CAPITAL UNIVERSITY OF SCIENCE AND TECHNOLOGY, ISLAMABAD

Fixed Point and Data Dependence Results for Non-linear Contractions in Controlled Metric Space

by

Abdul Daim

A thesis submitted in partial fulfillment for the degree of Master of Philosophy

in the

Faculty of Computing
Department of Mathematics

2023

Dedicated to my parents

CERTIFICATE OF APPROVAL

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Acknowledgements

In the name of **ALLAH**, the Compassionate, the Merciful, praise be to **ALLAH**, lord of the Universe, and peace and prayers be upon his final Prophet and Messenger.

I want to express my sincere gratitude to my kind supervisor **Dr. Dur-e-Shehwar Sagheer** for her motivation and continuous support in MPhil study and research, for her patience, motivation, enthusiasm, and immense knowledge. Her guidance helped me a lot at the time of research and writing of this thesis. Her kind effort and motivation will never be forgotten.

I thank the Mathematics Department at Capital University of Science and Technology for their help and support. I want to express my appreciation to the Head of the department, **Dr. Muhammad Sagheer**, for providing us with a learning and creative environment.

I am grateful to my grandmother and parents for their prayers, love, and motivation. It would not be justice if I didn't mention my brothers, sisters, and friends for their support in completing my degree program.

I thank my fellows, **Shahmir Ahmed** and **Aqib Saghir**, for supporting me during my degree programs.

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Abstract

The present research is aimed to analyse the existence of strict fixed points (SFP) and fixed points of multi-valued generalized contractions on the platform of controlled metric spaces (CMS). Wardowski type multivalued non-linear operators and Reich type (α, F)-contractions have been introduced by means of auxiliary functions, which modifies a new form of contractive requirements. Well-posedness of proved fixed point results is also established. Moreover, data dependence result for fixed point is provided. Some supporting examples are also available for better perception. Many existing results in literature are the special case of the results established in this dissertation.

Contents

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Symbols

(ξ, d)	Metric space
$(\xi, d, \mathfrak{f})$	Controlled metric space
Σ	Summation sign
Π	Pi
$\Rightarrow$	Implies that
$\forall$	For all
$\mathrm{Fix}\Omega$	The set containing fixed points of Ω
$\mathrm{SFix}\Omega$	The set containing strict fixed points of Ω
$\emptyset$	Empty set
$\in$	Belongs to
$\notin$	Does not belong to
H	Hausdorf metric function
$\mathbb{R}$	The set of real numbers
$\mathbb{R}^+$	The set of non-negative real numbers
$\mathbb{N}$	The set of natural numbers
$\longrightarrow$	Approaches to
$\iff$	if and only if

Chapter 1

Introduction

Functional analysis is the most interesting branch of classical mathematical analysis. It was initiated in the 19^{th} century and accepted from 1920 to 1930. Functional analysis deals with functionals and is vital in many mathematics and applied sciences fields. This branch of mathematics is a smart fusion of geometry, topology, and analysis which has remarkable importance in various branches of mathematics and the field of natural sciences.

The solution of non-linear problems has been an important issue in various mathematics and applied sciences disciplines. Fixed point theory deals with the existence of fixed points of certain mappings, which is, in fact, the solution to non-linear problems. Fixed point theorems are related to finding fixed points and investigating their uniqueness.

In 1866, Poincare [1] presented some initiative work on fixed point theory. Afterward, in 1912, Brouwer [2] proved a fixed point theorem on the unit sphere, confirming a fixed point's existence. Kakutani [3] generalized Brouwer fixed point result on set valued function. Fixed point theory is an integral part of functional analysis, used in various applications from general science to optimization, game theory, economics, and approximation theory.

In 1922, a prominent Polish mathematician Stefan Banach [4], did creditable work using contraction condition instead of continuity in his famous theorem named Banach contraction principle (BCP). This theorem assures the existence of a unique fixed point and provides a powerful tool for finding this unique fixed point as well. According to BCP, a self map Ω defined on complete metric space (MS) (ξ, d) satisfying

$$d\big(\Omega\varrho, \Omega\bar{\varrho}\big) \leq \ell d(\varrho, \bar{\varrho}), \quad \text{for all } \varrho, \bar{\varrho} \in \xi \text{ and } \ell \in [0, 1),$$

has a unique fixed point.

There are two major directions in which fixed point theory is developed. One is the generalization of fixed point theorems in the setting of different spaces, for example, metric spaces, Hilbert spaces, Banach spaces, topological spaces, and even by changing the structures of spaces. The second direction is to use the generalization of the contraction condition.

Edelstein gave the first generalization of the Banach contraction condition [5] in 1962 by taking constant $\ell = 1$ and using distinct points from the space ξ. In the same year, Rakotch [6] introduced a contractive condition, in which the constant ℓ of the contraction condition in BCP is replaced by a monotonic decreasing function $\ell : [0, \infty) \longrightarrow [0, 1]$. Kannan [7] introduced a contraction condition in 1968 that does not imply continuity like the Banach contraction. This contraction condition is as follows:

$$d\big(\Omega\varrho, \Omega\bar{\varrho}\big) \leq \ell\Big\{d(\varrho, \Omega\varrho) + d(\bar{\varrho}, \Omega\bar{\varrho})\Big\},$$

for all $\varrho, \bar{\varrho} \in \xi$ and $0 < \ell < \frac{1}{2}$.

In 1969, another extension of BCP was presented by Kannan [8] by omitting the completeness of the space. Following the work of Kannan [7, 8], another contraction condition was introduced by Chatterjee [9] in 1972, which is as follows:

$$d\big(\Omega\varrho, \Omega\bar{\varrho}\big) \leq \ell\Big\{d(\varrho, \Omega\bar{\varrho}) + d(\bar{\varrho}, \Omega\varrho)\Big\},$$

for all $\varrho, \bar{\varrho} \in \xi$ and $0 < \ell < 1$.

Nadler [10] is considered the founder of set-valued contraction. He first introduced the multi-valued contractive mappings and proved two fixed point theorems. The first theorem is the generalization of BCP for multi-valued contractive mapping. Edelstein's result is generalized for compact set-valued contractions in the second theorem. Massive research can be seen on such a generalization. For examples

Reich [11], Bianchini [12] and Caristi [13]. Due to BCP's importance and massive application, it has become a constructive procedure for many mathematicians. Wardowski [14] introduced another well-known contraction, namely F-contraction. In 2013, Sagroi et al. [15] proved some fixed point results on F-contraction. In 2016, Kamran et al. [16] presented an interesting generalization of F-contraction namely (α, F)-contraction. Hussain et al. [17] use this contraction for the multi-valued mappings. In 2020, Anwar et al. [18] proved some fixed point results for nonself multi-valued mappings using Wardowski type (α, F)-contractive approach. A data dependence problem is to estimate the distance between the sets of fixed point of two mappings. This idea is only meaningful if we have an assurance of non-empty fixed point sets of these two operators. The data dependence problem is mostly dealt with set-valued mappings since multivalued mappings often have larger fixed point sets than single-valued mappings. In August 2021, Iqbal et al. [19] introduced an interesting type of multi-valued generalized contraction and proved some fixed point results in the domain of complete MS.
In 1906, the notion of metric space was first presented by Maurice Frechet [20], the founder of metric space, as a generalized formulation of the Euclid distance. On the other hand, the concept of Hausdorff distance is due to Hausdorff [21]. Metric space plays a vital role in many areas of complex, functional, and real analysis. Due to this vital role in many fields, it is extended and generalized in many distinct directions. In 1989, Bakhtin [22] gave the first generalization of metric space, namely b-metric space (b-MS), by changing the triangular inequality of MS. Later on, the concept of b-MS was further used by Czerwick [23, 24] to establish different contraction results in b-MS. Certain literature can be seen on the extensions of existing fixed point results using the set-valued mappings. [25]-[27]. The study of b-MS endowed a prominent place in fixed point theory with multiple aspects. Many mathematicians led the foundation to improve fixed point theory in b-MS. In 1992, Matthews [28] highlighted a new idea of non-zero self distance and introduced the domain of partial metric space. Later, Altun et al. [29] proved some fixed point results for generalized contractive type mappings on partial metric space. In 2014, Shukla [30] gave a unique idea by blending b-MS

and partial metric space. He introduced a new domain of partial b-metric space. Ameer et al. [31] introduced generalized multi-valued (α_K^*, Y, Λ)-contraction and proved some fixed point results using the platform of partial b-metric space.

In 2007, Huang et al. [32] introduced an interesting idea of cone metric space by substituting an ordered Banach space for the set of real numbers. Afterwards, certain fixed point results are obtained using this new approach. In 2015, Ma et al. [33] presented a new concept of C^*-Algebra-valued b-metric spaces and established certain fixed point results for self-maps by using contractive conditions on these spaces. In the next year, Shehwar et al. [34] introduced the notion of partial order and proved the existence of fixed point by using the idea of the minimal element in C^*-Algebra-valued b-metric spaces. One can read [35–37] for more developments in this direction.

In 2017, Kamran et al. [38] introduced a new domain of extended b-metric space by further weakening the triangle inequality. Authors established certain fixed point theorems endowed with extended b-metric space [39–41]. In 2018, Mlaiki et al. [42] made another advancement by employing a control function on the right-hand side of the b-triangle inequality. They introduced an interesting domain of controlled metric space (CMS) and generalized BCP for this new space. Controlled metric spaces have become an exciting topic for researchers nowadays. In August 2021, Iqbal et al. [19] introduced Wardowski-type multi-valued nonlinear operators which satisfy certain contractive conditions. In this article, some fixed point results are established to prove the existence of fixed points and strict fixed points by using a new type of multi-valued generalized contractions. Data dependence and well-posedness of these contractions are also discussed by utilizing the platform of complete MS. Encouraged by the work of Iqbal et al. [19], we offered an idea of generalizing these results in the domain of CMS.

The layout of the thesis is briefly shown below:

Chapter 2 focuses on some basic ideas used in subsequent chapters. The major aim of this chapter is to review some essential definitions with suitable examples. Different types of mappings are portrayed with the help of various suitable examples and graphs. A quick review of some generalized metric spaces is also presented.

Chapter 3 gives an overview of different generalizations of contraction mapping. Certain fixed point results related to these contractions are also discussed. The major part of this chapter is the detailed review of the work of Iqbal et al. [19].

Chapter 4 consists of a discussion about controlled metric space (CMS). Fixed point results of [19] are generalized in the setting of CMS. A suitable example is given to support the new results. Data dependence and the well-posedness of some multi-valued generalized contractions are also articulated.

Chapter 5 concludes our work and unlocks further recommendations for others.

Chapter 2

Preliminaries

This chapter is devoted to discussing some basic concepts of functional analysis. This chapter is subdivided into five sections to better understand related literature.

2.1 Metric Spaces

Metric space is the generalization of the usual distance between two points on $\mathbb{R}$. The notion of the metric was first introduced by Maurice Frechet in 1906.

Definition 2.1. Metric Space.
"Suppose that ξ is a non-empty set and that d is a real valued function on $\xi \times \xi$ with the following three properties.

M1. $d(\varrho_1, \varrho_2) \geq 0 \ \forall \ \varrho_1, \varrho_2 \in \xi$ and $d(\varrho_1, \varrho_2) = 0 \iff \varrho_1 = \varrho_2$,

M2. $d(\varrho_1, \varrho_2) = d(\varrho_2, \varrho_1) \ \forall \ \varrho_1, \varrho_2 \in \xi$,

M3. $d(\varrho_1, \varrho_2) \leq d(\varrho_1, \varrho_3) + d(\varrho_3, \varrho_2) \ \forall \ \varrho_1, \varrho_2, \varrho_3 \in \xi$ (The triangle inequality).

The function d is called a metric on ξ and ξ, taken together with the metric d, is called a metric space which we denote by (ξ, d)." [43]

Example 2.1.

Let ξ denote the collection of all closed intervals in $\mathbb{R}$. For each $\varrho_1 = [\mathsf{u}_1, \mathsf{u}_2]$ and $\varrho_2 = [\mathsf{v}_1, \mathsf{v}_2]$ in ξ, define

$$d(\varrho_1, \varrho_2) = \max\Big\{ \mid \mathsf{v}_1 - \mathsf{u}_1 \mid, \mid \mathsf{v}_2 - \mathsf{u}_2 \mid \Big\}.$$

It is easy to verify that d is a metric on ξ.

Example 2.2.

Consider $\xi = \mathbb{R} \cup \{\infty\} \cup \{-\infty\}$. Define $h \colon \xi \longrightarrow \mathbb{R}$ by the rule

$$h(\varrho) = \begin{cases} \dfrac{\varrho}{1+|\varrho|} & \text{if } -\infty < \varrho < \infty, \\ 1 & \text{if } \varrho = \infty, \\ -1 & \text{if } \varrho = -\infty, \end{cases}$$

evidently, h is one-to-one and $-1 \leq h(\varrho) \leq 1$. Define d on $\xi \times \xi$ by

$$d(\varrho_1, \varrho_2) = \mid h(\varrho_1) - h(\varrho_2) \mid .$$

It is easy to verify that d is a metric on ξ.

Example 2.3.

Let ξ be the collection of all bounded and unbounded sequences of complex numbers. We define a metric function d on $\xi \times \xi$ as,

$$d(\varrho, \bar{\varrho}) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{\mid \varrho_{\mathsf{i}} - \bar{\varrho}_i \mid}{1 + \mid \varrho_{\mathsf{i}} - \bar{\varrho}_i \mid},$$

where $\varrho = \{\varrho_{\mathsf{i}}\}$ and $\bar{\varrho} = \{\bar{\varrho}_i\}$.

Definition 2.2. Open and Closed Ball.

"Let (ξ, d) be a metric space. Given a point $\varrho_0 \in \xi$ and a real number $a > 0$, an open ball is defined as

$$B(\varrho_0, a) = \{\varrho \in \xi : d(\varrho_0, \varrho) < a\},$$

and a closed ball is defined as

$$\overline{B(\varrho_0, a)} = \{\varrho \in \xi : d(\varrho_0, \varrho) \leq a\}.\text{"}[44]$$

Definition 2.3. Open and Closed Set.

"Let (ξ, d) be a metric space. A subset U of ξ is said to be an open set if it contains a ball about each of its points. A subset V of a metric space (ξ, d) is said to be a closed set if its complement in ξ is open, that is $\mathsf{V}^c = \xi - \mathsf{V}$ is open." [44]

Definition 2.4. Sequence in Metric Space.

"A sequence in a metric space ξ is a function $\varrho \colon \mathbb{N} \longrightarrow \xi$. We exhibit the sequence ϱ as $\{\varrho_{\mathfrak{s}}\}$ where $\varrho_{\mathfrak{s}} = \varrho(\mathfrak{s})$. Given a sequence ϱ in a metric space, a subsequence is the restriction of ϱ to an infinite subset $\mathsf{S} \subset \mathbb{N}$." [45]

Definition 2.5. Convergence of a Sequence.

"A sequence $\{\varrho_{\mathfrak{s}}\}$ in a metric space (ξ, d) is said to be convergent to $\varrho \in \xi$, if given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $\mathfrak{s} \geq N$, we have

$$\varrho_{\mathfrak{s}} \in B(\varrho, \epsilon),$$

ϱ is called limit of $\{\varrho_{\mathfrak{s}}\}$ and we write

$$\lim_{\mathfrak{s} \longrightarrow \infty} \varrho_{\mathfrak{s}} = \varrho \text{ or } \varrho_{\mathfrak{s}} \longrightarrow \varrho.\text{" [45]}$$

Example 2.4.

Let $\xi = \mathbb{R}^2$, define metric d on ξ by,

$$d(\varrho, \bar{\varrho}) = \sqrt{(\varrho_1 - \varrho_2)^2 + (\bar{\varrho}_1 - \bar{\varrho}_2)^2}.$$

Let

$$\varrho_{\mathfrak{s}} = \left(\frac{\mathfrak{s}}{2\mathfrak{s} + 1}, \frac{2\mathfrak{s}^2}{\mathfrak{s}^2 - 2}\right),$$

then, as $\mathfrak{s} \longrightarrow \infty$, $\varrho_{\mathfrak{s}} \longrightarrow (\frac{1}{2}, 2)$, that is,

$$\lim_{\mathfrak{s} \longrightarrow \infty} \varrho_{\mathfrak{s}} = (\frac{1}{2}, 1)$$

Remark 2.6.

A sequence in a MS cannot converge to two distinct limits.

Definition 2.7. Cauchy Sequence.

"A sequence $\{\varrho_{\mathfrak{s}}\}$ in a metric space (ξ, d) is said to be Cauchy sequence (or fundamental) if for every $\varepsilon > 0$ there is an $N = N(\varepsilon)$, such that

$$d(\varrho_{\mathsf{m}}, \varrho_{\mathfrak{s}}) < \varepsilon \text{ for every } \mathsf{m}, \mathfrak{s} \geq N." \text{ [44]}$$

Example 2.5.

Let $\xi = \mathbb{R}$. Define a metric d on ξ as

$$d(\varrho, \bar{\varrho}) = \mid \varrho - \bar{\varrho} \mid .$$

Let $\{\varrho_{\mathfrak{s}}\} \subset \xi$ be a sequence defined as

$$\varrho_{\mathfrak{s}} = \frac{\mathfrak{s}^2}{\mathfrak{s}^2 + 2}.$$

Observe that,

$$\begin{aligned}
|\varrho_{\mathfrak{r}} - \varrho_{\mathfrak{s}}| &= \left| \frac{\mathfrak{r}^2}{\mathfrak{r}^2 + 2} - \frac{\mathfrak{s}^2}{\mathfrak{s}^2 + 2} \right| \\
&= \left| \frac{\mathfrak{r}^2\mathfrak{s}^2 + 2\mathfrak{r}^2 - \mathfrak{r}^2\mathfrak{s}^2 - 2\mathfrak{s}^2}{(\mathfrak{r}^2 + 2)(\mathfrak{s}^2 + 2)} \right| \\
&= \left| \frac{2\mathfrak{r}^2 - 2\mathfrak{s}^2}{(\mathfrak{r}^2 + 2)(\mathfrak{s}^2 + 2)} \right| \\
&= \left| \frac{2\mathfrak{r}^2}{(\mathfrak{r}^2 + 2)(\mathfrak{s}^2 + 2)} + \frac{-2\mathfrak{s}^2}{(\mathfrak{r}^2 + 2)(\mathfrak{s}^2 + 2)} \right| \\
&\leq \left| \frac{2\mathfrak{r}^2}{(\mathfrak{r}^2 + 2)(\mathfrak{s}^2 + 2)} \right| + \left| \frac{2\mathfrak{s}^2}{(\mathfrak{r}^2 + 2)(\mathfrak{s}^2 + 2)} \right| \\
&< \left| \frac{2\mathfrak{r}^2}{(\mathfrak{r}^2)(\mathfrak{s}^2 + 2)} \right| + \left| \frac{2\mathfrak{s}^2}{(\mathfrak{r}^2 + 2)(\mathfrak{s}^2)} \right| \\
&= \frac{2}{\mathfrak{s}^2 + 2} + \frac{2}{\mathfrak{r}^2 + 2} \\
&< \frac{2}{\mathfrak{s}^2} + \frac{2}{\mathfrak{r}^2}
\end{aligned}$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2}$$
$$= \epsilon,$$

for each $\mathfrak{r}, \mathfrak{s} > \frac{2}{\sqrt{\epsilon}}$ and by Archimedean's property letting $N > \frac{2}{\sqrt{\epsilon}}$, it is concluded that $\{\varrho_{\mathfrak{s}}\}$ is Cauchy sequence in $\mathbb{R}$.

Remark 2.8.

Every convergent sequence in a MS eventually becomes a Cauchy sequence but the converse of this result may or may not be true. For instance , suppose $\xi = (0, 1)$ with metric d defined on ξ as

$$d(\varrho, \bar{\varrho}) = |\varrho - \bar{\varrho}|.$$

Now, the sequence $\{\varrho_{\mathfrak{s}}\}$ in ξ given by

$$\varrho_{\mathfrak{s}} = \frac{1}{\mathfrak{s}},$$

is Cauchy sequence but not convergent in ξ.

Definition 2.9. Complete Metric Space.

"A metric space (ξ, d) is said to be complete if every Cauchy sequence in ξ converges to an element in ξ." [45]

Example 2.6.

Suppose l^{∞} be the space of all bounded sequences of complex numbers, with the metric defined by

$$d(\varrho, \bar{\varrho}) = \sup_{\mathfrak{s} \in \mathbb{N}} |\varrho_{\mathfrak{s}} - \bar{\varrho}_{\mathfrak{s}}|,$$

where $\varrho = \{\varrho_{\mathfrak{s}}\}$ and $\bar{\varrho} = \{\bar{\varrho}_{\mathfrak{s}}\}$, then (l^{∞}, d) is a complete metric space.

Example 2.7.

Let ξ be a of all real-valued functions which are the functions of an independent real variable ℓ and are defined and continuous on a given closed interval $I = [\mathfrak{a}, \mathfrak{b}]$. Choose a metric d on ξ defined by

$$d(\varrho, \bar{\varrho}) = \max_{\ell \in I} |\varrho(\ell) - \bar{\varrho}(\ell)|,$$

then, (ξ, d) is a complete metric space.

Example 2.8. Let $\xi = \mathbb{R}$, define a metric d on ξ by

$$d(\varrho, \bar{\varrho}) = \mid \arctan(\varrho) - \arctan(\bar{\varrho}) \mid,$$

then, d is not a complete metric on ξ.

2.2 Mappings on Metric Spaces

This section is furnished with the idea of different type of mappings on metric spaces.

Definition 2.10. Continuous Mapping.
"Let (ξ, d_ξ) and $(\mathcal{E}, d_\mathcal{E})$ be metric spaces, and let $\Omega\colon \xi \longrightarrow \mathcal{E}$ be a function that maps ξ into $\mathcal{E}$. We say that Ω is continuous at a point $\varrho_0 \in \xi$ if for every $\epsilon > 0$ there exists $\delta > 0$, such that

$$d_\xi(\varrho, \varrho_0) < \delta \implies d_\mathcal{E}\big(\Omega\varrho, \Omega\varrho_0\big) < \varepsilon,$$

for all $\varrho \in \xi$." [46]

Theorem 2.11.
"A mapping $\Omega\colon \xi \longrightarrow \mathcal{E}$ of a metric space (ξ, d_ξ) into a metric space $(\mathcal{E}, d_\mathcal{E})$ is continuous at a point $\varrho_0 \in \xi$ if and only if

$$\varrho_\mathfrak{s} \longrightarrow \varrho_0 \implies \Omega\varrho_\mathfrak{s} \longrightarrow \Omega\varrho_0.\text{"} \ [44]$$

Definition 2.12. Lipschitz Maps.
"Let (ξ, d) be a metric space. A mapping $\Omega\colon \xi \longrightarrow \xi$ is called to be Lipschitzian if there exists a constant $\beta \geq 0$ with,

$$d(\Omega\varrho, \Omega\bar{\varrho}) \leq \beta d(\varrho, \bar{\varrho}),$$

for all $\varrho, \bar{\varrho} \in \xi$. The smallest β for which this condition holds is said to be the Lipschitz constant for Ω." [47]

Example 2.9.

Let $\xi = [0, \infty)$, define a metric $d\colon \xi \times \xi \longrightarrow \mathbb{R}^+$ by

$$d(\varrho_1, \varrho_2) \quad = \quad | \varrho_1 - \varrho_2 |,$$

then, the mapping $\Omega\colon \xi \longrightarrow \xi$ defined by

$$\Omega\varrho = 4\varrho - 15,$$

is Lipschitz map on Ω with Lipschitz constant 4.

Definition 2.13. Contraction.

"Let (ξ, d) be a metric space. A mapping $\Omega\colon \xi \longrightarrow \xi$ is called a contraction if and only if there exists a positive real number $0 \leq \beta < 1$, independent of ϱ_1, ϱ_2 in ξ, such that for all ϱ_1, ϱ_2 in ξ,

$$d\left(\Omega\varrho_1, \Omega\varrho_2\right) \leq \beta d\left(\varrho_1, \varrho_2\right).\text{" [48]}$$

Example 2.10.

Consider a metric space $(\mathbb{R}^+, d)$ with usual metric d on $\mathbb{R}^+$, then the mapping $\Omega\colon \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ defined by

$$\Omega\varrho = \frac{5}{7}\varrho,$$

is a contraction on $\mathbb{R}^+$.

Example 2.11.

Let $\Omega\colon \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable function and suppose that $| \Omega^{'}\varrho | \leq \beta < 1$ on $\mathbb{R}$. Then, the mapping Ω is a contraction on $\mathbb{R}$ with usual metric d on $\mathbb{R}$, since the mean value theorem gives

$$| \Omega\varrho - \Omega\bar{\varrho} | = | \Omega^{'}\varrho^{*} || \varrho - \bar{\varrho} | \leq \beta | \varrho - \bar{\varrho} |,$$

where $\varrho < \varrho^{*} < \bar{\varrho}$.

Definition 2.14. Contractive Mapping.

"Let (ξ, d) be a metric space. A mapping $\Omega\colon \xi \longrightarrow \xi$ is said to be contractive if for every $\varrho, \bar{\varrho} \in \xi$,

$$d\left(\Omega\varrho, \Omega\bar{\varrho}\right) < d\left(\varrho, \bar{\varrho}\right),$$

with $\varrho \neq \bar{\varrho}$." [49]

Example 2.12.

Consider $\xi = [1, \infty]$ with usual metric d on ξ. Let the self map Ω on ξ is defined as

$$\Omega\varrho = \varrho + \frac{1}{\varrho},$$

then Ω is a contractive mapping. Note that Ω is not a contraction.

Definition 2.15. Non-expensive Mapping.

"Let (ξ, d) be a metric space. A mapping $\Omega\colon \xi \longrightarrow \xi$ is said to be non-expensive if for every $\varrho, \bar{\varrho} \in \xi$,

$$d\left(\Omega\varrho, \Omega\bar{\varrho}\right) \leq d\left(\varrho, \bar{\varrho}\right)."\ [49]$$

Example 2.13.

Consider $\mathbb{R}$ with usual metric d. The self mapping Ω on $\mathbb{R}$ defined as

$$\Omega\varrho = \varrho,$$

is non-expensive but not contractive.

2.3 Fixed Points of a Mapping

The aim of this section is to introduce the notion of fixed point and clarify this concept with the help of a variety of suitable examples and graphs.

Definition 2.16. Fixed Point.

"A fixed point of a mapping $\Omega\colon \xi \longrightarrow \xi$ on a set ξ into itself is $\varrho \in \xi$ which is mapped onto itself, that is $\Omega\varrho = \varrho$, the image $\Omega\varrho$ coincides with ϱ. "[44]

Note that for real valued functions, fixed points are the points of intersection of the line $y = x$ and the curve $y = \Omega x$.

Example 2.14.

Define a mapping $\Omega\colon [0,1] \longrightarrow [0,1]$ by

$$\Omega\varrho = \frac{\varrho}{2},$$

then, $\varrho = 0$ is the only fixed point of Ω.

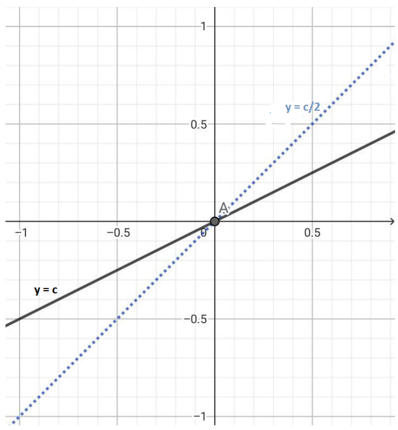

FIGURE 2.1: One fixed point.

Example 2.15.

Let us define a mapping $\Omega\colon \mathbb{R} \longrightarrow \mathbb{R}$ by

$$\Omega\varrho = 6\varrho^2 + 2\varrho - 1,$$

then, the fixed points of Ω are $\dfrac{1}{3}$ and $-\dfrac{1}{2}$.

Example 2.16.

Consider a mapping $\Omega\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by

$$\Omega(\varrho, \bar{\varrho}) = (\varrho, 0),$$

then, all points on x-axis are fixed points of this mapping. This mapping is projection of plane on x-axis.

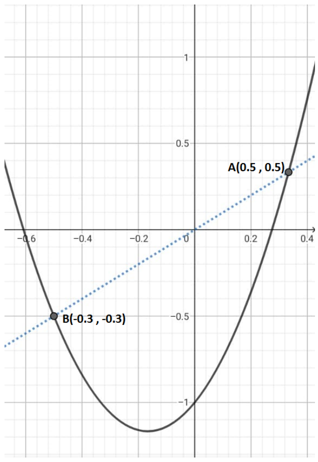

FIGURE 2.2: Two fixed points.

Example 2.17.

Consider a translation mapping $\Omega\colon \mathbb{R}^{+} \longrightarrow \mathbb{R}^{+}$ defined by

$$\Omega(\varrho) = \varrho + 4,$$

then, the mapping Ω has no fixed point.

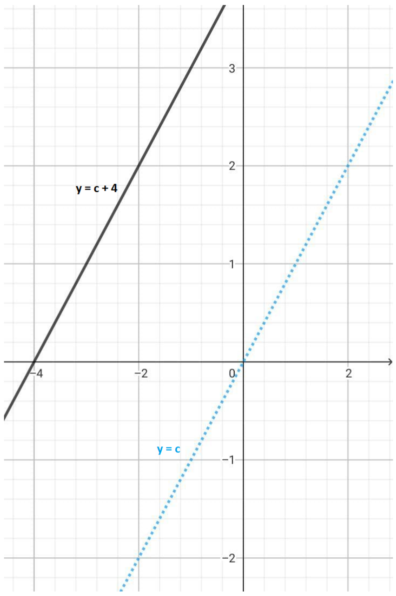

FIGURE 2.3: No fixed point.

Remark 2.17.

A mapping may or may not have a fixed point. Furthermore, the fixed point of a mapping, if exists, may or may not be unique.

2.4 Some Classical Fixed Point Theorems

A fixed point theorem is a statement that guarantees the existence of fixed points of a mapping under suitable conditions in any space. This section provides some important theorems, which are the milestones of the fixed point theory.
In 1912, the Brouwer theorem was presented, which assures the existence of the fixed point but does not provide any information about its location.

Theorem 2.18.
"Every continuous mapping from a closed ball of Euclidean space into itself has a fixed point." [50]

An improved version of above result was provided by Schauder in 1930.

Theorem 2.19.
"Every continuous function from a convex compact subset of Euclidean space to itself has a fixed point." [51]

The behavior of contraction mapping in complete metric space is of crucial importance. Stefan Banach (1892-1945) was a famous Polish mathematician and is considered one of the founders of functional analysis. BCP was formulated and proved in his PhD dissertation in 1920, which was published in 1922 [4]. In classical functional analysis, BCP is one of the pivotal results and is a source for researchers in the field of fixed point theory.

Theorem 2.20. ***Banach Contraction Principle***
"Let (ξ, d) be a complete metric space with a contraction mapping $\Omega\colon \xi \longrightarrow \xi$, then, Ω admits a unique fixed point in ξ." [4]

Example 2.18.
Suppose $\xi = \mathbb{R}$ and d be the usual metric defined on ξ. Let us define $\Omega\colon \xi \longrightarrow \xi$ as

$$\Omega\varrho = 7 + \frac{\varrho}{9},$$

then, all the conditions of BCP are satisfied and hence Ω has a unique fixed point.

Proof.

$$\begin{aligned} d\left(\Omega\varrho, \Omega\bar{\varrho}\right) &= d\left(7+\frac{\varrho}{9}, 7+\frac{\bar{\varrho}}{9}\right) \\ &= |7+\frac{\varrho}{9}-7-\frac{\bar{\varrho}}{9}| \\ &= |\frac{\varrho}{9}-\frac{\bar{\varrho}}{9}| \\ &= \frac{1}{9}|\varrho-\bar{\varrho}| \\ &= \frac{1}{9}d(\varrho,\bar{\varrho}). \end{aligned}$$

Here $\beta = \frac{1}{9} \in [0,1)$, so Ω is a contraction. $\mathbb{R}$ is a complete metric space with usual metric. Hence all conditions of BCP are satisfied, and $\varrho = \frac{63}{8}$ is the fixed point of Ω. □

Banach Contraction Principle (BCP) is furnished with the existence and uniqueness of fixed points of specific self maps on a MS. It also defines a constructive method to find these fixed points. Many researchers have extended the Banach Contraction Principle with various generalized metric spaces.

2.5 Some Generalizations of Metric Space

This section is devoted to state some generalized metric spaces in the light of suitable examples. Some important results of metric fixed point theory are also the part of this discussion.

The generalization of metric space as b-metric space is due to Bakhtin [22] and later on, Czerwik [23] provided more work in this space.

Definition 2.21. ***b*-Metric Space.**

"Let ξ be a non-empty set and $\beta \geq 1$ be a given real number. A function $d\colon \xi \times \xi \longrightarrow [0,\infty)$ is called b-metric if it satisfies the following properties for each $\varrho_1, \varrho_2, \varrho_3 \in \xi$:

B1 $d(\varrho_1, \varrho_2) = 0$ iff $\varrho_1 = \varrho_2$, B2 $d(\varrho_1, \varrho_2) = d(\varrho_2, \varrho_1)$,

B3 $d(\varrho_1, \varrho_3) \leq \beta\Big(d(\varrho_1, \varrho_2) + d(\varrho_2, \varrho_3)\Big)$.

The pair (ξ, d) is called b-metric space." [38]

Example 2.19. Consider $\xi = \mathbb{N} \cup \{\infty\}$. Define $d : \xi \times \xi \longrightarrow \mathbb{R}$ as

$$d(\varrho_1, \varrho_2) = \begin{cases} 0 & \text{if } \varrho_1 = \varrho_2 \\ \left|\dfrac{1}{\varrho_1} - \dfrac{1}{\varrho_2}\right| & \text{if } \varrho_1, \varrho_2 \text{ are even or } \varrho_1, \varrho_2 = \infty \\ 5 & \text{if } \varrho_1, \varrho_2 \text{ are odd and } \varrho_1 \neq \varrho_2 \\ 2 & \text{if else,} \end{cases}$$

it can be verified by taking $\varrho_1, \varrho_2, \varrho_3 \in \xi$ that

$$d(\varrho_1, \varrho_3) \leq 3\Big[d(\varrho_1, \varrho_2) + d(\varrho_2, \varrho_3)\Big],$$

showing that (ξ, d) is a b-metric space with $\beta = 3 > 1$.

Remark 2.22.

Every metric space (ξ, d) is b-metric space with $\beta = 1$, but its converse is not always true. For example, let $\xi = \{0, 1, 2\}$ and define $d \colon \xi \times \xi \longrightarrow \mathbb{R}^+$ by,

$$d(0, 2) = d(2, 0) = w \geq 2,$$
$$d(0, 1) = d(1, 2) = d(2, 1) = d(1, 0) = 1,$$
$$\text{and } d(0, 0) = d(1, 1) = d(2, 2) = 0,$$

then,

$$d(\varrho_1, \varrho_3) \leq \frac{w}{2}\Big(d(\varrho_1, \varrho_2) + d(\varrho_2, \varrho_3)\Big),$$

for all $\varrho_1, \varrho_2, \varrho_3 \in \xi$. Hence, (ξ, d) is a b-metric space. But for $w > 2$, the ordinary triangle inequality does not hold. For instance, $w = 3$ implies

$$d(2, 1) + d(1, 0) = 1 + 1 = 2 < d(2, 0),$$

showing that (ξ, d) is not a metric space.

Matthews [28] generalized the concept of a metric space and introduce Partial metric space.

Definition 2.23. Partial Metric Space.

"Let ξ be a non-empty set. A mapping $d_p\colon \xi \times \xi \longrightarrow [0,\infty)$ is said to be partial metric if d_p satisfies following axioms for all $\varrho_1, \varrho_2, \varrho_3 \in \xi$:

P1. $d_p(\varrho_1,\varrho_1) = d_p(\varrho_1,\varrho_2) = d_p(\varrho_2,\varrho_2)$ if and only if $\varrho_1 = \varrho_2$,

P2. $d_p(\varrho_1,\varrho_1) \leq d_p(\varrho_1,\varrho_2)$,

P3. $d_p(\varrho_1,\varrho_2) = d_p(\varrho_2,\varrho_1)$,

P4. $d_p(\varrho_1,\varrho_3) \leq d_p(\varrho_1,\varrho_2) + d_p(\varrho_2,\varrho_3) - d_p(\varrho_2,\varrho_2)$.

The pair (ξ, d_p) is said to be partial metric space." [28]

Example 2.20.

Suppose

$$\xi = \{[\varrho_1,\varrho_2] : \varrho_1,\varrho_2 \in \mathbb{R} \quad and \quad \varrho_1 \leq \varrho_2\}.$$

Define $d_p\colon \xi \times \xi \longrightarrow [0,\infty)$ by

$$d_p\Big([\varrho_1,\varrho_3],[\varrho_2,\varrho_4]\Big) = \max(\varrho_3,\varrho_4) - \min(\varrho_1,\varrho_2),$$

then d_p is a partial metric over ξ.

Remark 2.24.

Every metric space is a partial metric space, but every partial metric space need not to be a metric space. However, any partial metric space with zero self distance becomes a metric space.

In 2014, Shukla [30] introduced a new generalization of b-metric space namely partial b-metric space.

Definition 2.25. Partial b-Metric Space.

"A partial b-metric on a nonempty set ξ is a function $d : \xi \times \xi \longrightarrow \mathbb{R}^+$ such that for all $\varrho_1, \varrho_2, \varrho_3 \in \xi$:

Pb1. $\varrho_1 = \varrho_2$ iff $d(\varrho_1, \varrho_1) = d(\varrho_1, \varrho_2) = d(\varrho_2, \varrho_2)$,

Pb2. $d(\varrho_1, \varrho_1) \leq d(\varrho_1, \varrho_2)$,

Pb3. $d(\varrho_1, \varrho_2) = d(\varrho_2, \varrho_1)$,

Pb4. there exists $\beta \geq 1$ such that $d(\varrho_1, \varrho_2) \leq \beta\Big(d(\varrho_1, \varrho_3) + d(\varrho_3, \varrho_2)\Big) - d(\varrho_3, \varrho_3)$.

A partial b-metric space is a pair (ξ, d) such that ξ is a non-empty set and d is a partial b-metric on ξ. The number b is called the coefficient of (ξ, d)." [30]

Example 2.21.

Let $k > 1$ is a constant, $\xi = \mathbb{R}^+$ and $d : \xi \times \xi \longrightarrow \mathbb{R}^+$ is defined by

$$d(\varrho_1, \varrho_2) = [\max(\varrho_1, \varrho_2)]^k + \mid \varrho_1 - \varrho_2 \mid^k,$$

for all $\varrho_1, \varrho_2 \in \xi$, then (ξ, d) is a partial b-metric space with coefficient $\beta = 2^k > 1$. but it is neither a partial metric nor a b-metric space. Since for any $\varrho > 0$ we have

$$d(\varrho, \varrho) = \varrho^k \neq 0,$$

which implies that d is not b-metric on ξ.

Also, if $\varrho_1 = 5, \varrho_2 = 1, \varrho_3 = 4$, then,

$$d(\varrho_1, \varrho_2) = 5^k + 4^k.$$

and

$$\begin{aligned} d(\varrho_1, \varrho_3) + d(\varrho_3, \varrho_2) - d(\varrho_3, \varrho_3) &= 5^k + 1 + 4^k + 3^k - 4^k \\ &= 5^k + 1 + 3^k. \end{aligned}$$

Hence,

$$d(\varrho_1, \varrho_2) > d(\varrho_1, \varrho_3) + d(\varrho_3, \varrho_2) - d(\varrho_3, \varrho_3) \text{ for all } k > 1,$$

which implies that d is not partial metric on ξ.

Remark 2.26.

If (ξ, d) is a partial metric space, then it is partial b-metric space with coefficient $\beta = 1$ and every b-metric space is also a partial b-metric space with the same coefficient and zero self distance. However, the converse is not true in general.

Kamran et al. [38] generalize the concept of b-metric space and introduce extended b-metric space. The definition along with a suitable example is provided here.

Definition 2.27. Extended b-Metric Space.

"Let ξ be a non-empty set and $\rho\colon \xi\times\xi \longrightarrow [1,\infty)$. A function $d_\rho\colon \xi\times\xi \longrightarrow [0,\infty)$ is called an extended b-metric if for all $\varrho_1, \varrho_2, \varrho_3 \in \xi$ it satisfies:

$(d_\rho 1)$ $d_\rho(\varrho_1, \varrho_2) = 0$ if and only if $\varrho_1 = \varrho_2$,

$(d_\rho 2)$ $d_\rho(\varrho_1, \varrho_2) = d_\rho(\varrho_2, \varrho_1)$,

$(d_\rho 3)$ $d_\rho(\varrho_1, \varrho_3) \leq \rho(\varrho_1, \varrho_3)\left[d_\rho(\varrho_1, \varrho_2) + d_\rho(\varrho_2, \varrho_3)\right]$.

The pair (ξ, d_ρ) is called an extended b-metric space." [38]

Example 2.22.

Let $\xi = \{2, 1, -1\}$. Define $\rho\colon \xi \times \xi \longrightarrow [1,\infty)$ as

$$\rho(\varrho_1, \varrho_2) = |\varrho_1| + |\varrho_2|.$$

Also, define $d\colon \xi \times \xi \longrightarrow [0,\infty)$ as

$$d(2,2) = d(1,1) = d(-1,-1) = 0$$

$$d(1,2) = d(2,1) = \frac{1}{2},$$

and

$$d(1,-1) = d(-1,1) = d(2,-1) = d(-1,2) = \frac{1}{3},$$

then, (ξ, d_ρ) is an extended b-metric space.

Remark 2.28.

Let (ξ, d_ρ) be an extended b-metric space. Define $\rho\colon \xi \times \xi \longrightarrow [1, \infty)$ as

$$\rho(\varrho_1, \varrho_2) = \beta,$$

where $\beta \geq 1$. This reduces the definition of extended b-metric space to b-metric space.

The concept of distance between two closed sets was initiated by Pompeiu [52] $(1873, 1954)$, and established in the general setting of a metric space by Hausdorff since 1914.

Definition 2.29. Distance of a Point and a Set.

"The distance $D(\varrho, A)$ from a point ϱ to a non-empty subset A of metric space (ξ, d) is defined to be

$$D(\varrho, A) = \inf_{\bar{\varrho} \in A} d(\varrho, \bar{\varrho})." \text{ [44]}$$

Definition 2.30. Hausdorff Metric Space.

"Let (ξ, d) be a metric space and $CB(\xi)$ denotes the collection of all non-empty closed and bounded subsets of ξ. For $A, B \in CB(\xi)$, define

$$H(A, B) = \max \left\{ \sup_{\bar{\varrho} \in B} D(\varrho, B), \sup_{\varrho \in A} D(\bar{\varrho}, A) \right\},$$

where $D(\varrho, B)$ is the distance of a point ϱ to the set B. It is known that H is a metric on $CB(\xi)$, called the Hausdorff metric induced by the metric d." [53]

Chapter 3

Data Dependence and Well-Posedness of Fixed Point Theorems for Non-linear Contractions

This chapter is furnished with the exhaustive debate about the work of Iqbal et al. [19]. The article focuses on the existence of fixed points and strict fixed points for multi-valued nonlinear contraction in the domain of complete MS. Data dependence and well-posedness of the problems are also discussed.

3.1 Fixed Point in the Context of F-Contractions

In this section, a few ideas regarding F-contraction are recalled.

Definition 3.1. F-Mapping.

A function $F : (0, \infty) \longrightarrow \mathbb{R}$ is called F-Mapping, if it satisfies the following properties

($\mathbb{F}1$) F is strictly increasing,

($\mathbb{F}2$) For all sequences $\{\Psi_{\mathfrak{s}}\} \subseteq (0, \infty)$, $\lim_{\mathfrak{s} \longrightarrow \infty} \Psi_{\mathfrak{s}} = 0$ iff $\lim_{\mathfrak{s} \longrightarrow \infty} F(\Psi_{\mathfrak{s}}) = -\infty$,

($\mathbb{F}3$) There exists $\mathsf{k} \in (0, 1)$ such that $\lim_{\Psi \longrightarrow 0^+} \Psi^{\mathsf{k}} F(\Psi) = 0$. [14]

Example 3.1.

Define $F : (0, \infty) \longrightarrow \mathbb{R}$

$$F(\varrho) = \ln(\varrho^2 + \varrho).$$

Obviously, F satisfies ($\mathbb{F}1$)-($\mathbb{F}3$).

Definition 3.2. F-Contraction.

Let (ξ, d) be a MS and $F : (0, \infty) \longrightarrow \mathbb{R}$ be an F-Mapping. A mapping $\Omega : \xi \longrightarrow \xi$ is said to be F-contraction if there exists $\upsilon > 0$ s.t $d(\Omega\varrho, \Omega\bar{\varrho}) > 0$ implies

$$\upsilon + F(d(\Omega\varrho, \Omega\bar{\varrho})) \leq F(d(\varrho, \bar{\varrho})),$$

for all $\varrho, \bar{\varrho} \in \xi$. [14]

Let us denote the collection of all such functions F which satisfy ($\mathbb{F}1$), ($\mathbb{F}2$) and ($\mathbb{F}3$) by $\Delta(\mathbb{F})$. Also, assume that

$$\Delta\,(O*) = \left\{F \in \Delta(\mathbb{F}) : F \text{ satisfeis } (\mathbb{F}4)\right\},$$

where

($\mathbb{F}4$) $F(\inf \mathsf{X}) = \inf F(\mathsf{X})$ for all $\mathsf{X} \subset (0, \infty)$ with $\inf \mathsf{X} > 0$.

Turinici [54] replace ($\mathbb{F}2$) by ($\mathbb{F}2'$), where,

($\mathbb{F}2'$) $\lim_{\mathsf{t} \longrightarrow 0^+} F(\mathsf{t}) = -\infty$.

Denote the collection of all such functions F which satisfy ($\mathbb{F}1$), ($\mathbb{F}2'$), ($\mathbb{F}3$) and ($\mathbb{F}4$) by $\Delta\,(0*)$. In 2012, Wardowski proved the following result.

Theorem 3.3.

Suppose (ξ, d) is a MS and $\Omega : \xi \longrightarrow \xi$ is an F-contraction, then Ω has unique *fixed point* $\varrho^* \in \xi$. *Also, for every* $\varrho_0 \in \xi$, *a picard sequence* $\{\Omega^{\mathfrak{s}}\varrho_0\}_{\mathfrak{s}\in\mathbb{N}}$ *converges to* ϱ^*. *[14]*

Proposition 3.4.

Assume that $F : (0,\infty) \longrightarrow \mathbb{R}\cup\{-\infty\}$ *is a function which satisfies* $(\mathbb{F}1)$ *and* $(\mathbb{F}2)$, *then a countable subset* $\eth F$ *contained in* $(0,\infty)$ *exists such that*

$$\lim_{\mathsf{t}\longrightarrow\mathfrak{s}^-} F(\mathsf{t}) = F(\mathsf{t}) = \lim_{\mathsf{t}\longrightarrow\mathfrak{s}^+} F(\mathsf{t}),$$

for every $\mathfrak{s} \in (0,\infty)/\eth F$. *[54]*

Lemma 3.5.

Assume that $F : (0,\infty) \longrightarrow \mathbb{R}\cup\{-\infty\}$ *is a function which satisfies* $(\mathbb{F}1)$ *and* $(\mathbb{F}2)$, *then for each sequence* $\{\mathsf{t}_{\mathfrak{s}}\} \subseteq (0,\infty)$,

$$F(\mathsf{t}_{\mathfrak{s}}) \to -\infty \implies \mathsf{t}_{\mathfrak{s}} \to 0. \text{ [54]}$$

Proof.

Suppose that $F(\mathsf{t}_{\mathfrak{s}}) \to -\infty$ does not implies $\mathsf{t}_{\mathfrak{s}} \to 0$, there must exist some $\epsilon > 0$ such that for every $\mathfrak{s}$, there exists $\mathfrak{s}' \geq \mathfrak{s}$, such that

$$\mathsf{t}_{\mathfrak{s}'} > \epsilon.$$

So, we obtain a subsequence $\{\mathsf{t}'_{\mathfrak{s}}\}$ of $\{\mathsf{t}_{\mathfrak{s}}\}$ such that

$$\mathsf{t}'_{\mathfrak{s}} > \epsilon \quad \text{for all } \mathfrak{s},$$
$$\implies F(\mathsf{t}'_{\mathfrak{s}}) > F(\epsilon) \quad \text{for all } \mathfrak{s},$$

which is a contradiction to the property $F(\mathsf{t}'_{\mathfrak{s}}) \longrightarrow -\infty$.

Hence

$$F(\mathsf{t}_{\mathfrak{s}}) \to -\infty \implies \mathsf{t}_{\mathfrak{s}} \to 0.$$

□

Definition 3.6. Multivalued Mapping.

Assume that ξ and η are non-empty sets, a mapping $\Omega : \xi \longrightarrow P(\eta)$ is called multi-valued mapping (MVM) if every element of ξ corresponds to any subset of η. Here, $P(\eta)$ represents the collection of subsets of η. [10]

Example 3.2.

Consider $\xi = [0, 1]$ and $M(\xi) = \{\eta \subseteq \xi : \eta \neq \emptyset\}$. A mapping $\Omega : \xi \longrightarrow M(\xi)$ defined as

$$\Omega\varrho = [0, \varrho],$$

is a multivalued mapping. Figure 3.1 shows the graphical picture of this map.

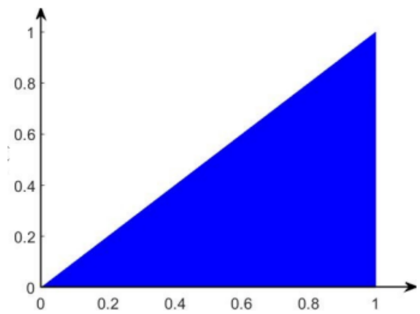

FIGURE 3.1: Multivalued mapping.

Assume that $CL(\xi)$ and $K(\xi)$ denote the set of all non-empty closed subsets of ξ and the set of all non-empty compact subsets of ξ respectively.

Definition 3.7. Fixed point of MVM.

Let $\Omega\colon \xi \longrightarrow P(\xi)$ be a MVM. An element $\varrho \in \xi$ is called fixed point of Ω if $\varrho \in \Omega\varrho$. The set of all fixed points of Ω is denoted as FixΩ. An element $\varrho \in \xi$ is called strict fixed point of Ω if $\Omega\varrho = \{\varrho\}$. The set of all strict fixed points of Ω is denoted by SFixΩ.

Altun et at. [55] generalized the idea of F-contraction with a flavor of multivalued mapping.

Definition 3.8. Multivalued F-Contraction.

Let (ξ, d) be a MS. A mapping $\Omega : \xi \longrightarrow CB(\xi)$ is said to be multivalued F-contraction, if there exists $\upsilon > 0$ and $F \in \Delta(\mathbb{F})$ such that

$$H(\Omega\varrho, \Omega\bar{\varrho}) > 0 \text{ implies } \upsilon + F(H(\Omega\varrho, \Omega\bar{\varrho})) \leq F(d(\varrho, \bar{\varrho})),$$

for all $\varrho, \bar{\varrho} \in \xi$. [55]

Theorem 3.9.

Suppose that (ξ, d) is a complete MS and $\Omega : \xi \longrightarrow K(\xi)$ is a multi-valued F-contraction, then Ω has fixed point in ξ. [55]

3.2 Fixed Point in the Setting of Non-linear F-Contraction

Klim and Wardowski [56] generalized F-contraction mapping to non-linear F-contraction through dynamic processes.

Definition 3.10. Dynamic Process of Mapping.

Let ξ be a non-empty set, S be the collection of non-empty subsets of ξ, and $\Omega : \xi \longrightarrow \mathsf{S}(\xi)$ be multi-valued mapping. Suppose ϱ_0 be an arbitrary element of ξ. Define

$$\mathfrak{D}(\Omega, \varrho_0) = \{(\varrho_{\mathfrak{s}})_{\mathfrak{s} \in \mathbb{N} \cup \{0\}} : \varrho_{\mathfrak{s}} \in \Omega \varrho_{\mathfrak{s}-1} \text{ for all } \mathfrak{s} \in \mathbb{N}\}.$$

Every element of $\mathfrak{D}(\Omega, \varrho_0)$ is known as dynamic process of Ω starting at ϱ_0. [56]

Definition 3.11. Set Valued F-contraction.

Let (ξ, d) be a MS, $\varrho_0 \in \xi$ and $F \in \Delta(\mathbb{F})$. A set valued F-contraction with respect to the dynamic process $(\varrho_{\mathfrak{s}})_{\mathfrak{s} \in \mathbb{N} \cup \{0\}}$ is defined as the mapping $\Omega : \xi \longrightarrow \mathsf{S}(\xi)$ if there exists a function $\beta : (0, \infty) \longrightarrow (0, \infty)$ such that

$$d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}) > 0 \text{ implies } \beta\big(d(\varrho_{\mathfrak{s}-1}, \varrho_{\mathfrak{s}})\big) F\big(d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1})\big) \leq F\big(d(\varrho_{\mathfrak{s}-1}, \varrho_{\mathfrak{s}})\big),$$

for all $\mathfrak{s} \in \mathbb{N}$. [56]

Wardowski [57] introduced non-linear F-contraction in 2017 as follows:

Definition 3.12. Non-linear F-Contraction.

Let (ξ, d) be a MS. A mapping $\Omega : \xi \longrightarrow \xi$ is called non-linear F-contraction, if there exists $F \in \Delta(\mathbb{F})$ and a function $\Xi : (0, \infty) \longrightarrow (0, \infty)$ that fulfill the followings,

(H_1) $\lim_{s\longrightarrow\Psi^+}\inf \Xi(s) > 0, \ \forall\ \Psi > 0.$

(H_2) $\Xi(d(\varrho,\bar{\varrho})) + F(d(\Omega\varrho,\Omega\bar{\varrho})) \leq F(d(\varrho,\bar{\varrho})) \quad \forall\ \varrho,\bar{\varrho}\in\xi$ s.t $\Omega\varrho \neq \Omega\bar{\varrho}$. [57]

Theorem 3.13.

Assume that (ξ, d) is a complete MS, and $\Omega : \xi \longrightarrow \xi$ is a non-linear F-contraction, then, Ω has unique fixed point in ξ. [57]

Non-linear case of Theorem 3.9 is presented by Olgun et al. [58] as follows:

Theorem 3.14.

Assume that (ξ, d) is a complete MS and $\Omega : \xi \longrightarrow K(\xi)$ is a multi-valued F-contraction, the mapping Ω has fixed points in ξ if $F \in \Delta(\mathbb{F})$ and $\zeta : (0,1) \longrightarrow (0,1)$ satisfy

$$\liminf_{s\to\Bbbk^+} \Omega\varrho > 0 \text{ for all } \Bbbk > 0,$$

$$\zeta\big(d(\varrho,\bar{\varrho})\big) + F\Big(H\big(\Omega\varrho,\Omega\bar{\varrho}\big)\Big) \leq F\big(d(\varrho,\bar{\varrho})\big),$$

for all $\varrho,\bar{\varrho}\in\xi$. [58]

3.3 Fixed Point For Generalized Multivalued Non-linear Contraction

This section is dedicated for the discussion of generalized multivalued non-linear contraction, presented by Iqbal et al. [19]. An example is also presented for support of the result.

Definition 3.15.

Suppose that P is the collection of all continuous mappings $\rho : [0,\infty)^5 \longrightarrow [0,\infty)$ which satify the following conditions,

(p_1) $\rho(1,1,1,2,0) \in (0,1]$,

(p_2) ρ is subhomogeneous, that is,

$$\rho(\gamma\varrho_1, \gamma\varrho_2, \gamma\varrho_3, \gamma\varrho_4, \gamma\varrho_5) \leq \gamma\rho(\varrho_1, \varrho_2, \varrho_3, \varrho_4, \varrho_5),$$

for all $(\varrho_1, \varrho_2, \varrho_3, \varrho_4, \varrho_5) \in [0, \infty)^5$ and $\gamma \geq 0$.

(p_3) ρ is non-decreasing , that is, for all $\varrho, \bar{\varrho} \in \mathbb{R}^+$ we have

$$\rho(\varrho_1, \varrho_2, \varrho_3, \varrho_4, \varrho_5) \leq \rho(\bar{\varrho}_1, \bar{\varrho}_2, \bar{\varrho}_3, \bar{\varrho}_4, \bar{\varrho}_5),$$

where $\varrho_i \leq \bar{\varrho}_i$ for $i = 1, 2, 3, 4, 5$.

If $\varrho_i, \bar{\varrho}_i \in \mathbb{R}^+$ such that $\varrho_i < \bar{\varrho}_i$ for $i = 1, 2, 3, 4$, then,

$$\rho(\varrho_1, \varrho_2, \varrho_3, \varrho_4, 0) < \rho(\bar{\varrho}_1, \bar{\varrho}_2, \bar{\varrho}_3, \bar{\varrho}_4, 0),$$

and

$$\rho(\varrho_1, \varrho_2, \varrho_3, 0, \varrho_4) < \rho(\bar{\varrho}_1, \bar{\varrho}_2, \bar{\varrho}_3, 0, \bar{\varrho}_4).$$

Also, suppose that

$$\mathcal{P} = \{\rho \in \mathsf{P} : \rho(1, 0, 0, 1, 1) \in (0, 1]\}.$$

Obviously, $\mathcal{P} \subseteq \mathsf{P}$.

Example 3.3.

Let $\rho_1 : [0, \infty)^5 \longrightarrow [0, \infty)$ is defined as

$$\rho_1(\varrho_1, \varrho_2, \varrho_3, \varrho_4, \varrho_5) = \varrho_1 + \eta\varrho_5,$$

where $\eta \in (0, 1)$. As

(p_1) $\rho_1(1, 1, 1, 2, 0) = 1 + \eta(0) = 1 \in (0, 1]$,

(p_2) ρ_1 is subhomogeneous, that is, for $\gamma \geq 0$,

$$\begin{aligned}\rho_1(\gamma\varrho_1, \gamma\varrho_2, \gamma\varrho_3, \gamma\varrho_4, \gamma\varrho_5) &= \gamma\varrho_1 + \eta\gamma\varrho_5 \\ &= \gamma\big(\varrho_1 + \eta\varrho_5\big) \\ &= \gamma\rho_1(\varrho_1, \varrho_2, \varrho_3, \varrho_4, \varrho_5).\end{aligned}$$

(p_3) Clearly, ρ_1 is non-decreasing.

Hence, $\rho_1 \in \mathsf{P}$. Also, since

$$\rho_1(1, 0, 0, 1, 1) = 1 + \eta > 1,$$

so, $\rho_1 \notin \mathcal{P}$.

Example 3.4.

Let $\rho_2 : [0, \infty)^5 \longrightarrow [0, \infty)$ is defined as

$$\rho_2(\varrho_1, \varrho_2, \varrho_3, \varrho_4, \varrho_5) = h \max\left\{\varrho_1, \frac{1}{2}(\varrho_2 + \varrho_3), \frac{1}{2}(\varrho_4 + \varrho_5)\right\},$$

where $h \in (0, 1)$. As

(p_1) $\rho_2(1, 1, 1, 2, 0) = h \max\left\{1, \frac{1}{2}(2), \frac{1}{2}(2)\right\} = h \in (0, 1]$,

(p_2) ρ_2 is subhomogeneous, that is, for $\gamma \geq 0$

$$\begin{aligned}\rho_2(\gamma\varrho_1, \gamma\varrho_2, \gamma\varrho_3, \gamma\varrho_4, \gamma\varrho_5) &= h \max\left\{\gamma\varrho_1, \frac{1}{2}(\gamma\varrho_2 + \gamma\varrho_3), \frac{1}{2}(\gamma\varrho_4 + \gamma\varrho_5)\right\} \\ &= \gamma h \max\left\{\varrho_1, \frac{1}{2}(\varrho_2 + \varrho_3), \frac{1}{2}(\varrho_4 + \varrho_5)\right\} \\ &= \gamma\rho_2(\varrho_1, \varrho_2, \varrho_3, \varrho_4, \varrho_5).\end{aligned}$$

(p_3) Clearly, ρ_2 is non-decreasing.

Also,

$$\rho_2(1, 0, 0, 1, 1) = h \max\{1, 0, 1\} = h \in (0, 1].$$

Hence, $\rho_2 \in \mathcal{P}$.

Example 3.5.

Let us define $\rho_3 : [0, \infty)^5 \longrightarrow [0, \infty)$ by

$$\rho_3(\varrho_1, \varrho_2, \varrho_3, \varrho_4, \varrho_5) = \beta_1\varrho_1 + \beta_2(\varrho_2 + \varrho_3) + \beta_3(\varrho_4 + \varrho_5),$$

where $\beta_1 + 2\beta_2 + 2\beta_3 < 1$. As

- (p_1) $\rho_3(1, 1, 1, 2, 0) = \beta_1 + 2\beta_2 + 2\beta_3 \in (0, 1]$,

- (p_2) ρ_3 is subhomogeneous, that is, for $\gamma \geq 0$

$$\begin{aligned} \rho_3(\gamma\varrho_1, \gamma\varrho_2, \gamma\varrho_3, \gamma\varrho_4, \gamma\varrho_5) &= \beta_1\big(\gamma\varrho_1\big) + \beta_2\big(\gamma\varrho_2 + \gamma\varrho_3\big) + \beta_3\big(\gamma\varrho_4 + \gamma\varrho_5\big) \\ &= \gamma\beta_1\varrho_1 + \beta_2(\varrho_2 + \varrho_3) + \beta_3(\varrho_4 + \varrho_5) \\ &= \gamma\rho_3(\varrho_1, \varrho_2, \varrho_3, \varrho_4, \varrho_5). \end{aligned}$$

- (p_3) Clearly, ρ_3 is non-decreasing.

Also,

$$\rho_3(1, 0, 0, 1, 1) = \beta_1 + 0 + 2\beta_3 < 1.$$

Hence, $\rho_3 \in \mathcal{P}$.

Lemma 3.16.

Suppose $\rho \in \mathsf{P}$ and $a, b \in [0, \infty)$ be such that

$$a \leq \max\left\{\rho(b, b, a, b + a, 0), \rho(b, b, a, 0, b + a), \rho(b, a, b, b + a, 0), \rho(b, a, b, 0, b + a)\right\},$$

then, $a \leq b$.

Proof.

With no loss of generality, assume that

$$a \leq \rho(b, b, a, b + a, 0). \tag{3.1}$$

On contrary suppose that, $b < a$. Now consider

$$\begin{aligned}\rho(b, b, a, b + a, 0) &< \rho(b, b, b, 2b, 0)\\ &\leq b\rho(1, 1, 1, 2, 0)\\ &\leq a.\end{aligned}$$

This implies

$$\rho(b, b, a, b + a, 0) < a,$$

which is contradiction of (3.1). Hence our supposition is wrong, so $a \leq b$. □

Following examples are provided to elaborate the properties $(\mathbb{F}1), (\mathbb{F}2), (\mathbb{F}3), (\mathbb{F}2')$ and the continuity.

Example 3.6.

Suppose that $F : (0, \infty) \longrightarrow (-\infty, \infty)$ is defined by

$$F(\varrho) = \frac{-1}{\varrho} \quad \forall \varrho \in (0, \infty),$$

then, F satisfies $(\mathbb{F}1)$ and $(\mathbb{F}2')$, but $(\mathbb{F}3)$ is not satisfied. Also, F is continuous.

Example 3.7.

Suppose that $F : (0, \infty) \longrightarrow (-\infty, \infty)$ is defined by

$$F(\varrho) = \begin{cases} -\dfrac{1}{\varrho} & \text{if } 0 < \varrho < 1 \\ 0 & \text{Otherwise,} \end{cases}$$

then, F fulfils $(\mathbb{F}1)$ and $(\mathbb{F}2')$. However, F is not continuous.

Example 3.8.

Suppose that $F : (0, \infty) \longrightarrow (-\infty, \infty)$ is defined by

$$F(\varrho) = -\frac{1}{(\varrho + [\varrho])^{\mathtt{t}}},$$

where, $[\varrho]$ denotes the integral part of ϱ, and $\mathtt{t} \in \left(0, \frac{1}{b}\right), b > 1$, then, F satisfies $(\mathbb{F}1), (\mathbb{F}2')$ and $(\mathbb{F}3)$. However, F is not continuous.

It is clear from above examples that there exist functions $F : (0, \infty) \longrightarrow (-\infty, \infty)$ for which $(\mathbb{F}1), (\mathbb{F}2), (\mathbb{F}3)$ and the continuity condition are not satisfied at the same time.

Definition 3.17.

Suppose that the set of all functions $\chi : (0, \infty) \longrightarrow (0, \infty)$ satisfying,

$$\lim_{g \longrightarrow t^+} \inf \chi > 0 \qquad \forall\, g \geq 0, \tag{3.2}$$

is denoted by Ψ.

Definition 3.18. (χF-Contraction.)

Let F_1, F_2 be real valued functions defined on $(0, \infty), \rho \in \mathcal{P}$ and $\chi \in \Psi$. The mapping Ω is called χF-contraction if

(Ni.) $F_1(\mathfrak{c}) \leq F_2(\mathfrak{c})$ for all $\mathfrak{c} > 0$,

(Nii.) $H(\Omega\varrho, \Omega\bar{\varrho}) > 0$ implies

$$\begin{aligned}\chi(d(\varrho, \bar{\varrho}))+&F_2(H(\Omega\varrho, \Omega\bar{\varrho}))\\ &\leq F_1\Big\{\rho\Big(d(\varrho, \bar{\varrho}), D(\varrho, \Omega\varrho), D(\bar{\varrho}, \Omega\bar{\varrho}), D(\varrho, \Omega\bar{\varrho}), D(\bar{\varrho}, \Omega\varrho)\Big)\Big\},\end{aligned}$$

for all $\varrho, \bar{\varrho} \in \xi$.

Theorem 3.19.

Suppose that $(\xi, d, \mathfrak{f})$ is a complete MS. Let $\Omega : \xi \longrightarrow K(\xi)$ be a MVM and F_1, F_2 are functions satisfying χF-contraction. Suppose F_1 is non-decreasing and F_2 satisfies the conditions $(\mathbb{F}2')$ and $(\mathbb{F}3)$.
Then, FixΩ is non-empty.

Proof.

Consider an arbitrary point $\varrho_0 \in \xi$ and $\varrho_1 \in \Omega\varrho_0$. If $\varrho_1 \in \Omega\varrho_1$. Then, ϱ_1 is fixed point of Ω. So, wo assume that $\varrho_1 \notin \Omega\varrho_1$.
Now,

$$D(\varrho_1, \Omega\varrho_1) > 0,$$

$$\Rightarrow H(\Omega\varrho_0, \Omega\varrho_1) > 0.$$

Since $\Omega(\varrho_1)$ is compact, so there exists $\varrho_2 \in \Omega(\varrho_1)$ such that

$$d(\varrho_1, \varrho_2) = D(\varrho_1, \Omega\varrho_1).$$

From $(Ni.)$ and $(Nii.)$, we have

$$\begin{aligned}
F_1\big(d(\varrho_1, \varrho_2)\big) &= F_1\big(D(\varrho_1, \Omega\varrho_1)\big) \\
&\leq F_1\big(H(\Omega\varrho_0, \Omega\varrho_1)\big) \\
&\leq F_2\big(H(\Omega\varrho_0, \Omega\varrho_1)\big) \\
&\leq F_1\Big(\rho\big(d(\varrho_0, \varrho_1), D(\varrho_0, \Omega\varrho_0), D(\varrho_1, \Omega\varrho_1), D(\varrho_0, \Omega\varrho_1), D(\varrho_1, \Omega\varrho_0)\big)\Big) \\
&\quad - \chi\big(d(\varrho_0, \varrho_1)\big) \\
&< F_1\Big(\rho\big(d(\varrho_0, \varrho_1), d(\varrho_0, \varrho_1), d(\varrho_1, \varrho_2), d(\varrho_0, \varrho_2), d(\varrho_1, \varrho_1)\big)\Big) \\
&= F_1\Big(\rho\big(d(\varrho_0, \varrho_1), d(\varrho_0, \varrho_1), d(\varrho_1, \varrho_2), d(\varrho_0, \varrho_2), 0\big)\Big).
\end{aligned}$$

$$\Rightarrow F_1\big(d(\varrho_1, \varrho_2)\big) < F_1\Big(\rho\big(d(\varrho_0, \varrho_1), d(\varrho_0, \varrho_1), d(\varrho_1, \varrho_2), d(\varrho_0, \varrho_2), 0\big)\Big). \tag{3.3}$$

As F_1 is nondecreasing, (3.3) and $(\mathtt{p}_3)$ implies

$$\begin{aligned}
d(\varrho_1, \varrho_2) &< \rho\Big(d(\varrho_0, \varrho_1), d(\varrho_0, \varrho_1), d(\varrho_1, \varrho_2), d(\varrho_0, \varrho_2), 0\Big) \\
&\leq \rho\Big(d(\varrho_0, \varrho_1), d(\varrho_0, \varrho_1), d(\varrho_1, \varrho_2), d(\varrho_0, \varrho_1) + d(\varrho_1, \varrho_2), 0\Big).
\end{aligned}$$

$$\Rightarrow d(\varrho_1, \varrho_2) < \rho\Big(d(\varrho_0, \varrho_1), d(\varrho_0, \varrho_1), d(\varrho_1, \varrho_2), d(\varrho_0, \varrho_1) + d(\varrho_1, \varrho_2), 0\Big). \tag{3.4}$$

Using Lemma 3.16, (3.4) implies

$$d(\varrho_1, \varrho_2) < d(\varrho_0, \varrho_1).$$

In the similar way, we get $\varrho_3 \in \Omega\varrho_2$, such that

$$\begin{aligned}
&d(\varrho_2, \varrho_3) = D\big(\varrho_2, \Omega\varrho_2\big), \\
&D\big(\varrho_2, \Omega\varrho_2\big) > 0,
\end{aligned}$$

$$d(\varrho_2, \varrho_3) < d(\varrho_1, \varrho_2).$$

Continuing in the same way, we obtain a sequence $\{\varrho_{\mathfrak{s}}\}$ in ξ such that $\varrho_{\mathfrak{s}+1} \in \Omega\varrho_{\mathfrak{s}}$ which satisfy

$$\begin{aligned} & d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}) = D\big(\varrho_{\mathfrak{s}}, \Omega\varrho_{\mathfrak{s}}\big) \quad \text{with} \quad D\big(\varrho_{\mathfrak{s}}, \Omega\varrho_{\mathfrak{s}}\big) > 0, \\ \text{and} \qquad & d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}) < d(\varrho_{\mathfrak{s}-1}, \varrho_{\mathfrak{s}}), \quad \forall\, \mathfrak{s} \in \mathbb{N}. \end{aligned}$$

Hence, $\{d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1})\}_{\mathfrak{s}\in\mathbb{N}}$ is a decreasing sequence of real numbers. Now, using $(Ni.)$ and $(Nii.)$

$$\begin{aligned} & \chi\big(d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1})\big) + F_2\big(H(\Omega\varrho_{\mathfrak{s}}, \Omega\varrho_{\mathfrak{s}+1})\big) \\ & \leq F_1\Big(\rho\big(d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}), D(\varrho_{\mathfrak{s}}, \Omega\varrho_{\mathfrak{s}}), D(\varrho_{\mathfrak{s}+1}, \Omega\varrho_{\mathfrak{s}+1}), D(\varrho_{\mathfrak{s}}, \Omega\varrho_{\mathfrak{s}+1}), D(\varrho_{\mathfrak{s}+1}, \Omega\varrho_{\mathfrak{s}})\big)\Big) \\ & \leq F_1\Big(\rho\big(d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}), d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}), d(\varrho_{\mathfrak{s}+1}, \varrho_{\mathfrak{s}+2}), d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+2}), d(\varrho_{\mathfrak{s}+1}, \varrho_{\mathfrak{s}+1})\big)\Big) \\ & \leq F_1\Big(\rho\big(d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}), d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}), d(\varrho_{\mathfrak{s}+1}, \varrho_{\mathfrak{s}+2}), d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}) + d(\varrho_{\mathfrak{s}+1}, \varrho_{\mathfrak{s}+2}), 0\big)\Big) \\ & \leq F_1\Big(\rho\big(d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}), d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}), d(\varrho_{\mathfrak{s}+1}, \varrho_{\mathfrak{s}+2}), d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}) + d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}), 0\big)\Big) \\ & \leq F_1\Big(\rho\big(d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}), d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}), d(\varrho_{\mathfrak{s}+1}, \varrho_{\mathfrak{s}+2}), 2d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}), 0\big)\Big) \\ & \leq F_1\Big(d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1})\rho\big(1, 1, 1, 2, 0\big)\Big) \\ & \leq F_1\big(d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1})\big) \\ & = F_1\big(D(\varrho_{\mathfrak{s}}, \Omega\varrho_{\mathfrak{s}})\big) \\ & \leq F_1\big(H(\Omega\varrho_{\mathfrak{s}-1}, \Omega\varrho_{\mathfrak{s}})\big) \\ & \leq F_2\big(H(\Omega\varrho_{\mathfrak{s}-1}, \Omega\varrho_{\mathfrak{s}})\big). \end{aligned}$$

$$\implies F_2\big(H(\Omega\varrho_{\mathfrak{s}}, \Omega\varrho_{\mathfrak{s}+1})\big) \leq F_2\big(H(\Omega\varrho_{\mathfrak{s}-1}, \Omega\varrho_{\mathfrak{s}})\big) - \chi\big(d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1})\big) \quad \forall\ \mathfrak{s} \in \mathbb{N}. \tag{3.5}$$

As $\chi \in \Psi$, so $\mathsf{h} > 0$ and $\mathfrak{s}_0 \in \mathbb{N}$ exist such that $\chi\big(d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1})\big) > \mathsf{h}, \quad \forall\ \mathfrak{s} \geq \mathfrak{s}_0$. Using (3.5), we have

$$F_2\big(H(\Omega\varrho_{\mathfrak{s}}, \Omega\varrho_{\mathfrak{s}+1})\big) = F_2\big(H(\Omega\varrho_{\mathfrak{s}-1}, \Omega\varrho_{\mathfrak{s}})\big) - \chi\big(d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1})\big)$$

$$
\begin{aligned}
&= F_2\big(H(\Omega\varrho_{\mathfrak{s}-2},\Omega\varrho_{\mathfrak{s}-1})\big) - \chi\big(d(\varrho_{\mathfrak{s}-1},\varrho_{\mathfrak{s}})\big)\\
&\quad - \chi\big(d(\varrho_{\mathfrak{s}},\varrho_{\mathfrak{s}+1})\big)\\
&\qquad \vdots\\
&\le F_2\big(H(\Omega\varrho_0,\Omega\varrho_1)\big) - \sum_{j=1}^{\mathfrak{s}} \chi\big(d(\varrho_j,\varrho_{j+1})\big)\\
&= F_2\big(H(\Omega\varrho_0,\Omega\varrho_1)\big) - \sum_{j=1}^{\mathfrak{s}_0-1} \chi\big(d(\varrho_j,\varrho_{j+1})\big)\\
&\quad - \sum_{j=\mathfrak{s}_0}^{\mathfrak{s}} \chi\big(d(\varrho_j,\varrho_{j+1})\big)\\
&\le F_2\big(H(\Omega\varrho_0,\Omega\varrho_1)\big) - (\mathfrak{s}-\mathfrak{s}_0)\mathsf{h} \quad \forall\, \mathfrak{s}\ge \mathfrak{s}_0.
\end{aligned}
$$

$$
F_2\big(H(\Omega\varrho_{\mathfrak{s}},\Omega\varrho_{\mathfrak{s}+1})\big) \le F_2\big(H(\Omega\varrho_0,\Omega\varrho_1)\big) - (\mathfrak{s}-\mathfrak{s}_0)\mathsf{h} \quad \forall\, \mathfrak{s}\ge \mathfrak{s}_0. \tag{3.6}
$$

Applying $\mathfrak{s}\longrightarrow\infty$ in (3.6), we obtain $F_2\big(H(\Omega\varrho_{\mathfrak{s}},\Omega\varrho_{\mathfrak{s}+1})\big)\longrightarrow -\infty$, then, by using ($\mathbb{F}2'$), we obtain

$$
\lim_{\mathfrak{s}\longrightarrow\infty} H\big(\Omega\varrho_{\mathfrak{s}},\Omega\varrho_{\mathfrak{s}+1}\big) = 0,
$$

Now

$$
\lim_{\mathfrak{s}\longrightarrow\infty} d(\varrho_{\mathfrak{s}},\varrho_{\mathfrak{s}+1}) = \lim_{\mathfrak{s}\longrightarrow\infty} D\big(\varrho_{\mathfrak{s}},\Omega\varrho_{\mathfrak{s}}\big) \le \lim_{\mathfrak{s}\longrightarrow\infty} H\big(\Omega\varrho_{\mathfrak{s}-1},\Omega\varrho_{\mathfrak{s}}\big) = 0. \tag{3.7}
$$

Using ($\mathbb{F}3$), we have $\mathsf{k}\in(0,1)$ such that

$$
\lim_{\mathfrak{s}\longrightarrow\infty} \Big(H\big(\Omega\varrho_{\mathfrak{s}},\Omega\varrho_{\mathfrak{s}+1}\big)\Big)^{\mathsf{k}} F_2\Big(H\big(\Omega\varrho_{\mathfrak{s}},\Omega\varrho_{\mathfrak{s}+1}\big)\Big) = 0. \tag{3.8}
$$

Using (3.6), we have for all $\mathfrak{s}\in\mathbb{N}$

$$
\begin{aligned}
&\Big(H\big(\Omega\varrho_{\mathfrak{s}},\Omega\varrho_{\mathfrak{s}+1}\big)\Big)^{\mathsf{k}} F_2\Big(H\big(\Omega\varrho_{\mathfrak{s}},\Omega\varrho_{\mathfrak{s}+1}\big)\Big)\\
&\quad - \Big(H\big(\Omega\varrho_{\mathfrak{s}},\Omega\varrho_{\mathfrak{s}+1}\big)\Big)^{\mathsf{k}} F_2\Big(H\big(\Omega\varrho_0,\Omega\varrho_1\big)\Big)\\
&\le \Big(H\big(\Omega\varrho_{\mathfrak{s}},\Omega\varrho_{\mathfrak{s}+1}\big)\Big)^{\mathsf{k}} \Big(F_2\big(H(\Omega\varrho_0,\Omega\varrho_1)\big) - (\mathfrak{s}-\mathfrak{s}_0)\mathsf{h}\Big)\\
&\quad - \Big(H\big(\Omega\varrho_{\mathfrak{s}},\Omega\varrho_{\mathfrak{s}+1}\big)\Big)^{\mathsf{k}} F_2\Big(H\big(\Omega\varrho_0,\Omega\varrho_1\big)\Big)\\
&= -\Big(H\big(\Omega\varrho_{\mathfrak{s}},\Omega\varrho_{\mathfrak{s}+1}\big)\Big)^{\mathsf{k}} (\mathfrak{s}-\mathfrak{s}_0)\mathsf{h} \le 0.
\end{aligned}
$$

Applying $\lim\limits_{\mathfrak{s}\longrightarrow\infty}$ and using (3.7) and (3.8),

$$\lim_{\mathfrak{s}\longrightarrow\infty}\mathfrak{s}\Big(H\big(\Omega\varrho_{\mathfrak{s}},\Omega\varrho_{\mathfrak{s}+1}\big)\Big)^{\mathsf{k}}=0,$$

so, there exists $\mathfrak{s}_1\in\mathbb{N}$ such that

$$\mathfrak{s}\Big(H\big(\Omega\varrho_{\mathfrak{s}},\Omega\varrho_{\mathfrak{s}+1}\big)\Big)^{\mathsf{k}}\leq 1\quad\forall\,\mathfrak{s}\geq\mathfrak{s}_1,$$

$$\Longrightarrow\ H\big(\Omega\varrho_{\mathfrak{s}},\Omega\varrho_{\mathfrak{s}+1}\big)\leq\frac{1}{\mathfrak{s}^{1/\mathsf{k}}}\quad\forall\,\mathfrak{s}\geq\mathfrak{s}_1,$$

$$\Longrightarrow\ d(\varrho_{\mathfrak{s}},\varrho_{\mathfrak{s}+1})=D\big(\varrho_{\mathfrak{s}},\Omega\varrho_{\mathfrak{s}+1}\big)\leq H\big(\Omega\varrho_{\mathfrak{s}-1},\Omega\varrho_{\mathfrak{s}}\big)\leq\frac{1}{\mathfrak{s}^{1/\mathsf{k}}}\quad\forall\,\mathfrak{s}\geq\mathfrak{s}_1.$$

Now, we prove that $\{\varrho_{\mathfrak{s}}\}_{\mathfrak{s}}$ is Cauchy sequence. Let $\mathsf{m},\mathfrak{s}\in\mathbb{N}$ such that $\mathsf{m}>\mathfrak{s}>\mathfrak{s}_1$. Consider

$$d(\varrho_{\mathsf{m}},\varrho_{\mathfrak{s}})\leq\sum_{i=\mathfrak{s}}^{\mathsf{m}-1}d(\varrho_i,\varrho_{i+1})\leq\sum_{i=\mathfrak{s}}^{\mathsf{m}-1}\frac{1}{i^{1/\mathsf{k}}}\leq\sum_{i=\mathfrak{s}}^{\infty}\frac{1}{i^{1/\mathsf{k}}}.$$

Since $\sum\limits_{i=\mathfrak{s}}^{\infty}\frac{1}{i^{1/\mathsf{k}}}$ is convergent series, hence, we can conclude that $\{\varrho_{\mathfrak{s}}\}_{\mathfrak{s}}$ is Cauchy sequence. Since (ξ,d) is complete MS, so there exists $\varrho*$ in ξ such that

$$\lim_{\mathfrak{s}\longrightarrow\infty}\varrho_{\mathfrak{s}}=\varrho*.$$

Consider

$$\begin{aligned}F_1\big(H(\Omega\varrho,\Omega\bar{\varrho})\big)&\leq F_2\big(H(\Omega\varrho,\Omega\bar{\varrho})\big)\leq\chi\big(d(\varrho,\bar{\varrho})\big)+F_2\big(H(\Omega\varrho,\Omega\bar{\varrho})\big)\\&\leq F_1\Big(\rho\big(d(\varrho,\bar{\varrho}),D(\varrho,\Omega\varrho),D(\bar{\varrho},\Omega\bar{\varrho}),D(\varrho,\Omega\bar{\varrho}),D(\bar{\varrho},\Omega\varrho)\big)\Big).\end{aligned}$$

Since F_1 is nondecreasing function, so $\forall\ \varrho,\bar{\varrho}\in\xi$, we have

$$H\big(\Omega\varrho,\Omega\bar{\varrho}\big)\leq\rho\big(d(\varrho,\bar{\varrho}),D(\varrho,\Omega\varrho),D(\bar{\varrho},\Omega\bar{\varrho}),D(\varrho,\Omega\bar{\varrho}),D(\bar{\varrho},\Omega\varrho)\big).\tag{3.9}$$

Now, we prove that $\varrho*$ is fixed point of ξ. On contrary, let $D(\varrho*,\Omega\varrho*)>0$ and using (3.9), we have

$D(\varrho*,\Omega\varrho*)=\inf\{d(\varrho*,\varrho):\varrho\in\Omega\varrho*\}$

$$\leq \inf\{d(\varrho*, \varrho_{\mathfrak{s}+1}) + d(\varrho_{\mathfrak{s}+1}, \varrho) : \varrho \in \Omega\varrho*\}$$
$$= d(\varrho*, \varrho_{\mathfrak{s}+1}) + \inf\{d(\varrho_{\mathfrak{s}+1}, \varrho) : \varrho \in \Omega\varrho*\}$$
$$= d(\varrho*, \varrho_{\mathfrak{s}+1}) + D(\varrho_{\mathfrak{s}+1}, \Omega\varrho*)$$
$$\leq d(\varrho*, \varrho_{\mathfrak{s}+1}) + H\big(\Omega\varrho_{\mathfrak{s}+1}, \Omega\varrho*\big)$$
$$\leq d(\varrho*, \varrho_{\mathfrak{s}+1}) + \rho\big(d(\varrho_{\mathfrak{s}}, \varrho*), D(\varrho_{\mathfrak{s}}, \Omega\varrho_{\mathfrak{s}}), D(\varrho*, \Omega\varrho*), D(\varrho_{\mathfrak{s}}, \Omega\varrho*), D(\varrho*, \Omega\varrho_{\mathfrak{s}})\big)$$
$$\leq d(\varrho*, \varrho_{\mathfrak{s}+1}) + \rho\big(d(\varrho_{\mathfrak{s}}, \varrho*), d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}), D(\varrho*, \Omega\varrho*), d(\varrho_{\mathfrak{s}}, \varrho*) + D(\varrho*, \Omega\varrho*), d(\varrho*, \varrho_{\mathfrak{s}+1})\big).$$

Hence,

$$D(\varrho*, \Omega\varrho*) \leq d(\varrho*, \varrho_{\mathfrak{s}+1}) + \rho\big(d(\varrho_{\mathfrak{s}}, \varrho*), d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}), D(\varrho*, \Omega\varrho*), d(\varrho_{\mathfrak{s}}, \varrho*) + D(\varrho*, \Omega\varrho*), d(\varrho*, \varrho_{\mathfrak{s}+1})\big).$$

Applying $\lim_{\mathfrak{s}\longrightarrow\infty}$ in above inequality,

$$D(\varrho*, \Omega\varrho*) \leq \rho\big(0, 0, D(\varrho*, \Omega\varrho*), 0 + D(\varrho*, \Omega\varrho*), 0).$$

Using Lemma 3.16 in above inequality,

$$D(\varrho*, \Omega\varrho*) \leq 0,$$

which is contradiction. Hence, $D(\varrho*, \Omega\varrho*) = 0$. As $\Omega\varrho*$ is closed, so $\varrho* \in \Omega\varrho*$ Hence, $\text{Fix}\Omega$ is non-empty. □

Remark 3.20.

If $F_1 = F_2 = F$ and $\rho(\varrho_1, \varrho_2, \varrho_3, \varrho_4, \varrho_5) = \varrho_1$ in Theorem 3.19, it becomes Theorem 2.3 of [58]

Example 3.9.

Suppose that $\xi = \left\{\varrho_{\mathfrak{s}} = \frac{\mathfrak{s}(\mathfrak{s}+1)}{2} : \mathfrak{s} \in \mathbb{N}\right\}$. Define a matric d on ξ as

$$d(\varrho, \bar{\varrho}) = \mid \varrho - \bar{\varrho} \mid,$$

then (ξ, d) is complete MS. Define $F_1, F_2 : (0,\infty) \longrightarrow \mathbb{R}$ by

$$F_1(\varrho) = \begin{cases} \dfrac{1}{\varrho} & \text{if } \varrho \in (0,1) \\ \varrho & \text{if } \varrho \in [1,\infty), \end{cases}$$

and

$$F_2(\varrho) = \ln(\varrho) + \varrho \quad \forall\, \varrho \in (0,\infty),$$

then, $F_1(\varrho) \leq F_2(\varrho)$ for all $\varrho > 0$. Also, F_1 is a non-decreasing function and F_2 satisfy $(\mathbb{F}2')$ and $(\mathbb{F}3)$.

Now, define $\Omega : \xi \longrightarrow K(\xi)$, $\rho : [0,\infty)^5 \longrightarrow [0,\infty)$ and $\chi : [0,\infty) \longrightarrow [0,\infty)$ by

$$\Omega\varrho = \begin{cases} \{\varrho_1\} & \text{if } \varrho = \varrho_1 \\ \{\varrho_1, \varrho_2\} & \text{if } \varrho = \varrho_{\mathfrak{s}},\ \mathfrak{s} \geq 2, \end{cases}$$

$$\rho(\varrho_1, \varrho_2, \varrho_3, \varrho_4, \varrho_5) = \varrho_1 + \eta\varrho_5, \eta \in (0,1),$$

and

$$\chi(t) = \frac{1}{t} \quad \forall\, t \in (0,\infty),$$

respectively. Clearly, $\chi \in \Psi$ and $\rho \in \mathsf{P}$. Note that for $\mathsf{m}, \mathfrak{s} \in \mathbb{N}$, we have

$$H\big(\Omega\varrho_{\mathsf{m}}, \Omega\mathfrak{s}\big) > 0 \iff \mathsf{m} > 2 \text{ and } \mathfrak{s} = 1.$$

Suppose that $H\big(\Omega\varrho_{\mathsf{m}}, \Omega\varrho_{\mathfrak{s}}\big) > 0, \mathsf{m} > 2$ and $\mathfrak{s} = 1$. For $\mathsf{m} > 2$, we have

$$\frac{2}{\mid \mathsf{m}^2 + \mathsf{m} - 2 \mid} + \ln \frac{\mid \mathsf{m}^2 - \mathsf{m} - 2 \mid}{2} + \frac{\mid \mathsf{m}^2 - \mathsf{m} - 2 \mid}{2} \leq \frac{\mid \mathsf{m}^2 + \mathsf{m} - 2 \mid}{2}. \qquad (3.10)$$

As $H\big(\Omega\varrho_{\mathsf{m}}, \Omega\varrho_1\big) = \mid \varrho_{\mathsf{m}-1} - 1 \mid$ and $D\big(\varrho_1, \Omega\varrho_{\mathsf{m}}\big) = 0$, so

$$\begin{aligned} \chi\Big(d(\varrho_{\mathsf{m}}, \varrho_1)\Big) + F_2\Big(H\big(\Omega\varrho_{\mathsf{m}}, \Omega\varrho_1\big)\Big) &= \frac{1}{\mid \varrho_{\mathsf{m}} - \varrho_1 \mid} + F_2\Big(\mid \varrho_{\mathsf{m}-1} - 1 \mid \Big) \\ &= \frac{2}{\mid \mathsf{m}^2 + \mathsf{m} - 2 \mid} + \ln \frac{\mid \mathsf{m}^2 - \mathsf{m} - 2 \mid}{2} \\ &\quad + \frac{\mid \mathsf{m}^2 - \mathsf{m} - 2 \mid}{2} \end{aligned}$$

$$\leq \frac{\mid \mathsf{m}^2 + \mathsf{m} - 2 \mid}{2}$$
$$= d(\varrho_{\mathsf{m}}, \varrho_1) + \eta D\big(\varrho_1, \Omega\varrho_{\mathsf{m}}\big)$$
$$= F_1\Big(d(\varrho_{\mathsf{m}}, \varrho_1)\Big) + \eta D\big(\varrho_1, \Omega\varrho_{\mathsf{m}}\big)$$
$$= F_1\Big\{\rho\Big(d(\varrho_{\mathsf{m}}, \varrho_1), D(\varrho_1, \Omega\varrho_1), D(\varrho_{\mathsf{m}}, \Omega\varrho_{\mathsf{m}}),$$
$$D(\varrho_{\mathsf{m}}, \Omega\varrho_1), D(\varrho_1, \Omega\varrho_{\mathsf{m}})\Big)\Big\}.$$

All conditions of Theorem 3.19 are fulfilled and $\text{Fix}\Omega = \{\varrho_1, \varrho_2\}$.

Corollary 3.21.

Suppose that (ξ, d) is a complete MS, $\Omega : \xi \longrightarrow K(\xi)$ is multivalued mapping, $\chi \in \Psi, F_1$ is real valued non-decreasing function defined on $(0, \infty)$ and F_2 is real valued function defined on $(0, \infty)$ which satisfy $(\mathbb{F}2')$ and $(\mathbb{F}3)$ such that $(Ni.)$and following condition holds:

$H(\Omega\varrho, \Omega\bar{\varrho}) > 0$ implies

$\chi(d(\varrho, \bar{\varrho})) + F_2\Big(H\big(\Omega\varrho, \Omega\bar{\varrho}\big)\Big) \leq F_1\big(\mathsf{M}(\varrho, \bar{\varrho})\big) \quad \forall\ \varrho, \bar{\varrho} \in \xi,$

where

$$\mathsf{M}(\varrho, \bar{\varrho}) = \max\left\{d(\varrho, \bar{\varrho}), D(\varrho, \Omega\varrho), D(\bar{\varrho}, \Omega\bar{\varrho}), \frac{D(\varrho, \Omega\bar{\varrho}), D(\bar{\varrho}, \Omega\varrho)}{2}\right\}$$

Then, FixΩ is non-empty.

Proof.

Let $\rho : [0, \infty)^5 \longrightarrow [0, \infty)$ be defined as

$$\rho(\varrho_1, \varrho_2, \varrho_3, \varrho_4, \varrho_5) = \max\left\{\varrho_1, \varrho_2, \varrho_3, \frac{(\varrho_4 + \varrho_5)}{2}\right\},$$

then, $\rho \in \mathsf{P}$ and result follows from Theorem 3.19. □

Remark 3.22.

In Corollary 3.21, Theorem 2.4 of [59] is generalized and improved.

Corollary 3.23.

Suppose that (ξ, d) is a complete MS, $\Omega : \xi \longrightarrow K(\xi)$ is multivalued mapping,

$\chi \in \Psi, F_1$ *is real valued non-decreasing function defined on* $(0, \infty)$ *and* F_2 *is real valued function defined on* $(0, \infty)$ *which satisfy* $(\mathbb{F}2')$ *and* $(\mathbb{F}3)$ *such that* $(Ni.)$*and following condition holds:*
$H(\Omega\varrho, \Omega\bar{\varrho}) > 0$ *implies*

$$\chi(d(\varrho, \bar{\varrho})) + F_2\Big(H\big(\Omega\varrho, \Omega\bar{\varrho}\big)\Big) \leq F_1\big(\mathfrak{w}(\varrho, \bar{\varrho})\big), \quad \forall\ \varrho, \bar{\varrho} \in \xi,$$

where

$$\mathfrak{w}(\varrho, \bar{\varrho}) = \beta_1 d(\varrho + \bar{\varrho}) + \beta_2 D(\varrho + \Omega\varrho) + \beta_3 D(\bar{\varrho} + \Omega\bar{\varrho}) + \beta_4\big(D(\varrho, \Omega\bar{\varrho}) + D(\bar{\varrho}, \Omega\varrho)\big),$$

with $\beta_1, \beta_2, \beta_3, \beta_4 \geq 0$ *and* $\beta_1 + \beta_2 + \beta_3 + 2\beta_4 < 1$, *then, Fix*$\Omega$ *is non-empty.*

Proof.
Let $\rho : [0, \infty)^5 \longrightarrow [0, \infty)$ be defined as

$$\rho(\varrho_1, \varrho_2, \varrho_3, \varrho_4, \varrho_5) = \beta_1\varrho_1 + \beta_2\varrho_2 + \beta_3\varrho_3 + \beta_4(\varrho_4 + \varrho_5),$$

where $\beta_1, \beta_2, \beta_3, \beta_4 \geq 0$ and $\beta_1 + \beta_2 + \beta_3 + 2\beta_4 < 1$. Then, $\rho \in \mathsf{P}$ and result follows from Theorem 3.19. □

Next, the condition $(\mathbb{F}3)$ of function F_2 is replaced by continuity of F_1 and a new result is obtained.

Theorem 3.24.
Suppose that (ξ, d) *is a complete MS,* $\Omega : \xi \longrightarrow K(\xi)$ *is a multivalued mapping and* $\chi \in \Psi$. *Let* F_1 *be a continuous and non-decreasing real valued function defined on* $(0, \infty)$ *and* F_2 *be a real valued function satisfying* $(\mathbb{F}2')$ *defined on* $(0, \infty)$ *such that* χF*-contraction is satisfied. Then, Fix*Ω *is non-empty.*

Proof.
Consider an arbitrary point $\varrho_0 \in \xi$ and $\varrho_1 \in \Omega\varrho_0$. Then, following the same steps as in the proof of Theorem 3.19, we have a sequence $\{\varrho_\mathfrak{s}\}$ in ξ such that $\varrho_{\mathfrak{s}+1} \in \Omega\varrho_\mathfrak{s}$

which satisfy

$$\begin{aligned} & d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}) = D\big(\varrho_{\mathfrak{s}}, \Omega\varrho_{\mathfrak{s}}\big) \text{ with } D\big(\varrho_{\mathfrak{s}}, \Omega\varrho_{\mathfrak{s}}\big) > 0, \\ \text{and} \qquad & d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}) < d(\varrho_{\mathfrak{s}-1}, \varrho_{\mathfrak{s}}), \quad \forall\, \mathfrak{s} \in \mathbb{N}. \end{aligned}$$

Also,

$$F_2\big(H(\Omega\varrho_{\mathfrak{s}}, \Omega\varrho_{\mathfrak{s}+1})\big) \leq F_2\big(H(\Omega\varrho_0, \Omega\varrho_1)\big) - (\mathfrak{s} - \mathfrak{s}_0)\mathsf{h}, \;\; \forall\, \mathfrak{s} \geq \mathfrak{s}_0. \tag{3.11}$$

Applying $\mathfrak{s} \longrightarrow \infty$ in (3.11) , we obtain $F_2\big(H(\Omega\varrho_{\mathfrak{s}}, \Omega\varrho_{\mathfrak{s}+1})\big) \longrightarrow -\infty$ and by using $(\mathbb{F}2')$,

$$\lim_{\mathfrak{s} \longrightarrow \infty} H\big(\Omega\varrho_{\mathfrak{s}}, \Omega\varrho_{\mathfrak{s}+1}\big) = 0.$$

Now,

$$\lim_{\mathfrak{s} \longrightarrow \infty} d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}) = \lim_{\mathfrak{s} \longrightarrow \infty} D\big(\varrho_{\mathfrak{s}}, \Omega\varrho_{\mathfrak{s}}\big) \leq \lim_{\mathfrak{s} \longrightarrow \infty} H\big(\Omega\varrho_{\mathfrak{s}-1}, \Omega\varrho_{\mathfrak{s}}\big) = 0. \tag{3.12}$$

Now, we prove that

$$\lim_{n,m \longrightarrow \infty} d(\varrho_{\mathfrak{s}}, \varrho_m) = 0. \tag{3.13}$$

If (3.13) does not hold, then we have some $\delta > 0$ such that $\forall\, \mathsf{r} \geq 0$, there exists $m_{\mathsf{k}} > n_{\mathsf{k}} > \mathsf{r}$ and

$$d(\varrho_{\mathfrak{s}}, \varrho_m) > \delta.$$

Further, there exists $\mathsf{r}_0 \in \mathbb{N}$ such that

$$\lambda_{\mathsf{r}_0} = d(\varrho_{\mathfrak{s}-1}, \varrho_{\mathfrak{s}}) < \delta \quad \forall\, \mathfrak{s} \geq \mathsf{r}_0.$$

Now, consider subsequences $\{\varrho_{n_{\mathsf{k}}}\}, \{\varrho_{m_{\mathsf{k}}}\}$ of $\{\varrho_{\mathfrak{s}}\}$ which satisfy

$$d(\varrho_{m_{\mathsf{k}}}, \varrho_{n_{\mathsf{k}}}) > \delta \quad \forall\;\; \mathsf{k} > 0 \text{ where } \mathsf{r}_0 \leq n_{\mathsf{k}} \leq m_{\mathsf{k}} + 1. \tag{3.14}$$

Note that

$$d(\varrho_{m_{\mathsf{k}}-1}, \varrho_{n_{\mathsf{k}}}) \leq \delta \quad \forall\;\; \mathsf{k}, \tag{3.15}$$

where m_{k} is taken as minimal index for which (3.15) holds. Further, we claim that

$$n_{\mathsf{k}} + 2 \leq m_{\mathsf{k}}.$$

If $m_{\mathsf{k}} < n_{\mathsf{k}} + 2$, then, (3.14) implies $d(\varrho_{m_{\mathsf{k}}}, \varrho_{n_{\mathsf{k}}+1}) > \delta$. Using (3.15), we have

$$d(\varrho_{m_{\mathsf{k}}}, \varrho_{n_{\mathsf{k}}+1}) < d(\varrho_{m_{\mathsf{k}}-1}, \varrho_{n_{\mathsf{k}}}) \leq \delta,$$

which is contradiction. Hence, $n_{\mathsf{k}} + 2 \leq m_{\mathsf{k}}$ for all k, which implies that

$$n_{\mathsf{k}} + 1 < m_{\mathsf{k}} < m_{\mathsf{k}} + 1 \quad \text{for all } \mathsf{k}.$$

Using (3.14), (3.15) and triangle inequality,

$$\delta < d(\varrho_{m_{\mathsf{k}}}, \varrho_{m_{\mathsf{k}}-1}) + d(\varrho_{m_{\mathsf{k}}-1}, \varrho_{n_{\mathsf{k}}}) \leq \lambda_{m_{\mathsf{k}}} + \delta.$$

Applying $\lim\limits_{\mathsf{k}\longrightarrow\infty}$, we obtain

$$\lim_{\mathsf{k}\longrightarrow\infty} d(\varrho_{m_{\mathsf{k}}}, \varrho_{n_{\mathsf{k}}}) = \delta. \tag{3.16}$$

Also,

$$\lim_{\mathsf{k}\longrightarrow\infty} d(\varrho_{m_{\mathsf{k}+1}}, \varrho_{n_{\mathsf{k}}+1}) = \delta. \tag{3.17}$$

Using $(Ni.), (Nii.)$ and the monotonicity of F_1,

$$\begin{aligned}
&\chi\big(d(\varrho_{m_{\mathsf{k}}}, \varrho_{n_{\mathsf{k}}})\big) + F_1\big(d(\varrho_{m_{\mathsf{k}+1}}, \varrho_{n_{\mathsf{k}+1}})\big)\\
&= \chi\big(d(\varrho_{m_{\mathsf{k}}}, \varrho_{n_{\mathsf{k}}})\big) + F_1\big(D(\varrho_{m_{\mathsf{k}+1}}, \Omega\varrho_{n_{\mathsf{k}+1}})\big)\\
&\leq \chi\big(d(\varrho_{m_{\mathsf{k}}}, \varrho_{n_{\mathsf{k}}})\big) + F_1\big(H(\Omega\varrho_{m_{\mathsf{k}}}, \Omega\varrho_{n_{\mathsf{k}}})\big)\\
&\leq \chi\big(d(\varrho_{m_{\mathsf{k}}}, \varrho_{n_{\mathsf{k}}})\big) + F_2\big(H(\Omega\varrho_{m_{\mathsf{k}}}, \Omega\varrho_{n_{\mathsf{k}}})\big)\\
&\leq F_1\Big(\rho\big(d(\varrho_{m_{\mathsf{k}}}, \varrho_{n_{\mathsf{k}}}), D(\varrho_{m_{\mathsf{k}}}, \Omega\varrho_{m_{\mathsf{k}}}), D(\varrho_{n_{\mathsf{k}}}, \Omega\varrho_{n_{\mathsf{k}}}), D(\varrho_{m_{\mathsf{k}}}, \Omega\varrho_{n_{\mathsf{k}}}), D(\varrho_{n_{\mathsf{k}}}, \Omega\varrho_{m_{\mathsf{k}}})\big)\Big)\\
&< F_1\Big(\rho\big(d(\varrho_{m_{\mathsf{k}}}, \varrho_{n_{\mathsf{k}}}), d(\varrho_{m_{\mathsf{k}}}, \varrho_{m_{\mathsf{k}}+1}), d(\varrho_{n_{\mathsf{k}}}, \varrho_{n_{\mathsf{k}}+1}), d(\varrho_{n_{\mathsf{k}}+1}, \varrho_{n_{\mathsf{k}}}) + d(\varrho_{n_{\mathsf{k}}}, \varrho_{m_{\mathsf{k}}}),\\
&\quad d(\varrho_{n_{\mathsf{k}}}, \varrho_{n_{\mathsf{k}}+1}) + d(\varrho_{n_{\mathsf{k}}+1}, \varrho_{m_{\mathsf{k}}+1})\big)\Big).
\end{aligned}$$

Applying $\lim_{\mathsf{k}\longrightarrow\infty}$ and then using (3.16), (3.17) and the continuity of F_1,

$$\lim_{\mathsf{k}\longrightarrow\infty}\chi\big(d(\varrho_{m_\mathsf{k}},\varrho_{n_\mathsf{k}})\big)+F_1(\delta)\leq F_1\big(\rho(\delta,0,0,\delta,\delta)\big)\leq F_1\big(\delta\rho(1,0,0,1,1)\big).$$

As, $\rho\in\mathsf{P}$, so $0<\rho(1,0,0,1,1)\leq 1$;

$$\Rightarrow\lim_{\mathsf{k}\longrightarrow\infty}\chi\big(d(\varrho_{m_\mathsf{k}},\varrho_{n_\mathsf{k}})\big)\leq 0.$$

Which contradicts (3.2). So, (3.13) holds and hence, $\{\varrho_\mathfrak{s}\}$ is a Cauchy sequence. Since (ξ,d) is complete MS so there exists $\varrho*$ in ξ such that

$$\lim_{\mathfrak{s}\longrightarrow\infty}\varrho_\mathfrak{s}=\varrho*.$$

In the same way, as in proof of Theorem 3.19, we have $\varrho*\in\Omega\varrho$. Hence, the proof is completed. □

Corollary 3.25.
Suppose that (ξ,d) is a complete MS, $\Omega:\xi\longrightarrow K(\xi)$ is a multivalued mapping and $\chi\in\Psi$. Let F_1 be a continuous real valued non-decreasing function defined on $(0,\infty)$ and F_2 be a real valued function satisfying $(\mathbb{F}2')$ defined on $(0,\infty)$ such that $(Ni.)$ and the following condition is satisfied:
For all $\varrho,\bar{\varrho}\in\xi$ and $\rho\in\mathsf{P}$, $H(\Omega\varrho,\Omega\bar{\varrho})>0$ implies
$\chi(d(\varrho,\bar{\varrho}))+F_2\Big(H\big(\Omega\varrho,\Omega\bar{\varrho}\big)\Big)$
$\leq F_1\Big(\beta_1 d(\varrho,\bar{\varrho})+\beta_2 D(\varrho,\Omega\varrho)+\beta_3 D(\bar{\varrho},\Omega\bar{\varrho})+\beta_4 D(\varrho,\Omega\bar{\varrho})+\beta_5 D(\bar{\varrho},\Omega\varrho)\Big),$
where $\beta_i\geq 0,\beta_1+\beta_2+\beta_3+2\beta_4=1$ and $\beta_1+\beta_3+\beta_4\leq 1$, then, $Fix\Omega$ is non-empty.

Proof.
Let $\rho:[0,\infty)^5\longrightarrow[0,\infty)$ be defined as

$$\rho(\varrho_1,\varrho_2,\varrho_3,\varrho_4,\varrho_5)=\beta_1\varrho_1+\beta_2\varrho_2+\beta_3\varrho_3+\beta_4\varrho_4+\beta_5\varrho_5,$$

where $\beta_i\geq 0,\beta_1+\beta_2+\beta_3+2\beta_4=1$ and $\beta_1+\beta_3+\beta_4\leq 1$. Hence, $\rho\in\mathsf{P}$ and the result follows from Theorem 4.5. □

Remark 3.26.

If $F_1 = F_2$ and $\Omega\varrho = \{\rho\}$ for all $\rho \in \xi$ in Corollary 3.25, then Theorem 1 of [60] is obtained.

Now, we take $\Omega\varrho$ as closed subset of ξ and obtain fixed point results.

Theorem 3.27.

Suppose that (ξ, d) is a complete MS, $\Omega : \xi \longrightarrow C(\xi)$ is a multivalued mapping and $\chi \in \Psi$. Let $F_1 \in \Delta(O*)$ and F_2 be a real valued function defined on $(0, \infty)$ such that χF-contraction is satisfied. Then, $Fix\Omega$ is non-empty.

Proof. Consider an arbitrary point $\varrho_0 \in \xi$ and $\varrho_1 \in \Omega\varrho_0$. If $\varrho_1 \in \Omega\varrho_1$, then ϱ_1 is fixed point of Ω, so, wo assume that $\varrho_1 \notin \Omega\varrho_1$.

Now

$$D(\varrho_1, \Omega\varrho_1) > 0,$$

Using $(\mathbb{F}4)$, we have

$$F_1\big(D(\varrho_1, \Omega\varrho_1)\big) = \inf_{\mathsf{z} \in \Omega\varrho_1} F_1\big(d(\varrho_1, \mathsf{z})\big).$$

Using $(Ni.)$ and $(Nii.)$, we have

$$\begin{aligned}
\inf_{\mathsf{z} \in \Omega\varrho_1} F_1\big(d(\varrho_1, \mathsf{z})\big) &= F_1\big(D(\varrho_1, \Omega\varrho_1)\big) \\
&\leq F_1\big(H(\Omega\varrho_0, \Omega\varrho_1)\big) \\
&\leq F_2\big(H(\Omega\varrho_0, \Omega\varrho_1)\big) \\
&\leq F_1\Big(\rho\big(d(\varrho_0, \varrho_1), D(\varrho_0, \Omega\varrho_0), D(\varrho_1, \Omega\varrho_1), D(\varrho_0, \Omega\varrho_1), D(\varrho_1, \Omega\varrho_0)\big)\Big) \\
&\quad - \chi\big(d(\varrho_0, \varrho_1)\big) \\
&< F_1\Big(\rho\big(d(\varrho_0, \varrho_1), d(\varrho_0, \varrho_1), d(\varrho_1, \varrho_2), d(\varrho_0, \varrho_2), d(\varrho_1, \varrho_1)\big)\Big) \\
&= F_1\Big(\rho\big(d(\varrho_0, \varrho_1), d(\varrho_0, \varrho_1), d(\varrho_1, \varrho_2), d(\varrho_0, \varrho_2), 0\big)\Big).
\end{aligned}$$

So, there exists $\varrho_2 \in \Omega\varrho_1$ such that

$$F_1\big(d(\varrho_1, \varrho_2)\big) < F_1\Big(\rho\big(d(\varrho_0, \varrho_1), d(\varrho_0, \varrho_1), d(\varrho_1, \varrho_2), d(\varrho_0, \varrho_2), 0\big)\Big).$$

As F_1 is nondecreasing, using (p_3) we have

$$\begin{aligned} d(\varrho_1,\varrho_2) &< \rho\Big(d(\varrho_0,\varrho_1), d(\varrho_0,\varrho_1), d(\varrho_1,\varrho_2), d(\varrho_0,\varrho_2), 0\Big) \\ &\leq \rho\Big(d(\varrho_0,\varrho_1), d(\varrho_0,\varrho_1), d(\varrho_1,\varrho_2), d(\varrho_0,\varrho_1)+d(\varrho_1,\varrho_2), 0\Big). \end{aligned}$$

$$\Rightarrow d(\varrho_1,\varrho_2) < \rho\Big(d(\varrho_0,\varrho_1), d(\varrho_0,\varrho_1), d(\varrho_1,\varrho_2), d(\varrho_0,\varrho_1)+d(\varrho_1,\varrho_2), 0\Big). \tag{3.18}$$

Using Lemma 3.16, (3.18) implies

$$d(\varrho_1,\varrho_2) < d(\varrho_0,\varrho_1).$$

In the similar way, we get $\varrho_3 \in \Omega\varrho_2$, such that

$$\begin{aligned} & d(\varrho_2,\varrho_3) = D\big(\varrho_2,\Omega\varrho_2\big) \text{ with } D\big(\varrho_2,\Omega\varrho_2\big) > 0, \\ \text{and} \qquad & d(\varrho_2,\varrho_3) < d(\varrho_1,\varrho_2). \end{aligned}$$

Continuing in the same way, we obtain a sequence $\{\varrho_\mathfrak{s}\}$ in ξ such that $\varrho_{\mathfrak{s}+1} \in \Omega\varrho_\mathfrak{s}$ which satisfy

$$\begin{aligned} & d(\varrho_\mathfrak{s},\varrho_{\mathfrak{s}+1}) = D\big(\varrho_\mathfrak{s},\Omega\varrho_\mathfrak{s}\big) \text{ with } D\big(\varrho_\mathfrak{s},\Omega\varrho_\mathfrak{s}\big) > 0, \\ \text{and} \qquad & d(\varrho_\mathfrak{s},\varrho_{\mathfrak{s}+1}) < d(\varrho_{\mathfrak{s}-1},\varrho_\mathfrak{s}) \quad \forall\, \mathfrak{s} \in \mathbb{N}. \end{aligned}$$

Hence, $\{d(\varrho_\mathfrak{s},\varrho_{\mathfrak{s}+1})\}_{\mathfrak{s}\in\mathbb{N}}$ is a decreasing sequence of real numbers. Next, Using $(Ni.)$ and $(Nii.)$,

$$\begin{aligned} \inf_{\mathsf{z}\in\Omega\varrho_\mathfrak{s}} F_1\big(d(\varrho_\mathfrak{s},\mathsf{z})\big) &= F_1\big(D(\varrho_\mathfrak{s},\Omega\varrho_\mathfrak{s})\big) \leq F_1\big(H(\Omega\varrho_{\mathfrak{s}-1},\Omega\varrho_\mathfrak{s}\big) \\ &\leq F_2\big(H(\Omega\varrho_{\mathfrak{s}-1},\Omega\varrho_\mathfrak{s})\big) \\ &\leq F_1\Big(\rho\big(d(\varrho_{\mathfrak{s}-1},\varrho_\mathfrak{s}), D(\varrho_{\mathfrak{s}-1},\Omega\varrho_{\mathfrak{s}-1}), D(\varrho_\mathfrak{s},\Omega\varrho_\mathfrak{s}), D(\varrho_{\mathfrak{s}-1},\Omega\varrho_\mathfrak{s}), \\ &\quad D(\varrho_\mathfrak{s},\Omega\varrho_{\mathfrak{s}-1})\big)\Big) - \chi\big(d(\varrho_{\mathfrak{s}-1},\varrho_\mathfrak{s})\big) \\ &\leq F_1\Big(\rho\big(d(\varrho_{\mathfrak{s}-1},\varrho_\mathfrak{s}), d(\varrho_{\mathfrak{s}-1},\varrho_\mathfrak{s}), d(\varrho_\mathfrak{s},\varrho_{\mathfrak{s}+1}), d(\varrho_{\mathfrak{s}-1},\varrho_{\mathfrak{s}+1}), d(\varrho_\mathfrak{s},\varrho_\mathfrak{s})\big)\Big) \\ &\quad - \chi\big(d(\varrho_{\mathfrak{s}-1},\varrho_\mathfrak{s})\big) \end{aligned}$$

$$\leq F_1\Big(\rho\big(d(\varrho_{\mathfrak{s}-1},\varrho_{\mathfrak{s}}),d(\varrho_{\mathfrak{s}-1},\varrho_{\mathfrak{s}}),d(\varrho_{\mathfrak{s}},\varrho_{\mathfrak{s}+1}),d(\varrho_{\mathfrak{s}-1},\varrho_{\mathfrak{s}})+d(\varrho_{\mathfrak{s}},\varrho_{\mathfrak{s}+1}),0\big)\Big)$$
$$-\chi\big(d(\varrho_{\mathfrak{s}-1},\varrho_{\mathfrak{s}})\big)$$
$$\leq F_1\Big(\rho\big(d(\varrho_{\mathfrak{s}-1},\varrho_{\mathfrak{s}}),d(\varrho_{\mathfrak{s}-1},\varrho_{\mathfrak{s}}),d(\varrho_{\mathfrak{s}},\varrho_{\mathfrak{s}+1}),d(\varrho_{\mathfrak{s}-1},\varrho_{\mathfrak{s}})+d(\varrho_{\mathfrak{s}-1},\varrho_{\mathfrak{s}}),0\big)\Big)$$
$$-\chi\big(d(\varrho_{\mathfrak{s}-1},\varrho_{\mathfrak{s}})\big)$$
$$\leq F_1\Big(\rho\big(d(\varrho_{\mathfrak{s}-1},\varrho_{\mathfrak{s}}),d(\varrho_{\mathfrak{s}-1},\varrho_{\mathfrak{s}}),d(\varrho_{\mathfrak{s}},\varrho_{\mathfrak{s}+1}),2d(\varrho_{\mathfrak{s}-1},\varrho_{\mathfrak{s}}),0\big)\Big)$$
$$-\chi\big(d(\varrho_{\mathfrak{s}-1},\varrho_{\mathfrak{s}})\big)$$
$$\leq F_1\Big(d(\varrho_{\mathfrak{s}-1},\varrho_{\mathfrak{s}})\rho\big(1,1,1,2,0\big)\Big)-\chi\big(d(\varrho_{\mathfrak{s}-1},\varrho_{\mathfrak{s}})\big)$$
$$\leq F_1\big(d(\varrho_{\mathfrak{s}-1},\varrho_{\mathfrak{s}})\big)-\chi\big(d(\varrho_{\mathfrak{s}-1},\varrho_{\mathfrak{s}})\big).$$

$$\Rightarrow \inf_{\mathsf{z}\in\Omega\varrho_{\mathfrak{s}}} F_1\big(d(\varrho_{\mathfrak{s}},\mathsf{z})\big)\leq F_1\big(d(\varrho_{\mathfrak{s}-1},\varrho_{\mathfrak{s}})\big)-\chi\big(d(\varrho_{\mathfrak{s}-1},\varrho_{\mathfrak{s}})\big).\quad \forall\,\mathfrak{s}\in\mathbb{N}$$

From above, it is clear that there exists $\varrho_{\mathfrak{s}+1}\in\Omega\varrho_{\mathfrak{s}}$ such that

$$\Rightarrow F_1\big(d(\varrho_{\mathfrak{s}},\varrho_{\mathfrak{s}+1})\big)\leq F_1\big(d(\varrho_{\mathfrak{s}-1},\varrho_{\mathfrak{s}})\big)-\chi\big(d(\varrho_{\mathfrak{s}-1},\varrho_{\mathfrak{s}})\big). \tag{3.19}$$

As $\chi\in\Psi$, so $\mathsf{h}>0$ and $\mathfrak{s}_0\in\mathbb{N}$ exist such that $\chi\big(d(\varrho_{\mathfrak{s}},\varrho_{\mathfrak{s}+1})\big)>\mathsf{h}\quad\forall\,\mathfrak{s}\geq\mathfrak{s}_0$. Using (3.19), we have

$$\begin{aligned}
F_1\big(d(\varrho_{\mathfrak{s}},\varrho_{\mathfrak{s}+1})\big)&\leq F_1\big(d(\varrho_{\mathfrak{s}-1},\varrho_{\mathfrak{s}})\big)-\chi\big(d(\varrho_{\mathfrak{s}-1},\varrho_{\mathfrak{s}})\big)\\
&\leq F_1\big(d(\varrho_{\mathfrak{s}-2},\varrho_{\mathfrak{s}-1})\big)-\chi\big(d(\varrho_{\mathfrak{s}-2},\varrho_{\mathfrak{s}-1})\big)-\chi\big(d(\varrho_{\mathfrak{s}-1},\varrho_{\mathfrak{s}})\big)\\
&\leq F_1\big(d(\varrho_0,\varrho_1)\big)-\sum_{j=1}^{n-1}\chi\big(d(\varrho_{\mathfrak{s}-1},\varrho_{\mathfrak{s}})\big)\\
&= F_1\big(d(\varrho_0,\varrho_1)\big)-\sum_{j=1}^{\mathfrak{s}_0-1}\chi\big(d(\varrho_{\mathfrak{s}-1},\varrho_{\mathfrak{s}})\big)-\sum_{\mathfrak{s}_0}^{n-1}\chi\big(d(\varrho_{\mathfrak{s}-1},\varrho_{\mathfrak{s}})\big)\\
&= F_1\big(d(\varrho_0,\varrho_1)\big)-(\mathfrak{s}-\mathfrak{s}_0)\mathsf{h},\quad\forall\,\mathfrak{s}\geq\mathfrak{s}_0.
\end{aligned}$$

$$F_1\big(d(\varrho_{\mathfrak{s}},\varrho_{\mathfrak{s}+1})\big)\leq F_1\big(d(\varrho_0,\varrho_1)\big)-(\mathfrak{s}-\mathfrak{s}_0)\mathsf{h},\quad\forall\,\mathfrak{s}\geq\mathfrak{s}_0. \tag{3.20}$$

Applying $\mathfrak{s}\longrightarrow\infty$, we obtain $F_1\big(d(\varrho_{\mathfrak{s}-1},\varrho_{\mathfrak{s}})\big)\longrightarrow-\infty$ and by using $(\mathbb{F}2')$, we obtain

$$\lim_{\mathfrak{s}\longrightarrow\infty} d(\varrho_{\mathfrak{s}-1},\varrho_{\mathfrak{s}})=0, \tag{3.21}$$

Using ($\mathbb{F}3$), we have $\mathsf{k} \in (0,1)$ such that

$$\lim_{\mathfrak{s}\longrightarrow\infty} \Big(d\big(\varrho_{\mathfrak{s}-1}, \varrho_{\mathfrak{s}}\big)\Big)^{\mathsf{k}} F_1\big(d(\varrho_{\mathfrak{s}-1}, \varrho_{\mathfrak{s}})\big) = 0. \tag{3.22}$$

Using (3.20), we have for all $\mathfrak{s} \in \mathbb{N}$

$$\begin{aligned}
&\Big(d\big(\varrho_{\mathfrak{s}-1}, \varrho_{\mathfrak{s}}\big)\Big)^{\mathsf{k}} F_1\Big(d\big(\varrho_{\mathfrak{s}-1}, \varrho_{\mathfrak{s}}\big)\Big) - \Big(d\big(\varrho_{\mathfrak{s}-1}, \varrho_{\mathfrak{s}}\big)\Big)^{\mathsf{k}} F_1\Big(d\big(\varrho_0, \varrho_1\big)\Big) \\
&\quad \le \Big(d\big(\varrho_{\mathfrak{s}-1}, \varrho_{\mathfrak{s}}\big)\Big)^{\mathsf{k}} \Big(F_1\big(d(\varrho_0, \varrho_1\big) - (\mathfrak{s} - \mathfrak{s}_0)\mathsf{h}\Big) - \Big(d\big(\varrho_{\mathfrak{s}-1}, \varrho_{\mathfrak{s}}\big)\Big)^{\mathsf{k}} F_1\Big(d\big(\varrho_0, \varrho_1\big)\Big) \\
&\quad = -\Big(d\big(\varrho_{\mathfrak{s}-1}, \varrho_{\mathfrak{s}}\big)\Big)^{\mathsf{k}} (\mathfrak{s} - \mathfrak{s}_0)\mathsf{h} \le 0.
\end{aligned}$$

Applying $\lim\limits_{\mathfrak{s}\longrightarrow\infty}$ and using (3.21) and (3.22),

$$\lim_{\mathfrak{s}\longrightarrow\infty} \mathfrak{s}\Big(d\big(\varrho_{\mathfrak{s}-1}, \varrho_{\mathfrak{s}}\big)\Big)^{\mathsf{k}} = 0.$$

So, there exists $\mathfrak{s}_1 \in \mathbb{N}$ such that

$$\mathfrak{s}\Big(d\big(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}\big)\Big)^{\mathsf{k}} \le 1 \quad \forall\, \mathfrak{s} \ge \mathfrak{s}_1,$$

$$\Rightarrow d\big(\varrho_{\mathfrak{s}-1}, \varrho_{\mathfrak{s}}\big) \le \frac{1}{\mathfrak{s}^{1/\mathsf{k}}} \quad \forall\, \mathfrak{s} \ge \mathfrak{s}_1,$$

Now, we prove that $\{\varrho_{\mathfrak{s}}\}_{\mathfrak{s}}$ is Cauchy sequence. Let $\mathsf{m}, \mathfrak{s} \in \mathbb{N}$ such that $\mathsf{m} > \mathfrak{s} > \mathfrak{s}_1$. Consider

$$d(\varrho_{\mathsf{m}}, \varrho_{\mathfrak{s}}) \le \sum_{i=\mathfrak{s}}^{\mathsf{m}-1} d(\varrho_i, \varrho_{i+1}) \le \sum_{i=\mathfrak{s}}^{\mathsf{m}-1} \frac{1}{i^{1/\mathsf{k}}} \le \sum_{i=\mathfrak{s}}^{\infty} \frac{1}{i^{1/\mathsf{k}}}.$$

Since $\sum\limits_{i=\mathfrak{s}}^{\infty} \dfrac{1}{i^{1/\mathsf{k}}}$ is convergent series, So, one we can conclude that $\{\varrho_{\mathfrak{s}}\}_{\mathfrak{s}}$ is Cauchy sequence. Since (ξ, d) is complete MS so there exists $\varrho*$ in ξ such that

$$\lim_{\mathfrak{s}\longrightarrow\infty} \varrho_{\mathfrak{s}} = \varrho * .$$

Now consider

$$F_1\big(H(\Omega\varrho, \Omega\bar{\varrho})\big) \le F_2\big(H(\Omega\varrho, \Omega\bar{\varrho})\big) \le \chi(d(\varrho, \bar{\varrho})) + F_2\big(H(\Omega\varrho, \Omega\bar{\varrho})\big)$$

$$\leq F_1\Big(\rho\big(d(\varrho,\bar{\varrho}), D(\varrho,\Omega\varrho), D(\bar{\varrho},\Omega\bar{\varrho}), D(\varrho,\Omega\bar{\varrho}), D(\bar{\varrho},\Omega\varrho)\big)\Big).$$

Since F_1 is nondecreasing function so, for all $\varrho, \bar{\varrho} \in \xi$, we have

$$H(\Omega\varrho,\Omega\bar{\varrho}) \leq \rho\big(d(\varrho,\bar{\varrho}), D(\varrho,\Omega\varrho), D(\bar{\varrho},\Omega\bar{\varrho}), D(\varrho,\Omega\bar{\varrho}), D(\bar{\varrho},\Omega\varrho)\big). \tag{3.23}$$

Now, we prove that $\varrho*$ is fixed point of ξ. On contrary, let $D(\varrho*,\Omega\varrho*) > 0$ and using (3.23), we have

$$\begin{aligned} D(\varrho*,\Omega\varrho*) &= \inf\{d(\varrho*,\varrho) : \varrho \in \Omega\varrho*\} \\ &\leq \inf\{d(\varrho*,\varrho_{\mathfrak{s}+1}) + d(\varrho_{\mathfrak{s}+1},\varrho) : \varrho \in \Omega\varrho*\} \\ &= d(\varrho*,\varrho_{\mathfrak{s}+1}) + \inf\{d(\varrho_{\mathfrak{s}+1},\varrho) : \varrho \in \Omega\varrho*\} \\ &= d(\varrho*,\varrho_{\mathfrak{s}+1}) + D(\varrho_{\mathfrak{s}+1},\Omega\varrho*) \\ &\leq d(\varrho*,\varrho_{\mathfrak{s}+1}) + H\left(\Omega\varrho_{\mathfrak{s}+1},\Omega\varrho*\right) \\ &\leq d(\varrho*,\varrho_{\mathfrak{s}+1}) + \rho\big(d(\varrho_{\mathfrak{s}},\varrho*), D(\varrho_{\mathfrak{s}},\Omega\varrho_{\mathfrak{s}}), D(\varrho*,\Omega\varrho*), D(\varrho_{\mathfrak{s}},\Omega\varrho*), \\ &\quad D(\varrho*,\Omega\varrho_{\mathfrak{s}})\big) \\ &\leq d(\varrho*,\varrho_{\mathfrak{s}+1}) + \rho\big(d(\varrho_{\mathfrak{s}},\varrho*), d(\varrho_{\mathfrak{s}},\varrho_{\mathfrak{s}+1}), D(\varrho*,\Omega\varrho*), d(\varrho_{\mathfrak{s}},\varrho*)+ \\ &\quad D(\varrho*,\Omega\varrho*), d(\varrho*,\varrho_{\mathfrak{s}+1})\big). \end{aligned}$$

Hence,

$$\begin{aligned} D(\varrho*,\Omega\varrho*) \leq d(\varrho*,\varrho_{\mathfrak{s}+1}) + \rho\big(&d(\varrho_{\mathfrak{s}},\varrho*), d(\varrho_{\mathfrak{s}},\varrho_{\mathfrak{s}+1}), D(\varrho*,\Omega\varrho*), \\ &d(\varrho_{\mathfrak{s}},\varrho*) + D(\varrho*,\Omega\varrho*), d(\varrho*,\varrho_{\mathfrak{s}+1})\big). \end{aligned}$$

Applying $\lim\limits_{\mathfrak{s}\longrightarrow\infty}$ in above inequality, we get

$$D(\varrho*,\Omega\varrho*) \leq \rho\big(0, 0, D(\varrho*,\Omega\varrho*), 0 + D(\varrho*,\Omega\varrho*), 0).$$

Using Lemma 3.16 in above inequality,

$$D(\varrho*,\Omega\varrho*) \leq 0,$$

which is contradiction. Hence, $D(\varrho*, \Omega\varrho*) = 0$. As $\Omega\varrho*$ is closed, so $\varrho* \in \Omega\varrho*$. Hence, FixΩ is non-empty. □

Corollary 3.28.

Suppose that (ξ, d) is a complete MS, $\Omega : \xi \longrightarrow C(\xi)$ is a multivalued mapping and $\chi \in \Psi$. Let $F_1 \in \Delta(O*)$ and F_2 be a real valued function defined on $(0, \infty)$ such that $(Ni.)$ and the following condition is satisfied:

For all $\varrho, \bar{\varrho} \in \xi$ and $\rho \in \mathsf{P}$, $H(\Omega\varrho, \Omega\bar{\varrho}) > 0$ implies ,

$\chi(d(\varrho, \bar{\varrho})) + F_2\Big(H\big(\Omega\varrho, \Omega\bar{\varrho}\big)\Big)$

$\leq F_1\Big(\beta_1 d(\varrho, \bar{\varrho}) + \beta_2 D(\varrho, \Omega\varrho) + \beta_3 D(\bar{\varrho}, \Omega\bar{\varrho}) + \beta_4 D(\varrho, \Omega\bar{\varrho}) + \beta_5 D(\bar{\varrho}, \Omega\varrho)\Big),$

where $\beta_i \geq 0, \beta_1 + \beta_2 + \beta_3 + 2\beta_4 = 1$.

Then, FixΩ is non-empty.

Proof.

Let $\rho : [0, \infty)^5 \longrightarrow [0, \infty)$ be defined as

$$\rho(\varrho_1, \varrho_2, \varrho_3, \varrho_4, \varrho_5) = \beta_1\varrho_1 + \beta_2\varrho_2 + \beta_3\varrho_3 + \beta_4\varrho_4 + \beta_5\varrho_5,$$

where $\beta_i \geq 0, \beta_1 + \beta_2 + \beta_3 + 2\beta_4 = 1$. Then $\rho \in \mathsf{P}$ and the result follows from Theorem 3.27. □

Corollary 3.29.

Suppose that (ξ, d) is a complete MS, $\Omega : \xi \longrightarrow C(\xi)$ is a multivalued mapping and $\chi \in \Psi$. Let $F_1 \in \Delta(O*)$ and F_2 be a real valued function defined on $(0, \infty)$ such that $(Ni.)$ and the following condition is satisfied:

For all $c, \bar{\varrho} \in \xi$ and $\rho \in \mathsf{P}$, $H(\Omega\varrho, \Omega\bar{\varrho}) > 0$ implies ,

$$\chi(d(\varrho, \bar{\varrho})) + F_2\Big(H\big(\Omega\varrho, \Omega\bar{\varrho}\big)\Big) \leq F_1\big(d(\varrho, \bar{\varrho}) + lD(\bar{\varrho}, \Omega\varrho)\big),$$

where $l \geq 0$. Then, FixΩ is non-empty.

Proof.

Let $\rho : [0, \infty)^5 \longrightarrow [0, \infty)$ be defined as

$$\rho(\varrho_1, \varrho_2, \varrho_3, \varrho_4, \varrho_5) = \varrho_1 + l\varrho_5,$$

where $l \geq 0$. Then, $\rho \in \mathsf{P}$ and the result follows from Theorem 3.27. □

Remark 3.30.

If $F_1 = F_2$ and $\chi(\mathsf{t}) = 2\tau$, where $\tau > 0$ in Corollary 3.28, then we obtain Theorem 3 of [15]. Also, if $F_1 = F_2$ and $\chi(\mathsf{t}) = \tau$, where $\tau > 0$ in Corollary 3.29, then we obtain Theorem 2.4 of [61].

Theorem 3.31.

Suppose that (ξ, d) is a complete MS, $\Omega : \xi \longrightarrow C(\xi)$ is a multivalued mapping and $\chi \in \Psi$. Let F_1 be a continuous and non-decreasing real valued function defined on $(0, \infty)$ satisfying $(\mathbb{F}2')$ and F_2 be a real valued function defined on $(0, \infty)$ such that χF-contraction is satisfied. Then, $Fix\Omega$ is nonempty.

Proof.

Consider an arbitrary point $\varrho_0 \in \xi$ and $\varrho_1 \in \Omega\varrho_0$. Then, following the same steps as in the proof of Theorem 3.27, we have a sequence $\{\varrho_\mathfrak{s}\}$ in ξ such that $\varrho_{\mathfrak{s}+1} \in \Omega\varrho_\mathfrak{s}$ which satisfy

$$\begin{aligned} &d(\varrho_\mathfrak{s}, \varrho_{\mathfrak{s}+1}) = D\big(\varrho_\mathfrak{s}, \Omega\varrho_\mathfrak{s}\big) \text{ with } D\big(\varrho_\mathfrak{s}, \Omega\varrho_\mathfrak{s}\big) > 0, \\ \text{and} \qquad &d(\varrho_\mathfrak{s}, \varrho_{\mathfrak{s}+1}) < d(\varrho_{\mathfrak{s}-1}, \varrho_\mathfrak{s}) \quad \forall\, \mathfrak{s} \in \mathbb{N}. \end{aligned}$$

also,

$$F_1\big(d(\varrho_{\mathfrak{s}-1}, \varrho_\mathfrak{s})\big) \leq F_1\big(d(\varrho_0, \varrho_1)\big) - (\mathfrak{s} - \mathfrak{s}_0)\mathsf{h}, \forall\, \mathfrak{s} \geq \mathfrak{s}_0. \tag{3.24}$$

Applying $\mathfrak{s} \longrightarrow \infty$ on (3.24) , we obtain $F_1\big(d(\varrho_{\mathfrak{s}-1}, \varrho_\mathfrak{s})\big) \longrightarrow -\infty$ and by using $(\mathbb{F}2')$, we obtain

$$\lim_{\mathfrak{s} \longrightarrow \infty} d\big(\varrho_{\mathfrak{s}-1}, \varrho_\mathfrak{s}\big) = 0,$$

Now, we prove that

$$\lim_{n,m \longrightarrow \infty} d(\varrho_\mathfrak{s}, \varrho_m) = 0. \tag{3.25}$$

If (3.25) does not hold, then we have some $\delta > 0$ such that $\forall\, \mathsf{r} \geq 0$, there exists $m_\mathsf{k} > n_\mathsf{k} > \mathsf{r}$ and

$$d(\varrho_\mathfrak{s}, \varrho_m) > \delta.$$

Further, there exists $\mathsf{r}_0 \in \mathbb{N}$ such that

$$\lambda_{\mathsf{r}_0} = d(\varrho_{\mathfrak{s}-1}, \varrho_{\mathfrak{s}}) < \delta \quad \forall\, \mathfrak{s} \geq \mathsf{r}_0$$

Now, consider subsequences $\{\varrho_{n_\mathsf{k}}\}, \{\varrho_{m_\mathsf{k}}\}$ of $\{\varrho_{\mathfrak{s}}\}$, then, as in proof of Theorem 4.5, we get

$$\lim_{\mathsf{k}\longrightarrow\infty} d(\varrho_{m_\mathsf{k}}, \varrho_{n_\mathsf{k}}) = \delta. \tag{3.26}$$

Also,

$$\lim_{\mathsf{k}\longrightarrow\infty} d(\varrho_{m_{\mathsf{k}+1}}, \varrho_{n_\mathsf{k}+1}) = \delta. \tag{3.27}$$

Using $(Ni.), (Nii.)$ and the monotonicity of F_1,

$$\begin{aligned}
&\chi\big(d(\varrho_{m_\mathsf{k}}, \varrho_{n_\mathsf{k}})\big) + F_1\big(d(\varrho_{m_{\mathsf{k}+1}}, \varrho_{n_{\mathsf{k}+1}})\big) \\
&= \chi\big(d(\varrho_{m_\mathsf{k}}, \varrho_{n_\mathsf{k}})\big) + F_1\big(D(\varrho_{m_{\mathsf{k}+1}}, \Omega\varrho_{n_{\mathsf{k}+1}}\big) \\
&\leq \chi\big(d(\varrho_{m_\mathsf{k}}, \varrho_{n_\mathsf{k}})\big) + F_1\big(H(\Omega\varrho_{m_\mathsf{k}}, \Omega\varrho_{n_\mathsf{k}})\big) \\
&\leq \chi\big(d(\varrho_{m_\mathsf{k}}, \varrho_{n_\mathsf{k}})\big) + F_2\big(H(\Omega\varrho_{m_\mathsf{k}}, \Omega\varrho_{n_\mathsf{k}})\big) \\
&\leq F_1\Big(\rho\big(d(\varrho_{m_\mathsf{k}}, \varrho_{n_\mathsf{k}}), D(\varrho_{m_\mathsf{k}}, \Omega\varrho_{m_\mathsf{k}}), D(\varrho_{n_\mathsf{k}}, \Omega\varrho_{n_\mathsf{k}}), D(\varrho_{m_\mathsf{k}}, \Omega\varrho_{n_\mathsf{k}}), D(\varrho_{n_\mathsf{k}}, \Omega\varrho_{m_\mathsf{k}})\big)\Big) \\
&< F_1\Big(\rho\big(d(\varrho_{m_\mathsf{k}}, \varrho_{n_\mathsf{k}}), d(\varrho_{m_\mathsf{k}}, \varrho_{m_\mathsf{k}+1}), d(\varrho_{n_\mathsf{k}}, \varrho_{n_\mathsf{k}+1}), d(\varrho_{n_\mathsf{k}+1}, \varrho_{n_\mathsf{k}}) + d(\varrho_{n_\mathsf{k}}, \varrho_{m_\mathsf{k}}), \\
&\quad d(\varrho_{n_\mathsf{k}}, \varrho_{n_\mathsf{k}+1}) + d(\varrho_{n_\mathsf{k}+1}, \varrho_{m_\mathsf{k}+1})\big)\Big).
\end{aligned}$$

Applying $\lim_{\mathsf{k}\longrightarrow\infty}$ and then using 3.26, 3.27 and the continuity of F_1,

$$\lim_{\mathsf{k}\longrightarrow\infty} \chi\big(d(\varrho_{m_\mathsf{k}}, \varrho_{n_\mathsf{k}})\big) + F_1(\delta) \leq F_1\big(\rho(\delta, 0, 0, \delta, \delta)\big) \leq F_1\big(\delta\rho(1, 0, 0, 1, 1)\big).$$

As, $\rho \in \mathsf{P}$, so $0 < \rho(1, 0, 0, 1, 1) \leq 1$;

$$\Rightarrow \lim_{\mathsf{k}\longrightarrow\infty} \chi\big(d(\varrho_{m_\mathsf{k}}, \varrho_{n_\mathsf{k}})\big) \leq 0,$$

which contradicts (3.2). So, (3.25) holds and hence, $\{\varrho_{\mathfrak{s}}\}$ is a Cauchy sequence. Since (ξ, d) is complete MS so there exists $\varrho*$ in ξ such that

$$\lim_{\mathfrak{s}\longrightarrow\infty} \varrho_{\mathfrak{s}} = \varrho * .$$

In the same way, as in proof of Theorem 3.27, we have $\varrho* \in \Omega\varrho$. Hence, the proof is completed. □

Corollary 3.32.

Suppose that (ξ, d) is a complete MS, $\Omega : \xi \longrightarrow C(\xi)$ is a multivalued mapping and $\chi \in \Psi$. Let F_1 be a continuous and non-decreasing real valued function defined on $(0, \infty)$ satisfying $(\mathbb{F}2')$ and F_2 be a real valued function defined on $(0, \infty)$ such that $(Ni.)$ and the following condition is satisfied:

For all $\varrho, \bar{\varrho} \in \xi$ and $\rho \in \mathsf{P}$, $H(\Omega\varrho, \Omega\bar{\varrho}) > 0$ implies ,

$\chi(d(\varrho, \bar{\varrho})) + F_2\Big(H\big(\Omega\varrho, \Omega\bar{\varrho}\big)\Big)$

$\leq F_1\Big(\beta_1 d(\varrho, \bar{\varrho}) + \beta_2 D(\varrho, \Omega\varrho) + \beta_3 D(\bar{\varrho}, \Omega\bar{\varrho}) + \beta_4 D(\varrho, \Omega\bar{\varrho}) + \beta_5 D(\bar{\varrho}, \Omega\varrho)\Big),$

where $\beta_i \geq 0, \beta_1 + \beta_2 + \beta_3 + 2\beta_4 = 1$ and $\beta_1 + \beta_3 + \beta_4 \leq 1$

Then, $Fix\Omega$ is non-empty.

Proof.

Let $\rho : [0, \infty)^5 \longrightarrow [0, \infty)$ be defined as

$$\rho(\varrho_1, \varrho_2, \varrho_3, \varrho_4, \varrho_5) = \beta_1\varrho_1 + \beta_2\varrho_2 + \beta_3\varrho_3 + \beta_4\varrho_4 + \beta_5\varrho_5,$$

where $\beta_i \geq 0, \beta_1 + \beta_2 + \beta_3 + 2\beta_4 = 1$ and $\beta_1 + \beta_3 + \beta_4 \leq 1$ Then $\rho \in \mathsf{P}$ and the result follows from Theorem 3.31. □

Corollary 3.33.

Suppose that (ξ, d) is a complete MS, $\Omega : \xi \longrightarrow C(\xi)$ is a multivalued mapping and $\chi \in \Psi$. Let F_1 be a continuous and non-decreasing real valued function defined on $(0, \infty)$ satisfying $(\mathbb{F}2')$ and F_2 be a real valued function defined on $(0, \infty)$ such that $(Ni.)$ and the following condition is satisfied:

For all $\varrho, \bar{\varrho} \in \xi$ and $\rho \in \mathsf{P}$, $H(\Omega\varrho, \Omega\bar{\varrho} > 0$ implies,

$$\chi(d(\varrho, \bar{\varrho})) + F_2\Big(H\big(\Omega\varrho, \Omega\bar{\varrho}\big)\Big) \leq F_1\big(d(\varrho, \bar{\varrho}) + lD(\bar{\varrho}, \Omega\varrho\big),$$

where $l \geq 0$. Then, $Fix\Omega$ is non-empty.

3.4 Data Dependence

This section is designed for the discussion of data dependence of the generalized multivalued contraction.

Consider a MS (ξ, d) and the mappings $\Omega_1, \Omega_2 : \xi \longrightarrow P(\xi)$ such that the fixed point sets $\text{Fix}\Omega_1$ and $\text{Fix}\Omega_2$ are non-empty. The problem of finding Pomeiu-Hausdroff distance H between $\text{Fix}\Omega_1$ and $\text{Fix}\Omega_2$ undr the condition that for $\ell > 0, H\big(\Omega_1\varrho, \Omega_2\varrho\big) < \ell, \forall\ \varrho \in \xi$, is addressed by many authors. In 2009, G. Mot and A. Petrusel [62] discussed certain basic problems including data dependence. Rus et al. [63] presented an interesting abstract notion as follows.

Definition 3.34. Multivalued Weakly Picard Operator.

Let us consider a metric space (ξ, d) and a multivalued operator $\Omega : \xi \longrightarrow CL(\xi)$. Ω is known as multivalued weakly picard operator(MWP Operator) if a sequence $\{\varrho_\mathfrak{s}\}$ exists for all $\varrho \in \xi$ and $\bar{\varrho} \in \Omega\varrho$ such that

(i) $\varrho_0 = \varrho,\ \varrho_1 = \bar{\varrho}$,

(ii) $\varrho_{\mathfrak{s}+1} = \Omega\varrho_\mathfrak{s}$, for all $\mathfrak{s} \in \mathbb{N}$,

(iii) The sequence $\{\varrho_\mathfrak{s}\}$ is converges to the fixed point of Ω.

If $\{\varrho_\mathfrak{s}\}$ satisfies only (i) and (ii) of Definition 3.34, then it is said to be a sequence of successive approximations of Ω starting from ϱ_0.

Theorem 3.35.

Suppose that (ξ, d) is a complete MS. $\Omega_1, \Omega_2 : \xi \longrightarrow K(\xi)$ are multivalued mappings and $\chi \in \Psi$. Let F_1 be real valued non-decreasing function defined on $(0, \infty)$ and F_2 be a real valued function satisfying $(\mathbb{F}2')$ and $(\mathbb{F}3)$ defined on $(0, \infty)$ such that χF-contraction is satisfied for Ω_i, where $\mathsf{i} \in \{1, 2\}$ and there exists $\lambda > 0$ such that $H\big(\Omega_1(\varrho), \Omega_2(\varrho)\big) \leq \lambda$, for all $\varrho \in \xi$.

Then,

(a.) Fix $\Omega_\mathsf{i} \in CL(\xi)$ for $\mathsf{i} \in \{1, 2\}$,

(b.) Ω_1, Ω_2 *are MWP Operators and*

$$H\Big(Fix(\Omega_1), Fix(\Omega_2)\Big) \leq \frac{\lambda}{1 - \max\{\rho_1(1,1,1,2,0), \rho_2(1,1,1,2,0)\}}.$$

Proof.

$(a.)$ Using Theorem 3.19, we have Fix Ω_{i} is not empty for $\mathsf{i} \in \{1,2\}$. Now, we prove that for $\mathsf{i} \in \{1,2\}$, the fixed point set of Ω_{i} is closed. Consider a sequence $\{\varrho_{\mathfrak{s}}\}$ in fixΩ_{i} such that $\varrho_{\mathfrak{s}} \longrightarrow \varrho$ as $\mathfrak{s} \longrightarrow \infty$. Now,

$$\begin{aligned} F_1\big(H(\Omega\varrho, \Omega\bar{\varrho}\big) &\leq F_2\big(H(\Omega\varrho, \Omega\bar{\varrho}\big) \leq \chi\big(d(\varrho,\bar{\varrho}) + F_2\big(H(\Omega\varrho, \Omega\bar{\varrho}\big) \\ &\leq F_1\Big(\rho\big(d(\varrho,\bar{\varrho}), D(\varrho,\Omega\varrho), D(\bar{\varrho},\Omega\bar{\varrho}), D(\varrho,\Omega\bar{\varrho}), D(\bar{\varrho},\Omega\varrho)\big)\Big). \end{aligned}$$

Since F_1 is nondecreasing function, so for all $\varrho, \bar{\varrho} \in \xi$,

$$H(\Omega\varrho, \Omega\bar{\varrho}) \leq \rho\big(d(\varrho,\bar{\varrho}), D(\varrho,\Omega\varrho), D(\bar{\varrho},\Omega\bar{\varrho}), D(\varrho,\Omega\bar{\varrho}), D(\bar{\varrho},\Omega\varrho)\big). \tag{3.28}$$

Let $D(\varrho,\Omega\varrho) > 0$ and using (4.33), we have

$$\begin{aligned} D(\varrho,\Omega\varrho) &= \inf\{d(\varrho,\bar{\varrho}) : \bar{\varrho} \in \Omega\varrho\} \\ &\leq \inf\{d(\varrho,\varrho_{\mathfrak{s}+1}) + d(\varrho_{\mathfrak{s}+1},\bar{\varrho}) : \bar{\varrho} \in \Omega\varrho\} \\ &= d(\varrho,\varrho_{\mathfrak{s}+1}) + \inf\{d(\varrho_{\mathfrak{s}+1},\bar{\varrho}) : \bar{\varrho} \in \Omega\varrho\} \\ &= d(\varrho,\varrho_{\mathfrak{s}+1}) + D(\varrho_{\mathfrak{s}+1},\Omega\varrho) \\ &\leq d(\varrho,\varrho_{\mathfrak{s}+1}) + H\big(\Omega(\varrho_{\mathfrak{s}+1}),\Omega\varrho\big) \\ &\leq d(\varrho,\varrho_{\mathfrak{s}+1}) + \rho\big(d(\varrho_{\mathfrak{s}},\varrho), D(\varrho_{\mathfrak{s}},\Omega(\varrho_{\mathfrak{s}})), D(\varrho,\Omega\varrho), D(\varrho_{\mathfrak{s}},\Omega\varrho), \\ &\quad D(\varrho,\Omega(\varrho_{\mathfrak{s}}))\big) \\ &\leq d(\varrho,\varrho_{\mathfrak{s}+1}) + \rho\big(d(\varrho_{\mathfrak{s}},\varrho), d(\varrho_{\mathfrak{s}},\varrho_{\mathfrak{s}+1}), D(\varrho,\Omega\varrho), d(\varrho_{\mathfrak{s}},\varrho) + \\ &\quad D(\varrho,\Omega\varrho), d(\varrho,\varrho_{\mathfrak{s}+1})\big). \end{aligned}$$

Hence

$D(\varrho,\Omega\varrho)$

$$\leq d(\varrho,\varrho_{\mathfrak{s}+1}) + \rho\big(d(\varrho_{\mathfrak{s}},\varrho), d(\varrho_{\mathfrak{s}},\varrho_{\mathfrak{s}+1}), D(\varrho,\Omega\varrho), d(\varrho_{\mathfrak{s}},\varrho) + D(\varrho,\Omega\varrho), d(\varrho,\varrho_{\mathfrak{s}+1})\big).$$

Applying $\lim_{\mathfrak{s}\longrightarrow\infty}$ in above inequality,

$$D(\varrho, \Omega\varrho) \leq \rho\big(0, 0, D(\varrho, \Omega\varrho), 0 + D(\varrho, \Omega\varrho), 0\big).$$

Using Lemma 3.16 in above inequality, we obtain

$$D(\varrho, \Omega\varrho) \leq 0,$$

which is contradiction. Hence, $D(\varrho, \Omega\varrho) = 0$. As $\Omega\varrho$ is closed, hence $\varrho \in \Omega\varrho$.

$(b.)$ Using Theorem 3.19, we get that Ω_1, Ω_2 are MWP Operators. So, We have to prove that

$$H\Big(\mathrm{Fix}(\Omega_1), \mathrm{Fix}(\Omega_2)\Big) \leq \frac{\lambda}{1 - \max\left\{\rho_1(1,1,1,2,0), \rho_2(1,1,1,2,0)\right\}}.$$

Suppose $\mathsf{q} > 1$ and $\varrho_0 \in Fix(\Omega_2)$, then, $\varrho_1 \in \Omega_2(\varrho_0)$ exists such that $d(\varrho_0, \varrho_1) = D(\varrho_0, \Omega_2(\varrho_0)$ and $d(\varrho_1, \varrho_2) \leq \mathsf{q}H(\Omega_1(\varrho_0), \Omega_2(\varrho_0)$. Now, $\varrho_2 \in \Omega_2(\varrho_1)$ exists such that $d(\varrho_0, \varrho_1) = D(\varrho_0, \Omega_2(\varrho_0)$ and $d(\varrho_1, \varrho_2) \leq \mathsf{q}H(\Omega_2(\varrho_0), \Omega_2(\varrho_1)$. Also, we get $d(\varrho_1, \varrho_2) \leq d(\varrho_0, \varrho_1)$ and

$$\begin{aligned}
&d(\varrho_1, \varrho_2) \leq \mathsf{q}H(\Omega_2(\varrho_0), \Omega_2(\varrho_1))\\
&\leq \mathsf{q}\rho\big(d(\varrho_0, \varrho_1), D(\varrho_0, \Omega\varrho_0), D(\varrho_1, \Omega\varrho_1), D(\varrho_0, \Omega\varrho_1), D(\varrho_1, \Omega\varrho_0)\big)\\
&\leq \mathsf{q}\rho\big(d(\varrho_0, \varrho_1), d(\varrho_0, \varrho_1), d(\varrho_1, \varrho_2), d(\varrho_0, \varrho_2), d(\varrho_1, \varrho_1)\big)\\
&\leq \mathsf{q}\rho\big(d(\varrho_0, \varrho_1), d(\varrho_0, \varrho_1), d(\varrho_1, \varrho_2), d(\varrho_0, \varrho_1) + d(\varrho_1, \varrho_2), 0\big)\\
&\leq \mathsf{q}\rho\big(d(\varrho_0, \varrho_1), d(\varrho_0, \varrho_1), d(\varrho_1, \varrho_2), d(\varrho_0, \varrho_1) + d(\varrho_0, \varrho_1), 0\big)\\
&\leq \mathsf{q}\big(d(\varrho_0, \varrho_1)\rho(1,1,1,2,0)\big)
\end{aligned}$$

Hence, we will get a sequence of successive approximations of Ω starting from ϱ_0, which satisfy the following

$$d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}) \leq \Big(\mathsf{q}\rho_1\big(1,1,1,2,0\big)\Big)^{\mathfrak{s}} d(\varrho_0, \varrho_1), \quad \forall\, \mathfrak{s} \in \mathbb{N}.$$

$$\Rightarrow d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+m}) \leq \frac{\Big(\mathsf{q}\rho_1\big(1,1,1,2,0\big)\Big)^{\mathfrak{s}}}{1 - \mathsf{q}\rho_1\big(1,1,1,2,0\big)} d(\varrho_0, \varrho_1), \quad \forall\, \mathfrak{s} \in \mathbb{N}. \tag{3.29}$$

Taking $\lim_{\mathfrak{s}\longrightarrow\infty}$, it is concluded that $\{\varrho_{\mathfrak{s}}\}$ is Cauchy sequence in (ξ, d) so converges to some $\mathsf{v} \in \xi$. Using the proof of Theorem 3.19, we have $\mathsf{v} \in \mathrm{Fix}\Omega_2$. Applying $\lim_{m\longrightarrow\infty}$, we get

$$d(\varrho_{\mathfrak{s}}, \mathsf{v}) \leq \frac{\Big(\mathsf{q}\rho_1\big(1,1,1,2,0\big)\Big)^{\mathfrak{s}}}{1-\mathsf{q}\rho_1\big(1,1,1,2,0\big)} d(\varrho_0, \varrho_1), \quad \forall\, \mathfrak{s} \in \mathbb{N}.$$

Choosing $\mathfrak{s} = 0$,

$$d(\varrho_0, \mathsf{v}) \leq \frac{1}{1-\mathsf{q}\rho_1\big(1,1,1,2,0\big)} d(\varrho_0, \varrho_1), \leq \frac{\mathsf{q}\lambda}{1-\mathsf{q}\rho_1\big(1,1,1,2,0\big)}.$$

Now, we interchange the role of Ω_1 and Ω_2, then for each $\mathsf{v}_0 \in \mathrm{Fix}\Omega_1$ such that

$$J(\mathsf{v}_0, c) \leq \frac{1}{1-\mathsf{q}\rho_2\big(1,1,1,2,0\big)} J(\mathsf{v}_0, \mathsf{v}_1) \leq \frac{\mathsf{q}\lambda}{1-\mathsf{q}\rho_2\big(1,1,1,2,0\big)}.$$

So,

$$H(\mathrm{Fix}\Omega_1, \mathrm{Fix}\Omega_2) \leq \frac{\mathsf{q}\lambda}{1-\max\Big(\mathsf{q}\rho_1\big(1,1,1,2,0\big), \mathsf{q}\rho_2\big(1,1,1,2,0\big)\Big)}\Big),$$

and suppose $\mathsf{q} \longrightarrow 1$, then the result is proved. □

3.5 Strict Fixed Point and Well Posedness

The aim of this section is to introduce the notion of well-posedness of the related fixed point results.

Definition 3.36.

Consider a MS (ξ, d), $\mathbb{B} \in P(\xi)$ and multi-valued mapping $\Omega\colon \mathbb{B} \longrightarrow C(\xi)$. A fixed point problem is said to be well posed for Ω with respect to D if

(a.) $\mathrm{Fix}\Omega = \{\varrho^*\}$,

(b.) If $\varrho_{\mathfrak{s}} \in \mathbb{B}$, $\mathfrak{s} \in \mathbb{N}$ and $D(\varrho_{\mathfrak{s}}, \Omega\varrho_{\mathfrak{s}}) \longrightarrow 0$ as $\mathfrak{s} \longrightarrow \infty$, then, $\varrho_{\mathfrak{s}} \longrightarrow \varrho^* \in \mathrm{Fix}\Omega$ as $\mathfrak{s} \longrightarrow \infty$. [64, 65]

Definition 3.37.

Consider a MS (ξ, d), $\mathbb{B} \in P(\xi)$ and multi-valued mapping $\Omega\colon \mathbb{B} \longrightarrow C(\xi)$. A fixed point problem is said to be well posed for Ω with respect to H if

(a.) $\mathsf{S}\mathrm{Fix}\Omega = \{\varrho^*\}$,

(b.) If $\varrho_{\mathfrak{s}} \in \mathbb{B}$, $\mathfrak{s} \in \mathbb{N}$ and $H(\varrho_{\mathfrak{s}}, \Omega\varrho_{\mathfrak{s}}) \longrightarrow 0$ as $\mathfrak{s} \longrightarrow \infty$, then, $\varrho_{\mathfrak{s}} \longrightarrow \varrho^* \in \mathsf{S}\mathrm{Fix}\Omega$ as $\mathfrak{s} \longrightarrow \infty$. [64, 65]

Theorem 3.38.

Assume that $(\xi, d, \mathfrak{f})$ is a complete CMS. Let $\Omega : \xi \longrightarrow K(\xi)$ be a MVM and F_1, F_2 are functions satisfying χF-contraction. Suppose F_1 is non-decreasing, F_2 satisfy condition $(\mathbb{F}2')$ with $\rho(1,0,0,1,1) \in (0,1)$ and $\mathsf{S}Fix\Omega \neq \emptyset$. Also suppose $\lim\limits_{\mathfrak{s}\longrightarrow\infty} \mathfrak{f}(\varrho_{\mathfrak{s}}, \varrho) \leq 1$. Then,

(a) *$Fix\Omega = \mathsf{S}Fix\Omega = \{\varrho^*\}$,*

(b) *The fixed point problem is well posed for MVM Ω with respect to H.*

Proof.

(a) Using Theorem 4.6 , we conclude that $\mathrm{Fix}\Omega \neq \emptyset$. Now, we prove that $\mathrm{Fix}\Omega = \{\varrho^*\}$. Using (Ni.) and (Nii.), we have

$$\begin{aligned} F_1(H(\Omega\varrho, \Omega\bar{\varrho})) &\leq F_2(H(\Omega\varrho, \Omega\bar{\varrho})) \leq \chi(d(\varrho, \bar{\varrho})) + F_2(H(\Omega\varrho, \Omega\bar{\varrho})) \\ &\leq F_1\Big\{\rho\Big(d(\varrho, \bar{\varrho}), D(\varrho, \Omega\varrho), D(\bar{\varrho}, \Omega\bar{\varrho}), D(\varrho, \Omega\bar{\varrho}), D(\bar{\varrho}, \Omega\varrho)\Big)\Big\} \end{aligned}$$

Since F_1 is non-decreasing function, we obtain for all $\varrho, \bar{\varrho} \in \xi$,

$$H(\Omega\varrho, \Omega\bar{\varrho}) \leq \rho\Big(d(\varrho, \bar{\varrho}), D(\varrho, \Omega\varrho), D(\bar{\varrho}, \Omega\bar{\varrho}), D(\varrho, \Omega\bar{\varrho}), D(\bar{\varrho}, \Omega\varrho)\Big).$$

let $\mathsf{v} \in \mathrm{Fix}\Omega$, with $\mathsf{v} \neq \varrho^*$, then, $D(\varrho^*, \Omega\mathsf{v}) > 0$. Now,

$$\begin{aligned} D(\varrho^*, \Omega\mathsf{v}) &= H(\Omega\varrho^*, \Omega\mathsf{v}) \\ &\leq \rho\Big(d(\varrho^*, \mathsf{v}), D(\varrho^*, \Omega\varrho^*), D(\mathsf{v}, \Omega\mathsf{v}), D(\varrho^*, \Omega\mathsf{v}), D(\mathsf{v}, \Omega\varrho^*)\Big) \\ &\leq \rho\Big(d(\varrho^*, \mathsf{v}), 0, 0, d(\varrho^*, \mathsf{v}), d(\mathsf{v}, \varrho^*)\Big) \end{aligned}$$

$$\leq d(\varrho^*, \mathsf{v})\rho(1,0,0,1,1).$$

As $\rho(1,0,0,1,1) \in (0,1)$, so

$$d(\varrho^*, \mathsf{v}) = D(\varrho^*, \Omega\mathsf{v}) < d(\varrho^*, \mathsf{v}),$$

which is contradiction, hence, $d(\varrho^*, \mathsf{v}) = 0$ and $\varrho^* = \mathsf{v}$.

(b) Let $\varrho_{\mathfrak{s}} \in \mathbb{B}, \mathfrak{s} \in \mathbb{N}$, such that

$$\lim_{\mathfrak{s}\longrightarrow\infty} D(\varrho_{\mathfrak{s}}, \Omega\varrho_{\mathfrak{s}}) = 0. \tag{3.30}$$

Now, we claim that $\lim_{\mathfrak{s}\longrightarrow\infty} d(\varrho_{\mathfrak{s}}, \varrho^*) = 0,$
where $\varrho^* \in \text{Fix}\Omega$. If the above equation is not true, then, for every $\mathfrak{s} \in \mathbb{N}$, there exists $\epsilon > 0$ such that

$$d(\varrho_{\mathfrak{s}}, \varrho^*) > \epsilon.$$

But (3.30) implies that there exists $\varrho_\epsilon \in \mathbb{N} - \{0\}$ such that

$$\lim_{\mathfrak{s}\longrightarrow\infty} D(\varrho_{\mathfrak{s}}, \Omega\varrho_{\mathfrak{s}}) < \epsilon,$$

for each $\mathfrak{s} > \mathfrak{s}_\epsilon$.Hence, for each $\mathfrak{s} > \mathfrak{s}_\epsilon$, we obtain

$$\begin{aligned}
d(\varrho_{\mathfrak{s}}, \varrho^*) = D(\varrho_{\mathfrak{s}}, \Omega\varrho^*) = D(\varrho_{\mathfrak{s}}, \Omega\varrho_{\mathfrak{s}}) + H(\Omega\varrho_{\mathfrak{s}}, \Omega\varrho^*) \\
&< D(\varrho_{\mathfrak{s}}, \Omega\varrho_{\mathfrak{s}}) + \rho\Big(d(\varrho_{\mathfrak{s}}, \varrho^*), D(\varrho_{\mathfrak{s}}, \Omega\varrho_{\mathfrak{s}}), D(\varrho^*, \Omega\varrho^*), \\
&\quad D(\varrho_{\mathfrak{s}}, \Omega\varrho^*), D(\varrho^*, \Omega\varrho_{\mathfrak{s}})\Big) \\
&\leq D(\varrho_{\mathfrak{s}}, \Omega\varrho_{\mathfrak{s}}) + \rho\Big(d(\varrho_{\mathfrak{s}}, \varrho^*), D(\varrho_{\mathfrak{s}}, \Omega\varrho_{\mathfrak{s}}), d(\varrho^*, \varrho^*), \\
&\quad d(\varrho_{\mathfrak{s}}, \varrho^*), d(\varrho^*, \varrho_{\mathfrak{s}}) + D(\varrho_{\mathfrak{s}}, \Omega\varrho_{\mathfrak{s}})\Big).
\end{aligned}$$

As $\lim_{\mathfrak{s}\longrightarrow\infty} \mathfrak{f}(\varrho_{\mathfrak{s}}, \varrho) \leq 1$ and $\rho(1,0,0,1,1) \in (0,1)$, so by applying limit $\mathfrak{s} \longrightarrow \infty$, we get $d(\varrho_{\mathfrak{s}}, \varrho^*) \longrightarrow 0$ as $\mathfrak{s} \longrightarrow \infty$, which is contradiction. Hence, fixed point problem is well posed for MVM Ω with respect to D. Also, $\text{Fix}\Omega = \mathsf{S}\text{Fix}\Omega$, hence the fixed point problem is well posed with respect to H. □

Chapter 4

Fixed Point in the Setting of Controlled Metric Space

This Chapter deals with fixed point results of [19] in the setting of CMS. Some important results are discussed regerding fixed points and strict fixed points.

4.1 Fixed Points in CMS

Mlaiki et al. [42] initiated the idea of controlled metric space as generalization of extended b-MS.

Definition 4.1.
Consider a nonempty set ξ and a function $\mathfrak{f} : \xi \times \xi \longrightarrow [1, \infty)$. The mapping $d : \xi \times \xi \longrightarrow [0, \infty)$ is said to be a CMS if $\forall\ \varrho_1, \varrho_2, \varrho_3 \in \xi$,

(i) $d(\varrho_1, \varrho_2) = 0 \quad \Leftrightarrow \quad \varrho_1 = \varrho_2,$

(ii) $d(\varrho_1, \varrho_2) = d(\varrho_2, \varrho_1),$

(iii) $d(\varrho_1, \varrho_2) \leq \mathfrak{f}(\varrho_1, \varrho_3)d(\varrho_1, \varrho_3) + \mathfrak{f}(\varrho_3, \varrho_2)d(\varrho_3, \varrho_2).$

The triplet $(\xi, d, \mathfrak{f})$ is called CMS.[42]

Example 4.1.

Assume that $\xi = \{1, 2, ...\}$. Define $d : \xi \times \xi \longrightarrow [0, \infty)$ by

$$d(\varrho_1, \varrho_2) = \begin{cases} 0 & \text{iff } \varrho_1 = \varrho_2, \\ \frac{1}{\varrho_1} & \text{if } \varrho_1 \text{ is even and } \varrho_2 \text{ is odd,} \\ \frac{1}{\varrho_2} & \text{if } \varrho_2 \text{ is even and } \varrho_1 \text{ is odd,} \\ 1 & \text{otherwise.} \end{cases}$$

Now, take $\rho : \xi \times \xi \longrightarrow [1, \infty)$ as

$$\rho(\varrho_1, \varrho_2) = \begin{cases} \varrho_1 & \text{if } \varrho_1 \text{ is even and } \varrho_2 \text{ is odd,} \\ \varrho_2 & \text{if } \varrho_2 \text{ is even and } \varrho_1 \text{ is odd,} \\ 1 & \text{otherwise,} \end{cases}$$

then (ξ, d, ρ) is a CMS.

Now, for $\varrho = 2, 3, ...$

$$d(2\varrho + 1, 4\varrho + 1) = 1 > \frac{1}{\varrho} = \rho(2\varrho + 1,\ 4\varrho + 1)\Big\{d(2\varrho + 1,\ 2\varrho) + d(2\varrho,\ 4\varrho + 1)\Big\},$$

hence (ξ, d_ρ) is not an extended b-MS.

Remark 4.2.

Assume that $\rho : \xi \times \xi \longrightarrow [1, \infty)$ is given as

$$\rho(\varrho_1, \varrho_2) = \beta \geq 1,$$

for all $\varrho_1, \varrho_2 \in \xi$. Then, (ξ, d, ρ) is a b-MS. Hence, b-MS is always a CMS. Furthermore, a CMS is not generally an extended b-MS with same function.

Definition 4.3.

Let us define a set $\mathcal{P}$ of all continuous mappings, $\rho : [0, \infty)^5 \longrightarrow [0, \infty)$ which satisfies the conditions:

(i) $\rho(1, 1, 1, \zeta + \eta, 0) \in (0, 1]$, where $\zeta, \eta \geq 1$,

(ii) ρ is sub-homogeneous, that is for all $(\varrho_1, \varrho_2, \varrho_3, \varrho_4, \varrho_5) \in (0, \infty]^5$ and $\lambda \geq 0$, we have
$\rho(\lambda\varrho_1, \lambda\varrho_2, \lambda\varrho_3, \lambda\varrho_4, \lambda\varrho_5) \leq \lambda\rho(\varrho_1, \varrho_2, \varrho_3, \varrho_4, \varrho_5)$,

(iii) ρ is non-decreasing function, i.e for $\varrho_i, \bar{\varrho}_i \in \mathbb{R}^+, \varrho_i \leq \bar{\varrho}_i$, $i = 1, 2, 3, 4, 5$, we have $\rho(\varrho_1, \varrho_2, \varrho_3, \varrho_4, \varrho_5) \leq \rho(\bar{\varrho}_1, \bar{\varrho}_2, \bar{\varrho}_3, \bar{\varrho}_4, \bar{\varrho}_5)$. If $\varrho_i, \bar{\varrho}_i \in \mathbb{R}^+$ such that $\varrho_i < \bar{\varrho}_i$,
for $i = 1, 2, 3, 4$, then,

$$\rho(\varrho_1, \varrho_2, \varrho_3, \varrho_4, 0) < \rho(\bar{\varrho}_1, \bar{\varrho}_2, \bar{\varrho}_3, \bar{\varrho}_4, 0),$$

and

$$\rho(\varrho_1, \varrho_2, \varrho_3, 0, \varrho_4) < \rho(\bar{\varrho}_1, \bar{\varrho}_2, \bar{\varrho}_3, 0, \bar{\varrho}_4).$$

Also define $\mathbb{P} = \Big\{\rho \in \mathcal{P} : \rho(1, 0, 0, \zeta, \eta) \in (0, 1]\Big\}$. Note that $\mathbb{P} \subseteq \mathcal{P}$.

Example 4.2.

Define $\rho_1 : [0, \infty)^5 \longrightarrow [0, \infty)$ by

$$\rho_1 (\varrho_1, \varrho_2, \varrho_3, \varrho_4, \varrho_5) = g \min\Big\{\varrho_1, \frac{1}{2}(\varrho_2, \varrho_3), \frac{1}{2}(\varrho_4, \varrho_5)\Big\},$$

where $g \in (0, 1)$. Then $\rho_1 \in \mathcal{P}$, as $\rho_1 (1, 0, 0, \zeta, \eta) = 0 \notin (0, 1]$. Hence, $\rho_1 \notin \mathbb{P}$.

Example 4.3.

Define $\rho_2 : [0, \infty)^5 \longrightarrow [0, \infty)$ by $\rho_2 (\varrho_1, \varrho_2, \varrho_3, \varrho_4, \varrho_5) = \dfrac{\varrho_1}{2} + \dfrac{\varrho_2 + \varrho_3}{4}$.
Then, $\rho_2 \in \mathbb{P}$.

Example 4.4.

Define $\rho_3 : [0, \infty)^5 \longrightarrow [0, \infty)$ by $\rho_3 (\varrho_1, \varrho_2, \varrho_3, \varrho_4, \varrho_5) = g \min\Big\{\frac{1}{2}(\varrho_1 + \varrho_3), \frac{1}{2}(\varrho_4 + \varrho_5)\Big\}$,
where $g \in (0, 1)$. Then, $\rho_3 \in \mathbb{P}$.

Lemma 4.4.

If $\rho \in \mathcal{P}$ and $\gamma, \delta \in [0, \infty)$ and $\zeta, \eta \in \mathbb{R}$ such that $\zeta, \eta \geq 1$. Also,

$$\gamma \leq max\Big\{\rho(\delta, \delta, \gamma, \eta\delta + \zeta\gamma, 0), \rho(\delta, \delta, \gamma, 0, \eta\delta + \zeta\gamma), \rho(\delta, \gamma, \delta, \eta\delta + \zeta\gamma, 0),$$

$$\rho(\delta, \gamma, \delta, 0, \eta\delta + \zeta\gamma)\Big\},$$

then $\gamma \leq \delta$.

Proof. With no loss of generality, assume that

$$\gamma \leq \rho(\delta, \delta, \gamma, \eta\delta + \zeta\gamma, 0). \tag{4.1}$$

On contrary suppose that, $\delta < \gamma$.

Now consider

$$\begin{aligned}
\rho(\delta, \delta, \gamma, \eta\delta + \zeta\gamma, 0) &< \rho(\gamma, \gamma, \gamma, \eta\gamma + \zeta\gamma, 0) \\
&\leq \gamma\rho(1, 1, 1, \eta + \zeta, 0) \\
&\leq \gamma(1) \\
\rho(\delta, \delta, \gamma, \eta\delta + \zeta\gamma, 0) &< \gamma
\end{aligned}$$

which is contradiction to (4.1). Hence our supposition is wrong, so $\gamma \leq \delta$. □

Theorem 4.5.

Suppose that $(\xi, d, \mathfrak{f})$ *is a complete CMS. Let* $\Omega : \xi \longrightarrow K(\xi)$ *be a* χF*-contraction. Suppose* F_1 *is non-decreasing, and* F_2 *satisfy conditions* $(\mathbb{F}2')$ *and* $(\mathbb{F}3)$ *For* $\varrho_0 \in \xi$, *define picard sequence* $\{\varrho_{\mathfrak{s}} = \Omega^{\mathfrak{s}}\varrho_0\}$, *so that*

$$\sup_{m \geq 1} \lim_{i \longrightarrow \infty} \frac{\mathfrak{f}(\varrho_{i+1}, \varrho_{i+2})\mathfrak{f}(\varrho_{i+1}, \varrho_m)}{\mathfrak{f}(\varrho_i, \varrho_{i+1})} < 1. \tag{4.2}$$

Also suppose

$$\lim_{\mathfrak{s} \longrightarrow \infty} \mathfrak{f}(\varrho_{\mathfrak{s}}, \varrho) \leq 1 \ \ \forall \ \varrho \in \xi. \tag{4.3}$$

Then, FixΩ *is non-empty.*

Proof. Let $\varrho_0 \in \xi$ and $\varrho_1 \in \Omega\varrho_0$. If $\varrho_1 \in \Omega\varrho_1$ then, $\varrho_1 \in \text{Fix}\Omega$. Suppose $\varrho_1 \notin \Omega\varrho_1$, it implies $D(\varrho_1, \Omega\varrho_1) > 0$ and consequently $H(\Omega\varrho_0, \Omega\varrho_1) > 0$. As $\Omega\varrho_1$ is compact, so there exists $\varrho_2 \in \Omega\varrho_1$ such that $d(\varrho_1, \varrho_2) = D(\varrho_1, \Omega\varrho_1)$. Now

$$\begin{aligned}F_1(d(\varrho_1, \varrho_2)) &= F_1(D(\varrho_1, \Omega\varrho_1)) \leq F_1(H(\Omega\varrho_0, \Omega\varrho_1)) \leq F_2(H(\Omega\varrho_0, \Omega\varrho_1))\\ &\leq F_1\Big\{\rho\Big(d(\varrho_0, \varrho_1), D(\varrho_0, \Omega\varrho_0), D(\varrho_1, \Omega\varrho_1), D(\varrho_0, \Omega\varrho_1), D(\varrho_1, \Omega\varrho_0)\Big)\Big\}\\ &\quad - \chi(d(\varrho_0, \varrho_1)\\ &< F_1\Big\{\rho\Big(d(\varrho_0, \varrho_1), d(\varrho_0, \varrho_1), d(\varrho_1, \varrho_2), d(\varrho_0, \varrho_2), d(\varrho_1, \varrho_1)\Big)\Big\}.\end{aligned}$$

As F_1 is non decreasing, so

$$\begin{aligned}d(\varrho_1, \varrho_2) &< \rho\Big(d(\varrho_0, \varrho_1), d(\varrho_0, \varrho_1), d(\varrho_1, \varrho_2), d(\varrho_0, \varrho_2), 0\Big)\\ &\leq \rho\Big(d(\varrho_0, \varrho_1), d(\varrho_0, \varrho_1), d(\varrho_1, \varrho_2), \mathfrak{f}(\varrho_0, \varrho_1))d(\varrho_0, \varrho_1) + \mathfrak{f}(\varrho_1, \varrho_2)d(\varrho_1, \varrho_2), 0\Big).\end{aligned}$$

By using Lemma 4.4,

$$d(\varrho_1, \varrho_2) < d(\varrho_0, \varrho_1).$$

Similarly we get $\varrho_3 \in \Omega\varrho_2$ such that $d(\varrho_2, \varrho_3) = D(\varrho_2, \Omega\varrho_2)$ with $D(\varrho_2, \Omega\varrho_2) > 0$ and we have,

$$d(\varrho_2, \varrho_3) < d(\varrho_1, \varrho_2),$$

By induction, we get a sequence $\{\varrho_\mathfrak{s}\}_{\mathfrak{s}\in\mathbb{N}} \subset \xi$ such that $\varrho_{\mathfrak{s}+1} \in \Omega\varrho_\mathfrak{s}$ satisfying $d(\varrho_\mathfrak{s}, \varrho_{\mathfrak{s}+1}) = D(\varrho_\mathfrak{s}, \Omega\varrho_\mathfrak{s})$ with $D(\varrho_\mathfrak{s}, \Omega\varrho_\mathfrak{s}) > 0$ and

$$d(\varrho_\mathfrak{s}, \varrho_{\mathfrak{s}+1}) < d(\varrho_{\mathfrak{s}-1}, \varrho_\mathfrak{s}) \quad \text{for all} \quad \mathfrak{s} \in \mathbb{N}.$$

So, $\{d(\varrho_\mathfrak{s}, \varrho_{\mathfrak{s}+1})\}_{\mathfrak{s}\in\mathbb{N}}$ is a decreasing sequence of real numbers. Now

$$\begin{aligned}&\chi(d(\varrho_\mathfrak{s}, \varrho_{\mathfrak{s}+1})) + F_2(H(\Omega\varrho_\mathfrak{s}, \Omega\varrho_{\mathfrak{s}+1}))\\ &\quad\leq F_1\Big\{\rho\Big(d(\varrho_\mathfrak{s}, \varrho_{\mathfrak{s}+1}), D(\varrho_\mathfrak{s}, \Omega\varrho_\mathfrak{s}), D(\varrho_{\mathfrak{s}+1}, \Omega\varrho_{\mathfrak{s}+1}), D(\varrho_\mathfrak{s}, \Omega\varrho_{\mathfrak{s}+1}), D(\varrho_{\mathfrak{s}+1}, \Omega\varrho_\mathfrak{s})\Big)\Big\}\\ &\quad= F_1\Big\{\rho\Big(d(\varrho_\mathfrak{s}, \varrho_{\mathfrak{s}+1})), d(\varrho_\mathfrak{s}, \varrho_{\mathfrak{s}+1}), d(\varrho_{\mathfrak{s}+1}, \varrho_{\mathfrak{s}+2}), d(\varrho_\mathfrak{s}, \varrho_{\mathfrak{s}+2}), d(\varrho_{\mathfrak{s}+1}, \varrho_{\mathfrak{s}+1})\Big)\Big\}\\ &\quad\leq F_1\Big\{\rho\Big(d(\varrho_\mathfrak{s}, \varrho_{\mathfrak{s}+1}), d(\varrho_\mathfrak{s}, \varrho_{\mathfrak{s}+1}), d(\varrho_{\mathfrak{s}+1}, \varrho_{\mathfrak{s}+2}), \mathfrak{f}(\varrho_\mathfrak{s}, \varrho_{\mathfrak{s}+1})d(\varrho_\mathfrak{s}, \varrho_{\mathfrak{s}+1}) +\\ &\qquad \mathfrak{f}(\varrho_{\mathfrak{s}+1}, \varrho_{\mathfrak{s}+2}d(\varrho_{\mathfrak{s}+1}, \varrho_{\mathfrak{s}+2}), 0\Big)\Big\}\\ &\quad< F_1\Big\{\rho\Big(d(\varrho_\mathfrak{s}, \varrho_{\mathfrak{s}+1}), d(\varrho_\mathfrak{s}, \varrho_{\mathfrak{s}+1}), d(\varrho_\mathfrak{s}, \varrho_{\mathfrak{s}+1}), \mathfrak{f}(\varrho_\mathfrak{s}, \varrho_{\mathfrak{s}+1})d(\varrho_\mathfrak{s}, \varrho_{\mathfrak{s}+1}) +\end{aligned}$$

$$
\begin{aligned}
&\mathfrak{f}(\varrho_{\mathfrak{s}+1}, \varrho_{\mathfrak{s}+2} d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}), 0\Big)\Big\} \\
&\leq F_1\Big\{d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1})\rho\Big(1, 1, 1, \mathfrak{f}(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}) + \mathfrak{f}(\varrho_{\mathfrak{s}+1}, \varrho_{\mathfrak{s}+2}), 0\Big)\Big\} \\
&\leq F_1\Big(d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1})\Big) \\
&= F_1\Big(D(\varrho_{\mathfrak{s}}, \Omega\varrho_{\mathfrak{s}})\Big) \\
&\leq F_1\Big(H(\Omega\varrho_{\mathfrak{s}-1}, \Omega\varrho_{\mathfrak{s}})\Big) \\
&\leq F_2\Big(H(\Omega\varrho_{\mathfrak{s}-1}, \Omega\varrho_{\mathfrak{s}})\Big).
\end{aligned}
$$

Hence, $\forall\ \mathfrak{s} \in \mathbb{N}$, we have

$$
F_2(H(\varrho_{\mathfrak{s}}, \Omega\varrho_{\mathfrak{s}+1})) < F_2(H((\Omega\varrho_{\mathfrak{s}-1}, \Omega\varrho_{\mathfrak{s}})) - \chi(d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1})). \tag{4.4}
$$

As $\chi \in \varPsi$, there exists $\mathsf{h} > 0$ and $\mathfrak{s}_0 \in \mathbb{N}$ such that $\chi(d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1})) > \mathsf{h}$ for all $\mathfrak{s} \geq \mathfrak{s}_0$. Now, from (4.4)

$$
\begin{aligned}
F_2(H(\Omega\varrho_{\mathfrak{s}}, \Omega\varrho_{\mathfrak{s}+1})) &< F_2(H((\Omega\varrho_{\mathfrak{s}-1}, \Omega\varrho_{\mathfrak{s}})) - \chi(d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1})) \\
&< F_2(H(\Omega\varrho_{\mathfrak{s}-2}, \Omega\varrho_{\mathfrak{s}-1})) - \chi(d(\varrho_{\mathfrak{s}-1}, \varrho_{\mathfrak{s}})) - \chi(d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1})) \\
&\ \vdots \\
&< F_2(H(\Omega\varrho_0, \Omega\varrho_1)) - \sum_{i=1}^{\mathfrak{s}} \chi(d(\varrho_i, \varrho_{i+1})) \\
&= F_2(H(\Omega\varrho_0, \Omega\varrho_1)) - \sum_{i=1}^{\mathfrak{s}_0-1} \chi(d(\varrho_i, \varrho_{i+1})) - \sum_{i=\mathfrak{s}_0}^{\mathfrak{s}} \chi(d(\varrho_i, \varrho_{i+1})) \\
&< F_2(H(\Omega\varrho_0, \Omega\varrho_1)) - (\mathfrak{s} - \mathfrak{s}_0)\mathsf{h}, \quad \mathfrak{s} \geq \mathfrak{s}_0
\end{aligned}
$$

$$
\implies F_2(H(\Omega\varrho_{\mathfrak{s}}, \Omega\varrho_{\mathfrak{s}+1})) < F_2(H((\Omega\varrho_0, \Omega\varrho_1)) - (\mathfrak{s} - \mathfrak{s}_0)\mathsf{h}, \quad \forall \quad \mathfrak{s} \geq \mathfrak{s}_0 \tag{4.5}
$$

Taking $\mathfrak{s} \longrightarrow \infty$ in (4.5), we get $F_2(H(\Omega\varrho_{\mathfrak{s}}, \Omega\varrho_{\mathfrak{s}+1})) \longrightarrow -\infty$ and then by $(\mathbb{F}2')$, we have

$$
\lim_{\mathfrak{s} \longrightarrow \infty} H(\Omega\varrho_{\mathfrak{s}}, \Omega\varrho_{\mathfrak{s}+1}) = 0,
$$

which further implies that

$$\lim_{\mathfrak{s}\longrightarrow\infty} d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}) = \lim_{\mathfrak{s}\longrightarrow\infty} D(\varrho_{\mathfrak{s}}, \Omega\varrho_{\mathfrak{s}}) \leq \lim_{\mathfrak{s}\longrightarrow\infty} H(\Omega\varrho_{\mathfrak{s}-1}, \Omega\varrho_{\mathfrak{s}}) = 0. \tag{4.6}$$

Now from ($\mathbb{F}3$), there exists $k \in (0, 1)$ such that

$$\lim_{\mathfrak{s}\longrightarrow\infty} (H(\Omega\varrho_{\mathfrak{s}}, \Omega\varrho_{\mathfrak{s}+1}))^k F_2(H(\Omega\varrho_{\mathfrak{s}}, \Omega\varrho_{\mathfrak{s}+1})) = 0. \tag{4.7}$$

Then from (4.5), for all $\mathfrak{s} \geq \mathfrak{s}_0$, we have

$$\begin{aligned}
&(H(\Omega\varrho_{\mathfrak{s}}, \Omega\varrho_{\mathfrak{s}+1}))^k F_2(H(\Omega\varrho_{\mathfrak{s}}, \Omega\varrho_{\mathfrak{s}+1})) - (H(\Omega\varrho_{\mathfrak{s}}, \Omega\varrho_{\mathfrak{s}+1}))^k F_2(H(\Omega\varrho_0, \Omega\varrho_1)) \\
&\leq (H(\Omega\varrho_{\mathfrak{s}}, \Omega\varrho_{\mathfrak{s}+1}))^k \Big(F_2(H(\Omega\varrho_0, \Omega\varrho_1)) - (\mathfrak{s} - \mathfrak{s}_0)h\Big) \\
&\quad - (H(\Omega\varrho_{\mathfrak{s}}, \Omega\varrho_{\mathfrak{s}+1}))^k F_2(H(\Omega\varrho_0, \Omega\varrho_1)) \\
&= -(H(\Omega\varrho_{\mathfrak{s}}, \Omega\varrho_{\mathfrak{s}+1}))^k (\mathfrak{s} - \mathfrak{s}_0)h \\
&\leq 0.
\end{aligned}$$

Taking limit $\mathfrak{s} \longrightarrow \infty$ and using (4.6) and (4.7)

$$\begin{aligned}
&0 \leq \lim_{\mathfrak{s}\longrightarrow\infty} \mathfrak{s}(H(\Omega\varrho_{\mathfrak{s}}, \Omega\varrho_{\mathfrak{s}+1}))^k \leq 0 \\
&\Longrightarrow \lim_{\mathfrak{s}\longrightarrow\infty} \mathfrak{s}(H(\Omega\varrho_{\mathfrak{s}}, \Omega\varrho_{\mathfrak{s}+1}))^k = 0.
\end{aligned}$$

From above equation, there exists $\mathfrak{s}_1 \in \mathbb{N}$ such that $\mathfrak{s}(H(\Omega\varrho_{\mathfrak{s}}, \Omega\varrho_{\mathfrak{s}+1}))^k \leq 1, \ \forall\, \mathfrak{s} \geq \mathfrak{s}_1$. Thus for all $\mathfrak{s} \geq \mathfrak{s}_1$, we have $H(\varrho_{\mathfrak{s}}, \Omega\varrho_{\mathfrak{s}+1}) \leq \dfrac{1}{\mathfrak{s}^{\frac{1}{k}}}$.
Now

$$d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}) = D(\varrho_{\mathfrak{s}}, \Omega\varrho_{\mathfrak{s}}) \leq H(\Omega\varrho_{\mathfrak{s}-1}, \Omega\varrho_{\mathfrak{s}}) \leq \frac{1}{\mathfrak{s}^{\frac{1}{k}}} \qquad \forall \quad \mathfrak{s} \geq \mathfrak{s}_1.$$

To prove that $\{\varrho_{\mathfrak{s}}\}_{\mathfrak{s}\in\mathbb{N}}$ is Cauchy sequence. Consider $\tau, \mathfrak{s} \in \mathbb{N}$ such that $\tau > \mathfrak{s} > \mathfrak{s}_1$. Now

$$\begin{aligned}
d(\varrho_{\mathfrak{s}}, \varrho_\tau) &\leq \mathfrak{f}(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}) d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}) + \mathfrak{f}(\varrho_{\mathfrak{s}+1}, \varrho_\tau) d(\varrho_{\mathfrak{s}+1}, \varrho_\tau) \\
&\leq \mathfrak{f}(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}) d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}) + \mathfrak{f}(\varrho_{\mathfrak{s}+1}, \varrho_\tau) \mathfrak{f}(\varrho_{\mathfrak{s}+1}, \varrho_{\mathfrak{s}+2}) d(\varrho_{\mathfrak{s}+1}, \varrho_{\mathfrak{s}+2}) \\
&\quad + \mathfrak{f}(\varrho_{\mathfrak{s}+1}, \varrho_\tau) \mathfrak{f}(\varrho_{\mathfrak{s}+2}, \varrho_\tau) d(\varrho_{\mathfrak{s}+2}, \varrho_\tau)
\end{aligned}$$

$$\leq \mathfrak{f}(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1})d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}) + \mathfrak{f}(\varrho_{\mathfrak{s}+1}, \varrho_{\tau})\mathfrak{f}(\varrho_{\mathfrak{s}+1}, \varrho_{\mathfrak{s}+2})d(\varrho_{\mathfrak{s}+1}, \varrho_{\mathfrak{s}+2}) + \mathfrak{f}(\varrho_{\mathfrak{s}+1}, \varrho_{\tau})$$
$$\mathfrak{f}(\varrho_{\mathfrak{s}+2}, \varrho_{\tau})\mathfrak{f}(\varrho_{\mathfrak{s}+2}, \varrho_{\mathfrak{s}+3})d(\varrho_{\mathfrak{s}+2}, \varrho_{\mathfrak{s}+3}) + \mathfrak{f}(\varrho_{\mathfrak{s}+1}, \varrho_{\tau})\mathfrak{f}(\varrho_{\mathfrak{s}+2}, \varrho_{\tau})\mathfrak{f}(\varrho_{\mathfrak{s}+3}, \varrho_{\tau})d(\varrho_{\mathfrak{s}+3}, \varrho_{\tau})$$
$$\vdots$$
$$\leq \mathfrak{f}(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1})d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}) + \sum_{i=\mathfrak{s}+1}^{\tau-2}\Big(\prod_{j=\mathfrak{s}+1}^{i} \mathfrak{f}(\varrho_j, \varrho_\tau)\Big)\mathfrak{f}(\varrho_i, \varrho_{i+1})d(\varrho_i, \varrho_{i+1})$$
$$+ \Big(\prod_{j=\mathfrak{s}+1}^{\tau-1} \mathfrak{f}(\varrho_i, \varrho_\tau)\Big)d(\varrho_{\tau-1}, \varrho_\tau)$$
$$\leq \mathfrak{f}(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1})d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}) + \sum_{i=\mathfrak{s}+1}^{\tau-2}\Big(\prod_{j=\mathfrak{s}+1}^{i} \mathfrak{f}(\varrho_j, \varrho_\tau)\Big)\mathfrak{f}(\varrho_i, \varrho_{i+1})d(\varrho_i, \varrho_{i+1})$$
$$+ \Big(\prod_{j=\mathfrak{s}+1}^{\tau-1} \mathfrak{f}(\varrho_i, \varrho_\tau)\Big)\mathfrak{f}(\varrho_{\tau-1}, \varrho_\tau)d(\varrho_{\tau-1}, \varrho_\tau)$$
$$= \mathfrak{f}(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1})d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}) + \sum_{i=\mathfrak{s}+1}^{\tau-1}\Big(\prod_{j=\mathfrak{s}+1}^{i} \mathfrak{f}(\varrho_j, \varrho_\tau)\Big)\mathfrak{f}(\varrho_i, \varrho_{i+1})d(\varrho_i, \varrho_{i+1})$$
$$\leq \mathfrak{f}(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1})d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}) + \sum_{i=\mathfrak{s}+1}^{\tau-1}\Big(\prod_{j=0}^{i} \mathfrak{f}(\varrho_j, \varrho_\tau)\Big)\mathfrak{f}(\varrho_i, \varrho_{i+1})d(\varrho_i, \varrho_{i+1}).$$

Therefore,

$$d(\varrho_{\mathfrak{s}}, \varrho_\tau) \leq \mathfrak{f}(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1})d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}) + \sum_{i=\mathfrak{s}+1}^{\tau-1}\Big(\prod_{j=0}^{i} \mathfrak{f}(\varrho_j, \varrho_\tau)\Big)\mathfrak{f}(\varrho_i, \varrho_{i+1})\frac{1}{i^{\frac{1}{k}}}. \tag{4.8}$$

Now

$$\sum_{i=\mathfrak{s}+1}^{\tau-1}\Big(\prod_{j=0}^{i} \mathfrak{f}(\varrho_j, \varrho_\tau)\Big)\mathfrak{f}(\varrho_i, \varrho_{i+1})\frac{1}{i^{\frac{1}{k}}} \leq \sum_{i=\mathfrak{s}+1}^{\infty}\frac{1}{i^{\frac{1}{k}}}\Big(\prod_{j=0}^{i} \mathfrak{f}(\varrho_j, \varrho_\tau)\Big)\mathfrak{f}(\varrho_i, \varrho_{i+1})$$
$$= \sum_{i=\mathfrak{s}+1}^{\infty} U_i V_i,$$

where $U_i = \frac{1}{i^{\frac{1}{k}}}$ and $V_i = \Big(\prod_{j=0}^{i} \mathfrak{f}(\varrho_j, \varrho_\tau)\Big)\mathfrak{f}(\varrho_i, \varrho_{i+1})$. Since $\frac{1}{k} > 0$, therefore $\sum_{i=\mathfrak{s}+1}^{\infty}\Big(\frac{1}{i^{\frac{1}{k}}}\Big)$ converges. Also $\{V_i\}_i$ is increasing and bounded above, so $\lim_{i\longrightarrow\infty}\{V_i\}_i$ (which is non-zero) exists. Hence $\{\lim_{i\longrightarrow\infty} U_i V_i\}_{\mathfrak{s}}$ converges.

Consider the partial sum $S_q = \sum_{i=0}^{q}\Big(\prod_{j=0}^{i} \mathfrak{f}(\varrho_j, \varrho_\tau)\Big)\mathfrak{f}(\varrho_i, \varrho_{i+1})\frac{1}{i^{\frac{1}{k}}}$. Now from (4.8), we have

$$d(\varrho_{\mathfrak{s}}, \varrho_{\tau}) \leq \mathfrak{f}(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1})d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}) + (S_{\tau-1} - S_{\mathfrak{s}}). \tag{4.9}$$

By using ratio test and the condition (4.2), we attain that $\lim_{\mathfrak{s}\longrightarrow\infty} \{S_{\mathfrak{s}}\}$ exists. By applying limit $\mathfrak{s} \longrightarrow \infty$ in (4.9), we get $\lim_{\mathfrak{s}\longrightarrow\infty} d(\varrho_{\mathfrak{s}}, \varrho_{\tau}) = 0$. Therefore $\{\varrho_{\mathfrak{s}}\}$ is a Cauchy sequence and the completeness of ξ implies that there exists $\varrho^* \in \xi$ such that,

$$\lim_{\mathfrak{s}\longrightarrow\infty} \varrho_{\mathfrak{s}} = \varrho^*.$$

Now

$$\begin{aligned}F_1(H(\Omega\varrho, \Omega\bar{\varrho})) &\leq F_2(H(\Omega\varrho, \Omega\bar{\varrho})) \leq \chi(d(\varrho, \bar{\varrho})) + F_2(H(\Omega\varrho, \Omega\bar{\varrho}))\\ &\leq F_1\Big\{\rho\Big(d(\varrho, \bar{\varrho}), D(\varrho, \Omega\varrho), D(\bar{\varrho}, \Omega\bar{\varrho}), D(\varrho, \Omega\bar{\varrho}), D(\bar{\varrho}, \Omega\varrho)\Big)\Big\}.\end{aligned}$$

Since F_1 is non-decreasing function, we obtain for all $\varrho, \bar{\varrho} \in \xi$.

$$H(\Omega\varrho, \Omega\bar{\varrho}) \leq \rho\Big(d(\varrho, \bar{\varrho}), D(\varrho, \Omega\varrho), D(\bar{\varrho}, \Omega\bar{\varrho}), D(\varrho, \Omega\bar{\varrho}), D(\bar{\varrho}, \Omega\varrho)\Big).$$

To prove that ϱ^* is fixed point of ξ. On contrary assume that $D(\varrho^*, \Omega\varrho^*) > 0$. Now, due to compactness of $\Omega\varrho^*$, there exists $\varrho \in \Omega\varrho^*$ such that

$$\begin{aligned}D(\varrho^*, \Omega\varrho^*) &= d(\varrho^*, \varrho)\\ &\leq \mathfrak{f}(\varrho^*, \varrho_{\mathfrak{s}+1})d(\varrho^*, \varrho_{\mathfrak{s}+1}) + \mathfrak{f}(\varrho_{\mathfrak{s}+1}, \varrho)d(\varrho_{\mathfrak{s}+1}, \varrho)\\ &= \mathfrak{f}(\varrho^*, \varrho_{\mathfrak{s}+1})d(\varrho^*, \varrho_{\mathfrak{s}+1}) + \mathfrak{f}(\varrho_{\mathfrak{s}+1}, \varrho)D(\varrho_{\mathfrak{s}+1}, \Omega\varrho^*)\\ &\leq \mathfrak{f}(\varrho^*, \varrho_{\mathfrak{s}+1})d(\varrho^*, \varrho_{\mathfrak{s}+1}) + \mathfrak{f}(\varrho_{\mathfrak{s}+1}, \varrho)H(\Omega\varrho_{\mathfrak{s}}, \Omega\varrho^*)\\ &\leq \mathfrak{f}(\varrho^*, \varrho_{\mathfrak{s}+1})d(\varrho^*, \varrho_{\mathfrak{s}+1}) + \mathfrak{f}(\varrho_{\mathfrak{s}+1}, \varrho)\rho\Big(d(\varrho_{\mathfrak{s}}, \varrho^*), D(\varrho_{\mathfrak{s}}, \Omega\varrho_{\mathfrak{s}}), D(\varrho^*, \Omega\varrho^*), D(\varrho_{\mathfrak{s}}, \Omega\varrho^*),\\ &\quad D(\varrho^*, \Omega\varrho_{\mathfrak{s}})\Big)\\ &\leq \mathfrak{f}(\varrho^*, \varrho_{\mathfrak{s}+1})d(\varrho^*, \varrho_{\mathfrak{s}+1}) + \mathfrak{f}(\varrho_{\mathfrak{s}+1}, \varrho)\rho\Big(d(\varrho_{\mathfrak{s}}, \varrho^*), d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}), D(\varrho^*, \Omega\varrho^*), \mathfrak{f}(\varrho_{\mathfrak{s}}, \varrho^*)d(\varrho_{\mathfrak{s}}, \varrho^*)\\ &+ \mathfrak{f}(\varrho^*, \varrho)D(\varrho^*, \Omega\varrho^*), d(\varrho^*, \varrho_{\mathfrak{s}+1})\Big).\end{aligned}$$

By applying limit $\mathfrak{s} \longrightarrow \infty$ in the above inequality and using (4.3)

$$D(\varrho^*, \Omega\varrho^*) \leq (1)\rho\Big(0, 0, D(\varrho^*, \Omega\varrho^*), 0 + \mathfrak{f}(\varrho^*, \bar{\varrho})D(\varrho^*, \Omega\varrho^*), 0\Big).$$

Using Lemma 4.4, we get $D(\varrho^*, \Omega\varrho^*) \leq 0$.

$$\implies 0 < D(\varrho^*, \Omega\varrho^*) \leq 0.$$

Hence $D(\varrho^*, \Omega\varrho^*) = 0$. As $\Omega\varrho^*$ is closed, so $\varrho^* \in \Omega\varrho^*$. Hence Fix$\Omega$ is non empty.

□

Example 4.5.

Let $\xi = \{0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}\}$. Define $d : \xi \times \xi \longrightarrow \mathbb{R}^+$ and $f : \xi \times \xi \longrightarrow [1, \infty)$ by $d(\varrho_1, \varrho_2) = (\varrho_1 - \varrho_2)^2$, also

$$\mathfrak{f}(\varrho_1, \varrho_2) = \begin{cases} 1 & \text{if } \varrho_1 = \varrho_2 = 0 \\ \dfrac{1}{(\varrho_1 + \varrho_2)^4} & \text{if } \varrho_1 \neq 0 \text{ or } \varrho_2 \neq 0. \end{cases}$$

Then, $(\xi, d, \mathfrak{f})$ is complete CMS.

Define $F_1, F_2 : (0, \infty) \longrightarrow \mathbb{R}$ by

$$F_1(\mathfrak{u}) = \begin{cases} -\frac{1}{\mathfrak{u}} & \text{if } \mathfrak{u} \in (0, 1) \\ \mathfrak{u} & \text{if } \mathfrak{u} \in [1, \infty)\ , \end{cases}$$

and $F_2(u) = ln(u) + u, \quad \forall\ u \in (0, \infty)$. Then, F_1 is non-decreasing, F_2 satisfies $(F2')$ and $(F3)$ and $F_1(u) \leq F_2(u) \quad \forall\ u > 0$. Now, define $\Omega : \xi \longrightarrow K(\xi)$ and $\rho : [0, \infty)^5 \longrightarrow [0, \infty)$ and $\chi : (0, \infty) \longrightarrow (0, \infty)$ by

$$\Omega\varrho = \begin{cases} \{0\} & \text{if } \varrho = 0 \\ \{0, \frac{1}{2}\} & \text{if } \varrho \neq 0, \end{cases}$$

$\rho(\varrho_1, \varrho_2, \varrho_3, \varrho_4, \varrho_5) = \frac{\varrho_1}{2} + 28\varrho_5$ and $\chi(t) = \frac{1}{t}$, $t \in (0, \infty)$. Then $\rho \in \mathcal{P}$ and $\chi \in \Psi$. Since $H(\Omega\varrho, \Omega\bar{\varrho}) > 0$ implies,

$$\chi(d(\varrho, \bar{\varrho})) + F_2(H(\Omega\varrho, \Omega\bar{\varrho})) \leq F_1\Big\{\rho\Big(d(\varrho, \bar{\varrho}), D(\varrho, \Omega\varrho), D(\bar{\varrho}, \Omega\bar{\varrho}), D(\varrho, \Omega\bar{\varrho}), D(\bar{\varrho}, \Omega\varrho)\Big)\Big\}.$$

Note that $\lim_{n \longrightarrow \infty} \mathfrak{f}(\varrho_n, \varrho) \leq 1$,
hence the assumptions of Theorem 4.5 are fulfilled and $\text{Fix}\Omega = \{0, \frac{1}{2}\}$.

Theorem 4.6.

Assume that $(\xi, d, \mathfrak{f})$ *is a complete CMS. Let* $\Omega : \xi \longrightarrow K(\xi)$ *be a MVM and* F_1, F_2 *are functions satisfying* χF*-contraction. Suppose* F_1 *is non-decreasing, and* F_2 *satisfy condition* $(\mathbb{F}2')$*. Also, suppose* $\lim_{k \longrightarrow \infty} \mathfrak{f}(\varrho_{\tau_k}, \varrho_{\mathfrak{s}_k}) \leq 1$*. Then, Fix*$\Omega$ *is non-empty.*

Proof. Let $\varrho_0 \in \xi$ and $\varrho_1 \in \Omega\varrho_0$. We obtain a sequence $\{\varrho_{\mathfrak{s}}\} \subset \xi$ as in proof of Theorem 4.5 such that $\varrho_{\mathfrak{s}+1} \in \Omega\varrho_{\mathfrak{s}}$. It satisfies $d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}) = D(\varrho_{\mathfrak{s}}, \Omega\varrho_{\mathfrak{s}})$ with $D(\varrho_{\mathfrak{s}}, \Omega\varrho_{\mathfrak{s}}) > 0$ and

$$d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}) < d(\varrho_{\mathfrak{s}-1}, \varrho_{\mathfrak{s}}) \qquad \forall\, \mathfrak{s} \in \mathbb{N} \tag{4.10}$$

$$F_2(H(\Omega\varrho_{\mathfrak{s}}, \Omega\varrho_{\mathfrak{s}+1})) < F_2(H(\Omega\varrho_0, \Omega\varrho_1)) - (\mathfrak{s} - \mathfrak{s}_0)h, \quad \forall\, \mathfrak{s} \geq \mathfrak{s}_0. \tag{4.11}$$

Taking $\mathfrak{s} \longrightarrow \infty$ in (4.11), we get $F_2(H(\Omega\varrho_{\mathfrak{s}}, \Omega\varrho_{\mathfrak{s}+1})) \longrightarrow -\infty$ and by $(\mathbb{F}2')$,

$$\lim_{\mathfrak{s} \longrightarrow \infty} H(\Omega\varrho_{\mathfrak{s}}, \Omega\varrho_{\mathfrak{s}+1}) = 0, \tag{4.12}$$

which further implies,

$$\lim_{\mathfrak{s} \longrightarrow \infty} d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}) = \lim_{\mathfrak{s} \longrightarrow \infty} D(\varrho_{\mathfrak{s}}, \Omega\varrho_{\mathfrak{s}}) \leq \lim_{\mathfrak{s} \longrightarrow \infty} H(\varrho_{\mathfrak{s}-1}, \Omega\varrho_{\mathfrak{s}}) = 0.$$

Also, we claim that

$$\lim_{\mathfrak{s}, \tau \longrightarrow \infty} d(\varrho_{\mathfrak{s}}, \varrho_{\tau}) = 0. \tag{4.13}$$

If (4.13) is not applicable, then there exists $\delta > 0$ such that $\forall\ r \geq 0$, there exists $\tau_k > \mathfrak{s}_k > r$,

$$d(\varrho_{\mathfrak{s}_k}, \varrho_{\tau_k}) > \delta.$$

Moreover there exists $r_0 \in \mathbb{N}$,

$$\lambda_{r_0} = d(\varrho_{\mathfrak{s}-1}, \varrho_{\mathfrak{s}}) < \delta \qquad \forall\ \mathfrak{s} \geq r_0.$$

There exists two sub sequences $\{\varrho_{\mathfrak{s}_k}\}$ and $\{\varrho_{\tau_k}\}$ of $\{\varrho_{\mathfrak{s}}\}$ which satisfies,

$$r_0 \leq \mathfrak{s}_k \leq \tau_k + 1 \quad \textit{and} \quad d(\varrho_{\mathfrak{s}_k}, \varrho_{\tau_k}) > \delta \qquad \forall\ k > 0. \tag{4.14}$$

Note that

$$d(\varrho_{\tau_k - 1}, \varrho_{\mathfrak{s}_k}) \leq \delta \quad \text{for all } k, \tag{4.15}$$

Also, τ_k is minimal index for which (4.15) is fulfilled.
Note that, $\mathfrak{s}_k + 2 \leq \tau_k\ \ \forall\ k$, because the case $\mathfrak{s}_k + 1 \leq \mathfrak{s}_k$ is impossible due to equations (4.14) and (4.15). It shows

$$\mathfrak{s}_k + 1 < \tau_k < \tau_k + 1 \quad \forall\ k.$$

By considering triangular inequality and using (4.14), (4.15), we have

$$\begin{aligned}\delta < d(\varrho_{\tau_k}, \varrho_{\mathfrak{s}_k}) &\leq \mathfrak{f}(\varrho_{\tau_k}, \varrho_{\tau_k - 1}) d(\varrho_{\tau_k}, \varrho_{\tau_k - 1}) + \mathfrak{f}(\varrho_{\tau_k - 1}, \varrho_{\mathfrak{s}_k}) d(\varrho_{\tau_k - 1}, \varrho_{\mathfrak{s}_k}) \\ &\leq \mathfrak{f}(\varrho_{\tau_k}, \varrho_{\tau_k - 1}) d(\varrho_{\tau_k}, \varrho_{\tau_k - 1}) + \delta \mathfrak{f}(\varrho_{\tau_k - 1}, \varrho_{\mathfrak{s}_k}).\end{aligned}$$

Taking limit $k \longrightarrow \infty$,

$$\begin{aligned}\delta &< \lim_{k \longrightarrow \infty} d(\varrho_{\tau_k}, \varrho_{\mathfrak{s}_k}) \leq 0 + \delta \lim_{k \longrightarrow \infty} \mathfrak{f}(\varrho_{\tau_k - 1}, \varrho_{\mathfrak{s}_k}). \\ \Longrightarrow\ \delta &< \lim_{k \longrightarrow \infty} d(\varrho_{\tau_k}, \varrho_{\mathfrak{s}_k}) \leq \delta \lim_{k \longrightarrow \infty} \mathfrak{f}(\varrho_{\tau_k - 1}, \varrho_{\mathfrak{s}_k}) \leq \delta\end{aligned}$$

$$\Longrightarrow \lim_{k\longrightarrow\infty} d(\varrho_{\tau_k}, \varrho_{\mathfrak{s}_k}) = \delta \tag{4.16}$$

Now, using (4.12) and (4.16), we get

$$\lim_{k\longrightarrow\infty} d(\varrho_{\tau_k+1}, \varrho_{\mathfrak{s}_k+1}) = \delta. \tag{4.17}$$

Consider

$$\begin{aligned}
\chi(d(\varrho_{\tau_k}, \varrho_{\mathfrak{s}_k})) + F_1(d(\varrho_{\tau_k+1}, \varrho_{\mathfrak{s}_k+1})) &= \chi(d(\varrho_{\tau_k}, \varrho_{\mathfrak{s}_k})) + F_1(D(\varrho_{\tau_k+1}, \Omega\varrho_{\mathfrak{s}_k}))\\
&\leq \chi(d(\varrho_{\tau_k}, \varrho_{\mathfrak{s}_k})) + F_1(H(\Omega\varrho_{\tau_k}, \Omega\varrho_{\mathfrak{s}_k}))\\
&\leq \chi(d(\varrho_{\tau_k}, \varrho_{\mathfrak{s}_k})) + F_2(H(\Omega\varrho_{\tau_k}, \Omega\varrho_{\mathfrak{s}_k}))
\end{aligned}$$

$$\begin{aligned}
&\leq F_1\Big\{\rho\Big(d(\varrho_{\tau_k}, \varrho_{\mathfrak{s}_k}), D(\varrho_{\tau_k}, \Omega\varrho_{\tau_k}), D(\varrho_{\mathfrak{s}_k}, \Omega\varrho_{\mathfrak{s}_k}), D(\varrho_{\tau_k}, \Omega\varrho_{\mathfrak{s}_k}), D(\varrho_{\mathfrak{s}_k}, \Omega\varrho_{\tau_k})\Big)\Big\}\\
&= F_1\Big\{\rho\Big(d(\varrho_{\tau_k}, \varrho_{\mathfrak{s}_k}), d(\varrho_{\tau_k}, \varrho_{\tau_{k+1}}), d(\varrho_{\mathfrak{s}_k}, \varrho_{\mathfrak{s}_{k+1}}), d(\varrho_{\tau_k}, \varrho_{\mathfrak{s}_{k+1}}), d(\varrho_{\mathfrak{s}_k}, \varrho_{\tau_{k+1}})\Big)\Big\}\\
&\leq F_1\Big\{\rho\Big(d(\varrho_{\tau_k}, \varrho_{\mathfrak{s}_k}), d(\varrho_{\tau_k}, \varrho_{\tau_{k+1}}), d(\varrho_{\mathfrak{s}_k}, \varrho_{\mathfrak{s}_{k+1}}), \mathfrak{f}(\varrho_{\mathfrak{s}_{k+1}}, \varrho_{\mathfrak{s}_k})d(\varrho_{\mathfrak{s}_{k+1}}, \varrho_{\mathfrak{s}_k}) +\\
&\quad \mathfrak{f}(\varrho_{\mathfrak{s}_k}, \varrho_{\tau_k})d(\varrho_{\mathfrak{s}_k}, \varrho_{\tau_k}), \mathfrak{f}(\varrho_{\mathfrak{s}_k}, \varrho_{\mathfrak{s}_{k+1}})d(\varrho_{\mathfrak{s}_k}, \varrho_{\mathfrak{s}_{k+1}}) + \mathfrak{f}(\varrho_{\mathfrak{s}_{k+1}}, \varrho_{\tau_{k+1}})d(\varrho_{\mathfrak{s}_{k+1}}, \varrho_{\tau_{k+1}})\Big)\Big\}.
\end{aligned}$$

As F_1 is continuous, then by applying the $\lim_{k\longrightarrow\infty}$ and using (4.16),(4.17), we obtain

$$\begin{aligned}
\lim_{k\longrightarrow\infty} \chi(d(\varrho_{\tau_k}, \varrho_{\mathfrak{s}_k})) + F_1(\delta) &\leq F_1\Big\{\rho\Big(\delta, 0, 0, 0 + \delta\mathfrak{f}(\varrho_{\mathfrak{s}_k}, \varrho_{\tau_k}), 0 + \delta\mathfrak{f}(\varrho_{\mathfrak{s}_{k+1}}, \varrho_{\tau_{k+1}})\Big)\Big\}\\
&\leq F_1\Big(\rho(\delta, 0, 0, \delta\mathfrak{f}(\varrho_{\mathfrak{s}_k}, \varrho_{\tau_k}), \delta\mathfrak{f}(\varrho_{\mathfrak{s}_{k+1}}, \varrho_{\tau_{k+1}})\Big)\Big\}\\
&\leq F_1\Big\{\delta\rho\Big(1, 0, 0, \mathfrak{f}(\varrho_{\mathfrak{s}_k}, \varrho_{\tau_k}), \mathfrak{f}(\varrho_{\mathfrak{s}_{k+1}}, \varrho_{\tau_{k+1}})\Big)\Big\}.
\end{aligned}$$

Since $\rho \in \mathbb{P}$, so $\rho\Big(1, 0, 0, \mathfrak{f}(\varrho_{\mathfrak{s}_k}, \varrho_{\tau_k}), \mathfrak{f}(\varrho_{\mathfrak{s}_{k+1}}, \varrho_{\tau_{k+1}})\Big) \in (0, 1]$.

$$\begin{aligned}
\Longrightarrow \lim_{k\longrightarrow\infty} \chi(d(\varrho_{\tau_k}, \varrho_{\mathfrak{s}_k})) + F_1(\delta) &\leq F_1(\delta),\\
\Longrightarrow \lim_{k\longrightarrow\infty} \chi(d(\varrho_{\tau_k}, \varrho_{\mathfrak{s}_k})) &\leq 0,\\
\Longrightarrow \lim_{S\longrightarrow\delta^+} \inf \chi(S) &\leq 0,
\end{aligned}$$

which is a contradiction, hence (4.13) holds. Therefore $\{\varrho_{\mathfrak{s}}\}$ is Cauchy sequence and there exists $\varrho^* \in \xi$ such that $\lim\limits_{\mathfrak{s}\longrightarrow\infty} \varrho_{\mathfrak{s}} = \varrho^*$. The rest of the proof follows from Theorem 4.5 and we get $\varrho^* \in \Omega\varrho^*$. □

Theorem 4.7.

Let $(\xi, d, \mathfrak{f})$ be a complete CMS and $\Omega : \xi \longrightarrow C(\xi)$ be a MVM. Assume that there exists $\chi \in \Psi, F \in \Delta\,(0*)$ and a real valued function L on $(0,\infty)$ such that following conditions hold:

(G_1) $F(\varrho) \leq L(\varrho) \quad \forall\ \varrho > 0,$

(G_2) $H(\Omega\varrho, \Omega\bar{\varrho}) > 0$ *implies,*

$$\chi(d(\varrho,\bar{\varrho}))+L(H(\Omega\varrho,\Omega\bar{\varrho})) \leq F\Big\{\rho\Big(d(\varrho,\bar{\varrho}), D(\varrho,\Omega\varrho), D(\bar{\varrho},\Omega\bar{\varrho}), D(\varrho,\Omega\bar{\varrho}), D(\bar{\varrho},\Omega\varrho)\Big)\Big\}.$$

For all $\varrho, \bar{\varrho} \in \xi$, $\rho \in \mathcal{P}$. Let $\varrho_0 \in \xi$, define the picard sequence $\{\varrho_{\mathfrak{s}} = \Omega^{\mathfrak{s}}\varrho_{\mathfrak{s}}\}$ such that

$$\sup_{m\geq 1} \lim_{i\longrightarrow\infty} \frac{\mathfrak{f}(\varrho_{i+1},\varrho_{i+2})\mathfrak{f}(\varrho_{i+1},\varrho_m)}{\mathfrak{f}(\varrho_i,\varrho_{i+1})} < 1. \tag{4.18}$$

Also suppose that $\lim\limits_{\mathfrak{s}\longrightarrow\infty} \mathfrak{f}(\varrho_{\mathfrak{s}}, \varrho) \leq 1$ for all $\varrho \in \xi$. Then, FixΩ is non empty.

Proof. Let $\varrho_0 \in \xi$ and $\varrho_1 \in \Omega\varrho_0$. If $\varrho_1 \in \Omega\varrho_1$ then, $\varrho_1 \in \text{fix}\Omega$. Suppose $\varrho_1 \notin \Omega\varrho_1$, it implies $D(\varrho_1, \Omega\varrho_1) > 0$ and consequently $H(\Omega\varrho_0, \Omega\varrho_1) > 0$. Due to ($\mathbb{F}4$), we obtain

$$F(D(\varrho_1,\Omega\varrho_1)) = \inf_{z\in\Omega\varrho_1} F(d(\varrho_1,z)). \tag{4.19}$$

Then, (4.19) with (G_1) and (G_2) gives

$$\begin{aligned}\inf_{z\in\Omega\varrho_1} F(d(\varrho_1,z)) &= F(D(\varrho_1,\Omega\varrho_1)) \\ &\leq F(H(\Omega\varrho_0,\Omega\varrho_1)) \\ &\leq L(H(\Omega\varrho_0,\Omega\varrho_1)) \\ &\leq F\Big\{\rho\Big(d(\varrho_0,\varrho_1), D(\varrho_0,\Omega\varrho_0), D(\varrho_1,\Omega\varrho_1), D(\varrho_0,\Omega\varrho_1), D(\varrho_1,\Omega\varrho_0)\Big)\Big\} \\ &\quad - \chi(d(\varrho_0,\varrho_1)).\end{aligned}$$

$$\implies \inf_{z\in\Omega\varrho_1} F(d(\varrho_1, z)) < F\Big\{\rho\Big(d(\varrho_0,\varrho_1), d(\varrho_0,\varrho_1), d(\varrho_1,\varrho_2), d(\varrho_0,\varrho_2), 0\Big)\Big\}.$$

Hence there exists $\varrho_2 \in \Omega\varrho_1$, such that

$$F(d(\varrho_1,\varrho_2)) < F\Big\{\rho\Big(d(\varrho_0,\varrho_1), d(\varrho_0,\varrho_1), d(\varrho_1,\varrho_2), d(\varrho_0,\varrho_2), 0\Big)\Big\}. \tag{4.20}$$

Since F is non-decreasing function, so (4.20) with (ρ_3) gives

$$\begin{aligned} d(\varrho_1,\varrho_2) &< \rho\Big(d(\varrho_0,\varrho_1), d(\varrho_0,\varrho_1), d(\varrho_1,\varrho_2), d(\varrho_0,\varrho_2), 0\Big) \\ &\leq \rho\Big(d(\varrho_0,\varrho_1), d(\varrho_0,\varrho_1), d(\varrho_1,\varrho_2), \mathfrak{f}(\varrho_0,\varrho_1))d(\varrho_0,\varrho_1) + \mathfrak{f}(\varrho_1,\varrho_2)d(\varrho_1,\varrho_2), 0\Big). \end{aligned}$$

By using Lemma 4.4

$$d(\varrho_1,\varrho_2) < d(\varrho_0,\varrho_1).$$

Next arguing as previous, we get $\varrho_3 \in \Omega\varrho_2$ with $D(\varrho_2, \Omega\varrho_2) > 0$. By considering Lemma 4.4, and using (G_1), (G_2)

$$d(\varrho_2,\varrho_3) < d(\varrho_1,\varrho_2).$$

By induction, we have a sequence $\{\varrho_\mathfrak{s}\} \subset \xi$ such that $\varrho_{\mathfrak{s}+1} \in \Omega\varrho_\mathfrak{s}$ with $D(\varrho_\mathfrak{s}, \Omega\varrho_\mathfrak{s}) > 0$ and

$$d(\varrho_\mathfrak{s},\varrho_{\mathfrak{s}+1}) < d(\varrho_{\mathfrak{s}-1},\varrho_\mathfrak{s}) \quad \text{for all} \quad \mathfrak{s} \in \mathbb{N}. \tag{4.21}$$

Now (4.21) implies that $\{d(\varrho_\mathfrak{s},\varrho_{\mathfrak{s}+1})\}_{\mathfrak{s}\in\mathbb{N}}$ is a decreasing sequence of positive real numbers. Hence from $(\mathbb{F}4)$

$$\begin{aligned} \inf_{z\in\Omega\varrho_\mathfrak{s}} F(d(\varrho_\mathfrak{s}, z)) &= F(D(\varrho_\mathfrak{s},\Omega\varrho_\mathfrak{s})) \leq F(H(\Omega\varrho_{\mathfrak{s}-1},\Omega\varrho_\mathfrak{s})) \leq L(H(\Omega\varrho_{\mathfrak{s}-1},\Omega\varrho_\mathfrak{s})) \\ &\leq F\{\rho\left(d(\varrho_{\mathfrak{s}-1},\varrho_\mathfrak{s}), D(\varrho_{\mathfrak{s}-1},\Omega\varrho_{\mathfrak{s}-1}), D(\varrho_\mathfrak{s},\Omega\varrho_\mathfrak{s}), D(\varrho_{\mathfrak{s}-1},\Omega\varrho_\mathfrak{s}), D(\varrho_\mathfrak{s},\Omega\varrho_{\mathfrak{s}-1})\right)\} \\ &- \chi(d(\varrho_{\mathfrak{s}-1},\varrho_\mathfrak{s})) \\ &\leq F\Big\{\rho\Big(d(\varrho_{\mathfrak{s}-1},\varrho_\mathfrak{s}), d(\varrho_{\mathfrak{s}-1},\varrho_\mathfrak{s}), d(\varrho_\mathfrak{s},\varrho_{\mathfrak{s}+1}), \mathfrak{f}(\varrho_{\mathfrak{s}-1},\varrho_\mathfrak{s})d(\varrho_{\mathfrak{s}-1},\varrho_\mathfrak{s}) + \mathfrak{f}(\varrho_\mathfrak{s},\varrho_{\mathfrak{s}+1}) \end{aligned}$$

$$
\begin{aligned}
&d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}), 0)\Big)\Big\} - \chi(d(\varrho_{\mathfrak{s}-1}, \varrho_{\mathfrak{s}}))\\
&\leq F\Big\{\rho\Big(d(\varrho_{\mathfrak{s}-1}, \varrho_{\mathfrak{s}}), d(\varrho_{\mathfrak{s}-1}, \varrho_{\mathfrak{s}}), d(\varrho_{\mathfrak{s}-1}, \varrho_{\mathfrak{s}}), \mathfrak{f}(\varrho_{\mathfrak{s}-1}, \varrho_{\mathfrak{s}})d(\varrho_{\mathfrak{s}-1}, \varrho_{\mathfrak{s}}) + \mathfrak{f}(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1})\\
&d(\varrho_{\mathfrak{s}-1}, \varrho_{\mathfrak{s}}), 0)\Big)\Big\} - \chi(d(\varrho_{\mathfrak{s}-1}, \varrho_{\mathfrak{s}}))\\
&\leq F\Big\{d(\varrho_{\mathfrak{s}-1}, \varrho_{\mathfrak{s}})\rho\Big(1, 1, 1, \mathfrak{f}(\varrho_{\mathfrak{s}-1}, \varrho_{\mathfrak{s}}) + \mathfrak{f}(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}), 0)\Big)\Big\} - \chi(d(\varrho_{\mathfrak{s}-1}, \varrho_{\mathfrak{s}})\\
&\leq F(d(\varrho_{\mathfrak{s}-1}, \varrho_{\mathfrak{s}})) - \chi(d(\varrho_{\mathfrak{s}-1}, \varrho_{\mathfrak{s}})).
\end{aligned}
$$

$$
\Longrightarrow \inf_{z\in\Omega\varrho_{\mathfrak{s}}} F(d(\varrho_{\mathfrak{s}}, z)) \leq F(d(\varrho_{\mathfrak{s}-1}, \varrho_{\mathfrak{s}})) - \chi(d(\varrho_{\mathfrak{s}-1}, \varrho_{\mathfrak{s}})) \quad \forall\ \mathfrak{s} \in \mathbb{N}. \tag{4.22}
$$

Since $\xi \in \Psi$, there exists $h > 0$ and $\mathfrak{s}_0 \in \mathbb{N}$ s.t $\chi(d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1})) < h, \ \forall \ \mathfrak{s} \geq \mathfrak{s}_0$. From (4.22)

$$
\begin{aligned}
F(d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1})) &\leq F(d(\varrho_{\mathfrak{s}-1}, \varrho_{\mathfrak{s}})) - \chi(d(\varrho_{\mathfrak{s}-1}, \varrho_{\mathfrak{s}}))\\
&\leq F(d(\varrho_{\mathfrak{s}-2}, \varrho_{\mathfrak{s}-1})) - \chi(d(\varrho_{\mathfrak{s}-2}, \varrho_{\mathfrak{s}-1})) - \chi(d(\varrho_{\mathfrak{s}-1}, \varrho_{\mathfrak{s}}))\\
&\vdots\\
&\leq F(d(\varrho_0, \varrho_1)) - \sum_{i=1}^{\mathfrak{s}-1} \chi(d(\varrho_{i-1}, \varrho_i))\\
&= F(d(\varrho_0, \varrho_1)) - \sum_{i=1}^{\mathfrak{s}_0-1} \chi(d(\varrho_{i-1}, \varrho_i)) - \sum_{i=\mathfrak{s}_0}^{\mathfrak{s}-1} \chi(d(\varrho_{i-1}, \varrho_i))
\end{aligned}
$$

$$
= F(d(\varrho_0, \varrho_1)) - (\mathfrak{s} - \mathfrak{s}_0)h, \qquad \mathfrak{s} \geq \mathfrak{s}_0. \tag{4.23}
$$

Applying limit $\mathfrak{s} \longrightarrow \infty$ in (4.23), we get $F(d(\varrho_{\mathfrak{s}-1}, \varrho_{\mathfrak{s}})) \longrightarrow -\infty$ and from ($\mathbb{F}2'$)

$$
\lim_{\mathfrak{s}\longrightarrow\infty} d(\varrho_{\mathfrak{s}-1}, \varrho_{\mathfrak{s}}) = 0. \tag{4.24}
$$

Now, from ($\mathbb{F}3$) there exists $0 < k < 1$ such that,

$$
\lim_{\mathfrak{s}\longrightarrow\infty} (d(\varrho_{\mathfrak{s}-1}, \varrho_{\mathfrak{s}}))^k F(d(\varrho_{\mathfrak{s}-1}, \varrho_{\mathfrak{s}})) = 0. \tag{4.25}
$$

Thus from (4.23) for all $\mathfrak{s} \geq \mathfrak{s}_0$, we have

$$\begin{aligned}(d(\varrho_{\mathfrak{s}-1}, \varrho_{\mathfrak{s}}))^k F(d(\varrho_{\mathfrak{s}-1}, \varrho_{\mathfrak{s}})) &- (d(\varrho_{\mathfrak{s}-1}, \varrho_{\mathfrak{s}}))^k F(d(\varrho_0, \varrho_1)) \\ &\leq (d(\varrho_{\mathfrak{s}-1}, \varrho_{\mathfrak{s}}))^k \Big(F(d(\varrho_0, \varrho_1)) - (\mathfrak{s} - \mathfrak{s}_0)h \Big) - (d(\varrho_{\mathfrak{s}-1}, \varrho_{\mathfrak{s}}))^k F(d(\varrho_0, \varrho_1))\end{aligned}$$

$$= -(d(\varrho_{\mathfrak{s}-1}, \varrho_{\mathfrak{s}}))^k (\mathfrak{s} - \mathfrak{s}_0)h \leq 0. \tag{4.26}$$

Taking limit $\mathfrak{s} \longrightarrow \infty$ in (4.26) and using (4.24) , (4.25)

$$0 \leq -\lim_{\mathfrak{s} \longrightarrow \infty} \mathfrak{s}(d(\varrho_{\mathfrak{s}-1}, \varrho_{\mathfrak{s}}))^k \leq 0.$$

$$\Longrightarrow \lim_{\mathfrak{s} \longrightarrow \infty} \mathfrak{s}(d(\varrho_{\mathfrak{s}-1}, \varrho_{\mathfrak{s}}))^k = 0 \tag{4.27}$$

Note that by using (4.27), there exists $\mathfrak{s}_1 \in \mathbb{N}$ s.t $\mathfrak{s}(d(\varrho_{\mathfrak{s}-1}, \varrho_{\mathfrak{s}}))^k \leq 1 \;\; \forall \;\; \mathfrak{s} \geq \mathfrak{s}_1$. We get

$$d(\varrho_{\mathfrak{s}-1}, \varrho_{\mathfrak{s}}) \leq \frac{1}{\mathfrak{s}^{\frac{1}{k}}} \qquad \forall \quad \mathfrak{s} \geq \mathfrak{s}_1.$$

Now to prove that $\{\varrho_{\mathfrak{s}}\}_{\mathfrak{s} \in \mathbb{N}}$ is Cauchy sequence. Consider $\tau, \mathfrak{s} \in \mathbb{N}$ such that $\tau > \mathfrak{s} > \mathfrak{s}_1$. The rest of the proof follows from Theorem 4.5 and by using (4.18) with ratio test, we deduce that $\{\varrho_{\mathfrak{s}}\}$ is Cauchy sequence and there exists $\varrho^* \in \xi$ such that

$$\lim_{\mathfrak{s} \longrightarrow \infty} \varrho_{\mathfrak{s}} = \varrho^*.$$

Now

$$\begin{aligned}F(H(\Omega\varrho, \Omega\bar{\varrho})) \leq L(H(\Omega\varrho, \Omega\bar{\varrho})) &\leq \chi(d(\varrho, \bar{\varrho})) + L(H(\Omega\varrho, \Omega\bar{\varrho})) \\ &\leq F\Big\{\rho\Big(d(\varrho, \bar{\varrho}), D(\varrho, \Omega\varrho), D(\bar{\varrho}, \Omega\bar{\varrho}), D(\varrho, \Omega\bar{\varrho}), D(\bar{\varrho}, \Omega\varrho)\Big)\Big\}.\end{aligned}$$

Since F is non-decreasing function, therefore

$$H(\Omega\varrho,\Omega\bar{\varrho}) \leq \rho\Big(d(\varrho,\bar{\varrho}), D(\varrho,\Omega\varrho), D(\bar{\varrho},\Omega\bar{\varrho}), D(\varrho,\Omega\bar{\varrho}), D(\bar{\varrho},\Omega\varrho)\Big) \quad \forall\ \varrho,\bar{\varrho}\in\xi.$$

Assume that ϱ^* is fixed point of ξ. On contrary, we have $D(\varrho^*,\Omega\varrho^*) > 0$. Then by following the proof of Theorem 4.5, $D(\varrho^*,\Omega\varrho^*) = 0$. Since $\Omega\varrho^*$ is closed, so $\varrho^* \in \Omega\varrho^*$. Hence Fix$\Omega$ is non empty. □

Theorem 4.8.
Let $(\xi, d, \mathfrak{f})$ be a complete CMS and $\Omega : \xi \longrightarrow C(\xi)$ be a MVM. Suppose there exists $\chi \in \Psi, \rho \in \mathbb{P}$ and a non decreasing and continuous real valued function $F : (0,\infty) \longrightarrow \mathbb{R}$ which satisfy $(\mathbb{F}2')$. Moreover a real valued function L on $(0,\infty)$ is such that given conditions hold:

(G_1) $F(\varrho) \leq L(\varrho) \quad \forall \quad \varrho > 0,$

(G_2) $H(\Omega\varrho,\Omega\bar{\varrho}) > 0$ *implies*

$$\chi(d(\varrho,\bar{\varrho})) + L(H(\Omega\varrho,\Omega\bar{\varrho})) \leq F\Big\{\rho\Big(d(\varrho,\bar{\varrho}), D(\varrho,\Omega\varrho), D(\bar{\varrho},\Omega\bar{\varrho}), D(\varrho,\Omega\bar{\varrho}), D(\bar{\varrho},\Omega\varrho)\Big)\Big\},$$

for all $\varrho,\bar{\varrho}\in\xi$.

Also

$$\lim_{\mathfrak{s}\longrightarrow\infty} \mathfrak{f}(\varrho_{\mathfrak{s}},\varrho) \leq 1 \textit{ for all } \varrho\in\xi.$$

Then, FixΩ is non empty.

Proof. Let $\varrho_0 \in \xi$ be an arbitrary point and $\varrho_1 \in \Omega\varrho_0$. As in proof of Theorem 4.5, we get a sequence $\{\varrho_{\mathfrak{s}}\} \subset \xi$, such that $\varrho_{\mathfrak{s}+1} \in \Omega\varrho_{\mathfrak{s}}$ with $D(\varrho_{\mathfrak{s}},\Omega\varrho_{\mathfrak{s}+1}) > 0$,

$$d(\varrho_{\mathfrak{s}},\varrho_{\mathfrak{s}+1}) < d(\varrho_{\mathfrak{s}-1},\varrho_{\mathfrak{s}}),$$

and

$$F(d(\varrho_{\mathfrak{s}-1},\varrho_{\mathfrak{s}})) \leq F(d(\varrho_0,\varrho_1)) - (\mathfrak{s}-\mathfrak{s}_0)h \qquad \forall\ \mathfrak{s}\geq\mathfrak{s}_0. \tag{4.28}$$

Taking $\mathfrak{s} \longrightarrow \infty$ in (4.28), $F(d(\varrho_{\mathfrak{s}-1}, \varrho_{\mathfrak{s}})) \longrightarrow -\infty$ and by ($\mathbb{F}2'$),

$$\lim_{\mathfrak{s} \longrightarrow \infty} d(\varrho_{\mathfrak{s}-1}, \varrho_{\mathfrak{s}}) = 0.$$

Now we claim that,

$$\lim_{\mathfrak{s}, \tau \longrightarrow \infty} d(\varrho_{\mathfrak{s}}, \varrho_{\tau}) = 0. \tag{4.29}$$

However if, (4.29) does not hold, then there exists $\delta > 0$ such that for all $r \geq 0$, we have $\tau_k > \mathfrak{s}_k > r$,

$$d(\varrho_{\mathfrak{s}}, \varrho_{\tau}) < \delta.$$

Also, there exists $r_0 \in \mathbb{N}$ such that

$$\lambda_{r_0} = d(\varrho_{\mathfrak{s}-1}, \varrho_{\mathfrak{s}}) < \delta \qquad \forall\ \mathfrak{s} \geq r_0.$$

There exists two sub sequences $\{\varrho_{\tau_k}\}$ and $\{\varrho_{\mathfrak{s}_k}\}$ of $\{\varrho_{\mathfrak{s}}\}$, then by following the proof of Theorem 4.6, we get $\lim_{k \longrightarrow \infty} d(\varrho_{\tau_k}, \varrho_{\mathfrak{s}_k}) = \delta$ also

$$\lim_{k \longrightarrow \infty} d(\varrho_{\tau_{k+1}}, \varrho_{\mathfrak{s}_{k+1}}) = \delta. \tag{4.30}$$

By monotonicity of F and using $(G_1), (G_2)$, we get

$$\begin{aligned}
&\chi(d(\varrho_{\tau_k}, \varrho_{\mathfrak{s}_k})) + F(d(\varrho_{\tau_{k+1}}, \Omega\varrho_{\mathfrak{s}_{k+1}})) = \chi(d(\varrho_{\tau_k}, \varrho_{\mathfrak{s}_k})) + F(D(\varrho_{\tau_{k+1}}, \Omega\varrho_{\mathfrak{s}_k})) \\
&\quad \leq \chi(d(\varrho_{\tau_k}, \varrho_{\mathfrak{s}_k})) + F(H(\Omega\varrho_{\tau_k}, \Omega\varrho_{\mathfrak{s}_k})) \\
&\quad \leq \chi(d(\varrho_{\tau_k}, \varrho_{\mathfrak{s}_k})) + L(H(\Omega\varrho_{\tau_k}, \Omega\varrho_{\mathfrak{s}_k})) \\
&\quad \leq F\Big\{\rho\Big(d(\varrho_{\tau_k}, \varrho_{\mathfrak{s}_k}), d(\varrho_{\tau_k}, \varrho_{\tau_{k+1}}), d(\varrho_{\mathfrak{s}_k}, \varrho_{\mathfrak{s}_{k+1}}), \mathfrak{f}(\varrho_{\mathfrak{s}_{k+1}}, \varrho_{\mathfrak{s}_k})d(\varrho_{\mathfrak{s}_{k+1}}, \varrho_{\mathfrak{s}_k}) + \\
&\qquad \mathfrak{f}(\varrho_{\mathfrak{s}_k}, \varrho_{\tau_k})d(\varrho_{\mathfrak{s}_k}, \varrho_{\tau_k}), \mathfrak{f}(\varrho_{\mathfrak{s}_k}, \varrho_{\mathfrak{s}_{k+1}})d(\varrho_{\mathfrak{s}_k}, \varrho_{\mathfrak{s}_{k+1}}) + \mathfrak{f}(\varrho_{\mathfrak{s}_{k+1}}, \varrho_{\tau_{k+1}})d(\varrho_{\mathfrak{s}_{k+1}}, \varrho_{\tau_{k+1}})\Big)\Big\}.
\end{aligned}$$

$$\Longrightarrow\ \chi(d(\varrho_{\tau_k}, \varrho_{\mathfrak{s}_k})) + F(d(\varrho_{\tau_{k+1}}, \Omega\varrho_{\mathfrak{s}_{k+1}}))$$

$$\leq \mathfrak{f}(\varrho_{\mathfrak{s}_k}, \varrho_{\tau_k})d(\varrho_{\mathfrak{s}_k}, \varrho_{\tau_k}), \mathfrak{f}(\varrho_{\mathfrak{s}_k}, \varrho_{\mathfrak{s}_{k+1}})d(\varrho_{\mathfrak{s}_k}, \varrho_{\mathfrak{s}_{k+1}}) + \mathfrak{f}(\varrho_{\mathfrak{s}_{k+1}}, \varrho_{\tau_{k+1}})d(\varrho_{\mathfrak{s}_{k+1}}, \varrho_{\tau_{k+1}})\Big)\Big\}. \tag{4.31}$$

By continuity of F and applying the limit $k \longrightarrow \infty$ as well as using (4.30), (4.31), we have

$$\lim_{k\longrightarrow\infty} \chi(d(\varrho_{\tau_k}, \varrho_{\mathfrak{s}_k})) + F(\delta) \leq F\Big\{\rho\Big(\delta, 0, 0, \delta \lim_{k\longrightarrow\infty} \mathfrak{f}(\varrho_{\mathfrak{s}_k}, \varrho_{\tau_k}), \delta \lim_{k\longrightarrow\infty} \mathfrak{f}(\varrho_{\mathfrak{s}_{k+1}}, \varrho_{\tau_{k+1}})\Big)\Big\}$$
$$\leq F\Big\{\delta\rho\Big(1, 0, 0, \delta V \lim_{k\longrightarrow\infty} \mathfrak{f}(\varrho_{\mathfrak{s}_k}, \varrho_{\tau_k}), \delta \lim_{k\longrightarrow\infty} \mathfrak{f}(\varrho_{\mathfrak{s}_{k+1}}, \varrho_{\tau_{k+1}})\Big)\Big\}.$$

Since $\rho \in \mathbb{P}$, we have $\rho\Big(1, 0, 0, \lim_{k\longrightarrow\infty} \mathfrak{f}(\varrho_{\mathfrak{s}_k}, \varrho_{\tau_k}), \lim_{k\longrightarrow\infty} \mathfrak{f}(\varrho_{\mathfrak{s}_{k+1}}, \varrho_{\tau_{k+1}})\Big) \in (0, 1]$. Hence

$$\lim_{S\longrightarrow\delta^+} \inf \chi(s) \leq 0$$

which is a contradiction to definition of $\varPsi$. Therefore (4.29) is fulfilled and ensures that $\{\varrho_{\mathfrak{s}}\}$ is Cauchy sequence. Hence there exists $\varrho^* \in \xi$ such that,

$$\lim_{\mathfrak{s}\longrightarrow\infty} \varrho_{\mathfrak{s}} = \varrho^*.$$

By following the proof of Theorem 4.7, we get $\varrho^* \in \Omega\varrho^*$. □

4.2 Data Depenence

The aim of this section is to present a data dependence result of the established result.

Theorem 4.9. *Suppose that* (ξ, d) *is a CMS,* $\Omega_1, \Omega_2 : \xi \longrightarrow K(\xi)$ *are multivalued mappings and* $\chi \in \varPsi$*. Let* F_1 *be real valued non-decreasing function defined on* $(0, \infty)$ *and* F_2 *be a real valued function satisfying* $(\mathbb{F}2')$ *and* $(\mathbb{F}3)$ *defined on* $(0, \infty)$ *such that* χF*-contraction is satisfied for* Ω_{i}*, where* $\mathsf{i} \in \{1, 2\}$ *and there exists* $\lambda > 0$ *such that* $H\big(\Omega_1(\varrho), \Omega_2(\varrho)\big) \leq \lambda$*, for all* $\varrho \in \xi$*. For* $\varrho_0 \in \xi$*, define picard sequence* $\{\varrho_{\mathfrak{s}} = \Omega^{\mathfrak{s}}\varrho_0\}$*, so that*

$$\sup_{m\geq 1} \lim_{i\longrightarrow\infty} \frac{\mathfrak{f}(\varrho_{i+1}, \varrho_{i+2})\mathfrak{f}(\varrho_{i+1}, \varrho_m)}{\mathfrak{f}(\varrho_i, \varrho_{i+1})} < 1. \tag{4.32}$$

Also suppose that $\lim_{\mathfrak{s}\longrightarrow\infty} \mathfrak{f}(\varrho_{\mathfrak{s}}, \varrho) \leq 1$ *for all* $\varrho \in \xi$. *Then,*

(*a*.) *Fix* $\Omega_{\mathfrak{i}} \in CL(\xi)$ *for* $\mathfrak{i} \in \{1, 2\}$,

(*b*.) Ω_1, Ω_2 *are MWP Operators and*

$$H\Big(Fix\Omega_1, Fix\Omega_2\Big) \leq \frac{\lambda}{1 - \max\{\rho_1(1,1,1,\zeta+\eta,0), \rho_2(1,1,1,\zeta+\eta,0)\}},$$

where $\zeta, \eta \geq 1$.

Proof. (*a*.) Using Theorem 4.5, we have Fix $\Omega_{\mathfrak{i}}$ is not empty for $\mathfrak{i} \in \{1, 2\}$. Now, we prove that for $\mathfrak{i} \in \{1, 2\}$, the fixed point set of $\Omega_{\mathfrak{i}}$ is closed. Consider a sequence $\{\varrho_{\mathfrak{s}}\}$ in Fix$\Omega_{\mathfrak{i}}$ such that $\varrho_{\mathfrak{s}} \longrightarrow \varrho$ as $\mathfrak{s} \longrightarrow \infty$. Now,

$$\begin{aligned} F_1\big(H(\Omega\varrho, \Omega\bar{\varrho}\big) &\leq F_2\big(H(\Omega\varrho, \Omega\bar{\varrho}\big) \leq \chi\big(d(\varrho, \bar{\varrho}) + F_2\Big(H\big(\Omega\varrho, \Omega\bar{\varrho}\big)\Big) \\ &\leq F_1\Big(\rho\big(d(\varrho, \bar{\varrho}), D(\varrho, \Omega\varrho), D(\bar{\varrho}, \Omega\bar{\varrho}), D(\varrho, \Omega\bar{\varrho}), D(\bar{\varrho}, \Omega\varrho)\big)\Big). \end{aligned}$$

Since F_1 is non-decreasing function so, for all $\varrho, \bar{\varrho} \in \xi$

$$H(\Omega\varrho, \Omega\bar{\varrho}) \leq \rho\big(d(\varrho, \bar{\varrho}), D(\varrho, \Omega\varrho), D(\bar{\varrho}, \Omega\bar{\varrho}), D(\varrho, \Omega\bar{\varrho}), D(\bar{\varrho}, \Omega\varrho)\big). \tag{4.33}$$

Assume that $D(\bar{\varrho}, \Omega\bar{\varrho}) > 0$. Now, there exists $\varrho \in \Omega\bar{\varrho}$ such that

$$\begin{aligned} D(\bar{\varrho}, \Omega\bar{\varrho}) &= d(\bar{\varrho}, \varrho) \\ &\leq \mathfrak{f}(\bar{\varrho}, \varrho_{\mathfrak{s}+1})d(\bar{\varrho}, \varrho_{\mathfrak{s}+1}) + \mathfrak{f}(\varrho_{\mathfrak{s}+1}, \varrho)d(\varrho_{\mathfrak{s}+1}, \varrho) \\ &= \mathfrak{f}(\bar{\varrho}, \varrho_{\mathfrak{s}+1})d(\bar{\varrho}, \varrho_{\mathfrak{s}+1}) + \mathfrak{f}(\varrho_{\mathfrak{s}+1}, \varrho)D(\varrho_{\mathfrak{s}+1}, \Omega\bar{\varrho}) \\ &\leq \mathfrak{f}(\bar{\varrho}, \varrho_{\mathfrak{s}+1})d(\bar{\varrho}, \varrho_{\mathfrak{s}+1}) + \mathfrak{f}(\varrho_{\mathfrak{s}+1}, \varrho)H(\Omega\varrho_{\mathfrak{s}}, \Omega\bar{\varrho}) \\ &\leq \mathfrak{f}(\bar{\varrho}, \varrho_{\mathfrak{s}+1})d(\bar{\varrho}, \varrho_{\mathfrak{s}+1}) + \mathfrak{f}(\varrho_{\mathfrak{s}+1}, \varrho)\rho\Big(d(\varrho_{\mathfrak{s}}, \bar{\varrho}), D(\varrho_{\mathfrak{s}}, \Omega\varrho_{\mathfrak{s}}), D(\bar{\varrho}, \Omega\bar{\varrho}), D(\varrho_{\mathfrak{s}}, \Omega\bar{\varrho}), \\ &\quad D(\bar{\varrho}, \Omega\varrho_{\mathfrak{s}})\Big) \\ &\leq \mathfrak{f}(\bar{\varrho}, \varrho_{\mathfrak{s}+1})d(\bar{\varrho}, \varrho_{\mathfrak{s}+1}) + \mathfrak{f}(\varrho_{\mathfrak{s}+1}, \varrho)\rho\Big(d(\varrho_{\mathfrak{s}}, \bar{\varrho}), d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}), D(\bar{\varrho}, \Omega\bar{\varrho}), \mathfrak{f}(\varrho_{\mathfrak{s}}, \bar{\varrho})d(\varrho_{\mathfrak{s}}, \bar{\varrho}) \\ &\quad + \mathfrak{f}(\bar{\varrho}, \varrho)D(\bar{\varrho}, \Omega\bar{\varrho}), d(\bar{\varrho}, \varrho_{\mathfrak{s}+1})\Big). \end{aligned}$$

By applying limit $\mathfrak{s} \longrightarrow \infty$ in the above inequality, we get

$$D(\bar{\varrho}, \Omega\bar{\varrho}) \leq (1)\rho\Big(0, 0, D(\bar{\varrho}, \Omega\bar{\varrho}), 0 + \mathfrak{f}(\bar{\varrho}, \varrho_1)D(\bar{\varrho}, \Omega\bar{\varrho}), 0\Big).$$

Using Lemma 4.4, $D(\bar{\varrho}, \Omega\bar{\varrho}) \leq 0$.

$$\Longrightarrow 0 < D(\bar{\varrho}, \Omega\bar{\varrho}) \leq 0.$$

Hence $D(\bar{\varrho}, \Omega\bar{\varrho}) = 0$. As $\Omega\bar{\varrho}$ is closed, so $\bar{\varrho} \in \Omega\bar{\varrho}$.

$(b.)$ Using Theorem 4.5, we get that Ω_1, Ω_2 are *MWP* Operators. So, We have to prove that

$$H\Big(\text{Fix}\Omega_1, \text{Fix}\Omega_2\Big) \leq \frac{\lambda}{1 - \max\{\rho_1(1,1,1,\zeta+\eta,0), \rho_2(1,1,1,\zeta+\eta,0)\}}.$$

Suppose $\mathsf{q} > 1$, and $\varrho_0 \in \text{Fix}\Omega_2$. Then, $\varrho_1 \in \Omega_2(\varrho_0)$ exists such that $d(\varrho_0, \varrho_1) = D(\varrho_0, \Omega_2(\varrho_0))$ and $d(\varrho_1, \varrho_2) \leq \mathsf{q}H\big(\Omega_1(\varrho_0), \Omega_2(\varrho_0)\big)$. Now, $\varrho_2 \in \Omega_2(\varrho_1)$ exists such that $d(\varrho_0, \varrho_1) = D(\varrho_0, \Omega_2(\varrho_0))$ and $d(\varrho_1, \varrho_2) \leq \mathsf{q}H\big(\Omega_2(\varrho_0), \Omega_2(\varrho_1)\big)$. Also, we get $d(\varrho_1, \varrho_2) \leq d(\varrho_0, \varrho_1)$ and

$$\begin{aligned}
d(\varrho_1, \varrho_2) &\leq \mathsf{q}H(\Omega_2(\varrho_0), \Omega_2(\varrho_1)) \\
&\leq \mathsf{q}\rho\big(d(\varrho_0, \varrho_1), D(\varrho_0, \Omega(\varrho_0)), D(\varrho_1, \Omega(\varrho_1)), D(\varrho_0, \Omega(\varrho_1)), D(\varrho_1, \Omega(\varrho_0))\big) \\
&\leq \mathsf{q}\rho\big(d(\varrho_0, \varrho_1), d(\varrho_0, \varrho_1), d(\varrho_1, \varrho_2), d(\varrho_0, \varrho_2), d(\varrho_1, \varrho_1)\big) \\
&\leq \mathsf{q}\rho\Big(d(\varrho_0, \varrho_1), d(\varrho_0, \varrho_1), d(\varrho_1, \varrho_2), \mathfrak{f}(\varrho_0, \varrho_1))d(\varrho_0, \varrho_1) + \mathfrak{f}(\varrho_1, \varrho_2) \\
&\qquad d(\varrho_1, \varrho_2), 0\Big) \\
&< \mathsf{q}\rho\Big(d(\varrho_0, \varrho_1), d(\varrho_0, \varrho_1), d(\varrho_0, \varrho_1), \mathfrak{f}(\varrho_0, \varrho_1))d(\varrho_0, \varrho_1) + \mathfrak{f}(\varrho_1, \varrho_2) \\
&\qquad d(\varrho_0, \varrho_1), 0\Big) \\
&\leq \mathsf{q}d(\varrho_0, \varrho_1)\rho\big(1, 1, 1, \mathfrak{f}(\varrho_0, \varrho_1)) + \mathfrak{f}(\varrho_1, \varrho_2), 0\big)
\end{aligned}$$

Hence, we will get a sequence of successive approximations of Ω starting from ϱ_0, which satisfy the following

$$d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}) \leq \Big(\mathsf{q}\rho_1\big(1, 1, 1, \zeta + \eta, 0\big)\Big)^{\mathfrak{s}} d(\varrho_0, \varrho_1), \quad \forall\, \mathfrak{s} \in \mathbb{N}.$$

$$\Rightarrow d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+m}) \leq \frac{\Big(\mathsf{q}\rho_1\big(1,1,1,\zeta+\eta,0\big)\Big)^{\mathfrak{s}}}{1-\mathsf{q}\rho_1\big(1,1,1,\zeta+\eta,0\big)} d(\varrho_0, \varrho_1), \quad \forall\, \mathfrak{s} \in \mathbb{N}. \tag{4.34}$$

Taking $\lim_{\mathfrak{s}\longrightarrow\infty}$, it is concluded that $\{\varrho_{\mathfrak{s}}\}$ is Cauchy sequence in (ξ, d) so converges to some $\mathsf{v} \in \xi$. Using the proof of Theorem 4.5, we have $\mathsf{v} \in \mathrm{Fix}\Omega_2$. Applying $\lim_{m\longrightarrow\infty}$, we get

$$d(\varrho_{\mathfrak{s}}, \mathsf{v}) \leq \frac{\Big(\mathsf{q}\rho_1\big(1,1,1,\zeta+\eta,0\big)\Big)^{\mathfrak{s}}}{1-\mathsf{q}\rho_1\big(1,1,1,\zeta+\eta,0\big)} d(\varrho_0, \varrho_1), \quad \forall\, \mathfrak{s} \in \mathbb{N}.$$

Letting $\mathfrak{s} = 0$

$$d(\varrho_0, \mathsf{v}) \leq \frac{1}{1-\mathsf{q}\rho_1\big(1,1,1,\zeta+\eta,0\big)} d(\varrho_0, \varrho_1) \leq \frac{\mathsf{q}\lambda}{1-\mathsf{q}\rho_1\big(1,1,1,\zeta+\eta,0\big)}.$$

Now, we interchange the role of Ω_1 and Ω_2, then for each $\mathsf{v}_0 \in \mathrm{Fix}\Omega_1$

$$d(\mathsf{v}_0, c) \leq \frac{1}{1-\mathsf{q}\rho_2\big(1,1,1,,\zeta+\eta,0\big)} d(\mathsf{v}_0, \mathsf{v}_1) \leq \frac{\mathsf{q}\lambda}{1-\mathsf{q}\rho_2\big(1,1,1,,\zeta+\eta,0\big)}.$$

So,

$$H(\mathrm{Fix}\Omega_1, \mathrm{Fix}\Omega_2) \leq \frac{\mathsf{q}\lambda}{1-\max\Big(\mathsf{q}\rho_1\big(1,1,1,,\zeta+\eta,0\big), \mathsf{q}\rho_2\big(1,1,1,,\zeta+\eta,0\big)\Big)}.$$

By taking $\mathsf{q} \longrightarrow 1$ the result is proved. □

4.3 Strict Fixed Point and Well Posedness

This section is furnished with fixed point results to assure the existence of strict fixed point and well-posedness of the multivalued generalized contractions in the setting of CMS.

Theorem 4.10. *Assume that $(\xi, d, \mathfrak{f})$ is a complete CMS. Let $\Omega : \xi \longrightarrow K(\xi)$ be a MVM and F_1, F_2 are functions satisfying χF-contraction. Suppose F_1 is non-decreasing, F_2 satisfy condition $(\mathbb{F}2')$ with $\rho(1,0,0,1,1) \in (0,1)$ and $\mathsf{S}Fix\Omega \neq \Psi$. Also suppose $\lim_{\mathfrak{s}\longrightarrow\infty} \mathfrak{f}(\varrho_{\mathfrak{s}}, \varrho) \leq 1$ for all $\varrho \in \xi$. Then,*

(a) $Fix\Omega = \mathsf{S}Fix\Omega = \{\varrho^*\}$,

(b) *The fixed point problem is well posed for MVM Ω with respect to H.*

Proof. (a) Using Theorem 4.6 , we conclude that $\text{Fix}\Omega \neq \Psi$. Now, we prove that $\text{Fix}\Omega = \{\varrho^*\}$. Using (Ni.) and (Nii.), we have

$$\begin{aligned} F_1(H(\Omega\varrho, \Omega\bar{\varrho})) &\leq F_2(H(\Omega\varrho, \Omega\bar{\varrho})) \leq \chi(d(\varrho, \bar{\varrho})) + F_2(H(\Omega\varrho, \Omega\bar{\varrho})) \\ &\leq F_1\Big\{\rho\Big(d(\varrho, \bar{\varrho}), D(\varrho, \Omega\varrho), D(\bar{\varrho}, \Omega\bar{\varrho}), D(\varrho, \Omega\bar{\varrho}), D(\bar{\varrho}, \Omega\varrho)\Big)\Big\}. \end{aligned}$$

Since F_1 is non-decreasing function, we obtain for all $\varrho, \bar{\varrho} \in \xi$,

$$H(\Omega\varrho, \Omega\bar{\varrho}) \leq \rho\Big(d(\varrho, \bar{\varrho}), D(\varrho, \Omega\varrho), D(\bar{\varrho}, \Omega\bar{\varrho}), D(\varrho, \Omega\bar{\varrho}), D(\bar{\varrho}, \Omega\varrho)\Big).$$

Let $\mathsf{v} \in \text{Fix}\Omega$, with $\mathsf{v} \neq \varrho^*$, then, $D(\varrho^*, \Omega\mathsf{v}) > 0$. Now, we have

$$\begin{aligned} D(\varrho^*, \Omega\mathsf{v}) &= H(\Omega\varrho^*, \Omega\mathsf{v}) \\ &\leq \rho\Big(d(\varrho^*, \mathsf{v}), D(\varrho^*, \Omega\varrho^*), D(\mathsf{v}, \Omega\mathsf{v}), D(\varrho^*, \Omega\mathsf{v}), D(\mathsf{v}, \Omega\varrho^*)\Big) \\ &\leq \rho\Big(d(\varrho^*, \mathsf{v}), 0, 0, d(\varrho^*, \mathsf{v}), d(\mathsf{v}, \varrho^*)\Big) \\ &\leq d(\varrho^*, \mathsf{v})\rho(1, 0, 0, 1, 1). \end{aligned}$$

As $\rho(1, 0, 0, 1, 1) \in (0, 1)$, so

$$d(\varrho^*, \mathsf{v}) = D(\varrho^*, \Omega\mathsf{v}) < d(\varrho^*, \mathsf{v}),$$

which is a contradiction, hence, $d(\varrho^*, \mathsf{v}) = 0$ and $\varrho^* = \mathsf{v}$.
(b) Let $\varrho_{\mathfrak{s}} \in \mathbb{B}, \mathfrak{s} \in \mathbb{N}$, such that

$$\lim_{\mathfrak{s} \longrightarrow \infty} D(\varrho_{\mathfrak{s}}, \Omega\varrho_{\mathfrak{s}}) = 0. \tag{4.35}$$

Now, we claim that

$$\lim_{\mathfrak{s} \longrightarrow \infty} d(\varrho_{\mathfrak{s}}, \varrho^*) = 0,$$

where $\varrho^* \in \text{Fix}\Omega$. If the above equation is not true, then, for every $\mathfrak{s} \in \mathbb{N}$, there exists $\epsilon > 0$ such that

$$d(\varrho_{\mathfrak{s}}, \varrho^*) > \epsilon.$$

But (4.35) implies that there exists $\mathfrak{s}_\epsilon \in \mathbb{N} - \{0\}$ such that

$$\lim_{\mathfrak{s} \longrightarrow \infty} D(\varrho_{\mathfrak{s}}, \Omega \varrho_{\mathfrak{s}}) < \epsilon,$$

for each $\mathfrak{s} > \mathfrak{s}_\epsilon$. Hence, for each $\mathfrak{s} > \mathfrak{s}_\epsilon$, we obtain

$$d(\varrho_{\mathfrak{s}}, \varrho^*) = D(\varrho_{\mathfrak{s}}, \Omega \varrho^*).$$

Compactness of $\Omega\varrho^*$ implies that there exists $\varrho \in \Omega\varrho^*$ such that

$$\begin{aligned}
d(\varrho_{\mathfrak{s}}, \varrho^*) &= D(\varrho_{\mathfrak{s}}, \Omega\varrho^*) = d(\varrho_{\mathfrak{s}}, \varrho) \\
&\leq \mathfrak{f}(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}) d(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}) + \mathfrak{f}(\varrho_{\mathfrak{s}+1}, \varrho) d(\varrho_{\mathfrak{s}+1}, \varrho) \\
&= \mathfrak{f}(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}) D(\varrho_{\mathfrak{s}}, \Omega\varrho_{\mathfrak{s}}) + \mathfrak{f}(\varrho_{\mathfrak{s}+1}, \varrho) D(\varrho_{\mathfrak{s}+1}, \Omega\varrho^*) \\
&\leq \mathfrak{f}(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}) D(\varrho_{\mathfrak{s}}, \Omega\varrho_{\mathfrak{s}}) + \mathfrak{f}(\varrho_{\mathfrak{s}+1}, \varrho) H(\Omega\varrho_{\mathfrak{s}}, \Omega\varrho^*) \\
&< \mathfrak{f}(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}) D(\varrho_{\mathfrak{s}}, \Omega\varrho_{\mathfrak{s}}) + \mathfrak{f}(\varrho_{\mathfrak{s}+1}, \varrho) \rho\Big(d(\varrho_{\mathfrak{s}}, \varrho^*), D(\varrho_{\mathfrak{s}}, \Omega\varrho_{\mathfrak{s}}), D(\varrho^*, \Omega\varrho^*), \\
&\quad D(\varrho_{\mathfrak{s}}, \Omega\varrho^*), D(\varrho^*, \Omega\varrho_{\mathfrak{s}}) \Big) \\
&\leq \mathfrak{f}(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}) D(\varrho_{\mathfrak{s}}, \Omega\varrho_{\mathfrak{s}}) + \mathfrak{f}(\varrho_{\mathfrak{s}+1}, \varrho) \rho\Big(d(\varrho_{\mathfrak{s}}, \varrho^*), D(\varrho_{\mathfrak{s}}, \Omega\varrho_{\mathfrak{s}}), d(\varrho^*, \varrho^*), \\
&\quad d(\varrho_{\mathfrak{s}}, \varrho^*), \mathfrak{f}(\varrho^*, \varrho_{\mathfrak{s}}) d(\varrho^*, \varrho_{\mathfrak{s}}) + \mathfrak{f}(\varrho_{\mathfrak{s}}, \varrho_{\mathfrak{s}+1}) D(\varrho_{\mathfrak{s}}, \Omega\varrho_{\mathfrak{s}}) \Big).
\end{aligned}$$

As $\lim_{\mathfrak{s} \longrightarrow \infty} \mathfrak{f}(\varrho_{\mathfrak{s}}, \varrho) \leq 1$ and $\rho(1, 0, 0, 1, 1) \in (0, 1)$, so by applying limit $\mathfrak{s} \longrightarrow \infty$, we get $d(\varrho_{\mathfrak{s}}, \varrho^*) \longrightarrow 0$ as $\mathfrak{s} \longrightarrow \infty$, which is a contradiction. Hence, the fixed point problem is well posed for MVM Ω with respect to D. Also, $\text{Fix}\Omega = \mathsf{S}\text{Fix}\Omega$, hence the fixed point problem is well posed with respect to H. □

Chapter 5

Conclusions

This dissertation arrives at its end in the following fashion:

- A quick history is presented for a concise discussion on the fixed point theory.
- A brief discussion of some fundamental ideas is referenced to provide a base for upcoming results.
- Some mappings are elaborated for a better understanding of contractions.
- A quick review of F-contraction mapping and its generalizations is highlighted.
- A segment dealing with generalizations of metric space is articulated.
- The work of Iqbal et al. [19] is reviewed in detail. Data dependence, the existence of fixed points, strict fixed points, and well posedness of some multivalued generalized contractions are discussed in the setting of complete metric space.
- Some fixed and strict fixed point results are established on controlled metric spaces. We followed the scheme of Iqbal et al. [19] and used the platform of CMS. We have also provided the well-posedness of the theorems. The data dependence problem of fixed points of the considered mappings is also established. A non-trivial example is provided for the authentication purpose.

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