact of G on the temperature profile.

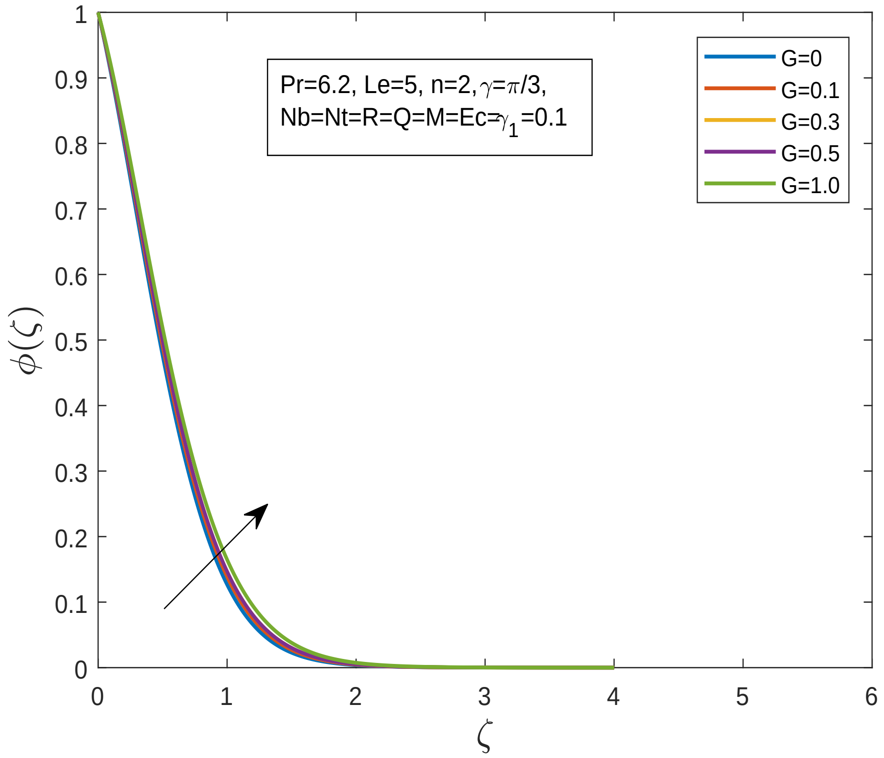

FIGURE 4.7: Impact of G on the concentration profile.

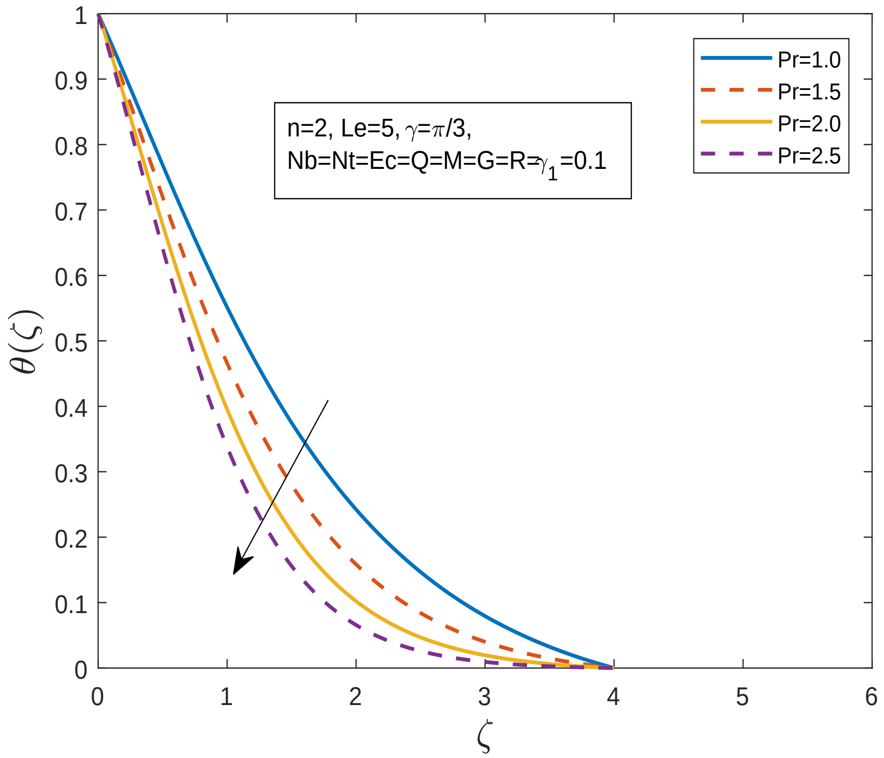

FIGURE 4.8: Impact of Pr on the temperature profile.

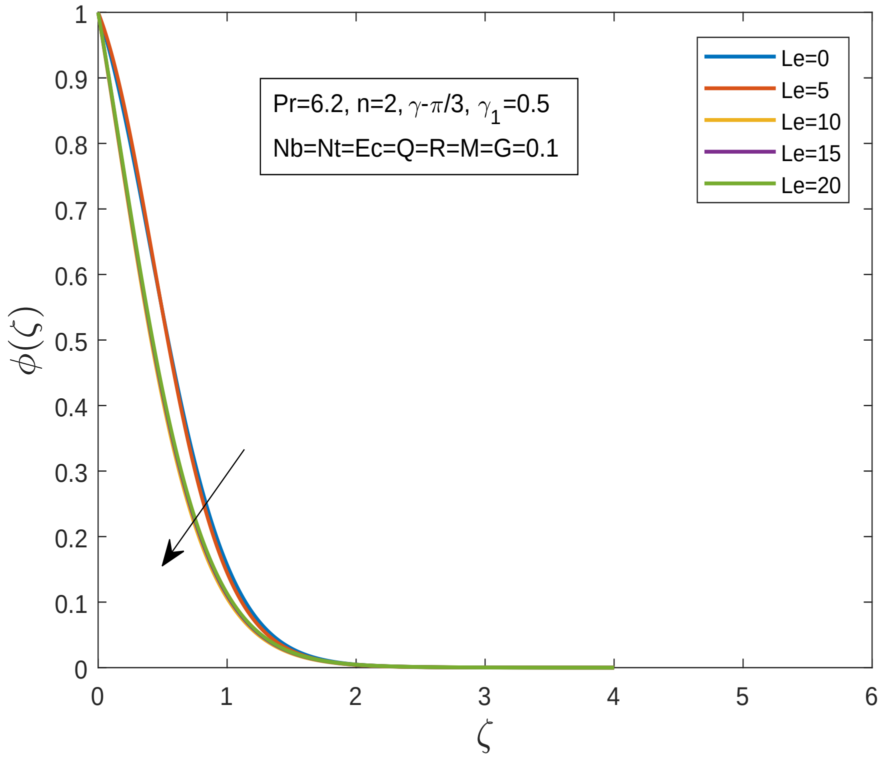

FIGURE 4.9: Impact of Le on the concentration profile.

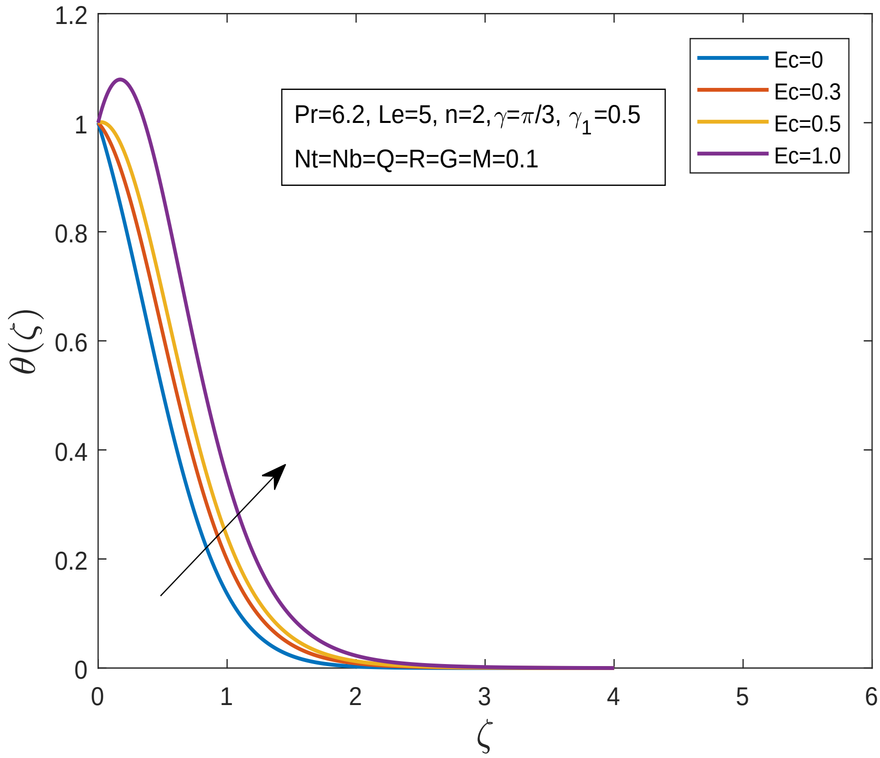

FIGURE 4.10: Impact of Ec on the temperature profile.

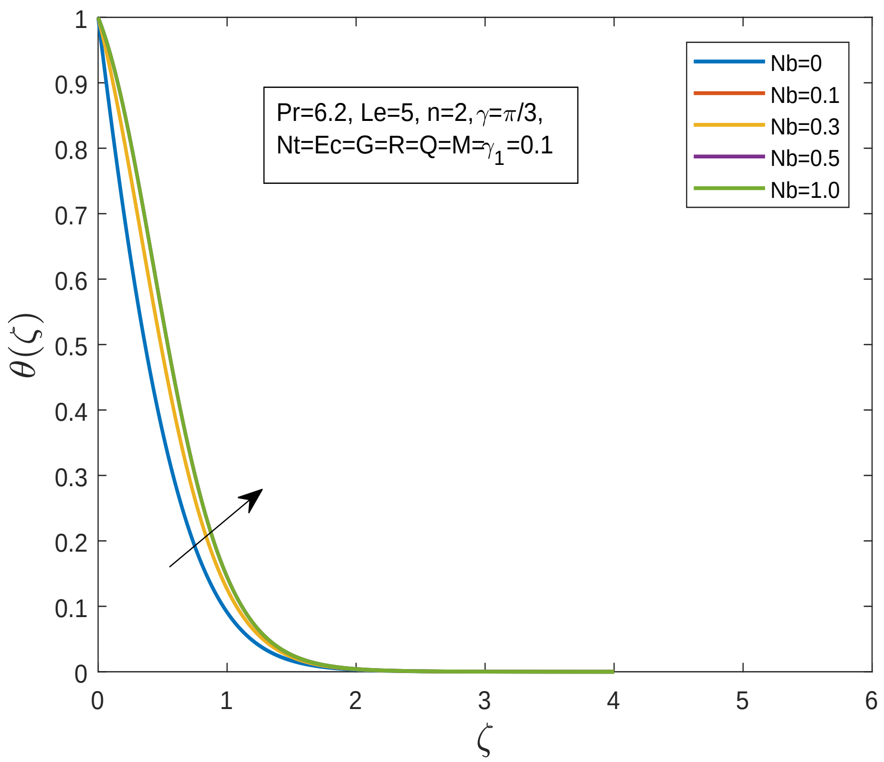

FIGURE 4.11: Impact of Nb on the temperature profile.

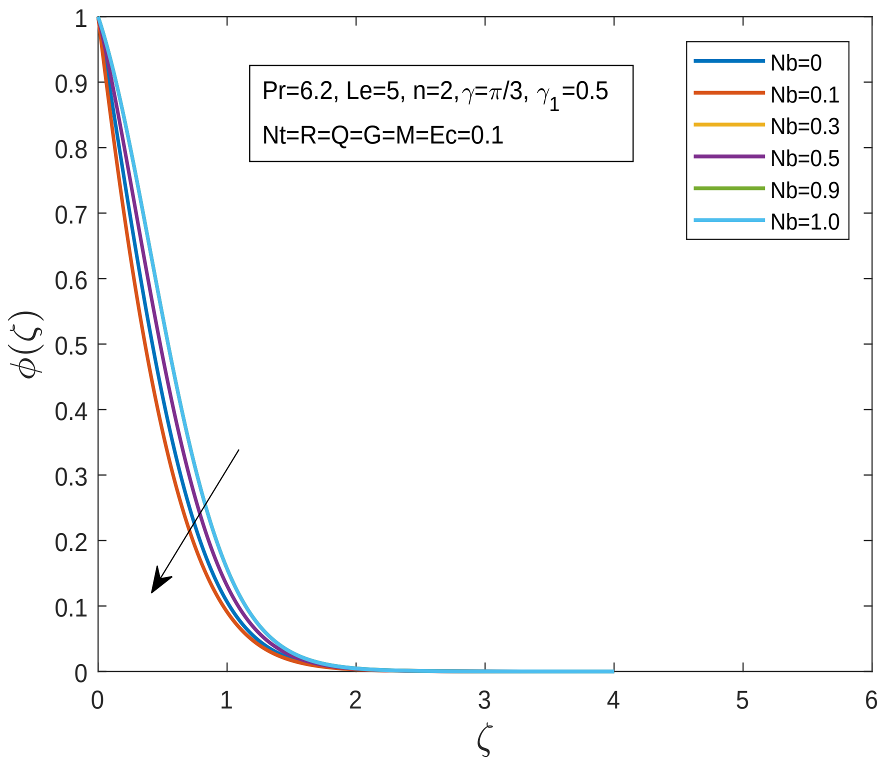

FIGURE 4.12: Impact of Nb on the concentration profile.

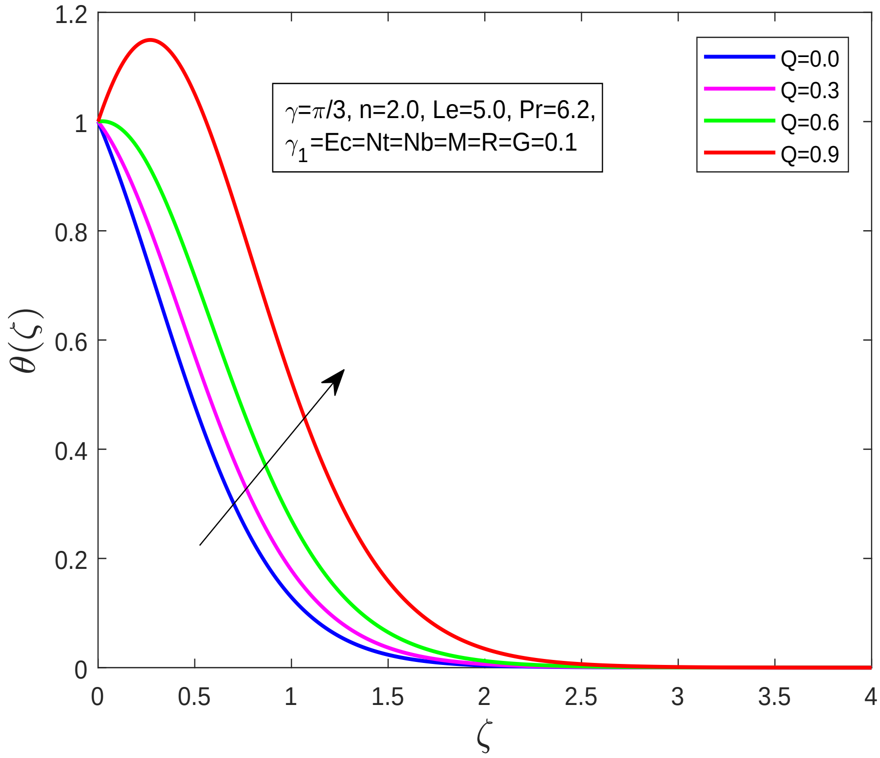

FIGURE 4.13: Impact of Q on the temperature profile.

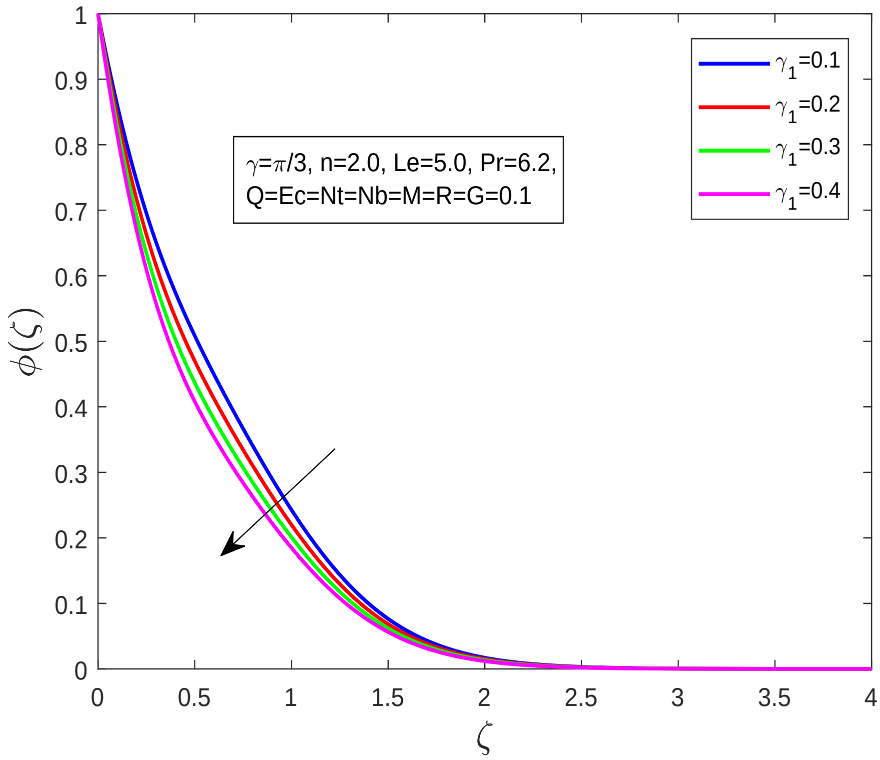

FIGURE 4.14: Impact of γ_1 on the concentration profile.

Chapter 5

Conclusion

In this thesis, the work of Rao et al is reviewed and extended with the effect of inclined magnetic field, heat generation and chemical reaction. First of all, momentum, energy and concentration equations are converted into the ODEs by using some similarity transformations. By using the shooting method, numerical solution has been found for the transformed ODEs. Using different values of the governing physical parameters, the results are presented in the form of tables and graphs for velocity, temperature and concentration profiles. The achievements of the current research can be summarized as below:

- Increasing the values of M, the velocity profile decreases while the temperature profile increases.
- A decrement is noticed in Nusselt number due to ascending values of Lewis number.
- The velocity profile is decreased due to the increasing values of the permeability paramater G.
- By increasing the values of chemical reaction γ_1, the concentration profile decreased.
- By increasing the values of M, the concentration profile increased.

- Rising the values of heat generation parameter Q results in increase the temperature profile.
- An increment is noticed in the temperature distribution by rising the values of Eckert number Ec.
- With a rise in Nb, the temperature profile increases.
- Due to the ascending values of Le, the numerical values of local Sh_x is increased.

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