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Results for Generalized Ćirić type α-F-Contractions in Double Controlled Metric Spaces

by

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Dedicated to my ***Parents*** *and* ***Teachers***

CERTIFICATE OF APPROVAL

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Abstract

Recently, Belhenniche et al. proved common fixed point results for Ćirić generalized contractions in extended b-metric spaces. In this thesis a notion of generalized Ćirić type α-F-contraction in setting of double controlled metric space has been introduced. Our main result is about the existence of common fixed points of generalized α-F-contraction mappings. Our results generalized the results of Belhenniche et al. Several existing results are special case of our results. The proposed approach is illustrated by some examples. For application purpose existence and uniqueness results of the solution of Bellman equations, Volterra integral equations and fractional differential equation has been established.

Contents

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Abbreviations

BCP	Banach contraction principle
***b*MS**	*b*-metric space
CFP	Common fixed point
CM	Controlled metric
CMS	Controlled metric space
cMS	Complete metric space
DCMS	Double controlled metric space
E*b*MS	Extended *b*-metric space
FDE	Fractional differential equation
FP	Fixed point
MS	Metric space
UFP	Unique fixed point

Symbols

$\aleph$	A non-empty set
$\Im$	Distance function
$C[\mathsf{c}_1, \mathsf{c}_2]$	Space of all continuous functions
l^p	A sequence space
$\mathbb{R}$	The set of real numbers
$\mathbb{N}$	The set of natural numbers
$\in$	Belongs to
$\exists$	There exist

Chapter 1

Introduction

1.1 Background

Mathematics is an essential subject of scientific knowledge with the large range of applications in all aspects of real life. It is regarded as the mother of sciences, since, it deals with quantitative calculations, logical reasoning, and its evolution in increasing degrees of idealization and abstraction of its subject matter. It is further divided into numerous subdivisions, with functional analysis being one of the most important branch of mathematics.

In functional analysis, one of the main part is fixed point (FP) theory, which concern with the existence of solution or we can say that existence of fixed point of certain problem. The fixed point theory has very large number of applications in various fields of sciences such as, mathematical economics, optimization theory and approximation theory. Fixed point theory has become one of the most rapidly increasing research area of mathematics in last 5-7 decades.

Poincare [1], in 1886, initiated the metric fixed point theory. In 1906, Maurice Frchet [2], a French mathematician, was the first who propose the notion of metric space. After that, in 1912, Brouwer [3] examined the fixed point problem and ingrained the fixed point theorem for solving the equation $\mathcal{T}(\mathsf{c}) = \mathsf{c}$.

The Banach Contraction Principle (BCP) [4], is established by Stefan Banach in

1922, is a major finding. The idea of BCP has grown to be a key component of fixed point theory. Using this concept, one can guarantee the existence and uniqueness of fixed points and also learn how to find the fixed point of a given problem. Since then, many researchers have established the fixed points thoery particularly in two main sides. Firstly, by stating the conditions on mapping $\mathcal{T}$ and secondly taking the set $\aleph$ as a more general structure. In 1961, Edelstein [5] firstly generalized the concept of BCP by considering the globally contractive mapping. In 1965, Presic S.B [6] generalized the BCP to operators defined on product spaces. In 1968, Kannan [7] changes the BCP mapping from contraction mapping to Kannan mapping and prove the fixed point results known as Banach-Kannan contraction principle. In 1969, E. Keeler and A. Meir [8] generalized the BCP which is stated as, let $(\aleph, \Im)$ be a complete metric space and $\mathcal{T} : \aleph \to \aleph$ be an operator. Suppose that for every $\epsilon > 0$, $\exists\ \delta(\epsilon) > 0$ such that $\forall\ \mathsf{c}_1, \mathsf{c}_2 \in \aleph$.

$$\epsilon \leq \Im(\mathsf{c}_1, \mathsf{c}_2) < \epsilon + \delta(\epsilon) \ \Rightarrow \ \Im(\mathcal{T}\mathsf{c}_1, \mathcal{T}\mathsf{c}_2) < \epsilon.$$

Then, F has unique fixed point. Another extension of BCP is done by Nadler [9] , in which he has used set valued contraction mapping instead of single valued contraction mapping. Also, Wardowski [10], generalized the BCP as, "Suppose $(\aleph, \Im)$ be a complete metric space (cMS) and self mapping $\mathcal{T}$ be an F-contraction. Then, $\mathcal{T}$ has unique fixed point (UFP)". In 1972, Chatterjea [11] also generalized the BCP where he used Chatterjea type contraction mapping instead of contraction mapping to prove the fixed point results. In 1974, Ćirić [12] proved a theorem for FP on cMS, which is generalized form of BCP. In [12], Ćirić presented a new category of contractive mappings in the setting of MS. In general, a Ćirić type mapping doesn't need to be continuous, however, it must be continuous at the fixed point. In 1975, BCP is further generalized by Dass and Gupta [13] as they used rational contraction mapping to prove fixed point results.

Metric space is very large space as while slightly change in axioms of metric space may result in, 2-metric spaces [14], cone metric space [15] and many more. In 1989, Bakhtin [16], generalized the concept of MS and introduced the b-metric spaces (bMS). Later on, in 1993, Czerwik [17], also work on b-metric space by considering

the weak triangular inequality to prove fixed points results. Many scholars have constructed numerous fixed point results using the b-metric. In 1994, Matthews [18] introduced the partial metric spaces which have the property that self-distance can't be zero. In 2017, Kamran et al. [19] generalized the concept of bMS known as extended b-metric space (EbMS) by further weakning the triangular inequality. In 2018, Mlaiki et al. [20] gave us novel type of EbMS, that is controlled metric space CMS and double controlled metric space DCMS [21].
Motivated from the work of Berinde [22, 23], in 2012, a new class of contractions introduced by Wardowski [10] named as F-contraction. Samet et al. [24], introduced the notion of α-admissible and α-F-contraction mapping, then they proved some fixed point results for such mappings. Using their idea some authors gave fixed points results for single and multivalued mappings [25]. Moreover, Wardowski introduced the α-F-contraction which is weaker than F-contraction. In 2022, Batul et al. [26], used α-F-contraction for finding fuzzy fixed point results in b-metric space. Also, Sagheer et al. [27], proved some fixed point results on α-F-contraction multi-valued mappings with uniform spaces.
In this thesis generalized Ćirić type α-F-contraction in the framework of DCMS has been introduced. Some significant results for this framework are established and proved. In the continuation, an example and three applications are presented to support the obtained results.

The rest of thesis is arranged as:

• **Chapter-2** is based on some basic definitions and results which will be used in subsequent chapters. Different generalizations of MS such as EbMS, CMS and DCMS are presented and elaborated with examples.

• **Chapter-3** gives a detailed review of Belhenniche et al. [28], where they used Ćirić type contraction in EbMS to prove some fixed points results. To elaborate our results, some interesting examples are given. Applications are also provided for the validity of main result.

• **Chapter-4** includes novel type of contraction known as generalized Ćirić type α-F-contraction in the setting of DCMS. Some significant FP results are established by using this idea. To elaborate the obtained results some examples and applications are given.

- **Chapter-5** includes the conclusion of the thesis.

Chapter 2

Preliminaries

In this chapter, some fundamental definitions from the functional analysis are presented that will be used in the subsequent chapters. We provide the concepts of metric space, b-metric space, extended b-metric space, controlled metric space and double controlled metric space with examples. Also, different type of contraction mappings are introduced with suitable examples. In the end, we give some classical fixed points results.

2.1 Metric Space

Metric is an extension of the Euclidean distance derived from the four well-known features of the Euclidean distance in mathematics. Euclidean metric determines the distance between two points on a straight line. However, distance other than straight lines, such as taxicab distances, may exist.
In 1906, *Frechet* developed the idea of metric space.

Definition 2.1.1.

"A metric space is a pair $(\aleph, \Im)$, where $\aleph$ is a non-empty set and $\Im$ is a metric on $\aleph$(or distance function on $\aleph$), that is, a function define on $\aleph \times \aleph$ such that $\forall\ \mathsf{c}_1, \mathsf{c}_2, \mathsf{c}_3 \in \aleph$ we have:
(M1): $\Im$ is real-valued, finite and non-negative,

(M2): $\Im(\mathsf{c}_1, \mathsf{c}_2) = 0$ if and only if $\mathsf{c}_1 = \mathsf{c}_2$,

(M3): $\Im(\mathsf{c}_1, \mathsf{c}_2) = \Im(\mathsf{c}_2, \mathsf{c}_1)$, (Symmetry)

(M4): $\Im(\mathsf{c}_1, \mathsf{c}_3) \leq \Im(\mathsf{c}_1, \mathsf{c}_2) + \Im(\mathsf{c}_2, \mathsf{c}_3)$. (Triangular inequality)" [2]

Example 2.1.2. Consider $\aleph = \mathbb{R}$ then the mapping $\Im : \aleph \times \aleph \to \mathbb{R}$, defined as:

$$\Im(\mathsf{c}_1, \mathsf{c}_2) = |\mathsf{c}_1 - \mathsf{c}_2| \quad \forall\ \mathsf{c}_1, \mathsf{c}_2 \in \aleph$$

is metric on $\mathbb{R}$ and $(\mathbb{R}, \aleph)$ is a MS.

Example 2.1.3. Consider a real number $p \geq 1$ and define a set of real sequences as

$$l^p = \{\psi = \{\psi_n\} :\mid \psi_1 \mid^p + \mid \psi_2 \mid^p + ... < \infty\}.$$

Define $\Im : l^p \times l^p \to \mathbb{R}$ as

$$\Im(\phi, \varphi) = \left(\sum_{i=1}^{\infty} \mid \phi_i - \varphi_i \mid^p \right)^{\frac{1}{p}}, \quad \{\phi\}, \{\varphi\} \in l^p$$

then, $(l^p, \Im)$ is a MS.

Example 2.1.4. Consider $\aleph = \mathbb{R}^2$ then the mapping $\Im : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined as:

$$\Im(\mathsf{c}_1, \mathsf{c}_2) = \sqrt{({\mathsf{c}_1}^1 - {\mathsf{c}_2}^1)^2 + ({\mathsf{c}_1}^2 - {\mathsf{c}_2}^2)^2}$$

$\forall\ \mathsf{c}_1 = ({\mathsf{c}_1}^1, {\mathsf{c}_1}^2), \mathsf{c}_2 = ({\mathsf{c}_2}^1, {\mathsf{c}_2}^2) \in \aleph$, is metric on $\mathbb{R}^2$ and $(\mathbb{R}^2, \Im)$ is a MS.

Definition 2.1.5.

"A mapping $\mathcal{T} : \aleph \to \mathbb{S}$ of a metric space $\aleph = (\aleph, \Im)$ to $\mathbb{S} = (\mathbb{S}, \Im_1)$ is continuous at a point $\mathsf{c} \in \aleph$ if and only if

$$\mathsf{c}_n \to \mathsf{c}_0 \quad \text{implies} \quad \mathcal{T}\mathsf{c}_n \to \mathcal{T}\mathsf{c}_0."$$ [2]

Definition 2.1.6.

"A sequence $\{\mathsf{c}_n\}$ in a metric space $\aleph = (\aleph, \Im)$ is said to converge or to be convergent if there is a $\mathsf{c} \in \aleph$ such that

$$\lim_{n \to \infty} \Im(\mathsf{c}_n, \mathsf{c}) = 0.$$

c is called limit of $\{\mathsf{c}_n\}$ and we write

$$\lim_{n\to\infty} \mathsf{c}_n = \mathsf{c} \quad \text{or} \quad \mathsf{c}_n \to \mathsf{c}$$

We say that $\{\mathsf{c}_n\}$ converges to c or has the limit c. If $\{\mathsf{c}_n\}$ is not converges, it is said to be divergent."[2]

Example 2.1.7. Consider the set of real numbers $\mathbb{R}$ with metric $\Im(\mathsf{c}_1, \mathsf{c}_2) = |\mathsf{c}_1 - \mathsf{c}_2|$ then, the sequence $\{\mathsf{c}_n\} = \frac{1}{n}$ in $\aleph$ is a convergent sequence.

Definition 2.1.8.

"A sequence $\{\mathsf{c}_n\}$ in a metric space $\aleph = (\aleph, \Im)$ is said to be Cauchy (or fundamental) if for every $\epsilon > 0$ there is an $N = N(\epsilon)$ such that,

$$\Im(\mathsf{c}_m, \mathsf{c}_n) < \epsilon \qquad \text{for every } \; m, n > N."[2]$$

Definition 2.1.9.

"A space $\aleph$ is said to be complete if every Cauchy sequence in $\aleph$ converges (that is, has a limit which is an element of $\aleph$).[2]"

Example 2.1.10. With usual metric on $\mathbb{R}$ the closed interval $[0, 1]$ is complete.

2.2 Some Generalizations of Metric Space

In 1989, Bakhtin [29] introduce the concept of bMS.

Definition 2.2.1.

"Let $\aleph$ be a non-empty set and let $b \geq 1$ be a given real number. A function $\Im_b$: $\aleph \times \aleph \to [0, \infty)$ is called a b-metric if for all $\mathsf{c}_1, \mathsf{c}_2, \mathsf{c}_3 \in \aleph$ the following conditions are satisfied,

(b_1) : $\Im_b(\mathsf{c}_1, \mathsf{c}_2) = 0 \iff \mathsf{c}_1 = \mathsf{c}_2$,

(b_2) : $\Im_b(\mathsf{c}_1, \mathsf{c}_2) = \Im_b(\mathsf{c}_2, \mathsf{c}_1)$,

(b_3) : $\Im_b(\mathsf{c}_1, \mathsf{c}_3) \leq b[\Im_b(\mathsf{c}_1, \mathsf{c}_2) + \Im_b(\mathsf{c}_2, \mathsf{c}_3)]$.

The pair $(\aleph, \Im_b)$ is called a b-metric space."

Example 2.2.2. Consider $\aleph = \mathbb{R}$, the mapping $\Im_b : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by;

$$\Im_b(\mathtt{c}_1, \mathtt{c}_2) = |\mathtt{c}_1 - \mathtt{c}_2|^2$$

is a b-metric on $\mathbb{R}$ with $b = 2$.

Example 2.2.3. Let $(\aleph, \Im)$ be a MS and;

$$\Im_b(\mathtt{c}_1, \mathtt{c}_2) = (\Im(\mathtt{c}_1, \mathtt{c}_2))^p$$

$\forall$ $\mathtt{c}_1, \mathtt{c}_2 \in \aleph$ and $p > 1$ in $\mathbb{R}$. Then,
$(\aleph, \Im_b)$ is bMS with $b = 2^{p-1}$.

In 2017, Kamran et al. [19] generalized the concept of bMS known as EbMS.

Definition 2.2.4.

"Let $\aleph$ be a non-empty set and $\Theta : \aleph \times \aleph \to [1, \infty)$ be a mapping. A function $\Im_\Theta : \aleph \times \aleph \to [0, \infty)$ is called an extended b-metric if $\forall$ $\mathtt{c}_1, \mathtt{c}_2, \mathtt{c}_3 \in \aleph$, it satisfies:
1): $\Im_\Theta(\mathtt{c}_1, \mathtt{c}_2) \geq 0$,
2): $\Im_\Theta(\mathtt{c}_1, \mathtt{c}_2) \iff \mathtt{c}_1 = \mathtt{c}_2$,
3): $\Im_\Theta(\mathtt{c}_1, \mathtt{c}_2) = \Im_\Theta(\mathtt{c}_2, \mathtt{c}_1)$,
4): $\Im_\Theta(\mathtt{c}_1, \mathtt{c}_3) \leq \Theta(\mathtt{c}_1, \mathtt{c}_3)[\Im_\Theta(\mathtt{c}_1, \mathtt{c}_2) + \Im_\Theta(\mathtt{c}_2, \mathtt{c}_3)]$.
The pair $(\aleph, \Im_\Theta)$ is called an extended b-metric space."

Remark:

(i): If $\Theta(\mathtt{c}_1, \mathtt{c}_3) = k$ for $k \geq 1$ in Definition 2.2.4, then Definition 2.2.4 coincides with b-metric space.

(ii): If $\Theta(\mathtt{c}_1, \mathtt{c}_3) = 1$ in Definition 2.2.4, then Definition 2.2.4 becomes metric space.

Example 2.2.5. Consider $\aleph = \{1, 2, 3, ...\}$. Define $\Theta : \aleph \times \aleph \to [1, \infty)$ and $\Im_\Theta : \aleph \times \aleph \to \mathbb{R}^+$ respectively as:

$$\Theta(\mathtt{c}_1, \mathtt{c}_2) = \begin{cases} |\ \mathtt{c}_1 - \mathtt{c}_2\ |,^3 \text{if } \ \mathtt{c}_1 \neq \mathtt{c}_2 \\ 1, \quad \mathtt{c}_1 = \mathtt{c}_2 \end{cases}$$

and

$$\Im_{\Theta} = (\mathtt{c}_1 - \mathtt{c}_2)^4.$$

Then, $(\aleph, \Im_{\Theta})$ is an EbMS.

In 2018, Mlaiki et al. [20] gave us new type of EbMS.

Definition 2.2.6.

"Given $\kappa : \aleph \times \aleph \to [1, \infty)$, where $\aleph$ is non-empty. Let $\Im : \aleph \times \aleph \to [0, \infty)$. Suppose that

$(\Im 1) : \Im(\mathtt{c}_1, \mathtt{c}_2) = 0 \iff \mathtt{c}_1 = \mathtt{c}_2$,

$(\Im 2) : \Im(\mathtt{c}_1, \mathtt{c}_2) = \Im(\mathtt{c}_2, \mathtt{c}_1)$,

$(\Im 3) : \Im(\mathtt{c}_1, \mathtt{c}_3) \leq \kappa(\mathtt{c}_1, \mathtt{c}_2)\Im(\mathtt{c}_1, \mathtt{c}_2) + \kappa(\mathtt{c}_2, \mathtt{c}_3)\Im(\mathtt{c}_2, \mathtt{c}_3)$.

For all $\mathtt{c}_1, \mathtt{c}_2, \mathtt{c}_3 \in \aleph$. Then $\Im$ is called a CM and $(\aleph, \Im)$ is called controlled metric space."

Remark: Every CMS is generalization of bMS.

Example 2.2.7. Consider $\aleph = \mathbb{R}$, the metric $\Im$ given as:

$$\Im(0, 0) = \Im(1, 1) = \Im(2, 2) = 0,$$

and

$$\Im(0, 1) = \Im(1, 0) = 1, \;\; \Im(0, 2) = \Im(2, 0) = \frac{1}{2} \;\; \Im(1, 2) = \Im(2, 1) = \frac{2}{5}.$$

Define $\kappa : \aleph \times \aleph \to [1, \infty)$ by;

$$\kappa(0, 0) = 1, \kappa(1, 1) = \kappa(2, 2) = 1, \kappa(0, 2) = 1, \;\; \kappa(1, 2) = \frac{5}{4}, \;\; \kappa(0, 1) = \frac{11}{10}.$$

Then, $\Im$ is a CMS.

Definition 2.2.8.

"Given non-comparable functions κ, $\varkappa : \aleph \times \aleph \to [1, \infty)$. If $\Im : \aleph \times \aleph \to [0, \infty)$ satisfies:

$(\Im 1) : \Im(\mathsf{c}_1, \mathsf{c}_2) = 0 \iff \mathsf{c}_1 = \mathsf{c}_2,$

$(\Im 2) : \Im(\mathsf{c}_1, \mathsf{c}_2) = \Im(\mathsf{c}_2, \mathsf{c}_1),$

$(\Im 3) : \Im(\mathsf{c}_1, \mathsf{c}_3) \leq \kappa(\mathsf{c}_1, \mathsf{c}_2)\Im(\mathsf{c}_1, \mathsf{c}_2) + \varkappa(\mathsf{c}_2, \mathsf{c}_3)\Im(\mathsf{c}_2, \mathsf{c}_3),$

for all $\mathsf{c}_1, \mathsf{c}_2, \mathsf{c}_3 \in \aleph$. Then $\Im$ is called DCMS by κ and $\varkappa$." [21]

Example 2.2.9. Let $\aleph = [0, \infty)$. Define $\Im$ by,

$$\Im(\mathsf{c}_1, \mathsf{c}_2) = \begin{cases} 0, & \text{iff } \mathsf{c}_1 = \mathsf{c}_2 \\ \frac{1}{\mathsf{c}_1}, & \text{if } \mathsf{c}_1 \geq 1 \text{ and } \mathsf{c}_2 \in [0,1) \\ \frac{1}{\mathsf{c}_2}, & \text{if } \mathsf{c}_2 \geq 1 \text{ and } \mathsf{c}_1 \in [0,1) \\ 1, & \text{if not.} \end{cases}$$

Consider $\kappa, \varkappa : \aleph^2 \to [1, \infty)$ as,

$$\kappa(\mathsf{c}_1, \mathsf{c}_2) = \begin{cases} \mathsf{c}_1, & \text{if } \mathsf{c}_1, \mathsf{c}_2 \geq 1 \\ 1 & \text{if not.} \end{cases}$$

And

$$\varkappa(\mathsf{c}_1, \mathsf{c}_2) = \begin{cases} 1, & \text{if } \mathsf{c}_1, \mathsf{c}_2 < 1 \\ \max\{\mathsf{c}_1, \mathsf{c}_2\}, & \text{if not.} \end{cases}$$

The conditions $(\Im 1)$ and $(\Im 2)$ hold. Suppose that $(\Im 3)$ is satisfied.

(i): If $\mathsf{c}_1 = \mathsf{c}_2$ then $(\Im 3)$ is satisfied.

(ii): Now suppose $\mathsf{c}_1 \neq \mathsf{c}_2$ then,

if $\mathsf{c}_1 \geq 1$ and $\mathsf{c}_2 \in [0,1)$ or $\mathsf{c}_2 \geq 1$ and $\mathsf{c}_1 \in [0,1)$, it is easy to see that $(\Im 3)$ hold. Here we have;

Subcase-1: $\mathsf{c}_1, \mathsf{c}_2 \geq 1$.

If $\mathsf{c}_3 \geq 1$, $(\Im 3)$ holds, if $\mathsf{c}_1 \in [0,1)$ then

$$1 \leq \frac{1}{\mathsf{c}_1} + \mathsf{c}_2 \frac{1}{\mathsf{c}_2},$$

that is, $(\Im 3)$ is satisfied.

Subcase-2: $\mathsf{c}_1, \mathsf{c}_2 < 1$. If $\mathsf{c}_3 \in [0,1)$, $(\Im 3)$ holds, if $\mathsf{c}_3 \geq 1$ then

$$1 \leq \frac{1}{\mathsf{c}_3} + \mathsf{c}_3 \frac{1}{\mathsf{c}_3'}$$

that is, ($\Im 3$) is satisfied.

In all above cases we deduce that $\Im$ is double controlled metric.

Remark:

The class of DCMS is larger than the class of controlled metric space, further, in turn, is larger than EbMS. Moreover, the class of EbMS is larger than class of b-metric space and all above classes of metric spaces are larger than standard metric space.

Obviously, every CMS is DCMS but converse is not true. And every EbMS is a controlled metric and DCMS but converse not hold (Fig. 2.1).

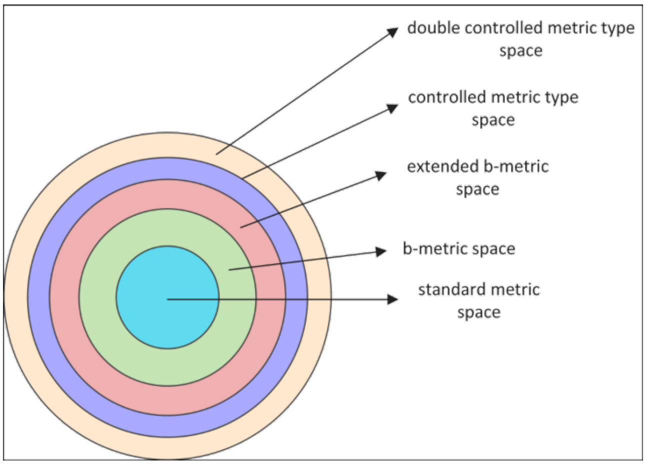

FIGURE 2.1: Relation between different metric type spaces

2.3 Banach Contraction Principle (BCP) and Some of it's Generalizations

A contraction mapping result known as the BCP was first presented by Polish mathematician Banach in 1922.

Definition 2.3.1.

"A fixed point of a mapping $\mathcal{T} : \aleph \to \aleph$ of a set $\aleph$ into itself is an $\mathsf{c}_1 \in \aleph$ which is mapped onto itself (is "kept fixed" by $\mathcal{T}$), that is,

$$\mathcal{T}\mathtt{c}_1 = \mathtt{c}_1,$$

the image $\mathcal{T}\mathtt{c}_1$ coincides with $\mathtt{c}_1$." [2]

In general a mapping may or may not have fixed points, and a fixed point may or may not be unique.

Example 2.3.2. Let $\aleph = \mathbb{R}$. Define the mapping $\mathcal{T} : \mathbb{R} \to \mathbb{R}$ by $\mathcal{T}\mathtt{c}_1 = {\mathtt{c}_1}^3$.

$$\Rightarrow \quad \mathtt{c}_1 = 0, \pm 1$$

are the fixed points of $\mathcal{T}$. Figure (??) represents the graphical picture of this mapping.

Graphical representation:

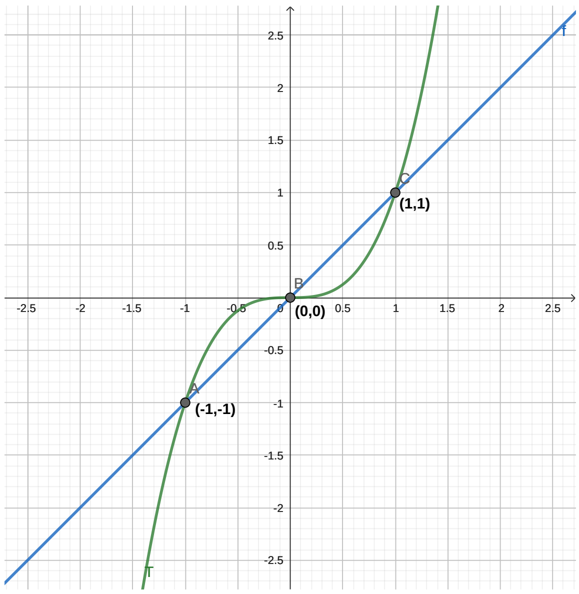

FIGURE 2.2: Mapping having more than one fixed points

Example 2.3.3. Suppose $\aleph = C[0, \frac{1}{2}]$, the mapping $\mathcal{T} : \mathbb{R} \to \mathbb{R}$, defined by;

$$\mathcal{T}(\mathtt{c}_1) = {\mathtt{c}_1}^2 + \mathtt{c}_1 + 1, \;\; \forall \;\; \mathtt{c}_1 \in \mathbb{R}$$

has no fixed point.

Following is graphical representation of functions having no fixed point of the mapping, $\mathcal{T}(\mathsf{c}_1) = {\mathsf{c}_1}^2 + \mathsf{c}_1 + 1, \ \forall \ \mathsf{c}_1 \in \mathbb{R}$

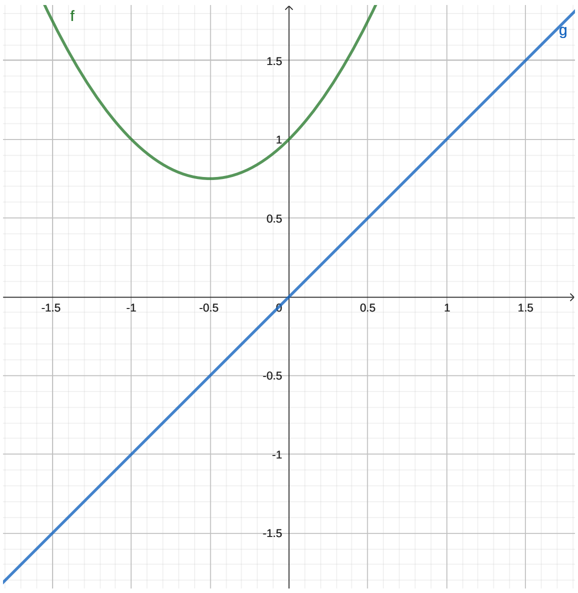

FIGURE 2.3: Mapping having no fixed point

Example 2.3.4. Let $\aleph = \mathbb{R}$. Define the mapping $\mathcal{T} : \mathbb{R} \to \mathbb{R}$ by

$$\mathcal{T}\mathsf{c}_1 = \frac{\mathsf{c}_1}{2} \quad \forall \ \mathsf{c}_1 \in \mathbb{R}$$

has a unique fixed point (Fig 2.4).

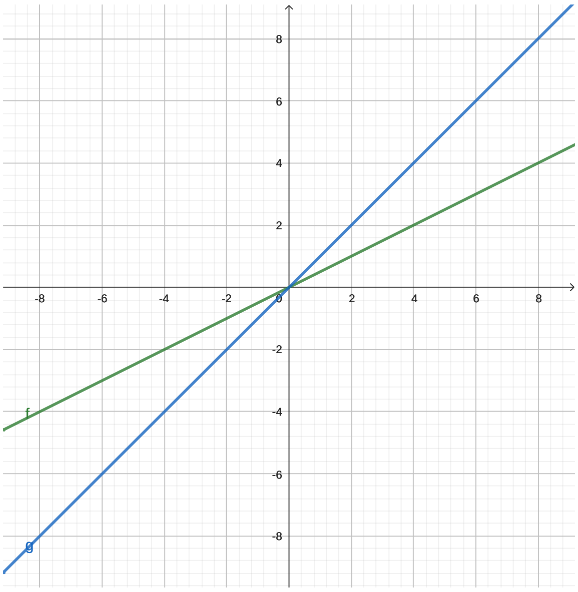

FIGURE 2.4: Mapping having unique fixed point

Example 2.3.5. Consider the following trigonometric function

$$\mathcal{T}\mathsf{c}_1 = \tan(\mathsf{c}_1).$$

This function has infinitely many fixed points that are shown in graph given below:

Graphical representation

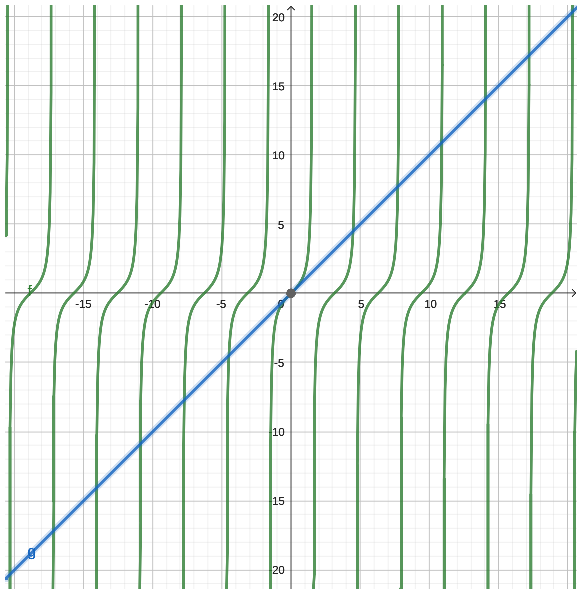

FIGURE 2.5: Mapping having infinitely many fixed point

Definition 2.3.6.

"Let $(\aleph, \Im)$ be a complete metric space. A mapping $\mathcal{T} : \aleph \to \aleph$ is a contraction mapping, or contraction, if, $\exists\, v \in [0, 1)$ such that

$$\Im(\mathcal{T}\mathsf{c}_1, \mathcal{T}\mathsf{c}_2) \leq v\, \Im(\mathsf{c}_1, \mathsf{c}_2) \qquad \forall \qquad \mathsf{c}_1, \mathsf{c}_2 \in \aleph\text{''}\,[30]$$

Example 2.3.7. Let $\aleph = C[0, \frac{1}{2}]$ with metric given by

$$\Im\left(\mathsf{c}_1(t), \mathsf{c}_2(t)\right) = \max_{t\in[o,\frac{1}{2}]} \mid \mathsf{c}_1(t) - \mathsf{c}_2(t) \mid .$$

Then $\mathcal{T} : \aleph \to \aleph$ defined by;

$$\mathcal{T}\mathsf{c}_1(t) = t(\mathsf{c}_1(t) + 1) \quad \forall \quad \mathsf{c}_1 \in \aleph$$

is contraction mapping with contraction constant $\frac{1}{2}$.

Definition 2.3.8. Banach Contraction Principle

"Consider the metric space $\aleph = (\aleph, \Im)$, where $\aleph \neq \emptyset$. Suppose that $\aleph$ is complete and let $\mathcal{T} : \aleph \to \aleph$ be a contraction mapping on $\aleph$. Then $\mathcal{T}$ has precisely one fixed point." [2]

Definition 2.3.9.

"A mapping $\mathcal{T} : \aleph \to \aleph$ is said to be contractive if for $\mathsf{c}_1 \neq \mathsf{c}_2$, we have,

$$\Im(\mathcal{T}(\mathsf{c}_1), \mathcal{T}(\mathsf{c}_2)) < \Im(\mathsf{c}_1, \mathsf{c}_2)$$

for all $\mathsf{c}_1, \mathsf{c}_2 \in \aleph$." [31]

Example 2.3.10. Let $\aleph = [1, \infty)$ with usual metric. Define $\mathcal{T} : \aleph \to \aleph$ by $\mathcal{T}(\mathsf{c}_1) = \mathsf{c}_1 = \frac{1}{\mathsf{c}_1}$,
since, $\lim_{n\to\infty} |1 - \dfrac{1}{\mathsf{c}_1 \mathsf{c}_2}| = 1$
then, $\mathcal{T}$ is a contractive mapping.

In [12], Ćirić presented a new category of contractive mappings in the setting of MS. In general, a Ćirić type mapping doesn't need to be continuous, however, it must appear to be continuous at a fixed point.

Definition 2.3.11.

"A self-mapping $\mathcal{T} : \aleph \to \aleph$ on a metric space $(\aleph, \Im)$ is said to be a Ćirić mapping if, for some $\Upsilon \in (0, 1)$, it satisfies the following inequality, for all $\mathsf{c}_1, \mathsf{c}_2 \in \aleph$,

$$\Im(\mathcal{T}\mathsf{c}_1, \mathcal{T}\mathsf{c}_2) \leq \Upsilon \max\left\{\Im(\mathsf{c}_1, \mathsf{c}_2), \Im(\mathsf{c}_1, \mathcal{T}\mathsf{c}_1), \Im(\mathsf{c}_2, \mathcal{T}\mathsf{c}_2), \frac{1}{2}\left(\Im(\mathsf{c}_1, \mathcal{T}\mathsf{c}_2) + \Im(\mathsf{c}_2, \mathcal{T}\mathsf{c}_1)\right)\right\}."$$

A novel type of contraction, known as an F-contraction, was defined by Wasrdowski in 2012.

Definition 2.3.12.

"Suppose, $\mathbf{F} : \mathbb{R}^+ \to \mathbb{R}$ be a function that satisfying the following:

($\mathbf{F}$-1): F is increasing, i.e., $\forall \mathsf{c}_1, \mathsf{c}_2 \in \mathbb{R}^+$ such that $\mathsf{c}_1 < \mathsf{c}_2, \implies \mathbf{F}(\mathsf{c}_1) < \mathbf{F}(\mathsf{c}_2)$.

($\mathbf{F}$-2): For any sequence $\{\mathsf{c}_n\}_{n=1}^{\infty}$ of positive real numbers,

$$\lim_{n\to\infty} \mathsf{c}_n = 0 \iff \lim_{n\to\infty} \mathbf{F}(\mathsf{c}_n) = -\infty$$

.

(**F**-3): There exist $k \in (0,1)$ such that $\lim_{\mathsf{c}\to 0^+} (\mathsf{c}_n)^k \mathbf{F}(z) = 0$." [10]

Then, the collection of F satisfying above three properties said to be F-mappings.

Definition 2.3.13. "Let $(\aleph, \Im)$ be a metric space. A mapping $\mathcal{T} : \aleph \to \aleph$ is said to be an F-contraction if there exists $\zeta > 0$ such that,

$$\Im(\mathcal{T}\mathsf{c}_1, \mathcal{T}\mathsf{c}_2) > 0 \implies \zeta + F(\Im(\mathcal{T}\mathsf{c}_1, \mathcal{T}\mathsf{c}_2)) \leq F(\Im(\mathsf{c}_1, \mathsf{c}_1) \quad \forall\ \mathsf{c}_1, \mathsf{c}_2 \in \aleph \ \text{ and } F \ \in \ \mathsf{F}."$$

A key result proved by Wardowski's [10] generalized the BCP in this manner;

Theorem 2.3.14. Let $(\aleph, \Im)$ be a cMS and $\mathcal{T} : \aleph \to \aleph$ be an F-contraction. Then, $\mathcal{T}$ has a UFP.

Example 2.3.15. A mapping $F : \mathbb{R}^+ \to \mathbb{R}$ be an F-mapping for:
$F(\mathsf{c}_1) = \ln(\mathsf{c}_1)$ and $\mathsf{c}_1 > 0$ satisfies all conditions of F-mapping (2.3.12) and contraction condition takes the form:

$$\Im(\mathcal{T}\mathsf{c}_1, \mathcal{T}\mathsf{c}_2) \leq e^{-\tau}\Im(\mathsf{c}_1, \mathsf{c}_2),$$

for all $\mathsf{c}_1, \mathsf{c}_2 \in \mathbb{R}$ and $\mathcal{T}\mathsf{c}_1 \neq \mathcal{T}\mathsf{c}_2$.

Samet et al.[24] in 2012 introduced the notion of α-admissible mapping.

Definition 2.3.16.

"For a non-empty set $\aleph$, let $F : \aleph \to \aleph$ and $\alpha : \aleph \times \aleph \to [0, \infty)$ be given mappings, then we say that F is α-admissible if for all $\mathsf{c}_1, \mathsf{c}_2 \in \aleph$, we have

$$\alpha(\mathsf{c}_1, \mathsf{c}_2) \geq 1 \implies \alpha(F\mathsf{c}_1, F\mathsf{c}_2) \geq 1."$$

Later in 2016, Aydi[32] generalized the term "α-admissible" for the pair of mapping in the manner described below:

Definition 2.3.17.

"For a non-empty set $\aleph$, let $F_1, F_2 : \aleph \to \aleph$ and $\alpha : \aleph \times \aleph \to [0, \infty)$ be given mappings, then we say that (F_1, F_2) is generalized α-admissible if for all $\mathsf{c}_1, \mathsf{c}_2 \in \aleph$, we have

$$\alpha(\mathsf{c}_1, \mathsf{c}_2) \geq 1 \implies \alpha(F_1\mathsf{c}_1, F_2\mathsf{c}_2) \geq 1 \quad \text{and} \quad \alpha(F_2\mathsf{c}_2, F_1\mathsf{c}_1) \geq 1."$$

Definition 2.3.18.

"Let $(\aleph, \Im)$ be a metric space and $F_1, F_2 : \aleph \to \aleph$ be self mappings. The pair (F_1, F_2) is α-F-contraction if there exists $\zeta > 0$ such that for all $\mathsf{c}_1, \mathsf{c}_2 \in \aleph$ with $\alpha(\mathsf{c}_1, \mathsf{c}_2) \geq 1$

$$\Im(F_1\mathsf{c}_1, F_2\mathsf{c}_2) > 0 \implies \zeta + F(\Im(F_1\mathsf{c}_1, F_2\mathsf{c}_2)) \leq F(M(\mathsf{c}_1, \mathsf{c}_2)$$

where,

$$M(\mathsf{c}_1, \mathsf{c}_2) = \max\left\{\Im(\mathsf{c}_1, \mathsf{c}_2), \Im(\mathsf{c}_1, F_1\mathsf{c}_1), \Im(\mathsf{c}_2, F_2\mathsf{c}_2), \frac{1}{2}\left[\Im(\mathsf{c}_1, F_2\mathsf{c}_2) + \Im(\mathsf{c}_2, F_1\mathsf{c}_1)\right]\right\}."[10]$$

Theorem 2.3.19. "Let $(\aleph, \Im)$ be a complete metric space and $F_1, F_2 : \aleph \to \aleph$ be such that (F_1, F_2) is α-F-contraction. Suppose that

(i): (F_1, F_2) is a generalized α-admissible pair;

(ii): $\exists\ \mathsf{c}_0 \in \aleph$ such that $\alpha(\mathsf{c}_0, F_1\mathsf{c}_0) \geq 1$ and $\alpha(F_1\mathsf{c}_0, \mathsf{c}_0) \geq 1$;

(iii): F_1 and F_2 are continuous.

Then, F_1 and F_2 have a common fixed point."[33]

Chapter 3

The Fixed-Point Technique to Solve Dynamic Problems in Extended b-Metric Spaces

3.1 Introduction

In this chapter a detailed review of Belhenniche et al. [12] is presented, which is based on FP results for generalized Ćirić type contraction mappings in EbMS. By using this contraction some FP results are established.

Lemma 3.1.1. ([34]) Every sequence $\{\mathsf{c_n}\}_{n\in\mathbb{N}}$ consisting of elements from an EbMS $(\aleph, \Im_\Theta)$, satisfies,

$$\Im_\Theta(\mathsf{c}_0, \mathsf{c}_k) \leq \sum_{i=0}^{k-1} \Im_\Theta(\mathsf{c}_i, \mathsf{c}_{i+1}) \prod_{j=0}^{i} \Theta(\mathsf{c}_j, \mathsf{c}_k)$$

for each $k \in \mathbb{N}$.

Lemma 3.1.2. ([34]) Each sequence $\{\mathsf{c}_n\}_{n\in\mathbb{N}}$ of elements from an EbMS $(\aleph, \Im_\Theta)$, satisfies the inequality,

$$\Im_\Theta(\mathsf{c}_{n+1}, \mathsf{c}_n) \leq \Upsilon \Im_\Theta(\mathsf{c}_n, \mathsf{c}_{n-1}) \text{ for every } n \in \mathbb{N} \text{ and } \Upsilon \in [0, 1),$$

is a Cauchy sequence.

Theorem 3.1.3. Suppose $(\aleph, \Im_\Theta)$ be a complete EbMS with $(\Im_\Theta)$ a continuous functional and let $F_1, F_2 : \aleph \to \aleph$ be two continuous self-operators satisfying,

$$\Im_\Theta(F_1\mathsf{c}_1, F_2\mathsf{c}_2) \leq \Upsilon \max\left\{\Im_\Theta(\mathsf{c}_1,\mathsf{c}_2), \Im_\Theta(\mathsf{c}_1, F_1\mathsf{c}_1), \Im_\Theta(\mathsf{c}_2, F_2\mathsf{c}_2), \frac{1}{2}\left[\Im_\Theta(\mathsf{c}_1, F_2\mathsf{c}_2) + \Im_\Theta(\mathsf{c}_2, F_1\mathsf{c}_1)\right]\right\} \tag{3.1}$$

$\forall \mathsf{c}_1, \mathsf{c}_2 \in \aleph$, where $\Upsilon \in (0,1)$ such that,

$$\Upsilon \lim_{n,m\to\infty} \Theta(\mathsf{c}_n, \mathsf{c}_m) < 1,$$

for any convergent sequence $\{\mathsf{c}_n\}$. Then, the operators F_1, F_2 have a unique CFP.

Proof. Suppose $\mathsf{c}_0 \in \aleph$, and define a sequence $\{\mathsf{c}_n\}$ as:

$$\mathsf{c}_{2n+1} = F_1\mathsf{c}_{2n} \text{ and } \mathsf{c}_{2n+2} = F_2\mathsf{c}_{2n+1}, \qquad n = 0,1,2,3,... \tag{3.2}$$

From (3.1) and (3.2),

$$\begin{aligned}
\Im_\Theta(\mathsf{c}_{2n+1}, \mathsf{c}_{2n+2}) &= \Im_\Theta(F_1\mathsf{c}_{2n}, F_2\mathsf{c}_{2n+1}) \\
&\leq \Upsilon \max\Big\{\Im_\Theta(\mathsf{c}_{2n}, \mathsf{c}_{2n+1}), \Im_\Theta(\mathsf{c}_{2n}, F_1\mathsf{c}_{2n}), \Im_\Theta(\mathsf{c}_{2n+1}, F_2\mathsf{c}_{2n+1}), \\
&\qquad \frac{1}{2}\left[\Im_\Theta(\mathsf{c}_{2n}, F_2\mathsf{c}_{2n+1}) + \Im_\Theta(\mathsf{c}_{2n+1}, F_1\mathsf{c}_{2n})\right]\Big\} \\
&\leq \Upsilon \max\Big\{\Im_\Theta(\mathsf{c}_{2n}, \mathsf{c}_{2n+1}), \Im_\Theta(\mathsf{c}_{2n}, \mathsf{c}_{2n+1}), \Im_\Theta(\mathsf{c}_{2n+1}, \mathsf{c}_{2n+2}) \\
&\qquad \frac{1}{2}\left[\Im_\Theta(\mathsf{c}_{2n}, \mathsf{c}_{2n+2}) + \Im_\Theta(\mathsf{c}_{2n+1}, \mathsf{c}_{2n+1})\right]\Big\} \\
&= \Upsilon \max\Big\{\Im_\Theta(\mathsf{c}_{2n}, \mathsf{c}_{2n+1}), \Im_\Theta(\mathsf{c}_{2n+1}, \mathsf{c}_{2n+2}), \frac{1}{2}\Im_\Theta(\mathsf{c}_{2n}, \mathsf{c}_{2n+2})\Big\}.
\end{aligned} \tag{3.3}$$

Consider the following cases:

Case-1

If

$$\max\Big\{\Im_\Theta(\mathsf{c}_{2n}, \mathsf{c}_{2n+1}), \Im_\Theta(\mathsf{c}_{2n+1}, \mathsf{c}_{2n+2}), \frac{1}{2}\Im_\Theta(\mathsf{c}_{2n}, \mathsf{c}_{2n+2})\Big\} = \Im_\Theta(\mathsf{c}_{2n+1}, \mathsf{c}_{2n+2}),$$

then (3.3) $\Rightarrow$

$$\Im_\Theta(\mathsf{c}_{2n+1},\mathsf{c}_{2n+2}) \leq \Upsilon\Im_\Theta(\mathsf{c}_{2n+1},\mathsf{c}_{2n+2}),$$

which contradict the fact that $\Upsilon < 1$.

Case-2

If

$$\max\left\{\Im_\Theta(\mathsf{c}_{2n},\mathsf{c}_{2n+1}),\Im_\Theta(\mathsf{c}_{2n+1},\mathsf{c}_{2n+2}),\frac{1}{2}\Im_\Theta(\mathsf{c}_{2n},\mathsf{c}_{2n+2})\right\} = \Im_\Theta(\mathsf{c}_{2n},\mathsf{c}_{2n+1}),$$

From (3.3)

$$\Im_\Theta(\mathsf{c}_{2n+1},\mathsf{c}_{2n+2}) \leq \Upsilon\Im_\Theta(\mathsf{c}_{2n},\mathsf{c}_{2n+1}). \tag{3.4}$$

Now by (3.1)

$$\Im_\Theta(\mathsf{c}_{2n+2},\mathsf{c}_{2n+3}) \leq \Upsilon\max\left\{\Im_\Theta(\mathsf{c}_{2n+1},\mathsf{c}_{2n+2})\Im_\Theta(\mathsf{c}_{2n+2},\mathsf{c}_{2n+3}),\frac{1}{2}\Im_\Theta(\mathsf{c}_{2n+1},\mathsf{c}_{2n+3})\right\}.$$

Then, we must take into account the following scenarios:

Case-2a

If

$$\max\left\{\Im_\Theta(\mathsf{c}_{2n+1},\mathsf{c}_{2n+2}),\Im_\Theta(\mathsf{c}_{2n+2},\mathsf{c}_{2n+3}),\frac{1}{2}\Im_\Theta(\mathsf{c}_{2n+1},\mathsf{c}_{2n+3})\right\} = \Im_\Theta(\mathsf{c}_{2n+2},\mathsf{c}_{2n+3}),$$

then

$$\Im_\Theta(\mathsf{c}_{2n+2},\mathsf{c}_{2n+3}) \leq \Upsilon\Im_\Theta(\mathsf{c}_{2n+2},\mathsf{c}_{2n+3}),$$

contradicting the fact that $\Upsilon < 1$.

Case-2b

If

$$\max\left\{\Im_\Theta(\mathsf{c}_{2n+1},\mathsf{c}_{2n+2}),\Im_\Theta(\mathsf{c}_{2n+2},\mathsf{c}_{2n+3}),\frac{1}{2}\Im_\Theta(\mathsf{c}_{2n+1},\mathsf{c}_{2n+3})\right\} = \Im_\Theta(\mathsf{c}_{2n+1},\mathsf{c}_{2n+2}),$$

then,

$$\Im_\Theta(\mathsf{c}_{2n+2}, \mathsf{c}_{2n+3}) \leq \Upsilon \Im_\Theta(\mathsf{c}_{2n+1}, \mathsf{c}_{2n+2}). \tag{3.5}$$

Continuing in the same way

$$\Im_\Theta(\mathsf{c}_{n+1}, \mathsf{c}_{n+2}) \leq \Upsilon \Im_\Theta(\mathsf{c}_{n}, \mathsf{c}_{n+1}) \quad \forall n.$$

Therefore, by Lemma 3.1.2, the above sequence is Cauchy.

Case-2c

If

$$\max\left\{\Im_\Theta(\mathsf{c}_{2n+1}, \mathsf{c}_{2n+2}), \Im_\Theta(\mathsf{c}_{2n+2}, \mathsf{c}_{2n+3}), \frac{1}{2}\Im_\Theta(\mathsf{c}_{2n+1}, \mathsf{c}_{2n+3})\right\} = \frac{1}{2}\Im_\Theta(\mathsf{c}_{2n+1}, \mathsf{c}_{2n+3}),$$

then,

$$\Im_\Theta(\mathsf{c}_{2n+2}, \mathsf{c}_{2n+3}) \leq \Upsilon \frac{1}{2}\Im_\Theta(\mathsf{c}_{2n+1}, \mathsf{c}_{2n+3}),$$

By triangular inequality

$$\begin{aligned}\frac{1}{2}\Im_\Theta(\mathsf{c}_{2n+1}, \mathsf{c}_{2n+3}) \leq \frac{1}{2}\Theta(\mathsf{c}_{2n+1}, \mathsf{c}_{2n+3})(&\Im_\Theta(\mathsf{c}_{2n+1}, \mathsf{c}_{2n+2}) \\ &+ \Im_\Theta(\mathsf{c}_{2n+2}, \mathsf{c}_{2n+3})).\end{aligned}$$

In this case, we obtain

$$\begin{aligned}\Im_\Theta(\mathsf{c}_{2n+2}, \mathsf{c}_{2n+3}) &\leq \Upsilon \frac{1}{2}\Theta(\mathsf{c}_{2n+1}, \mathsf{c}_{2n+3})\Big((\Im_\Theta(\mathsf{c}_{2n+1}, \mathsf{c}_{2n+2})) \\ &\qquad + \Im_\Theta(\mathsf{c}_{2n+2}, \mathsf{c}_{2n+3})\Big), \\ \implies \Im_\Theta(\mathsf{c}_{2n+2}, \mathsf{c}_{2n+3}) &- \frac{1}{2}\Theta(\mathsf{c}_{2n+1}, \mathsf{c}_{2n+3})\Upsilon(\Im_\Theta(\mathsf{c}_{2n+2}, \mathsf{c}_{2n+3})) \\ &\leq \frac{1}{2}\Theta(\mathsf{c}_{2n+1}, \mathsf{c}_{2n+3})\Upsilon(\Im_\Theta(\mathsf{c}_{2n+1}, \mathsf{c}_{2n+2})),\end{aligned}$$

and hence,

$$\begin{aligned}&\left(1 - \frac{1}{2}\Upsilon\Theta(\mathsf{c}_{2n+1}, \mathsf{c}_{2n+3})\right)\Im_\Theta(\mathsf{c}_{2n+2}, \mathsf{c}_{2n+3}) \\ &\leq \frac{1}{2}\Upsilon\Theta(\mathsf{c}_{2n+1}, \mathsf{c}_{2n+3})\Im_\Theta(\mathsf{c}_{2n+1}, \mathsf{c}_{2n+2}).\end{aligned}$$

Thus, we conclude that

$$\Im_{\Theta}(c_{2n+2}, c_{2n+3}) \leq \frac{\Upsilon\Theta(c_{2n+1}, c_{2n+3})}{2 - \Upsilon\Theta(c_{2n+1}, c_{2n+3})}\Im_{\Theta}(c_{2n+1}, c_{2n+2}). \tag{3.6}$$

Continuing in the same way, $\Im_{\Theta}(c_{n+1}, c_{n+2}) \leq \varsigma\Im_{\Theta}(c_n, c_{n+1})\ \forall\ n \in \mathbb{N}$, where $\varsigma := \max\{\frac{\Upsilon\Theta(c_n,c_{n+2})}{2-\Upsilon\Theta(c_n,c_{n+2})}, \Upsilon\}$. Now, we have to show that $\exists\ N_{\varsigma} \in \mathbb{N}$ such that $\varsigma = \varsigma(N_{\varsigma}) \leq 1\ \forall\ n \geq N_{\varsigma}$.

By assumption, $\Upsilon \lim_{n,m\to\infty} \Theta(c_n, c_m) < 1$, we have $2 - \Upsilon \lim_{n,m\to\infty} \Theta(c_n, c_m > 1$. From this,

$$\Upsilon \lim_{n,m\to\infty} \Theta(c_n, c_m) \leq 2 - \Upsilon \lim_{n,m\to\infty} \Theta(c_n, c_m).$$

This implies $\varsigma \leq 1$. Hence, $\Im_{\Theta}(c_{n+1}, c_{n+2}) \leq \varsigma\Im_{\Theta}(c_n, c_{n+1})$ such that $\varsigma \in [0, 1]$. Using Lemma 3.1.2, the sequence $\{c_n\}_{n\in\mathbb{N}}$ is Cauchy.

Case-3

If $\max\left\{\Im_{\Theta}(c_{2n}, c_{2n+1}), \Im_{\Theta}(c_{2n+1}, c_{2n+2}), \frac{1}{2}\Im_{\Theta}(c_{2n}, c_{2n+2})\right\} = \frac{1}{2}\Im_{\Theta}(c_{2n}, c_{2n+2})$,
By (3.5)

$$\Im_{\Theta}(c_{2n+1}, c_{2n+2}) \leq \Upsilon\frac{1}{2}\Im_{\Theta}(c_{2n}, c_{2n+2}) \tag{3.7}$$

Using triangular inequality

$$\frac{1}{2}\Im_{\Theta}(c_{2n}, c_{2n+2}) \leq \frac{1}{2}\Theta(c_{2n}, c_{2n+2})\Big(\Im_{\Theta}(c_{2n}, c_{2n+1}) + \Im_{\Theta}(c_{2n+1}, c_{2n+2})\Big),$$

in this case, we obtain

$$\Im_{\Theta}(c_{2n+1}, c_{2n+2}) \leq \frac{\Theta(c_{2n}, c_{2n+2})\Upsilon}{2}\big(\Im_{\Theta}(c_{2n}, c_{2n+1}) + \Im_{\Theta}(c_{2n+1}, c_{2n+2})\big).$$

$$\Im_{\Theta}(c_{2n+1}, c_{2n+2}) - \frac{\Theta(c_{2n}, c_{2n+2})\Upsilon}{2}\Im_{\Theta}(c_{2n+1}, c_{2n+2}) \leq \frac{\Theta(c_{2n}, c_{2n+2})\Upsilon}{2}(\Im_{\Theta}(c_{2n}, c_{2n+1}),$$

and hence,

$$\left(1 - \frac{\Theta(c_{2n}, c_{2n+2})\Upsilon}{2}\right)\Im_{\Theta}(c_{2n+1}, c_{2n+2}) \leq \frac{\Theta(c_{2n}, c_{2n+2})\Upsilon}{2}\Im_{\Theta}(c_{2n}, c_{2n+1}).$$

Thus, we conclude that

$$\Im_{\Theta}(\mathsf{c}_{2n+1}, \mathsf{c}_{2n+2}) \leq \frac{\Theta(\mathsf{c}_{2n}, \mathsf{c}_{2n+2})\Upsilon}{2 - \Theta(\mathsf{c}_{2n}, \mathsf{c}_{2n+2})\Upsilon} \Im_{\Theta}(\mathsf{c}_{2n}, \mathsf{c}_{2n+1}). \tag{3.8}$$

Now, by using (3.1)

$$\Im_{\Theta}(\mathsf{c}_{2n+2}, \mathsf{c}_{2n+3}) \leq \Upsilon \max\left\{\Im_{\Theta}(\mathsf{c}_{2n+1}, \mathsf{c}_{2n+2})\Im_{\Theta}(\mathsf{c}_{2n+2}, \mathsf{c}_{2n+3}), \frac{1}{2}\Im_{\Theta}(\mathsf{c}_{2n+1}, \mathsf{c}_{2n+3})\right\}.$$

Then, we have three cases:

Case-3a

If

$$\max\left\{\Im_{\Theta}(\mathsf{c}_{2n+1}, \mathsf{c}_{2n+2}), \Im_{\Theta}(\mathsf{c}_{2n+2}, \mathsf{c}_{2n+3}), \frac{1}{2}\Im_{\Theta}(\mathsf{c}_{2n+1}, \mathsf{c}_{2n+3})\right\} = \Im_{\Theta}(\mathsf{c}_{2n+2}, \mathsf{c}_{2n+3}),$$

then

$$\Im_{\Theta}(\mathsf{c}_{2n+2}, \mathsf{c}_{2n+3}) \leq \Upsilon \Im_{\Theta}(\mathsf{c}_{2n+2}, \mathsf{c}_{2n+3}),$$

which contradicts the fact that $\Upsilon < 1$.

Case-3b

If

$$\max\left\{\Im_{\Theta}(\mathsf{c}_{2n+1}, \mathsf{c}_{2n+2}), \Im_{\Theta}(\mathsf{c}_{2n+2}, \mathsf{c}_{2n+3}), \frac{1}{2}\Im_{\Theta}(\mathsf{c}_{2n+1}, \mathsf{c}_{2n+3})\right\} = \Im_{\Theta}(\mathsf{c}_{2n+1}, \mathsf{c}_{2n+2}),$$

then

$$\Im_{\Theta}(\mathsf{c}_{2n+2}, \mathsf{c}_{2n+3}) \leq \Upsilon \Im_{\Theta}(\mathsf{c}_{2n+1}, \mathsf{c}_{2n+2}), \tag{3.9}$$

hence by Lemma 3.1.2, above sequence is Cauchy.

Case-3c

If

$$\max\left\{\Im_{\Theta}(\mathsf{c}_{2n+1}, \mathsf{c}_{2n+2}), \Im_{\Theta}(\mathsf{c}_{2n+2}, \mathsf{c}_{2n+3}), \frac{1}{2}\Im_{\Theta}(\mathsf{c}_{2n+1}, \mathsf{c}_{2n+3})\right\} = \frac{1}{2}\Im_{\Theta}(\mathsf{c}_{2n+1}, \mathsf{c}_{2n+3}),$$

then

$$\Im_\Theta(\mathtt{c}_{2n+2},\mathtt{c}_{2n+3}) \leq \Upsilon \frac{1}{2} \Im_\Theta(\mathtt{c}_{2n+1},\mathtt{c}_{2n+3}).$$

After calculation,

$$\Im_\Theta(\mathtt{c}_{2n+2},\mathtt{c}_{2n+3}) \leq \frac{\Theta(\mathtt{c}_{2n+1},\mathtt{c}_{2n+3})\Upsilon}{2-\Theta(\mathtt{c}_{2n+1},\mathtt{c}_{2n+3})\Upsilon} \Im_\Theta(\mathtt{c}_{2n+1},\mathtt{c}_{2n+2}). \tag{3.10}$$

By proceeding similarly and using (3.8) and (3.10), it follows that
$\Im_\Theta(\mathtt{c}_{n+1},\mathtt{c}_{n+2}) \leq \varsigma \Im_\Theta(\mathtt{c}_n,\mathtt{c}_{n+1})\ \forall\ n \in \mathbb{N}$, where

$$0 < \varsigma(n) := \frac{\Theta(\mathtt{c}_n,\mathtt{c}_{n+2})\Upsilon}{2-\Theta(\mathtt{c}_n,\mathtt{c}_{n+2})\Upsilon} < 1.$$

Again by Lemma 3.1.2, the above sequence $\{\mathtt{c}_n\}$ is Cauchy.
Moreover, in all above cases, the sequence $\{\mathtt{c}_n\}$ is Cauchy. By completeness of $\aleph$, $\exists\ \mathtt{c}_1{}^* \in \aleph$ such that $\Im(\mathtt{c}_n,\mathtt{c}_1{}^*) \to 0$ as $n \to \infty$.
Then, it follows that $\Im(\mathtt{c}_{2n},\mathtt{c}_1{}^*) \to 0$ as $n \to \infty$.

From the continuity of F_1, we have that $\mathtt{c}_{2n+1} = F_1\mathtt{c}_{2n}$ as $n \to \infty$ then, $\mathtt{c}_1{}^* = F_1\mathtt{c}_1{}^*$.

At the same time, we have $\Im(\mathtt{c}_{2n+1},\mathtt{c}_1{}^*) \to 0$ as $n \to \infty$.
Using continuity of F_2, we have $\mathtt{c}_{2n+1} = F_2\mathtt{c}_{2n}$ as $n \to \infty$ then, $\mathtt{c}_1{}^* = F_2\mathtt{c}_1{}^*$.
Hence, $\mathtt{c}_1{}^*$ is CFP of the pair (F_1,F_2).

Now, to check uniqueness of $\mathtt{c}_1{}^*$. Suppose that $\mathtt{c}_2{}^* \in \aleph$ is another CFP of pair (F_1,F_2), then,

$$\begin{aligned}
\Im_\Theta(\mathtt{c}_1{}^*,\mathtt{c}_2{}^*) =& \Im_\Theta(F_1\mathtt{c}_1{}^*,F_2\mathtt{c}_2{}^*)\\
\leq& \Upsilon \max\Big\{\Im_\Theta(\mathtt{c}_1{}^*,\mathtt{c}_2{}^*),\Im_\Theta(\mathtt{c}_1{}^*,F_1\mathtt{c}_1{}^*),\Im_\Theta(\mathtt{c}_2{}^*,F_2\mathtt{c}_2{}^*),\\
&\frac{1}{2}\big[\Im_\Theta(\mathtt{c}_1{}^*,F_2\mathtt{c}_2{}^*)+\Im_\Theta(\mathtt{c}_2{}^*,F_1\mathtt{c}_1{}^*)\big]\Big\}\\
\leq& \Upsilon \max\Big\{\Im_\Theta(\mathtt{c}_1{}^*,\mathtt{c}_2{}^*),\Im_\Theta(\mathtt{c}_1{}^*,\mathtt{c}_1{}^*),\Im_\Theta(\mathtt{c}_2{}^*,\mathtt{c}_2{}^*),\frac{1}{2}\big[\Im_\Theta(\mathtt{c}_1{}^*,\mathtt{c}_2{}^*)+\Im_\Theta(\mathtt{c}_2{}^*,\mathtt{c}_1{}^*)\big]\Big\}\\
=& \Upsilon\Im_\Theta(\mathtt{c}_1{}^*,\mathtt{c}_2{}^*).
\end{aligned}$$

This implies that $\mathtt{c}_1{}^* = \mathtt{c}_2{}^*$. □

If we consider $\Theta(\mathtt{c}_1,\mathtt{c}_1) = b \geq 1$, then we have the following;

Corollary 3.1.4. Suppose $(\aleph, \Im_\Theta)$ be a complete bMS with $(\Im_\Theta)$ a continuous functional and let $F_1, F_2 : \aleph \to \aleph$ be two continuous self-operators satisfying,

$$\Im_\Theta(F_1\mathsf{c}_1, F_2\mathsf{c}_2) \leq \Upsilon \max\left\{\Im_\Theta(\mathsf{c}_1, \mathsf{c}_2), \Im_\Theta(\mathsf{c}_1, F_1\mathsf{c}_1), \Im_\Theta(\mathsf{c}_2, F_2\mathsf{c}_2), \frac{1}{2}\left[\Im_\Theta(\mathsf{c}_1, F_2\mathsf{c}_2) + \Im_\Theta(\mathsf{c}_2, F_1\mathsf{c}_1)\right]\right\} \tag{3.11}$$

$\forall \mathsf{c}_1, \mathsf{c}_2 \in \aleph$, where $\Upsilon \in (0, 1)$ such that,

$$\Upsilon\, b < 1,$$

for any convergent sequence $\{\mathsf{c}_n\}$. Then, the operators F_1, F_2 have a unique CFP.

Now, Theorem 3.1.3 can be proved by dropping the continuity of operators in the following way:

Theorem 3.1.5. Suppose $(\aleph, \Im_\Theta)$ be a complete EbMS where $(\Im_\Theta)$ is continuous and let $F_1, F_2 : \aleph \to \aleph$ be two self-operators satisfying,

$$\Im_\Theta(F_1\mathsf{c}_1, F_2\mathsf{c}_2) \leq \Upsilon \max\left\{\Im_\Theta(\mathsf{c}_1, \mathsf{c}_2), \Im_\Theta(\mathsf{c}_1, F_1\mathsf{c}_1), \Im_\Theta(\mathsf{c}_2, F_2\mathsf{c}_2), \frac{1}{2}\left[\Im_\Theta(\mathsf{c}_1, F_2\mathsf{c}_2) + \Im_\Theta(\mathsf{c}_2, F_1\mathsf{c}_1)\right]\right\} \tag{3.12}$$

$\forall \mathsf{c}_1, \mathsf{c}_2 \in \aleph$, where $\Upsilon \in (0, 1)$ is such that,

$$\Upsilon \lim_{n,m\to\infty} \Theta(\mathsf{c}_n, \mathsf{c}_m) < 1,$$

for any convergent sequence $\{\mathsf{c}_n\}$. Then, the operators F_1, F_2 have a unique CFP.

Proof. Preceding as in Theorem 3.1.3 we can prove sequence $\{\mathsf{c}_n\}$ converges to ${\mathsf{c}_1}^*$.

Since, the operators F_1 and F_2 are not continuous. Suppose, $\Im_\Theta({\mathsf{c}_1}^*, F_1{\mathsf{c}_1}^*) = h > 0$. We have following:

$$\begin{aligned} h = {} & \Im_\Theta({\mathsf{c}_1}^*, F_1{\mathsf{c}_1}^*) \\ \leq {} & \Theta({\mathsf{c}_1}^*, F_1{\mathsf{c}_1}^*)(\Im_\Theta({\mathsf{c}_1}^*, \mathsf{c}_{2n+2}) + \Im_\Theta(\mathsf{c}_{2n+2}, F_1{\mathsf{c}_1}^*)) \\ \leq {} & \Theta({\mathsf{c}_1}^*, F_1{\mathsf{c}_1}^*)\Im_\Theta({\mathsf{c}_1}^*, \mathsf{c}_{2n+2}) + \Theta({\mathsf{c}_1}^*, F_1{\mathsf{c}_1}^*)\Im_\Theta(F_2\mathsf{c}_{2n+1}, F_1{\mathsf{c}_1}^*) \end{aligned} \tag{3.13}$$

As,

$$\Im_\Theta(F_2\mathsf{c}_{2n+1}, F_1{\mathsf{c}_1}^*) \leq \Upsilon \max\{\Im_\Theta(\mathsf{c}_{2n+1}, {\mathsf{c}_1}^*), \Im_\Theta(\mathsf{c}_{2n+1}, F_2\mathsf{c}_{2n+1}), \Im_\Theta({\mathsf{c}_1}^*, F_1{\mathsf{c}_1}^*),$$

$$\frac{\left[\Im_{\Theta}(\mathtt{c}_{2n+1}, F_1\mathtt{c}_1^*) + \Im_{\Theta}(\mathtt{c}_1^*, F_2\mathtt{c}_{2n+1})\right]}{2}\Big\}$$

$$\leq \Upsilon \max\Big\{\Im_{\Theta}(\mathtt{c}_{2n+1}, \mathtt{c}_1^*), \Im_{\Theta}(\mathtt{c}_{2n+1}, \mathtt{c}_{2n+2}), \Im_{\Theta}(\mathtt{c}_1^*, F_1\mathtt{c}_1^*),$$
$$\frac{\left[\Im_{\Theta}(\mathtt{c}_{2n+1}, F_1\mathtt{c}_1^*) + \Im_{\Theta}(\mathtt{c}_1^*, \mathtt{c}_{2n+2})\right]}{2}\Big\}.$$

Using this in (3.13) and taking limit $n \to \infty$

$$\begin{aligned}\Im_{\Theta}(\mathtt{c}_1^*, F_1\mathtt{c}_1^*) \leq\ & \Theta(\mathtt{c}_1^*, F_1\mathtt{c}_1^*)\Im_{\Theta}(\mathtt{c}_1^*, \mathtt{c}_{2n+2}) + \Upsilon\Theta(\mathtt{c}_1^*, F_1\mathtt{c}_1^*)\Im_{\Theta}(\mathtt{c}_1^*, F_1\mathtt{c}_1^*)\\ \leq\ & \Theta(\mathtt{c}_1^*, F_1\mathtt{c}_1^*)\Im_{\Theta}(\mathtt{c}_1^*, \mathtt{c}_{2n+2}) + \Theta(\mathtt{c}_1^*, F_1\mathtt{c}_1^*)\Upsilon h.\end{aligned}$$

From the last inequality, we obtain

$$h \leq\ \Theta(\mathtt{c}_1^*, F_1\mathtt{c}_1^*)\Im_{\Theta}(\mathtt{c}_1^*, \mathtt{c}_{2n+2}) + \Theta(\mathtt{c}_1^*, F_1\mathtt{c}_1^*)\Upsilon h.$$

Consider $\Theta(\mathtt{c}_1^*, F_1\mathtt{c}_1^*) = 1$ and since $\lim_{n\to\infty} \Im_{\Theta}(\mathtt{c}_1^*, \mathtt{c}_{2n+2}) = 0$, we have

$$h \leq \Upsilon h$$

it follows that $\Upsilon \geq 1$ and, hence, a contraction. Therefore $\mathtt{c}_1^* = F_1\mathtt{c}_1^*$.

In the same way, we obtain $\mathtt{c}_1^* = F_2\mathtt{c}_1^*$.

Hence, $\mathtt{c}_1^*$ is common fixed point for the pair (F_1, F_2).

Uniqueness can be proved in the same way as in Theorem 3.1.3. □

If we consider $\Theta(\mathtt{c}_1, \mathtt{c}_1) = b \geq 1$, then we have the following;

Corollary 3.1.6. Suppose $(\aleph, \Im_{\Theta})$ be a complete bMS where $(\Im_{\Theta})$ is continuous and let $F_1, F_2 : \aleph \to \aleph$ be two self-operators satisfying,

$$\Im_{\Theta}(F_1\mathtt{c}_1, F_2\mathtt{c}_2) \leq \Upsilon \max\Bigg\{\Im_{\Theta}(\mathtt{c}_1, \mathtt{c}_2), \Im_{\Theta}(\mathtt{c}_1, F_1\mathtt{c}_1), \Im_{\Theta}(\mathtt{c}_2, F_2\mathtt{c}_2),$$
$$\frac{1}{2}\left[\Im_{\Theta}(\mathtt{c}_1, F_2\mathtt{c}_2) + \Im_{\Theta}(\mathtt{c}_2, F_1\mathtt{c}_1)\right]\Bigg\} \tag{3.14}$$

$\forall\ \mathtt{c}_1, \mathtt{c}_2 \in \aleph$, where $\Upsilon \in (0, 1)$ is such that,

$$\Upsilon\, b < 1,$$

for any convergent sequence $\{\mathtt{c}_n\}$. Then, the operators F_1, F_2 have a unique CFP.

Now, by considering $F_1 = F_2 = F$, a generalization of Ćirić type contraction is established in EbMS.

Theorem 3.1.7. Suppose $(\aleph, \Im_\Theta)$ be a complete EbMS such that $(\Im_\Theta)$ is continuous and let $F : \aleph \to \aleph$ is continuous mapping such that:

$$\Im_\Theta(F\mathsf{c}_1, F\mathsf{c}_2) \leq \Upsilon \max\left\{\Im_\Theta(\mathsf{c}_1, \mathsf{c}_2), \Im_\Theta(\mathsf{c}_1, F\mathsf{c}_1), \Im_\Theta(\mathsf{c}_2, F\mathsf{c}_2), \frac{1}{2}\left[\Im_\Theta(\mathsf{c}_1, F\mathsf{c}_2) + \Im_\Theta(\mathsf{c}_2, F\mathsf{c}_1)\right]\right\} \tag{3.15}$$

$\forall\ \mathsf{c}_1, \mathsf{c}_2 \in \aleph$, where $\Upsilon \in (0, 1)$, and, for each $\mathsf{c}_0 \in \aleph$, $\Upsilon \lim_{n,m\to\infty} \Theta(\mathsf{c}_n, \mathsf{c}_m) < 1$.

Then, F has a UFP.

Proof. Can be followed by taking $F_1 = F_2 = F$ in Theorem 3.1.3. □

If we consider $\Theta(\mathsf{c}_1, \mathsf{c}_1) = b \geq 1$, then we have the following;

Corollary 3.1.8. Suppose $(\aleph, \Im_\Theta)$ be a complete bMS such that $(\Im_\Theta)$ is continuous and let $F : \aleph \to \aleph$ is continuous mapping such that:

$$\Im_\Theta(F\mathsf{c}_1, F\mathsf{c}_2) \leq \Upsilon \max\left\{\Im_\Theta(\mathsf{c}_1, \mathsf{c}_2), \Im_\Theta(\mathsf{c}_1, F\mathsf{c}_1), \Im_\Theta(\mathsf{c}_2, F\mathsf{c}_2), \frac{1}{2}\left[\Im_\Theta(\mathsf{c}_1, F\mathsf{c}_2) + \Im_\Theta(\mathsf{c}_2, F\mathsf{c}_1)\right]\right\} \tag{3.16}$$

$\forall\ \mathsf{c}_1, \mathsf{c}_2 \in \aleph$, where $\Upsilon \in (0, 1)$, and, for each $\mathsf{c}_0 \in \aleph$,

$$\Upsilon b < 1.$$

Then, F has a UFP.

Example 3.1.9. Suppose $\aleph = [0, \infty)$, and define $\Im_\Theta : \aleph \times \aleph \to \mathbb{R}$, and $\Theta : \aleph \times \aleph \to [1, \infty)$ by:

$\Im_\Theta(\mathsf{c}_1, \mathsf{c}_2) := (\mathsf{c}_1 - \mathsf{c}_2)^2$, $\Theta(\mathsf{c}_1, \mathsf{c}_2) := \mathsf{c}_1 + \mathsf{c}_2 + 1$,

then, $(\aleph, \Im_\Theta)$ is a complete (EbMS).

Define F_1 and $F_2 : \aleph \to \aleph$ by $F_1\mathsf{c}_1 = \frac{\mathsf{c}_1}{2}$, $F_2\mathsf{c}_1 = \frac{\mathsf{c}_1}{4}$ respectively.

$$\text{Now} \quad \Im_\Theta(F_1\mathsf{c}_1, F_2\mathsf{c}_2) = \Im_\Theta\left(\frac{\mathsf{c}_1}{2}, \frac{\mathsf{c}_2}{4}\right)^2 = \frac{{\mathsf{c}_1}^2}{4} + \frac{{\mathsf{c}_2}^2}{16} - \frac{\mathsf{c}_1\mathsf{c}_2}{4}.$$

$$\text{Also,} \quad M(\mathsf{c}_1, \mathsf{c}_2) = \max\left\{\Im_\Theta(\mathsf{c}_1, \mathsf{c}_2), \Im_\Theta(\mathsf{c}_1, F_1\mathsf{c}_1), \Im_\Theta(\mathsf{c}_2, F_2\mathsf{c}_2),\right.$$

$$\left.\frac{1}{2}\left[\Im_{\Theta}(\mathtt{c}_1, F_2\mathtt{c}_2) + \Im_{\Theta}(\mathtt{c}_2, F_1\mathtt{c}_1)\right]\right\}$$

$$= \Im_{\Theta}(\mathtt{c}_1, \mathtt{c}_2).$$

Since, $\frac{1}{2}\left[\Im_{\Theta}(\mathtt{c}_1, F_2\mathtt{c}_2) + \Im_{\Theta}(\mathtt{c}_2, F_1\mathtt{c}_1)\right] = \frac{5\mathtt{c}_1{}^2}{8} + \frac{17\mathtt{c}_2{}^2}{32} - \frac{3\mathtt{c}_1\mathtt{c}_2}{4}$, we may write

$$\Im_{\Theta}(F_1\mathtt{c}_1, F_2\mathtt{c}_2) = \frac{\mathtt{c}_1{}^2}{4} + \frac{\mathtt{c}_2{}^2}{16} - \frac{\mathtt{c}_1\mathtt{c}_2}{4} = \frac{1}{2}\left(\frac{\mathtt{c}_1{}^2}{2} + \frac{\mathtt{c}_2{}^2}{8} - \frac{\mathtt{c}_1\mathtt{c}_2}{2}\right)$$

$$\leq \tfrac{1}{2}\left(\tfrac{1}{2}(\Im_{\Theta}(\mathtt{c}_1, F_2\mathtt{c}_2) + \Im_{\Theta}(\mathtt{c}_2, F_1\mathtt{c}_1))\right) \leq \tfrac{1}{2}M(\mathtt{c}_1, \mathtt{c}_2).$$

Therefore, all axioms of Theorem 3.1.3 are satisfied, hence 0 is CFP of F_1 and F_2.

3.2 Applications

Current section include some applications of above theorem to show the fixed point existence and uniqueness of Volterra-type integral equation, system of non-linear fractional differential equation (FDE) and dynamic programming Bellman's equation.

3.2.1 The Existence of a Solution for Integral Equations of the Volterra type

The concept of integral equations is very essential in applied mathematics. At the end of 19th century Vito Volterra proposed the concept of integral equation of Volterra type, later on, Traian Lalescu, worked on it in 1912 [35].

Integral equations of Volterra type are used in many physical areas, including demography, actuarial studeis, radiative equilibrium [36, 37]. Recently, numerous intriguing approaches for solving Volterra integral equations, such as the power-series approach [38], homotopy perturbation approach, block by block approach, and method of expansion [39], have been developed.

Consider the integral equation of Volterra type:

$$\mathsf{c}_1(\mathsf{q}) = \int_0^{\mathsf{q}} J(\mathsf{q}, s, \mathsf{c}_1(s))ds + w(\mathsf{q}), \quad \mathsf{q} \in [0,1]. \tag{3.17}$$

Define the operator $L : C([0,1], \mathbb{R}^n) \to C([0,1], \mathbb{R}^n)$,

$$L\mathsf{c}_1(\mathsf{q}) = \int_o^{\mathsf{q}} J(\mathsf{q}, s, \mathsf{c}_1(s))ds + w(\mathsf{q}), \quad \mathsf{q} \in [0,1].$$

Theorem 3.2.1. Suppose the Equation (3.17) meets the conditions, given below:

(i): $J : [0,1] \times [0,1] \times \mathbb{R}^n \to \mathbb{R}^n$ and $w : [0,1] \to \mathbb{R}^n$ are continuous;

(ii): $J(\mathsf{q}, s, .) : \mathbb{R}^n \to \mathbb{R}^n$, is increasing for each $\mathsf{q} \in [0,1]$and

$0 \leq s \leq 1$;

(iii): $\forall\ \mathsf{q}$, and $s \in [0,1]$, $\exists\ 0 < \Upsilon < 1$;

$$|J(\mathsf{q}, s, \mathsf{c}_1) - J(\mathsf{q}, s, \mathsf{c}_2)| \leq \Upsilon M(\mathsf{c}_1, \mathsf{c}_2),$$

such that, $M(\mathsf{c}_1, \mathsf{c}_2) = \max\Big\{ \|\mathsf{c}_1 - \mathsf{c}_2\|, \|\mathsf{c}_1 - L\mathsf{c}_1\|, \|\mathsf{c}_2 - L\mathsf{c}_2\|, \frac{1}{2}\big(\|\mathsf{c}_1 - L\mathsf{c}_2\| + \|\mathsf{c}_2 - L\mathsf{c}_1\|\big)\Big\}$ for $\mathsf{q}, s \in [0,1]$.

Then, (3.17) has a unique solution.

Proof. Suppose $\aleph = C([0,1], \mathbb{R}^n)$ is equipped with extended b-metric $\Im(\mathsf{c}_1, \mathsf{c}_2) = \|\mathsf{c}_1 - \mathsf{c}_2\|_C = \sup_{\mathsf{q} \in [0,1]} |\mathsf{c}_1(\mathsf{q}) - \mathsf{c}_2(\mathsf{q})|^2$, where $\Theta : \aleph \times \aleph \to [1, \infty)$ defined as;

$$\Theta(\mathsf{c}_1, \mathsf{c}_2) = 2\|\mathsf{c}_1(\mathsf{q})\| + 3\|\mathsf{c}_2(\mathsf{q})\| + 1.$$

Assume that $\Upsilon \lim_{n,m\to\infty} \Theta(\mathsf{c}_n, \mathsf{c}_m) < 1$

From $(\mathsf{c}_1 - \mathsf{c}_2)^2 \geq 0$, we obtain;

$$\frac{1}{4}({\mathsf{c}_1}^2 + {\mathsf{c}_2}^2) \geq \frac{1}{2}\mathsf{c}_1\mathsf{c}_2. \tag{3.18}$$

Now

$$|L\mathsf{c}_1(\mathsf{q}) - L\mathsf{c}_2(\mathsf{q})|^2 \leq \int_0^{\mathsf{q}} |J(\mathsf{q}, s, \mathsf{c}_1(s)) - J(\mathsf{q}, s, \mathsf{c}_2(s))|^2 ds$$

$$
\begin{aligned}
&\leq \Upsilon^2 \int_0^{\mathsf{q}} \max\Big\{|\mathsf{c}_1(s)-\mathsf{c}_2(s)|^2, |\mathsf{c}_1(s)-L\mathsf{c}_1(s)|^2, |\mathsf{c}_2(s)-L\mathsf{c}_2(s)|^2,\\
&\quad \frac{1}{4}\big[|\mathsf{c}_1(s)-L\mathsf{c}_{2(}s)|+|\mathsf{c}_2(s)-L\mathsf{c}_1(s)|\big]^2\Big\}ds\\
|L\mathsf{c}_1(\mathsf{q})-L\mathsf{c}_2(\mathsf{q})|^2 &\leq \Upsilon^2 \int_0^{\mathsf{q}} \max\Big\{|\mathsf{c}_1(s)-\mathsf{c}_2(s)|^2, |\mathsf{c}_1(s)-L\mathsf{c}_{1(s)}|^2, |\mathsf{c}_2(s)-L\mathsf{c}_2(s)|^2,\\
&\quad \frac{1}{4}\big(\mathsf{c}_1(s)-\mathsf{c}_2(s)\big)^2+\frac{1}{2}\big((\mathsf{c}_1(s)-L\mathsf{c}_{2(}s))(\mathsf{c}_2(s)-L\mathsf{c}_1(s))+\\
&\quad \frac{1}{4}\big(\mathsf{c}_2(s)-L\mathsf{c}_1(s)\big)^2\Big\}ds\\
&\leq \Upsilon^2 \int_0^{t} \max\Big\{|\mathsf{c}_1(s)-\mathsf{c}_2(s)|^2, |\mathsf{c}_1(s)-L\mathsf{c}_2(s)|^2, |\mathsf{c}_2(s)-L\mathsf{c}_2(s)|^2,\\
&\quad \frac{1}{2}\big[|\mathsf{c}_1(s)-L\mathsf{c}_2(s)|^2+|\mathsf{c}_2(s)-L\mathsf{c}_1(s)|^2\big]\Big\}ds.
\end{aligned}
$$

Since, $\|\mathsf{c}_1\|_C = \sup_{\mathsf{q}\in[0,1]}\{|\mathsf{c}_1(\mathsf{q})|^2\}$, taking $\sup_{\mathsf{q}\in[0,1]}$ in above inequality,

$$
\begin{aligned}
\sup_{\mathsf{q}\in[0,1]}|L\mathsf{c}_1(\mathsf{q})-L\mathsf{c}_2(\mathsf{q})|^2 &\leq \Upsilon^2 \int_0^{\mathsf{q}} \max \sup_{\mathsf{q}\in[0,1]}\Big\{|\mathsf{c}_1-\mathsf{c}_2|^2, |\mathsf{c}_1-L\mathsf{c}_1|^2, |\mathsf{c}_2-L\mathsf{c}_2|^2,\\
&\quad \frac{1}{2}\big[|\mathsf{c}_1-L\mathsf{c}_2|^2+|\mathsf{c}_2-L\mathsf{c}_1|^2\big]\Big\}ds\\
\|L\mathsf{c}_1(\mathsf{q})-L\mathsf{c}_2(\mathsf{q})\|_C &\leq \Upsilon^2 \int_0^{\mathsf{q}} \max\Big\{\|\mathsf{c}_1-\mathsf{c}_2\|_C, \|\mathsf{c}_1-L\mathsf{c}_1\|_C, \|\mathsf{c}_2-L\mathsf{c}_2\|_C,\\
&\quad \frac{1}{2}\big[\|\mathsf{c}_1-L\mathsf{c}_2\|_C+\|\mathsf{c}_2-L\mathsf{c}_1\|_C\big]\Big\}ds\\
\Im_\Theta(L\mathsf{c}_1, L\mathsf{c}_2) &\leq \Upsilon^2 \max\Big\{\Im(\mathsf{c}_1,\mathsf{c}_2), \Im(\mathsf{c}_1, L\mathsf{c}_1), \Im(\mathsf{c}_2, L\mathsf{c}_2),\\
&\quad \frac{1}{2}\big(\Im(\mathsf{c}_1, L\mathsf{c}_2)+\Im(\mathsf{c}_2, L\mathsf{c}_1)\big)\Big\}\\
\Im_\Theta(L\mathsf{c}_1, L\mathsf{c}_2) &\leq \Upsilon^2 M(\mathsf{c}_1,\mathsf{c}_2)
\end{aligned}
$$

Now $0 < \alpha = \Upsilon^2 < 1$, therefore

$$
\Im_\Theta(L\mathsf{c}_1, L\mathsf{c}_2) \leq \alpha M(\mathsf{c}_1,\mathsf{c}_2)
$$

for each $\mathsf{c}_1, \mathsf{c}_2 \in \aleph$. Conclusion follows from Theorem 3.1.3. □

3.2.2 The Occurrence of a Common Solution to a Set of Nonlinear Fractional Differential Equation

Because of the demands of many real-world applications, fractional calculus of FDE is a powerful instrument in the domain of mathematics.

In current application, we use Theorem 3.1.3 to show the existence of solution for a nonlinear FDE system of the Caputo type derivative.

Let $y : [0, \infty) \to \mathbb{R}$ be a continuous function. The Caputo derivative of order $\Psi > 0$ of the function y is:

$$ {}^{c}D^{\Psi}(y(\mathsf{q})) := \frac{1}{\Gamma(k-\Psi)} \int_{o}^{\mathsf{q}} (\mathsf{q}-s)^{k-\Psi-1} g^{(k)}(s) ds \quad (k-1 < \Psi < k, k = [\Psi]+1), \tag{3.19} $$

where $[\Psi]$, Γ denote the integer component of $\mathbb{R}^{+}$, and the Gamma function respectively ([40]).

In current section, we demonstrate how Theorem 3.1.3 can be used to demonstrate the presence of one common solution for all nonlinear FDE system,

$$ \begin{cases} {}^{c}D^{\Psi}(\mathsf{c}_1(\mathsf{q})) + p_1(\mathsf{q}, \mathsf{c}_1(\mathsf{q})) = 0 \\ {}^{c}D^{\Psi}(\mathsf{c}_2(\mathsf{q})) + p_2(\mathsf{q}, \mathsf{c}_2(\mathsf{q})) = 0 \end{cases} \tag{3.20} $$

for $\mathsf{q} \in [0, 1], \Psi < 1$, with the conditions at boundary:

$$ \begin{cases} \mathsf{c}_1(0) = 0 = \mathsf{c}_1(1), \\ \mathsf{c}_2(0) = 0 = \mathsf{c}_2(1), \end{cases} \tag{3.21} $$

where $\mathsf{c}_1 \in C([0, 1], \mathbb{R})$, $p_1, p_2 : [0, 1] \times \mathbb{R} \to \mathbb{R}$ are continuous functions, and ${}^{c}D^{\Psi}$ is the Caputo derivative of order Ψ. Also, Green function of (3.20) is given in [41] as:

$$ \mathcal{G}(\mathsf{q}, s) = \begin{cases} \frac{(\mathsf{q}(1-s))^{\gamma-1} - (\mathsf{q}-s)^{\gamma-1}}{\Gamma(\gamma)}, & \text{if } 0 \le s \le \mathsf{q} \le 1, \\ \frac{(\mathsf{q}(1-s))^{\gamma-1}}{\Gamma(\gamma)}, & \text{if } 0 \le \mathsf{q} \le s \le 1. \end{cases} $$

Theorem 3.2.2. Given a system of nonlinear FDE (3.20), which satisfies:
(i): $F_1, F_2 : C([0, 1], \mathbb{R}) \to C([0, 1], \mathbb{R})$

are defined as follows:

$$\begin{cases} F_1\mathsf{c}_1 = & \int_0^1 \mathcal{G}(\mathsf{q},s)p_1(s,\mathsf{c}_1(s))ds, \\ F_2\mathsf{c}_2 = & \int_0^1 \mathcal{G}(\mathsf{q},s)p_2(s,\mathsf{c}_2(s))ds; \end{cases} \tag{3.22}$$

(ii): $|p(\mathsf{q},\mathsf{c}_1)-p(\mathsf{q},\mathsf{c}_2)| \leq \frac{1}{\Omega}M(\mathsf{c}_1,\mathsf{c}_2) \quad \forall \quad \mathsf{q} \in [0,1], \Omega > 1,\ \mathsf{c}_1,\mathsf{c}_2 \in \mathbb{R}$
where,

$$M(\mathsf{c}_1,\mathsf{c}_2) \leq \Upsilon \max\left\{\Im(\mathsf{c}_1,\mathsf{c}_2), \Im(\mathsf{c}_1,F_1\mathsf{c}_1), \Im(\mathsf{c}_2,F_2\mathsf{c}_2), \frac{1}{2}\left(\Im(\mathsf{c}_1,F_2\mathsf{c}_2)+\Im(\mathsf{c}_2,F_1\mathsf{c}_1)\right)\right\}.$$

Then, the Equation (3.20) has unique solution.

Proof. Suppose $\aleph = C([0,1],\mathbb{R})$ with the Bielecki norm

$$\Im_\Theta(\mathsf{c}_1,\mathsf{c}_2) = \|\mathsf{c}_1\| = \sup_{\mathsf{q}\in[0,1]} \left\{|\mathsf{c}_1(\mathsf{q})|e^{-\Omega\mathsf{q}}\right\} \quad \text{with} \quad \Omega > 1$$

and $\Theta : \aleph \times \aleph \to [1,\infty)$ is given by $\Theta(\mathsf{c}_1,\mathsf{c}_2) = |\mathsf{c}_1(\mathsf{q})| + 2|\mathsf{c}_2(\mathsf{q})| + 1$. Assume that $\Upsilon \lim_{n,m\to\infty} \Theta(\mathsf{c}_n,\mathsf{c}_m) < 1$.

It is easy to conclude that $(\aleph, \Im_\Theta)$ is complete EbMS.
It is obvious that ${\mathsf{c}_1}^* \in \aleph$ is a common solution for Caputo derivative (3.20) iff ${\mathsf{c}_1}^* \in \aleph$ is a common solution for the (3.22), $\forall\ \mathsf{q} \in [0,1]$. Then, (3.20) can be solved to find an element ${\mathsf{c}_1}^* \in \aleph$, such that ${\mathsf{c}_1}^*$ is a CFP for the operators F_1 and F_2.
By (i) and (ii);

$$\begin{aligned}
|F_1\mathsf{c}_1(\mathsf{q}) - F_2\mathsf{c}_1(\mathsf{q})|^2 &= \left|\int_0^1 \mathcal{G}\left[p_1(\mathsf{q},\mathsf{c}_1(\mathsf{q})) - p_2(\mathsf{q},\mathsf{c}_2(\mathsf{q}))\right]\Im\mathsf{q}\right|^2 \\
&\leq \left(\int_0^1 \mathcal{G}(\mathsf{q},s)\Im\mathsf{q}\right)^2 \int_0^1 |p_1(\mathsf{q},\mathsf{c}_1(\mathsf{q})) - p_2(\mathsf{q},\mathsf{c}_2(\mathsf{q}))|^2\Im\mathsf{q} \\
&\leq \frac{1}{\Omega^2}|M(\mathsf{c}_1,\mathsf{c}_2)e^{-\Omega\mathsf{q}}|^2 e^{2\Omega\mathsf{q}} \left(\int_0^1 \mathcal{G}(\mathsf{q},s)\Im\mathsf{q}\right)^2.
\end{aligned}$$

Now

$$|[F_1\mathsf{c}_1(\mathsf{q}) - F_2\mathsf{c}_1(\mathsf{q})]e^{-\Omega\mathsf{q}}|^2 \leq \frac{1}{\Omega^2}|M(\mathsf{c}_1,\mathsf{c}_2)e^{-\Omega\mathsf{q}}|^2 \left(\int_0^1 \mathcal{G}(\mathsf{q},s)\Im s\right)^2. \tag{3.23}$$

By taking $\sup_{\mathsf{q}\in[0,1]}$ in above inequality, we get

$$|\sup_{\mathsf{q}\in[0,1]}\{(F_1\mathtt{c}_1(\mathsf{q}) - F_2\mathtt{c}_1(\mathsf{q}))e^{\Omega\mathsf{q}}\}|^2 \leq \frac{1}{\Omega^2}\sup_{\mathsf{q}\in[0,1]}|M(\mathtt{c}_1,\mathtt{c}_2))e^{\Omega\mathsf{q}}|^2 \sup_{\mathsf{q}\in[0,1]}\left\{\left(\int_0^1 \mathcal{G}(\mathsf{q},s)\Im s\right)^2\right\}$$
$$\leq \frac{1}{\Omega^2}\sup_{\mathsf{q}\in[0,1]}|M(\mathtt{c}_1,\mathtt{c}_2))e^{\Omega\mathsf{q}}|^2.$$

Then,

$$\|F_1\mathtt{c}_1 - F_2\mathtt{c}_2\| \leq \frac{1}{\Omega}\|M(\mathtt{c}_1,\mathtt{c}_2)\|. \tag{3.24}$$

For $0 < \Upsilon = \frac{1}{\Omega} < 1$, using Theorem 3.1.3, there exist ${\mathtt{c}_1}^* \in \aleph$ as a CFP of the operators F_1 and F_2. □

3.2.3 An Existence of Solution to the Dynamic Programming Equation

Suppose that $\aleph$ is a state space and set of control values $\mathsf{U}(\mathtt{c}_1) \subset \mathsf{U}$. Let $\mathtt{M}$ be the set of all functions $\varrho : \aleph \to \mathsf{U}$ with $\varrho(\mathtt{c}_1) \in \mathsf{U}(\mathtt{c}_1)\ \forall\ \mathtt{c}_1 \in \aleph$, and $\mathtt{M}$ is said to be "stationary policy". Suppose $B(\aleph)$ be the set of real-valued bounded functions $\mathrm{S} : \aleph \to \mathbb{R}$ and $\Theta : \aleph \times \aleph \to [1,\infty)$. For each policy $\varrho \in \mathtt{M}$, assume that the mapping $F_\varrho : B(\aleph) \to B(\aleph)$ defined as:

$$F_\varrho \mathrm{S}(\mathtt{c}_1) = \mathsf{H}(\mathtt{c}_1, \varrho(\mathtt{c}_1), \mathrm{S}) \text{ for all } \mathtt{c}_1 \in \aleph.$$

Where $\mathsf{H} : \aleph \times \mathsf{U} \times B(\aleph) \to \mathbb{R}$.

We also suppose mapping $F : B(\aleph) \to B(\aleph)$ defined as:

$$F\mathrm{S}(\mathtt{c}_1) = \inf_{u\in\mathsf{U}(\mathtt{c}_1)}\{\mathsf{H}(\mathtt{c}_1, u, \mathrm{S})\} = \min_{\varrho\in\mathbb{M}} F_\varrho \mathrm{S}(\mathtt{c}_1) \text{ for all } \mathtt{c}_1 \in \aleph.$$

Now, the pair $(B(\aleph), \|.\|_\Theta)$, such that

$$\|\mathrm{S}\|_\Theta = \sup_{\mathtt{c}_1\in\aleph}|\mathrm{S}(\mathtt{c}_1)|^2, \quad \mathrm{S} \in B(\aleph) \tag{3.25}$$

is complete EbMS.

We find the optimal cost of the function $\mathbb{J}* \in B(\aleph)$ as,

$$\mathrm{S}(\mathtt{c}_1) = \inf_{u \in \mathsf{U}(\mathtt{c}_1)} \{\mathsf{H}(\mathtt{c}_1, u, \mathrm{S})\} \quad \forall \mathtt{c}_1 \in \aleph. \tag{3.26}$$

This is known as the Bellman equation [42]. Our goal of this section is to apply Theorem (3.1.3) to determine the UFP of F within $B(\aleph)$. The following suppositions are required:

A1): (Well posedness). $\forall$ S $\in B(\aleph)$, and for all $\varrho \in \mathtt{M}$, we have that $F_\varrho \mathrm{S} \in B(\aleph)$ and $F\mathrm{S} \in B(\aleph)$.

A2): (Monotonicity). If $\mathrm{S}, \mathrm{S}' \in B(\aleph)$, and $\mathrm{S} \leq \mathrm{S}'$, then

$$\mathsf{H}(\mathtt{c}_1, u, \mathrm{S}) \leq \mathsf{H}(\mathtt{c}_1, u, \mathrm{S}') \quad \forall \ \mathtt{c}_1 \in \aleph, u \in \mathsf{U}$$

A3): (Attainability). $\forall$ S $\in B(\aleph)$, $\exists$ $\varrho \in \mathtt{M}$, such that $F_\varrho \mathrm{S} = F\mathrm{S}$.

Theorem 3.2.3. Suppose that the Bellman equation meets the following criteria:

(i):F_ϱ and F are monotone;

(ii):$F_\varrho : B(\aleph) \to B(\aleph)$ is a Ćirić type contraction mapping.

Then, there is just one solution to the Bellman equation in $(B(\aleph), \Im)$.

Proof. Suppose $B(\aleph)$ be an EbMS with $\|J\| = \sup_{\mathtt{c}_1 \in \aleph}\{|\mathrm{S}(\mathtt{c}_1)|^2\}$ and $\Theta : \aleph \times \aleph \to [1, \infty)$ defined by $\Theta(\mathrm{S}, \mathrm{S}') := 2|\mathrm{S}(\mathtt{c}_1)| + 3|\mathrm{S}'(\mathtt{c}_1)| + 1$ with the assumption that $\Upsilon \lim_{n,m\to\infty} \Theta(\mathrm{S}_n, \mathrm{S}'_m) < 1$.

Suppose the operator $F : B(\aleph) \to B(\aleph)$ defined as:

$$F\mathrm{S}(\mathtt{c}_1) = \inf_{\varrho \in \mathsf{U}(\mathtt{c}_1)} \{\mathsf{H}(\mathtt{c}_1, u, \mathrm{S})\} \qquad \forall \ \mathtt{c}_1 \in \aleph.$$

As, $(a-b)^2 \geq 0$, we can write,

$$\frac{1}{4}(a^2 + b^2) \geq \frac{1}{2}ab. \tag{3.27}$$

Now, for operator F;

$$|F\mathrm{S}(\mathtt{c}_1) - F\mathrm{S}'(\mathtt{c}_1)|^2 \leq \ |\mathsf{H}(\mathtt{c}_1, u, \mathrm{S}) - \mathsf{H}(\mathtt{c}_1, u, \mathrm{S}')|^2$$

$$
\begin{aligned}
&\leq \Upsilon^2 \Big| \max\Big\{ |\mathrm{S}(\mathtt{c}_1) - \mathrm{S}'(\mathtt{c}_1)|, |\mathrm{S}(\mathtt{c}_1) - F_\varrho \mathrm{S}(\mathtt{c}_1)|, |\mathrm{S}'(\mathtt{c}_1) - F_\varrho \mathrm{S}'(\mathtt{c}_1)|, \\
&\quad \frac{1}{2}\big(|\mathrm{S}(\mathtt{c}_1) - F_\varrho \mathrm{S}'(\mathtt{c}_1)| + |\mathrm{S}'(\mathtt{c}_1) - F_\varrho \mathrm{S}(\mathtt{c}_1)|\big) \Big\} \Big|^2 \\
&\leq \Upsilon^2 \max\Big\{ |\mathrm{S}(\mathtt{c}_1) - \mathrm{S}'(\mathtt{c}_1)|^2, |\mathrm{S}(\mathtt{c}_1) - F_\varrho \mathrm{S}(\mathtt{c}_1)|^2, |\mathrm{S}'(\mathtt{c}_1) - F_\varrho \mathrm{S}'(\mathtt{c}_1)|^2, \\
&\quad |\frac{1}{2}\big(|\mathrm{S}(\mathtt{c}_1) - F_\varrho \mathrm{S}'(\mathtt{c}_1)| + |\mathrm{S}'(\mathtt{c}_1) - F_\varrho \mathrm{S}(\mathtt{c}_1)|\big)|^2 \Big\} \\
&\leq \Upsilon^2 \max\Big\{ |\mathrm{S}(\mathtt{c}_1) - \mathrm{S}'(\mathtt{c}_1)|^2, |\mathrm{S}(\mathtt{c}_1) - F_\varrho \mathrm{S}(\mathtt{c}_1)|^2, |\mathrm{S}'(\mathtt{c}_1) - F_\varrho \mathrm{S}'(\mathtt{c}_1)|^2, \\
&\quad \frac{1}{4}\big[(\mathrm{S}(\mathtt{c}_1) - F_\varrho \mathrm{S}'(\mathtt{c}_1))^2 + (\mathrm{S}'(\mathtt{c}_1) - F_\varrho \mathrm{S}(\mathtt{c}_1))^2 \\
&\quad + 2\big((\mathrm{S}(\mathtt{c}_1) - F_\varrho \mathrm{S}'(\mathtt{c}_1))(\mathrm{S}'(\mathtt{c}_1) - F_\varrho \mathrm{S}(\mathtt{c}_1))\big)\big] \Big\} \\
|F\mathrm{S}(\mathtt{c}_1) - F\mathrm{S}'(\mathtt{c}_1)|^2 &\leq \Upsilon^2 \max\Big\{ |\mathrm{S}(\mathtt{c}_1) - \mathrm{S}'(\mathtt{c}_1)|^2, |\mathrm{S}(\mathtt{c}_1) - F_\varrho \mathrm{S}(\mathtt{c}_1)|^2, |\mathrm{S}'(\mathtt{c}_1) - F_\varrho \mathrm{S}'(\mathtt{c}_1)|^2, \\
&\quad \frac{1}{4}\big[(\mathrm{S}(\mathtt{c}_1) - F_\varrho \mathrm{S}'(\mathtt{c}_1))^2 + (\mathrm{S}'(\mathtt{c}_1) - F_\varrho \mathrm{S}(\mathtt{c}_1))^2\big] \\
&\quad + \frac{1}{2}\big[\big((\mathrm{S}(\mathtt{c}_1) - F_\varrho \mathrm{S}'(\mathtt{c}_1))(\mathrm{S}'(\mathtt{c}_1) - F_\varrho \mathrm{S}(\mathtt{c}_1))\big)\big] \Big\}.
\end{aligned}
$$

Moreover, from the above, by using $F_\varrho \mathrm{S}(\mathtt{c}_1) \geq F\mathrm{S}(\mathtt{c}_1)$, and (3.27), we obtain

$$
\begin{aligned}
|F\mathrm{S}(\mathtt{c}_1) - F\mathrm{S}'(\mathtt{c}_1)|^2 &\leq \Upsilon^2 \max\Big\{ |\mathrm{S}(\mathtt{c}_1) - \mathrm{S}'(\mathtt{c}_1)|^2, |\mathrm{S}(\mathtt{c}_1) - F_\varrho \mathrm{S}(\mathtt{c}_1)|^2, |\mathrm{S}'(\mathtt{c}_1) - F_\varrho \mathrm{S}'(\mathtt{c}_1)|^2, \\
&\quad + \frac{1}{2}\big[|\mathrm{S}(\mathtt{c}_1) - F_\varrho \mathrm{S}'(\mathtt{c}_1)|^2 + |\mathrm{S}'(\mathtt{c}_1) - F_\varrho \mathrm{S}(\mathtt{c}_1)|^2\big] \Big\} \\
&\leq \Upsilon^2 \max\Big\{ \|\mathrm{S} - \mathrm{S}'\|_\Theta, \|\mathrm{S} - F_\varrho \mathrm{S}\|_\Theta, \|\mathrm{S}' - F_\varrho \mathrm{S}'\|_\Theta, \\
&\quad + \frac{1}{2}\big[\|\mathrm{S} - F_\varrho \mathrm{S}'\|_\Theta + \|\mathrm{S}' - F_\varrho \mathrm{S}\|_\Theta\big] \Big\} \\
&\leq \Upsilon^2 \max\Big\{ \|\mathrm{S} - \mathrm{S}'\|_\Theta, \|\mathrm{S} - F\mathrm{S}\|_\Theta, \|\mathrm{S}' - F\mathrm{S}'\|_\Theta, \\
&\quad + \frac{1}{2}\big[\|\mathrm{S} - F\mathrm{S}'\|_\Theta + \|\mathrm{S}' - F\mathrm{S}\|_\kappa\big] \Big\} \\
&\leq \Upsilon^2 \max\Big\{ \|\mathrm{S} - \mathrm{S}'\|_\Theta, \|\mathrm{S} - F\mathrm{S}\|_\Theta, \|\mathrm{S}' - F\mathrm{S}'\|_\Theta, \\
&\quad + \frac{1}{2}\big[\|\mathrm{S} - F\mathrm{S}'\|_\Theta + \|\mathrm{S}' - F\mathrm{S}\|_\Theta\big] \Big\} \\
|F\mathrm{S}(\mathtt{c}_1) - F\mathrm{S}'(\mathtt{c}_1)|^2 &\leq \Upsilon^2 M(\mathrm{S}, \mathrm{S}'),
\end{aligned}
$$

for any $\mathtt{c}_1 \in \aleph$. For α such that $0 < \alpha = \Upsilon^2 < 1$,

$$\|F\mathrm{S} - F\mathrm{S}'\| \leq \alpha M(\mathrm{S}, \mathrm{S}')$$

$$\Im(F\mathrm{S}, F\mathrm{S}') \leq \alpha M(\mathrm{S}, \mathrm{S}')$$

$\forall \quad \mathrm{S}, \mathrm{S}' \in B(\aleph)$.

Conclusion follows from Theorem 3.1.3. □

Chapter 4

The Fixed Point Result for Generalized α-F-Contractions in Double Controlled Metric Spaces

4.1 Preliminaries

Current chapter includes FP results in DCMS via generalized α-F-contraction mappings. An example is provided to validate the main result. Also some applications are provided for implementation of the main result.

Definition 4.1.1.
Given a non-empty set $\aleph$, let
$F_1, F_2 : \aleph \to \aleph$ and $\alpha : \aleph \times \aleph \to [0, \infty)$ be given mappings, then we say that (F_1, F_2) is generalized α-admissible if $\forall\ \mathsf{c}_1, \mathsf{c}_2 \in \aleph$,

$$\alpha(\mathsf{c}_1, \mathsf{c}_2) \geq 1 \implies \alpha(F_1\mathsf{c}_1, F_2\mathsf{c}_2) \geq 1 \quad \text{and} \quad \alpha(F_2\mathsf{c}_2, F_1\mathsf{c}_1) \geq 1. \tag{4.1}$$

Definition 4.1.2.
Suppose that $(\aleph, \Im)$ be a DCMS. A pair of self mapping $F_1, F_2 : \aleph \to \aleph$ is called generalized Ćirić type α-$\mathcal{F}$-contraction if $\exists\ \zeta > 0$ such that $\forall\ \mathsf{c}_1, \mathsf{c}_2 \in \aleph$,

$$\Im(F_1\mathsf{c}_1, F_2\mathsf{c}_2) > 0 \implies \zeta + \mathcal{F}\left(\alpha(\mathsf{c}_1, \mathsf{c}_2)\Im(F_1\mathsf{c}_1, F_2\mathsf{c}_2)\right) \leq \mathcal{F}(M(\mathsf{c}_1, \mathsf{c}_2)) \tag{4.2}$$

where

$$M(\mathsf{c}_1, \mathsf{c}_2) = \max\left\{\Im(\mathsf{c}_1, \mathsf{c}_2), \Im(\mathsf{c}_1, F_1\mathsf{c}_1), \Im(\mathsf{c}_2, F_2\mathsf{c}_2), \frac{1}{2}\left[\Im(\mathsf{c}_1, F_2\mathsf{c}_2)+\Im(\mathsf{c}_2, F_1\mathsf{c}_1)\right]\right\},$$

and $\mathcal{F} \in \mathsf{F}$.

Theorem 4.1.3. Suppose that $(\aleph, \Im)$ be a complete DCMS such that $\Im$ is continuous and $F_1, F_2 : \aleph \to \aleph$ be two self-operators satisfying:
(i): The pair (F_1, F_2) is generalized Ćirić type α-F-contraction,
(ii): $\exists\ \mathsf{c}_0 \in \aleph$ such that $\alpha(\mathsf{c}_0, F_1\mathsf{c}_0) \geq 1$ and $\alpha(F_1\mathsf{c}_0, \mathsf{c}_0) \geq 1$,
(iii): F_1 and F_2 are continuous,
(iv): for any convergent sequence $\{\mathsf{c}_n\}$,

$$\sup_{l\geq 1} \lim_{i\to\infty} \frac{\kappa(\mathsf{c}_{i+1}, \mathsf{c}_{i+2})\varkappa(\mathsf{c}_{i+1}, \mathsf{c}_{2n+l})}{\kappa(\mathsf{c}_i, \mathsf{c}_{i+1})} < 1. \tag{4.3}$$

Also, suppose

$$\lim_{n,m\to\infty} \kappa(\mathsf{c}_n, \mathsf{c}_m) \leq 1 \quad \text{and} \quad \lim_{n,m\to\infty} \varkappa(\mathsf{c}_n, \mathsf{c}_m) \leq 1.$$

Then, F_1 and F_2 have a unique CFP.

Proof. Consider $\mathsf{c}_0 \in \aleph$ such that $\alpha(\mathsf{c}_0, F\mathsf{c}_0) \geq 1$ and $\alpha(F\mathsf{c}_0, \mathsf{c}_0) \geq 1$. Take $\mathsf{c}_1 = F_1\mathsf{c}_0$ and $\mathsf{c}_2 = F_2\mathsf{c}_1$. By the induction, construct the sequence $\{\mathsf{c}_n\}$, defined as follows:

$$\mathsf{c}_{2n+1} = F_1\mathsf{c}_{2n} \quad \text{and} \quad \mathsf{c}_{2n+2} = F_2\mathsf{c}_{2n+1} \qquad \forall\ n = 0, 1, 2, 3, ... \tag{4.4}$$

Suppose that $z_n = \Im(\mathsf{c}_n, \mathsf{c}_{n+1})$ for $n \geq 0$.
Now, divide the proof into three parts:

Part-1

To prove $\alpha(\mathsf{c}_n, \mathsf{c}_{n+1}) \geq 1$ and $\alpha(\mathsf{c}_{n+1}, \mathsf{c}_n) \geq 1 \quad \forall\, n \geq 0$.

Since $\alpha(\mathsf{c}_0, \mathsf{c}_1) \geq 1$ and $\alpha(\mathsf{c}_1, \mathsf{c}_0) \geq 1$ and (F_1, F_2) is a generalized α-admissible pair of mapping. So

$$\alpha(\mathsf{c}_1, \mathsf{c}_2) = \alpha(F_1\mathsf{c}_0, F_2\mathsf{c}_1) \geq 1 \quad \text{and} \quad \alpha(\mathsf{c}_2, \mathsf{c}_1) = \alpha(F_2\mathsf{c}_1, F_1\mathsf{c}_0) \geq 1.$$

Also,

$$\alpha(\mathsf{c}_3, \mathsf{c}_2) = \alpha(F_1\mathsf{c}_2, F_2\mathsf{c}_1) \geq 1 \quad \text{and} \quad \alpha(\mathsf{c}_2, \mathsf{c}_3) = \alpha(F_2\mathsf{c}_1, F_1\mathsf{c}_2) \geq 1.$$

Proceeding in this way

$$\alpha(\mathsf{c}_n, \mathsf{c}_{n+1}) \geq 1 \quad \text{and} \quad \alpha(\mathsf{c}_{n+1}, \mathsf{c}_n) \geq 1 \quad \forall\, n = 0, 1, 2, ...$$

Part-2

We have to prove that

$$\lim_{n\to\infty} z_n = 0 \qquad \forall\, n \in \aleph.$$

If $\Im(\mathsf{c}_{2n+1}, \mathsf{c}_{2n+2}) = 0$ for some n, then we have to prove that $\Im(\mathsf{c}_{2n+2}, \mathsf{c}_{2n+3}) = 0$. Argue by the contradiction that,

$$\Im(\mathsf{c}_{2n+2}, \mathsf{c}_{2n+3}) = \Im(F_2\mathsf{c}_{2n+1}, F_1\mathsf{c}_{2n+2}) > 0.$$

Since, $\alpha(\mathsf{c}_{2n+1}, \mathsf{c}_{2n+2}) \geq 1$,

from contraction condition (4.2)

$$\begin{aligned}\zeta + \mathcal{F}(\Im(\mathsf{c}_{2n+2}, \mathsf{c}_{2n+3})) &= \zeta + \mathcal{F}(\Im(F_2\mathsf{c}_{2n+1}, F_1\mathsf{c}_{2n+2})) \\ &\leq \zeta + \mathcal{F}(\alpha(\mathsf{c}_{2n+1}, \mathsf{c}_{2n+2})\Im(F_2\mathsf{c}_{2n+1}, F_1\mathsf{c}_{2n+2})) \\ &\leq \mathcal{F}(M(\mathsf{c}_{2n+1}, \mathsf{c}_{2n+2})). \end{aligned} \tag{4.5}$$

Now $M(\mathsf{c}_{2n+1}, \mathsf{c}_{2n+2}) = \max\{\Im(\mathsf{c}_{2n+1}, \mathsf{c}_{2n+2}), \Im(\mathsf{c}_{2n+1}, F_2\mathsf{c}_{2n+1}), \Im(\mathsf{c}_{2n+2}, F_1\mathsf{c}_{2n+2}),$

$$\begin{aligned}
&\frac{1}{2}[\Im(\mathsf{c}_{2n+1}, F_1\mathsf{c}_{2n+2}) + \Im(\mathsf{c}_{2n+2}, F_2\mathsf{c}_{2n+1})]\} \\
M(\mathsf{c}_{2n+1}, \mathsf{c}_{2n+2}) =& \max\{\Im(\mathsf{c}_{2n+1}, \mathsf{c}_{2n+2}), \Im(\mathsf{c}_{2n+1}, \mathsf{c}_{2n+2}), \Im(\mathsf{c}_{2n+2}, \mathsf{c}_{2n+3}), \\
&\frac{1}{2}[\Im(\mathsf{c}_{2n+1}, \mathsf{c}_{2n+3}) + \Im(\mathsf{c}_{2n+2}, \mathsf{c}_{2n+2})]\} \\
M(\mathsf{c}_{2n+1}, \mathsf{c}_{2n+2}) =& \max\{0, 0, \Im(\mathsf{c}_{2n+2}, \mathsf{c}_{2n+3}), \frac{1}{2}[\Im(\mathsf{c}_{2n+1}, \mathsf{c}_{2n+3}) + 0]\} \\
=&\Im(\mathsf{c}_{2n+2}, \mathsf{c}_{2n+3}).
\end{aligned}$$

Then, from (4.5)

$$\zeta + \mathcal{F}(\Im(\mathsf{c}_{2n+2}, \mathsf{c}_{2n+3})) \leq \mathcal{F}(\Im(\mathsf{c}_{2n+2}, \mathsf{c}_{2n+3})),$$

which leads to contradiction, so $\Im(\mathsf{c}_{2n+2}, \mathsf{c}_{2n+3}) = 0$.

Finally, we have,

$$\mathsf{c}_{2n+1} = \mathsf{c}_{2n+2} = F_2\mathsf{c}_{2n+1} \quad \text{and} \quad \mathsf{c}_{2n+1} = \mathsf{c}_{2n+3} = F_1\mathsf{c}_{2n+2} = F_1\mathsf{c}_{2n+1}.$$

Hence, c_{2n+1} is a CFP of F_1 and F_2.

Similarly, if $\Im(\mathsf{c}_{2n+2}, \mathsf{c}_{2n+3}) = 0$, then c_{2n+2} is CFP of F_1 and F_2 and this complete the proof.

Now, let $\Im(\mathsf{c}_{2n+1}, \mathsf{c}_{2n+2}) > 0$

Since, $\alpha(\mathsf{c}_{2n}, \mathsf{c}_{2n+1}) \geq 1$ and $\Im(\mathsf{c}_{2n+1}, \mathsf{c}_{2n+2}) = \Im(F_1\mathsf{c}_{2n}, F_2\mathsf{c}_{2n+1})) > 0$.

(4.2) $\implies$

$$\begin{aligned}
\zeta + \mathcal{F}(\Im(\mathsf{c}_{2n+1}, \mathsf{c}_{2n+2})) &= \zeta + \mathcal{F}(\Im(F_1\mathsf{c}_{2n}, F_2\mathsf{c}_{2n+1})) \\
&\leq \zeta + \mathcal{F}(\alpha(\mathsf{c}_{2n}, \mathsf{c}_{2n+1})\Im(F_1\mathsf{c}_{2n}, F_2\mathsf{c}_{2n+1})) \\
&\leq \mathcal{F}(M(\mathsf{c}_{2n}, \mathsf{c}_{2n+1})). \qquad (4.6)
\end{aligned}$$

Now

$$M(\mathsf{c}_{2n}, \mathsf{c}_{2n+1}) = \max\Big\{\Im(\mathsf{c}_{2n}, \mathsf{c}_{2n+1}), \Im(\mathsf{c}_{2n}, F_1\mathsf{c}_{2n}), \Im(\mathsf{c}_{2n+1}, F_2\mathsf{c}_{2n+1}), \frac{\{\Im(\mathsf{c}_{2n}, F_2\mathsf{c}_{2n+1}) + \Im(\mathsf{c}_{2n+1}, F_1\mathsf{c}_{2n})\}}{2}\Big\}$$

$$M(c_{2n}, c_{2n+1}) = \max\Big\{\Im(c_{2n}, c_{2n+1}), \Im(c_{2n}, c_{2n+1}), \Im(c_{2n+1}, c_{2n+2}), \frac{\{\Im(c_{2n}, c_{2n+2}) + \Im(c_{2n+1}, c_{2n+1})\}}{2}\Big\}$$
$$= \max\{\Im(c_{2n}, c_{2n+1}), \Im(c_{2n+1}, c_{2n+2})\}.$$

If $M(c_{2n}, c_{2n+1}) = \Im(c_{2n+1}, c_{2n+2})$ then, (4.6) leads to contradiction.

If $M(c_{2n}, c_{2n+1}) = \Im(c_{2n}, c_{2n+1})$ then, from (4.6)

$$\zeta + \mathcal{F}(\Im(c_{2n+1}, c_{2n+2})) \leq \mathcal{F}(\Im(c_{2n}, c_{2n+1})). \tag{4.7}$$

Now, suppose that $\Im(c_{2n+2}, c_{2n+3}) > 0$

Since, $\Im(c_{2n+2}, c_{2n+3}) = \Im(F_2 c_{2n+1}, F_1 c_{2n+2})) > 0$.

Using contraction condition (4.2)

$$\begin{aligned}\zeta + \mathcal{F}(\Im(c_{2n+2}, c_{2n+3})) &= \zeta + \mathcal{F}(\Im(F_2 c_{2n+1}, F_1 c_{2n+2}))\\ &\leq \zeta + \mathcal{F}(\alpha(c_{2n+1}, c_{2n+2})\Im(F_2 c_{2n+1}, F_1 c_{2n+2}))\\ &\leq \mathcal{F}(M(c_{2n+1}, c_{2n+2})).\end{aligned} \tag{4.8}$$

Now

$$M(c_{2n+1}, c_{2n+2}) = \max\Big\{\Im(c_{2n+1}, c_{2n+2}), \Im(c_{2n+1}, F_2 c_{2n+1}), \Im(c_{2n+2}, F_1 c_{2n+2}), \frac{\Im(c_{2n+1}, F_1 c_{2n+2}) + \Im(c_{2n+2}, F_2 c_{2n+1})}{2}\Big\}$$
$$M(c_{2n+1}, c_{2n+2}) = \max\Big\{\Im(c_{2n+1}, c_{2n+2}), \Im(c_{2n+1}, c_{2n+2}), \Im(c_{2n+2}, c_{2n+3}), \frac{\Im(c_{2n+1}, c_{2n+3}) + \Im(c_{2n+2}, c_{2n+2})}{2}\Big\}$$
$$= \max\{\Im(c_{2n+1}, c_{2n+2}), \Im(c_{2n+2}, c_{2n+3})\}.$$

If $M(c_{2n+1}, c_{2n+2}) = \Im(c_{2n+2}, c_{2n+3})$ then, (4.8) leads to contradiction.

If $M(c_{2n+1}, c_{2n+2}) = \Im(c_{2n+1}, c_{2n+2})$ then, from (4.8)

$$\zeta + \mathcal{F}(\Im(c_{2n+2}, c_{2n+3})) \leq \mathcal{F}(\Im(c_{2n+1}, c_{2n+2})). \tag{4.9}$$

Combining (4.7) and (4.9),

$$\mathcal{F}(z_n) \leq \mathcal{F}(z_{n-1}) - \zeta \quad \text{for all} \quad n \geq 1.$$

Continuing in this way,

$$\mathcal{F}(z_n) \ \leq \ \mathcal{F}(z_{n-1}) - \zeta \ \leq \ \mathcal{F}(z_{n-2}) - 2\zeta \ \leq ... \leq \ \mathcal{F}(z_0) - n\zeta \quad \forall \quad n \geq 1 \quad (4.10)$$

By taking $\lim_{n\to\infty}$ in above, we obtain $\lim_{n\to\infty} \mathcal{F}(z_n) = -\infty$.
Using (**F**-2) from Definition (2.3.12),

$$\lim_{n\to\infty} (z_n) = 0. \quad (4.11)$$

Part-3

Now, we shall prove that $\{z_n\}$ is a Cauchy sequence.
By (**F**-3) from Definition (2.3.12) and (4.11), $\exists\ k \in (0,1)$ such that

$$\lim_{n\to\infty} (z_n)^k \mathcal{F}(z_n) = 0. \quad (4.12)$$

By (4.10) $\forall\, n = 1, 2, 3, ...$

$$\begin{aligned} (z_n)^k \mathcal{F}(z_n) &\leq (z_n)^k \mathcal{F}(z_0) - (z_n)^k n\zeta \\ (z_n)^k \mathcal{F}(z_n) - (z_n)^k \mathcal{F}(z_0) &\leq (z_n)^k n\zeta \leq 0. \end{aligned} \quad (4.13)$$

Taking limit as $n \to \infty$

$$\lim_{n\to\infty} (z_n)^k n = 0$$

$\exists\ n_0 \in \mathbb{N}$ such that, $\forall\, n \geq n_0$

$$\begin{aligned} & (z_n)^k n \leq 1 \\ \Longrightarrow \qquad & z_n \leq \frac{1}{n^{\frac{1}{k}}} \qquad \forall\, n \geq n_0. \end{aligned} \quad (4.14)$$

In order to demonstrate $\{z_n\}$ is a Cauchy sequence, suppose $l \in \mathbb{N}$ is such that $l \geq n_0$.

By triangular inequality and taking $l \geq 2$,

$$\begin{aligned}
\Im(\mathsf{c}_{2n+1}, \mathsf{c}_{2n+l}) &\leq \kappa(\mathsf{c}_{2n+1}, \mathsf{c}_{2n+2})\Im(\mathsf{c}_{2n+1}, \mathsf{c}_{2n+2})+ \\
&\quad \varkappa(\mathsf{c}_{2n+2}, \mathsf{c}_{2n+l})\Im(\mathsf{c}_{2n+2}, \mathsf{c}_{2n+l}) \\
&\leq \kappa(\mathsf{c}_{2n+1}, \mathsf{c}_{2n+2})\Im(\mathsf{c}_{2n+1}, \mathsf{c}_{2n+2})+ \\
&\quad \varkappa(\mathsf{c}_{2n+2}, \mathsf{c}_{2n+l})\kappa(\mathsf{c}_{2n+2}, \mathsf{c}_{2n+3})\Im(\mathsf{c}_{2n+2}, \mathsf{c}_{2n+3}) \\
&\quad + \varkappa(\mathsf{c}_{2n+2}, \mathsf{c}_{2n+l})\varkappa(\mathsf{c}_{2n+3}, \mathsf{c}_{2n+l})\Im(\mathsf{c}_{2n+3}, \mathsf{c}_{2n+l}) \\
&\leq \kappa(\mathsf{c}_{2n+1}, \mathsf{c}_{2n+2})\Im(\mathsf{c}_{2n+1}, \mathsf{c}_{2n+2})+ \\
&\quad \varkappa(\mathsf{c}_{2n+2}, \mathsf{c}_{2n+l})\kappa(\mathsf{c}_{2n+2}, \mathsf{c}_{2n+3})\Im(\mathsf{c}_{2n+2}, \mathsf{c}_{2n+3}) \\
&\quad + \varkappa(\mathsf{c}_{2n+2}, \mathsf{c}_{2n+l})\varkappa(\mathsf{c}_{2n+3}, \mathsf{c}_{2n+l}) \\
&\quad \kappa(\mathsf{c}_{2n+3}, \mathsf{c}_{2n+4})\Im(\mathsf{c}_{2n+3}, \mathsf{c}_{2n+4}) \\
&\quad + \varkappa(\mathsf{c}_{2n+2}, \mathsf{c}_{2n+l})\varkappa(\mathsf{c}_{2n+3}, \mathsf{c}_{2n+l}) \\
&\quad \varkappa(\mathsf{c}_{2n+4}, \mathsf{c}_{2n+l})\Im(\mathsf{c}_{2n+4}, \mathsf{c}_{2n+l}) \\
&\quad \vdots \\
&\leq \kappa(\mathsf{c}_{2n+1}, \mathsf{c}_{2n+2})\Im(\mathsf{c}_{2n+1}, \mathsf{c}_{2n+2})+ \\
&\quad \sum_{i=2n+2}^{2n+l-2}\left(\prod_{j=2n+2}^{i}\varkappa(\mathsf{c}_j, \mathsf{c}_{2n+l})\right)\kappa(\mathsf{c}_i, \mathsf{c}_{i+1})\Im(\mathsf{c}_i, \mathsf{c}_{i+1})+ \\
&\quad \left(\prod_{j=2n+2}^{2n+l-1}\varkappa(\mathsf{c}_j, \mathsf{c}_{2n+l})\right)\Im(\mathsf{c}_{2n+l-1}, \mathsf{c}_{2n+l}) \\
&\leq \kappa(\mathsf{c}_{2n+1}, \mathsf{c}_{2n+2})\Im(\mathsf{c}_{2n+1}, \mathsf{c}_{2n+2})+ \\
&\quad \sum_{i=2n+2}^{2n+l-2}\left(\prod_{j=2n+2}^{i}\varkappa(\mathsf{c}_j, \mathsf{c}_{2n+l})\right)\kappa(\mathsf{c}_i, \mathsf{c}_{i+1})\Im(\mathsf{c}_i, \mathsf{c}_{i+1})+ \\
&\quad \left(\prod_{j=2n+2}^{2n+l-1}\varkappa(\mathsf{c}_j, \mathsf{c}_{2n+l})\right)\kappa(\mathsf{c}_{2n+l-1}, \mathsf{c}_{2n+l})\Im(\mathsf{c}_{2n+l-1}, \mathsf{c}_{2n+l}).
\end{aligned}$$

This implies that,

$$\begin{aligned}
\Im(\mathsf{c}_{2n+1}, \mathsf{c}_{2n+l}) &\leq \kappa(\mathsf{c}_{2n+1}, \mathsf{c}_{2n+2})\Im(\mathsf{c}_{2n+1}, \mathsf{c}_{2n+2}) \\
&\quad + \sum_{i=2n+2}^{2n+l-1}\left(\prod_{j=2n+2}^{i}\varkappa(\mathsf{c}_j, \mathsf{c}_{2n+l})\right)\kappa(\mathsf{c}_i, \mathsf{c}_{i+1})\Im(\mathsf{c}_i, \mathsf{c}_{i+1})
\end{aligned}$$

Using (4.14), above can be written as

$$\begin{aligned}\Im(\mathtt{c}_{2n+1},\mathtt{c}_{2n+l}) &\leq \kappa(\mathtt{c}_{2n+1},\mathtt{c}_{2n+2})\Im(\mathtt{c}_{2n+1},\mathtt{c}_{2n+2})+\\ &\quad\sum_{i=2n+2}^{2n+l-1}\left(\prod_{j=2n+2}^{i}\varkappa(\mathtt{c}_j,\mathtt{c}_{2n+l})\right)\kappa(\mathtt{c}_i,\mathtt{c}_{i+1})\frac{1}{i^{\frac{1}{k}}}\\ &\leq \kappa(\mathtt{c}_{2n+1},\mathtt{c}_{2n+2})\Im(\mathtt{c}_{2n+1},\mathtt{c}_{2n+2})+\\ &\quad\sum_{i=2n+2}^{2n+l-1}\left(\prod_{j=0}^{i}\varkappa(\mathtt{c}_j,\mathtt{c}_{2n+l})\right)\kappa(\mathtt{c}_i,\mathtt{c}_{i+1})\frac{1}{i^{\frac{1}{k}}}. \qquad (4.15)\end{aligned}$$

Let $S_p = \sum_{i=0}^{p}\left(\prod_{j=0}^{i}\varkappa(\mathtt{c}_j,\mathtt{c}_{2n+l})\right)\kappa(\mathtt{c}_i,\mathtt{c}_{i+1})\frac{1}{i^{\frac{1}{k}}}$,
then applying ratio test, we have

$$a_n = \left(\prod_{j=0}^{i}\varkappa(\mathtt{c}_j,\mathtt{c}_{2n+l})\right)\kappa(\mathtt{c}_i,\mathtt{c}_{i+1})\frac{1}{i^{\frac{1}{k}}}$$

$$\frac{a_{n+1}}{a_n} = \frac{\kappa(\mathtt{c}_{i+1},\mathtt{c}_{i+2})\varkappa(\mathtt{c}_{\mathtt{i}+1},\mathtt{c}_{2n+l})}{\kappa(\mathtt{c}_i,\mathtt{c}_{i+1})}\left(\frac{i}{i+1}\right)^{\frac{1}{k}}$$

Since $\lim_{n,m\to\infty}\kappa(\mathtt{c}_n,\mathtt{c}_m) < 1$ and $\frac{1}{k} < 1$. Therefore under condition (4.3) series $\sum_n a_n$ converges. Therefore, $\lim_{p\to\infty} S_p$ exists. So the real sequence S_p is Cauchy. Thus we obtained the following inequality

$$\Im(\mathtt{c}_{2n+1},\mathtt{c}_{2n+l}) \leq \kappa(\mathtt{c}_{2n+1},\mathtt{c}_{2n+2})\Im(\mathtt{c}_{2n+1},\mathtt{c}_{2n+2}) + [S_{2n+l-1} - S_{2n+1}]. \qquad (4.16)$$

By applying limit $n\to\infty$ in (4.16), then $\lim_{n\to\infty}\Im(\mathtt{c}_{2n+1},\mathtt{c}_{2n+l}) = 0$. Then sequence $\{c_n\}$ is Cauchy.

Since $\aleph$ is complete, $\exists\ \mathtt{c_1}^* \in \aleph$ such that

$$\lim_{n\to\infty}\mathtt{c}_n = \mathtt{c_1}^*.$$

Then, it follows that $\Im(\mathtt{c}_{2n},\mathtt{c_1}^*) \to 0$ as $n\to\infty$.

Now, $\mathtt{c}_{2n+1} = F_1\mathtt{c}_{2n}$, taking $n\to\infty$ and by continuity of F_1, $\mathtt{c_1}^* = F_1\mathtt{c_1}^*$.

At the same time, $\Im(\mathtt{c}_{2n+1},\mathtt{c_1}^*) \to 0$ as $n\to\infty$.

Similarly $\mathtt{c}_{2n+2} = F_2\mathtt{c}_{2n+1}$, taking $n\to\infty$ and by continuity of F_2,

$\mathsf{c_1}^* = F_2\mathsf{c_1}^*$.

Hence $\mathsf{c_1}^*$ is CFP of (F_1, F_2).

Now to show uniqueness of $\mathsf{c_1}^*$, suppose that $\mathsf{c_2}^* \in \aleph$ is another CFP of pair (F_1, F_2), then,

$$\begin{aligned}\zeta + \mathcal{F}(\Im(\mathsf{c_1}^*, \mathsf{c_2}^*) &= \zeta + \mathcal{F}(\Im(F_1\mathsf{c_1}^*, F_2\mathsf{c_2}^*)) \\ &\leq \zeta + \mathcal{F}(\alpha(\mathsf{c_1}^*, \mathsf{c_2}^*)\Im(F_1\mathsf{c_1}^*, F_2\mathsf{c_2}^*)) \\ &\leq \mathcal{F}(M(\mathsf{c_1}^*, \mathsf{c_2}^*)) \qquad (4.17)\end{aligned}$$

where,

$$M(\mathsf{c_1}^*, \mathsf{c_2}^*) = \max\left\{\Im(\mathsf{c_1}^*, \mathsf{c_2}^*), \Im(\mathsf{c_1}^*, F_1\mathsf{c_1}^*), \Im(\mathsf{c_2}^*, F_2\mathsf{c_2}^*), \frac{1}{2}[\Im(\mathsf{c_1}^*, F_2\mathsf{c_2}^*) + \Im(\mathsf{c_2}^*, F_1\mathsf{c_1}^*)]\right\}$$
$$M(\mathsf{c_1}^*, \mathsf{c_2}^*) = \max\left\{\Im(\mathsf{c_1}^*, \mathsf{c_2}^*), \Im(\mathsf{c_1}^*, \mathsf{c_1}^*), \Im(\mathsf{c_2}^*, \mathsf{c_2}^*), \frac{1}{2}[\Im(\mathsf{c_1}^*, \mathsf{c_2}^*) + \Im(\mathsf{c_2}^*, \mathsf{c_1}^*)]\right\}$$
$$M(\mathsf{c_1}^*, \mathsf{c_2}^*) = \max\left\{\Im(\mathsf{c_1}^*, \mathsf{c_2}^*), 0, 0, \frac{1}{2}[\Im(\mathsf{c_1}^*, \mathsf{c_2}^*) + \Im(\mathsf{c_2}^*, \mathsf{c_1}^*)]\right\}$$
$$M(\mathsf{c_1}^*, \mathsf{c_2}^*) = \max\left\{\Im(\mathsf{c_1}^*, \mathsf{c_2}^*), 0, 0, \frac{1}{2}[2\Im(\mathsf{c_1}^*, \mathsf{c_2}^*)]\right\}$$
$$M(\mathsf{c_1}^*, \mathsf{c_2}^*) = \max\left\{\Im(\mathsf{c_1}^*, \mathsf{c_2}^*), 0, 0, \Im(\mathsf{c_1}^*, \mathsf{c_2}^*)\right\}$$
$$M(\mathsf{c_1}^*, \mathsf{c_2}^*) = \Im(\mathsf{c_1}^*, \mathsf{c_2}^*).$$

so, (4.17) become,

$$\zeta + \mathcal{F}(\Im(\mathsf{c_1}^*, \mathsf{c_2}^*) \leq \mathcal{F}(\Im(\mathsf{c_1}^*, \mathsf{c_2}^*))$$

This implies that $\mathsf{c_1}^* = \mathsf{c_2}^*$. □

If $\kappa(\mathsf{c_1}, \mathsf{c_2}) = \varkappa(\mathsf{c_1}, \mathsf{c_2}) = \Theta(\mathsf{c_1}, \mathsf{c_2}) = b \geq 1$, then we have the following;

Corollary 4.1.4. Suppose that $(\aleph, \Im)$ be a complete bMS such that $\Im$ is continuous and $F_1, F_2 : \aleph \to \aleph$ be two self-operators satisfying:

(i): The pair (F_1, F_2) is generalized Ćirić type α-F-contraction,

(ii): $\exists\ \mathsf{c_0} \in \aleph$ such that $\alpha(\mathsf{c_0}, F_1\mathsf{c_0}) \geq 1$ and $\alpha(F_1\mathsf{c_0}, \mathsf{c_0}) \geq 1$,

(iii): F_1 and F_2 are continuous,

Then, F_1 and F_2 have a unique CFP.

Theorem 4.1.5. Suppose that $(\aleph, \Im)$ be a complete DCMS such that $\Im$ is continuous and $F_1, F_2 : \aleph \to \aleph$ be two self-operators satisfying:

(i): The pair (F_1, F_2) is generalized Ćirić type α-F-contraction,

(ii): $\exists\ \mathsf{c}_0 \in \aleph$ such that $\alpha(\mathsf{c}_0, F_1\mathsf{c}_0) \geq 1$ and $\alpha(F_1\mathsf{c}_0, \mathsf{c}_0) \geq 1$,

(iii): for any convergent sequence $\{\mathsf{c}_n\}$,

$$\sup_{l\geq 1} \lim_{i\to\infty} \frac{\kappa(\mathsf{c}_{i+1}, \mathsf{c}_{i+2})\varkappa(\mathsf{c}_{i+1}, \mathsf{c}_{2n+l})}{\kappa(\mathsf{c}_i, \mathsf{c}_{i+1})} < 1. \tag{4.18}$$

Also assume

$$\lim_{n,m\to\infty} \kappa(\mathsf{c}_n, \mathsf{c}_m) \leq 1 \quad \text{and} \quad \lim_{n,m\to\infty} \varkappa(\mathsf{c}_n, \mathsf{c}_m) \leq 1.$$

Then, F_1 and F_2 have a unique CFP.

Proof. Adopting the same procedure as in Theorem 4.1.3. We can prove that $\mathsf{c}_n \to \mathsf{c_1}^*$.

Since the operators F_1 and F_2 are discontinuous. Suppose, $\Im_\Theta(\mathsf{c_1}^*, F_1\mathsf{c_1}^*) = h > 0$.

Now,

$$\begin{aligned} h =\ & \Im(\mathsf{c_1}^*, F_1\mathsf{c_1}^*) \\ \leq\ & \kappa(\mathsf{c_1}^*, \mathsf{c}_{2n+2})\Im(\mathsf{c_1}^*, \mathsf{c}_{2n+2}) + \varkappa(\mathsf{c}_{2n+2}, F_1\mathsf{c_1}^*)\Im(\mathsf{c}_{2n+2}, F_1\mathsf{c_1}^*) \\ \leq\ & \kappa(\mathsf{c_1}^*, \mathsf{c}_{2n+2})\Im(\mathsf{c_1}^*, \mathsf{c}_{2n+2}) + \varkappa(\mathsf{c}_{2n+2}, F_1\mathsf{c_1}^*)\Im(F_2\mathsf{c}_{2n+1}, F_1\mathsf{c_1}^*) \end{aligned} \tag{4.19}$$

Now,

$$\begin{aligned} \mathcal{F}(\Im(F_2\mathsf{c}_{2n+1}, F_1\mathsf{c_1}^*)) \leq & \Big[\mathcal{F}\Big(\max\Big\{\Im_\Theta(\mathsf{c}_{2n+1}, \mathsf{c_1}^*), \Im(\mathsf{c}_{2n+1}, F_2\mathsf{c}_{2n+1}), \\ & \Im(\mathsf{c_1}^*, F_1\mathsf{c_1}^*), \frac{\big[\Im(\mathsf{c}_{2n+1}, F_1\mathsf{c_1}^*) + \Im(\mathsf{c_1}^*, F_2\mathsf{c}_{2n+1})\big]}{2}\Big\}\Big) - \zeta\Big] \\ \mathcal{F}(\Im(F_2\mathsf{c}_{2n+1}, F_1\mathsf{c_1}^*)) \leq & \Big[\mathcal{F}\Big(\max\Big\{\Im(\mathsf{c}_{2n+1}, \mathsf{c_1}^*), \Im(\mathsf{c}_{2n+1}, \mathsf{c}_{2n+2}), \\ & \Im(\mathsf{c_1}^*, F_1\mathsf{c_1}^*), \frac{\big[\Im(\mathsf{c}_{2n+1}, F_1\mathsf{c_1}^*) + \Im(\mathsf{c_1}^*, \mathsf{c}_{2n+2})\big]}{2}\Big\}\Big) - \zeta\Big]. \end{aligned} \tag{4.20}$$

Using (4.20) in (4.19) to obtain

$$h \leq \kappa(\mathtt{c_1}^*, \mathtt{c}_{2n+2})\Im(\mathtt{c_1}^*, \mathtt{c}_{2n+2}) + \varkappa(\mathtt{c}_{2n+2}, F_1\mathtt{c_1}^*)\big[\mathcal{F}\big(\Im(\mathtt{c_1}^*, F_1\mathtt{c_1}^*)\big) - \zeta\big].$$

Taking limit $n \to \infty$

$$h \leq \lim_{n\to\infty} \kappa(\mathtt{c_1}^*, \mathtt{c}_{2n+2})0 + \lim_{n\to\infty} \varkappa(\mathtt{c}_{2n+2}, F_1\mathtt{c_1}^*)\big[\mathcal{F}\big(h\big) - \zeta\big]$$

Since, $\mathcal{F}$ is non-decreasing,

$$h \leq \lim_{n\to\infty} \varkappa(\mathtt{c_1}^*, F_1\mathtt{c_1}^*)[h - \zeta].$$

Since $\lim_{n\to\infty} \kappa(\mathtt{c_1}^*, F_1\mathtt{c_1}^*) = 1$ and $\lim_{n\to\infty} \Im(\mathtt{c_1}^*, \mathtt{c}_{2n+2}) = 0 \implies \mathtt{c_1}^* = F_1\mathtt{c_1}^*$.

In the same way, we obtain $\mathtt{c_1}^* = F_2\mathtt{c_1}^*$. Hence, $\mathtt{c_1}^*$ is CFP for pair (F_1, F_2). Uniqueness can be proved similarly as in Theorem 4.1.3. □

If $\kappa(\mathtt{c}_1, \mathtt{c}_2) = \varkappa(\mathtt{c}_1, \mathtt{c}_2) = \Theta(\mathtt{c}_1, \mathtt{c}_2) = b \geq 1$, then we have the following;

Corollary 4.1.6. Suppose that $(\aleph, \Im)$ be a complete bMS such that $\Im$ is continuous and $F_1, F_2 : \aleph \to \aleph$ be two self-operators satisfying:

(i): The pair (F_1, F_2) is generalized Ćirić type α-F-contraction,

(ii): $\exists\ \mathtt{c}_0 \in \aleph$ such that $\alpha(\mathtt{c}_0, F_1\mathtt{c}_0) \geq 1$ and $\alpha(F_1\mathtt{c}_0, \mathtt{c}_0) \geq 1$,

Then, F_1 and F_2 have a unique CFP.

Now by considering $F_1 = F_2 = F$, a generalization of generalized Ćirić type α-F-contraction is established in DCMS.

Theorem 4.1.7. Suppose that $(\aleph, \Im)$ be a complete DCMS, $\Im$ is continuous and $F : \aleph \to \aleph$ be a self-operators satisfying:

(i): F is generalized Ćirić type α-F-contraction,

(ii): $\exists\ \mathtt{c}_0 \in \aleph$ such that $\alpha(\mathtt{c}_0, F\mathtt{c}_0) \geq 1$ and $\alpha(F\mathtt{c}_0, \mathtt{c}_0) \geq 1$,

(iii): F is continuous,

(iv): for any convergent sequence $\{\mathtt{c}_n\}$,

$$\sup_{l\geq 1} \lim_{i\to\infty} \frac{\kappa(\mathtt{c}_{i+1}, \mathtt{c}_{i+2})\varkappa(\mathtt{c}_{i+1}, \mathtt{c}_{2n+l})}{\kappa(\mathtt{c}_i, \mathtt{c}_{i+1})} < 1.$$

Also, suppose

$$\lim_{n,m\to\infty} \kappa(\mathsf{c}_n, \mathsf{c}_m) \leq 1 \quad \text{and} \quad \lim_{n,m\to\infty} \varkappa(\mathsf{c}_n, \mathsf{c}_m) \leq 1.$$

Then, F has UFP.

Remark: If $\kappa(\mathsf{c}_1, \mathsf{c}_2) = \varkappa(\mathsf{c}_1, \mathsf{c}_2) = \Theta(\mathsf{c}_1, \mathsf{c}_2)$, then the results of Belhenniche et al. is the special case of our main result with $\mathcal{F}(\mathsf{c}) = \ln(\mathsf{c})$ and $\alpha(\mathsf{c}_1, \mathsf{c}_2) = 1$.

Example 4.1.8.

Suppose $\aleph = [0,1]$, define $\Im_\Theta : \aleph \times \aleph \to \mathbb{R}$, and $\kappa, \varkappa : \aleph \times \aleph \to [1, \infty)$ by:
$\Im(\mathsf{c}_1, \mathsf{c}_2) := (\mathsf{c}_1 - \mathsf{c}_2)^2$,

$$\kappa(\mathsf{c}_1, \mathsf{c}_2) := 2\mathsf{c}_1 + \mathsf{c}_2 + 3 \quad \text{and} \quad \varkappa(\mathsf{c}_1, \mathsf{c}_2) := 3\mathsf{c}_1 + \mathsf{c}_2 + 2.$$

Then, $(\aleph, \Im)$ is a complete DCMS.
Define F_1 and $F_2 : \aleph \to \aleph$ by $F_1\mathsf{c}_1 = \frac{\mathsf{c}_1}{2}$, $F_2\mathsf{c}_1 = \frac{\mathsf{c}_1}{4}$.

Define the mapping $\alpha : \aleph \times \aleph \to [0, \infty)$ by

$$\alpha(\mathsf{c}_1, \mathsf{c}_2) = \begin{cases} 2 + \cos({\mathsf{c}_1}^2 + \mathsf{c}_2), & \text{if } \mathsf{c}_1, \mathsf{c}_2 \in [0,1] \\ 0, & \text{otherwise.} \end{cases}$$

Suppose $\mathsf{c}_1, \mathsf{c}_2 \in \aleph$ such that $\alpha(\mathsf{c}_1, \mathsf{c}_2) \geq 1$.
Since, $\alpha(F_1\mathsf{c}_1, F_2\mathsf{c}_2) = \alpha(\frac{\mathsf{c}_1}{2}, \frac{\mathsf{c}_2}{4}) = 2 + \cos(\frac{{\mathsf{c}_1}^2}{4}, \frac{\mathsf{c}_2}{4}) \geq 1$ and $2 + \cos(\frac{{\mathsf{c}_2}^2}{16}, \frac{\mathsf{c}_1}{2}) \geq 1$.
Then (F_1, F_2) is generalized α-admissible pair.
Now

$$\Im(F_1\mathsf{c}_1, F_2\mathsf{c}_2) = \Im\big(\frac{\mathsf{c}_1}{2}, \frac{\mathsf{c}_2}{4}\big)^2 = \frac{{\mathsf{c}_1}^2}{4} + \frac{{\mathsf{c}_2}^2}{16} - \frac{\mathsf{c}_1\mathsf{c}_2}{4}.$$

Now, $M(\mathsf{c}_1, \mathsf{c}_2) = \max\Big\{\Im(\mathsf{c}_1, \mathsf{c}_2), \Im(\mathsf{c}_1, F_1\mathsf{c}_1), \Im(\mathsf{c}_2, F_2\mathsf{c}_2), \frac{1}{2}\big[\Im(\mathsf{c}_1, F_2\mathsf{c}_2) + \Im(\mathsf{c}_2, F_1\mathsf{c}_1)\big]\Big\}$.
Since, $\frac{1}{2}\big[\Im(\mathsf{c}_1, F_2\mathsf{c}_2) + \Im(\mathsf{c}_2, F_1\mathsf{c}_1)\big] = \frac{5{\mathsf{c}_1}^2}{8} + \frac{17{\mathsf{c}_2}^2}{32} - \frac{3\mathsf{c}_1\mathsf{c}_2}{4}$, we may write

$$\Im(F_1\mathsf{c}_1, F_2\mathsf{c}_2) = \frac{{\mathsf{c}_1}^2}{4} + \frac{{\mathsf{c}_2}^2}{16} - \frac{\mathsf{c}_1\mathsf{c}_2}{4} = \frac{1}{2}\bigg(\frac{{\mathsf{c}_1}^2}{2} + \frac{{\mathsf{c}_2}^2}{8} - \frac{\mathsf{c}_1\mathsf{c}_2}{2}\bigg)$$

$$\leq \tfrac{1}{2}\Big(\tfrac{1}{2}(\Im_\Theta(\mathsf{c}_1, F_2\mathsf{c}_2) + \Im(\mathsf{c}_2, F_1\mathsf{c}_1))\Big) \leq \tfrac{1}{2}M(\mathsf{c}_1, \mathsf{c}_2).$$

By contraction condition (4.2), the above inequality,

$$\zeta + \mathcal{F}\big(\Im(F_1\mathsf{c}_1, F_2\mathsf{c}_2)\big) \leq \mathcal{F}\big(\frac{1}{2}M(\mathsf{c}_1, \mathsf{c}_2)\big).$$

But $M(\mathsf{c}_1, \mathsf{c}_2) = \Im(\mathsf{c}_1, \mathsf{c}_2)$. Where $\zeta \in \left(0, \frac{M(\mathsf{c}_1,\mathsf{c}_2)}{2\Im(F_1\mathsf{c}_1, F_2\mathsf{c}_2)}\right)$. So all the axioms of Theorem 4.1.3 are satisfied. Hence, 0 is CFP of F_1 and F_2.

4.2 Application

In this section, we give some application of our results to prove the existence and uniqueness of solution of Volterra-type integral equation, nonlinear fractional differential equation and dynamic programming equation (Bellman's equation).

4.2.1 The Existence of a Solution for Integral Equations of the Volterra type

Vito Volterra, at the end of 19th century, introduced a new type of integral equation named as integral equation of Volterra type, in which upper limit of integral sign is unknown and lower limit is fixed.

Later on, Traian Lalescu, worked on it in 1912 [35]. Volterra integral equations have many applications in different domains of sciences such as potential theory, Dirichlet problem, actuarial sciences, mathematical problems and radiative heat transfer problems.

Consider the integral equation of Volterra type:

$$\mathsf{c}_1(\mathsf{q}) = \int_0^{\mathsf{q}} J(\mathsf{q}, s, \mathsf{c}_1(s))ds + w(\mathsf{q}), \quad \mathsf{q} \in [0, 1]. \tag{4.21}$$

In current section we use Theorem 4.1.3 to show the presence of a solution of the above equation.

Define the operator $L : C([0,1],\mathbb{R}^n) \to C([0,1],\mathbb{R}^n)$ as,

$$L\mathsf{c}_1(\mathsf{q}) = \int_o^{\mathsf{q}} J(\mathsf{q}, s, \mathsf{c}_1(s))ds + w(\mathsf{q}), \quad \mathsf{q} \in [0,1].$$

Theorem 4.2.1. Suppose that Equation (4.21) fulfills the properties given below:

(i) : $J : [0,1] \times [0,1] \times \mathbb{R}^n \to \mathbb{R}^n$ and $w : [0,1] \to \mathbb{R}^n$ are continuous;

(ii) : $J(\mathsf{q}, s, .) : \mathbb{R}^n \to \mathbb{R}^n$ is increasing for each q and $s \in [0,1]$;

(iii) : $\exists\ \zeta > 0$ and $\alpha : \aleph \times \aleph \to [0,\infty)$ such that

$$|J(\mathsf{q}, s, \mathsf{c}_1) - J(\mathsf{q}, s, \mathsf{c}_2)| \leq \frac{e^{-\frac{\zeta}{2}}}{(\alpha(\mathsf{c}_1,\mathsf{c}_2))^{\frac{1}{2}}}(M(\mathsf{c}_1,\mathsf{c}_2)), \quad \forall\ \mathsf{c}_1,\mathsf{c}_2 \in \aleph,\ \mathsf{q} \text{ and } s \in [0,1]$$

where,
$M(\mathsf{c}_1,\mathsf{c}_2) = \max\Big\{\Im(\mathsf{c}_1,\mathsf{c}_2), \Im(\mathsf{c}_1, L\mathsf{c}_1), \Im(\mathsf{c}_2, L\mathsf{c}_2), \frac{1}{2}\big(\Im(\mathsf{c}_1, L\mathsf{c}_2) + \Im(\mathsf{c}_2, L\mathsf{c}_1)\big)\Big\}$.
Then the integral Equation (4.21) has a unique solution.

Proof. Suppose $\aleph = C([0,1],\mathbb{R}^n)$ equipped with double controlled metric $\Im(\mathsf{c}_1,\mathsf{c}_2) = \|\mathsf{c}_1 - \mathsf{c}_2\|_C = \sup_{\mathsf{q}\in[0,1]} |\mathsf{c}_1(\mathsf{q}) - \mathsf{c}_2(\mathsf{q})|^2$ and $\kappa, \varkappa : \aleph \times \aleph \to [1,\infty)$ defined as;

$$\kappa = 2\|\mathsf{c}_1\| + 3\|\mathsf{c}_2\| + 2 \quad \text{and} \quad \varkappa = 2\|\mathsf{c}_1\| + 1$$

with

$$\sup_{l\geq 1} \lim_{i\to\infty} \frac{\kappa(\mathsf{c}_{i+1},\mathsf{c}_{i+2})\varkappa(\mathsf{c}_{i+1},\mathsf{c}_{2n+l})}{\kappa(\mathsf{c}_i,\mathsf{c}_{i+1})} < 1.$$

From, $(\mathsf{c}_1 - \mathsf{c}_2)^2 \geq 0$ we have;

$$\frac{1}{4}({\mathsf{c}_1}^2 + {\mathsf{c}_2}^2) \geq \frac{1}{2}\mathsf{c}_1\mathsf{c}_2. \tag{4.22}$$

Now,

$$\begin{aligned}|L\mathsf{c}_1(\mathsf{q}) - L\mathsf{c}_2(\mathsf{q})|^2 &= \int_0^{\mathsf{q}} |J(\mathsf{q}, s, (\mathsf{c}_1(s))) - J(\mathsf{q}, s, (\mathsf{c}_2(s)))|^2 ds \\ &\leq \Big(\frac{e^{-\frac{\zeta}{2}}}{(\alpha(\mathsf{c}_1,\mathsf{c}_2))^{\frac{1}{2}}}\Big)^2 \int_0^{\mathsf{q}} \max\Big\{|\mathsf{c}_1(s) - \mathsf{c}_2(s)|^2, |\mathsf{c}_1(s) - L\mathsf{c}_1(s)|^2, |\mathsf{c}_2(s) - L\mathsf{c}_2(s)|^2, \\ &\quad \frac{1}{4}\big[|\mathsf{c}_1(s) - L\mathsf{c}_2(s)| + |\mathsf{c}_2(s) - L\mathsf{c}_1(s)|\big]^2\Big\} ds\end{aligned}$$

$$
\begin{aligned}
|L\mathsf{c}_1(\mathsf{q}) - L\mathsf{c}_2(\mathsf{q})|^2 \leq\ & \frac{e^{-\zeta}}{\alpha(\mathsf{c}_1,\mathsf{c}_2)} \int_0^{\mathsf{q}} \max\Big\{ |\mathsf{c}_1(s)-\mathsf{c}_2(s)|^2, |\mathsf{c}_1(s)-L\mathsf{c}_1(s)|^2, |\mathsf{c}_2(s)-L\mathsf{c}_2(s)|^2, \\
& \frac{1}{4}\big(\mathsf{c}_1(s)-\mathsf{c}_2(s)\big)^2 + \frac{1}{2}\big((\mathsf{c}_1(s)-L\mathsf{c}_2(s))(\mathsf{c}_2(s)-L\mathsf{c}_1(s)) + \\
& \frac{1}{4}\big(\mathsf{c}_2(s)-L\mathsf{c}_1(s)\big)^2 \Big\} ds \\
\leq\ & \frac{e^{-\zeta}}{\alpha(\mathsf{c}_1,\mathsf{c}_2)} \int_0^{\mathsf{q}} \max\Big\{ |\mathsf{c}_1(s)-\mathsf{c}_2(s)|^2, |\mathsf{c}_1(s)-L\mathsf{c}_2(s)|^2, |\mathsf{c}_2(s)-L\mathsf{c}_2(s)|^2, \\
& \frac{1}{2}\big[|\mathsf{c}_1(s)-L\mathsf{c}_2(s)|^2 + |\mathsf{c}_2(s)-L\mathsf{c}_1(s)|^2\big] \Big\} ds.
\end{aligned}
$$

Then,

$$
\begin{aligned}
\sup_{\mathsf{q}\in[0,1]} |L\mathsf{c}_1(\mathsf{q}) - L\mathsf{c}_2(\mathsf{q})|^2 \leq \frac{e^{-\zeta}}{\alpha(\mathsf{c}_1,\mathsf{c}_2)} \sup_{\mathsf{q}\in[0,1]} \int_0^{\mathsf{q}} \max\Big\{ |\mathsf{c}_1(s)-\mathsf{c}_2(s)|^2, |\mathsf{c}_1(s)-L\mathsf{c}_2(s)|^2, \\
|\mathsf{c}_2(s)-L\mathsf{c}_2(s)|^2, \frac{1}{2}\big[|\mathsf{c}_1(s)-L\mathsf{c}_2(s)|^2 + |\mathsf{c}_2(s)-L\mathsf{c}_1(s)|^2\big] \Big\} ds.
\end{aligned}
$$

Since, $\|\mathsf{c}_1\|_C = \sup_{\mathsf{q}\in[0,1]} \{|\mathsf{c}_1(\mathsf{q})|^2\}$, then above inequality become,

$$
\begin{aligned}
\|L\mathsf{c}_1(\mathsf{q}) - L\mathsf{c}_2(\mathsf{q})\|_C \leq\ & \frac{e^{-\zeta}}{\alpha(\mathsf{c}_1,\mathsf{c}_2)} \int_0^{\mathsf{q}} \max\Big\{ \|\mathsf{c}_1-\mathsf{c}_2\|_C, \|\mathsf{c}_1-L\mathsf{c}_2\|_C, \|\mathsf{c}_2-L\mathsf{c}_2\|_C, \\
& \frac{1}{2}\big[\|\mathsf{c}_1-L\mathsf{c}_2\|_C + \|\mathsf{c}_2-L\mathsf{c}_1\|_C\big] \Big\} ds \\
\implies \quad \Im(L\mathsf{c}_1, L\mathsf{c}_2) \leq\ & \frac{e^{-\zeta}}{\alpha(\mathsf{c}_1,\mathsf{c}_2)} \max\Big\{ \Im(\mathsf{c}_1,\mathsf{c}_2), \Im(\mathsf{c}_1,L\mathsf{c}_1), \Im(\mathsf{c}_2,L\mathsf{c}_2), \\
& \frac{1}{2}\big(\Im(\mathsf{c}_1,L\mathsf{c}_2) + \Im(\mathsf{c}_2,L\mathsf{c}_1)\big) \Big\} \\
\leq\ & \frac{e^{-\zeta}}{\alpha(\mathsf{c}_1,\mathsf{c}_2)} M(\mathsf{c}_1,\mathsf{c}_2) \\
\alpha(\mathsf{c}_1,\mathsf{c}_2)\Im(L\mathsf{c}_1, L\mathsf{c}_2) \leq\ & e^{-\zeta} M(\mathsf{c}_1,\mathsf{c}_2).
\end{aligned}
$$

Taking natural log on both sides,

$$\ln(\alpha(\mathsf{c}_1,\mathsf{c}_2)\Im(L\mathsf{c}_1, L\mathsf{c}_2)) \leq \ln(e^{-\zeta} M(\mathsf{c}_1,\mathsf{c}_2))$$

$$\ln(\alpha(\mathsf{c}_1,\mathsf{c}_2)\Im(L\mathsf{c}_1, L\mathsf{c}_2)) \leq \ln(e^{-\zeta}) + \ln(M(\mathsf{c}_1,\mathsf{c}_2))$$

$$\ln(\alpha(\mathsf{c}_1,\mathsf{c}_2)\Im(L\mathsf{c}_1, L\mathsf{c}_2)) \leq -\zeta + \ln(M(\mathsf{c}_1,\mathsf{c}_2)).$$

Therefore

$$\zeta + \mathcal{F}\big(\alpha(\mathsf{c}_1,\mathsf{c}_2)\Im(L\mathsf{c}_1, L\mathsf{c}_2)\big) \leq \mathcal{F}(M(\mathsf{c}_1,\mathsf{c}_2)),$$

where, $\mathcal{F}(\mathsf{c}_1) = \ln(\mathsf{c}_1)$ and $\zeta > 0$.

Hence conclusion follows from Theorem 4.1.3. □

4.2.2 Application Regarding System of Nonlinear Fractional Differential Equations

In the last century, fractional calculus helped in many challenges application in control, modeling and optimization in wide range of domains. At the same time, the FP theory is also used for to show the existence of solution of these fractional differential equations.

In current application, we use Theorem 4.1.3 to show the presence and distinct theorems for a nonlinear FDE system of the Caputo type derivative.

Let $y : [0, \infty) \to \mathbb{R}$ be a continuous function. The Caputo derivative of order $\Psi > 0$ of the function y is:

$$ {}^{c}D^{\Psi}(y(\mathsf{q})) := \frac{1}{\Gamma(k-\Psi)} \int_{o}^{\mathsf{q}} (\mathsf{q}-s)^{k-\Psi-1} g^{(k)}(s) \Im(s) \quad (n-1 < \Psi < n, n = [\Psi]+1), \tag{4.23} $$

such that $[\Psi]$, is integer part of $\mathbb{R}^+$ and Γ is the Gamma function.

Consider a system of non-linear FDE of Caputo type, this section is about the existence and uniqueness of solution of;

$$ \begin{cases} {}^{c}D^{\Psi}(\mathsf{c}_1(\mathsf{q})) + p_1(\mathsf{q}, \mathsf{c}_1(\mathsf{q})) = 0, \\ {}^{c}D^{\Psi}(\mathsf{c}_2(\mathsf{q})) + p_2(\mathsf{q}, \mathsf{c}_2(\mathsf{q})) = 0, \end{cases} \tag{4.24} $$

for $\mathsf{q} \in [0, 1], \Psi < 1$, with the boundary condition,

$$ \begin{cases} \mathsf{c}_1(0) = 0 = \mathsf{c}_1(1), \\ \mathsf{c}_2(0) = 0 = \mathsf{c}_2(1), \end{cases} \tag{4.25} $$

where, $\mathsf{c}_1 \in C([0, 1], \mathbb{R})$, $p_1, p_2 : [0, 1] \times \mathbb{R} \to \mathbb{R}$ are continuous functions and ${}^{c}D^{\Psi}$ is the Caputo derivative of order Ψ. Also, Green function associated with (4.24) is given in [41] as follows:

$$\mathcal{G}(\mathsf{q}, s) = \begin{cases} \frac{(\mathsf{q}(1-s))^{\gamma-1} - (\mathsf{q}-s)^{\gamma-1}}{\Gamma(\gamma)} & \text{if } 0 \leq s \leq \mathsf{q} \leq 1, \\ \frac{(\mathsf{q}(1-s))^{\gamma-1}}{\Gamma(\gamma)} & \text{if } 0 \leq s \leq \mathsf{q} \leq 1. \end{cases}$$

Theorem 4.2.2. Given the nonlinear FDE with the given below properties: (i): $F_1, F_2 : C([0,1], \mathbb{R}) \to C([0,1], \mathbb{R})$ are defined as:

$$\begin{cases} F_1 \mathsf{c}_1 = & \int_0^1 \mathcal{G}(t, s) p_1(s, \mathsf{c}_1(s)) ds, \\ F_2 \mathsf{c}_2 = & \int_0^1 \mathcal{G}(\mathsf{q}, s) p_2(s, \mathsf{c}_2(s)) ds; \end{cases} \tag{4.26}$$

(ii): $\exists\ \zeta > 0$ and $\alpha : \aleph \times \aleph \to [0, \infty)$

$$|p_1(\mathsf{q}, \mathsf{c}_1) - p_2(\mathsf{q}, \mathsf{c}_2)| \leq \frac{e^{\frac{-\zeta}{2}}}{(\alpha(\mathsf{c}_1, \mathsf{c}_2))^{\frac{1}{2}}} M(\mathsf{c}_1, \mathsf{c}_2), \quad \forall \ \ \mathsf{q} \in [0,1], \Omega > 1, \mathsf{c}_1, \mathsf{c}_2 \in \mathbb{R}$$

where,

$$M(\mathsf{c}_1, \mathsf{c}_2) \leq \max \left\{ \Im(\mathsf{c}_1, \mathsf{c}_2), \Im(\mathsf{c}_1, F_1\mathsf{c}_1), \Im(\mathsf{c}_2, F_2\mathsf{c}_2), \frac{1}{2} \left(\Im(\mathsf{c}_1, F_2\mathsf{c}_2) + \Im(\mathsf{c}_2, F_1\mathsf{c}_1) \right) \right\}.$$

Then, the (4.24) has unique solution.

Proof. Suppose $\aleph = C([0,1], \mathbb{R})$ and Bielecki norm,

$$\Im_\Theta(\mathsf{c}_1, \mathsf{c}_2) = \|\mathsf{c}_1 - \mathsf{c}_2\| = | \sup_{\mathsf{q} \in [0,1]} \left\{ |\mathsf{c}_1(\mathsf{q})| e^{-\Omega \mathsf{q}} \right\} |^2 \quad \text{with} \quad \Omega > 1$$

and $\kappa, \varkappa : \aleph \times \aleph \to [1, \infty)$ is given by $\kappa(\mathsf{c}_1, \mathsf{c}_2) = |\mathsf{c}_1(\mathsf{q})| + 2|\mathsf{c}_2(\mathsf{q})| + 1$ and $\varkappa(\mathsf{c}_1, \mathsf{c}_2) = |2\mathsf{c}_1(\mathsf{q})| + 3|\mathsf{c}_2(\mathsf{q})| + 1$, with

$$\sup_{l \geq 1} \lim_{i \to \infty} \frac{\kappa(\mathsf{c}_{i+1}, \mathsf{c}_{i+2}) \varkappa(\mathsf{c}_{i+1}, \mathsf{c}_{2n+1})}{\kappa(\mathsf{c}_i, \mathsf{c}_{i+1})} < 1.$$

Then $(\aleph, \Im_\Theta)$ is DCMS.

Obvious ${\mathsf{c}_1}^* \in \aleph$ is solution for the (4.24) iff ${\mathsf{c}_1}^* \in \aleph$ is a common solution for the Equation (4.25), $\forall\ \mathsf{q} \in [0,1]$. Then, the (4.24) can be reduced to find an element ${\mathsf{c}_1}^* \in \aleph$ which is a CFP for the operators F_1 and F_2.

Suppose $\mathsf{c}_1, \mathsf{c}_2 \in \aleph$ such that $\varrho(\mathsf{c}_1(\mathsf{q}), \mathsf{c}_2(\mathsf{q})) \geq 0\ \forall\ \mathsf{q} \in [0,1]$. By (i) and (ii),

$$
\begin{aligned}
|F_1\mathsf{c}_1(\mathsf{q}) - F_2\mathsf{c}_1(\mathsf{q})|^2 &= |\int_0^1 \mathcal{G}\left[p_1(\mathsf{q}, \mathsf{c}_1(s)) - p_2(\mathsf{q}, r(s)\right] ds|^2 \\
&\leq \left(\int_0^1 \mathcal{G}(\mathsf{q}, s) ds\right)^2 \int_0^1 |p_1(\mathsf{q}, \mathsf{c}_1(s)) - p_2(\mathsf{q}, \mathsf{c}_2(s)|^2 ds \\
&\leq \left(\frac{e^{\frac{-\zeta}{2}}}{(\alpha(\mathsf{c}_1, \mathsf{c}_2))^{\frac{1}{2}}}\right)^2 |M(\mathsf{c}_1, \mathsf{c}_2) e^{-\Omega\mathsf{q}}|^2 e^{2\Omega\mathsf{q}} \left(\int_0^1 \mathcal{G}(\mathsf{q}, s) ds\right)^2.
\end{aligned}
$$

Then,

$$
|[F_1\mathsf{c}_1(\mathsf{q}) - F_2\mathsf{c}_1(\mathsf{q})]e^{-\Omega\mathsf{q}}|^2 \leq \frac{e^{-\zeta}}{\alpha(\mathsf{c}_1, \mathsf{c}_2)} |M(\mathsf{c}_1, \mathsf{c}_2) e^{-\Omega\mathsf{q}}|^2 \left(\int_0^1 \mathcal{G}(\mathsf{q}, s) ds\right)^2. \tag{4.27}
$$

Taking $\sup\limits_{\mathsf{q}\in[0,1]}$ in above inequality,

$$
\begin{aligned}
|\sup_{\mathsf{q}\in[0,1]} \{(F_1\mathsf{c}_1(t) - F_2\mathsf{c}_1(\mathsf{q}))e^{-\Omega\mathsf{q}}\}|^2 &\leq \frac{e^{-\zeta}}{\alpha(\mathsf{c}_1, \mathsf{c}_2)} \sup_{\mathsf{q}\in[0,1]} |M(\mathsf{c}_1, \mathsf{c}_2))e^{-\Omega\mathsf{q}}|^2 \sup_{\mathsf{q}\in[0,1]} \left\{\left(\int_0^{\mathsf{q}} \mathcal{G}(\mathsf{q}, s) ds\right)^2\right\} \\
&\leq \frac{e^{-\zeta}}{\alpha(\mathsf{c}_1, \mathsf{c}_2)} \sup_{\mathsf{q}\in[0,1]} |M(\mathsf{c}_1, \mathsf{c}_2))e^{-\Omega\mathsf{q}}|^2.
\end{aligned}
$$

Since $\Im_\Theta(\mathsf{c}_1, \mathsf{c}_2) = \|\mathsf{c}_1 - \mathsf{c}_2\| = |\sup\limits_{\mathsf{q}\in[0,1]} \left\{|\mathsf{c}_1(\mathsf{q})|e^{-\Omega\mathsf{q}}\right\}|^2$, then above become

$$
\begin{aligned}
\Im(F_1\mathsf{c}_1, F_2\mathsf{c}_2) \leq &\ \frac{e^{-\zeta}}{\alpha(\mathsf{c}_1, \mathsf{c}_2)} M(\mathsf{c}_1, \mathsf{c}_2) \\
\alpha(\mathsf{c}_1, \mathsf{c}_2)\Im(F_1\mathsf{c}_1, F_2\mathsf{c}_2) \leq &\ e^{-\zeta} M(\mathsf{c}_1, \mathsf{c}_2).
\end{aligned}
$$

Taking natural log then,

$$
\ln(\alpha(\mathsf{c}_1, \mathsf{c}_2)\Im(F_1\mathsf{c}_1, F_2\mathsf{c}_2)) \leq \ \ln(e^{-\zeta} M(\mathsf{c}_1, \mathsf{c}_2)))
$$

$$
\begin{aligned}
\ln(\alpha(\mathsf{c}_1, \mathsf{c}_2)\Im(F_1\mathsf{c}_1, F_2\mathsf{c}_2)) \leq &\ \ln(e^{-\zeta}) + \ln(M(\mathsf{c}_1, \mathsf{c}_2)) \\
\ln(\alpha(\mathsf{c}_1, \mathsf{c}_2)\Im(F_1\mathsf{c}_1, F_2\mathsf{c}_2)) \leq &\ -\zeta + \ln(M(\mathsf{c}_1, \mathsf{c}_2)).
\end{aligned}
$$

Therefore

$$
\zeta + \mathcal{F}(\alpha(\mathsf{c}_1, \mathsf{c}_2)\Im(F_1\mathsf{c}_1, F_2\mathsf{c}_2)) \leq \ \mathcal{F}(M(\mathsf{c}_1, \mathsf{c}_2)),
$$

where $\mathcal{F}(\mathtt{c}_1) = \ln(\mathtt{c}_1)$ and $\zeta > 0$.

Theorem 4.1.3 yields the existence of $\mathtt{c}^* \in \aleph$ as a CFP of F_1 and F_2. □

4.2.3 An Existence of Solution to the Dynamic Programming Equation

Suppose that the state space is $\aleph$ and set of control values $\mathsf{U}(\mathtt{c}_1) \subset \mathsf{U}$. Let $\mathtt{M}$ be the set of all functions $\varrho : \aleph \to \mathsf{U}$ with $\varrho(\mathtt{c}_1) \in \mathsf{U}(\mathtt{c}_1)\ \forall\ \mathtt{c}_1 \in \aleph$, and $\mathtt{M}$ is said to be "stationary policy". Suppose $B(\aleph)$ be the set of real-valued bounded functions $\mathrm{S} : \aleph \to \mathbb{R}$. For each policy $\varrho \in \mathtt{M}$, assume that the mapping $F_\varrho : B(\aleph) \to B(\aleph)$ defined as:

$$F_\varrho \mathrm{S}(\mathtt{c}_1) = \mathsf{H}(\mathtt{c}_1, \varrho(\mathtt{c}_1), \mathrm{S}) \text{ for all } \mathtt{c}_1 \in \aleph.$$

Where $\mathsf{H} : \aleph \times \mathsf{U} \times B(\aleph) \to \mathbb{R}$.

We also suppose mapping $F : B(\aleph) \to B(\aleph)$ defined as:

$$F\mathrm{S}(\mathtt{c}_1) = \inf_{u \in \mathsf{U}(\mathtt{c}_1)} \{\mathsf{H}(\mathtt{c}_1, u, \mathrm{S})\} = \min_{\varrho \in \mathbb{M}} F_\varrho \mathrm{S}(\mathtt{c}_1) \text{ for all } \mathtt{c}_1 \in \aleph.$$

Now, the pair $(B(\aleph), \|.\|_\Theta)$, such that

$$\|\mathrm{S}\|_\Theta = \sup_{\mathtt{c}_1 \in \aleph} |\mathrm{S}(\mathtt{c}_1)|^2, \quad \mathrm{S} \in B(\aleph) \tag{4.28}$$

is complete DCMS.

We find the optimal cost of the function $\mathbb{J}* \in B(\aleph)$ as,

$$\mathrm{S}(\mathtt{c}_1) = \inf_{u \in \mathsf{U}(\mathtt{c}_1)} \{\mathsf{H}(\mathtt{c}_1, u, \mathrm{S})\} \quad \forall \mathtt{c}_1 \in \aleph. \tag{4.29}$$

This is known as the Bellman equation with given below properties:

A1): (Well posedness). $\forall$ $\mathrm{S} \in B(\aleph)$, and $\forall$ $\varrho \in \mathcal{M}$, we have that $F_\varrho \mathrm{S} \in B(\aleph)$ and $F\mathrm{S} \in B(\aleph)$.

A2): (Monotonicity). If $\mathrm{S}, \mathrm{S}' \in B(\mathtt{c}_1)$, and $\mathrm{S} \le \mathrm{S}'$, then

$$\mathsf{H}(\mathtt{c}_1, u, \mathrm{S}) \le \mathsf{H}(\mathtt{c}_1, u, \mathrm{S}') \ \ \forall\ \mathtt{c}_1 \in \aleph, u \in \mathsf{U}.$$

A3): (Attainability). $\forall\ \mathrm{S} \in B(\aleph)$, $\exists\ \varrho \in \mathcal{M}$, such that $F_\varrho \mathrm{S} = F\mathrm{S}$.

Theorem 4.2.3. Suppose that the Bellman equation meets the following:

(i):F_ϱ and F are monotone;

(ii):$F_\varrho : B(\aleph) \to B(\aleph)$ is generalized Ćirić type α-F-contraction.

(iii): $\exists\ \zeta > 0$ such that

$$|H(\mathtt{c}_1, u, \mathrm{S}) - H(\mathtt{c}_1, u, \mathrm{S}')| \leq e^{\frac{-\zeta}{2}} M(\mathrm{S}, \mathrm{S}').$$

Then, (4.29) has a unique solution.

Proof. Let $B(\aleph)$ denote the set of all bounded real-valued function with $\|\mathrm{S}\| = \sup_{\mathtt{c}_1 \in \aleph}\{|\mathrm{S}(\mathtt{c}_1)^2|\}$. Then $B(\aleph)$ is DCMS and $\kappa : \aleph \times \aleph \to [1, \infty)$ and $\varkappa : \aleph \times \aleph \to [1, \infty)$ defined as, $\kappa(\mathrm{S}, \mathrm{S}') := 2|\mathrm{S}(\mathtt{c}_1)| + 3|\mathrm{S}'(\mathtt{c}_1)| + 2$ and $\varkappa(\mathrm{S}, \mathrm{S}') := 2|\mathrm{S}(\mathtt{c}_1)| + |\mathrm{S}'(\mathtt{c}_1)| + 1$.

With assumption

$$\sup_{l \geq 1} \lim_{i \to \infty} \frac{\kappa(S_{i+1}, S'_{i+2}) \varkappa(S_{i+1}, S'_{2n+l})}{\kappa(S_i, S'_{i+1})} < 1.$$

Consider $F : B(\aleph) \to B(\aleph)$ as;

$$F\mathrm{S}(\mathtt{c}_1) = \inf_{u \in U(\mathtt{c}_1)} \{H(\mathtt{c}_1, u, \mathrm{S})\}. \quad \forall\ \mathtt{c}_1 \in \aleph$$

Also consider $\zeta > 0$ and $(a - b)^2 \geq 0$, then following holds,

$$\frac{1}{4}(a^2 + b^2) \geq \frac{1}{2}(ab), \tag{4.30}$$

We will show that F meets all the requirements of Theorem 4.1.3.

Now

$$\begin{aligned}
\alpha(\mathrm{S}, \mathrm{S}')|F\mathrm{S}(\mathtt{c}_1) - F\mathrm{S}'(\mathtt{c}_1)|^2 &= \alpha(\mathrm{S}, \mathrm{S}')|H(\mathtt{c}_1, u, \mathrm{S}) - H(\mathtt{c}_1, u, \mathrm{S}')|^2 \\
&\leq \bigg| e^{\frac{-\zeta}{2}} \max\bigg\{|\mathrm{S}(\mathtt{c}_1) - \mathrm{S}'(\mathtt{c}_1)|, |\mathrm{S}(\mathtt{c}_1) - F_\varrho \mathrm{S}(\mathtt{c}_1)|, |\mathrm{S}'(\mathtt{c}_1) - F_\varrho \mathrm{S}'(\mathtt{c}_1)|, \\
&\quad \frac{1}{4}\big(|\mathrm{S}(\mathtt{c}_1) - F_\varrho \mathrm{S}'(\mathtt{c}_1)| + |\mathrm{S}'(\mathtt{c}_1) - F_\varrho \mathrm{S}(\mathtt{c}_1)|\big)\bigg\}\bigg|^2 \\
&\leq e^{-\zeta} \max\bigg\{|\mathrm{S}(\mathtt{c}_1) - \mathrm{S}'(\mathtt{c}_1)|^2, |\mathrm{S}(\mathtt{c}_1) - F_\varrho \mathrm{S}(\mathtt{c}_1)|^2, |\mathrm{S}'(\mathtt{c}_1) - F_\varrho \mathrm{S}'(\mathtt{c}_1)|^2,
\end{aligned}$$

$$
\begin{aligned}
&|\frac{1}{2}\big(|\mathrm{S}(\mathtt{c}_1) - F_\varrho \mathrm{S}'(\mathtt{c}_1)| + |\mathrm{S}'(\mathtt{c}_1) - F_\varrho \mathrm{S}(\mathtt{c}_1)|\big)|^2\Big\} \\
&\leq\ e^{-\zeta}\max\Big\{|\mathrm{S}(\mathtt{c}_1) - \mathrm{S}'(\mathtt{c}_1)|^2, |\mathrm{S}(\mathtt{c}_1) - F_\varrho \mathrm{S}(\mathtt{c}_1)|^2, |\mathrm{S}'(\mathtt{c}_1) - F_\varrho \mathrm{S}'(\mathtt{c}_1)|^2, \\
&\frac{1}{4}\big[(\mathrm{S}(\mathtt{c}_1) - \mathbb{F}_\varrho \mathrm{S}'(\mathtt{c}_1))^2 + (\mathrm{S}'(\mathtt{c}_1) - F_\varrho \mathrm{S}(\mathtt{c}_1))^2 \\
&+ 2\big((\mathrm{S}(\mathtt{c}_1) - F_\varrho \mathrm{S}'(\mathtt{c}_1))(\mathrm{S}'(\mathtt{c}_1) - F_\varrho \mathrm{S}(\mathtt{c}_1))\big)\big]\Big\}
\end{aligned}
$$

$$
\begin{aligned}
\alpha(\mathrm{S},\mathrm{S}')|F\mathrm{S}(\mathtt{c}_1) - F\mathrm{S}'(\mathtt{c}_1)|^2 \leq\ & e^{-\zeta}\max\Big\{|\mathrm{S}(\mathtt{c}_1) - \mathrm{S}'(\mathtt{c}_1)|^2, |\mathrm{S}(\mathtt{c}_1) - F_\varrho \mathrm{S}(\mathtt{c}_1)|^2, |\mathrm{S}'(\mathtt{c}_1) - F_\varrho \mathrm{S}'(\mathtt{c}_1)|^2, \\
& \frac{1}{4}\big[(\mathrm{S}(\mathtt{c}_1) - F_\varrho \mathrm{S}'(\mathtt{c}_1))^2 + (\mathrm{S}'(\mathtt{c}_1) - F_\varrho \mathrm{S}(\mathtt{c}_1))^2\big] \\
& + \frac{1}{2}\big[\big((\mathrm{S}(\mathtt{c}_1) - F_\varrho \mathrm{S}'(\mathtt{c}_1))(\mathrm{S}'(\mathtt{c}_1) - F_\varrho \mathrm{S}(\mathtt{c}_1))\big)\big]\Big\}.
\end{aligned}
$$

Moreover, by using $F_\varrho \mathrm{S}(\mathtt{c}_1) \geq F\mathrm{S}(\mathtt{c}_1)$ and (4.30),

$$
\begin{aligned}
\alpha(\mathrm{S},\mathrm{S}')|F\mathrm{S}(\mathtt{c}_1) - F\mathrm{S}'(\mathtt{c}_1)|^2 \leq\ & e^{-\zeta}\max\Big\{|\mathrm{S}(\mathtt{c}_1) - \mathrm{S}'(\mathtt{c}_1)|^2, |\mathrm{S}(\mathtt{c}_1) - F_\varrho \mathrm{S}(\mathtt{c}_1)|^2, |\mathrm{S}'(\mathtt{c}_1) - F_\varrho \mathrm{S}'(\mathtt{c}_1)|^2, \\
& + \frac{1}{2}\big[|\mathrm{S}(\mathtt{c}_1) - F_\varrho \mathrm{S}'(\mathtt{c}_1)|^2 + |\mathrm{S}'(\mathtt{c}_1) - F_\varrho \mathrm{S}(\mathtt{c}_1)|^2\big]\Big\} \\
\alpha(\mathrm{S},\mathrm{S}')\|F\mathrm{S} - F\mathrm{S}'\| \leq\ & e^{-\zeta}\max\Big\{\|\mathrm{S} - \mathrm{S}'\|, \|\mathrm{S} - F_\varrho \mathrm{S}\|, \|\mathrm{S}' - F_\varrho \mathrm{S}'\|, \\
& + \frac{1}{2}\big[\|\mathrm{S} - F_\varrho \mathrm{S}'\| + \|\mathrm{S}' - F_\varrho \mathrm{S}\|\big]\Big\} \\
\leq\ & e^{-\zeta}\max\Big\{\|\mathrm{S} - \mathrm{S}'\|, \|\mathrm{S} - F\mathrm{S}\|, \|\mathrm{S}' - F\mathrm{S}'\|, \\
& + \frac{1}{2}\big[\|\mathrm{S} - F\mathrm{S}'\| + \|\mathrm{S}' - F\mathrm{S}\|\big]\Big\} \\
\leq\ & e^{-\zeta}\max\Big\{\|\mathrm{S} - \mathrm{S}'\|, \|\mathrm{S} - F\mathrm{S}\|, \|\mathrm{S}' - F\mathrm{S}'\|, \\
& + \frac{1}{2}\big[\|\mathrm{S} - F\mathrm{S}'\| + \|\mathrm{S}' - F\mathrm{S}\|\big]\Big\} \\
\alpha(\mathrm{S},\mathrm{S}')\|F\mathrm{S} - F\mathrm{S}'\| \leq\ & e^{-\zeta}M(\mathrm{S},\mathrm{S}').
\end{aligned}
$$

Taking natural log

$$
\begin{aligned}
\ln(\alpha(\mathrm{S},\mathrm{S}')\|F\mathrm{S} - F\mathrm{S}'\|) &\leq \ln(e^{-\zeta}M(\mathrm{S},\mathrm{S}')) \\
\ln(\alpha(\mathrm{S},\mathrm{S}')\|F\mathrm{S} - F\mathrm{S}'\|) &\leq \ln(e^{-\zeta}) + \ln(M(\mathrm{S},\mathrm{S}')) \\
\ln(\alpha(\mathrm{S},\mathrm{S}')\|F\mathrm{S} - F\mathrm{S}'\|) &\leq -\zeta + \ln(M(\mathrm{S},\mathrm{S}')).
\end{aligned}
$$

Therefore,

$\forall\ \ \mathrm{S},\mathrm{S}' \in B(\aleph)$

$$\zeta + \mathcal{F}(\|F\mathrm{S} - F\mathrm{S}'\|) \leq \mathcal{F}(\beta M(\mathrm{S}, \mathrm{S}')),$$
$$\zeta + \mathcal{F}(\Im(F\mathrm{S}, F\mathrm{S}') \leq \mathcal{F}(\beta M(\mathrm{S}, \mathrm{S}')), \quad \forall \quad \mathrm{S}, \mathrm{S}' \in B(\aleph)$$

for any $\mathtt{c}_1 \in \aleph$ and by considering $\ln(\mathtt{c}_1) = \mathcal{F}(\mathtt{c}_1)$. Conclusion follows from Theorem 4.1.3. □

Chapter 5

Conclusion

• A detailed review of Belhenniche et al. [28] on "Solving nonlinear and dynamic programming equations on EbMS with the fixed point technique" is given and elaborated.

• Existence and uniqueness of solution for Volterra integral equations, system of non-linear fractional differential equations and dynamic programming equations like Bellman's equation, has been established by using Ćirić contraction mapping in the setting of EbMS.

• Motivated by the above work, the notion of generalized Ćirić type α-F-contraction in setting of DCMS has been introduced. Some fixed point results are established for generalized Ćirić type α-F-contraction in the framework of DCMS. An example is provided to elaborate our main result.

• For application purpose:

(i): We prove the existence and uniqueness of a solution for the Volterra type integral equation by using the proven result (main results) in the framework of DCMS.

(ii): The existence and uniqueness of solution of system of FDE involving Caputo derivative has been given using Theorem (4.1.3).

(iii): We used our main result to show the uniqueness and existence of the dynamic programming Bellman's equation.

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