Complex Synchronization and Parameter Identification of Nonlinear Complex Chaotic Systems Using Adaptive Integral Sliding Mode Control

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Dedication

This thesis is dedicated to my Parents, Teachers and Friends for their endless love, support and encouragement.

CAPITAL UNIVERSITY OF SCIENCE & TECHNOLOGY
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CERTIFICATE OF APPROVAL

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DECLARATION

It is declared that this is an original piece of my own work, except where otherwise acknowledged in text and references. This work has not been submitted in any form for another degree or diploma at any university or other institution for tertiary education and shall not be submitted by me in future for obtaining any degree from this or any other University or Institution.

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ABSTRACT

This thesis presents three different kinds of complex synchronization (CS), (i) Complex Complete Synchronization (CCS), (ii) Complex Projective Synchronization (CPS), (iii) Complex Generalized Synchronization (CGS) of Identical and Non-identical Nonlinear Complex Systems with unknown parameters. Based on adaptive integral sliding mode control, an adaptive controller and parameter update laws are designed to realize CCS, CPS and CGS. To employ the adaptive integral sliding mode control, the error system is transformed into a special structure containing nominal part and some unknown terms. The unknown terms are computed adaptively. Then the error system is stabilized using adaptive integral sliding mode control. The stabilizing controller for the error system is constructed which consists of the nominal control plus some compensator control. The compensator controller and the adapted law are derived in such a way that the time derivative of a Lyapunov function becomes strictly negative. The proposed scheme is successfully applied to complex chaotic nonlinear systems with unknown parameters for the realization of (i) Complex Complete Synchronization (CCS),(ii) Complex Projective Synchronization (CPS),(iii) Complex Generalized synchronization (CGS).

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LIST OF ACRONYMS

CS	Complete Synchronization
AS	Anti Synchronization
CCS	Complex Complete Synchronization
CP	Projective Synchronization
MPS	Modified Projective Synchronization
MHPS	Modified Hybrid Projective Synchronization
CPS	Complex Projective Synchronization
CMPS	Complex Modified Projective Synchronization
CMHPS	Complex Modified Hybrid Projective Synchronization
CMHFPS	Complex Modified Hybrid Function Projective Synchronization
CGS	Complex Generalize Synchronization
SMC	Sliding Mode Control
ISMC	Integral Sliding Mode Control
RP	Reaching Phase
SM	Sliding Mode
SS	Sliding Surface
SOSM	Second Order Sliding Mode
HOSM	Higher Order Sliding Mode

Chapter 1

INTRODUCTION

1.1 Introduction

All physical systems are nonlinear by nature. In order to attain better understanding about the dynamical behavior of different nonlinear systems, an interesting and important phenomenon is to investigate synchronization between these dynamical systems. Synchronization, observed as naturally occurring process, has significant impact in diverse areas of engineering, sciences and even in the social life. Synchronization of nonlinear systems is an attractive area among researchers of different disciplines due to its numerous applications in the fields of engineering and technology. Noteworthy efforts by researchers have been devoted to investigate the problem of synchronization of nonlinear systems. To address the problem of complex synchronization of nonlinear systems, the estimation of different unknown parameters associated with nonlinear system is crucial. The unknown parameters have strong influence on complex synchronization.

1.1.1 Overview

Synchronization of complex chaotic systems is the rudimentary determination of this research work. We need to stabilize the error system for any initial condition. The technique used is Adaptive Integral Sliding Mode Control. Appropriate Hurwitz sliding surface and Lyapunov function are selected to stabilize the error system. Adaptive laws are obtained using Lyapunov stability theory.

1.1.2 Motivation

The complex chaotic synchronization has been a topic of interest for the researchers over the last two decades. It is hardly possible to avoid contact with complex chaotic systems. Such problems arise in our daily life. Some of these problems are simple to solve but there are control problems with more complications. Synchronization of nonlinear systems contains diverse area of application in almost every field of life.

1.2 Problem statement

The purpose of this study is to develop appropriate synchronization schemes for different nonlinear complex chaotic systems working according to master-slave principal that addresses

- Complex Complete Synchronization of two identical nonlinear complex chaotic systems.
- Complex Projective Synchronization of two non-identical nonlinear complex chaotic systems.
- Complex Generalized Synchronization of two identical, non-identical and hyperchaotic nonlinear complex chaotic systems.

1.3 Application of Research

As we are dealing with the complex chaotic systems, there are many examples of these systems in our daily life. We have begun to understand that the tools of chaotic theory can be applied on the way to understanding, manipulation, and control of a variety of systems. Complex chaotic system is applicable in actual-world as epileptic seizure, heart fibrillation, neural process, chemical reactions, climate, industrial control processes, and many more.

1.4 Structure of the Thesis

The rest of this thesis is organized as follows:

Chapter 2: Literature review

This chapter will give us a review of the literature published about the chaotic systems and synchronization of complex chaotic systems.

Chapter 3: Complex Complete Synchronization (CCS)

This chapter contains the proposed algorithm for complex complete synchronization (CCS) systems. Adaptive Integral Sliding Mode Control law is developed to investigate the problem of synchronization of nonlinear systems with unknown parameters.

Chapter 4: Complex Projective Synchronization (CPS)

This chapter contains the proposed algorithm for complex projective synchronization (CPS) systems. Adaptive Integral Sliding Mode Control verify the proposed scheme.

Chapter 5: Complex Generalized Synchronization (CGS)

This chapter contains the proposed algorithm for complex generalized synchronization (CGS) systems. Adaptive Integral Sliding Mode Control is used to prove the proposed scheme.

Chapter 6: Conclusion and future work

A brief conclusion of thesis is outlined in this Chapter. Moreover, some future research work is suggested for the researchers to work in the area of complex synchronization of nonlinear systems.

Chapter 2

LITRATURE REVIEW

2.1 Introduction

This chapter presents literature review of complex chaotic systems, synchronization types, sliding mode control and integral sliding mode control.

2.2 Chaotic Systems

Chaos is the irregular motion of a dynamical system; it is deterministic, sensitive to initial conditions, and impossible to predict in the long term. It is neither harmonic nor random. Chaos is characterized by the way a dynamical system does not repeat itself even though the system is governed by deterministic equations [1]. Phase plane and correlation is used to identify the attractor and randomness of the chaotic system. The attractor is a region of the state space from which there are no exit paths. For chaotic systems, the attractor does not settle to one of these but span the state space around the attractor for all time without ever repeating. It does not come back to previously points in the state space, this describes the stretching and folding properties [2], which can be seen when plotting the states of the system against each other. Figure 2.1 plots the trajectory of the Lorenz attractor in the phase space, depicting the stretching and folding properties [3], which can be seen when plotting the states of the system against each other.

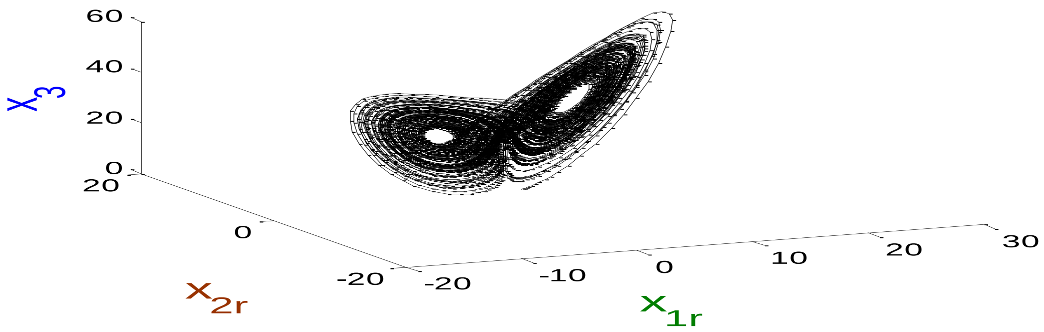

Figure 2.1: The phase portrait of Lorenz system x_{1r}, x_{2r}, x_3.

2.3 Chaos Synchronization

Dutch researcher Christian Huygens was probably the first scientist who observed and described the synchronization phenomena in seventeenth century. In 1658, Christian Huygens investigated the synchronization between two coupled pendulum clocks [4]. Despite the study of the first synchronization phenomena, the actual work on synchronization of nonlinear systems was started late in 1920. After a few years, in 1927, Balthasar Vander Pol extended the efforts of W. H. Reck and J. H. Vincent by obtaining theoretical and practical results for synchronization [5]. In the literature Peccora and Carrol first introduced the idea of synchronization of nonlinear chaotic systems, by investigating the properties of two nonlinear systems and described that two nonlinear systems can be synchronized by linking them with a common signal. After the inspirational work of Peccora and Carrol, on synchronization of dynamical systems, this problem attracted a great number of researchers from different fields of engineering and sciences [6]. Considerable research work has been carried out to investigate the synchronization phenomena in different nonlinear systems and different control strategies have been developed [7]. Since after the pioneer work on synchronization of two identical nonlinear systems, namely, response and drive systems [8], the problem of synchronization of nonlinear systems has been extensively studied in both theoretical and practical systems.

2.4 Types of Synchronization

There are some main types of synchronization:

(1) Complete Synchronization: When driven and response meet to be exactly same.

$$\lim_{t\to+\infty}\|e(t)\| = \lim_{t\to+\infty}\|y(t)-x(t)\| = 0.$$

(2) Generalized synchronization: Synchronization between the states of two systems by a functional relation is defined as generalized synchronization.

$$\lim_{t\to+\infty}\|e(t)\| = \lim_{t\to+\infty}\|y(t)-D(x)\| = 0.$$

(3) Phase synchronization: When their phase difference remains bounded and amplitudes remain uncorrelated. $\|\varphi_1(t)-\varphi_2(t)\|=0$. Where, $\varphi_1(t)$ and $\varphi_2(t)$ indicate the phases of two coupled oscillators.

(4) Lag synchronization: When dynamics is described by delay differential equations. One of the oscillators follows of other. $\|X_1(t)-X_1(t+\tau)\|=0$, Where τ is delay.

(5) Projective synchronization: The states of master $x(t)$ and response system $y(t)$ synchronize with respect to scaling factor α. i.e. $\lim_{t\to+\infty}\|e(t)\|=\lim_{t\to+\infty}\|y(t)-\alpha\ x(t)\|=0$.

2.4.1 Complex Complete Synchronization (CCS)

Chaos synchronization, as an important topic in nonlinear science, has been widely investigated in many fields, such as physics, chemistry and ecological science [9, 10, 11]. Chaotic system is deterministic, as long-term a periodic behavior, and sensitive dependence on the initial conditions. If the system has one positive Lyapunov-exponent, then the system is called chaotic. For more details see synchronization of chaotic systems [12-14]. In the literature reference some results on chaos synchronization are derived by using the known parameters of master and slave systems, and the controller is constructed by those known parameters. The synchronization will be destroyed with the effects of these uncertainties. On the other hand, in real physical systems or experimental situations, chaotic systems may have some uncertain parameters and may change from time to time [9-12]. Thus, it is a very important problem to realize chaos synchronization for these uncertain chaotic systems. The adaptive control is one of popular and useful approaches to control and synchronize nonlinear systems with uncertain parameters [9-21]. In early 1950s, research on the adaptive control was first proposed to design the autopilot for high-performance aircrafts, which operate at a wide range of speeds and altitudes [22]. In last few years researcher has been introduced and studied several examples of chaotic nonlinear systems with complex variables [23-29]. These systems which involving complex variables are used to describe the physics of a detuned laser, rotating fluids, disk dynamos, electronic circuits, and particle beam dynamics in high energy accelerators.

Consider the two complex chaotic systems:

$$\dot{x} = f(x) + F(x)\theta \tag{2.1}$$

$$\dot{y} = g(y) + G(y)\theta + u \tag{2.2}$$

are said to be complex complete synchronization if:

$$\lim_{t\to+\infty}\|e(t)\| = \lim_{t\to+\infty}\|y(t) - x(t)\| = 0 \tag{2.3}$$

2.4.2 Complex Projective Synchronization (CPS)

Projective synchronization is a form of chaos synchronization recently observed in coupled partially linear systems of three dimensions [31]. In projective synchronization, the phases are locked and the amplitudes of the two coupled systems synchronize up to a scaling factor. The scaling factor is a constant transformation between the synchronized variables of the master and slave systems.

In the literature Zhang et al. [32] discussed modified projective synchronization (MPS) with complex scaling factors of uncertain real chaos and complex chaos. Mahmoud et al. [33] achieved complex modified projective synchronization (CMPS) of two certain chaotic complex systems. Sun et al. [34] introduced combination synchronization with complex scaling matrix. Liu and Zhang [35] discussed function projective synchronization (FPS) with complex function matrix of coupled chaotic complex system with known real parameters.

It should be noted that the above papers only consider complex chaotic synchronization of the same dimensional, and the states of the drive and response systems synchronize by a diagonal matrix, so each state variable of response system synchronizes one of drive system by a special scaling factor. As a matter of fact, the synchronization can be carried out through different dimensional oscillators, especially biological science and social science [36], where the drive and response systems could synchronize by a desired transformation matrix, not a square matrix. Therefore, Luo and Wang [37] introduced hybrid modified function projective synchronization (MHFPS) of two different dimensional complex chaotic systems. Moreover, as the complex function transformation matrix is more unpredictable than real function transformation matrix in [38], it will greatly increase the complexity and diversity of the synchronization.

Consider master/slave complex chaotic systems:

$$\dot{x} = f(x) + F(x)\theta \tag{2.4}$$

$$\dot{y} = g(y) + G(y)\vartheta + u \tag{2.5}$$

are said to be complex projective synchronization if:

$$\lim_{t\to+\infty}\|e(t)\| = \lim_{t\to+\infty}\|y(t) - D(t)x(t)\| = 0 \tag{2.6}$$

2.4.3 Complex Generalized Synchronization (CGS)

In complex projective synchronization the response complex systems has been synchronized with the drive complex systems up to the desired complex scaling matrices. Rulkov et al. firstly proposed the generalized synchronization, where two chaotic systems are said to be synchronized if a given functional relation can be realized between the variables of drive and response systems [40]. In the literature generalized synchronization of chaotic or hyperchaotic real systems has been widely investigated in last two decade. For instance, [41, 66] realized generalized synchronization of different chaotic and hyperchaotic systems, while [47, 48] achieved adaptive generalized synchronization (AGS) and parameter identification of different chaotic systems with unknown parameters. There were few published achievements on CGS of non-identical nonlinear complex systems. So, it is meaningful and challenging to extend GS from real systems to complex systems, and to realize CGS and parameter identification of chaotic and hyperchaotic complex systems with unknown parameters using adaptive integral sliding mode. In [49], the author presented adaptive control scheme and parameter update laws for two non-identical and hyperchaotic complex chaotic systems with unknown parameters.

Consider master/slave complex chaotic systems:

$$\dot{x} = f(x) + F(x)\theta \tag{2.7}$$

$$\dot{y} = g(y) + G(y)\vartheta + u \tag{2.8}$$

are said to be complex generalized synchronization if:

$$\lim_{t\to+\infty}\|e(t)\| = \lim_{t\to+\infty}\|y(t) - \phi(x)\| = 0 \tag{2.9}$$

The above types of synchronization can be summarized by the following table.

Settings the matrix M	**Type of synchronization**
$M = diag(1,1,...,1)$	Complete Synchronization (CS)
$M = diag(-1,-1,...,-1)$	Anti Synchronization (AS)
$M = diag(d, d,.....d) \in R^{n\times n}$	Projective synchronization (PS)
$M = diag(d_1, d_2,.....d_n) \in R^{n\times n}$	Modified Projective Synchronization (MPS)
$M \in R^{m\times n}$	Modified Hybrid Projective Synchronization (MHPS)
$M = diag(d, d,.....d) \in C^{n\times n}$	Complex Projective Synchronization (CPS)
$M = diag(d_1, d_2,.....d_n) \in C^{n\times n}$	Complex Modified Projective Synchronization (CMPS)
$M \in C^{m\times n}$	Complex Modified Hybrid Projective Synchronization (CMHPS)
$M = M_r(t) + jM_i(t) \in C^{m\times n}$	Complex Modified Hybrid Function Projective Synchronization (CMHFPS)

$$\lim_{t\to+\infty}\|e(t)\| = \lim_{t\to+\infty}\|y(t) - M(x)\| = 0$$

Table 2.1: Types of function synchronization

2.5 Sliding Mode Control

Sliding mode control (SMC) is a nonlinear control technique featuring remarkable properties of accuracy, robustness, and easy tuning and implementation. Nonlinear control laws are designed to drive the system states onto a particular surface in the state space, named sliding surface. Once the sliding surface is reached, sliding mode control keeps the states on the close neighborhood of the sliding surface. Hence the sliding mode control is a two part controller design. The first part involves the design of a sliding surface so that the sliding motion satisfies design specifications. The second is concerned with the selection of a control law that will make the switching surface attractive to the system state [50]. There are two main advantages of sliding mode control. First is that the dynamic behavior of the system may be tailored by the particular choice of the sliding surface. Secondly, the closed loop response becomes totally insensitive to some particular uncertainties like parameter uncertainties, disturbance.

2.5.1 Sliding Surface Design

This section explores variable structure control as a speedy swapped feedback control causing in sliding mode. The reason for transferring control function is to make the nonlinear system state onto a pre-indicated plane in the state space and to keep up system state path on this surface for consequent time. The surface is known as a switching surface. The feedback track has gained one when the plant states route is “above” the surface and different gains if the path is “beneath” the surface. This surface characterizes the principle for proper switching. This surface is similarly named the sliding surface.

The uncertainties and disturbances are always present in practical system and in such cases, discontinuous control ensures robustness. Figure 2.2 shows the reaching phase (RP), sliding mode (SM) and sliding surface (SS).

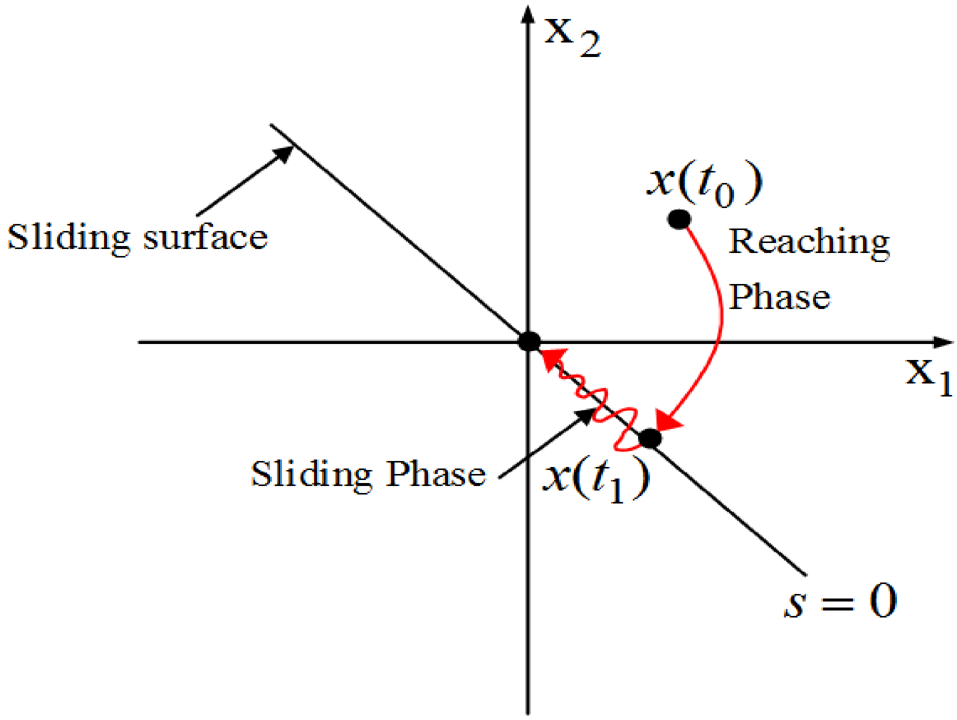

Figure 2.2: The Sliding Phase, Reaching Phase and Sliding Surface

2.5.2 Chattering Phenomenon

In sliding mode scheme the control signal exhibits high frequency oscillation called chattering. Such chattering has much effect in real world applications. This phenomenon may lead to large unwanted oscillations that degrade performance of the system. In order to avoid chattering effect, various solutions of this problem have been proposed. i.e. the boundary layer design. A new design scheme based on estimation of sliding variable was presented [51]. The method based on the describing function approach was developed for chattering analysis of the system in the presence of the un-modeled dynamics. Another way to reduce chattering effect is by means of Second Order Sliding Mode (SOSM) and the Higher Order Sliding Mode (HOSM) control techniques. Figure 2.3 shows the chattering effect.

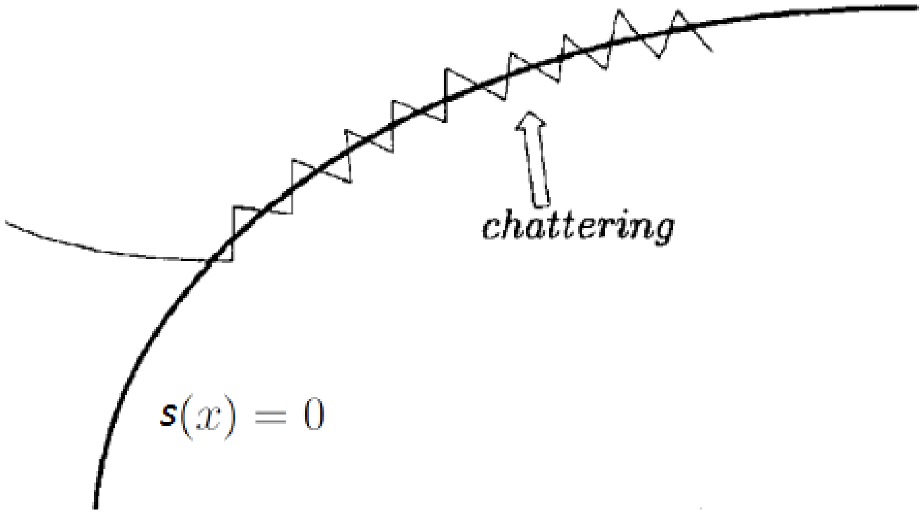

Figure 2.3: The Chattering Effect

2.6 Integral Sliding Mode Control

The basic idea of ISMC was initially proposed to enforce a sliding mode from the beginning of the system response, which means a controller based on ISMC ideas can provide compensation to matched uncertainties throughout the entire system response. In ISMC, it is assumed that there exists a nominal plant, for which a properly designed state feedback controller has already been designed to ensure asymptotic stability of the closed-loop system, and to satisfy predefined performance specifications. Integral term is ‘added’ to the nominal state feedback controller to ensure the nominal performance is maintained, and the system is insensitive to external disturbances. This design philosophy provides the opportunity to retro-fit an ISM to the existing baseline controller to compensate for the matched uncertainties and external disturbances throughout the system response.

2.6.1 Properties of Integral Sliding Mode

The properties of integral sliding mode are listed below:

• There is no reaching phase and a sliding mode is enforced throughout the entire system response.

- During sliding, the order of the motion is the same as the original system.
- By a suitable choice of sliding surface, the effect of unmatched uncertainty can be ameliorated.
- During the sliding mode, the system motion is invariant to matched uncertainties.
- The ISM approach has the ability to be retro-fitted to an existing feedback controller.

Chapter 3

Complex Complete Synchronization (CCS)

3.1 Introduction

In this chapter we present a new control design methodology to achieve Complex Complete Synchronization (CCS) in complex chaotic systems with unknown parameters. The proposed design methodology is based on Adaptive Integral Sliding Mode Control. First, the design methodology is presented for the general case of complex chaotic systems. Then, to illustrate the design procedure, to verify its validity, and to show its effectiveness, the proposed design approach is applied on two identical complex Lorenz systems [30] with unknown parameters.

3.2 Problem formulation

The aim of this chapter is to study and investigate the complete synchronization of two identical complex Lorenz systems with unknown parameters. We design adaptive integral sliding mode control and prove the effectiveness of this scheme for these complex systems.

Consider the following two complex chaotic nonlinear systems as:

$$\dot{x} = f(x) + F(x)\theta \qquad (3.1)$$

$$\dot{y} = g(y) + G(y)\theta + u \qquad (3.2)$$

Where $x = (x_1, x_2, ..., x_n)^T$ and $y = (y_1, y_2, ..., y_n)^T$ are complex state vector, and $u = (u_r + ju_i) \in R^n$ is the control input.

Definition: For the drive system (3.1) and the response system (3.2), it is said to be complete synchronization between $y(t)$ and $x(t)$, if there exists a controller $u(x, y)$ such that:

$$\lim_{t\to+\infty}\|e(t)\| = \lim_{t\to+\infty}\|y(t) - x(t)\| = 0 \qquad (3.3)$$

for all initial conditions.

3.3 General Proposed Algorithm for Complex Complete Synchronization

Consider the following two systems

$$\dot{x} = f(x) + F(x)\theta \tag{3.4}$$

$$\dot{y} = g(y) + G(y)\theta + u(x, y) \tag{3.5}$$

Where $x = (x_1, x_2, ..., x_n)^T \in R^n$ and $y = (y_1, y_2, ..., y_n)^T \in R^m$ are complex state vectors of the drive system (3.4) and response system (3.5) respectively, where $x_k = x_{kr} + jx_{ki}, k = 1,2,...,n$, $y_l = y_{lr} + jy_{li}, l = 1,2,...,n$, $j = \sqrt{-1}$, the subscripts r and i denote the real and imaginary parts of the complex variables, vectors and matrices throughout this paper. $\theta \in \Re^p$ and $\vartheta \in \Re^q$ are real vectors of unknown parameters. $F(x) \in C^{n\times p}$ and $G(y) \in C^{n\times q}$ are complex matrices, $F(x) = F_r(x) + jF_i(x)$, $G(x) = G_r(x) + jG_i(x)$. $f(x) \in C^n$ and $g(y) \in C^n$ are vectors of nonlinear complex functions, and $f(x) = f_r(x) + jf_i(x)$, $g(y) = g_r(y) + jg_i(y)$. $u(x, y) \in C^n$ is the complex control vector, and $u(x, y) = u_r(x, y) + ju_i(x, y)$.

Assume that $m = n$ define the complex CCS error vector gives as:

$$\begin{aligned} e = y - x = e_r + je_i &= (y_r + jy_i) - (x_r + jx_i) \\ e_r + je_i &= (y_r - x_r) + j(y_i - x_i) \end{aligned} \tag{3.6}$$

Where $e = (e_1, e_2, ..., e_n)^T \in C^n$, $e_r = (e_{1r}, e_{2r}, ..., e_{nr})^T \in \Re^n$ $e_i = (e_{1i}, e_{2i}, ..., e_{ni})^T \in \Re^n$

By taking the derivative of equation (3.6) with respect time, the error dynamic can be written as:

$$\begin{aligned} \dot{e} = \dot{e}_r + j\dot{e}_i &= \dot{y} - \dot{x} = g(y) + G(y)\theta + u(x, y) - j\{f(x) + F(x)\theta\} \\ &= g_r(y) + jg_i(y) + G_r(y)\theta + jG_i(y)\theta + u_r(x, y) + ju_i(x, y) \\ &\quad - \{f_r(x) + jf_i(x) + F_r(x)\theta + jF_i(x)\theta\} \\ &= g_r(y) + G_r(y)\theta + u_r(x, y) - \{f_r(x) + F_r(x)\theta\} \\ &\quad + j[g_i(y) + G_i(y)\theta + u_i(x, y) - \{f_i(x) + F_i(x)\theta\}] \end{aligned} \tag{3.7}$$

The complex error system (3.7) can be written in real form as:

$$\begin{aligned} \dot{e}_r &= g_r(y) + G_r(y)\theta + u_r(x, y) - \{f_r(x) + F_r(x)\theta\} \\ \dot{e}_i &= \{g_i(y) + G_i(y)\theta + u_i(x, y) - \{f_i(x) + F_i(x)\theta\} \end{aligned} \tag{3.8}$$

Let $\hat{\theta}$ be estimate of θ respectively and let $\tilde{\theta} = \theta - \hat{\theta}$ be error in estimating θ respectively. Then equation (3.8) can be written as:

$$\dot{e}_r = g_r(y) + G_r(y)\hat{\theta} + G_r(y)\tilde{\theta} + u_r(x,y) - \{f_r(x) + F_r(x)\hat{\theta} + F_r(x)\tilde{\theta}\}$$
$$\dot{e}_i = g_i(y) + G_i(y)\hat{\theta} + G_i(y)\tilde{\theta} + u_i(x,y) - \{f_i(x) + F_i(x)\hat{\theta} + F_i(x)\tilde{\theta}\} \quad (3.9)$$

That can be written in vector form:

$$\begin{bmatrix}\dot{e}_r\\ \dot{e}_i\end{bmatrix} = \begin{bmatrix} g_r(y) + G_r(y)\hat{\theta} - f_r(x) - F_r(x)\hat{\theta} \\ g_i(y) + G_i(y)\hat{\theta} - f_i(x) - F_i(x)\hat{\theta}\end{bmatrix} + \begin{bmatrix} u_r(x,y)\\ u_i(x,y)\end{bmatrix} + \begin{bmatrix} G_r(y)\tilde{\theta} - F_r(x)\tilde{\theta} \\ G_i(y)\tilde{\theta} - F_i(x)\tilde{\theta}\end{bmatrix} \quad (3.10)$$

By choosing

$$\begin{bmatrix} u_r(x,y)\\ u_i(x,y)\end{bmatrix} = \begin{bmatrix} ee_r\\ ee_i\end{bmatrix} - \begin{bmatrix} g_r(y) + G_r(y)\hat{\theta} - f_r(x) - F_r(x)\hat{\theta} \\ g_i(y) + G_i(y)\hat{\theta} - f_i(x) - F_i(x)\hat{\theta}\end{bmatrix} \quad (3.11)$$

Where $ee_r = \begin{bmatrix} e_{2r}\\ e_{3r}\\ \vdots \\ e_{nr}\\ e_{1i}\end{bmatrix}, ee_i = \begin{bmatrix} e_{2i}\\ e_{3i}\\ \vdots \\ e_{ni}\\ v\end{bmatrix}$

v is the new input, then system (3.10) becomes:

$$\begin{bmatrix}\dot{e}_r\\ \dot{e}_i\end{bmatrix} = \begin{bmatrix} ee_r\\ ee_i\end{bmatrix} + \begin{bmatrix} G_r(y)\tilde{\theta} - F_r(x)\tilde{\theta} \\ G_i(y)\tilde{\theta} - F_r(x)\tilde{\theta}\end{bmatrix}$$

or

$$\begin{bmatrix}\dot{e}_{1r}\\ \dot{e}_{2r}\\ \vdots \\ \dot{e}_{nr}\\ \dot{e}_{1i}\\ \dot{e}_{2i}\\ \vdots \\ \dot{e}_{ni}\end{bmatrix} = \begin{bmatrix} e_{2r}\\ \vdots \\ e_{nr}\\ e_{1i}\\ e_{2i}\\ \vdots \\ e_{ni}\\ v\end{bmatrix} + \begin{bmatrix} G_r(y)\tilde{\theta} - F_r(x)\tilde{\theta} \\ G_i(y)\tilde{\theta} - F_i(x)\tilde{\theta}\end{bmatrix} \quad (3.12)$$

To employ the integral sliding mode control, choose the nominal system for (3.12) as:

$$\begin{bmatrix} \dot{e}_{1r} \\ \dot{e}_{2r} \\ \vdots \\ \dot{e}_{nr} \\ \dot{e}_{1i} \\ \dot{e}_{2i} \\ \vdots \\ \dot{e}_{ni} \end{bmatrix} = \begin{bmatrix} e_{2r} \\ \vdots \\ e_{nr} \\ e_{1i} \\ e_{2i} \\ \vdots \\ e_{ni} \\ v_o \end{bmatrix} \qquad (3.13)$$

Define the Hurwitz sliding surface for nominal system (3.13) as: $\sigma_0 = C\,e = C_1 e_r + C_2 e_i$

$$C_1 = [1, c_1, \ldots, c_{n-1}], C_2 = [c_n, c_{n+1}, \ldots, c_{2n-2}, 1]$$

$$\sigma_0 = e_{1r} + \sum_{i=1}^{n-1} c_i e_{(i+1)r} + \sum_{k=0}^{n-2} c_{(n+k)} e_{(k+2)i} + e_{ni}$$

$$\dot{\sigma}_0 = \dot{e}_{1r} + \sum_{i=1}^{n-1} c_i \dot{e}_{(i+1)r} + \sum_{k=0}^{n-2} c_{(n+k)} \dot{e}_{(k+2)i} + \dot{e}_{ni}$$

$$\dot{\sigma}_0 = e_{2r} + \sum_{i=1}^{n-1} c_i e_{(i+2)r} + \sum_{k=0}^{n-2} c_{(n+k)} e_{(k+2)i} + v_o$$

By choosing $v_o = -e_{2r} - \sum_{i=1}^{n-1} c_i e_{(i+2)r} - \sum_{k=0}^{n-2} c_{(n+k)} e_{(k+2)i} - k\,\sigma_0 - k\,sign(\sigma_0),\ k > 0$, we have $\dot{\sigma}_0 = -k\,\sigma_0 - k\,sign(\sigma_0)$. Therefore the nominal system (3.13) is asymptotically stable.

Now choose the sliding surface for the system (3.12) as:

$$\sigma = \sigma_0 + z = Ce + z$$
$$\sigma = C_1 e_r + C_2 e_i + z$$

Where z is some integral term computed later. To avoid the reaching phase, choose $z(0)$ such that $\sigma(0) = 0$. Choose $v = v_0 + v_s$ where, v_0 is the nominal input and v_s is compensator term computed later. Then

$$
\begin{aligned}
\dot{\sigma} &= C_1\dot{e}_r + C_2\dot{e}_i + \dot{z} \\
\dot{\sigma} &= e_{2r} + \sum_{i=1}^{n-1} c_i e_{(i+2)r} + \sum_{k=0}^{n-2} c_{(n+k)} e_{(k+2)i} + v_0 + v_s + \dot{z} \\
&+ C_1[G_r(y)\tilde{\theta} - F_r(x)\tilde{\theta}] + C_2[G_i(y)\tilde{\theta} - F_r(x)\tilde{\theta}] \\
&= e_{2r} + \sum_{i=1}^{n-1} c_i e_{(i+2)r} + \sum_{k=0}^{n-2} c_{(n+k)} e_{(k+2)i} + v_0 + v_s + \dot{z} \\
&+ C_1 G_r(y)\tilde{\theta} - C_1 F_r(x)\tilde{\theta} + C_2 G_i(y)\tilde{\theta} - C_2 F_r(x)\tilde{\theta} \\
&= [e_{2r} + \sum_{i=1}^{n-1} c_i e_{(i+2)r} + \sum_{k=0}^{n-2} c_{(n+k)} e_{(k+2)i} + v_0 + v_s + \dot{z}] \\
&+ \{C_1 G_r(y) + C_2 G_i(y)\}\tilde{\theta} - \{C_1 F_r(x) + C_2 F_i(x)\}\tilde{\theta}
\end{aligned}
\tag{3.14}
$$

By choosing a Lyapunov function: $V = \frac{1}{2}\sigma^2 + \frac{1}{2}\tilde{\theta}^T\tilde{\theta}$, design the adaptive laws for $\tilde{\theta}, \hat{\theta}, \tilde{\vartheta}, \hat{\vartheta}$ and compute v_s such that $\dot{V} < 0$.

$$
\begin{aligned}
\dot{z} &= -e_{2r} + \sum_{i=1}^{n-1} c_i e_{(i+2)r} - \sum_{k=0}^{n-2} c_{(n+k)} e_{(k+2)i} - v_0, \\
v_s &= -k\sigma - k\, sign(\sigma) \\
\dot{\tilde{\theta}} &= -\sigma\{G_r(y)^T C_1^T + G_i(y)^T C_2^T\} + \sigma\,\{F_r(x)^T C_1^T \\
&+ F_i(x)^T C_2^T\} - k_1\tilde{\theta}, \text{where}, k, k_1 > 0
\end{aligned}
\tag{3.15}
$$

Proof:

Since

$$
\begin{aligned}
\dot{V} &= \sigma\dot{\sigma} + \tilde{\theta}^T\dot{\tilde{\theta}} \\
&= \sigma\{[e_{2r} + \sum_{i=1}^{n-1} c_i e_{(i+2)r} + \sum_{k=0}^{n-2} c_{(n+k)} e_{(k+2)i} + v_0 + v_s + \dot{z}] \\
&+ \{C_1 G_r(y) + C_2 G_i(y)\}\tilde{\theta} - \{C_1 F_r(x) + C_2 F_i(x)\}\tilde{\theta}\} \\
&+ \tilde{\theta}^T\dot{\tilde{\theta}} \\
&= \sigma\{[e_{2r} + \sum_{i=1}^{n-1} c_i e_{(i+2)r} + \sum_{k=0}^{n-2} c_{(n+k)} e_{(k+2)i} + v_0 + v_s + \dot{z}]\} \\
&+ \tilde{\theta}^T[\dot{\tilde{\theta}} - \sigma\{C_1 G_r(y) + C_2 G_i(y)\} + \sigma\{C_1 F_r(x) + C_2 F_i(x)\}]
\end{aligned}
$$

By using

$$\dot{z} = -e_{2r} + \sum_{i=1}^{n-1} c_i e_{(i+2)r} - \sum_{k=0}^{n-2} c_{(n+k)} e_{(k+2)i} - v_0,$$
$$v_s = -k\sigma - k\, sign(\sigma)$$
$$\dot{\tilde{\theta}} = -\sigma\{G_r(y)^T C_1^T + G_i(y)^T C_2^T\}\sigma\ \{F_r(x)^T C_1^T$$
$$+ F_i(x)^T C_2^T\} - k_1\tilde{\theta}, \text{where}, k, k_1, k_2 > 0$$

We have

$$\dot{V} = -k\sigma^2 - k|\sigma| - k_1\tilde{\theta}^T\tilde{\theta}.$$

From this we conclude that $\sigma, \tilde{\theta} \to 0$. Since $\sigma \to 0$, therefore $e_r, e_i \to 0$.

3.4 Numerical Example

The following example is taken from [30] where CCS problem was solved by adaptive control scheme. We have achieved CCS using adaptive integral sliding mode control.

Consider the Master system given in [30] as:

$$\begin{aligned} \dot{x}_1 &= a(x_2 - x_1) \\ \dot{x}_2 &= bx_1 - x_2 - x_1x_3 \\ \dot{x}_3 &= 0.5(\bar{x}_1 x_2 + x_1\bar{x}_2) - cx_3 \end{aligned} \qquad (3.16)$$

Where, $x_1 = x_{1r} + jx_{1i}$, $x_2 = x_{2r} + jx_{2i}$ are complex and $x_3 = x_{3r}$, are real. $\bar{x}_1, \bar{x}_2$ denote the complex conjugate variables of x_1, x_2. a, b and c are unknown real parameters. When $a = 14$, $b = 35$, $c = 3.7$ and $x(0) = [2+1j,\ 5+3j,\ 4]^T$,the chaotic attractor is plotted in Fig 3.1.

The slave system consider as [30]:

$$\begin{aligned} \dot{y}_1 &= a(y_2 - y_1) + u_1 \\ \dot{y}_2 &= by_1 - y_2 - y_1y_3 + u_2 \\ \dot{y}_3 &= 0.5(\bar{y}_1 y_2 + y_1\bar{y}_2) - cy_3 + u_3 \end{aligned} \qquad (3.17)$$

Where, $y_1 = y_{1r} + jy_{1i}$, $y_2 = y_{2r} + jy_{2i}$ are complex and $y_3 = y_{3r}$, are real. $\bar{y}_1, \bar{y}_2$ denote the complex conjugate variables of y_1, y_2 and u_1, u_2, u_3 controllers.

We investigate CCS of two identical complex systems with the same orders.

Let $\hat{a}, \hat{b}, \hat{c}$ be estimates of a, b, c and $\tilde{a} = a_i - \hat{a}, \tilde{b} = b - \hat{b}, \tilde{c} = c - \hat{c}$, be the errors in estimations of a, b, c , respectively. Then systems (3.16) and (3.17) can be written as:

$$\begin{aligned} \dot{x}_1 &= \hat{a}(x_2 - x_1) + \tilde{a}(x_2 - x_1) \\ \dot{x}_2 &= \hat{b}x_1 + \tilde{b}x_1 - x_2 - x_1 x_3 \\ \dot{x}_3 &= 0.5(\bar{x}_1 x_2 + x_1 \bar{x}_2) - \hat{c}x_3 - \tilde{c}x_3 \end{aligned} \tag{3.18}$$

$$\begin{aligned} \dot{y}_1 &= \hat{a}(y_2 - y_1) + \tilde{a}(y_2 - y_1) + u_1 \\ \dot{y}_2 &= \hat{b}y_1 + \tilde{b}y_1 - y_2 - y_1 y_3 + u_2 \\ \dot{y}_3 &= 0.5(\bar{y}_1 y_2 + y_1 \bar{y}_2) - \hat{c}y_3 - \tilde{c}y_3 + u_3 \end{aligned} \tag{3.19}$$

The 3-dimensional complex systems (3.18)-(3.19) can be written into 5-dimensional real systems:

$$\begin{aligned} \dot{x}_{1r} &= \hat{a}(x_{2r} - x_{1r}) + \tilde{a}(x_{2r} - x_{1r}) \\ \dot{x}_{1i} &= \hat{a}(x_{2i} - x_{1i}) + \tilde{a}(x_{2i} - x_{1i}) \\ \dot{x}_{2r} &= \hat{b}x_{1r} + \tilde{b}x_{1r} - x_{2r} - x_{1r} x_3 \\ \dot{x}_{2i} &= \hat{b}x_{1i} + \tilde{b}x_{1i} - x_{2i} - x_{1i} x_3 \\ \dot{x}_3 &= (x_{1r} x_{2r} + x_{1i} x_{2i}) - \hat{c}x_3 - \tilde{c}x_3 \end{aligned} \tag{3.20}$$

$$\begin{aligned} \dot{y}_{1r} &= \hat{a}(y_{2r} - y_{1r}) + \tilde{a}(y_{2r} - y_{1r}) + u_{1r} \\ \dot{y}_{1i} &= \hat{a}(y_{2i} - y_{1i}) + \tilde{a}(y_{2i} - y_{1i}) + u_{1i} \\ \dot{y}_{2r} &= \hat{b}y_{1r} + \tilde{b}y_{1r} - y_{2r} - y_{1r} y_3 + u_{2r} \\ \dot{y}_{2i} &= \hat{b}y_{1i} + \tilde{b}y_{1i} - y_{2i} - y_{1i} y_3 + u_{2i} \\ \dot{y}_3 &= y_{1r} y_{2r} + y_{1i} y_{2i} - \hat{c}y_3 - \tilde{c}y_3 + u_3 \end{aligned} \tag{3.21}$$

Where $u_{1r}, u_{1i}, u_{2r}, u_{2i}$ and u_3 in Equation (3.21) are the control functions to be determined.

The complete complex synchronization error signals are defined as:

$$\begin{aligned} &e_{1r} = y_{1r} - x_{1r}, e_{1i} = y_{1i} - x_{1i}, e_{2r} = y_{2r} - x_{2r}, \\ &e_{2i} = y_{2i} - x_{2i}, e_3 = y_3 - x_3 \end{aligned} \tag{3.22}$$

Then the dynamics of the error system becomes:

$$\begin{aligned} &\dot{e}_{1r} = \dot{y}_{1r} - \dot{x}_{1r}, \dot{e}_{1i} = \dot{y}_{1i} - \dot{x}_{1i}, \dot{e}_{2r} = \dot{y}_{2r} - \dot{x}_{2r}, \\ &\dot{e}_{2i} = \dot{y}_{2i} - \dot{x}_{2i}, \dot{e}_3 = \dot{y}_3 - \dot{x}_3 \end{aligned} \tag{3.23}$$

$$
\begin{aligned}
\dot{e}_{1r} &= \dot{y}_{1r} - \dot{x}_{1r} = \hat{a}(y_{2r} - y_{1r}) + \tilde{a}(y_{2r} - y_{1r}) + u_{1r} - (\hat{a}(x_{2r} - x_{1r}) + \tilde{a}(x_{2r} - x_{1r})) \\
\dot{e}_{1i} &= \dot{y}_{1i} - \dot{x}_{1i} = \hat{a}(y_{2i} - y_{1i}) + \tilde{a}(y_{2i} - y_{1i}) + u_{1i} - (\hat{a}(x_{2i} - x_{1i}) + \tilde{a}(x_{2i} - x_{1i})) \\
\dot{e}_{2r} &= \dot{y}_{2r} - \dot{x}_{2r} = \hat{b}y_{1r} + \tilde{b}y_{1r} - y_{2r} - y_{1r}y_3 + u_{2r} - (\hat{b}x_{1r} + \tilde{b}x_{1r} - x_{2r} - x_{1r}x_3) \\
\dot{e}_{2i} &= \dot{y}_{2i} - \dot{x}_{2i} = \hat{b}y_{1i} + \tilde{b}y_{1i} - y_{2i} - y_{1i}y_3 + u_{2i} - (\hat{b}x_{1i} + \tilde{b}x_{1i} - x_{2i} - x_{1i}x_3) \\
\dot{e}_3 &= \dot{y}_3 - \dot{x}_3 = y_{1r}\, y_{2r} + y_{1i}y_{2i} - \hat{c}y_3 - \tilde{c}y_3 + u_3 - ((x_{1r}\, x_{2r} + x_{1i}x_{2i}) - \hat{c}x_3 - \tilde{c}x_3)
\end{aligned} \tag{3.24}
$$

By choosing

$$
\begin{aligned}
u_{1r} &= -\hat{a}(y_{2r} - y_{1r}) + \hat{a}(x_{2r} - x_{1r}) + e_{1i} \\
u_{1i} &= -\hat{a}(y_{2i} - y_{1i}) + \hat{a}(x_{2i} - x_{1i}) + e_{2r} \\
u_{2r} &= -(\hat{b}y_{1r} - y_{2r} - y_{1r}y_3) + (\hat{b}x_{1r} - x_{2r} - x_{1r}x_3) + e_{2i} \\
u_{2i} &= -(\hat{b}y_{1i} - y_{2i} - y_{1i}y_3) + (\hat{b}x_{1i} - x_{2i} - x_{1i}x_3) + e_3 \\
u_3 &= -(y_{1r}\, y_{2r} + y_{1i}y_{2i} - \hat{c}y_3) + ((x_{1r}\, x_{2r} + x_{1i}x_{2i}) - \hat{c}x_3) + v
\end{aligned} \tag{3.25}
$$

Where v is the new input, the system (3.24) can be written as:

$$
\begin{aligned}
\dot{e}_{1r} &= \tilde{a}(y_{2r} - y_{1r}) - \tilde{a}(x_{2r} - x_{1r}) + e_{1i} \\
\dot{e}_{1i} &= \tilde{a}(y_{2i} - y_{1i}) - \tilde{a}(x_{2i} - x_{1i}) + e_{2r} \\
\dot{e}_{2r} &= \tilde{b}y_{1r} - \tilde{b}x_{1r} + e_{2i} \\
\dot{e}_{2i} &= \tilde{b}y_{1i} - \tilde{b}x_{1i} + e_3 \\
\dot{e}_3 &= -\tilde{c}y_3 + \tilde{c}x_3 + v
\end{aligned} \tag{3.26}
$$

Choose the nominal system for (3.26) as:

$$
\begin{aligned}
\dot{e}_{1r} &= e_{1i} \\
\dot{e}_{1i} &= e_{2r} \\
\dot{e}_{2r} &= e_{2i} \\
\dot{e}_{2i} &= e_3 \\
\dot{e}_3 &= v_0
\end{aligned} \tag{3.27}
$$

Define the sliding surface for nominal system (3.27) as:

$$
\sigma_0 = (1 + \frac{d}{dt})^4 e_{1r} = e_{1r} + 4e_{1i} + 6e_{2r} + 4e_{2i} + e_3
$$

Then

$$
\dot{\sigma}_0 = \dot{e}_{1r} + 4\dot{e}_{1i} + 6\dot{e}_{2r} + 4\dot{e}_{2i} + \dot{e}_3 = e_{1i} + 4e_{2r} + 6e_{2i} + 4e_3 + v_0
$$

By choosing $v_0 = -e_{1i} - 4e_{2r} - 6e_{2i} - 4e_3 - k\,\sigma_0 - k\,sign(\sigma_0),\ k > 0$, we have $\dot{\sigma}_0 = -k\,\sigma_0$. Therefore the nominal system (3.27) is asymptotically stable.

Define the sliding surface for system (3.26) as:

$$\sigma = \sigma_0 + z = e_{1r} + 4e_{1i} + 6e_{2r} + 4e_{2i} + e_3 + z$$

Where, z is some integral term computed later. To avoid the reaching phase, choose $z(0)$ such that $\sigma(0) = 0$. Choose $v = v_0 + v_s$ where, v_0 is the nominal input and v_s is compensator term computed later. Then

$$\begin{aligned}\dot{\sigma} &= \dot{e}_{1r} + 4\dot{e}_{1i} + 6\dot{e}_{2r} + 4\dot{e}_{2i} + \dot{e}_3 + \dot{z} \\ &= \tilde{a}(y_{2r} - y_{1r}) - \tilde{a}(x_{2r} - x_{1r}) + e_{1i} + 4\tilde{a}(y_{2i} - y_{1i}) - 4\tilde{a}(x_{2i} - x_{1i}) + 4e_{2r} \\ &\quad + 6\tilde{b}y_{1r} - 6\tilde{b}x_{1r} + 6e_{2i} + 4\tilde{b}y_{1i} - 4\tilde{b}x_{1i} + 4e_3 - \tilde{c}y_3 + \tilde{c}x_3 + v_0 + v_s + \dot{z}\end{aligned} \tag{3.28}$$

By choosing a Lyapunov function: $V = \frac{1}{2}\sigma^2 + \frac{1}{2}(\tilde{a}^2 + \tilde{b}^2 + \tilde{c}^2)$, design the adaptive laws for $\tilde{a}, \hat{a}, , \tilde{b}, \hat{b}, \tilde{c}, \hat{c}$ and compute v_s such that $\dot{V} < 0$.

$$\begin{aligned}&\dot{z} = -e_{1i} - 4e_{2r} - 6e_{2i} - 4e_3 - v_0, \\ &v_s = -k\sigma - k\,sign(\sigma) \\ &\dot{\tilde{a}} = -\sigma(y_{2r} - y_{1r}) - 4\sigma(y_{2i} - y_{1i}) + \sigma(x_{2r} - x_{1r}) \\ &+ 4\sigma(x_{2i} - x_{1i}) - k_1\tilde{a}\ , \quad \dot{\hat{a}} = -\dot{\tilde{a}} \\ &\dot{\tilde{b}} = -6\sigma\, y_{1r} + 6\sigma\, x_{1r} - 4\sigma\, y_{1i} + 4\sigma\, x_{1i} - k_2\tilde{b}\ , \quad \dot{\hat{b}} = -\dot{\tilde{b}} \\ &\dot{\tilde{c}} = \sigma y_3 - \sigma x_3 - k_3\tilde{c}\ , \ \dot{\hat{c}} = -\dot{\tilde{c}} \quad k_i, k_i > 0, i = 1,\ldots,3\end{aligned} \tag{3.29}$$

Proof:

Since

$$\begin{aligned}\dot{V} &= \sigma\dot{\sigma} + \tilde{a}\dot{\tilde{a}} + \tilde{b}\dot{\tilde{b}} + \tilde{c}\dot{\tilde{c}} \\ &= \sigma\{\tilde{a}(y_{2r} - y_{1r}) - \tilde{a}(x_{2r} - x_{1r}) + e_{1i} + 4\tilde{a}(y_{2i} - y_{1i}) - 4\tilde{a}(x_{2i} - x_{1i}) + 4e_{2r} \\ &\quad + 6\tilde{b}y_{1r} - 6\tilde{b}x_{1r} + 6e_{2i} + 4\tilde{b}y_{1i} - 4\tilde{b}x_{1i} + 4e_3 - \tilde{c}y_3 + \tilde{c}x_3 + v_0 + v_s + \dot{z}\} \\ &\quad + \tilde{a}\dot{\tilde{a}} + \tilde{b}\dot{\tilde{b}} + \tilde{c}\dot{\tilde{c}} \\ &= \sigma\{e_{1i} + 4e_{2r} + 6e_{2i} + 4e_3 + v_0 + v_s + \dot{z}\} \\ &\quad + \tilde{a}\{\dot{\tilde{a}} + \sigma(y_{2r} - y_{1r}) + 4\sigma(y_{2i} - y_{1i}) - \sigma(x_{2r} - x_{1r}) - 4\sigma(x_{2i} - x_{1i})\} \\ &\quad + \tilde{b}\{\dot{\tilde{b}} + 6\sigma\, y_{1r} - 6\sigma\, x_{1r} + 4\sigma\, y_{1i} - 4\sigma\, x_{1i}\} + \tilde{c}\{\dot{\tilde{c}} - \sigma\, y_3 + \sigma\, x_3\}\end{aligned}$$

By using

$$\dot{z} = -e_{1i} - 4e_{2r} - 6e_{2i} - 4e_3 - v_0, \quad v_s = -k\sigma - k\, sign(\sigma)$$
$$\dot{\tilde{a}} = -\sigma(y_{2r} - y_{1r}) - 4\sigma(y_{2i} - y_{1i}) + \sigma(x_{2r} - x_{1r})$$
$$+ 4\sigma(x_{2i} - x_{1i}) - k_1\tilde{a} \ , \quad \dot{\hat{a}} = -\dot{\tilde{a}}$$
$$\dot{\tilde{b}} = -6\sigma\, y_{1r} + 6\sigma\, x_{1r} - 4\sigma\, y_{1i} + 4\sigma\, x_{1i} - k_2\tilde{b} \ , \quad \dot{\hat{b}} = \dot{\tilde{b}}$$
$$\dot{\tilde{c}} = \sigma y_3 - \sigma x_3 - k_3\tilde{c} \ , \ \dot{\hat{c}} = \dot{\tilde{c}} \quad k_i, k_i > 0, i = 1,\ldots,3$$

We have

$$\dot{V} = -k\sigma^2 - k|\sigma| - k_1\tilde{a}^2 - k_2\tilde{b}^2 - k_3\tilde{c}^2 .$$

From this we conclude that $\sigma, \tilde{a}, \tilde{b}, \tilde{c} \to 0$. Since $\sigma \to 0$, therefore $e = (e_{1r}, e_{1i}, e_{2r}, e_{2i}, e_3) \to 0$.

In simulations, the initial conditions are chosen as: $x(0) = (2 + 1j, 5 + 3j, 4)$ and $y(0) = (2.001 + 1j, 5 + 3j, 4.01)$. The true value of the unknown parameters are chosen as:: $a = 14$, $b = 35, c = 3.7$.

Simulation results:

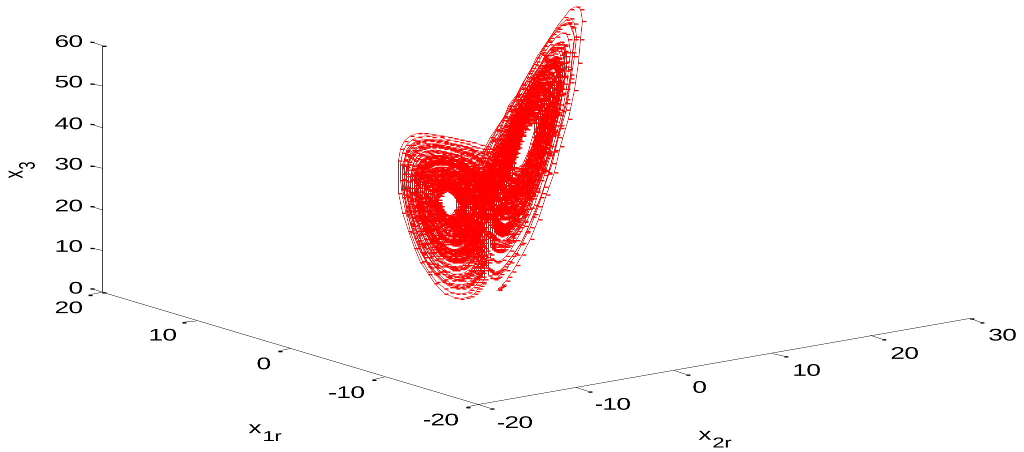

Figure 3.1 3D Phase portrait of complex Lorenz system

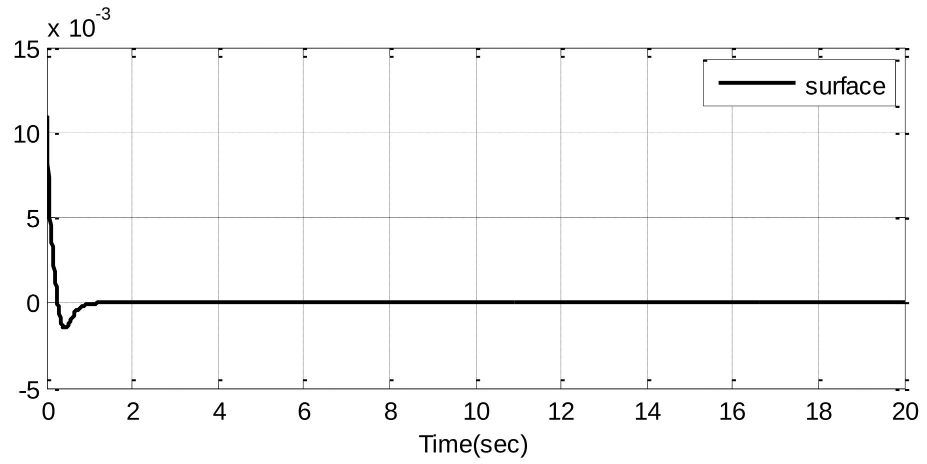

Figure 3.2: Time Response of surface

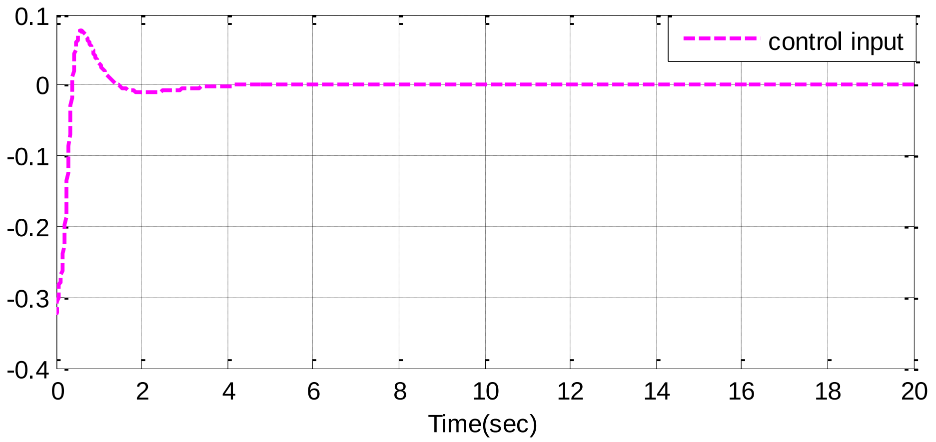

Figure 3.3: Time Response of control input

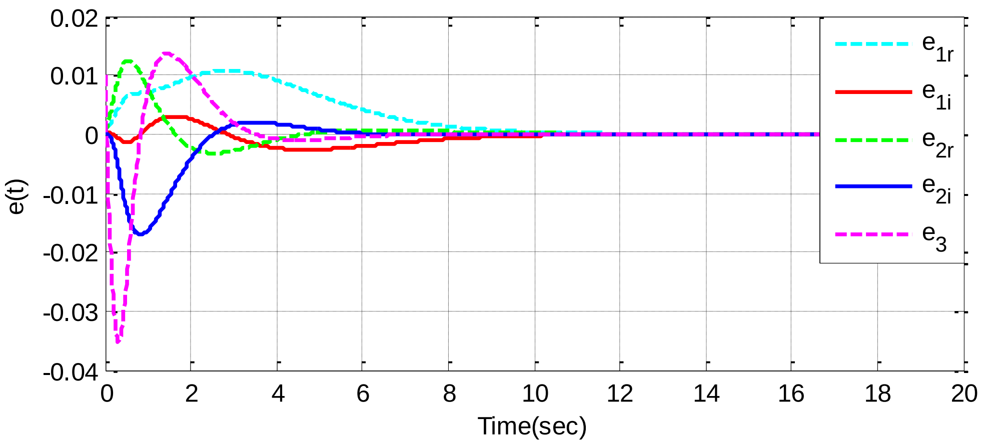

Figure 3.4: Time Response of error $e_{1r}, e_{1i}, e_{2r}, e_{2i} \& e_3$

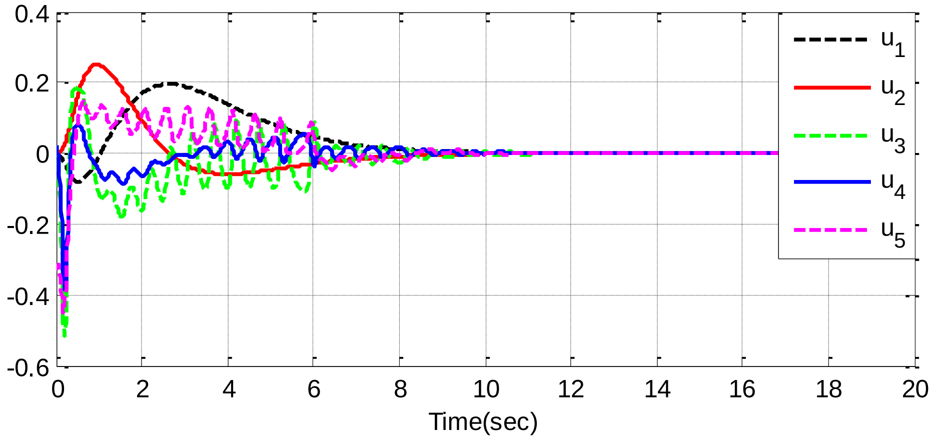

Figure 3.5: Time Response of adaptive controllers $u_1, u_2, u_3, u_4 \& u_5$

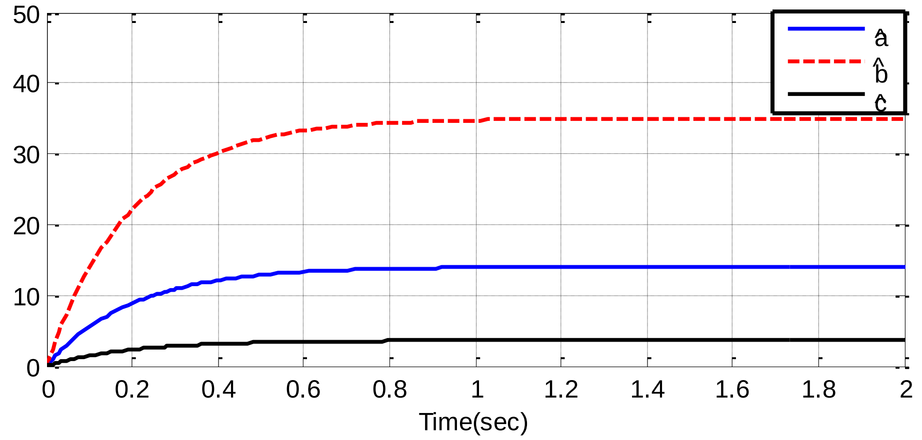

Figure 3.6: Estimation parameter of $\hat{a}, \hat{b} \& \hat{c}$

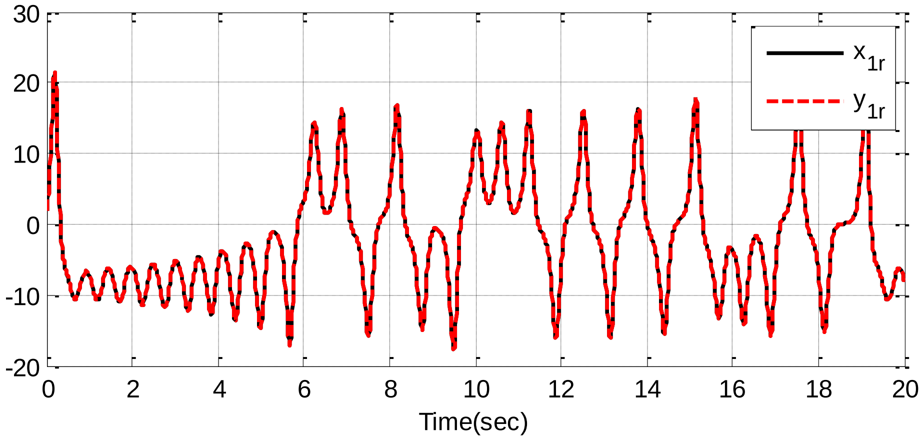

Figure 3.7: Time Response of $x_{1r} \& y_{1r}$ with IC (-2, 2.001)

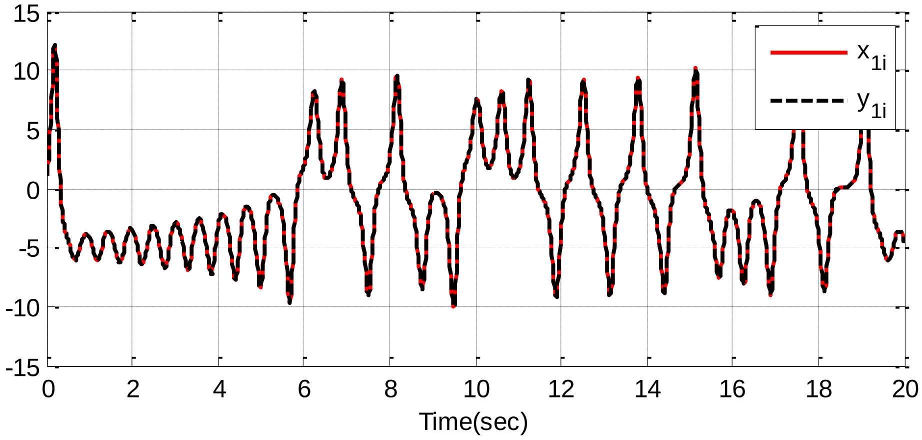

Figure 3.8: Time Response of x_{1i}&y_{1i} with IC (1, 1)

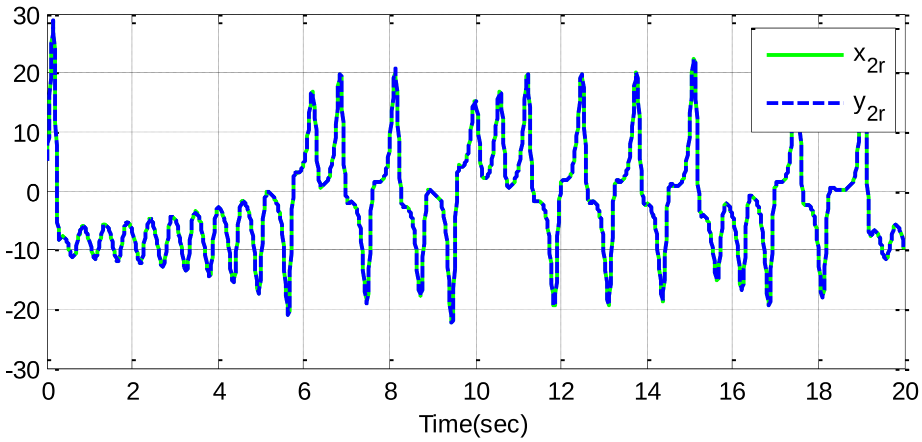

Figure 3.9: Time Response of x_{2r}&$y_{2r}$with IC (5,5)

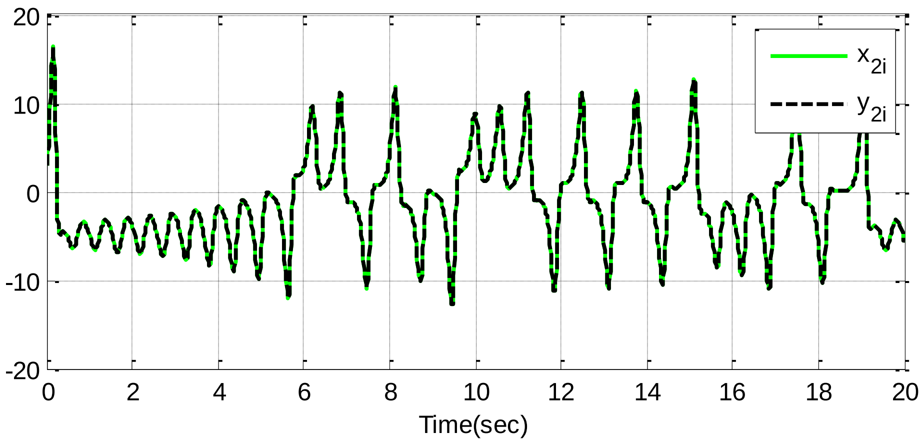

Figure 3.10: Time Response of x_{2r}&$y_{2r}$with IC (3,3)

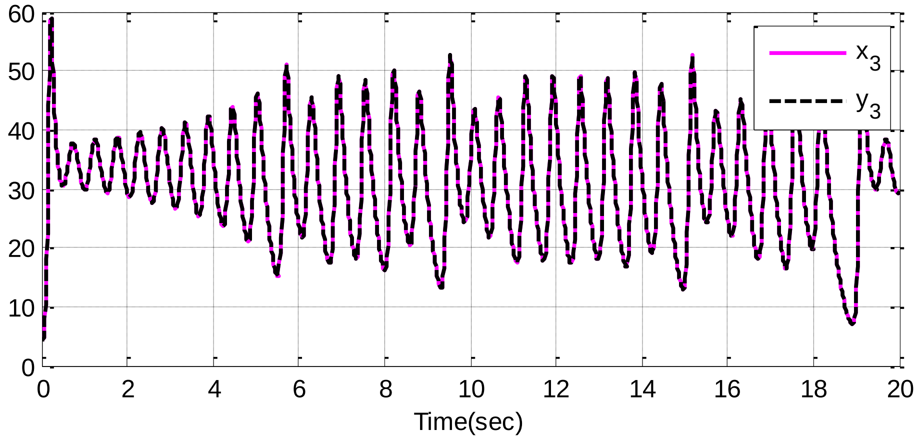

Figure 3.11: Time Response of x_3&y_3 with IC (4,4.001)

Chapter 4

Complex Projective Synchronization (CPS)

4.1 Introduction

In this chapter we present the extended version of control design strategy proposed in the previous chapter to achieve Projective Synchronization (PS) in complex chaotic systems with unknown parameters. The proposed design methodology is based on Adaptive Integral Sliding Mode Control. The proposed design approach is applied on different dimensional complex chaotic systems with unknown complex parameters.

4.2 Problem formulation

We consider the following general *m*-dimensional complex chaotic (hyperchaotic) drive system

$$\dot{x} = f(x) + F(x)\theta \qquad (4.1)$$

and *n*-dimensional complex chaotic (hyperchaotic) response system

$$\dot{y} = g(y) + G(y)\vartheta + u \qquad (4.2)$$

$x = (x_1, x_2, ..., x_m)^T \in R^m$ and $y = (y_1, y_2, ..., y_n)^T \in R^n$ are complex state vector, and $u = (u_r + ju_i) \in R^n$ is the control input.

Next, we introduce the definition of projective synchronization with complex function transformation matrix of complex chaotic (hyperchaotic) systems with complex parameters as follows.

Definition: For the drive system (4.1) and the response system (4.2), it is said to be CMHFPS with $D(t)$ between $y(t)$ and $x(t)$, if there exists a controller $u(x, y, t)$ such that :

$$\lim_{t \to +\infty} \|e(t)\| = \lim_{t \to +\infty} \|y(t) - D(t)x(t)\| = 0 \qquad (4.3).$$

Where $D(t) = D_r(t) + jD_i(t)$, the elements of $D(t)$ should be continuously differential functions with bounded.

4.3 General Proposed Algorithm for Complex Projective Synchronization

Consider the following non-identical drive and response complex system with fully unknown parameters

$$\dot{x} = f(x) + F(x)\theta \qquad (4.4)$$

$$\dot{y} = g(y) + G(y)\vartheta + u(x,y) \qquad (4.5)$$

Where $x = (x_1, x_2, ..., x_m)^T \in R^m$ and $y = (y_1, y_2, ..., y_n)^T \in R^n$ are complex state vectors of the drive system (4.4) and response system (4.5) respectively. $x_k = x_{kr} + jx_{ki}, k = 1,2,...,m$, $y_l = y_{lr} + jy_{li}, l = 1,2,...,n$, $j = \sqrt{-1}$, the subscripts r and i denote the real and imaginary parts of the complex variables, vectors and matrices throughout this paper. $\theta \in \Re^p$ and $\vartheta \in \Re^q$ are real vectors of unknown parameters. $F(x) \in C^{m \times p}$ and $G(y) \in C^{n \times q}$ are complex matrices, $F(x) = F_r(x) + jF_i(x)$, $G(x) = G_r(x) + jG_i(x)$. $f(x) \in C^m$ and $g(y) \in C^n$ are vectors of nonlinear complex functions, and $f(x) = f_r(x) + jf_i(x)$, $g(y) = g_r(y) + jg_i(y)$. $u(x,y) \in C^n$ is the complex control vector, and $u(x,y) = u_r(x,y) + ju_i(x,y)$.

For the drive system (4.4) and response system (4.5), CPS is achieved if there exist a complex controller $u(x,y)$ and a complex matrix $D(t) \in C^{m \times n}$ such that $\lim_{t \to \infty} \|e(t)\| = 0$ where $\| \ \|$ represent the matrix norm

Define the complex CMHFPS error vector as

$$\begin{aligned} e &= y - Dx \\ e_r + je_i &= (y_r + jy_i) - (D_r + jD_i)(x_r + jx_i) = (y_r + jy_i) - \{(D_r x_r + D_i x_i) \\ &\quad + j(D_r x_i + D_i x_r) = y_r - (D_r x_r + D_i x_i) + j\{y_i - (D_r x_i + D_i x_r)\} \end{aligned} \qquad (4.6)$$

Where $e = (e_1, e_2, ..., e_n)^T \in C^n$, $e_r = (e_{1r}, e_{2r}, ..., e_{nr})^T \in \Re^n$ $e_i = (e_{1i}, e_{2i}, ..., e_{ni})^T \in \Re^n$

By taking the derivative of equation (4.6) with respect time we obtain the error dynamic system as:

$$\dot{e} = \dot{y} - (\dot{D}\,x + D\,\dot{x}) = g(y) + G(y)\vartheta + u(x,y) - D\{f(x) + F(x)\theta\} - \dot{D}\,x$$

$$\begin{aligned}
\dot{e}_r + j\dot{e}_i &= g_r(y) + jg_i(y) + (G_r(y) + jG_i(y)(\vartheta_r + \vartheta_i) + u_r(x,y) \\
&\quad + ju_i(x,y) - [(D_r + D_i)\{f_r(x) + jf_r(x) + (F_r(x) + jF_i(x))(\theta_r + j\theta_i)\} \\
&\quad + (\dot{D}_r + j\dot{D}_i)(x_r + jx_i)] \\
&= g_r(y) + jg_i(y) + (G_r(y)\vartheta_r + G_i(y\vartheta_i) + j(G_i(y)\vartheta_r + G_r(y)\vartheta_i) \\
&\quad + u_r(x,y) + j + u_i(x,y) - [D_r\{f_r(x) + (F_r(x)\theta_r + F_i(x)\theta_i)\} \\
&\quad + D_i\{f_i(x) + (F_i(x)\theta_r + F_r(x)\theta_i)\} + jD_r\{f_i(x) + (F_r(x)\theta_i + F_i(x)\theta_r)\} \\
&\quad + jD_i\{f_r(x) + (F_r(x)\theta_r + F_i(x)\theta_i)\} + \dot{D}_r x_r + \dot{D}_i x_i + j\dot{D}_i x_r + j\dot{D}_r x_i] \\
&= g_r(y) + (G_r(y)\vartheta_r + G_i(y\vartheta_i) + u_r(x,y) - [D_r\{f_r(x) + (F_r(x)\theta_r \\
&\quad + F_i(x)\theta_i)\} + D_i\{f_i(x) + (F_i(x)\theta_r + F_r(x)\theta_i) + \dot{D}_r x_r + \dot{D}_i x_i\}] + j\{g_i(y) \\
&\quad + (G_i(y)\vartheta_r + G_r(y)\vartheta_i) + u_i(x,y) - [D_r\{f_i(x) + (F_r(x)\theta_i + F_i(x)\theta_r)\} \\
&\quad + D_i\{f_r(x) + (F_r(x)\theta_r + F_i(x)\theta_i)\} + \dot{D}_i x_r + \dot{D}_r x_i]\}
\end{aligned} \tag{4.7}$$

The complex error system (4.7) can be written in real form as:

$$\begin{aligned}
\dot{e}_r &= g_r(y) + (G_r(y)\vartheta_r + G_i(y\vartheta_i) + u_r(x,y) - [D_r\{f_r(x) + (F_r(x)\theta_r \\
&\qquad + F_i(x)\theta_i)\} + D_i\{f_i(x) + (F_i(x)\theta_r + F_r(x)\theta_i) + \dot{D}_r x_r + \dot{D}_i x_i\}] \\
\dot{e}_i &= \{g_i(y) + (G_i(y)\vartheta_r + G_r(y)\vartheta_i) + u_i(x,y) - [D_r\{f_i(x) + (F_r(x)\theta_i \\
&\qquad + F_i(x)\theta_r)\} + D_i\{f_r(x) + (F_r(x)\theta_r + F_i(x)\theta_i)\} + \dot{D}_i x_r + \dot{D}_r x_i]\}
\end{aligned} \tag{4.8}$$

Let $\hat{\theta}, \hat{\vartheta}$ be estimate of θ, ϑ respectively and let $\tilde{\theta} = \theta - \hat{\theta}, \tilde{\vartheta} = \vartheta - \hat{\vartheta}$ be error in estimating θ, ϑ respectively. Then error system (4.8) becomes:

$$\begin{aligned}
\dot{e}_r &= g_r(y) + (G_r(y)\hat{\vartheta}_r + G_r(y)\tilde{\vartheta}_r + G_i(y)\hat{\vartheta}_i + G_i(y)\tilde{\vartheta}_i + u_r(x,y) \\
&\qquad - [D_r\{f_r(x) + (F_r(x)\hat{\theta}_r + F_r(x)\tilde{\theta}_r + F_i(x)\hat{\theta}_i + F_i(x)\tilde{\theta}_i)\} + D_i\{f_i(x) \\
&\qquad + (F_i(x)\hat{\theta}_r + F_i(x)\tilde{\theta}_r + F_r(x)\hat{\theta}_i + F_r(x)\tilde{\theta}_i) + \dot{D}_r x_r + \dot{D}_i x_i\}] \\
\dot{e}_i &= \{g_i(y) + (G_i(y)\hat{\vartheta}_r + G_i(y)\tilde{\vartheta}_r + G_r(y)\vartheta_i + G_r(y)\tilde{\vartheta}_i) + u_i(x,y) \\
&\qquad - [D_r\{f_i(x) + (F_r(x)\hat{\theta}_i + F_r(x)\tilde{\theta}_i + F_i(x)\hat{\theta}_r + F_i(x)\tilde{\theta}_r)\} + D_i\{f_r(x) \\
&\qquad + (F_r(x)\hat{\theta}_r + F_r(x)\tilde{\theta}_r + F_i(x)\hat{\theta}_i + F_i(x)\tilde{\theta}_i)\} + \dot{D}_i x_r + \dot{D}_r x_i\}]
\end{aligned} \tag{4.9}$$

That can be written in vector form:

$$\begin{bmatrix} \dot{e}_r \\ \dot{e}_i \end{bmatrix} = \begin{bmatrix} g_r(y) + G_r(y)\hat{\vartheta}_r + G_i(y)\hat{\vartheta}_i - [D_r f_r(x) + D_i f_i(x) + \{D_i F_r(x) \\ + D_r F_i(x)\hat{\theta}_i\} + \{D_r F_r(x) + D_r F_r(x)\hat{\theta}_r\} + \dot{D}_r x_r + \dot{D}_i x_i] \\ g_i(y) + G_i(y)\hat{\vartheta}_r + G_r(y)\hat{\vartheta}_i - [D_r f_i(x) + D_i f_r(x) + \{D_i F_r(x) \\ + D_r F_i(x)\hat{\theta}_r\} + \{D_r F_r(x) + D_r F_r(x)\hat{\theta}_i\} + \dot{D}_r x_i + \dot{D}_i x_r] \end{bmatrix}$$

$$+ \begin{bmatrix} u_r(x,y) \\ u_i(x,y) \end{bmatrix} + \begin{bmatrix} G_r(y)\tilde{\vartheta}_r + G_i(y)\tilde{\vartheta}_i - [\{D_i F_r(x) + D_r F_i(x)\tilde{\theta}_i\} \\ + \{D_r F_r(x) + D_r F_r(x)\tilde{\theta}_r\}] \\ G_i(y)\tilde{\vartheta}_r + G_r(y)\tilde{\vartheta}_i - [\{D_r F_i(x) + D_i F_{ri}(x)\tilde{\theta}_r\} \\ + \{D_r F_r(x) + D_r F_r(x)\tilde{\theta}_i\}] \end{bmatrix} \quad (4.10)$$

By choosing

$$\begin{bmatrix} u_r(x,y) \\ u_i(x,y) \end{bmatrix} = \begin{bmatrix} ee_r \\ ee_i \end{bmatrix} + \begin{bmatrix} g_r(y) + G_r(y)\hat{\vartheta}_r + G_i(y)\hat{\vartheta}_i - [D_r f_r(x) + D_i f_i(x) \\ + \{D_i F_r(x) + D_r F_i(x)\hat{\theta}_i\} + \{D_r F_r(x) + D_r F_r(x)\hat{\theta}_r\} \\ + \dot{D}_r x_r + \dot{D}_i x_i] \\ g_i(y) + G_i(y)\hat{\vartheta}_r + G_r(y)\hat{\vartheta}_i - [D_r f_i(x) + D_i f_r(x) \\ + \{D_i F_r(x) + D_r F_i(x)\hat{\theta}_r\} + \{D_r F_r(x) + D_r F_r(x)\hat{\theta}_i\} \\ + \dot{D}_r x_i + \dot{D}_i x_r] \end{bmatrix} \quad (4.11)$$

Where $ee_r = \begin{bmatrix} e_{2r} \\ e_{3r} \\ \vdots \\ e_{nr} \\ e_{1i} \end{bmatrix}, ee_i = \begin{bmatrix} e_{2i} \\ e_{3i} \\ \vdots \\ e_{ni} \\ v \end{bmatrix}$

v is the new input vector, then system (4.10) becomes:

$$\begin{bmatrix} \dot{e}_r \\ \dot{e}_i \end{bmatrix} = \begin{bmatrix} ee_r \\ ee_i \end{bmatrix} + \begin{bmatrix} G_r(y)\tilde{\vartheta}_r + G_i(y)\tilde{\vartheta}_i - [\{D_i F_r(x) + D_r F_i(x)\tilde{\theta}_i\} + \{D_r F_r(x) + D_r F_r(x)\tilde{\theta}_r\}] \\ G_i(y)\tilde{\vartheta}_r + G_r(y)\tilde{\vartheta}_i - [\{D_r F_i(x) + D_i F_{ri}(x)\tilde{\theta}_r\} + \{D_r F_r(x) + D_r F_r(x)\tilde{\theta}_i\}] \end{bmatrix}$$

Or

$$\begin{bmatrix} \dot{e}_{1r} \\ \dot{e}_{2r} \\ \vdots \\ \dot{e}_{nr} \\ \dot{e}_{1i} \\ \dot{e}_{2i} \\ \vdots \\ \dot{e}_{ni} \end{bmatrix} = \begin{bmatrix} e_{2r} \\ \vdots \\ e_{nr} \\ e_{1i} \\ e_{2i} \\ \vdots \\ e_{ni} \\ v \end{bmatrix} + \begin{bmatrix} G_r(y)\tilde{\vartheta}_r + G_i(y)\tilde{\vartheta}_i - [\{D_i F_r(x) + D_r F_i(x)\tilde{\theta}_i\} \\ + \{D_r F_r(x) + D_r F_r(x)\tilde{\theta}_r\}] \\ G_i(y)\tilde{\vartheta}_r + G_r(y)\tilde{\vartheta}_i - [\{D_r F_i(x) + D_i F_{ri}(x)\tilde{\theta}_r\} \\ + \{D_r F_r(x) + D_r F_r(x)\tilde{\theta}_i\}] \end{bmatrix} \quad (4.12)$$

To employ the integral sliding mode control, choose the nominal system for (4.12) as:

$$\begin{bmatrix} \dot{e}_{1r} \\ \dot{e}_{2r} \\ \vdots \\ \dot{e}_{nr} \\ \dot{e}_{1i} \\ \dot{e}_{2i} \\ \vdots \\ \dot{e}_{ni} \end{bmatrix} = \begin{bmatrix} e_{2r} \\ \vdots \\ e_{nr} \\ e_{1i} \\ e_{2i} \\ \vdots \\ e_{ni} \\ v_0 \end{bmatrix} \quad (4.13)$$

Define the sliding surface for nominal system (4.13) as:

$$\sigma_0 = C[e_r + e_i]^T = e_{1r} + \sum_{k=2}^{n-1} c_k e_{kr} + \sum_{k=1}^{n-1} c_{(n+k)} e_{ki}$$

$C = [1, c_1, \ldots, c_{n-1}, c_n, \ldots, c_{2n-1}, 1]$ is chosen in such a way that σ_0 becomes Hurwitz polynomial.

$$\dot{\sigma}_0 = C[\dot{e}_r + \dot{e}_r]^T$$

$$\dot{\sigma}_0 = e_{2r} + \sum_{k=2}^{n-1} c_k e_{(k+1)r} + \sum_{k=0}^{n-1} c_{(n+k)} e_{(k+1)i} + v_o$$

By choosing $v_o = -e_{2r} - \sum_{k=2}^{n-1} c_k e_{(k+1)r} - \sum_{k=0}^{n-1} c_{(n+k)} e_{(k+1)i} - k\,\sigma_0 - k\,sign(\sigma_0),\ k > 0$, we have $\dot{\sigma}_0 = -k\,\sigma_0$. Therefore the nominal system (4.13) is asymptotically stable.

Now choose the sliding surface for the system (4.12) as:

$$\sigma = \sigma_0 + z = Ce + z$$
$$\sigma = C_1 e_r + C_2 e_i + z$$

Where z is some integral term computed later. To avoid the reaching phase, choose $z(0)$ such that $\sigma(0) = 0$. Choose $v = v_0 + v_s$ where, v_0 is the nominal input and v_s is compensator term computed later. Then

$$\dot{\sigma} = C_1\dot{e}_r + C_2\dot{e}_i + \dot{z}$$

$$\begin{aligned}
\dot{\sigma} &= e_{2r} + \sum_{k=2}^{n-1} c_k e_{(k+1)r} + \sum_{k=0}^{n-1} c_{(n+k)} e_{(k+1)i} + v_o + v_s + \dot{z} \\
&\quad + C_1 G_r(y)\tilde{\vartheta} + C_2 G_i(y)\tilde{\vartheta} - C_1 D(t) F_r(x)\tilde{\theta} - C_1 D(t) F_i(x)\tilde{\theta} \\
\dot{\sigma} &= e_{2r} + \sum_{k=2}^{n-1} c_k e_{(k+1)r} + \sum_{k=0}^{n-1} c_{(n+k)} e_{(k+1)i} + v_o + v_s + \dot{z} \\
&\quad + C_1 G_r(y)\tilde{\vartheta}_r + C_1 G_i(y)\tilde{\vartheta}_i + C_2 G_i(y)\tilde{\vartheta}_r + C_2 G_r(y)\tilde{\vartheta}_i \\
&\quad - [C_1(D_r F_r(x) + D_i F_i(x))\tilde{\theta}_r + C_1(D_r F_i(x) + D_i F_r(x))\tilde{\theta}_i \\
&\quad + C_2(D_r F_i(x) + D_i F_r(x))\tilde{\theta}_r + C_2(D_i F_i(x) + D_r F_r(x))\tilde{\theta}_i
\end{aligned} \tag{4.14}$$

$$C_1 = [1, c_1, \ldots, c_{n-1}], C_2 = [c_n, c_{n+1}, \ldots, c_{2n-1}, 1]$$

By choosing a Lyapunov function: $V = \frac{1}{2}\sigma^2 + \frac{1}{2}\tilde{\theta}_r^T\tilde{\theta}_r + \frac{1}{2}\tilde{\theta}_i^T\tilde{\theta}_i + \frac{1}{2}\tilde{\vartheta}_r^T\tilde{\vartheta}_r + \frac{1}{2}\tilde{\vartheta}_i^T\tilde{\vartheta}_i$,

design the adaptive laws for $\tilde{\theta}, \hat{\theta}, \tilde{\vartheta}, \hat{\vartheta}$ and compute v_s such that $\dot{V} < 0$.

$$\begin{aligned}
&\dot{z} = -e_{2r} - \sum_{k=2}^{n-1} c_k e_{(k+1)r} - \sum_{k=0}^{n-1} c_{(n+k)} e_{(k+1)i} + v_o, \quad v_s = -k\sigma - k\,sign(\sigma) \\
&\dot{\tilde{\theta}}_r = \sigma(D_r F_r(x) + D_i F_i(x))^T C_1^T + (D_r F_i(x) + D_i F_r(x))^T C_2^T - k_1\tilde{\theta}_r \\
&\dot{\tilde{\theta}}_i = \sigma(D_r F_i(x) + D_i F_r(x))^T C_1^T + (D_r F_r(x) + D_i F_i(x))^T C_2^T - k_2\tilde{\theta}_i \\
&\dot{\tilde{\vartheta}}_r = -\sigma G_r(y)^T C_1^T - \sigma G_i(y)^T C_2^T - k_3\tilde{\vartheta}_r \\
&\dot{\tilde{\vartheta}}_i = -\sigma G_i(y)^T C_1^T - \sigma G_r(y)^T C_2^T - k_4\tilde{\vartheta}_i, \text{where}, k, k_1, k_2, k_3, k_4 > 0
\end{aligned} \tag{4.15}$$

Proof:

Since

$$\begin{aligned}
\dot{V} &= \sigma\dot{\sigma} + \tilde{\theta}_r^T\dot{\tilde{\theta}}_r + \tilde{\theta}_i^T\dot{\tilde{\theta}}_i + \tilde{\vartheta}_r^T\dot{\tilde{\vartheta}}_r + \tilde{\vartheta}_i^T\dot{\tilde{\vartheta}}_i \\
&= \sigma\{e_{2r} + \sum_{k=2}^{n-1} c_k e_{(k+1)r} + \sum_{k=0}^{n-1} c_{(n+k)} e_{(k+1)i} + v_o + v_s + \dot{z} \\
&\quad + C_1 G_r(y)\tilde{\vartheta} + C_2 G_i(y)\tilde{\vartheta} - C_1 D(t) F_r(x)\tilde{\theta} - C_1 D(t) F_i(x)\tilde{\theta}\} \\
&\quad + \tilde{\theta}_r^T\dot{\tilde{\theta}}_r + \tilde{\theta}_i^T\dot{\tilde{\theta}}_i + \tilde{\vartheta}_r^T\dot{\tilde{\vartheta}}_r + \tilde{\vartheta}_i^T\dot{\tilde{\vartheta}}_i \\
&= \sigma\{e_{2r} + \sum_{k=2}^{n-1} c_k e_{(k+1)r} + \sum_{k=0}^{n-1} c_{(n+k)} e_{(k+1)i} + v_o + v_s + \dot{z}\} \\
&\quad + \tilde{\theta}_r^T\{\dot{\tilde{\theta}}_r - \sigma(D_r F_r(x) + D_i F_i(x))^T C_1^T - (D_r F_i(x) + D_i F_r(x))^T C_2^T\} \\
&\quad + \tilde{\theta}_i^T\{\dot{\tilde{\theta}}_i - \sigma(D_r F_i(x) + D_i F_r(x))^T C_1^T - (D_r F_r(x) + D_i F_i(x))^T C_2^T\} \\
&\quad + \tilde{\vartheta}_r^T\{\dot{\tilde{\vartheta}}_r + \sigma G_r(y)^T C_1^T + \sigma G_i(y)^T C_2^T\} + \tilde{\vartheta}_i^T\{\dot{\tilde{\vartheta}}_i + \sigma G_i(y)^T C_1^T + \sigma G_r(y)^T C_2^T\}
\end{aligned}$$

By using

$$\dot{z} = -e_{2r} - \sum_{k=2}^{n-1} c_k e_{(k+1)r} - \sum_{k=0}^{n-1} c_{(n+k)} e_{(k+1)i} + v_o, \quad v_s = -k\sigma - k\, sign(\sigma)$$
$$\dot{\tilde{\theta}}_r = \sigma(D_r F_r(x) + D_i F_i(x))^T C_1^T + (D_r F_i(x) + D_i F_r(x))^T C_2^T - k_1 \tilde{\theta}_r$$
$$\dot{\tilde{\theta}}_i = \sigma(D_r F_i(x) + D_i F_r(x))^T C_1^T + (D_r F_r(x) + D_i F_i(x))^T C_2^T - k_2 \tilde{\theta}_i$$
$$\dot{\tilde{\vartheta}}_r = -\sigma G_r(y)^T C_1^T - \sigma G_i(y)^T C_2^T - k_3 \tilde{\vartheta}_r$$
$$\dot{\tilde{\vartheta}}_i = -\sigma G_i(y)^T C_1^T - \sigma G_r(y)^T C_2^T - k_4 \tilde{\vartheta}_i, \text{where}, k, k_1, k_2, k_3, k_4 > 0$$

We have

$$\dot{V} = -k\sigma^2 - k|\sigma| - k_1 \tilde{\theta}_r^T \tilde{\theta}_r - k_2 \tilde{\theta}_i^T \tilde{\theta}_i - k_3 \tilde{\vartheta}_r^T \tilde{\vartheta}_r - k_4 \tilde{\vartheta}_i^T \tilde{\vartheta}_i.$$

From this we conclude that $\sigma, \tilde{\theta}_r, \tilde{\theta}_i, \tilde{\vartheta}_r, \tilde{\vartheta}_i \to 0$. Since $\sigma \to 0$, therefore $e_r, e_i \to 0$.

4.4 Numerical Example

The following example is taken from [39], where CPS problem was solved by adaptive control scheme. We have achieved CCS using adaptive integral sliding mode control.

Case 1: m<n

Consider the Master system given in [39] as:

$$\begin{aligned} \dot{x}_1 &= a_1(x_2 - x_1) \\ \dot{x}_2 &= a_2 x_1 - a_3 x_2 - x_1 x_3 \\ \dot{x}_3 &= 0.5(\bar{x}_1 x_2 + x_1 \bar{x}_2) - a_4 x_3 \end{aligned} \qquad (4.16)$$

Where, $x_1 = x_{1r} + jx_{1i}$, $x_2 = x_{2r} + jx_{2i}$ are complex and $x_3 = x_{3r}$ are real. $\bar{x}_1, \bar{x}_2$ denote the complex conjugate variables of x_1, x_2. a_1, a_2, a_3 & a_4 are unknown real parameters .When $a_4 = 2,\ a_2 = 60 + 0.02j$, $a_3 = 1 - 0.06j$ & $a_4 = 0.8$ and $x(0) = [2 + 0.02j,\ 1 + 0.2j, -1]^T$ the hyperchaotic attractor is plotted in Fig 4.1.

The Slave system given in [39] as:

$$
\begin{aligned}
\dot{y}_1 &= b_1(y_2 - y_1) + u_1 \\
\dot{y}_2 &= b_2 y_1 - y_2 - y_1 y_3 + y_4 + u_2 \\
\dot{y}_3 &= 0.5(\bar{y}_1 y_2 + y_1 \bar{y}_2) - b_3 y_3 + u_3 \\
\dot{y}_4 &= b_4 y_1 + b_5 y_2 + u_4
\end{aligned} \tag{4.17}
$$

Where, $y_1 = y_{1r} + jy_{1i}$, $y_2 = y_{2r} + jy_{2i}$, $y_4 = y_{4r} + jy_{4i}$ are complex and $y_3 = y_{3r}$ is real. $\bar{y}_1, \bar{y}_2$ denote the complex conjugate variables of y_1, y_2. b_1, b_2, b_3, b_4 and b_5 are known real parameters. u_1, u_2, u_3 and u_4 are controllers. When $b_1 = 14, b_2 = 35, b_3 = 3, b_4 = -5, b_5 = -4$ $y(0) = [-1-2j, -3-4j, -5, -6-7j]^T$, the hyperchaotic attractor is plotted in Fig4.2.

We investigate CPS of two non-identical complex systems with the different orders.

Let $\hat{a}_i, \hat{b}_f, i = 1,..,4, f = 1,...,5$ be estimates of $a_i, b_f, i = 1,..,4, f = 1,...,5$ and $\tilde{a}_i = a_i - \hat{a}_i, \tilde{b}_f = b_f - \hat{b}_f, i = 1,....,4, f = 1,...,5$, be the errors in estimations of $a_i, b_f, i = 1,..,4, f = 1,...,5$, respectively. Then systems (4.16) and (4.17) can be written as:

$$
\begin{aligned}
\dot{x}_1 &= \hat{a}_1(x_2 - x_1) + \tilde{a}_1(x_2 - x_1) \\
\dot{x}_2 &= \hat{a}_2 x_1 + \tilde{a}_2 x_1 - \hat{a}_3 x_2 - \tilde{a}_3 x_2 - x_1 x_3 \\
\dot{x}_3 &= 0.5(\bar{x}_1 x_2 + x_1 \bar{x}_2) - \hat{a}_4 x_3 - \tilde{a}_4 x_3
\end{aligned} \tag{4.18}
$$

$$
\begin{aligned}
\dot{y}_1 &= \hat{b}_1(y_2 - y_1) + \tilde{b}_1(y_2 - y_1) + u_1 \\
\dot{y}_2 &= \hat{b}_2 y_1 + \tilde{b}_2 y_1 - y_2 - y_1 y_3 + y_4 + u_2 \\
\dot{y}_3 &= 0.5(\bar{y}_1 y_2 + y_1 \bar{y}_2) - \hat{b}_3 y_3 - \tilde{b}_3 y_3 + u_3 \\
\dot{y}_4 &= \hat{b}_4 y_1 + \tilde{b}_4 y_1 + \hat{b}_5 y_2 + \tilde{b}_5 y_2 + u_4
\end{aligned} \tag{4.19}
$$

The 3-D complex systems (4.18) can be into 5-dimensional real system:

$$
\begin{aligned}
\dot{x}_{1r} &= \hat{a}_1(x_{2r} - x_{1r}) + \tilde{a}_1(x_{2r} - x_{1r}) \\
\dot{x}_{1i} &= \hat{a}_1(x_{2i} - x_{1i}) + \tilde{a}_1(x_{2i} - x_{1i}) \\
\dot{x}_{2r} &= \hat{a}_{2r} x_{1r} + \tilde{a}_{2r} x_{1r} - \hat{a}_{3r} x_{2r} - \tilde{a}_{3r} x_{2r} - x_{1r} x_3 \\
\dot{x}_{2i} &= \hat{a}_{2i} x_{1i} + \tilde{a}_{2i} x_{1i} - \hat{a}_{3i} x_{2i} - \tilde{a}_{3i} x_{2i} - x_{1i} x_3 \\
\dot{x}_3 &= (x_{1r} x_{2r} + x_{1i} x_{2i}) - \hat{a}_4 x_3 - \tilde{a}_4 x_3
\end{aligned} \tag{4.20}
$$

The 4-dimensional complex systems (4.19) can be into 7-dimensional real systems:

$$
\begin{aligned}
\dot{y}_{1r} &= \hat{b}_1(y_{2r} - y_{1r}) + \tilde{b}_1(y_{2r} - y_{1r}) + u_{1r} \\
\dot{y}_{1i} &= \hat{b}_1(y_{2i} - y_{1i}) + \tilde{b}_1(y_{2i} - y_{1i}) + u_{1i} \\
\dot{y}_{2r} &= \hat{b}_2 y_{1r} + \tilde{b}_2 y_{1r} - y_{2r} - y_{1r} y_3 + y_{4r} + u_{2r} \\
\dot{y}_{2i} &= \hat{b}_2 y_{1i} + \tilde{b}_2 y_{1i} - y_{2i} - y_{1i} y_3 + y_{4i} + u_{2i} \\
\dot{y}_3 &= y_{1r}\, y_{2r} + y_{1i} y_{2i} - \hat{b}_3 y_3 - \tilde{b}_3 y_3 + y_4 + u_3 \\
\dot{y}_{4r} &= \hat{b}_4 y_{1r} + \tilde{b}_4 y_{1r} + \hat{b}_5 y_{2r} + \tilde{b}_5 y_{2r} + u_{4r} \\
\dot{y}_{4i} &= \hat{b}_4 y_{1i} + \tilde{b}_4 y_{1i} + \hat{b}_5 y_{2i} + \tilde{b}_5 y_{2i} + u_{4i}
\end{aligned}
\tag{4.21}
$$

Where $u_{1r}, u_{1i}, u_{2r}, u_{2i}, u_3 \;\; u_{4r}$ and u_{4i} in Equation (4.21) are the control functions to be determined.

The complex function transformation matrix is taken as:

$$
D = \begin{pmatrix} 0.5\exp(j\pi t/5) & 0 & 0 \\ 0 & \exp(j\pi t/10) & 0 \\ 0 & 0 & 1.5\cos(j\pi t/15) \\ 2\exp(j\pi t/20) & 0 & 0 \end{pmatrix} \tag{4.22}
$$

Where $\beta \exp(j\theta t) = \beta(\cos\theta t + j\sin\theta t)$.

The error signals are defined as:

$$
\begin{aligned}
& e_{1r} = y_{1r} - (D_{1r} x_{1r} + D_{1i} x_{1i}), e_{1i} = y_{1i} - (D_{1r} x_{1i} + D_{1i} x_{1r}) \\
& e_{2r} = y_{2r} - (D_{2r} x_{2r} + D_{2i} x_{2i}), e_{2i} = y_{2i} - (D_{2r} x_{2i} + D_{2i} x_{2r}) \\
& e_3 = y_3 - (D_3 x_3 + D_3 x_3), e_{4r} = y_{4r} - (D_{4r} x_{4r} + D_{4i} x_{4i}), \\
& e_{4i} = y_{4i} - (D_{4r} x_{4i} + D_{4i} x_{4r})
\end{aligned}
\tag{4.23}
$$

Then the dynamics of the error system becomes:

$$
\begin{aligned}
\dot{e}_{1r} &= \dot{y}_{1r} - (\dot{D}_{1r} x_{1r} + D_{1r} \dot{x}_{1r} + \dot{D}_{1i} x_{1i} + D_{1i} \dot{x}_{1i}), \\
\dot{e}_{1i} &= \dot{y}_{1i} - (\dot{D}_{1r} x_{1i} + D_{1r} \dot{x}_{1i} + \dot{D}_{1i} x_{1r} + D_{1i} \dot{x}_{1r}), \\
\dot{e}_{2r} &= \dot{y}_{2r} - (\dot{D}_{2r} x_{2r} + D_{2r} \dot{x}_{2r} + \dot{D}_{2i} x_{2i} + D_{2i} \dot{x}_{2i}), \\
\dot{e}_{2i} &= \dot{y}_{2i} - (\dot{D}_{2r} x_{2i} + D_{2r} \dot{x}_{2i} + \dot{D}_{2i} x_{2r} + D_{2i} \dot{x}_{2r}), \\
\dot{e}_3 &= \dot{y}_3 - (\dot{D}_3 x_3 + D_3 \dot{x}_3), \\
\dot{e}_{4r} &= \dot{y}_{4r} - (\dot{D}_{4r} x_{4r} + D_{4r} \dot{x}_{4r} + \dot{D}_{4i} x_{4i} + D_{4i} \dot{x}_{4i}), \\
\dot{e}_{4i} &= \dot{y}_{4i} - (\dot{D}_{4r} x_{4i} + D_{4r} \dot{x}_{4i} + \dot{D}_{4i} x_{4r} + D_{4i} \dot{x}_{4r})
\end{aligned}
\tag{4.24}
$$

Then

$$
\begin{aligned}
&\dot{e}_{1r} = \dot{y}_{1r} - (\dot{D}_{1r}x_{1r} + D_{1r}\dot{x}_{1r} + \dot{D}_{1i}x_{1i} + D_{1i}\dot{x}_{1i}) = \hat{b}_1(y_{2r} - y_{1r}) \\
&+ \tilde{b}_1(y_{2r} - y_{1r}) + u_{1r} - (D_{1r}(\hat{a}_1(x_{2r} - x_{1r}) + \tilde{a}_1(x_{2r} - x_{1r})) \\
&+ D_{1i}(\hat{a}_1(x_{2i} - x_{1i}) + \tilde{a}_1(x_{2i} - x_{1i})) + \dot{D}_{1r}x_{1r} + \dot{D}_{1i}x_{1i}), \\
&\dot{e}_{1i} = \dot{y}_{1i} - (\dot{D}_{1r}x_{1i} + D_{1r}\dot{x}_{1i} + \dot{D}_{1i}x_{1r} + D_{1i}\dot{x}_{1r}) = \hat{b}_1(y_{2i} - y_{1i}) \\
&+ \tilde{b}_1(y_{2i} - y_{1i}) + u_{1i} - (D_{1r}(\hat{a}_1(x_{2i} - x_{1i}) + \tilde{a}_1(x_{2i} - x_{1i})) \\
&+ D_{1i}(\hat{a}_1(x_{2r} - x_{1r}) + \tilde{a}_1(x_{2r} - x_{1r})) + \dot{D}_{1r}x_{1i} + \dot{D}_{1i}x_{1r}), \\
&\dot{e}_{2r} = \dot{y}_{2r} - (\dot{D}_{2r}x_{2r} + D_{2r}\dot{x}_{2r} + \dot{D}_{2i}x_{2i} + D_{2i}\dot{x}_{2i}) = \\
&(\hat{b}_2 y_{1r} + \tilde{b}_2 y_{1r} - y_{2r} - y_{1r}y_3 + y_{4r}) + u_{2r} - (D_{2r}(\hat{a}_{2r}x_{1r} + \tilde{a}_{2r}x_{1r} - \hat{a}_{3r}x_{2r} \\
&- \tilde{a}_{3r}x_{2r} - x_{1r}x_3) + D_{2i}(\hat{a}_{2i}x_{1i} + \tilde{a}_{2i}x_{1i} - \hat{a}_{3i}x_{2i} - \tilde{a}_{3i}x_{2i} - x_{1i}x_3) + \dot{D}_{2r}x_{2r} + \dot{D}_{2i}x_{2i}), \\
&\dot{e}_{2i} = \dot{y}_{2i} - (\dot{D}_{2r}x_{2i} + D_{2r}\dot{x}_{2i} + \dot{D}_{2i}x_{2r} + D_{2i}\dot{x}_{2r}) = \\
&(\hat{b}_2 y_{1i} + \tilde{b}_2 y_{1i} - y_{2i} - y_{1i}y_3 + y_{4i}) + u_{2i} - (D_{2r}(\hat{a}_{2i}x_{1i} + \tilde{a}_{2i}x_{1i} \\
&- \hat{a}_{3i}x_{2i} - \tilde{a}_{3i}x_{2i} - x_{1i}x_3) + D_{2i}(\hat{a}_{2r}x_{1r} + \tilde{a}_{2r}x_{1r} - \hat{a}_{3r}x_{2r} - \tilde{a}_{3r}x_{2r} \\
&- x_{1r}x_3) + \dot{D}_{2r}x_{2i} + \dot{D}_{2i}x_{2r}), \\
&\dot{e}_3 = \dot{y}_3 - (\dot{D}_3 x_3 + D_3\dot{x}_3) = (y_{1r}\, y_{2r} + y_{1i}y_{2i} - \hat{b}_3 y_3 - \tilde{b}_3 y_3 + y_4) \\
&+ u_3 - (D_3((x_{1r}\, x_{2r} + x_{1i}x_{2i}) - \hat{a}_4 x_3 - \tilde{a}_4 x_3) + \dot{D}_3 x_3), \\
&\dot{e}_{4r} = \dot{y}_{4r} - (\dot{D}_{4r}x_{1r} + D_{4r}\dot{x}_{1r} + \dot{D}_{4i}x_{1i} + D_{4i}\dot{x}_{1i}) = (\hat{b}_4 y_{1r} + \tilde{b}_4 y_{1r} + \hat{b}_5 y_{2r} \\
&+ \tilde{b}_5 y_{2r}) + u_{4r} - (D_{4r}(\hat{a}_1(x_{2r} - x_{1r}) + \tilde{a}_1(x_{2r} - x_{1r})) + D_{4i}(\hat{a}_1(x_{2i} - x_{1i}) \\
&+ \tilde{a}_1(x_{2i} - x_{1i})) + \dot{D}_{4r}x_{1r} + D_{4i}\dot{x}_{1i}), \\
&\dot{e}_{4i} = \dot{y}_{4i} - (\dot{D}_{4r}x_{1i} + D_{4r}\dot{x}_{1i} + \dot{D}_{4i}x_{1r} + D_{4i}\dot{x}_{1r}) = (\hat{b}_4 y_{1i} + \tilde{b}_4 y_{1i} + \hat{b}_5 y_{2i} \\
&+ \tilde{b}_5 y_{2i}) + u_{4i} - (D_{4r}(\hat{a}_1(x_{2i} - x_{1i}) + \tilde{a}_1(x_{2i} - x_{1i})) + D_{4i}(\hat{a}_1(x_{2r} - x_{1r}) \\
&+ \tilde{a}_1(x_{2r} - x_{1r})) + \dot{D}_{4r}x_{1i} + \dot{D}_{4i}x_{1r})
\end{aligned}
\tag{4.25}
$$

By choosing

$$
\begin{aligned}
u_{1r} &= -\hat{b}_1(y_{2r} - y_{1r}) + (D_{1r}\hat{a}_1(x_{2r} - x_{1r}) + D_{1i}\hat{a}_1(x_{2i} - x_{1i}) \\
&+ \dot{D}_{1r}x_{1r} + \dot{D}_{1i}x_{1i}) + e_{1i}, \\
u_{1i} &= -\hat{b}_1(y_{2i} - y_{1i}) + (D_{1r}\hat{a}_1(x_{2i} - x_{1i}) + D_{1i}\hat{a}_1(x_{2r} - x_{1r}) \\
&+ \dot{D}_{1r}x_{1i} + \dot{D}_{1i}x_{1r}) + e_{2r}, \\
u_{2r} &= -(\hat{b}_2 y_{1r} - y_{2r} - y_{1r}y_3 + y_{4r}) + (D_{2r}(\hat{a}_{2r}x_{1r} - \hat{a}_{3r}x_{2r} \\
&- x_{1r}x_3) + D_{2i}(\hat{a}_{2i}x_{1i} - \hat{a}_{3i}x_{2i} - x_{1i}x_3) + \dot{D}_{2r}x_{2r} + \dot{D}_{2i}x_{2i}) + e_{21}, \\
u_{2i} &= -(\hat{b}_2 y_{1i} - y_{2i} - y_{1i}y_3 + y_{4i}) + (D_{2r}(\hat{a}_{2i}x_{1i} - \hat{a}_{3i}x_{2i} - x_{1i}x_3) \\
&+ D_{2i}(\hat{a}_{2r}x_{1r} - \hat{a}_{3r}x_{2r} - x_{1r}x_3) + \dot{D}_{2r}x_{2i} + \dot{D}_{2i}x_{2r}) + e_3, \\
u_3 &= (y_{1r}\, y_{2r} + y_{1i}y_{2i} - \hat{b}_3 y_3 + y_4) + (D_3((x_{1r}\, x_{2r} + x_{1i}x_{2i}) \\
&- \hat{a}_4 x_3) + \dot{D}_3 x_3) + e_{4r}, \\
u_{4r} &= -(\hat{b}_4 y_{1r} + \hat{b}_5 y_{2r}) + (D_{4r}\hat{a}_1(x_{2r} - x_{1r}) + D_{4i}\hat{a}_1(x_{2i} - x_{1i}) \\
&+ \dot{D}_{4r}x_{1r} + D_{4i}\dot{x}_{1i}) + e_{4i}, \\
u_{4i} &= (\hat{b}_4 y_{1i} + \hat{b}_5 y_{2i}) + (D_{4r}\hat{a}_1(x_{2i} - x_{1i}) + D_{4i}\hat{a}_1(x_{2r} - x_{1r}) \\
&+ \dot{D}_{4r}x_{1i} + \dot{D}_{4i}x_{1r}) + v
\end{aligned} \tag{4.26}
$$

where v is the new input, the system (4.25) can be written as:

$$
\begin{aligned}
\dot{e}_{1r} &= \tilde{b}_1(y_{2r} - y_{1r}) - (D_{1r}\tilde{a}_1(x_{2r} - x_{1r}) + D_{1i}\tilde{a}_1(x_{2i} - x_{1i})) + e_{1i}, \\
\dot{e}_{1i} &= \tilde{b}_1(y_{2i} - y_{1i}) - (D_{1r}\tilde{a}_1(x_{2i} - x_{1i}) + D_{1i}\tilde{a}_1(x_{2r} - x_{1r})) + e_{2r}, \\
\dot{e}_{2r} &= \tilde{b}_2 y_{1r} - (D_{2r}(\tilde{a}_{2r}x_{1r} - \tilde{a}_{3r}x_{2r}) + D_{2i}(\tilde{a}_{2i}x_{1i} - \tilde{a}_{3i}x_{2i})) + e_{2i}, \\
\dot{e}_{2i} &= \tilde{b}_2 y_{1i} - (D_{2r}(\tilde{a}_{2i}x_{1i} - \tilde{a}_{3i}x_{2i}) + D_{2i}(\tilde{a}_{2r}x_{1r} - \tilde{a}_{3r}x_{2r})) + e_3, \\
\dot{e}_3 &= -\tilde{b}_3 y_3 + D_3\tilde{a}_4 x_3) + e_{4i}, \\
\dot{e}_{4r} &= (\tilde{b}_4 y_{1r} + \tilde{b}_5 y_{2r}) - (D_{4r}\tilde{a}_1(x_{2r} - x_{1r}) + D_{4i}\tilde{a}_1(x_{2i} - x_{1i})) + e_{4r}, \\
\dot{e}_{4i} &= (\tilde{b}_4 y_{1i} + \tilde{b}_5 y_{2i}) - (D_{4r}\tilde{a}_1(x_{2i} - x_{1i}) + D_{4i}\tilde{a}_1(x_{2r} - x_{1r})) + v
\end{aligned} \tag{4.27}
$$

Choose the nominal system for (4.27) as:

$$
\begin{aligned}
\dot{e}_{1r} &= e_{1i} \\
\dot{e}_{1i} &= e_{2r} \\
\dot{e}_{2r} &= e_{2i} \\
\dot{e}_{2i} &= e_3 \\
\dot{e}_3 &= e_{4r} \\
\dot{e}_{4r} &= e_{4i} \\
\dot{e}_{4i} &= v_0
\end{aligned} \tag{4.28}
$$

Define the sliding surface for nominal system (4.28) as:

$$\sigma_0 = (1+\frac{d}{dt})^6 e_{1r} = e_{1r} + 6e_{1i} + 15e_{2r} + 20e_{2i} + 15e_3 + 6e_{4r} + e_{4i}$$

Then

$$\dot{\sigma}_0 = \dot{e}_{1r} + 6\dot{e}_{1i} + 15\dot{e}_{2r} + 20\dot{e}_{2i} + 15\dot{e}_3 + 6\dot{e}_{4r} + \dot{e}_{4i} = e_{1i} + 6e_{2r} + 15e_{2i} + 20e_3 + 15e_{4r} + 6e_{4i} + v_0$$

By choosing $v_0 = -e_{1i} - 6e_{2r} - 15e_{2i} - 20e_3 - 15e_{4r} - 6e_{4i} - k\,\sigma_0 - k\,sign(\sigma_0),\ k > 0$, we have $\dot{\sigma}_0 = -k\,\sigma_0 - k\,sign(\sigma_0)$. Therefore the nominal system (4.28) is asymptotically stable.

Define the sliding surface for system (4.27) as:

$$\sigma = \sigma_0 + z = e_{1r} + 6e_{1i} + 15e_{2r} + 20e_{2i} + 15e_3 + 6e_{4r} + e_{4i} + z$$

Where, z is some integral term computed later. To avoid the reaching phase, choose $z(0)$ such that $\sigma(0) = 0$. Choose $v = v_0 + v_s$ where, v_0 is the nominal input and v_s is compensator term computed later. Then

$$\begin{aligned}
\dot{\sigma} &= \dot{e}_{1r} + 6\dot{e}_{1i} + 15\dot{e}_{2r} + 20\dot{e}_{2i} + 15\dot{e}_3 + 6\dot{e}_{4r} + \dot{e}_{4i} + \dot{z} \\
&= \tilde{b}_1(y_{2r} - y_{1r}) - D_{1r}\tilde{a}_1(x_{2r} - x_{1r}) - D_{1i}\tilde{a}_1(x_{2i} - x_{1i}) + e_{1i} \\
&+ 6\tilde{b}_1(y_{2i} - y_{1i}) - 6D_{1r}\tilde{a}_1(x_{2i} - x_{1i}) - 6D_{1i}\tilde{a}_1(x_{2r} - x_{1r}) + 6e_{2r} \\
&+ 15\tilde{b}_2 y_{1r} - 15D_{2r}\tilde{a}_{2r}x_{1r} + 15D_{2r}\tilde{a}_{3r}x_{2r} - 15D_{2i}\tilde{a}_{2i}x_{1i} + 15D_{2i}\tilde{a}_{3i}x_{2i} + 15e_{2i} \\
&+ 20\tilde{b}_2 y_{1i} - 20D_{2r}\tilde{a}_{2i}x_{1i} + 20D_{2r}\tilde{a}_{3i}x_{2i} - 20D_{2i}\tilde{a}_{2r}x_{1r} + 20D_{2i}\tilde{a}_{3r}x_{2r} + 20e_3 \\
&- 15\tilde{b}_3 y_3 + 15D_3\tilde{a}_4 x_3 + 15e_{4i} + 6\tilde{b}_4 y_{1r} + 6\tilde{b}_5 y_{2r}) - 6D_{4r}\tilde{a}_1(x_{2r} - x_{1r}) \\
&- 6D_{4i}\tilde{a}_1(x_{2i} - x_{1i}) + 6e_{4r} + \tilde{b}_4 y_{1i} + \tilde{b}_5 y_{2i} - D_{4r}\tilde{a}_1(x_{2i} - x_{1i}) \\
&- D_{4i}\tilde{a}_1(x_{2r} - x_{1r}) + v_0 + v_s + \dot{z}
\end{aligned} \qquad (4.29)$$

By choosing a Lyapunov function:

$V = \frac{1}{2}\sigma^2 + \frac{1}{2}(\tilde{a}_1^2 + \tilde{a}_{2r}^2 + \tilde{a}_{2i}^2 + \tilde{a}_{3r}^2 + \tilde{a}_{3i}^2 + \tilde{a}_4^2 + \tilde{b}_1^2 + \tilde{b}_2^2 + \tilde{b}_3^2 + \tilde{b}_4^2 + \tilde{b}_5^2)$, design the adaptive laws for $\tilde{a}_r, \hat{a}_r, r = 1,...4, \tilde{a}_i, \hat{a}_i, i = 1,...2, \tilde{b}_f, \hat{b}_f, i = 1,...,5$ and compute v_s such that $\dot{V} < 0$.

$$\dot{z} = -e_{1i} - 6e_{2r} - 15e_{2i} - 20e_3 - 15e_{4r} - 6e_{4i} - v_0,$$
$$v_s = -k\sigma - k\,sign(\sigma)$$
$$\dot{\tilde{a}}_1 = \sigma D_{1r1}(x_{2r} - x_{1r}) + \sigma D_{1i}(x_{2i} - x_{1i}) + 6\sigma D_{1r}(x_{2i} - x_{1i})$$
$$+ 6\sigma D_{1i}(x_{2r} - x_{1r}) + 6\sigma D_{4r}(x_{2r} - x_{1r}) + 6\sigma D_{4i}(x_{2i} - x_{1i})$$
$$+ \sigma D_{4r}(x_{2i} - x_{1i}) + \sigma D_{4i}(x_{2r} - x_{1r}) - k_1\tilde{a}_1 \ , \quad \dot{\hat{a}}_1 = -\dot{\tilde{a}}_1$$
$$\dot{\tilde{a}}_{2r} = 15\sigma D_{2r}x_{1r} + 20\sigma D_{2i}x_{1r} - k_2\tilde{a}_{2r} \ , \quad \dot{\hat{a}}_{2r} = -\dot{\tilde{a}}_{2r}$$
$$\dot{\tilde{a}}_{2i} = 15\sigma D_{2i}x_{1i} + 20\sigma D_{2r}x_{1i} - k_3\tilde{a}_{2i} \ , \quad \dot{\hat{a}}_{2i} = -\dot{\tilde{a}}_{2i}$$
$$\dot{\tilde{a}}_{3r} = -15\sigma D_{2r}x_{2r} - 20\sigma D_{2i}x_{2r} - k_4\tilde{a}_{3r} \ , \quad \dot{\hat{a}}_{3r} = -\dot{\tilde{a}}_{3r}$$
$$\dot{\tilde{a}}_{3i} = -15\sigma D_{2i}x_{2i} - 20\sigma D_{2r}x_{2i} - k_5\tilde{a}_{3r} \ , \quad \dot{\hat{a}}_{3r} = -\dot{\tilde{a}}_{3r}$$
$$\dot{\tilde{a}}_4 = -15\sigma D_3 x_3 - k_6\tilde{a}_4 \ , \quad \dot{\hat{a}}_4 = -\dot{\tilde{a}}_4$$
$$\dot{\tilde{b}}_1 = -\sigma(y_{2r} - y_{1r}) - 6\sigma(y_{2i} - y_{1i}) - k_7\tilde{b}_1, \quad \dot{\hat{b}}_1 = -\dot{\tilde{b}}_1$$
$$\dot{\tilde{b}}_2 = -15\sigma y_{1r} - 50\sigma y_{1i} - k_8\tilde{b}_2, \quad \dot{\hat{b}}_2 = -\dot{\tilde{b}}_2$$
$$\dot{\tilde{b}}_3 = 15\sigma y_3 - k_9\tilde{b}_3, \quad \dot{\hat{b}}_3 = -\dot{\tilde{b}}_3,$$
$$\dot{\tilde{b}}_4 = -6\sigma y_{1r} - \sigma y_{1i} - k_{10}\tilde{b}_4, \quad \dot{\hat{b}}_4 = -\dot{\tilde{b}}_4, \tag{4.30}$$
$$\dot{\tilde{b}}_5 = -6\sigma y_{2r} - \sigma y_{2i} - k_{11}\tilde{b}_4, \quad \dot{\hat{b}}_4 = -\dot{\tilde{b}}_4, \quad k, k_i > 0, i = 1,\ldots,11$$

Proof:

Since

$$\dot{V} = \sigma\dot{\sigma} + \tilde{a}_1 \dot{\tilde{a}}_1 + \tilde{a}_{2r}\dot{\tilde{a}}_{2r} + \tilde{a}_{2i}\dot{\tilde{a}}_{2i} + \tilde{a}_{3r}\dot{\tilde{a}}_{3r} + \tilde{a}_{3i}\dot{\tilde{a}}_{3i}$$
$$+ \tilde{a}_4 \dot{\tilde{a}}_4 + \tilde{b}_1 \dot{\tilde{b}}_1 + \tilde{b}_2 \dot{\tilde{b}}_{21} + \tilde{b}_3 \dot{\tilde{b}}_{31} + \tilde{b}_4 \dot{\tilde{b}}_4 + \tilde{b}_5 \dot{\tilde{b}}_5$$
$$= \sigma\{\tilde{b}_1(y_{2r} - y_{1r}) - D_{1r}\tilde{a}_1(x_{2r} - x_{1r}) - D_{1i}\tilde{a}_1(x_{2i} - x_{1i}) + e_{1i}$$
$$+ 6\tilde{b}_1(y_{2i} - y_{1i}) - 6D_{1r}\tilde{a}_1(x_{2i} - x_{1i}) - 6D_{1i}\tilde{a}_1(x_{2r} - x_{1r}) + 6e_{2r}$$
$$+ 15\tilde{b}_2 y_{1r} - 15D_{2r}\tilde{a}_{2r}x_{1r} + 15D_{2r}\tilde{a}_{3r}x_{2r} - 15D_{2i}\tilde{a}_{2i}x_{1i} + 15D_{2i}\tilde{a}_{3i}x_{2i} + 15e_{2i}$$
$$+ 20\tilde{b}_2 y_{1i} - 20D_{2r}\tilde{a}_{2i}x_{1i} + 20D_{2r}\tilde{a}_{3i}x_{2i} - 20D_{2i}\tilde{a}_{2r}x_{1r} + 20D_{2i}\tilde{a}_{3r}x_{2r} + 20e_3$$
$$- 15\tilde{b}_3 y_3 + 15D_3\tilde{a}_4 x_3 + 15e_{4i} + 6\tilde{b}_4 y_{1r} + 6\tilde{b}_5 y_{2r}) - 6D_{4r}\tilde{a}_1(x_{2r} - x_{1r})$$
$$- 6D_{4i}\tilde{a}_1(x_{2i} - x_{1i}) + 6e_{4r} + \tilde{b}_4 y_{1i} + \tilde{b}_5 y_{2i} - D_{4r}\tilde{a}_1(x_{2i} - x_{1i})$$
$$- D_{4i}\tilde{a}_1(x_{2r} - x_{1r}) + v_0 + v_s + \dot{z}\} + \tilde{a}_1 \dot{\tilde{a}}_1 + \tilde{a}_{2r}\dot{\tilde{a}}_{2r} + \tilde{a}_{2i}\dot{\tilde{a}}_{2i} + \tilde{a}_{3r}\dot{\tilde{a}}_{3r}$$
$$+ \tilde{a}_{3i}\dot{\tilde{a}}_{3i} + \tilde{a}_4 \dot{\tilde{a}}_4 + \tilde{b}_1 \dot{\tilde{b}}_1 + \tilde{b}_2 \dot{\tilde{b}}_2 + \tilde{b}_3 \dot{\tilde{b}}_3 + \tilde{b}_4 \dot{\tilde{b}}_4 + \tilde{b}_5 \dot{\tilde{b}}_5$$
$$= \sigma\{e_{1i} + 6e_{2r} + 15e_{2i} + 20e_3 + 15e_{4r} + 6e_{4i} + v_0 + v_s + \dot{z}\}$$
$$+ \tilde{a}_1 \{\dot{\tilde{a}}_1 - \sigma D_{1r1}(x_{2r} - x_{1r}) - \sigma D_{1i}(x_{2i} - x_{1i}) - 6\sigma D_{1r}(x_{2i} - x_{1i})$$
$$- 6\sigma D_{1i}(x_{2r} - x_{1r}) - 6\sigma D_{4r}(x_{2r} - x_{1r}) - 6\sigma D_{4i}(x_{2i} - x_{1i})$$
$$- \sigma D_{4r}(x_{2i} - x_{1i}) - \sigma D_{4i}(x_{2r} - x_{1r})\} + \tilde{a}_{2r}\{\dot{\tilde{a}}_{2r} - 15\sigma D_{2r}x_{1r} - 20\sigma D_{2i}x_{1r}\}$$
$$+ \tilde{a}_{2i}\{\dot{\tilde{a}}_{2i} - 15\sigma D_{2i}x_{1i} - 20\sigma D_{2r}x_{1i}\} + \tilde{a}_{3r}\{\dot{\tilde{a}}_{3r} + 15\sigma D_{2r}x_{2r} + 20\sigma D_{2i}x_{2r}\}$$
$$+ \tilde{a}_{3i}\{\dot{\tilde{a}}_{3i} + 15\sigma D_{2i}x_{2i} + 20\sigma D_{2r}x_{2i}\} + \tilde{a}_4 \{\dot{\tilde{a}}_4 + 15\sigma D_3 x_3\}$$
$$+ \tilde{b}_1 \{\dot{\tilde{b}}_1 + \sigma(y_{2r} - y_{1r}) + 6\sigma(y_{2i} - y_{1i})\} + \tilde{b}_2 \{\dot{\tilde{b}}_2 + 15\sigma y_{1r} + 50\sigma y_{1i}\}$$
$$+ \tilde{b}_3 \{\dot{\tilde{b}}_3 - 15\sigma y_3\} + \tilde{b}_4 \{\dot{\tilde{b}}_4 + 6\sigma y_{1r} + \sigma y_{1i}\} + \tilde{b}_5 \{\dot{\tilde{b}}_5 + 6\sigma y_{2r} + \sigma y_{2i}\}$$

By using

$$\dot{z} = -e_{1i} - 6e_{2r} - 15e_{2i} - 20e_3 - 15e_{4r} - 6e_{4i} - v_0 , \quad v_s = -k\sigma - k\, sign(\sigma)$$

$$\dot{\tilde{a}}_1 = \sigma D_{1r1}(x_{2r} - x_{1r}) + \sigma D_{1i}(x_{2i} - x_{1i}) + 6\sigma D_{1r}(x_{2i} - x_{1i})$$
$$+ 6\sigma D_{1i}(x_{2r} - x_{1r}) + 6\sigma D_{4r}(x_{2r} - x_{1r}) + 6\sigma D_{4i}(x_{2i} - x_{1i})$$
$$+ \sigma D_{4r}(x_{2i} - x_{1i}) + \sigma D_{4i}(x_{2r} - x_{1r}) - k_1\tilde{a}_1 , \quad \dot{\hat{a}}_1 = -\dot{\tilde{a}}_1$$

$$\dot{\tilde{a}}_{2r} = 15\sigma D_{2r}x_{1r} + 20\sigma D_{2i}x_{1r} - k_2\tilde{a}_{2r} , \quad \dot{\hat{a}}_{2r} = -\dot{\tilde{a}}_{2r}$$

$$\dot{\tilde{a}}_{2i} = 15\sigma D_{2i}x_{1i} + 20\sigma D_{2r}x_{1i} - k_3\tilde{a}_{2i} , \quad \dot{\hat{a}}_{2i} = -\dot{\tilde{a}}_{2i}$$

$$\dot{\tilde{a}}_{3r} = -15\sigma D_{2r}x_{2r} - 20\sigma D_{2i}x_{2r} - k_4\tilde{a}_{3r} , \quad \dot{\hat{a}}_{3r} = -\dot{\tilde{a}}_{3r}$$

$$\dot{\tilde{a}}_{3i} = -15\sigma D_{2i}x_{2i} - 20\sigma D_{2r}x_{2i} - k_5\tilde{a}_{3r} , \quad \dot{\hat{a}}_{3r} = -\dot{\tilde{a}}_{3r}$$

$$\dot{\tilde{a}}_4 = -15\sigma D_3x_3 - k_6\tilde{a}_4 , \quad \dot{\hat{a}}_4 = -\dot{\tilde{a}}_4$$

$$\dot{\tilde{b}}_1 = -\sigma(y_{2r} - y_{1r}) - 6\sigma(y_{2i} - y_{1i}) - k_7\tilde{b}_1 , \quad \dot{\hat{b}}_1 = -\dot{\tilde{b}}_1$$

$$\dot{\tilde{b}}_2 = -15\sigma y_{1r} - 50\sigma y_{1i} - k_8\tilde{b}_2 , \quad \dot{\hat{b}}_2 = -\dot{\tilde{b}}_2$$

$$\dot{\tilde{b}}_3 = 15\sigma y_3 - k_9\tilde{b}_3 , \quad \dot{\hat{b}}_3 = -\dot{\tilde{b}}_3 ,$$

$$\dot{\tilde{b}}_4 = -6\sigma y_{1r} - \sigma y_{1i} - k_{10}\tilde{b}_4 , \quad \dot{\hat{b}}_4 = -\dot{\tilde{b}}_4 ,$$

$$\dot{\tilde{b}}_5 = -6\sigma y_{2r} - \sigma y_{2i} - k_{11}\tilde{b}_4 , \quad \dot{\hat{b}}_5 = -\dot{\tilde{b}}_5 , \quad k, k_i > 0, i = 1,...,11$$

We have

$$\dot{V} = -k\sigma^2 - k|\sigma| - k_1\tilde{a}_1^2 - k_2\tilde{a}_{2r}^2 - k_3\tilde{a}_{2i}^2 - k_4\tilde{a}_{3r}^2$$
$$- k_5\tilde{a}_{3i}^2 - k_6\tilde{a}_4^2 - k_7\tilde{b}_1^2 - k_8\tilde{b}_2^2 - k_9\tilde{b}_3^2 - k_{10}\tilde{b}_4^2 - k_{11}\tilde{b}_5^2$$

From this we conclude that $\sigma, \tilde{a}_r, \tilde{a}_i, \tilde{b}_f \to 0$. Since $\sigma \to 0$, therefore $e = (e_{1r}, e_{1i}, e_{2r}, e_{2i}, e_3, e_{4r}, e_{4i}) \to 0$.

In simulations, the initial conditions are chosen as: $x(0) = [2 + 0.02j, 1 + 0.2j, -1]^T$ and $y(0) = [-1 - 2j, -3 - 4j, -5, -6 - 7j]^T$. The true values of the unknown parameters are chosen as: $a_4 = 2,\ a_2 = 60 + 0.02j\ a_3 = 1 - 0.06j\ a_4 = 0.8$ $b_1 = 14, b_2 = 35, b_3 = 3, b_4 = -5, b_5 = -4$

Simulation results of Case 1:

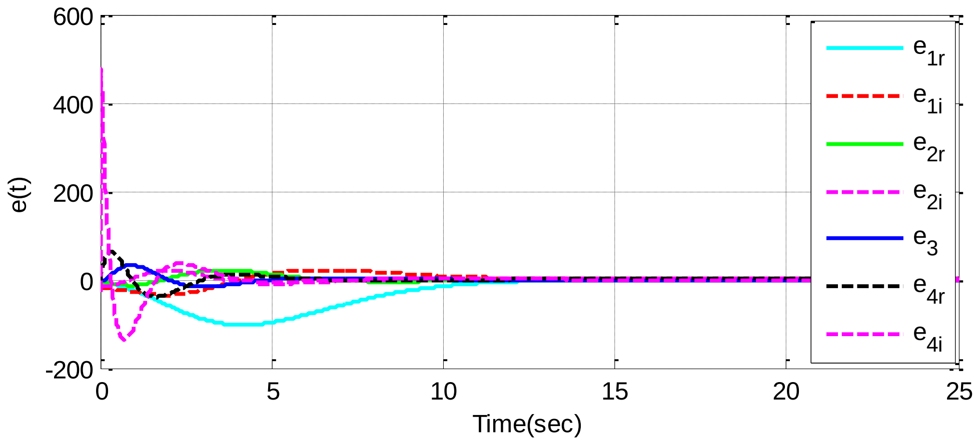

Figure 4.1: Time Response of error $e_{1r}, e_{1i}, e_{2r}, e_{2i}, e_3, e_{4r}$ & e_{4i}

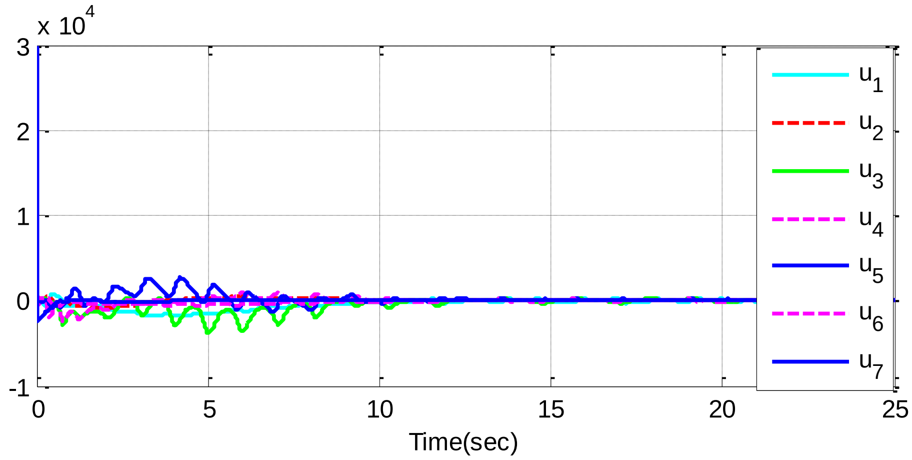

Figure 4.2: Time Response of adaptive controller $u_1, u_2, u_3, u_4, u_5, u_6$ & u_7

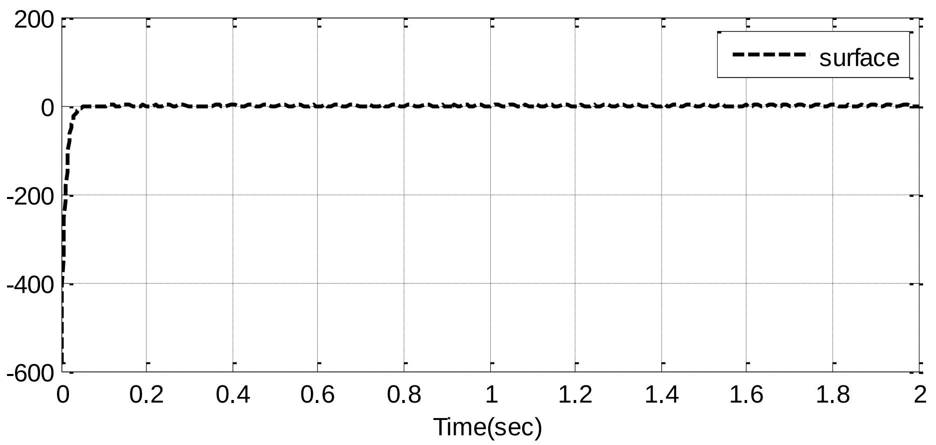

Figure 4.3: Time Response of surface

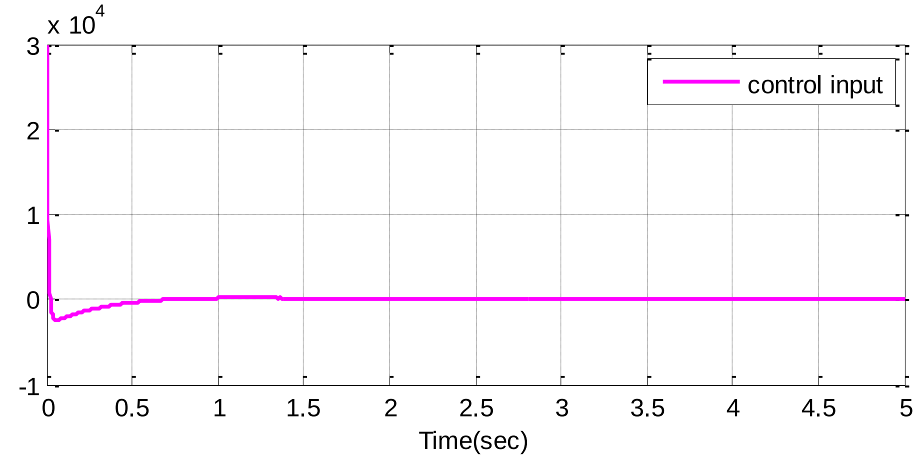

Figure 4.4: Time Response of control input

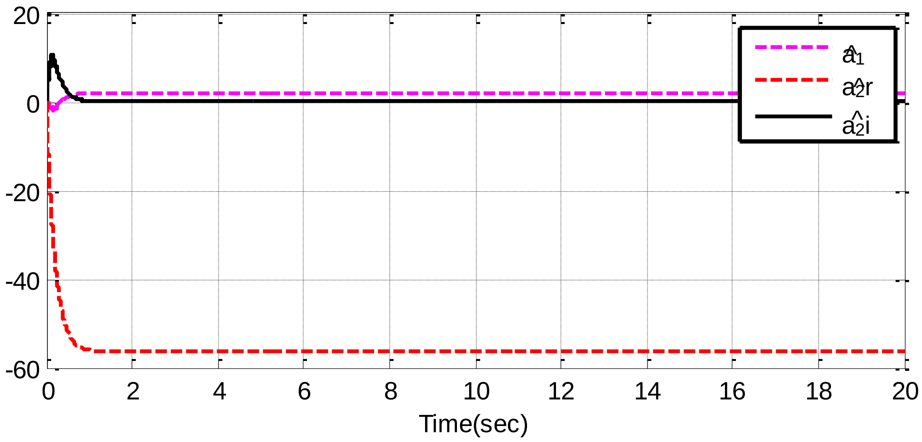

Figure 4.5: Estimation parameter of $\hat{a}_1, \hat{a}_{2r}$ & $\hat{a}_{1i}$

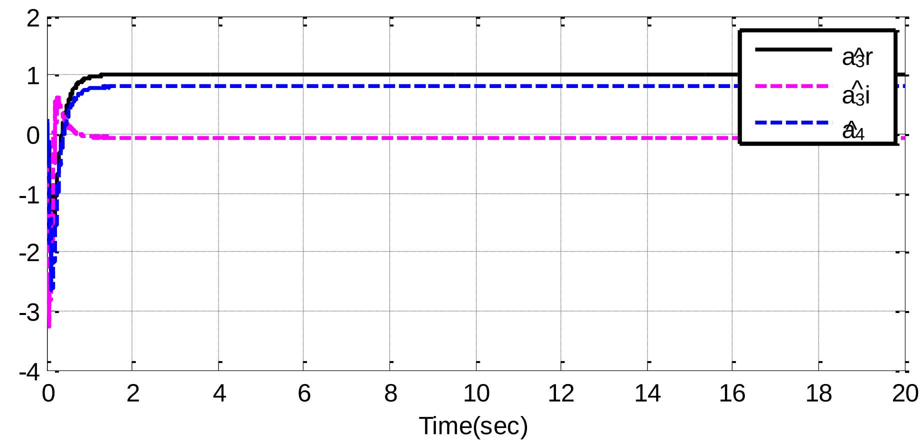

Figure 4.6: Estimation parameter of $\hat{a}_{3r}, \hat{a}_{3i}$ & $\hat{c}_4$

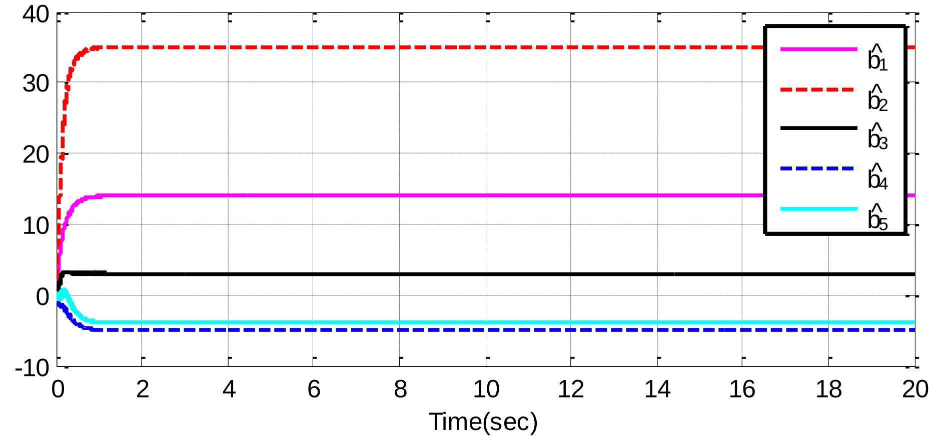

Figure 4.7: Estimation parameter of $\hat{b}_1, \hat{b}_2, \hat{b}_3, \hat{b}_4$ & $\hat{b}_5$

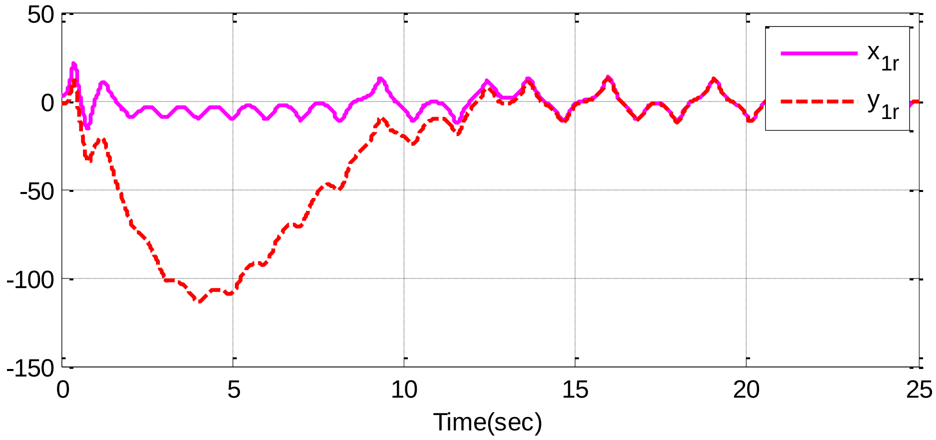

Figure 4.8: Time Response of x_{1r}&$y_{1r}$with IC (2, -1)

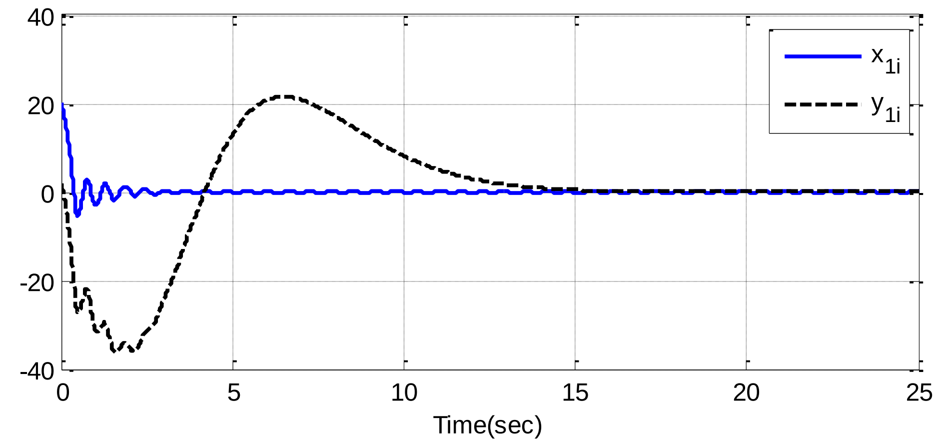

Figure 4.9: Time Response of x_{1i}&$y_{1i}$with IC (20, -2)

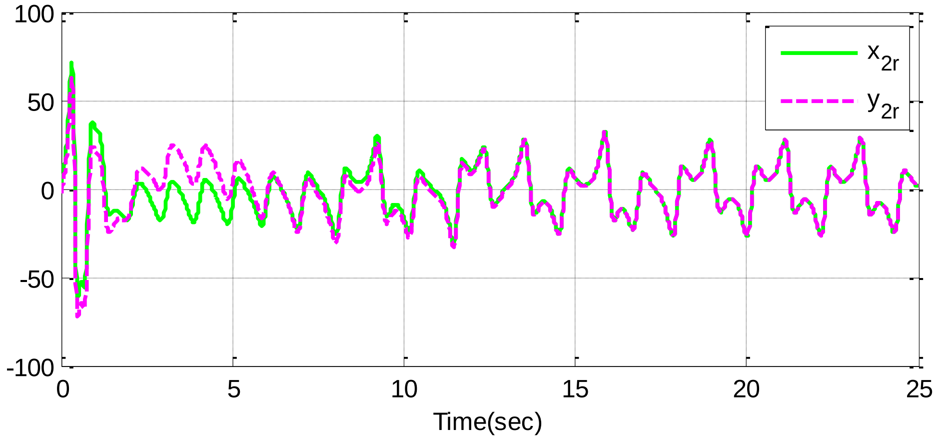

Figure 4.10: Time Response of x_{2r}&y_{2r} with IC (1, -3)

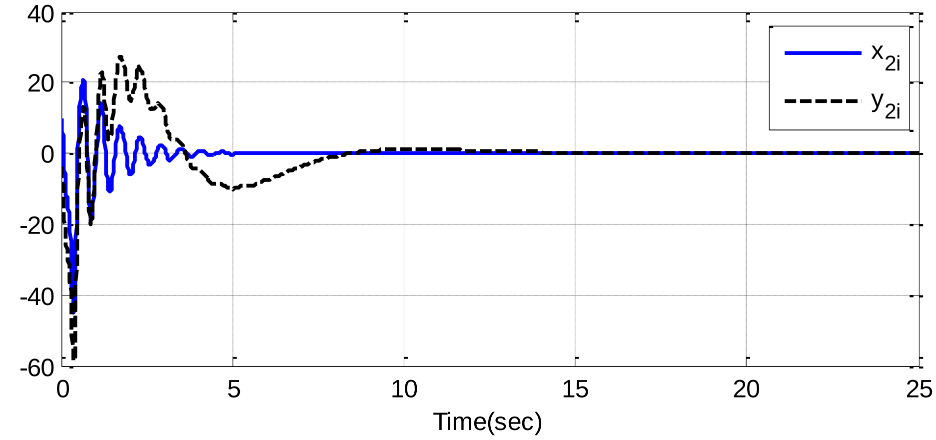

Figure 4.11: Time Response of x_{2i}&y_{2i} with IC (10, -4)

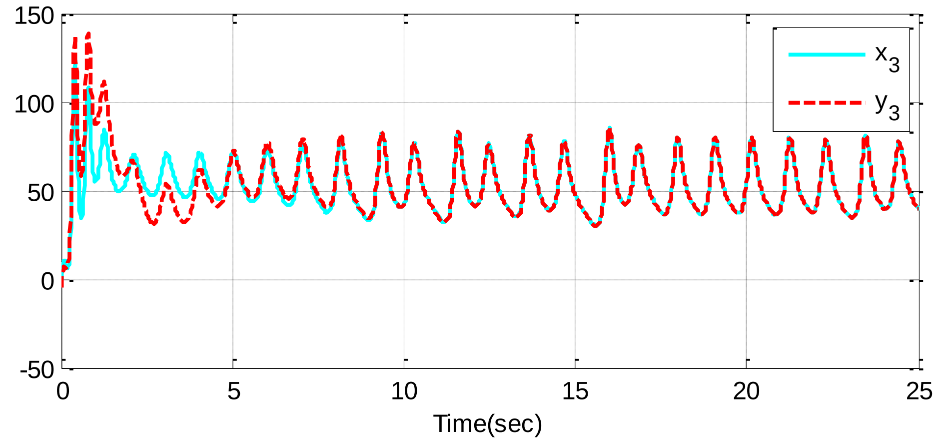

Figure 4.12: Time Response of x_3&y_3 with IC (-1, -5)

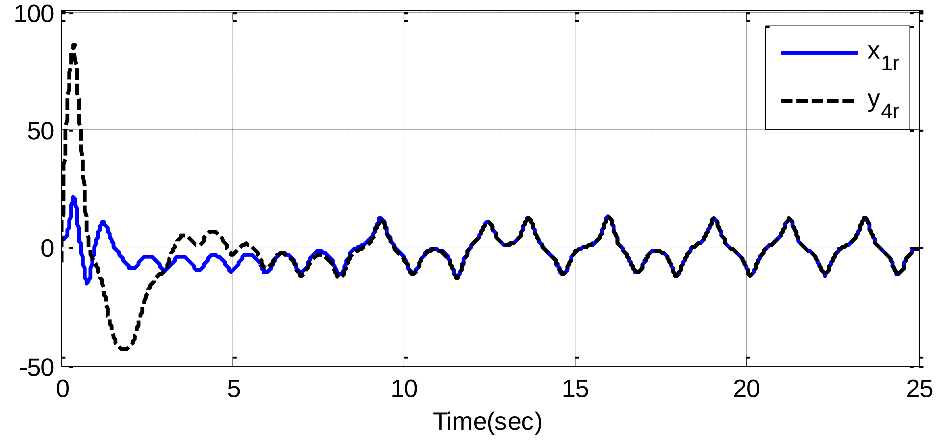

Figure 4.13: Time Response of x_{1r}&$y_{4r}$with IC (2, -6)

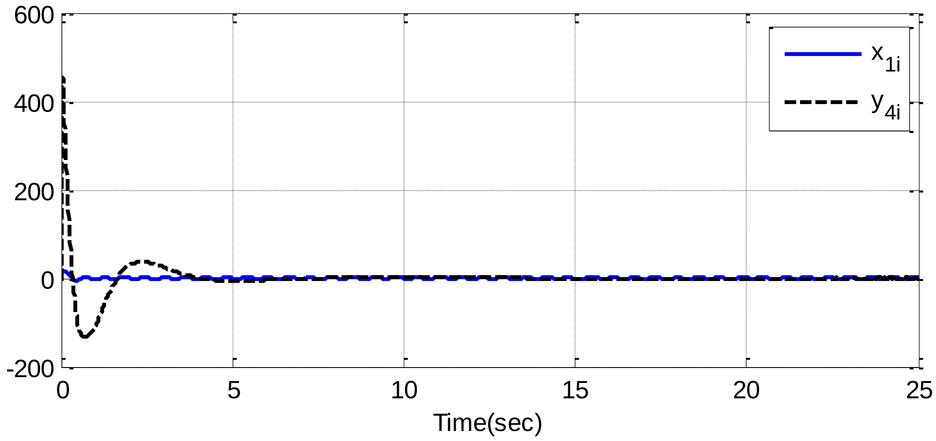

Figure 4.14: Time Response of x_{1i}&$y_{4i}$with IC (2, -1)

Case 2: m>n

Consider the Master system given in [39] as:

$$\begin{aligned}
\dot{x}_1 &= a_1(x_2 - x_1) + x_4 \\
\dot{x}_2 &= a_2 x_2 - x_1 x_3 + x_4 \\
\dot{x}_3 &= 0.5(\bar{x}_1 x_2 + x_1 \bar{x}_2) - a_3 x_3 \\
\dot{x}_4 &= 0.5(\bar{x}_1 x_2 + x_1 \bar{x}_2) - a_4 x_4
\end{aligned} \qquad (4.31)$$

Where, $x_1 = x_{1r} + jx_{1i}$, $x_2 = x_{2r} + jx_{2i}$ are complex $x_3 = x_{3r}$ and $x_4 = x_{4r}$ are real. $\bar{x}_1, \bar{x}_2$ denote the complex conjugate variables of x_1, x_2. a_1, a_2, a_3 & a_4 are unknown real parameters.

When $a_1 = 42$, $a_2 = 25$, $a_3 = 6$ & $a_4 = 10$

$x(0) = [10 + 5j, 10 + 6j, 2, 12]^T$.

Consider the Slave system given in [39] as:

$$\begin{aligned}
\dot{y}_1 &= b_1(y_2 - y_1) + u_1 \\
\dot{y}_2 &= b_2 y_1 - b_3 y_2 - y_1 y_3 + u_2 \\
\dot{y}_3 &= 0.5(\bar{y}_1 y_2 + y_1 \bar{y}_2) - b_4 y_3 + u_3
\end{aligned} \qquad (4.32)$$

Where, $y_1 = y_{1r} + jy_{1i}$, $y_2 = y_{2r} + jy_{2i}$ are complex and $y_3 = y_{3r}$ is real. $\bar{y}_1, \bar{y}_2$ denote the complex conjugate variables of y_1, y_2. b_1, b_2, b_3, b_4 are known real parameters. u_1, u_2, u_3 and u_4 are controllers. When $b_1 = 20, b_2 = 60 + 0.02j, b_3 = 1 - 0.06, b_4 = 0.8$ $y(0) = [2 + 0.02j, 1 + 0.2j, -1]^T$

We investigate CPS of two non-identical complex systems with the different orders.

Let $\hat{a}_i, \hat{b}_i, i = 1,..,4$ be estimates of $a_i, b_i, i = 1,..,4$ and $\tilde{a}_i = a_i - \hat{a}_i, \tilde{b}_i = b_i - \hat{b}_i, i = 1,....,4$, be the errors in estimations of $a_i, b_i, i = 1,..,4$, respectively. Then systems (4.31) and (4.32) can be written as:

$$\begin{aligned}
\dot{x}_1 &= \hat{a}_1(x_2 - x_1) + \tilde{a}_1(x_2 - x_1) + x_4 \\
\dot{x}_2 &= \hat{a}_2 x_2 + \tilde{a}_2 x_2 - x_1 x_3 + x_4 \\
\dot{x}_3 &= 0.5(\bar{x}_1 x_2 + x_1 \bar{x}_2) - \hat{a}_3 x_3 - \tilde{a}_3 x_3 \\
\dot{x}_4 &= 0.5(\bar{x}_1 x_2 + x_1 \bar{x}_2) - \hat{a}_4 x_4 - \tilde{a}_4 x_4
\end{aligned} \qquad (4.33)$$

$$
\begin{aligned}
\dot{y}_1 &= \hat{b}_1(y_2 - y_1) + \tilde{b}_1(y_2 - y_1) + u_1 \\
\dot{y}_2 &= \hat{b}_2 y_1 + \tilde{b}_2 y_1 - \hat{b}_3 y_2 - \tilde{b}_3 y_2 - y_1 y_3 + u_2 \\
\dot{y}_3 &= 0.5(\bar{y}_1 y_2 + y_1 \bar{y}_2) - \hat{b}_4 y_3 - \tilde{b}_4 y_3 + u_3
\end{aligned}
\tag{4.34}
$$

The 4-dimensional complex systems (4.33) can be into 6-dimensional real system:

$$
\begin{aligned}
\dot{x}_{1r} &= \hat{a}_1(x_{2r} - x_{1r}) + \tilde{a}_1(x_{2r} - x_{1r}) + x_4 \\
\dot{x}_{1i} &= \hat{a}_1(x_{2i} - x_{1i}) + \tilde{a}_1(x_{2i} - x_{1i}) + x_4 \\
\dot{x}_{2r} &= \hat{a}_2 x_{2r} + \tilde{a}_2 x_{2r} - x_{1r} x_3 + x_4 \\
\dot{x}_{2i} &= \hat{a}_2 x_{2i} + \tilde{a}_2 x_{2i} - x_{1i} x_3 + x_4 \\
\dot{x}_3 &= x_{1r} x_{2r} + x_{1i} x_{2i} - \hat{a}_3 x_3 - \tilde{a}_3 x_3 \\
\dot{x}_4 &= x_{1r} x_{2r} + x_{1i} x_{2i} - \hat{a}_4 x_4 - \tilde{a}_4 x_4
\end{aligned}
\tag{4.35}
$$

The 3-D complex systems (4.34) can be into 5-dimensional real systems:

$$
\begin{aligned}
\dot{y}_{1r} &= \hat{b}_1(y_{2r} - y_{1r}) + \tilde{b}_1(y_{2r} - y_{1r}) + u_{1r} \\
\dot{y}_{1i} &= \hat{b}_1(y_{2i} - y_{1i}) + \tilde{b}_1(y_{2i} - y_{1i}) + u_{1i} \\
\dot{y}_{2r} &= \hat{b}_{2r} y_{1r} + \tilde{b}_{2r} y_{1r} - \hat{b}_{3r} y_{2r} - \tilde{b}_{3r} y_{2r} - y_{1r} y_3 + u_{2r} \\
\dot{y}_{2i} &= \hat{b}_{2i} y_{1i} + \tilde{b}_{2i} y_{1i} - \hat{b}_{3i} y_{2i} - \tilde{b}_{3i} y_{2i} - y_{1i} y_3 + u_{2i} \\
\dot{y}_3 &= y_{1r} y_{2r} + y_{1i} y_{2i} - \hat{b}_4 y_3 - \tilde{b}_4 y_3 + u_3
\end{aligned}
\tag{4.36}
$$

Where $u_{1r}, u_{1i}, u_{2r}, u_{2i}$ and u_3 in Equation (4.36) are the control functions to be determined.

The complex function transformation matrix is taken as:

$$
D = \begin{pmatrix} 0.5\exp(j\pi t/5) & 0 & 0 & 0 \\ 0 & \exp(j\pi t/10) & 0 & 0 \\ 0 & 0 & 1.2 + \sin t & 1.2 + \cos t \end{pmatrix}
\tag{4.37}
$$

Where $\beta \exp(j\theta t) = \beta(\cos\theta t + j\sin\theta t)$.

The projective synchronization error signals are defined as:

$$
\begin{aligned}
& e_{1r} = y_{1r} - (D_{1r} x_{1r} + D_{1i} x_{1i}), e_{1i} = y_{1i} - (D_{1r} x_{1i} + D_{1i} x_{1r}) \\
& e_{2r} = y_{2r} - (D_{2r} x_{2r} + D_{2i} x_{2i}), e_{2i} = y_{2i} - (D_{2r} x_{2i} + D_{2i} x_{2r}) \\
& e_3 = y_3 - (D_3 x_3)
\end{aligned}
\tag{4.38}
$$

Then the dynamics of the error system becomes:

$$\begin{aligned}
\dot{e}_{1r} &= \dot{y}_{1r} - (\dot{D}_{1r}x_{1r} + D_{1r}\dot{x}_{1r} + \dot{D}_{1i}x_{1i} + D_{1i}\dot{x}_{1i}),\\
\dot{e}_{1i} &= \dot{y}_{1i} - (\dot{D}_{1r}x_{1i} + D_{1r}\dot{x}_{1i} + \dot{D}_{1i}x_{1r} + D_{1i}\dot{x}_{1r}),\\
\dot{e}_{2r} &= \dot{y}_{2r} - (\dot{D}_{2r}x_{2r} + D_{2r}\dot{x}_{2r} + \dot{D}_{2i}x_{2i} + D_{2i}\dot{x}_{2i}),\\
\dot{e}_{2i} &= \dot{y}_{2i} - (\dot{D}_{2r}x_{2i} + D_{2r}\dot{x}_{2i} + \dot{D}_{2i}x_{2r} + D_{2i}\dot{x}_{2r}),\\
\dot{e}_{3} &= \dot{y}_{3} - (\dot{D}_{3}x_{3} + D_{3}\dot{x}_{3})
\end{aligned} \tag{4.39}$$

Then

$$\begin{aligned}
&\dot{e}_{1r} = \dot{y}_{1r} - (\dot{D}_{1r}x_{1r} + D_{1r}\dot{x}_{1r} + \dot{D}_{1i}x_{1i} + D_{1i}\dot{x}_{1i}) = \hat{b}_1(y_{2r} - y_{1r})\\
&+ \tilde{b}_1(y_{2r} - y_{1r}) + u_{1r} - (D_{1r}(\hat{a}_1(x_{2r} - x_{1r}) + \tilde{a}_1(x_{2r} - x_{1r}) + x_4)\\
&+ D_{1i}(\hat{a}_1(x_{2i} - x_{1i}) + \tilde{a}_1(x_{2i} - x_{1i}) + x_4) + \dot{D}_{1r}x_{1r} + \dot{D}_{1i}x_{1i}),\\
&\dot{e}_{1i} = \dot{y}_{1i} - (\dot{D}_{1r}x_{1i} + D_{1r}\dot{x}_{1i} + \dot{D}_{1i}x_{1r} + D_{1i}\dot{x}_{1r}) = \hat{b}_1(y_{2i} - y_{1i})\\
&+ \tilde{b}_1(y_{2i} - y_{1i}) + u_{1i} - (D_{1r}(\hat{a}_1(x_{2i} - x_{1i}) + \tilde{a}_1(x_{2i} - x_{1i}) + x_4)\\
&+ D_{1i}(\hat{a}_1(x_{2r} - x_{1r}) + \tilde{a}_1(x_{2r} - x_{1r}) + x_4) + \dot{D}_{1r}x_{1i} + \dot{D}_{1i}x_{1r}),\\
&\dot{e}_{2r} = \dot{y}_{2r} - (\dot{D}_{2r}x_{2r} + D_{2r}\dot{x}_{2r} + \dot{D}_{2i}x_{2i} + D_{2i}\dot{x}_{2i}) = (\hat{b}_{2r}y_{1r} + \tilde{b}_{2r}y_{1r}\\
&- \hat{b}_{3r}y_{2r} - \tilde{b}_{3r}y_{2r} - y_{1r}y_3) + u_{2r} - (D_{2r}(\hat{a}_2x_{2r} + \tilde{a}_2x_{2r} - x_{1r}x_3 + x_4)\\
&+ D_{2i}(\hat{a}_2x_{2i} + \tilde{a}_2x_{2i} - x_{1i}x_3 + x_4) + \dot{D}_{2r}x_{2r} + \dot{D}_{2i}x_{2i}),\\
&\dot{e}_{2i} = \dot{y}_{2i} - (\dot{D}_{2r}x_{2i} + D_{2r}\dot{x}_{2i} + \dot{D}_{2i}x_{2r} + D_{2i}\dot{x}_{2r}) = (\hat{b}_{2i}y_{1i} + \tilde{b}_{2i}y_{1i}\\
&- \hat{b}_{3i}y_{2i} - \tilde{b}_{3i}y_{2i} - y_{1i}y_3) + u_{2i} - (D_{2r}(\hat{a}_2x_{2i} + \tilde{a}_2x_{2i} - x_{1i}x_3 + x_4)\\
&+ D_{2i}(\hat{a}_2x_{2r} + \tilde{a}_2x_{2r} - x_{1r}x_3 + x_4) + \dot{D}_{2r}x_{2i} + \dot{D}_{2i}x_{2r}),\\
&\dot{e}_3 = \dot{y}_3 - (\dot{D}_3x_3 + D_3\dot{x}_3 + \dot{D}_4x_4 + D_4\dot{x}_4) = (y_{1r}\,y_{2r} + y_{1i}y_{2i}\\
&- \hat{b}_4y_3 - \tilde{b}_4y_3) + u_3 - (D_3(x_{1r}\,x_{2r} + x_{1i}x_{2i} - \hat{a}_3x_3 - \tilde{a}_3x_3)\\
&+ D_4(x_{1r}\,x_{2r} + x_{1i}x_{2i} - \hat{a}_4x_4 - \tilde{a}_4x_4) + \dot{D}_3x_3 + \dot{D}_4x_4)
\end{aligned} \tag{4.40}$$

By choosing

$$
\begin{aligned}
u_{1r} &= -\hat{b}_1(y_{2r} - y_{1r}) + (D_{1r}(\hat{a}_1(x_{2r} - x_{1r}) + x_4) + D_{1i}(\hat{a}_1(x_{2i} - x_{1i}) \\
&+ x_4) + \dot{D}_{1r}x_{1r} + \dot{D}_{1i}x_{1i}) + e_{1i}, \\
u_{1i} &= -\hat{b}_1(y_{2i} - y_{1i}) + (D_{1r}(\hat{a}_1(x_{2i} - x_{1i}) + x_4) + D_{1i}(\hat{a}_1(x_{2r} - x_{1r}) \\
&+ x_4) + \dot{D}_{1r}x_{1i} + \dot{D}_{1i}x_{1r}) + e_{2r}, \\
u_{2r} &= -(\hat{b}_{2r}y_{1r} - \hat{b}_{3r}y_{2r} - y_{1r}y_3) + (D_{2r}(\hat{a}_2x_{2r} - x_{1r}x_3 + x_4) \\
&+ D_{2i}(\hat{a}_2x_{2i} - x_{1i}x_3 + x_4) + \dot{D}_{2r}x_{2r} + \dot{D}_{2i}x_{2i}) + e_{2i}, \\
u_{2i} &= -(\hat{b}_{2i}y_{1i} - \hat{b}_{3i}y_{2i} - y_{1i}y_3) + (D_{2r}(\hat{a}_2x_{2i} - x_{1i}x_3 + x_4) \\
&+ D_{2i}(\hat{a}_2x_{2r} - x_{1r}x_3 + x_4) + \dot{D}_{2r}x_{2i} + \dot{D}_{2i}x_{2r}) + e_3, \\
u_3 &= -(y_{1r}\, y_{2r} + y_{1i}y_{2i} - \hat{b}_4y_3) + (D_3(x_{1r}\, x_{2r} + x_{1i}x_{2i} - \hat{a}_3x_3) \\
&+ D_4(x_{1r}\, x_{2r} + x_{1i}x_{2i} - \hat{a}_4x_4) + \dot{D}_3x_3 + \dot{D}_4x_4) + v
\end{aligned}
\tag{4.41}
$$

Where v is the new input, the system (4.40) can be written as:

$$
\begin{aligned}
\dot{e}_{1r} &= \tilde{b}_1(y_{2r} - y_{1r}) - D_{1r}\tilde{a}_1(x_{2r} - x_{1r} - D_{1i}\tilde{a}_1(x_{2i} - x_{1i}) + e_{1i}, \\
\dot{e}_{1i} &= \tilde{b}_1(y_{2i} - y_{1i}) - D_{1r}\tilde{a}_1(x_{2i} - x_{1i}) - D_{1i}\tilde{a}_1(x_{2r} - x_{1r}) + e_{2r}, \\
\dot{e}_{2r} &= (\tilde{b}_{2r}y_{1r} - \tilde{b}_{3r}y_{2r}) - D_{2r}\tilde{a}_2x_{2r} - D_{2i}\tilde{a}_2x_{2i} + e_{2i}, \\
\dot{e}_{2i} &= (\tilde{b}_{2i}y_{1i} - \tilde{b}_{3i}y_{2i}) - D_{2r}\tilde{a}_2x_{2i} - D_{2i}\tilde{a}_2x_{2r} + e_3, \\
\dot{e}_3 &= -\tilde{b}_4y_3 + D_3\tilde{a}_3x_3 + D_4\tilde{a}_4x_4 + v
\end{aligned}
\tag{4.42}
$$

Choose the nominal system for (4.42) as:

$$
\begin{aligned}
\dot{e}_{1r} &= e_{1i} \\
\dot{e}_{1i} &= e_{2r} \\
\dot{e}_{2r} &= e_{2i} \\
\dot{e}_{2i} &= e_3 \\
\dot{e}_3 &= v_0
\end{aligned}
\tag{4.43}
$$

Define the sliding surface for nominal system (4.43) as:

$$\sigma_0 = (1 + \frac{d}{dt})^4 e_{1r} = e_{1r} + 4e_{1i} + 6e_{2r} + 4e_{2i} + e_3$$

Then

$$\dot{\sigma}_0 = \dot{e}_{1r} + 4\dot{e}_{1i} + 6\dot{e}_{2r} + 4\dot{e}_{2i} + \dot{e}_3 = e_{1i} + 4e_{2r} + 6e_{2i} + 4e_3 + v_0$$

By choosing $v_0 = -e_{1i} - 4e_{2r} - 6e_{2i} - 4e_3 - k\,\sigma_0 - k\,sign(\sigma_0),\ k > 0$, we have $\dot{\sigma}_0 = -k\,\sigma_0 - k\,sign(\sigma_0)$. Therefore the nominal system (4.43) is asymptotically stable.

Define the sliding surface for system (4.42) as:

$$\sigma = \sigma_0 + z = e_{1r} + 4e_{1i} + 6e_{2r} + 4e_{2i} + e_3 + z$$

Where, z is some integral term computed later. To avoid the reaching phase, choose $z(0)$ such that $\sigma(0) = 0$. Choose $v = v_0 + v_s$ where, v_0 is the nominal input and v_s is compensator term computed later. Then

$$\begin{aligned}
\dot{\sigma} &= \dot{e}_{1r} + 4\dot{e}_{1i} + 6\dot{e}_{2r} + 4\dot{e}_{2i} + \dot{e}_3 + \dot{z} \\
&= \tilde{b}_1(y_{2r} - y_{1r}) - D_{1r}\tilde{a}_1(x_{2r} - x_{1r}) - D_{1i}\tilde{a}_1(x_{2i} - x_{1i}) + e_{1i} \\
&+ 4\tilde{b}_1(y_{2i} - y_{1i}) - 4D_{1r}\tilde{a}_1(x_{2i} - x_{1i}) - 4D_{1i}\tilde{a}_1(x_{2r} - x_{1r}) + 4e_{2r} \\
&+ 6\tilde{b}_{2r}y_{1r} - 6\tilde{b}_{3r}y_{2r} - 6D_{2r}\tilde{a}_2x_{2r} - 6D_{2i}\tilde{a}_2x_{2i} + 6e_{2i} + 4\tilde{b}_{2i}y_{1i} - 4\tilde{b}_{3i}y_{2i} \\
&- 4D_{2r}\tilde{a}_2x_{2i} - 4D_{2i}\tilde{a}_2x_{2r} + 4e - \tilde{b}_4y_3 + D_3\tilde{a}_3x_3 + D_4\tilde{a}_4x_4 + v_0 + v_s + \dot{z}
\end{aligned} \tag{4.44}$$

By choosing a Lyapunov function:

$V = \frac{1}{2}\sigma^2 + \frac{1}{2}(\tilde{a}_1^2 + \tilde{a}_2^2 + \tilde{a}_3^2 + \tilde{a}_4^2 + \tilde{b}_1^2 + \tilde{b}_{2r}^2 + \tilde{b}_{2i}^2 + \tilde{b}_{3r}^2 + \tilde{b}_{3i}^2 + \tilde{b}_4^2)$, design the adaptive laws for $\tilde{a}_r, \hat{a}_r, r = 1,...4, \tilde{a}_i, \hat{a}_i, i = 1,...2, \tilde{b}_f, \hat{b}_f, i = 1,...,5$ and compute v_s such that $\dot{V} < 0$.

$$\begin{aligned}
&\dot{z} = -e_{1i} - 4e_{2r} - 6e_{2i} - 4e_3 - v_0, \\
&v_s = -k\sigma - k\, sign(\sigma) \\
&\dot{\tilde{a}}_1 = \sigma D_{1r}(x_{2r} - x_{1r}) + \sigma D_{1i}(x_{2i} - x_{1i}) + 4D_{1r}(x_{2i} - x_{1i}) \\
&+ 4D_{1i1}(x_{2r} - x_{1r}) - k_1\tilde{a}_1 , \qquad \dot{\hat{a}}_1 = -\dot{\tilde{a}}_1 \\
&\dot{\tilde{a}}_2 = 6\sigma D_{2r}x_{2r} + 6\sigma D_{2i}x_{2i} + 4\sigma D_{2r}x_{2i} \\
&+ 4\sigma D_{2i}x_{2r} - k_2\tilde{a}_{2r} , \qquad \dot{\hat{a}}_2 = -\dot{\tilde{a}}_2 \\
&\dot{\tilde{a}}_3 = -\sigma D_3x_3 - k_3\tilde{a}_{3r} , \qquad \dot{\hat{a}}_3 = -\dot{\tilde{a}}_3 \\
&\dot{\tilde{a}}_4 = -\sigma D_4x_4 - k_4\tilde{a}_4 , \qquad \dot{\hat{a}}_4 = -\dot{\tilde{a}}_4 \\
&\dot{\tilde{b}}_1 = -\sigma(y_{2r} - y_{1r}) - 4\sigma(y_{2i} - y_{1i}) - k_5\tilde{b}_1, \qquad \dot{\hat{b}}_1 = -\dot{\tilde{b}}_1 \\
&\dot{\tilde{b}}_{2r} = -6\sigma y_{1r} - k_6\tilde{b}_{2r}, \qquad \dot{\hat{b}}_{2r} = -\dot{\tilde{b}}_{2r} \\
&\dot{\tilde{b}}_{2i} = -4\sigma y_{1i} - k_7\tilde{b}_{2i}, \qquad \dot{\hat{b}}_{2i} = -\dot{\tilde{b}}_{2i} \\
&\dot{\tilde{b}}_{3r} = 6\sigma y_{2r} - k_8\tilde{b}_{3r}, \qquad \dot{\hat{b}}_{3r} = -\dot{\tilde{b}}_{3r}, \\
&\dot{\tilde{b}}_{3i} = 4\sigma y_{2i} - k_9\tilde{b}_{3i}, \qquad \dot{\hat{b}}_{3i} = -\dot{\tilde{b}}_{3i}, \\
&\dot{\tilde{b}}_4 = \sigma y_3 - k_{10}\tilde{b}_4, \qquad \dot{\hat{b}}_4 = -\dot{\tilde{b}}_4, \qquad k, k_i > 0, i = 1,...,10
\end{aligned} \tag{4.45}$$

Proof:

Since

$$\dot{V} = \sigma\dot{\sigma} + \tilde{a}_1 \dot{\tilde{a}}_1 + \tilde{a}_2 \dot{\tilde{a}}_2 + \tilde{a}_3 \dot{\tilde{a}}_3 + \tilde{a}_4 \dot{\tilde{a}}_4 + \tilde{b}_1 \dot{\tilde{b}}_1 + \tilde{b}_{2r} \dot{\tilde{b}}_{2r} + \tilde{b}_{2i} \dot{\tilde{b}}_{2i} + \tilde{b}_{3r} \dot{\tilde{b}}_{3r} + \tilde{b}_{3i} \dot{\tilde{b}}_{3i} + \tilde{b}_4 \dot{\tilde{b}}_4$$
$$= \sigma\{\tilde{b}_1(y_{2r} - y_{1r}) - D_{1r}\tilde{a}_1(x_{2r} - x_{1r}) - D_{1i}\tilde{a}_1(x_{2i} - x_{1i}) + e_{1i} + 4\tilde{b}_1(y_{2i} - y_{1i})$$
$$- 4D_{1r}\tilde{a}_1(x_{2i} - x_{1i}) - 4D_{1i}\tilde{a}_1(x_{2r} - x_{1r}) + 4e_{2r} + 6\tilde{b}_{2r} y_{1r} - 6\tilde{b}_{3r} y_{2r} - 6D_{2r}\tilde{a}_2 x_{2r}$$
$$- 6D_{2i}\tilde{a}_2 x_{2i} + 6e_{2i} + 4\tilde{b}_{2i} y_{1i} - 4\tilde{b}_{3i} y_{2i} - 4D_{2r}\tilde{a}_2 x_{2i} - 4D_{2i}\tilde{a}_2 x_{2r} + 4e - \tilde{b}_4 y_3$$
$$+ D_3\tilde{a}_3 x_3 + D_4\tilde{a}_4 x_4 + v_0 + v_s + \dot{z}\} + \tilde{a}_1 \dot{\tilde{a}}_1 + \tilde{a}_2 \dot{\tilde{a}}_2 + \tilde{a}_3 \dot{\tilde{a}}_3 + \tilde{a}_4 \dot{\tilde{a}}_4 + \tilde{b}_1 \dot{\tilde{b}}_1 + \tilde{b}_{2r} \dot{\tilde{b}}_{2r}$$
$$+ \tilde{b}_{2i} \dot{\tilde{b}}_{2i} + \tilde{b}_{3r} \dot{\tilde{b}}_{3r} + \tilde{b}_{3i} \dot{\tilde{b}}_{3i} + \tilde{b}_4 \dot{\tilde{b}}_4$$
$$= \sigma\{e_{1i} + 4e_{2r} + 6e_{2i} + 4e_3 + v_0 + v_s + \dot{z}\} + \tilde{a}_1 \{\dot{\tilde{a}}_1 - \sigma D_{1r}(x_{2r} - x_{1r}) - \sigma D_{1i}(x_{2i} - x_{1i})$$
$$- 4D_{1r}(x_{2i} - x_{1i}) - 4D_{1i1}(x_{2r} - x_{1r})\} + \tilde{a}_2 \{\dot{\tilde{a}}_2 - 6\sigma D_{2r} x_{2r} - 6\sigma D_{2i} x_{2i} - 4\sigma D_{2r} x_{2i}$$
$$- 4\sigma D_{2i} x_{2r}\} + \tilde{a}_3 \{\dot{\tilde{a}}_3 + \sigma D_3 x_3\} + \tilde{a}_4 \{\dot{\tilde{a}}_4 + \sigma D_4 x_4\} + \tilde{b}_1 \{\dot{\tilde{b}}_1 + \sigma(y_{2r} - y_{1r})$$
$$+ 4\sigma(y_{2i} - y_{1i})\} + \tilde{b}_{2r}\{\dot{\tilde{b}}_{2r} + 6\sigma y_{1r}\} + \tilde{b}_{2i}\{\dot{\tilde{b}}_{2i} + 4\sigma y_{1i}\} + \tilde{b}_{3r}\{\dot{\tilde{b}}_{3r} - 6\sigma y_{2r}\}$$
$$+ \tilde{b}_{3i}\{\dot{\tilde{b}}_{3i} - 4\sigma y_{2i}\} + \tilde{b}_4 \{\dot{\tilde{b}}_4 - \sigma y_3\}$$

By using

$$\dot{z} = -e_{1i} - 4e_{2r} - 6e_{2i} - 4e_3 - v_0,$$
$$v_s = -k\sigma - k\, sign(\sigma)$$
$$\dot{\tilde{a}}_1 = \sigma D_{1r}(x_{2r} - x_{1r}) + \sigma D_{1i}(x_{2i} - x_{1i}) + 4D_{1r}(x_{2i} - x_{1i})$$
$$+ 4D_{1i1}(x_{2r} - x_{1r}) - k_1\tilde{a}_1 , \quad \dot{\hat{a}}_1 = -\dot{\tilde{a}}_1$$
$$\dot{\tilde{a}}_2 = 6\sigma D_{2r} x_{2r} + 6\sigma D_{2i} x_{2i} + 4\sigma D_{2r} x_{2i}$$
$$+ 4\sigma D_{2i} x_{2r} - k_2\tilde{a}_{2r} , \quad \dot{\hat{a}}_2 = -\dot{\tilde{a}}_2$$
$$\dot{\tilde{a}}_3 = -\sigma D_3 x_3 - k_3\tilde{a}_{3r} , \quad \dot{\hat{a}}_3 = -\dot{\tilde{a}}_3$$
$$\dot{\tilde{a}}_4 = -\sigma D_4 x_4 - k_4\tilde{a}_4 , \quad \dot{\hat{a}}_4 = -\dot{\tilde{a}}_4$$
$$\dot{\tilde{b}}_1 = -\sigma(y_{2r} - y_{1r}) - 4\sigma(y_{2i} - y_{1i}) - k_5\tilde{b}_1, \quad \dot{\hat{b}}_1 = -\dot{\tilde{b}}_1$$
$$\dot{\tilde{b}}_{2r} = -6\sigma y_{1r} - k_6\tilde{b}_{2r}, \quad \dot{\hat{b}}_{2r} = -\dot{\tilde{b}}_{2r}$$
$$\dot{\tilde{b}}_{2i} = -4\sigma y_{1i} - k_7\tilde{b}_{2i}, \quad \dot{\hat{b}}_{2i} = -\dot{\tilde{b}}_{2i}$$
$$\dot{\tilde{b}}_{3r} = 6\sigma y_{2r} - k_8\tilde{b}_{3r}, \quad \dot{\hat{b}}_{3r} = -\dot{\tilde{b}}_{3r},$$
$$\dot{\tilde{b}}_{3i} = 4\sigma y_{2i} - k_9\tilde{b}_{3i}, \quad \dot{\hat{b}}_{3i} = -\dot{\tilde{b}}_{3i},$$
$$\dot{\tilde{b}}_4 = \sigma y_3 - k_{10}\tilde{b}_4, \quad \dot{\hat{b}}_4 = -\dot{\tilde{b}}_4, \quad k, k_i > 0, i = 1,\ldots,10$$

We have

$$\dot{V} = -k\sigma^2 - k|\sigma| - k_1\tilde{a}_1^2 - k_2\tilde{a}_2^2 - k_3\tilde{a}_3^2 - k_4\tilde{a}_4^2 - k_5\tilde{b}_1^2 - k_6\tilde{b}_{2r}^2 - k_7\tilde{b}_{2i}^2 - k_8\tilde{b}_{3r}^2 - k_9\tilde{b}_{3i}^2 - k_{10}\tilde{b}_4^2$$

From this we conclude that $\sigma, \tilde{a}_r, \tilde{a}_i, \tilde{b}_f \rightarrow 0$. Since $\sigma \rightarrow 0$, therefore

$e = (e_{1r}, e_{1i}, e_{2r}, e_{2i}, e_3) \rightarrow 0$.

In simulations, the initial conditions are chosen as: $x(0) = [10+5j, 10+6j, 2, 12]^T$ and $y(0) = [2+0.02j, 1+0.2j, -1]^T$. The true values of the unknown parameters are chosen as:

$a_1 = 42,\ a_2 = 25, a_3 = 6, a_4 = 10, b_1 = 20, b_2 = 60+0.02j, b_3 = 1-0.06, b_4 = 0.8$

Simulation results of Case 2:

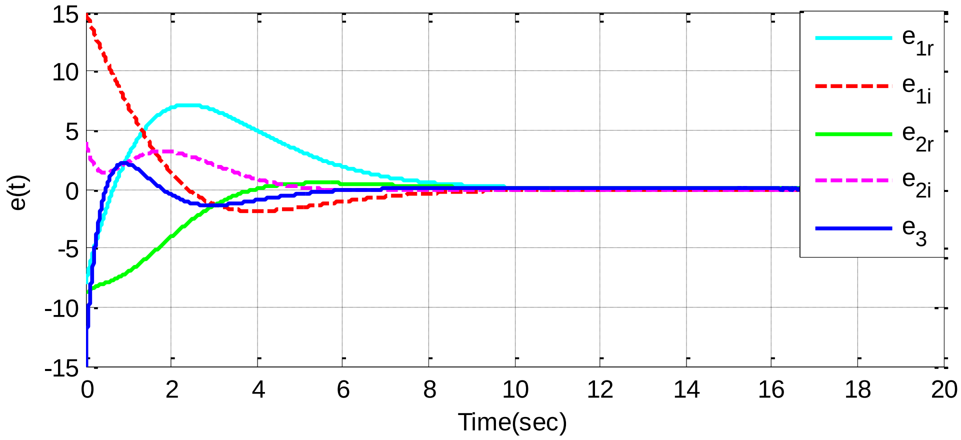

Figure 4.15: Time Response of error $e_{1r}, e_{1i}, e_{2r}, e_{2i}$ & e_3

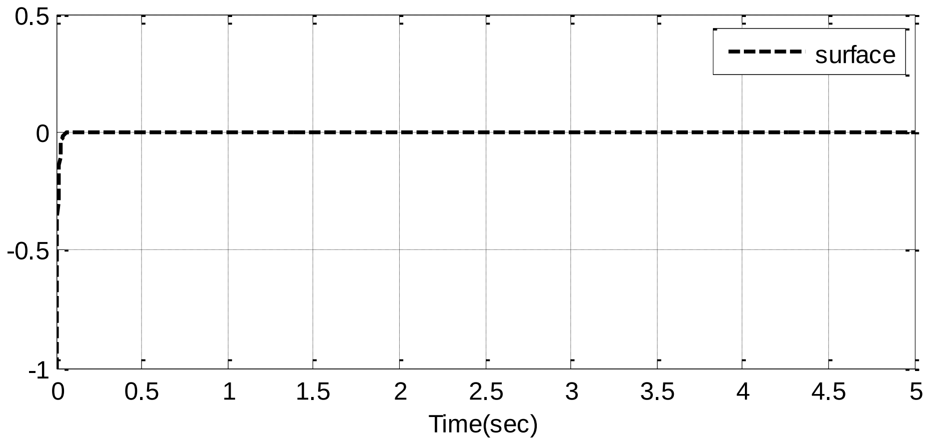

Figure 4.16: Time Response of surface

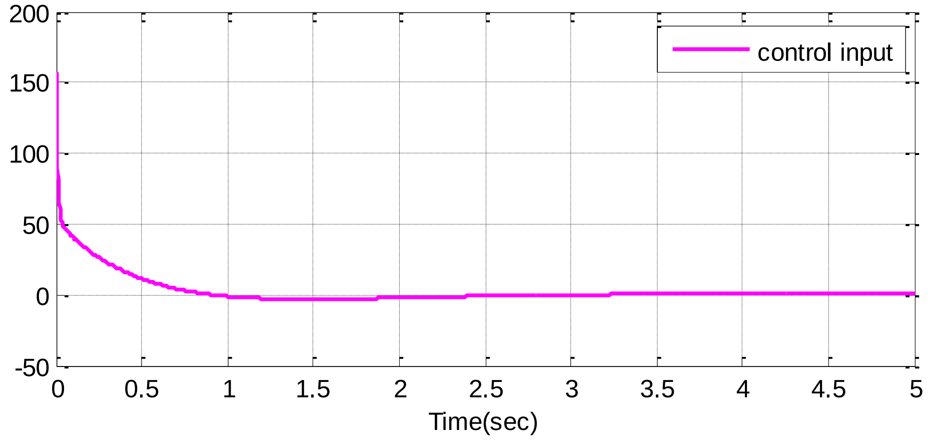

Figure 4.17: Time Response of control input

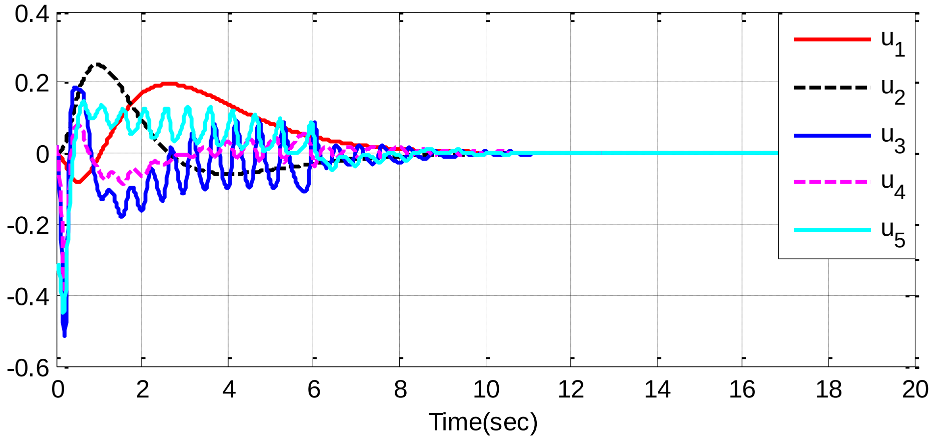

Figure 4.18: Time Response of adaptive controller u_1, u_2, u_3, u_4 & u_5

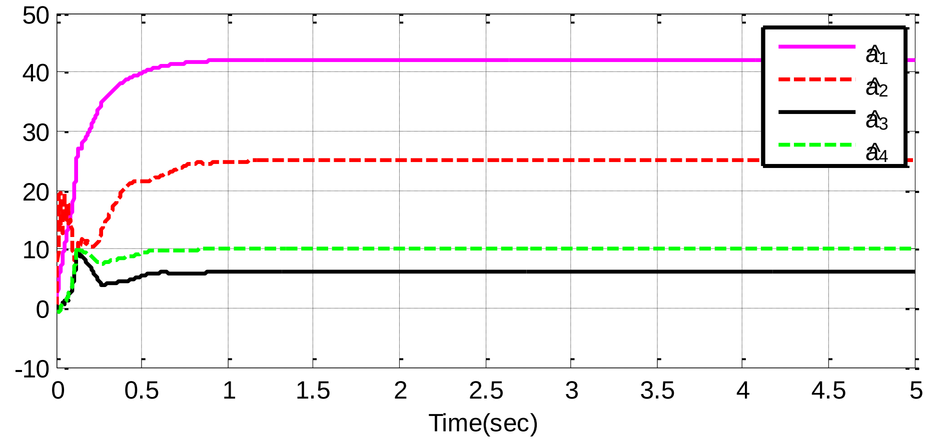

Figure 4.19: Estimation parameter of $\hat{a}_1, \hat{a}_2, \hat{a}_3$ & $\hat{a}_4$

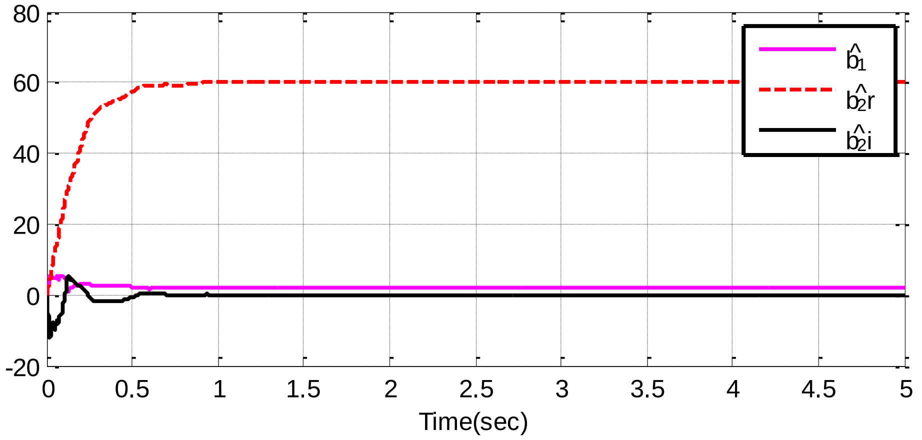

Figure 4.20: Estimation parameter of $\hat{b}_1, \hat{b}_{2r}$ & $\hat{b}_{2i}$

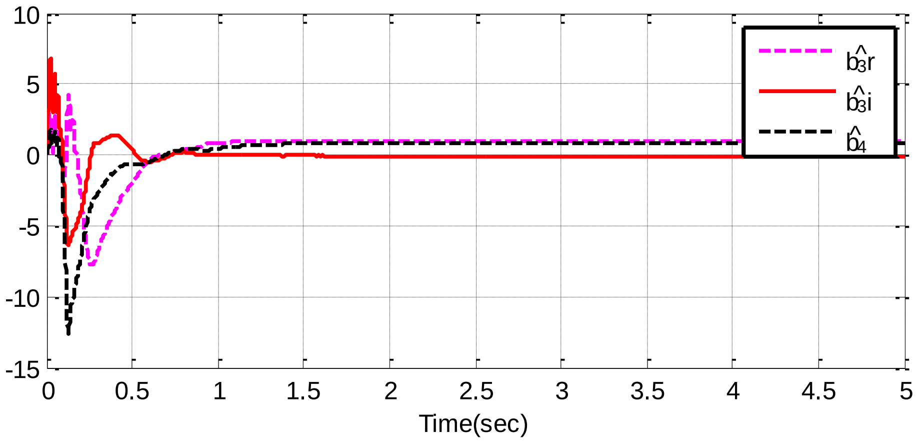

Figure 4.21: Estimation parameter of $\hat{b}_{3r}, \hat{b}_{3i}$ & $\hat{b}_4$

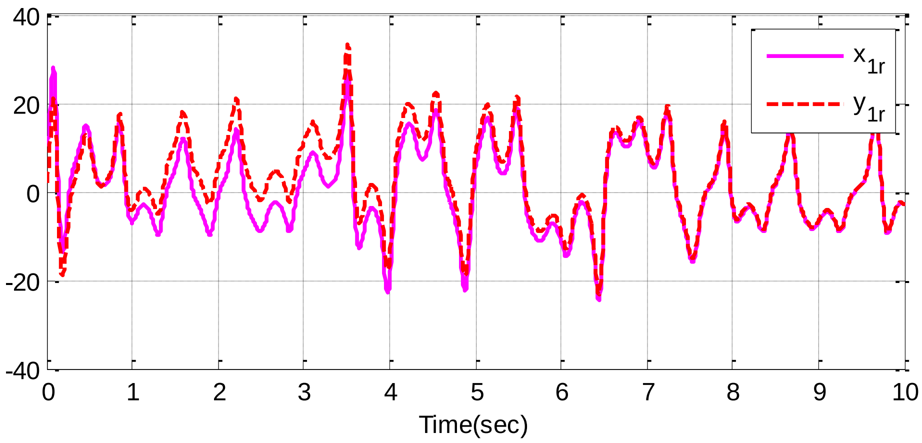

Figure 4.22: Time Response of x_{1r}&$y_{1r}$with IC (10, 2)

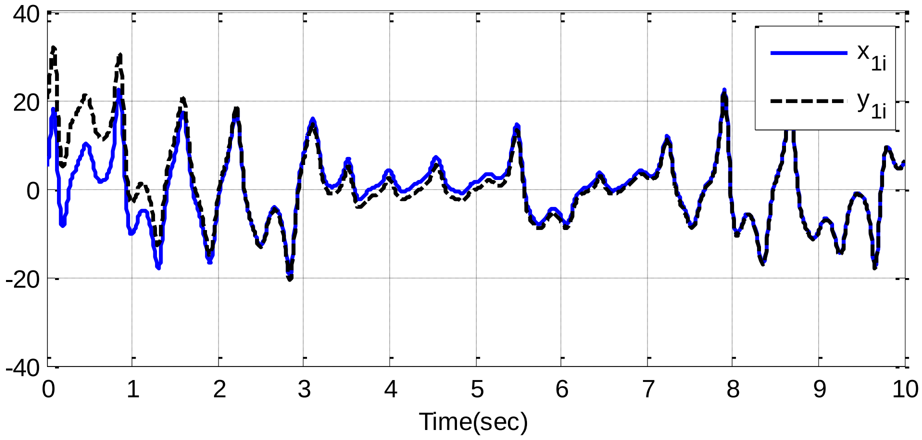

Figure 4.23: Time Response of x_{1i}&$y_{1i}$with IC (5, 20)

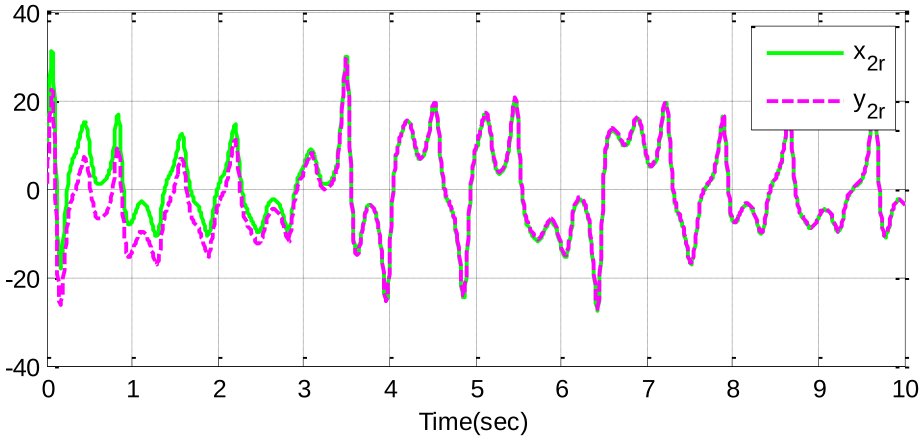

Figure 4.24: Time Response of x_{2r}&$y_{2r}$with IC (10, 1)

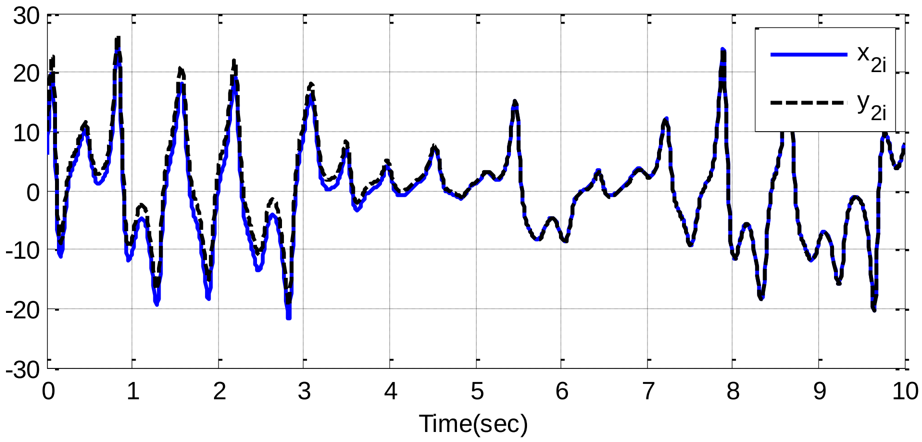

Figure 4.25: Time Response of x_{2i}&$y_{2i}$with IC (6, 10)

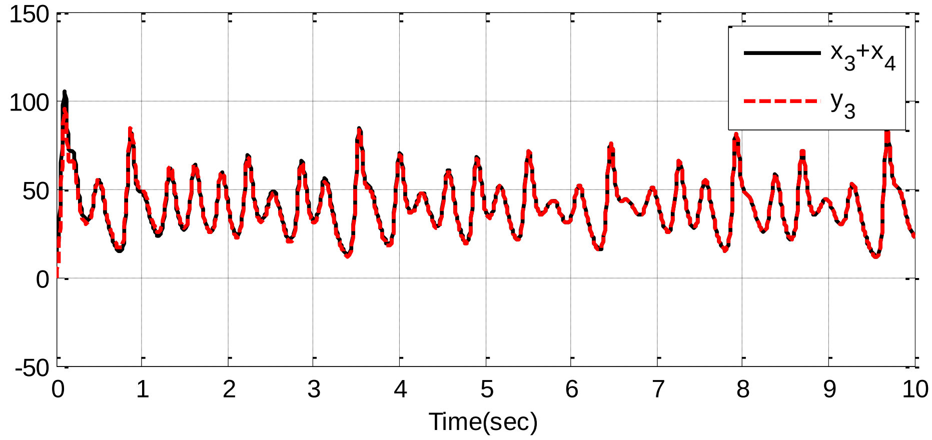

Figure 4.26: Time Response of $x_3 + x_4$&$y_3$with IC (2, 12, -1)

Chapter 5

Complex Generalized Synchronization (CGS)

5.1 Introduction

In this chapter we present the extended version of control design strategy proposed in the previous chapter to achieve Complex Generalized Synchronization (CGS) in complex chaotic systems with unknown parameters. The proposed design methodology is based on Adaptive Integral Sliding Mode Control. The proposed design approach is applied on synchronize a memristor-based hyperchaotic complex Lü system and a memristor-based chaotic complex Lorenz system, a chaotic complex Chen system and a memristorbased chaotic complex Lorenz system, as well as a memristor-based hyperchaotic complex Lü system and a chaotic complex Lü systems with fully unknown parameters.

5.2 Problem formulation

Consider the following non-identical drive and response complex systems with fully unknown parameters

$$\dot{x} = f(x) + F(x)\theta \qquad (5.1)$$

$$\dot{y} = g(y) + G(y)\vartheta + u \qquad (5.2)$$

$x = (x_1, x_2, ..., x_m)^T \in R^m$ and $y = (y_1, y_2, ..., y_n)^T \in R^n$ are complex state vector, and $u = (u_r + ju_i) \in R^n$ is the control input.

Some nonlinear complex systems can be formed as system (5.1), such as complex Lorenz system, complex Chen system, complex Lü system, memristor-based complex Lorenz system, memristor-based complex Lü system, and so on. For synchronizing such complex systems, the complex variables and functions could be divided into the real parts and imaginary parts.

Definition: For the drive system (5.1) and the response system (5.2), CGS is achieved if there exist

a complex controller $u(x, y)$ and a given complex map $\phi(x): C^m \to C^n$ such that

$$\lim_{t\to+\infty}\|e(t)\| = \lim_{t\to+\infty}\|y(t) - \phi(x)\| = 0 \qquad (5.3)$$

Where $\phi(x) = [\phi_1(x), \phi_2(x) \cdots \phi_n(x)]^T$ a nonzero complex map vector is whose elements are continuously differentiable complex functions of x, and $\phi(x) = \phi_r(x) + j\phi_i(x)$.

5.3 Proposed Algorithm for Complex Generalization Synchronization

Consider the Master system (5.4)

$$\dot{x} = f(x) + F(x)\theta \qquad (5.4)$$

Where $x = (x_1, x_2, ..., x_n)^T$ are complex states vectors of the drive system (5.4) $x_k = x_{kr} + jx_{ki}, k = 1,2,...,n$, $j = \sqrt{-1}$, the subscripts r and i denote the real and imaginary parts of the complex variables, $\theta \in \Re^p$, real vectors of known parameters. $F(x) \in C^{n\times p}$ are complex matrices, $F(x) = F_r(x) + jF_i(x)$, $f(x) \in C^n$ vectors of nonlinear complex functions, and $f(x) = f_r(x) + jf_i(x)$, is the complex control vector.

Consider the slave system as (5.5):

$$\dot{y} = g(y) + G(y)\vartheta + u(x, y) \qquad (5.5)$$

$y = (y_1, y_2, ..., y_m)^T$ are complex state vectors of the slave system (5.5) $y_l = y_{lr} + jy_{li}, l = 1,2,...,m$, $j = \sqrt{-1}$, the subscripts r and i denote the real and image parts of the complex variables $, \vartheta \in \Re^q$ are real vectors of unknown parameters. $G(y) \in C^{m\times q}$ are complex matrices, $G(x) = G_r(x) + jG_i(x)$. $g(y) \in C^m$ are vectors of nonlinear complex functions, $g(y) = g_r(y) + jg_i(y)$. $u(x, y) \in C^m$ is the complex control vector, and $u(x, y) = u_r(x, y) + ju_i(x, y)$.

For the drive system (5.4) and response system (5.5), Complex generalized synchronization (CGS) is achieved if there exist a complex controller $u(x, y)$) and a given complex map $\phi(x) : C^n \to C^m$ such that:

$$\lim_{t\to+\infty} \|e(t)\| = \lim_{t\to+\infty} \|y(t) - \phi(x)\| = 0 \qquad (5.6\ )$$

where $\| \quad \|$ represent the matrix norm, $\phi(x) = [\phi_2(x) \quad \phi_2(x) \quad \cdots \quad \phi_m(x)]^T$ is a nonzero complex map vector whose elements are continuously differentiable complex functions of x, and $\phi(x) = \phi_r(x) + j\phi_i(x)$.

Define the complex CGS error vector as

$$\begin{aligned} e &= y - \phi(x) = e_r + je_i \\ &= (y_r + jy_i) - (\phi_r(x) + j\phi_i(x)) \\ &= (y_r - \phi_r(x)) + j(y_i - \phi_i(x)) \end{aligned} \tag{5.7}$$

Where $e = (e_1, e_2, ..., e_m)^T \in C^m$ $e_r = (e_{1r}, e_{2r}, ..., e_{mr})^T \in \Re^m$ $e_i = (e_{1i}, e_{2i}, ..., e_{mi})^T \in \Re^m$

From equation (4) we have

$$\begin{aligned} e &= y - \phi(x) = e_r + je_i \\ &= (y_r + jy_i) - (\phi_r(x) + j\phi_i(x)) \\ &= (y_r - \phi_r(x)) + j(y_i - \phi_i(x)) \end{aligned} \tag{5.8}$$

Choose: $J = \dfrac{\partial \phi(x)}{\partial (x)}$

By taking the derivative of equation (5.8) with respect time, the CGS error dynamical system is obtained as:

$$\begin{aligned} \dot{e} &= \dot{e}_r + j\dot{e}_i = \dot{y} - \frac{\partial \phi(x)}{\partial x}\dot{x} = g(y) + G(y)\vartheta + u(x,y) - J\{f(x) + F(x)\theta\} \\ &= g_r(y) + G_r(y)\vartheta + u_r(x,y) - J_r\{f_r(x) + F_r(x)\theta\} + J_i\{f_i(x) + F_i(x)\theta\} \\ &+ \{g_i(y) + G_i(y)\vartheta + u_i(x,y) - J_i\{f_r(x) + F_r(x)\theta\} - J_r\{f_i(x) + F_i(x)\theta\}\} \end{aligned} \tag{5.9}$$

Or

$$\begin{aligned} \dot{e}_r &= g_r(y) + G_r(y)\vartheta + u_r(x,y) - J_r\{f_r(x) + F_r(x)\theta\} + J_i\{f_i(x) + F_i(x)\theta\}\} \\ \dot{e}_i &= g_i(y) + G_i(y)\vartheta + u_i(x,y) - J_i\{f_r(x) + F_r(x)\theta\} - J_r\{f_i(x) + F_i(x)\theta\} \end{aligned} \tag{5.10}$$

Let $\hat{\theta}, \hat{\vartheta}$ be estimate of θ, ϑ respectively and let $\tilde{\theta} = \theta - \hat{\theta}, \tilde{\vartheta} = \vartheta - \hat{\vartheta}$ be error in estimating θ, ϑ respectively. Then error system (5.10) becomes:

$$\begin{aligned} \dot{e}_r &= g_r(y) + G_r(y)\hat{\vartheta} + G_r(y)\tilde{\vartheta} + u_r(x,y) - J_r\{f_r(x) + F_r(x)\hat{\theta} + F_r(x)\tilde{\theta}\} \\ &+ J_i\{f_i(x) + F_i(x)\hat{\theta} + F_i(x)\tilde{\theta}\}\} \\ \dot{e}_i &= g_i(y) + G_i(y)\hat{\vartheta} + G_i(y)\tilde{\vartheta} + u_i(x,y) - J_i\{f_r(x) + F_r(x)\hat{\theta} + F_r(x)\tilde{\theta}\} \\ &- J_r\{f_i(x) + F_i(x)\hat{\theta} + F_i(x)\tilde{\theta}\} \end{aligned} \tag{5.11}$$

That can be written in vector form:

$$\begin{bmatrix} \dot{e}_r \\ \dot{e}_i \end{bmatrix} = \begin{bmatrix} g_r(y) + G_r(y)\hat{\vartheta} - J_r\{f_r(x) + F_r(x)\hat{\theta}\} + J_i\{f_i(x) + F_i(x)\hat{\theta}\}\} \\ g_i(y) + G_i(y)\hat{\vartheta} - J_i\{f_r(x) + F_r(x)\hat{\theta}\} + J_i\{f_i(x) + F_i(x)\hat{\theta}\}\} \end{bmatrix}$$
$$+ \begin{bmatrix} u_r(x,y) \\ u_i(x,y) \end{bmatrix} + \begin{bmatrix} G_r(y)\tilde{\vartheta} - J_r F_r(x)\tilde{\theta} - J_i F_i(x)\tilde{\theta} \\ G_i(y)\tilde{\vartheta} - J_i F_r(x)\tilde{\theta} - J_r F_i(x)\tilde{\theta} \end{bmatrix} \tag{5.12}$$

By choosing

$$\begin{bmatrix} u_r(x,y) \\ u_i(x,y) \end{bmatrix} = \begin{bmatrix} ee_r \\ ee_i \end{bmatrix} - \begin{bmatrix} g_r(y) + G_r(y)\hat{\vartheta} - J_r\{f_r(x) + F_r(x)\hat{\theta}\} \\ + J_i\{f_i(x) + F_i(x)\hat{\theta}\}\} \\ g_i(y) + G_i(y)\hat{\vartheta} - J_i\{f_r(x) + F_r(x)\hat{\theta}\} \\ + J_i\{f_i(x) + F_i(x)\hat{\theta}\}\} \end{bmatrix} \tag{5.13}$$

Where $ee_r = \begin{bmatrix} e_{2r} \\ e_{3r} \\ \vdots \\ e_{nr} \\ e_{1i} \end{bmatrix}, ee_i = \begin{bmatrix} e_{2i} \\ e_{3i} \\ \vdots \\ e_{ni} \\ v \end{bmatrix}$

v is the new input vector, then system (5.12) becomes:

$$\begin{bmatrix} \dot{e}_r \\ \dot{e}_i \end{bmatrix} = \begin{bmatrix} ee_r \\ ee_i \end{bmatrix} + \begin{bmatrix} G_r(y)\tilde{\vartheta} - J_r F_r(x)\tilde{\theta} - J_i F_i(x)\tilde{\theta} \\ G_i(y)\tilde{\vartheta} - J_i F_r(x)\tilde{\theta} - J_r F_i(x)\tilde{\theta} \end{bmatrix}$$

Or

$$\begin{bmatrix} \dot{e}_{1r} \\ \dot{e}_{2r} \\ \vdots \\ \dot{e}_{nr} \\ \dot{e}_{1i} \\ \dot{e}_{2i} \\ \vdots \\ \dot{e}_{ni} \end{bmatrix} = \begin{bmatrix} e_{2r} \\ \vdots \\ e_{nr} \\ e_{1i} \\ e_{2i} \\ \vdots \\ e_{ni} \\ v \end{bmatrix} + \begin{bmatrix} G_r(y)\tilde{\vartheta} - J_r F_r(x)\tilde{\theta} - J_i F_i(x)\tilde{\theta} \\ G_i(y)\tilde{\vartheta} - J_i F_r(x)\tilde{\theta} - J_r F_i(x)\tilde{\theta} \end{bmatrix} \tag{5.14}$$

To employ the integral sliding mode control, choose the nominal system for (5.14) as:

$$\begin{bmatrix} \dot{e}_{1r} \\ \dot{e}_{2r} \\ \vdots \\ \dot{e}_{nr} \\ \dot{e}_{1i} \\ \dot{e}_{2i} \\ \vdots \\ \dot{e}_{ni} \end{bmatrix} = \begin{bmatrix} e_{2r} \\ \vdots \\ e_{nr} \\ e_{1i} \\ e_{2i} \\ \vdots \\ e_{ni} \\ v_0 \end{bmatrix} \qquad (5.15)$$

Define the Hurwitz sliding surface for nominal system (5.15) as: $\sigma_0 = C[e_r + e_i]^T$

$$\sigma_0 = C[e_r + e_i]^T = e_{1r} + \sum_{k=2}^{n-1} c_k e_{kr} + \sum_{k=1}^{n-1} c_{(n+k)} e_{ki}$$

$C = [1, c_1, \ldots, c_{n-1}, c_n, \ldots, c_{2n-1}, 1]$ is chosen in such a way that σ_0 becomes Hurwitz polynomial.

$$\dot{\sigma}_0 = C[\dot{e}_r + \dot{e}_r]^T$$

$$\dot{\sigma}_0 = e_{2r} + \sum_{k=2}^{n-1} c_k e_{(k+1)r} + \sum_{k=0}^{n-1} c_{(n+k)} e_{(k+1)i} + v_o$$

By choosing $v_o = -e_{2r} - \sum_{k=2}^{n-1} c_k e_{(k+1)r} - \sum_{k=0}^{n-1} c_{(n+k)} e_{(k+1)i} - k\,\sigma_0 - k\, sign(\sigma_0),\ k > 0$, we have $\dot{\sigma}_0 = -k\,\sigma_0 - k\, sign(\sigma_0)$. Therefore the nominal system (5.15) is asymptotically stable.

Now choose the sliding surface for the system (5.14) as:

$$\sigma = \sigma_0 + z = Ce + z$$
$$\sigma = C_1 e_r + C_2 e_i + z$$

Where z is some integral term computed later. To avoid the reaching phase, choose $z(0)$ such that $\sigma(0) = 0$. Choose $v = v_0 + v_s$ where, v_0 is the nominal input and v_s is compensator term computed later. Then

$$
\begin{aligned}
&\dot{\sigma} = C_1\dot{e}_r + C_2\dot{e}_i + \dot{z} \\
&\dot{\sigma} = e_{2r} + \sum_{k=2}^{n-1} c_k e_{(k+1)r} + \sum_{k=0}^{n-1} c_{(n+k)} e_{(k+1)i} + v_o + v_s + \dot{z} \\
&+ C_1 G_r(y)\tilde{\vartheta} + C_2 G_i(y)\tilde{\vartheta} - C_1\{ j_r F_r(x)\tilde{\theta} + j_i F_i(x)\tilde{\theta}\} \\
&- C_2\{ j_i C_2 F_r(x)\tilde{\theta} + j_r F_i(x)\tilde{\theta}\} \\
&= e_{2r} + \sum_{k=2}^{n-1} c_k e_{(k+1)r} + \sum_{k=0}^{n-1} c_{(n+k)} e_{(k+1)i} + v_o + v_s + \dot{z} \\
&+ C_1 G_r(y)\tilde{\vartheta} - j_r C_1 F_r(x)\tilde{\theta} - j_i C_1 F_i(x)\tilde{\theta} \\
&+ C_2 G_i(y)\tilde{\vartheta} - j_i C_2 F_r(x)\tilde{\theta} - j_r C_2 F_i(x)\tilde{\theta}
\end{aligned} \tag{5.16}
$$

$$
C_1 = [1, c_1, \ldots, c_{n-1}, c_n], C_2 = [c_{n+1}, c_{n+2}, \ldots, c_{2n-1}, 1]
$$

By choosing a Lyapunov function: $V = \frac{1}{2}\sigma^2 + \frac{1}{2}\tilde{\theta}^T\tilde{\theta} + \frac{1}{2}\tilde{\vartheta}^T\tilde{\vartheta}$, design the adaptive laws for $\tilde{\theta}, \hat{\theta}, \tilde{\vartheta}, \hat{\vartheta}$ and compute v_s such that $\dot{V} < 0$.

$$
\begin{aligned}
&\dot{z} = -e_{2r} - \sum_{k=2}^{n-1} c_k e_{(k+1)r} - \sum_{k=0}^{n-1} c_{(n+k)} e_{(k+1)i} + v_o, \quad v_s = -k\sigma - k\, sign(\sigma) \\
&\dot{\tilde{\theta}} = \sigma\{ j_r F_r(x)^T C_1^T + j_i F_i(x)^T C_1^T + j_{1r} F_r(x)^T C_2^T + j_r F_i(x)^T C_2^T\} - k_1\tilde{\theta} \\
&\dot{\tilde{\vartheta}} = -\sigma G_r(y)^T C_1^T - \sigma G_i(y)^T C_2^T - k_2\tilde{\vartheta} \quad \text{, where, } k, k_1, k_2 > 0
\end{aligned} \tag{5.17}
$$

Proof:

Since

$$
\begin{aligned}
&\dot{V} = \sigma\dot{\sigma} + \tilde{\theta}^T\dot{\tilde{\theta}} + \tilde{\vartheta}^T\dot{\tilde{\vartheta}} \\
&= \sigma\{e_{2r} + \sum_{k=2}^{n-1} c_k e_{(k+1)r} + \sum_{k=0}^{n-1} c_{(n+k)} e_{(k+1)i} + v_o + v_s + \dot{z} \\
&+ C_1 G_r(y)\tilde{\vartheta} - j_r C_1 F_r(x)\tilde{\theta} - j_i C_1 F_i(x)\tilde{\theta} \\
&+ C_2 G_i(y)\tilde{\vartheta} - j_i C_2 F_r(x)\tilde{\theta} - j_r C_2 F_i(x)\tilde{\theta} \\
&+ \tilde{\theta}^T\dot{\tilde{\theta}} + \tilde{\vartheta}^T\dot{\tilde{\vartheta}} \\
&= \sigma\{e_{2r} + \sum_{k=2}^{n-1} c_k e_{(k+1)r} + \sum_{k=0}^{n-1} c_{(n+k)} e_{(k+1)i} + v_o + v_s + \dot{z}\} \\
&+ \tilde{\theta}^T\{\dot{\tilde{\theta}} - \sigma\{(j_r F_r(x)^T C_1^T + j_i F_i(x)^T C_1^T + j_{1r} F_r(x)^T C_2^T + j_r F_i(x)^T C_2^T\}\} \\
&+ \tilde{\vartheta}^T\{\dot{\tilde{\vartheta}} + \sigma G_r(y)^T C_1^T + \sigma G_i(y)^T C_2^T\}
\end{aligned}
$$

By using

$$\dot{z} = -e_{2r} - \sum_{k=2}^{n-1} c_k e_{(k+1)r} - \sum_{k=0}^{n-1} c_{(n+k)} e_{(k+1)i} + v_o, \quad v_s = -k\sigma - k\,sign(\sigma)$$
$$\dot{\tilde{\theta}} = \sigma(j_r F_r(x)^T C_1^T + j_i F_i(x)^T C_1^T + j_{1r} F_r(x)^T C_2^T + j_r F_i(x)^T C_2^T - k_1\tilde{\theta}$$
$$\dot{\tilde{\vartheta}} = -\sigma G_r(y)^T C_1^T - \sigma G_i(y)^T C_2^T - k_2\tilde{\vartheta} \quad \text{, where, } k, k_1, k_2 > 0$$

We have

$$\dot{V} = -k\sigma^2 - k|\sigma| - k_1\tilde{\theta}^T\tilde{\theta} - k_3\tilde{\vartheta}^T\tilde{\vartheta}.$$

From this we conclude that $\sigma, \tilde{\theta}_r, \tilde{\theta}_i, \tilde{\vartheta}_r, \tilde{\vartheta}_i \to 0$. Since $\sigma \to 0$, therefore $e_r, e_i \to 0$.

5.4 Numerical Example

The following example is taken from [49], where CPS problem was solved by adaptive control scheme. We have achieved CCS using adaptive integral sliding mode control.

Case1 when *n*=*m*:

Consider the Master system given in [49] as:

$$\begin{aligned}
\dot{x}_1 &= a_1(x_2 - x_1) \\
\dot{x}_2 &= -x_1x_3 + a_2x_2 - a_3(\alpha_1 + 3\beta_1x_4^2)x_1 \\
\dot{x}_3 &= 0.5(\bar{x}_1 x_2 + x_1\bar{x}_2) - a_4x_3 \\
\dot{x}_4 &= 0.5(\bar{x}_1 + x_1)
\end{aligned} \tag{5.18}$$

Where, $x_1 = x_{1r} + jx_{1i}$, $x_2 = x_{2r} + jx_{2i}$ are complex and $x_3 = x_{3r}$, $x_4 = x_{4r}$ are real. $\bar{x}_1, \bar{x}_2$ denote the complex conjugate variables of x_1, x_2. a_1, a_2, a_3 and a_4 are unknown real parameters α_1 and β_1 are considered as known positive constants. When $\alpha_1 = 4$, $\beta_1 = 0.01$, $a_1 = 36$, $a_2 = 20$, $a_3 = 3.2$, $a_4 = 3$ and $x(0) = [-1+2j, 1+j, 2, -1]^T$, and a hyperchaotic attractor is plotted in Fig1.

The Slave system given in [49] as:

$$\begin{aligned}
\dot{y}_1 &= -b_1y_1 + b_2y_2 - (\alpha_2 + 3\beta_2y_4^2)y_1 + u_1 \\
\dot{y}_2 &= b_3y_1 - y_2 - y_1y_3 + u_2 \\
\dot{y}_3 &= 0.5(\bar{y}_1 y_2 + y_1\bar{y}_2) - b_4y_3 + u_3 \\
\dot{y}_4 &= -0.5(\bar{y}_1 + y_1) + u_4
\end{aligned} \tag{5.19}$$

Where, $y_1 = y_{1r} + jy_{1i}$, $y_2 = y_{2r} + jy_{2i}$ are complex and $y_3 = y_{3r}$, $y_4 = y_{4r}$ are real. $\bar{y}_1, \bar{y}_2$ denote the complex conjugate variables of y_1, y_2. b_1, b_2, b_3 and b_4 are unknown real parameters α_2 and β_2 are considered as known positive constants. u_1, u_2, u_3 and u_4 arecontrollers.When $\alpha_2 = 0.67\times 10^{-3}$, $\beta_2 = 0.02\times 10^{-3}$,and $b_1 = 8, b_2 = 11, b_3 = 50, b_4 = 8/3$ $y(0) = [2,\ 1+4j,\ 0.1, 0]^T$, the system (5.19) operates in chaotic orbits without control, as shown in Fig 2.

In this section, we investigate CGS of two nonidentical complex systems with the same orders.

Let $\hat{a}_i, \hat{b}_i, i = 1,..,4$ be estimates of $a_i, b_i, i = 1,..,4$

and $\tilde{a}_i = a_i - \hat{a}_i, \tilde{b}_i = b_i - \hat{b}_i, i = 1,....,4$, be the errors in estimations of $a_i, b_i, i = 1,..,4$, respectively. Then systems (5.18) and (5.19) can be written as:

$$\begin{aligned}
\dot{x}_1 &= \hat{a}_1(x_2 - x_1) + \tilde{a}_1(x_2 - x_1) \\
\dot{x}_2 &= -x_1 x_3 + \hat{a}_2 x_2 + \tilde{a}_2 x_2 - \hat{a}_3(\alpha_1 + 3\beta_1 x_4^2)x_1 - \tilde{a}_3(\alpha_1 + 3\beta_1 x_4^2)x_1 \\
\dot{x}_3 &= 0.5(\bar{x}_1 x_2 + x_1 \bar{x}_2) - \hat{a}_4 x_3 - \tilde{a}_4 x_3 \\
\dot{x}_4 &= 0.5(\bar{x}_1 + x_1)
\end{aligned} \tag{5.20}$$

$$\begin{aligned}
\dot{y}_1 &= \hat{b}_1 y_1 + \tilde{b}_1 y_1 + \hat{b}_2 y_2 + \tilde{b}_2 y_2 - (\alpha_2 + 3\beta_2 y_4^2)y_1 + u_1 \\
\dot{y}_2 &= \hat{b}_3 y_1 + \tilde{b}_3 y_1 - y_2 - y_1 y_3 + u_2 \\
\dot{y}_3 &= 0.5(\bar{y}_1 y_2 + y_1 \bar{y}_2) - \hat{b}_4 y_3 - \tilde{b}_4 y_3 + u_3 \\
\dot{y}_4 &= -0.5(\bar{y}_1 + y_1) + u_4
\end{aligned} \tag{5.21}$$

The 4-dimensional complex systems (5.20)-(5.21) can be into 6-dimensional real systems:

$$\begin{aligned}
\dot{x}_{1r} &= \hat{a}_1(x_{2r} - x_{1r}) + \tilde{a}_1(x_{2r} - x_{1r}) \\
\dot{x}_{1i} &= \hat{a}_1(x_{2i} - x_{1i}) + \tilde{a}_1(x_{2i} - x_{1i}) \\
\dot{x}_{2r} &= -x_{1r} x_3 + \hat{a}_2 x_{2r} + \tilde{a}_2 x_{2r} - \hat{a}_3(\alpha_1 + 3\beta_1 x_4^2)x_{1r} - \tilde{a}_3(\alpha_1 + 3\beta_1 x_4^2)x_{1r} \\
\dot{x}_{2i} &= -x_{1i} x_3 + \hat{a}_2 x_{2i} + \tilde{a}_2 x_{2i} - \hat{a}_3(\alpha_1 + 3\beta_1 x_4^2)x_{1i} - \tilde{a}_3(\alpha_1 + 3\beta_1 x_4^2)x_{1i} \\
\dot{x}_3 &= (x_{1r} x_{2r} + x_{1i} x_{2i}) - \hat{a}_4 x_3 - \tilde{a}_4 x_3 \\
\dot{x}_4 &= x_{1r}
\end{aligned} \tag{5.22}$$

$$\begin{aligned}
\dot{y}_{1r} &= \hat{b}_1 y_{1r} + \tilde{b}_1 y_{1r} + \hat{b}_2 y_{2r} + \tilde{b}_2 y_{2r} - (\alpha_2 + 3\beta_2 y_4^2) y_{1r} + u_{r1} \\
\dot{y}_{1i} &= \hat{b}_1 y_{1i} + \tilde{b}_1 y_{1i} + \hat{b}_2 y_{2i} + \tilde{b}_2 y_{2i} - (\alpha_2 + 3\beta_2 y_4^2) y_{1j} + u_{1j} \\
\dot{y}_{2r} &= \hat{b}_3 y_{1r} + \tilde{b}_3 y_{1r} - y_{2r} - y_{1r} y_3 + u_{2r} \\
\dot{y}_{2i} &= \hat{b}_3 y_{1i} + \tilde{b}_3 y_{1i} - y_{2i} - y_{1i} y_3 + u_{2i} \\
\dot{y}_3 &= y_{1r}\, y_{2r} + y_{1i} y_{2i} - \hat{b}_4 y_3 - \tilde{b}_4 y_3 + u_3 \\
\dot{y}_4 &= -y_{1r} + u_4
\end{aligned} \qquad (5.23)$$

The complex map vector is given by

$$\phi(x) = \begin{bmatrix} x_1 + jx_2 & 2x_2 - jx_2 & x_3 + x_4 & x_4^2 \end{bmatrix}^T \qquad (5.24)$$

This gives:

$$\begin{aligned}
&\phi_{1r}(x) = x_{1r} - x_{2i}\,, \phi_{1i}(x) = x_{1i} + x_{2i}\,, \phi_{2r}(x) = 2x_{2r} + x_{2i}\,, \\
&\phi_{2i}(x) = 2x_{2i} - x_{2r}\,, \phi_3(x) = x_3 + x_4\,, \phi_4(x) = x_4^2
\end{aligned} \qquad (5.25)$$

The error signals are defined as:

$$\begin{aligned}
&e_{1r} = y_{1r} - \phi_{1r}(x), e_{1i} = y_{1i} - \phi_{1i}(x), e_{2r} = y_{2r} - \phi_{2i}(x), \\
&e_{2i} = y_{2i} - \phi_{2i}(x), e_3 = y_3 - \phi_3(x), e_4 = y_4 - \phi_4(x)
\end{aligned} \qquad (5.26)$$

Then the error dynamics becomes:

$$\begin{aligned}
&\dot{e}_{1r} = \dot{y}_{1r} - \dot{\phi}_{1r} = \dot{y}_{1r} - (\dot{x}_{1r} - \dot{x}_{2i}) = \hat{b}_1 y_{1r} + \tilde{b}_1 y_{1r} + \hat{b}_2 y_{2r} \\
&+ \tilde{b}_2 y_{2r} - (\alpha_2 + 3\beta_2 y_4^2) y_{1r} + u_{r1} - \hat{a}_1(x_{2r} - x_{1r}) - \tilde{a}_1(x_{2r} - x_{1r}) \\
&- x_{1r} x_3 + \hat{a}_2 x_{2r} + \tilde{a}_2 x_{2r} - \hat{a}_3(\alpha_1 + 3\beta_1 x_4^2) x_{1r} - \tilde{a}_3(\alpha_1 + 3\beta_1 x_4^2) x_{1r} \\
&\dot{e}_{1i} = \dot{y}_{1i} - \dot{\phi}_{1i} = \dot{y}_{1i} - (\dot{x}_{1i} + \dot{x}_{2i}) = \hat{b}_1 y_{1i} + \tilde{b}_1 y_{1i} + \hat{b}_2 y_{2i} + \tilde{b}_2 y_{2i} \\
&- (\alpha_2 + 3\beta_2 y_4^2) y_{1j} + u_{1i} - \hat{a}_1(x_{2i} - x_{1i}) - \tilde{a}_1(x_{2i} - x_{1i}) + x_{1i} x_3 - \hat{a}_2 x_{2i} \\
&- \tilde{a}_2 x_{2i} + \hat{a}_3(\alpha_1 + 3\beta_1 x_4^2) x_{1i} + \tilde{a}_3(\alpha_1 + 3\beta_1 x_4^2) x_{1i} \\
&\dot{e}_{2r} = \dot{y}_{2r} - \dot{\phi}_{2r} = \dot{y}_{2r} - (2\dot{x}_{2r} + \dot{x}_{2i}) = \hat{b}_3 y_{1r} + \tilde{b}_3 y_{1r} - y_{2r} - y_{1r} y_3 + u_{2r} \\
&+ 2x_{1r} x_3 - 2\hat{a}_2 x_{2r} - 2\tilde{a}_2 x_{2r} + 2\hat{a}_3(\alpha_1 + 3\beta_1 x_4^2) x_{1r} + 2\tilde{a}_3(\alpha_1 + 3\beta_1 x_4^2) x_{1r} \\
&+ x_{1i} x_3 - \hat{a}_2 x_{2i} - \tilde{a}_2 x_{2i} + \hat{a}_3(\alpha_1 + 3\beta_1 x_4^2) x_{1i} + \tilde{a}_3(\alpha_1 + 3\beta_1 x_4^2) x_{1i} \\
&\dot{e}_{2i} = \dot{y}_{2i} - \dot{\phi}_{2i} = \dot{y}_{2i} - (2\dot{x}_{2i} - \dot{x}_{2r}) = \hat{b}_3 y_{1i} + \tilde{b}_3 y_{1i} - y_{2i} - y_{1i} y_3 + u_{2i} \\
&+ 2x_{1i} x_3 - 2\hat{a}_2 x_{2i} - 2\tilde{a}_2 x_{2i} + 2\hat{a}_3(\alpha_1 + 3\beta_1 x_4^2) x_{1i} + 2\tilde{a}_3(\alpha_1 + 3\beta_1 x_4^2) x_{1i} \\
&- x_{1r} x_3 + \hat{a}_2 x_{2r} + \tilde{a}_2 x_{2r} - \hat{a}_3(\alpha_1 + 3\beta_1 x_4^2) x_{1r} - \tilde{a}_3(\alpha_1 + 3\beta_1 x_4^2) x_{1r} \\
&\dot{e}_3 = \dot{y}_3 - \dot{\phi}_3 = \dot{y}_3 - (\dot{x}_3 + \dot{x}_4) = y_{1r}\, y_{2r} + y_{1i} y_{2i} - \hat{b}_4 y_3 - \tilde{b}_4 y_3 + u_3 \\
&- x_{1r}\, x_{2r} - x_{1i} x_{2i} + \hat{a}_4 x_3 + \tilde{a}_4 x_3 - x_{1r} \\
&\dot{e}_4 = \dot{y}_4 - \dot{\phi}_4 = \dot{y}_4 - 2x_4 \dot{x}_4 = -y_{1r} + u_4 + 2x_4 x_{1r}
\end{aligned} \qquad (5.27)$$

By choosing

$$
\begin{aligned}
u_{r1} &= e_{1i} - \hat{b}_1 y_{1r} - \hat{b}_2 y_{2r} + (\alpha_2 + 3\beta_2 y_4^2) y_{1r} + \hat{a}_1 (x_{2r} - x_{1r}) \\
&+ x_{1r} x_3 - \hat{a}_2 x_{2r} + \hat{a}_3 (\alpha_1 + 3\beta_1 x_4^2) x_{1r} \\
u_{1i} &= e_{2r} - \hat{b}_1 y_{1i} - \hat{b}_2 y_{2i} + (\alpha_2 + 3\beta_2 y_4^2) y_{1j} + \hat{a}_1 (x_{2i} - x_{1i}) \\
&- x_{1i} x_3 + \hat{a}_2 x_{2i} - \hat{a}_3 (\alpha_1 + 3\beta_1 x_4^2) x_{1i} \\
u_{2r} &= e_{2i} - \hat{b}_3 y_{1r} + y_{2r} + y_{1r} y_3 - 2x_{1r} x_3 + 2\hat{a}_2 x_{2r} - 2\hat{a}_3 (\alpha_1 \\
&+ 3\beta_1 x_4^2) x_{1r} - x_{1i} x_3 + \hat{a}_2 x_{2i} - \hat{a}_3 (\alpha_1 + 3\beta_1 x_4^2) x_{1i} \\
u_{2i} &= e_3 - \hat{b}_3 y_{1i} + y_{2i} + y_{1i} y_3 - 2x_{1i} x_3 + 2\hat{a}_2 x_{2i} - 2\hat{a}_3 (\alpha_1 \\
&+ 3\beta_1 x_4^2) x_{1i} + x_{1r} x_3 - \hat{a}_2 x_{2r} + \hat{a}_3 (\alpha_1 + 3\beta_1 x_4^2) x_{1r} \\
u_3 &= e_4 - y_{1r}\, y_{2r} - y_{1i} y_{2i} + \hat{b}_4 y_3 + x_{1r}\, x_{2r} + x_{1i} x_{2i} - \hat{a}_4 x_3 + x_{1r} \\
u_4 &= v + y_{1r} - 2x_4 x_{1r}
\end{aligned}
\qquad (5.28)
$$

Where v is the new input, the system (5.27) can be written as:

$$
\begin{aligned}
\dot{e}_{1r} &= e_{1i} + \tilde{b}_1 y_{1r} + \tilde{b}_2 y_{2r} - \tilde{a}_1 (x_{2r} - x_{1r}) + \tilde{a}_2 x_{2r} - \tilde{a}_3 (\alpha_1 + 3\beta_1 x_4^2) x_{1r} \\
\dot{e}_{1i} &= e_{2r} + \tilde{b}_1 y_{1i} + \tilde{b}_2 y_{2i} - \tilde{a}_1 (x_{2i} - x_{1i}) - \tilde{a}_2 x_{2i} + \tilde{a}_3 (\alpha_1 + 3\beta_1 x_4^2) x_{1i} \\
\dot{e}_{2r} &= e_{2i} + \tilde{b}_3 y_{1r} - 2\tilde{a}_2 x_{2r} + 2\tilde{a}_3 (\alpha_1 + 3\beta_1 x_4^2) x_{1r} - \tilde{a}_2 x_{2i} + \tilde{a}_3 (\alpha_1 + 3\beta_1 x_4^2) x_{1i} \\
\dot{e}_{2i} &= e_3 + \tilde{b}_3 y_{1i} - 2\tilde{a}_2 x_{2i} + 2\tilde{a}_3 (\alpha_1 + 3\beta_1 x_4^2) x_{1i} + \tilde{a}_2 x_{2r} - \tilde{a}_3 (\alpha_1 + 3\beta_1 x_4^2) x_{1r} \\
\dot{e}_3 &= e_4 - \tilde{b}_4 y_3 + \tilde{a}_4 x_3 \\
\dot{e}_4 &= v
\end{aligned}
\qquad (5.29)
$$

To employ the integral sliding mode control, choose the nominal system for (5.29) as:

$$
\begin{aligned}
\dot{e}_{1r} &= e_{1i} \\
\dot{e}_{1i} &= e_{2r} \\
\dot{e}_{2r} &= e_{2i} \\
\dot{e}_{2i} &= e_3 \\
\dot{e}_3 &= e_4 \\
\dot{e}_4 &= v_0
\end{aligned}
\qquad (5.30)
$$

Define the sliding surface for nominal system (5.30) as:

$$
\sigma_0 = \dot{e}_{1r} + 5\dot{e}_{1i} + 10\dot{e}_{2r} + 10\dot{e}_{2i} + 5\dot{e}_3 + \dot{e}_4 = e_{1i} + 5e_{2r} + 10e_{2i} + 10e_3 + 5e_4 + v_0
$$

By choosing $v_0 = -e_{1i} - 5e_{2r} - 10e_{2i} - 10e_3 - 5e_4 - k\,\sigma_0 - k\,sign(\sigma_0),\ k > 0$, we have $\dot{\sigma}_0 = -k\,\sigma_0 - k\,sign(\sigma_0)$. Therefore the nominal system (5.30) is asymptotically stable.

Now choose the sliding surface for the system (5.29) as:

$$\sigma = \sigma_0 + z = e_{1r} + 5e_{1i} + 10e_{2r} + 10e_{2i} + 5e_3 + e_4 + z$$

where, z is some integral term computed later. To avoid the reaching phase, choose $z(0)$ such that $\sigma(0) = 0$. Choose $v = v_0 + v_s$ where, v_0 is the nominal input and v_s is compensator term computed later. Then

$$\begin{aligned}
\dot{\sigma} &= \dot{e}_{1r} + 5\dot{e}_{1i} + 10\dot{e}_{2r} + 10\dot{e}_{2i} + 5\dot{e}_3 + \dot{e}_4 + \dot{z} \\
&= e_{1i} + \tilde{b}_1 y_{1r} + \tilde{b}_2 y_{2r} - \tilde{a}_1(x_{2r} - x_{1r}) + \tilde{a}_2 x_{2r} - \tilde{a}_3(\alpha_1 + 3\beta_1 x_4^2)x_{1r} \\
&+ 5e_{2r} + 5\tilde{b}_1 y_{1i} + 5\tilde{b}_2 y_{2i} - 5\tilde{a}_1(x_{2i} - x_{1i}) - 5\tilde{a}_2 x_{2i} + 5\tilde{a}_3(\alpha_1 + 3\beta_1 x_4^2)x_{1i} \\
&+ 10e_{2i} + 10\tilde{b}_3 y_{1r} - 20\tilde{a}_2 x_{2r} + 20\tilde{a}_3(\alpha_1 + 3\beta_1 x_4^2)x_{1r} - 10\tilde{a}_2 x_{2i} + 10\tilde{a}_3(\alpha_1 \\
&+ 3\beta_1 x_4^2)x_{1i} + +10e_3 + 10\tilde{b}_3 y_{1i} - 20\tilde{a}_2 x_{2i} + 20\tilde{a}_3(\alpha_1 + 3\beta_1 x_4^2)x_{1i} + 10\tilde{a}_2 x_{2r} \\
&- 10\tilde{a}_3(\alpha_1 + 3\beta_1 x_4^2)x_{1r} + 5e_4 - 5\tilde{b}_4 y_3 + 5\tilde{a}_4 x_3 + v_0 + v_s + \dot{z}
\end{aligned} \tag{5.31}$$

By choosing a Lyapunov function:

$V = \frac{1}{2}\sigma^2 + \frac{1}{2}(\tilde{a}_1^2 + \tilde{a}_2^2 + \tilde{a}_3^2 + \tilde{a}_4^2 + \tilde{b}_1^2 + \tilde{b}_2^2 + \tilde{b}_3^2 + \tilde{b}_4^2)$, design the adaptive laws for $\tilde{a}_i, \hat{a}_i, , \tilde{b}_i, \hat{b}_i, i = 1,...,4$ and compute v_s such that $\dot{V} < 0$.

$$\begin{aligned}
&\dot{z} = -e_{1i} - 5e_{2r} - 10e_{2i} - 10e_3 - 5e_4 - v_0, \quad v_s = -k\sigma - k\, sign(\sigma) \\
&\dot{\tilde{a}}_1 = \sigma(x_{2r} - x_{1r}) + 5\sigma(x_{2i} - x_{1i}) - k_1\tilde{a}_1, \quad \dot{\hat{a}}_1 = -\dot{\tilde{a}}_1 \\
&\dot{\tilde{a}}_2 = 9\sigma x_{2r} + 35\sigma x_{2i} - k_2\tilde{a}_2, \quad \dot{\hat{a}}_2 = -\dot{\tilde{a}}_2 \\
&\dot{\tilde{a}}_3 = -9\sigma(\alpha_1 + 3\beta_1 x_4^2)x_{1r} - 35\sigma(\alpha_1 + 3\beta_1 x_4^2)x_{1i} - k_3\tilde{a}_3, \quad \dot{\hat{a}}_3 = -\dot{\tilde{a}}_3 \\
&\dot{\tilde{a}}_4 = -5\sigma x_3 - k_4\tilde{a}_4, \quad \dot{\hat{a}}_4 = -\dot{\tilde{a}}_4 \\
&\dot{\tilde{b}}_1 = -\sigma y_{1r} - 5\sigma y_{1i} - k_5\tilde{b}_1, \quad \dot{\hat{b}}_1 = -\dot{\tilde{b}}_1 \\
&\dot{\tilde{b}}_2 = -\sigma y_{2r} - 5\sigma y_{2i} - k_6\tilde{b}_2, \quad \dot{\hat{b}}_2 = -\dot{\tilde{b}}_2 \\
&\dot{\tilde{b}}_3 = -10\sigma y_{1r} - 10\sigma y_{1i} - k_7\tilde{b}_3, \quad \dot{\hat{b}}_3 = -\dot{\tilde{b}}_3, \\
&\dot{\tilde{b}}_4 = 5\sigma y_3 - k_8\tilde{b}_4, \quad \dot{\hat{b}}_4 = -\dot{\tilde{b}}_4, \quad k, k_i > 0, i = 1,...,8
\end{aligned} \tag{5.32}$$

Proof:

Since

$$
\begin{aligned}
\dot V &= \sigma\dot\sigma + \tilde a_1\dot{\tilde a}_1 + \tilde a_2\dot{\tilde a}_2 + \tilde a_3\dot{\tilde a}_3 + \tilde a_4\dot{\tilde a}_4 + \tilde b_1\dot{\tilde b}_1 + \tilde b_2\dot{\tilde b}_2 + \tilde b_3\dot{\tilde b}_3 + \tilde b_4\dot{\tilde b}_4 \\
&= \sigma\{e_{1i} + \tilde b_1 y_{1r} + \tilde b_2 y_{2r} - \tilde a_1(x_{2r} - x_{1r}) + \tilde a_2 x_{2r} - \tilde a_3(\alpha_1 + 3\beta_1 x_4^2)x_{1r} \\
&+ 5e_{2r} + 5\tilde b_1 y_{1i} + 5\tilde b_2 y_{2i} - 5\tilde a_1(x_{2i} - x_{1i}) - 5\tilde a_2 x_{2i} + 5\tilde a_3(\alpha_1 + 3\beta_1 x_4^2)x_{1i} \\
&+ 10e_{2i} + 10\tilde b_3 y_{1r} - 20\tilde a_2 x_{2r} + 20\tilde a_3(\alpha_1 + 3\beta_1 x_4^2)x_{1r} - 10\tilde a_2 x_{2i} + 10\tilde a_3(\alpha_1 + 3\beta_1 x_4^2)x_{1i} + \\
&+ 10e_3 + 10\tilde b_3 y_{1i} - 20\tilde a_2 x_{2i} + 20\tilde a_3(\alpha_1 + 3\beta_1 x_4^2)x_{1i} + 10\tilde a_2 x_{2r} - 10\tilde a_3(\alpha_1 + 3\beta_1 x_4^2)x_{1r} \\
&+ 5e_4 - 5\tilde b_4 y_3 + 5\tilde a_4 x_3 + v_0 + v_s + \dot z\} \\
&+ \tilde a_1\dot{\tilde a}_1 + \tilde a_2\dot{\tilde a}_2 + \tilde a_3\dot{\tilde a}_3 + \tilde a_4\dot{\tilde a}_4 + \tilde b_1\dot{\tilde b}_1 + \tilde b_2\dot{\tilde b}_2 + \tilde b_3\dot{\tilde b}_3 + \tilde b_4\dot{\tilde b}_4 \\
&= \sigma\{e_{1i} + 5e_{2r} + 10e_{2i} + 10e_3 + 5e_4 + v_0 + v_s + \dot z\} \\
&+ \tilde a_1\{\dot{\tilde a}_1 - \sigma(x_{2r} - x_{1r}) - 5\sigma(x_{2i} - x_{1i})\} + \tilde a_2\{\dot{\tilde a}_2 - 9\sigma\, x_{2r} - 35\sigma\, x_{2i}\} \\
&+ \tilde a_3\{\dot{\tilde a}_3 + 9\sigma(\alpha_1 + 3\beta_1 x_4^2)x_{1r} + 35\sigma(\alpha_1 + 3\beta_1 x_4^2)x_{1i}\} + \tilde a_4\{\dot{\tilde a}_4 + 5\sigma x_3\} \\
&+ \tilde b_1\{\dot{\tilde b}_1 + \sigma\, y_{1r} + 5\sigma\, y_{1i}\} + \tilde b_2\{\dot{\tilde b}_2 + \sigma\, y_{2r} + 5\sigma\, y_{2i}\} + \tilde b_3\{\dot{\tilde b}_3 + 10\sigma\, y_{1r} + 10\sigma y_{1i}\} + \tilde b_4\{\dot{\tilde b}_4 - 5\sigma y_3\}
\end{aligned}
$$

By using

$$
\begin{aligned}
&\dot z = -e_{1i} - 5e_{2r} - 10e_{2i} - 10e_3 - 5e_4 - v_0, \quad v_s = -k\sigma - k\, sign(\sigma) \\
&\dot{\tilde a}_1 = \sigma(x_{2r} - x_{1r}) + 5\sigma(x_{2i} - x_{1i}) - k_1\tilde a_1 , \quad \dot{\hat a}_1 = -\dot{\tilde a}_1 \\
&\dot{\tilde a}_2 = 9\sigma\, x_{2r} + 35\sigma\, x_{2i} - k_2\tilde a_2 , \quad \dot{\hat a}_2 = -\dot{\tilde a}_2 \\
&\dot{\tilde a}_3 = -9\sigma(\alpha_1 + 3\beta_1 x_4^2)x_{1r} - 35\sigma(\alpha_1 + 3\beta_1 x_4^2)x_{1i} - k_3\tilde a_3 , \quad \dot{\hat a}_3 = -\dot{\tilde a}_3 \\
&\dot{\tilde a}_4 = -5\sigma x_3 - k_4\tilde a_4 , \quad \dot{\hat a}_4 = -\dot{\tilde a}_4 \\
&\dot{\tilde b}_1 = -\sigma\, y_{1r} - 5\sigma\, y_{1i} - k_5\tilde b_1, \quad \dot{\hat b}_1 = -\dot{\tilde b}_1 \\
&\dot{\tilde b}_2 = -\sigma\, y_{2r} - 5\sigma\, y_{2i} - k_6\tilde b_2, \quad \dot{\hat b}_2 = -\dot{\tilde b}_2 \\
&\dot{\tilde b}_3 = -10\,\sigma\, y_{1r} - 10\sigma y_{1i} - k_7\tilde b_3, \quad \dot{\hat b}_3 = -\dot{\tilde b}_3, \\
&\dot{\tilde b}_4 = 5\sigma y_3 - k_8\tilde b_4, \quad \dot{\hat b}_4 = -\dot{\tilde b}_4, \quad k, k_i > 0, i = 1,\dots,8
\end{aligned}
$$

We have

$$
\dot V = -k\sigma^2 - k|\sigma| - k_1\tilde a_1^2 - k_2\tilde a_2^2 - k_3\tilde a_3^2 - k_4\tilde a_4^2 - k_5\tilde b_1^2 - k_6\tilde b_2^2 - k_7\tilde b_3^2 - k_8\tilde b_4^2 .
$$

From this we conclude that $\sigma, \tilde a_i, \tilde b_i \to 0$. Since $\sigma \to 0$, therefore

$$
e = (e_{1r}, e_{1i}, e_{2r}, e_{2i}, e_3, e_4) \to 0 .
$$

In simulations, the initial conditions are chosen as: $(x_1(0), x_2(0), x_3(0), x_4(0)) = (-1+2j, 1+j, 2, -1)$,and $(y_1(0), y_2(0), y_3(0), y_4(0)) = (10-8j, 4-3j, 6, 5)$. The values of known parameters are: $\alpha_1 = 4, \beta_1 = 0.01$, $\alpha_2 = 0.67\times10^{-3}, \beta_2 = 0.02\times10^{-3}$. The true values of unknown parameters are chosen as:

$$a_1 = 36, a_2 = 20, a_3 = 3.2, a_4 = 3, b_1 = 8, b_2 = 11, b_3 = 50, b_4 = 8/3.$$

Generalized Synchronization Case 1 results:

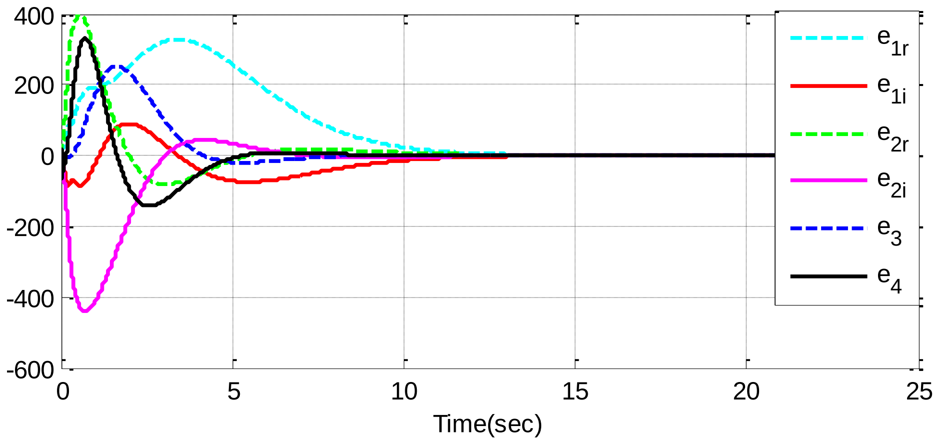

Figure 5.1: Time Response of error $e_{1r}, e_{1i}, e_{2r}, e_{2i}, e_3$ & e_4

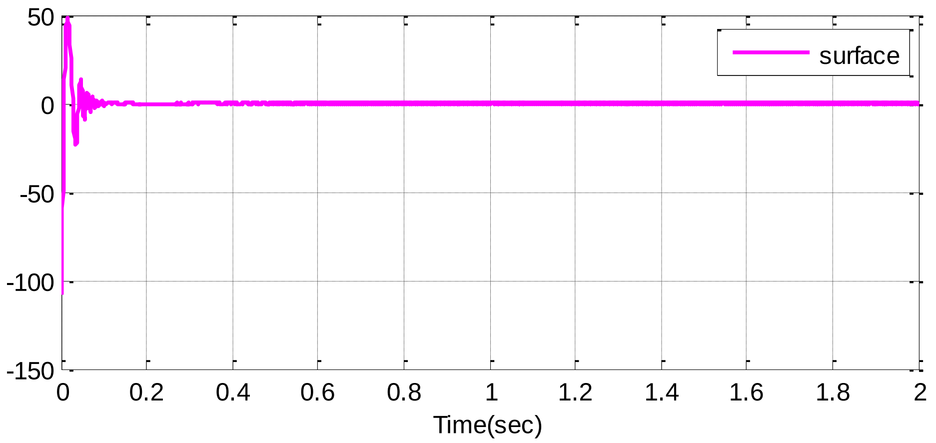

Figure 5.2: Time Response of surface

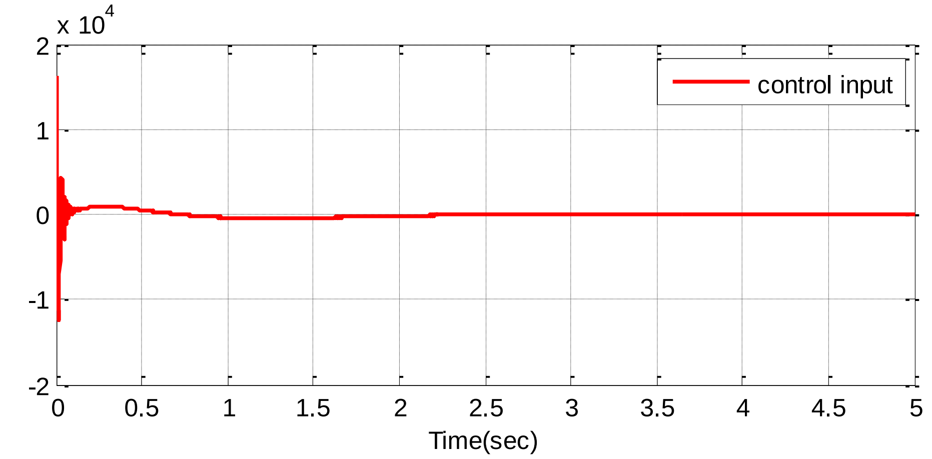

Figure 5.3: Time Response of control input

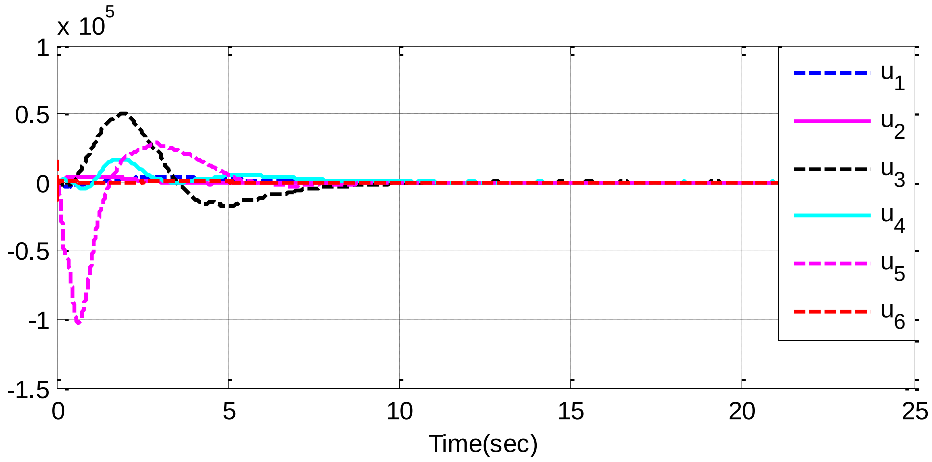

Figure 5.4: Time Response of adaptive controller u_1, u_2, u_3, u_4, u_5 & u_6

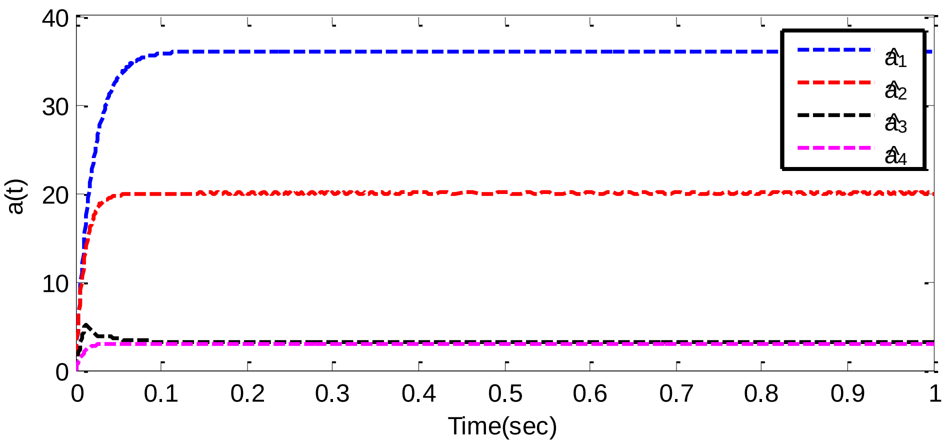

Figure 5.5: Estimation parameter of $\hat{a}_1, \hat{a}_2, \hat{a}_3$ & $\hat{a}_4$

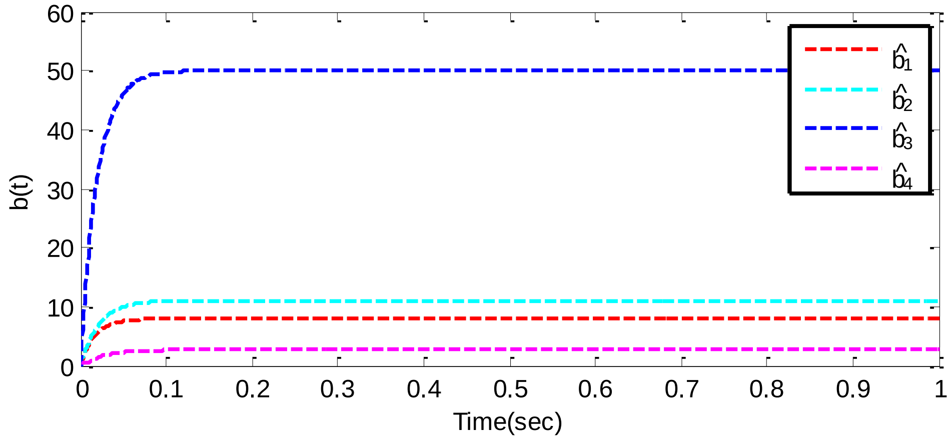

Figure 5.6: Estimation parameter of $\hat{b}_1, \hat{b}_2, \hat{b}_3$ & $\hat{b}_4$

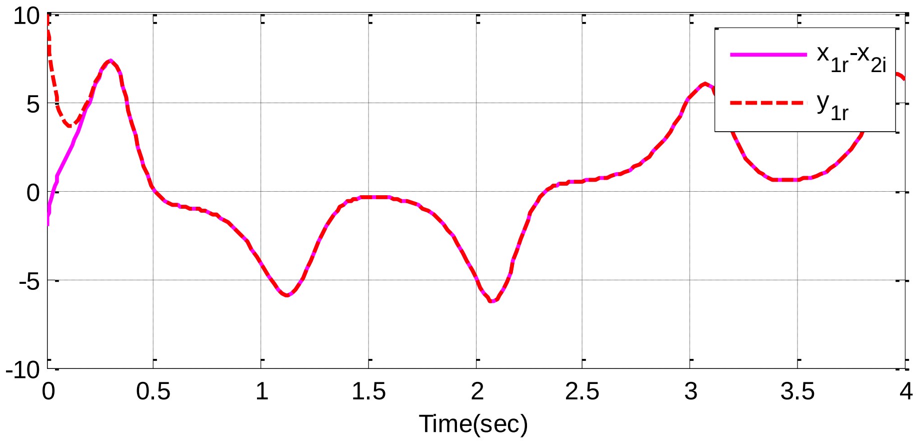

Figure 5.7: Time Response of $x_{1r} - x_{2i}$&y_{1r} with IC (-1,1,10)

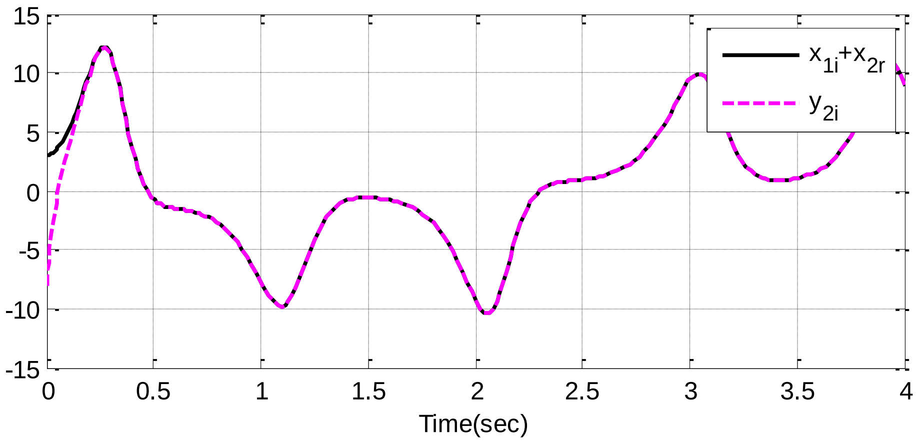

Figure 5.8.: Time Response of $x_{1i} + x_{2r}$&y_{1i} with IC (2, 1, -8)

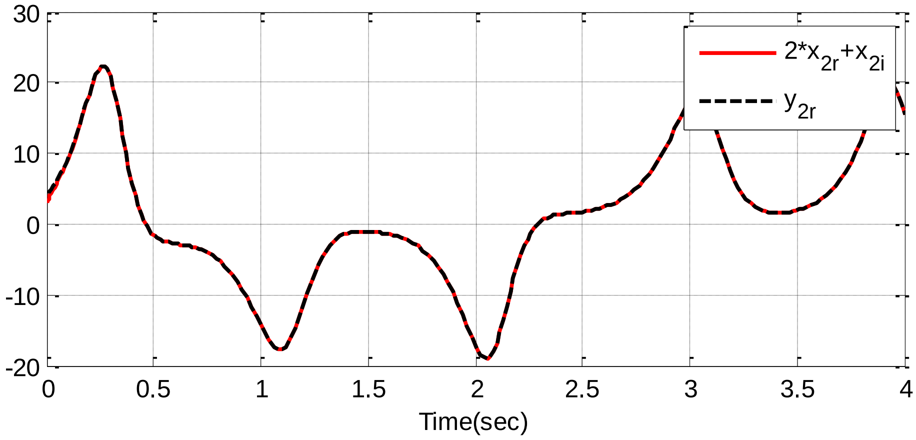

Figure 5.9: Time Response of $2x_{2r} + x_{2i}$&y_{2r} with IC (1, 1, 4)

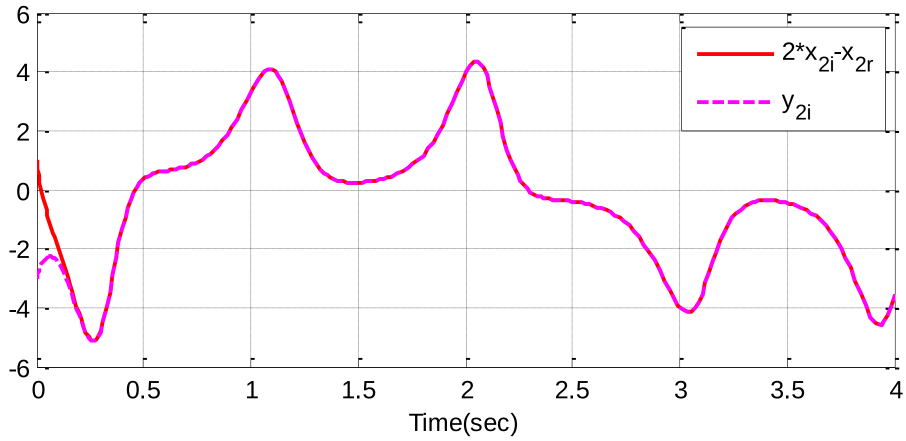

Figure 5.10: Time Response of $2x_{2i} - x_{2r}$&y_{2i} with IC (1, 1, -3)

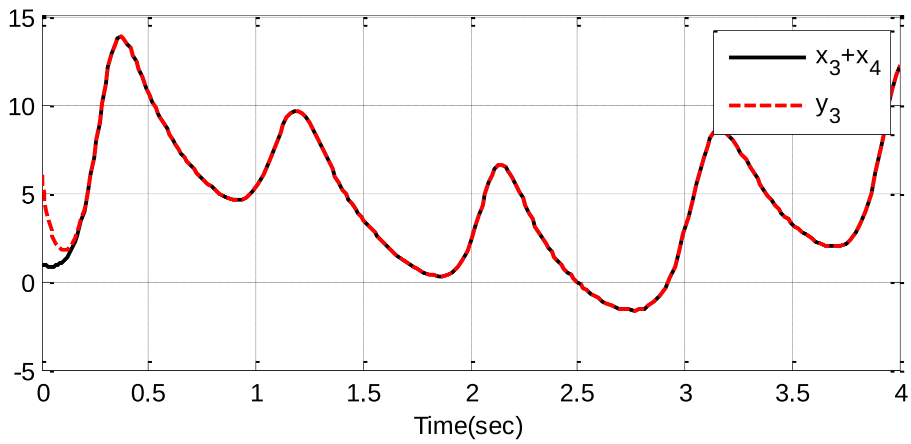

Figure 5.11: Time Response of $x_3 + x_4$&$y_3$with IC (2, -1, 6)

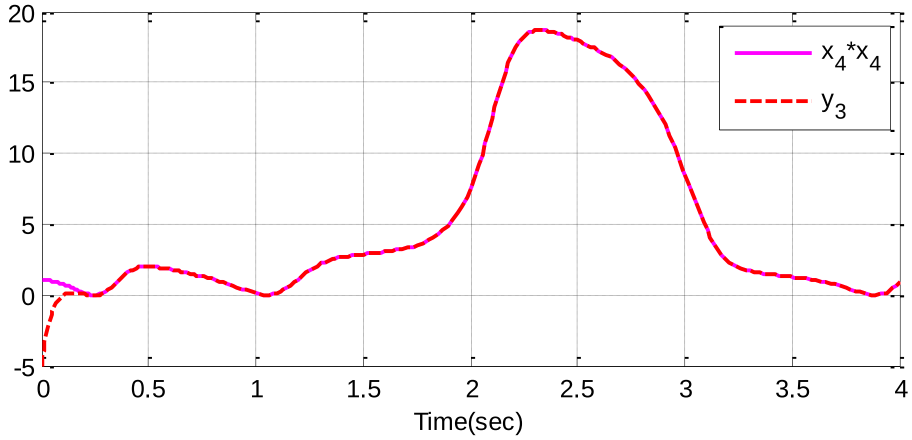

Figure 5.12: Time Response of x_4^2&$y_3$with IC (2, 6)

Case 2 when (*n*<*m*):

Consider the Master system given in [49] as:

$$\begin{aligned}
\dot{x}_1 &= c_1(x_2 - x_1) \\
\dot{x}_2 &= (c_2 - c_1)x_1 - x_1x_3 + c_2x_2 \\
\dot{x}_3 &= 0.5(\bar{x}_1\, x_2 + x_1\bar{x}_2) - c_3x_3
\end{aligned} \qquad (5.33)$$

Where $x_1 = x_{1r} + jx_{1i}$, $x_2 = x_{2r} + jx_{2i}$ are complex and $x_3 = x_{3r}$ is real. $\bar{x}_1, \bar{x}_2$ denote the complex conjugate variables of x_1, x_2. d_1, d_2 and d_3 are unknown real parameters. u_1, u_2 and u_3 are controllers. When $c_1 = 27, c_2 = 23,, c_3 = 1,$ and $x(0) = [-3-2j,\ 1-5j, -4]^T$

The Slave system given in [49] as:

$$\begin{aligned}
\dot{y}_1 &= -b_1y_1 + b_2y_2 - (\alpha_2 + 3\beta_2 y_4^2)y_1 + u_1 \\
\dot{y}_2 &= b_3y_1 - y_2 - y_1y_3 + u_2 \\
\dot{y}_3 &= 0.5(\bar{y}_1\, y_2 + y_1\bar{y}_2) - b_4y_3 + u_3 \\
\dot{y}_4 &= -0.5(\bar{y}_1 + y_1) + u_4
\end{aligned} \qquad (5.34)$$

Where, $y_1 = y_{1r} + jy_{1i}$, $y_2 = y_{2r} + jy_{2i}$ are complex and $y_3 = y_{3r}$, $y_4 = y_{4r}$ are real. $\bar{y}_1, \bar{y}_2$ denote the complex conjugate variables of y_1, y_2. b_1, b_2, b_3 and b_4 are unknown real parameters α_2 and β_2 are considered as known positive constants. u_1, u_2, u_3 and u_4 arecontrollers.When $\alpha_2 = 0.67\times10^{-3}$, $\beta_2 = 0.02\times10^{-3}$,and $b_1 = 8, b_2 = 11, b_3 = 50, b_4 = 8/3$ $y(0) = [2,\ 1+4j,\ 0.1, 0]^T$

In this section, we investigate CGS of two nonidentical complex systems with the different orders.

Let $\hat{c}_f, f = 1,..,3,\ \hat{b}_i, i = 1,..,4$ be estimates of $c_f, f = 1,...,3,\ b_i, i = 1,..,4$ and $\tilde{c}_f = c_f - \hat{c}_f, f = 1,...,3,, \tilde{b}_i = b_i - \hat{b}_i, i = 1,....,4$, be the errors in estimations of $c_f, f = 1,...,3,\ b_i, i = 1,..,4$, respectively. Then systems (5.33) and (5.34) can be written as:

$$\begin{aligned}
\dot{x}_1 &= \hat{c}_1(x_2 - x_1) + \tilde{c}_1(x_2 - x_1) \\
\dot{x}_2 &= (\hat{c}_2 - \hat{c}_1)x_1 + (\tilde{c}_2 - \tilde{c}_1)x_1 - x_1x_3 + \hat{c}_2x_2 + \tilde{c}_2x_2 \\
\dot{x}_3 &= 0.5(\bar{x}_1\, x_2 + x_1\bar{x}_2) - \hat{c}_3x_3 - \tilde{c}_3x_3
\end{aligned} \qquad (5.35)$$

The 3-D complex system (5.35) can be written into 5-D real system as:

$$\begin{aligned}
\dot{x}_{1r} &= \hat{c}_1(x_{2r} - x_{1r}) + \tilde{c}_1(x_{2r} - x_{1r}) \\
\dot{x}_{1i} &= \hat{c}_1(x_{2i} - x_{1i}) + \tilde{c}_1(x_{2i} - x_{1i}) \\
\dot{x}_{2r} &= (\hat{c}_2 - \hat{c}_1)x_{1r} + (\tilde{c}_2 - \tilde{c}_1)x_{1r} - x_{1r}x_3 + \hat{c}_2 x_{2r} + \tilde{c}_2 x_{2r} \\
\dot{x}_{2i} &= (\hat{c}_2 - \hat{c}_1)x_{1i} + (\tilde{c}_2 - \tilde{c}_1)x_{1i} - x_{1i}x_3 + \hat{c}_2 x_{2i} + \tilde{c}_2 x_{2i} \\
\dot{x}_3 &= (x_{1r}\, x_{2r} + x_{1i}x_{2i}) - \hat{c}_3 x_3 - \tilde{c}_3 x_3
\end{aligned} \tag{5.36}$$

We consider the system (5.36) as a drive system and the following system (5.34) as a response system.

$$\begin{aligned}
\dot{y}_{1r} &= \hat{b}_1 y_{1r} + \tilde{b}_1 y_{1r} + \hat{b}_2 y_{2r} + \tilde{b}_2 y_{2r} - (\alpha_2 + 3\beta_2 y_4^2) y_{1r} + u_{r1} \\
\dot{y}_{1i} &= \hat{b}_1 y_{1i} + \tilde{b}_1 y_{1i} + \hat{b}_2 y_{2i} + \tilde{b}_2 y_{2i} - (\alpha_2 + 3\beta_2 y_4^2) y_{1j} + u_{1j} \\
\dot{y}_{2r} &= \hat{b}_3 y_{1r} + \tilde{b}_3 y_{1r} - y_{2r} - y_{1r} y_3 + u_{2r} \\
\dot{y}_{2i} &= \hat{b}_3 y_{1i} + \tilde{b}_3 y_{1i} - y_{2i} - y_{1i} y_3 + u_{2i} \\
\dot{y}_3 &= y_{1r}\, y_{2r} + y_{1i} y_{2i} - \hat{b}_4 y_3 - \tilde{b}_4 y_3 + u_3 \\
\dot{y}_4 &= -0.5 y_{1r} + u_4
\end{aligned} \tag{5.37}$$

The complex map vector is given by

$\phi(x) = \begin{bmatrix} -jx_1 & -jx_2 & -x_3 & x_3 \end{bmatrix}^T$. This gives:

$$\begin{aligned}
&\phi_{1r}(x) = x_{1i}\,, \phi_{1i}(x) = -x_{1r}\,, \phi_{2r}(x) = x_{2i}\,, \\
&\phi_{2i}(x) = -x_{2r}\,, \phi_3(x) = -x_3\,, \phi_4(x) = x_3
\end{aligned} \tag{5.38}$$

The error signals are defined as:

$$\begin{aligned}
&e_{1r} = y_{1r} - \phi_{1r}, e_{1i} = y_{1i} - \phi_{1i}, e_{2r} = y_{2r} - \phi_{2r}, \\
&e_{2i} = y_{2i} - \phi_{2i}, e_3 = y_3 - \phi_3, e_4 = y_4 - \phi_4
\end{aligned} \tag{5.39}$$

Then the error dynamics becomes:

$$
\begin{aligned}
\dot{e}_{1r} &= \dot{y}_{1r} - \dot{\phi}_{1r} = \dot{y}_{1r} - \dot{x}_{1i} = \hat{b}_1 y_{1r} + \tilde{b}_1 y_{1r} + \hat{b}_2 y_{2r} + \tilde{b}_2 y_{2r} - (\alpha_2 + 3\beta_2 y_4^2) y_{1r} + u_{r1} \\
&- \hat{c}_1 (x_{2i} - x_{1i}) - \tilde{c}_1 (x_{2i} - x_{1i}) \\
\dot{e}_{1i} &= \dot{y}_{1i} - \dot{\phi}_{1i} = \dot{y}_{1i} + \dot{x}_{1r} = \hat{b}_1 y_{1i} + \tilde{b}_1 y_{1i} + \hat{b}_2 y_{2i} + \tilde{b}_2 y_{2i} - (\alpha_2 + 3\beta_2 y_4^2) y_{1j} + u_{1j} \\
&+ \hat{c}_1 (x_{2r} - x_{1r}) + \tilde{c}_1 (x_{2r} - x_{1r}) \\
\dot{e}_{2r} &= \dot{y}_{2r} - \dot{\phi}_{2r} = \dot{y}_{2r} - \dot{x}_{2i} = \hat{b}_3 y_{1r} + \tilde{b}_3 y_{1r} - y_{2r} - y_{1r} y_3 + u_{2r} \\
&- (\hat{c}_2 - \hat{c}_1) x_{1i} - (\tilde{c}_2 - \tilde{c}_1) x_{1i} + x_{1i} x_3 - \hat{c}_2 x_{2i} - \tilde{c}_2 x_{2i} \\
\dot{e}_{2i} &= \dot{y}_{2i} - \dot{\phi}_{2i} = \dot{y}_{2i} + \dot{x}_{2r} = \hat{b}_3 y_{1i} + \tilde{b}_3 y_{1i} - y_{2i} - y_{1i} y_3 + u_{2i} \\
&+ (\hat{c}_2 - \hat{c}_1) x_{1r} + (\tilde{c}_2 - \tilde{c}_1) x_{1r} - x_{1r} x_3 + \hat{c}_2 x_{2r} + \tilde{c}_2 x_{2r} \\
\dot{e}_3 &= \dot{y}_3 - \dot{\phi}_3 = \dot{y}_3 + \dot{x}_3 = y_{1r} \, y_{2r} + y_{1i} y_{2i} - \hat{b}_4 y_3 - \tilde{b}_4 y_3 + u_3 \\
&+ (x_{1r} \, x_{2r} + x_{1i} x_{2i}) - \hat{c}_3 x_3 - \tilde{c}_3 x_3 \\
\dot{e}_4 &= \dot{y}_4 - \dot{\phi}_4 = \dot{y}_4 - \dot{x}_3 = -0.5 y_{1r} + u_4 - (x_{1r} \, x_{2r} + x_{1i} x_{2i}) + \hat{c}_3 x_3 + \tilde{c}_3 x_3
\end{aligned} \tag{5.40}
$$

By choosing:

$$
\begin{aligned}
u_{1r} &= e_{1i} - \hat{b}_1 y_{1r} - \hat{b}_2 y_{2r} + (\alpha_2 + 3\beta_2 y_4^2) y_{1r} + \hat{c}_1 (x_{2i} - x_{1i}) \\
u_{1i} &= e_{2r} - \hat{b}_1 y_{1i} - \hat{b}_2 y_{2i} + (\alpha_2 + 3\beta_2 y_4^2) y_{1j} - \hat{c}_1 (x_{2r} - x_{1r}) \\
u_{2r} &= e_{2i} - \hat{b}_3 y_{1r} + y_{2r} + y_{1r} y_3 + (\hat{c}_2 - \hat{c}_1) x_{1i} - x_{1i} x_3 + \hat{c}_2 x_{2i} \\
u_{2i} &= e_3 - \hat{b}_3 y_{1i} + y_{2i} + y_{1i} y_3 - (\hat{c}_2 - \hat{c}_1) x_{1r} + x_{1r} x_3 - \hat{c}_2 x_{2r} \\
u_3 &= e_4 - y_{1r} \, y_{2r} - y_{1i} y_{2i} + \hat{b}_4 y_3 - (x_{1r} \, x_{2r} + x_{1i} x_{2i}) + \hat{c}_3 x_3 \\
u_4 &= v + 0.5 y_{1r} + (x_{1r} \, x_{2r} + x_{1i} x_{2i}) - \hat{c}_3 x_3
\end{aligned} \tag{5.41}
$$

where v is the new input, the system (40) can be written as:

$$
\begin{aligned}
\dot{e}_{1r} &= e_{1i} + \tilde{b}_1 y_{1r} + \tilde{b}_2 y_{2r} - \tilde{c}_1 (x_{2i} - x_{1i}) \\
\dot{e}_{1i} &= e_{2r} + \tilde{b}_1 y_{1i} + \tilde{b}_2 y_{2i} + \tilde{c}_{11} (x_{2r} - x_{1r}) \\
\dot{e}_{2r} &= e_{2i} + \tilde{b}_3 y_{1r} - (\tilde{c}_2 - \tilde{c}_1) x_{1i} - \tilde{c}_2 x_{2i} \\
\dot{e}_{2i} &= e_3 + \tilde{b}_3 y_{1i} + (\tilde{c}_2 - \tilde{c}_1) x_{1r} + \tilde{c}_2 x_{2r} \\
\dot{e}_3 &= e_4 - \tilde{b}_4 y_3 - \tilde{c}_3 x_3 \\
\dot{e}_4 &= v + \tilde{c}_3 x_3
\end{aligned} \tag{5.42}
$$

To employ the integral sliding mode control, choose the nominal system for (5.42) as:

$$\begin{aligned}
\dot{e}_{1r} &= e_{1i} \\
\dot{e}_{1i} &= e_{2r} \\
\dot{e}_{2r} &= e_{2i} \\
\dot{e}_{2i} &= e_3 \\
\dot{e}_3 &= e_4 \\
\dot{e}_4 &= v_0
\end{aligned} \qquad (5.43)$$

Define the sliding surface for nominal system (5.43) as:

$$\sigma_0 = (1+\frac{d}{dt})^5 e_{1r} = e_{1r} + 5e_{1i} + 10e_{2r} + 10e_{2i} + 5e_3 + e_4$$

Then

$$\dot{\sigma}_0 = \dot{e}_{1r} + 5\dot{e}_{1i} + 10\dot{e}_{2r} + 10\dot{e}_{2i} + 5\dot{e}_3 + \dot{e}_4 = e_{1i} + 5e_{2r} + 10e_{2i} + 10e_3 + 5e_4 + v_0$$

By choosing $v_0 = -e_{1i} - 5e_{2r} - 10e_{2i} - 10e_3 - 5e_4 - k\,\sigma_0 - k\,sign(\sigma_0),\ k > 0$, we have $\dot{\sigma}_0 = -k\,\sigma_0 - k\,sign(\sigma_0)$. Therefore the nominal system (5.43) is asymptotically stable.

Now choose the sliding surface for the system (5.42) as:

$$\sigma = \sigma_0 + z = e_{1r} + 5e_{1i} + 10e_{2r} + 10e_{2i} + 5e_3 + e_4 + z$$

where, z is some integral term computed later. To avoid the reaching phase, choose $z(0)$ such that $\sigma(0) = 0$. Choose $v = v_0 + v_s$ where, v_0 is the nominal input and v_s is compensator term computed later. Then

$$\begin{aligned}
\dot{\sigma} &= \dot{e}_{1r} + 5\dot{e}_{1i} + 10\dot{e}_{2r} + 10\dot{e}_{2i} + 5\dot{e}_3 + \dot{e}_4 + \dot{z} \\
&= e_{1i} + \tilde{b}_1 y_{1r} + \tilde{b}_2 y_{2r} - \tilde{c}_1(x_{2i} - x_{1i}) + 5e_{2r} + 5\tilde{b}_1 y_{1i} + 5\tilde{b}_2 y_{2i} + 5\tilde{c}_1(x_{2r} - x_{1r}) \\
&\quad + 10e_{2i} + 10\tilde{b}_3 y_{1r} - 10(\tilde{c}_2 - \tilde{c}_1)x_{1i} - 10\tilde{c}_2 x_{2i} + 10e_3 + 10\tilde{b}_3 y_{1i} + 10(\tilde{c}_2 - \tilde{c}_1)x_{1r} \\
&\quad + 10\tilde{c}_2 x_{2r} + 5e_4 - 5\tilde{b}_4 y_3 - 5\tilde{c}_3 x_3 + \tilde{c}_3 x_3 + v_0 + v_s + \dot{z} \\
&= e_{1i} + 5e_{2r} + 10e_{2i} + 10e_3 + 5e_4 + \tilde{c}_1\{-x_{2i} + x_{1i} + 5x_{2r} - 5x_{1r} + 10x_{1i} - 10x_{1r}\} \\
&\quad + \tilde{c}_2\{-10x_{1i} + 10x_{1r} - 10x_{2i} + 10x_{2r}\} + \tilde{c}_3\{-5x_3 + x_3\} + \tilde{b}_1\{y_{1r} + 5y_{1i}\} \\
&\quad + \tilde{b}_2\{y_{2r} + 5y_{2i}\} + \tilde{b}_3\{10y_{1r} + 10y_{1i}\} - \tilde{b}_4\{5y_3\} + v_0 + v_s + \dot{z}
\end{aligned} \qquad (5.44)$$

By choosing a Lyapunov function: $V = \frac{1}{2}\sigma^2 + \frac{1}{2}(\tilde{b}_1^2 + \tilde{b}_2^2 + \tilde{b}_3^2 + \tilde{b}_4^2 + \tilde{c}_1^2 + \tilde{c}_2^2 + \tilde{c}_3^2)$,

design the adaptive laws for $\tilde{b}_i, \hat{b}_i, i = 1,...,4, \hat{c}_f, \tilde{c}_f, f = 1,2,3$ and compute v_s such that $\dot{V} < 0$.

$$\begin{aligned}
&\dot{z} = -e_{1i} - 5e_{2r} - 10e_{2i} - 10e_3 - 5e_4 - v_0, \quad v_s = -k\sigma - k\, sign(\sigma) \\
&\dot{\tilde{b}}_1 = -\sigma y_{1r} - 5\sigma y_{1i} - k_1\tilde{b}_1, \quad \dot{\hat{b}}_1 = -\dot{\tilde{b}}_1 \\
&\dot{\tilde{b}}_2 = -\sigma y_{2r} - 5\sigma y_{2i} - k_2\tilde{b}_2, \quad \dot{\hat{b}}_2 = -\dot{\tilde{b}}_2 \\
&\dot{\tilde{b}}_3 = -10\sigma y_{1r} - 10\sigma y_{1i} - k_3\tilde{b}_3, \quad \dot{\hat{b}}_3 = -\dot{\tilde{b}}_3 \\
&\dot{\tilde{b}}_4 = 5\sigma y_3 - k_4\tilde{b}_4, \quad \dot{\hat{b}}_4 = -\dot{\tilde{b}}_4 \\
&\dot{\tilde{c}}_1 = \sigma x_{2i} - \sigma x_{1i} - 5\sigma x_{2r} + 5\sigma x_{1r} - 10\sigma x_{1i} + 10\sigma x_{1r} - k_5\tilde{d}_1, \quad \dot{\hat{c}}_1 = -\dot{\tilde{c}}_1 \\
&\dot{\tilde{c}}_2 = 10\sigma x_{1i} - 10\sigma x_{1r} + 10\sigma x_{2i} - 10\sigma x_{2r} - k_6\tilde{d}_2, \quad \dot{\hat{c}}_2 = -\dot{\tilde{c}}_2 \\
&\dot{\tilde{c}}_3 = 5\sigma x_3 - \sigma x_3 - k_7\tilde{d}_3, \quad \dot{\hat{c}}_3 = -\dot{\tilde{c}}_3, \quad k, k_i > 0, i = 1,...,7
\end{aligned} \tag{5.45}$$

Proof:

Since

$$\begin{aligned}
\dot{V} &= \sigma\dot{\sigma} + \tilde{b}_1\dot{\tilde{b}}_1 + \tilde{b}_2\dot{\tilde{b}}_2 + \tilde{b}_3\dot{\tilde{b}}_3 + \tilde{b}_4\dot{\tilde{b}}_4 + \tilde{c}_1\dot{\tilde{c}}_1 + \tilde{c}_2\dot{\tilde{c}}_2 + \tilde{c}_3\dot{\tilde{c}}_3 \\
&= \sigma\{e_{1i} + 5e_{2r} + 10e_{2i} + 10e_3 + 5e_4 \\
&+ \tilde{c}_1(-x_{2i} + x_{1i} + 5x_{2r} - 5x_{1r} + 10x_{1i} - 10x_{1r}) + \tilde{c}_2(-10x_{1i} + 10x_{1r} \\
&- 10x_{2i} + 10x_{2r}\} + \tilde{c}_3\{-5x_3 + x_3) + \tilde{b}_1(y_{1r} + 5y_{1i}) + \tilde{b}_2(y_{2r} + 5y_{2i}) \\
&+ \tilde{b}_3(10y_{1r} + 10y_{1i}) - \tilde{b}_4(5y_3) + v_0 + v_s + \dot{z}\} + \tilde{b}_1\dot{\tilde{b}}_1 + \tilde{b}_2\dot{\tilde{b}}_2 + \tilde{b}_3\dot{\tilde{b}}_3 + \tilde{b}_4\dot{\tilde{b}}_4 + \tilde{c}_1\dot{\tilde{c}}_1 + \tilde{c}_2\dot{\tilde{c}}_2 + \tilde{c}_3\dot{\tilde{c}}_3 \\
&= \sigma\{e_{1i} + 5e_{2r} + 10e_{2i} + 10e_3 + 5e_4 + v_0 + v_s + \dot{z}\} \\
&+ \tilde{b}_1\{\dot{\tilde{b}}_1 + \sigma y_{1r} + 5\sigma y_{1i}\} + \tilde{b}_2\{\dot{\tilde{b}}_2 + \sigma y_{2r} + 5\sigma y_{2i}\} \\
&+ \tilde{b}_3\{\dot{\tilde{b}}_3 + 10\sigma y_{1r} + 10\sigma y_{1i}\} + \tilde{b}_4\{\dot{\tilde{b}}_4 - 5\sigma y_3\} \\
&+ \tilde{c}_1\{\dot{\tilde{c}}_1 - x_{2i} + x_{1i} + 5x_{2r} - 5x_{1r} + 10x_{1i} - 10x_{1r}\} \\
&+ \tilde{c}_2\{\dot{\tilde{c}}_2 - 10x_{1i} + 10x_{1r} - 10x_{2i} + 10x_{2r}\} + \tilde{c}_3\{\dot{\tilde{c}}_3 - 5x_3 + x_3\}
\end{aligned}$$

By using

$$\dot{z} = -e_{1i} - 5e_{2r} - 10e_{2i} - 10e_3 - 5e_4 - v_0, \quad v_s = -k\sigma - k\, sign(\sigma)$$
$$\dot{\tilde{b}}_1 = -\sigma y_{1r} - 5\sigma y_{1i} - k_1\tilde{b}_1 , \quad \dot{\hat{b}}_1 = -\dot{\tilde{b}}_1$$
$$\dot{\tilde{b}}_2 = -\sigma\, y_{2r} - 5\,\sigma\, y_{2i} - k_2\tilde{b}_2 , \quad \dot{\hat{b}}_2 = -\dot{\tilde{b}}_2$$
$$\dot{\tilde{b}}_3 = -10\sigma y_{1r} - 10\sigma y_{1i} - k_3\tilde{b}_3 , \quad \dot{\hat{b}}_3 = -\dot{\tilde{b}}_3$$
$$\dot{\tilde{b}}_4 = 5\sigma y_3 - k_4\tilde{b}_4 , \quad \dot{\hat{b}}_4 = -\dot{\tilde{b}}_4$$
$$\dot{\tilde{c}}_1 = \sigma x_{2i} - \sigma x_{1i} - 5\sigma x_{2r} + 5\sigma x_{1r} - 10\sigma x_{1i} + 10\sigma x_{1r} - k_5\tilde{d}_1, \quad \dot{\hat{c}}_1 = -\dot{\tilde{c}}_1$$
$$\dot{\tilde{c}}_2 = 10\sigma x_{1i} - 10\sigma x_{1r} + 10\sigma x_{2i} - 10\sigma x_{2r} - k_6\tilde{d}_2, \quad \dot{\hat{c}}_2 = -\dot{\tilde{c}}_2$$
$$\dot{\tilde{c}}_3 = 5\sigma x_3 - \sigma x_3 - k_7\tilde{d}_3, \quad \dot{\hat{c}}_3 = -\dot{\tilde{c}}_3, \quad k, k_i > 0, i = 1,...,7$$

We have

$$\dot{V} = -k\sigma^2 - k|\sigma| - k_1\tilde{b}_1^2 - k_2\tilde{b}_2^2 - k_3\tilde{b}_3^2 - k_4\tilde{b}_4^2 - k_5\tilde{c}_1^2 - k_6\tilde{c}_2^2 - k_7\tilde{c}_3^2 .$$

From this we conclude that $\sigma, \tilde{b}_i, \tilde{d}_i \to 0$. Since $\sigma \to 0$, therefore $e = (e_{1r}, e_{1i}, e_{2r}, e_{2i}, e_3, e_4) \to 0$.

In simulations, the initial conditions are chosen as:, and $x(0) = [-3-2j,\, 1-5j, -4]^T$,,and $(y_1(0), y_2(0), y_3(0), y_4(0)) = (10-8j, 4-3j, 6,5)$. The values of known parameters are: $\alpha_2 = 0.67\times10^{-3}$, $\beta_2 = 0.02\times10^{-3}$. The true value of the known parameters are : $c_1 = 27, c_2 = 23,, c_3 = 1, b_1 = 8, b_2 = 11, b_3 = 50, b_4 = 8/3$.

Generalized Synchronization results of Case 2:

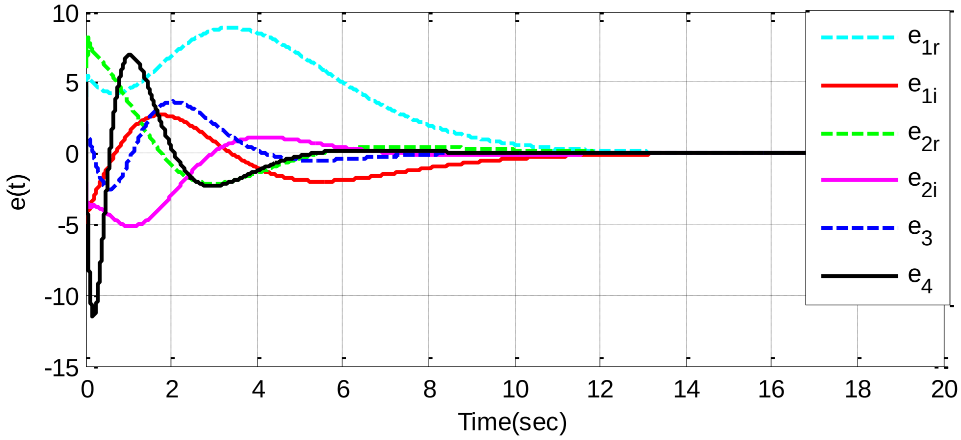

Figure 5.13: Time Response of error $e_{1r}, e_{1i}, e_{2r}, e_{2i}, e_3$ & e_4

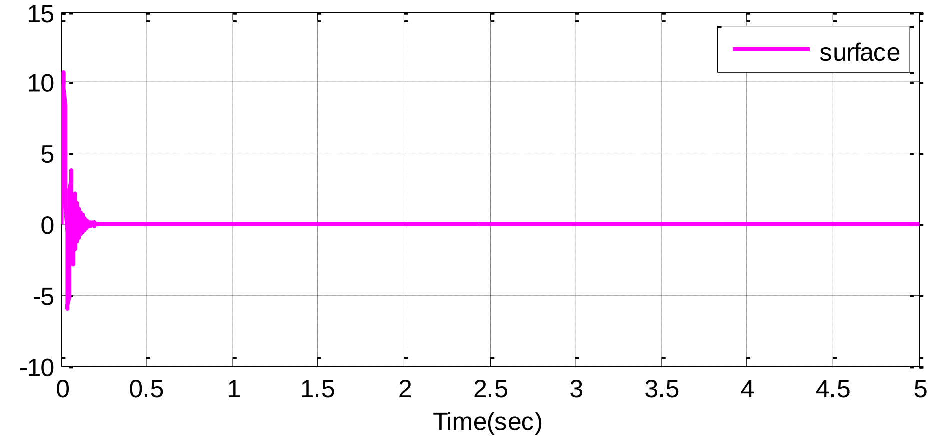

Figure 5.14: Time Response of surface

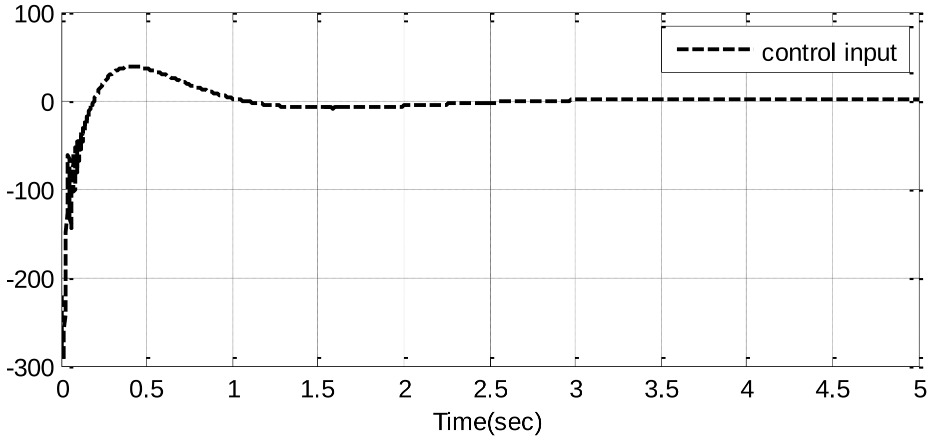

Figure 5.15: Time Response of control input

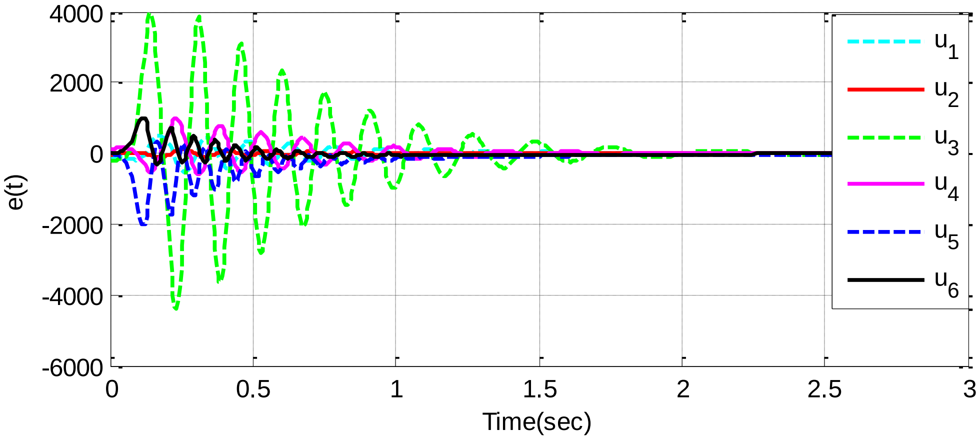

Figure 5.16: Time Response of adaptive controller u_1, u_2, u_3, u_4, u_5 & u_6

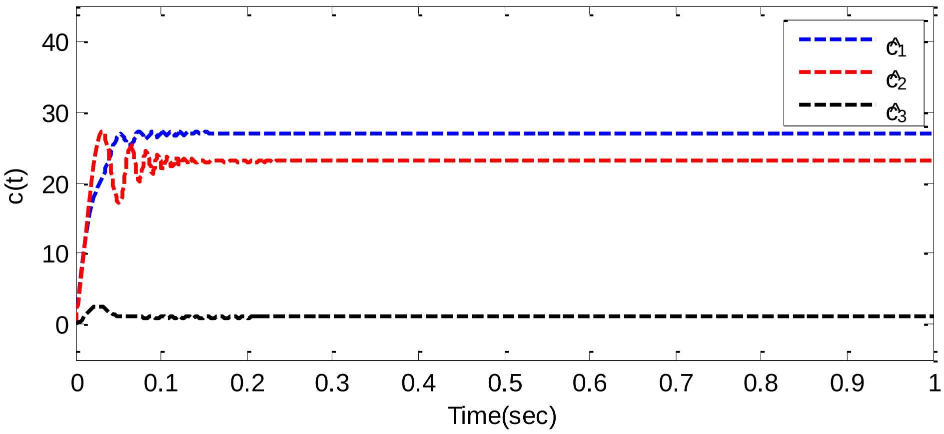

Figure 5.17: Estimation parameter of $\hat{c}_1, \hat{c}_2$ & $\hat{c}_3$

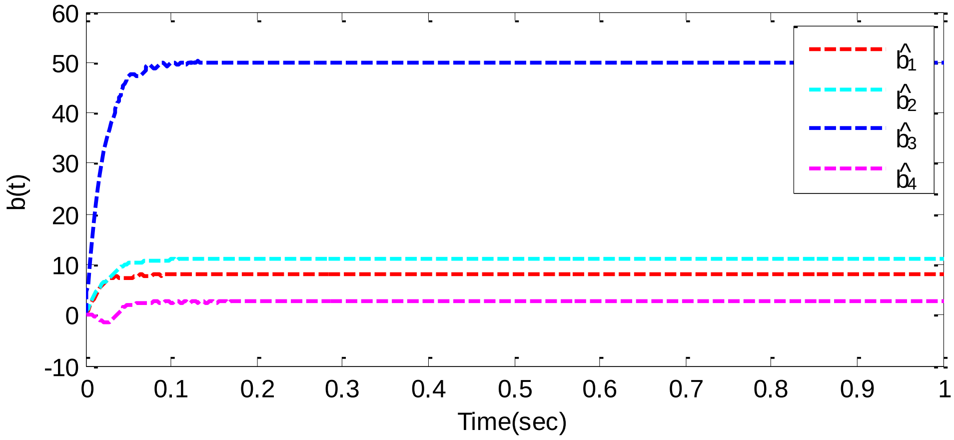

Figure 5.18: Estimation parameter of $\hat{b}_1, \hat{b}_2, \hat{b}_3$ & $\hat{b}_4$

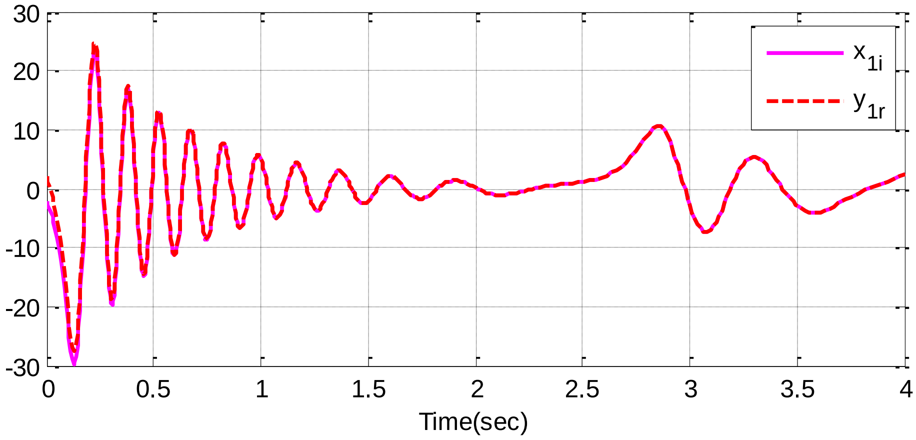

Figure 5.19: Time Response of x_{1i}&y_{1r} with IC (-2, 2)

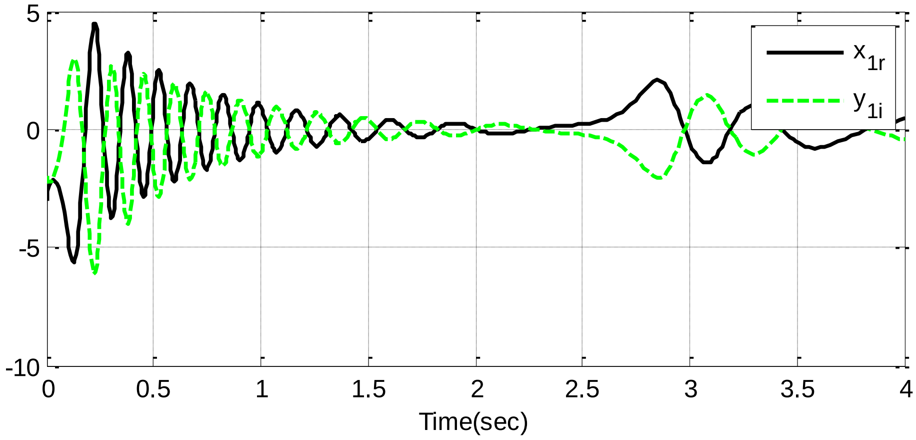

Figure 5.20.: Time Response of x_{1r}&y_{1i} with IC (-3, -2)

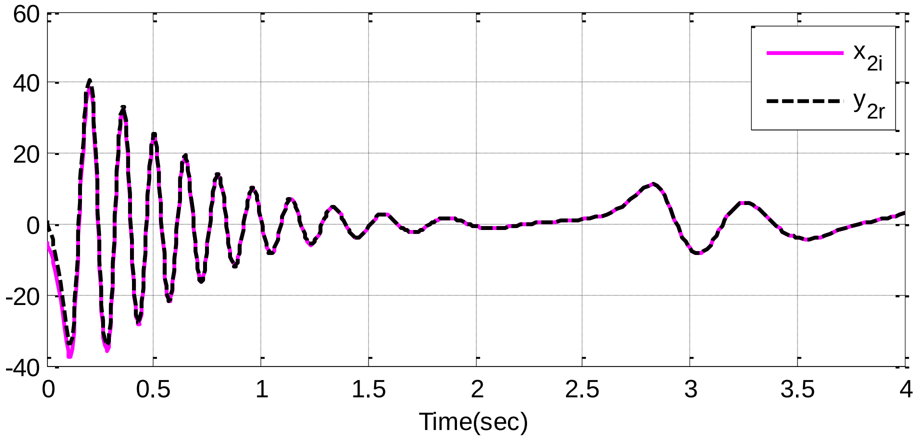

Figure 5.21: Time Response of x_{2i}&y_{2r} with IC (-2, 1)

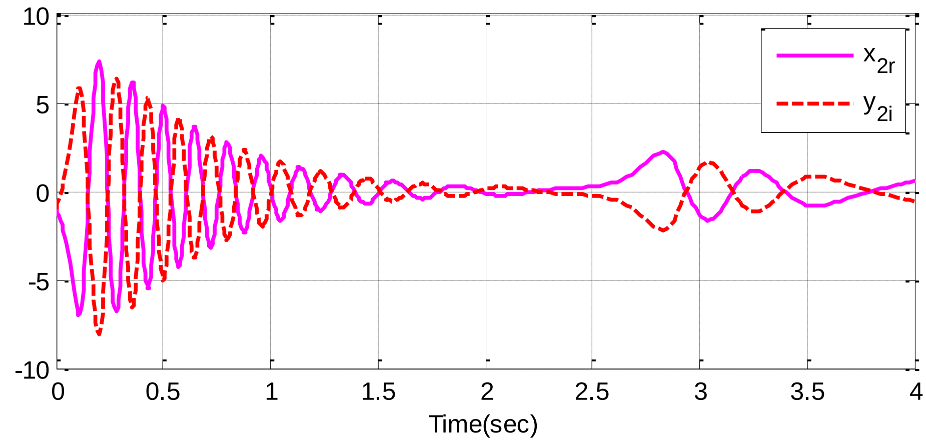

Figure 5.22: Time Response of x_{2r} & y_{2i} with IC (-5, 1)

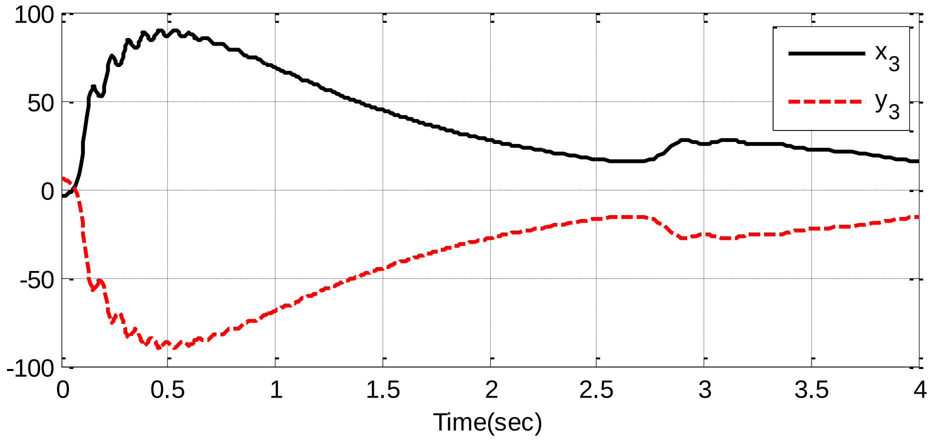

Figure 5.23: Time Response of x_3 & y_3 with IC (-4, 6)

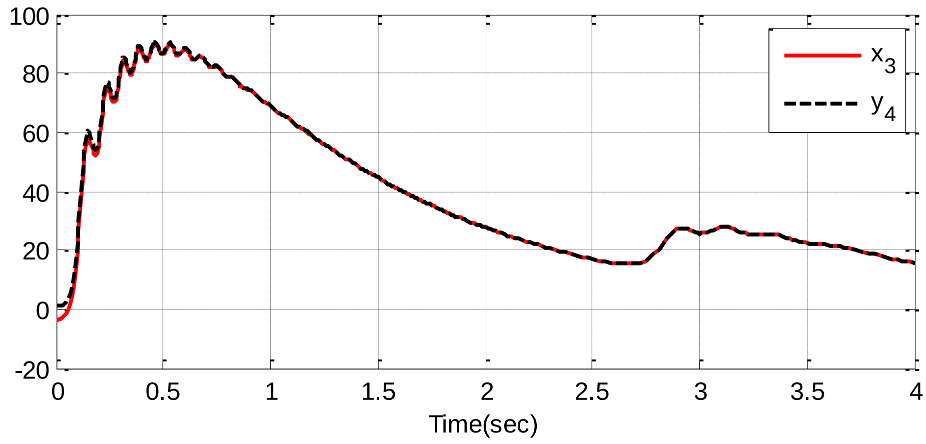

Figure 5.24: Time Response of x_3 & y_4 with IC (-4, 1)

Case 3 when (*n*>*m*):

Consider the Master system given in [49] as:

$$\begin{aligned}
\dot{x}_1 &= a_1(x_2 - x_1) \\
\dot{x}_2 &= -x_1x_3 + a_2x_2 - a_3(\alpha_1 + 3\beta_1 x_4^2)x_1 \\
\dot{x}_3 &= 0.5(\bar{x}_1\, x_2 + x_1\bar{x}_2) - a_4x_3 \\
\dot{x}_4 &= 0.5(\bar{x}_1 + x_1)
\end{aligned} \qquad (5.46)$$

Where, $x_1 = x_{1r} + jx_{1i}$, $x_2 = x_{2r} + jx_{2i}$ are complex and $x_3 = x_{3r}$, $x_4 = x_{4r}$ are real. $\bar{x}_1, \bar{x}_2$ denote the complex conjugate variables of x_1, x_2. a_1, a_2, a_3 and a_4 are unknown real parameters α_1 and β_1 are considered as known positive constants. When $\alpha_1 = 4$, $\beta_1 = 0.01$, $a_1 = 36$, $a_2 = 20$, $a_3 = 3.2$, $a_4 = 3$ and $x(0) = [-1+2j,\ 1+j,\ 2, -1]^T$

The Slave system given in [49] as:

$$\begin{aligned}
\dot{y}_1 &= d_1(y_2 - y_1) + u_1 \\
\dot{y}_2 &= -y_1y_3 + d_2y_2 + u_2 \\
\dot{y}_3 &= 0.5(\bar{y}_1\, y_2 + y_1\bar{y}_2) - d_3y_3 + u_3
\end{aligned} \qquad (5.47)$$

Where, $y_1 = y_{1r} + jy_{1i}$, $y_2 = y_{2r} + jy_{2i}$ are complex and $y_3 = y_{3r}$, are real. $\bar{y}_1, \bar{y}_2$ denote the complex conjugate variables of y_1, y_2. d_1, d_2, d_3 are unknown.. u_1, u_2, u_3 are controllers.When $d_1 = 29,\ d_2 = 21,\ d_3 = 2$ $y(0) = [4+10j,\ 6+10j,\ 12]^T$

In this section, we investigate CGS of two nonidentical complex systems with the different orders.

Let $\hat{a}_i, \hat{d}_f, i = 1,..,4, f = 1,...,3$ be estimates of $a_i, d_f, i = 1,..,4, f = 1,...,3$ and $\tilde{a}_i = a_i - \hat{a}_i, \tilde{d}_f = d_f - \hat{d}_f, i = 1,....,4, f = 1,...,3$, be the errors in estimations of $a_i, b_f, i = 1,..,4, f = 1,...3$, respectively.

Then systems (5.46) and (5.47) can be written as:

$$\begin{aligned}
\dot{x}_1 &= \hat{a}_1(x_2 - x_1) + \tilde{a}_1(x_2 - x_1) \\
\dot{x}_2 &= -x_1x_3 + \hat{a}_2x_2 + \tilde{a}_2x_2 - \hat{a}_3(\alpha_1 + 3\beta_1 x_4^2)x_1 - \tilde{a}_3(\alpha_1 + 3\beta_1 x_4^2)x_1 \\
\dot{x}_3 &= 0.5(\bar{x}_1\, x_2 + x_1\bar{x}_2) - \hat{a}_4x_3 - \tilde{a}_4x_3 \\
\dot{x}_4 &= 0.5(\bar{x}_1 + x_1)
\end{aligned} \qquad (5.48)$$

$$\dot{y}_1 = \hat{d}_1(y_2 - y_1) + \tilde{d}_1(y_2 - y_1) + u_1$$
$$\dot{y}_2 = -y_1 y_3 + \hat{d}_2 y_2 + \tilde{d}_2 y_2 + u_2 \tag{5.49}$$
$$\dot{y}_3 = 0.5(\bar{y}_1 y_2 + y_1 \bar{y}_2) - \hat{d}_3 y_3 - \tilde{d}_3 y_3 + u_3$$

The 4-dimensional complex systems (4.48) can be into 6-dimensional real systems:

$$\begin{aligned}
\dot{x}_{1r} &= \hat{a}_1(x_{2r} - x_{1r}) + \tilde{a}_1(x_{2r} - x_{1r}) \\
\dot{x}_{1i} &= \hat{a}_1(x_{2i} - x_{1i}) + \tilde{a}_1(x_{2i} - x_{1i}) \\
\dot{x}_{2r} &= -x_{1r} x_3 + \hat{a}_2 x_{2r} + \tilde{a}_2 x_{2r} - \hat{a}_3(\alpha_1 + 3\beta_1 x_4^2) x_{1r} - \tilde{a}_3(\alpha_1 + 3\beta_1 x_4^2) x_{1r} \\
\dot{x}_{2i} &= -x_{1i} x_3 + \hat{a}_2 x_{2i} + \tilde{a}_2 x_{2i} - \hat{a}_3(\alpha_1 + 3\beta_1 x_4^2) x_{1i} - \tilde{a}_3(\alpha_1 + 3\beta_1 x_4^2) x_{1i} \\
\dot{x}_3 &= (x_{1r} x_{2r} + x_{1i} x_{2i}) - \hat{a}_4 x_3 - \tilde{a}_4 x_3 \\
\dot{x}_4 &= 0.5 x_{1r}
\end{aligned} \tag{5.50}$$

The 3-D complex system (5.49) can be written into 5-D real system as:

$$\begin{aligned}
\dot{y}_{1r} &= \hat{d}_1(y_{2r} - y_{1r}) + \tilde{d}_1(y_{2r} - y_{1r}) + u_{1r} \\
\dot{y}_{1i} &= \hat{d}_1(y_{2i} - y_{1i}) + \tilde{d}_1(y_{2i} - y_{1i}) + u_{1i} \\
\dot{y}_{2r} &= -y_{1r} y_3 + \hat{d}_2 y_{2r} + \tilde{d}_2 y_{2r} + u_{2r} \\
\dot{y}_{2i} &= -y_{1i} y_3 + \hat{d}_2 y_{2i} + \tilde{d}_2 y_{2i} + u_{2i} \\
\dot{y}_3 &= y_{1r} y_{2r} + y_{1i} y_{2i} - \hat{d}_3 y_3 - \tilde{d}_3 y_3 + u_3
\end{aligned} \tag{5.51}$$

The complex map vector is given by

$\phi(x) = \begin{bmatrix} jx_2 & jx_1 & x_3 & x_4^2 \end{bmatrix}^T$. This gives:

$$\begin{aligned}
&\phi_{1r}(x) = x_{2i}, \phi_{1i}(x) = -x_{2r}, \phi_{2r}(x) = x_{1i}, \\
&\phi_{2i}(x) = -x_{1r}, \phi_3(x) = -x_3 + x_4^2
\end{aligned} \tag{5.52}$$

The error signals are defined as:

$$\begin{aligned}
&e_{1r} = y_{1r} - \phi_{1r}, e_{1i} = y_{1i} - \phi_{1i}, e_{2r} = y_{2r} - \phi_{2r}, \\
&e_{2i} = y_{2i} - \phi_{2i}, e_3 = y_3 - \phi_3
\end{aligned} \tag{5.53}$$

Then the error dynamics becomes:

$$
\begin{aligned}
&\dot{e}_{1r} = \dot{y}_{1r} - \dot{\phi}_{1r} = \dot{y}_{1r} + \dot{x}_{2i} = \hat{d}_1(y_{2r} - y_{ir}) + \tilde{d}_1(y_{2r} - y_{ir}) + u_{r1} - x_{1i}x_3 \\
&+ \hat{a}_2 x_{2i} + \tilde{a}_2 x_{2i} - \hat{a}_3(\alpha_1 + 3\beta_1 x_4^2)x_{1i} - \tilde{a}_3(\alpha_1 + 3\beta_1 x_4^2)x_{1i} \\
&\dot{e}_{1i} = \dot{y}_{1i} - \dot{\phi}_{1i} = \dot{y}_{1i} - \dot{x}_{2r} = \hat{d}_1(y_{2i} - y_{1i}) + \tilde{d}_1(y_{2i} - y_{1i}) + u_{1i} + x_{1r}x_3 \\
&- \hat{a}_2 x_{2r} - \tilde{a}_2 x_{2r} + \hat{a}_3(\alpha_1 + 3\beta_1 x_4^2)x_{1r} + \tilde{a}_3(\alpha_1 + 3\beta_1 x_4^2)x_{1r} \\
&\dot{e}_{2r} = \dot{y}_{2r} - \dot{\phi}_{2r} = \dot{y}_{2r} + \dot{x}_{1i} = -y_{1r}y_3 + \hat{d}_2 y_{2r} + \tilde{d}_2 y_{2r} + u_{2r} \\
&+ \hat{a}_1(x_{2i} - x_{1i}) + \tilde{a}_1(x_{2i} - x_{1i}) \\
&\dot{e}_{2i} = \dot{y}_{2i} - \dot{\phi}_{2i} = \dot{y}_{2i} - \dot{x}_{1r} = -y_{1i}y_3 + \hat{d}_2 y_{1i} + \tilde{d}_2 y_{1i} + u_{2i} \\
&- \hat{a}_1(x_{2r} - x_{1r}) - \tilde{a}_1(x_{2r} - x_{1r}) \\
&\dot{e}_3 = \dot{y}_3 - \dot{\phi}_3 = \dot{y}_3 - (\dot{x}_3 - 2x_4\dot{x}_4) = y_{1r}\, y_{2r} + y_{1i}y_{2i} - \hat{d}_3 y_3 - \tilde{d}_3 y_3 + u_3 \\
&- x_{1r}\, x_{2r} - x_{1i}x_{2i} + \hat{a}_4 x_3 + \tilde{a}_4 x_3 + x_4 x_{1r}
\end{aligned}
\tag{5.54}
$$

By choosing:

$$
\begin{aligned}
&u_{1r} = -\hat{d}_1(y_{2r} - y_{ir}) + x_{1i}x_3 - \hat{a}_2 x_{2i} + \hat{a}_3(\alpha_1 + 3\beta_1 x_4^2)x_{1i} + e_{1i} \\
&u_{1i} = -\hat{d}_1(y_{2i} - y_{1i}) - x_{1r}x_3 + \hat{a}_2 x_{2r} - \hat{a}_3(\alpha_1 + 3\beta_1 x_4^2)x_{1r} + e_{2r} \\
&u_{2r} = y_{1r}y_3 - \hat{d}_2 y_{2r} - \hat{a}_1(x_{2i} - x_{1i}) + e_{2i} \\
&u_{2i} = y_{1i}y_3 - \hat{d}_2 y_{1i} + \hat{a}_1(x_{2r} - x_{1r}) + e_3 \\
&u_3 = -y_{1r}\, y_{2r} - y_{1i}y_{2i} + \hat{d}_3 y_3 + x_{1r}\, x_{2r} + x_{1i}x_{2i} - \hat{a}_4 x_3 - x_4 x_{1r} + v
\end{aligned}
\tag{5.55}
$$

where v is the new input, the system (5.54) can be written as:

$$
\begin{aligned}
&\dot{e}_{1r} = \tilde{d}_1(y_{2r} - y_{ir}) + \tilde{a}_2 x_{2i} - \tilde{a}_3(\alpha_1 + 3\beta_1 x_4^2)x_{1i} + e_{1i} \\
&\dot{e}_{1i} = \tilde{d}_1(y_{2i} - y_{1i}) - \tilde{a}_2 x_{2r} + \tilde{a}_3(\alpha_1 + 3\beta_1 x_4^2)x_{1r} + e_{2r} \\
&\dot{e}_{2r} = \tilde{d}_2 y_{2r} + \tilde{a}_1(x_{2i} - x_{1i}) + e_{2i} \\
&\dot{e}_{2i} = \tilde{d}_2 y_{2i} - \tilde{a}_1(x_{2r} - x_{1r}) + e_3 \\
&\dot{e}_3 = -\tilde{d}_3 y_3 + \tilde{a}_4 x_3 + v
\end{aligned}
\tag{5.56}
$$

To employ the integral sliding mode control, choose the nominal system for (5.56) as:

$$
\begin{aligned}
&\dot{e}_{1r} = e_{1i} \\
&\dot{e}_{1i} = e_{2r} \\
&\dot{e}_{2r} = e_{2i} \\
&\dot{e}_{2i} = e_3 \\
&\dot{e}_3 = v_0
\end{aligned}
\tag{5.57}
$$

Define the sliding surface for nominal system (5.57) as:

$$
\sigma_0 = (1 + \frac{d}{dt})^4 e_{1r} = e_{1r} + 4e_{1i} + 6e_{2r} + 4e_{2i} + e_3
$$

Then

$$\dot{\sigma}_0 = \dot{e}_{1r} + 4\dot{e}_{1i} + 6\dot{e}_{2r} + 4\dot{e}_{2i} + \dot{e}_3 = e_{1i} + 4e_{2r} + 6e_{2i} + 4e_3 + v_0$$

By choosing $v_0 = -e_{1i} - 4e_{2r} - 6e_{2i} - 4e_3 - k\,\sigma_0 - k\,sign(\sigma_0),\ k > 0$, we have $\dot{\sigma}_0 = -k\,\sigma_0 - k\,sign(\sigma_0)$. Therefore the nominal system (5.57) is asymptotically stable.

Now choose the sliding surface for the system (5.56) as:

$$\sigma = \sigma_0 + z = e_{1r} + 4e_{1i} + 6e_{2r} + 4e_{2i} + e_3 + z$$

where, z is some integral term computed later. To avoid the reaching phase, choose $z(0)$ such that $\sigma(0) = 0$. Choose $v = v_0 + v_s$ where, v_0 is the nominal input and v_s is compensator term computed later. Then

$$\begin{aligned}
\dot{\sigma} &= \dot{e}_{1r} + 4\dot{e}_{1i} + 6\dot{e}_{2r} + 4\dot{e}_{2i} + \dot{e}_3 + \dot{z} \\
&= \tilde{d}_1(y_{2r} - y_{ir}) + \tilde{a}_2 x_{2i} - \tilde{a}_3(\alpha_1 + 3\beta_1 x_4^2)x_{1i} + e_{1i} + 4\tilde{d}_1(y_{2i} - y_{1i}) - 4\tilde{a}_2 x_{2r} \\
&+ 4\tilde{a}_3(\alpha_1 + 3\beta_1 x_4^2)x_{1r} + 4e_{2r} + 6\tilde{d}_2 y_{2r} + 6\tilde{a}_1(x_{2i} - x_{1i}) + 6e_{2i} + 4\tilde{d}_2 y_{2i} \\
&- 4\tilde{a}_1(x_{2r} - x_{1r}) + 4e_3 - \tilde{d}_3 y_3 + \tilde{a}_4 x_3 + v_0 + v_s + \dot{z}
\end{aligned} \qquad (5.58)$$

By choosing a Lyapunov function: $V = \frac{1}{2}\sigma^2 + \frac{1}{2}(\tilde{a}_1^2 + \tilde{a}_2^2 + \tilde{a}_3^2 + \tilde{a}_4^2 + \tilde{d}_1^2 + \tilde{d}_2^2 + \tilde{d}_3^2)$

design the adaptive laws for $\tilde{a}_i, \hat{a}_i, i = 1,...,4, \tilde{d}_f, \hat{d}_f, f = 1,...,3$ and compute v_s such that $\dot{V} < 0$.

$$\begin{aligned}
&\dot{z} = -e_{1i} - 4e_{2r} - 6e_{2i} - 4e_3 - v_0, \quad v_s = -k\sigma - k\,sign(\sigma) \\
&\dot{\tilde{a}}_1 = 4\sigma(x_{2r} - x_{1r}) - 6\sigma(x_{2i} - x_{1i}) - k_1\tilde{a}_1\ , \quad \dot{\hat{a}}_1 = -\dot{\tilde{a}}_1 \\
&\dot{\tilde{a}}_2 = -\sigma\, x_{2i} + 4\sigma\, x_{2r} - k_2\tilde{a}_2\ , \quad \dot{\hat{a}}_2 = -\dot{\tilde{a}}_2 \\
&\dot{\tilde{a}}_3 = \sigma(\alpha_1 + 3\beta_1 x_4^2)x_{1i} - 4\sigma(\alpha_1 + 3\beta_1 x_4^2)x_{1r} - k_3\tilde{a}_3\ , \quad \dot{\hat{a}}_3 = -\dot{\tilde{a}}_3 \\
&\dot{\tilde{a}}_4 = -\sigma x_3 - k_4\tilde{a}_4\ , \quad \dot{\hat{a}}_4 = -\dot{\tilde{a}}_4 \\
&\dot{\tilde{d}}_1 = -\sigma(y_{2r} - y_{1r}) - 4\sigma(y_{2i} - y_{1i}) - k_5\tilde{b}_1, \quad \dot{\hat{d}}_1 = -\dot{\tilde{d}}_1 \\
&\dot{\tilde{d}}_2 = -6\sigma\, y_{2r} - 4\sigma\, y_{2i} - k_6\tilde{b}_2, \quad \dot{\hat{d}}_2 = -\dot{\tilde{d}}_2 \\
&\dot{\tilde{d}}_3 = \sigma\, y_3 - k_7\tilde{b}_3, \quad \dot{\hat{d}}_3 = -\dot{\tilde{d}}_3,\ k, k_i > 0, i = 1,...,7
\end{aligned} \qquad (5.59)$$

Proof:

Since

$$
\begin{aligned}
\dot V &= \sigma\dot\sigma + \tilde a_1\dot{\tilde a}_1 + \tilde a_2\dot{\tilde a}_2 + \tilde a_3\dot{\tilde a}_3 + \tilde a_4\dot{\tilde a}_4 + \tilde d_1\dot{\tilde d}_1 + \tilde d_2\dot{\tilde d}_2 + \tilde d_3\dot{\tilde d}_3 \\
&= \sigma\{\tilde d_1(y_{2r} - y_{ir}) + \tilde a_2 x_{2i} - \tilde a_3(\alpha_1 + 3\beta_1 x_4^2)x_{1i} + e_{1i} + 4\tilde d_1(y_{2i} - y_{1i}) - 4\tilde a_2 x_{2r} \\
&+ 4\tilde a_3(\alpha_1 + 3\beta_1 x_4^2)x_{1r} + 4e_{2r} + 6\tilde d_2 y_{2r} + 6\tilde a_1(x_{2i} - x_{1i}) + 6e_{2i} + 4\tilde d_2 y_{2i} \\
&- 4\tilde a_1(x_{2r} - x_{1r}) + 4e_3 - \tilde d_3 y_3 + \tilde a_4 x_3 + v_0 + v_s + \dot z\} \\
&+ \tilde a_1\dot{\tilde a}_1 + \tilde a_2\dot{\tilde a}_2 + \tilde a_3\dot{\tilde a}_3 + \tilde a_4\dot{\tilde a}_4 + \tilde d_1\dot{\tilde d}_1 + \tilde d_2\dot{\tilde d}_2 + \tilde d_3\dot{\tilde d}_3 \\
&= \sigma\{e_{1i} + 4e_{2r} + 6e_{2i} + 4e_3 + v_0 + v_s + \dot z\} \\
&+ \tilde a_1\{\dot{\tilde a}_1 - 4\sigma(x_{2r} - x_{1r}) + 6\sigma(x_{2i} - x_{1i})\} + \tilde a_2\{\dot{\tilde a}_2 + \sigma x_{2i} - 4\sigma x_{2r}\} \\
&+ \tilde a_3\{\dot{\tilde a}_3 - \sigma(\alpha_1 + 3\beta_1 x_4^2)x_{1i} + 4\sigma(\alpha_1 + 3\beta_1 x_4^2)x_{1r}\} + \tilde a_4\{\dot{\tilde a}_4 + \sigma x_3\} \\
&+ \tilde d_1\{\dot{\tilde d}_1 + \sigma(y_{2r} - y_{1r}) + 4\sigma(y_{2i} - y_{1i})\} + \tilde d_2\{\dot{\tilde d}_2 + 6\sigma y_{2r} + 4\sigma y_{2i}\} + \tilde d_3\{\dot{\tilde d}_3 - \sigma y_3\}
\end{aligned}
$$

By using

$$
\begin{aligned}
&\dot z = -e_{1i} - 4e_{2r} - 6e_{2i} - 4e_3 - v_0, \quad v_s = -k\sigma - k\,sign(\sigma) \\
&\dot{\tilde a}_1 = 4\sigma(x_{2r} - x_{1r}) - 6\sigma(x_{2i} - x_{1i}) - k_1\tilde a_1\ , \quad \dot{\hat a}_1 = -\dot{\tilde a}_1 \\
&\dot{\tilde a}_2 = -\sigma x_{2i} + 4\sigma x_{2r} - k_2\tilde a_2\ , \quad \dot{\hat a}_2 = -\dot{\tilde a}_2 \\
&\dot{\tilde a}_3 = \sigma(\alpha_1 + 3\beta_1 x_4^2)x_{1i} - 4\sigma(\alpha_1 + 3\beta_1 x_4^2)x_{1r} - k_3\tilde a_3\ , \quad \dot{\hat a}_3 = -\dot{\tilde a}_3 \\
&\dot{\tilde a}_4 = -\sigma x_3 - k_4\tilde a_4\ , \quad \dot{\hat a}_4 = -\dot{\tilde a}_4 \\
&\dot{\tilde d}_1 = -\sigma(y_{2r} - y_{1r}) - 4\sigma(y_{2i} - y_{1i}) - k_5\tilde d_1, \quad \dot{\hat d}_1 = -\dot{\tilde d}_1 \\
&\dot{\tilde d}_2 = -6\sigma y_{2r} - 4\sigma y_{2i} - k_6\tilde d_2, \quad \dot{\hat d}_2 = -\dot{\tilde d}_2 \\
&\dot{\tilde d}_3 = \sigma y_3 - k_7\tilde d_3, \quad \dot{\hat d}_3 = -\dot{\tilde d}_3,\ k, k_i > 0, i = 1,\ldots,7
\end{aligned}
$$

We have

$$\dot V = -k\sigma^2 - k|\sigma| - k_1\tilde a_1^2 - k_2\tilde a_2^2 - k_3\tilde a_3^2 - k_4\tilde a_4^2 - k_5\tilde d_1^2 - k_6\tilde d_2^2 - k_7\tilde d_3^2 .$$

From this we conclude that $\sigma, \tilde a_i, \tilde b_i \to 0$. Since $\sigma \to 0$, therefore $e = (e_{1r}, e_{1i}, e_{2r}, e_{2i}, e_3) \to 0$.

In simulations, the initial conditions are chosen as: $x(0) = [-1+2j, 1+j, 2, -1]^{\mathrm T}$,and $y(0) = [4+16j, 6+10j, 12]^{\mathrm T}$. The values of known parameters are: $\alpha_1 = 4, \beta_1 = 0.01$. The true value of the unknown parameters are chosen as: $a_1 = 36$, $a_2 = 20$, $a_3 = 3.2$, $a_4 = 3, d_1 = 29, d_2 = 21, d_3 = 2.$

Generalized Synchronization results of unknown parameter:

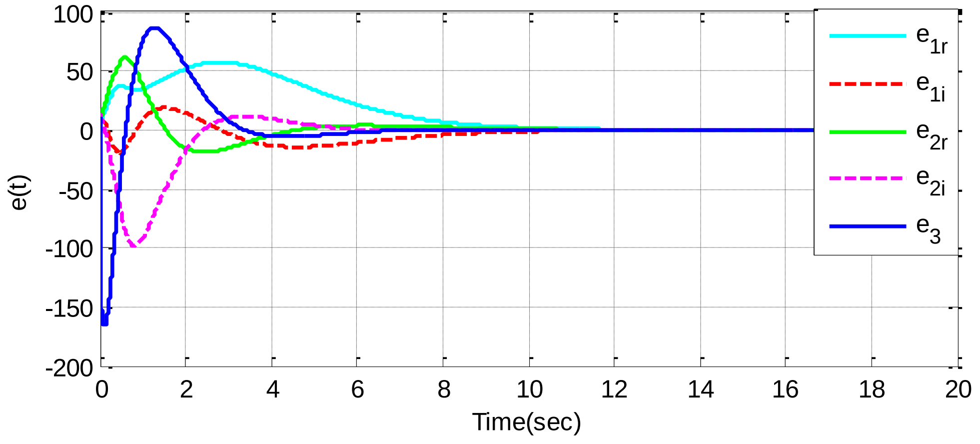

Figure 5.25: Time Response of error $e_{1r}, e_{1i}, e_{2r}, e_{2i}$ & e_3

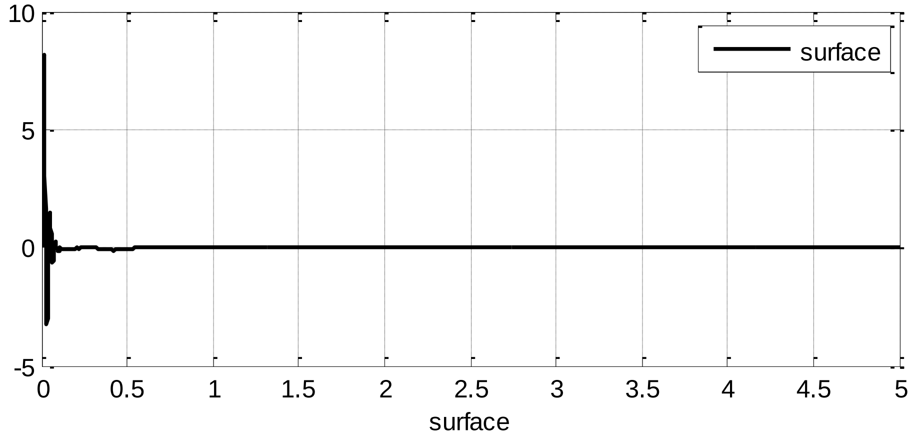

Figure 5.26: Time Response of surface

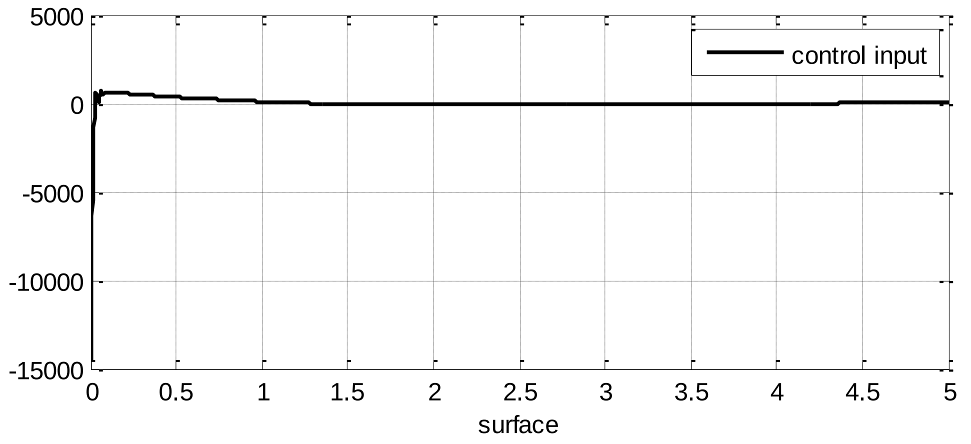

Figure 5.27: Time Response of control input

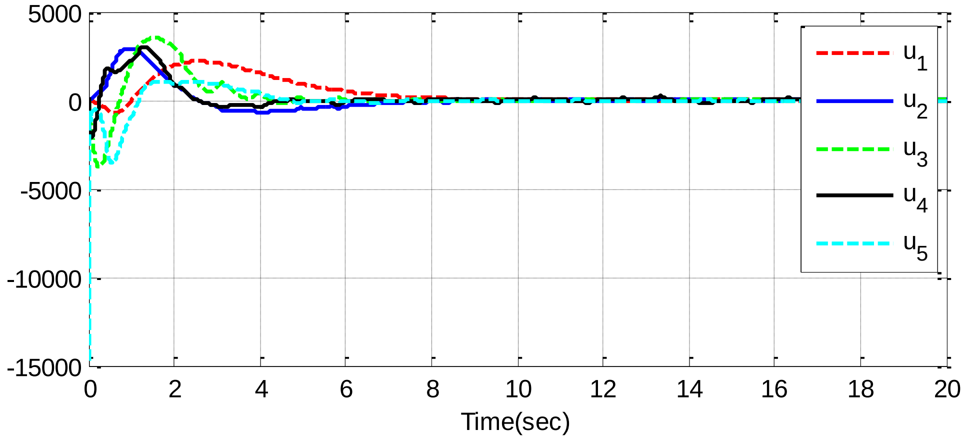

Figure 5.28: Time Response of adaptive controller u_1, u_2, u_3, u_4 & u_5

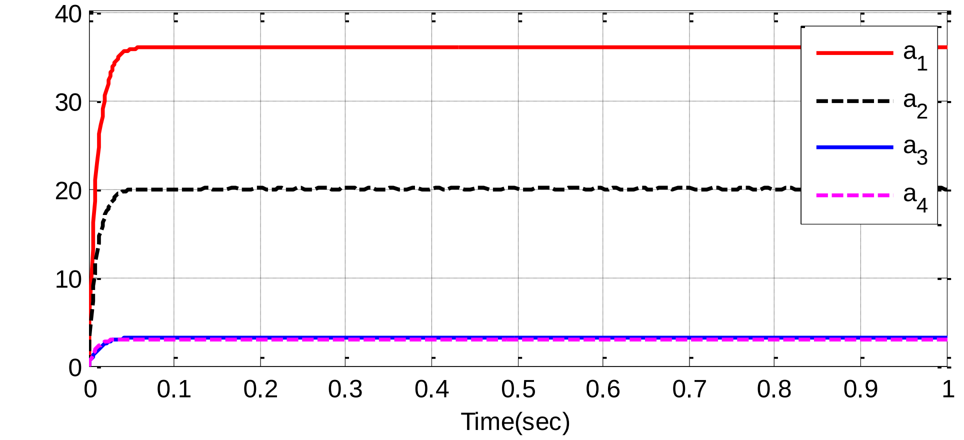

Figure 5.29: Estimation parameter of $\hat{a}_1, \hat{a}_2, \hat{a}_3$ & $\hat{a}_4$

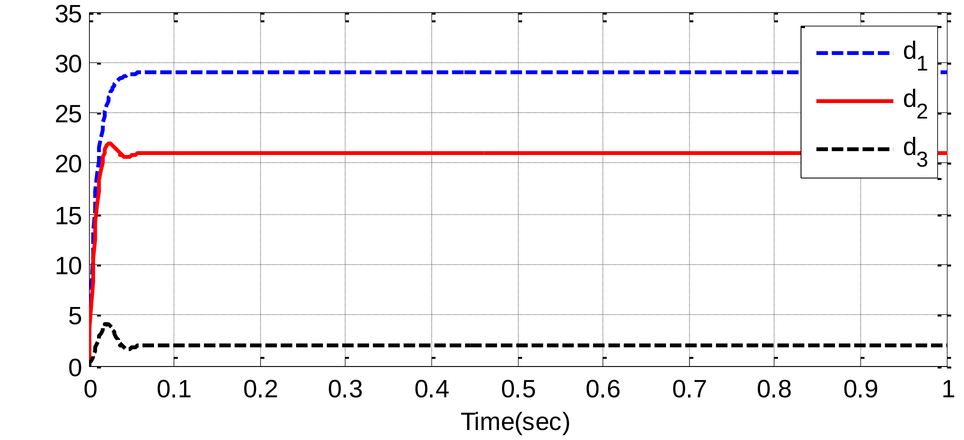

Figure 5.30: Estimation parameter of $\hat{d}_1, \hat{d}_2$ & $\hat{d}_3$

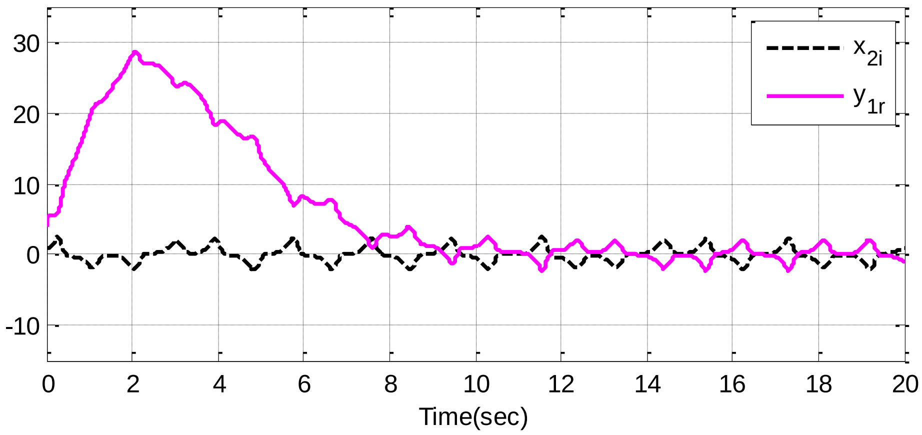

Figure 5.31: Time Response of x_{2i}&y_{1r} with IC (1, 4)

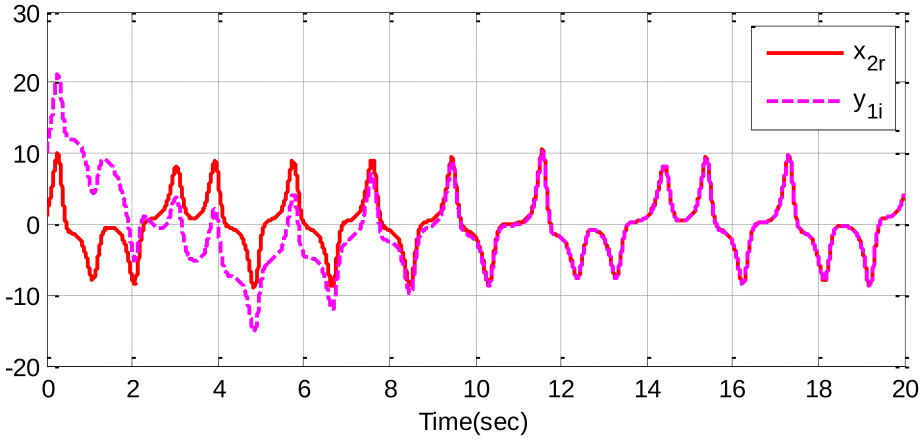

Figure 5.32: Time Response of x_{2r}&$y_{1i}$with IC (1, 10)

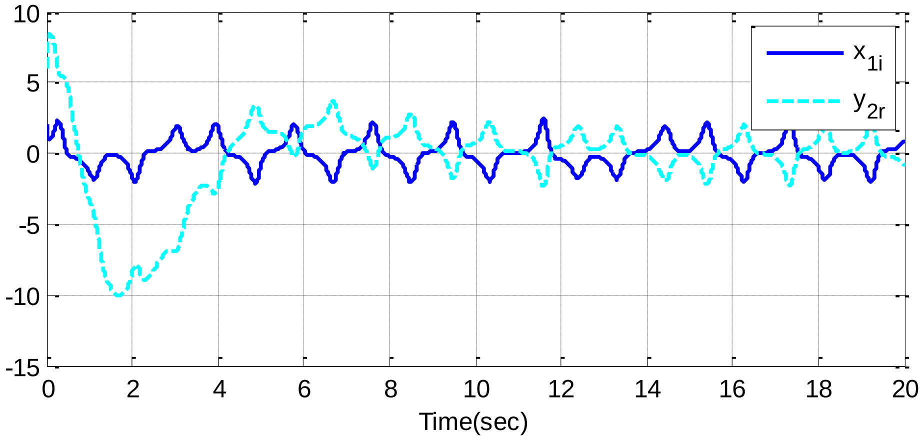

Figure 5.33: Time Response of x_{1i}&y_{2r} with IC (2, 6)

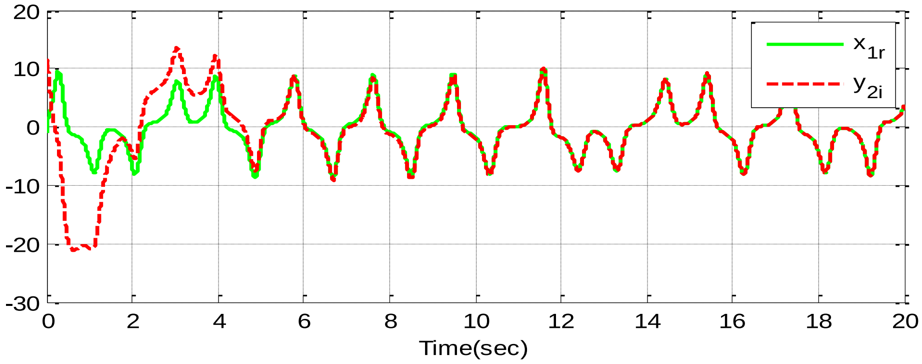

Figure 5.34: Time Response of $x_{1r}\&y_{2i}$ with IC (-1, 10)

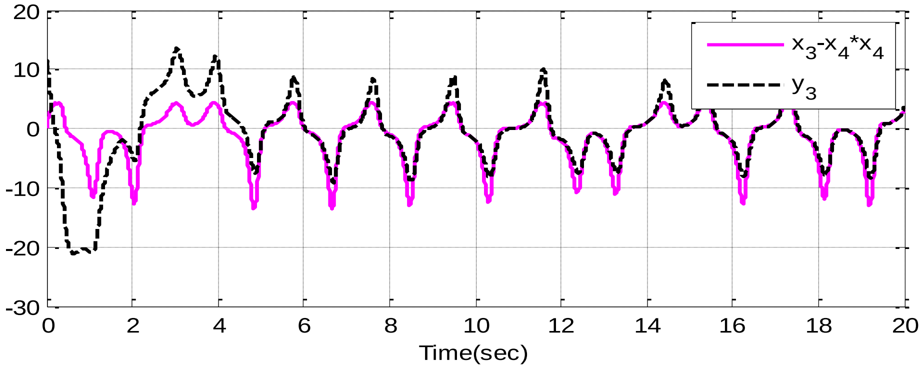

Figure 5.35: Time Response of $x_3 - x_4^2\&y_3$ with IC (2, -1, 12)

Chapter 6

CONCLUSION AND FUTURE WORK

6.1 Introduction

The work done by human being can never be complete. Taking into account this reality, this chapter is aimed to explain the results and conclusion of this research thesis.

6.2 Conclusion

This thesis presents three different kinds of complex synchronization (CS), (i) Complex Complete Synchronization (CCS), (ii) Complex Projective Synchronization (CPS), (iii) Complex Generalized Synchronization (CGS) of Identical and Non-identical Nonlinear Complex Systems with unknown parameters. Based on adaptive integral sliding mode control, an adaptive controller and parameter update laws are designed to realize CCS, CPS and CGS. To employ the adaptive integral sliding mode control, the error system is transformed into a special structure containing nominal part and some unknown terms. The unknown terms are computed adaptively. Then the error system is stabilized using adaptive integral sliding mode control. The stabilizing controller for the error system is constructed which consists of the nominal control plus some compensator control. The compensator controller and the adapted law are derived in such a way that the time derivative of a Lyapunov function becomes strictly negative. The proposed scheme is successfully applied to complex chaotic nonlinear systems with unknown parameters for the realization of (i) Complex Complete Synchronization (CCS),(ii) Complex Projective Synchronization (CPS),(iii) Complex Generalized synchronization (CGS).Numerical results have verified the effectiveness and feasibility of the presented method.

6.3 Future Work

In this thesis Synchronization of different complex chaotic systems is considered. This work can be extended for complex chaotic systems, such as the typical multi-scroll chaotic systems by some effective design methods using piecewise-linear functions, cellular neural networks, nonlinear modulating functions, circuit component design, switching manifolds, etc.

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