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Numerical study of viscous dissipation and Joule heating effects in MHD flow of nanofluid through porous media

by

Abida Begum

A thesis submitted in partial fulfillment for the
degree of Master of Philosophy

in the

Faculty of Computing
Department of Mathematics

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Dedicated to my beloved **parents** and **Mahin Ali**

CAPITAL UNIVERSITY OF SCIENCE & TECHNOLOGY
ISLAMABAD

CERTIFICATE OF APPROVAL

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Abstract

A numerical analysis is performed for the mathematical model of boundary layer flow with three different kinds of nanoparticles. Heat and mass transfer are analysed for an incompressible electrically conducting fluid with viscous dissipations and Joule heating past a porous plate embedded in a porous medium. An appropriate set of similarity transformations are used to transform the governing partial differential equations (PDEs) into a system of nonlinear ordinary differential equations (ODEs). The resulting system of ODEs is solved numerically by using shooting method and obtained numerical results are compared with Matlab bvp4c built in function. Furthermore, we compared our results with the existing results for especial cases. which are in an excellent agreement. The numerical values obtained for various non-dimensional physical quantities together with velocity and temperature profiles are presented through graphs and tables. The effects of different physical parameters on the flow and heat transfer characteristics are discussed in detail.

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Abbreviations

PDEs Partial Differential Equations

MHD Magnetohydrodynamics

ODEs Ordinary Differential Equations

Symbols

B	magnetic field
C	nanoparticles concentration
C_∞	free stream concentration
C_f	skin friction coefficient
D	mass diffusivity
D_a	Darcy's number
Ec	Eckert number
k	porosity
$K(x)$	variable reaction rate
L	reference length
Nu	Nusselt number
P	permeability parameter
p	pressure
Pr	Prandtl number
q_r	radiative heat flux
R	radiation parameter
Re	Reynolds number
S	suction/blowing parameter
Sc	Schmidt number
T	temperature
U	free stream velocity
(u, v)	velocity components
$v_w(x)$	suction/blowing velocity

Greek Symbols

ρ	fluid density
μ	viscosity
ν	kinematic viscosity
α	thermal diffusivity
σ_s	electrical conductivity
ϕ	solid volume fraction
η	dimensionless similarity variable
ψ	stream function
α_{nf}	thermal diffusivity
θ	dimensionless temperature
δ	velocity slip parameter
β	thermal slip parameter
λ	chemical reaction parameter
σ_f	electrical conductivity
ϵ	thermal conductivity parameter

Chapter 1

Introduction

The boundary layer is the region adjacent to the surface of an object around which the fluid is flowing. Flow of the boundary layer play an important role in fluid mechanics and has been extensively studied in the literature. Prandtl [1] was the first who presented the concept of boundary layer. Makinde and Motsumi [2] studied the boundary layer flow of nanofluids over a continuously moving surface. Ibrahim and Makinde [3] investigated the thermal boundary layer flow with effects of double stratification over a vertical sheet. In addition to that in a series of article, Makinde [4, 5] studied the boundary layer flow of nanofluids passing over a flat plate. They further analysed the impacts of viscous dissipation and Newtonian heating for various types of geometry including permeable surface. The main purpose of their study is the computation of mathematical models of nanofluid over steady/unsteady stretching sheet. The boundary layer flow over a moving surface have a number of applications in engineering and industrial fields. Sakiadis *et al.* [6, 7] presented the concept of the boundary layer flow through a stretching surface.

The small solid particle is known as nanoparticle, these nanoparticles ranges from 1-100 nanometers in size. The nanofluid is defined as the homogenous mixture of the base fluid and nanoparticle. In 1995, Choi [8] in his pioneering work introduced the terminology of nanofluids. Since then an extensive research is carried out on this topic by many researchers due to its potential industrial applications. In the

current progress in the field of science and technology, the nanotechnology has a wide range of applications in differen fields. In the last couple of decades, the development in nanotechnology is exponentially increasing. The effects of nanoparticle migration on force convection of nanofluid in a channel are studied for alumina [9] critical analysis of thermophysical characteristics of nanofluids are investigated by Khanafer and Vafai [10]. Moreover, *Copper*-water nanofluid in a porous and moving plates are discussed by Sureshkumar and Muthtamilselvan [11]. Nagarajana and Akbarb [12] discussed the heat transfer enhancement of *Copper*-water nanofluid in a porous square enclosure driven by moving flat plate. Study for the driven cavity flow with different characteristics of heat transfer in nanofluid can be found in [13–16]. Ho *et al.* [17] provided numerical solutions using finite volume and finite difference methods for convective heat transfer in nanofluid. The numerical simulations are performed for nanofluid in a square enclosure. They discussed the effects of viscosity and thermal conductivity in nanofluid.
The heat transfer in the boundary layer flow of nanofluid has been an interesting topic for researchers. Masuda *et al.* [18] found that nanofluids are enhancing thermal conductivity, they further noted the potential applications of nanofluid in nuclear technology. They studied the characteristics of nanofluid by dispersing ultra-fine particles in base fluid with varying viscosity and thermal conductivity. Boungiorno [19] developed a model with analytic solution for convective heat transfer in a Brownainan diffusion of nanofluid. He observed the effects of diffusion and thermophoresis in nanofluid. A cavity flow with heat transfer and entropy generation are analysed by Hiemenz *et al.* [20].

MHD stand for magneto-hydrodynamics or hydromagnetics. It is the study of magnetic properties of electrically conducting fluids. Magneto-hydrodynamics (MHD) of an electrically conducting fluid has wide range of applications in the fields of geophysics, astrophysics, engineering and many other areas described by Khaleque and Samad [21]. Due to their wide range of applications many researchers tend to apply MHD flow into their problem. Ibrahim *et al.* [22] discussed the effects of time dependent MHD with viscous dissipation and radiation. Zhang *et al.* [23] numerically studied the effects of radiation, heat flux and chemical reaction on heat transfer of nanofluid through porous media. They further considered the influence of magnetohydrodynamics in the thermal boundary layer flow of viscous fluid. Omowaye and Animasaum [24] discussed the MHD upper convected Maxwell fluid over melting surface subject to thermal stratification with variable thermo-physical properties. Mansur *et al.* [25] studied MHD stagnation point flow of a nanofluid over a stretching sheet with suction. Ashikin *et al.* [26] discussed the issue of heat transfer for the MHD Maxwell nanofluid flow past a vertical permeable sheet. MHD slip flow with convective boundary conditions over stretching surface for carbon nanotubes was analysed by Rizwan *et al.* [27].

In the process of heat transfer, viscous dissipations mean heating up the fluid via different source. In short, in this mechanism the viscosity of the fluid will absorb heat from the kinetic energy and transform it into internal energy of the system. Moreover, the process in which the electric current through a conductor produce heat is known as Joule heating. Eldahab *et al.* [28] studied the viscous dissipation and Joule heating effects on MHD-free convection from a vertical plate. Viscous dissipations play an important role in the natural convection in various devices. Viscous dissipation effects and the effects of Joule heating on thermal boundary layer flow are studied in [29, 30]. There, an electrically conducting fluid flow due to free convection studied from a vertical plate. The analysis of thermal radiation on MHD boundary layer with different geometries, Joule heating and viscous dissipations are discussed for Newtonian and non-Newtonian fluids. Alam *et al.* [31] studied the Joule heating and viscous dissipation on an inclined isothermal permeable surface with MHD and thermophoresis. Hayat *et al.* [32] discussed the incompressible unsteady 3-dimensional MHD flow over stretching sheet with viscous dissipation and Joule heating.

1.1 Thesis contributions

The main contribution of present work is to perform numerical analysis of heat and mass transfer in MHD flow of two dimensional nanofluid over a flat surface. The fluid is electrically conducting with a magnetic field B_0. The system of non-linear PDEs are transformed into the system of ODEs and solved using shooting technique. The numerical results are verified using bvp4c built-in function of Matlab, which are in excellent agreement. The influence of different parameters on the velocity, temperature and concentration profiles are shown graphically and discussed in detail.

1.2 Outline of the dissertation

This thesis is further partitioned into following chapters:

Chapter 2 contains useful definitions, basic concepts and laws of conservations which are essential for understanding the work in upcoming chapters.

Chapter 3 describes the numerical investigation of heat and mass transfer in nanoparticles of an electrically conducting boundary layer flow of two dimensional incompressible fluid. This chapter is the review work of Zhang *et al.* [23]

Chapter 4 is extended for numerical analysis of heat and mass transfer in an electrically conducting fluid through permeable surface with radiation, viscous dissipation and Joule heating. This chapter consists of general introduction of the topic, mathematical formulation for the extended model, development of numerical solution and results. The nonlinear coupled system of PDEs are transformed into system of ODEs. The shooting method is utilized to solve the mathematical model. Numerical values are calculated and analysed for different ranges of physical parameters.

Chapter 5 complete this study with summary and concluding remarks.

Chapter 2

Basic concepts and definitions

This chapter includes some basic laws and terminologies of the fluids. Which will be essential for understanding and continuation of the preceding chapters regarding this work [33]

2.1 Basics definition

Definition 2.1.1 (Fluid)

"It is a physical substance that deforms continuously under the influence of an applied shear stress. Fluid yields easily to shear stress and repeatedly deforms its shape as long as the shear stress acts. Fluid has no fixed shape and conforms to the shape of a container in which it is placed."

Definition 2.1.2. (Fluid mechanics)

"The branch of mechanics, which deals with the properties of fluids either in a rest or a moving states, respectively. Fluid mechanics is mainly divided into two categories. i.e, fluid dynamics and fluid statics. "

Definition 2.1.3. (Fluid dynamics)

"It is the branch of fluid mechanics, in which we study the movement of the fluids."

Definition 2.1.4. (Fluid statics)

"It is the branch of fluid mechanics, which depict the properties of the fluid at a rest. "

Definition 2.1.5. (Viscosity)

"Viscosity of a fluid is defined as the measure of resistance to steady distortion by shear/tensile stress. A notation used for viscosity is μ and its mathematical expression is,

$$Viscosity = \mu = \frac{\text{shear stress}}{\text{rate of shear strain}}$$

where μ is called the coefficient of absolute viscosity/dynamics viscosity or simple viscosity. The dimension of viscosity is $[\frac{M}{LT}]$."

Definition 2.1.6. (Kinematic viscosity)

" It is the ratio of dynamic viscosity to the density of the fluid is known as kinematic viscosity and it is denoted by ν. Mathematically it is defined by $\nu = \frac{\mu}{\rho}$ where ρ is the density of the fluid and dimension of kinematic viscosity is given by $[\frac{L^2}{T}]$."

2.2 Classification of fluids

Definition 2.2.1. (Compressible and incompressible fluids)

"A flow is incompressible in which the density remains constant within the fluid.

Therefore, the volume of every portion of the fluid remain unchanged. Mathematically,

$$\frac{D\rho}{Dt} = 0,$$

where ρ is called fluid density and $\frac{D}{Dt}$ is the material derivative given by

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla. \tag{2.1}$$

In above equation, $\mathbf{V}$ denotes the velocity of the flow and ∇ is the differential operator. In Cartesian coordinate system ∇ is given as

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}.$$

If the density of the fluids varies then the fluid is known compressible fluid. Mathematically,

$$\rho = \rho(x, y, z, t)\text{''}$$

Definition 2.2.2. (Newtonian and non-Newtonian fluids)
"The fluids, which fulfill Newton's law of viscosity are known as Newtonian fluid. Mathematically,

$$\tau_{yx} = \mu \left(\frac{du}{dx} \right), \tag{2.2}$$

where τ_{yx} is the shear stress and μ is called the constant of proportionality. The most common example of Newtonian fluids is water. Those fluids, which do not obey the Newton's law of viscosity are known as non-Newtonian fluids. Mathematically,

$$\tau_{yx} = k \left(\frac{du}{dx} \right)^n, \tag{2.3}$$

where $n \neq 1$ is flow behaviour index. For $n = 1$ with $k = \mu$, the above equation reduce to the Newton's law of viscosity. Paints, blood, biological fluids and polymer melts etc, are good examples of non-Newtonian fluids."

Definition 2.2.3. (Real fluid)
"The fluids, which have non-zero viscosity are called real fluids. These fluids may

be compressible or incompressible. It depends upon the relationship between the shear stress and rate of shear strain."

Definition 2.2.4. (Nanofluid)

"The nanofluid is defined as the homogenous mixture of the base fluid and nanoparticles."

Definition 2.2.5. (Flow)

"It is the deformation of material under the influence of different forces. If this deformation increase continuously without any limit, then this process is known as flow."

2.3 Types of flows

Definition 2.3.2. (Uniform and non-uniform flows)

"The flow, in which magnitude as well as the direction of the fluid velocity is the same at each points of the flow. In case of non-uniform flow, the velocity is not same at each point of the flow at any given instant."

Definition 2.3.3. (Steady and unsteady flows)

"The flow properties does not change with respect to time is known as steady flow. Mathematically,

$$\frac{\partial \zeta}{\partial t} = 0,$$

where ζ is any fluid property. In case of unsteady flow, fluid property change with respect to time. It is known as time-dependent flow."

Definition 2.3.4. (Laminar and turbulent flows)

"A flow in which the particles of the fluid have specific path and individual particle does not intersect each other is known as laminar flow. In such flow, the particles move along well-defined path. The flow in which fluid particles have no specific paths and they move randomly is called Turbulent flow."

Definition 2.3.5. (Internal flow)

"Internal flow are those where fluids flow through confined spaces, e.g, flow in pipe."

Definition 2.3.6. (External flow)

"The flow which is not confined by the solid surface, is known as external flow. The flow of water in the river is an example of the external flow. "

2.4 Mechanism of heat transfer

"Due to temperature difference energy transfer is called heat transfer. Heat transfer occur through different mechanism."

Definition 2.4.1. (Conduction)

"Due to collision of molecules in contact form, heat is transferred from one objects to another objects is called conduction. Such types of heat transfer occurs in the solid."

Definition 2.4.2. (Convection)

"It is a mechanism in which heat transfer occurs due to the motion of molecules within the fluid such as air and water. A mathematical expression for convection phenomena is

$$q = hA\left(T_s - T_\infty\right), \tag{2.4}$$

where h, A, T_s and T_∞ denote the heat transfer coeffcient, the area, the temperature of the surface and the temperature away from the surface respectively. Further, it is subdivided into the following three categories."

Definition 2.4.3. (Natural convection)

"It is the process, in which heat transfer is caused by the temperature differences. It effects the density of the fluids and the fluid motion is not developed by an external source. It occurs only in the presence of gravitational force and also known as free convection."

Definition 2.4.4. (Forced convection)

"It is a type of heat transfer in which an external source is used to produce motion of the fluid. e.g. fan or a pump."

Definition 2.4.5. (Mixed convection)

"It is the combination of both forced convection and natural convection and both occur simultaneously."

Definition 2.4.6. (Radiation)

"In radiation process, heat is transferred through electromagnetic rays and waves. It takes place in liquids and gasses. An example of radiation would be atmosphere, the atmosphere is heated by the radiation of the sun."

Definition 2.4.7. (Thermal conductivity)

"It is the property of a substance which measures the ability to transfer heat. Fourier's law of conduction which relates the flow rate of heat by conduction to the temperature gradient is

$$\frac{dQ}{dt} = -kA\frac{dT}{dx},$$

where A, k, $\frac{dT}{dx}$ and $\frac{dQ}{dt}$ are the area, the thermal conductivity, the temperature and the rate of heat transfer, respectively. The SI unit of thermal conductivity is $\frac{Kgm}{s^3}$ and the dimension of thermal conductivity is $[\frac{ML}{T^3}]$."

Definition 2.4.8. (Thermal diffusivity)

"The ratio of the unsteady heat conduction κ, of a substance to the product of specific heat capacity C_p and density ρ is called thermal diffusivity. It quantify the ability of a substance to transfer heat rather to store it. Mathematically, it can be written as"

$$\alpha = \frac{\kappa}{\rho C_p},$$

The unit and dimension of thermal diffusivity in SI system are m^2s^{-1} and $[LT^{-1}]$ respectively."

Definition 2.4.9. (Joule heating)

"It is the procedure in which heat is generated by passing an electric current through a conductor. It is also known as Ohmic heating and resistive heating."

Definition 2.4.10. (Viscous dissipation)

"In viscous fluid flow, the viscosity of the fluid will take energy from the kinetic energy and transform it into internal energy of the fluids. This process is called viscous dissipation."

2.5 Dimensionless numbers

Definition 2.5.1. (Reynolds number (Re)**)**

"It is the ratio of inertial forces to viscous forces. The behaviour of the different kinds of flow will be identify like laminar or turbulent flow. Mathematically,"

$$Re = \frac{\frac{\rho U^2}{L}}{\frac{\mu U}{L^2}} \quad \Longrightarrow \quad Re = \frac{LU}{\nu},$$

where U denotes the free stream velocity, L is the characteristics length and μ stands for kinematic viscosity."

Definition 2.5.2. (Prandtl number (P_r)**)**

"The ratio of kinematic diffusivity to heat diffusivity is said to be Prandtl number. It is denoted by P_r Mathematically it can be written as

$$P_r = \frac{\nu}{\alpha} \quad \Longrightarrow \quad \frac{\mu/\rho}{k/c_p} \quad \Longrightarrow \quad \frac{\mu c_p}{\rho k},$$

where μ and α denote the momentum diffusivity or kinetic diffusivity and thermal diffusivity respectively. c_p denotes the specific heat and κ stands for thermal conductivity."

Definition 2.5.3. (Nusselt number (Nu)**)**

"It is the relationship between the convective to the conductive heat transfer through the boundary of the surface. It is a dimensionless number which was first introduced by the German mathematician Nusselt. Mathematically, it is defined as:

$$Nu = \frac{hL}{\kappa},$$

where h stands for convective heat transfer, L stands for characteristics length and κ stands for thermal conductivity."

Definition 2.5.4. (Schmidt number (Sc)**)**

"Schmidt number (Sc) is a dimensionless number characterized as the proportion of momentum diffusivity (viscosity) to mass diffusivity and is utilized to describe fluid flows in which there are simultaneous momentum and mass diffusion convection."

Definition 2.5.5. (Eckert number (Ec))
"It is the proportion of the kinetic energy dissipated in the flow to the thermal energy conducted into or away from the fluid."

2.6 Basic governing equations

2.6.1 Law of conservation of mass

"The law of conservation of mass [34] states that, mass can neither be created nor destroyed inside a control volume. Mathematically,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0, \tag{2.5}$$

where ρ denotes fluid density. For incompressible fluids, Eq. (2.5) reduces to

$$\nabla \cdot \mathbf{V} = 0, \tag{2.6}$$

with V as velocity vector."

2.6.2 Law of conservation of momentum

"Law of conservation of momentum [34] is state that, the net force $\mathbf{F}$ acting on the particles is equal to the time rate of change of linear momentum. Equation of linear momentum can be obtained from the Newton's second law of motion. In vector notation this law is given by

$$\frac{\partial}{\partial t} (\rho \mathbf{V}) + \mathbf{V} \cdot \nabla (\rho \mathbf{v}) - \nabla \cdot \tau - \rho g = 0. \tag{2.7}$$

The Cauchy stress tensor is written as:

$$\tau = -p\mathbf{I} + \mathbf{S}, \tag{2.8}$$

The momentum equation becomes

$$\rho\left(\frac{\partial \mathbf{V}}{\partial t}+\mathbf{V}\cdot(\nabla \mathbf{V})\right)=\nabla\cdot(-p\mathbf{I}+\mathbf{S})+\rho\mathbf{g}, \tag{2.9}$$

where g is the body force, p the pressure, $\mathbf{S}$ denotes the extra stress tensor and μ the dynamic viscosity. In the tensor form the Cauchy stress τ is written as

$$\tau=\begin{bmatrix}\sigma_{xx} & \tau_{xy} & \tau_{xz}\\ \tau_{yx} & \sigma_{yy} & \tau_{yz}\\ \tau_{zx} & \tau_{zy} & \sigma_{zz}\end{bmatrix}. \tag{2.10}$$

Eq. (2.9) is a vector equation and can be decomposed further into three scalar components by taking the scalar product with the basis vectors of an appropriate orthogonal coordinate system. By setting $\mathbf{g}=-g\nabla z$, where z is the distance from an arbitrary reference elevation in the direction of gravity, Eq. (2.9) can also be expressed as

$$\rho\frac{D\mathbf{V}}{Dt}=\rho\left(\frac{\partial \mathbf{V}}{\partial t}+\mathbf{V}\cdot(\nabla \mathbf{V})\right)=\nabla\cdot(-p\mathbf{I}+\mathbf{S})+\nabla\left(-\rho g z\right), \tag{2.11}$$

where $\frac{D}{Dt}$ represent the substantial derivative. In Cartesian coordinates, by using the velocity field $\mathbf{V}=\left[u\left(x,y,t\right),v\left(x,y,t\right),w\left(x,y,t\right)\right]$, the momentum equation then becomes

$$\begin{aligned}
\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)&=\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}+\frac{\partial \tau_{xz}}{\partial z}+\rho b_x,\\
\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)&=\frac{\partial \tau_{yx}}{\partial x}+\frac{\partial \tau_{yy}}{\partial y}+\frac{\partial \tau_{yz}}{\partial z}+\rho b_y,\\
\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)&=\frac{\partial \tau_{zx}}{\partial x}+\frac{\partial \tau_{zy}}{\partial y}+\frac{\partial \tau_{zz}}{\partial z}+\rho b_z.
\end{aligned} \tag{2.12}$$

In which, $\tau_{xx}, \tau_{xy}, \tau_{xz}, \tau_{yx}, \tau_{yy}, \tau_{yz}, \tau_{zx}, \tau_{zy}$ and τ_{zz} denote the components of Cauchy stress tensors and b_x, b_y and b_z denote the components of the body force.”

2.6.3 Energy equation

"Law of conservation of energy [34] states that the total energy of the system remains conserved, but it can be transferred from one form to another form. Mathematically it can be written as,

$$\rho\left[\frac{\partial \mathbf{U}}{\partial t}+\mathbf{V}\cdot\nabla\mathbf{U}\right]=\left[\tau:\nabla\mathbf{V}+p\nabla\cdot\mathbf{V}\right]+\nabla\left(k\nabla T\right)\pm\hat{H}r, \tag{2.13}$$

where $\hat{H}r$ is the heat of reaction and $\mathbf{U}$ is the internal energy per unit mass. By using the definition of the internal energy, $d\mathbf{U}\equiv C_v dT$, Eq. (2.13) becomes

$$\rho C_v\left[\frac{\partial T}{\partial t}+\mathbf{V}\cdot\nabla T\right]=\left[\tau:\nabla\mathbf{V}+p\nabla\cdot\mathbf{V}\right]+\nabla\left(k\nabla T\right)\pm\hat{H}r. \tag{2.14}$$

For heat conduction in solids, i.e., when $\mathbf{V}=0$, $\nabla\cdot\mathbf{V}=0$, and $C_v=C$, the resulting equation is written in the form of

$$\rho C\frac{\partial T}{\partial t}=\nabla\left(k\nabla T\right)\pm\hat{H}r." \tag{2.15}$$

2.7 Solution methodology

"Shooting method [35] used to solved the higher order nonlinear ordinary differential equations. To implement this technique, first convert the higher order ODEs to the system of first order ODEs. In the shooting method, first we assume the missing initial conditions and the differential equations are then integrated numerically through Runge-Kutta method as an initial value problem. The accuracy of the assumed missing initial condition is then checked by comparing the calculated values of the dependent variables at the terminal point with their given value there. If the boundary conditions are not fulfilled upto the required accuracy, with the new set of initial conditions, which are modified by Newton's method. The method is repeated again until the required accuracy is achieved. To explain the shooting method, we consider a general second order boundary value problem,

$$y''(x) \;=\; f\left(x, y, y'(x)\right) \tag{2.16}$$

subject to the boundary conditions

$$y\left(0\right) = 0, \quad y\left(L\right) = A. \tag{2.17}$$

By denoting y by y_1 and y_1' by y_2, Eq. (2.16) can be written in the form of following system of first order equations.

$$\left.\begin{aligned} y_1' &= y_2, & y_1(0) &= 0, \\ y_2' &= f\left(x, y_1, y_2\right), & y_1(L) &= A. \end{aligned}\right\} \tag{2.18}$$

Denote the missing initial condition $y_2(0)$ by s, to have

$$\left.\begin{aligned} y_1' &= y_2, & y_1(0) &= 0, \\ y_2' &= f\left(x, y_1, y_2\right), & y_2(0) &= s. \end{aligned}\right\} \tag{2.19}$$

Now the problem is to find s such that the solution of the IVP (2.19) satisfies the boundary condition $y(L) = A$. In other words, if the solutions of the initial value problem (2.19) are denoted by $y_1(x,s)$ and $y_2(x,s)$, one should search for that value of s which is an approximate root the the equation.

$$y_1(L,s) - A = \phi(s) = 0. \tag{2.20}$$

To find an approximate root of the Eq. (2.20) by the Newton's method, the iteration formula is given by

$$s_{n+1} = s_n - \frac{\phi(s_n)}{d\phi(s_n)/ds}, \tag{2.21}$$

or

$$s_{n+1} = s_n - \frac{y_1(L,s_n) - A}{dy_1(L,s_n)/ds}. \tag{2.22}$$

To find the derivative of y_1 with respect of s, differentiate (2.19) with respect to s. For simplification, use the following notations

$$\frac{dy_1}{ds} = y_3, \quad \frac{dy_2}{ds} = y_4. \tag{2.23}$$

This process results in the following IVP.

$$\left.\begin{aligned} y_3' &= y_4, & y_3(0) &= 0,\\ y_4' &= \frac{\partial f}{\partial y_1}y_3 + \frac{\partial f}{\partial y_2}y_4, & y_4(0) &= 1. \end{aligned}\right\} \tag{2.24}$$

Now, solving the IVP Eq. ((2.24)), the value of y_3 at L can be computed. This value is actually the derivative of y_1 with respect to s computed at L. Setting the value of $y_3(L,s)$ in Eq. (2.22), the modified value of s can be achieved. This new value of s is used to solve the Eq. (2.19) and the process is repeated until the value of s is within a described degree of accuracy."

Chapter 3

Chemical reaction and MHD flow of nanofluids through porous media

This chapter included the MHD flow, radiative heat and mass transfer of two dimensional laminar flow of an incompressible viscous fluid over a permeable plate in a porous media. The effects of heat flux and chemical reactions are taken under consideration. The modeled boundary layer equations for momentum, energy and mass transfer are obtained using boundary layer appoximations. Using similarity transformations we convert the nonlinear partial differential equations into system of ordinary differential equations (ODEs). Numerical scheme based on the shooting methods is developed for system of ODEs. The effects of different physical parameters are discussed through graphs and tables. Numerical results are further computed for Nusselt number and Sherwood number.

3.1 Mathematical model

We consider the steady, viscous and incompressible nanofluid flow. The flow is two dimensional passed over a permeable plate in a porous media. In addition, a magnetic field is applied to the plate as shown in Figure 3.1. The temperature at surface is T_w, U denotes velocity in the free stream condition and the ambient temperature is T_∞ respectively. The increasing temperature along the sheet is T_0. The following system of equations are incorporated for mathematical model Zhang *et al.* [23]

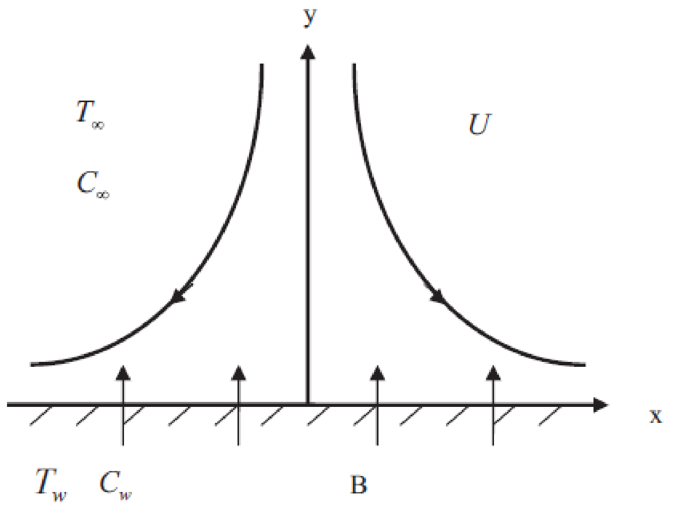

FIGURE 3.1: A schematic diagram of flow.

The following system of equations are incorporated for mathematical model Zhang [23].

Continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{3.1}$$

Momentum equation:

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} = U\frac{dU}{dx} - \frac{\mu_{nf}}{\rho_{nf}}\frac{\partial^2 u}{\partial y^2} - \frac{\mu_{nf}}{\rho_{nf}k}(u - U) - \frac{\sigma_{nf}}{\rho_{nf}}B^2(u - U), \tag{3.2}$$

Energy equation:

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha_{nf}\frac{\partial^2 T}{\partial y^2} - \frac{1}{\rho_{nf}}\frac{\partial q_r}{\partial y}, \tag{3.3}$$

Mass equation:

$$u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}=D\frac{\partial^2 C}{\partial y^2}-K(C-C_\infty). \tag{3.4}$$

The associated boundary conditions:

$$\left.\begin{aligned}&u=0,\ v=v_w,\ T=T_w=T_\infty+T_0\,e^{\frac{x}{2l}},\ C=C_\infty+C_0\,e^{\frac{x}{2L}}\quad \text{at}\quad y=0,\\ &u\to U=a\,e^{\frac{x}{L}},\ T\to T_\infty,\ C\to C_\infty\quad \text{at}\quad y\to\infty,\end{aligned}\right\} \tag{3.5}$$

where u is the velocity components along x direction and v represent velocity components along the y directions respectively. T is the temperature of the nanofluids, p is the fluid pressure, σ_s is the electrical conductivity of the base fluid and σ_f denotes the electrical conductivity of nanofluid respectively. In above equations, C is nanoparticles concentration, C_∞ shows the free stream concentration, D is known as the mass diffusivity, U represents the velocity, which is far away from the surface, a is any constant number, μ_{nf} is called the viscosity of nanofluids, ρ_{nf} is the density of nanofluids, σ_{nf} denotes the thermal diffusivity and K is called the variable reaction rate. Mathematically,

$$K(x)=k_0\,e^{\frac{x}{L}},$$

where L is called the reference length and k_0 is the constant. The suction velocity is given by:

$$v_w(x)=v_0\,e^{\frac{x}{2L}},$$

here v_0 a constant. In the solution of the problem, first of all we converted the system of Eqs. (3.1) - (3.4) along with the boundary conditions (3.5) into dimensionless forms using suitable similarity transformation. The similarity variable is defined as:

$$\eta=y\sqrt{\frac{a}{2v_fL}}e^{\frac{x}{2L}}. \tag{3.6}$$

The temperature dimensionless function θ, g and dimensionless stream function ψ are given in the form of

$$\psi=y\sqrt{2aLv_f}f(\eta)e^{\frac{x}{2L}},\quad \theta(\eta)=\frac{T-T_\infty}{T_w-T_\infty},\quad g(\eta)=\frac{C-C_\infty}{C_w-C_\infty}. \tag{3.7}$$

The stream function $\psi = \psi(x, y)$ is identically satisfying continuity equation. Mathematically,

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}. \tag{3.8}$$

Using similarity transformation from Eqs. (3.6)-(3.7) in momentum Eq. (3.2), energy Eq. (3.3) and concentration Eq. (3.4) along the boundary conditions (3.5) we get the following ODEs.

Nanoparticles	$\rho(kg/m^3)$	$Cp(J/kgK)$	$k(W/mK)$	$\sigma(S/m)$
Fluid phase (H_2O)	4179	997.1	0.613	5.5×10^{-6}
Silver (Ag)	10.50	235	429	63.01×10^{-6}
Copper (Cu)	8933	385	400	59.6×10^{-6}
Aluminium oxide (Al_2O_3)	3970	765	40	35×10^{-6}

TABLE 3.1: Thermo-physical properties of H_2O and nanoparticles.

$$\frac{1}{\phi_1} f''' + ff'' + 2(1 - f'^2) + \left(\frac{1}{\phi_1} P + \frac{\phi_4}{\phi_2} M \right) (1 - f') = 0, \tag{3.9}$$

$$\left[\frac{K_{nf}}{\phi_3 K_f} + \frac{R}{\phi_3} \right] \theta'' + P_r \left(f\theta' - f'\theta \right) = 0, \tag{3.10}$$

$$g'' + Sc(fg' - f'g - \lambda g) = 0. \tag{3.11}$$

The transformed boundary conditions are:

$$\left.\begin{aligned} &f = S, \ f' = 0, \ \theta = 1, \ g = 1, \quad \text{at} \quad \eta = 0, \\ &f' = 1, \ \theta = 0, \ g = 0, \quad \text{at} \quad \eta \to \infty, \end{aligned}\right\} \tag{3.12}$$

The parameters are:

$$\left.\begin{aligned} &P = \frac{2Lv_f}{ak_0}, \ M = \frac{2\sigma_f B_0^2 L}{a\rho_f}, \ R = \frac{16\sigma_1 T_\infty^3}{3kk_f}, \\ &S = \frac{v_0}{\sqrt{\frac{av_f}{2L}}}, \ Sc = \frac{v_f}{D}, \ \lambda = \frac{K_0}{a}, \ P_r = \frac{v_f}{\alpha_f}, \\ &\phi_1 = (1 - \phi)^{2.5} \left[(1 - \phi) + \phi \frac{\rho_s}{\rho_f} \right], \quad \phi_2 = (1 - \phi) + \phi \frac{\rho_s}{\rho_f}, \\ &\phi_3 = (1 - \phi) + \phi \frac{(\rho C_p)_s}{(\rho C_p)_f}, \quad \phi_4 = (1 - \phi) + \phi \frac{\sigma_s}{\sigma_f}, \end{aligned}\right\} \tag{3.13}$$

In the above equations P_r denotes the Prandtl number, p is permeability parameter, M and R represent the magnetic parameter and radiation parameter respectively. S is known as suction parameter, Sc represents the Schmidt number and λ is the notation of chemical reaction coefficient. The quantities of interest in this studies are the Nusselt number Nu_x and the Sherwood number Sh_w respectively. which are defined as:

$$Nu_x = \frac{(xq_w)}{(K_f(T_w - T_\infty)}, \quad Sh_x = \frac{(xp_m)}{(D(C_w - C_\infty)}, \tag{3.14}$$

q_w is notation of the heat flux from the sheet and P_w denotes the mass flux. Mathematically,

$$q_w = K_{nf}\frac{\partial T}{\partial y}\Big|_{y=0}, \quad p_w = D\frac{\partial C}{\partial y}\Big|_{y=0}. \tag{3.15}$$

$$Nu_x Re_x^{\frac{-1}{2}} \frac{K_f}{K_{nf}} = -\theta''(0), \quad Sh_x Re_x^{\frac{-1}{2}} = -g'(0). \tag{3.16}$$

In Eq. (3.16), Re_x represents the local Reynolds numbers. It is written as

$$Re_x = \frac{Ux^2}{2L\upsilon_f}. \tag{3.17}$$

3.2 Method of the solution

The numerical solution of the following Eqs. (3.9)-(3.11) with corresponding boundary conditions (3.12) can be obtained by shooting technique.
To use the shooting method, first we convert these Eqs. (3.9)-(3.11) into a system of first order ODEs. For these purpose, we denote f by y_1, f' by y_2, f'' by y_3, θ by y_4 and g by y_6 . The coupled nonlinear momentum, heat and concentration equations are converted into system of seven first order ODEs as into the following

form:

$$
\begin{aligned}
y_1' &= y_2, \\
y_2' &= y_3, \\
y_3' &= \phi_1\left(2y_2^2 - y_1 y_3 - 2\right) + \left(P + \frac{\phi_1}{\phi_2}\phi_4 M\right)(y_2 - 1), \\
y_4' &= y_5, \\
y_5' &= -\frac{P_r \phi_3}{k_{nf}/k_f + R}(y_1 y_5 - y_2 y_4), \\
y_6' &= y_7, \\
y_7' &= -Sc\,(y_1 y_7 - y_2 y_6 - \lambda y_6).
\end{aligned}
$$

The boundary conditions make the following forms

$$f_1(0) = S, \quad f_2(0) = 0, \quad f_3(0) = u,, \quad f_4(0) = 1, \quad f_5(0) = v, \quad f_6(0) = 1, \quad f_7(0) = w$$

To solve the above system of eqs. numerically, we replace the domain $[0,\infty)$ by the bounded domain $[0, \eta_{max}]$, where η_{max} is some suitable real number. The stoping criteria is give by

$$max|y_2(7)| - 1, |y_4(7)|, |y_6(7)| < \epsilon,$$

where $\epsilon > 0$ represents any positive number. In this thesis the numerical results are achieved with $\epsilon = 10^{-6}$

TABLE 3.2: Numerical results of Nusselt number -$\theta'(0)$ for various values of ϕ and R.

		$Cu - H_2O$			$Al_2O_3 - H_2O$			$Ag - H_2O$		
ϕ	R	[23]	Shooting	bvp4c	[23]	Shooting	bvp4c	[23]	Shooting	bvp4c
0	1	2.9174	2.9430	2.9430	2.9174	2.9430	2.9430	2.9174	2.9430	2.9430
	2	2.0294	2.0510	2.0510	2.0294	2.0510	2.0510	2.0294	2.0510	2.0510
	4	1.3733	1.3906	1.3906	1.3733	1.3906	1.3906	1.3733	1.3906	1.3906
0.1	1	2.6876	2.7048	2.7048	2.6269	2.6487	2.6487	2.4348	2.4572	2.4572
	2	1.9518	1.9600	1.9600	1.9007	1.9189	1.9189	1.7688	1.7873	1.7873
	4	1.3597	1.3709	1.3709	1.3211	1.3357	1.3357	1.2351	1.2499	1.2499
0.2	1	2.4201	2.4344	2.4344	2.3358	2.3560	2.3560	2.0037	2.0243	2.0243
	2	1.8277	1.8395	1.8395	1.7551	1.7721	1.7721	2.0037	1.5360	1.5360
	4	1.3101	1.3195	1.3195	1.2542	1.2680	1.2680	1.0966	1.1107	1.1107

Table 3.3: Numerical results of Sherwood number -$g'(0)$ for the values of ϕ and Sc.

		$Cu-H_2O$			$Al_2O_3-H_2O$			$Ag-H_2O$		
ϕ	Sc	[23]	Shooting	bvp4c	[23]	Shooting	bvp4c	[23]	Shooting	bvp4c
0	0.25	0.8438	0.8607	0.8607	0.8438	0.8606	0.8606	0.8438	0.8607	0.8607
	0.5	1.2530	1.2627	1.2627	1.2530	1.2627	1.2627	1.2530	1.2617	1.2617
	1	1.9172	1.9288	1.9288	1.9172	1.9288	1.9288	1.9172	1.9288	1.9288
0.1	0.25	0.8428	0.8736	0.8736	0.8472	0.8628	0.8628	0.8397	0.8548	0.8548
	0.5	1.2776	1.9598	1.9598	1.2580	1.9333	1.9333	1.2453	1.9158	1.9158
	1	1.9525	1.2836	1.2836	1.9241	1.2661	1.2661	1.9055	1.2535	1.2535
0.2	0.25	0.8525	0.8768	0.8768	0.8464	0.8631	0.8631	1.8340	0.8484	0.8484
	0.5	1.2838	1.2888	1.2888	1.2561	1.2636	1.2636	1.2359	1.2438	1.2438
	1	1.9616	1.9678	1.9678	1.9211	1.9312	1.9312	1.0966	1.8023	1.8023

Table 3.2 shows the effects of different parameters on Nusselt number. Both the physical parameters R (Radiation parameter), volume fraction coefficient ϕ and Nu_x (local Nusselt number) are of great interest for engineers. In Table 3.2, the numerical analysis of different physical parameters such as ϕ, R and their effects on Nusselt number under discussion is displayed. Furthermore, our results are compared with existing literature [23], which show excellent agreement of numerical results. It also investigated that by increasing radiation parameter R, the Nusselt number increases. Moreover, when we increase the volume fraction ϕ, the Nusselt number also decreases. In Table 3.3, we discussed the effects of Schmidt number Sc and volume fraction coefficient ϕ on Sherwood number. We compare the results obtained by shooting method with Matlab built-in function bvp4c solver and both found in excellent agreement. The results are compared with existing work of Zhang [23]. Which are in good agreement. It is investigated that an increase in the values of volume fraction coefficient ϕ, it also increase the Sherwood number for nanoparticles Cu, Al_2O_3 and Ag. Moreover, when we increase the Schmidt number Sc, Sherwood number also increases.

3.3 Graphical representation

The effects of different parameters on the velocity, temperature and concentration profiles are discussed through graphs. Figure 3.2 shows the effects of volume

fraction ϕ w.r.t velocity profile for different physical parameters. By increasing the values of ϕ, velocity profile as well as the thermal conductivity of nanofluid increases. It also increases in the thickness of boundary layer. Figure 3.3 indicates the effects of volume fraction ϕ on temperature profile. By increasing the values of ϕ, temperature profile increases and it also enhance the thermal conductivity. Figure 3.4 shows the effect of suction/blowing parameter S on the velocity profile. It is inferred that when we increment the suction parameter, the velocity profile is increasing significantly. Where the fluid velocity decreases due to blowing, which are depicted in the graph. The suction $(S > 0)$ indicates decreasing in the boundary layer thickness and increasing in the fluid velocity. When $(S = 0)$, the result suggests that the case is non porous plate and when $(S < 0)$, it is noted for blowing. The outcome demonstrates that the temperature profile decreases when we increase in the suction parameter, while it increases due to increase in blowing. Figure 3.5 indicates that when we increase in the suction parameter, the temperature profile is increasing whereas, it increases due to increase in blowing. Figure 3.6 shows the impact of suction/blowing parameter S on concentration profile. From this figure, we conclude that by increasing in the suction parameter S, the concentration profile and thickness of concentration boundary layer are reduced.
Figure 3.7 indicates that by increasing magnetic parameter, velocity profile increases. Figure 3.8 represents that by increasing in permeability parameter enhances the velocity profile.

Figure 3.9 shows the influence of R on the temperature profile. By increasing R, temperature profile increases significantly. The figure indicates that the thermal radiation is responsible for enhancement of the thermal boundary layer thickness. Figure 3.10 indicates by increasing in Schmidt number Sc, concentration as well as the thickness of concentration decrease. The effects of chemical reaction coefficient λ for concentration profile are observed in the Figure 3.11, which show that by increasing λ, the concentration profile increases and it also increases in the thickness of concentration.

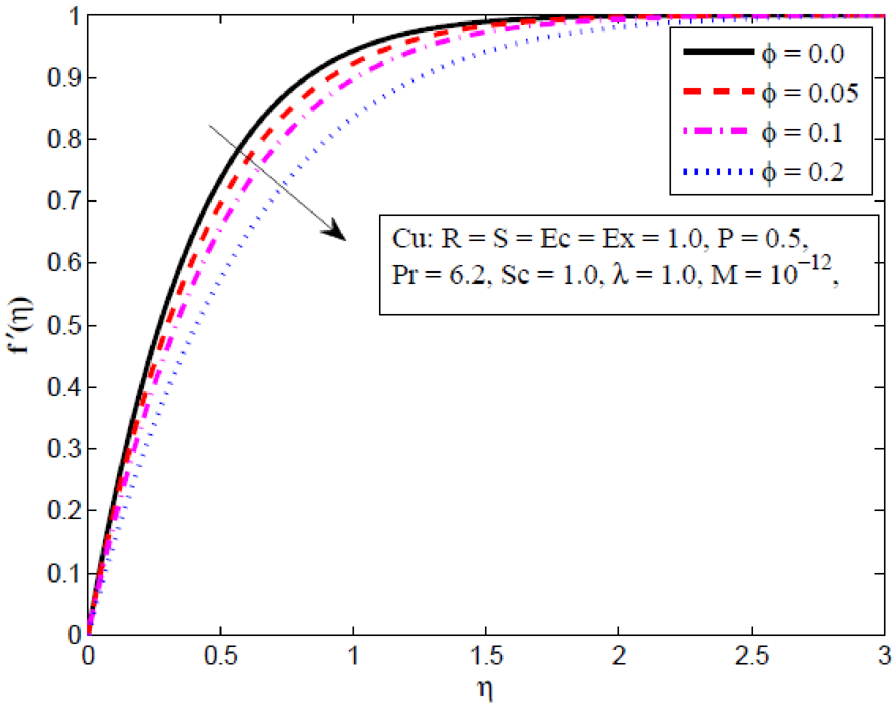

FIGURE 3.2: Influence of volume fraction ϕ on dimensionless velocity.

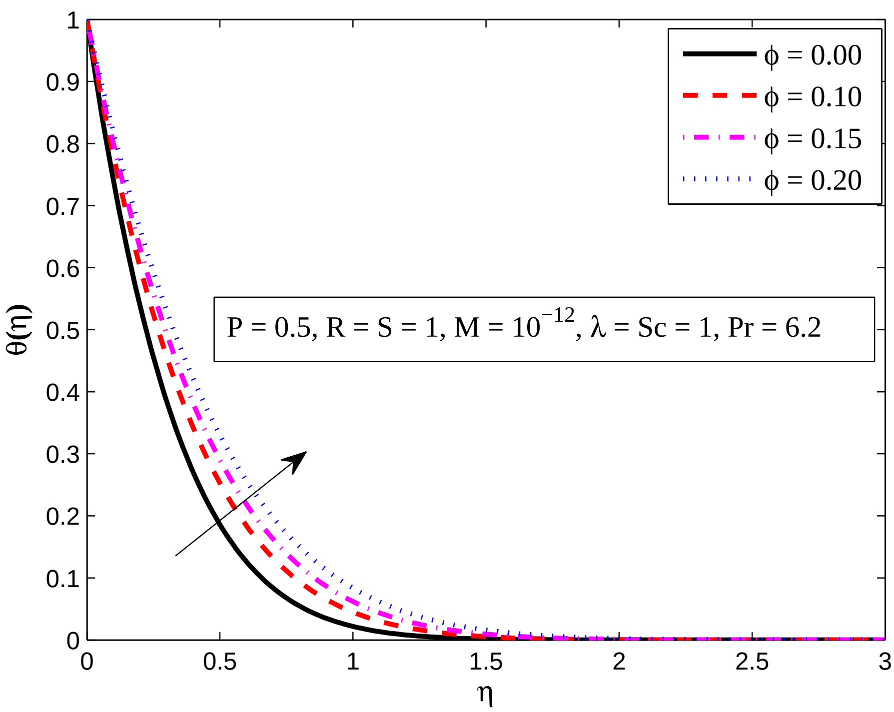

FIGURE 3.3: Influence of ϕ on dimensionless temperature profile.

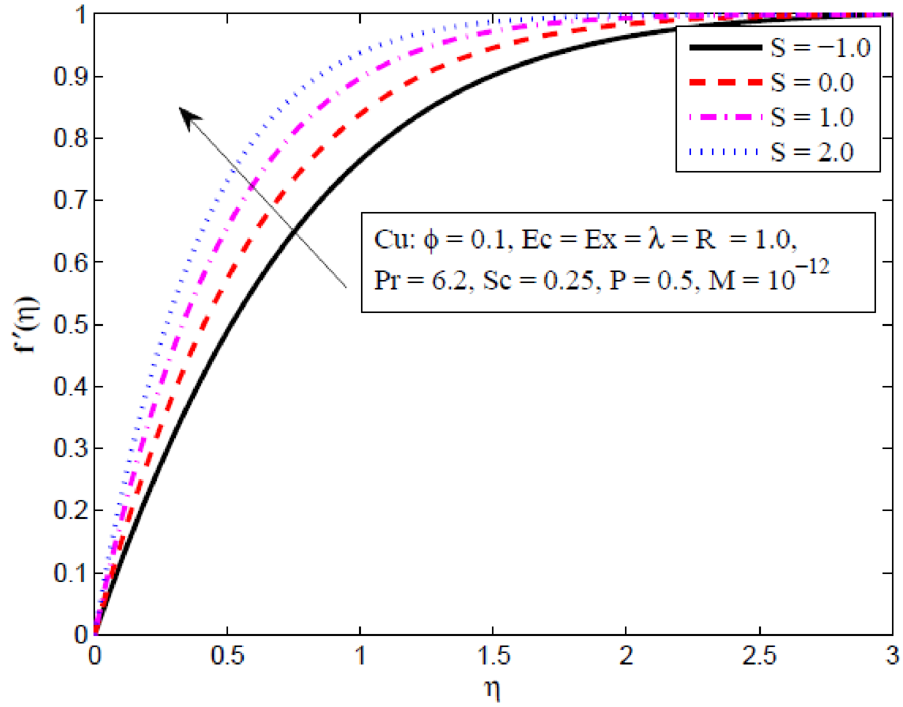

Figure 3.4: Impact of suction/blowing parameter on velocity profile.

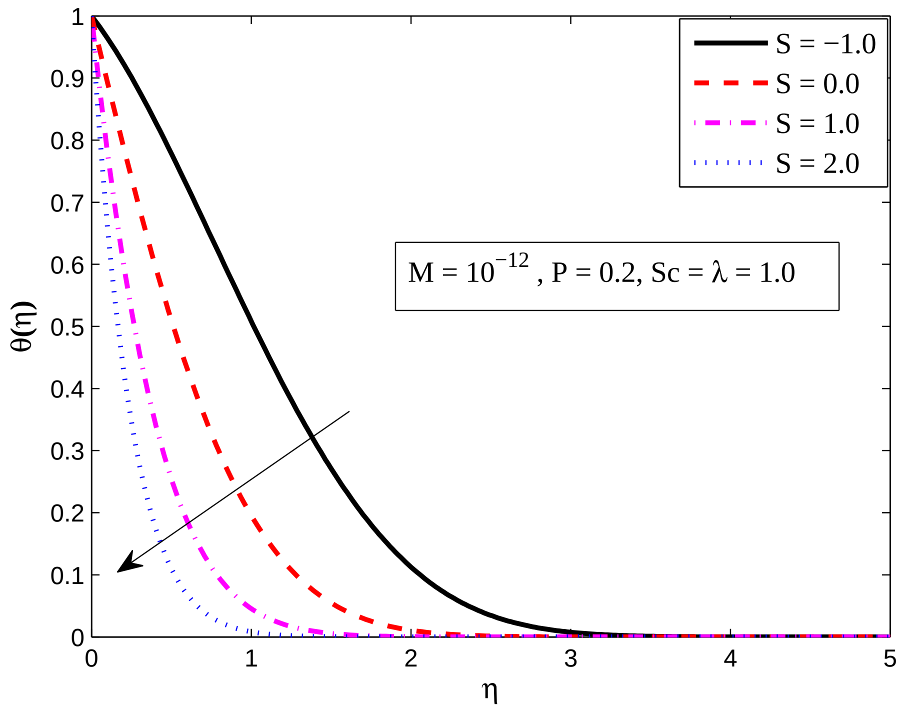

Figure 3.5: Impact of suction/blowing parameter on temperature profile.

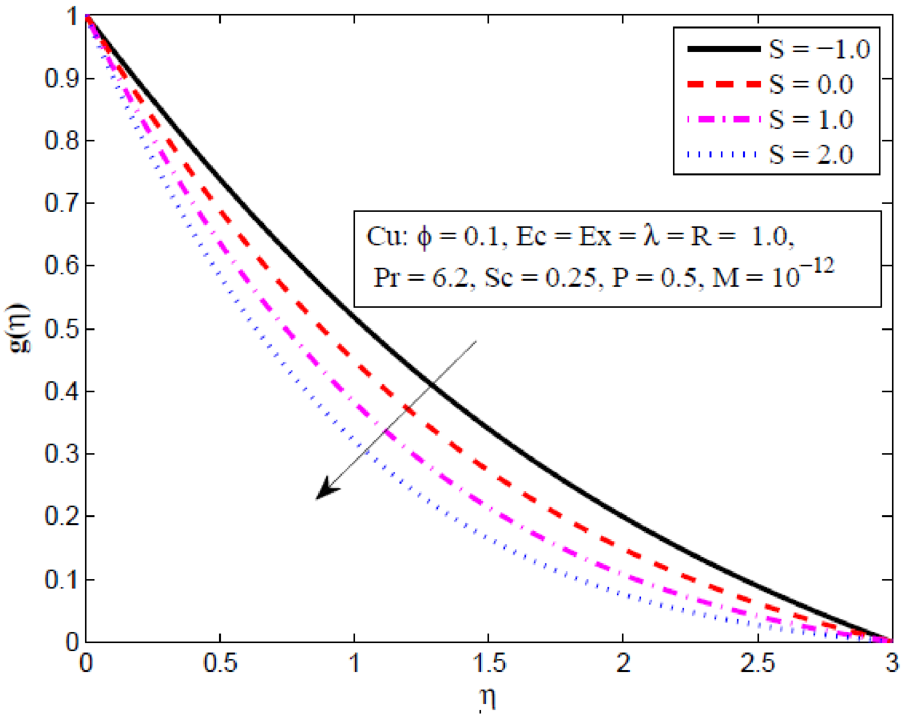

FIGURE 3.6: Impact of suction/blowing parameters on concentration profile.

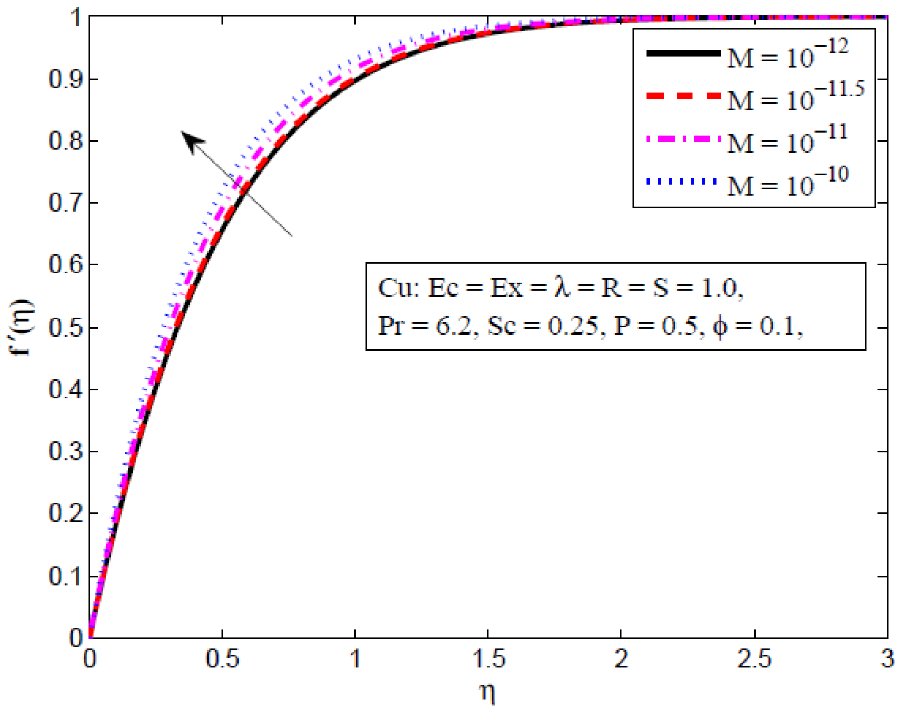

FIGURE 3.7: Impact of M on dimensionless velocity.

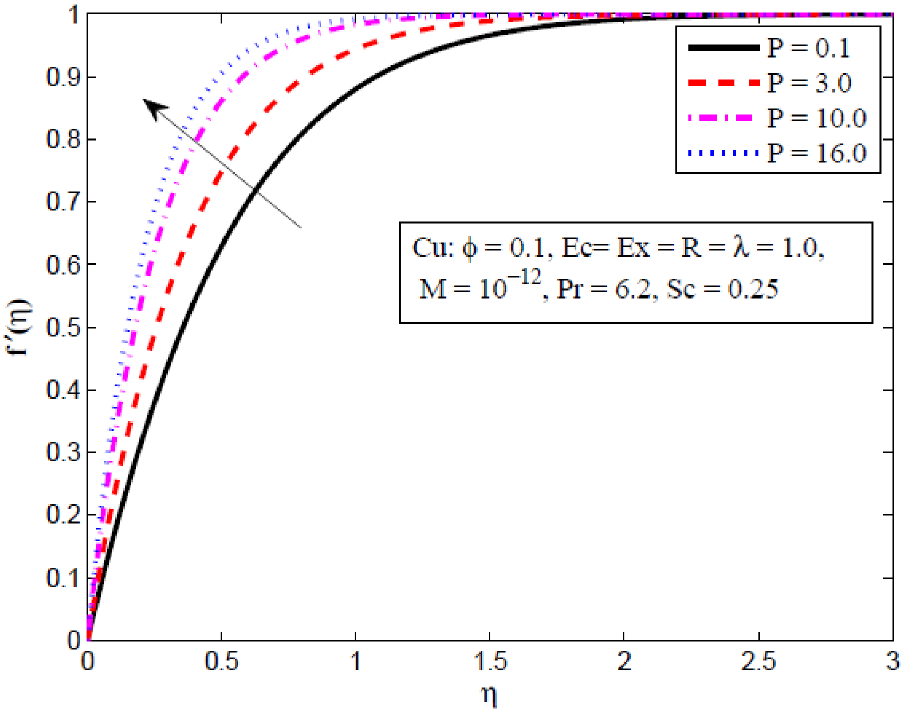

FIGURE 3.8: Impact of permeability parameter P on velocity profile.

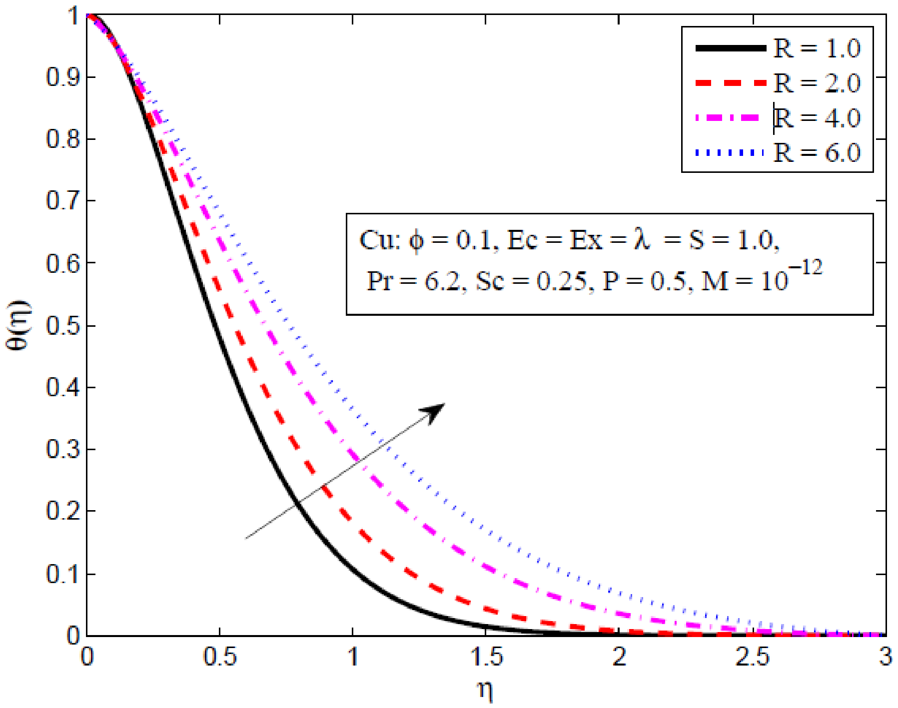

FIGURE 3.9: Impact of radiation parameter R on temperature profile.

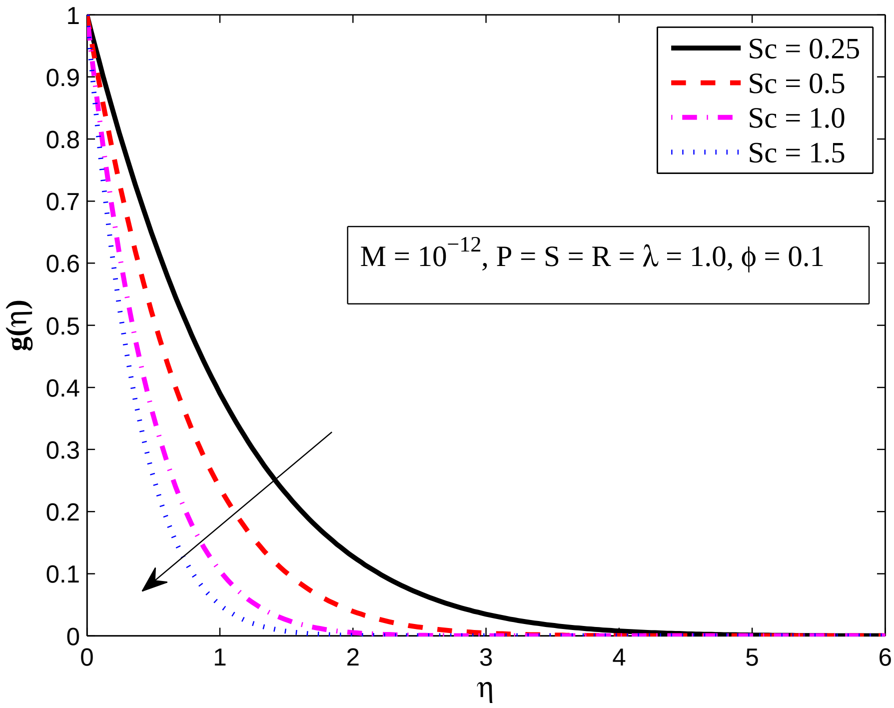

FIGURE 3.10: Influence of Schmidt number Sc on concentration profile.

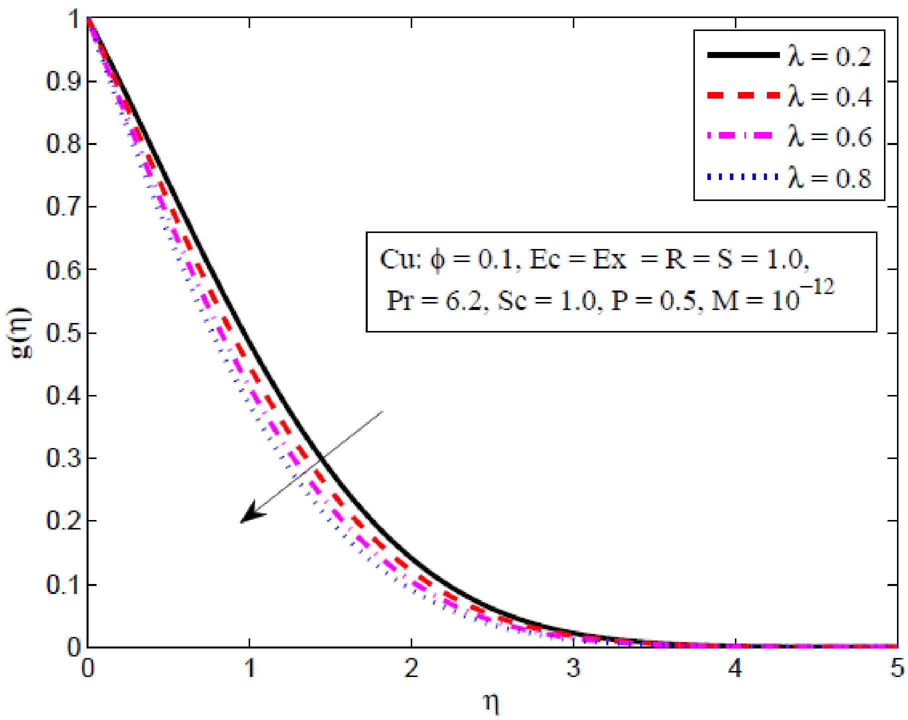

FIGURE 3.11: Influence of λ on the concentration profile.

Chapter 4

Numerical solution of viscous dissipation and Joule heating in nanofluids

In this chapter we extended the study of MHD boundary layer flow, radiative heat and mass transfer of laminar incompressible two dimensional viscous fluid over a permeable plate in a porous media with viscous dissipation and Joule heating. Using appropriate similarity transformation we converted nonlinear partail differential equations into system of ODEs. The numerical solution of these modeled ODEs are obtained by using shooting technique.The effects of different physical parameters are discussed through graphs and tables.

4.1 Mathematical analysis of governing equations

We consider the steady, viscous and incompressible nanofluid flow. The flow is two dimensional passed over a permeable plate in a porous media. In addition, a magnetic field B is applied normally to the plate as shown in Figure 4.1. The temperature at surface is T_w, free stream velocity is denoted by U and the ambient temperature is T_∞ respectively.

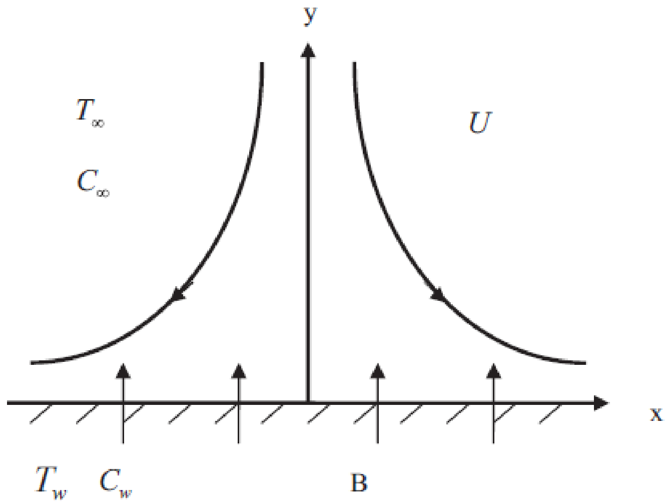

FIGURE 4.1: A schematic diagram of flow.

Continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{4.1}$$

Momentum equation:

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} = U\frac{dU}{dx} - \frac{\mu_{nf}}{\rho_{nf}}\frac{\partial^2 u}{\partial y^2} - \frac{\mu_{nf}}{\rho_{nf}k}(u-U) - \frac{\sigma_{nf}}{\rho_{nf}}B^2(u-U), \tag{4.2}$$

Energy equation:

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha_{nf}\frac{\partial^2 T}{\partial y^2} - \frac{1}{\rho_{nf}}\frac{\partial q_r}{\partial y} + \frac{\mu_{nf}}{\rho_{nf}}\left(\frac{\partial u}{\partial y}\right)^2 + \frac{\sigma_{nf}}{(\rho C_p)_{nf}}B^2u^2, \tag{4.3}$$

Mass equation:

$$u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} = D\frac{\partial^2 C}{\partial y^2} - K(C - C_\infty). \tag{4.4}$$

The associated boundary conditions:

$$\left.\begin{array}{l} u=0,\ v=v_w,\ T=T_w=T_\infty+T_0\,e^{\frac{x}{2l}},\ C=C_\infty+C_0\,e^{\frac{x}{2l}}, \quad \text{at} \quad y=0, \\ u\to U=a\,e^{\frac{x}{l}},\ T\to T_\infty,\ C\to C_\infty, \quad \text{at} \quad y\to\infty, \end{array}\right\} \tag{4.5}$$

In the above equations, u and v are the velocity components along x and y directions, respectively. T is the temperature of the nanofluids, p denotes pressure of the fluids, σ_s denotes the electrical conductivity of the base fluid and σ_f denotes the electrical conductivity of nanofluid respectively. In above equations, C is called the concentration of nanoparticles , C_∞ shows the free stream concentration, D is known as the mass diffusivity, U denotes the velocity in the free stream component, a is any constant number, μ_{nf} represents the viscosity of nanofluids, ρ_{nf} denotes density of nanofluids and σ_{nf} is called the thermal diffusivity and K denotes the variable reaction rate. Mathematically,

$$K = k_0 e^{\frac{x}{L}}, \tag{4.6}$$

where L is called the reference length and k_0 is any constant number. The suction velocity is written mathematically as:

$$v_w(x) = v_0\, e^{\frac{x}{2L}}, \tag{4.7}$$

here v_0 is any constant number. In the solution of the problem, first of all we converted the system of Eqs. (4.1)-(4.4) along with (4.5) into dimensionless forms using suitable similarity transformation. The similarity variable are defined as:

$$\eta = y\sqrt{\frac{a}{2v_f L}}e^{\frac{x}{2L}}, \tag{4.8}$$

The temperature dimensionless function θ, g and dimensionless stream function ψ are given in the form of

$$\psi = y\sqrt{2aLv_f}f(\eta)e^{\frac{x}{2L}}, \quad \theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty}, \quad g(\eta) = \frac{C - C_\infty}{C_w - C_\infty}. \tag{4.9}$$

The stream function $\psi = \psi(x, y)$ is identically satisfying continuity equation. Mathematically, the components of velocity are written as:

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}. \tag{4.10}$$

The equation of continuity (4.1) is satisfied identically. Using similarity transformation in momentum Eq. (4.2), energy Eq. (4.3) and concentration Eq. (4.4) along the boundary conditions (4.5). We get the following ODEs.

$$\frac{1}{\phi_1}f''' + ff'' + 2(1-f'^2) + \left(\frac{1}{\phi_1}P + \frac{\phi_4}{\phi_2}M\right)(1-f') = 0, \tag{4.11}$$

$$\left[\frac{K_n f}{\phi_3 K_f} + \frac{R}{\phi_3}\right]\theta'' + P_r\left(f\theta' - f'\theta\right) + P_r Ec Ex\left[\frac{f''^2}{\phi_1} + \frac{Mf'^2}{\phi_2}\right] = 0, \tag{4.12}$$

$$g'' + Sc(fg' - f'g - \lambda g) = 0. \tag{4.13}$$

The BCs are:

$$\left.\begin{aligned} &f = S, f' = 0, \theta = 1, g = 1, \quad \text{at} \quad \eta = 0,\\ &f' = 1, \theta = 0, g = 0, \quad \text{at} \quad \eta \to \infty. \end{aligned}\right\} \tag{4.14}$$

The parameters in the above set of equations are:

$$\left.\begin{aligned} &P = \frac{2Lv_f}{ak_0},\ M = \frac{2\sigma_f B_0^2 L}{a\rho_f},\ R = \frac{16\sigma_1 T_\infty^3}{3kk_f},\ S = \frac{v_0}{\sqrt{\frac{av_f}{2L}}},\ Sc = \frac{v_f}{D},\\ &E_c = \frac{a^2 e^{\frac{2x}{L}}}{T_0\,(C_p)_{nf}},\ Ex = e^{-\frac{x}{2L}},\ P_r = \frac{v_f}{\alpha_f},\ \lambda = \frac{K_0}{a},\\ &\phi_1 = (1-\phi)^{2.5}\left[(1-\phi) + \phi\frac{\rho_s}{\rho_f}\right],\quad \phi_2 = (1-\phi) + \phi\frac{\rho_s}{\rho_f},\\ &\phi_3 = (1-\phi) + \phi\frac{(\rho C_p)_s}{(\rho C_p)_f},\quad \phi_4 = (1-\phi) + \phi\frac{\sigma_s}{\sigma_f}, \end{aligned}\right\} \tag{4.15}$$

P_r represents the Prandtl number, P is known as permeability parameter, M is the notation used for magnetic parameter, R is called the radiation parameter, S is used for suction or blowing parameter, Sc is Schmidt number and λ is the notation of chemical reaction parameter. The quantities of practical interest in these studies are the local Nusselt number Nu_x, the local Sherwood number Sh_w, Ec Eckert number and the exponential function Ex. Which are defined as:

$$Nu_x = \frac{(xq_w)}{(K_f(T_w - T_\infty))},\quad Sh_x = \frac{(xp_w)}{(D(C_w - C_\infty))}. \tag{4.16}$$

q_w is the known as heat flux and p_w is called the mass flux. Mathematically it can be written as

$$q_w = K_{nf}\frac{\partial T}{\partial y}\bigg|_{y=0},\quad p_w = D\frac{\partial C}{\partial y}\bigg|_{y=0}. \tag{4.17}$$

In Eq. (4.18), Re_x represents the local Reynolds numbers. It is written as

$$Re_x = \frac{Ux^2}{2Lv_f}. \tag{4.18}$$

4.2 Method of the solution

The solution of the system of eqs. (4.11)-(4.13)) with corresponding boundary conditions (4.14) can obtained by shooting technique.

To use the shooting method, first we converted these Eqs. (4.11)-(4.13) into a system of first order differential equations. For these purpose, we denote f by y_1, f' by y_2, f'' by y_3, θ by y_4 and g by y_6 . The coupled nonlinear momentum, heat and concentration equations are written in the form of seven first order order ODEs:

$$\begin{aligned}
y_1' &= y_2, \\
y_2' &= y_3, \\
y_3' &= \phi_1 \left(2y_2^2 - y_1 y_3 - 2\right) + \left(P + \frac{\phi_1}{\phi_2}\phi_4 M\right)(y_2 - 1), \\
y_4' &= y_5, \\
y_5' &= -\frac{P_r \phi_3}{K_{nf}/k_f + R}(y_1 y_5 - y_2 y_4) + P_r Ec Ex \left[\frac{y_3^2}{\phi_1} + \frac{My_2^2}{\phi_2}\right], \\
y_6' &= y_7, \\
y_7' &= -Sc\left(y_1 y_7 - y_2 y_6 - \lambda y_6\right).
\end{aligned}$$

The boundary conditions make the following forms

$$f_1(0) = S, \quad f_2(0) = 0, \quad f_3(0) = u, \quad f_4(0) = 1, \quad f_5(0) = v, \quad f_6(0) = 1, \quad f_7(0) = w.$$

The shooting method requires the initial guess for $f_3(\eta)$, $f_5(\eta)$ and $f_7(\eta)$ at $\eta = 0$, and through Newton's method we need to vary each guess until we obtain an appropriate solution for our problem. To check accuracy of the obtained numerical results by shooting method, we compare them by the numerical results acquired by Matlab built-in function bvp4c solver and found them in excellent agreement. The stoping criteria is given as

$$max|y_2(7)| - 1, |y_4(7)|, |y_6(7)| < \epsilon,$$

where $\epsilon > 0$ represents any positive number. In this thesis the numerical results are achieved with $\epsilon = 10^{-6}$

4.3 Graphical and tabular representation

In this section we analyzed the effects of velocity, temperature and concentration profiles for different parameters such as Permeability parameter P, magnetic parameter M, radiation parameter R, suction/blowing parameter S, Schmidt number Sc, Eckert number Ec and exponential function Ex through graphs and tables. Tables 4.1, represents the effects of different parameters on Nusselt and Sherwood numbers. We compare the results obtained by shooting method with Matlab built-in function bvp4c solver and found both to be in excellent agreement. By increasing Schmidt number Sc, there is no effect on Nusselt number but the Sherwood number is increasing. An increase in exponential function Ex shows an increases in Nusselt number but has no effect on Sherwood number. It can be noticed from the table that by increasing Eckert number, Nusselt number increases but no effect on Sherwood number. Due to increases in permeability parameter P, both Nusselt and Sherwood numbers increase. It is investigated that increase in magnetic field M enhance both Nusselt and Sherwood numbers. Increase in the radiation parameter R causes Nusselt number to increase and there is no effect on local Sherwood number. By increasing chemical reaction coefficient λ, Nusselt numbers increase but no effect on Sherwood numbers. It is concluded from this table that increase in suction/blowing parameter, Nusselt number decreases and it rises Sherwood number. By increasing volume fraction ϕ, both Nusselt and Sherwood numbers decrease.

Parameters									**Shooting method**		**Matlab bvp4c**	
Sc	Ex	Ec	P	M	R	λ	S	ϕ	-$\theta'(0)$	-g$'(0)$	-$\theta'(0)$	-g$'(0)$
0.25	1	1	0.5	10^{-12}	0.4	1	0.1	0.1	0.4969	0.7284	0.4969	0.7284
0.6									0.4969	1.0766	0.4969	1.0766
0.8									0.4969	1.2302	0.4969	1.2302
	1.2								0.9160	1.2302	0.9160	1.2302
	1.4								1.3350	1.2302	1.3350	1.2302
	1.6								1.7541	1.2302	1.7541	1.2302
		0.4							0.7603	1.2301	0.7603	1.2301
		0.6							0.3412	1.2301	0.3412	1.2301
		0.8							0.0779	1.2301	0.0779	1.2301
			0.2						0.5153	1.2312	0.5153	1.2312
			0.3						0.5336	1.2322	0.5336	1.2322
			0.4						0.5518	1.2332	0.5518	1.2332
				10^{-11}					1.6926	1.2786	1.6926	1.2786
				$10^{-11.5}$					0.8620	1.2463	0.8620	1.2463
				10^{-10}					7.6089	1.3700	7.6089	1.3700
					0.8				0.2481	1.2302	0.2481	1.2302
					0.6				0.3769	1.2302	0.3769	1.2302
					1				0.4725	1.2302	0.4725	1.2302
						1.2			0.5518	1.2894	0.5518	1.2894
						1.4			0.5518	1.3465	0.5518	1.3465
						1.6			0.5518	1.4016	0.5518	1.4016
							0.4		0.3002	0.7745	0.3002	0.7745
							0.6		0.0849	0.8060	0.0849	0.8060
							0.8		-0.1609	0.8382	-0.1609	0.8382
								0.2	0.3002	0.7321	0.3002	0.7321
								0.3	0.1822	0.7309	0.1822	0.7309
								0.4	0.1447	0.7261	0.1447	0.7261

TABLE 4.1: Numerical results of $g'(0)$, -$\theta'(0)$ for different values of Sc, Ex, Ec, P, M, R, λ, S and ϕ.

Figure 4.2 shows the influence of Ec on dimensionless temperature profile. Temperature profile increases with an increase in the values of Eckert number Ec. As Eckert number is the ratio of the kinetic energy dissipated in the flow to the thermal energy conducted into or away from the fluid. This figure shows that when we increase the values of Eckert number for temperature profile, the profile of temperature distribution also increase. We may conclude that the presence of viscous dissipation and Joule heating increase the tangential velocity f' and the temperature θ. This rise in the temperature is due to the heat created by viscous dissipation.

Figure 4.3 indicates the effect of exponential function for temperature profile. By increasing exponential parameter Ex, temperature profile also increases. It is noted that the thermal boundary layer thickness increases with the increase of Ex but shows a significant effect near the stretching sheet.

Figure 4.4 represents the influence of P on velocity profile. By increasing permeability parameter P, velocity profile also increases. The results show that increase in Permeability parameter leads an increases in velocity profile.
Figure 4.5 represents the effect of magnetic parameter on velocity. By increasing the values of M, velocity profile increases but it decreases the thickness of boundary layer.

Figure 4.6 shows the radiation parameter R effect on temperature profile. It is seen that temperature profile decreases significantly with increasing the radiation parameter R. This figure indicates that an increases in radiation parameter increases in boundary layer thickness.

Figure 4.7 shows that by increasing suction/blowing parameter S, concentration profile decreases. By an increasing in the suction, concentration profile as well as boundary layer thickness decrease. Figure 4.8 indicates the effect of chemical reaction coefficient λ on concentration profile. By increasing chemical reaction coefficient λ, concentration profile decreases. Figure 4.9 reprenents that when we increase in the volume fraction ϕ, thermal conductivity and thickness of boundary layer increase. Figure 4.10 depicts that by increasing suction parameter, velocity profile decreases significantly, whereas fluid velocity is increases with blowing.

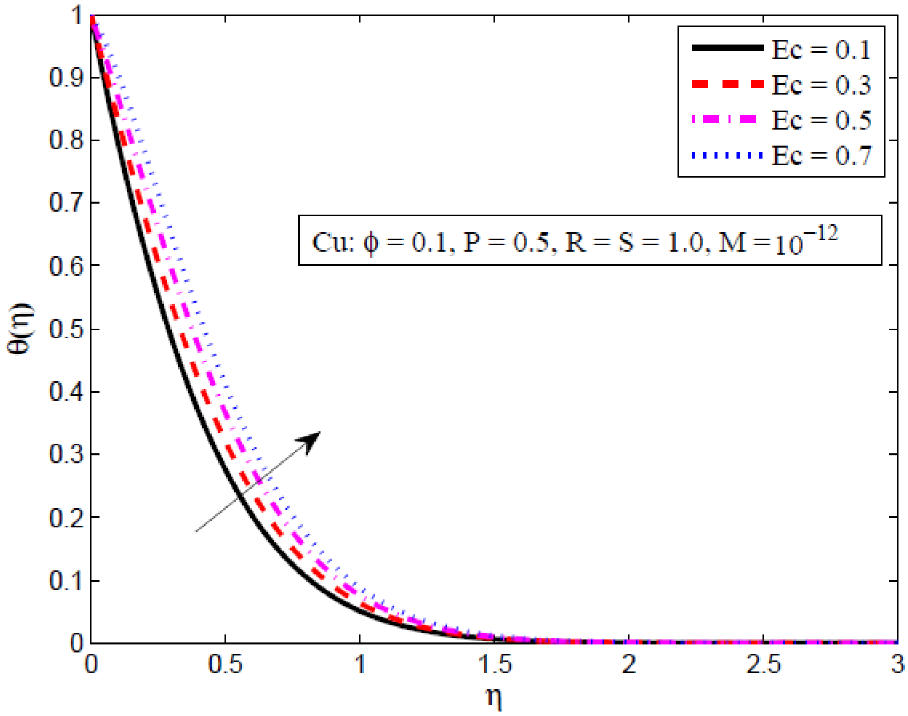

FIGURE 4.2: Eckert number influence on dimensionless temperature.

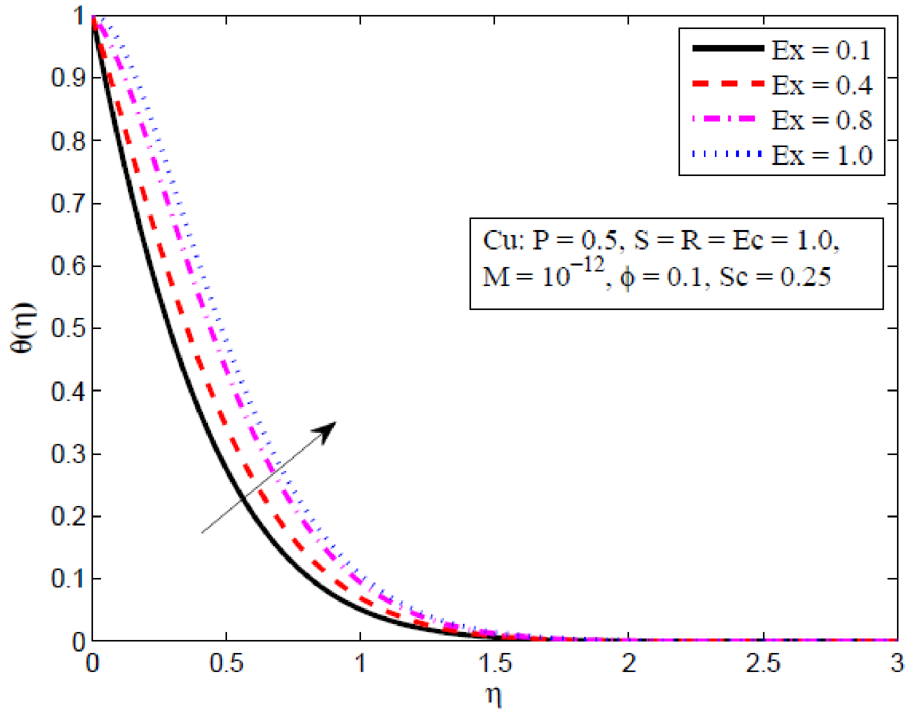

FIGURE 4.3: Influence of exponential function on dimensionless temperature.

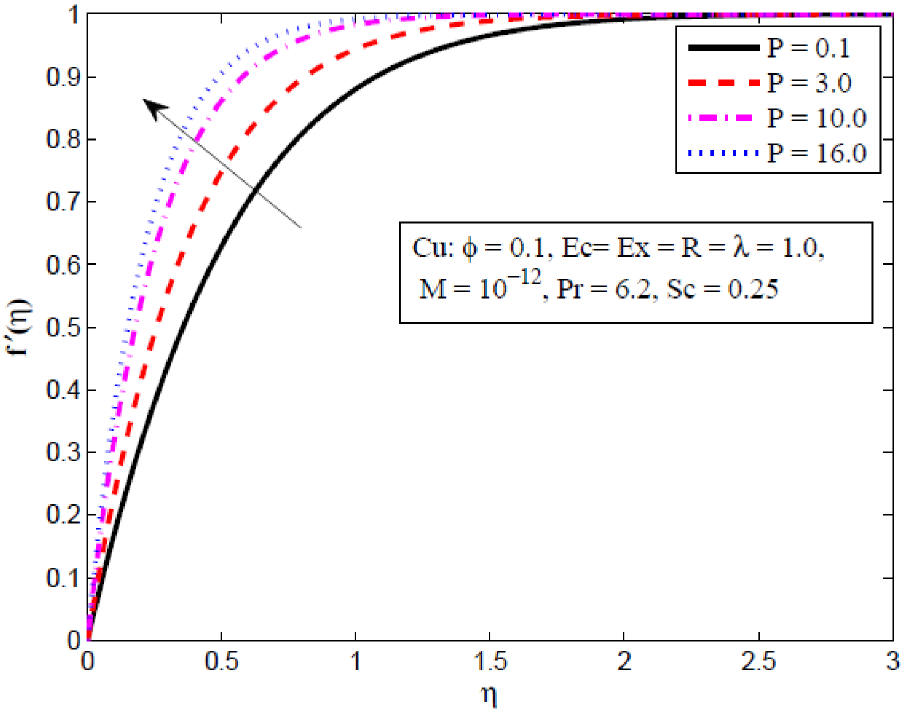

FIGURE 4.4: Influence of P on dimensionless velocity .

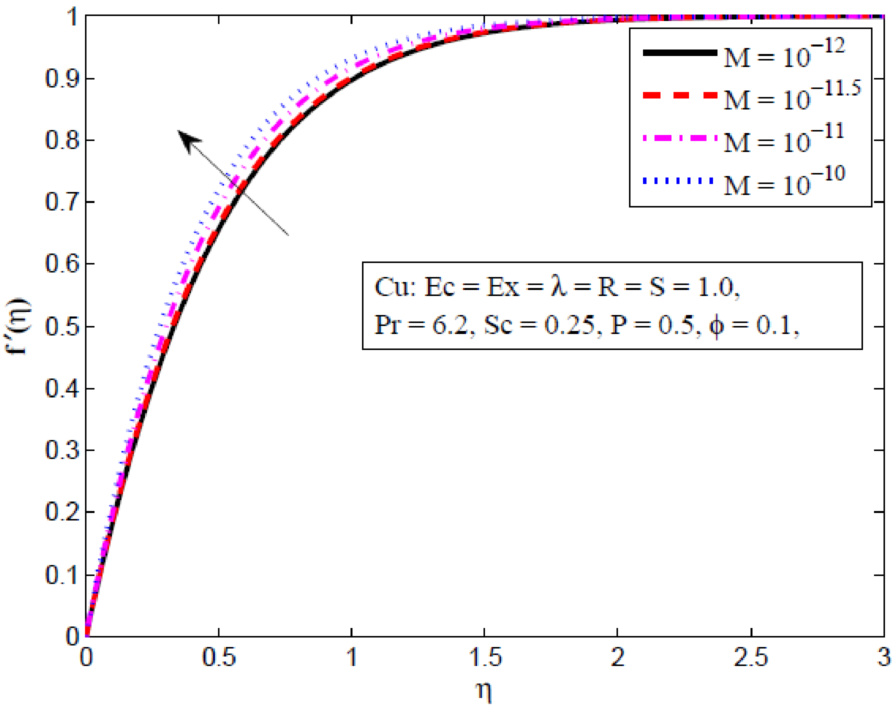

FIGURE 4.5: Magnetic parameter M influence on dimensionless velocity.

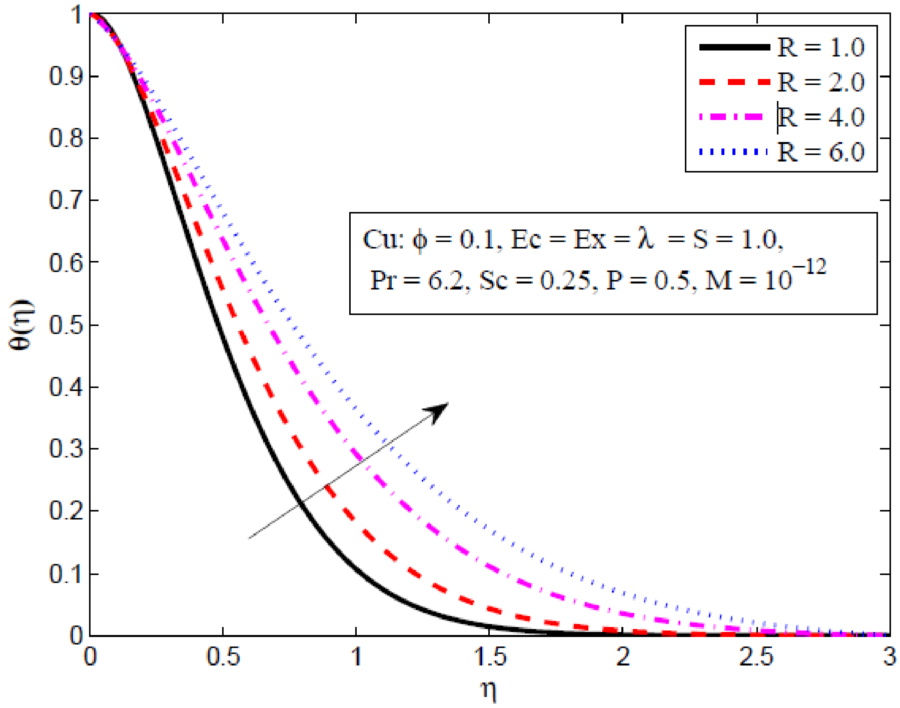

FIGURE 4.6: Influence of radiation parameter on dimensionless temperature.

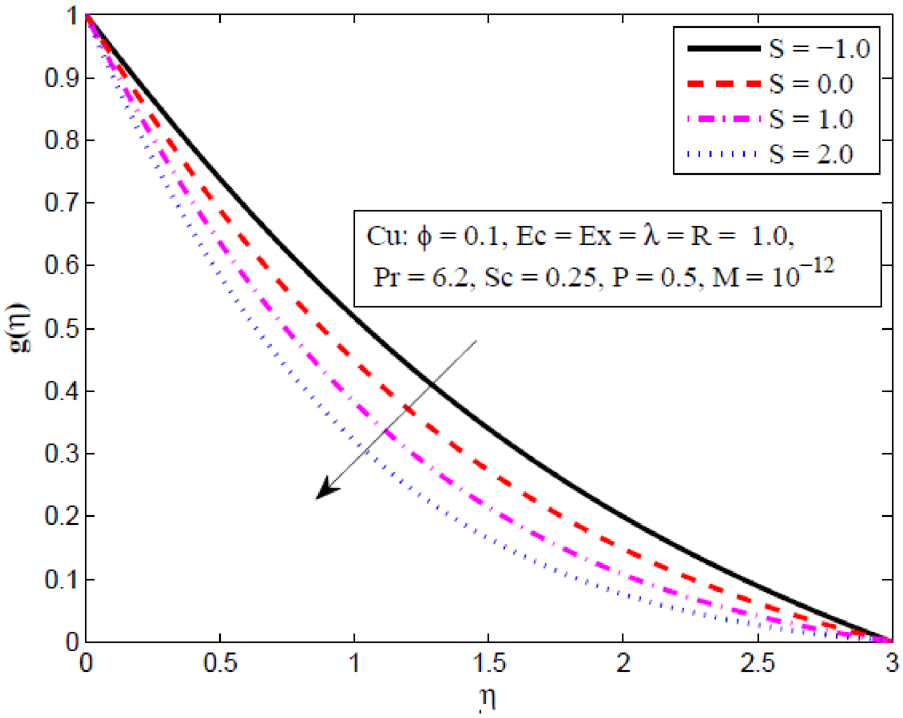

FIGURE 4.7: Influence of suction/blowing S parameter on concentration.

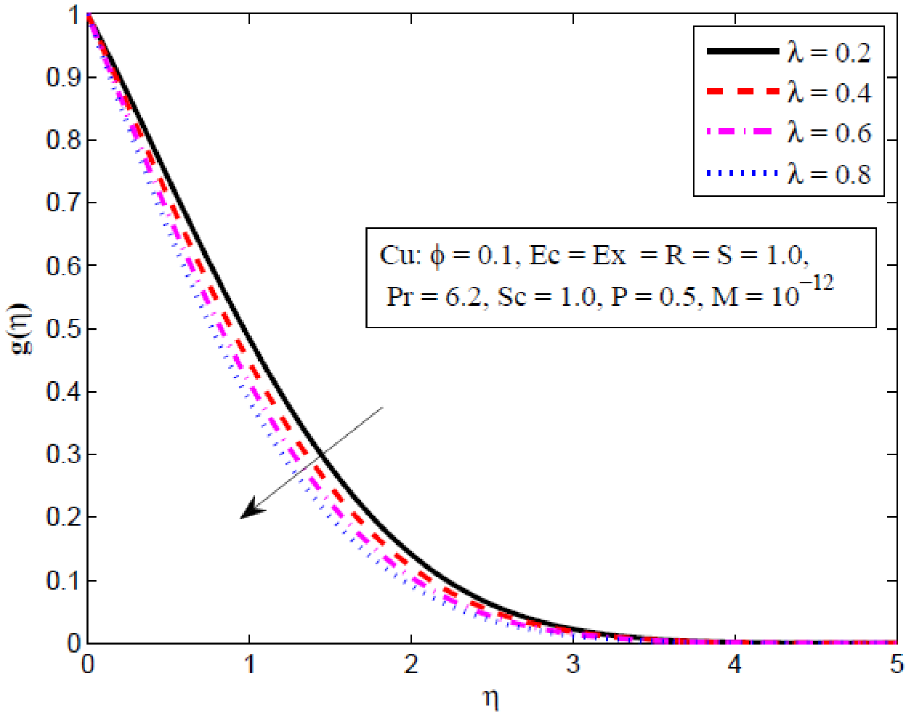

FIGURE 4.8: Influence of λ on dimensionless concentration.

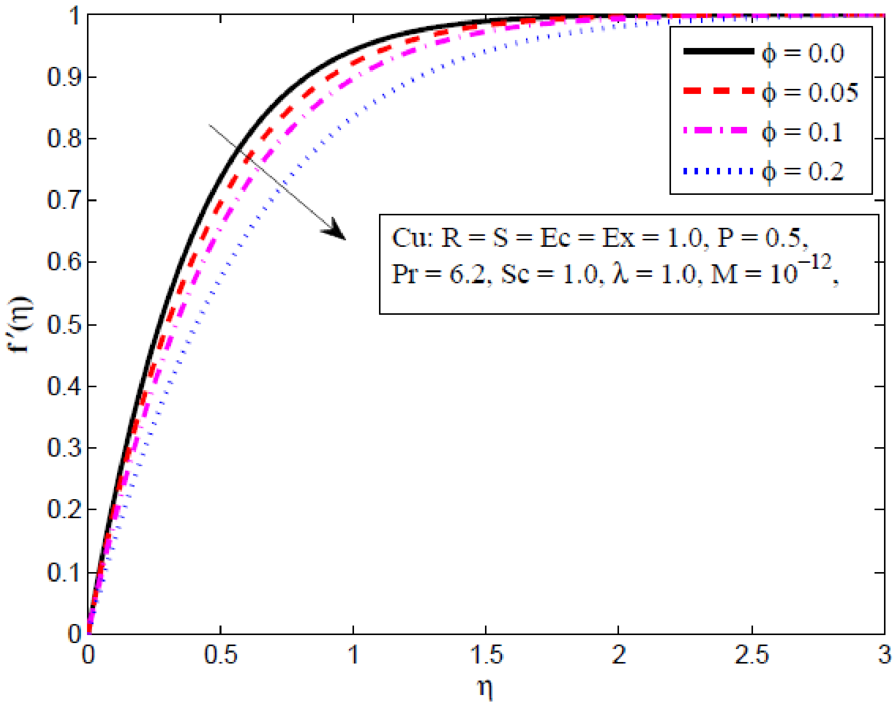

FIGURE 4.9: Influence ϕ on dimensionless velocity.

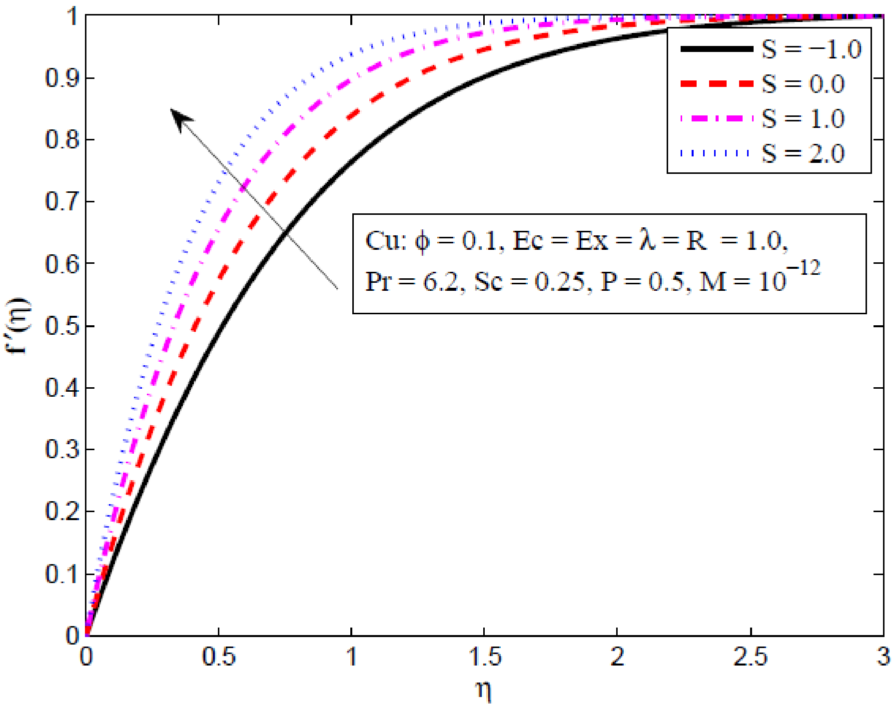

FIGURE 4.10: Influence of suction/blowing parameter on dimensionless velocity.

Chapter 5

Conclusion and outlook

Conclusion of this thesis contains the heat and mass transfer in MHD laminar boundary layer flow of incompressible viscous fluid over a permeable surface through porous media. Furthermore, the impacts of radiation, viscous dissipation and Joule heating are under consideration. The influence of heat flux and chemical reaction coefficient λ are considered. By provoking a suitable set of similarity transformations the governing nonlinear PDEs of momentum, heat and mass equations are converted into system of ODEs. Shooting method are used to solve the system of equations. The numerical results are in good agreement with Matlab built-in function bvp4c solver. Physical significance of different parameters are discussed w.r.t. dimensionless velocity, temperature and concentrations profiles.

- By increasing volume fraction ϕ, velocity profile also increases.
- By increasing volume fraction ϕ, Temperature field θ also increases.
- By increasing in the suction parameter S, velocity profile increases whereas it decreases in the blowing parameter.
- Temperature profile decreases with an increase in suction while due to increasing blowing, it increases.

- The thickness of concentration boundary layers reduce due to increasing in the suction parameter.
- Velocity profile f' increases due to increases in the magnetic parameter M.
- By increasing the Schmidt number Sc, concentration boundary layer thickness decreases.
- Temperature profile will increase due to increases in the Eckert number .

5.1 Future recommendations

Three dimensional flow can be considered for further studies. Non Newtonian fluids such as Oldryd B fluid and Maxwell fluid can be used. Some other parameters such as mixed convection, inclined magnetic field and non linear thermal radiation can be incorporated.

Bibliography

[1] L. Prandtl, "Uber flussigkeitsbewegung bei sehr kleiner reibung," *Verhandl. 3rd Int. Math. Kongr. Heidelberg (1904), Leipzig*, 1905.

[2] T. Motsumi and O. Makinde, "Effects of thermal radiation and viscous dissipation on boundary layer flow of nanofluids over a permeable moving flat plate," *Physica Scripta*, vol. 86, no. 4, p. 045003, 2012.

[3] W. Ibrahim and O. Makinde, "The effect of double stratification on boundary-layer flow and heat transfer of nanofluid over a vertical plate," *Computers & Fluids*, vol. 86, pp. 433–441, 2013.

[4] O. Makinde, "Analysis of Sakiadis flow of nanofluids with viscous dissipation and Newtonian heating," *Applied Mathematics and Mechanics*, pp. 1–10, 2012.

[5] O. Daniel Makinde, "Computational modeling of nanofluids flow over a convectively heated unsteady stretching sheet," *Current Nanoscience*, vol. 9, no. 5, pp. 673–678, 2013.

[6] B. Sakiadis, "Boundary-layer behavior on continuous solid surfaces: I. boundary-layer equations for two-dimensional and axisymmetric flow," *AIChE Journal*, vol. 7, no. 1, pp. 26–28, 1961.

[7] ——, "Boundary-layer behavior on continuous solid surfaces: II. the boundary layer on a continuous flat surface," *AiChE journal*, vol. 7, no. 2, pp. 221–225, 1961.

[8] S. U. S. Choi and J. A. Eastman, “Enhancing thermal conductivity of fluids with nanoparticles,” 1995.

[9] Malvandi and Ganji*et al.*, “Effects of nanoparticle migration on force convection of alumina/water nanofluid in a cooled parallel-plate channel,” *Adv. Powder Technol*, vol. 25, no. 4, pp. 1369–1375, 2014.

[10] K. Khanafer and K. Vafai, “A critical synthesis of thermophysical characteristics of nanofluids,” *International Journal of Heat and Mass Transfer*, vol. 54, no. 19, pp. 4410–4428, 2011.

[11] S. Sureshkumar and M. Muthtamilselvan, “A slanted porous enclosure filled with Cu-water nanofluid,” *The European Physical Journal Plus*, vol. 131, no. 4, p. 95, 2016.

[12] N. Nagarajan and S. Akbar, “Heat transfer enhancement of Cu-water nanofluid in a porous square enclosure driven by an incessantly moving flat plate,” *Procedia Engineering*, vol. 127, pp. 279–286, 2015.

[13] X. Chen, J.-M. Li, W.-T. Dai, and B.-X. Wang, “Enhancing convection heat transfer in mini tubes with nanoparticle suspensions,” *Journal of Engineering Thermophysics.*, vol. 25, no. 4, pp. 643–645, 2004.

[14] M. M. Rahman, H. F. Öztop, R. Saidur, S. Mekhilef, and K. Al-Salem, “Unsteady mixed convection in a porous media filled lid-driven cavity heated by a semi-circular heaters,” *Thermal Science*, vol. 19, no. 5, pp. 1761–1768, 2015.

[15] A. A. Hassan and M. A. Ismael, “Mixed convection in superposed nanofluid and porous layers inside lid-driven square cavity,” *Int. J. of Thermal & Environmental Engineering*, vol. 10, no. 2, pp. 93–104, 2015.

[16] E. Vishnuvardhanarao and M. K. Das, “Laminar mixed convection in a parallel two-sided lid-driven differentially heated square cavity filled with a fluid-saturated porous medium,” *Numerical Heat Transfer, Part A: Applications*, vol. 53, no. 1, pp. 88–110, 2007.

[17] C. Ho, M. Chen, and Z. Li, "Effect of natural convection heat transfer of nanofluid in an enclosure due to uncertainties of viscosity and thermal conductivity," vol. 1, pp. 833–841, 2007.

[18] H. Masuda, A. Ebata, and K. Teramae, "Alteration of thermal conductivity and viscosity of liquid by dispersing ultra-fine particles. dispersion of Al_2O_3, SiO_2 and TiO_2 ultra-fine particles," 1993.

[19] J. Buongiorno, "Convective transport in nanofluids," *Journal of Heat Transfer*, vol. 128, no. 3, pp. 240–250, 2006.

[20] Z. Mehrez, M. Bouterra, A. El Cafsi, and A. Belghith, "Heat transfer and entropy generation analysis of nanofluids flow in an open cavity," *Computers & Fluids*, vol. 88, pp. 363–373, 2013.

[21] T. S. Khaleque and M. Samad, "Effects of radiation, heat generation and viscous dissipation on MHD free convection flow along a stretching sheet," *Research Journal of Applied Sciences, Engineering and Technology*, vol. 2, no. 4, pp. 368–377, 2010.

[22] F. Ibrahim, A. Elaiw, and A. Bakr, "Influence of viscous dissipation and radiation on unsteady MHD mixed convection flow of micropolar fluids," *Appl. Math. Inf. Sci*, vol. 2, pp. 143–162, 2008.

[23] C. Zhang, L. Zheng, X. Zhang, and G. Chen, "MHD flow and radiation heat transfer of nanofluids in porous media with variable surface heat flux and chemical reaction," *Applied Mathematical Modelling*, vol. 39, no. 1, pp. 165–181, 2015.

[24] A. Omowaye and I. Animasaun, "Upper-convected maxwell fluid flow with variable thermo-physical properties over a melting surface situated in hot environment subject to thermal stratification," *Journal of Applied Fluid Mechanics*, vol. 9, no. 4, pp. 1777–1790, 2016.

[25] A. Ishak, R. Nazar, and I. Pop, "Magnetohydrodynamics stagnation point flow towards a stretching vertical sheet," *Magnetohydrodynamics*, vol. 42, no. 1, pp. 77–90, 2006.

[26] N. A. A. Bakar, K. Zaimi, and R. A. Hamid, "MHD boundary layer flow of a maxwell nanofluid over a permeable vertical surface," in *AIP Conference Proceedings*, vol. 1605, no. 1. AIP, 2014, pp. 422–427.

[27] R. U. Haq, S. Nadeem, Z. H. Khan, and N. Noor, "Convective heat transfer in MHD slip flow over a stretching surface in the presence of carbon nanotubes," *Physica B: condensed matter*, vol. 457, pp. 40–47, 2015.

[28] E. M. Abo-Eldahab and M. A. El Aziz, "Viscous dissipation and Joule heating effects on MHD-free convection from a vertical plate with power-law variation in surface temperature in the presence of Hall and ion-slip currents," *Applied Mathematical Modelling*, vol. 29, no. 6, pp. 579–595, 2005.

[29] B. Sahoo, "Effects of partial slip, viscous dissipation and Joule heating on von kármán flow and heat transfer of an electrically conducting non-newtonian fluid," *Communications in Nonlinear Science and Numerical Simulation*, vol. 14, no. 7, pp. 2982–2998, 2009.

[30] P. Sreenivasulu, T. Poornima, and N. B. Reddy, "Thermal radiation effects on MHD boundary layer slip flow past a permeable exponential stretching sheet in the presence of Joule heating and viscous dissipation." *Journal of Applied Fluid Mechanics*, vol. 9, no. 1, 2016.

[31] M. Alam, M. Rahman, and M. Sattar, "On the effectiveness of viscous dissipation and Joule heating on steady magnetohydrodynamic heat and mass transfer flow over an inclined radiate isothermal permeable surface in the presence of thermophoresis," *Communications in Nonlinear Science and Numerical Simulation*, vol. 14, no. 5, pp. 2132–2143, 2009.

[32] T. Hayat, A. Naseem, M. Farooq, and A. Alsaedi, "Unsteady MHD three-dimensional flow with viscous dissipation and Joule heating," *The European Physical Journal Plus*, vol. 128, no. 12, p. 158, 2013.

[33] F. M. White and I. Corfield, *Viscous fluid flow.* McGraw-Hill Higher Education Boston, 2006, vol. 3.

[34] T. Papanastasiou, G. Georgiou, and A. N. Alexandrou, *Viscous fluid flow.* CRC Press, 1999.

[35] T. Y. Na, *Computational methods in engineering boundary value problems.* Academic Press, 1980, vol. 145.