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TENTH EDITION

# BASIC TECHNICAL MATHEMATICS

with Calculus **SI VERSION**

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ALLYN J. WASHINGTON • MICHELLE BOUÉ



Basic Technical  
Mathematics  
with Calculus  
SI Version

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TENTH EDITION

# Basic Technical Mathematics with Calculus

## SI Version

**Allyn J. Washington**

*Dutchess Community College*

*Michelle Boué*

PEARSON

Toronto



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*To Douglas, Julia and Andrea ~Michelle Boué*

*In memory of my loving wife, Millie ~Allyn J. Washington*

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# Contents

Preface	xi		
<b>1 Basic Algebraic Operations</b>	<b>1</b>		
1.1 Numbers	2	4.3 Values of the Trigonometric Functions	122
1.2 Fundamental Operations of Algebra	6	4.4 The Right Triangle	126
1.3 Measurement, Calculation, and Approximate Numbers	11	4.5 Applications of Right Triangles	131
1.4 Exponents	21	<b>Equations, Review Exercises, and Practice Test</b>	<b>136</b>
1.5 Scientific Notation	26	<b>5 Systems of Linear Equations; Determinants</b>	<b>142</b>
1.6 Roots and Radicals	30	5.1 Linear Equations	143
1.7 Addition and Subtraction of Algebraic Expressions	32	5.2 Graphs of Linear Functions	146
1.8 Multiplication of Algebraic Expressions	36	5.3 Solving Systems of Two Linear Equations in Two Unknowns Graphically	149
1.9 Division of Algebraic Expressions	38	5.4 Solving Systems of Two Linear Equations in Two Unknowns Algebraically	153
1.10 Solving Equations	41	5.5 Solving Systems of Two Linear Equations in Two Unknowns by Determinants	160
1.11 Formulas and Literal Equations	45	5.6 Solving Systems of Three Linear Equations in Three Unknowns Algebraically	166
1.12 Applied Word Problems	48	5.7 Solving Systems of Three Linear Equations in Three Unknowns by Determinants	170
<b>Equations, Review Exercises, and Practice Test</b>	<b>51</b>	<b>Equations, Review Exercises, and Practice Test</b>	<b>176</b>
<b>2 Geometry</b>	<b>55</b>	<b>6 Factoring and Fractions</b>	<b>181</b>
2.1 Lines and Angles	56	6.1 Special Products	182
2.2 Triangles	60	6.2 Factoring: Common Factor and Difference of Squares	185
2.3 Quadrilaterals	66	6.3 Factoring Trinomials	190
2.4 Circles	69	6.4 The Sum and Difference of Cubes	196
2.5 Measurement of Irregular Areas	74	6.5 Equivalent Fractions	197
2.6 Solid Geometric Figures	78	6.6 Multiplication and Division of Fractions	202
<b>Equations, Review Exercises, and Practice Test</b>	<b>81</b>	6.7 Addition and Subtraction of Fractions	206
<b>3 Functions and Graphs</b>	<b>86</b>	6.8 Equations Involving Fractions	212
3.1 Introduction to Functions	87	<b>Equations, Review Exercises, and Practice Test</b>	<b>216</b>
3.2 More about Functions	91	<b>7 Quadratic Equations</b>	<b>220</b>
3.3 Rectangular Coordinates	95	7.1 Quadratic Equations; Solution by Factoring	221
3.4 The Graph of a Function	97	7.2 Completing the Square	225
3.5 More about Graphs	104	7.3 The Quadratic Formula	228
3.6 Graphs of Functions Defined by Tables of Data	109	7.4 The Graph of the Quadratic Function	232
<b>Review Exercises and Practice Test</b>	<b>112</b>	<b>Equations, Review Exercises, and Practice Test</b>	<b>236</b>
<b>4 The Trigonometric Functions</b>	<b>115</b>		
4.1 Angles	116		
4.2 Defining the Trigonometric Functions	119		

---

## 8 Trigonometric Functions of Any Angle 240

- 8.1 Signs of the Trigonometric Functions 241  
 8.2 Trigonometric Functions of Any Angle 243  
 8.3 Radians 249  
 8.4 Applications of Radian Measure 253

**Equations, Review Exercises, and Practice Test 260**

---

## 9 Vectors and Oblique Triangles 264

- 9.1 Introduction to Vectors 265  
 9.2 Components of Vectors 269  
 9.3 Vector Addition by Components 273  
 9.4 Applications of Vectors 277  
 9.5 Oblique Triangles, the Law of Sines 282  
 9.6 The Law of Cosines 288

**Equations, Review Exercises, and Practice Test 292**

---

## 10 Graphs of The Trigonometric Functions 296

- 10.1 Graphs of  $y = a \sin x$  and  $y = a \cos x$  297  
 10.2 Graphs of  $y = a \sin bx$  and  $y = a \cos bx$  300  
 10.3 Graphs of  $y = a \sin (bx + c)$  and  $y = a \cos (bx + c)$  303  
 10.4 Graphs of  $y = \tan x$ ,  $y = \cot x$ ,  $y = \sec x$ ,  $y = \csc x$  307  
 10.5 Applications of the Trigonometric Graphs 310  
 10.6 Composite Trigonometric Curves 313

**Equations, Review Exercises, and Practice Test 317**

---

## 11 Exponents and Radicals 320

- 11.1 Simplifying Expressions with Integral Exponents 321  
 11.2 Fractional Exponents 325  
 11.3 Simplest Radical Form 329  
 11.4 Addition and Subtraction of Radicals 333  
 11.5 Multiplication and Division of Radicals 335

**Equations, Review Exercises, and Practice Test 339**

---

## 12 Complex Numbers 341

- 12.1 Basic Definitions 342  
 12.2 Basic Operations with Complex Numbers 345  
 12.3 Graphical Representation of Complex Numbers 348  
 12.4 Polar Form of a Complex Number 350  
 12.5 Exponential Form of a Complex Number 352  
 12.6 Products, Quotients, Powers, and Roots of Complex Numbers 355

- 12.7 An Application to Alternating-Current (ac) Circuits 361

**Equations, Review Exercises, and Practice Test 366**

---

## 13 Exponential and Logarithmic Functions 370

- 13.1 Exponential Functions 371  
 13.2 Logarithmic Functions 373  
 13.3 Properties of Logarithms 377  
 13.4 Logarithms to the Base 10 382  
 13.5 Natural Logarithms 385  
 13.6 Exponential and Logarithmic Equations 388  
 13.7 Graphs on Logarithmic and Semilogarithmic Paper 392

**Equations, Review Exercises, and Practice Test 396**

---

## 14 Additional Types of Equations and Systems of Equations 399

- 14.1 Graphical Solution of Systems of Equations 400  
 14.2 Algebraic Solution of Systems of Equations 403  
 14.3 Equations in Quadratic Form 407  
 14.4 Equations with Radicals 410

**Review Exercises and Practice Test 414**

---

## 15 Equations of Higher Degree 417

- 15.1 The Remainder and Factor Theorems; Synthetic Division 418  
 15.2 The Roots of an Equation 423  
 15.3 Rational and Irrational Roots 427

**Equations, Review Exercises, and Practice Test 433**

---

## 16 Matrices; Systems of Linear Equations 435

- 16.1 Matrices: Definitions and Basic Operations 436  
 16.2 Multiplication of Matrices 439  
 16.3 Finding the Inverse of a Matrix 445  
 16.4 Matrices and Linear Equations 449  
 16.5 Gaussian Elimination 454  
 16.6 Higher-Order Determinants 457

**Equations, Review Exercises, and Practice Test 463**

---

## 17 Inequalities 467

- 17.1 Properties of Inequalities 468  
 17.2 Solving Linear Inequalities 472  
 17.3 Solving Nonlinear Inequalities 476

17.4	Inequalities Involving Absolute Values	482	22.2	Summarizing Data	620
17.5	Graphical Solution of Inequalities with Two Variables	485	22.3	Normal Distributions	628
17.6	Linear Programming	488	22.4	Confidence Intervals	634
	<b>Equations, Review Exercises, and Practice Test</b>	<b>492</b>	22.5	Statistical Process Control	640
<b>18</b>	<b>Variation</b>	<b>495</b>	22.6	Linear Regression	646
18.1	Ratio and Proportion	496	22.7	Nonlinear Regression	651
18.2	Variation	500		<b>Equations, Review Exercises, and Practice Test</b>	<b>654</b>
	<b>Equations, Review Exercises, and Practice Test</b>	<b>506</b>	<b>23</b>	<b>The Derivative</b>	<b>659</b>
<b>19</b>	<b>Sequences and The Binomial Theorem</b>	<b>510</b>	23.1	Limits	660
19.1	Arithmetic Sequences	511	23.2	The Slope of a Tangent to a Curve	669
19.2	Geometric Sequences	516	23.3	The Derivative	673
19.3	Infinite Geometric Series	520	23.4	The Derivative as an Instantaneous Rate of Change	677
19.4	The Binomial Theorem	523	23.5	Derivatives of Polynomials	682
	<b>Equations, Review Exercises, and Practice Test</b>	<b>528</b>	23.6	Derivatives of Products and Quotients of Functions	686
<b>20</b>	<b>Additional Topics in Trigonometry</b>	<b>531</b>	23.7	The Derivative of a Power of a Function	690
20.1	Fundamental Trigonometric Identities	532	23.8	Differentiation of Implicit Functions	699
20.2	The Sum and Difference Formulas	537	23.9	Higher Derivatives	702
20.3	Double-Angle Formulas	542		<b>Equations, Review Exercises, and Practice Test</b>	<b>706</b>
20.4	Half-Angle Formulas	545	<b>24</b>	<b>Applications of the Derivative</b>	<b>711</b>
20.5	Solving Trigonometric Equations	548	24.1	Tangents and Normals	712
20.6	The Inverse Trigonometric Functions	553	24.2	Newton's Method for Solving Equations	714
	<b>Equations, Review Exercises, and Practice Test</b>	<b>558</b>	24.3	Curvilinear Motion	718
<b>21</b>	<b>Plane Analytic Geometry</b>	<b>562</b>	24.4	Related Rates	722
21.1	Basic Definitions	563	24.5	Using Derivatives in Curve Sketching	727
21.2	The Straight Line	567	24.6	More on Curve Sketching	732
21.3	The Circle	573	24.7	Applied Maximum and Minimum Problems	737
21.4	The Parabola	578	24.8	Differentials and Linear Approximations	743
21.5	The Ellipse	582		<b>Equations, Review Exercises, and Practice Test</b>	<b>747</b>
21.6	The Hyperbola	587	<b>25</b>	<b>Integration</b>	<b>752</b>
21.7	Translation of Axes	593	25.1	Antiderivatives	753
21.8	The Second-Degree Equation	596	25.2	The Indefinite Integral	755
21.9	Rotation of Axes	599	25.3	The Area Under a Curve	760
21.10	Polar Coordinates	603	25.4	The Definite Integral	765
21.11	Curves in Polar Coordinates	606	25.5	Numerical Integration: The Trapezoidal Rule	768
	<b>Equations, Review Exercises, and Practice Test</b>	<b>610</b>	25.6	Simpson's Rule	771
<b>22</b>	<b>Introduction to Statistics</b>	<b>615</b>		<b>Equations, Review Exercises, and Practice Test</b>	<b>774</b>
22.1	Tabular and Graphical Representation of Data	616	<b>26</b>	<b>Applications of Integration</b>	<b>777</b>
			26.1	Applications of the Indefinite Integral	778
			26.2	Areas by Integration	782



26.3	Volumes by Integration	788
26.4	Centroids	793
26.5	Moments of Inertia	799
26.6	Other Applications	804

**Equations, Review Exercises, and Practice Test 809**

**27 Differentiation of Transcendental Functions 814**

27.1	Derivatives of the Sine and Cosine Functions	815
27.2	Derivatives of the Other Trigonometric Functions	819
27.3	Derivatives of the Inverse Trigonometric Functions	822
27.4	Applications	825
27.5	Derivative of the Logarithmic Function	830
27.6	Derivative of the Exponential Function	834
27.7	L'Hospital's Rule	837
27.8	Applications	841

**Equations, Review Exercises, and Practice Test 844**

**28 Methods of Integration 849**

28.1	The General Power Formula	850
28.2	The Basic Logarithmic Form	852
28.3	The Exponential Form	856
28.4	Basic Trigonometric Forms	859
28.5	Other Trigonometric Forms	863
28.6	Inverse Trigonometric Forms	867
28.7	Integration by Parts	871
28.8	Integration by Trigonometric Substitution	876
28.9	Integration by Partial Fractions: Nonrepeated Linear Factors	879
28.10	Integration by Partial Fractions: Other Cases	883
28.11	Integration by Use of Tables	888

**Equations, Review Exercises, and Practice Test 891**

**29 Partial Derivatives and Double Integrals 895**

29.1	Functions of Two Variables	896
29.2	Curves and Surfaces in Three Dimensions	899

29.3	Partial Derivatives	905
29.4	Double Integrals	909

**Equations, Review Exercises, and Practice Test 913**

**30 Expansion of Functions in Series 915**

30.1	Infinite Series	916
30.2	Maclaurin Series	919
30.3	Operations with Series	923
30.4	Computations by Use of Series Expansions	928
30.5	Taylor Series	931
30.6	Introduction to Fourier Series	934
30.7	More About Fourier Series	940

**Equations, Review Exercises, and Practice Test 945**

**31 Differential Equations 949**

31.1	Solutions of Differential Equations	950
31.2	Separation of Variables	952
31.3	Integrating Combinations	955
31.4	The Linear Differential Equation of the First Order	957
31.5	Numerical Solutions of First-Order Equations	960
31.6	Elementary Applications	963
31.7	Higher-Order Homogeneous Equations	969
31.8	Auxiliary Equation with Repeated or Complex Roots	973
31.9	Solutions of Nonhomogeneous Equations	977
31.10	Applications of Higher-Order Equations	982
31.11	Laplace Transforms	989
31.12	Solving Differential Equations by Laplace Transforms	994

**Equations, Review Exercises, and Practice Test 998**

Appendix A	Solving Word Problems	A.1
Appendix B	A Table of Integrals	A.2
	Answer to Odd-Numbered Exercises	B.1
	Solutions to Practice Test Problems	C.1
	Index of Applications	D.1
	Index of Writing Exercises	D.10
	Index	D.12

## Scope of the Book

*Basic Technical Mathematics with Calculus, SI Version*, tenth edition, is intended primarily for students in technical and pre-engineering technology programs or other programs for which coverage of basic mathematics is required. Chapters 1 through 20 provide the necessary background for further study, with an integrated treatment of algebra and trigonometry. Chapter 21 covers the basic topics of analytic geometry, and Chapter 22 gives an introduction to statistics. Fundamental topics of calculus are covered in Chapters 23 through 31. In the examples and exercises, numerous applications from many fields of technology are included, primarily to indicate where and how mathematical techniques are used. However, it is not necessary that the student have a specific knowledge of the technical area from which any given problem is taken.

Most students using this text will have a background that includes some algebra and geometry. However, the material is presented in adequate detail for those who may need more study in these areas. The material presented here is sufficient for three to four semesters.

One of the principal reasons for the arrangement of topics in this text is to present material in an order that allows a student to take courses concurrently in allied technical areas, such as physics and electricity. These allied courses normally require a student to know certain mathematical topics by certain definite times; yet the traditional order of topics in mathematics courses makes it difficult to attain this coverage without loss of continuity. However, the material in this book can be rearranged to fit any appropriate sequence of topics. Another feature of this text is that certain topics traditionally included for mathematical completeness have been covered only briefly or have been omitted. The approach used in this text is not unduly rigorous mathematically, although all appropriate terms and concepts are introduced as needed and given an intuitive or algebraic foundation. The aim is to help the student develop an understanding of mathematical methods without simply providing a collection of formulas. The text material has been developed with the recognition that it is essential for the student to have a sound background in algebra and trigonometry in order to understand and succeed in any subsequent work in mathematics.

## New Features

In this tenth edition of *Basic Technical Mathematics with Calculus, SI Version*, we have retained all the basic features of successful previous editions and have also introduced a number of improvements, described here.

### NEW AND REVISED COVERAGE

The topics of units and measurement covered in an appendix in the ninth edition have been expanded and integrated into Chapter 1, together with new discussions on rounding and on engineering notation. Interval notation is introduced in Chapter 3 and is used in several sections throughout the text. Chapter 31 includes a new subsection on solving nonhomogeneous differential equations using Fourier series.

Chapter 22 has been revised and expanded; a new section on summarizing data covers measures of central tendency, measures of spread, and new material on Chebychev's theorem; the section on normal distributions now includes a subsection on sampling distributions. In addition, the chapter now includes a completely new section on confidence intervals.

### EXPANDED PEDAGOGY

- NEW “Common Error” boxes appear throughout the text. A fresh design emphasizes valuable warnings against common mistakes or areas where students frequently have difficulty. These boxes replace the notes flagged by a “Caution” indicator in the previous edition.

- NEW “Learning Tip” boxes appear in the margin throughout the text. These colourful boxes highlight the underlying rationale of using specific mathematical functions and encourage students to think strategically about how and why specific mathematical concepts are needed and applied. They also focus attention on material that is of particular importance in understanding the topic under discussion. These boxes replace the notes flagged by a “Notes” indicator in the previous edition.
- NEW “Procedure” boxes include step-by-step instructions on how to perform select calculations.

### FEWER CALCULATOR SCREENS

Many figures involving screens from a graphic calculator have been either removed from the text or replaced by regular graphs. The calculator displays that remain are, for the most part, related to topics that require the use of technology (such as the graphical solution of systems of equations) or topics where technology can greatly simplify a process (such as obtaining the inverse of a large matrix). The appendix on graphing calculators from the previous edition dedicated to the graphing calculator will be available in Chapter 34 of the Study Plan in both MyMathLab and MathXL versions of this course. Students will also have easy access to it through the eText in MyMathLab.

### FUNCTIONAL USE OF COLOUR

The new full-colour design of this edition uses colour effectively for didactical purposes. Many figures and graphs have been enhanced with colour. Moreover, colour is used to identify and focus attention on the text’s new pedagogical features. Colour is also used to highlight the question numbers of writing exercises so that students and instructors can identify them easily.

### NOTATION

Symbols used in accordance with professional Canadian standards are applied consistently throughout the text.

### INCREASED BREADTH OF APPLICATIONS

New examples and exercises have been added in order to increase the range of applications covered by the text. New material can be found involving statics, fluid mechanics, optics, acoustics, cryptography, forestry, reliability, and quality control, to name but a few.

### INTERNATIONAL AND CANADIAN CONTENT

New Canadian content appears either in the form of examples within the text (some of which are linked to chapter openers, so they are accompanied by a full colour image), or as exercises at the end of a section or chapter. All material of global interest has been retained or updated, and some new exercises were also added.

### LEARNING OUTCOMES

A list of Learning Outcomes appears on the introductory page of each chapter, replacing the list of key topics for each section in the previous edition. This new learning tool reflects the current emphasis on learning outcomes and gives the student and instructor a quick way of checking that they have covered key contents of the chapter.

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## Continuing Features

### EXAMPLE DESCRIPTIONS

A brief descriptive title is given with each example number. This gives an easy reference for the example, which is particularly helpful when a student is reviewing the contents of the section.

### PRACTICE EXERCISES

Throughout the text, there are *practice exercises* in the margin. Most sections have at least one (and up to as many as four) of these basic exercises. They are included so that a student is more actively involved in the learning process and can check his or her understanding of

the material to that point in the section. They can also be used for classroom exercises. The answers to these exercises are given at the end of the exercise set for the section.

### **NEW EXERCISES**

More than 300 exercises are new or have been updated. This tenth edition contains a total of about 12 500 exercises.

### **CHAPTER INTRODUCTIONS**

Each chapter introduction illustrates specific examples of how the development of technology has been related to the development of mathematics. These introductions show that past discoveries in technology led to some of the methods in mathematics, whereas in other cases mathematical topics already known were later very useful in bringing about advances in technology.

### **SPECIAL EXPLANATORY COMMENTS**

Throughout the book, special explanatory comments in colour have been used in the examples to emphasize and clarify certain important points. Arrows are often used to indicate clearly the part of the example to which reference is made.

### **IMPORTANT FORMULAS**

Throughout the book, important formulas are set off and displayed so that they can be easily referenced.

### **SUBHEADS AND KEY TERMS**

Many sections include subheads to indicate where the discussion of a new topic starts within the section. Key terms are noted in the margin for emphasis and easy reference.

### **EXERCISES DIRECTLY REFERENCED TO TEXT EXAMPLES**

The first few exercises in most of the text sections are referenced directly to a specific example of the section. These exercises are worded so that it is necessary for the student to refer to the example in order to complete the required solution. In this way, the student should be able to review and understand the text material better before attempting to solve the exercises that follow.

### **WRITING EXERCISES**

One specific writing exercise is included at the end of each chapter. These exercises give the students practice in explaining their solutions. Also, there are more than 400 additional exercises throughout the book (at least 8 in each chapter) that require at least a sentence or two of explanation as part of the answer. The question numbers of writing exercises are highlighted in colour. A special “Index of Writing Exercises” is included at the back of the book.

### **WORD PROBLEMS**

There are more than 120 examples throughout the text that show the complete solutions of word problems. There are also more than 850 exercises in which word problems are to be solved.

### **CHAPTER EQUATIONS, REVIEW EXERCISES, AND PRACTICE TESTS**

At the end of each chapter, all important equations are listed together for easy reference. Each chapter is also followed by a set of review exercises that covers all the material in the chapter. Following the chapter equations and review exercises is a chapter practice test that students can use to check their understanding of the material. Solutions to all practice test problems are given in the back of the book.

### **APPLICATIONS**

Examples and exercises illustrate the application of mathematics in all fields of technology. Many relate to modern technology such as computer design, electronics, solar energy, lasers, fibre optics, the environment, and space technology. A special “Index of Applications” is included near the end of the book.

**EXAMPLES**

There are more than 1400 worked examples in this text. Of these, more than 300 illustrate technical applications.

**FIGURES**

There are more than 1300 figures in the text. Approximately 20% of the figures are new or have been modified for this edition.

**MARGIN NOTES**

Throughout the text, some margin notes briefly point out relevant historical events in mathematics and technology. Other margin notes are used to make specific comments related to the text material. Also, where appropriate, equations from earlier material are shown for reference in the margin.

**ANSWERS TO EXERCISES**

The answers to all odd-numbered exercises (except the end-of-chapter writing exercises) are given at the back of the book.

**FLEXIBILITY OF MATERIAL COVERAGE**

The order of material coverage can be changed in many places, and certain sections may be omitted without loss of continuity of coverage. Users of earlier editions have indicated the successful use of numerous variations in coverage. Any changes will depend on the type of course and completeness required.

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## Supplements

**SUPPLEMENTS FOR THE STUDENT**

Extensively updated by text author Michelle Boué, the *Students Solutions Manual* contains revised solutions for every other odd-numbered exercise. These step-by-step solutions have been expanded for even greater accuracy, clarity, and consistency to improve student problem-solving skills. The *Students Solutions Manual* is included in MyMathLab and is also available as a printed supplement via the Pearson Custom Library. (Please contact your local Pearson representative to learn more about this option.)

**SUPPLEMENTS FOR THE INSTRUCTOR**

Instructor's resources include the following supplements.

**Instructor's Solutions Manual**

The *Instructor's Solution Manual* contains detailed solutions to every section exercise, including review exercises. These in-depth, step-by-step solutions have been thoroughly revised by text author Michelle Boué for greater clarity and consistency; note that this expansion has been carried through to the Student Solutions Manual as well. The *Instructors Solutions Manual* can be downloaded from Pearson's online catalogue at [www.pearsoned.ca](http://www.pearsoned.ca). The *Instructor's Solution Manual* contains solutions for all section exercises.

**Animated PowerPoint Presentations**

More than 150 animated slides are available for download from a protected location on Pearson Education's online catalogue, at [www.pearsoned.ca](http://www.pearsoned.ca).

Each slide offers a step-by-step mini lesson on an individual section, or key concept, formula, or equation from the first 28 chapters of the book. For instance, 15 steps for using the "General Power Formula for Integration" are beautifully illustrated in the animated slide for Chapter 28. There are two sets of slides for "Operations with Complex Numbers" for section 2 of Chapter 12; the 9 steps to perform addition are shown on one slide, and the 13 steps to perform subtraction appear on the second slide.

These animated slides offer bite-sized chunks of key information for students to review and process prior to going to the homework questions for practice. Please note that not every section in every chapter is accompanied by an animated slide as some topics lend themselves to this approach more than others. These PowerPoint slide are also integrated in the Pearson eText within MyMathLab.

### TestGen with Algorithmically Generated Questions

Instructors can easily create tests from textbook section objectives. Algorithmically generated questions allow unlimited versions. Instructors can edit problems or create their own by using the built-in question editor to generate graphs; import graphics; and insert math notation, variable numbers, or text. Tests can be printed or administered online via the Web or other network.

### MyMathLab<sup>®</sup> Online Course

MyMathLab delivers **proven results** in helping individual students succeed:

- MyMathLab has a consistently positive impact on the quality of learning in higher education math instruction. MyMathLab can be successfully implemented in any environment—lab-based, hybrid, fully online, traditional—and demonstrates the quantifiable difference that integrated usage has on student retention, subsequent success, and overall achievement.
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# Basic Algebraic Operations

Interest in things such as the land on which they lived, the structures they built, and the motion of the planets led people in early civilizations to keep records and to create methods of counting and measuring. In turn, some of the early ideas of arithmetic, geometry, and trigonometry were developed. From such beginnings, mathematics has played a key role in the great advances in science and technology.

Often, mathematical methods were developed from studies made in sciences, such as astronomy and physics, to better describe, measure, and understand the subject being studied. Some of these methods resulted from the needs in a particular area of application.

Many people were interested in the mathematics itself and added to what was then known. Although this additional mathematical knowledge may not have been related to applications at the time it was developed, it often later became useful in applied areas.

In the chapter introductions that follow, examples of the interaction of technology and mathematics are given. From these examples and the text material, it is hoped you will better understand the important role that mathematics has had and still has in technology. In this text, there are applications from technologies including (but not limited to) aeronautical, business, communications, electricity, electronics, engineering, environmental, heat and air conditioning, mechanical, medical, meteorology, petroleum, product design, solar, and space. To solve the applied problems in this text will require a knowledge of the mathematics presented but will *not* require prior knowledge of the field of application.

We begin by reviewing the concepts that deal with numbers and symbols. This will enable us to develop topics in algebra, an understanding of which is essential for progress in other areas such as geometry, trigonometry, and calculus.

► **The Great Pyramid at Giza in Egypt was built about 4500 years ago.**



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◄ **In the 1500s, 1600s, and 1700s, discoveries in astronomy and the need for more accurate maps and instruments in navigation were very important in leading scientists and mathematicians to develop useful new ideas and methods in mathematics.**



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## LEARNING OUTCOMES

**After completion of this chapter, the student should be able to:**

- Identify real, imaginary, rational, and irrational numbers
- Perform mathematical operations on integers, decimals, fractions, and radicals
- Use the fundamental laws of algebra in numeric and algebraic equations
- Employ mathematical order of operations
- Understand technical measurement and approximation, as well as the use of significant digits and rounding
- Use scientific and engineering notations
- Convert units of measurement
- Rearrange and solve basic algebraic expressions
- Interpret word problems using algebraic symbols

◄ **Late in the 1800s, scientists were studying the nature of light. This led to a mathematical prediction of the existence of radio waves, now used in many types of communication. Also, in the 1900s and 2000s, mathematics has been vital to the development of electronics and space travel.**

## 1.1 Numbers

Real Number System • Number Line • Absolute Value • Signs of Inequality • Reciprocal • Denominate Numbers • Literal Numbers

■ Irrational numbers were discussed by the Greek mathematician Pythagoras in about 540 B.C.E.

### LEARNING TIP

A notation that is often used for repeating decimals is to place a bar over the digits that repeat. Using this notation we can write

$$\frac{1121}{1665} = 0.673\overline{2} \text{ and } \frac{2}{3} = 0.\overline{6}$$

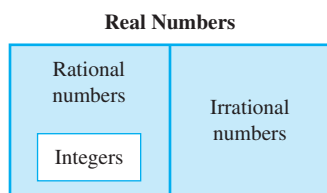


Fig. 1.1

In technology and science, as well as in everyday life, we use the very familiar **counting numbers** 1, 2, 3, and so on. They are also called **natural numbers** or **positive integers**. The **negative integers**  $-1, -2, -3,$  and so on are also very necessary and useful in mathematics and its applications. *The integers include the positive integers and the negative integers and zero, which is neither positive nor negative.* This means the integers are the numbers  $\dots, -3, -2, -1, 0, 1, 2, 3,$  and so on.

To specify parts of a quantity, *rational numbers* are used. A **rational number** is any number that can be represented by the division of one integer by another nonzero integer. Another type of number, an **irrational number**, cannot be written as the division of one integer by another.

### EXAMPLE 1 Identifying rational numbers and irrational numbers

The numbers 5 and  $-19$  are integers. They are also rational numbers since they can be written as  $\frac{5}{1}$  and  $\frac{-19}{1}$ , respectively. Normally, we do not write the 1's in the denominators.

The numbers  $\frac{5}{8}$  and  $\frac{-11}{3}$  are rational numbers because the numerator and the denominator of each are integers.

The numbers  $\sqrt{2}$  and  $\pi$  are irrational numbers. It is not possible to find two integers, one divided by the other, to represent either of these numbers. It can be shown that square roots (and other roots) that cannot be expressed exactly in decimal form are irrational. Also,  $\frac{22}{7}$  is sometimes used as an *approximation* for  $\pi$ , but it is not equal *exactly* to  $\pi$ . We must remember that  $\frac{22}{7}$  is rational and  $\pi$  is irrational.

The decimal number 1.5 is rational since it can be written as  $\frac{3}{2}$ . Any such *terminating decimal* is rational. The number  $0.6666\dots$ , where the 6's continue on indefinitely, is rational since we may write it as  $\frac{2}{3}$ . In fact, any *repeating decimal* (in decimal form, a specific sequence of digits is repeated indefinitely) is rational. The decimal number  $0.673\ 273\ 273\ 2\dots$  is a repeating decimal where the sequence of digits 732 is repeated indefinitely ( $0.673\ 273\ 273\ 2\dots = \frac{1121}{1665}$ ). ■

*The integers, the rational numbers, and the irrational numbers, including all such numbers that are positive, negative, or zero, make up the real number system* (see Fig. 1.1). There are times we will encounter an **imaginary number**, *the name given to the square root of a negative number*. Imaginary numbers are not real numbers and will be discussed in Chapter 12. However, unless specifically noted, we will use real numbers. Until Chapter 12, it will be necessary to only *recognize* imaginary numbers when they occur.

Also in Chapter 12, we will consider **complex numbers**, which include both the real numbers and imaginary numbers. See Exercise 37 of this section.

### EXAMPLE 2 Identifying real numbers and imaginary numbers

The number 7 is an integer. It is also rational since  $7 = \frac{7}{1}$ , and it is a real number since the real numbers include all the rational numbers.

The number  $3\pi$  is irrational, and it is real since the real numbers include all the irrational numbers.

The numbers  $\sqrt{-10}$  and  $-\sqrt{-7}$  are imaginary numbers.

The number  $\frac{-3}{7}$  is rational and real. The number  $-\sqrt{7}$  is irrational and real.

The number  $\frac{\pi}{6}$  is irrational and real. The number  $\frac{\sqrt{-3}}{2}$  is imaginary. ■

A **fraction** may contain any number or symbol representing a number in its numerator or in its denominator. The fraction indicates the division of the numerator by the denominator, as we previously indicated in writing rational numbers. Therefore, a fraction may be a number that is rational, irrational, or imaginary. A fraction can represent

■ Real numbers and imaginary numbers are both included in the *complex number system*. See Exercise 37.

■ Fractions were used by early Egyptians and Babylonians. They were used for calculations that involved parts of measurements, property, and possessions.

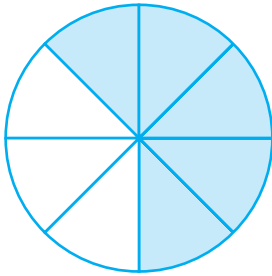


Fig. 1.2

a part of a whole, and sometimes it can represent the number of equal-sized parts that a whole is divided into. For example, in Fig. 1.2, a whole circle has been divided into eight equal pieces. The shaded portion represents five of those eight pieces, or  $5/8$  of the whole circle.

**EXAMPLE 3 Fractions**

The numbers  $\frac{2}{7}$  and  $\frac{-3}{2}$  are fractions, and they are rational.

The numbers  $\frac{\sqrt{2}}{9}$  and  $\frac{6}{\pi}$  are fractions, but they are not rational numbers. It is not possible to express either as one integer divided by another integer.

The number  $\frac{\sqrt{-5}}{6}$  is a fraction, and it is an imaginary number. ■

**The Number Line**

Real numbers may be represented by points on a line. We draw a horizontal line and designate some point on it by  $O$ , which we call the **origin** (see Fig. 1.3). The integer *zero* is located at this point. Equal intervals are marked to the right of the origin, and the positive integers are placed at these positions. The other positive rational numbers are located between the integers. The points that cannot be defined as rational numbers represent irrational numbers. We cannot tell whether a given point represents a rational number or an irrational number unless it is specifically marked to indicate its value.

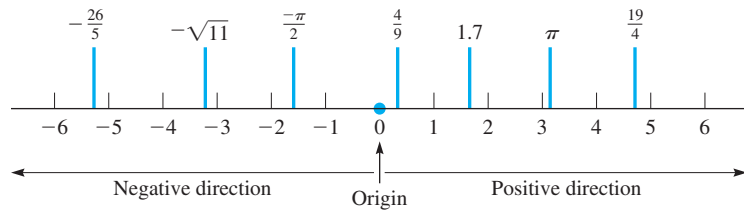


Fig. 1.3

The negative numbers are located on the number line by starting at the origin and marking off equal intervals *to the left*, which is the **negative direction**. As shown in Fig. 1.3, *the positive numbers are to the right of the origin and the negative numbers are to the left of the origin*. Representing numbers in this way is especially useful for graphical methods.

We next define another important concept of a number. The **absolute value** of a number is the numerical value (magnitude) of the number without regard to its sign. The absolute value of a positive number is the number itself, and the absolute value of a negative number is just the number, without the negative sign. On the number line, we may interpret the absolute value of a number as the distance (which is always positive) between the origin and the number. Absolute value is denoted by writing the number between vertical lines, as shown in the following example.

**EXAMPLE 4 Absolute value**

The absolute value of 6 is 6, and the absolute value of  $-7$  is 7. We write these as  $|6| = 6$  and  $|-7| = 7$ . See Fig. 1.4.

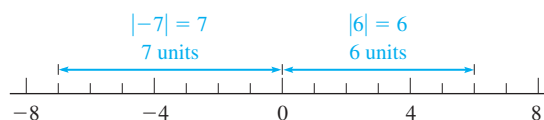


Fig. 1.4

Other examples are  $|\frac{7}{5}| = \frac{7}{5}$ ,  $|\sqrt{-2}| = \sqrt{2}$ ,  $|0| = 0$ ,  $|\pi| = \pi$ ,  $|-5.29| = 5.29$ ,  $-|-9| = -9$  since  $|-9| = 9$ . ■

**Practice Exercises**

1.  $|-4.2| = ?$
2.  $|\frac{-3}{4}| = ?$

■ The symbols  $=$ ,  $<$ , and  $>$  were introduced by English mathematicians in the late 1500s.

### Practice Exercises

Place the correct sign of inequality ( $<$  or  $>$ ) between the given numbers.

3.  $-5$  4    4.  $0$   $-3$

On the number line, if a first number is to the right of a second number, then the first number is said to be **greater than** the second. If the first number is to the left of the second, it is **less than** the second number. The symbol  $>$  designates “is greater than,” and the symbol  $<$  designates “is less than.” These are called **signs of inequality**. See Fig. 1.5.

### EXAMPLE 5 Signs of inequality

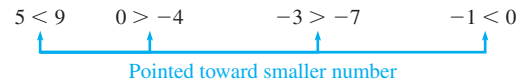
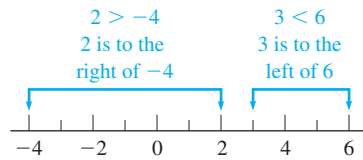


Fig. 1.5

Every number, except zero, has a **reciprocal**. The reciprocal of a number is 1 divided by the number.

### EXAMPLE 6 Reciprocal

The reciprocal of 7 is  $\frac{1}{7}$ . The reciprocal of  $\frac{2}{3}$  is

$$\frac{1}{\frac{2}{3}} = 1 \times \frac{3}{2} = \frac{3}{2} \quad \text{invert denominator and multiply (from arithmetic)}$$

The reciprocal of 0.5 is  $\frac{1}{0.5} = 2$ . The reciprocal of  $-\pi$  is  $-\frac{1}{\pi}$ . Note that the negative sign is retained in the reciprocal of a negative number.

We showed the multiplication of 1 and  $\frac{3}{2}$  as  $1 \times \frac{3}{2}$ . We could also show it as  $1 \cdot \frac{3}{2}$  or  $1\left(\frac{3}{2}\right)$ . We will often find the form with parentheses is preferable.

In applications, numbers that represent a measurement and are written with units of measurement are called **denominate numbers**. The next example illustrates the use of units and the symbols that represent them.

### EXAMPLE 7 Denominate numbers

To show that a certain HDTV set has mass of 28 kilograms, we write the mass as 28 kg.

To show that a giant redwood tree is 110 metres high, we write the height as 110 m.

To show that the speed of a rocket is 1500 metres per second, we write the speed as 1500 m/s. (Note the use of s for second. We use s rather than sec.)

To show that the area of a computer chip is 0.75 square centimetres, we write the area as  $0.75 \text{ cm}^2$ . (We will not use sq cm.)

To show that the volume of water in a glass tube is 25 cubic centimetres, we write the volume as  $25 \text{ cm}^3$ . (We will not use cu cm or cc.)

### Literal Numbers

It is usually more convenient to state definitions and operations on numbers in a general form. To do this, we represent the numbers by letters, called **literal numbers**. For example, if we want to say “If a first number is to the right of a second number on the number line, then the first number is greater than the second number,” we can write “If  $a$  is to the right of  $b$  on the number line, then  $a > b$ .” Another example of using a literal number is “The reciprocal of  $n$  is  $1/n$ .”

Certain literal numbers may take on any allowable value, whereas other literal numbers represent the same value throughout the discussion. Those literal numbers that may vary in a given problem are called **variables**, and those literal numbers that are held fixed are called **constants**.

■ For reference, see Section 1.3 for units of measurement and the symbols used for them.

**EXAMPLE 8** Variables and constants

- (a) The resistance of an electric resistor is  $R$ . The current  $I$  in the resistor equals the voltage  $V$  divided by  $R$ , written as  $I = V/R$ . For this resistor,  $I$  and  $V$  may take on various values, and  $R$  is fixed. This means  $I$  and  $V$  are variables and  $R$  is a constant. For a *different* resistor, the value of  $R$  may differ.
- (b) The fixed cost for a calculator manufacturer to operate a certain plant is  $b$  dollars per day, and it costs  $a$  dollars to produce each calculator. The total daily cost  $C$  to produce  $n$  calculators is

$$C = an + b$$

Here,  $C$  and  $n$  are variables, and  $a$  and  $b$  are constants, and the product of  $a$  and  $n$  is shown as  $an$ . For *another* plant, the values of  $a$  and  $b$  would probably differ.

If specific numerical values of  $a$  and  $b$  are known, say  $a = \$7$  per calculator and  $b = \$3000$ , then  $C = 7a + 3000$ . Thus, constants may be numerical or literal. ■

**EXERCISES 1.1**

In Exercises 1–4, make the given changes in the indicated examples of this section, and then answer the given questions.

- In the first line of Example 1, change the 5 to  $-3$  and the  $-19$  to 14. What other changes must then be made in the first paragraph?
- In Example 4, change the 6 to  $-6$ . What other changes must then be made in the first paragraph?
- In the left figure of Example 5, change the 2 to  $-6$ . What other changes must then be made?
- In Example 6, change the  $\frac{2}{3}$  to  $\frac{3}{2}$ . What other changes must then be made?

In Exercises 5 and 6, designate each of the given numbers as being an integer, rational, irrational, real, or imaginary. (More than one designation may be correct.)

5. 3,  $\sqrt{-4}$ ,  $-\frac{\pi}{6}$                       6.  $-\sqrt{-6}$ ,  $-2.33$ ,  $\frac{\sqrt{7}}{3}$

In Exercises 7 and 8, find the absolute value of each number.

7. 3,  $-4$ ,  $-\frac{\pi}{2}$                       8.  $-0.857$ ,  $\sqrt{2}$ ,  $-\frac{19}{4}$

In Exercises 9–16, insert the correct sign of inequality ( $>$  or  $<$ ) between the given numbers.

- |                                   |                         |
|-----------------------------------|-------------------------|
| 9. 6      8                       | 10. 7      5            |
| 11. $\pi$ $-3.2$                  | 12. $-4$ 0              |
| 13. $-4$ $- -3 $                  | 14. $-\sqrt{2}$ $-1.42$ |
| 15. $-\frac{1}{3}$ $-\frac{1}{2}$ | 16. $-0.6$ 0.2          |

In Exercises 17 and 18, find the reciprocal of each number.

17. 3,  $-\frac{4}{\sqrt{3}}$ ,  $\frac{y}{b}$                       18.  $\frac{1}{3}$ ,  $-0.25$ ,  $x$

In Exercises 19 and 20, locate each number on a number line, as in Fig. 1.3.

19. 2.5,  $-\frac{12}{5}$ ,  $\sqrt{3}$                       20.  $-\frac{\sqrt{2}}{2}$ ,  $2\pi$ ,  $\frac{123}{19}$

In Exercises 21–44, solve the given problems. Refer to Fig. 1.9 for units of measurement and their symbols.

- Is an absolute value always positive? Explain.
- Is 2.17 rational? Explain.
- What is the reciprocal of the reciprocal of any positive or negative number?
- Find a rational number between  $-0.9$  and  $-1.0$  that can be written with a denominator of 11 and an integer in the numerator.
- Find a rational number between 0.13 and 0.14 that can be written with a numerator of 3 and an integer in the denominator.
- If  $b > a$  and  $a > 0$ , is  $|b - a| < |b| - |a|$ ?
- List the following numbers in numerical order, starting with the smallest:  $-1$ , 9,  $\pi$ ,  $\sqrt{5}$ ,  $|-8|$ ,  $-|-3|$ ,  $-3.5$ .
- List the following numbers in numerical order, starting with the smallest:  $\frac{-1}{5}$ ,  $-\sqrt{10}$ ,  $-|-6|$ ,  $-4$ , 0.25,  $|\pi|$ .
- If  $a$  and  $b$  are positive integers and  $b > a$ , what type of number is represented by the following?
 

(a) $b - a$	(b) $a - b$	(c) $\frac{b - a}{b + a}$
-------------	-------------	---------------------------
- If  $a$  and  $b$  represent positive integers, what kind of number is represented by (a)  $a + b$ , (b)  $a/b$ , and (c)  $a \times b$ ?
- For any positive or negative integer: (a) Is its absolute value always an integer? (b) Is its reciprocal always a rational number?
- For any positive or negative rational number: (a) Is its absolute value always a rational number? (b) Is its reciprocal always a rational number?
- Describe the location of a number  $x$  on the number line when (a)  $x > 0$  and (b)  $x < -4$ .



34. Describe the location of a number  $x$  on the number line when (a)  $|x| < 1$  and (b)  $|x| > 2$ .
35. For a number  $x > 1$ , describe the location on the number line of the reciprocal of  $x$ .
36. For a number  $x < 0$ , describe the location on the number line of the number with a value of  $|x|$ .
37. A *complex number* is defined as  $a + bj$ , where  $a$  and  $b$  are real numbers and  $j = \sqrt{-1}$ . For what values of  $a$  and  $b$  is the complex number  $a + bj$  a real number? (All real numbers and all imaginary numbers are also complex numbers.)
38. A sensitive gauge measures the total weight  $w$  of a container and the water that forms in it as vapor condenses. It is found that  $w = c\sqrt{0.1t + 1}$ , where  $c$  is the weight of the container and  $t$  is the time of condensation. Identify the variables and constants.
39. In an electric circuit, the reciprocal of the total capacitance of two capacitors in series is the sum of the reciprocals of the capacitances. Find the total capacitance of two capacitances of 0.0040 F and 0.0010 F connected in series.
40. Alternating-current (ac) voltages change rapidly between positive and negative values. If a voltage of 100 V changes to  $-200$  V, which is greater in absolute value?
41. The memory of a certain computer has  $a$  bits in each byte. Express the number  $N$  of bits in  $n$  kilobytes in an equation. (A *bit* is a single digit, and bits are grouped in *bytes* in order to represent special characters. Generally, there are 8 bits per byte. If necessary, see Fig. 1.10 for the meaning of *kilo*.)
42. The computer design of the base of a truss is  $x$  m long. Later it is redesigned and shortened by  $y$  cm. Give an equation for the length  $L$ , in centimetres, of the base in the second design.
43. In a laboratory report, a student wrote “ $-20^\circ\text{C} > -30^\circ\text{C}$ .” Is this statement correct? Explain.
44. After 5 s, the pressure on a valve is less than 600 kPa. Using  $t$  to represent time and  $p$  to represent pressure, this statement can be written “for  $t > 5$  s,  $p < 600$  kPa.” In this way, write the statement “when the current  $I$  in a circuit is less than 4 A, the voltage  $V$  is greater than 12 V.”

### Answers to Practice Exercises

1. 4.2    2.  $-\frac{3}{4}$     3.  $<$     4.  $>$

## 1.2 Fundamental Operations of Algebra

Fundamental Laws of Algebra •  
Operations on Positive and Negative  
Numbers • Order of Operations •  
Operations with Zero

### The Commutative and Associative Laws

If two numbers are added, it does not matter in which order they are added. (For example,  $5 + 3 = 8$  and  $3 + 5 = 8$ , or  $5 + 3 = 3 + 5$ .) This statement, generalized and accepted as being correct for all possible combinations of numbers being added, is called the **commutative law** for addition. It states that *the sum of two numbers is the same, regardless of the order in which they are added*. We make no attempt to prove this law in general, but accept that it is true.

In the same way, we have the **associative law** for addition, which states that *the sum of three or more numbers is the same, regardless of the way in which they are grouped for addition*. For example,  $3 + (5 + 6) = (3 + 5) + 6$ .

The laws just stated for addition are also true for multiplication. Therefore, *the product of two numbers is the same, regardless of the order in which they are multiplied, and the product of three or more numbers is the same, regardless of the way in which they are grouped for multiplication*. For example,  $2 \times 5 = 5 \times 2$ , and  $5 \times (4 \times 2) = (5 \times 4) \times 2$ .

### The Distributive Law

Another very important law is the **distributive law**. It states that *the product of one number and the sum of two or more other numbers is equal to the sum of the products of the first number and each of the other numbers of the sum*. For example,

$$5(4 + 2) = 5 \times 4 + 5 \times 2$$

In this case, it can be seen that the total is 30 on each side.

In practice, these **fundamental laws of algebra** are used naturally without thinking about them, except perhaps for the distributive law.

Not all operations are commutative and associative. For example, division is not commutative, since the order of division of two numbers does matter. For instance,  $\frac{6}{5} \neq \frac{5}{6}$  ( $\neq$  is read “does not equal”). (Also, see Exercise 50.)

Using literal numbers, the fundamental laws of algebra are as follows:

**Commutative law of addition:**  $a + b = b + a$

**Associative law of addition:**  $a + (b + c) = (a + b) + c$

**Commutative law of multiplication:**  $ab = ba$

■ Note carefully the difference:  
associative law:  $5 \times (4 \times 2)$   
distributive law:  $5 \times (4 + 2)$

**Associative law of multiplication:**  $a(bc) = (ab)c$

**Distributive law:**  $a(b + c) = ab + ac$

Each of these laws is an example of an *identity*, in that the expression to the left of the = sign equals the expression to the right for any value of each of  $a$ ,  $b$ , and  $c$ .

### OPERATIONS ON POSITIVE AND NEGATIVE NUMBERS

When using the basic operations (addition, subtraction, multiplication, division) on positive and negative numbers, we determine the result to be either positive or negative according to the following rules.

**Addition of two numbers of the same sign** *Add their absolute values and assign the sum their common sign.*

#### EXAMPLE 1 Adding numbers of the same sign

- (a)  $2 + 6 = 8$  the sum of two positive numbers is positive  
 (b)  $-2 + (-6) = -(2 + 6) = -8$  the sum of two negative numbers is negative

■ From Section 1.1, we recall that a positive number is preceded by no sign. Therefore, in using these rules, we show the “sign” of a positive number by simply writing the number itself.

The negative number  $-6$  is placed in parentheses since it is also preceded by a plus sign showing addition. It is not necessary to place the  $-2$  in parentheses. ■

**Addition of two numbers of different signs** *Subtract the number of smaller absolute value from the number of larger absolute value and assign to the result the sign of the number of larger absolute value.* Alternatively, one can visualize addition using the number line concept discussed in Section 1.1. Start with the number line location of the first number in the addition problem. Then, if you add a positive number, move *right* along the number line to the total. If you add a negative number, move *left* along the number line until you arrive at the solution.

#### EXAMPLE 2 Adding numbers of different signs

- (a)  $2 + (-6) = -(6 - 2) = -4$  ← the negative 6 has the larger absolute value  
 (b)  $-6 + 2 = -(6 - 2) = -4$  ←  
 (c)  $6 + (-2) = 6 - 2 = 4$  ← the positive 6 has the larger absolute value  
 (d)  $-2 + 6 = 6 - 2 = 4$  ← the subtraction of absolute values ■

**Subtraction of one number from another** *Change the sign of the number being subtracted and change the subtraction to addition. Perform the addition.*

#### EXAMPLE 3 Subtracting positive and negative numbers

- (a)  $2 - 6 = 2 + (-6) = -(6 - 2) = -4$

Note that after changing the subtraction to addition, and changing the sign of 6 to make it  $-6$ , we have precisely the same illustration as Example 2(a).

- (b)  $-2 - 6 = -2 + (-6) = -(2 + 6) = -8$

Note that after changing the subtraction to addition, and changing the sign of 6 to make it  $-6$ , we have precisely the same illustration as Example 1(b).

- (c)  $-a - (-a) = -a + a = 0$

This shows that subtracting a number from itself results in zero, even if the number is negative. Therefore, *subtracting a negative number is equivalent to adding a positive number of the same absolute value.* ■

**Subtraction of a Negative Number**

**Multiplication and division of two numbers** *The product (or quotient) of two numbers of the same sign is positive. The product (or quotient) of two numbers of different signs is negative.*

**EXAMPLE 4** Multiplying and dividing positive and negative numbers

- (a)  $3(12) = 3 \times 12 = 36$        $\frac{12}{3} = 4$       result is positive if both numbers are positive
- (b)  $-3(-12) = 3 \times 12 = 36$        $\frac{-12}{-3} = 4$       result is positive if both numbers are negative
- (c)  $3(-12) = -(3 \times 12) = -36$        $\frac{-12}{3} = -\frac{12}{3} = -4$       result is negative if one number is positive and the other is negative
- (d)  $-3(12) = -(3 \times 12) = -36$        $\frac{12}{-3} = -\frac{12}{3} = -4$       ■

**ORDER OF OPERATIONS**

When mathematical operation symbols separate a series of numbers in an expression, it is important to follow an unambiguous *order* for completing those operations.

**Order of Operations**

1. Perform operations within specific groupings first—that is, inside parentheses ( ), brackets [ ], or absolute values | |.
2. Exponents and roots/radicals are evaluated next.  
**These will be discussed in Section 1.4 and Section 1.6, respectively.**
3. Perform multiplications and divisions (from left to right).
4. Perform additions and subtractions (from left to right).

**EXAMPLE 5** Order of operations

■ Note that  $20 \div (2 + 3) = \frac{20}{2+3}$ , whereas  $20 \div 2 + 3 = \frac{20}{2} + 3$ .

- (a)  $20 \div (2 + 3)$  is evaluated by first adding  $2 + 3$  and then dividing. The grouping of  $2 + 3$  is clearly shown by the parentheses. Therefore,  
 $20 \div (2 + 3) = 20 \div 5 = 4$ .
- (b)  $20 \div 2 + 3$  is evaluated by first dividing 20 by 2 and then adding. No specific grouping is shown, and therefore the division is done before the addition. This means  $20 \div 2 + 3 = 10 + 3 = 13$ .
- (c)  $16 - 2 \times 3$  is evaluated by **first multiplying 2 by 3** and then subtracting. We **do not first subtract 2 from 16**. Therefore,  $16 - 2 \times 3 = 16 - 6 = 10$ .
- (d)  $16 \div 2 \times 4$  is evaluated by first dividing 16 by 2 and then multiplying. From left to right, the division occurs first. Therefore,  $16 \div 2 \times 4 = 8 \times 4 = 32$ .
- (e)  $|3 - 5| - |-3 - 6|$  is evaluated by first performing the subtractions within the absolute value vertical bars, then evaluating the absolute values, and then subtracting. This means that  $|3 - 5| - |-3 - 6| = |-2| - |-9| = 2 - 9 = -7$ . ■

**Practice Exercises**

- Evaluate: 1.  $12 - 6 \div 2$   
2.  $16 \div (2 \times 4)$

**COMMON ERROR**

Remember that order of operations takes precedence over perceived left-to-right sequences of operators.

$20 + 10 \div 5$  is evaluated by *first dividing 10 by 5, then adding the result to 20*.

$20 + 10 \div 5 = 20 + 2 = 22$  is evaluated correctly.

A common error would be to perform the addition first:

$$20 + 10 \div 5 \neq 30 \div 5 = 6$$

When evaluating expressions, it is generally more convenient to change the operations and numbers so that the result is found by the addition and subtraction of positive numbers. When this is done, we must remember that

$$a + (-b) = a - b \quad (1.1)$$

$$a - (-b) = a + b \quad (1.2)$$

### Practice Exercises

Evaluate: 3.  $2(-3) - \frac{4-8}{2}$

4.  $\frac{|5-15|}{2} - \frac{-9}{3}$

### EXAMPLE 6 Evaluating numerical expressions

(a)  $7 + (-3) - 6 = 7 - 3 - 6 = 4 - 6 = -2$  using Eq. (1.1)

(b)  $\frac{18}{-6} + 5 - (-2)(3) = -3 + 5 - (-6) = 2 + 6 = 8$  using Eq. (1.2)

(c)  $\frac{|3-15|}{-2} - \frac{8}{4-6} = \frac{12}{-2} - \frac{8}{-2} = -6 - (-4) = -6 + 4 = -2$

(d)  $\frac{-12}{2-8} + \frac{5-1}{2(-1)} = \frac{-12}{-6} + \frac{4}{-2} = 2 + (-2) = 2 - 2 = 0$

In illustration (b), we see that the division and multiplication were done before the addition and subtraction. In (c) and (d), we see that the groupings were evaluated first. Then we did the divisions, and finally the subtraction and addition. ■

### EXAMPLE 7 Evaluating in an application

A 1500-kg van going at 40 km/h ran head-on into a 1000-kg car going at 20 km/h. An insurance investigator determined the velocity of the vehicles immediately after the collision from the following calculation. See Fig. 1.6.

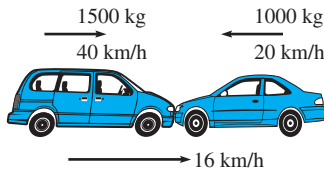


Fig. 1.6

$$\begin{aligned} \frac{1500(40) + (1000)(-20)}{1500 + 1000} &= \frac{60\,000 + (-20\,000)}{1500 + 1000} = \frac{60\,000 - 20\,000}{2500} \\ &= \frac{40\,000}{2500} = 16 \text{ km/h} \end{aligned}$$

The numerator and the denominator must be evaluated before the division is performed. The multiplications in the numerator are performed first, followed by the addition in the denominator and the subtraction in the numerator. ■

### OPERATIONS WITH ZERO

Since operations with zero tend to cause some difficulty, we will show them here.

If  $a$  is a real number, the operations of addition, subtraction, multiplication, and division with zero are as follows:

$$a + 0 = a$$

$$a - 0 = a \quad 0 - a = -a$$

$$a \times 0 = 0$$

$$0 \div a = \frac{0}{a} = 0 \quad (\text{if } a \neq 0) \quad (\neq \text{ means "is not equal to"})$$

### LEARNING TIP

Division by zero is undefined because no real value can be associated with that division.

If  $c = \frac{4}{0}$ , then  $c \times 0 = 4$ , which is *not true*, since  $c \times 0 = 0$  for any value of  $c$ .

There is a special case of division by zero termed *indeterminate* because no specific value can be determined from the division, but many real values are indeed possible.

If  $c = \frac{0}{0}$ , then  $c \times 0 = 0$ , which is true, since  $c \times 0 = 0$  for any value of  $c$ .

Thus,  $c = 13$ ,  $c = -4.76$ , and  $c = 0$  are *all* valid solutions for the division.

### EXAMPLE 8 Operations with zero

(a)  $5 + 0 = 5$     (b)  $-6 - 0 = -6$     (c)  $0 - 4 = -4$

(d)  $\frac{0}{6} = 0$     (e)  $\frac{0}{-3} = 0$     (f)  $\frac{5 \times 0}{7} = \frac{0}{7} = 0$  ■

Note that there is no result defined for division by zero. To understand the reason for this, consider the results for  $\frac{6}{2}$  and  $\frac{6}{0}$ .

$$\frac{6}{2} = 3 \quad \text{since} \quad 2 \times 3 = 6$$

If  $\frac{6}{0} = b$ , then  $0 \times b = 6$ . This cannot be true because  $0 \times b = 0$  for any value of  $b$ .

**EXAMPLE 9** Division by zero is undefined

$$\frac{2}{5} \div 0 \text{ is undefined} \quad \frac{8}{0} \text{ is undefined} \quad \frac{7 \times 0}{0 \times 6} \text{ is indeterminate}$$

↙ see bottom of page 9

The operations with zero will not cause any difficulty if we remember never to divide by zero. Division by zero is the only undefined basic operation. All the other operations with zero may be performed as for any other number.

**COMMON ERROR**

Division by zero is an undefined operation in mathematics. Even when trying to solve equations, every time you perform a division, you must specify that you are *not* committing a division by zero error.

For example, to solve  $x \cdot x = 3 \cdot x$  one might be tempted to divide both sides of the equation by  $x$ . This is fine as long as  $x \neq 0$ .

$$\begin{aligned} \frac{x \cdot x}{x} &= \frac{3 \cdot x}{x} \\ x &= 3 \end{aligned}$$

Notice, however, that a solution to the equation has been missed.  $x = 0$  is also a valid solution ( $0 \cdot 0 = 3 \cdot 0$ ), yet it was missed because when  $x = 0$ , an invalid division by  $x$  took place.

**EXERCISES 1.2**

In Exercises 1–4, make the given changes in the indicated examples of this section and then solve the resulting problems.

- In Example 5(c), change 3 to  $(-3)$  and then evaluate.
- In Example 6(b), change 18 to  $-18$  and then evaluate.
- In Example 6(d), interchange the 2 and 8 in the first denominator and then evaluate.
- In the rightmost illustration in Example 9, interchange the 6 and the 0 above the 6. Is any other change needed?

In Exercises 5–36, evaluate each of the given expressions by performing the indicated operations.

- $8 + (-4)$
- $-4 + (-7)$
- $-3 + 9$
- $18 - 21$
- $-19 - (-16)$
- $8 - (-4)$
- $8(-3)$
- $-9(3)$
- $-7(-5)$
- $\frac{-9}{3}$
- $\frac{-6(20 - 10)}{-3}$
- $\frac{28}{-7(6 - 5)}$
- $-2(4)(-5)$
- $3(-4)(6)$
- $2(2 - 7) \div 10$
- $\frac{-64}{-2|4 - 8|}$
- $-9 - |2 - 10|$
- $(7 - 7) \div (5 - 7)$
- $\frac{17 - 7}{7 - 7}$
- $\frac{7 - 7}{7 - 7}$
- $8 - 3(-4)$
- $20 + 8 \div 4$
- $-2(-6) + \left| \frac{8}{-2} \right|$
- $-10 - (-6)(-8)$
- $30(-6)(-2) \div (0 - 40)$
- $\frac{7 - |-5|}{-1(-2)}$
- $\frac{24}{3 + (-5)} - 4(-9)$
- $\frac{-18}{3} - \frac{4 - 6}{-1}$

- $-7 - \frac{|-14|}{2(2 - 3)} - 3|6 - 8|$
- $-7(-3) + \frac{6}{-3} - (-9)$
- $\frac{3(-9) - 2(-3)}{3 - 10}$
- $\frac{20(-12) - 40(-15)}{98 - |-98|}$

In Exercises 37–44, determine which of the fundamental laws of algebra is demonstrated.

- $6(7) = 7(6)$
- $6 + 8 = 8 + 6$
- $6(3 + 1) = 6(3) + 6(1)$
- $4(5 \times \pi) = (4 \times 5)(\pi)$
- $3 + (5 + 9) = (3 + 5) + 9$
- $8(3 - 2) = 8(3) - 8(2)$
- $(\sqrt{5} \times 3) \times 9 = \sqrt{5} \times (3 \times 9)$
- $(3 \times 6) \times 7 = 7 \times (3 \times 6)$

In Exercises 45–48, for numbers  $a$  and  $b$ , determine which of the following expressions equals the given expression.

- (a)  $a + b$  (b)  $a - b$  (c)  $b - a$  (d)  $-a - b$
- $-a + (-b)$
  - $b - (-a)$
  - $-b - (-a)$
  - $-a - (-b)$

In Exercises 49–64, answer the given questions. Refer to Fig. 1.9 for units of measurement and their symbols.

- (a) What is the sign of the product of an even number of negative numbers? (b) What is the sign of the product of an odd number of negative numbers?
- Is subtraction commutative? Explain.

51. Explain why the following definition of the absolute value of a real number  $x$  is either correct or incorrect (the symbol  $\geq$  means “is equal to or greater than”): If  $x \geq 0$ , then  $|x| = x$ ; if  $x < 0$ , then  $|x| = -x$ .
52. Explain what the error is if the expression  $24 - 6 \div 2 \cdot 3$  is evaluated as 27. What is the correct value?
53. Describe the values of  $x$  and  $y$  for which (a)  $-xy = 1$  and (b)  $\frac{x-y}{x-y} = 1$ .
54. Describe the values of  $x$  and  $y$  for which (a)  $|x+y| = |x| + |y|$  and (b)  $|x-y| = |x| + |y|$ .
55. Some solar energy systems are used to supplement the utility company power supplied to a home such that the meter runs backward if the solar energy being generated is greater than the energy being used. With such a system, if the solar power averages 1.5 kW for a 3.0-h period and only 2.1 kW · h is used during this period, what will be the change in the meter reading for this period?
56. A baseball player’s batting average (total number of hits divided by total number of at-bats) is expressed in decimal form from 0.000 (no hits for all at-bats) to 1.000 (one hit for each at-bat). A player’s batting average is often shown as 0.000 before the first at-bat of the season. Is this a correct batting average? Explain.
57. The daily high temperatures (in °C) in the Falkland Islands in the southern Atlantic Ocean during the first week in July were recorded as 7, 3, -2, -3, -1, 4, and 6. What was the average daily temperature for the week? (Divide the algebraic sum of the readings by the number of readings.)
58. A flare is shot up from the top of a tower. Distances above the flare gun are positive and those below it are negative. After 5 s the vertical distance (in m) of the flare from the flare gun is found by evaluating  $(20)(5) + (-5)(25)$ . Find this distance.

59. Find the sum of the voltages of the batteries shown in Fig. 1.7. Note the directions in which they are connected.

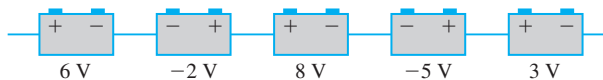


Fig. 1.7

60. The electric current was measured in a given ac circuit at equal intervals as 0.7 mA, -0.2 mA, -0.9 mA, and -0.6 mA. What was the change in the current between (a) the first two readings, (b) the middle two readings, and (c) the last two readings?
61. One oil-well drilling rig drills 100 m deep the first day and 200 m deeper the second day. A second rig drills 200 m deep the first day and 100 m deeper the second day. In showing that the total depth drilled by each rig was the same, state what fundamental law of algebra is illustrated.
62. A water tank leaks 12 L each hour for 7 h, and a second tank leaks 7 L each hour for 12 h. In showing that the total amount leaked is the same for the two tanks, what fundamental law of algebra is illustrated?
63. Each of four persons spends 8 min browsing one website and 6 min browsing a second website. Set up the expression for the total time these persons spent browsing these websites. What fundamental law of algebra is illustrated?
64. A jet travels 600 km/h relative to the air. The wind is blowing at 50 km/h. If the jet travels with the wind for 3 h, set up the expression for the distance travelled. What fundamental law of algebra is illustrated?

#### Answers to Practice Exercises

1. 9    2. 2    3. -4    4. 8

## 1.3 Measurement, Calculation, and Approximate Numbers

Graphing Calculators • Approximate Numbers • Measurement • Units • Significant Digits • Accuracy and Precision • Rounding Off • Operations with Approximate Numbers • Estimating Results

■ The calculator screens shown with text material are for a TI-83 or TI-84. They are intended only as an illustration of a calculator screen for the particular operation. Screens for other models may differ.

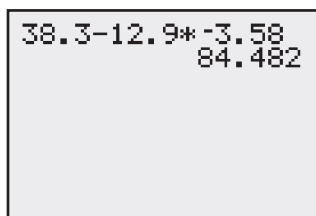


Fig. 1.8

### CALCULATORS

You will be doing many of your calculations on a calculator, and a *graphing calculator* can be used for these calculations and many other operations. In this text, we will restrict our coverage of calculator use to graphing calculators because a *scientific calculator* cannot perform many of the required operations we will cover.

A discussion regarding the use of a graphing calculator can be found at the text’s companion web site. Since there are many models of graphing calculators, the notation and screen appearance for many operations will differ from one model to another. Therefore, although we include some calculator screens throughout the book, not every calculator discussion will be accompanied by a sample screen.

You should *practice using your calculator and review its manual to be sure how it is used*. Following is an example of a basic calculation done on a graphing calculator.

#### EXAMPLE 1 Calculating on a graphing calculator

In order to calculate the value of  $38.3 - 12.9(-3.58)$ , the numbers are entered as follows. The calculator will perform the multiplication first, following the order of operations shown in Section 1.2. The sign of  $-3.58$  is entered using the  $(-)$  key, before 3.58 is entered. The display on the calculator screen is shown in Fig. 1.8.

38.3  $(-)$  12.9  $\times$   $(-)$  3.58  $(\text{ENTER})$     keystrokes



This means that  $38.3 - 12.9(-3.58) = 84.482$ .

Note in the display that the negative sign of  $-3.58$  is smaller and a little higher to distinguish it from the minus sign for subtraction. Also note the \* shown for multiplication; the asterisk is the standard computer symbol for multiplication. ■

■ Some calculator keys on different models are labelled differently. For example, on some models, the EXE key is equivalent to the ENTER key.

■ Calculator keystrokes will generally not be shown, except as they appear in the display screens. They may vary from one model to another.

Looking back into Section 1.2, we see that *the minus sign is used in two different ways*: (1) to indicate subtraction and (2) to designate a negative number. This is clearly shown on a graphing calculator because there is a key for each purpose. The  $\boxed{-}$  key is used for subtraction, and the  $\boxed{(-)}$  key is used before a number to make it negative.

We will first use a graphing calculator for the purpose of graphing in Section 3.5. Before then, we will show some calculational uses of a graphing calculator.

## UNITS OF MEASUREMENT

Most scientific and technical calculations involve numbers that represent a measurement or count of a specific physical quantity. A **measurement** represents an *estimate* of the value of the physical quantity that exists in reality, and is usually accompanied by an **uncertainty** or **error** in that measured value. To report a measurement in a meaningful way, the **units of measurement**, which indicate a specific size or magnitude of a physical measurement, have to be expressed. For example, if the length of an object is measured to be 12.5, it is critical to know if that is measured in centimetres, metres, feet, or some other unit of length.

The definition and practical use of units of measurement has spawned many different systems of counting and units throughout human history. Many of the ancient systems invented were largely based on dimensions of the human body. Consequently, measurements varied from place to place, and communication of the measured values was inconsistent since each unit did not have a universally recognized size. The metric system, first adopted in France in the late 1700s, incorporated the feature of standardization of units, wherein everyone using the system agreed to a specific size for each unit. The SI metric system of units (International System of Units) has been agreed upon by international committees of scientists and engineers and was established in 1960. Most scientific endeavours worldwide employ the SI system of units. It is important for scientists, engineers, and technologists to be able to communicate measurements to each other easily and without confusion.

The SI system consists of seven base units (from which all other units are constructed), supplementary units (used for measuring plane and solid angles), and derived units (which are formed by multiplication and division of the seven base units).

Each unit measures a specific physical quantity, has a standard symbol, and has a single spelling when written out in full. (Exception: The United States has different spellings for deca, metre, and litre, writing them as deka, meter, and liter.)

Fig. 1.9 summarizes some SI physical quantities and common variable symbols, their unit names and SI unit symbols, and any re-expression of a derived unit in terms of more fundamental base units.

Among the units for time, for which the standard unit is the second, other units like minute (min), hour (h), day (d), and year (y or yr) are also acceptable. For angles, divisions such as the degree, minute of arc, and second of arc are also permitted.

The kilogram is the SI unit for mass (not weight). It is different because it also contains an SI prefix kilo, which denotes a power of  $10^3$ . Please note that weight and mass are different: Mass is the amount of material in an object (in kg), and weight is the gravitational force (in N) exerted on that mass. Weight changes with the local strength of the gravity field, whereas mass remains constant.

Originally the metre was defined as one ten-millionth of the length along the globe from the North Pole to the equator. Today it is defined as the distance travelled by light in a vacuum in  $1/299\,792\,458$  s. Similarly, the second was once defined as the fraction  $1/86\,400$  of the mean solar day. It is now defined as the time required for  $9\,192\,631\,770$  cycles of the radiation corresponding to the transition between the two lowest energy states of the cesium-133 atom.

Fig. 1.9 Physical Quantities and Their SI Units

	Quantity	Quantity Symbol	Unit Name	Unit Symbol	In Terms of Other SI Units
<b>Base Units</b>	Length	$s$	metre	m	
	Mass	$m$	kilogram	kg	
	Time	$t$	second	s	
	Electric current	$I, i$	ampere	A	
	Thermodynamic temperature	$T$	kelvin	K	
	Amount of substance	$n$	mole	mol	
	Luminous intensity	$I$	candela	cd	
<b>Supplementary Units</b>	Plane angle	$q$	radian	rad	unitless
	Solid angle	$q$	steradian	sr	unitless
<b>Derived Units</b>	Area	$A$		$m^2$	
	Volume	$V$		$m^3$	
	Volume	$V$	litre	L	(1000 L = 1 m <sup>3</sup> )
	Velocity	$v$		m/s	
	Acceleration	$a$		m/s <sup>2</sup>	
	Force	$F$	newton	N	kg · m/s <sup>2</sup>
	Density	$r$		kg/m <sup>3</sup>	
	Pressure	$p$	pascal	Pa	N/m <sup>2</sup> = kg/(m · s <sup>2</sup> )
	Energy, work	$E, W$	joule	J	N · m = kg · m <sup>2</sup> /s <sup>2</sup>
	Power	$P$	watt	W	J/s = kg · m <sup>2</sup> /s <sup>3</sup>
	Frequency	$f$	hertz	Hz	1/s
	Electric charge	$q$	coulomb	C	A · s
	Electric potential	$V, E$	volt	V	J/(A · s) = kg · m <sup>2</sup> /(A · s <sup>3</sup> )
	Capacitance	$C$	farad	F	s/Ω = s · A/V
	Inductance	$L$	henry	H	Ω · s = V · s/A
	Resistance	$R$	ohm	Ω	V/A
	Heat	$Q$	joule	J	N · m = kg · m <sup>2</sup> /s <sup>2</sup>
Temperature	$T$	degrees Celsius	°C	A change of 1°C = 1 K	

### WRITING UNITS

When writing units, there are several conventions that one must follow:

- Unit names are never capitalized (e.g., newtons, square metres, pascals). Exception: Degrees Celsius.
- The only unit symbols that are capitalized are named after people (e.g., N for Isaac Newton, Pa for Blaise Pascal). Exception: The litre symbol is L, which is not named for a person. It used to be l or l but it was easily confused with the digit 1 (one) so it was altered. The l symbol still has some international acceptance. Both °C and L were added to the SI system due to their practical importance.
- Ensure that a · symbol appears between units that are multiplied (e.g., kg · m<sup>2</sup>/s<sup>2</sup> not kgm<sup>2</sup>/s<sup>2</sup>). This will prevent confusion between units and SI prefixes, some of which use the same symbol (e.g., mm is millimetres, but m · m is metres squared).

- Do not use more than one division symbol in a single unit (e.g., use  $\text{m/s}^2$  not  $\text{m/s/s}$ ).
- Unit symbols are upright (non-italicized), and physical quantity variable symbols are italicized (e.g.,  $V$  is the quantity of electrical potential, and  $\text{V}$  is the unit volts).
- Unit symbols are never pluralized (e.g., 312 kg, not 312 kgs).
- Spaces may be used to separate thousands to avoid confusion with many international practices of different interpretations of commas (e.g., 10 585 is acceptable, while 10,585 means 10.585 in some countries).

### SI PREFIXES

In science, it is common to deal with measurements that consist of very large numbers, or very small numbers. In order to avoid the problem of having to write many zeros in a decimal (whether trailing or leading zeros), one can utilize some common unit prefixes allowing for a quick way to write a specific multiple of 10 applied to the unit. These prefixes have specific names and symbols, just like units, but are written *preceding* the unit, as a normal prefix. There can never be more than one prefix for a single unit. Scientific and engineering notations, which are used to report very large or very small measurements using these prefixes, will be discussed in Section 1.5.

Fig. 1.10 SI Prefixes

Multiple of 10	Prefix	Symbol
$10^{12}$ , or trillion	tera	T
$10^9$ , or billion	giga	G
$10^6$ , or million	mega	M
$10^3$ , or thousand	kilo	k
$10^2$ , or hundred	hecto	h
$10^1$ , or ten	deca	da
$10^{-1}$ , or one tenth	deci	d
$10^{-2}$ , or one hundredth	centi	c
$10^{-3}$ , or one thousandth	milli	m
$10^{-6}$ , or one millionth	micro	$\mu$
$10^{-9}$ , or one billionth	nano	n
$10^{-12}$ , or one trillionth	pico	p

### EXAMPLE 2 Using SI prefixes

(a) Using SI prefixes, we can rewrite the following measurements:

$$\begin{array}{lll}
 123\,000\,000\text{ s} & = 123 \times 10^6\text{ s} & = 123\text{ Ms, or }0.123\text{ Gs} \\
 0.000\,005\,0\text{ m} & = 5.0 \times 10^{-6}\text{ m} & = 5.0\ \mu\text{m} \\
 85\,300\ \Omega & = 85.3 \times 10^3\ \Omega & = 85.3\ \text{k}\Omega
 \end{array}$$

(b) We also use the definitions of the SI prefixes to give the name and meaning of the units corresponding to the following symbols:

$$\begin{array}{lll}
 \text{ks} & = \text{kiloseconds} & = 1000\text{ s} \\
 \text{mC} & = \text{millicoulombs} & = 0.001\text{ C} \\
 \text{GHz} & = \text{gigahertz} & = 1\,000\,000\text{ Hz}
 \end{array}$$

### UNIT CONVERSION

When using measurements, it may be necessary to convert from one set of units to another. To change a given set of units into another set of units, we perform mathematical operations with the units in the same manner as we do with any other algebraic symbol.

This process is more fully discussed in Sections 1.7 to 1.12, but it is important to discuss the principle here, since measurements and units have a fundamental role in most subsequent applied problems.

To convert a set of units, you multiply the measurement by a fraction equal to one, where the fraction represents the equivalency ratio between the two units. You put the units you want to eliminate on the opposite side of the fraction of the converting ratio from where they are in the original measurement when you multiply. By multiplying by a fraction equal to one, the measurement is not changing. To convert multiple units at the same time, just use more than one conversion fraction multiplication. This is illustrated in Example 3.

### EXAMPLE 3 Converting units

Convert the following units.

- (a) 1350 m into km:

$$1350 \text{ m} \times \left( \frac{1 \text{ km}}{1000 \text{ m}} \right) = 1.35 \text{ km}$$

- (b) 25.2 kg into g:

$$25.2 \text{ kg} \times \left( \frac{1000 \text{ g}}{1 \text{ kg}} \right) = 25\,200 \text{ g}$$

- (c) 72.0 km/h into m/s:

$$72.0 \frac{\text{km}}{\text{h}} \times \left( \frac{1000 \text{ m}}{1 \text{ km}} \right) \times \left( \frac{1 \text{ h}}{3600 \text{ s}} \right) = 20.0 \text{ m/s}$$

- (d) 8.75 g/cm<sup>3</sup> into kg/m<sup>3</sup>:

$$8.75 \frac{\text{g}}{\text{cm}^3} \times \left( \frac{1 \text{ kg}}{1000 \text{ g}} \right) \times \left( \frac{100 \text{ cm}}{1 \text{ m}} \right)^3 = 8750 \text{ kg/m}^3$$

- (e) 62.8 kPa into N/cm<sup>2</sup>:

$$62.8 \text{ kPa} \times \left( \frac{1000 \text{ N/m}^2}{1 \text{ kPa}} \right) \times \left( \frac{1 \text{ m}}{100 \text{ cm}} \right)^2 = 6.28 \text{ N/cm}^2$$

- (f) 32 500 ft to km (1 ft = 0.3048 m):

$$32\,500 \text{ ft} \times \left( \frac{0.3048 \text{ m}}{1 \text{ ft}} \right) \times \left( \frac{1 \text{ km}}{1000 \text{ m}} \right) = 9.91 \text{ km}$$

### APPROXIMATE NUMBERS AND SIGNIFICANT DIGITS

Most numbers in technical and scientific work are **approximate numbers**, having been determined by some *measurement*. Certain other numbers are **exact numbers**, having been determined by a *definition* or *counting process*.

### EXAMPLE 4 Approximate numbers and exact numbers

If a voltage on a voltmeter is read as 116 V, the 116 is approximate. Another voltmeter might show the voltage as 115.7 V. However, the voltage cannot be determined *exactly*.

If a computer prints out the number of names on a list of 97, this 97 is exact. We know it is not 96 or 98. Since 97 was found from precise counting, it is exact.

By definition, 60 s = 1 min, and the 60 and the 1 are exact.

**Significant digits** are digits in a measurement or result that you can confidently estimate. That is to say, those digits that are not swamped by the error or uncertainty in the measurement are significant. The **accuracy** of a measurement refers to the number of significant digits it has.

The measurements 5.00 m and 5.000 m may not seem to be very different, but to a scientist, an engineer, or a technologist, they are *not* the same thing. The first measurement has been measured to the nearest centimetre and the second measurement to the nearest millimetre. The **precision** of a measurement is defined as the last decimal place

to which the measurement is expressed, or the decimal place corresponding to the last measured (significant) digit in the measurement. For instance, 5.00 m has precision 0.01 m = 1 cm, and 5.000 m has precision 0.001 m = 1 mm. Therefore, the second measurement is *more precise* (it has a smaller precision). The concept of precision is important when finding the proper significant digits in a calculated result.

To find the number of significant digits in a single measurement, you start counting at the first nonzero digit, and finish counting once the precision of the measurement is reached. Some rules to remember are:

- All nonzero digits *are* significant
- Zeros between nonzero digits *are* significant
- Zeros to the left of the first nonzero digit are *not* significant
- Trailing zeros after a decimal *are* significant
- Trailing zeros before a decimal are generally *not* significant

Measurement	# Significant Digits
23.0 m	3
55.001 cm	5
0.0034 m	2
125 000 s	3 <small>In this case, you don't know the precision—is it 1000 s, 100 s, 10 s, or 1 s? It is ambiguous, so you must assume the worst, i.e., nearest 1000 s</small>
0.0120 s	3
120 cm	2 <small>ambiguous again</small>

### EXAMPLE 5 Accuracy and precision

- (a) Suppose that an electric current is measured to be 0.31 A on one ammeter and 0.312 A on another ammeter. The measurement 0.312 A is measured to the nearest thousandth ampere, so it is more precise than 0.31 A, which is measured to the nearest hundredth ampere. 0.312 A is also more accurate, since it contains three significant digits, whereas 0.31 A contains only two.
- (b) If a concrete driveway is measured to be 135 m long and 0.1 m thick, the measurement 0.1 m (measured to the nearest tenth metre) is more precise than the measurement 135 m (measured to the nearest metre). On the other hand, 135 m is more accurate, since it contains three significant digits, whereas 0.1 m contains only one. ■

### EXAMPLE 6 Significant digits

All numbers in this example are assumed to be approximate.

34.7 has three significant digits.

0.039 has two significant digits. The zeros properly locate the decimal point.

706.1 has four significant digits. The zero is not used for the location of the decimal point. It shows the number of tens in 706.1.

5.90 has three significant digits.

1400 has two significant digits, unless information is known about the number that makes either or both zeros significant. (A temperature shown as 1400°C has two significant digits. If a price list gives all costs in dollars, a price shown as \$1400 has four significant digits.) Without such information, we assume that the zeros are placeholders for proper location of the decimal point.

Other approximate numbers with the number of significant digits are 0.0005 (one), 960 000 (two), 0.0709 (three), 1.070 (four), and 700.00 (five). ■

■ To show that zeros at the end of a whole number are significant, a notation that can be used is to place a bar over the last significant zero. Using this notation,  $\overline{78\,000}$  is shown to have four significant digits.

### COMMON ERROR

Do not write trailing zeros if they are not significant. The measurement 15 m is different from 15.0 m because the precision is different.

From Example 6, we see that *all nonzero digits are significant. Also, zeros not used as placeholders (for location of the decimal point) are significant.*

The last significant digit of an approximate number is not exact. It has usually been determined by estimating or *rounding off*. However, it is not off by more than one-half of a unit in its place value.

### EXAMPLE 7 Meaning of the last digit of an approximate number

When we write the voltage in Example 4 as 115.7 V, we are saying that the voltage is more than 115.65 V and less than 115.75 V. Any such value, rounded off to tenths, would be expressed as 115.7 V.

In changing the fraction  $\frac{2}{3}$  to the approximate decimal value 0.667, we are saying that the value is strictly between 0.6665 and 0.6675. ■

■ On graphing calculators, it is possible to set the number of decimal places (to the right of the decimal point) to which results will be rounded off. Note that calculators round half up.

The method of **unbiased rounding** (also known as round half to even) for **rounding off** any measurement to a specific precision, or a number to a specified number of significant digits, consists of three simple rules. Locate the last significant digit (the digit to be rounded). Then:

- if the next digit is 6 or more, or a 5 followed by one or more nonzero digits, **round up** (increase the rounded digit by one, discard the rest);
- if the next digit is 4 or less, **round down** (leave the rounded digit as is, discard the rest);
- if the next digit is a 5 followed by no more digits other than 0, **round to the nearest even** (make the rounded digit the *nearest even number* and discard the rest).

This last rule ensures proper statistical treatment of all the measurements falling in this category, as half will round up, and half will round down. This technique will not statistically bias your measurements to be consistently larger upon rounding.

We will use unbiased rounding throughout the text. However, there are many different rules that can be followed when rounding. For example, in the common method of **round half up**, if the first discarded digit is 5, then the number is always rounded up. It can be seen that the two methods are identical except for their treatment of those numbers where the digit following the rounding digit is a five and has no nonzero digits after it.

### EXAMPLE 8 Rounding off

70 360 rounded off to three significant digits is 70 400. Here, 3 is the third significant digit, and the next digit is 6. Since  $6 > 5$ , we add 1 to 3 and the result, 4, becomes the third significant digit of the approximation. The 6 is then replaced with a zero in order to keep the decimal point in the proper position.

70 430 rounded off to three significant digits, or to the nearest hundred, is 70 400. Here the 3 is replaced with a zero.

187.35 rounded off to four significant digits, or to tenths, is 187.4, because 4 is the nearest even to 3.5.

187.349 rounded off to four significant digits is 187.3. *We do not round up the 4 and then round up the 3.*

35.003 rounded off to four significant digits is 35.00. *We do not discard the zeros since they are significant* and are not used only to properly place the decimal point.

187.45 rounded off to four significant digits is 187.4 since 4 is the nearest even to 4.5. ■

#### Practice Exercises

Round off each number to three significant digits.

1. 2015    2. 0.3004

OPERATIONS WITH APPROXIMATE NUMBERS

COMMON ERROR

When performing operations on approximate numbers or measurements, we must not express the result to an accuracy or precision that is not valid. Measurement uncertainty restricts how many significant digits can exist in a calculated result.

Consider the following examples.

EXAMPLE 9 Application of precision

A pipe is made in two sections. One is measured as 16.3 m long and the other as 0.927 m long. What is the total length of the two sections together?

It may appear that we simply add the numbers as shown at the left. However, both numbers are approximate, and adding the smallest possible values and the largest possible values, the result differs by 0.1 (17.2 and 17.3) when rounded off to tenths. Rounded off to hundredths (17.18 and 17.28), they do not agree at all since the tenths digit is different. Thus, we get a good approximation for the total length if it is rounded off to *tenths*, the precision of the least precise length, and it is written as 17.2 m. ■

	smallest values	largest values
16.3 m	16.25 m	16.35 m
0.927 m	0.9265 m	0.9275 m
17.227 m	17.1765 m	17.2775 m

EXAMPLE 10 Application of accuracy

We find the area of the rectangular piece of land in Fig. 1.11 by multiplying the length, 207.54 m, by the width, 81.4 m. Using a calculator, we find that  $(207.54)(81.4) = 16\,893.756$ . This apparently means the area is  $16\,893.756\text{ m}^2$ .

However, the area should not be expressed with this accuracy. Since the length and width are both approximate, we have

$$(207.535\text{ m})(81.35\text{ m}) = 16\,882.972\,25\text{ m}^2 \quad \text{least possible area}$$

$$(207.545\text{ m})(81.45\text{ m}) = 16\,904.540\,25\text{ m}^2 \quad \text{greatest possible area}$$

These values agree when rounded off to three significant digits ( $16\,900\text{ m}^2$ ) but do not agree when rounded off to a greater accuracy. Thus, we conclude that the result is accurate only to *three* significant digits, the accuracy of the least accurate measurement, and that the area is written as  $16\,900\text{ m}^2$ . ■

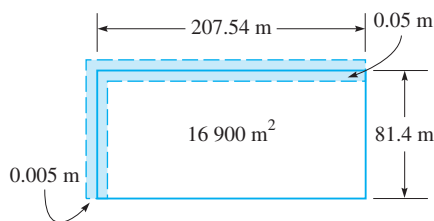


Fig. 1.11

Following are the rules used in expressing the result when we perform basic operations on approximate numbers. They are based on reasoning similar to that shown in Examples 9 and 10.

Operations with Approximate Numbers

1. When approximate numbers are added or subtracted, the result is expressed with the precision of the least precise number.
2. When approximate numbers are multiplied or divided, the result is expressed with the accuracy of the least accurate number.
3. When the root of an approximate number is found, the result is expressed with the accuracy of the number.
4. When approximate numbers and exact numbers are involved, the accuracy of the result is limited only by the approximate numbers.

LEARNING TIP

Always express the result of a calculation with the proper accuracy or precision. When using a calculator, if additional digits are displayed, round off the final result (do not round off in any of the intermediate steps).



■ When rounding off a number, it may seem difficult to discard the extra digits. However, if you keep those digits, you show a number with too great an accuracy, and it is incorrect to do so.

### EXAMPLE 11 Adding approximate numbers

Find the sum of the approximate numbers 73.2, 8.0627, and 93.57.

Showing the addition in the standard way and using a calculator, we have

$$\begin{array}{r} 73.2 \leftarrow \text{least precise number (expressed to tenths)} \\ 8.0627 \\ 93.57 \\ \hline 174.8327 \leftarrow \text{final display must be rounded to tenths} \end{array}$$

Therefore, the sum of these approximate numbers is 174.8. ■

### EXAMPLE 12 Combined operations

In finding the product of the approximate numbers 2.4832 and 30.5 on a calculator, the final display shows 75.7376. However, since 30.5 has only three significant digits, the product is 75.7.

In Example 1, we calculated that  $38.3 - 12.9(-3.58) = 84.482$ . We know that  $38.3 - 12.9(-3.58) = 38.3 + 46.182 = 84.482$ . If these numbers are approximate, we must round off the result to tenths, which means the sum is 84.5. We see that when there is a combination of operations, we must examine the individual steps of the calculation and determine how many significant digits can carry through to the final result. ■

### EXAMPLE 13 Operations with exact numbers and approximate numbers

Using the exact number 600 and the approximate number 2.7, we express the result to tenths if the numbers are added or subtracted. If they are multiplied or divided, we express the result to two significant digits. Since 600 is exact, the accuracy of the result depends only on the approximate number 2.7.

$$\begin{array}{ll} 600 + 2.7 = 602.7 & 600 - 2.7 = 597.3 \\ 600 \times 2.7 = 1600 & 600 \div 2.7 = 220 \end{array}$$

You should *make a rough estimate* of the result when using a calculator. An estimation may prevent accepting an incorrect result after using an incorrect calculator sequence, particularly if the calculator result is far from the estimated value. ■

### EXAMPLE 14 Estimating results

In Example 1, we found that

$$38.3 - 12.9(-3.58) = 84.482 \quad \text{using exact numbers}$$

When using the calculator, if we forgot to make 3.58 negative, the display would be  $-7.882$ , or if we incorrectly entered 38.3 as 83.3, the display would be 129.482.

However, if we estimate the result as

$$40 - 10(-4) = 80$$

we know that a result of  $-7.882$  or 129.482 cannot be correct.

When estimating, we can often use one-significant-digit approximations. If the calculator result is far from the estimate, we should do the calculation again. ■

#### Practice Exercises

Evaluate using a calculator.

3.  $40.5 + \frac{3275}{-60.041}$  (Numbers are approximate.)

#### LEARNING TIP

A note regarding the equal sign (=) is in order. We will use it for its defined meaning of "equals exactly" and when the result is an approximate number that has been properly rounded off. Although  $\sqrt{27.8} \approx 5.27$ , where  $\approx$  means "equals approximately," we write  $\sqrt{27.8} = 5.27$ , since 5.27 has been properly rounded off.

## EXERCISES 1.3

In Exercises 1–4, make the given changes in the indicated examples of this section, and then solve the given problems.

- In Example 6, change 0.039 (the second number discussed) to 0.390. Is there any change in the conclusion?
- In the next-to-last paragraph of Example 8, change 35.003 to 35.303 and then find the result.
- In the first paragraph of Example 12, change 2.4832 to 2.483 and then find the result.
- In Example 14, change 12.9 to 21.9 and then find the estimated value.



In Exercises 5–8, give the symbol and the meaning for the given unit.

5. megahertz    6. kilowatt    7. millimetre    8. picosecond

In Exercises 9–12, give the name and the meaning for the units whose symbols are given.

9. kV    10. GΩ    11. mA    12. pF

In Exercises 13–44, make the indicated changes in units.

13. 1 km to centimetres.
14. 1 kg to milligrams.
15. 20 s to megaseconds.
16. 800 Pa to kilopascals.
17. 250 mm<sup>2</sup> to square metres.
18. 1.75 m<sup>2</sup> to square centimetres.
19. 80.0 m<sup>3</sup> to L.
20. 0.125 L to millilitres.
21. 45.0 m/s to centimetres per second.
22. 1.32 km/h to metres per second.
23. 9.80 m/s<sup>2</sup> to centimetres per minute squared.
24. 5.10 g/cm<sup>3</sup> to kilograms per cubic metre.
25. 25 h to milliseconds.
26. 5.25 mV to watts per ampere.
27. 15.0 μF to millicoulombs per volt.
28. Determine how many metres light travels in one year.
29. Determine the speed (in km/h) of the earth moving around the sun. Assume it is a circular path of radius 150 000 000 km.
30. At sea level, atmospheric pressure is about 101 300 Pa. How many kilopascals is this?
31. A car's gasoline tank holds 56 L. What is this capacity in cubic centimetres?
32. A hockey puck has a mass of about 0.160 kg. What is its mass in milligrams?
33. The velocity of some seismic waves is 6800 m/s. What is this velocity in kilometres per hour?
34. The memory of a 1985 computer was 64 kB (B is the symbol for byte), and the memory of a 2012 computer is 1.50 TB. How many times greater is the memory of the 2012 computer?
35. The recorded surface area of a DVD is 112 cm<sup>2</sup>. What is this area in square metres?
36. A solar panel can generate 0.024 MW · h each day. Convert this to joules.
37. The density of water is 1000 kg/m<sup>3</sup>. Change this to grams per litre.
38. Water flows from a kitchen faucet at the rate of 8500 mL/min. What is this rate in cubic metres per second?
39. The speed of sound is about 332 m/s. Change this speed to kilometres per hour.
40. Fifteen grams of a medication are to be dissolved in 0.060 L of water. Express this concentration in milligrams per decilitre.
41. The earth's surface receives energy from the sun at the rate of 1.35 kW/m<sup>2</sup>. Reduce this to joules per second per square centimetre.
42. The moon travels about 2 400 000 km in about 28 d in one rotation about the earth. Express its velocity in metres per second.

43. A typical electric current density in a wire is  $1.2 \times 10^6$  A/m<sup>2</sup>. Express this in milliamperes per square centimetre.
44. A certain car travels 24 km on 2.0 L of gas. Express the fuel consumption in litres per 100 kilometres.

In Exercises 45–48, determine whether the given numbers are approximate or exact.

45. A car with 8 cylinders travels at 55 km/h.
46. A computer chip 0.002 mm thick is priced at \$7.50.
47. In 24 h there are 1440 min.
48. A calculator has 50 keys, and its battery lasted for 50 h of use.

In Exercises 49–54, determine the number of significant digits in each of the given approximate numbers.

49. 107; 3004    50. 3600; 730    51. 6.80; 6.08  
 52. 0.8735; 0.0075    53. 3000; 3000.1    54. 1.00; 0.01

In Exercises 55–60, determine which of the pair of approximate numbers is (a) more precise and (b) more accurate.

55. 30.8; 0.01    56. 0.041; 7.673    57. 0.1; 78.0  
 58. 7040; 0.004    59. 7000; 0.004    60. 50.060; 8.914

In Exercises 61–68, round off the given approximate numbers (a) to three significant digits and (b) to two significant digits.

61. 4.936    62. 80.53    63. 50 893    64. 7.005  
 65. 9549    66. 30.96    67. 0.9445    68. 0.9999

In Exercises 69–76, assume that all numbers are approximate. (a) Estimate the result and (b) perform the indicated operations on a calculator and compare with the estimate.

69.  $12.78 + 1.0495 - 1.633$     70.  $3.64(17.06)$   
 71.  $0.0350 - \frac{0.0450}{1.909}$     72.  $\frac{0.3275}{1.096 \times 0.50085}$   
 73.  $\frac{23.962 \times 0.01537}{10.965 - 8.249}$     74.  $\frac{0.69378 + 0.04997}{257.4 \times 3.216}$   
 75.  $\frac{3872}{503.1} - \frac{2.056 \times 309.6}{395.2}$     76.  $\frac{1.00}{0.5926} + \frac{3.6957}{2.935 - 1.054}$

In Exercises 77–80, perform the indicated operations. The first number is approximate, and the second number is exact.

77.  $0.9788 + 14.9$     78.  $17.311 - 22.98$   
 79.  $3.142(65)$     80.  $8.62 \div 1728$

In Exercises 81–84, answer the given questions.

81. The manual for a heart monitor lists the frequency of the ultrasound wave as 2.75 MHz. What are the least possible and the greatest possible frequencies?
82. A car manufacturer states that the engine displacement for a certain model is 2400 cm<sup>3</sup>. What should be the least possible and greatest possible displacements?
83. A flash of lightning struck a tower 5.23 km from a person. The thunder was heard 15 s later. The person calculated the speed of sound and reported it as 348.7 m/s. What is wrong with this conclusion?
84. A technician records 4.4 s as the time for a robot arm to swing from the extreme left to the extreme right, 2.72 s as the time for the return swing, and 1.68 s as the difference in these times. What is wrong with this conclusion?

In Exercises 85–100, perform the calculations on a calculator.

85. Evaluate: (a)  $2.2 + 3.8 \times 4.5$  (b)  $(2.2 + 3.8) \times 4.5$   
 86. Evaluate: (a)  $6.03 \div 2.25 + 1.77$  (b)  $6.03 \div (2.25 + 1.77)$   
 87. Evaluate: (a)  $2 + 0$  (b)  $2 - 0$  (c)  $0 - 2$  (d)  $2 \times 0$  (e)  $2 \div 0$   
 Compare with operations with zero in Section 1.2.  
 88. Evaluate: (a)  $2 \div 0.0001$  and  $2 \div 0$  (b)  $0.0001 \div 0.0001$  and  $0 \div 0$  (c) Explain why the displays differ.  
 89. Enter a positive integer  $x$  (five or six digits is suggested) and then rearrange the same digits to form another integer  $y$ . Evaluate  $(x - y) \div 9$ . What type of number is the result?  
 90. Enter the digits in the order 9, 8, 7, 6, 5, 4, 3, 2, 1, 0, using between them any of the operations (+, −, ×, ÷) that will lead to a result of 100.  
 91. Show that  $\pi$  is not equal exactly to (a) 3.1416, or (b) 22/7.  
 92. At some point in the decimal equivalent of a rational number, some sequence of digits will start repeating endlessly. An irrational number never has an endlessly repeating sequence of digits. Find the decimal equivalents of (a)  $8/33$  and (b)  $\pi$ . Note the repetition for  $8/33$  and that no such repetition occurs for  $\pi$ .  
 93. Following Exercise 92, show that the decimal equivalents of the following fractions indicate they are rational: (a)  $1/3$  (b)  $5/11$  (c)  $2/5$ . What is the repeating part of the decimal in (c)?  
 94. Following Exercise 92, show that the decimal equivalent of the fraction  $124/990$  indicates that it is rational. Why is the last digit different?

95. In 3 successive days, a home solar system produced 32.4 MJ, 26.704 MJ, and 36.23 MJ of energy. What was the total energy produced in these 3 days?  
 96. Two jets flew at 938 km/h and 1450 km/h, respectively. How much faster was the second jet?  
 97. If 1 K of computer memory has 1024 bytes, how many bytes are there in 256 K of memory? (All numbers are exact.)  
 98. Find the voltage in a certain electric circuit by multiplying the sum of the resistances 15.2  $\Omega$ , 5.64  $\Omega$ , and 101.23  $\Omega$  by the current 3.55 A.  
 99. The percent of alcohol in a certain car engine coolant is found by performing the calculation  $\frac{100(40.63 + 52.96)}{105.30 + 52.96}$ . Find this percent of alcohol. The number 100 is exact.  
 100. The tension (in N) in a pulley cable lifting a certain crate was found by calculating the value of  $\frac{50.45(9.80)}{1 + 100.9 \div 23}$ , where the 1 is exact. Calculate the tension.

### Answers to Practice Exercises

1. 2020 2. 0.300 3. −14.0

## 1.4 Exponents

Positive Integer Exponents • Zero and Negative Exponents • Order of Operations • Evaluating Algebraic Expressions

In mathematics and its applications, we often have a number multiplied by itself several times. To show this type of product, we use the notation  $a^n$ , where  $a$  is the number and  $n$  is the number of times it appears. In the expression  $a^n$ , the number  $a$  is called the **base**, and  $n$  is called the **exponent**; in words,  $a^n$  is read as “the  $n$ th power of  $a$ .”

### EXAMPLE 1 Meaning of exponents

- (a)  $4 \times 4 \times 4 \times 4 \times 4 = 4^5$  the fifth power of 4  
 (b)  $(-2)(-2)(-2)(-2) = (-2)^4$  the fourth power of  $-2$   
 (c)  $a \times a = a^2$  the second power of  $a$ , called “ $a$  squared”  
 (d)  $\left(\frac{1}{5}\right)\left(\frac{1}{5}\right)\left(\frac{1}{5}\right) = \left(\frac{1}{5}\right)^3$  the third power of  $\frac{1}{5}$ , called “ $\frac{1}{5}$  cubed” ■

We now state the basic operations with exponents using positive integers as exponents. Therefore, with  $m$  and  $n$  as positive integers, we have the following operations:

$$a^m \times a^n = a^{m+n} \quad (1.3)$$

$$\frac{a^m}{a^n} = a^{m-n} \quad (m > n, a \neq 0) \quad \frac{a^m}{a^n} = \frac{1}{a^{n-m}} \quad (m < n, a \neq 0) \quad (1.4)$$

$$(a^m)^n = a^{mn} \quad (1.5)$$

$$(ab)^n = a^n b^n \quad \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n} \quad (b \neq 0) \quad (1.6)$$

■ Two forms are shown for Eq. (1.4) in order that the resulting exponent is a positive integer. We consider negative and zero exponents after the next three examples.

■ In  $a^3$ , which equals  $a \times a \times a$ , each  $a$  is called a factor. A more general definition of factor is given in Section 1.7.

■ Here we are using the fact that  $a$  (not zero) divided by itself equals 1, or  $a/a = 1$ .

**EXAMPLE 2** Illustrating Eqs. (1.3) and (1.4)

Using Eq. (1.3):

$$a^3 \times a^5 = a^{3+5} = a^8$$

add exponents

Using the meaning of exponents:

$$a^3 \times a^5 = (a \times a \times a)(a \times a \times a \times a \times a) = a^8$$

(3 factors of a)(5 factors of a) → 8 factors of a

Using first form Eq. (1.4):

$$\frac{a^5}{a^3} = a^{5-3} = a^2$$

$5 > 3$

Using the meaning of exponents:

$$\frac{a^5}{a^3} = \frac{\underset{1}{a} \times \underset{1}{a} \times \underset{1}{a} \times a \times a}{\underset{1}{a} \times \underset{1}{a} \times \underset{1}{a}} = a^2$$

Using second form Eq. (1.4):

$$\frac{a^3}{a^5} = \frac{1}{a^{5-3}} = \frac{1}{a^2}$$

$5 > 3$

Using the meaning of exponents:

$$\frac{a^3}{a^5} = \frac{\underset{1}{a} \times \underset{1}{a} \times \underset{1}{a}}{\underset{1}{a} \times \underset{1}{a} \times \underset{1}{a} \times a \times a} = \frac{1}{a^2}$$

**EXAMPLE 3** Illustrating Eqs. (1.5) and (1.6)

Using Eq. (1.5):

$$(a^5)^3 = a^{5(3)} = a^{15}$$

multiply exponents

Using the meaning of exponents:

$$(a^5)^3 = (a^5)(a^5)(a^5) = a^{5+5+5} = a^{15}$$

Using first form Eq. (1.6):

$$(ab)^3 = a^3b^3$$

Using the meaning of exponents:

$$(ab)^3 = (ab)(ab)(ab) = a^3b^3$$

Using second form Eq. (1.6):

$$\left(\frac{a}{b}\right)^3 = \frac{a^3}{b^3}$$

Using the meaning of exponents:

$$\left(\frac{a}{b}\right)^3 = \left(\frac{a}{b}\right)\left(\frac{a}{b}\right)\left(\frac{a}{b}\right) = \frac{a^3}{b^3}$$

**LEARNING TIP**

When an expression involves a product or a quotient of different bases, **only exponents of the same base may be combined.**

**Practice Exercises**

Use Eqs. (1.3)–(1.6) to simplify the given expressions.

1.  $ax^3(-ax)^2$     2.  $\frac{(2c)^5}{(3cd)^2}$

**EXAMPLE 4** Other illustrations of exponents

(a)  $(-x^2)^3 = [(-1)x^2]^3 = (-1)^3(x^2)^3 = -x^6$

exponent of 1      add exponents of a

(b)  $ax^2(ax)^3 = ax^2(a^3x^3) = a^4x^5$  ← add exponents of x

(c)  $\frac{(3 \times 2)^4}{(3 \times 5)^3} = \frac{3^4 2^4}{3^3 5^3} = \frac{3 \times 2^4}{5^3}$

(d)  $\frac{(ry^3)^2}{r(y^2)^4} = \frac{r^2 y^6}{r y^8} = \frac{r}{y^2}$

**COMMON ERROR**

Note that  $ax^2$  means *a times the square of x* and does not mean  $a^2x^2$ , whereas  $(ax)^3$  does mean  $a^3x^3$ .

**EXAMPLE 5** Exponents in an application

In the analysis of the deflection of a beam (the amount it bends), the expression that follows is simplified as shown.

$$\begin{aligned}\frac{1}{2}\left(\frac{PL}{4EI}\right)\left(\frac{2}{3}\right)\left(\frac{L}{2}\right)^2 &= \frac{1}{2}\left(\frac{PL}{4EI}\right)\left(\frac{2}{3}\right)\left(\frac{L^2}{2^2}\right) \\ &= \frac{\overset{1}{2}PL(L^2)}{\underset{1}{2}(3)(4)(4)EI} = \frac{PL^3}{48EI}\end{aligned}$$

$L$  is the length of the beam, and  $P$  is the force applied to it.  $E$  and  $I$  are constants related to the beam. In *simplifying* this expression, we combined exponents of  $L$  and divided out the 2 that was in the numerator and in the denominator. ■

**ZERO AND NEGATIVE EXPONENTS**

If we let  $n = m$  in Eq. (1.4), we would have  $a^m/a^m = a^{m-m} = a^0$ . Also,  $a^m/a^m = 1$ , since any nonzero quantity divided by itself equals 1. Therefore, for Eq. (1.4) to hold, when  $m = n$ , we have

$$a^0 = 1 \quad (a \neq 0) \quad (1.7)$$

Eq. (1.7) states that *any nonzero expression raised to the zero power is 1*. Zero exponents can be used with any of the operations for exponents.

**EXAMPLE 6** Zero as an exponent

(a)  $5^0 = 1$  (b)  $(-3)^0 = 1$  (c)  $-(-3)^0 = -1$  (d)  $(2x)^0 = 1$   
 (e)  $(ax + b)^0 = 1$  (f)  $(a^2b^0c)^2 = a^4c^2$  (g)  $2t^0 = 2(1) = 2$   
 (Note: In (f), an arrow points from  $b^0 = 1$  to the  $b^0$  term in the base. In (g), an arrow points from  $t^0 = 1$  to the  $t^0$  term in the base.)

We note in illustration (g) that *only  $t$  is raised to the zero power*. If the quantity  $2t$  were raised to the zero power, it would be written as  $(2t)^0$ . ■

If we apply the first form of Eq. (1.4) to the case where  $n > m$ , the resulting exponent is negative. This leads to the definition of a negative exponent.

**EXAMPLE 7** Basis for negative exponents

Applying both forms of Eq. (1.4) to  $a^2/a^7$ , we have

$$\frac{a^2}{a^7} = a^{2-7} = a^{-5} \quad \text{and} \quad \frac{a^2}{a^7} = \frac{1}{a^{7-2}} = \frac{1}{a^5}$$

If these results are to be consistent, then  $a^{-5} = \frac{1}{a^5}$ . ■

Following the reasoning of Example 7, if we define

$$a^{-n} = \frac{1}{a^n} \quad (a \neq 0) \quad (1.8)$$

then all of the laws of exponents will hold for negative integers.

**LEARNING TIP**

Although positive exponents are generally preferred in a final result, there are some cases in which zero or negative exponents are to be used. Also, negative exponents are very useful in some operations that we will use later.

Practice Exercises

Simplify: 3.  $\frac{-7^0}{c^{-3}}$  4.  $\frac{(3x)^{-1}}{2a^{-2}}$

LEARNING TIP

From the statement  $3^{-1} = \frac{1}{3}$ , we see that the reciprocal of any number  $x$  different than zero is  $x^{-1} = \frac{1}{x}$ .

EXAMPLE 8 Negative exponents

(a)  $3^{-1} = \frac{1}{3}$  (b)  $4^{-2} = \frac{1}{4^2} = \frac{1}{16}$  (c)  $\frac{1}{a^{-3}} = a^3$  change signs of exponents

(d)  $\left(\frac{a^3t}{b^2x}\right)^{-2} = \frac{(a^3t)^{-2}}{(b^2x)^{-2}} = \frac{(b^2x)^2}{(a^3t)^2} = \frac{b^4x^2}{a^6t^2}$  (e)  $3x^{-1} = 3\left(\frac{1}{x}\right) = \frac{3}{x}$  ■

COMMON ERROR

In the statement  $3x^{-1} = \frac{3}{x}$ , the exponent only applies to  $x$ , not to 3. Do not confuse this with the statement  $(3x)^{-1} = 3^{-1}x^{-1} = \frac{1}{3x}$ .

■ The use of exponents is taken up in more detail in Chapter 11.

ORDER OF OPERATIONS

In Section 1.2, we saw that it is necessary to follow a particular order of operations when performing the basic operations on numbers. Since raising a number to a power is a form of multiplication, this operation is performed before additions and subtractions. In fact, it is performed before multiplications and divisions.

Order of Operations

1. Operations within specific groupings first—that is, inside parentheses ( ), brackets [ ], or absolute values | |
2. Exponents and roots/radicals
3. Multiplications and divisions (from left to right)
4. Additions and subtractions (from left to right)

EXAMPLE 9 Order of operations

$$\begin{aligned} 8 - (-1)^2 - 2(-3)^2 &= 8 - 1 - 2(9) \\ &= 8 - 1 - 18 \\ &= -11 \end{aligned}$$

Since there were no specific groupings, we first squared  $-1$  and  $-3$ . Next, we found the product  $2(9)$  in the last term. Finally, the subtractions were performed. Note that *we did not change the sign of  $-1$  before we squared it.* ■

■ On many calculators, there is a specific key or key sequence to evaluate  $x^3$ .

EVALUATING ALGEBRAIC EXPRESSIONS

An algebraic expression is **evaluated** by **substituting** given values of the literal numbers in the expression and calculating the result. On a calculator, the  $x^2$  key is used to square numbers, and the  $\wedge$  or  $x^y$  key is used for other powers.

EXAMPLE 10 Evaluating using exponents

The distance (in m) that an object falls in 4.2 s is found by substituting 4.2 for  $t$  in the expression  $4.90t^2$ . We show this as

$$4.90(4.2)^2 = 86 \text{ m} \quad \begin{array}{l} \leftarrow \text{substituting } t = 4.2 \text{ s} \\ \leftarrow \text{estimation } \rightarrow 5(4)^2 = 80 \end{array}$$

The result is rounded off to two significant digits (the accuracy of  $t$ ). The calculator will square 4.2 before multiplying. See Fig. 1.12. ■

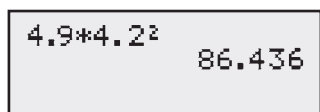


Fig. 1.12

■ When less than half of a calculator screen is needed, the figure for that calculator screen will show only a partial screen.

**LEARNING TIP**

Note that a negative number raised to an even power gives a positive value and a negative number raised to an odd power gives a negative value.

As stated in Section 1.3, in evaluating an expression on a calculator, we should also estimate its value as in Example 10.

**EXAMPLE 11 Power of a number**

Using the meaning of a power of a number, we have

$$\begin{aligned} (-2)^2 &= (-2)(-2) = 4 & (-2)^3 &= (-2)(-2)(-2) = -8 \\ (-2)^4 &= 16 & (-2)^5 &= -32 & (-2)^6 &= 64 & (-2)^7 &= -128 \end{aligned}$$

**EXAMPLE 12 Evaluating in an application**

A wire made of a special alloy has an electric resistance  $R$  (in  $\Omega$ ) given by  $R = a + 0.0115T^3$ , where  $T$  (in  $^{\circ}\text{C}$ ) is the temperature (between  $-4^{\circ}\text{C}$  and  $4^{\circ}\text{C}$ ). Find  $R$  for  $a = 0.838 \Omega$  and  $T = -2.87^{\circ}\text{C}$ .

Substituting these values, we have

$$\begin{aligned} R &= 0.838 + 0.0115(-2.87)^3 & \text{estimation:} \\ &= 0.566 \Omega & 0.8 + 0.01(-3)^3 = 0.8 + 0.01(-27) = 0.53 \end{aligned}$$

Note in the estimation that  $(-3)^3 = -27$ .

Graphing calculators generally use computer symbols in the display for some of the operations to be performed. These symbols are as follows:

Multiplication: \*    Division: /    Powers: ^

Therefore, to calculate the value of  $20 \times 6 + 200/5 - 3^4$ , we use the key sequence

$$20 \text{ [}\times\text{]} 6 \text{ [}+\text{]} 200 \text{ [}\div\text{]} 5 \text{ [}-\text{]} 3 \text{ [}\wedge\text{]} 4$$

with a result of 79. **Note carefully that 200 is divided only by 5.** If it were divided by  $5 - 3^4$ , then we would use parentheses and show the expression to be evaluated as  $20 \times 6 + 200/(5 - 3^4)$ .

**COMMON ERROR**

When using a calculator, ensure that you use brackets to indicate grouped division in a denominator. To evaluate the expression  $\frac{12}{3+9}$  on a calculator, use the key sequence  $12 \div (3 + 9) = 1$ .

If you forget the brackets and write  $12 \div 3 + 9 = 13$ , an error will result (the division will occur before the addition).

**EXERCISES 1.4**

In Exercises 1–4, make the given changes in the indicated examples of this section, and then simplify the resulting expression.

- In Example 4(a), change  $(-x^2)^3$  to  $(-x^3)^2$ .
- In Example 6(d), change  $(2x)^0$  to  $2x^0$ .
- In Example 8(d), interchange the  $a^3$  and  $b^2$ .
- In Example 9, change  $(-1)^2$  to  $(-1)^3$ .

In Exercises 5–48, simplify the given expressions. Express results with positive exponents only.

- |                      |                      |                        |                      |                                    |                                    |                                  |                                     |
|----------------------|----------------------|------------------------|----------------------|------------------------------------|------------------------------------|----------------------------------|-------------------------------------|
| 5. $x^3x^4$          | 6. $y^2y^7$          | 7. $2b^4b^2$           | 8. $3k^5k$           | 17. $(aT^2)^{30}$                  | 18. $(3r^2)^3$                     | 19. $\left(\frac{2}{b}\right)^3$ | 20. $\left(\frac{F}{t}\right)^{20}$ |
| 9. $\frac{m^5}{m^3}$ | 10. $\frac{2x^6}{x}$ | 11. $\frac{n^5}{7n^9}$ | 12. $\frac{3s}{s^4}$ | 21. $\left(\frac{x^2}{2}\right)^4$ | 22. $\left(\frac{3}{n^3}\right)^3$ | 23. $(8a)^0$                     | 24. $6v^0$                          |
| 13. $(P^2)^4$        | 14. $(x^8)^3$        | 15. $(2\pi)^3$         | 16. $(ax)^5$         | 25. $-3x^0$                        | 26. $-(-2)^0$                      | 27. $6^{-1}$                     | 28. $-w^{-5}$                       |
|                      |                      |                        |                      | 29. $\frac{1}{R^{-2}}$             | 30. $\frac{1}{-t^{-48}}$           | 31. $(-t^2)^7$                   | 32. $(-y^3)^5$                      |
|                      |                      |                        |                      | 33. $(2v^2)^{-6}$                  | 34. $-(-c^4)^{-4}$                 | 35. $\frac{L^{-3}}{L^{-5}}$      | 36. $2i^{40}i^{-70}$                |
|                      |                      |                        |                      | 37. $\frac{2v^4}{(2v)^4}$          | 38. $\frac{x^2x^3}{(x^2)^3}$       | 39. $\frac{(n^2)^4}{(n^4)^2}$    | 40. $\frac{(3t)^{-1}}{3t^{-1}}$     |
|                      |                      |                        |                      | 41. $(\pi^0x^2a^{-1})^{-1}$        | 42. $(3m^{-2}n^4)^{-2}$            | 43. $(-8g^{-1}s^3)^2$            |                                     |



44.  $ax^{-2}(-a^2x)^3$       45.  $\left(\frac{4x^{-1}}{a^{-1}}\right)^{-3}$       46.  $\left(\frac{2b^2}{y^5}\right)^{-2}$   
 47.  $\frac{15n^2T^5}{3n^{-1}T^6}$       48.  $\frac{(nRT^{-2})^{32}}{R^{-2}T^{32}}$

In Exercises 49–56, evaluate the given expressions. In Exercises 51–56, all numbers are approximate.

49.  $7(-4) - (-5)^2$       50.  $6 + (-2)^5 - (-2)(8)$   
 51.  $-(-26.5)^2 - (-9.85)^3$       52.  $-0.711^2 - (-0.809)^6$   
 53.  $\frac{3.07(-1.86)}{(-1.86)^4 + 1.596}$       54.  $\frac{15.66^2 - (-4.017)^4}{1.044(-3.68)}$   
 55.  $2.38(-60.7)^2 - 2540/1.17^3 + 0.806^5(26.1^3 - 9.88^4)$   
 56.  $0.513(-2.778) - (-3.67)^3 + 0.889^4/(1.89 - 1.09^2)$

In Exercises 57–68, perform the indicated operations.

57. Does  $\left(\frac{1}{x-1}\right)^{-1}$  represent the reciprocal of  $x$ ?  
 58. Does  $\left(\frac{0.2 - 5^{-1}}{10^{-2}}\right)^0$  equal 1? Explain.  
 59. If  $a^3 = 5$ , then what does  $a^{12}$  equal?  
 60. Is  $a^{-2} < a^{-1}$  for any negative value of  $a$ ? Explain.  
 61. If  $a$  is a positive integer, simplify  $(x^a \cdot x^{-a})^5$ .  
 62. If  $a$  and  $b$  are positive integers, simplify  $(y^{a-b} \cdot y^{a+b})^2$ .

63. In developing the “big bang” theory of the origin of the universe, the expression  $(kT/(hc))^3(GkThc)^2c$  arises. Simplify this expression.  
 64. In studying planetary motion, the expression  $(GmM)(mr)^{-1}(r^{-2})$  arises. Simplify this expression.  
 65. In designing a cam for a pump, the expression  $\pi\left(\frac{r}{2}\right)^3\left(\frac{4}{3\pi r^2}\right)$  is used. Simplify this expression.  
 66. For a certain integrated electric circuit, it is necessary to simplify the expression  $\frac{gM}{2\pi fC(2\pi fM)^2}$ . Perform this simplification.  
 67. If \$2500 is invested at 4.2% interest, compounded quarterly, the amount in the account after 6 years is  $2500(1 + 0.042/4)^{24}$ . Calculate this amount. (The values 1, 4, and 24 are exact.)  
 68. In designing a building, it was determined that the forces acting on an I beam would deflect the beam an amount (in cm) given by  $\frac{x(1000 - 20x^2 + x^3)}{1850}$ , where  $x$  is the distance (in m) from one end of the beam. Find the deflection for  $x = 6.85$  m. (The 1000 and 20 are exact.)

Answers to Practice Exercises

1.  $a^3x^5$     2.  $\frac{2^5c^3}{3^2d^2} = \frac{32c^3}{9d^2}$     3.  $-c^3$     4.  $\frac{a^2}{6x}$

## 1.5 Scientific Notation

Meaning of Scientific Notation • Changing Numbers to and from Scientific Notation • Scientific Notation on a Calculator • Engineering Notation

■ Television was invented in the 1920s and first used commercially in the 1940s.

The use of fibre optics was developed in the 1950s.

X-rays were discovered by Roentgen in 1895.

In technical and scientific work, we often encounter numbers that are either very large or very small. Examples of such numbers are: (a) Radio and television signals travel at about 30 000 000 000 cm/s; (b) The mass of the earth is about 6 000 000 000 000 000 000 000 kg; (c) A typical individual fibre in a fibre-optic communications cable has a diameter of 0.000 005 m; (d) some X-rays have a wavelength of about 0.000 000 095 cm.

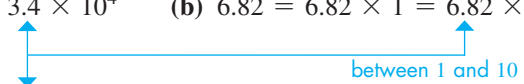
As we can see, writing numbers such as these is inconvenient in ordinary notation. Although calculators and computers can often handle such numbers, there is a more efficient way of expressing numbers, in order to work more easily with them. A convenient and useful notation, called *scientific notation*, is used to represent these or any other numbers.

A number in **scientific notation** is expressed as the product of a number greater than or equal to 1 and less than 10, and a power of 10, and is written as

$$P \times 10^k$$

where  $1 \leq P < 10$  and  $k$  is an integer. (The symbol  $\leq$  means “is less than or equal to.”)

### EXAMPLE 1 Scientific notation

(a)  $34\,000 = 3.4 \times 10\,000 = 3.4 \times 10^4$       (b)  $6.82 = 6.82 \times 1 = 6.82 \times 10^0$   
  
 (c)  $0.005\,03 = \frac{5.03}{1000} = \frac{5.03}{10^3} = 5.03 \times 10^{-3}$  ■

From Example 1, we see how a number is changed from ordinary notation to scientific notation. *The decimal point is moved so that only one nonzero digit is to its left. The number of places moved is the power of 10 (k), which is positive if the decimal point is moved to the left and negative if moved to the right.*

**EXAMPLE 2** Changing numbers to scientific notation

(a)  $34\,000 = 3.4 \times 10^4$     (b)  $6.82 = 6.82 \times 10^0$     (c)  $0.005\,03 = 5.03 \times 10^{-3}$

To change a number from scientific notation to ordinary notation, we reverse the procedure used in Example 2.

**EXAMPLE 3** Changing numbers to ordinary notation

- (a) To change  $5.83 \times 10^6$  to ordinary notation, we move the decimal point six places to the right, including additional zeros to properly locate the decimal point.
- (b) To change  $8.06 \times 10^{-3}$  to ordinary notation, we must move the decimal point three places to the left, again including additional zeros to properly locate the decimal point.

$$5.83 \times 10^6 = 5\,830\,000$$

$$8.06 \times 10^{-3} = 0.008\,06$$

Scientific notation provides a practical way to handle calculations with very large or very small numbers, particularly products, quotients, and powers. First, all numbers are expressed in scientific notation. Then the calculation can be performed on numbers between 1 and 10, using the laws of exponents to find the power of 10 in the result. In this way, scientific notation gives an important application of the use of exponents.

**EXAMPLE 4** Scientific notation in calculations

In designing a computer, it was determined that it would be able to process 803 000 bits of data in 0.000 005 25 s. (See Exercise 41 of Exercises 1.1 for a brief note on computer data.) The rate of processing the data is

$$\frac{803\,000}{0.000\,005\,25} = \frac{8.03 \times 10^5}{5.25 \times 10^{-6}} = \left(\frac{8.03}{5.25}\right) \times 10^{5 - (-6)} = 1.53 \times 10^{11} \text{ bits/s}$$

As shown, it is proper to leave the result (*rounded off*) in scientific notation. This method is useful when using a calculator and then estimating the result. In this case, the estimate is  $(8 \times 10^5) \div (5 \times 10^{-6}) = 1.6 \times 10^{11}$ .

Another advantage of scientific notation is that the precise number of significant digits of a number can be shown directly, even when the final significant digit is 0.

**EXAMPLE 5** Scientific notation and significant digits

In evaluating  $750\,000\,000\,000^2$ , if we know that 750 000 000 000 has *three* significant digits, we can show the significant digits by writing

$$750\,000\,000\,000^2 = (7.50 \times 10^{11})^2 = 7.50^2 \times 10^{2 \times 11} = 56.3 \times 10^{22}$$

If the answer is to be written in scientific notation, we should write it as the product of a number between 1 and 10 and a power of 10. This means we should rewrite it as

$$56.3 \times 10^{22} = (5.63 \times 10) (10^{22}) = 5.63 \times 10^{23}$$

We can enter numbers in scientific notation on a calculator, as well as have the calculator give results automatically in scientific notation. See the next example.

**Practice Exercises**

- Change  $2.35 \times 10^{-3}$  to ordinary notation.
- Change 235 to scientific notation.

■ The number  $7.50 \times 10^{11}$  cannot be entered on a scientific calculator in the form 750 000 000 000. It could be entered in this form on a graphing calculator.

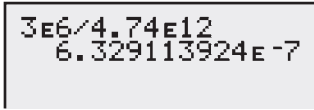


Fig. 1.13

**EXAMPLE 6 Scientific notation on a calculator**

The wavelength  $\lambda$  (in m) of the light in a red laser beam can be found from the following calculation. Note the significant digits in the numerator.

$$\lambda = \frac{3\,000\,000}{4\,740\,000\,000\,000} = \frac{3.00 \times 10^6}{4.74 \times 10^{12}} = 6.33 \times 10^{-7} \text{ m}$$

The key sequence is 3 **(EE)** 6 **(÷)** 4.74 **(EE)** 12 **(ENTER)**. See Fig. 1.13. ■

**COMMON ERROR**

When using scientific notation for SI measurements, powers of 10 might end up being mixed with SI prefixes. Generally, try to avoid mixing SI prefixes with scientific notation.

**EXAMPLE 7 Scientific notation with SI prefixes**

A measurement of a small distance is quoted in units of km as 0.000 045 km. Convert this unit to scientific notation and with a single SI prefix.

$$\begin{aligned} 0.000\,045 \text{ km} &= 4.5 \times 10^{-5} \text{ km} \\ &= 4.5 \times 10^{-5} \times 10^3 \text{ m} \\ &= 4.5 \times 10^{-2} \text{ m} \\ &= 4.5 \text{ cm} \end{aligned}$$

A number in **engineering notation** is expressed as a power of 10 where the exponent is a multiple of 3. It is written as

$$P \times 10^k$$

where  $k$  is a multiple of 3 and  $1 \leq P < 1000$ , so that  $P$  can have one, two, or three digits to the left of the decimal point. To change a number from ordinary notation to engineering notation, the decimal point is moved three spaces at a time, until you have a number from 1 to 999. The number of places moved is the power of 10 ( $k$ ), positive if moved to the left, negative if moved to the right.

**EXAMPLE 8 Engineering notation**

- (a)  $19\,680\,000\,000 = 19.68 \times 10^9$       (b)  $0.45 = 450 \times 10^{-3}$
- 
- (c)  $5.36 \times 10^4 = 5.36 \times 10 \times 10^3 = 53.6 \times 10^3$
- 

Engineering notation, when combined with SI prefixes, allows us to replace the power of 10 notation altogether, as can be seen in the following example.

**EXAMPLE 9 Engineering notation and SI prefixes**

Write each of the following quantities in engineering notation and replace the power of 10 with the corresponding SI prefix.

- (a)  $0.000\,009 \text{ F} = 9 \times 10^{-6} \text{ F} = 9 \mu\text{F}$
- (b)  $62\,900 \text{ W} = 62.9 \times 10^3 \text{ W} = 62.9 \text{ kW}$  ■

**Practice Exercises**

3. Write 0.000 053 5 in engineering notation.
4. Write  $9.1 \times 10^{-8}$  in engineering notation.

## EXERCISES 1.5

In Exercises 1 and 2, make the given changes in the indicated examples of this section and then rewrite the number as directed.

- In Example 3(b), change the exponent  $-3$  to  $3$  and then write the number in ordinary notation.
- In Example 5, change the exponent  $2$  to  $-1$  and then write the result in scientific notation.

In Exercises 3–10, change the numbers from scientific notation to ordinary notation.

- $4.5 \times 10^4$
- $6.8 \times 10^7$
- $2.01 \times 10^{-3}$
- $9.61 \times 10^{-5}$
- $3.23 \times 10^0$
- $8 \times 10^0$
- $1.86 \times 10$
- $1 \times 10^{-1}$

In Exercises 11–20, change the numbers from ordinary notation to scientific notation.

- 4000
- 56 000
- 0.0087
- 0.7
- 6.09
- 100
- 0.063
- 0.000 090 8
- 1
- 20

In Exercises 21–24, perform the indicated calculations using a calculator and by first expressing all numbers in scientific notation.

- $28\ 000(2\ 000\ 000\ 000)$
- $50\ 000(0.006)$
- $\frac{88\ 000}{0.0004}$
- $\frac{0.000\ 03}{6\ 000\ 000}$

In Exercises 25–28, perform the indicated calculations and then check the result using a calculator. Assume that all numbers are exact.

- $2 \times 10^{-35} + 3 \times 10^{-34}$
- $5.3 \times 10^{12} - 3.7 \times 10^{10}$
- $(1.2 \times 10^{29})^3$
- $(2 \times 10^{-16})^{-5}$

In Exercises 29–36, perform the indicated calculations using a calculator. All numbers are approximate.

- $1280(865\ 000)(43.8)$
- $0.000\ 056\ 9(3\ 190\ 000)$
- $\frac{0.0732(6710)}{0.001\ 34(0.0231)}$
- $\frac{0.004\ 52}{2430(97\ 100)}$
- $(3.642 \times 10^{-8})(2.736 \times 10^5)$
- $\frac{(7.309 \times 10^{-1})^2}{5.9843(2.5036 \times 10^{-20})}$
- $\frac{(3.69 \times 10^{-7})(4.61 \times 10^{21})}{0.0504}$
- $\frac{(9.907 \times 10^7)(1.08 \times 10^{12})^2}{(3.603 \times 10^{-5})(2054)}$

In Exercises 37–44, change numbers in ordinary notation to scientific notation or change numbers in scientific notation to ordinary notation.

- The Sir Adam Beck Generating Stations in Niagara Falls produce  $2\ 000\ 000$  kW of power.
- A certain computer has  $85\ 000\ 000\ 000$  bytes (85 gigabytes) of memory.
- A fibre-optic system requires  $0.000\ 003$  W of power.
- A red blood cell measures  $0.0075$  mm across.
- The frequency of a certain TV signal is  $2\ 000\ 000\ 000$  Hz.
- A *parsec*, a unit used in astronomy, is about  $3.086 \times 10^{16}$  m.

- The power of the signal of a laser beam probe is  $1.6 \times 10^{-12}$  W.
- The electrical force between two electrons is about  $2.4 \times 10^{-43}$  times the gravitational force between them.

In Exercises 45–49, write the numbers from Exercises 37–41 in engineering notation and replace the power of 10 with the corresponding SI prefix.

In Exercises 50–53, solve the given problems.

- Write the following numbers in engineering notation. (a) 2300 (b) 0.23 (c) 23
- Write the following numbers in engineering notation. (a) 8 090 000 (b) 809 000 (c) 0.0809
- A *googol* is defined as 1 followed by 100 zeros. (a) Write this number in scientific notation. (b) A *googolplex* is defined as 10 to the googol power. Write this number using powers of 10, and not the word *googol*. (Note the name of the Internet company.)
- The number of electrons in the universe has been estimated at  $10^{79}$ . How many times greater is a googol than the estimated number of electrons in the universe? (See Exercise 52.)

In Exercises 54–57, perform the indicated calculations.

- A computer can do an addition in  $7.5 \times 10^{-15}$  s. How long does it take to perform  $5.6 \times 10^6$  additions?
- Uranium is used in nuclear reactors to generate electricity. About  $0.000\ 000\ 039\%$  of the uranium disintegrates each day. How much of  $0.085$  mg of uranium disintegrates in a day?
- A TV signal travels at  $3.00 \times 10^5$  km/s for a total of  $7.37 \times 10^4$  km from the station transmitter to a satellite and then to a receiver dish. How long does it take the signal to go from the transmitter to the dish?
- (a) Determine the number of seconds in a day in scientific notation. (b) Using the result of part (a), determine the number of seconds in a century (assume 365.25 days/year).

In Exercises 58–61, perform the indicated calculations by first expressing all numbers in scientific notation.

- One *atomic mass unit* (amu) is  $1.66 \times 10^{-27}$  kg. If one oxygen atom has 16 amu (an exact number), what is the mass of  $125\ 000\ 000$  oxygen atoms?
- The rate of energy radiation (in W) from an object is found by evaluating the expression  $kT^4$ , where  $T$  is the thermodynamic temperature. Find this value for the human body, for which  $k = 0.000\ 000\ 057$  W/K<sup>4</sup> and  $T = 303$  K.
- In a microwave receiver circuit, the resistance  $R$  of a wire 1 m long is given by  $R = k/d^2$ , where  $d$  is the diameter of the wire. Find  $R$  if  $k = 0.000\ 000\ 021\ 96$   $\Omega \cdot \text{m}^2$  and  $d = 0.000\ 079\ 98$  m.
- The average distance between the sun and earth is about  $149\ 600\ 000$  km, and this distance is called an *astronomical unit* (AU). It takes light about  $499.0$  s to travel 1 AU. What is the speed of light? Compare this with the speed of the TV signal in Exercise 56.

## Answers to Practice Exercises

- 0.002 35
- $2.35 \times 10^2$
- $53.5 \times 10^{-6}$
- $91 \times 10^{-9}$

## 1.6 Roots and Radicals

Principal  $n$ th Root • Simplifying Radicals • Using a Calculator • Imaginary Numbers

At times, we have to find the *square root* of a number, or maybe some other root of a number, such as a *cube root*. This means we must find a number that when squared, or cubed, and so on equals some given number. For example, to find the square root of 9, we must find a number that when squared equals 9. In this case, either 3 or  $-3$  is an answer. Therefore, *either 3 or  $-3$  is a square root of 9 since  $3^2 = 9$  and  $(-3)^2 = 9$ .*

To have a general notation for the square root and have it represent *one* number, we define the **principal square root** of  $a$  to be positive if  $a$  is positive and represent it by  $\sqrt{a}$ . This means  $\sqrt{9} = 3$  and not  $-3$ .

■ Unless we state otherwise, when we refer to the root of a number, it is the principal root.

The general notation for the **principal  $n$ th root** of  $a$  is  $\sqrt[n]{a}$ . (When  $n = 2$ , do not write the 2 for  $n$ .) The  $\sqrt{\quad}$  sign is called a **radical sign**.

### EXAMPLE 1 Roots of numbers

(a)  $\sqrt{2}$  (the square root of 2)

(b)  $\sqrt[3]{2}$  (the cube root of 2)

(c)  $\sqrt[4]{2}$  (the fourth root of 2)

(d)  $\sqrt[7]{6}$  (the seventh root of 6) ■

To have a single defined value for all roots (not just square roots) and to consider only real-number roots, we define the **principal  $n$ th root** of  $a$  to be positive if  $a$  is positive and to be negative if  $a$  is negative and  $n$  is odd. (If  $a$  is negative and  $n$  is even, the roots are not real.)

### EXAMPLE 2 Principal $n$ th root

(a)  $\sqrt{169} = 13$  ( $\sqrt{169} \neq -13$ )

(b)  $-\sqrt{64} = -8$

(c)  $\sqrt[3]{27} = 3$  since  $3^3 = 27$

(d)  $\sqrt{0.04} = 0.2$  since  $0.2^2 = 0.04$

(e)  $-\sqrt[4]{256} = -4$

(f)  $\sqrt[3]{-27} = -3$

(g)  $-\sqrt[3]{27} = -(+3) = -3$  ■

Another property of square roots is developed by noting illustrations such as  $\sqrt{36} = \sqrt{4 \times 9} = \sqrt{4} \times \sqrt{9} = 2 \times 3 = 6$ . In general, this property states that *the square root of a product of positive numbers is the product of their square roots.*

$$\sqrt{ab} = \sqrt{a}\sqrt{b} \quad (a \text{ and } b \text{ positive real numbers}) \quad (1.9)$$

This property is used in simplifying radicals. It is most useful if either  $a$  or  $b$  is a **perfect square**, which is the square of a rational number.

### EXAMPLE 3 Simplifying square roots

(a)  $\sqrt{8} = \sqrt{(4)(2)} = \sqrt{4}\sqrt{2} = 2\sqrt{2}$  ← simplest form

(b)  $\sqrt{75} = \sqrt{(25)(3)} = \sqrt{25}\sqrt{3} = 5\sqrt{3}$  ← simplest form

(c)  $\sqrt{4 \times 10^2} = \sqrt{4}\sqrt{10^2} = 2(10) = 20$

(Note that the square root of the square of a positive number is that number.) ■

■ Try this one on your calculator:  
 $\sqrt{12345678987654321}$

In order to represent the square root of a number *exactly*, use Eq. (1.9) to write it in simplest form. However, a decimal *approximation* is often acceptable, and we use the  $\sqrt{x}$  key on a calculator. (We will show another way of finding the root of a number in Chapter 11.)

**EXAMPLE 4 Square root application**

After reaching its greatest height, the time (in s) for a rocket to fall  $h$  m is found by evaluating  $0.45\sqrt{h}$ . Find the time for the rocket to fall 360 m.

We calculate  $0.45\sqrt{360} = 8.5$ . Therefore, the rocket takes 8.5 s to fall 360 m. The result is rounded off to two significant digits, the accuracy of 0.45 (an approximate number).

In simplifying a radical, *all operations under a radical sign must be done before finding the root*. The horizontal bar groups the numbers under it.

**EXAMPLE 5 More on simplifying square roots**

(a)  $\sqrt{16 + 9} = \sqrt{25}$  first perform the addition  $16 + 9$   
 $= 5$

**COMMON ERROR**

However,  $\sqrt{16 + 9}$  is *not*  $\sqrt{16} + \sqrt{9} = 4 + 3 = 7$ .

(b)  $\sqrt{2^2 + 6^2} = \sqrt{4 + 36} = \sqrt{40} = \sqrt{4}\sqrt{10} = 2\sqrt{10}$ , but  
 $\sqrt{2^2 + 6^2}$  is *not*  $\sqrt{2^2} + \sqrt{6^2} = 2 + 6 = 8$ .

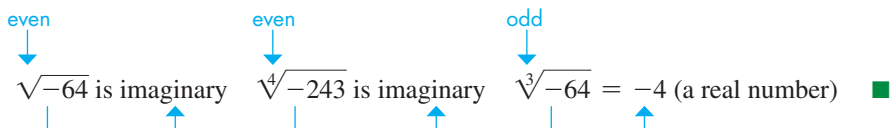
**Practice Exercises**

Simplify:

1.  $\sqrt{12}$  2.  $\sqrt{36 + 144}$

In defining the principal square root, we did not define the square root of a negative number. However, in Section 1.1, we defined the square root of a negative number to be an **imaginary number**. More generally, *the even root of a negative number is an imaginary number, and the odd root of a negative number is a negative real number*.

**EXAMPLE 6 Imaginary roots and real roots**



A much more detailed coverage of imaginary numbers, roots, and radicals is taken up in Chapters 11 and 12.

**EXERCISES 1.6**

In Exercises 1–4, make the given changes in the indicated examples of this section and then solve the given problems.

- In Example 2(b), change the square root to a cube root and then evaluate.
- In Example 3(b), change  $\sqrt{(25)(3)}$  to  $\sqrt{(15)(5)}$  and explain whether or not this would be a better expression to use.
- In Example 5(a), change the  $+$  to  $\times$  and then evaluate.
- In the first illustration of Example 6, place a  $-$  sign before the radical. Is there any other change in the statement?

In Exercises 5–36, simplify the given expressions. In each of 5–9 and 12–21, the result is an integer.

- |                         |                            |   |  |
|-------------------------|----------------------------|---|--|
| 5. $\sqrt{81}$          | 6. $\sqrt{225}$            | 7. $-\sqrt{121}$                        | 8. $-\sqrt{36}$                            |
| 9. $-\sqrt{49}$         | 10. $\sqrt{0.25}$          | 11. $\sqrt{0.09}$                       | 12. $-\sqrt{900}$                          |
| 13. $\sqrt[3]{125}$     | 14. $\sqrt[4]{16}$         | 15. $\sqrt[3]{-216}$                    | 16. $\sqrt[5]{-32}$                        |
| 17. $(\sqrt{5})^2$      | 18. $(\sqrt[3]{31})^3$     | 19. $(-\sqrt[3]{-47})^3$                | 20. $(\sqrt[5]{-23})^5$                    |
| 21. $(-\sqrt[4]{53})^4$ | 22. $-\sqrt{32}$           | 23. $\sqrt{1200}$                       | 24. $\sqrt{50}$                            |
| 25. $2\sqrt{84}$        | 26. $\frac{\sqrt{108}}{2}$ | 27. $\sqrt{\frac{80}{7-3}}$             | 28. $\sqrt{81 \times 10^2}$                |
| 29. $\sqrt[3]{8^2}$     | 30. $\sqrt[4]{9^2}$        | 31. $\frac{7^2\sqrt{81}}{3^2\sqrt{49}}$ | 32. $\frac{2^5\sqrt[3]{243}}{3\sqrt{144}}$ |



33.  $\sqrt{36 + 64}$                       34.  $\sqrt{25 + 144}$   
 35.  $\sqrt{3^2 + 9^2}$                       36.  $\sqrt{8^2 - 4^2}$

In Exercises 37–44, find the value of each square root by use of a calculator. Each number is approximate.

37.  $\sqrt{85.4}$       38.  $\sqrt{3762}$       39.  $\sqrt{0.4729}$       40.  $\sqrt{0.0627}$   
 41. (a)  $\sqrt{1296 + 2304}$                       (b)  $\sqrt{1296} + \sqrt{2304}$   
 42. (a)  $\sqrt{10.6276 + 2.1609}$                       (b)  $\sqrt{10.6276} + \sqrt{2.1609}$   
 43. (a)  $\sqrt{0.0429^2 - 0.0183^2}$                       (b)  $\sqrt{0.0429^2} - \sqrt{0.0183^2}$   
 44. (a)  $\sqrt{3.625^2 + 0.614^2}$                       (b)  $\sqrt{3.625^2} + \sqrt{0.614^2}$

In Exercises 45–56, solve the given problems.

45. The speed (in km/h) of a car that skids to a stop on dry pavement is often estimated by  $\sqrt{207s}$ , where  $s$  is the length (in m) of the skid marks. Estimate the speed if  $s = 46$  m.  
 46. The resistance in an amplifier circuit is found by evaluating  $\sqrt{Z^2 - X^2}$ . Find the resistance for  $Z = 5.362 \Omega$  and  $X = 2.875 \Omega$ .  
 47. The speed (in m/s) of sound in seawater is found by evaluating  $\sqrt{B/d}$  for  $B = 2.18 \times 10^9$  Pa and  $d = 1.03 \times 10^3$  kg/m<sup>3</sup>. Find this speed, which is important in locating underwater objects using sonar.  
 48. The terminal speed (in m/s) of a skydiver can be approximated by  $\sqrt{40m}$ , where  $m$  is the mass (in kg) of the skydiver. Calculate the terminal speed (after reaching this speed, the skydiver's speed

remains fairly constant before opening the parachute) of a 75-kg skydiver.

49. An HDTV screen is 93.0 cm wide and 52.1 cm high. The length of a diagonal (the dimension used to describe it—from one corner to the opposite corner) is found by evaluating  $\sqrt{w^2 + h^2}$ , where  $w$  is the width and  $h$  is the height. Find the diagonal.  
 50. A car costs \$22 000 new and is worth \$15 000 2 years later. The annual rate of depreciation is found by evaluating  $100(1 - \sqrt{V/C})$ , where  $C$  is the cost and  $V$  is the value after 2 years. At what rate did the car depreciate? (100 and 1 are exact.)  
 51. Is it always true that  $\sqrt{a^2} = a$ ? Explain.  
 52. For what values of  $x$  is (a)  $x > \sqrt{x}$ , (b)  $x = \sqrt{x}$ , and (c)  $x < \sqrt{x}$ ?  
 53. A graphing calculator has a specific key sequence to find cube roots. Using a calculator, find  $\sqrt[3]{2140}$  and  $\sqrt[3]{-0.214}$ .  
 54. A graphing calculator has a specific key sequence to find  $n$ th roots. Using a calculator, find  $\sqrt[4]{0.382}$  and  $\sqrt[4]{-382}$ .  
 55. Determine if the following numbers are real or imaginary: (a)  $\sqrt[4]{-81}$  (b)  $\sqrt{-128}$   
 56. Determine if the following numbers are real or imaginary: (a)  $\sqrt[5]{-32}$  (b)  $\sqrt[4]{-64}$

Answers to Practice Exercises

1.  $2\sqrt{3}$     2.  $6\sqrt{5}$

## 1.7 Addition and Subtraction of Algebraic Expressions

Algebraic Expressions • Terms • Factors • Polynomials • Similar Terms • Simplifying • Symbols of Grouping

Since we use letters to represent numbers, we can see that all operations that can be used on numbers can also be used on literal numbers. In this section, we discuss the methods for adding and subtracting literal numbers.

Addition, subtraction, multiplication, division, and taking of roots are known as **algebraic operations**. Any combination of numbers and literal symbols that results from algebraic operations is known as an **algebraic expression**.

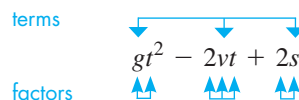
When an algebraic expression consists of several parts connected by plus signs and minus signs, each part (along with its sign) is known as a **term** of the expression. If a given expression is made up of the product of a number of quantities, each of these quantities, or any product of them, is called a **factor** of the expression.

**COMMON ERROR**

It is very important to *distinguish clearly between terms and factors*, because some operations that are valid for terms are not valid for factors, and conversely. Some of the common errors in handling algebraic expressions occur because these operations are not handled properly.

**EXAMPLE 1** Terms and factors

In the study of the motion of a rocket, the following algebraic expression may be used.



This expression has three terms:  $gt^2$ ,  $-2vt$ , and  $2s$ . The first term,  $gt^2$ , has a factor of  $g$  and two factors of  $t$ . Any product of these factors is also a factor of  $gt^2$ . This means other factors are  $gt$ ,  $t^2$ , and  $gt^2$  itself. ■

**EXAMPLE 2** Terms and factors

$7a(x^2 + 2y) - 6x(5 + x - 3y)$  is an expression with terms  $7(x^2 + 2y)$  and  $-6x(5 + x - 3y)$ .

The term  $7a(x^2 + 2y)$  has individual factors of 7,  $a$ , and  $(x^2 + 2y)$ , as well as products of these factors. The factor  $x^2 + 2y$  has two terms,  $x^2$  and  $2y$ .

The term  $-6x(5 + x - 3y)$  has factors 2, 3,  $x$ , and  $(5 + x - 3y)$ . The negative sign in front can be treated as a factor of  $-1$ . The factor  $5 + x - 3y$  has three terms, 5,  $x$ , and  $-3y$ . ■

■ In Chapter 11, we will see that roots are equivalent to noninteger exponents.

A **polynomial** is an algebraic expression with only nonnegative integer exponents on one or more variables, and has no variable in a denominator. The **degree of a term** is the sum of the exponents of the variables of that term, and the **degree of the polynomial** is the degree of the term of highest degree.

A **multinomial** is any algebraic expression of more than one term. Terms like  $1/x$  and  $\sqrt{x}$  can be included in a multinomial, but not in a polynomial. (Since  $1/x = x^{-1}$ , the exponent is negative.)

**EXAMPLE 3** Polynomials

Some examples of polynomials are as follows:

- (a)  $4x^2 - 5x + 3$  (degree 2)    (b)  $2x^6 - x$  (degree 6)    (c)  $3x$  (degree 1)  
 (d)  $xy^3 + 7x - 3$  (degree 4) (add exponents of  $x$  and  $y$ )  
 (e)  $-6$  (degree 0) ( $-6 = -6x^0$ )

From (c), note that a single term can be a polynomial, and from (e), note that a constant can be a polynomial. The expressions in (a), (b), and (d) are also multinomials.

The expression  $x^2 + \sqrt{y} + 2 - 8$  is a multinomial, but not a polynomial because of the square root term. ■

■ In practice, the numerical coefficient is usually called the coefficient. However, a more general definition is: any factor of a product is the coefficient of the other factors.

A polynomial with one term is called a **monomial**. A polynomial with two terms is called a **binomial**, and one with three terms is called a **trinomial**. The numerical factor is called the **numerical coefficient** (or simply **coefficient**) of the term. All terms that differ at most in their numerical coefficients are known as **similar** or **like** terms. That is, similar terms have the same variables with the same exponents.

**EXAMPLE 4** Monomial, binomial, trinomial

- (a)  $7x^4$  is a monomial. The numerical coefficient is 7.  
 (b)  $3ab - 6a$  is a binomial. The numerical coefficient of the first term is 3, and the numerical coefficient of the second term is  $-6$ . Note that the sign is attached to the coefficient.  
 (c)  $8cx^3 - x + 2$  is a trinomial. The coefficients of the first two terms are 8 and  $-1$ .  
 (d)  $x^2y^2 - 2x + 3y - 9$  is a polynomial with four terms (no special name). ■

**EXAMPLE 5** Similar terms

- (a)  $8b - 6ab + 81b$  is a trinomial. The first and third terms are similar since they differ only in their numerical coefficients. The middle term is not similar to the others since it has a factor of  $a$ .  
 (b)  $4x^2 - 3x$  is a binomial. The terms are not similar since the first term has two factors of  $x$ , and the second term has only one factor of  $x$ .  
 (c)  $3x^2y^3 - 5y^3x^2 + x^2 - 2y^3$  is a polynomial. The commutative law tells us that  $x^2y^3 = y^3x^2$ , which means the first two terms are similar. ■

**LEARNING TIP**

In adding and subtracting algebraic expressions, we combine similar terms into a single term. The **simplified** expression will contain only terms that are not similar.

**EXAMPLE 6** Simplifying expressions

- (a)  $3x + 2x - 5y = 5x - 5y$  Add similar terms—result has unlike terms
- (b)  $6a^2 - 7a + 8ax$  cannot be simplified since there are no like terms.
- (c)  $6a + 5c + 2a - c = 6a + 2a + 5c - c$  commutative law  
 $= 8a + 4c$  add like terms

■ Some calculators can display algebraic expressions and perform algebraic operations.

To group terms in an algebraic expression, we use **symbols of grouping**. In this text, we use **parentheses**,  $()$ , **brackets**,  $[\ ]$ , and **braces**,  $\{\}$ . The **bar**, which is used with radicals and fractions, also groups terms. In earlier sections, we used parentheses and the bar.

When adding and subtracting algebraic expressions, it may be necessary to remove symbols of grouping. To do so, we must *change the sign of every term within the symbols if the grouping is preceded by a minus sign. If the symbols of grouping are preceded by a plus sign, each term within the symbols retains its original sign.* This is a result of the distributive law,  $a(b + c) = ab + ac$ .

**COMMON ERROR**

Do not forget to multiply using the distributive law if there is a minus sign in front of a factor with multiple terms.

$$2b + 6a - (3a - 4b) = 2b + 6a - 3a + 4b = 3a + 6b$$

A common error is to forget to change the signs to all the terms in the factor.

$$2b + 6a - (3a - 4b) \neq 2b + 6a - 3a - 4b = 3a - 2b$$

**EXAMPLE 7** Symbols of grouping

- (a)  $2(a + 2x) = 2a + 2(2x)$  use distributive law  
 $= 2a + 4x$
- (b)  $- (+a - 3c) = (-1)(+a - 3c)$  treat  $-$  sign as  $-1$   
 $= (-1)(+a) + (-1)(-3c)$   
 $= -a + 3c$  note change of signs

Normally,  $+a$  would be written simply as  $a$ .

In Example 7(a), we see that  $2(a + 2x) = 2a + 4x$ . This is true for any values of  $a$  and  $x$ , and is therefore an *identity* (see Section 1.2). In fact, an expression and any proper form to which it may be changed form an identity when they are shown to be equal.

**EXAMPLE 8** Simplifying: signs before parentheses

- (a)  $3c + (2b - c) = 3c + 2b - c = 2b + 2c$  use distributive law  
+ sign before parentheses  
 $2b = +2b$  signs retained
- (b)  $3c - (2b - c) = 3c - 2b + c = -2b + 4c$  use distributive law  
- sign before parentheses  
 $2b = +2b$  signs changed
- (c)  $3c - (-2b + c) = 3c + 2b - c = 2b + 2c$  use distributive law  
signs changed

**Practice Exercises**

Use the distributive law.

1.  $3(2a + y)$     2.  $-3(-2r + s)$

**Practice Exercise**

3. Simplify:  $2x - 3(4y - x)$

■ Note in each case that the parentheses are removed and the sign before the parentheses is also removed.

**EXAMPLE 9** Simplifying in an application

In designing a certain machine part, it is necessary to perform the following simplification.

$$16(8 - x) - 2(8x - x^2) - (64 - 16x + x^2) = 128 - 16x - 16x + 2x^2 - 64 + 16x - x^2$$

$$= 64 - 16x + x^2$$

**LEARNING TIP**

At times, we have expressions in which more than one symbol of grouping is to be removed in the simplification. Normally, when several symbols of grouping are to be removed, it is more convenient to remove the innermost symbols first.

**EXAMPLE 10** Several symbols of grouping

$$(a) \quad 3ax - [ax - (5s - 2ax)] = 3ax - [ax - 5s + 2ax] \leftarrow \text{remove parentheses}$$

$$= 3ax - ax + 5s - 2ax \leftarrow \text{remove brackets}$$

$$= 5s$$

$$(b) \quad 3a^2b - \{[a - (2a^2b - a)] + 2b\} = 3a^2b - \{[a - 2a^2b + a] + 2b\} \leftarrow \text{remove parentheses}$$

$$= 3a^2b - \{a - 2a^2b + a + 2b\} \leftarrow \text{remove brackets}$$

$$= 3a^2b - a + 2a^2b - a - 2b$$

$$= 5a^2b - 2a - 2b \leftarrow \text{remove braces}$$

Calculators use only parentheses for grouping symbols, and we often need to use one set of parentheses within another set. These are called **nested parentheses**. In the next example, note that the innermost parentheses are removed first.

**EXAMPLE 11** Nested parentheses

$$2 - (3x - 2(5 - (7 - x))) = 2 - (3x - 2(5 - 7 + x))$$

$$= 2 - (3x - 10 + 14 - 2x)$$

$$= 2 - 3x + 10 - 14 + 2x$$

$$= -x - 2$$

**EXERCISES 1.7**

In Exercises 1–4, make the given changes in the indicated examples of this section, and then solve the resulting problems.

- In Example 6(a), change  $2x$  to  $2y$ .
- In Example 8(a), change the sign before  $(2b - c)$  from  $+$  to  $-$ .
- In Example 10(a), change  $[ax - (5s - 2ax)]$  to  $[(ax - 5s) - 2ax]$ .
- In Example 10(b), change  $\{[a - (2a^2b - a)] + 2b\}$  to  $\{a - [2a^2b - (a + 2b)]\}$ .

In Exercises 5–46, simplify the given algebraic expressions.

- $5x + 7x - 4x$
- $2y - y + 4x$
- $2F - 2T - 2 + 3F - T$
- $a^2b - a^2b^2 - 2a^2b$
- $s + (3s - 4 - s)$
- $v - (4 - 5x + 2v)$
- $2 - 3 - (4 - 5a)$
- $(a - 3) + (5 - 6a)$
- $-(t - 2u) + (3u - t)$
- $6t - 3t - 4t$
- $4C + L - 6C$
- $x - 2y + 3x - y + z$
- $xy^2 - 3x^2y^2 + 2xy^2$
- $5 + (3 - 4n + p)$
- $2a - (b - a)$
- $\sqrt{A} + (h - 2\sqrt{A}) - 3\sqrt{A}$
- $(4x - y) - (-2x - 4y)$
- $2(x - 2y) + (5x - y)$
- $3(2r + s) - (-5s - r)$
- $-7(6 - 3j) - 2(j + 4)$
- $-[(6 - n) - (2n - 3)]$
- $2[4 - (t^2 - 5)]$
- $-2[-x - 2a - (a - x)]$
- $aZ - [3 - (aZ + 4)]$
- $9v - [6 - (v - 4) + 4v]$
- $8c - \{5 - [2 - (3 + 4c)]\}$
- $7y - \{y - [2y - (x - y)]\}$
- $5p - (q - 2p) - [3q - (p - q)]$
- $-(4 - \sqrt{LC}) - [(5\sqrt{LC} - 7) - (6\sqrt{LC} + 2)]$
- $-2\{-(4 - x^2) - [3 + (4 - x^2)]\}$
- $-\{-[-(x - 2a) - b] - (a - x)\}$
- $5V^2 - (6 - (2V^2 + 3))$
- $-2F + 2((2F - 1) - 5)$
- $-(3t - (7 + 2t - (5t - 6)))$
- $a^2 - 2(x - 5 - (7 - 2(a^2 - 2x) - 3x))$
- $-4[4R - 2.5(Z - 2R) - 1.5(2R - Z)]$
- $3\{2.1e - 1.3[f - 2(e - 5f)]\}$
- $3(a - b) - 2(a - 2b)$
- $-(5t + a^2) - 2(3a^2 - 2st)$
- $-[(A - B) - (B - A)]$
- $3[-3 - (a - 4)]$
- $-2[-3(x - 2y) + 4y]$

47. In determining the size of a V belt to be used with an engine, the expression  $3D - (D - d)$  is used. Simplify this expression.
48. When finding the current in a transistor circuit, the expression  $i_1 - (2 - 3i_2) + i_2$  is used. Simplify this expression. (The numbers below the  $i$ 's are *subscripts*. Different subscripts denote different variables.)
49. Research on a plastic building material leads to  $[(B + \frac{4}{3}\alpha) + 2(B - \frac{2}{3}\alpha)] - [(B + \frac{4}{3}\alpha) - (B - \frac{2}{3}\alpha)]$ . Simplify this expression.
50. One car goes 30 km/h for  $t - 1$  hours, and a second car goes 40 km/h for  $t + 2$  hours. Find the expression for the sum of the distances travelled by the two cars.
51. A shipment contains  $x$  film cartridges for 15 exposures each and  $x + 10$  cartridges for 25 exposures each. What is the total number of photographs that can be taken with the film from this shipment?
52. Each of two stores has  $2n + 1$  mouse pads costing \$3 each and  $n - 2$  mouse pads costing \$2 each. How much more is the total

value of the \$3 mouse pads than the \$2 mouse pads in the two stores?

53. For the expressions  $2x^2 - y + 2a$  and  $3y - x^2 - b$  find (a) the sum, and (b) the difference if the second is subtracted from the first.
54. For the following expressions, subtract the third from the sum of the first two:  $3a^2 + b - c^3$ ,  $2c^3 - 2b - a^2$ ,  $4c^3 - 4b + 3$ .

In Exercises 55 and 56, answer the given questions.

55. For any real numbers  $a$  and  $b$ , is it true that  $|a - b| = |b - a|$ ? Explain.
56. Is subtraction associative? That is, in general, does  $(a - b) - c$  equal  $a - (b - c)$ ? Explain.

Answers to Practice Exercises

1.  $6a + 3y$    2.  $6r - 3s$    3.  $5x - 12y$

## 1.8 Multiplication of Algebraic Expressions

Multiplying Monomials • Products of Monomials and Polynomials • Powers of Polynomials

To find the product of two or more monomials, we use the laws of exponents as given in Section 1.4 and the laws for multiplying signed numbers as stated in Section 1.2. We first multiply the numerical coefficients to find the numerical coefficient of the product. Then we multiply the literal numbers, remembering that *the exponents may be combined only if the base is the same*.

### EXAMPLE 1 Multiplying monomials

(a)  $3c^5(-4c^2) = -12c^7$    multiply numerical coefficients and add exponents of  $c$

(b)  $(-2b^2y^3)(-9aby^5) = 18ab^3y^8$    add exponents of same base

(c)  $2xy(-6cx^2)(3xcy^2) = -36c^2x^4y^3$    ■

If a product contains a monomial that is raised to a power, we must first raise it to the indicated power before proceeding with the multiplication.

### EXAMPLE 2 Product containing a power of a monomial

(a)  $3(2a^2x)^3(-ax) = 3(8a^6x^3)(-ax) = -24a^7x^4$

(b)  $2s^3(-st^4)^2(4s^2t) = 2s^3(s^2t^8)(4s^2t) = 8s^7t^9$    ■

We find the product of a monomial and a polynomial by using the distributive law, which states that we *multiply each term of the polynomial by the monomial*. In doing so, we must be careful to give the correct sign to each term of the product.

### EXAMPLE 3 Product of a monomial and a polynomial

(a)  $2ax(3ax^2 - 4yz) = 2ax(3ax^2) + (2ax)(-4yz) = 6a^2x^3 - 8axyz$

(b)  $5cy^2(-7cx - ac) = (5cy^2)(-7cx) + (5cy^2)(-ac) = -35c^2xy^2 - 5ac^2y^2$    ■

#### Practice Exercises

Perform the indicated multiplications.

1.  $2a^3b(-6ab^2)$    2.  $-5x^2y^3(2xy - y^4)$

It is generally not necessary to write out the middle step as it appears in the preceding example. We write the answer directly. For instance, Example 3(a) would appear as  $2ax(3ax^2 - 4yz) = 6a^2x^3 - 8axyz$ .

We find the product of two polynomials by using the distributive law. The result is that we multiply each term of one polynomial by each term of the other and add the results. Again we must be careful to give each term of the product its correct sign.

#### EXAMPLE 4 Product of polynomials

$$\begin{aligned}(x - 2)(x + 3) &= x(x) + x(3) + (-2)(x) + (-2)(3) \\ &= x^2 + 3x - 2x - 6 \\ &= x^2 + x - 6\end{aligned}$$

■ Note that, using the distributive law,  $(x - 2)(x + 3) = (x - 2)(x) + (x - 2)(3)$  leads to the same result.

Finding the power of a polynomial is equivalent to using the polynomial as a factor the number of times indicated by the exponent. It is sometimes convenient to write the power of a polynomial in this form before multiplying.

#### EXAMPLE 5 Power of a polynomial

$$\begin{aligned}\text{(a)} \quad (x + 5)^2 &= (x + 5)(x + 5) = x^2 + 5x + 5x + 25 \\ &= x^2 + 10x + 25 \\ \text{(b)} \quad (2a - b)^3 &= (2a - b)(2a - b)(2a - b) \quad \text{the exponent 3 indicates three factors} \\ &= (2a - b)(4a^2 - 2ab - 2ab + b^2) \\ &= (2a - b)(4a^2 - 4ab + b^2) \\ &= 8a^3 - 8a^2b + 2ab^2 - 4a^2b + 4ab^2 - b^3 \\ &= 8a^3 - 12a^2b + 6ab^2 - b^3\end{aligned}$$

#### Practice Exercises

Perform the indicated multiplications.

3.  $(2s - 5t)(s + 4t)$     4.  $(3u + 2v)^2$

#### COMMON ERROR

$$(x + 5)^2 \neq x^2 + 25$$

A common error is to forget to include the middle term of  $10x$ . By following proper procedure, you must expand the relation as

$$\begin{aligned}(x + 5)^2 &= (x + 5)(x + 5) \\ &= x \cdot x + x \cdot 5 + 5 \cdot x + 5 \cdot 5 \\ &= x^2 + 10x + 25\end{aligned}$$

Thus, you cannot simply square each of the terms within the parentheses.

#### EXAMPLE 6 Products in an application

An expression used with a lens of a certain telescope is simplified as shown.

$$\begin{aligned}a(a + b)^2 + a^3 - (a + b)(2a^2 - s^2) \\ &= a(a + b)(a + b) + a^3 - (2a^3 - as^2 + 2a^2b - bs^2) \\ &= a(a^2 + ab + ab + b^2) + a^3 - 2a^3 + as^2 - 2a^2b + bs^2 \\ &= a^3 + a^2b + a^2b + ab^2 - a^3 + as^2 - 2a^2b + bs^2 \\ &= ab^2 + as^2 + bs^2\end{aligned}$$



## EXERCISES 1.8

In Exercises 1–4, make the given changes in the indicated examples of this section, and then solve the resulting problems.

- In Example 2(b), change the factor  $(-st^4)^2$  to  $(-st^4)^3$ .
- In Example 3(a), change the factor  $2ax$  to  $-2ax$ .
- In Example 4, change the factor  $(x + 3)$  to  $(x - 3)$ .
- In Example 5(b), change the exponent 3 to 2.

In Exercises 5–50, perform the indicated multiplications.

- $(a^2)(ax)$
  - $(2xy)(x^2y^3)$
  - $-ac^2(acx^3)$
  - $-2cs^2(-4cs)^2$
  - $(2ax^2)^2(-2ax)$
  - $6pq^3(3pq^2)^2$
  - $i^2(R + 2r)$
  - $2x(p - q)$
  - $-3s(s^2 - 5t)$
  - $-3b(2b^2 - b)$
  - $5m(m^2n + 3mn)$
  - $a^2bc(2ac - 3b^2c)$
  - $3M(-M - N + 2)$
  - $-4e^2(-9gc - 2c + g^2)$
  - $ax(cx^2)(x + y^3)$
  - $-2(-3st^3)(3s - 4t)$
  - $(x - 3)(x + 5)$
  - $(a + 7)(a + 1)$
  - $(x + 5)(2x - 1)$
  - $(4t_1 + t_2)(2t_1 - 3t_2)$
  - $(2a - b)(3a - 2b)$
  - $(4w^2 - 3)(3w^2 - 1)$
  - $(2s + 7t)(3s - 5t)$
  - $(5p - 2q)(p + 8q)$
  - $(x^2 - 1)(2x + 5)$
  - $(3y^2 + 2)(2y - 9)$
  - $(x - 2y - 4)(x - 2y + 4)$
  - $(2a + 3b + 1)(2a + 3b - 1)$
  - $2(a + 1)(a - 9)$
  - $-5(y - 3)(y + 6)$
  - $-3(3 - 2T)(3T + 2)$
  - $2n(5 - n)(6n + 5)$
  - $2L(L + 1)(4 - L)$
  - $ax(x + 4)(7 - x^2)$
  - $(2x - 5)^2$
  - $(x - 3y)^2$
  - $(x_1 + 3x_2)^2$
  - $(7m + 1)^2$
  - $(xyz - 2)^2$
  - $(b - 6x^2)^2$
  - $2(x + 8)^2$
  - $3(3R + 4)^2$
  - $(2 + x)(3 - x)(x - 1)$
  - $(3x - c^2)^3$
  - $3T(T + 2)(2T - 1)$
  - $[(x - 2)^2(x + 2)]^2$
51. Let  $x = 3$  and  $y = 4$  to show that (a)  $(x + y)^2 \neq x^2 + y^2$  and (b)  $(x - y)^2 \neq x^2 - y^2$ . ( $\neq$  means “does not equal.”)
52. Evaluate the product  $(98)(102)$  by expressing it as  $(100 - 2)(100 + 2)$ .

- Square an integer between 1 and 9 and subtract 1 from the result. Explain why the result is the product of the integer before and the integer after the one you chose.
- Explain how, by appropriate grouping, the product  $(x - 2)(x + 3)(x + 2)(x - 3)$  is easier to find. Find the product.
- By multiplication, show that  $(x + y)^3$  is not equal to  $x^3 + y^3$ .
- By multiplication, show that  $(x + y)(x^2 - xy + y^2) = x^3 + y^3$ .
- In finding the value of a certain savings account, the expression  $P(1 + 0.01r)^2$  is used. Multiply out this expression.
- The force due to the liquid in a parabolic container leads to the expression  $w(1 - x)(4 - x^2)$ . Multiply out this expression.
- In using aircraft radar, the expression  $(2R - X)^2 - (R^2 + X^2)$  arises. Simplify this expression.
- In calculating the temperature variation of an industrial area, the expression  $(2T^3 + 3)(T^2 - T - 3)$  arises. Perform the indicated multiplication.
- In a particular computer design containing  $n$  circuit elements,  $n^2$  switches are needed. Find the expression for the number of switches needed for  $n + 100$  circuit elements.
- Simplify the expression  $(T^2 - 100)(T - 10)(T + 10)$ , which arises when analysing the energy radiation from an object.
- In finding the maximum power in part of a microwave transmitter circuit, the expression  $(R_1 + R_2)^2 - 2R_2(R_1 + R_2)$  is used. Multiply and simplify.
- In determining the deflection of a certain steel beam, the expression  $27x^2 - 24(x - 6)^2 - (x - 12)^3$  is used. Multiply and simplify.

### Answers to Practice Exercises

- $-12a^4b^3$
- $-10x^3y^4 + 5x^2y^7$
- $2s^2 + 3st - 20t^2$
- $9u^2 + 12uv + 4v^2$

## 1.9 Division of Algebraic Expressions

**Dividing Monomials • Dividing by a Monomial • Dividing One Polynomial by Another**

To find the quotient of one monomial divided by another, we use the laws of exponents and the laws for dividing signed numbers. Again, the *exponents may be combined only if the base is the same*.

### EXAMPLE 1 Dividing monomials

$$(a) \frac{3c^7}{c^2} = 3c^{7-2} = 3c^5$$

$$(c) \frac{-6a^2xy^2}{2axy^4} = -\left(\frac{6}{2}\right)\frac{a^{2-1}x^{1-1}}{y^{4-2}} = -\frac{3a}{y^2}$$

$$(b) \frac{16x^3y^5}{4xy^2} = \frac{16}{4}(x^{3-1})(y^{5-2}) = 4x^2y^3$$

divide
↑
subtract
↑

coefficients
↑
exponents
↑

As shown in illustration (c), we use only positive exponents in the final result unless there are specific instructions otherwise. ■

From arithmetic, we may show how a multinomial is to be divided by a monomial.

When adding fractions (say  $\frac{2}{7}$  and  $\frac{3}{7}$ ), we have  $\frac{2}{7} + \frac{3}{7} = \frac{2+3}{7}$ .

Looking at this from *right to left*, we see that *the quotient of a multinomial divided by a monomial is found by dividing each term of the multinomial by the monomial and adding the results*. This can be shown as

$$\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$$

■ This is an identity and is valid for all values of  $a$  and  $b$ , and all values of  $c$  except zero (which would make it undefined).

### COMMON ERROR

Be careful: Although  $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$ , we must note that  $\frac{c}{a+b}$  is *not*  $\frac{c}{a} + \frac{c}{b}$ .

### EXAMPLE 2 Dividing by a monomial

$$(a) \frac{4a^2 + 8a}{2a} = \frac{4a^2}{2a} + \frac{8a}{2a} = 2a + 4$$

$$(b) \frac{4x^3y - 8x^3y^2 + 2x^2y}{2x^2y} = \frac{4x^3y}{2x^2y} - \frac{8x^3y^2}{2x^2y} + \frac{2x^2y}{2x^2y} = 2x - 4xy + 1$$

each term of numerator divided by denominator

### Practice Exercise

Divide: 1.  $\frac{4ax^2 - 6a^2x}{2ax}$

■ Until you are familiar with the method, it is recommended that you do write out the middle step.

We usually do not write out the middle step as shown in these illustrations. The divisions of the terms of the numerator by the denominator are usually done by inspection (mentally), and the result is shown as it appears in the next example. ■

### EXAMPLE 3 Application of dividing by a monomial

The expression  $\frac{2p + v^2d + 2ydg}{2dg}$  is used when analysing the operation of an irrigation pump. Performing the indicated division, we have

$$\frac{2p + v^2d + 2ydg}{2dg} = \frac{p}{dg} + \frac{v^2}{2g} + y$$

## DIVISION OF ONE POLYNOMIAL BY ANOTHER

To divide one polynomial by another, use the following steps.

■ This is similar to long division of numbers.

### Algebraic Division

1. Arrange the dividend (the polynomial to be divided) and the divisor in descending powers of the variable.
2. Divide the first term of the dividend by the first term of the divisor. The result will be the first term in the quotient.
3. Multiply the entire divisor by the first term of the quotient and **subtract** this product from the dividend.
4. Divide the first term of this remainder by the first term of the divisor. This gives the second term of the quotient.
5. Multiply the entire divisor by the second term of the quotient and **subtract** this product from the remainder.
6. Repeat this process until the remainder is zero or until a term of lower degree than the divisor is found.
7. Write the quotient solution as the quotient terms obtained **plus** any remainder **divided by the divisor**.

**EXAMPLE 4** Dividing one polynomial by another

Perform the division  $(6x^2 + x - 2) \div (2x - 1)$ .

(This division can also be indicated in the fractional form  $\frac{6x^2 + x - 2}{2x - 1}$ .)

We set up the division as we would for long division in arithmetic. Then, following the procedure outlined above, we have the following:

$$\begin{array}{r}
 \phantom{2x - 1} \overline{) 6x^2 + x - 2} \\
 \underline{6x^2 - 3x} \phantom{- 2} \\
 4x - 2 \\
 \underline{4x - 2} \\
 0
 \end{array}$$

$3x + 2$   $\frac{6x^2}{2x}$  divide first term of dividend by first term of divisor

$3x(2x - 1)$

subtract  $6x^2 - 6x^2 = 0$

$x - (-3x) = 4x$

subtract

**Practice Exercise**

Divide: **2.**  $(6x^2 + 7x - 3) \div (3x - 1)$

The remainder is zero and the quotient is  $3x + 2$ . Note that when we subtracted  $-3x$  from  $x$ , we obtained  $4x$ . Therefore,  $\frac{6x^2 + x - 2}{2x - 1} = 3x + 2$ . ■

**EXAMPLE 5** Quotient with a remainder

Perform the division  $\frac{8x^3 + 4x^2 + 3}{4x^2 - 1}$ . Since there is no  $x$ -term in the dividend, we should leave space for any  $x$ -terms that might arise (which we will show as  $0x$ ).

$$\begin{array}{r}
 \phantom{4x^2 - 1} \overline{) 8x^3 + 4x^2 + 0x + 3} \\
 \underline{8x^3 - 2x} \phantom{+ 3} \\
 4x^2 + 2x + 3 \\
 \underline{4x^2 - 1} \\
 2x + 4
 \end{array}$$

$2x + 1$   $\frac{8x^3}{4x^2} = 2x$   $\frac{4x^2}{4x^2} = 1$

divisor dividend subtract

subtract remainder

Since the degree of the remainder  $2x + 4$  is less than that of the divisor, we now show the result in this case as  $\frac{8x^3 + 4x^2 + 3}{4x^2 - 1} = 2x + 1 + \frac{2x + 4}{4x^2 - 1}$ . ■

**EXERCISES 1.9**

In Exercises 1–4, make the given changes in the indicated examples of this section and then perform the indicated divisions.

1. In Example 1(c), change the denominator to  $-2a^2xy^5$ .
2. In Example 2(b), change the denominator to  $2xy^2$ .
3. In Example 4, change the dividend to  $6x^2 - 7x + 2$ .
4. In Example 5, change the sign of the middle term of the numerator from  $+$  to  $-$ .

In Exercises 5–24, perform the indicated divisions.

- |  |  |                               |                              |
|--|--|-------------------------------|------------------------------|
| 5. $\frac{8x^3y^2}{-2xy}$                    | 6. $\frac{-18b^7c^3}{bc^2}$                    | 7. $\frac{-16r^3t^5}{-4r^5t}$ | 8. $\frac{51mn^5}{17m^2n^2}$ |
| 9. $\frac{(15x^2)(4bx)(2y)}{30bxy}$          | 10. $\frac{(5sT)(8s^2T^3)}{10s^3T^2}$          | 11. $\frac{6(ax)^2}{-ax^2}$   |                              |
| 12. $\frac{12a^2b}{(3ab^2)^2}$               | 13. $\frac{3a^2x + 6xy}{3x}$                   | 14. $\frac{2m^2n - 6mn}{2m}$  |                              |
| 15. $\frac{3rst - 6r^2st^2}{3rs}$            | 16. $\frac{-5a^2n - 10an^2}{5an}$              |                               |                              |
| 17. $\frac{4pq^3 + 8p^2q^2 - 16pq^5}{4pq^2}$ | 18. $\frac{a^2x_1x_2^2 + ax_1^3 - ax_1}{ax_1}$ |                               |                              |
| 19. $\frac{2\pi fL - \pi fR^2}{\pi fR}$      | 20. $\frac{9(aB)^4 - 6aB^4}{3aB^3}$            |                               |                              |

21. 
$$\frac{3ab^2 - 6ab^3 + 9a^2b^2}{9a^2b^2}$$

22. 
$$\frac{2x^{n+2} + 4ax^n}{2x^n}$$

23. 
$$\frac{6y^{2n} - 4ay^{n+1}}{2y^n}$$

24. 
$$\frac{3a(F + T)b^2 - (F + T)}{a(F + T)}$$

In Exercises 25–42, perform the indicated divisions. Express the answer as shown in Example 5 when applicable.

25.  $(2x^2 + 7x + 3) \div (x + 3)$

26.  $(3t^2 - 7t + 4) \div (t - 1)$

27.  $(x^2 - 3x + 2) \div (x - 2)$

28.  $(2x^2 - 5x - 7) \div (x + 1)$

29.  $(x - 14x^2 + 8x^3) \div (2x - 3)$

30.  $(6 + 7y + 6y^2) \div (2y + 1)$

31.  $(4Z^2 - 5Z - 7) \div (4Z + 3)$

32.  $(6x^2 - 5x - 9) \div (3x - 4)$

33. 
$$\frac{x^3 + 3x^2 - 4x - 12}{x + 2}$$

34. 
$$\frac{3x^3 + 19x^2 + 13x - 20}{3x - 2}$$

35. 
$$\frac{2a^4 + 4a^2 - 16}{a^2 - 2}$$

36. 
$$\frac{6T^3 + T^2 + 2}{3T^2 - T + 2}$$

37. 
$$\frac{x^3 + 8}{x + 2}$$

38. 
$$\frac{D^3 - 1}{D - 1}$$

39. 
$$\frac{x^2 - 2xy + y^2}{x - y}$$

40. 
$$\frac{3r^2 - 5rR + 2R^2}{r - 3R}$$

41. 
$$\frac{x^2 - y^2 + 2yz - z^2}{x + y - z}$$

42. 
$$\frac{a^4 + b^4}{a^2 - 2ab + 2b^2}$$

In Exercises 43–52, perform the indicated divisions.

43. When  $2x^2 - 9x - 5$  is divided by  $x + c$ , the quotient is  $2x + 1$ . Find  $c$ .

44. When  $6x^2 - x + k$  is divided by  $3x + 4$ , the remainder is zero. Find  $k$ .

45. By division show that  $\frac{x^4 + 1}{x + 1}$  is not equal to  $x^3$ .

46. By division show that  $\frac{x^3 + y^3}{x + y}$  is not equal to  $x^2 + y^2$ .

47. In the optical theory dealing with lasers, the following expression arises: 
$$\frac{8A^5 + 4A^3\mu^2E^2 - A\mu^4E^4}{8A^4}$$
. ( $\mu$  is the Greek letter mu.)

Perform the division.

48. In finding the total resistance of the resistors shown in Fig. 1.14, the following expression is used.

$$\frac{6R_1 + 6R_2 + R_1R_2}{6R_1R_2}$$

Perform the division.

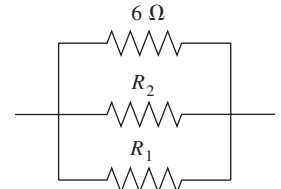


Fig. 1.14

49. When analysing the potential energy associated with gravitational forces, the expression 
$$\frac{GMm[(R + r) - (R - r)]}{2rR}$$
 arises. Perform the indicated division.

50. A computer model shows that the temperature change  $T$  in a certain freezing unit is found by using the expression 
$$\frac{3T^3 - 8T^2 + 8}{T - 2}$$
. Perform the indicated division.

51. In analysing the displacement of a certain valve, the expression 
$$\frac{s^2 - 2s - 2}{s^4 + 4}$$
 is used. Find the reciprocal of this expression and then perform the indicated division.

52. The voltage and resistance in a certain electric circuit vary with time such that the current is given by the expression 
$$\frac{2t^3 + 94t^2 - 290t + 500}{2t + 100}$$
. By performing the indicated division, find the expression for the current.

### Answers to Practice Exercises

1.  $2x - 3a$     2.  $2x + 3$

## 1.10 Solving Equations

Types of Equations • Solving Basic Types of Equations • Checking the Solution • First Steps • Ratio and Proportion

In this section, we show how algebraic operations are used in solving equations. In the following sections, we show some of the important applications of equations.

An **equation** is an algebraic statement that two algebraic expressions are equal. Any value of the literal number representing the **unknown** that produces equality when **substituted** in the equation is said to **satisfy** the equation.

### EXAMPLE 1 Valid values for equations

The equation  $3x - 5 = x + 1$  is true only if  $x = 3$ . Substituting 3 for  $x$  in the equation, we have  $3(3) - 5 = 3 + 1$ , or  $4 = 4$ ; substituting  $x = 2$ , we have  $1 = 3$ , which is not correct.

This equation is valid for only one value of the unknown. An *equation valid only for certain values of the unknown* is a **conditional equation**. In this section, nearly all equations we solve will be conditional equations that are satisfied by only one value of the unknown. ■

■ We have previously noted *identities* on pages 6, 7 and 30.

■ Equations can be solved on most graphing calculators. An estimate (or guess) of the answer may be required to find the solution. See Exercises 43 and 44.

■ Although we may multiply both sides of an equation by zero, this produces  $0 = 0$ , which is not useful in finding the solution.

■ The word *algebra* comes from Arabic and means “a restoration.” It refers to the fact that when a number has been added to one side of an equation, the same number must be added to the other side to maintain equality.

#### LEARNING TIP

The solution to any equation can be checked by substitution into the original equation.

#### EXAMPLE 2 Identity and contradiction

- (a) The equation  $x^2 - 4 = (x - 2)(x + 2)$  is true for all values of  $x$ . For example, substituting  $x = 3$  in the equation, we have  $3^2 - 4 = (3 - 2)(3 + 2)$ , or  $5 = 5$ . Substituting  $x = -1$ , we have  $(-1)^2 - 4 = (-1 - 2)(-1 + 2)$ , or  $-3 = -3$ . An equation valid for all values of the unknown is an **identity**.
- (b) The equation  $x + 5 = x + 1$  is not true for any value of  $x$ . For any value of  $x$  we try, we find that the left side is 4 greater than the right side. Such an equation is called a **contradiction**. ■

To solve an equation, we find the values of the unknown that satisfy it. There is one basic rule to follow when solving an equation:

*Perform the same operation on both sides of the equation.*

We do this to isolate the unknown and thus to find its value.

By performing the same operation on both sides of an equation, the two sides remain equal. Thus,

*we may add the same number to both sides, subtract the same number from both sides, multiply both sides by the same number, or divide both sides by the same number (not zero).*

#### EXAMPLE 3 Basic operations used in solving

In each of the following equations, we may isolate  $x$ , and thereby solve the equation, by performing the indicated operation.

$x - 3 = 12$ <i>add 3 to both sides</i>	$x + 3 = 12$ <i>subtract 3 from both sides</i>	$\frac{x}{3} = 12$ <i>multiply both sides by 3</i>	$3x = 12$ <i>divide both sides by 3</i>
$x - 3 + 3 = 12 + 3$ $x = 15$	$x + 3 - 3 = 12 - 3$ $x = 9$	$3\left(\frac{x}{3}\right) = 3(12)$ $x = 36$	$\frac{3x}{3} = \frac{12}{3}$ $x = 4$ ■

#### Procedure for Solving Equations

1. Remove grouping symbols (*distributive law*).
2. Combine any like terms on each side.
3. Perform the same operations on both sides to simplify, with the goal of isolating  $x$ .
4. Repeat Steps 1–3 until  $x =$  result is obtained.
5. Check the solution in the original equation.

#### EXAMPLE 4 Operations used for solution; checking

Solve the equation  $2t - 7 = 9$ .

We are to perform basic operations to both sides of the equation to finally isolate  $t$  on one side. The steps to be followed are suggested by the form of the equation.

$2t - 7 = 9$	<i>original equation</i>
$2t - 7 + 7 = 9 + 7$	<i>add 7 to both sides</i>
$2t = 16$	<i>combine like terms</i>
$\frac{2t}{2} = \frac{16}{2}$	<i>divide both sides by 2</i>
$t = 8$	<i>simplify</i>

■ Note that the solution generally requires a combination of basic operations.

Therefore, we conclude that  $t = 8$ . Checking *in the original equation*, we have

$$\begin{aligned} 2t - 7 &= 9 \\ 2(8) - 7 &= 9 \\ 16 - 7 &= 9 \\ 9 &= 9 \end{aligned}$$

The solution checks. ■

### EXAMPLE 5 First remove parentheses

Solve the equation  $x - 7 = 3x - (6x - 8)$ .

$$\begin{aligned} x - 7 &= 3x - 6x + 8 && \text{parentheses removed} \\ x - 7 &= -3x + 8 && \text{x-terms combined on right} \\ 4x - 7 &= 8 && \text{3x added to both sides \quad -by inspection} \\ 4x &= 15 && \text{7 added to both sides \quad -by inspection} \\ x &= \frac{15}{4} && \text{both sides divided by 4 \quad -by inspection} \end{aligned}$$

Checking in the original equation, we obtain (after simplifying)  $-\frac{13}{4} = -\frac{13}{4}$ . ■

Note that we *always check in the original equation*. This is done since errors may have been made in finding the later equations.

If an equation contains numbers not easily combined by inspection, the best procedure is to *first solve for the unknown algebraically and then perform the calculation*.

### EXAMPLE 6 First solve for unknown and then calculate

When finding the current  $i$  (in A) in a certain radio circuit, the following equation and solution are used.

$$\begin{aligned} 0.0595 - 0.525i - 8.85(i + 0.00316) &= 0 \\ 0.0595 - 0.525i - 8.85i - 8.85(0.00316) &= 0 && \text{note how the above procedure is followed} \\ (-0.525 - 8.85)i &= 8.85(0.00316) - 0.0595 \\ i &= \frac{8.85(0.00316) - 0.0595}{-0.525 - 8.85} && \text{evaluate} \\ &= 0.00336 \text{ A} \end{aligned}$$

When doing the calculation indicated above in the solution for  $i$ , be careful to group the numbers in the denominator for the division. Also, be sure to round off the result as shown above, but do not round off values before the final calculation. ■

The quotient  $a/b$  is also called the **ratio** of  $a$  to  $b$ . An equation stating that two ratios are equal is called a **proportion**. Since a proportion is an equation, if one of the numbers is unknown, we can solve for its value as with any equation. Usually, this is done by noting the denominators and multiplying each side by a number that will clear the fractions.

### EXAMPLE 7 Ratio

If the ratio of  $x$  to 8 equals the ratio of 3 to 4, we have the proportion

$$\frac{x}{8} = \frac{3}{4}$$

We can solve this equation by multiplying both sides by 8. This gives

$$\begin{aligned} 8\left(\frac{x}{8}\right) &= 8\left(\frac{3}{4}\right) \\ x &= \frac{24}{4} = 6 \end{aligned}$$

■ With simpler numbers, many basic steps are done by inspection and not actually written down.

#### Practice Exercises

Solve for  $x$ .

- $3x + 4 = x - 6$
- $2(5 - x) = x - 8$

■ Many other types of equations require more advanced methods for solving. These are considered in later chapters.

#### Practice Exercise

- If the ratio of 2 to 5 equals the ratio of  $x$  to 30, find  $x$ .



Substituting  $x = 6$  into the original proportion gives the proportion  $\frac{6}{8} = \frac{3}{4}$ . Since these ratios are equal, the solution checks. ■

### EXAMPLE 8 Applied proportion

The ratio of electric current  $I$  (in A) to the voltage  $V$  across a resistor is constant. If  $I = 1.52$  A for  $V = 60.0$  V, find  $I$  for  $V = 82.0$  V.

Since the ratio of  $I$  and  $V$  is constant, we have the following solution.

$$\begin{aligned}\frac{1.52 \text{ A}}{60.0 \text{ V}} &= \frac{I}{82.0 \text{ V}} \\ 82.0 \text{ V} \left( \frac{1.52 \text{ A}}{60.0 \text{ V}} \right) &= 82.0 \text{ V} \left( \frac{I}{82.0 \text{ V}} \right) \\ 2.08 \text{ A} &= I \\ I &= 2.08 \text{ A}\end{aligned}$$

■ Generally, units of measurement will not be shown in intermediate steps. The proper units will be shown with the data and final result.

Checking in the original equation, we have  $0.0253 \text{ A/V} = 0.0254 \text{ A/V}$ . Since the value of  $I$  is rounded up, the solution checks. ■

We will use the terms *ratio* and *proportion* (particularly ratio) when studying trigonometry in Chapter 4. A detailed discussion of ratio and proportion is found in Chapter 18. A general method of solving equations involving fractions is given in Chapter 6.

## EXERCISES 1.10

In Exercises 1–4, make the given changes in the indicated examples of this section and then solve the resulting problems.

- In Example 3, change 12 to  $-12$  in each of the four illustrations and then solve.
- In Example 4, change  $2t - 7$  to  $7 - 2t$  and then solve.
- In Example 5, change  $(6x - 8)$  to  $(8 - 6x)$  and then solve.
- In Example 8, for the same given values of  $I$  and  $V$ , find  $I$  for  $V = 48.0$  V.

In Exercises 5–40, solve the given equations.

- $x - 2 = 7$
- $x - 4 = -1$
- $x + 5 = 4$
- $s + 6 = -3$
- $\frac{t}{2} = -5$
- $\frac{x}{-4} = 2$
- $4E = -20$
- $2x = 12$
- $3t + 5 = -4$
- $5D - 2 = 13$
- $5 - 2y = -3$
- $8 - 5t = 18$
- $3x + 7 = x$
- $6 + 4L = 5 - 3L$
- $2(s - 4) = s$
- $3(4 - n) = -n$
- $6 - (r - 4) = 2r$
- $5 - (x + 2) = 5x$
- $2(x - 3) = -x$
- $4(7 - F) = -7$
- $0.1x - 0.5(x - 2) = 2$
- $1.5x - 0.3(x - 4) = 6$
- $7 - 3(1 - 2p) = 4 + 2p$
- $3 - 6(2 - 3t) = t - 5$
- $\frac{4x - 2(x - 4)}{3} = 8$
- $2x = \frac{3 - 5(7 - 3x)}{4}$
- $|x| - 1 = 8$
- $2 - |x| = 4$

In Exercises 33–40, all numbers are approximate.

- $5.8 - 0.3(x - 6.0) = 0.5x$
- $1.9t = 0.5(4.0 - t) - 0.8$

- $-0.24(C - 0.50) = 0.63$
- $27.5(5.17 - 1.44x) = 73.4$
- $\frac{x}{2.0} = \frac{17}{6.0}$
- $\frac{3.0}{7.0} = \frac{R}{42}$
- $\frac{165}{223} = \frac{13V}{15}$
- $\frac{276x}{17.0} = \frac{1360}{46.4}$

In Exercises 41–52, solve the given problems.

- Identify each of the following equations as a conditional equation, an identity, or a contradiction.  
(a)  $2x + 3 = 3 + 2x$       (b)  $2x - 3 = 3 - 2x$
- For what values of  $a$  is the equation  $2x + a = 2x$  a conditional equation? Explain.
- Solve the equation of Example 5 by using the Equation Solver of a graphing calculator.
- Solve the equation of Example 6 by using the Equation Solver of a graphing calculator.
- In finding the speed  $v$  (in km/h) at which a certain person was travelling, the equation  $2.0v + 40 = 2.5(v + 5.0)$  is to be solved. Find the speed.
- In finding the rate  $v$  (in km/h) at which a polluted stream is flowing, the equation  $15(5.5 + v) = 24(5.5 - v)$  is used. Find  $v$ .
- In finding the maximum operating temperature  $T$  (in  $^{\circ}\text{C}$ ) for a computer integrated circuit, the equation  $1.1 = (T - 76)/40$  is used. Find the temperature.
- To find the voltage  $V$  in a circuit in a TV remote-control unit, the equation  $1.12V - 0.67(10.5 - V) = 0$  is used. Find  $V$ .
- In blending two gasolines of different octanes, in order to find the number  $n$  of litres of one octane needed, the equation

$0.14n + 0.06(2000 - n) = 0.09(2000)$  is used. Find  $n$ , given that 0.06 and 0.09 are exact and the first zero of 2000 is significant.

50. In order to find the distance  $x$  such that the weights are balanced on the lever shown in Fig. 1.15, the equation  $215(3x) = 55.3x + 38.5(8.25 - 3x)$  must be solved. Find  $x$ . (3 is exact.)

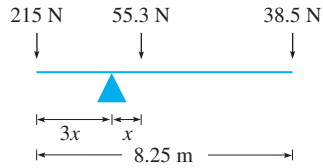


Fig. 1.15

51. The manufacturer of a certain car powered partly by gas and partly by batteries (a hybrid car) claims it can go 2050 km on one full tank of 55 L. How far can the car go on 22 L?
52. A person 1.8 m tall is photographed with a 35-mm camera, and the film image is 20 mm. Under the same conditions, how tall is a person whose film image is 16 mm?

Answers to Practice Exercises

1. -5   2. 6   3. 12

## 1.11 Formulas and Literal Equations

**Formulas • Literal Equations • Subscripts • Solve for Symbol before Substituting Numerical Values**

■ Einstein published his first paper on relativity in 1905.

### LEARNING TIP

We can solve a formula for a particular symbol just as we solve any equation. That is, we *isolate the required symbol by using algebraic operations on literal numbers*.

■ The subscript  $_0$  makes  $v_0$  a different literal symbol from  $v$ . (We have used subscripts in a few of the earlier exercises.)

An important application of equations is in the use of *formulas* that are found in geometry and nearly all fields of science and technology. A **formula** is an equation that expresses the relationship between two or more related quantities. For example, Einstein’s famous formula  $E = mc^2$  shows the equivalence of energy  $E$  to the mass  $m$  of an object and the speed of light squared,  $c^2$ .

### EXAMPLE 1 Solving for a symbol in a formula

In Einstein’s formula  $E = mc^2$ , solve for  $m$ .

$$\frac{E}{c^2} = m \quad \text{divide both sides by } c^2$$

$$m = \frac{E}{c^2} \quad \text{switch sides to place } m \text{ at left}$$

The required symbol is usually placed on the left, as shown. ■

### EXAMPLE 2 Symbol with subscript in a formula

A formula relating acceleration  $a$ , velocity  $v$ , initial velocity  $v_0$ , and time  $t$  is  $v = v_0 + at$ . Solve for  $t$ .

$$v - v_0 = at \quad v_0 \text{ subtracted from both sides}$$

$$t = \frac{v - v_0}{a} \quad \text{both sides divided by } a \text{ and then sides switched} \quad \blacksquare$$

### EXAMPLE 3 Symbol in capital and in lowercase

In the study of the forces on a certain beam, the equation  $W = \frac{L(wL + 2P)}{8}$  is used. Solve for  $P$ .

$$8W = \frac{8L(wL + 2P)}{8} \quad \text{multiply both sides by 8}$$

$$8W = L(wL + 2P) \quad \text{simplify right side}$$

$$8W = wL^2 + 2LP \quad \text{remove parentheses}$$

$$8W - wL^2 = 2LP \quad \text{subtract } wL^2 \text{ from both sides}$$

$$P = \frac{8W - wL^2}{2L} \quad \text{divide both sides by } 2L \text{ and switch sides} \quad \blacksquare$$

**COMMON ERROR**

Be careful! Just as subscripts can indicate different literal numbers, a capital letter and the same letter in lowercase are different literal numbers. In Example 3,  $W$  and  $w$  are different. This is shown in several of the exercises for this section.

**EXAMPLE 4 Formula with groupings**

The effect of temperature upon measurements is important when measurements must be made with great accuracy. The volume  $V$  of a special precision container at temperature  $T$  in terms of the volume  $V_0$  at temperature  $T_0$  is given by  $V = V_0[1 + b(T - T_0)]$ , where  $b$  depends on the material of which the container is made. Solve for  $T$ .

Since we are to solve for  $T$ , we must isolate the term containing  $T$ . This can be done by first removing the grouping symbols.

$$V = V_0[1 + b(T - T_0)] \quad \text{original equation}$$

$$V = V_0[1 + bT - bT_0] \quad \text{remove parentheses}$$

$$V = V_0 + bTV_0 - bT_0V_0 \quad \text{remove brackets}$$

$$V - V_0 + bT_0V_0 = bTV_0 \quad \text{subtract } V_0 \text{ and add } bT_0V_0 \text{ to both sides}$$

$$T = \frac{V - V_0 + bT_0V_0}{bV_0} \quad \text{divide both sides by } bV_0 \text{ and switch sides} \quad \blacksquare$$

**Practice Exercises**

Solve for the indicated letter. Each comes from the indicated area of study.

1.  $\theta = kA + \lambda$ , for  $\lambda$  (robotics)
2.  $P = n(p - c)$ , for  $p$  (economics)

**LEARNING TIP**

To determine the values of any literal number in an expression for which we know values of the other literal numbers, we should *first solve for the required symbol and then evaluate*. It is also a good engineering practice to solve equations algebraically before substituting in variable values for two reasons:

- (1) it generates a general formula, and
- (2) algebra is easier to troubleshoot than numbers.

**EXAMPLE 5 Solve for a symbol before substituting**

The electric resistance  $R$  (in  $\Omega$ ) of a resistor changes with the temperature  $T$  (in  $^\circ\text{C}$ ) according to  $R = R_0 + R_0\alpha T$ , where  $R_0$  is the resistance at  $0^\circ\text{C}$ . For a given resistor,  $R_0 = 712 \Omega$  and  $\alpha = 0.00455/^\circ\text{C}$ . Find the value of  $T$  for  $R = 825 \Omega$ .

We first solve for  $T$  and then substitute the given values.

$$R = R_0 + R_0\alpha T$$

$$R - R_0 = R_0\alpha T$$

$$T = \frac{R - R_0}{\alpha R_0}$$

Now substituting, we have

$$T = \frac{825 - 712}{(0.00455)(712)} \quad \text{estimation:}$$

$$= 34.9^\circ\text{C} \quad \text{rounded off} \quad \frac{800 - 700}{0.005(700)} = \frac{1}{0.035} = 30 \quad \blacksquare$$

**EXERCISES 1.11**

In Exercises 1–4, solve for the given letter from the indicated example of this section.

1. For the formula in Example 2, solve for  $a$ .
2. For the formula in Example 3, solve for  $w$ .
3. For the formula in Example 4, solve for  $T_0$ .
4. For the formula in Example 5, solve for  $\alpha$ . (Do not evaluate.)

In Exercises 5–40, each of the given formulas arises in the technical or scientific area of study shown. Solve for the indicated letter.

5.  $E = IR$ , for  $R$  (electricity)
6.  $pV = nRT$ , for  $T$  (chemistry)
7.  $rL = g_2 - g_1$ , for  $g_1$  (surveying)
8.  $W = S_dT - Q$ , for  $Q$  (air conditioning)
9.  $Q = SLd^2$ , for  $L$  (machine design)
10.  $P = 2\pi T f$ , for  $T$  (mechanics)
11.  $p = p_a + dgh$ , for  $h$  (hydrodynamics)
12.  $2Q = 2I + A + S$ , for  $I$  (nuclear physics)
13.  $A = \frac{Rt}{pV}$ , for  $t$  (jet engine design)
14.  $u = -\frac{eL}{2m}$ , for  $L$  (spectroscopy)
15.  $ct^2 = 0.3t - ac$ , for  $a$  (medical technology)
16.  $2p + dv^2 = 2d(C - W)$ , for  $C$  (fluid flow)
17.  $T = \frac{c + d}{v}$ , for  $d$  (traffic flow)

18.  $L = \frac{N\Phi}{i}$ , for  $\Phi$  (electricity)
19.  $\frac{K_1}{K_2} = \frac{m_1 + m_2}{m_1}$ , for  $m_2$  (kinetic energy)
20.  $f = \frac{F}{d - F}$ , for  $d$  (photography)
21.  $a = \frac{2mg}{M + 2m}$ , for  $M$  (pulleys)
22.  $v = \frac{V(m + M)}{m}$ , for  $M$  (ballistics)
23.  $C_0^2 = C_1^2(1 + 2V)$ , for  $V$  (electronics)
24.  $A_1 = A(M + 1)$ , for  $M$  (photography)
25.  $N = r(A - s)$ , for  $s$  (engineering)
26.  $T = 3(T_2 - T_1)$ , for  $T_1$  (oil drilling)
27.  $T_2 = T_1 - \frac{h}{100}$ , for  $h$  (air temperature)
28.  $p_2 = p_1 + rp_1(1 - p_1)$ , for  $r$  (population growth)
29.  $Q_1 = P(Q_2 - Q_1)$ , for  $Q_2$  (refrigeration)
30.  $p - p_a = dg(y_2 - y_1)$ , for  $y_2$  (pressure gauges)
31.  $N = N_1T - N_2(1 - T)$ , for  $N_1$  (machine design)
32.  $t_a = t_c + (1 - h)t_m$ , for  $h$  (computer access time)
33.  $L = \pi(r_1 + r_2) + 2x_1 + x_2$ , for  $r_1$  (pulleys)
34.  $I = \frac{VR_2 + VR_1(1 + \mu)}{R_1R_2}$ , for  $\mu$  (electronics)
35.  $P = \frac{V_1(V_2 - V_1)}{gJ}$ , for  $V_2$  (jet engine power)
36.  $W = T(S_1 - S_2) - Q$ , for  $S_2$  (refrigeration)
37.  $C = \frac{2EAk_1k_2}{d(k_1 + k_2)}$ , for  $E$  (electronics)
38.  $d = \frac{3LPx^2 - Px^3}{6EI}$ , for  $L$  (beam deflection)
39.  $V = C\left(1 - \frac{n}{N}\right)$ , for  $n$  (property depreciation)
40.  $\frac{p}{P} = \frac{AI}{B + AI}$ , for  $B$  (atomic theory)

In Exercises 41–46, rearrange the equation for the indicated values and then solve.

41. The efficiency  $\eta$  (in %) of a certain engine is given by the formula  $\eta = T_2/(T_1 + T_2)$ . Find  $T_1$  if  $\eta = 45.0$  and  $T_2 = 875$  K.
42. A formula used in determining the total transmitted power  $P_t$  in an AM radio signal is  $P_t = P_c(1 + 0.500m^2)$ . Find  $P_c$  if  $P_t = 685$  W and  $m = 0.925$ .

43. A formula relating the Fahrenheit temperature  $F$  and the Celsius temperature  $C$  is  $F = \frac{9}{5}C + 32$ . Find the Celsius temperature that corresponds to  $90.2^\circ\text{F}$ . (32 and  $\frac{9}{5}$  are exact.)
44. In forestry, a formula used to determine the volume  $V$  of a log is  $V = \frac{1}{2}L(B + b)$ , where  $L$  is the length of the log and  $B$  and  $b$  are the areas of the ends. Find  $b$  (in  $\text{m}^2$ ) if  $V = 1.09$   $\text{m}^3$ ,  $L = 4.91$  m, and  $B = 0.244$   $\text{m}^2$ . See Fig. 1.16.

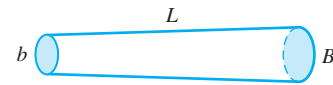


Fig. 1.16

45. The voltage  $V_1$  across resistance  $R_1$  is  $V_1 = \frac{VR_1}{R_1 + R_2}$ , where  $V$  is the voltage across resistances  $R_1$  and  $R_2$ . See Fig. 1.17. Find  $R_2$  (in  $\Omega$ ) if  $R_1 = 3.56$   $\Omega$ ,  $V_1 = 6.30$  V, and  $V = 12.0$  V.

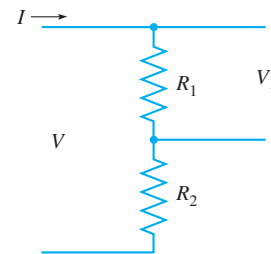


Fig. 1.17

46. The efficiency  $\eta$  of a computer multiprocessor compilation is given by  $\eta = \frac{1}{q + p(1 - q)}$ , where  $p$  is the number of processors and  $q$  is the fraction of the compilation that can be performed by the available parallel processors. Find  $p$  for  $\eta = 0.66$  and  $q = 0.83$ .

In Exercises 47 and 48, set up the required formula and solve for the indicated letter.

47. One missile travels at a speed of  $v_2$  km/h for 4 h, and another missile travels at a speed of  $v_1$  for  $t + 2$  hours. If they travel a total of  $d$  km, solve the resulting formula for  $t$ .
48. A microwave transmitter can handle  $x$  telephone communications, and 15 separate cables can handle  $y$  connections each. If the combined system can handle  $C$  connections, solve for  $y$ .

Answers to Practice Exercises

1.  $\theta - kA$     2.  $\frac{P + nc}{n}$

## 1.12 Applied Word Problems

Procedure for Solving Word Problems •  
 Identifying the Unknown Quantities •  
 Setting Up the Proper Equation •  
 Examples of Solving Word Problems

■ See Appendix A, page A-1, for a variation to the method outlined in these steps. You might find it helpful.

■ Be sure to carefully identify your choice for the unknown. In most problems, there is really a choice. Using the word *let* clearly shows that a specific choice has been made.

■ The statement after “let  $x$  (or some other appropriate letter) =” should be clear. It should completely define the chosen unknown.

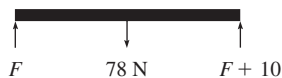


Fig. 1.18

Many applied problems are at first word problems, and we must put them into mathematical terms for solution. Usually, the most difficult part in solving a word problem is identifying the information needed for setting up the equation that leads to the solution. To do this, you must read the problem carefully to be sure that you understand all of the terms and expressions used. Following is an approach you should use.

### Procedure for Solving Word Problems

1. *Read the statement of the problem.* First, read it quickly for a general overview. Then *reread* slowly and carefully, *listing the information* given.
2. *Clearly identify the unknown quantities* and then *assign an appropriate letter to represent one of them*, stating this choice clearly.
3. *Specify the other unknown quantities* in terms of the one in step 2.
4. *If possible, make a sketch* using the known and unknown quantities.
5. *Analyse the statement of the problem and write the necessary equation.* This is often the most difficult step because ***some of the information may be implied and not explicitly stated.*** Again, a very careful reading of the statement is necessary.
6. *Solve the equation*, clearly stating the solution.
7. *Check the solution with the original statement* of the problem.

Read the following examples very carefully and note just how the outlined procedure is followed.

#### EXAMPLE 1 Sum of forces on a beam

A 78-N beam is supported at each end. The supporting force at one end is 10 N more than at the other end. Find the forces.

Since the force at each end is required, we write

$$\text{let } F = \text{the smaller force (in N)} \quad \text{step 2}$$

as a way of establishing the unknown for the equation. Any appropriate letter could be used, and we could have let it represent the larger force.

Also, since the other force is 10 N more, we write

$$F + 10 = \text{the larger force (in N)} \quad \text{step 3}$$

We now draw the sketch in Fig. 1.18. step 4

Since the forces at each end of the beam support the weight of the beam, we have the equation

$$F + (F + 10) = 78 \quad \text{step 5}$$

This equation can now be solved:  $2F = 68$

$$F = 34 \text{ N} \quad \text{step 6}$$

Thus, the smaller force is 34 N, and the larger force is 44 N. This checks with the original statement of the problem. step 7 ■

#### EXAMPLE 2 Total resistance of electric resistors

In designing an electric circuit, it is found that 34 resistors in series with a total resistance of  $56 \Omega$  are required. Two different resistances,  $1.5 \Omega$  and  $2.0 \Omega$ , are used. How many of each are in the circuit?

■ “Let  $x = 1.5\text{-}\Omega$  resistors” is incomplete. We want to find out how many of them there are.

Since we want to find the number of each resistance, we

$$\text{let } x = \text{number of } 1.5\text{-}\Omega \text{ resistors}$$

Also, since there are 34 resistors in all,

$$34 - x = \text{number of } 2.0\text{-}\Omega \text{ resistors}$$

We also know that the total resistance of all resistors is  $56\ \Omega$ . This means that

$$\begin{array}{rcl}
 \begin{array}{l} 1.5\ \Omega \\ \text{each} \end{array} \downarrow & \begin{array}{l} \text{number} \\ \downarrow \end{array} & \\
 1.5x & & \\
 \text{total resistance} & & \\
 \text{of } 1.5\text{-}\Omega \text{ resistors} & & \\
 + & \begin{array}{l} 2.0\ \Omega \\ \text{each} \end{array} \downarrow & \begin{array}{l} \text{number} \\ \downarrow \end{array} & = & 56 & \leftarrow \text{total resistance of all resistors} \\
 & 2.0(34 - x) & & & & \\
 \text{total resistance} & \text{of } 2.0\text{-}\Omega \text{ resistors} & & & & \\
 1.5x + 68 - 2.0x = 56 & & & & & \\
 -0.5x = -12 & & & & & \\
 x = 24 & & & & & 
 \end{array}$$

■ See Appendix A, page A-1, for a “sketch” that might be used with this example.

Therefore, there are 24  $1.5\text{-}\Omega$  resistors and 10  $2.0\text{-}\Omega$  resistors. The total resistance of these is  $24(1.5) + 10(2.0) = 36 + 20 = 56\ \Omega$ . We see that this checks with the statement of the problem. ■

**EXAMPLE 3** Number of medical slides

A medical researcher finds that a given sample of an experimental drug can be divided into 4 more slides with 5 mg each than with 6 mg each. How many slides with 5 mg each can be made up?

We are asked to find the number of slides with 5 mg, and therefore we

$$\text{let } x = \text{number of slides with } 5 \text{ mg}$$

Since the sample may be divided into 4 more slides with 5 mg each than with 6 mg each, we know that

$$x - 4 = \text{number of slides with } 6 \text{ mg}$$

Since *it is the same sample that is to be divided*, the total mass of the drug on each type is the same. This means

$$\begin{array}{rcl}
 \begin{array}{l} 5 \text{ mg} \\ \text{each} \end{array} \downarrow & \begin{array}{l} \text{number} \\ \downarrow \end{array} & \\
 5x & & \\
 \text{total mass} & & \\
 \text{5-mg slides} & & \\
 = & & \\
 \begin{array}{l} 6 \text{ mg} \\ \text{each} \end{array} \downarrow & \begin{array}{l} \text{number} \\ \downarrow \end{array} & \\
 6(x - 4) & & \\
 \text{total mass} & & \\
 \text{6-mg slides} & & \\
 5x = 6x - 24 & & \\
 -x = -24 \text{ or } x = 24 & & 
 \end{array}$$

Therefore, the sample can be divided into 24 slides with 5 mg each, or 20 slides with 6 mg each. Since the total mass, 120 mg, is the same for each set of slides, the solution checks with the statement of the problem. ■

**EXAMPLE 4** Distance travelled—space travel

A space shuttle maneuvers so that it may “capture” an already orbiting satellite that is 6000 km ahead. If the satellite is moving at 27 000 km/h and the shuttle is moving at 29 500 km/h, how long will it take the shuttle to reach the satellite? (All digits shown are significant.)

First, we let  $t =$  the time for the shuttle to reach the satellite. Then, using the fact that the shuttle must go 6000 km farther in the same time, we draw the sketch in Fig. 1.19.

**Practice Exercise**

- Solve the problem in Example 3 by letting  $y =$  number of slides with 6 mg.

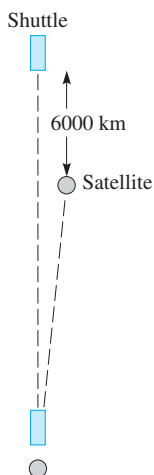


Fig. 1.19

■ A maneuver similar to the one in this example was used to repair the Hubble space telescope in 1994 and 2001.

Next, we use the formula  $distance = rate \times time$  ( $d = rt$ ). This leads to the following equation and solution.

$$\begin{array}{rcc}
 \begin{array}{l} \text{speed of} \\ \text{shuttle} \end{array} \swarrow \quad \searrow \begin{array}{l} \text{time} \\ \text{time} \end{array} & & \begin{array}{l} \text{speed of} \\ \text{satellite} \end{array} \swarrow \quad \searrow \begin{array}{l} \text{time} \\ \text{time} \end{array} \\
 29\,500t & = & 6000 & + & 27\,000t \\
 \begin{array}{l} \text{distance travelled} \\ \text{by shuttle} \end{array} & & \begin{array}{l} \text{distance between} \\ \text{at beginning} \end{array} & & \begin{array}{l} \text{distance travelled} \\ \text{by satellite} \end{array} \\
 2500t = 6000 & & & & \\
 t = 2.400 \text{ h} & & & & 
 \end{array}$$

This means that it will take the shuttle 2.400 h to reach the satellite. In 2.400 h, the shuttle will travel 70 800 km, and the satellite will travel 64 800 km. We see that the solution checks with the statement of the problem. ■

**EXAMPLE 5 Mixture—gasoline and methanol**

■ “Let  $x = \text{methanol}$ ” is incomplete. We want to find out the volume (in L) that is to be added.

A refinery has 7250 L of a gasoline-methanol blend that is 6.00% methanol. How much pure methanol must be added so that the resulting blend is 10.0% methanol?

First, let  $x =$  the number of litres of methanol to be added. The total volume of methanol in the final blend is the volume in the original blend plus that which is added. This total volume is to be 10.0% of the final blend. See Fig. 1.20.

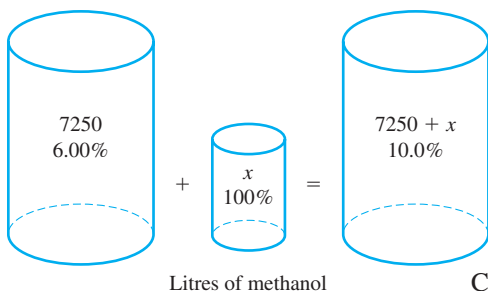


Fig. 1.20

$$\begin{array}{rcc}
 6.00\% \swarrow \quad \searrow \begin{array}{l} \text{original volume} \\ \text{volume} \end{array} & & 10\% \swarrow \quad \searrow \begin{array}{l} \text{final volume} \\ \text{volume} \end{array} \\
 0.0600(7250) & + & x & = & 0.100(7250 + x) \\
 \begin{array}{l} \text{methanol in} \\ \text{original mixture} \end{array} & & \begin{array}{l} \text{methanol} \\ \text{added} \end{array} & & \begin{array}{l} \text{methanol in} \\ \text{final mixture} \end{array} \\
 435 + x = 725 + 0.100x & & & & \\
 0.900x = 290, & x = 322 \text{ L} & \text{to be added} & & 
 \end{array}$$

Checking (to three sig. digits), there would be 757 L of methanol of a total 7570 L. ■

**EXERCISES 1.12**

In Exercises 1–4, make the given changes in the indicated examples of this section and then solve the resulting problems.

- In Example 2, in the second line, change  $2.0 \Omega$  to  $2.5 \Omega$ .
- In Example 3, in the second line, change “4 more slides” to “3 more slides.”
- In Example 4, in the second line, change 27 000 km/h to 27 500 km/h.
- In Example 5, change “pure methanol” to “of a blend with 50.0% methanol.”

In Exercises 5–32, set up an appropriate equation and solve. Data are accurate to two sig. digits unless greater accuracy is given.

- A certain new car costs \$5000 more than the same model new car cost 6 years ago. Together a new model today and six years ago cost \$49 000. What was the cost of each?
- The flow of one stream into a lake is  $45 \text{ m}^3/\text{s}$  more than the flow of a second stream. In 1 h,  $4.14 \times 10^5 \text{ m}^3$  flow into the lake from the two streams. What is the flow rate of each?

- Approximately 4.5 million wrecked cars are recycled in two consecutive years. There were 700 000 more recycled the second year than the first year. How many are recycled each year?
- A business website was accessed as often on the first day of a promotion as on the next two days combined. The first day hits were four times those on the third. The second day hits were 4000 more than the third. How many were there each day?
- Petroleum rights to 140 hectares of land are leased for \$37 000. Part of the land leases for \$200 per hectare, and the remainder for \$300 per hectare. How much is leased at each price?
- A vial contains 2000 mg, which is to be used for two dosages. One patient is to be administered 660 mg more than another. How much should be administered to each?
- After installing a pollution control device, a car’s exhaust contained the same amount of pollutant after 5.0 h as it had in 3.0 h. Before the installation the exhaust contained 150 ppm/h (parts per million per hour) of the pollutant. By how much did the device reduce the emission?



12. Three meshed spur gears have a total of 107 teeth. If the second gear has 13 more teeth than the first and the third has 15 more teeth than the second, how many teeth does each have?
13. In the design of a bridge, an engineer determines that four fewer 18-m girders are needed for the span length than 15-m girders. How many 18-m girders are needed?
14. A fuel oil storage depot had an 8-week supply on hand. However, cold weather caused the supply to be used in 6 weeks when 20 kL extra were used each week. How many litres were in the original supply?
15. The sum of three electric currents that come together at a point in a circuit is zero. If the second current is twice the first and the third current is  $9.2 \mu\text{A}$  more than the first, what are the currents? (The sign indicates the direction of flow.)
16. A delivery firm uses one fleet of trucks on daily routes of 8 h. A second fleet, with five more trucks than the first, is used on daily routes of 6 h. Budget allotments allow for 198 h of daily delivery time. How many trucks are in each fleet?
17. A natural gas pipeline feeds into three smaller pipelines, each of which is 2.6 km longer than the main pipeline. The total length of the four pipelines is 35.4 km. How long is each section?
18. At 100% efficiency two generators would produce 750 MW of power. At efficiencies of 65% and 75%, they produce 530 MW. At 100% efficiency, what power would each produce?
19. A person has a total of 54 CDs and DVDs, which cost a total of \$876. Each CD cost \$15 and each DVD cost \$18. How many of each does the person have?
20. A person won a lottery prize of \$20 000, from which 25% was deducted for taxes. The remainder was invested, partly for a 40% gain, and the rest for a 10% loss. How much was each investment if there was a \$2000 net investment gain?
21. A ski lift takes a skier up a slope at 50 m/min. The skier then skis down the slope at 150 m/min. If a round trip takes 24 min, how long is the slope?
22. Before being put out of service, the supersonic jet Concorde made a trip averaging 100 km/h less than the speed of sound for 1.00 h, and 400 km/h more than the speed of sound for 3.00 h. If the trip covered 5740 km, what is the speed of sound?
23. Trains at each end of the 50.0-km-long Eurotunnel under the English Channel start at the same time into the tunnel. Find their speeds if the train from France travels 8.0 km/h faster than the train from England and they pass in 17.0 min. See Fig. 1.21.

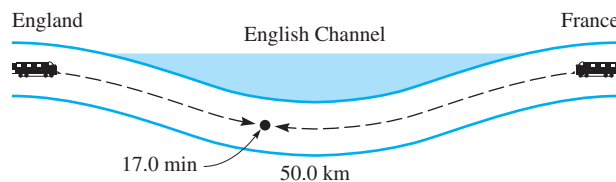


Fig. 1.21

24. An executive would arrive 10.0 min early for an appointment if travelling at 60 km/h, or 5.0 min early if travelling at 45.0 km/h. How much time is there until the appointment?
25. One lap at the Canadian Grand Prix in Montreal is 4.36 km. In a race, a car stalls and then starts 30.0 s after a second car. The first car travels at 79.0 m/s, and the second car travels at 73.0 m/s. How long does it take the first car to overtake the second, and which car will be ahead after eight laps?
26. A computer chip manufacturer produces two types of chips. In testing a total of 6100 chips of both types, 0.50% of one type and 0.80% of the other type were defective. If a total of 38 defective chips were found, how many of each type were tested?
27. Two gasoline distributors, A and B, are 367 km apart on a highway. A charges \$1.80/L and B charges \$1.68/L. Each charges 0.16¢/L per kilometre for delivery. Where on this highway is the cost to the customer the same?
28. An outboard engine uses a gasoline-oil fuel mixture in the ratio of 15 to 1. How much gasoline must be mixed with a gasoline-oil mixture, which is 75% gasoline, to make 8.0 L of the mixture for the outboard engine?
29. A car's radiator contains 12.0 L of antifreeze at a 25% concentration. How many litres must be drained and then replaced by pure antifreeze to bring the concentration to 50% (the manufacturer's "safe" level)?
30. How much sand must be added to 250 kg of a cement mixture that is 22% sand to have a mixture that is 25% sand?
31. To pass a 20-m long semitrailer travelling at 70 km/h in 10 s, how fast must a 5.0-m long car go?
32. An earthquake emits primary waves moving at 8.0 km/s and secondary waves moving at 5.0 km/s. How far from the epicentre of the earthquake is the seismic station if the two waves arrive at the station 2.0 min apart?

## CHAPTER 1 EQUATIONS

**Commutative law of addition:**  $a + b = b + a$

**Associative law of addition:**  $a + (b + c) = (a + b) + c$

**Commutative law of multiplication:**  $ab = ba$

**Associative law of multiplication:**  $a(bc) = (ab)c$

**Distributive law:**  $a(b + c) = ab + ac$

$$a + (-b) = a - b$$

(1.1)

$$a - (-b) = a + b \quad (1.2)$$

$$a^m \times a^n = a^{m+n} \quad (1.3)$$

$$\frac{a^m}{a^n} = a^{m-n} \quad (m > n, a \neq 0), \quad \frac{a^m}{a^n} = \frac{1}{a^{n-m}} \quad (m < n, a \neq 0) \quad (1.4)$$

$$(a^m)^n = a^{mn} \quad (1.5)$$

$$(ab)^n = a^n b^n, \quad \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n} \quad (b \neq 0) \quad (1.6)$$

$$a^0 = 1 \quad (a \neq 0) \quad (1.7)$$

$$a^{-n} = \frac{1}{a^n} \quad (a \neq 0) \quad (1.8)$$

$$\sqrt{ab} = \sqrt{a}\sqrt{b} \quad (a \text{ and } b \text{ positive real numbers}) \quad (1.9)$$

## CHAPTER 1 REVIEW EXERCISES

In Exercises 1–12, evaluate the given expressions.

1.  $(-2) + (-5) - 3$

2.  $6 - 8 - (-4)$

3.  $\frac{(-5)(6)(-4)}{(-2)(3)}$

4.  $\frac{(-9)(-12)(-4)}{24}$

5.  $-5 - |2(-6)| + \frac{-15}{3}$

6.  $3 - 5|-3 - 2| - \frac{12}{-4}$

7.  $\frac{18}{3 - 5} - (-4)^2$

8.  $-(-3)^2 - \frac{-8}{(-2) - |-4|}$

9.  $\sqrt{16} - \sqrt{64}$

10.  $-\sqrt{81} + 144$

11.  $(\sqrt{7})^2 - \sqrt[3]{8}$

12.  $-\sqrt[4]{16} + (\sqrt{6})^2$

In Exercises 13–20, simplify the given expressions. Where appropriate, express results with positive exponents only.

13.  $(-2rt^2)^2$

14.  $(3a^0b^{-2})^3$

15.  $-3mn^{-5}t(8m^{-3}n^4)$

16.  $\frac{15p^4q^2r}{5pq^5r}$

17.  $\frac{-16N^{-2}(NT^2)}{-2N^0T^{-1}}$

18.  $\frac{-35x^{-1}y(x^2y)}{5xy^{-1}}$

19.  $\sqrt{45}$

20.  $\sqrt{9 + 36}$

In Exercises 21–24, for each number, (a) determine the number of significant digits and (b) round off each to two significant digits.

21. 8840

22. 21 450

23. 9.040

24. 0.700

In Exercises 25–28, evaluate the given expressions. All numbers are approximate.

25.  $37.3 - 16.92(1.067)^2$

26.  $\frac{8.896 \times 10^{-12}}{3.5954 + 6.0449}$

27.  $\frac{\sqrt{0.1958 + 2.844}}{3.142(65)^2}$

28.  $\frac{1}{0.03568} + \frac{37,466}{29.63^2}$

In Exercises 29–60, perform the indicated operations.

29.  $a - 3ab - 2a + ab$

30.  $xy - y - 5y - 4xy$

31.  $6LC - (3 - LC)$

32.  $-(2x - b) - 3(-x - 5b)$

33.  $(2x - 1)(x + 5)$

34.  $(C - 4D)(2C - D)$

35.  $(x + 8)^2$

36.  $(2r - 9s)^2$

37.  $\frac{2h^3k^2 - 6h^4k^5}{2h^2k}$

38.  $\frac{4a^2x^3 - 8ax^4}{-2ax^2}$

39.  $4R - [2r - (3R - 4r)]$

40.  $3b - [3a - (a - 3b)] + 4a$

41.  $2xy - \{3z - [5xy - (7z - 6xy)]\}$

42.  $x^2 + 3b + [(b - y) - 3(2b - y + z)]$

43.  $(2x + 1)(x^2 - x - 3)$

44.  $(x - 3)(2x^2 - 3x + 1)$

45.  $-3y(x - 4y)^2$

46.  $-s(4s - 3t)^2$

47.  $3p[(q - p) - 2p(1 - 3q)]$

48.  $3x[2y - r - 4(s - 2r)]$

49.  $\frac{12p^3q^2 - 4p^4q + 6pq^5}{2p^4q}$

50.  $\frac{27s^3t^2 - 18s^4t + 9s^2t}{9s^2t}$

51.  $(2x^2 + 7x - 30) \div (x + 6)$

52.  $(4x^2 - 41) \div (2x + 7)$

53.  $\frac{3x^3 - 7x^2 + 11x - 3}{3x - 1}$

54.  $\frac{w^3 - 4w^2 + 7w - 12}{w - 3}$

55.  $\frac{4x^4 + 10x^3 + 18x - 1}{x + 3}$

56.  $\frac{8x^3 - 14x + 3}{2x + 3}$

57.  $-3\{(r + s - t) - 2[(3r - 2s) - (t - 2s)]\}$

58.  $(1 - 2x)(x - 3) - (x + 4)(4 - 3x)$

59.  $\frac{2y^3 + 9y^2 - 7y + 5}{2y - 1}$

60.  $\frac{6x^2 + 5xy - 4y^2}{2x - y}$

In Exercises 61–72, solve the given equations.

61.  $3x + 1 = x - 8$

62.  $4y - 3 = 5y + 7$

63.  $\frac{5x}{7} = \frac{3}{2}$

64.  $\frac{2(N - 4)}{3} = \frac{5}{4}$

65.  $6x - 5 = 3(x - 4)$

66.  $-2(-4 - y) = 3y$

67.  $2s + 4(3 - s) = 6$

68.  $2|x| - 1 = 3$

69.  $3t - 2(7 - t) = 5(2t + 1)$

70.  $-(8 - x) = x - 2(2 - x)$

71.  $2.7 + 2.0(2.1x - 3.4) = 0.1$

72.  $0.250(6.721 - 2.44x) = 2.08$

In Exercises 73–82, (a) change numbers in ordinary notation to scientific notation or change numbers in scientific notation to ordinary notation. (b) Write the numbers in engineering notation and replace the power of 10 with the corresponding SI prefix.

73. A certain computer has 60 000 000 000 bytes of memory.
74. The escape velocity (the velocity required to leave the earth's gravitational field) is about 40 000 km/h.
75. When pictures of the surface of Mars were transmitted to the earth from the Pathfinder mission in 1997, Mars was about 192 000 000 km from earth.
76. Police radar has a frequency of  $1.02 \times 10^9$  Hz.
77. Among the stars nearest the earth, Centaurus A is about  $4.05 \times 10^{13}$  km away.
78. Before its destruction in 2001, the World Trade Center had nearly  $10^6$  m<sup>2</sup> of office space. (See Exercise 32, page 135.)
79. The faintest sound that can be heard has an intensity of about  $10^{-12}$  W/m<sup>2</sup>.
80. An optical coating on glass to reduce reflections is about 0.000 000 15 m thick.
81. The maximum safe level of radiation in the air of a home due to radon gas is  $1.5 \times 10^{-1}$  Bq/L. (Bq is the symbol for bequerel, the metric unit of radioactivity, where 1 Bq = 1 decay/s.)
82. A certain virus was measured to have a diameter of about 0.000 000 18 m.

In Exercises 83–96, solve for the indicated letter. Where noted, the given formula arises in the technical or scientific area of study.

83.  $R = n^2Z$ , for  $Z$  (electricity)
84.  $R = \frac{2GM}{c^2}$ , for  $G$  (astronomy: black holes)
85.  $P = \frac{\pi^2 EI}{L^2}$ , for  $E$  (mechanics)
86.  $f = p(c - 1) - c(p - 1)$ , for  $p$  (thermodynamics)
87.  $Pp + Qq = Rr$ , for  $q$  (moments of forces)
88.  $V = IR + Ir$ , for  $R$  (electricity)
89.  $d = (n - 1)A$ , for  $n$  (optics)
90.  $mu = (m + M)v$ , for  $M$  (physics: momentum)
91.  $N_1 = T(N_2 - N_3) + N_3$ , for  $N_2$  (mechanics: gears)
92.  $q = \frac{KA(B - C)}{L}$ , for  $B$  (solar heating)
93.  $R = \frac{A(T_2 - T_1)}{H}$ , for  $T_2$  (thermal resistance)
94.  $Z^2 \left(1 - \frac{\lambda}{2a}\right) = k$ , for  $\lambda$  (radar design)
95.  $d = kx^2[3(a + b) - x]$ , for  $a$  (mechanics: beams)
96.  $V = V_0[1 + 3a(T_2 - T_1)]$ , for  $T_2$  (thermal expansion)

In Exercises 97–102, perform the indicated calculations.

97. A computer's memory is  $5.25 \times 10^{10}$  bytes, and that of a model 20 years older is  $6.4 \times 10^4$  bytes. What is the ratio of the newer computer's memory to the older computer's memory?
98. The time (in s) for an object to fall  $h$  metres is given by the expression  $0.45\sqrt{h}$ . How long does it take a person to fall

22 m from a sixth-floor window into a net while escaping a fire?

99. The CN Tower in Toronto is 0.553 km high. The Sears Tower in Chicago is 443 m high. How much taller is the CN Tower than the Sears Tower?
100. The time (in s) it takes a computer to check  $n$  memory cells is found by evaluating  $(n/2650)^2$ . Find the time to check 48 cells.
101. The combined electric resistance of two parallel resistors is found by evaluating the expression  $\frac{R_1 R_2}{R_1 + R_2}$ . Evaluate this for  $R_1 = 0.0275 \Omega$  and  $R_2 = 0.0590 \Omega$ .
102. The distance (in m) from the earth for which the gravitational force of the earth on a spacecraft equals the gravitational force of the sun on it is found by evaluating  $1.5 \times 10^{11} \sqrt{m/M}$ , where  $m$  and  $M$  are the masses of the earth and sun, respectively. Find this distance for  $m = 5.98 \times 10^{24}$  kg and  $M = 1.99 \times 10^{30}$  kg.

In Exercises 103–106, simplify the given expressions.

103. One transmitter antenna is  $(x - 2a)$  cm long, and another is  $(x + 2a)$  m long. What is the sum, in centimetres, of their lengths?
104. In finding the value of an annuity, the expression  $(Ai - R)(1 + i)^2$  is used. Multiply out this expression.
105. A computer analysis of the velocity of a link in an industrial robot leads to the expression  $4(t + h) - 2(t + h)^2$ . Simplify this expression.
106. When analysing the motion of a communications satellite, the expression  $\frac{k^2 r - 2h^2 k + h^2 r v^2}{k^2 r}$  is used. Perform the indicated division.

In Exercises 107–114, solve the given problems.

107. Does the value of  $3 \times 18 \div (9 - 6)$  change if the parentheses are removed?
108. Does the value of  $(3 \times 18) \div 9 - 6$  change if the parentheses are removed?
109. In solving the equation  $x - (3 - x) = 2x - 3$ , what conclusion can be made?
110. In solving the equation  $7 - (2 - x) = x + 2$ , what conclusion can be made?
111. Show that  $(x - y)^3 = -(y - x)^3$ .
112. Is division associative? That is, is it true (if  $b \neq 0, c \neq 0$ ) that  $(a \div b) \div c = a \div (b \div c)$ ?
113. What is the ratio of  $8 \times 10^{-3}$  to  $2 \times 10^4$ ?
114. What is the ratio of  $\sqrt{4 + 36}$  to  $\sqrt{4}$ ?

In Exercises 115–128, solve the given problems. All data are accurate to two significant digits unless greater accuracy is given.

115. Two computer software programs cost \$190 together. If one costs \$72 more than the other, what is the cost of each?
116. A sponsor pays a total of \$9500 to run a commercial on two different TV stations. One station charges \$1100 more than the other. What does each charge to run the commercial?

117. Three chemical reactions each produce oxygen. If the first produces twice that of the second, the third produces twice that of the first, and the combined total is  $560 \text{ cm}^3$ , what volume is produced by each?
118. In testing the rate at which a polluted stream flows, a boat that travels at  $5.5 \text{ km/h}$  in still water took  $5.0 \text{ h}$  to go downstream between two points, and it took  $8.0 \text{ h}$  to go upstream between the same two points. What is the rate of flow of the stream?
119. The voltage across a resistor equals the current times the resistance. In a microprocessor series circuit, one resistor is  $1200 \Omega$  greater than another. The sum of the voltages across them is  $12.0 \text{ mV}$ . Find the resistances if the current is  $2.4 \mu\text{A}$  in each.
120. An air sample contains  $4.0 \text{ ppm}$  (parts per million) of two pollutants. The concentration of one is four times the other. What are the concentrations?
121. One road crew constructs  $450 \text{ m}$  of road bed in  $12 \text{ h}$ . How long will it take them to construct another  $250 \text{ m}$  of road bed?
122. The fuel for a two-cycle motorboat engine is a mixture of gasoline and oil in the ratio of  $15$  to  $1$ . How many litres of each are in  $6.6 \text{ L}$  of mixture?
123. A ship enters the Red Sea from the Suez Canal, moving south at  $28.0 \text{ km/h}$ . Two hours later, a second ship enters the Red Sea at the southern end, moving north at  $35.0 \text{ km/h}$ . If the Red Sea is  $2230 \text{ km}$  long, when will the ships pass?
124. A helicopter used in fighting a forest fire travels at  $175 \text{ km/h}$  from the fire to a pond and  $115 \text{ km/h}$  with water from the pond to the fire. If a round trip takes  $30 \text{ min}$ , how long does it take from the pond to the fire? See Fig. 1.22.

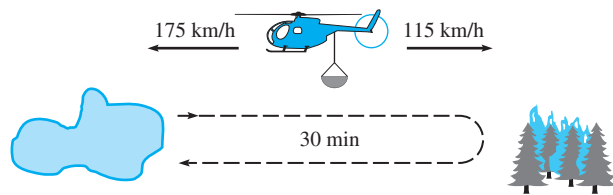


Fig. 1.22

125. One grade of oil has  $0.50\%$  of an additive, and a higher grade has  $0.75\%$  of the additive. How many litres of each must be used to have  $1000 \text{ L}$  of a mixture with  $0.65\%$  of the additive?
126. How much water must be added to  $20.0 \text{ mL}$  of a  $60.0\%$  saline solution in order to make a  $45.0\%$  saline solution?
127. An architect plans to have  $25\%$  of the floor area of a house in ceramic tile. In all but the kitchen and entry, there are  $205 \text{ m}^2$  of floor area,  $15\%$  of which is tile. What area can be planned for the kitchen and entry if each has an all-tile floor?
128. A karat equals  $1/24$  part of gold in an alloy (for example, 9-karat gold is  $9/24$  gold). How many grams of 9-karat gold must be mixed with 18-karat gold to get  $200 \text{ g}$  of 14-karat gold?

### Writing Exercise

129. In calculating the simple interest earned by an investment, the equation  $P = P_0 + P_0rt$  is used, where  $P$  is the value after an initial principal  $P_0$  is invested for  $t$  years at interest rate  $r$ . Solve for  $r$ , and then evaluate  $r$  for  $P = \$7625$ ,  $P_0 = \$6250$ , and  $t = 4.000$  years. Write a paragraph or two explaining (a) your method for solving for  $r$ , and (b) the calculator steps used to evaluate  $r$ , noting the use of parentheses.

## CHAPTER 1 PRACTICE TEST

In Problems 1–5, evaluate the given expressions. In Problems 3 and 5, the numbers are approximate.

- $\sqrt{9 + 16}$
- $\frac{(7)(-3)(-2)}{(-6)(0)}$
- $\frac{3.372 \times 10^{-3}}{7.526 \times 10^{12}}$
- $\frac{(+6)(-2) - 3(-1)}{|2 - 5|}$
- $\frac{346.4 - 23.5}{287.7} - \frac{0.944^3}{(3.46)(0.109)}$

In Problems 6–12, perform the indicated operations and simplify. When exponents are used, use only positive exponents in the result.

- $(2a^0b^{-2}c^3)^{-3}$
- $(2x + 3)^2$
- $3m^2(am - 2m^3)$
- $\frac{8a^3x^2 - 4a^2x^4}{-2ax^2}$
- $\frac{6x^2 - 13x + 7}{2x - 1}$
- $(2x - 3)(x + 7)$
- $3x - [4x - (3 - 2x)]$
- Solve for  $y$ :  $5y - 2(y - 4) = 7$
- Solve for  $x$ :  $3(x - 3) = x - (2 - 3d)$

- Express  $0.000\,003\,6$  in scientific notation.
- List the numbers  $-3$ ,  $|-4|$ ,  $-\pi$ ,  $\sqrt{2}$ , and  $0.3$  in numerical order.
- What fundamental law is illustrated by  $3(5 + 8) = 3(5) + 3(8)$ ?
- (a) How many significant digits are in the number  $3.0450$ ?  
(b) Round it off to two significant digits.
- If  $P$  dollars is deposited in a bank that compounds interest  $n$  times a year, the value of the account after  $t$  years is found by evaluating  $P(1 + i/n)^{nt}$ , where  $i$  is the annual interest rate. Find the value of an account for which  $P = \$1000$ ,  $i = 5\%$ ,  $n = 2$ , and  $t = 3$  years (values are exact).
- In finding the illuminance from a light source, the expression  $8(100 - x)^2 + x^2$  is used. Simplify this expression.
- The equation  $L = L_0[1 + \alpha(t_2 - t_1)]$  is used when studying thermal expansion. Solve for  $t_2$ .
- An alloy weighing  $20 \text{ N}$  is  $30\%$  copper. How many newtons of another alloy, which is  $80\%$  copper, must be added for the final alloy to be  $60\%$  copper?



# Geometry

## 2

**M**assive structures like the pyramids in Egypt and Stonehenge in the United Kingdom were built by ancient civilizations and required the ability to measure the size and shape of objects, as well as an understanding of alignment to celestial phenomena. Today, MRI (magnetic resonance imaging) can be used to determine the location, size, and shape of a tumour in a human being. Geometry deals with size, shape, angles, and surfaces, and geometric methods are important in many technological applications.

Many of the methods of measuring geometric objects were known in ancient times, and most of the geometry used in technology has been known for hundreds of years. In about 300 B.C.E., the Greek mathematician Euclid (who lived and taught in Alexandria, Egypt) organized what was known in geometry. He added many new ideas in a 13-volume set of writings known as the *Elements*. Centuries later it was translated into various languages and was one of the earliest mathematical works to be printed in mass production with a printing press.

The study of Euclidean geometry includes the properties and measurements of angles, lines, surfaces, and the basic figures they form. In this chapter, we review the methods and formulas for calculating important geometric measures such as area and volume. Technical applications are included from architecture, construction, instrumentation, surveying and civil engineering, mechanical design, product design of various types, and other areas of engineering.

Geometric figures and concepts are also fundamental to the development of many areas of mathematics, such as graphing and trigonometry. We will start our study of graphs in Chapter 3 and trigonometry in Chapter 4.

### LEARNING OUTCOMES

**After completion of this chapter, the student should be able to:**

- Identify perpendicular and parallel lines
- Identify supplementary, complementary, vertical, and corresponding angles
- Determine interior angles and sides in various triangles
- Use the Pythagorean theorem
- Identify and analyse different types of quadrilaterals and polygons
- Identify and analyse circles, arcs, and interior angles of a circle
- Calculate area and perimeter of a geometric shape
- Use an approximation method to estimate an irregular area
- Identify and analyse three-dimensional geometric figures, including volume, angles, and dimensions



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◀ In Section 2.5, we see how to find an excellent approximation of the area of an irregular geometric figure, such as a lake. At the left is a satellite photograph of Lake Ontario, one of the Great Lakes between the United States and Canada.

## 2.1 Lines and Angles

Basic Angles • Parallel Lines and Perpendicular Lines • Supplementary Angles and Complementary Angles • Angles Between Intersecting Lines • Segments of Transversals

■ The use of 360 comes from the ancient Babylonians and their number system based on 60 rather than 10, as we use today. However, the specific reason for the choice of 360 is not known. (One theory is that 360 is divisible by many smaller numbers and is also close to the number of days in a year.)

It is not possible to define every term we use. In geometry, *the meanings of point, line, and plane are accepted without being defined*. These terms give us a starting point for the definitions of other useful geometric terms. Even though we may not have an independent definition of these fundamental concepts, it may be useful to give operational definitions or analogies such that they can be understood. A **point** is a *location in space* and does not possess any width, length, or height. Often, a point like the “origin” is used as a reference position for defining graphs of functions (discussed in Chapter 3). A “dot” is often used to represent a point. A **line** extends indefinitely along a *single dimension* (in both directions), having a length without limit. A line contains an infinite number of points. A **plane** is a *flat surface*, extending infinitely in all directions along its flat orientation. A plane contains an infinite number of lines and points.

The amount of rotation of a **ray** (or *half-line*) about its endpoint is called an **angle**. A ray is that part of a line to *one side of a fixed point* on the line. The fixed point is the **vertex of the angle**. One complete rotation of a ray is an angle with a measure of 360 degrees, written as  $360^\circ$ . Some special types of angles are as follows:

Name of angle	Measure of angle
Right angle	$90^\circ$
Straight angle	$180^\circ$
Acute angle	Between $0^\circ$ and $90^\circ$
Obtuse angle	Between $90^\circ$ and $180^\circ$
Reflex angle	Between $180^\circ$ and $360^\circ$

### EXAMPLE 1 Basic angles

Fig. 2.1(a) shows a right angle (marked as  $\square$ ). The vertex of the angle is point  $B$ , and the ray is the half-line  $BA$ . Fig. 2.1(b) shows a straight angle. Fig. 2.1(c) shows an acute angle, denoted as  $\angle E$  (or  $\angle DEF$  or  $\angle FED$ ). In Fig. 2.1(d),  $\angle G$  is an obtuse angle. In Fig. 2.1 (e),  $\angle H$  is a reflex angle.

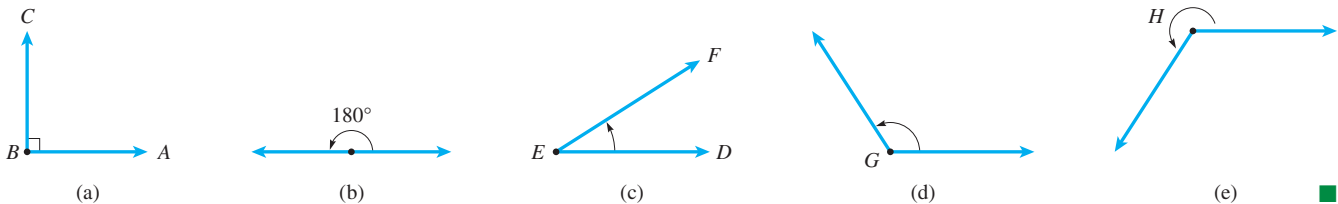


Fig. 2.1

A **line segment** is a line that links two points without extending beyond them. Often a line segment is named for the two endpoints. For example, if a line segment joins point  $A$  and point  $B$ , it is often named line segment  $AB$ . Another symbol might also be used denoting length of the line segment. For example,  $d$  might be the length of line segment  $AB$ . If a line passes through two points  $A$  and  $B$ , but also extends beyond them, it can also be named line  $AB$ . The context of the use of the name is important. If two lines **intersect** so that there is a non-zero angle between them, they *cross at a single point*. If the angle between the intersecting lines is a right angle, the lines are **perpendicular**. Lines in the same plane that do not intersect are **parallel**. These concepts are illustrated in the following example.

### EXAMPLE 2 Parallel lines and perpendicular lines

In Fig. 2.2(a), lines  $AC$  and  $DE$  are perpendicular (which is shown as  $AC \perp DE$ ) since they meet in a right angle (again, shown as  $\square$ ) at  $B$ .

In Fig. 2.2(b), lines  $AB$  and  $CD$  are drawn so they do not meet, even if extended. Therefore, these lines are parallel (which can be shown as  $AB \parallel CD$ ).

In Fig. 2.2(c), line segments  $AB \perp BC$ ,  $DC \perp BC$ , and  $AB \parallel DC$ .

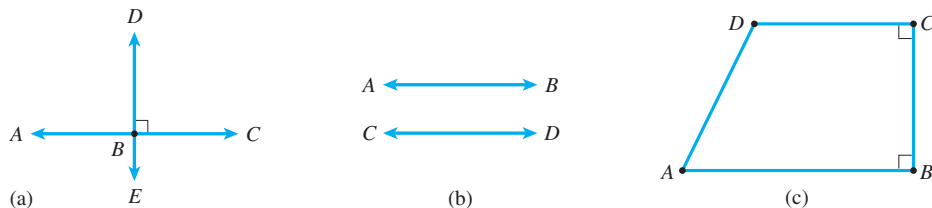


Fig. 2.2

It will be important to recognize perpendicular sides and parallel sides in many of the geometric figures in later sections. ■

■ Recognizing complementary angles is important in trigonometry.

If the sum of the measures of two angles is  $180^\circ$ , then the angles are called **supplementary angles**. Each angle is the **supplement** of the other. If the sum of the measures of two angles is  $90^\circ$ , the angles are called **complementary angles**. Each is the **complement** of the other.

**EXAMPLE 3** Supplementary angles and complementary angles

- (a) In Fig. 2.3,  $\angle BAC = 55^\circ$  and  $\angle DAC = 125^\circ$ . Since  $55^\circ + 125^\circ = 180^\circ$ ,  $\angle BAC$  and  $\angle DAC$  are supplementary angles.  $\angle BAD$  is a straight angle.
- (b) In Fig. 2.4, we see that  $\angle POQ$  is a right angle, or  $\angle POQ = 90^\circ$ . Since  $\angle POR + \angle ROQ = \angle POQ = 90^\circ$ ,  $\angle POR$  is the complement of  $\angle ROQ$  (or  $\angle ROQ$  is the complement of  $\angle POR$ ).

**Practice Exercise**

1. What is the measure of the complement of  $\angle BAC$  in Fig. 2.3?

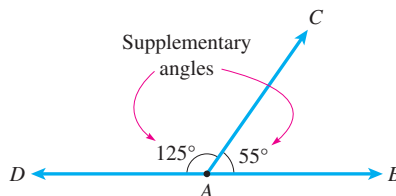


Fig. 2.3

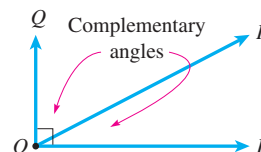


Fig. 2.4

It is often necessary to refer to certain special pairs of angles. *Two angles that have a common vertex and a side or ray common between them are known as **adjacent angles***. If two lines cross to form equal angles on opposite sides of the point of intersection, which is the common vertex, these angles are called **vertical angles**, or **vertically opposite angles**.

**EXAMPLE 4** Adjacent angles and vertical angles

- (a) In Fig. 2.5,  $\angle BAC$  and  $\angle CAD$  have a common vertex at  $A$  and the common ray  $AC$  between them. This means that  $\angle BAC$  and  $\angle CAD$  are adjacent angles.
- (b) In Fig. 2.6, lines  $AB$  and  $CD$  intersect at point  $O$ . Here,  $\angle AOC$  and  $\angle BOD$  are vertical angles, and they are equal. Also,  $\angle BOC$  and  $\angle AOD$  are vertical angles and are equal.

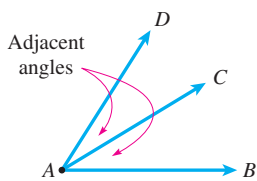


Fig. 2.5

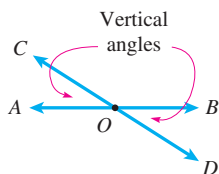


Fig. 2.6

We should also be able to identify the sides or rays of an angle that are adjacent to the angle. In Fig. 2.5, rays  $AB$  and  $AC$  are adjacent to  $\angle BAC$ , and in Fig. 2.6, rays  $OB$  and  $OD$  are adjacent to  $\angle BOD$ . Identifying sides adjacent and opposite an angle in a triangle is important in trigonometry.



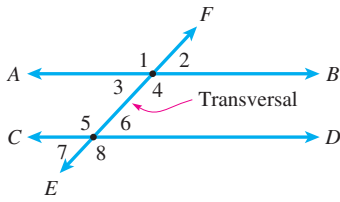


Fig. 2.7

**Practice Exercise**

2. In Fig. 2.7, if  $\angle 2 = 42^\circ$ , then  $\angle 5 = ?$

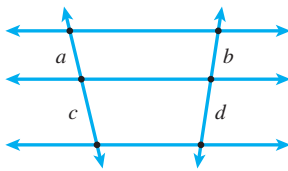


Fig. 2.8

$$\frac{a}{b} = \frac{c}{d} \tag{2.1}$$

**EXAMPLE 5 Segments of transversals**

In Fig. 2.9, part of the beam structure within a building is shown. The vertical beams are parallel. From the distances between beams that are shown, determine the distance  $x$  between the middle and right vertical beams.

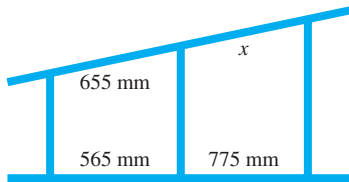


Fig. 2.9

Using Eq. (2.1), we have

$$\begin{aligned} \frac{655}{565} &= \frac{x}{775} \\ x &= \frac{655(775)}{565} \\ x &= 898 \text{ mm} \end{aligned}$$

**EXERCISES 2.1**

In Exercises 1–4, answer the given questions about the indicated examples of this section.

- In Example 2, what is the measure of  $\angle ABE$  in Fig. 2.2(a)?
- In Example 3(b), if  $\angle POR = 32^\circ$  in Fig. 2.4, what is the measure of  $\angle QOR$ ?
- In Example 4, how many different pairs of adjacent angles are there in Fig. 2.6?
- In Example 5, if the segments of 655 mm and 775 mm are interchanged, (a) what is the answer, and (b) is the beam along which  $x$  is measured more nearly vertical or more nearly horizontal?

In Exercises 5–12, identify the indicated angles and sides in Fig. 2.10. In Exercises 9 and 10, also find the measures of the indicated angles.

- Two acute angles
- Two right angles

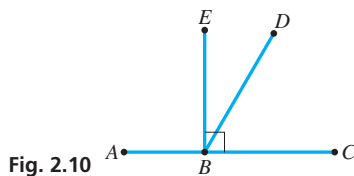


Fig. 2.10

- The straight angle
- The obtuse angle
- If  $\angle CBD = 65^\circ$ , find its complement.
- If  $\angle CBD = 65^\circ$ , find its supplement.
- The sides adjacent to  $\angle DBC$
- The acute angle adjacent to  $\angle DBC$

In Exercises 13–15, use Fig. 2.11. In Exercises 16–18, use Fig. 2.12. Find the measures of the indicated angles.

- $\angle AOB$
- $\angle AOC$
- $\angle BOD$
- $\angle 3$
- $\angle 4$
- $\angle 5$

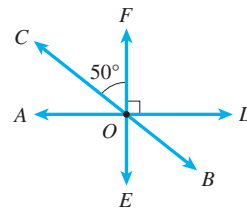


Fig. 2.11

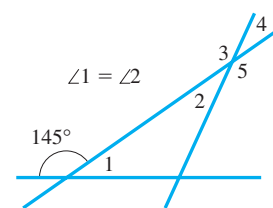


Fig. 2.12

In a plane, if a line crosses two or more parallel or nonparallel lines, it is called a **transversal**. In Fig. 2.7,  $AB \parallel CD$ , and the transversal of these two parallel lines is the line  $EF$ .

When a transversal crosses a pair of parallel lines, certain pairs of equal angles result. In Fig. 2.7, the **corresponding angles** are equal (that is,  $\angle 1 = \angle 5$ ,  $\angle 2 = \angle 6$ ,  $\angle 3 = \angle 7$ , and  $\angle 4 = \angle 8$ ). Also, the **alternate-interior angles** are equal ( $\angle 3 = \angle 6$  and  $\angle 4 = \angle 5$ ), and the **alternate-exterior angles** are equal ( $\angle 1 = \angle 8$  and  $\angle 2 = \angle 7$ ).

When more than two parallel lines are crossed by two transversals, such as is shown in Fig. 2.8, the segments of the transversals between the same two parallel lines are called **corresponding segments**. A useful theorem is that the ratios of corresponding segments of the transversals are equal. In Fig. 2.8, this means that

In Exercises 19–22, find the measures of the angles in Fig. 2.13.

19.  $\angle 1$       20.  $\angle 2$       21.  $\angle 3$       22.  $\angle 4$

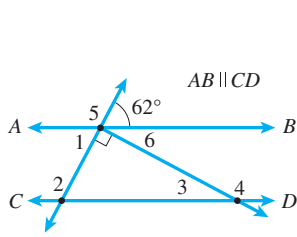


Fig. 2.13

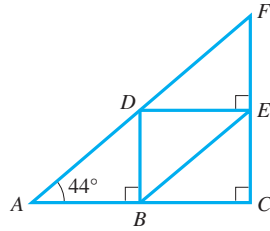


Fig. 2.14

In Exercises 23–28, find the measures of the angles in the truss shown in Fig. 2.14. A truss is a rigid support structure that is used in the construction of buildings and bridges.

23.  $\angle BDF$       24.  $\angle ABE$       25.  $\angle DEB$   
 26.  $\angle DBE$       27.  $\angle DFE$       28.  $\angle ADE$

In Exercises 29–32, find the indicated distances between the straight irrigation ditches shown in Fig. 2.15. The vertical ditches are parallel.

29.  $a$       30.  $b$       31.  $c$       32.  $d$

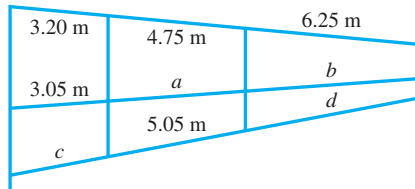


Fig. 2.15

In Exercises 33–36, solve the given problems.

33. A steam pipe is connected in sections  $AB$ ,  $BC$ , and  $CD$  as shown in Fig. 2.16. What is the angle between  $BC$  and  $CD$  if  $AB \parallel CD$ ?

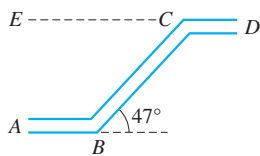


Fig. 2.16

34. Part of a laser beam striking a surface is reflected at the same angle as the original beam, and the remainder is refracted (passes into the new material) at a new angle (see Fig. 2.17). Find the total angle between the reflected beam and the refracted beam and specify the type of angle it is (acute, right, obtuse, straight, or reflex).

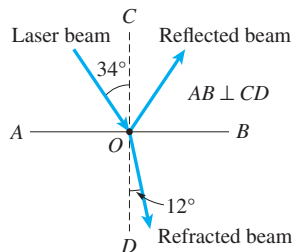


Fig. 2.17

35. Find the distance on Dundas St. W between Dufferin St. and Ossington Ave. in Toronto, as shown in Fig. 2.18. The north–south streets are parallel.

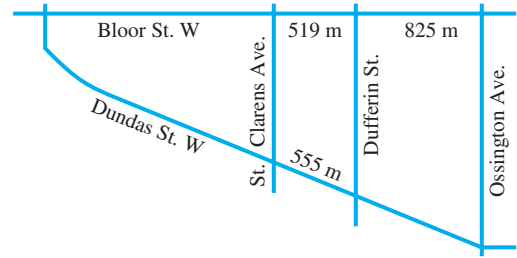


Fig. 2.18

36. An electric circuit board has equally spaced parallel wires with connections at points  $A$ ,  $B$ , and  $C$ , as shown in Fig. 2.19. How far is  $A$  from  $C$ , if  $BC = 2.15$  cm?

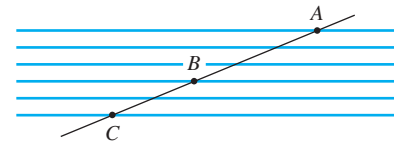


Fig. 2.19

In Exercises 37–40, solve the given problems related to Fig. 2.20.

37.  $\angle 1 + \angle 2 + \angle 3 = ?$   
 38.  $\angle 4 + \angle 2 + \angle 5 = ?$

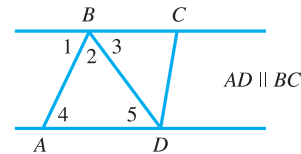


Fig. 2.20

39. Based on Exercise 38, what conclusion can be drawn about a closed geometric figure like the one with vertices at  $A$ ,  $B$ , and  $D$ ?
40. Based on Exercises 38 and 39, what conclusion can be drawn about a closed geometric figure like the one with vertices  $A$ ,  $B$ ,  $C$ , and  $D$ ?

Answers to Practice Exercises

1.  $35^\circ$       2.  $138^\circ$

## 2.2 Triangles

Types and Properties of Triangles • Perimeter and Area • Hero's Formula • Pythagorean Theorem • Similar Triangles

■ The properties of triangles are important in the study of trigonometry, which we start in Chapter 4.

When part of a plane is bounded and closed by straight-line segments, it is called a **polygon**, and it is named according to the number of sides it has. A **triangle** has three sides, a **quadrilateral** has four sides, a **pentagon** has five sides, a **hexagon** has six sides, and so on. The most important polygons are the triangle, which we consider in this section, and the quadrilateral, which we study in the next section.

### TYPES AND PROPERTIES OF TRIANGLES

In a **scalene triangle**, no two sides are equal in length. In an **isosceles triangle**, two of the sides are equal in length, and the two *base angles* (the angles opposite the equal sides) are equal. In an **equilateral triangle**, the three sides are equal in length, and each of the three angles is  $60^\circ$ .

The most important triangle in technical applications is the **right triangle**. In a right triangle, one of the angles is a right angle. The side opposite the right angle is the **hypotenuse**, and the other two sides are called **legs**.

#### EXAMPLE 1 Types of triangles

Fig. 2.21(a) shows a scalene triangle; each side is of a different length. Fig. 2.21(b) shows an isosceles triangle with two equal sides of 2 m and equal base angles of  $40^\circ$ . Fig. 2.21(c) shows an equilateral triangle with equal sides of 5 cm and equal angles of  $60^\circ$ . Fig. 2.21(d) shows a right triangle. The hypotenuse is side AB.

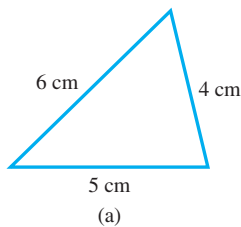
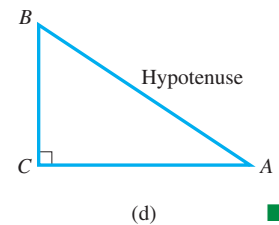
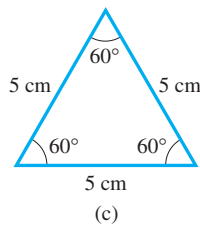
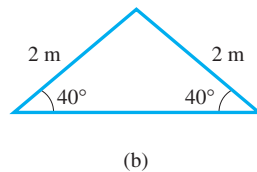


Fig. 2.21



One very important property of a triangle is that

*the sum of the measures of the three interior angles of a triangle is  $180^\circ$ .*

In the next example, we show this property by using material from Section 2.1.

#### EXAMPLE 2 Sum of angles of a triangle

In Fig. 2.22, since  $\angle 1$ ,  $\angle 2$ , and  $\angle 3$  constitute a straight angle,

$$\angle 1 + \angle 2 + \angle 3 = 180^\circ$$

Also, by noting alternate interior angles, we see that  $\angle 1 = \angle 4$  and  $\angle 3 = \angle 5$ . Therefore, by substitution, we have

$$\angle 4 + \angle 2 + \angle 5 = 180^\circ$$

Therefore, if two of the angles of a triangle are known, the third may be found by subtracting the sum of the first two from  $180^\circ$ .

#### EXAMPLE 3 Application of sum of angles in a triangle

An airplane is flying north and then makes a  $90^\circ$  turn to the west. Later, it makes another left turn of  $150^\circ$ . What is the angle of a third left turn that will cause the plane to again fly north? See Fig. 2.23.

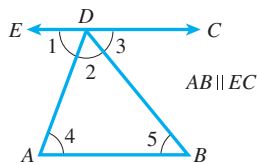


Fig. 2.22

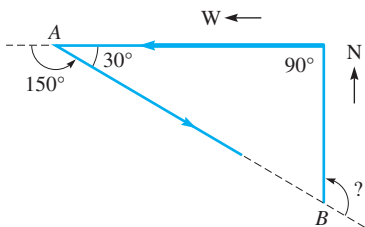


Fig. 2.23

From Fig. 2.23, we see that the interior angle of the triangle at  $A$  is the supplement of  $150^\circ$ , or  $30^\circ$ . Since the sum of the measures of the interior angles of the triangle is  $180^\circ$ , the interior angle at  $B$  is

$$\angle B = 180^\circ - (90^\circ + 30^\circ) = 60^\circ$$

The required angle is the supplement of  $60^\circ$ , which is  $120^\circ$ . ■

A line segment drawn from a vertex of a triangle to the *midpoint* of the opposite side is called a **median** of the triangle. Each median also divides the triangle into two smaller triangles each having the same area. A basic property of a triangle is that *the three medians meet at a single point, called the centroid* (or centre of mass) of the triangle. See Fig. 2.24. Centroids are calculated as an application of calculus in Chapter 26. Also, *the three angle bisectors* (lines from the vertices that divide the angles in two) *meet at a common point*. See Fig. 2.25.

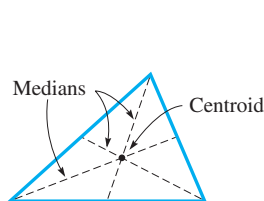


Fig. 2.24

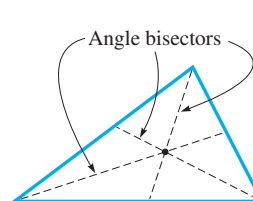


Fig. 2.25

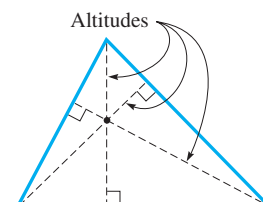


Fig. 2.26

### Practice Exercise

1. If the triangle in Fig. 2.24 is isosceles and the vertex angle (at the left) is  $30^\circ$ , what are the base angles?

### LEARNING TIP

The three common points of the medians, angle bisectors, and altitudes are generally not the same point for a given triangle.

An **altitude** (or **height**) of a triangle is the line segment drawn from a vertex perpendicular to the opposite side (or its extension), which is called the **base** of the triangle. The three altitudes of a triangle meet at a common point. See Fig. 2.26.

### PERIMETER AND AREA OF A TRIANGLE

We now consider two of the most basic measures of a plane geometric figure. The first of these is its **perimeter**, which is the total length of the boundary around the figure. In the following example, we find the perimeter of a triangle.

#### EXAMPLE 4 Perimeter of a triangle

Find the perimeter  $p$  of a triangle with sides 2.56 m, 3.22 m, and 4.89 m. See Fig. 2.27.

Using the definition of perimeter, for this triangle we have

$$p = 2.56 + 3.22 + 4.89 = 10.67 \text{ m}$$

Therefore, the distance around the triangle is 10.67 m. We express the results to hundredths since each side is given to hundredths. ■

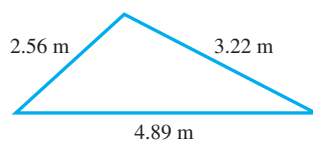


Fig. 2.27

The second important measure of a geometric figure is its **area**. Although the concept of area is primarily intuitive, it is easily defined and calculated for the basic geometric figures. *Area gives a measure of the size of the surface of the figure*, just as perimeter gives the measure of the distance around it.

The area  $A$  of a triangle of base  $b$  and altitude  $h$  is

$$A = \frac{1}{2}bh \quad (2.2)$$

The following example illustrates the use of Eq. (2.2).

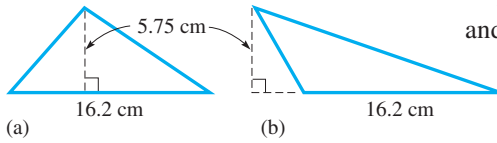


Fig. 2.28

■ Hero’s formula is named for Hero (or Heron), a first-century Greek mathematician.

**EXAMPLE 5 Area of a triangle**

Find the areas of the triangles in Fig. 2.28(a) and Fig. 2.28(b).

Even though the triangles are of different shapes, the base  $b$  of each is 16.2 cm, and the altitude  $h$  of each is 5.75 cm. Therefore, the area of each triangle is

$$A = \frac{1}{2}bh = \frac{1}{2}(16.2)(5.75) = 46.6 \text{ cm}^2$$

Another formula for the area of a triangle that is particularly useful when we have a triangle with three known sides and no right angle is **Hero’s formula**, which is

$$A = \sqrt{s(s-a)(s-b)(s-c)}, \text{ where } s = \frac{1}{2}(a+b+c) \tag{2.3}$$

In Eq. (2.3),  $a$ ,  $b$ , and  $c$  are the lengths of the sides, and  $s$  is one-half of the perimeter.

**EXAMPLE 6 Application of Hero’s formula**

A surveyor measures the sides of a triangular parcel of land between two intersecting straight roads to be 206 m, 293 m, and 187 m. Find the area of this parcel (see Fig. 2.29).

To use Eq. (2.3), we first find  $s$ :

$$\begin{aligned} s &= \frac{1}{2}(206 + 293 + 187) \\ &= \frac{1}{2}(686) = 343 \text{ m} \end{aligned}$$

Now, substituting in Eq. (2.3), we have

$$\begin{aligned} A &= \sqrt{343(343 - 206)(343 - 293)(343 - 187)} \\ &= 19\,144.968\,01 \text{ m}^2 \\ &= 19\,100 \text{ m}^2 \text{ (rounded to 3 significant digits)} \end{aligned}$$

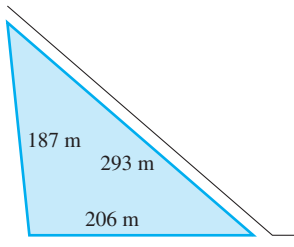


Fig. 2.29

**Practice Exercise**

- What is the perimeter of the triangle in Fig. 2.29?

■ The Pythagorean theorem is named for the Greek mathematician Pythagoras (sixth century B.C.E.)

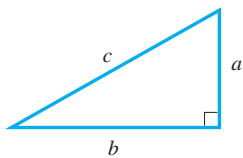


Fig. 2.30

**THE PYTHAGOREAN THEOREM**

As we have noted, one of the most important geometric figures in technical applications is the right triangle. A very important property of a right triangle is given by the **Pythagorean theorem**, which states that

*in a right triangle, the square of the length of the hypotenuse equals the sum of the squares of the lengths of the other two sides.*

If  $c$  is the length of the hypotenuse and  $a$  and  $b$  are the lengths of the other two sides (see Fig. 2.30), the Pythagorean theorem is

$$c^2 = a^2 + b^2 \tag{2.4}$$

**EXAMPLE 7 Application of the Pythagorean theorem**

A pole is perpendicular to the level ground around it. A guy wire is attached 3.20 m up the pole and at a point on the ground 2.65 m from the pole. How long is the guy wire?

We sketch the pole and guy wire as shown in Fig. 2.31. Using the Pythagorean theorem, and then substituting, we have

$$\begin{aligned} AC^2 &= AB^2 + BC^2 \\ AC^2 &= 2.65^2 + 3.20^2 \\ AC &= \sqrt{2.65^2 + 3.20^2} = 4.15 \text{ m} \end{aligned}$$

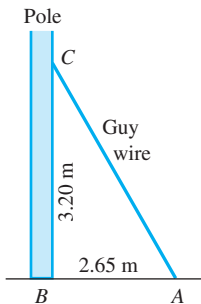


Fig. 2.31

The guy wire is 4.15 m long.

**LEARNING TIP**

For two triangles that are similar, if one property is true, then the other is also true. In two similar triangles, the **corresponding sides** are the sides, one in each triangle, that are between the same pair of equal corresponding angles.

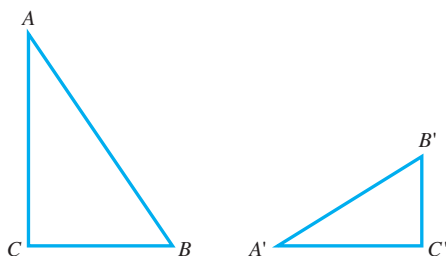


Fig. 2.32

**SIMILAR TRIANGLES**

The perimeter and area of a triangle are measures of its *size*. We now consider the shape of triangles.

Two triangles are **similar** if they have the same shape (but not necessarily the same size). There are two very important properties of similar triangles.

**Properties of Similar Triangles**

1. The corresponding angles of similar triangles are equal.
2. The corresponding sides of similar triangles are proportional.

**EXAMPLE 8 Similar triangles**

In Fig. 2.32, a pair of similar triangles are shown. They are similar even though the corresponding parts are not in the same position relative to the page. Using standard symbols, we can write  $\triangle ABC \sim \triangle A'B'C'$ , where  $\triangle$  means “triangle” and  $\sim$  means “is similar to.”

The pairs of corresponding angles are  $A$  and  $A'$ ,  $B$  and  $B'$ , and  $C$  and  $C'$ . This means  $A = A'$ ,  $B = B'$ , and  $C = C'$ .

The pairs of corresponding sides are  $AB$  and  $A'B'$ ,  $BC$  and  $B'C'$ , and  $AC$  and  $A'C'$ . In order to show that these corresponding sides are *proportional*, we write

$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{AC}{A'C'} \quad \begin{array}{l} \leftarrow \text{sides of } \triangle ABC \\ \leftarrow \text{sides of } \triangle A'B'C' \end{array}$$

If we know that two triangles are similar, we can use the two basic properties of similar triangles to find the unknown parts of one triangle from the known parts of the other triangle. The next example illustrates this in a practical application.

**EXAMPLE 9 Application of similar triangles**

On level ground, a silo casts a shadow 24.0 m long. At the same time, a nearby vertical pole 4.00 m high casts a shadow 3.00 m long. How tall is the silo? See Fig. 2.33.

The rays of the sun are essentially parallel. The two triangles in Fig. 2.33 are similar since *each has a right angle and the angles at the tops are equal*. The other angles must be equal since the sum of the angles is  $180^\circ$ . The lengths of the hypotenuses are of no importance in this problem, so we use only the other sides in stating the ratios of corresponding sides. Denoting the height of the silo as  $h$ , we have

$$\begin{aligned} \frac{h}{4.00} &= \frac{24.0}{3.00} \\ h &= 32.0 \text{ m} \end{aligned}$$

We conclude that the silo is 32.0 m high.

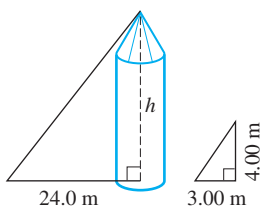


Fig. 2.33

**Practice Exercise**

3. In Fig. 2.33, knowing the value of  $h$ , what is the distance between the top of the silo and the end of its shadow?

One of the most practical uses of similar geometric figures is for **scale drawings**. Maps, charts, architectural blueprints, engineering sketches, and most drawings that appear in books (including many that have already appeared, and will appear, in this text) are familiar examples of scale drawings.

In any scale drawing, all distances are drawn at a certain ratio of the distances that they represent, and all angles are drawn equal to the angles they represent. Note the distances and angles shown in Fig. 2.34 in the following example.

**EXAMPLE 10 Scale drawing**

In drawing a map of the area shown in Fig. 2.34, a scale of  $1 \text{ cm} = 200 \text{ km}$  is used. In measuring the distance between Chicago and Toronto on the map, we find it to be 3.5 cm. The actual distance  $x$  between Chicago and Toronto is found from the proportion

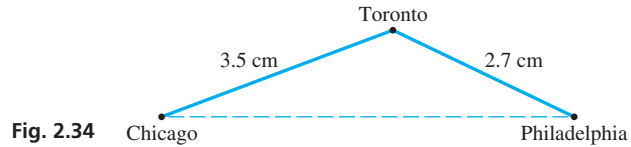


Fig. 2.34

**LEARNING TIP**

This calculation can also be done exactly like a unit conversion (see Section 1.3). We are converting between map units and real units, using the scale of the drawing.

$$x = 3.5 \text{ cm} \cdot \left( \frac{200 \text{ km}}{1 \text{ cm}} \right) = 700 \text{ km}$$

Scale conversion

scale  
↓

$$\begin{array}{l} \text{actual distance} \longrightarrow \frac{x}{3.5 \text{ cm}} = \frac{200 \text{ km}}{1 \text{ cm}} \text{ or } x = 700 \text{ km} \\ \text{distance on map} \longrightarrow \end{array}$$

If we did not have the scale but knew that the distance between Chicago and Toronto is 700 km, then by measuring distances on the map between Chicago and Toronto (3.5 cm) and between Toronto and Philadelphia (2.7 cm), we could find the distance between Toronto and Philadelphia. It is found from the following proportion, determined by use of similar triangles:

$$\begin{aligned} \frac{700 \text{ km}}{3.5 \text{ cm}} &= \frac{y}{2.7 \text{ cm}} \\ y &= \frac{2.7(700)}{3.5} = 540 \text{ km} \end{aligned}$$

Similarity requires *equal* angles and *proportional* sides. If the corresponding angles and the corresponding sides of two triangles are equal, the two triangles are **congruent**. As a result of this definition, the areas and perimeters of congruent triangles are also equal. Informally, we can say that similar triangles have the same shape, whereas congruent triangles have the same shape and same size.

**EXAMPLE 11 Similar and congruent triangles**

A right triangle with legs of 2 m and 4 m is congruent to any other right triangle with legs of 2 m and 4 m. It is also similar to any right triangle with legs of 5 m and 10 m, or any other right triangle that has legs in the ratio of 1 to 2, since the corresponding sides are proportional. See Fig. 2.35.

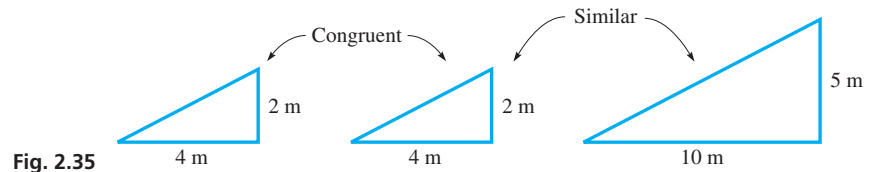


Fig. 2.35

**EXERCISES 2.2**

In Exercises 1–4, answer the given questions about the indicated examples of this section.

1. In Example 2, if  $\angle 1 = 70^\circ$  and  $\angle 5 = 45^\circ$  in Fig. 2.22, what is the measure of  $\angle 2$ ?
2. In Example 5, change 16.2 cm to 61.2 cm. What is the answer?
3. In Example 7, change 2.65 m to 6.25 m. What is the answer?
4. In Example 9, interchange 4.00 m and 3.00 m. What is the answer?

In Exercises 5–8, determine  $\angle A$  in the indicated figures.

5. Fig. 2.36(a)
6. Fig. 2.36(b)

7. Fig. 2.36(c)
8. Fig. 2.36(d)

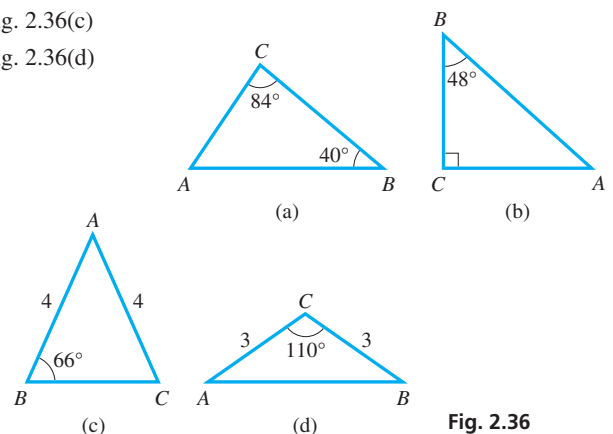


Fig. 2.36



In Exercises 9–16, find the area of each triangle.

9. Fig. 2.37(a)                      10. Fig. 2.37(b)  
 11. Fig. 2.37(c)                    12. Fig. 2.37(d)

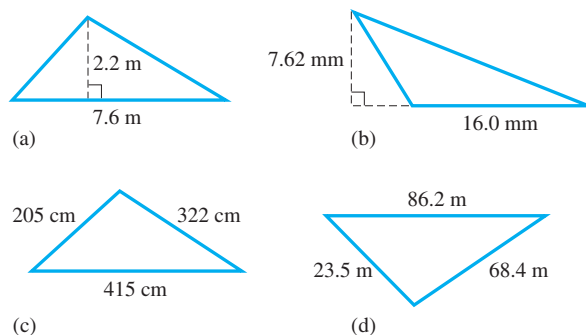


Fig. 2.37

13. Right triangle with legs 3.46 cm and 2.55 cm  
 14. Right triangle with legs 234 mm and 343 mm  
 15. Isosceles triangle, equal sides of 0.986 m, third side of 0.884 m  
 16. Equilateral triangle of sides 322 dm

In Exercises 17–20, find the perimeter of each triangle.

17. Fig. 2.37(c)                      18. Fig. 2.37(d)  
 19. An equilateral triangle of sides 21.5 cm  
 20. Isosceles triangle, equal sides of 2.45 mm, third side of 3.22 mm

In Exercises 21–24, find the third side of the right triangle shown in Fig. 2.38 for the given values.

21.  $a = 13.8$  mm,  $b = 22.7$  mm  
 22.  $a = 2.48$  m,  $b = 1.45$  m  
 23.  $a = 175$  cm,  $c = 551$  cm  
 24.  $b = 0.474$  km,  $c = 0.836$  km

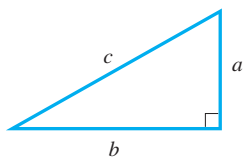


Fig. 2.38

In Exercises 25–28, use the right triangle in Fig. 2.39.

25. Find  $\angle B$ .  
 26. Find side  $c$ .  
 27. Find the perimeter.  
 28. Find the area.

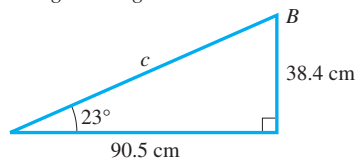


Fig. 2.39

In Exercises 29–54, solve the given problems.

29. What is the angle between the bisectors of the acute angles of a right triangle?  
 30. If the midpoints of the sides of an isosceles triangle are joined, another triangle is formed. What do you conclude about this inner triangle?  
 31. For what type of triangle is the centroid the same as the intersection of altitudes and the intersection of angle bisectors?  
 32. Is it possible that the altitudes of a triangle meet, when extended, outside the triangle? Explain.  
 33. The altitude to the hypotenuse of a right triangle divides the triangle into two smaller triangles. What do you conclude about these two triangles?

34. The altitude to the hypotenuse of a right triangle divides the triangle into two smaller triangles. What do you conclude about the original triangle and the two new triangles? Explain.  
 35. In Fig. 2.40, show that  $\triangle MKL \sim \triangle MNO$ .  
 36. In Fig. 2.41, show that  $\triangle ACB \sim \triangle ADC$ .

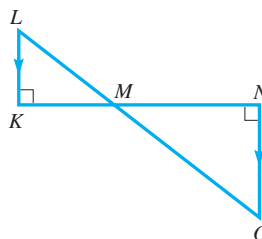


Fig. 2.40

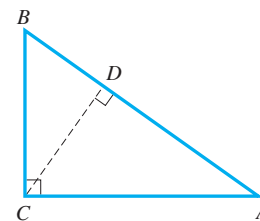


Fig. 2.41

37. In Fig. 2.40, if  $KN = 15$ ,  $MN = 9$ , and  $MO = 12$ , find  $LM$ .  
 38. In Fig. 2.41, if  $AD = 9$  and  $AC = 12$ , find  $AB$ .  
 39. The angle between the roof sections of an A-frame house is  $50^\circ$ . What is the angle between either roof section and a horizontal rafter?  
 40. A transmitting tower is supported by a wire that makes an angle of  $52^\circ$  with the level ground. What is the angle between the tower and the wire?  
 41. A wall pennant is in the shape of an isosceles triangle. If each equal side is 76.6 cm long and the third side is 30.6 cm, what is the area of the pennant?  
 42. The Bermuda Triangle is sometimes defined as an equilateral triangle 1600 km on a side, with vertices in Bermuda, Puerto Rico, and the Florida coast. Assuming it is flat, what is its approximate area?  
 43. The sail of a sailboat is in the shape of a right triangle with sides of 3.2 m, 6.0 m, and 6.8 m. What is the area of the sail?  
 44. An observer is 550 m horizontally from the launch pad of a rocket. After the rocket has ascended 750 m, how far is it from the observer?  
 45. The base of a 6.0-m ladder is 1.8 m from a wall. How far up on the wall does the ladder reach?

46. The beach shade shown in Fig. 2.42 is made of  $30^\circ$ - $60^\circ$ - $90^\circ$  triangular sections. Find  $x$ . (In a  $30^\circ$ - $60^\circ$ - $90^\circ$  triangle, the side opposite the  $30^\circ$  angle is one-half the hypotenuse.)

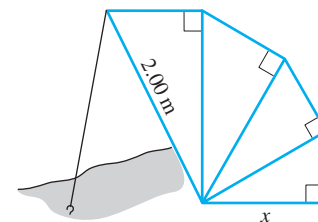


Fig. 2.42

47. A rectangular room is 18.0 m long, 12.0 m wide, and 8.00 m high. What is the length of the longest diagonal from one corner to another corner of the room?  
 48. On a blueprint, a hallway is 45.6 cm long. The scale is 1.20 cm = 1.00 m. How long is the hallway?  
 49. Two parallel guy wires are attached to a vertical pole 4.5 m and 5.4 m above the ground. They are secured on the level ground at points 1.2 m apart. How long are the guy wires?  
 50. The two sections of a folding door, hinged in the middle, are each 0.85 m wide. What width of the doorway is taken up when the sections are at right angles to each other?

51. A 1.5-m wall stands 0.75 m from a building. The ends of a straight pole touch the building and the ground 2.0 m from the wall. A point on the pole touches the top of the wall. How long is the pole? See Fig. 2.43.

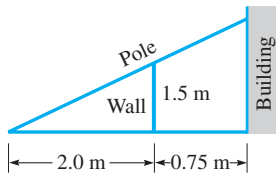


Fig. 2.43

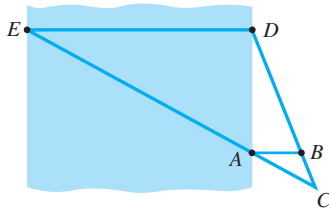


Fig. 2.44

52. To find the width  $ED$  of a river, a surveyor places markers at  $A$ ,  $B$ ,  $C$ , and  $D$ , as shown in Fig. 2.44. The markers are placed such that  $AB \parallel ED$ ,  $BC = 50.0$  m,  $DC = 312$  m, and  $AB = 80.0$  m. How wide is the river?
53. A water pumping station is to be built on a river at point  $P$  in order to deliver water to points  $A$  and  $B$ . See Fig. 2.45. The design requires that  $\angle APD = \angle BPC$  so that the total length of piping that will be needed is a minimum. Find this minimum length of pipe.

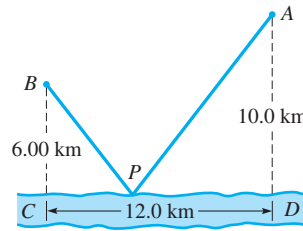


Fig. 2.45

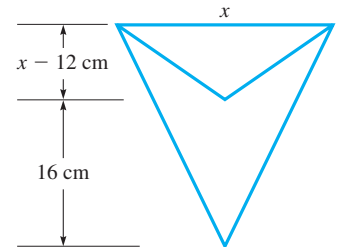


Fig. 2.46

54. The cross section of a drainage trough has the shape of an isosceles triangle whose depth is 12 cm less than its width. If the depth is increased by 16 cm and the width remains the same, the area of the cross section is increased by  $160 \text{ cm}^2$ . Find the original depth and width. See Fig. 2.46.

Answers to Practice Exercises

1.  $75^\circ$  2. 686 m 3. 40.0 m

## 2.3 Quadrilaterals

### Types and Properties of Quadrilaterals • Perimeter and Area

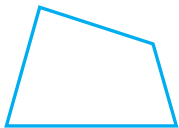


Fig. 2.47

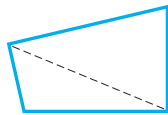


Fig. 2.48

A **quadrilateral** is a closed plane figure with four sides, and these four sides form four interior angles. A general quadrilateral is shown in Fig. 2.47.

A **diagonal** of a polygon is a straight line segment joining any two nonadjacent vertices. The dashed line is one of two diagonals of the quadrilateral in Fig. 2.48.

### TYPES OF QUADRILATERALS

A **parallelogram** is a quadrilateral in which opposite sides are parallel. In a parallelogram, opposite sides are equal and opposite angles are equal. A **rhombus** is a parallelogram with four equal sides.

A **rectangle** is a parallelogram in which intersecting sides are perpendicular, which means that all four interior angles are right angles. In a rectangle, the longer side is usually called the **length**, and the shorter side is called the **width**. A **square** is a rectangle with four equal sides.

A **trapezoid** is a quadrilateral in which two sides are parallel. The parallel sides are called the **bases** of the trapezoid.

### EXAMPLE 1 Types of quadrilaterals

A parallelogram is shown in Fig. 2.49(a). Opposite sides  $a$  are equal in length, as are opposite sides  $b$ . A rhombus with equal sides  $s$  is shown in Fig. 2.49(b). A rectangle is shown in Fig. 2.49(c). The length is labelled  $l$ , and the width is labelled  $w$ . A square with equal sides  $s$  is shown in Fig. 2.49(d). A trapezoid with bases  $b_1$  and  $b_2$  is shown in Fig. 2.49(e).

Practice Exercise

1. Develop a formula for the length  $d$  of a diagonal for the rectangle in Fig. 2.49(c).

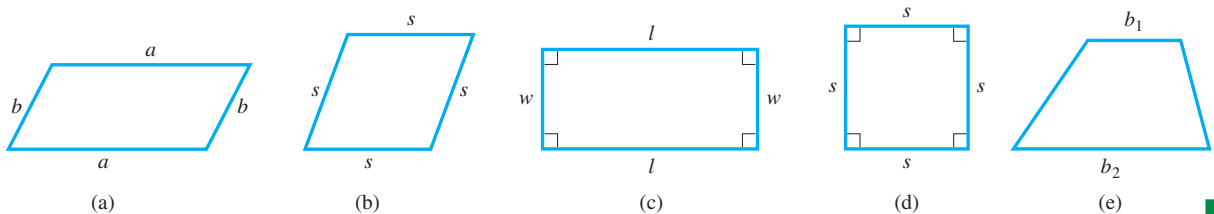


Fig. 2.49

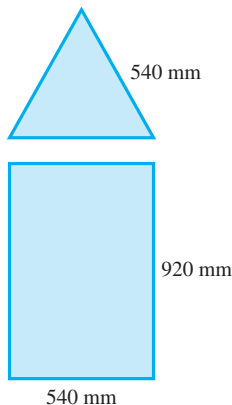


Fig. 2.50

**LEARNING TIP**

We could write down formulas for the perimeters of the different kinds of triangles and quadrilaterals. However, if we *remember the meaning of perimeter as being the total distance around a geometric figure*, such formulas are not necessary.

**Practice Exercise**

2. If we did develop perimeter formulas, what would be the formula for the perimeter  $p$  of a rhombus of side  $s$ ?

**PERIMETER AND AREA OF A QUADRILATERAL**

The perimeter of a quadrilateral is the sum of the lengths of the four sides.

**EXAMPLE 2 Application of perimeter**

An architect designs a room with a rectangular window 920 mm high and 540 mm wide, with another window above in the shape of an equilateral triangle, 540 mm on a side. See Fig. 2.50. How much moulding is needed for these windows?

The length of moulding is the sum of the perimeters of the windows. For the rectangular window, the opposite sides are equal, which means the perimeter is twice the length  $l$  plus twice the width  $w$ . For the equilateral triangle, the perimeter is three times the side  $s$ . Therefore, the length  $L$  of moulding is

$$L = 2l + 2w + 3s$$

$$L = 2(920) + 2(540) + 3(540)$$

$$L = 4540 \text{ mm}$$

For the areas of the square, rectangle, parallelogram, and trapezoid, we have the following formulas:

$$A = s^2 \quad \text{Square of side } s \text{ (Fig. 2.51)} \quad (2.5)$$

$$A = lw \quad \text{Rectangle of length } l \text{ and width } w \text{ (Fig. 2.52)} \quad (2.6)$$

$$A = bh \quad \text{Parallelogram of base } b \text{ and height } h \text{ (Fig. 2.53)} \quad (2.7)$$

$$A = \frac{1}{2}h(b_1 + b_2) \quad \text{Trapezoid of bases } b_1 \text{ and } b_2 \text{ and height } h \text{ (Fig. 2.54)} \quad (2.8)$$

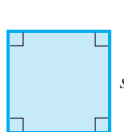


Fig. 2.51

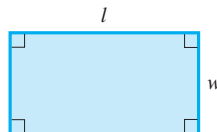


Fig. 2.52

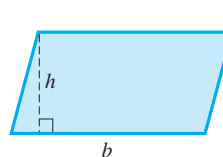


Fig. 2.53

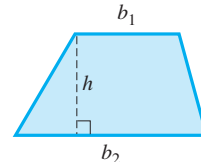


Fig. 2.54

Since a rectangle, a square, and a rhombus are special types of parallelograms, the area of these figures can be found from Eq. (2.7). The area of a trapezoid is of importance when we find areas of irregular geometric figures in Section 2.5.

**EXAMPLE 3 Application of area**

A city park is designed with lawn areas in the shape of a right triangle, a parallelogram, and a trapezoid, as shown in Fig. 2.55, with walkways between them. Find the area of each section of lawn and the total lawn area.

$$A_1 = \frac{1}{2}bh = \frac{1}{2}(72)(45) = 1600 \text{ m}^2$$

$$A_2 = bh = (72)(45) = 3200 \text{ m}^2$$

$$A_3 = \frac{1}{2}h(b_1 + b_2) = \frac{1}{2}(45)(72 + 35) = 2400 \text{ m}^2$$

The total lawn area is about 7200 m<sup>2</sup>.

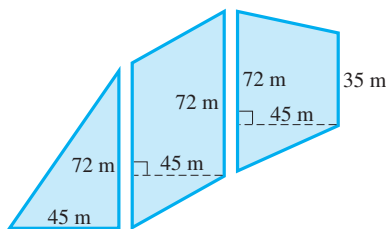


Fig. 2.55

■ The computer microprocessor chip was first commercially available in 1971.

**EXAMPLE 4 Perimeter in a word problem**

The length of a rectangular computer chip is 2.0 mm longer than its width. Find the dimensions of the chip if its perimeter is 26.4 mm.

Since the dimensions, the length and the width, are required, let  $w$  = the width of the chip. Since the length is 2.0 mm more than the width, we know that  $w + 2.0$  = the length of the chip. See Fig. 2.56.

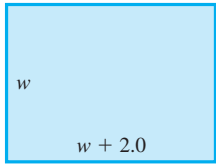


Fig. 2.56

Since the perimeter of a rectangle is twice the length plus twice the width, we have the equation

$$2(w + 2.0) + 2w = 26.4$$

since the perimeter is given as 26.4 mm. This is the equation we need.

Solving this equation, we have

$$2w + 4.0 + 2w = 26.4$$

$$4w = 22.4$$

$$w = 5.6 \text{ mm} \quad \text{and} \quad w + 2.0 = 7.6 \text{ mm}$$

Therefore, the length is 7.6 mm and the width is 5.6 mm. These values check with the statements of the original problem. ■

### EXERCISES 2.3

In Exercises 1–4, make the given changes in the indicated examples of this section and then solve the given problems.

1. In Example 1, interchange the lengths of  $b_1$  and  $b_2$  in Fig. 2.49(e). What type of quadrilateral is the resulting figure?
2. In Example 2, change the equilateral triangle of side 540 mm to a square of side 540 mm and then find the length of moulding.
3. In Example 3, change the dimension of 45 m to 55 m in each figure and then find the area.
4. In Example 4, change 2.0 mm to 3.0 mm, and then solve.

In Exercises 5–12, find the perimeter of each figure.

5. Square: side of 65 m
6. Rhombus: side of 2.46 km
7. Rectangle:  $l = 0.920$  mm,  $w = 0.742$  mm
8. Rectangle:  $l = 142$  cm,  $w = 126$  cm
9. Parallelogram in Fig. 2.57
10. Parallelogram in Fig. 2.58
11. Trapezoid in Fig. 2.59
12. Trapezoid in Fig. 2.60

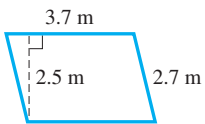


Fig. 2.57

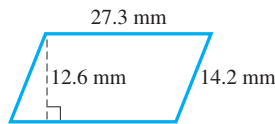


Fig. 2.58

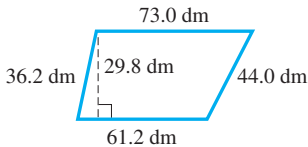


Fig. 2.59

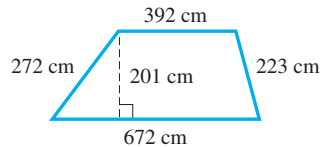


Fig. 2.60

In Exercises 13–20, find the area of each figure.

13. Square:  $s = 2.7$  mm
14. Square:  $s = 15.6$  m
15. Rectangle:  $l = 0.920$  km,  $w = 0.742$  km
16. Rectangle:  $l = 142$  cm,  $w = 126$  cm
17. Parallelogram in Fig. 2.57
18. Parallelogram in Fig. 2.58
19. Trapezoid in Fig. 2.59
20. Trapezoid in Fig. 2.60

In Exercises 21–24, set up a formula for the indicated perimeter or area. (Do not include dashed lines.)

21. The perimeter of the figure in Fig. 2.61 (a parallelogram and a square attached)
22. The perimeter of the figure in Fig. 2.62 (two trapezoids attached)
23. Area of figure in Fig. 2.61
24. Area of figure in Fig. 2.62

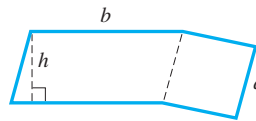


Fig. 2.61

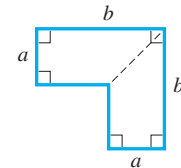


Fig. 2.62

In Exercises 25–40, solve the given problems.

25. If the angle between adjacent sides of a parallelogram is  $90^\circ$ , what conclusion can you make about the parallelogram?
26. What conclusion can you make about the two triangles formed by the sides and diagonal of a parallelogram? Explain.
27. Find the area of a square whose diagonal is 24.0 cm.
28. In a trapezoid, find the angle between the bisectors of the two angles formed by the bases and one nonparallel side.
29. Noting how a diagonal of a rhombus divides an interior angle, explain why the automobile jack in Fig. 2.63 is in the shape of a rhombus.

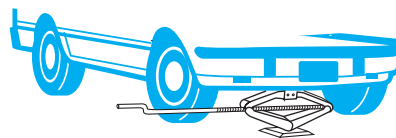


Fig. 2.63

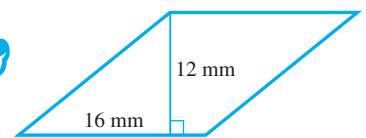


Fig. 2.64

30. Part of an electric circuit is wired in the configuration of a rhombus and one of its altitudes, as shown in Fig. 2.64. What is the length of wire in this part of the circuit?

31. A walkway 3.00 m wide is constructed along the outside edge of a square courtyard. If the perimeter of the courtyard is 324 m, what is the perimeter of the square formed by the outer edge of the walkway?
32. An architect designs a rectangular window such that the width of the window is 450 mm less than the height. If the perimeter of the window is 4500 mm, what are its dimensions?
33. A designer plans the top of a rectangular workbench to be four times as long as it is wide and then determines that if the width were 1500 mm greater and the length were 4500 mm less, it would be a square. What are its dimensions?
34. A beam support in a building is in the shape of a parallelogram, as shown in Fig. 2.65. Find the area of the side of the beam shown.

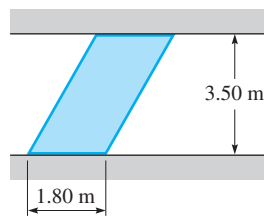


Fig. 2.65

35. Each of two walls (with rectangular windows) of an A-frame house has the shape of a trapezoid, as shown in Fig. 2.66. If a litre of paint covers 12 m<sup>2</sup>, how much paint is required to paint these walls? (All data are accurate to two significant digits.)

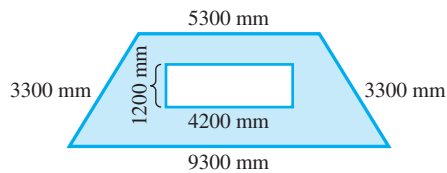


Fig. 2.66

36. Six equal trapezoidal sections form a conference table in the shape of a hexagon, with a hexagonal opening in the middle. See Fig. 2.67. From the dimensions shown, find the area of the table top.

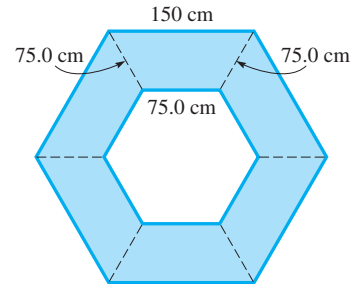


Fig. 2.67

37. A fenced section of a ranch is in the shape of a quadrilateral whose sides are 1.74 km, 1.46 km, 2.27 km, and 1.86 km, the last two sides being perpendicular to each other. Find the area of the section.
38. A rectangular security area is enclosed on one side by a wall, and the other sides are fenced. The length of the wall is twice the width of the area. The total cost of building the wall and fence is \$13 200. If the wall costs \$50.00/m and the fence costs \$5.00/m, find the dimensions of the area.
39. What is the sum of the measures of the interior angles of a quadrilateral? Explain.
40. Find a formula for the area of a rhombus in terms of its diagonals  $d_1$  and  $d_2$ . (See Exercise 29.)

Answers to Practice Exercises

1.  $d = \sqrt{l^2 + w^2}$
2.  $p = 4s$

## 2.4 Circles

Properties of Circles • Tangent and Secant Lines • Circumference and Area • Circular Arcs and Angles • Radian Measure of an Angle

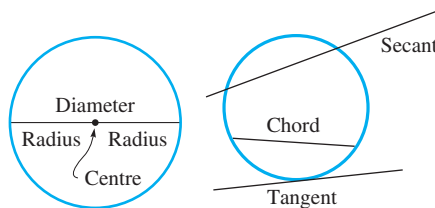


Fig. 2.68

Fig. 2.69

The next geometric figure we consider is the circle. All points on a **circle** are at the same distance from a fixed point, the **centre** of the circle. The distance from the centre to a point on the circle is the **radius** of the circle. The distance between two points on the circle on a line through the centre is the **diameter**. Therefore, the diameter  $d$  is twice the radius  $r$ , or  $d = 2r$ . See Fig. 2.68.

There are also certain types of lines associated with a circle. A **chord** is a line segment having its endpoints on the circle. A **tangent** is a line that touches (does not pass through) the circle at one point. A **secant** is a line that passes through two points of the circle. See Fig. 2.69.

An important property of a tangent is that a *tangent is perpendicular to the radius drawn to the point of contact*. This is illustrated in the following example.

### EXAMPLE 1 Tangent line perpendicular to the radius

In Fig. 2.70,  $O$  is the centre of the circle, and  $AB$  is tangent at  $B$ . If  $\angle OAB = 25^\circ$ , find  $\angle AOB$ .

Since the centre is  $O$ ,  $OB$  is a radius of the circle. A tangent is perpendicular to a radius at the point of tangency, which means  $\angle ABO = 90^\circ$ .

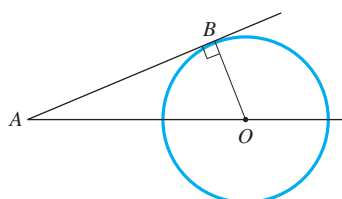


Fig. 2.70

Since the sum of the angles of a triangle is  $180^\circ$ , we have

$$\begin{aligned}\angle OAB + \angle ABO + \angle AOB &= 180^\circ \\ \angle AOB &= 180^\circ - 90^\circ - 25^\circ \\ \angle AOB &= 65^\circ\end{aligned}$$

### CIRCUMFERENCE AND AREA OF A CIRCLE

The perimeter of a circle is called the **circumference**. The formulas for the circumference and area of a circle are as follows:

$$c = 2\pi r \quad \text{Circumference of a circle of radius } r \quad (2.9)$$

$$A = \pi r^2 \quad \text{Area of a circle of radius } r \quad (2.10)$$

■ The symbol  $\pi$  (the Greek letter pi), which we use as a number, was first used in this way as a number in the 1700s.

#### LEARNING TIP

The formulas for the circumference and the area of a circle can also be written in terms of the diameter  $d$ :

$$c = \pi d$$

$$A = \frac{\pi d^2}{4}$$

The most convenient form for the parameter given (radius or diameter) can be used.

Here,  $\pi$  equals approximately 3.1416. In using a calculator,  $\pi$  can be entered to a much greater accuracy by using the  $\pi$  key.

#### EXAMPLE 2 Application of the area of a circle

A circular oil spill has a diameter of 2.4 km. It is to be enclosed within special flexible tubing. What is the area of the spill, and how long must the tubing be? See Fig. 2.71.

Since  $d = 2r$ ,  $r = d/2 = 1.2$  km. Using Eq. (2.10), the area is

$$\begin{aligned}A &= \pi r^2 = \pi (1.2)^2 \\ A &= 4.5 \text{ km}^2\end{aligned}$$

The length of tubing needed is the circumference of the circle. Therefore,

$$\begin{aligned}c &= 2\pi r = 2\pi (1.2) \\ c &= 7.5 \text{ km} \quad \text{rounded off}\end{aligned}$$

Many applied problems involve a combination of geometric figures. The following example illustrates one such combination.

#### EXAMPLE 3 Application of perimeter and area for circles

A machine part is a square of side 3.25 cm with a quarter-circle removed (see Fig. 2.72). Find the perimeter and the area of one side of the part.

Setting up a formula for the perimeter, we add the two sides of length  $s$  to *one-fourth of the circumference of a circle with radius  $s$* . For the area, we *subtract the area of one-fourth of a circle from the area of the square*. This gives

$$p = 2s + \frac{2\pi s}{4} = 2s + \frac{\pi s}{2} \quad A = s^2 - \frac{\pi s^2}{4}$$

bottom and left
circular section
square
quarter circle

where  $s$  is the side of the square and the radius of the circle. Evaluating, we have

$$\begin{aligned}p &= 2(3.25) + \frac{\pi(3.25)}{2} = 11.6 \text{ cm} \\ A &= 3.25^2 - \frac{\pi(3.25)^2}{4} = 2.27 \text{ cm}^2\end{aligned}$$

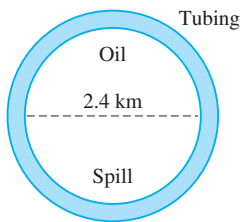


Fig. 2.71

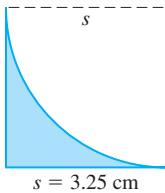


Fig. 2.72

#### Practice Exercises

- Find the circumference of a circle with a radius of 20.0 cm.
- Find the area of the circle in Exercise 1.

### CIRCULAR ARCS AND ANGLES

An **arc** is part of a circle, and an angle formed at the centre by two radii is a **central angle**. The **measure of an arc** is the same as the central angle between the ends of the radii that define the arc. A **sector** of a circle is the region bounded by two radii and the



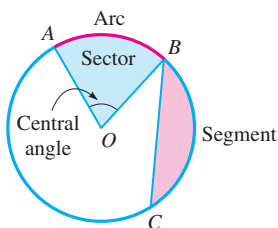


Fig. 2.73

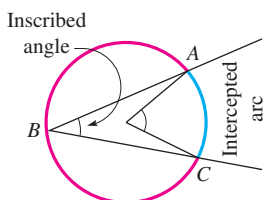


Fig. 2.74

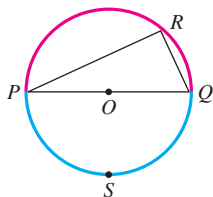


Fig. 2.75

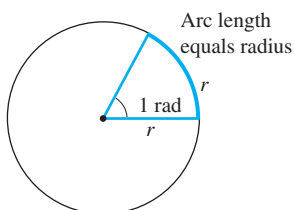


Fig. 2.76

arc they intercept. A **segment** of a circle is the region bounded by a chord and its arc. (There are two possible segments for a given chord. The smaller region is a *minor segment*, and the larger region is a *major segment*.) These are illustrated in the following example.

#### EXAMPLE 4 Sector and segment

In Fig. 2.73, a sector of the circle is between radii  $OA$  and  $OB$  and arc  $AB$  (denoted as  $\widehat{AB}$ ). If the measure of the central angle at  $O$  between the radii is  $70^\circ$ , the measure of  $\widehat{AB}$  is  $70^\circ$ .

A segment of the circle is the region between chord  $BC$  and arc  $BC$  ( $\widehat{BC}$ ). ■

An **inscribed angle** of an arc is one for which the endpoints of the arc are points on the sides of the angle and for which the vertex is a point (not an endpoint) on the arc. An important property of a circle is that the measure of an inscribed angle is one-half the measure of its intercepted arc.

#### EXAMPLE 5 Inscribed angle

- (a) In the circle shown in Fig. 2.74,  $\angle ABC$  is inscribed in  $\widehat{AC}$ , and it intercepts  $\widehat{AC}$ . If the measure of  $\widehat{AC} = 60^\circ$ , then  $\angle ABC = 30^\circ$ .
- (b) In the circle shown in Fig. 2.75,  $PQ$  is a diameter, and  $\angle PRQ$  is inscribed in the semicircular  $\widehat{PRQ}$ . Since the measure of  $\widehat{PSQ} = 180^\circ$ ,  $\angle PRQ = 90^\circ$ . From this we conclude that *an angle inscribed in a semicircle is a right angle*. ■

### RADIAN MEASURE OF AN ANGLE

To this point, we have measured all angles in degrees. There is another measure of an angle, the *radian*, that is defined in terms of an arc of a circle. We will find it of importance when we study trigonometry.

If a central angle of a circle intercepts an arc equal in length to the radius of the circle, the measure of the central angle is defined as **1 radian**. See Fig. 2.76. The radius can be marked off along the circumference  $2\pi$  times (about 6.283 times). Thus,  $2\pi \text{ rad} = 360^\circ$  (where rad is the symbol for radian), and the basic relationship between radians and degrees is

$$\pi \text{ rad} = 180^\circ \quad (2.11)$$

This ratio can be used as a unit conversion factor, as discussed in Section 1.3.

#### EXAMPLE 6 Radian measure of an angle

- (a) If we divide each side of Eq. (2.11) by  $\pi$ , we get

$$1 \text{ rad} = 57.3^\circ$$

where the result has been rounded off.

- (b) To change an angle of  $118.2^\circ$  to radian measure, we have

$$118.2^\circ = 118.2^\circ \left( \frac{\pi \text{ rad}}{180^\circ} \right) = 2.06 \text{ rad}$$

Multiplying  $118.2^\circ$  by  $\pi \text{ rad}/180^\circ$ , the unit that remains is rad since degrees “cancel.” We will review radian measure again when we study trigonometry. ■

Radian measure is especially useful when determining the length of an arc formed by a central angle in a circle. Since the arc length for 1 radian of central angle is equivalent to 1 radius, any multiple of radian measure for the central angle will result in the

#### Practice Exercise

3. Express the angle  $85.0^\circ$  in radian measure.



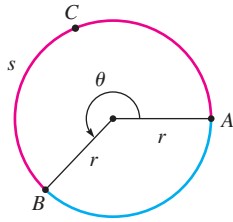


Fig. 2.77

same multiple for the length of the resulting arc (see Fig. 2.77). In general, the arc length  $s$  (the length of  $\widehat{ABC}$ ) can be determined from the product of the central angle  $\theta$  (if expressed in radians) and the radius  $r$ .

$$s = \theta r \tag{2.12}$$

**LEARNING TIP**

Notice that a radian, although named, is “unitless.” It represents a multiple of a radius and possesses no real units of its own. For this reason, when multiplying the radius by the angle to obtain arc length, we obtain the same unit of length as the radius.

**EXAMPLE 7 Arc length with radian measure**

If a circle of radius 2.75 m has a central angle of 3.76 radians, determine the length of the arc formed by the central angle.

$$\begin{aligned} s &= \theta r \\ s &= (3.76)(2.75 \text{ m}) \\ s &= 10.3 \text{ m} \end{aligned}$$

**EXERCISES 2.4**

In Exercises 1–4, answer the given questions about the indicated examples of this section.

1. In Example 1, if  $\angle AOB = 72^\circ$  in Fig. 2.70, then what is the measure of  $\angle OAB$ ?
2. In the first line of Example 2, if “diameter” is changed to “radius,” what are the results?
3. In Example 3, if the machine part is the unshaded part (rather than the shaded part) of Fig. 2.72, what are the results?
4. In Example 5(a), if  $\angle ABC = 25^\circ$  in Fig. 2.74, then what is the measure of  $\widehat{AC}$ ?

In Exercises 5–8, refer to the circle with centre at  $O$  in Fig. 2.78. Identify the following.

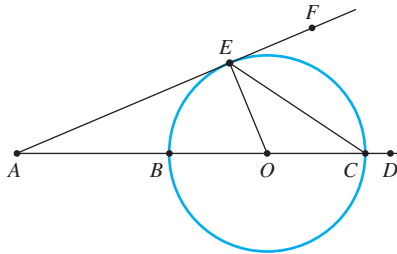


Fig. 2.78

5. (a) A secant line  
(b) A tangent line
6. (a) Two chords  
(b) An inscribed angle
7. (a) Two perpendicular line segments  
(b) An isosceles triangle
8. (a) A segment  
(b) A sector with an acute central angle

In Exercises 9–12, find the circumference of the circle with the given radius or diameter.

9.  $r = 275 \text{ cm}$
10.  $r = 0.563 \text{ m}$
11.  $d = 23.1 \text{ mm}$
12.  $d = 8.2 \text{ dm}$

In Exercises 13–16, find the area of the circle with the given radius or diameter.

13.  $r = 0.0952 \text{ km}$
14.  $r = 45.8 \text{ cm}$
15.  $d = 2.33 \text{ m}$
16.  $d = 1256 \text{ mm}$

In Exercises 17–20, refer to Fig. 2.79, where  $AB$  is a diameter,  $TB$  is a tangent line at  $B$ , and  $\angle ABC = 65^\circ$ . Determine the indicated angles.

17.  $\angle CBT$
18.  $\angle BCT$
19.  $\angle CAB$
20.  $\angle BTC$

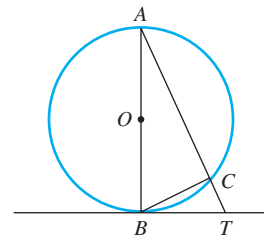


Fig. 2.79

In Exercises 21–24, refer to Fig. 2.80. Determine the indicated arcs and angles.

21.  $\widehat{BC}$  (angle measure)
22.  $\widehat{AB}$  (angle measure)
23.  $\angle ABC$
24.  $\angle ACB$

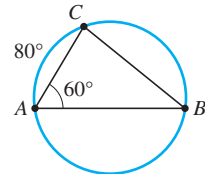


Fig. 2.80

In Exercises 25–28, change the given angles to radian measure.

25.  $22.5^\circ$
26.  $60.0^\circ$
27.  $125.2^\circ$
28.  $323.0^\circ$

In Exercises 29–32, find a formula for the indicated perimeter or area.

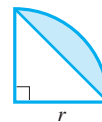


Fig. 2.81

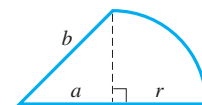


Fig. 2.82

29. The perimeter of the quarter-circle in Fig. 2.81
30. The perimeter of the figure in Fig. 2.82. A quarter-circle is attached to a triangle.

- 31. The area of the segment of the quarter-circle in Fig. 2.81
- 32. The area of the figure in Fig. 2.82

In Exercises 33–49, solve the given problems.

- 33. Describe the location of the midpoints of a set of parallel chords of a circle.
- 34. The measure of  $\widehat{AB}$  on a circle of radius  $r$  is  $45^\circ$ . What is the length of the arc in terms of  $r$  and  $\pi$ ?
- 35. In a circle, a chord connects the ends of two perpendicular radii of 6.00 cm. What is the area of the minor segment?
- 36. In Fig. 2.83, chords  $AB$  and  $DE$  are parallel. What is the relationship between  $\triangle ABC$  and  $\triangle CDE$ ? Explain.

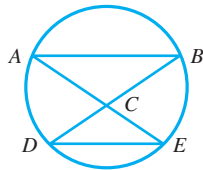


Fig. 2.83

- 37. The radius of the earth's equator is 6375 km. What is the circumference?
- 38. As a ball bearing rolls along a straight track, it makes 11.0 revolutions while travelling a distance of 109 mm. Find its radius.
- 39. The rim on a basketball hoop has an inside diameter of 45.7 cm. The largest cross section of a basketball has a diameter of 30.5 cm. What is the ratio of the cross-sectional area of the basketball to the area of the hoop?
- 40. With no change in the speed of flow, by what factor should the diameter of a pipe be increased in order to double the amount of water that flows through the pipe?
- 41. Using a tape measure, the circumference of a tree is found to be 112 cm. What is the diameter of the tree (assuming a circular cross section)?
- 42. What is the area of the largest circle that can be cut from a rectangular plate 21.2 cm by 15.8 cm?
- 43. A window designed between semicircular regions is shown in Fig. 2.84. Find the area of the window.

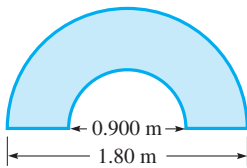


Fig. 2.84

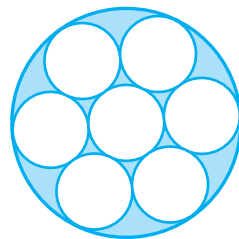


Fig. 2.85

- 44. The cross section of a large circular conduit has seven smaller equal circular conduits within it. The conduits are tangent to each other, as shown in Fig. 2.85. What fraction of the large conduit is occupied by the seven smaller conduits?

- 45. Find the area of the room in the plan shown in Fig. 2.86.

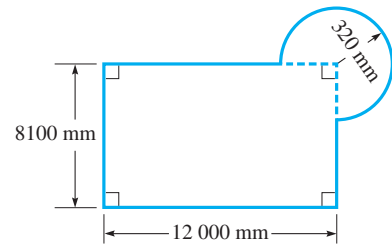


Fig. 2.86

- 46. Find the length of the pulley belt shown in Fig. 2.87 if the belt crosses at right angles. The radius of each pulley wheel is 5.50 cm.

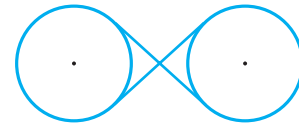


Fig. 2.87

- 47. The velocity of an object moving in a circular path is directed tangent to the circle in which it is moving. A stone on a string moves in a vertical circle, and the string breaks after 5.5 revolutions. If the string was initially in a vertical position, in what direction does the stone move after the string breaks? Explain.
- 48. Part of a circular gear with 24 teeth is shown in Fig. 2.88. Find the indicated angle.

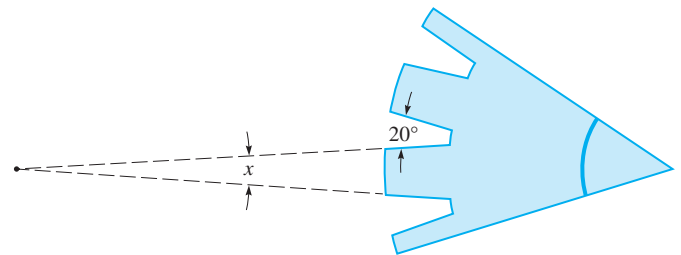


Fig. 2.88

- 49. The Natapoka Arc is a geological feature located on the eastern shore of Hudson Bay. It is a nearly perfect semicircular arc, covering about 2.8 radians of a circle 450 km in diameter. Determine the length of the arc.

Answers to Practice Exercises

- 1. 126 cm    2. 1260 cm<sup>2</sup>    3. 1.48 rad

## 2.5 Measurement of Irregular Areas

Trapezoidal Rule • Simpson's Rule

The figures for which we have found areas are well defined, and the areas can be found by direct use of a specific formula. In practice, however, it may be necessary to find the area of a figure with an irregular perimeter or one for which there is no specific formula. In this section, we show two methods of finding a very good *approximation* of such an area. These methods are particularly useful in technical areas such as surveying, architecture, and mechanical design.

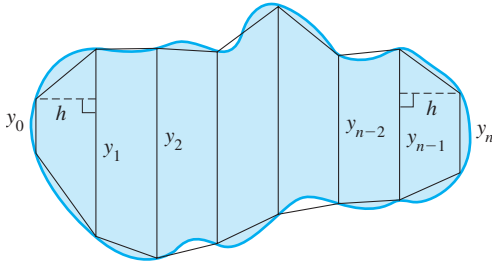


Fig. 2.89

### THE TRAPEZOIDAL RULE

The first method is based on dividing the required area into trapezoids with equal heights. Considering the area shown in Fig. 2.89, we draw parallel lines at *n* equal intervals between the edges of the area. We then join the ends of these parallel line segments to form adjacent trapezoids. The sum of the areas of the trapezoids gives a reasonable approximation to the required area.

Calling the lengths of the parallel lines *y*<sub>0</sub>, *y*<sub>1</sub>, *y*<sub>2</sub>, . . . , *y*<sub>*n*</sub> and the height of each trapezoid *h* (the distance between the parallel lines), the total area *A* is the sum of areas of all the trapezoids. This gives us

$$A = \frac{h}{2}(y_0 + y_1) + \frac{h}{2}(y_1 + y_2) + \frac{h}{2}(y_2 + y_3) + \cdots + \frac{h}{2}(y_{n-2} + y_{n-1}) + \frac{h}{2}(y_{n-1} + y_n)$$

$$A = \frac{h}{2}(y_0 + y_1 + y_1 + y_2 + y_2 + y_3 + \cdots + y_{n-2} + y_{n-1} + y_{n-1} + y_n)$$

Therefore, the approximate area is

$$A = \frac{h}{2}(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n) \tag{2.13}$$

Equation (2.13) is known as the **trapezoidal rule**. We will encounter it again in Section 25.5 as an approximation for areas under curves.

**COMMON ERROR** Note carefully that the values of *y*<sub>0</sub> and *y*<sub>*n*</sub> are *not* multiplied by 2. All the lengths in the trapezoidal rule are multiplied by 2 **except** *y*<sub>0</sub> and *y*<sub>*n*</sub>.

**EXAMPLE 1** Application of the trapezoidal rule

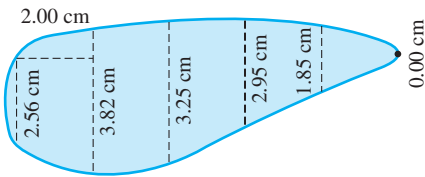


Fig. 2.90

A plate cam for opening and closing a valve is shown in Fig. 2.90. Widths of the face of the cam are shown at 2.00-cm intervals from one end of the cam. Find the area of the face of the cam.

From the figure, we see that

$$y_0 = 2.56 \text{ cm} \quad y_1 = 3.82 \text{ cm} \quad y_2 = 3.25 \text{ cm}$$

$$y_3 = 2.95 \text{ cm} \quad y_4 = 1.85 \text{ cm} \quad y_5 = 0.00 \text{ cm}$$

(In making such measurements, often a *y*-value at one end—or both ends—is zero. In such a case, the end “trapezoid” is actually a triangle.) From the given information in this example, *h* = 2.00 cm. Therefore, using the trapezoidal rule, Eq. (2.13), we have

$$A = \frac{2.00}{2} [2.56 + 2(3.82) + 2(3.25) + 2(2.95) + 2(1.85) + 0.00]$$

$$A = 26.3 \text{ cm}^2$$

The area of the face of the cam is approximately 26.3 cm<sup>2</sup>. ■

When approximating the area with trapezoids, we omit small parts of the area for some trapezoids and include small extra areas for other trapezoids. The omitted areas often approximate the extra areas, which makes the approximation better. Also, the use of smaller intervals improves the approximation since the total omitted area or total extra area is smaller.

**EXAMPLE 2 Application of the trapezoidal rule**

From a satellite photograph of Lake Ontario (as shown on page 55), one of the Great Lakes between the United States and Canada, measurements of the width of the lake were made along its length, starting at the west end, at 26.0-km intervals. The widths are shown in Fig. 2.91 and are given in the following table.

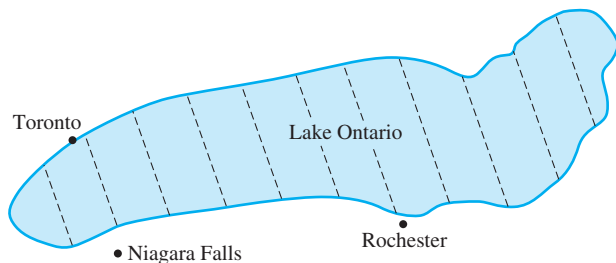


Fig. 2.91

Distance from West End (km)	0.0	26.0	52.0	78.0	104	130	156
Width (km)	0.0	46.7	52.1	59.2	60.4	65.7	73.9
Distance from West End (km)	182	208	234	260	286	312	
Width (km)	87.0	75.5	66.4	86.1	77.0	0.0	

Here, we see that  $y_0 = 0.0$  km,  $y_1 = 46.7$  km,  $y_2 = 52.1$  km,  $\dots$ , and  $y_n = 0.0$  km. Therefore, using the trapezoidal rule, the approximate area of Lake Ontario is found as follows:

$$A = \frac{26.0}{2} [0.0 + 2(46.7) + 2(52.1) + 2(59.2) + 2(60.4) + 2(65.7) + 2(73.9) + 2(87.0) + 2(75.5) + 2(66.4) + 2(86.1) + 2(77.0) + 0.0]$$

$$A = 19\,500 \text{ km}^2$$

The area of Lake Ontario is actually  $19\,477 \text{ km}^2$ . ■

■ See the chapter introduction.

**Practice Exercise**

- In Example 2, use only the distances from west end of (in km) 0.0, 52.0, 104, 156, 208, 260, and 312. Calculate the area and compare with the answer in Example 2.

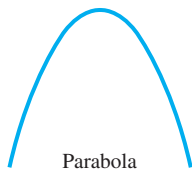


Fig. 2.92

**SIMPSON'S RULE**

For an improved method of measuring an irregular area, we also draw parallel lines at equal intervals between the edges of the area. We then join the ends of these parallel lines with curved arcs. This takes into account the fact that the perimeters of most figures are curved. The arcs used in this method are not arcs of a circle, but arcs of a parabola. A parabola is shown in Fig. 2.92 and is discussed in detail in Chapter 21. (Examples of parabolas are (1) the path of a ball that has been thrown and (2) the cross section of a microwave satellite “dish.”)

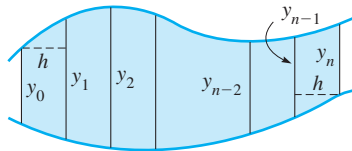


Fig. 2.93

The development of this method requires advanced mathematics. Therefore, we will simply state the formula to be used. It might be noted that the form of the equation is similar to that of the trapezoidal rule.

The approximate area of the geometric figure shown in Fig. 2.93 is given by

$$A = \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + \dots + 2y_{n-2} + 4y_{n-1} + y_n) \tag{2.14}$$

■ Simpson’s rule is named for the English mathematician Thomas Simpson (1710–1761).

Equation (2.14) is known as **Simpson’s rule**. In using Eq. (2.14), the number  $n$  of intervals of width  $h$  must be even. Simpson’s rule will be encountered again in Section 25.14 as an approximation to areas under curves.

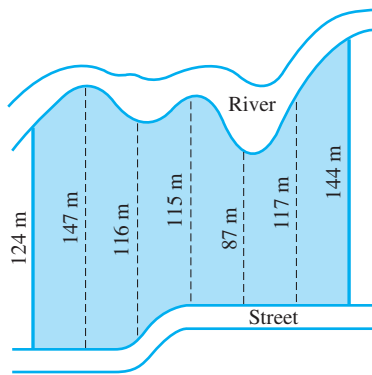


Fig. 2.94

**EXAMPLE 3** Application of Simpson's rule

A parking lot is proposed for a riverfront area in a town. The town engineer measured the widths of the area at 30.0-m intervals, as shown in Fig. 2.94. Find the area available for parking.

First, we see that there are six intervals, which means Eq. (2.14) may be used. With  $y_0 = 124$  m,  $y_1 = 147$  m,  $\dots$ ,  $y_6 = 144$  m, and  $h = 30.0$  m, we have

$$A = \frac{30.0}{3} [124 + 4(147) + 2(116) + 4(115) + 2(87) + 4(117) + 144]$$

$$A = 21\,900 \text{ m}^2$$

For most areas, Simpson's rule gives a somewhat better approximation than the trapezoidal rule. The accuracy of Simpson's rule is also usually improved by using smaller intervals.

**EXAMPLE 4** Application of Simpson's rule

From an aerial photograph, a cartographer determines the widths of Easter Island (in the Pacific Ocean) at 1.50-km intervals as shown in Fig. 2.95. The widths found are as follows:

Distance from South End (km)	0	1.50	3.00	4.50	6.00	7.50	9.00	10.5	12.0	13.5	15.0
Width (km)	0	4.8	5.7	10.5	15.2	18.5	18.8	17.9	11.3	8.8	3.1

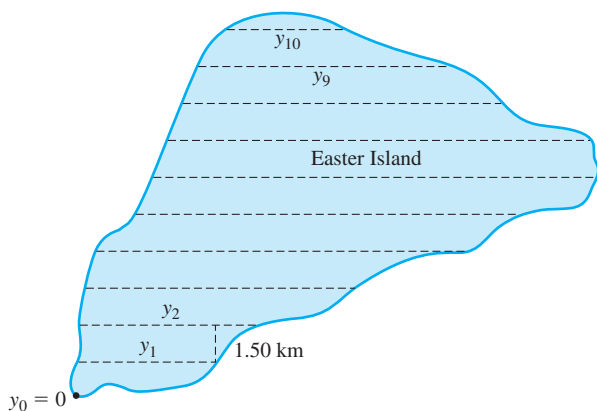


Fig. 2.95

Since there are 10 intervals, Simpson's rule may be used. From the table, we have the following values:  $y_0 = 0$ ,  $y_1 = 4.8$ ,  $y_2 = 5.7$ ,  $\dots$ ,  $y_9 = 8.8$ ,  $y_{10} = 3.1$ , and  $h = 1.5$ . Using Simpson's rule, the cartographer would approximate the area of Easter Island as follows:

$$A = \frac{1.50}{3} (0 + 4(4.8) + 2(5.7) + 4(10.5) + 2(15.2) + 4(18.5) + 2(18.8) + 4(17.9) + 2(11.3) + 4(8.8) + 3.1)$$

$$A = 174 \text{ km}^2$$

The actual area is around  $165 \text{ km}^2$ .

**EXERCISES 2.5**

In Exercises 1 and 2, answer the given questions related to the indicated examples of this section.

- In Example 1, if widths of the face of the same cam were given at 1.00-cm intervals (five more widths would be included), from the methods of this section, what is probably the most accurate way of finding the area? Explain.
- In Example 4, if you use only the data from the south end of (in km) 0, 3.00, 6.00, 9.00, 12.0, and 15.0, would you choose the trapezoidal rule or Simpson's rule to calculate the area? Explain. Do not calculate the area for these data.

In Exercises 3 and 4, answer the given questions related to Fig. 2.96.

- Which should be more accurate for finding the area, the trapezoidal rule or Simpson's rule? Explain.
- If the trapezoidal rule is used to find the area, will the result probably be too high, about right, or too little? Explain.

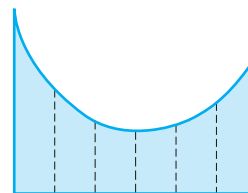


Fig. 2.96

In Exercises 5–16, calculate the indicated areas. All data are accurate to at least two significant digits.

5. The widths of a kidney-shaped swimming pool were measured at 2.0-m intervals, as shown in Fig. 2.97. Calculate the surface area of the pool, using the trapezoidal rule.

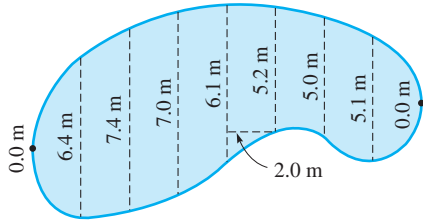


Fig. 2.97

6. Calculate the surface area of the swimming pool in Fig. 2.97, using Simpson's rule.  
 7. The widths of a cross section of an airplane wing are measured at 0.30-m intervals, as shown in Fig. 2.98. Calculate the area of the cross section, using Simpson's rule.

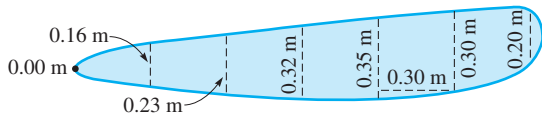


Fig. 2.98

8. Calculate the area of the cross section of the airplane wing in Fig. 2.98, using the trapezoidal rule.  
 9. Using aerial photography, the widths of an area burned by a forest fire were measured at 0.5-km intervals, as shown in the following table:

Distance (km)	0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
Width (km)	0.6	2.2	4.7	3.1	3.6	1.6	2.2	1.5	0.8

Determine the area burned by the fire by using the trapezoidal rule.

10. Find the area burned by the forest fire of Exercise 9, using Simpson's rule.  
 11. A cartographer measured the width of Kruger National Park (and game reserve) in South Africa at 6.0-km intervals on a map, as shown in Fig. 2.99. The widths are shown in the list that follows. Find the area of the park if the scale of the map is 1.0 mm = 6.0 km.

- $y_0 = 7 \text{ mm}$      $y_1 = 15 \text{ mm}$
- $y_2 = 7 \text{ mm}$      $y_3 = 11 \text{ mm}$
- $y_4 = 13 \text{ mm}$     $y_5 = 10 \text{ mm}$
- $y_6 = 9 \text{ mm}$      $y_7 = 12 \text{ mm}$
- $y_8 = 8 \text{ mm}$      $y_9 = 3 \text{ mm}$

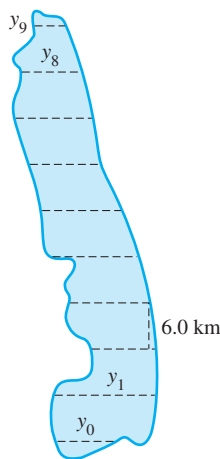


Fig. 2.99

12. The widths of an oval-shaped floor were measured at 1.5-m intervals, as shown in the following table:

Distance (m)	0.0	1.5	3.0	4.5	6.0	7.5	9.0	10.5	12.0
Width (m)	0.0	5.0	7.2	8.3	8.6	8.3	7.2	5.0	0.0

Find the area of the floor by using Simpson's rule.

13. The widths of Kejimikujik National Park (and National Historic Site) in Nova Scotia, measured at 2.0-km intervals, are shown in Fig. 2.100. Find the area of the park using the trapezoidal rule.

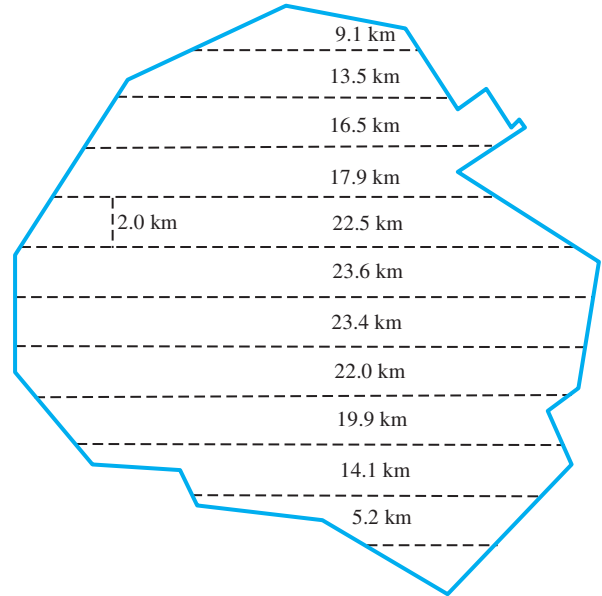


Fig. 2.100

14. Find the area of Kejimikujik National Park (see Exercise 13) using Simpson's rule.  
 15. Soundings taken across a river channel give the following depths, with the corresponding distances from one shore.

Distance (m)	0	50	100	150	200	250	300	350	400	450	500
Depth (m)	5	12	17	21	22	25	26	16	10	8	0

Find the area of the cross section of the channel using Simpson's rule.

16. The widths of a bell crank are measured at 2.0-cm intervals, as shown in Fig. 2.101. Find the area of the bell crank if the two connector holes are each 2.50 cm in diameter.

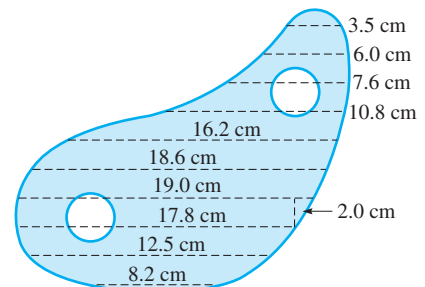


Fig. 2.101

In Exercises 17–20, calculate the area of the circle by the indicated method.

The lengths of parallel chords of a circle that are 0.250 cm apart are given in the following table. The diameter of the circle is 2.000 cm. The distance shown is the distance from one end of a diameter.

Distance (cm)	0.000	0.250	0.500	0.750	1.000	1.250	1.500	1.750	2.000
Length (cm)	0.000	1.323	1.732	1.936	2.000	1.936	1.732	1.323	0.000

Using the formula  $A = \pi r^2$ , the area of the circle is  $3.14 \text{ cm}^2$ .

17. Find the area of the circle using the trapezoidal rule and only the values of distance of 0.000 cm, 0.500 cm, 1.000 cm, 1.500 cm, and 2.000 cm with the corresponding values of the chord lengths. Explain why the value found is less than  $3.14 \text{ cm}^2$ .

18. Find the area of the circle using the trapezoidal rule and all values in the table. Explain why the value found is closer to  $3.14 \text{ cm}^2$  than the value found in Exercise 17.
19. Find the area of the circle using Simpson’s rule and the same table values as in Exercise 17. Explain why the value found is closer to  $3.14 \text{ cm}^2$  than the value found in Exercise 17.
20. Find the area of the circle using Simpson’s rule and all values in the table. Explain why the value found is closer to  $3.14 \text{ cm}^2$  than the value found in Exercise 19.

**Answer to Practice Exercise**

1.  $18\,100 \text{ km}^2$

## 2.6 Solid Geometric Figures

Rectangular Solid • Cylinder • Prism • Cone • Pyramid • Sphere • Frustum

We now review the formulas for the *volume* and *surface area* of some basic solid geometric figures. Just as area is a measure of the surface of a plane geometric figure, **volume** is a measure of the space occupied by a solid geometric figure.

One of the most common solid figures is the **rectangular solid**. This figure has six sides (**faces**), and opposite sides are rectangles. All intersecting sides are perpendicular to each other. The **bases** of the rectangular solid are the top and bottom faces. A **cube** is a rectangular solid with all six faces being equal squares.

A **right circular cylinder** is *generated* by rotating a rectangle about one of its sides. Each **base** is a circle, and the *cylindrical surface* is perpendicular to each of the bases. The **height** is one side of the rectangle, and the **radius** of the base is the other side.

A **right circular cone** is generated by rotating a right triangle about one of its legs. The **base** is a circle, and the **slant height** is the hypotenuse of the right triangle. The **height** is one leg of the right triangle, and the **radius** of the base is the other leg.

The bases of a **right prism** are equal and parallel polygons, and the sides are rectangles. The **height** of a prism is the perpendicular distance between bases. The base of a **pyramid** is a polygon, and the other faces, the **lateral faces**, are triangles that meet at a common point, the **vertex**. A **regular pyramid** has congruent triangles for its lateral faces.

A **sphere** is generated by rotating a circle about a diameter. The **radius** is a line segment joining the centre and a point on the sphere. The **diameter** is a line segment through the centre and having its endpoints on the sphere.

In the following formulas,  $V$  represents the *volume*,  $A$  represents the *total surface area*,  $S$  represents the *lateral surface area* (bases not included),  $B$  represents the *area of the base*, and  $p$  represents the *perimeter of the base*:

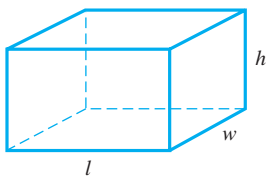


Fig. 2.102

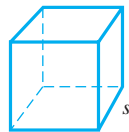


Fig. 2.103

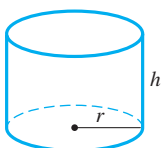


Fig. 2.104

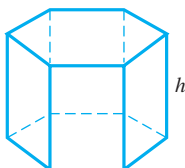


Fig. 2.105

$V = lwh$	Rectangular solid (Fig. 2.102)	(2.15)
$A = 2lw + 2lh + 2wh$		(2.16)
$V = s^3$	Cube (Fig. 2.103)	(2.17)
$A = 6s^2$		(2.18)
$V = \pi r^2 h$	Right circular cylinder (Fig. 2.104)	(2.19)
$A = 2\pi r^2 + 2\pi rh$		(2.20)
$S = 2\pi rh$		(2.21)
$V = Bh$	Right prism (Fig. 2.105)	(2.22)
$S = ph$		(2.23)



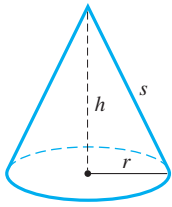


Fig. 2.106

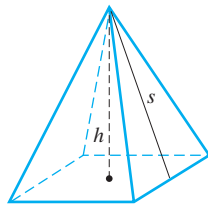


Fig. 2.107

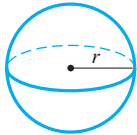


Fig. 2.108

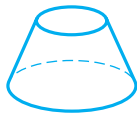


Fig. 2.109

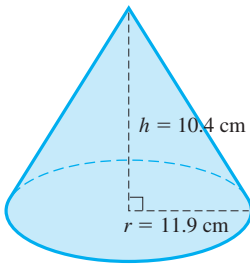


Fig. 2.110

**Practice Exercise**

1. Find the volume of the cone in Example 2.

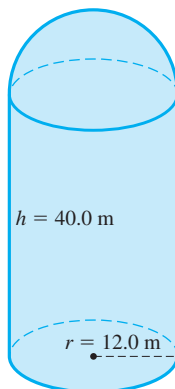


Fig. 2.111

$V = \frac{1}{3}\pi r^2 h$	Right circular cone (Fig. 2.106)	(2.24)
$A = \pi r^2 + \pi r s$		(2.25)
$S = \pi r s$		(2.26)
$V = \frac{1}{3} B h$	Regular pyramid (Fig. 2.107)	(2.27)
$S = \frac{1}{2} p s$		(2.28)
$V = \frac{4}{3}\pi r^3$	Sphere (Fig. 2.108)	(2.29)
$A = 4\pi r^2$		(2.30)

Equation (2.22) is valid for any prism, and Eq. (2.27) is valid for any pyramid. There are other types of cylinders and cones, but we restrict our attention to right circular cylinders and right circular cones, and we will often use “cylinder” or “cone” when referring to them.

The **frustum** of a cone or pyramid is the solid figure that remains after the top is cut off by a plane parallel to the base. Fig. 2.109 shows the frustum of a cone.

**EXAMPLE 1 Volume of a rectangular solid**

What volume of concrete is needed for a driveway 25.0 m long, 2.75 m wide, and 0.100 m thick?

The driveway is a rectangular solid for which  $l = 25.0$  m,  $w = 2.75$  m, and  $h = 0.100$  m. Using Eq. (2.15), we have

$$V = (25.0)(2.75)(0.100)$$

$$V = 6.88 \text{ m}^3$$

**EXAMPLE 2 Total surface area of a right circular cone**

Calculate the total surface area of a right circular cone for which the radius is  $r = 11.9$  cm and the height is  $h = 10.4$  cm. See Fig. 2.110.

To find the total surface area using Eq. (2.25), we need the radius and the slant height  $s$  of the cone. Therefore, we must first find  $s$ . The radius and height are legs of a right triangle, and the slant height is the hypotenuse. To find  $s$ , we use the *Pythagorean theorem*:

$$s^2 = r^2 + h^2 \quad \text{Pythagorean theorem}$$

$$s = \sqrt{r^2 + h^2} \quad \text{solve for } s$$

$$s = \sqrt{11.9^2 + 10.4^2}$$

$$s = 15.8 \text{ cm}$$

Now, calculating the total surface area, we have

$$A = \pi r^2 + \pi r s \quad \text{Eq. (2.25)}$$

$$A = \pi(11.9)^2 + \pi(11.9)(15.8) \quad \text{substituting}$$

$$A = 1040 \text{ cm}^2$$

**EXAMPLE 3 Volume of a combination of solids**

A grain storage building is in the shape of a cylinder surmounted by a hemisphere (*half a sphere*). See Fig. 2.111. Find the volume of grain that can be stored if the height of the cylinder is 40.0 m and its radius is 12.0 m.

The total volume of the structure is the volume of the cylinder plus the volume of the hemisphere. By the construction we see that the radius of the hemisphere is the same as the radius of the cylinder. Therefore,

cylinder hemisphere

$$V = \pi r^2 h + \frac{1}{2} \left( \frac{4}{3} \pi r^3 \right)$$

$$V = \pi r^2 h + \frac{2}{3} \pi r^3$$

$$V = \pi (12.0)^2 (40.0) + \frac{2}{3} \pi (12.0)^3$$

$$V = 21\,700 \text{ m}^3$$

## EXERCISES 2.6

In Exercises 1–4, answer the given questions about the indicated examples of this section.

- In Example 1, if the length is doubled and the thickness is doubled, by what factor is the volume changed?
- In Example 2, if the value of the slant height  $s = 17.5$  cm is given instead of the height, what is the height?
- In Example 2, if the radius is halved and the height is doubled, what is the volume?
- In Example 3, if  $h$  is halved, what is the volume?

In Exercises 5–20, find the volume or area of each solid figure for the given values. See Figs. 2.102 to 2.108.

- Volume of cube:  $s = 7.15$  cm
- Volume of right circular cylinder:  $r = 23.5$  cm,  $h = 48.4$  cm
- Total surface area of right circular cylinder:  $r = 689$  m,  $h = 233$  m
- Area of sphere:  $r = 0.067$  mm
- Volume of sphere:  $r = 0.877$  m
- Volume of right circular cone:  $r = 25.1$  mm,  $h = 5.66$  mm
- Lateral area of right circular cone:  $r = 78.0$  cm,  $s = 83.8$  cm
- Lateral area of regular pyramid:  $p = 345$  m,  $s = 272$  m
- Volume of regular pyramid: square base of side 76 cm,  $h = 130$  cm
- Volume of right prism: square base of side 29.0 cm,  $h = 11.2$  cm
- Lateral area of regular prism: equilateral triangle base of side 1.092 m,  $h = 1.025$  m
- Lateral area of right circular cylinder: diameter = 250 mm,  $h = 347$  mm
- Volume of hemisphere: diameter = 0.83 cm
- Volume of regular pyramid: square base of side 22.4 m,  $s = 14.2$  m
- Total surface area of right circular cone:  $r = 3.39$  cm,  $h = 0.274$  cm
- Total surface area of pyramid: All faces and base are equilateral triangles of side 3.67 dm.

In Exercises 21–40, solve the given problems.

- Eq. (2.29) expresses the volume  $V$  of a sphere in terms of the radius  $r$ . Express  $V$  in terms of the diameter  $d$ .
- Derive a formula for the total surface area  $A$  of a hemispherical volume of radius  $r$  (curved surface and flat surface).

- The radius of a cylinder is twice as long as the radius of a cone, and the height of the cylinder is half as long as the height of the cone. What is the ratio of the volume of the cylinder to that of the cone?
- The base area of a cone is one-fourth of the total area. Find the ratio of the radius to the slant height.
- In designing a weather balloon, it is decided to double the diameter of the balloon so that it can carry a heavier instrument load. What is the ratio of the final surface area to the original surface area?
- During a rainfall of 3.00 cm, what weight of water falls on an area of 1.00 km<sup>2</sup>? Each cubic metre of water weighs 9800 N.
- A rectangular box is to be used to store radioactive materials. The inside of the box is 12.0 cm long, 9.50 cm wide, and 8.75 cm deep. What is the area of sheet lead that must be used to line the inside of the box?
- A swimming pool is 15.0 m wide, 24.0 m long, 1.00 m deep at one end, and 2.60 m deep at the other end. How many cubic metres of water can it hold? (The slope on the bottom is constant.) See Fig. 2.112.

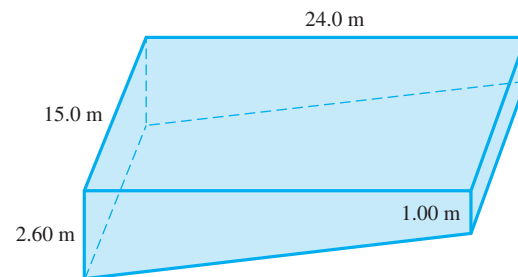


Fig. 2.112

- The Athabasca oil pipeline in Alberta is 540 km long and has a diameter of 0.76 m. What is the maximum volume of the pipeline?
- A glass prism used in the study of optics has a right triangular base. The legs of the triangle are 3.00 cm and 4.00 cm. The prism is 8.50 cm high. What is the total surface area of the prism? See Fig. 2.113.

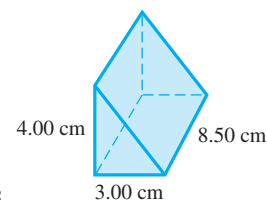


Fig. 2.113

31. The Great Pyramid of Egypt has a square base approximately 230 m on a side. The height of the pyramid is about 150 m. What is its volume? See Fig. 2.114.

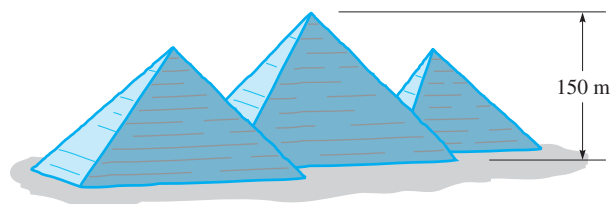


Fig. 2.114

32. A paper cup is in the shape of a cone, as shown in Fig. 2.115. What is the surface area of the cup?

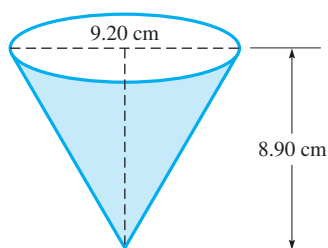


Fig. 2.115

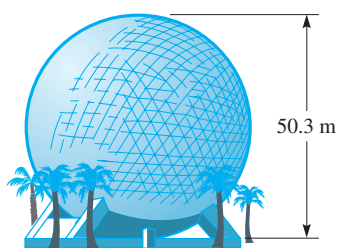


Fig. 2.116

33. *Spaceship Earth* (shown in Fig. 2.116) at Epcot Center in Florida is a sphere 50.3 m in diameter. What is the volume of *Spaceship Earth*?
34. A propane tank is constructed in the shape of a cylinder with a hemisphere at each end, as shown in Fig. 2.117. Find the volume of the tank.

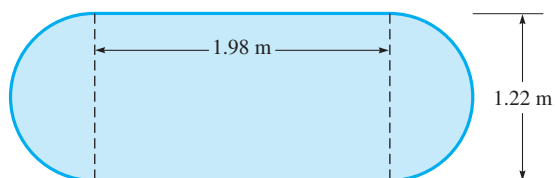


Fig. 2.117

35. A special wedge in the shape of a regular pyramid has a square base 16.0 mm on a side. The height of the wedge is 40.0 mm. What is the total surface area of the wedge?
36. A lawn roller is a cylinder 0.96 m long and 0.60 m in diameter. How many revolutions of the roller are needed to roll 76 m<sup>2</sup> of lawn?
37. The circumference of a basketball is about 75.7 cm. What is its volume?
38. What is the area of a paper label that is to cover the lateral surface of a cylindrical can 8.50 cm in diameter and 11.5 cm high? The ends of the label will overlap 0.50 cm, when the label is placed on the can.
39. The side view of a rivet is shown in Fig. 2.118. It is a conical part on a cylindrical part. Find the volume of the rivet.

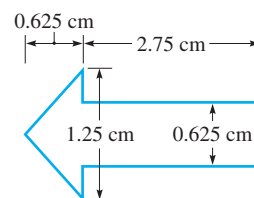


Fig. 2.118

40. A dipstick is made to measure the volume remaining in the conical container shown in Fig. 2.119. How far below the full mark (at the top of the container) on the stick should the mark for half-full be placed?

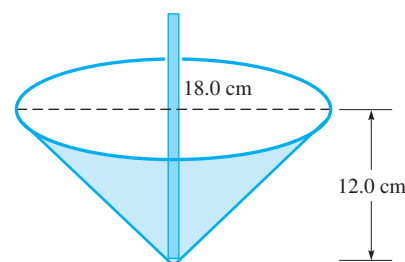


Fig. 2.119

## Answer to Practice Exercise

1. 1540 cm<sup>3</sup>

## CHAPTER 2 EQUATIONS

Line segments	Fig. 2.8	$\frac{a}{b} = \frac{c}{d}$	(2.1)
Triangle		$A = \frac{1}{2}bh$	(2.2)
Hero's formula		$A = \sqrt{s(s-a)(s-b)(s-c)}$ , where $s = \frac{1}{2}(a+b+c)$	(2.3)
Pythagorean theorem	Fig. 2.30	$c^2 = a^2 + b^2$	(2.4)
Square	Fig. 2.51	$A = s^2$	(2.5)

<b>Rectangle</b>	Fig. 2.52	$A = lw$	(2.6)
<b>Parallelogram</b>	Fig. 2.53	$A = bh$	(2.7)
<b>Trapezoid</b>	Fig. 2.54	$A = \frac{1}{2}h(b_1 + b_2)$	(2.8)
<b>Circle</b>		$c = 2\pi r$	(2.9)
		$A = \pi r^2$	(2.10)
<b>Radians</b>	Fig. 2.76	$\pi \text{ rad} = 180^\circ$	(2.11)
	Fig. 2.77	$s = \theta r$	(2.12)
<b>Trapezoidal rule</b>	Fig. 2.89	$A = \frac{h}{2}(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n)$	(2.13)
<b>Simpson's rule</b>	Fig. 2.93	$A = \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n)$	(2.14)
<b>Rectangular solid</b>	Fig. 2.102	$V = lwh$	(2.15)
		$A = 2lw + 2lh + 2wh$	(2.16)
<b>Cube</b>	Fig. 2.103	$V = s^3$	(2.17)
		$A = 6s^2$	(2.18)
<b>Right circular cylinder</b>	Fig. 2.104	$V = \pi r^2 h$	(2.19)
		$A = 2\pi r^2 + 2\pi rh$	(2.20)
		$S = 2\pi rh$	(2.21)
<b>Right prism</b>	Fig. 2.105	$V = Bh$	(2.22)
		$S = ph$	(2.23)
<b>Right circular cone</b>	Fig. 2.106	$V = \frac{1}{3}\pi r^2 h$	(2.24)
		$A = \pi r^2 + \pi rs$	(2.25)
		$S = \pi rs$	(2.26)
<b>Regular pyramid</b>	Fig. 2.107	$V = \frac{1}{3}Bh$	(2.27)
		$S = \frac{1}{2}ps$	(2.28)
<b>Sphere</b>	Fig. 2.108	$V = \frac{4}{3}\pi r^3$	(2.29)
		$A = 4\pi r^2$	(2.30)

## CHAPTER 2 REVIEW EXERCISES

In Exercises 1–4, use Fig. 2.120. Determine the indicated angles.

1.  $\angle CGE$
2.  $\angle EGF$
3.  $\angle DGH$
4.  $\angle EGI$

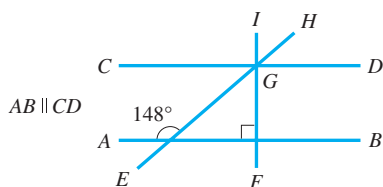


Fig. 2.120

In Exercises 5–12, find the indicated sides of the right triangle shown in Fig. 2.121.

5.  $a = 9, b = 40, c = ?$
6.  $a = 14, b = 48, c = ?$
7.  $a = 400, b = 580, c = ?$
8.  $b = 56, c = 65, a = ?$
9.  $a = 6.30, b = 3.80, c = ?$
10.  $a = 126, b = 25.1, c = ?$

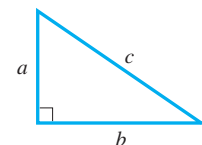


Fig. 2.121

11.  $b = 29.3, c = 36.1, a = ?$
12.  $a = 0.782, c = 0.885, b = ?$

In Exercises 13–20, find the perimeter or area of the indicated figure.

13. Perimeter: equilateral triangle of side 8.5 mm
14. Perimeter: rhombus of side 15.2 cm
15. Area: triangle,  $b = 3.25$  m,  $h = 1.88$  m
16. Area: triangle of sides 175 cm, 138 cm, 119 cm
17. Circumference of circle:  $d = 98.4$  mm
18. Perimeter: rectangle,  $l = 2.98$  dm,  $w = 1.86$  dm
19. Area: trapezoid,  $b_1 = 67.2$  cm,  $b_2 = 126.7$  cm,  $h = 34.2$  cm
20. Area: circle,  $d = 32.8$  m

In Exercises 21–24, find the volume of the indicated solid geometric figure.

21. Prism: base is right triangle with legs 26.0 cm and 34.0 cm, height is 14.0 cm
22. Cylinder: base radius 36.0 cm, height 2.40 cm
23. Pyramid: base area 3850 m<sup>2</sup>, height 125 m
24. Sphere: diameter 22.1 mm

In Exercises 25–28, find the surface area of the indicated solid geometric figure.

25. Total area of cube of edge 0.520 m
26. Total area of cylinder: base diameter 1.20 cm, height 5.80 cm
27. Lateral area of cone: base radius 1.82 mm, height 11.5 mm
28. Total area of sphere:  $d = 12$  760 km

In Exercises 29–32, use Fig. 2.122. Line  $CT$  is tangent to the circle with centre at  $O$ . Find the indicated angles.

29.  $\angle BTA$
30.  $\angle TAB$
31.  $\angle BTC$
32.  $\angle ABT$

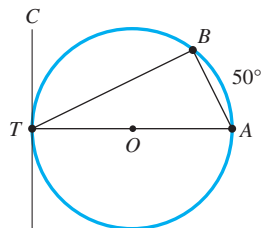


Fig. 2.122

In Exercises 33–36, use Fig. 2.123. Given that  $AB = 4, BC = 4, CD = 6$ , and  $\angle ADC = 53^\circ$ , find the indicated angle and lengths.

33.  $\angle ABE$
34.  $AD$
35.  $BE$
36.  $AE$

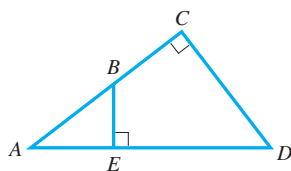


Fig. 2.123

In Exercises 37–40, find the formulas for the indicated perimeters and areas.

37. Perimeter of Fig. 2.124 (a right triangle and semicircle attached)
38. Perimeter of Fig. 2.125 (a square with a quarter circle at each end)
39. Area of Fig. 2.124
40. Area of Fig. 2.125

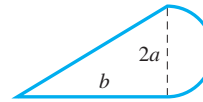


Fig. 2.124

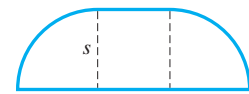


Fig. 2.125

In Exercises 41–46, answer the given questions.

41. Is a square also a rectangle, a parallelogram, and a rhombus?
42. If the measures of two angles of one triangle equal the measures of two angles of a second triangle, are the two triangles similar?
43. If the dimensions of a plane geometric figure are each multiplied by  $n$ , by how much is the area multiplied? Explain, using a circle to illustrate.
44. If the dimensions of a solid geometric figure are each multiplied by  $n$ , by how much is the volume multiplied? Explain, using a cube to illustrate.
45. What is an equation relating chord segments  $a, b, c$ , and  $d$  shown in Fig. 2.126? The dashed chords are an aid in the solution.
46. From a common point, two line segments are tangent to the same circle. If the angle between the line segments is  $36^\circ$ , what is the angle between the two radii of the circle drawn from the points of tangency?

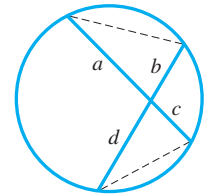


Fig. 2.126

In Exercises 47–72, solve the given problems.

47. A tooth on a saw is in the shape of an isosceles triangle. If the angle at the point is  $38^\circ$ , find the two base angles.
48. A lead sphere 1.50 cm in diameter is flattened into a circular sheet 14.0 cm in diameter. How thick is the sheet?
49. A ramp for the disabled is designed so that it rises 1.20 m over a horizontal distance of 7.80 m. How long is the ramp?
50. An airplane is 640 m directly above one end of a 3200-m runway. How far is the plane from the glide-slope indicator on the ground at the other end of the runway?
51. A machine part is in the shape of a square with equilateral triangles attached to two sides (see Fig. 2.127). Find the perimeter of the machine part.

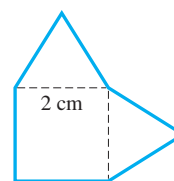


Fig. 2.127

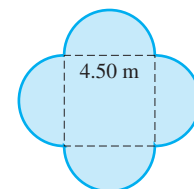


Fig. 2.128

52. A patio is designed with semicircular areas attached to a square, as shown in Fig. 2.128. Find the area of the patio.

53. A radio transmitting tower is supported by guy wires. The tower and three parallel guy wires are shown in Fig. 2.129. Find the distance  $AB$  along the tower.

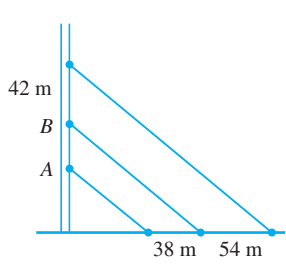


Fig. 2.129

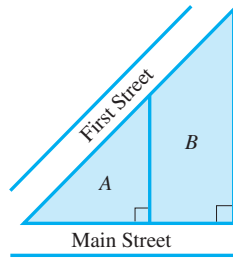


Fig. 2.130

54. Find the areas of lots  $A$  and  $B$  in Fig. 2.130.  $A$  has a frontage on Main St. of 145 m, and  $B$  has a frontage on Main St. of 84.0 m. The boundary between lots is 125 m.
55. To find the height of a flagpole, a person places a mirror at  $M$ , as shown in Fig. 2.131. The person's eyes at  $E$  are 160 cm above the ground at  $A$ . From physics, it is known that  $\angle AME = \angle BMF$ . If  $AM = 120$  cm and  $MB = 4.5$  m, find the height  $BF$  of the flagpole.

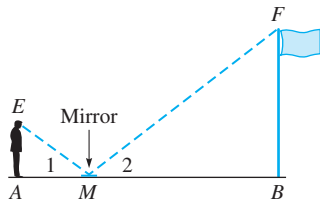


Fig. 2.131

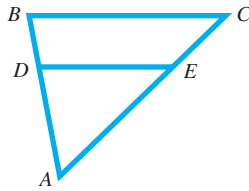


Fig. 2.132

56. The metal support in the form of  $\triangle ABC$  shown in Fig. 2.132 is strengthened by brace  $DE$ , which is parallel to  $BC$ . How long is the brace if  $AB = 24.0$  cm,  $AD = 16.0$  cm, and  $BC = 33.0$  cm?
57. A typical scale for an aerial photograph is  $1/18\,450$ . In a 20.0-by-25.0-cm photograph with this scale, what is the longest distance (in km) between two locations in the photograph?
58. For a hydraulic press, the mechanical advantage is the ratio of the large piston area to the small piston area. Find the mechanical advantage if the pistons have diameters of 3.10 cm and 2.25 cm.
59. The diameter of the earth is 12 700 km, and a satellite is in orbit at an altitude of 345 km. How far does the satellite travel in one rotation about the earth?
60. BC Place Stadium in Vancouver, British Columbia, is the world's largest cable-supported retractable-roof stadium. Its opening is rectangular, having length 15.0 m longer than its width, and it has a diagonal of 131 m. Determine the area of the roof opening. *Hint:* You do not need to solve a quadratic equation; the area of the opening appears in the diagonal equation.
61. A rectangular piece of wallboard with two holes cut out for heating ducts is shown in Fig. 2.133. What is the area of the remaining piece?

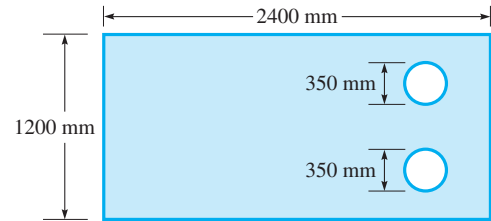


Fig. 2.133

62. The diameter of the sun is  $1.38 \times 10^6$  km, the diameter of the earth is  $1.27 \times 10^4$  km, and the distance from the earth to the sun (centre to centre) is  $1.50 \times 10^8$  km. What is the distance from the centre of the earth to the end of the shadow due to the rays from the sun?
63. Using aerial photography, the width of an oil spill is measured at 250-m intervals, as shown in Fig. 2.134. Using Simpson's rule, find the area of the oil spill.

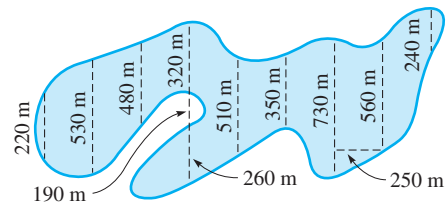


Fig. 2.134

64. To build a highway, it is necessary to cut through a hill. A surveyor measured the cross-sectional areas at 250-m intervals through the cut, as shown in the following table. Using the trapezoidal rule, determine the volume of soil to be removed.

Dist. (m)	0	250	500	750	1000	1250	1500	1750
Area (m <sup>2</sup> )	560	1780	4650	6730	5600	6280	2260	230

65. The Hubble space telescope is within a cylinder 4.32 m in diameter and 13.2 m long. What is the volume within this cylinder?
66. A horizontal cross section of a concrete bridge pier is a regular hexagon (six sides, all equal in length, and all internal angles are equal), each side of which is 2.50 m long. If the height of the pier is 6.75 m, what is the volume of concrete in the pier?
67. A railroad track 1000.00 m long expands 0.20 m (20 cm) during the afternoon (due to an increase in temperature of about 17°C). Assuming that the track cannot move at either end and that the increase in length causes a bend straight up in the middle of the track, how high is the top of the bend?
68. Two persons are talking to each other on cellular phones. If the angle between their signals at the tower is  $90^\circ$ , as shown in Fig. 2.135, how far apart are they?

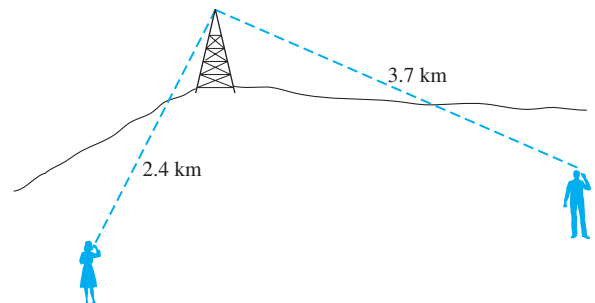


Fig. 2.135

69. A hot-water tank is in the shape of a right circular cylinder surmounted by a hemisphere, as shown in Fig. 2.136. How many litres does the tank hold? (1.00 m<sup>3</sup> contains 1000 L.)
70. A tent is in the shape of a regular pyramid surmounted on a cube. If the edge of the cube is 2.50 m and the total height of the tent is 3.25 m, find the area of the material used in making the tent (not including any floor area).
71. The diagonal of a HDTV screen is 107 cm. If the ratio of the width of the screen to the height of the screen is 16 to 9, what are the width and height?

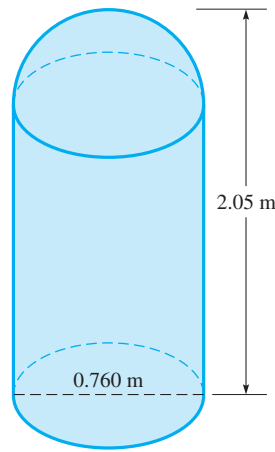


Fig. 2.136

72. A satellite is 1590 km directly above the centre of the eye of a circular (approximately) hurricane that has formed in the Atlantic Ocean. The distance from the satellite to the edge of the hurricane is 1620 km. What area does the hurricane cover? Neglect the curvature of the Earth and any possible depth of the hurricane.

**Writing Exercise**

73. The Pentagon, headquarters of the U.S. Department of Defense, is the world's largest office building. It is a regular pentagon (five sides, all equal in length, and all interior angles are equal) 281 m on a side, with a diagonal of length 454 m. Using these data, draw a sketch and write one or two paragraphs to explain how to find the area covered within the outside perimeter of the Pentagon. (What is the area?)

**CHAPTER 2 PRACTICE TEST**

1. In Fig. 2.137, determine  $\angle 1$ .
2. In Fig. 2.137, determine  $\angle 2$ .

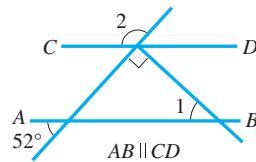


Fig. 2.137

3. A tree is 2.4 m high and casts a shadow 3.0 m long. At the same time, a telephone pole casts a shadow 7.6 m long. How tall is the pole?
4. Find the area of a triangle with sides of 2.46 cm, 3.65 cm, and 4.07 cm.
5. What is the diagonal distance between corners of a rectangular room 3810 mm wide and 5180 mm long?
6. What is the area of the trapezoid shown in Fig. 2.138?

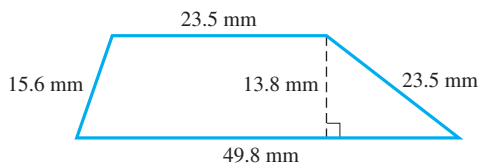


Fig. 2.138

7. The edge of a cube is 4.50 cm. What are (a) the surface area and (b) the volume of the cube?

8. Find the surface area of a tennis ball whose circumference is 21.0 cm.
9. Find the volume of a right circular cone of radius 2.08 m and height 1.78 m.
10. In Fig. 2.139, find  $\angle 1$ .
11. In Fig. 2.139, find  $\angle 2$ .

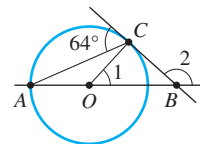


Fig. 2.139

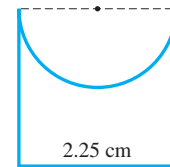


Fig. 2.140

12. In Fig. 2.140, find the perimeter of the figure shown. It is a square with a semicircle removed.
13. In Fig. 2.140, find the area of the figure shown.
14. The width of a marshy area is measured at 50-m intervals, with the results shown in the following table. Using the trapezoidal rule, find the area of the marsh. (All data accurate to two or more significant digits.)

Distance (m)	0	50	100	150	200	250	300
Width (m)	0	90	145	260	205	110	20



# 3

## LEARNING OUTCOMES

After completion of this chapter, the student should be able to:

- Define a function and distinguish between dependent and independent variables
- Use mathematical functional notation
- Interpret a function as a process
- Determine domain and range for a function
- Graph a function using the rectangular coordinate system
- Use a graphing calculator to solve an equation graphically
- Graph and interpret sets of data values

# Functions and Graphs

Reasoning that heavier objects contain more “earth” than lighter ones, the Greek philosopher Aristotle (in about 350 B.C.E.) concluded that heavier objects fall faster in order to return to their “natural” place upon the ground. More of a philosophical argument than a scientific one, this argument was supported by the observational evidence that some lighter objects (such as feathers) do fall more slowly than heavier ones (such as stones). Almost 2000 years later, the Italian scientist Galileo reasoned that if mass were important to the speed at which an object falls, then two identical objects should fall faster when they are connected together than when they are not, which is not the case. Instead, Galileo claimed that it is the interaction of an object with the medium through which it falls that is responsible for the different rates of motion for falling objects. Subsequent experiments showed that Galileo was indeed correct, including the famous 1971 experiment, during the *Apollo 15* mission, when astronaut David Scott simultaneously dropped a hammer and a feather on the moon and observed identical motions in the absence of air resistance.

Controlled experiments and observations of natural phenomena form part of the *scientific method*. Galileo was one of the first scientists to realize that claims about how the universe works can be tested mathematically, using mathematical formulas to describe the phenomena. He realized that such formulas provide a way of showing a compact and precise relationship between the observed variables.

In technology and science, determining how one quantity depends on others is a primary goal. A rule that shows such a relation is of great importance, and in mathematics such a rule is called a *function*. The concept of a function is introduced in this chapter.

Examples of such relations in technology and science, as well as in everyday life, are numerous. Plant growth depends on sunlight and rainfall; traffic flow depends on roadway design; the sales tax on an item depends on the cost of an item; the time to access the Internet depends

► The electric power  $P$  produced by a wind turbine depends on the velocity  $v$  of the wind. In Section 3.4 we draw the graph of  $P = 0.5C_p\rho Av^3 = 1.304v^3$  for a specific 3.00-m diameter turbine to see this type of relationship.

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on how fast a computer processes data; distance travelled depends on time and speed of travel; electric voltage depends on the current and resistance. These are but a few of the innumerable possibilities.

A way of visualizing how one quantity depends on another is by means of a *graph*. The basic method of graphing was devised by the French mathematicians Descartes and Fermat in the 1630s from their work to combine the methods of algebra and geometry. Their work was very influential in later developments in mathematics and technology. In this chapter, the basic methods of graphing a function and visualizing the relationship between variables will be introduced.

## 3.1 Introduction to Functions

Definition of Function •  
Independent and Dependent Variables •  
Functional Notation

An important method of finding formulas such as those in Section 1.11 is through scientific observation and experimentation, like that of Galileo. For example, if we were to perform an experiment to determine the relationship between the distance an object falls and the time it falls, we should find (approximately, at least) that  $s = 4.9t^2$ , where  $s$  is the distance in metres and  $t$  is the time in seconds. Another example is that electrical measurements of voltage  $V$  and current  $I$  through a particular resistor would show that  $V = kI$ , where  $k$  is a constant.

Considerations such as these lead us to one of the basic concepts in mathematics, that of a *function*.

### Definition of a Function

*Whenever a relationship exists between two variables—such that for every value of the first, there is only one corresponding value of the second—we say that the second variable is a **function** of the first variable.*

*The first variable is called the **independent variable**, and the second variable is called the **dependent variable**. The first variable is termed *independent* since permissible values can be assigned to it arbitrarily, and the second variable is termed *dependent* since its value is determined by the choice of the independent variable. In the context of experimental science, the independent variable is called the **manipulated variable**, since it represents the physical quantity that is varied through choosing its values. The dependent variable is called the **responding variable**, since it represents the value of the measured quantity that reacts in response to changes in the value of the manipulated variable. The relationship between the manipulated and responding variables may be unknown in real experiments, and determining the function that describes such a relationship is important.*

*Values of the independent variable and dependent variable are to be real numbers. Therefore, there may be restrictions on their possible values. This is discussed in the following section.*

### EXAMPLE 1 Examples of functions

- (a) In the equation  $y = 2x$ ,  $y$  is a function of  $x$ , since for each value of  $x$  there is only one value of  $y$ . For example, if  $x = 3$ , then  $y = 6$  and no other value. By arbitrarily assigning values to  $x$ , and substituting, we see that the values of  $y$  we obtain *depend* on the values of  $x$ . Therefore,  $x$  is the independent variable, and  $y$  is the dependent variable.
- (b) The power  $P$  developed in a certain resistor by a current  $I$  is given by  $P = 4I^2$ . Here,  $P$  is a function of  $I$ . The dependent variable is  $P$  and the independent variable is  $I$ . ■

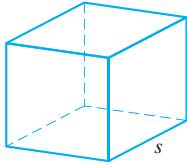


Fig. 3.1

**EXAMPLE 2** Independent and dependent variables

Figure 3.1 shows a cube of edge  $s$ . From Chapter 2, we know that the volume  $V$  in terms of the edge  $s$  is  $V = s^3$ . Here,  $V$  is a function of  $s$ , since for each value of  $s$ , there is only one value of  $V$ . The dependent variable is  $V$ , and the independent variable is  $s$ .

If the equation relating the volume and the edge of a cube is written as  $s = \sqrt[3]{V}$ , that is, if the edge is expressed in terms of the volume,  $s$  is a function of  $V$ . In this case,  $s$  is the dependent variable, and  $V$  is the independent variable. ■

**EXAMPLE 3** Manipulated and responding variables

In an experiment, the distance  $s$  (in metres) that is travelled by a falling object is suspected to depend on the time elapsed  $t$  (in seconds) during the fall. During the experiment, measurements will be taken every 0.1 s, and the distance  $s$  will be measured at those times. Therefore, time  $t$  is the manipulated variable, since the values 0.0 s, 0.1 s, 0.2s, 0.3 s, . . . are selected by the experimenter. The displacement  $s$  is the responding variable, as its value will be measured during the fall in response to the changes in time. ■

There are many ways to express functions. Formulas, tables, charts, and graphs can also define functions. Functions will be of importance throughout the book, and we will use a number of different types of functions in later chapters.

**FUNCTIONAL NOTATION**

For convenience of notation, the phrase

*“function of  $x$ ”* is written as  $f(x)$ .

This means that “ $y$  is a function of  $x$ ” may be written as  $y = f(x)$ . Here,  $f$  denotes *dependence* and does not represent a quantity or a variable.

**COMMON ERROR**

$f(x)$  does not mean  $f$  multiplied by  $x$ .  $f(x)$  describes a variable dependence, in that the function value returned will be determined by the value of variable  $x$ .

**EXAMPLE 4** Functional notation  $f(x)$ 

If  $y = 6x^3 - 5x$ , we say that  $y$  is a function of  $x$ . This functional relation is  $6x^3 - 5x$ . It is also common to write such a function as  $f(x) = 6x^3 - 5x$ . However,  $y$  and  $f(x)$  represent the same expression,  $6x^3 - 5x$ . Using  $y$ , the dependent variable values are indicated, and using  $f(x)$ , the functional dependence is shown. ■

One of the most important uses of functional notation is to designate the value of the function for a particular value of the independent variable. That is,

*the value of the function  $f(x)$  when  $x = a$  is written as  $f(a)$ .*

**EXAMPLE 5** Meaning of  $f(a)$ 

For a function  $f(x)$ , the value of  $f(x)$  for  $x = 2$  may be expressed as  $f(2)$ . Thus, substituting 2 for  $x$  in  $f(x) = 3x - 7$ , we have

$$f(2) = 3(2) - 7 = -1 \quad \text{substitute 2 for } x$$

The value of  $f(x)$  for  $x = -1.4$  is

$$f(-1.4) = 3(-1.4) - 7 = -11.2 \quad \text{substitute } -1.4 \text{ for } x$$

■ A graphing calculator can be used to evaluate a function in several ways. One is to directly substitute the value into the function. A second is to enter the function as  $Y_1$  and evaluate. A third way, which is very useful when many values are to be used, is to enter the function as  $Y_1$  and use the *table* feature.

**LEARNING TIP**

Whatever number  $a$  may represent, in order to evaluate  $f(a)$ , we substitute the value of  $a$  for  $x$  in the function  $f(x)$ . This is true even if  $a$  is a literal number or is itself a function composed of another variable.

**Practice Exercise**

- For the function in Example 6, find  $g(-1)$ .

**EXAMPLE 6** Meaning of  $f(a)$ 

If  $g(t) = \frac{t^2}{2t + 1}$ , to find  $g(a^3)$  we substitute  $a^3$  for  $t$  in  $g(t)$ :

$$g(a^3) = \frac{(a^3)^2}{2a^3 + 1} = \frac{a^6}{2a^3 + 1}$$

For the same function,  $g(3) = \frac{3^2}{2(3) + 1} = \frac{9}{7}$ . In both cases, to obtain the value of  $g(t)$ , we simply substitute the value within the parentheses for  $t$ . ■

**EXAMPLE 7** Application of functional notation

The electric resistance  $R$  of a certain resistor as a function of the temperature  $T$  (in  $^{\circ}\text{C}$ ) is given by  $R = 10.0 + 0.01T + 0.001T^2$ . If a given temperature  $T$  is increased by  $10^{\circ}\text{C}$ , what is the value of  $R$  for the increased temperature as a function of  $T$ ?

We are to determine  $R$  for a temperature of  $T + 10$ . Since

$$R(T) = 10.0 + 0.01T + 0.001T^2$$

then

$$\begin{aligned} R(T + 10) &= 10.0 + 0.01(T + 10) + 0.001(T + 10)^2 && \text{substitute } T + 10 \text{ for } T \\ R(T + 10) &= 10.0 + 0.01T + 1.0 + 0.001T^2 + 0.02T + 0.1 \\ R(T + 10) &= 11.1 + 0.02T + 0.001T^2 \end{aligned}$$

At times, we need to define more than one function. We then use different symbols, such as  $f(x)$  and  $g(x)$ , to denote these functions.

**EXAMPLE 8** Multiple functions notation

For the functions  $f(x) = 5x - 3$  and  $g(x) = ax^2 + x$ , where  $a$  is a constant, we have

$$\begin{aligned} f(-4) &= 5(-4) - 3 = -23 && \text{substitute } -4 \text{ for } x \text{ in } f(x) \\ g(-4) &= a(-4)^2 + (-4) = 16a - 4 && \text{substitute } -4 \text{ for } x \text{ in } g(x) \end{aligned}$$

**Practice Exercise**

- For the function in Example 8, find  $f(-a^2x)$  and  $g(-a^2x)$ .

**LEARNING TIP**

A function may be looked upon as a process, or set of instructions. These instructions tell us how to obtain the value of the dependent variable for a particular value of the independent variable, even if the instructions are expressed in literal symbols.

**LEARNING TIP**

The functions  $f(t) = t^2 - 3t$  and  $f(n) = n^2 - 3n$  are the same as the function  $f(x) = x^2 - 3x$ , since the operations performed on the independent variable are the same. **Although different literal symbols appear, this does not change the processes in the function.**

**EXAMPLE 9** Function as a set of instructions

The function  $f(x) = x^2 - 3x$  tells us to “square the value of the independent variable, multiply the value of the independent variable by 3, and subtract the second result from the first.” An analogy would be a computer that was programmed so that when a number was entered into the program, it would square the number, then multiply the number by 3, and finally subtract the second result from the first. This is diagrammed in Fig. 3.2.

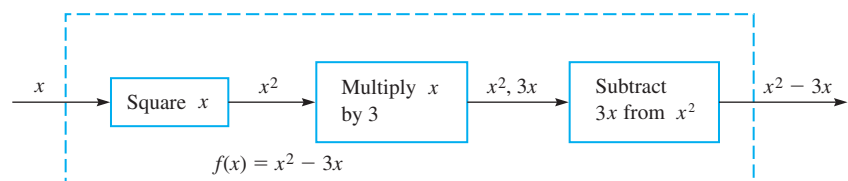


Fig. 3.2

## EXERCISES 3.1

In Exercises 1–4, solve the given problems related to the indicated examples of this section.

- In Example 5, find the value of  $f(-2)$ .
- In Example 6, evaluate  $g(-a^2)$ .
- In Example 7, change “increased” to “decreased” in the second and third lines and then evaluate the function to find the proper expression.
- In Example 9, change  $x^2 - 3x$  to  $x^3 + 4x$  and then determine the statements that should be placed in the three boxes in Fig. 3.2.

In Exercises 5–12, find the indicated functions.

- Express the area  $A$  of a circle as a function of (a) its radius  $r$  and (b) its diameter  $d$ .
- Express the circumference  $c$  of a circle as a function of (a) its radius  $r$  and (b) its diameter  $d$ .
- Express the diameter  $d$  of a sphere as a function of its volume  $V$ .
- Express the edge  $s$  of a cube as a function of its surface area  $A$ .
- Express the area  $A$  of a square as a function of its side  $s$ ; express the side  $s$  of a square as a function of its area  $A$ .
- Express the perimeter  $p$  of a square as a function of its side  $s$ ; express the side  $s$  of a square as a function of its perimeter  $p$ .
- A circle is inscribed in a square (the circle is tangent to each side of the square). Express the area  $A$  of the four corner regions bounded by the circle and the square as a function of the radius  $r$  of the circle.
- Express the area  $A$  of an equilateral triangle as a function of its side  $s$ .

In Exercises 13–24, evaluate the given functions.

- $f(x) = 2x + 1$ ; find  $f(1)$  and  $f(-1)$ .
- $f(x) = 5x - 9$ ; find  $f(2)$  and  $f(-2)$ .
- $f(x) = 5$ ; find  $f(-2)$  and  $f(0.4)$ .
- $f(T) = 7.2 - 2.5|T|$ ; find  $f(2.6)$  and  $f(-4)$ .
- $\phi(x) = \frac{6 - x^2}{2x}$ ; find  $\phi(2\pi)$  and  $\phi(-2)$ .
- $H(q) = \frac{8}{q} + 2\sqrt{q}$ ; find  $H(4)$  and  $H(0.16)$ .
- $g(t) = at^2 - a^2t$ ; find  $g(-\frac{1}{2})$  and  $g(a)$ .
- $s(y) = 6\sqrt{y + 11} - 3$ ; find  $s(-2)$  and  $s(a^2)$ .
- $K(s) = 3s^2 - s + 6$ ; find  $K(-s)$  and  $K(2s)$ .
- $T(t) = 5t + 7$ ; find  $T(-2t)$  and  $T(t + 1)$ .
- $f(x) = 2x + 4$ ; find  $f(3x) - 3f(x)$ .
- $f(x) = 2x^2 + 1$ ; find  $f(x + 2) - [f(x) + 2]$ .

In Exercises 25–28, evaluate the given functions. The values of the independent variable are approximate.

- Given  $f(x) = 5x^2 - 3x$ , find  $f(3.86)$  and  $f(-6.92)$ .
- Given  $g(t) = \sqrt{t + 1.0604} - 6t^3$ , find  $g(0.9261)$ .
- Given  $F(H) = \frac{2H^2}{H + 36.85}$ , find  $F(-84.466)$ .
- Given  $f(x) = \frac{x^4 - 2.0965}{6x}$ , find  $f(1.9654)$ .

In Exercises 29–36, state the instructions of the function in words as in Example 9.

- $f(x) = x^2 + 2$
- $g(y) = 6y - y^3$
- $R(r) = 3(2r + 5) - 1$
- $p(t) = \frac{2t - 3}{t + 2}$
- $f(x) = 2x - 6$
- $\phi(s) = 8 - 5s + s^2$
- $f(z) = \frac{4z}{5 - z}$
- $Y(y) = 2 + \frac{5y}{2(y - 3)}$

In Exercises 37–40, write the equation as given by the statement. Then write the indicated function using functional notation.

- The surface area  $A$  of a cubical open-top aquarium equals 5 times the square of an edge  $s$  of the aquarium.
- A helicopter is at an altitude of 1000 m and is  $x$  m horizontally from a fire. Its distance  $d$  from the fire is the square root of the sum of 1000 squared and  $x$  squared.
- The area  $A$  of the Athabasca Glacier in Jasper National Park, Canada, given that its present area is 6.00 km<sup>2</sup> and that it is melting at the rate of  $0.25t$  km<sup>2</sup> where  $t$  is the time in centuries.
- The electrical resistance  $R$  of a certain ammeter, in which the resistance of the coil is  $R_c$ , is the product of 10 and  $R_c$  divided by the sum of 10 and  $R_c$ .

In Exercises 41–46, solve the given problems and round all results to 3 significant digits.

- A demolition ball is used to tear down a building. Its distance  $s$  (in m) above the ground as a function of time  $t$  (in s) after it is dropped is  $s = 17.5 - 4.9t^2$ . Since  $s = f(t)$ , find  $f(1.2)$ .
- The length  $L$  (in m) of a cable hanging between equal supports 100 m apart is  $L = 100(1 + 0.0003s^2)$ , where  $s$  is the sag (in m) in the middle of the cable. Since  $L = f(s)$ , find  $f(15)$ .
- The stopping distance  $d$  (in m) of a car going  $v$  km/h is given by  $d = 0.2v + 0.008v^2$ . Since  $d = f(v)$ , find  $f(30)$ ,  $f(2v)$ , and  $f(60)$ , using both  $f(v)$  and  $f(2v)$ .
- The electric power  $P$  (in W) dissipated in a resistor of resistance  $R$  (in  $\Omega$ ) is given by the function  $P = \frac{200R}{(100 + R)^2}$ . Since  $P = f(R)$ , find  $f(R + 10)$ .
- A motorist travels at 55 km/h for  $t$  hours. Express the distance  $d$  travelled as a function of  $t$ .
- The sales tax in a city is 7%. The price tag on an item shows  $D$  dollars. Express the total cost  $C$  of the item as a function of  $D$ .
- (a) Explain the meaning of  $f[f(x)]$ . (b) Find  $f[f(x)]$  for  $f(x) = 2x^2$ .
- If  $f(x) = x$  and  $g(x) = x^2$ , find (a)  $f[g(x)]$ , and (b)  $g[f(x)]$ . What does this suggest about  $f[g(x)]$  and  $g[f(x)]$ ?

## Answers to Practice Exercises

- $f(-1) = -1$
- $f(-a^2x) = -5a^2x - 3$   
 $g(-a^2x) = a^5x^2 - a^2x$

## 3.2 More about Functions

Domain and Range • Functions from Verbal Statements • Relation

For a given function, the complete set of possible values of the independent variable is called the **domain** of the function, and the complete set of all possible resulting values of the dependent variable is called the **range** of the function.

As noted earlier, we will be using only **real numbers** when using functions. This means there may be restrictions as to the values that may be used since

*values that lead to **division by zero** or to **imaginary numbers** may not be included in the domain or the range.*

### EXAMPLE 1 Domain and range

The function  $f(x) = x^2 + 2$  is defined for all real values of  $x$ . This means its domain is written as *all real numbers*. However, since  $x^2$  is never negative,  $x^2 + 2$  is never less than 2. We then write the range as *all real numbers*  $f(x) \geq 2$ , where the symbol  $\geq$  means “is greater than or equal to.”

The function  $f(t) = \frac{1}{t+2}$  is not defined for  $t = -2$ , for this value would require division by zero. Also, no matter how large  $t$  becomes,  $f(t)$  will never exactly equal zero. Therefore, the domain of this function is *all real numbers except  $-2$* , and the range is *all real numbers except 0*. ■

### EXAMPLE 2 Domain and range

The function  $g(s) = \sqrt{3-s}$  is not defined for real numbers greater than 3, since such values make  $3-s$  negative and would result in imaginary values for  $g(s)$ . This means that the domain of this function is *all real numbers*  $s \leq 3$ , where the symbol  $\leq$  means “is less than or equal to.”

Also, since  $\sqrt{3-s}$  means the principal square root of  $3-s$  (see Section 1.6), we know that  $g(s)$  cannot be negative. This tells us that the range of the function is *all real numbers*  $g(s) \geq 0$ . ■

■ Note that  $f(2) = 6$  and  $f(-2) = 6$ . This is all right. Two different values of  $x$  may give the same value of  $y$ , but there may not be two different values of  $y$  for one value of  $x$ .

#### Practice Exercise

- Find the domain and range of the function  $f(x) = \sqrt{x+4}$ .

### INTERVAL NOTATION

There is a short way to write both the domain and range of a function: using interval notation. Interval notation lists two values within brackets, between which the possible values of the variable lie. Square brackets mean “include the value in the interval,” and rounded brackets mean “exclude the value in the interval.”

In Example 2 above, the domain for the function was the set of all real numbers less than or equal to 3. This can be written out in full as in the example, or using interval notation it would be indicated as  $(-\infty, 3]$ . This domain is interpreted as “all real values of  $s$  greater than negative infinity but less than or equal to 3.” By a similar argument, the range of  $g(s)$  can be written as  $[0, \infty)$ , which means “all real values of  $g(s)$  greater than or equal to zero but less than positive infinity.” Remember that  $\infty$  means “increases without limit” and is not an actual number. Interval notation convention always uses rounded brackets with infinity, since it can never really be included in the set.

In Example 1 above, the domain of the function  $f(x)$  was all real values of  $x$ , or  $(-\infty, \infty)$ . The range of  $f(x)$  was all real numbers  $f(x) \geq 2$ , or in interval notation,  $[2, \infty)$ .

In Examples 1 and 2, we found the domains by looking for values of the independent variable that cannot be used. The range was found through an inspection of the function. We generally use this procedure, although it may be necessary to use more advanced methods to find the range. Therefore, until we develop other methods, we will look only for the domain of some functions.

■ Later in this chapter, we will see that the graphing calculator is useful in finding the range of a function.



**EXAMPLE 3 Find domain only**

Find the domain of the function  $f(x) = 16\sqrt{x} + \frac{1}{x}$ .

From the term  $16\sqrt{x}$ , we see that  $x$  must be greater than or equal to zero in order to have real values. The term  $\frac{1}{x}$  indicates that  $x$  cannot be zero, because of division by zero. Thus, putting these together, the domain is *all real numbers*  $x > 0$ , or  $(0, \infty)$ .

As for the range, it is *all real numbers*  $f(x) \geq 12$ , or  $[12, \infty)$ . More advanced methods are needed to determine this, however. ■

We have seen that the domain may be restricted since we do not use imaginary numbers or divide by zero. The domain may also be restricted by the definition of the function, or by practical considerations in an application.

**EXAMPLE 4 Restricted domain**

A function defined as

$$f(x) = x^2 + 4 \quad (\text{for } x > 2)$$

has a domain restricted to real numbers greater than 2 by definition. Thus,  $f(5) = 29$ , but  $f(1)$  is not defined, since 1 is not in the domain. Also, the range is all real numbers greater than 8, so domain:  $(2, \infty)$ , range:  $(8, \infty)$ .

The height  $h$  (in m) of a certain projectile as a function of the time  $t$  (in s) is

$$h = 20t - 4.9t^2$$

Negative values of time have no real meaning in this case. Therefore, the domain is  $t \geq 0$ . Also, since we know the projectile will not continue in flight indefinitely, there is some upper limit on the value of  $t$ . These restrictions are not usually stated unless it affects the solution.

There could be negative values of  $h$  if it is possible that the projectile is *below* the launching point at some time (such as a stone thrown from the top of a cliff). ■

The following example illustrates a function that is defined differently for different intervals of the domain.

**EXAMPLE 5 Function defined for intervals of the domain**

In a certain electric circuit, the current  $i$  (in mA) is a function of the time  $t$  (in s), which means  $i = f(t)$ . The function is

$$f(t) = \begin{cases} 8 - 2t & (\text{for } 0 \leq t \leq 4 \text{ s}) \\ 0 & (\text{for } t > 4 \text{ s}) \end{cases}$$

Since negative values of  $t$  are not usually meaningful,  $f(t)$  is not defined for  $t < 0$ . Find the current for  $t = 3$  s,  $t = 6$  s, and  $t = -1$  s.

We are to find  $f(3)$ ,  $f(6)$ , and  $f(-1)$ , and we see that values of this function are determined differently depending on the value of  $t$ . Since 3 is between 0 and 4,

$$f(3) = 8 - 2(3) = 2 \quad \text{or} \quad i = 2 \text{ mA}$$

Since 6 is greater than 4,  $f(6) = 0$ , or  $i = 0$  mA. We see that  $i = 0$  mA for all values of  $t$  that are 4 or greater.

Since  $f(t)$  is not defined for  $t < 0$ ,  $f(-1)$  is not defined. ■

**FUNCTIONS FROM VERBAL STATEMENTS**

To find a mathematical function from a verbal statement, we use methods like those for setting up equations in Chapter 1. This is shown in the following examples.

**EXAMPLE 6** Function from a verbal statement with cost

The fixed cost for a company to operate a certain plant is \$3000 per day. It also costs \$4 for each unit produced in the plant. Express the daily cost  $C$  of operating the plant as a function of the number  $n$  of units produced.

The daily total cost  $C$  equals the fixed cost of \$3000 plus the cost of producing  $n$  units. Since the cost of producing one unit is \$4, the cost of producing  $n$  units is  $4n$ . Thus, the total cost  $C$ , where  $C = f(n)$ , is

$$C = 3000 + 4n$$

Here, we know that the domain is all values of  $n \geq 0$ , with some upper limit on  $n$  based on the production capacity of the plant. ■

**EXAMPLE 7** Function from a verbal statement with mixture percentage

A metallurgist melts and mixes  $m$  grams (g) of solder that is 40% tin with  $n$  grams of another solder that is 20% tin to get a final solder mixture that contains 200 g of tin. Express  $n$  as a function of  $m$ . See Fig. 3.3.

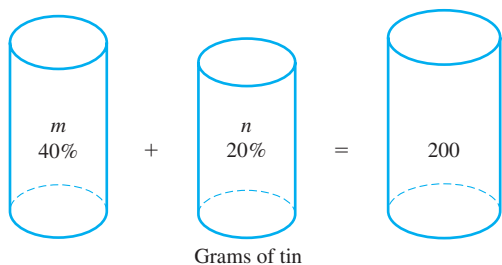


Fig. 3.3

The statement leads to the following equation:

tin in first solder	+	tin in second solder	=	total amount of tin
$0.40m$		$0.20n$		$200$

Since we want  $n = f(m)$ , we now solve for  $n$ :

$$\begin{aligned} 0.20n &= 200 - 0.40m \\ n &= 1000 - 2m \end{aligned}$$

This is the required function. Since neither  $m$  nor  $n$  can be negative, the domain is all values  $0 \leq m \leq 500$  g, which means that  $m$  is greater than or equal to 0 g and less than or equal to 500 g, or  $[0 \text{ g}, 500 \text{ g}]$ . The range is all values  $0 \leq n \leq 1000$  g, or  $[0 \text{ g}, 1000 \text{ g}]$ . ■

**EXAMPLE 8** Function from a verbal statement with perimeter

An architect designs a window in the shape of a rectangle with a semicircle on top, as shown in Fig. 3.4. The base of the window is 10 cm less than the height of the rectangular part. Express the perimeter  $p$  of the window as a function of the radius  $r$  of the circular part.

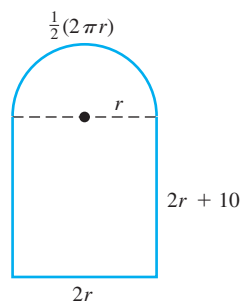


Fig. 3.4

The perimeter is the distance around the window. Since the top part is a semicircle, the length of this top circular part is  $\frac{1}{2}(2\pi r)$ , and the base of the window is  $2r$  because it is equal in length to the dashed line (the diameter of the circle). Finally, the base being 10 cm less than the height of the rectangular part tells us that each vertical side of the rectangle is  $2r + 10$ . Therefore, the perimeter  $p$ , where  $p = p(r)$ , is

$$\begin{aligned} p &= \frac{1}{2}(2\pi r) + 2r + 2(2r + 10) \\ p &= \pi r + 2r + 4r + 20 \\ p &= \pi r + 6r + 20 \end{aligned}$$

We see that the required function is  $p = \pi r + 6r + 20$ . Since the radius cannot be negative and there would be no window if  $r = 0$ , the domain of the function is all values  $0 < r \leq R$ , where  $R$  is a maximum possible value of  $r$  determined by design considerations, or  $(0, R]$ . ■

**Practice Exercise**

2. In Example 8, find  $p$  as a function of  $r$  if there is to be a semicircular section of the window at the bottom, as well as at the top.

From the definition of a function, we know that any value of the independent variable must yield only one value of the dependent variable. *If a value of the independent variable yields one or more values of the dependent variable, the relationship is called a **relation**.* A function is a relation for which each value of the independent variable yields only one value of the dependent variable. Therefore, a function is a special type of relation. There are also relations that are not functions.

**EXAMPLE 9** Relation

For  $y^2 = 4x^2$ , if  $x = 2$ , then  $y$  can be either 4 or  $-4$ . Since a value of  $x$  yields more than one value for  $y$ , we see that  $y^2 = 4x^2$  is a relation, not a function. ■

**EXERCISES 3.2**

In Exercises 1–4, solve the given problems related to the indicated examples of this section.

- In Example 1, in the first line, change  $x^2$  to  $-x^2$ . What other changes must be made in the rest of the paragraph?
- In Example 3, in the first line, change  $\frac{1}{x}$  to  $\frac{1}{x-1}$ . What other changes must be made in the first paragraph?
- In Example 5, find  $f(2)$  and  $f(5)$ .
- In Example 7, interchange 40% and 20% and then find the function.

In Exercises 5–14, find the domain and range of the given functions. In Exercises 11 and 12, explain your answers.

- $f(x) = x + 5$
- $G(R) = \frac{3.2}{R}$
- $f(s) = \frac{2}{s^2}$
- $H(h) = 2h + \sqrt{h} + 1$
- $y = |x - 3|$
- $g(u) = 3 - u^2$
- $F(r) = \sqrt{r + 4}$
- $T(t) = 2t^4 + t^2 - 1$
- $f(x) = \frac{-6}{\sqrt{2-x}}$
- $y = x + |x|$

In Exercises 15–18, find the domain of the given functions.

- $Y(y) = \frac{y+1}{\sqrt{y-2}}$
- $f(n) = \frac{n}{6-2n}$
- $f(D) = \frac{D}{D-2} + \frac{4}{D+4}$
- $g(x) = \frac{\sqrt{x-2}}{x-3}$

In Exercises 19–22, evaluate the indicated functions.

$$F(t) = 3t - t^2 \quad (\text{for } t \leq 2) \quad h(s) = \begin{cases} 2s & (\text{for } s < -1) \\ s + 1 & (\text{for } s \geq -1) \end{cases}$$

$$f(x) = \begin{cases} x + 1 & (\text{for } x < 1) \\ \sqrt{x + 3} & (\text{for } x \geq 1) \end{cases} \quad g(x) = \begin{cases} \frac{1}{x} & (\text{for } x \neq 0) \\ 0 & (\text{for } x = 0) \end{cases}$$

- Find  $F(2)$  and  $F(3)$ .
- Find  $h(-8)$  and  $h(-0.5)$ .
- Find  $f(1)$  and  $f(-0.25)$ .
- Find  $g(0.2)$  and  $g(0)$ .

In Exercises 23–36, determine the appropriate functions.

- A motorist travels at 60 km/h for 2 h and then at 80 km/h for  $t$  h. Express the distance  $d$  travelled as a function of  $t$ .
- Express the cost  $C$  of insulating a cylindrical water tank of height 2 m as a function of its radius  $r$ , if the cost of insulation is \$3 per square metre.
- A rocket burns up at the rate of 2 Mg/min after falling out of orbit into the atmosphere. If the rocket weighed 5500 Mg before reentry, express its weight  $w$  as a function of the time  $t$ , in minutes, of reentry.

- A computer part costs \$3 to produce and distribute. Express the profit  $p$  made by selling 100 of these parts as a function of the price of  $c$  dollars each.
- Upon ascending, a weather balloon ices up at the rate of 0.5 kg/m after reaching an altitude of 1000 m. If the mass of the balloon below 1000 m is 110 kg, express its mass  $m$  as a function of its altitude  $h$  if  $h > 1000$  m.
- A chemist adds  $x$  L of a solution that is 50% alcohol to 100 L of a solution that is 70% alcohol. Express the number  $n$  of litres of alcohol in the final solution as a function of  $x$ .
- A company installs underground cable at a cost of \$500 for the first 50 m (or up to 50 m) and \$5 for each metre thereafter. Express the cost  $C$  as a function of the length  $l$  of underground cable if  $l > 50$  m.
- The *mechanical advantage* of an inclined plane is the ratio of the length of the plane to its height. Express the mechanical advantage  $MA$  of a plane of length 8 m as a function of its height  $h$ .
- The capacities (in L) of two oil-storage tanks are  $x$  and  $y$ . The tanks are initially full; 1200 L is removed from them by taking 10% of the contents of the first tank and 40% of the contents of the second tank. (a) Express  $y$  as a function of  $x$ . (b) Find  $y(400)$ .
- A city does not tax the first \$30 000 of a resident's income but taxes any amount over \$30 000 at 5%. (a) Find the tax  $T$  as a function of a resident's income  $I$  and (b) find  $T(25\,000)$  and  $T(45\,000)$ .
- A company finds that it earns a profit of \$15 on each cell phone and a profit of \$25 on each DVD player that it sells. If  $x$  cell phones and  $y$  DVD players are sold, the profit is \$2750. Express  $y$  as a function of  $x$ .
- A satellite telephone is leased at a cost  $C$  of \$200 plus \$10 per minute  $t$  the phone is used. Solve for  $t$  as a function of  $C$  and find  $t(500)$ .
- For studying the electric current that is induced in wire rotating through a magnetic field, a piece of wire 60 cm long is cut into two pieces. One of these is bent into a circle and the other into a square. Express the total area  $A$  of the two figures as a function of the perimeter  $p$  of the square.
- The cross section of an air-conditioning duct is in the shape of a square with semicircles on each side. See Fig. 3.5. Express the area  $A$  of this cross section as a function of the diameter  $d$  (in cm) of the circular part.

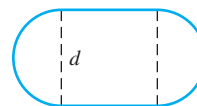


Fig. 3.5

In Exercises 37–50, solve the given problems.

37. A helicopter 120 m from a person takes off vertically. Express the distance  $d$  from the person to the helicopter as a function of the height  $h$  of the helicopter. What are the domain and the range of  $d(h)$ ? See Fig. 3.6.

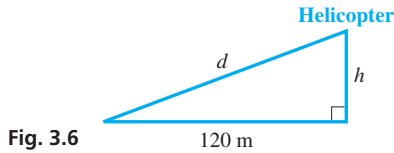


Fig. 3.6

38. A computer program displays a circular image of radius 6 cm. If the radius is decreased by  $x$  cm, express the area of the image as a function of  $x$ . What are the domain and range of  $A(x)$ ?
39. A truck travels 300 km in  $t$  h. Express the average speed  $v$  of the truck as a function of  $t$ . What are the domain and range of  $v(t)$ ?
40. A rectangular grazing range with an area of  $8 \text{ km}^2$  is to be fenced. Express the length  $l$  of the field as a function of its width  $w$ . What are the domain and range of  $l(w)$ ?
41. The resonant frequency  $f$  of a certain electric circuit as a function of the capacitance is  $f = \frac{1}{2\pi\sqrt{C}}$ . Describe the domain.
42. A jet is travelling directly between Calgary, Alberta, and Halifax, Nova Scotia, which are 3760 km apart. If the jet is  $x$  km from Calgary and  $y$  km from Halifax, find the domain of  $y = f(x)$ .

43. Express the mass  $m$  of the weather balloon in Exercise 27 as a function of any height  $h$  in the same manner as the function in Example 5 (and Exercises 19–22) was expressed.
44. Express the cost  $C$  of installing any length  $l$  of the underground cable in Exercise 29 in the same manner as the function in Example 5 (and Exercises 19–22) was represented.
45. A rectangular piece of cardboard twice as long as wide is to be made into an open box by cutting 5-cm squares from each corner and bending up the sides. (a) Express the volume  $V$  of the box as a function of the width  $w$  of the piece of cardboard. (b) Find the domain of the function.
46. A spherical buoy 36.0 cm in diameter is floating in a lake and is more than half above the water. (a) Express the circumference  $c$  of the circle of intersection of the buoy and water surface as a function of the depth  $d$  to which the buoy sinks. (b) Find the domain and range of the function.
47. For  $f(x - 1) = |x|$ , find  $f(0)$ .
48. For  $f(x) = x^2$ , find  $\frac{f(x + h) - f(x)}{h}$ .
49. What is the range of the function  $f(x) = |x| + |x - 2|$ ?
50. For  $f(x) = \sqrt{x - 1}$  and  $g(x) = x^2$ , find the domain of  $g[f(x)]$ . Explain.

#### Answers to Practice Exercises

1. Domain: all real numbers  $x \geq -4$ ; Range: all real numbers  $f(x) \geq 0$     2.  $p = 2\pi r + 4r + 20$

## 3.3 Rectangular Coordinates

### Rectangular Coordinate System

■ Rectangular (Cartesian) coordinates were developed by the French mathematician René Descartes (1596–1650).

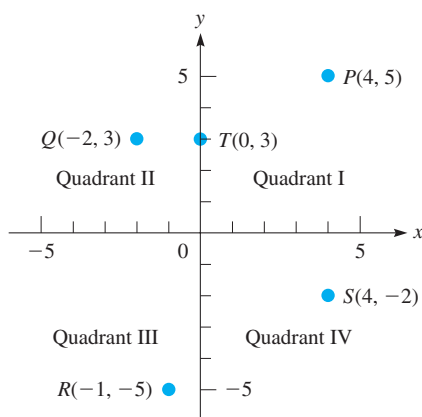


Fig. 3.7

One of the most valuable ways of representing a function is by graphical representation. By using graphs, we can obtain a “picture” of the function; by using this picture, we can learn a great deal about the function.

To make a graphical representation of a function, recall from Chapter 1 that numbers can be represented by points on a line. For a function, we have values of the independent variable as well as the corresponding values of the dependent variable. Therefore, it is necessary to use two different lines to represent the values from each of these sets of numbers. We do this by placing the lines perpendicular to each other.

Place one line horizontally and label it the **x-axis**. The values of the independent (or manipulated) variable are normally placed on this axis. The other line is placed vertically and labelled the **y-axis**. Normally, the y-axis is used for values of the dependent (or responding) variable. The point of intersection is called the **origin**. This is the **rectangular coordinate system**.

On the x-axis, positive values are to the right of the origin, and negative values are to the left of the origin. On the y-axis, positive values are above the origin, and negative values are below it. The positive direction is also indicated by an arrow on the axis line. The four parts into which the plane is divided are called **quadrants**, which are numbered as in Fig. 3.7.

A point  $P$  in the plane is designated by the pair of numbers  $(x, y)$ , where  $x$  is the value of the independent variable and  $y$  is the value of the dependent variable. The  $x$ -value is the perpendicular distance of  $P$  from the y-axis, and the  $y$ -value is the perpendicular distance of  $P$  from the x-axis. The values of  $x$  and  $y$ , written as  $(x, y)$ , are the **coordinates** of the point  $P$ .

**LEARNING TIP**

Note very carefully that the ***x*-coordinate is always written first, and the *y*-coordinate is always written second**. It is very important to keep the proper order when writing the coordinates of a point. This is why the coordinates of a point  $P(x, y)$  are called an **ordered pair**. (The *x*-coordinate is also known as the *abscissa*, and the *y*-coordinate is also known as the *ordinate*.)

**EXAMPLE 1 Locating points**

- (a) Locate the points  $A(2, 1)$  and  $B(-4, -3)$  on the rectangular coordinate system.

The coordinates  $(2, 1)$  for  $A$  mean that the point is 2 units to the *right* of the *y*-axis and 1 unit *above* the *x*-axis, as shown in Fig. 3.8. The coordinates  $(-4, -3)$  for  $B$  mean that the point is 4 units to the *left* of the *y*-axis and 3 units *below* the *x*-axis, as shown. The *x*-coordinate of  $A$  is 2, and the *y*-coordinate of  $A$  is 1. For point  $B$ , the *x*-coordinate is  $-4$ , and the *y*-coordinate is  $-3$ .

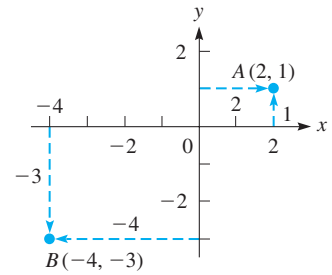


Fig. 3.8

- (b) The positions of points  $P(4, 5)$ ,  $Q(-2, 3)$ ,  $R(-1, -5)$ ,  $S(4, -2)$ , and  $T(0, 3)$  are shown in Fig. 3.7. We see that this representation allows for *one point for any pair of values*  $(x, y)$ . Also note that the point  $T(0, 3)$  is on the *y*-axis. Any such point that is on either axis is not *in* any of the four quadrants. ■

**EXAMPLE 2 Coordinates of the vertices of a rectangle**

Three vertices of the rectangle in Fig. 3.9 are  $A(-3, -2)$ ,  $B(4, -2)$ , and  $C(4, 1)$ . What is the fourth vertex?

We use the fact that opposite sides of a rectangle are equal and parallel to find the solution. Since both vertices of the base  $AB$  of the rectangle have a *y*-coordinate of  $-2$ , the base is parallel to the *x*-axis. Therefore, the top of the rectangle must also be parallel to the *x*-axis. Thus, the vertices of the top must both have a *y*-coordinate of 1, since one of them has a *y*-coordinate of 1. In the same way, the *x*-coordinates of the left side must both be  $-3$ . Therefore, the fourth vertex is  $D(-3, 1)$ . ■

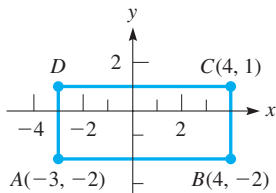


Fig. 3.9

**EXAMPLE 3 Locating sets of points**

- (a) Where are all the points whose *y*-coordinates are 2?

We can see that the question could be stated as: “Where are all the points for which  $y = 2$ ?” Since all such points are two units above the *x*-axis, the answer could be stated as “on a line 2 units above the *x*-axis.” See Fig. 3.10.

- (b) Where are all points  $(x, y)$  for which  $x < 0$  and  $y < 0$ ?

Noting that  $x < 0$  means “*x* is less than zero,” or “*x* is negative,” and that  $y < 0$  means the same for *y*, we want to determine where both *x* and *y* are negative. Our answer is “in the third quadrant,” since both coordinates are negative for all points in the third quadrant, and this is the only quadrant for which this is true. See Fig. 3.11. ■

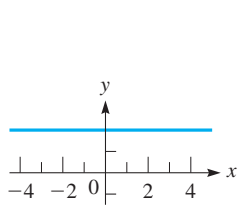


Fig. 3.10

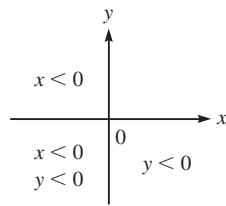


Fig. 3.11

**Practice Exercise**

1. Where are all points for which  $x = 0$  and  $y > 0$ ?

**EXERCISES 3.3**

In Exercises 1 and 2, make the given changes in the indicated examples of this section and then solve the resulting problems.

- In Example 2, change  $A(-3, -2)$  to  $A(-1, -2)$  and then find the fourth vertex.
- In Example 3(b), change  $y < 0$  to  $y > 0$  and then find the location of the points  $(x, y)$ .

In Exercises 3 and 4, determine (at least approximately) the coordinates of the points shown in Fig. 3.12.

3.  $A, B, C$       4.  $D, E, F$

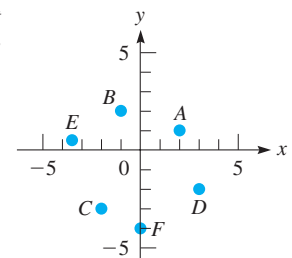


Fig. 3.12

In Exercises 5 and 6, plot the given points.

5.  $A(2, 7), B(-1, -2), C(-4, 2)$   
 6.  $A(3, \frac{1}{2}), B(-6, 0), C(-\frac{5}{2}, -5)$

In Exercises 7–10, plot the given points and then join these points, in the order given, by straight-line segments. Name the geometric figure formed.

7.  $A(-1, 4), B(3, 4), C(1, -2)$   
 8.  $A(0, 3), B(0, -1), C(4, -1)$   
 9.  $A(-2, -1), B(3, -1), C(3, 5), D(-2, 5)$   
 10.  $A(-5, -2), B(4, -2), C(6, 3), D(-3, 3)$

In Exercises 11–14, find the indicated coordinates.

11. Three vertices of a rectangle are  $(5, 2)$ ,  $(-1, 2)$ , and  $(-1, 4)$ . What are the coordinates of the fourth vertex?  
 12. Two vertices of an equilateral triangle are  $(7, 1)$  and  $(2, 1)$ . What is the  $x$ -coordinate of the third vertex?  
 13.  $P$  is the point  $(3, 2)$ . Locate point  $Q$  such that the  $x$ -axis is the perpendicular bisector of the line segment joining  $P$  and  $Q$ .  
 14.  $P$  is the point  $(-4, 1)$ . Locate point  $Q$  such that the line segment joining  $P$  and  $Q$  is bisected by the origin.

In Exercises 15–34, answer the given questions.

15. Where are all points whose  $x$ -coordinates are 1?  
 16. Where are all points whose  $y$ -coordinates are  $-3$ ?  
 17. Where are all points such that  $y = 3$ ?  
 18. Where are all points such that  $|x| = 2$ ?

19. Where are all points whose  $x$ -coordinates equal their  $y$ -coordinates?  
 20. Where are all points whose  $x$ -coordinates equal the negative of their  $y$ -coordinates?  
 21. What is the  $x$ -coordinate of all points on the  $y$ -axis?  
 22. What is the  $y$ -coordinate of all points on the  $x$ -axis?  
 23. Where are all points for which  $x > 0$ ?  
 24. Where are all points for which  $y < 0$ ?  
 25. Where are all points for which  $x < -1$ ?  
 26. Where are all points for which  $y > 4$ ?  
 27. Where are all points for which  $xy > 0$ ?  
 28. Where are all points for which  $y/x < 0$ ?  
 29. Where are all points for which  $xy = 0$ ?  
 30. Where are all points for which  $x < y$ ?  
 31. If the point  $(a, b)$  is in the second quadrant, in which quadrant is  $(a, -b)$ ?  
 32. The points  $(3, -1)$ ,  $(3, 0)$ , and  $(x, y)$  are on the same straight line. Describe this line.  
 33. Find the distance (a) between  $(3, -2)$  and  $(-5, -2)$  and (b) between  $(3, -2)$  and  $(3, 4)$ .  
 34. Find the distance between  $(-5, -2)$  and  $(3, 4)$ . Explain your method. (*Hint*: See Exercise 33.)

#### Answer to Practice Exercise

1. On the positive  $y$ -axis

## 3.4 The Graph of a Function

Table of Values • Plotting Points •  
 Be Careful about the Domain •  
 Vertical-Line Test

Now that the concepts of a function and the rectangular coordinate system have been introduced, the graph of a function can be determined. In this way, a visual representation of a function will be obtained.

The graph of a function is the set of all points whose coordinates  $(x, y)$  satisfy the functional relationship  $y = f(x)$ . Since  $y = f(x)$ , we can write the coordinates of the points on the graph as  $(x, f(x))$ . Writing the coordinates in this manner tells us exactly how to find them. *We assume a certain value for  $x$  and then find the value of the function of  $x$ . These two numbers are the coordinates of the point.*

Since there is no limit to the possible number of points that can be chosen, we normally select a few values of  $x$ , obtain the corresponding values of the function, plot these points, and then join them. Therefore, we use the following basic procedure in plotting a graph.

### Procedure for Plotting the Graph of a Function

1. Let  $x$  take on several values and calculate the corresponding values of  $y$ .
2. Tabulate these values, arranging the table so that **values of  $x$  are increasing**.
3. Construct the  $x$ - and  $y$ -rectangular coordinate axes, properly labelled, with an appropriate scale.
4. Plot the points and join them from left to right by a **smooth curve** (not short straight-line segments).

■ As to just what values of  $x$  to choose and how many to choose, with a little experience you will usually be able to tell if you have enough points to plot an accurate graph.



**EXAMPLE 1** Graphing a function by plotting points

Graph the function  $f(x) = 3x - 5$ .

For purposes of graphing, let  $y = f(x)$ , or  $y = 3x - 5$ . Then, let  $x$  have various values and determine the corresponding values of  $y$ . Note that once we choose a given value of  $x$ , we have no choice about the corresponding  $y$ -value, as it is determined by evaluating the function. If  $x = 0$ , we find that  $y = -5$ . This means that the point  $(0, -5)$  is on the graph of the function  $3x - 5$ . Choosing another value of  $x$ , like  $x = 1$ , we find that  $y = -2$ . This means that the point  $(1, -2)$  is on the graph of the function  $3x - 5$ . Continuing to choose a few other values of  $x$ , we tabulate the results, as shown in Fig. 3.13. It is best to arrange the table so that the values of  $x$  increase; then there is no doubt how they are to be connected, for they are connected in the order shown. Finally, we connect the points in Fig. 3.13 and see that the graph of the function  $3x - 5$  is a straight line.

■ We will show how a graph is displayed on a graphing calculator in the next section.

■ The table of values for a graph can be seen on a graphing calculator using the *table* feature.

Select values of  $x$  and calculate corresponding values of  $y$

$$\begin{aligned} f(x) &= 3x - 5 \\ f(-1) &= 3(-1) - 5 = -8 \\ f(0) &= 3(0) - 5 = -5 \\ f(1) &= 3(1) - 5 = -2 \\ f(2) &= 3(2) - 5 = 1 \\ f(3) &= 3(3) - 5 = 4 \end{aligned}$$

Tabulate with values of  $x$  increasing

$x$	$y$
-1	-8
0	-5
1	-2
2	1
3	4

Draw the axes with an appropriate scale, plot points, and join from left to right with smooth curve

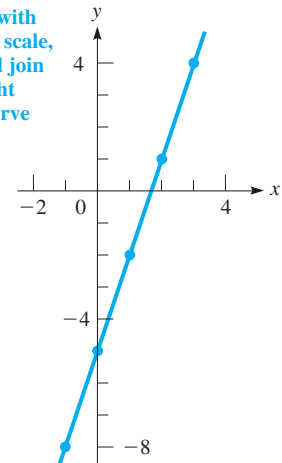


Fig. 3.13

**EXAMPLE 2** Be careful plotting with negative numbers

Graph the function  $f(x) = 2x^2 - 4$ .

First, let  $y = 2x^2 - 4$  and tabulate the values as shown in Fig. 3.14. In determining the values in the table, take particular care to obtain the correct values of  $y$  for negative values of  $x$ . We must carefully use the laws for signed numbers. For example, if  $x = -2$ , we have  $y = 2(-2)^2 - 4 = 2(4) - 4 = 8 - 4 = 4$ . Once the values are obtained, plot and connect the points with a smooth curve, as shown.

**COMMON ERROR**

Mistakes are common in evaluating  $f(x)$  for negative values of  $x$ . Ensure that you are careful!

As in Example 2, if  $f(x) = 2x^2 - 4$ , then

$$f(-1) = 2(-1)^2 - 4$$

$$f(-1) = 2(1) - 4$$

$$f(-1) = -2$$

But if  $f(x) = -5x^3 + 7x$ , then

$$f(-2) = -5(-2)^3 + 7(-2)$$

$$f(-2) = -5(-8) - 14$$

$$f(-2) = 40 - 14$$

$$f(-2) = 26$$

Select values of  $x$  and calculate corresponding values of  $y$

$$\begin{aligned} f(x) &= 2x^2 - 4 \\ f(-2) &= 2(-2)^2 - 4 = 4 \\ f(-1) &= 2(-1)^2 - 4 = -2 \\ f(0) &= 2(0)^2 - 4 = -4 \\ f(1) &= 2(1)^2 - 4 = -2 \\ f(2) &= 2(2)^2 - 4 = 4 \end{aligned}$$

Tabulate with values of  $x$  increasing

$x$	$y$
-2	4
-1	-2
0	-4
1	-2
2	4

Draw axes using an appropriate scale, plot points, and join from left to right with smooth curve

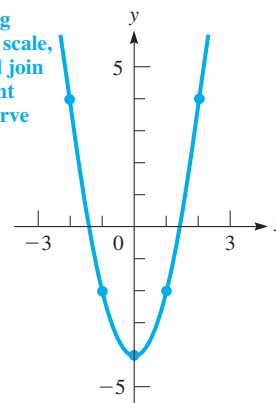


Fig. 3.14

When graphing a function, we must use extra care with certain parts of the graph. These include the following:

### Special Notes on Graphing

1. Since the graphs of most common functions are smooth, *any place that the graph changes in a way that is not expected should be checked with care.* It usually helps to check values of  $x$  between which the question arises.
2. The domain of the function may not include all values of  $x$ . Remember, *division by zero is not defined, and only real values of the variables may be used.*
3. In applications, we must *use only values of the variables that have meaning.* In particular, *negative values of some variables, such as time, can have no meaning.*

The following examples illustrate these points.

### EXAMPLE 3 Be careful when graphing; additional points may be needed

Graph the function  $y = x - x^2$ .

First, we determine the values in the table, as shown with Fig. 3.15. Again, we must be careful when dealing with negative values of  $x$ . For the value  $x = -1$ , we have  $y = (-1) - (-1)^2 = -1 - 1 = -2$ . Once all values in the table have been found and plotted, note that  $y = 0$  for both  $x = 0$  and  $x = 1$ . The question arises—*what happens between these values?* Trying  $x = \frac{1}{2}$ , we find that  $y = \frac{1}{4}$ . Using this point completes the information needed to complete an accurate graph.

Note that in plotting these graphs, we do not stop the graph with the points we found, but indicate that the curve continues by drawing the graph past these points. However, this is not true for all graphs. As we will see in later examples, the graph should not include points for values of  $x$  not in the domain, and therefore may not continue past a particular point.

$x$	$y$
-2	-6
-1	-2
0	0
1	0
2	-2
3	-6

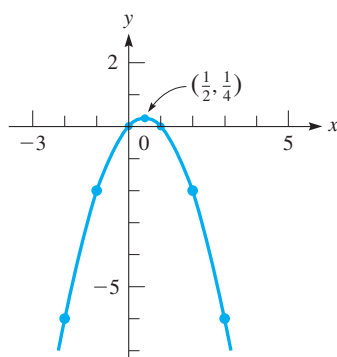


Fig. 3.15

**LEARNING TIP**

An **asymptote** is any curve or line that the graph of a function approaches but does not touch. The term was first introduced by Apollonius of Perga (262–190 B.C.E.) to mean any line that does not intersect a given function, but the modern meaning is two curves that “do not fall together” but can be arbitrarily close to each other. Asymptotes for curves are commonly **vertical asymptotes** or **horizontal asymptotes** but can be **oblique asymptotes** (angled lines) or **curvilinear asymptotes** (where the graph of a known curved function is approached by that of the function of interest).

**EXAMPLE 4** Be careful when graphing; watch for division by zero errors

Graph the function  $y = 1 + \frac{1}{x}$ .

In finding the points on this graph, as shown in Fig. 3.16, note that  $y$  is not defined for  $x = 0$ , due to division by zero. Thus,  $x = 0$  is not in the domain, and **we must be careful not to have any part of the curve cross the  $y$ -axis** ( $x = 0$ ). Although we cannot let  $x = 0$ , we can choose other values for  $x$  between  $-1$  and  $1$  that are close to zero. In doing so, we find that as  $x$  gets closer to zero, the points get closer to the  $y$ -axis, although they do not reach or touch it. In this case, the  $y$ -axis is called an **asymptote** of the curve.

We see that the curve is smooth, except when  $x = 0$ , where it is not defined. ■

$x$	$y$	$x$	$y$
-4	3/4	1/3	4
-3	2/3	1/2	3
-2	1/2	1	2
-1	0	2	3/2
-1/2	-1	3	4/3
-1/3	-2	4	5/4

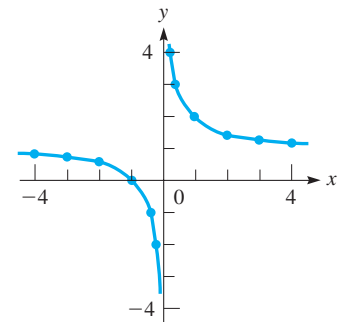


Fig. 3.16

**EXAMPLE 5** Be careful when graphing; watch for imaginary values

Graph the function  $y = \sqrt{x + 1}$ .

When finding the points for the graph, we may not let  $x$  take on any value less than  $-1$ , for **all such values would lead to imaginary values for  $y$**  and are not in the domain. Also, since we have the positive square root indicated, the range consists of all values of  $y$  that are positive or zero ( $y \geq 0$ ). See Fig. 3.17. Note that the graph starts at  $(-1, 0)$ .

Although not defined for all values of  $x$ , the curve is smooth. ■

$x$	$y$	$x$	$y$
-1	0	4	2.2
0	1	5	2.4
1	1.4	6	2.6
2	1.7	7	2.8
3	2	8	3

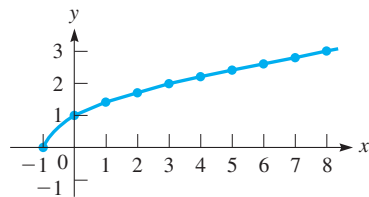


Fig. 3.17

**EXAMPLE 6** Graphing application with wind turbines

The electric power  $P$  (in W) produced by a wind turbine depends on the velocity  $v$  (in m/s) of the wind, the power coefficient  $C_p$ , which is related to the efficiency of the turbine, the air density  $\rho$  (in  $\text{kg}/\text{m}^3$ ), and the sweep area  $A$  (in  $\text{m}^2$ ) of the turbine according to  $P = 0.5C_p\rho Av^3$ . For a specific 3.00-m diameter turbine of  $C_p = 30\%$  and  $A = 7.069 \text{ m}^2$ , with air density  $\rho = 1.23 \text{ kg}/\text{m}^3$ , the power produced is  $P = 1.304v^3$ . Plot  $P$  as a function of  $v$ .

Since negative values of velocity are not relevant, we only plot values of  $v > 0$ . Fig. 3.18 shows the table and the smooth curve obtained. Note on the graph how the

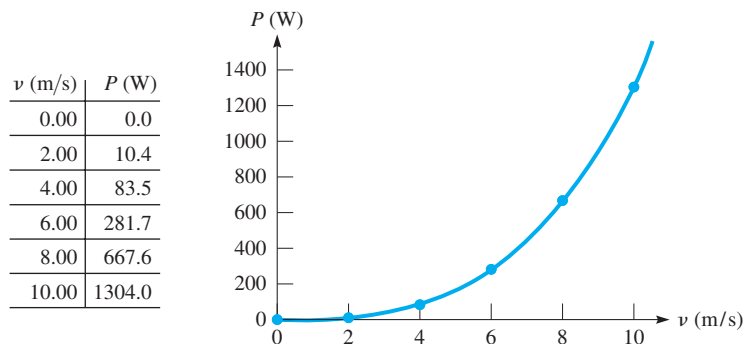


Fig. 3.18

■ See the chapter introduction.

scale on the  $P$ -axis differs from that of the  $v$ -axis. Different scales are normally used when the variables differ in magnitudes and ranges. The power produced by the wind turbine increases rapidly as wind velocity increases. ■

**Practice Exercises**

Determine for what value(s) of  $x$  there are no points on the graph.

1.  $y = \frac{2x}{x + 5}$     2.  $y = \sqrt{x - 3}$

Functions of a particular type have graphs with a certain basic shape, and many have been named. Three examples of this are the *straight line* (Example 1), the *parabola* (Examples 2 and 3), the *hyperbola* (Example 4), and the *cubic* (Example 6). We consider the straight line again in Chapter 5 and the parabola in Chapter 7. All of these graphs and others are studied in detail in Chapter 21. Other types of graphs are found in many of the later chapters.

The following example illustrates an application of graphing in which a different scale is used for each variable.

**EXAMPLE 7 Graphing application with different scales for each variable**

The electric power  $P$  (in W) delivered by a certain fuel cell as a function of the resistance  $R$  (in  $\Omega$ ) in the circuit is given by  $P = \frac{100R}{(0.50 + R)^2}$ . Plot  $P$  as a function of  $R$ .

■ The unit of power, the watt (W), is named for James Watt (1736–1819), a British engineer.

Since negative values for the resistance have no physical significance, we should not plot any values of  $P$  for negative values of  $R$ . The following table is obtained:

$R$ ( $\Omega$ )	0	0.25	0.50	1.0	2.0	3.0	4.0	5.0	10.0
$P$ (W)	0.0	44.4	50.0	44.4	32.0	24.5	19.8	16.5	9.1

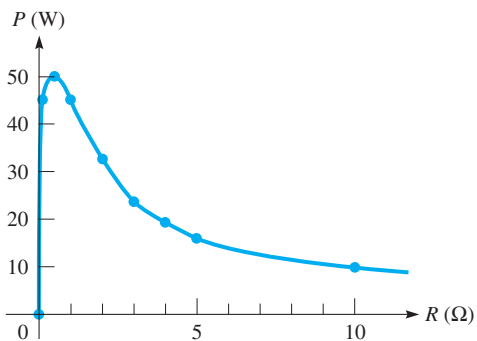


Fig. 3.19

The  $R$  values of 0.25 and 0.50 are used since  $P$  is less for  $R = 2 \Omega$  than for  $R = 1 \Omega$ . The sharp change in direction should be checked by using these additional points to get a smoother curve near  $R = 1 \Omega$ . See Fig. 3.19.

Note the scale on the  $P$ -axis differs from that on the  $R$ -axis. The graph gives us information about  $P$  and  $R$ . For example, the maximum power of 50 W occurs for  $R = 0.50 \Omega$ . Also,  $P$  decreases as  $R$  increases beyond  $0.50 \Omega$ . Reading information from a graph is discussed in the next section. ■

**EXAMPLE 8 Function defined for intervals of the domain**

Graph the function  $f(x) = \begin{cases} 2x + 1 & \text{(for } x \leq 1) \\ 6 - x^2 & \text{(for } x > 1) \end{cases}$ .

$x$	$y$
-2	-3
-1	-1
0	1
1	3
2	2
3	-3

$f(-2) = 2(-2) + 1 = -3$   
 $f(-1) = 2(-1) + 1 = -1$   
 $f(0) = 2(0) + 1 = 1$   
 $f(1) = 2(1) + 1 = 3$   
 $f(2) = 6 - 2^2 = 2$   
 $f(3) = 6 - 3^2 = -3$

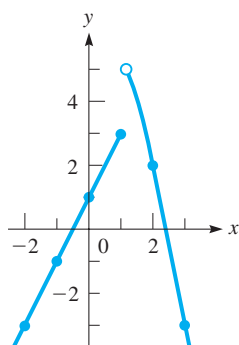


Fig. 3.20

First, let  $y = f(x)$  and then tabulate the necessary values. In evaluating  $f(x)$ , we must be careful to use the proper part of the definition. To see where to start the curve for  $x > 1$ , we evaluate  $6 - x^2$  for  $x = 1$ , but we must realize that **the curve does not include this point** (1, 5) and starts immediately to its right. To show that it is not part of the curve, draw it as an open circle. See Fig. 3.20.

A function such as this one, with a “break” in it, is called *discontinuous*. Curve continuity and discontinuity will be discussed in Section 23.1. ■

A function is defined such that there is only one value of the dependent variable for each value of the independent variable, but a *relation* may have more than one such value of the dependent variable. The following example illustrates how to use a graph to see whether or not a relation is also a function by using the **vertical-line test**.

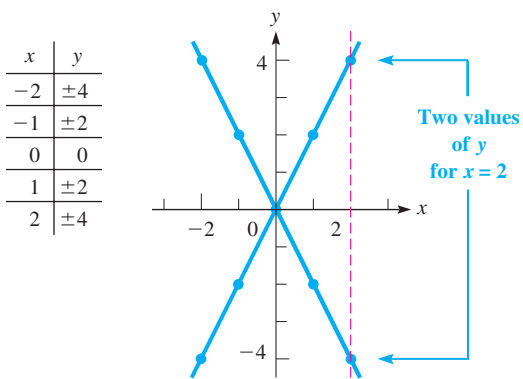
**EXAMPLE 9** Vertical-line test for a function

Fig. 3.21

In Example 9 of Section 3.2, we noted that  $y^2 = 4x^2$  is a relation, but not a function. Since  $y = 4$  or  $y = -4$ , normally written as  $y = \pm 4$ , for  $x = 2$ , we have two values of  $y$  for  $x = 2$ . Therefore, it is not a function. Making the table as shown at the left, we then draw the graph in Fig. 3.21. When making the table, we also note that there are two values of  $y$  for every value of  $x$ , except  $x = 0$ .

If any *vertical line* that crosses the  $x$ -axis in the domain intersects the graph in more than one point, it is the graph of a relation that is not a function. Any such vertical line in any of the previous graphs of this section would intersect the graph in only one point. This shows that they are graphs of functions. ■

The use of graphs is extensive in mathematics and its applications. For example, in the next section, we use graphs to solve equations. For another example, the parabola has applications in microwave-dish design, suspension bridge design, and the path (trajectory) of a baseball. We will note many more applications throughout the book.

**EXERCISES 3.4**

In Exercises 1–4, make the given changes in the indicated examples of this section and then plot the graphs.

- In Example 1, change the  $-$  sign to  $+$ .
- In Example 2, change  $2x^2 - 4$  to  $4 - 2x^2$ .
- In Example 4, change the  $x$  in the denominator to  $x - 1$ .
- In Example 5, change the  $+$  sign to  $-$ .

In Exercises 5–36, graph the given functions.

- $y = 3x$
- $y = -2x$
- $y = 2x - 4$
- $y = 3x + 5$
- $s = 7 - 2t$
- $y = -3$
- $y = \frac{1}{2}x - 2$
- $A = 6 - \frac{1}{3}r$
- $y = x^2$
- $y = -2x^2$
- $y = 3 - x^2$
- $y = x^2 - 3$
- $y = \frac{1}{2}x^2 + 2$
- $y = 2x^2 + 1$
- $y = x^2 + 2x$
- $h = 20t - 5t^2$
- $y = x^2 - 3x + 1$
- $y = 2 + 3x + x^2$
- $V = s^3$
- $y = -2x^3$
- $y = x^3 - x^2$
- $L = 3e - e^3$
- $D = v^4 - 4v^2$
- $y = x^3 - x^4$
- $P = \frac{1}{V} + 1$
- $y = \frac{2}{x + 2}$
- $y = \frac{4}{x^2}$
- $p = \frac{1}{n^2 + 0.5}$
- $y = \sqrt{x}$
- $y = \sqrt{4 - x}$
- $v = \sqrt{16 - h^2}$
- $y = \sqrt{x^2 - 16}$

In Exercises 37–62, graph the indicated functions.

- In blending gasoline, the number of litres  $n$  of 85-octane gas to be blended with  $m$  litres of 92-octane gas is given by the equation  $n = 0.40m$ . Plot  $n$  as a function of  $m$ .

- The consumption of fuel  $c$  (in L/h) of a certain engine is determined as a function of the number  $r$  of r/min of the engine, to be  $c = 0.011r + 40$ . This formula is valid for 500 r/min to 3000 r/min. Plot  $c$  as a function of  $r$ . ( $r$  is the symbol for revolution.)
- For a certain model of truck, its resale value  $V$  (in dollars) as a function of the distance  $m$  it has been driven is  $V = 50\,000 - 0.2m$ . Plot  $V$  as a function of  $m$  for  $m \leq 100\,000$  km.
- The resistance  $R$  (in  $\Omega$ ) of a resistor as a function of the temperature  $T$  (in  $^{\circ}\text{C}$ ) is given by  $R = 250(1 + 0.0032T)$ . Plot  $R$  as a function of  $T$ .
- The rate  $H$  (in W) at which heat is developed in the filament of an electric light bulb as a function of the electric current  $I$  (in A) is  $H = 240I^2$ . Plot  $H$  as a function of  $I$ .
- The total annual fraction  $f$  of energy supplied by solar energy to a home as a function of the area  $A$  (in  $\text{m}^2$ ) of the solar collector is  $f = 0.065\sqrt{A}$ . Plot  $f$  as a function of  $A$ .
- The maximum speed  $v$  (in km/h) at which a car can safely travel around a circular turn of radius  $r$  (in m) is given by  $r = 0.55v^2$ . Plot  $r$  as a function of  $v$ .
- The height  $h$  (in m) of a rocket as a function of the time  $t$  (in s) is given by the function  $h = 1500t - 4.9t^2$ . Plot  $h$  as a function of  $t$ , assuming level terrain.
- The power  $P$  (in W) that a certain windmill generates is given by  $P = 0.004v^3$ , where  $v$  is the wind speed (in km/h). Plot the graph of  $P$  vs.  $v$ .
- An astronaut weighs 750 N at sea level. The astronaut's weight at an altitude of  $x$  km above sea level is given by  $w = 750\left(\frac{6400}{6400 + x}\right)^2$ . Plot  $w$  as a function of  $x$  for  $x = 0$  to  $x = 8000$  km.

47. A formula used to determine the amount of lumber  $V$  (in  $\text{dm}^3$ ) that can be cut from a 2-m section of a log of diameter  $d$  (in mm) is  $V = 0.0013d^2 - 0.043d$ . Plot  $V$  as a function of  $d$  for values of  $d$  from 300 mm to 1000 mm.
48. A copper electrode with a mass of 25.0 g is placed in a solution of copper sulfate. An electric current is passed through the solution, and 1.6 g of copper is deposited on the electrode each hour. Express the total mass  $m$  on the electrode as a function of the time  $t$  and plot the graph.
49. A land developer is considering several options of dividing a large tract into rectangular building lots, many of which would have perimeters of 200 m. For these, the minimum width would be 30 m and the maximum width would be 70 m. Express the areas  $A$  of these lots as a function of their widths  $w$  and plot the graph.
50. The distance  $p$  (in m) from a camera with a 50-mm lens to the object being photographed is a function of the magnification  $m$  of the camera, given by  $p = \frac{0.05(1+m)}{m}$ . Plot the graph for positive values of  $m$  up to 0.50.
51. A measure of the light beam that can be passed through an optic fibre is its numerical aperture  $N$ . For a particular optic fibre,  $N$  is a function of the index of refraction  $n$  of the glass in the fibre, given by  $N = \sqrt{n^2 - 1.69}$ . Plot the graph for  $n \leq 2.00$ .
52.  $F = x^4 - 12x^3 + 46x^2 - 60x + 25$  is the force (in N) exerted by a cam on the arm of a robot, as shown in Fig. 3.22. Noting that  $x$  varies from 1 cm to 5 cm, plot the graph.

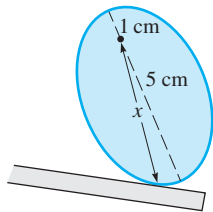


Fig. 3.22

53. The number of times  $S$  that a certain computer can perform a computation faster with a multiprocessor than with a uniprocessor is given by  $S = \frac{5n}{4+n}$ , where  $n$  is the number of processors. Plot  $S$  as a function of  $n$ .
54. The voltage  $V$  (in Volts) across a capacitor in a certain electric circuit for a 2-s interval is  $V = 2t$  during the first second and  $V = 4 - 2t$  during the second second. Here,  $t$  is the time (in s). Plot  $V$  as a function of  $t$ .
55. Given that the point  $(1, 2)$  is on the graph of  $y = f(x)$ , must it be true that  $f(2) = 1$ ? Explain.

56. In Fig. 3.23, part of the graph of  $y = 2x^2 + 0.5$  is shown. What is the area of the rectangle, if its vertex is on the curve, as shown?

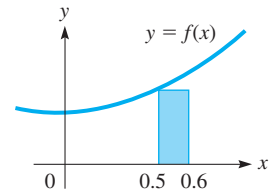


Fig. 3.23

57. Plot the graphs of  $y = x$  and  $y = |x|$  on the same coordinate system. Explain why the graphs differ.
58. Plot the graphs of  $y = 2 - x$  and  $y = |2 - x|$  on the same coordinate system. Explain why the graphs differ.
59. Plot the graph of  $f(x) = \begin{cases} 3 - x & (\text{for } x < 1) \\ x^2 + 1 & (\text{for } x \geq 1) \end{cases}$ .
60. Plot the graph of  $f(x) = \begin{cases} \frac{1}{x-1} & (\text{for } x < 0) \\ \sqrt{x+1} & (\text{for } x \geq 0) \end{cases}$ .
61. Plot the graphs of (a)  $y = x + 2$  and (b)  $y = \frac{x^2 - 4}{x - 2}$ . Explain the difference between the graphs.
62. Plot the graphs of (a)  $y = x^2 - x + 1$  and (b)  $y = \frac{x^3 + 1}{x + 1}$ . Explain the difference between the graphs.

In Exercises 63–66, determine whether or not the indicated graph is that of a function.

63. Fig. 3.24(a)  
 64. Fig. 3.24(b)  
 65. Fig. 3.24(c)  
 66. Fig. 3.24(d)

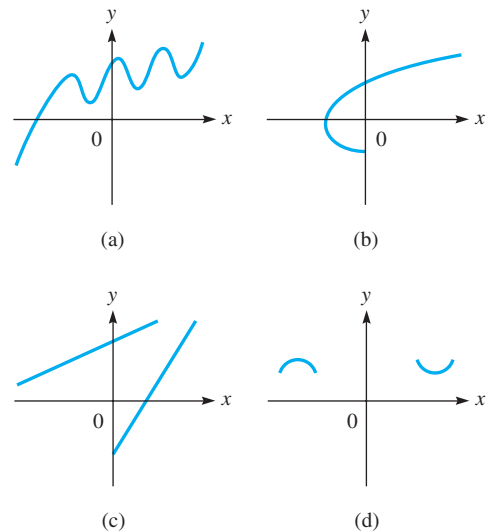


Fig. 3.24

Answers to Practice Exercises

1.  $x = -5$     2.  $x < 3$



## 3.5 More about Graphs

Graphs on a Graphing Calculator • Solving Equations Graphically • Finding the Range of a Function • Shifting a Graph

### Practice Exercise

- Use your calculator to determine good window settings for graphing  $y = 6 - 3x$ .

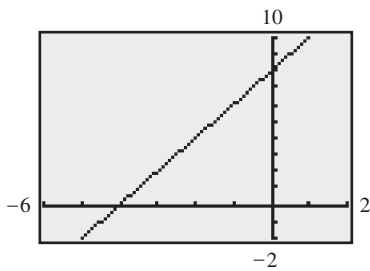


Fig. 3.25

■ The specific detail shown in any nongraphic display depends on the model. A graph with the same window settings should appear about the same on all models.

### GRAPHS ON A GRAPHING CALCULATOR

A graphing calculator can display a graph quickly and easily. As we have pointed out, to use your calculator effectively, *you must know the sequence of keys* for any operation you intend to use. The manual for any particular model should be used for a detailed coverage of its features.

#### EXAMPLE 1 Entering the function: window settings

To graph the function  $y = 2x + 8$ , first display  $Y_1 =$  and then enter the  $2x + 8$ . Next use the *window* (or *range*) feature to set the part of the domain and the range that will be seen in the *viewing window*. For this function, set

$$X_{\min} = -6, X_{\max} = 2, X_{\text{scl}} = 1, Y_{\min} = -2, Y_{\max} = 10, Y_{\text{scl}} = 1$$

in order to get a good view of the graph. Then display the graph, using the *graph* (or *exe*) key. The display showing the graph of  $y = 2x + 8$  is shown in Fig. 3.25.

Note that the calculator makes the graph just as we have been doing—by plotting points (square dots called *pixels*). It just does it a lot faster. ■

In Example 1, the *window* settings were chosen to give a good view of the graph of  $y = 2x + 8$ . However, if we had chosen  $X_{\min} = -2$ ,  $X_{\max} = 3$ ,  $Y_{\min} = -3$ , and  $Y_{\max} = 3$ , we would have had a view where no part of the graph can be seen. Unless the settings are appropriate, you may not get a good view of a graph.

### COMMON ERROR

When using a graphing calculator to view a graph, it is possible that very little or no part of a graph can be seen with the chosen settings. Therefore, it is necessary to be careful when choosing the settings.

Since the settings are easily changed, to get the general location of a graph, it is usually best to begin by choosing intervals that are greater than is probably necessary. Then they can be reduced as needed.

### SOLVING EQUATIONS GRAPHICALLY

An equation can be solved by use of a graph. Most of the time, the solution will be approximate, but with a graphing calculator, it is possible to get good accuracy in the result. The procedure used is as follows:

#### Procedure for Solving an Equation Graphically

- Collect all terms on one side of the equal sign. This gives us the equation  $f(x) = 0$ .
- Set  $y = f(x)$  and graph this function.
- Find the points where the graph crosses the  $x$ -axis. These points are called the  **$x$ -intercepts** of the graph. At these points,  $y = 0$ .
- The values of  $x$  for which  $y = 0$  are the solutions of the equation. (These values are called the **zeros** of the function  $f(x)$ .)

The next example illustrates the method.

**EXAMPLE 2 Graphical solution of an equation**

Graphically solve the equation  $x^2 - 2x = 1$ .

Following the above procedure, first rewrite the equation as  $x^2 - 2x - 1 = 0$  and then set  $y = x^2 - 2x - 1$ . Next, plot the graph of this function, as shown in Fig. 3.26.

For this curve, we see that it crosses the  $x$ -axis between  $x = -1$  and  $x = 0$  and between  $x = 2$  and  $x = 3$ . This means that there are *two* solutions of the equation, and they can be estimated from the graph as

$$x = -0.5 \quad \text{and} \quad x = 2.5$$

For these values,  $y = 0$ , which means  $x^2 - 2x - 1 = 0$ , which in turn means  $x^2 - 2x = 1$ . ■

We now show the use of a graphing calculator in solving equations graphically. In the next example, we solve the same equation as in Example 2.

**EXAMPLE 3 Graphing calculator solution of an equation**

Using a graphing calculator, solve the equation  $x^2 - 2x = 1$ .

Again, rewrite the equation as  $x^2 - 2x - 1 = 0$  and set  $y = x^2 - 2x - 1$ . Then display this function on the calculator, as shown in Fig. 3.27(a).

To get values of  $x$  for which  $y = 0$ , we can use the *trace* and *zoom* features or the *zero* feature. Using the *trace* feature, we move the *cursor* (the blinking pixel) with the arrow keys to find the value of  $y$  closest to zero (generally not *exactly* zero). We then have the display in Fig. 3.27(b), which shows solutions of about  $x = -0.4$  and  $x = 2.4$ . More accurate values can be found by using the *zoom* feature with which any region of the screen can be magnified. In Fig. 3.27(c), we see that the solutions are about  $x = -0.415$  and  $x = 2.415$ .

In using the *zero* feature, it is necessary to give the part of the domain where you want to find the zero, and make a reasonable guess as to its value. In Fig. 3.27(d), this has been done, and the zero is shown to be  $x = -0.4142136$ . The other zero is at  $x = 2.4142136$ .

Using the *value* feature, we can check these solutions. In all cases, we find that the value of  $y$  is very near zero, which means that these solutions check. Obviously, the values found using the *zero* feature are the most accurate.

$x$	$y$
-2	7
-1	2
0	-1
1	-2
2	-1
3	2
4	7

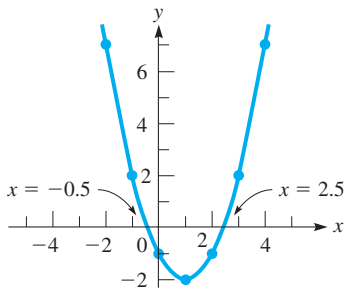
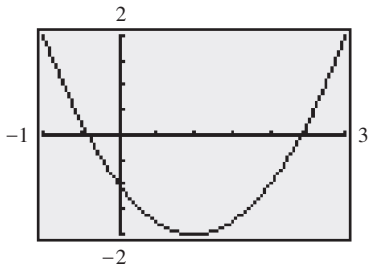
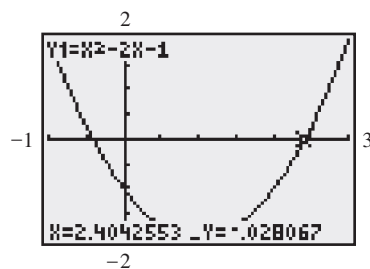


Fig. 3.26

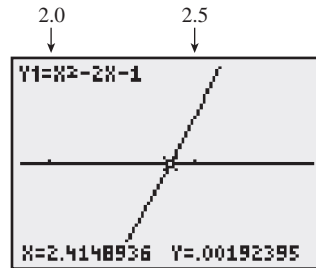
■ Most calculators have an *intersect* feature. The equation in Example 3 can be solved using this feature and finding points where the curves  $y = x^2 - 2x$  and  $y = 1$  cross.



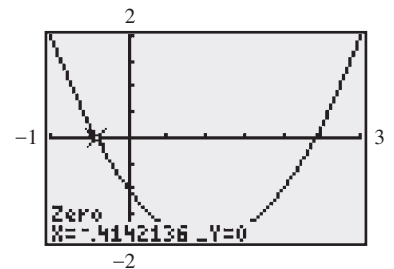
(a)



(b)



(c)



(d)

Fig. 3.27

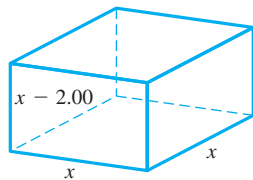


Fig. 3.28

**EXAMPLE 4 Graphical solution of a word problem**

A rectangular box whose volume is  $30.0 \text{ cm}^3$  is made with a square base and a height that is  $2.00 \text{ cm}$  less than the length of a side of the base. Find the dimensions of the box by first setting up the necessary equation and then solving it graphically.

Let  $x =$  the length of the side of the square base (see Fig. 3.28); the height is then  $x - 2.00$ . This means that the volume is  $(x)(x)(x - 2.00)$ , or  $x^3 - 2.00x^2$ . Since the volume is  $30.0 \text{ cm}^3$ , we have the equation

$$x^3 - 2.00x^2 = 30.0$$

To solve it graphically, first rewrite the equation as  $x^3 - 2.00x^2 - 30.0 = 0$  and then set  $y = x^3 - 2.00x^2 - 30.0$ . The graph of this function is shown in Fig. 3.29. Note that only positive values of  $x$  have meaning in the context of the problem. On a graphing calculator, using the *trace*, *zero*, or *intersect* feature, we find that  $x = 3.94 \text{ cm}$  is the approximate solution. Therefore, the dimensions are  $3.94 \text{ cm}$ ,  $3.94 \text{ cm}$ , and  $1.94 \text{ cm}$ . Checking, these dimensions give a volume of  $30.1 \text{ cm}^3$ . The difference from  $30.0 \text{ cm}^3$  is due to the use of the approximate value of  $x$ . A better check can be made by using the unrounded calculator value for  $x$ . ■

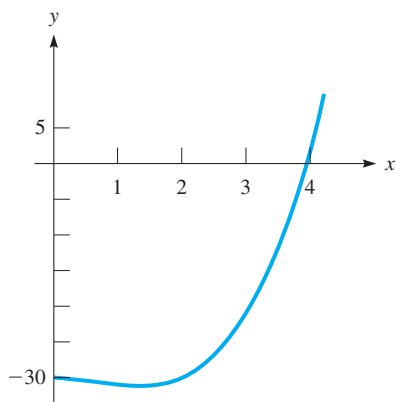


Fig. 3.29

**FINDING THE RANGE OF A FUNCTION**

In Section 3.2, we stated that advanced methods are often necessary to find the range of a function and that we would see that a graphing calculator is useful for this purpose. As with solving equations, we can get very good approximations of the range for most functions by using a graphing calculator.

The method is simply to graph the function and see for what values of  $y$  there is a point on the graph. These values of  $y$  give us the range of the function.

**EXAMPLE 5 Graphically finding the range of a function**

Graphically find the range of the function  $y = 16\sqrt{x} + 1/x$ .

If we start by using the default *window* settings of  $X_{\min} = -10$ ,  $X_{\max} = 10$ ,  $Y_{\min} = -10$ ,  $Y_{\max} = 10$ , we see no part of the graph. Looking back to the function, note that  $x$  cannot be zero because that would require dividing by zero in the  $1/x$  term. Also,  $x > 0$  since  $\sqrt{x}$  is not defined for  $x < 0$ . Thus, we should try  $X_{\min} = 0$  and  $Y_{\min} = 0$ .

We also can easily see that  $y = 17$  for  $x = 1$ . This means we should try a value more than 17 for  $Y_{\max}$ . Using  $Y_{\max} = 20$ , after two or three *window* settings, we use the settings shown in Fig. 3.30. The display shows the range to be all real numbers greater than about 12. (Actually, the range is all real numbers  $y \geq 12$ .) Alternatively, one could use the *minimum* feature on a graphing calculator to determine the lowest possible value of the function. Similarly, the *maximum* feature can determine the largest possible value of the function (if it exists). ■

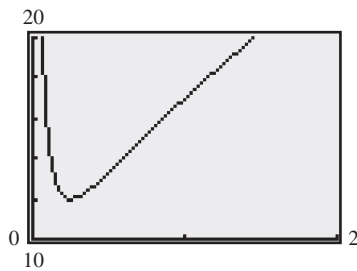


Fig. 3.30

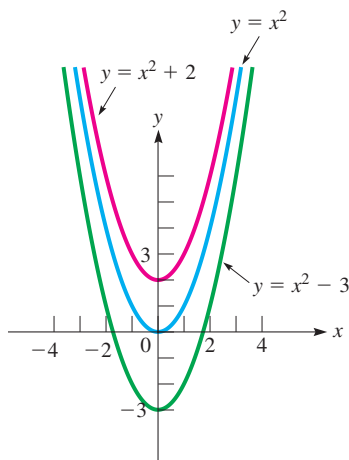


Fig. 3.31

**SHIFTING A GRAPH**

By adding a positive constant to the right side of the function  $y = f(x)$ , the graph of the function is *shifted* straight up. If a negative constant is added to the right side of the function, the graph is shifted straight down.

**EXAMPLE 6 Shifting a graph vertically**

Fig. 3.31 shows the graphs of the functions  $y = x^2$ ,  $y = x^2 + 2$ , and  $y = x^2 - 3$ . We see that adding 2 to the function  $y = x^2$  shifts the graph up 2 units, whereas adding  $-3$  shifts the graph down 3 units. ■

By adding a constant to  $x$  in the function  $y = f(x)$ , the graph of the function is shifted to the right or to the left. *Adding a positive constant to  $x$  shifts the graph to the left, and adding a negative constant to  $x$  shifts the graph to the right.*

**COMMON ERROR**

Because an added **positive** constant to the **function** shifts the graph of the function **upward**, it is a common error to think that adding a positive constant to the variable of the function shifts the graph to the right (in the positive direction). However, adding a **positive** constant to the **variable** results in a shift to the **left** (to the negative direction), and adding a **negative** constant to the **variable** results in a shift to the **right** (to the positive direction).

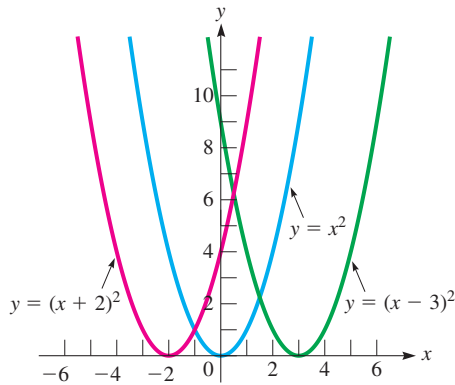


Fig. 3.32

**EXAMPLE 7** Shifting a graph horizontally

For the function  $y = x^2$ , if we add 2 to  $x$ , we get  $y = (x + 2)^2$ , or if we add  $-3$  to  $x$ , we get  $y = (x - 3)^2$ . The graphs of these three functions are shown in Fig. 3.32.

We see that the graph of  $y = (x + 2)^2$  is 2 units to the *left* of  $y = x^2$  and that the graph of  $y = (x - 3)^2$  is 3 units to the *right* of  $y = x^2$ . ■

Be very careful when shifting graphs horizontally. To check the direction and magnitude of a horizontal shift, find the value of  $x$  that makes the expression in parentheses equal to 0. For example, in Example 7,  $y = (x - 3)^2$  is 3 units to the *right* of  $y = x^2$  because  $x = +3$  makes  $x - 3$  equal to zero. The point  $(3, 0)$  on  $y = (x - 3)^2$  is equivalent to the point  $(0, 0)$  on  $y = x^2$ . In the same way  $x + 2 = 0$  for  $x = -2$ , and the graph of  $y = (x + 2)^2$  is shifted 2 units to the *left* of  $y = x^2$ .

Summarizing how a graph is shifted, we have the following:

**Shifting a Graph**

**Vertical shifts:**  $y = f(x) + k$  shifts the graph of  $y = f(x)$   
up  $k$  units if  $k > 0$  and down  $k$  units if  $k < 0$ .

**Horizontal shifts:**  $y = f(x + k)$  shifts the graph of  $y = f(x)$   
*left*  $k$  units if  $k > 0$  and *right*  $k$  units if  $k < 0$ .

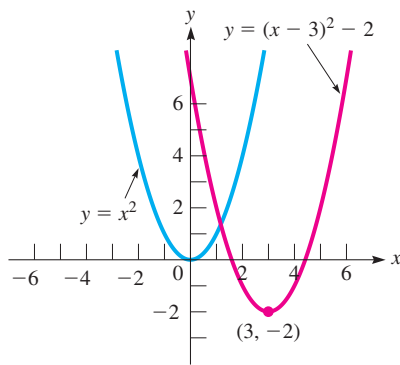


Fig. 3.33

**EXAMPLE 8** Shifting vertically and horizontally

The graph of a function can be shifted both vertically and horizontally. To shift the graph of  $y = x^2$  to the right 3 units and down two units, add  $-3$  to  $x$  and add  $-2$  to the resulting function. In this way, we get  $y = (x - 3)^2 - 2$ . The graph of this function and the graph of  $y = x^2$  are shown in Fig. 3.33.

Note that the point  $(3, -2)$  on the graph of  $y = (x - 3)^2 - 2$  is equivalent to the point  $(0, 0)$  on the graph of  $y = x^2$ . Checking, by setting  $x - 3 = 0$ , we get  $x = 3$ . This means the graph of  $y = x^2$  has been shifted 3 units to the *right*. The vertical shift of  $-2$  is clear. ■

**Practice Exercise**

2. Describe the graph of  $y = (x + 2)^2 + 3$  relative to the graph of  $y = x^2$ .

**EXERCISES 3.5**

In Exercises 1–4, make the indicated changes in the given examples of this section and then solve.

- In Example 2, change the sign on the left side of the equation from  $-$  to  $+$ .
- In Example 4, change 2.00 cm to 3.00 cm.
- In Example 5, change the denominator of the second term on the right side of the equation to  $x^2$ .
- In Example 8, in the second line, change “to the right 3 units and down 2 units” to “to the left 2 units and down 3 units.”

In Exercises 5–18, display the graphs of the given functions on a graphing calculator. Use appropriate window settings.

- $y = 3x - 1$
- $y = 4 - 0.5x$
- $y = x^2 - 4x$
- $y = 8 - 2x^2$
- $y = 6 - x^3$
- $y = x^4 - 6x^2$
- $y = x^4 - 2x^3 - 5$
- $y = x^2 - 3x^5 + 3$
- $y = \frac{2x}{x - 2}$
- $y = \frac{3}{x^2 - 4}$

15.  $y = x + \sqrt{x+3}$       16.  $y = \sqrt[3]{2x+1}$   
 17.  $y = 3 + \frac{2}{x}$       18.  $y = \frac{x^2}{\sqrt{3-x}}$

In Exercises 19–28, solve the given equations graphically to the nearest 0.001.

19.  $x^2 + x - 5 = 0$       20.  $v^2 - 2v - 4 = 0$   
 21.  $x^3 - 3 = 3x$       22.  $x^4 - 2x = 0$   
 23.  $s^3 - 4s^2 = 6$       24.  $3x^2 - x^4 = 2 + x$   
 25.  $\sqrt{5R+2} = 3$       26.  $\sqrt{x} + 3x = 7$   
 27.  $\frac{1}{x^2+1} = 0$       28.  $T - 2 = \frac{1}{T}$

In Exercises 29–36, find the range of the given functions graphically. (The functions of Exercises 33–36 are the same as the functions of Exercises 15–18 of Section 3.2.)

29.  $y = \frac{4}{x^2 - 4}$       30.  $y = \frac{x+1}{x^2}$   
 31.  $y = \frac{x^2}{x+1}$       32.  $y = \frac{x}{x^2 - 4}$   
 33.  $Y(y) = \frac{y+1}{\sqrt{y-2}}$       34.  $f(n) = \frac{n}{6-2n}$   
 35.  $f(D) = \frac{D}{D-2} + \frac{4}{D+4}$       36.  $g(x) = \frac{\sqrt{x-2}}{x-3}$

In Exercises 37–44, a function and how it is to be shifted is given. Find the shifted function and then graph both functions on the same coordinate system.

37.  $y = 3x$ , up 1      38.  $y = x^3$ , down 2  
 39.  $y = \sqrt{x}$ , right 3      40.  $y = \frac{2}{x}$ , left 4  
 41.  $y = -2x^2$ , down 3, left 2      42.  $y = -4x$ , up 4, right 3  
 43.  $y = \sqrt{2x+1}$ , up 1, left 1  
 44.  $y = \sqrt{x^2+4}$ , down 2, right 2

In Exercises 45–52, solve the indicated equations graphically. Assume all data are accurate to three significant digits unless greater accuracy is given.

45. In an electric circuit, the current  $i$  (in A) as a function of voltage  $v$  is given by  $i = 0.01v - 0.06$ . Find  $v$  for  $i = 0$ .  
 46. For tax purposes, a corporation assumes that one of its computers depreciates according to the equation  $V = 90\,000 - 12\,000t$ , where  $V$  is the value (in dollars) of the computer after  $t$  years. According to this formula, when will the computer be fully depreciated (no value)?  
 47. Two cubical coolers together hold 40.0 L (40 000 cm<sup>3</sup>). If the inside edge of one is 5.00 cm greater than the inside edge of the other, what is the inside edge of each?  
 48. The height  $h$  (in m) of a rocket as a function of time  $t$  (in s) of flight is given by  $h = 15 + 86t - 4.9t^2$ . Determine when the rocket is at ground level.

49. The length of a rectangular solar panel is 12 cm more than its width. If its area is 520 cm<sup>2</sup>, find its dimensions.  
 50. A computer model shows that the cost (in dollars) to remove  $x$  percent of a pollutant from a lake is  $C = \frac{8000x}{100-x}$ . What percent can be removed for \$25 000?  
 51. In finding the illumination at a point  $x$  m from one of two light sources that are 100 m apart, it is necessary to solve the equation  $9x^3 - 2400x^2 + 240\,000x - 8\,000\,000 = 0$ . Find  $x$ .

52. A rectangular storage bin is to be made from a rectangular piece of sheet metal 12.0 cm by 10.0 cm, by cutting out equal corners of side  $x$  and bending up the sides. See Fig. 3.34. Find  $x$  if the storage bin is to hold 90.0 cm<sup>3</sup>.

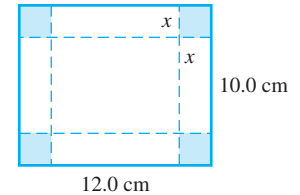


Fig. 3.34

In Exercises 53–60, solve the given problems.

53. In Example 4, assume the data are good to five significant digits and then find the dimensions of the box.  
 54. Explain how to show the graph of the relation  $y^2 = 4x^2$  on a graphing calculator, and then display it on the calculator. See Example 9 in Section 3.4.  
 55. The cutting speed  $s$  (in cm/s) of a saw in cutting a particular type of metal piece is given by  $s = \sqrt{t - 4t^2}$ , where  $t$  is the time in seconds. What is the maximum cutting speed in this operation (to three significant digits)? (Hint: Find the range.)  
 56. Referring to Exercise 52, explain how to determine the maximum possible capacity for a storage bin constructed in this way. What is the maximum possible capacity (to three significant digits)?  
 57. A balloon is being blown up at a constant rate. (a) Sketch a reasonable graph of the radius of the balloon as a function of time. (b) Compare to a typical situation that can be described by  $r = \sqrt[3]{3t}$ , where  $r$  is the radius (in cm) and  $t$  is the time (in s).  
 58. A hot-water faucet is turned on. (a) Sketch a reasonable graph of the water temperature as a function of time. (b) Compare to a typical situation described by  $T = \frac{t^3 + 80}{0.015t^3 + 4}$ , where  $T$  is the water temperature (in °C) and  $t$  is the time (in s).  
 59. Display the graph of  $y = cx^3$  with  $c = -2$  and with  $c = 2$ . Describe the effect of the value of  $c$ .  
 60. Display the graph of  $y = cx^4$ , with  $c = 4$  and with  $c = 1/4$ . Describe the effect of the value of  $c$ .

Answers to Practice Exercises

1.  $x: -1$  to  $3$ ;  $y: -2$  to  $8$  probably best  
 2. shifted 2 units to the left, 3 units up

## 3.6 Graphs of Functions Defined by Tables of Data

Straight-Line Segments Graph •  
 Smooth Curve Graph •  
 Reading a Graph • Linear Interpolation

As we noted in Section 3.1, there are ways other than formulas to show functions. One important way to show the relationship between variables is by means of a table of values found by observation or from an experiment.

Statistical data often give values that are taken for certain intervals or are averaged over various intervals, and there is no meaning to the intervals *between* the points. Such points should be connected by straight-line segments only to highlight changes in the value of the function and to aid in visualization. See Example 1.

### EXAMPLE 1 Graph using straight-line segments

The electric energy usage (in kW·h) for a certain all-electric house for each month of a year is shown in the following table. Plot these data.

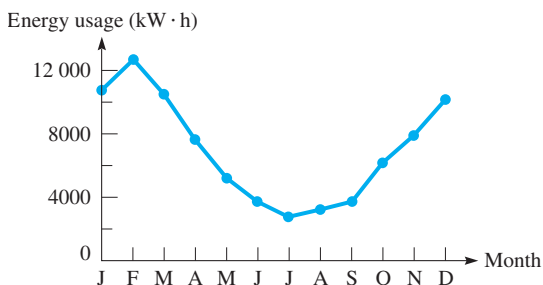


Fig. 3.35

■ Graphs such as the one shown in Fig. 3.35 can be obtained with a spread sheet, or by using the statistical plotting features of a graphing calculator.

We know that there is no meaning to the intervals between the month categories, since we have the total number of kilowatt-hours for each month. Therefore, we use straight-line segments, but only to highlight changes in the function. See Fig. 3.35. ■

Data from experiments in science and technology often indicate that the variables could have a formula relating them, although the formula may not be known. In this case, when plotting the graph, the points should be connected by a smooth curve.

### EXAMPLE 2 Graph using a smooth curve

Steam in a boiler was heated to 150°C and then allowed to cool. Its temperature  $T$  (in °C) was recorded each minute, as shown in the following table. Plot the graph.

■ The Celsius degree is named for the Swedish astronomer Andres Celsius (1701–1744). He designated 100° as the freezing point of water and 0° as the boiling point. These were later reversed.

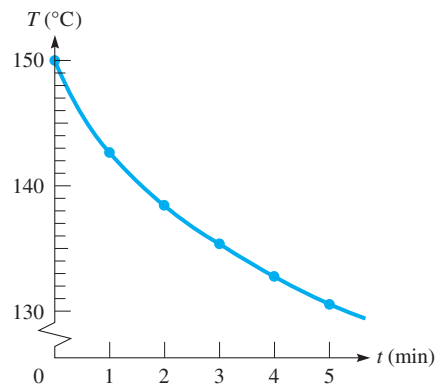


Fig. 3.36

■ In Chapter 22, we see how to find an equation that approximates the function relating the variables in a table of values, a process called curve-fitting.

Time (min)	0.0	1.0	2.0	3.0	4.0	5.0
Temperature (°C)	150.0	142.8	138.5	135.2	132.7	130.8

Since the temperature changes in a continuous way, there is meaning to the values in the intervals between points. Therefore, these points are joined by a smooth curve, as in Fig. 3.36. Also, note that most of the data is far from the zero point on the Celsius scale, presented with the indicated break in the scale between 0 and 130. ■

If the graph relating two variables is known, values can be obtained directly from the graph. In finding such a value through the inspection of the graph, we are *reading the graph*.



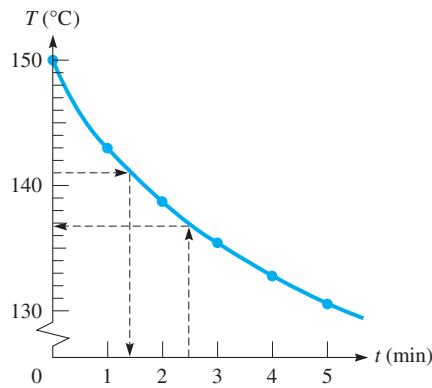


Fig. 3.37

**EXAMPLE 3** Reading a graph

For the cooling steam in Example 2, we can estimate values of one variable for given values of the other variable.

If we want to know the temperature after 2.5 min, estimate 0.5 of the interval between 2 and 3 on the  $t$ -axis and mark this point. See Fig. 3.37. Then draw a vertical line from this point to the curve. From the point where it intersects the curve, draw a horizontal line to the  $T$ -axis. We now estimate that the line crosses at  $T = 136.7^\circ\text{C}$ . Obviously, the number of tenths is a rough estimate.

In the same way, if we want to determine how long the steam took to cool down to  $141.0^\circ\text{C}$ , go from the  $141.0$  marker on the  $T$ -axis to the curve and then down to the  $t$ -axis. This crosses at about  $t = 1.4$  min.

Note that we cannot read values between points on the graph in Example 1, since there is no real meaning to values between those listed. However, in this example we assume that the general trend shown by the graph is probably valid for values between those in the table. ■

**LINEAR INTERPOLATION**

In Example 3, we see that we can estimate values from a graph. However, unless a very accurate graph is drawn with expanded scales for both variables, only very approximate values can be found. There is a method, called *linear interpolation*, that uses the table itself to get more accurate results.

**Linear interpolation** assumes that if a particular value of one variable lies between two of those listed in the table, then the corresponding value of the other variable is at the same proportional distance between the listed values. On the graph, linear interpolation assumes that two points defined in the table are connected by a straight line. Although this is generally not correct, it is a good approximation if the values in the table are sufficiently close together.

**EXAMPLE 4** Linear interpolation

For the cooling steam in Example 2, we can use interpolation to find its temperature after 1.4 min. Since 1.4 min is  $\frac{4}{10}$  of the way from 1.0 min to 2.0 min, we will assume that the value of  $T$  we want is  $\frac{4}{10}$  of the way between 142.8 and 138.5, the values of  $T$  for 1.0 min and 2.0 min, respectively. The difference between these values is 4.3, and  $\frac{4}{10}$  of 4.3 is 1.7 (rounded off to tenths). Subtracting (the values of  $T$  are decreasing) 1.7 from 142.8, we obtain 141.1. Thus, the required value of  $T$  is about  $141.1^\circ\text{C}$ . (Note that this agrees well with the result in Example 3.)

Another method of indicating the interpolation is shown in Fig. 3.38. From the figure, we have the proportion

$$\frac{0.4}{1.0} = \frac{x}{-4.3}$$

$$x = -1.7 \quad \text{rounded off}$$

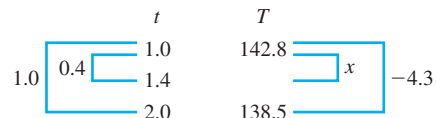


Fig. 3.38

Therefore,

$$142.8 + (-1.7) = 141.1^\circ\text{C}$$

is the required value of  $T$ . If the values of  $T$  had been increasing, we would have added 1.7 to the value of  $T$  for 1.0 min. ■

■ Before the use of electronic calculators, interpolation was used extensively in mathematics textbooks for finding values from mathematics tables. It is still of use when using scientific, technical, and statistical tables.

**Practice Exercise**

- In Example 4, interpolate to find  $T$  for  $t = 4.2$  min.

## EXERCISES 3.6

In Exercises 1–8, represent the data graphically.

1. The diesel fuel production (in 1000's of litres) at a certain refinery during an 8-wk period was as follows:

Week	1	2	3	4	5	6	7	8
Production	765	780	840	850	880	840	760	820

2. The annual average exchange rate for the number of Canadian dollars equal to one U.S. dollar for 1998–2012 is as follows:

Year	1998	2000	2002	2004	2006	2008	2010	2012
Can. Dol.	1.48	1.49	1.57	1.30	1.13	1.07	1.03	1.00

3. The amount of material necessary to make a cylindrical litre container depends on the diameter, as shown in this table:

Diameter (cm)	6.0	8.0	10.0	12.0	14.0	16.0
Material (cm <sup>2</sup> )	723	601	557	560	594	652

4. An oil burner propels air that has been heated to 90°C. The temperature then drops as the distance from the burner increases, as shown in the following table:

Distance (m)	0.0	1.0	2.0	3.0	4.0	5.0	6.0
Temperature (°C)	90	84	76	66	54	46	41

5. A changing electric current in a coil of wire will induce a voltage in a nearby coil. Important in the design of transformers, the effect is called *mutual inductance*. For two coils, the mutual inductance (in H) as a function of the distance between them is given in the following table:

Distance (cm)	0.0	2.0	4.0	6.0	8.0	10.0	12.0
M. ind. (H)	0.77	0.75	0.61	0.49	0.38	0.25	0.17

6. The temperatures felt by the body as a result of the *wind-chill factor* for an outside temperature of 20°F (as determined by the National Weather Service) are given in the following table:

Wind speed (km/h)	10	20	30	40	50	60	70
Temp. felt (°C)	-4	-7	-8	-9	-10	-10	-11

7. The time required for a sum of money to double in value, when compounded annually, is given as a function of the interest rate in the following table:

Rate (%)	4	5	6	7	8	9	10
Time (years)	17.7	14.2	11.9	10.2	9.0	8.0	7.3

8. The torque  $T$  of an engine, as a function of the frequency  $f$  of rotation, was measured as follows:

$f$ (r/min)	500	1000	1500	2000	2500	3000	3500
$T$ (N·m)	175	90	62	45	34	31	27

In Exercises 9 and 10, use the graph in Fig. 3.36, which relates the temperature of cooling steam and the time. Find the indicated values by reading the graph.

9. (a) For  $t = 4.3$  min, find  $T$ . (b) For  $T = 145.0^\circ\text{C}$ , find  $t$ .

10. (a) For  $t = 1.8$  min, find  $T$ . (b) For  $T = 133.5^\circ\text{C}$ , find  $t$ .

In Exercises 11 and 12, use the following table, which gives the valve lift  $L$  (in mm) of a certain cam as a function of the angle  $\theta$  (in degrees) through which the cam is turned. Plot the values. Find the indicated values by reading the graph.

$\theta$ (°)	0	20	40	60	80	100	120	140
$L$ (mm)	0	1.2	2.3	3.3	3.8	3.0	1.6	0

11. (a) For  $\theta = 25^\circ$ , find  $L$ . (b) For  $\theta = 96^\circ$ , find  $L$ .

12. For  $L = 2.0$  mm, find  $\theta$ .

In Exercises 13–16, find the indicated values by means of linear interpolation.

13. In Exercise 5, find the inductance for  $d = 9.2$  cm.

14. In Exercise 6, find the temperature for  $s = 12$  km/h.

15. In Exercise 7, find the rate for  $t = 10.0$  years.

16. In Exercise 8, find the torque for  $t = 2300$  r/min.

In Exercises 17–20, use the following table that gives the rate  $R$  of discharge from a tank of water as a function of the height  $H$  of water in the tank. For Exercises 17 and 18, plot the graph and find the values from the graph. For Exercises 19 and 20, find the indicated values by linear interpolation.

Height (cm)	0	50	100	200	300	400	600
Rate (m <sup>3</sup> /s)	0	1.0	1.5	2.2	2.7	3.1	3.5

17. (a) for  $R = 2.0$  m<sup>3</sup>/s, find  $H$ . (b) For  $H = 240$  cm, find  $R$ .

18. (a) For  $R = 3.4$  m<sup>3</sup>/s, find  $H$ . (b) For  $H = 320$  cm, find  $R$ .

19. Find  $R$  for  $H = 80$  cm.      20. Find  $H$  for  $R = 2.5$  m<sup>3</sup>/s.

In Exercises 21–24, use the following table, which gives the fraction (as a decimal) of the total heating load of a certain system that will be supplied by a solar collector of area  $A$  (in m<sup>2</sup>). Find the indicated values by linear interpolation.

$f$	0.22	0.30	0.37	0.44	0.50	0.56	0.61
$A$ (m <sup>2</sup> )	20	30	40	50	60	70	80

21. For  $A = 36 \text{ m}^2$ , find  $f$ .      22. For  $A = 52 \text{ m}^2$ , find  $f$ .  
 23. For  $f = 0.59$ , find  $A$ .      24. For  $f = 0.27$ , find  $A$ .

In Exercises 25–28, a method of finding values beyond those given is considered. By using a straight-line segment to extend a graph beyond the last known point, we can estimate values from the extension of the graph. The method is known as **linear extrapolation**. Use this method to estimate the required values from the given graphs.

25. Using Fig. 3.37, estimate  $T$  for  $t = 5.3$  min.  
 26. Using the graph for Exercise 7, estimate  $t$  for  $R = 10.4\%$ .  
 27. Using the graph for Exercises 17–20, estimate  $R$  for  $H = 700$  cm.  
 28. Using the graph for Exercises 17–20, estimate  $R$  for  $H = 800$  cm.

#### Answer to Practice Exercise

1.  $T = 132.3^\circ\text{C}$

## CHAPTER 3 REVIEW EXERCISES

In Exercises 1–4, determine the appropriate function.

- The radius of a circular water wave increases at the rate of 2 m/s. Express the area of the circle as a function of the time  $t$  (in s).
- A conical sheet-metal hood is to cover an area 6 m in diameter. Find the total surface area  $A$  of the hood as a function of its height  $h$ .
- A person invests  $x$  dollars at 5% APR (annual percentage rate) and  $y$  dollars at 4% APR. If the total annual income is \$2000, solve for  $y$  as a function of  $x$ .
- Fencing around a rectangular storage depot area costs twice as much along the front as along the other three sides. The back costs \$10 per metre. Express the cost  $C$  of the fencing as a function of the width  $w$  if the length (along the front) is 20 m longer than the width.

In Exercises 5–12, evaluate the given functions.

- $f(x) = 7x - 5$ ; find  $f(3)$  and  $f(-6)$ .
- $g(t) = 8 - 3t$ ; find  $g(\frac{1}{6})$  and  $g(-4)$ .
- $H(h) = \sqrt{1 - 2h}$ ; find  $H(-4)$  and  $H(2h)$ .
- $\phi(v) = \frac{3v - 2}{|v + 1|}$ ; find  $\phi(-2)$  and  $\phi(v + 1)$ .
- $F(x) = x^3 + 2x^2 - 3x$ ; find  $F(3 + h) - F(3)$ .
- $f(x) = 3x^2 - 2x + 4$ ; find  $\frac{f(x + h) - f(x)}{h}$ .
- $f(x) = 3 - 2x$ ; find  $f(2x) - 2f(x)$ .
- $f(x) = 1 - x^2$ ; find  $[f(x)]^2 - f(x^2)$ .

In Exercises 13–16, evaluate the given functions. Values of the independent variable are approximate.

- $f(x) = 8.07 - 2x$ ; find  $f(5.87)$  and  $f(-4.29)$ .
- $g(x) = 7x - x^2$ ; find  $g(45.81)$  and  $g(-21.85)$ .
- $G(S) = \frac{S - 0.087629}{3.0125S}$ ; find  $G(0.17427)$  and  $G(0.053206)$ .
- $h(t) = \frac{t^2 - 4t}{t^3 + 564}$ ; find  $h(8.91)$  and  $h(-4.91)$ .

In Exercises 17–22, determine the domain and the range of the given functions.

- $f(x) = x^4 + 1$
- $G(z) = \frac{4}{z^3}$

- $g(t) = \frac{2}{\sqrt{t + 4}}$
- $f(n) = 1 + \frac{2}{(n - 5)^2}$
- $F(y) = 1 - 2\sqrt{y}$
- $F(x) = 3 - |x|$

In Exercises 23–32, plot the graphs of the given functions.

- $y = 4x + 2$
- $s = 4t - t^2$
- $A = 2 - s^4$
- $y = \frac{x}{x + 1}$
- $y = 5x - 10$
- $y = x^2 - 8x - 5$
- $V = 3 - 0.5s^3$
- $y = x^4 - 4x$
- $Z = \sqrt{25 - 2R^2}$

In Exercises 33–40, solve the given equations graphically to the nearest 0.01.

- $7x - 3 = 0$
- $x^2 + 1 = 6x$
- $x^3 - x^2 = 2 - x$
- $\frac{1}{v} = 2v$
- $3x + 11 = 0$
- $3t - 2 = t^2$
- $5 - x^3 = 2x^2$
- $\sqrt{x} = 2x - 1$

In Exercises 41–44, find the range of the given function graphically.

- $y = x^4 - 5x^2$
- $A = w + \frac{2}{w}$
- $y = x\sqrt{4 - x^2}$
- $y = 2x + \frac{3}{\sqrt{x}}$

In Exercises 45–62, solve the given problems.

- Explain how  $A(a, b)$  and  $B(b, a)$  may be in different quadrants.
- Determine the distance from the origin to the point  $(a, b)$ .
- Two vertices of an equilateral triangle are  $(0, 0)$  and  $(2, 0)$ . What is the third vertex?
- The points  $(1, 2)$  and  $(1, -3)$  are two adjacent vertices of a square. Find the other vertices.
- Where are all points for which  $|y/x| > 0$ ?
- Describe the values of  $x$  and  $y$  for which  $(1, -2)$ ,  $(-1, -2)$ , and  $(x, y)$  are on the same straight line.
- Sketch the graph of a function for which the domain is  $0 \leq x \leq 4$  and the range is  $1 \leq y \leq 3$ .
- Sketch the graph of a function for which the domain is all values of  $x$  and the range is  $-2 < y < 2$ .

53. If the function  $y = \sqrt{x-1}$  is shifted left 2 and up 1, what is the resulting function?
54. If the function  $y = 3 - 2x$  is shifted right 1 and down 3, what is the resulting function?
55. For  $f(x) = 2x^3 - 3$ , display the graphs of  $f(x)$  and  $f(-x)$  on a graphing calculator. Describe the graphs in relation to the  $y$ -axis.
56. For  $f(x) = 2x^3 - 3$ , display the graphs of  $f(x)$  and  $-f(x)$  on a graphing calculator. Describe the graphs in relation to the  $x$ -axis.
57. Is it possible that the points  $(2, 3)$ ,  $(5, -1)$ , and  $(-1, 3)$  are all on the graph of the same function? Explain.
58. For the functions  $f(x) = |x|$  and  $g(x) = \sqrt{x^2}$ , use a graphing calculator to determine whether or not  $f(x) = g(x)$  for all real  $x$ .
59. An equation used in electronics with a transformer antenna is  $I = 12.5\sqrt{1 + 0.5m^2}$ . For  $I = f(m)$ , find  $f(0.550)$ .
60. The percent  $p$  of wood lost in cutting it into boards 38.0 mm thick due to the thickness  $t$  (in mm) of the saw blade is  $p = \frac{100t}{t + 38}$ . Find  $p$  if  $t = 5.00$  mm. That is, since  $p = f(t)$ , find  $f(5.00)$ .
61. The angle  $A$  (in degrees) of a robot arm with the horizontal as a function of time  $t$  (for 0.0 s to 6.0 s) is given by  $A = 8.0 + 12t^2 - 2.0t^3$ . What is the greatest value of  $A$  to the nearest  $0.1^\circ$ ? See Fig. 3.39. (Hint: Find the range.)
62. The electric power  $P$  (in W) produced by a certain battery is  $P = \frac{24R}{R^2 + 1.40R + 0.49}$ , where  $R$  is the resistance (in  $\Omega$ ) in the circuit. What is the maximum power produced? (Hint: Find the range.) See Example 7 in Section 3.4.

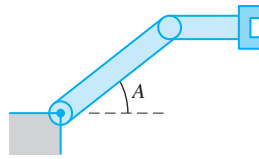


Fig. 3.39

In Exercises 63–78, plot the graphs of the indicated functions.

63. When El Niño, a Pacific Ocean current, moves east and warms the water off South America, weather patterns in many parts of the world change significantly. Special buoys along the equator in the Pacific Ocean send data via satellite to monitoring stations. If the temperature  $T$  (in  $^\circ\text{C}$ ) at one of these buoys is  $T = 28.0 + 0.15t$ , where  $t$  is the time in weeks between Jan. 1 and Aug. 1 (30 weeks), plot the graph of  $T = f(t)$ .
64. The change  $C$  (in cm) in the length of a 100-m steel bridge girder from its length at  $10^\circ\text{C}$  as a function of the temperature  $T$  is given by  $C = 0.12(T - 10)$ . Plot the graph for  $T = -20^\circ\text{C}$  to  $T = 40^\circ\text{C}$ .
65. The profit  $P$  (in dollars) a retailer makes in selling 50 cell phones is given by  $P = 50(p - 50)$ , where  $p$  is the selling price. Plot  $P$  as a function of  $p$  for  $p = \$30$  to  $p = \$150$ .
66. There are 500 L of oil in a tank that has the capacity of 100 000 L. It is filled at the rate of 7000 L/h. Determine the function relating the number of litres  $N$  and the time  $t$  while the tank is being filled. Plot  $N$  as a function of  $t$ .
67. The length  $L$  (in cm) of a pulley belt is 12 cm longer than the circumference of one of the pulley wheels. Express  $L$  as a function of the radius  $r$  of the wheel and plot  $L$  as a function of  $r$ .
68. The pressure loss  $P$  (in kPa per 100 m) in a fire hose is given by  $P = 0.00012Q^2 + 0.0055Q$ , where  $Q$  is the rate of flow (in L/min). Plot the graph of  $P$  as a function of  $Q$ .
69. A thermograph measures the infrared radiation from each small area of a person's skin. Since the skin over a tumor radiates more than skin from nearby areas, a thermograph can help detect cancer cells. The emissivity  $\varepsilon$  (in %) of radiation as a function of skin temperature  $T$  (in K) is  $\varepsilon = f(T) = 100(T^4 - 307^4)/307^4$ , if nearby skin is at  $34^\circ\text{C}$  (307K). Find  $\varepsilon$  for  $T = 309\text{K}$ .
70. A company buys a new copier for \$1000 and determines that it costs \$10 per day to use it (for paper, toner, etc.). Plot the total cost  $C$  of the copier as a function of the number  $n$  of days of use.
71. For a certain laser device, the laser output power  $P$  (in mW) is negligible if the drive current  $i$  is less than 80 mA. From 80 mA to 140 mA,  $P = 1.5 \times 10^{-6}i^3 - 0.77$ . Plot the graph of  $P = f(i)$ .
72. It is determined that a good approximation for the cost  $C$  (in cents/km) of operating a certain car at a constant speed  $v$  (in km/h) is given by  $C = 0.025v^2 - 1.4v + 35$ . Plot  $C$  as a function of  $v$  for  $v = 10$  km/h to  $v = 60$  km/h.
73. A medical researcher exposed a virus culture to an experimental vaccine. It was observed that the number of live cells  $N$  in the culture as a function of the time  $t$  (in h) after exposure was given by  $N = \frac{1000}{\sqrt{t+1}}$ . Plot the graph of  $N = f(t)$ .
74. The electric field  $E$  (in V/m) from a certain electric charge is given by  $E = 25/r^2$ , where  $r$  is the distance (in m) from the charge. Plot the graph of  $E = f(r)$  for values of  $r$  up to 10 cm.
75. To draw the approximate shape of an irregular shoreline, a surveyor measured the distances  $d$  from a straight wall to the shoreline at 20-m intervals along the wall, as shown in the following table. Plot the graph of distance  $d$  as a function of the distance  $D$  along the wall.

$D$ (m)	0	20	40	60	80	100	120	140	160
$d$ (m)	15	32	56	33	29	47	68	31	52

76. The percent  $p$  of a computer network that is in use during a particular loading cycle as a function of the time  $t$  (in s) is given in the following table. Plot the graph of  $p = f(t)$ .

$t$ (s)	0.0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6
$P$ (%)	0	45	85	90	85	85	60	10	0

77. The vertical sag  $s$  (in m) at the middle of an 800-m power line as a function of the temperature  $T$  (in  $^\circ\text{C}$ ) is given in the following table. See Fig. 3.40. For the function  $s = f(T)$ , find  $f(14)$  by linear interpolation.

$T$ ( $^\circ\text{C}$ )	-10	0	10	20
$s$ (m)	3.11	3.23	3.38	3.57

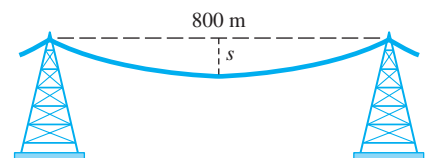


Fig. 3.40

78. In an experiment measuring the pressure  $p$  (in kPa) at a given depth  $d$  (in m) of seawater, the results in the following table were found. Plot the graph of  $p = f(d)$  and from the graph determine  $f(10)$ .

$d$ (m)	0.0	3.0	6.0	9.0	12	15
$p$ (kPa)	101	131	161	193	225	256

In Exercises 79–86, solve the indicated equations graphically.

79. A person 350 km from home starts toward home and travels at 90 km/h for the first 2.0 h and then slows down to 60 km/h for the rest of the trip. How long does it take the person to be 80 km from home?
80. One industrial cleaner contains 30% of a certain solvent, and another contains 10% of the solvent. To get a mixture containing 50 L of the solvent, 120 L of the first cleaner is used. How much of the second must be used?
81. The solubility  $s$  (in  $\text{kg}/\text{m}^3$  of water) of a certain type of fertilizer is given by  $s = 135 + 4.9T + 0.19T^2$ , where  $T$  is the temperature (in  $^\circ\text{C}$ ). Find  $T$  for  $s = 500 \text{ kg}/\text{m}^3$ .
82. A 2.00-L ( $2000\text{-cm}^3$ ) metal container is to be made in the shape of a right circular cylinder. Express the total area  $A$  of metal necessary as a function of the radius  $r$  of the base. Then find  $A$  for  $r = 6.00 \text{ cm}$ ,  $7.00 \text{ cm}$ , and  $8.00 \text{ cm}$ .
83. In an oil pipeline, the velocity  $v$  (in m/s) of the oil as a function of the distance  $x$  (in m) from the wall of the pipe is given by  $v = 9.6x - 7.5x^2$ . Find  $x$  for  $v = 2.6 \text{ m/s}$ . The diameter of the pipe is 1.20 m.
84. One ball bearing is 1.00 mm more in radius and has twice the volume of another ball bearing. What is the radius of each?
85. A computer, using data from a refrigeration plant, estimates that in the event of a power failure, the temperature (in  $^\circ\text{C}$ ) in the freezers would be given by  $T = \frac{4t^2}{t + 2} - 20$ , where  $t$  is the number of hours after the power failure. How long would it take for the temperature to reach  $0^\circ\text{C}$ ?
86. Two electrical resistors in parallel (see Fig. 3.41) have a combined resistance  $R_T$  given by  $R_T = \frac{R_1 R_2}{R_1 + R_2}$ . If  $R_2 = R_1 + 2.00$ , express  $R_T$  as a function of  $R_1$  and find  $R_1$  if  $R_T = 6.00 \Omega$ .

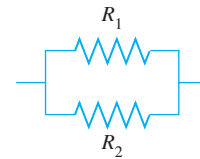


Fig. 3.41

### Writing Exercise

87. Find the inner surface area  $A$  of a cylindrical  $250.0\text{-cm}^3$  cup as a function of the radius  $r$  of the base. Then if  $A = 175.0 \text{ cm}^2$ , solve for  $r$ . Write one or two paragraphs explaining your method for setting up the function, and how you used a graphing calculator to solve for  $r$  with the given value of  $A$ .

## CHAPTER 3 PRACTICE TEST

- Given  $f(x) = 2x - x^2 + \frac{8}{x}$ , find  $f(-4)$  and  $f(2.385)$ .
- A rocket has a mass of 2000 Mg at liftoff. If the first-stage engines burn fuel at the rate of 10 Mg/s, find the mass  $m$  of the rocket as a function of the time  $t$  (in s) while the first-stage engines operate.
- Plot the graph of the function  $f(x) = 4 - 2x$ .
- Solve the equation  $2x^2 - 3 = 3x$  graphically to the nearest 0.1.
- Plot the graph of the function  $y = \sqrt{4 + 2x}$ .
- Locate all points  $(x, y)$  for which  $x < 0$  and  $y = 0$ .
- Find the domain and the range of the function  $f(x) = \sqrt{6 - x}$ .
- If the function  $y = 2x^2 - 3$  is shifted right 1 and up 3, what is the resulting function?
- Find the range of the function  $y = \frac{x^2 + 2}{x + 2}$  graphically.

- A window has the shape of a semicircle over a square, as shown in Fig. 3.42. Express the area of the window as a function of the radius of the circular part.
- The voltage  $V$  and current  $i$  (in mA) for a certain electrical experiment were measured as shown in the following table. Plot the graph of  $i = f(V)$  and from the graph find  $f(45.0)$ .

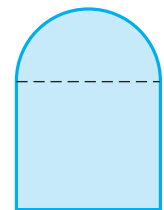


Fig. 3.42

Voltage (V)	10.0	20.0	30.0	40.0	50.0	60.0
Current (mA)	145	188	220	255	285	315

- From the table in Problem 11, find the voltage for  $i = 200 \text{ mA}$ .



# The Trigonometric Functions

One of the earliest uses of measurements with triangles was performed by the Greek mathematician and astronomer Eratosthenes (276–194 B.C.E.), while trying to determine the circumference and radius of earth. Mathematicians at the time were just developing techniques to determine a physical quantity that is impossible to measure directly. Eratosthenes knew that during the summer solstice, the sun is directly overhead at noon in the city of Syene (present-day Aswan), causing a vertical rod to cast no shadow. In Alexandria, a city 800 km approximately due north of Syene, an identical vertical rod cast a shadow whose length could be measured. Using the length of this shadow and the approximate distance between the two cities, Eratosthenes was able to estimate the circumference and radius of a presumably spherical earth.

Hipparchus (190–120 B.C.E.) was a Greek astronomer and geographer who attempted to determine the distance between earth and the moon. To complete this task, he expanded upon earlier mathematical works by collecting and developing various mathematical techniques that involve triangles into a coherent field of study now known as **trigonometry**.

Trigonometry involves measuring the sides and angles of triangles and determining and using the relationships between those quantities. These relationships have application to geometric problems and applications involving triangles directly, such as navigation, surveying, structural design, and astronomy. Further developments in mathematics have shown how trigonometric relationships are valuable in applications in which an actual triangle is not involved, including electric circuits, analysis of vibrations, acoustics, optics, seismology, and mechanical engineering, to name but a few. Because trigonometry has a great number of applications in so many areas, it is considered one of the most practical branches of mathematics.

In this chapter, basic trigonometric functions will be introduced, and many applications from science and technology using right triangles will be presented. Other types of triangles and their applications will be studied in later chapters.



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## 4

### LEARNING OUTCOMES

After completion of this chapter, the student should be able to:

- Define positive angle and negative angle
- Express an angle in degrees or radians and convert between the two measurements
- Determine a standard position angle
- Define, calculate, and use the six trigonometric functions
- Use the Pythagorean theorem and trigonometric functions to solve a right triangle
- Employ the inverse trigonometric functions to solve for a missing angle
- Solve application problems involving right triangles

◀ Using trigonometry, it is often possible to calculate distances that may not be directly measured. In Section 4.5, we show how we can measure the height of Horseshoe Falls on the Canadian side of Niagara Falls. Also in Section 4.5, using a method similar to one used by Eratosthenes, the circumference and radius of earth can be estimated.



## 4.1 Angles

Positive and Negative Angles • Coterminal Angles • Angle Conversions • Standard Position of an Angle

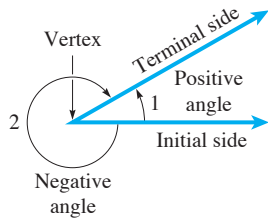


Fig. 4.1

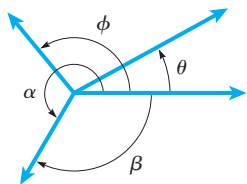


Fig. 4.2

■ The division of a degree ( $1/360$  of a rotation) into 60 minutes of arc and each minute into 60 seconds of arc comes from the ancient Sumerians, who employed a sexagesimal (base-60) counting system. The Babylonians extended this counting system to divide the hours of a day into 60 minutes of time, each minute containing 60 seconds of time.

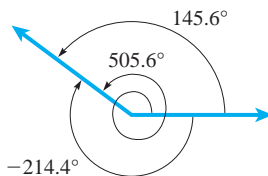


Fig. 4.3

In Chapter 2, we gave a basic definition of an *angle*. In this section, we extend this definition and also give some other important definitions related to angles.

An **angle** is generated by rotating a ray about its fixed endpoint from an **initial position** to a **terminal position**. The initial position is called the **initial side** of the angle, the terminal position is called the **terminal side**, and the fixed endpoint is the **vertex**. The angle itself is the amount of rotation from the initial side to the terminal side.

If the rotation of the terminal side from the initial side is **counterclockwise**, the angle is defined as **positive**. If the rotation is **clockwise**, the angle is **negative**. In Fig. 4.1,  $\angle 1$  is positive and  $\angle 2$  is negative.

Many symbols are used to designate angles. Among the most widely used are certain Greek letters such as  $\theta$  (theta),  $\phi$  (phi),  $\alpha$  (alpha), and  $\beta$  (beta). Capital letters representing the vertex (e.g.,  $\angle A$  or simply  $A$ ) and other literal symbols, such as  $x$  and  $y$ , are also used commonly.

In Chapter 2, we introduced two measurements of an angle. These are the *degree* and the *radian*. Since degrees and radians are both used on calculators and computers, we will briefly review the relationship between them in this section. We will make use of radians again in Chapter 8.

From Section 2.1, we recall that a **degree** is  $1/360$  of one complete rotation. In Fig. 4.2,  $\theta = 30^\circ$ ,  $\phi = 140^\circ$ ,  $\alpha = 240^\circ$ , and  $\beta = -120^\circ$ . Note that  $\beta$  is drawn in a clockwise direction to show that it is negative. The other angles are drawn in a counterclockwise direction to show that they are positive angles.

In Chapter 2, we used degrees and decimal parts of a degree. Most calculators use degrees in this decimal form. Another traditional way is to divide a degree into 60 equal parts called **minutes**; each minute is divided into 60 equal parts called **seconds**. The symbols ' and " are used to designate minutes and seconds, respectively.

In Fig. 4.2, we note that angles  $\alpha$  and  $\beta$  have the same initial and terminal sides. Such angles are called **coterminal angles**. An understanding of coterminal angles will be important in the use of trigonometric concepts in later chapters.

### EXAMPLE 1 Coterminal angles

Determine the measures of two angles that are coterminal with an angle of  $145.6^\circ$ .

Since there are  $360^\circ$  in one complete rotation, we can find a coterminal angle by adding  $360^\circ$  to the given angle of  $145.6^\circ$  to get  $505.6^\circ$ . Another coterminal angle can be found by subtracting  $360^\circ$  from  $145.6^\circ$  to get  $-214.4^\circ$ . See the angles in Fig. 4.3. We could continue to add  $360^\circ$ , or subtract  $360^\circ$ , as many times as needed to get as many additional coterminal angles as may be required. ■

### ANGLE CONVERSIONS

We will primarily use degrees as a measure of angles in this chapter. When using a calculator, *be sure to use the mode feature to set the calculator for degrees or radians, as is appropriate*. We can change between degrees and radians by using a calculator feature or by performing a unit conversion using the definition (see Section 2.4)  $\pi \text{ rad} = 180^\circ$ .

#### COMMON ERROR

When computing the values of trigonometric functions, or when solving for angles using inverse trigonometric functions (see Sections 4.2 and 4.3), **you must know what mode your calculator is in** to ensure that you are using or obtaining the angle with the correct unit. It is a common error to use the values given by a calculator without ensuring the proper mode is set (radians or degrees).

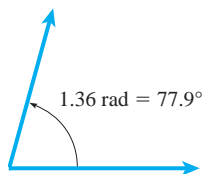


Fig. 4.4

■ In some graphing calculators, functions that can convert radians to degrees (while in degree mode) and degrees to radians (while in radian mode) can be found under the *Angle* menu.

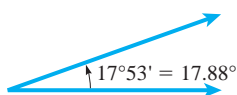


Fig. 4.5

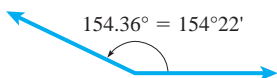


Fig. 4.6

#### Practice Exercise

1. Change  $17^\circ 24'$  to decimal form, to the nearest  $0.01^\circ$ .

#### Practice Exercise

2. In Fig. 4.7, which terminal side is that of a standard position angle of  $240^\circ$ ?

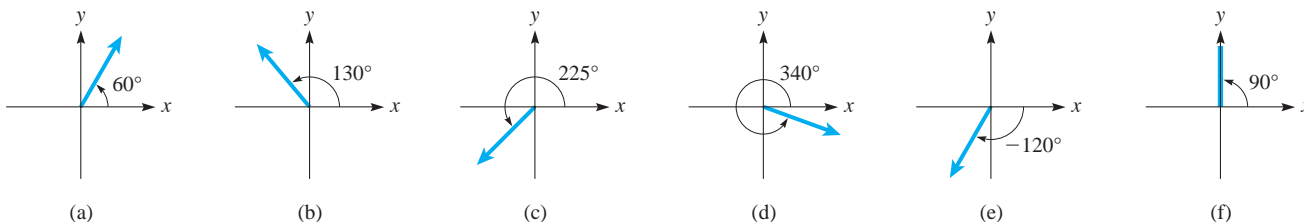


Fig. 4.7

Before the extensive use of calculators, it was common to use degrees and minutes in tables, whereas calculators use degrees and decimal parts of a degree. Changing from one form to another can be done directly on a calculator by use of the *dms* (*degree-minute-second*) feature. The following examples illustrate angle conversions by using the definition.

#### EXAMPLE 2 Convert radians to degrees

Express  $1.36 \text{ rad}$  in degrees.

We know that  $\pi \text{ rad} = 180^\circ$ , which becomes our unit conversion fraction (see Section 1.3). Therefore,

$$1.36 \text{ rad} = 1.36 \text{ rad} \left( \frac{180^\circ}{\pi \text{ rad}} \right) = 77.9^\circ \quad \text{to nearest } 0.1^\circ$$

This angle is shown in Fig. 4.4. We again note that degrees and radians are simply two different ways of measuring an angle. ■

#### EXAMPLE 3 Converting angles between degrees-minutes and decimal forms

To change  $17^\circ 53'$  to decimal form, we use the fact that  $1^\circ = 60'$ . Therefore, at this point we change the  $53'$  to degrees, which gives us  $53' = \left(\frac{53}{60}\right)^\circ = 0.88^\circ$  (to the nearest  $0.01^\circ$ ). This means that  $17^\circ 53' = 17.88^\circ$ . This angle is shown in Fig. 4.5.

To change  $154.36^\circ$  to an angle measured to the nearest minute, we use the definition  $1^\circ = 60'$  as our unit conversion fraction, and

$$0.36^\circ = 0.36^\circ \left( \frac{60'}{1^\circ} \right) = 22'$$

This means that  $154.36^\circ = 154^\circ 22'$ . See Fig. 4.6. ■

#### STANDARD POSITION OF AN ANGLE

If the initial side of the angle is the positive  $x$ -axis and the vertex is the origin, the angle is said to be in **standard position**. The angle is then determined by the position of the terminal side. If the terminal side is in the first quadrant, the angle is called a **first-quadrant angle**. Similar terms are used when the terminal side is in the other quadrants. If the terminal side coincides with one of the axes, the angle is a **quadrantal angle**. For an angle in standard position, the terminal side can be determined if we know any point, except the origin, on the terminal side.

#### EXAMPLE 4 Angles in standard position

A standard position angle of  $60^\circ$  is a first-quadrant angle with its terminal side  $60^\circ$  from the  $x$ -axis. See Fig. 4.7(a).

A second-quadrant angle of  $130^\circ$  is shown in Fig. 4.7(b).

A third-quadrant angle of  $225^\circ$  is shown in Fig. 4.7(c).

A fourth-quadrant angle of  $340^\circ$  is shown in Fig. 4.7(d).

A standard-position angle of  $-120^\circ$  is shown in Fig. 4.7(e). Since the terminal side is in the third quadrant, it is a third-quadrant angle.

A standard-position angle of  $90^\circ$  is a quadrantal angle since its terminal side is the positive  $y$ -axis. See Fig. 4.7(f).

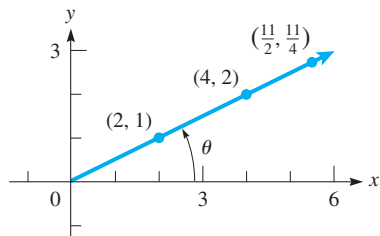


Fig. 4.8

**EXAMPLE 5** Standard position—terminal side

In Fig. 4.8,  $\theta$  is in standard position, and the terminal side is uniquely determined by knowing that it passes through the point  $(2, 1)$ . The same terminal side passes through the points  $(4, 2)$  and  $(\frac{11}{2}, \frac{11}{4})$ , among an unlimited number of other points. Knowing that the terminal side passes through any one of these points makes it possible to determine the terminal side of the angle. The method to calculate this angle  $\theta$  is shown in Section 4.3. ■

**EXERCISES 4.1**

In Exercises 1–4, find the indicated angles in the given examples of this section.

- In Example 1, find another angle that is coterminal with the given angle.
- In Example 3, change  $53'$  to  $35'$  and then find the decimal form.
- In Example 4, find another standard-position angle that has the same terminal side as the angle in Fig. 4.7(c).
- In Example 4, find another standard-position angle that has the same terminal side as the angle in Fig. 4.7(e).

In Exercises 5–8, draw the given angles.

- $60^\circ$ ,  $120^\circ$ ,  $-90^\circ$
- $330^\circ$ ,  $-150^\circ$ ,  $450^\circ$
- $50^\circ$ ,  $-360^\circ$ ,  $-30^\circ$
- $45^\circ$ ,  $245^\circ$ ,  $-250^\circ$

In Exercises 9–16, determine one positive and one negative coterminal angle for each angle given.

- $45^\circ$
- $73^\circ$
- $-150^\circ$
- $462^\circ$
- $70^\circ 30'$
- $153^\circ 47'$
- $278.1^\circ$
- $-197.6^\circ$

In Exercises 17–20, by means of the definition of a radian, change the given angles in radians to equal angles expressed in degrees to the nearest  $0.01^\circ$ .

- $0.265$  rad
- $0.838$  rad
- $1.447$  rad
- $-3.642$  rad

In Exercises 21–24, use a calculator conversion sequence to change the given angles in radians to equal angles expressed in degrees to the nearest  $0.01^\circ$ .

- $0.329$  rad
- $2.089$  rad
- $-4.110$  rad
- $6.705$  rad

In Exercises 25–28, use a calculator conversion sequence to change the given angles to equal angles expressed in radians to three significant digits.

- $56.0^\circ$
- $137.4^\circ$
- $384.8^\circ$
- $-17.5^\circ$

In Exercises 29–32, change the given angles to equal angles expressed to the nearest minute.

- $47.50^\circ$
- $715.80^\circ$
- $-5.62^\circ$
- $142.87^\circ$

In Exercises 33–36, change the given angles to equal angles expressed in decimal form to the nearest  $0.01^\circ$ .

- $15^\circ 12'$
- $517^\circ 39'$
- $301^\circ 16'$
- $-4^\circ 47'$

In Exercises 37–44, draw angles in standard position such that the terminal side passes through the given point.

- $(4, 2)$
- $(-3, 8)$
- $(-3, -5)$
- $(6, -1)$
- $(-7, 5)$
- $(-4, -2)$
- $(-2, 0)$
- $(0, 6)$

In Exercises 45–52, the given angles are in standard position. Designate each angle by the quadrant in which the terminal side lies, or as a quadrantal angle.

- $31^\circ, 310^\circ$
- $180^\circ, 92^\circ$
- $435^\circ, -270^\circ$
- $-5^\circ, 265^\circ$
- $1$  rad,  $2$  rad
- $3$  rad,  $\pi$  rad
- $4$  rad,  $\pi/3$  rad
- $12$  rad,  $-2$  rad

In Exercises 53 and 54, change the given angles to equal angles expressed in decimal form to the nearest  $0.001^\circ$ . In Exercises 55 and 56, change the given angles to equal angles expressed to the nearest second.

- $21^\circ 42' 36''$
- $7^\circ 16' 23''$
- $86.274^\circ$
- $257.019^\circ$

**Answers to Practice Exercises**

- $17.40^\circ$
- (e)

## 4.2 Defining the Trigonometric Functions

Similar Triangles • Definitions of the Trigonometric Functions • Evaluating the Trigonometric Functions

Important to the definitions and development in this section are the right triangle, the Pythagorean theorem, and the properties of similar triangles. We now briefly review similar triangles and their properties.

As stated in Section 2.2, two triangles are **similar** if they have the same shape (but not necessarily the same size). Similar triangles have the following important properties.

### Properties of Similar Triangles

1. Corresponding angles are equal.
2. Corresponding sides are proportional.

The **corresponding sides** are the sides, one in each triangle, that are between the same pair of equal corresponding angles.

### EXAMPLE 1 Similar triangles

In Fig. 4.9, the triangles are similar and are lettered so that corresponding sides and angles have the same letters. That is, angles  $A_1$  and  $A_2$ , angles  $B_1$  and  $B_2$ , and angles  $C_1$  and  $C_2$  are pairs of corresponding angles. The pairs of corresponding sides are  $a_1$  and  $a_2$ ,  $b_1$  and  $b_2$ , and  $c_1$  and  $c_2$ . From the properties of similar triangles, we know that the corresponding angles are equal, or

$$\angle A_1 = \angle A_2 \quad \angle B_1 = \angle B_2 \quad \angle C_1 = \angle C_2$$

Also, the corresponding sides are proportional, which we can show as

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} \quad \frac{a_1}{a_2} = \frac{c_1}{c_2} \quad \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

In Example 1, if we multiply both sides of  $a_1/a_2 = b_1/b_2$  by  $a_2/b_1$ , we get

$$\frac{a_1}{a_2} \left( \frac{a_2}{b_1} \right) = \frac{b_1}{b_2} \left( \frac{a_2}{b_1} \right), \text{ or } \frac{a_1}{b_1} = \frac{a_2}{b_2}$$

Using this, we now proceed to the definitions of the trigonometric functions.

An angle  $\theta$  is placed in standard position and perpendiculars from points on the terminal side are dropped to the  $x$ -axis, as shown in Fig. 4.10. In doing this, similar triangles are obtained, each with one vertex at the origin and one side along the  $x$ -axis.

### EXAMPLE 2 Equality of ratios of corresponding sides

In Fig. 4.10, we can see that triangles  $ORP$  and  $OSQ$  are similar since their corresponding angles are equal (each has the same angle at  $O$ , a right angle, and therefore equal angles at  $P$  and  $Q$ ). This means ratios of the lengths of corresponding sides are equal. For example,

$$\frac{RP}{OR} = \frac{SQ}{OS}, \text{ which is the same as } \frac{y}{x} = \frac{b}{a} \quad \text{using the coordinates of points } P \text{ and } Q$$

For any position (except at the origin) of  $Q$  on the terminal side of  $\theta$ , the ratio of  $b/a$  of the ordinate to the abscissa will equal  $y/x$ .

For an angle in standard position, we may set up six different ratios of the values of  $x$ ,  $y$ , and  $r$ , as shown in Fig. 4.11. Because of the properties of similar triangles, for a given angle, any of these ratios has the same value for any point on the terminal side. This means the values of the ratios depend on the size of the angle, and there is only one value for each ratio. This means the ratios are **functions** of the angle, and they are

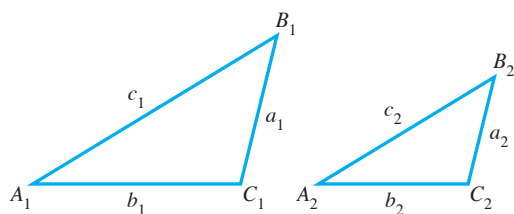


Fig. 4.9

### LEARNING TIP

When two triangles are similar, the ratio of one side to another side in one triangle is the same as the ratio of the corresponding sides in the other triangle.

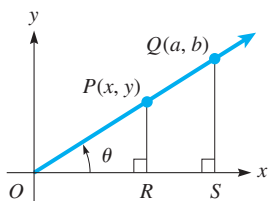


Fig. 4.10

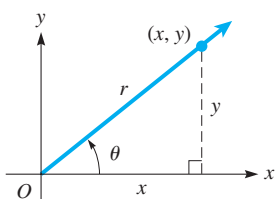


Fig. 4.11

called the **trigonometric functions**. We now define them, giving their names and the abbreviations that are used when using them.

### Trigonometric Functions

$$\begin{aligned}
 \text{sine of } \theta: \sin \theta &= \frac{y}{r} & \text{cosecant of } \theta: \csc \theta &= \frac{1}{\sin \theta} = \frac{r}{y} \\
 \text{cosine of } \theta: \cos \theta &= \frac{x}{r} & \text{secant of } \theta: \sec \theta &= \frac{1}{\cos \theta} = \frac{r}{x} \\
 \text{tangent of } \theta: \tan \theta &= \frac{y}{x} & \text{cotangent of } \theta: \cot \theta &= \frac{1}{\tan \theta} = \frac{x}{y}
 \end{aligned} \tag{4.1}$$

Notice that three of the trigonometric functions (cosecant, secant, and cotangent) are reciprocals of the other three trigonometric functions (sine, cosine, and tangent, respectively).

Here, the distance  $r$  from the origin to the point  $(x, y)$  on the terminal side is called the **radius vector**, and it is assumed that  $r > 0$  (if  $r = 0$ , there would be no terminal side and therefore no angle).

A given function is not defined when the denominator is zero, and if either  $x = 0$  or  $y = 0$ , this does affect the domain of some of the trigonometric functions. We will discuss the domains and ranges of these functions in Chapter 10, when we discuss their graphs.

In this chapter, we use the trigonometric functions of acute angles (angles between  $0^\circ$  and  $90^\circ$ ). However, the definitions above are general and may be used with angles of any size. We will discuss these functions in general in Chapters 8 and 20.

### EVALUATING THE TRIGONOMETRIC FUNCTIONS

The definitions in Eqs. (4.1) are used in evaluating the trigonometric functions. Also, we often use the *Pythagorean theorem*, which is discussed in Section 2.2, and which we restate here for reference. For the right triangle in Fig. 4.12, with hypotenuse  $c$  and legs  $a$  and  $b$ ,

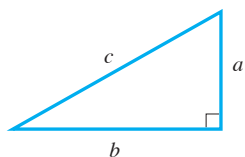


Fig. 4.12

$$c^2 = a^2 + b^2 \tag{4.2}$$

### EXAMPLE 3 Values of trigonometric functions as fractions

Find the values of the trigonometric functions of the standard-position angle  $\theta$  with its terminal side passing through the point  $(3, 4)$ .

By placing the angle in standard position, as shown in Fig. 4.13, and drawing the terminal side through  $(3, 4)$ , we find by use of the Pythagorean theorem that

$$r = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

Using the values  $x = 3$ ,  $y = 4$ , and  $r = 5$ , we find that

$$\begin{aligned}
 \sin \theta &= \frac{4}{5} & \cos \theta &= \frac{3}{5} & \tan \theta &= \frac{4}{3} \\
 \cot \theta &= \frac{3}{4} & \sec \theta &= \frac{5}{3} & \csc \theta &= \frac{5}{4}
 \end{aligned}$$

We have left each of these results in the form of a fraction, which is considered to be an *exact form* in that there has been no approximation made. In writing decimal values, we find that  $\tan \theta = 1.33$  and  $\sec \theta = 1.67$ , where these values have been rounded off to 3 significant digits and are therefore *approximate*. ■

#### Practice Exercise

- In Example 3, change  $(3, 4)$  to  $(4, 3)$  and then find  $\tan \theta$  and  $\sec \theta$ .

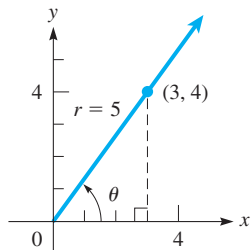


Fig. 4.13

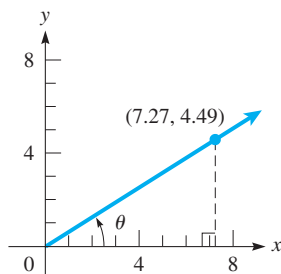


Fig. 4.14

**EXAMPLE 4** Values of trigonometric functions as decimals

Find the values of the trigonometric functions of the standard-position angle whose terminal side passes through  $(7.27, 4.49)$ . The coordinates are approximate.

We show the angle and the given point in Fig. 4.14. From the Pythagorean theorem, we have

$$r = \sqrt{7.27^2 + 4.49^2} = 8.545$$

(Here, we show a rounded-off value of  $r$ . It is not actually necessary to record the value of  $r$  since its value can be stored in the memory of a calculator. The reason for recording it here is to show the values used in the calculation of each of the trigonometric functions.) Therefore, we have the following values:

$$\begin{aligned} \sin \theta &= \frac{4.49}{8.545} = 0.525 & \cos \theta &= \frac{7.27}{8.545} = 0.851 \\ \tan \theta &= \frac{4.49}{7.27} = 0.618 & \cot \theta &= \frac{7.27}{4.49} = 1.62 \\ \sec \theta &= \frac{8.545}{7.27} = 1.18 & \csc \theta &= \frac{8.545}{4.49} = 1.90 \end{aligned}$$

Since the coordinates are approximate, the results are rounded off to 3 significant digits. ■

**COMMON ERROR**

When one expresses the result of a trigonometric function in form  $\sin \theta = 0.525$ , it is a common error to omit the angle and report the value as  $\sin = 0.525$ . This is a meaningless expression because **the angle for which the trigonometric expression is evaluated must be given**. The trigonometric function itself indicates which pair of sides in the right triangle to divide, but without an angle for reference, the ratio has no meaning.

If one of the trigonometric functions is known, it is possible to find the values of the other functions. The following example illustrates the method.

**EXAMPLE 5** Given one trigonometric function, find others

If we know that  $\sin \theta = 3/7$ , and that  $\theta$  is a first-quadrant angle, we know the ratio of the ordinate to the radius vector ( $y$  to  $r$ ) is 3 to 7. Therefore, the point on the terminal side for which  $y = 3$  can be found by use of the Pythagorean theorem. The  $x$ -value for this point is

$$x = \sqrt{7^2 - 3^2} = \sqrt{49 - 9} = \sqrt{40} = 2\sqrt{10}$$

Therefore, the point  $(2\sqrt{10}, 3)$  is on the terminal side, as shown in Fig. 4.15.

Therefore, using the values  $x = 2\sqrt{10}$ ,  $y = 3$ , and  $r = 7$ , we have the other trigonometric functions of  $\theta$ . They are

$$\cos \theta = \frac{2\sqrt{10}}{7} \quad \tan \theta = \frac{3}{2\sqrt{10}} \quad \cot \theta = \frac{2\sqrt{10}}{3} \quad \sec \theta = \frac{7}{2\sqrt{10}} \quad \csc \theta = \frac{7}{3}$$

These values are *exact*. *Approximate* decimal values found on a calculator are

$$\begin{aligned} \cos \theta &= 0.904 & \tan \theta &= 0.474 & \cot \theta &= 2.11 \\ \sec \theta &= 1.11 & \csc \theta &= 2.33 \end{aligned}$$

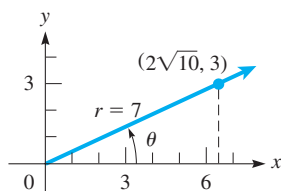


Fig. 4.15

**Practice Exercise**

2. In Example 5, change  $\sin \theta = 3/7$  to  $\cos \theta = 3/7$ , and then find approximate values of  $\sin \theta$  and  $\cot \theta$ .



## EXERCISES 4.2

In Exercises 1 and 2, answer the given questions about the indicated examples of this section.

- In Example 3, if the point  $(4, 3)$  replaces the point  $(3, 4)$ , what are the values of the trigonometric functions in exact form?
- In Example 5, if  $4/7$  replaces  $3/7$ , what are the values of the trigonometric functions in exact form?

In Exercises 3–16, find values of the trigonometric functions of the angle (in standard position) whose terminal side passes through the given points. For Exercises 3–14, give answers in exact form. For Exercises 15 and 16, the coordinates are approximate.

- $(6, 8)$
- $(5, 12)$
- $(15, 8)$
- $(240, 70)$
- $(0.09, 0.40)$
- $(3.2, 6.0)$
- $(1, \sqrt{15})$
- $(\sqrt{3}, 2)$
- $(1, 1)$
- $(6, 5)$
- $(50, 20)$
- $(1, \frac{1}{2})$
- $(0.687, 0.943)$
- $(37.65, 21.87)$

In Exercises 17–24, find the values of the indicated functions. In Exercises 17–20, give answers in exact form. In Exercises 21–24, the values are approximate. Assume all angles are acute.

- Given  $\cos \theta = 12/13$ , find  $\sin \theta$  and  $\cot \theta$ .
- Given  $\sin \theta = 1/2$ , find  $\cos \theta$  and  $\csc \theta$ .
- Given  $\tan \theta = 2$ , find  $\sin \theta$  and  $\sec \theta$ .
- Given  $\sec \theta = \sqrt{5}/2$ , find  $\tan \theta$  and  $\cos \theta$ .
- Given  $\sin \theta = 0.750$ , find  $\cot \theta$  and  $\csc \theta$ .
- Given  $\cos \theta = 0.0326$ , find  $\sin \theta$  and  $\tan \theta$ .
- Given  $\cot \theta = 0.254$ , find  $\cos \theta$  and  $\tan \theta$ .
- Given  $\csc \theta = 1.20$ , find  $\sec \theta$  and  $\cos \theta$ .

In Exercises 25–28, each point listed is on the terminal side of an angle. Show that each of the indicated functions is the same for each of the points.

- $(3, 4)$ ,  $(6, 8)$ ,  $(4.5, 6)$ ,  $\sin \theta$  and  $\tan \theta$
- $(5, 12)$ ,  $(15, 36)$ ,  $(7.5, 18)$ ,  $\cos \theta$  and  $\cot \theta$
- $(0.3, 0.1)$ ,  $(9, 3)$ ,  $(33, 11)$ ,  $\tan \theta$  and  $\sec \theta$
- $(40, 30)$ ,  $(8, 6)$ ,  $(36, 27)$ ,  $\csc \theta$  and  $\cos \theta$

In Exercises 29–36, answer the given questions.

- If  $\tan \theta = 3/4$ , what is the value of  $\sin^2 \theta + \cos^2 \theta$ ? [ $\sin^2 \theta = (\sin \theta)^2$ ]
- If  $\sin \theta = 2/3$ , what is the value of  $\sec^2 \theta - \tan^2 \theta$ ?
- If  $y = \sin \theta$ , what is  $\cos \theta$  in terms of  $y$ ?
- If  $x = \cos \theta$ , what is  $\tan \theta$  in terms of  $x$ ?
- What is  $x$  if  $(x + 1, 4)$  and  $(-2, 6)$  are on the same terminal side of a standard-position angle?
- What is  $x$  if  $(2, 5)$  and  $(7, x)$  are on the same terminal side of a standard-position angle?
- From the definitions of the trigonometric functions, it can be seen that  $\csc \theta$  is the reciprocal of  $\sin \theta$ . What function is the reciprocal of  $\cos \theta$ ?
- Refer to the definitions of the trigonometric functions in Eqs. (4.1). Is the quotient of one of the functions divided by  $\cos \theta$  equal to  $\tan \theta$ ? Explain.

## Answers to Practice Exercises

- $\tan \theta = 3/4$ ,  $\sec \theta = 5/4$
- $\sin \theta = 0.904$ ,  $\cot \theta = 0.474$

## 4.3 Values of the Trigonometric Functions

Function Values Using Geometry •  
Function Values from Calculator •  
Inverse Trigonometric Functions •  
Accuracy of Trigonometric Functions •  
Reciprocal Trigonometric Functions

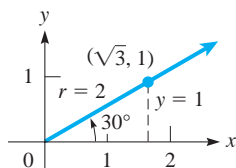


Fig. 4.16

In practice, we often need the values of the trigonometric functions for angles measured in degrees. One way to find some of these values is to use certain basic facts from geometry to find the functions of particular common angles. This is illustrated in the next two examples.

EXAMPLE 1 Function values of  $30^\circ$  and  $60^\circ$ 

From geometry, we find that the side opposite a  $30^\circ$  angle in a right triangle is one-half of the hypotenuse. Therefore, letting  $y = 1$  and  $r = 2$  (see Fig. 4.16), and using the Pythagorean theorem, we have  $x = \sqrt{2^2 - 1^2} = \sqrt{3}$ . Now, with  $x = \sqrt{3}$ ,  $y = 1$ , and  $r = 2$ ,

$$\sin 30^\circ = \frac{1}{2} \quad \cos 30^\circ = \frac{\sqrt{3}}{2} \quad \tan 30^\circ = \frac{1}{\sqrt{3}}$$

Using this same method, we find the functions of  $60^\circ$  to be

$$\sin 60^\circ = \frac{\sqrt{3}}{2} \quad \cos 60^\circ = \frac{1}{2} \quad \tan 60^\circ = \sqrt{3}$$

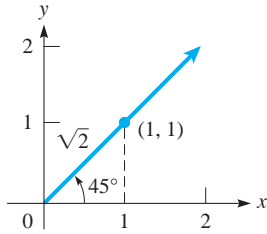


Fig. 4.17

**EXAMPLE 2** Trigonometric function values of  $45^\circ$ 

Find  $\sin 45^\circ$ ,  $\cos 45^\circ$ , and  $\tan 45^\circ$ .

If we place an isosceles right triangle with one of its  $45^\circ$  angles in standard position and hypotenuse along the radius vector (see Fig. 4.17), the terminal side passes through  $(1, 1)$ , since the legs of the triangle are equal. Using this point,  $x = 1$ ,  $y = 1$ , and  $r = \sqrt{2}$ . Thus,

$$\sin 45^\circ = \frac{1}{\sqrt{2}} \quad \cos 45^\circ = \frac{1}{\sqrt{2}} \quad \tan 45^\circ = 1$$

In Examples 1 and 2, we have given *exact* values. Decimal approximations are also given in the following table that summarizes the results for  $30^\circ$ ,  $45^\circ$ , and  $60^\circ$ .

**Trigonometric Functions of  $30^\circ$ ,  $45^\circ$ , and  $60^\circ$** 

$\theta$	<i>(exact values)</i>			<i>(decimal approximations)</i>		
	$30^\circ$	$45^\circ$	$60^\circ$	$30^\circ$	$45^\circ$	$60^\circ$
$\sin \theta$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	0.500	0.707	0.866
$\cos \theta$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0.866	0.707	0.500
$\tan \theta$	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	0.577	1.000	1.732

■ It is helpful to be familiar with these values, as they are used in later sections.

Another way to find values of the functions is to use a scale drawing. Measure the angle with a protractor, then measure directly the values of  $x$ ,  $y$ , and  $r$  for some point on the terminal side, and finally use the proper ratios to evaluate the functions. However, this method is only approximate, and geometric methods work only for a limited number of angles. As it turns out, it is possible to find these values to any required accuracy through more advanced methods (using calculus and what are known as *power series*).

Values of  $\sin \theta$ ,  $\cos \theta$ , and  $\tan \theta$  are programmed into graphing calculators. The following examples illustrate using a calculator to find trigonometric values.

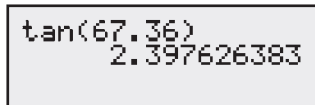


Fig. 4.18

**EXAMPLE 3** Trigonometric values from a calculator

Using a graphing calculator to find the value of  $\tan 67.36^\circ$ , first enter the function and then the angle, just as it is written. The resulting display is shown in Fig. 4.18.

Therefore,  $\tan 67.36^\circ = 2.397\ 626\ 383$ . This also means the ratio  $y/x$  is approximately 2.398 for this angle. ■

**LEARNING TIP**

In most examples, we will round off trigonometric function values to 3 significant digits. However, *if the angle is approximate* (meaning it is a **measured value**), then we round the trigonometric function values to the same number of significant digits as there are in the angle.

Not only are we able to find values of the trigonometric functions if we know the angle, but we can also find the angle if we know that value of a function. In doing this, we are actually using another important type of mathematical function, an **inverse function**. *The inverse of a function, in general, undoes the process that the function performs.* A trigonometric function operates on an angle and returns the value of the ratio of two sides of a triangle. An **inverse trigonometric function** does the opposite: *it operates on the ratio of sides* (a number) *and returns an angle* (in degrees or radians, depending on the calculator mode). They are discussed in detail in Chapter 20. For the purpose of using a calculator at this point, it is sufficient to recognize and understand the notation that is used.

The notation for “the angle whose sine is  $x$ ” is  $\sin^{-1} x$ . This is called the *inverse sine function*. Equivalent meanings are given to  $\cos^{-1} x$  (the angle whose cosine is  $x$ ) and  $\tan^{-1} x$  (the angle whose tangent is  $x$ ).

**COMMON ERROR**

With trigonometric functions, a raised  $-1$  shows that the *inverse of the function is intended*, it is **not a negative exponent**.

$\sin^{-1}x$  represents an angle whose sine value is  $x$ . It is a common error to confuse this with cosecant, the reciprocal of sine.

$$\sin^{-1}x = \text{inverse sine of ratio } x$$

$$\sin^{-1}x \neq \frac{1}{\sin x} = \csc x$$

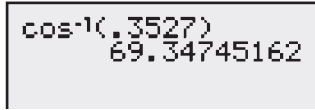


Fig. 4.19

The following example illustrates the use of the  $\cos^{-1}$  key.

**EXAMPLE 4 Inverse trigonometric function value from a calculator**

If  $\cos \theta = 0.3527$ , which means that  $\theta = \cos^{-1} 0.3527$  ( $\theta$  is the angle whose cosine is 0.3527), we can use a graphing calculator to find  $\theta$ . The display is shown in Fig. 4.19.

Therefore,  $\theta = 69.35^\circ$  (rounded off). ■

Although we can usually set up the solution of a problem in terms of the sine, cosine, or tangent, there are times when a value of the cotangent, secant, or cosecant is used. We now show how values of these functions are found on a calculator.

From the definitions in Eqn. (4.1),  $\csc \theta = r/y$  and  $\sin \theta = y/r$ . This means *the value of  $\csc \theta$  is the reciprocal of the value of  $\sin \theta$* . Again, using the definitions, we find that *the value of  $\sec \theta$  is the reciprocal of  $\cos \theta$  and the value of  $\cot \theta$  is the reciprocal of  $\tan \theta$* . Since the reciprocal of  $x$  equals  $x^{-1}$ , we use the  $x^{-1}$  key along with the sin, cos, and tan keys, to find the values of  $\csc \theta$ ,  $\sec \theta$ , and  $\cot \theta$ .

**EXAMPLE 5 Reciprocal trigonometric function value from a calculator**

To find the value of  $\sec 27.82^\circ$ , we use the fact that

$$\sec 27.82^\circ = \frac{1}{\cos 27.82^\circ} \quad \text{or} \quad \sec 27.82^\circ = (\cos 27.82^\circ)^{-1}$$

The right-hand side of either of these two expressions can be entered into a calculator to find that  $\sec 27.82^\circ = 1.131$  (rounded off).

Note that we calculated the reciprocal  $(\cos 27.82^\circ)^{-1}$ , and not the angle that would be denoted by using the  $\cos^{-1}$  notation. ■

**EXAMPLE 6 Given a reciprocal trigonometric function, solve for  $\theta$** 

To find the value of  $\theta$  if  $\cot \theta = 0.354$ , we use the fact that

$$\tan \theta = \frac{1}{\cot \theta} = \frac{1}{0.354}$$

This means that  $\theta = \tan^{-1}(0.354^{-1})$ . Therefore,  $\theta = 70.5^\circ$  (rounded off). ■

**EXAMPLE 7 Given one trigonometric function, find another**

Find  $\sin \theta$  if  $\sec \theta = 2.504$ .

Since the value of  $\sec \theta$  is known, we know that  $\cos \theta = 2.504^{-1}$  (or  $1/2.504$ ). This in turn tells us that  $\theta = \cos^{-1}(2.504^{-1})$ . Since we are to find the value of  $\sin \theta$ , we can see that

$$\sin \theta = \sin(\cos^{-1}(2.504^{-1}))$$

Therefore,  $\sin \theta = 0.9168$  (rounded off). Instead of using compounded functions,  $\theta$  could be solved numerically first, then  $\sin \theta$  evaluated, rounding only after the final step. ■

**LEARNING TIP**

Because the use of a raised  $-1$  in trigonometric function notation to indicate the inverse is often confused with reciprocal trigonometric functions, a common practice to avoid such confusion is to use the prefix “arc” in front of the trigonometric relation instead of the  $-1$  notation. For example,

$$\arcsin x = \sin^{-1}x$$

$$\arccos x = \cos^{-1}x$$

$$\arctan x = \tan^{-1}x$$

Both notations are commonly used, and are interchangeable.

**Reciprocal Trigonometric Functions****Practice Exercises**

1. Find the value of  $\sin 12.5^\circ$ .
2. Find  $\theta$  if  $\tan \theta = 1.039$ .

**Practice Exercises**

3. Find the value of  $\cot 56.4^\circ$ .
4. Find  $\theta$  if  $\csc \theta = 1.904$ .

The following example illustrates the use of the value of a trigonometric function in an applied problem. We consider various types of applications later in the chapter.

### EXAMPLE 8 Applied problem with a trigonometric function

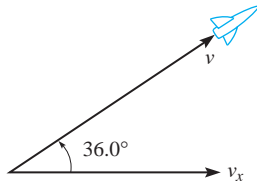


Fig. 4.20

When a rocket is launched, its horizontal velocity  $v_x$  is related to the velocity  $v$  with which it is fired by the equation  $v_x = v \cos \theta$  (which means  $v(\cos \theta)$ ). Here,  $\theta$  is the angle between the horizontal and the direction in which it is fired (see Fig. 4.20). Find  $v_x$  if  $v = 1250$  m/s and  $\theta = 36.0^\circ$ .

Substituting the given values of  $v$  and  $\theta$  in  $v_x = v \cos \theta$ , we have

$$v_x = 1250 \cos 36.0^\circ$$

$$v_x = 1010 \text{ m/s}$$

Therefore, the horizontal velocity is 1010 m/s. ■

### ORDER OF OPERATIONS AND NOTATION WITH TRIGONOMETRIC FUNCTIONS

Some standard conventions have been adopted for trigonometric functions regarding order of operations and how the independent variable is written. First of all, it is customary to write trigonometric functions without the brackets surrounding the independent variable, so that  $\sin(x) = \sin x$ .

This is also acceptable when the argument of the function is an expression containing a single term. For example,  $\sin(2x) = \sin 2x$  is the common shorthand convention, meaning that  $2x$  forms the value that the sine function operates on. Similarly, it is understood that  $\sin(x^2) = \sin x^2$ . However, if the argument of the function involves multiple terms, then the brackets must be included. For example,  $\sin(x + 5) \neq \sin x + 5 = 5 + \sin x$ . ***If confusion arises because of the shorthand convention for trigonometric arguments, the brackets around the value that the trigonometric function is supposed to operate upon must be included to remove the confusion.***

For exponents of a trigonometric function, a shorthand convention is also applied. The expression  $(\sin x)^2 = \sin^2 x$  is useful since it allows one to avoid writing the extra set of brackets. Order of operations indicates that  $\sin x$  is evaluated first, and then the result is squared. We must be careful, however, since  $\sin^2 x \neq \sin x^2$ . The left side of this equation requires that  $\sin x$  be evaluated first, and then the result is squared. The right side requires that  $x$  be squared first, and then the sine function is applied to the result.

## EXERCISES 4.3

In Exercises 1–4, make the given changes in the indicated examples of this section and then find the indicated values.

- In Example 4, change  $\cos \theta$  to  $\sin \theta$  and then find the angle.
- In Example 5, change  $\sec 27.82^\circ$  to  $\csc 27.82^\circ$  and then find the value.
- In Example 6, change 0.354 to 0.345 and then find the angle.
- In Example 7, change  $\sin \theta$  to  $\tan \theta$  and then find the value.

In Exercises 5–8, use a protractor to draw the given angle. Measure off 10 units (centimetres are convenient) along the radius vector. Then measure the corresponding values of  $x$  and  $y$ . From these values, determine the trigonometric functions of the angle.

- $40^\circ$
- $75^\circ$
- $15^\circ$
- $53^\circ$

In Exercises 9–24, find the values of the trigonometric functions.

- |                         |                        |
|-------------------------|------------------------|
| 9. $\sin 22.4^\circ$    | 10. $\cos 72.5^\circ$  |
| 11. $\tan 57.6^\circ$   | 12. $\sin 36.0^\circ$  |
| 13. $\cos 15.71^\circ$  | 14. $\tan 8.653^\circ$ |
| 15. $\sin 89.0^\circ$   | 16. $\cos 0.700^\circ$ |
| 17. $\cot 67.78^\circ$  | 18. $\csc 22.81^\circ$ |
| 19. $\sec 50.4^\circ$   | 20. $\cot 41.8^\circ$  |
| 21. $\csc 0.4900^\circ$ | 22. $\sec 7.80^\circ$  |
| 23. $\cot 85.96^\circ$  | 24. $\csc 76.30^\circ$ |

In Exercises 25–40, find  $\theta$  for each of the given trigonometric functions.

- |                            |                            |
|----------------------------|----------------------------|
| 25. $\cos \theta = 0.3261$ | 26. $\tan \theta = 2.470$  |
| 27. $\sin \theta = 0.9114$ | 28. $\cos \theta = 0.0427$ |

29.  $\tan \theta = 0.207$                       30.  $\sin \theta = 0.109$   
 31.  $\cos \theta = 0.650\ 07$                 32.  $\tan \theta = 5.7706$   
 33.  $\csc \theta = 1.245$                       34.  $\sec \theta = 2.045$   
 35.  $\cot \theta = 0.06060$                 36.  $\csc \theta = 1.002$   
 37.  $\sec \theta = 3.65$                       38.  $\cot \theta = 14.4$   
 39.  $\csc \theta = 8.26$                       40.  $\cot \theta = 0.1519$

In Exercises 41–44, use a calculator to verify the given relationships or statements. [ $\sin^2 \theta = (\sin \theta)^2$ ]

41.  $\frac{\sin 43.7^\circ}{\cos 43.7^\circ} = \tan 43.7^\circ$             42.  $\sin^2 77.5^\circ + \cos^2 77.5^\circ = 1$   
 43.  $\tan 70^\circ = \frac{\tan 30^\circ + \tan 40^\circ}{1 - (\tan 30^\circ)(\tan 40^\circ)}$   
 44.  $\sin 78.4^\circ = 2(\sin 39.2^\circ)(\cos 39.2^\circ)$

In Exercises 45–48, explain why the given statements are true for an acute angle  $\theta$ .

45.  $\sin \theta$  is always between 0 and 1.  
 46.  $\tan \theta$  can equal any positive real number.  
 47.  $\cos \theta$  decreases in value from  $0^\circ$  to  $90^\circ$ .  
 48. The value of  $\sec \theta$  is never less than 1.

In Exercises 49–52, find the values of the indicated trigonometric functions.

49. Find  $\sin \theta$ , given  $\tan \theta = 1.936$ .  
 50. Find  $\cos \theta$ , given  $\sin \theta = 0.6725$ .  
 51. Find  $\tan \theta$ , given  $\sec \theta = 1.3698$ .  
 52. Find  $\csc \theta$ , given  $\cos \theta = 0.1063$ .

In Exercises 53–58, solve the given problems.

53. Find the first quadrant angle between the line  $y = 3x$  and the  $x$ -axis.  
 54. Find the angle in the first quadrant between the lines  $y = 3x$  and  $y = 2x$ .

55. The sound produced by a jet engine was measured at a distance of 100 m in all directions. The loudness  $L$  of the sound (in decibels) was found to be  $L = 70.0 + 30.0 \cos \theta$ , where the  $0^\circ$  line was directed in front of the engine. Calculate  $L$  for  $\theta = 54.5^\circ$ .  
 56. A brace is used in the structure shown in Fig. 4.21. Its length is  $l = a(\sec \theta + \csc \theta)$ . Find  $l$  if  $a = 28.0$  cm and  $\theta = 34.5^\circ$ .

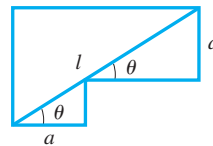


Fig. 4.21

57. The signal from an AM radio station with two antennas  $d$  metres apart has a wavelength  $\lambda$  (in m). The intensity of the signal depends on the angle  $\theta$  as shown in Fig. 4.22. An angle of minimum intensity is given by  $\sin \theta = 1.50 \lambda/d$ . Find  $\theta$  if  $\lambda = 200$  m and  $d = 400$  m.

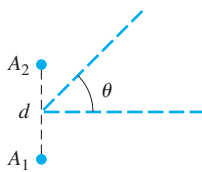


Fig. 4.22

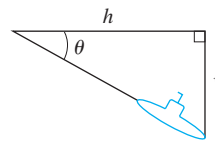


Fig. 4.23

58. A submarine dives such that the horizontal distance  $h$  it moves and the vertical distance  $v$  it dives are related by  $v = h \tan \theta$ . Here,  $\theta$  is the angle of the dive, as shown in Fig. 4.23. Find  $\theta$  if  $h = 2.35$  km and  $v = 1.52$  km.

Answers to Practice Exercises

1. 0.216    2.  $46.10^\circ$     3. 0.664    4.  $31.68^\circ$

## 4.4 The Right Triangle

Solving a Triangle • Cofunctions • Procedure for Solving a Right Triangle • Calculator Solutions



Fig. 4.24

We know that a triangle has three sides and three angles. If one side and any other two of these six parts are known, we can find the other three parts. One of the known parts must be a side, for if we know only the three angles, we know only that any triangle with these angles is similar to any other triangle with these angles.

### EXAMPLE 1 Parts of a triangle

Assume that one side and two angles are known, such as the side of 5 and the angles of  $35^\circ$  and  $90^\circ$  in the triangle in Fig. 4.24. Then we may determine the third angle  $\alpha$  by the fact that the sum of the angles of a triangle is always  $180^\circ$ . Of all possible similar triangles having the three angles of  $35^\circ$ ,  $90^\circ$ , and  $55^\circ$  (which is  $\alpha$ ), we have the one with the particular side of 5 between angles of  $35^\circ$  and  $90^\circ$ . Only *one* triangle with these parts is possible. ■

To solve a triangle means that, when we are given three parts of a triangle (at least one a side), we are to find the other three parts. In this section, we are going to demonstrate the method of solving a right triangle. Since one angle of the triangle is  $90^\circ$ , it is

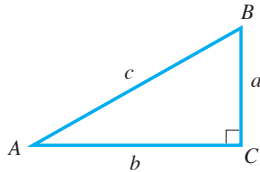


Fig. 4.25

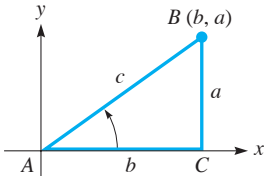


Fig. 4.26

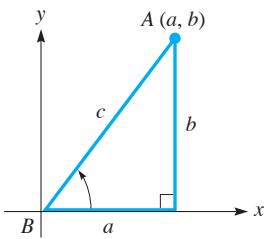


Fig. 4.27

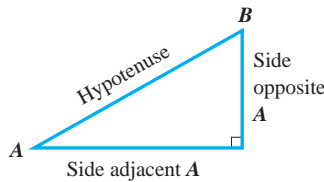


Fig. 4.28

**LEARNING TIP**

Which side is adjacent or opposite depends on the angle being considered. In Fig. 4.28, the side opposite A is adjacent to B, and the side adjacent to A is opposite B.

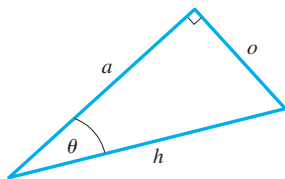


Fig. 4.29

necessary to know one side and one other part. Also, since the sum of the three angles is  $180^\circ$ , we know that the sum of the other two angles is  $90^\circ$ , and they are acute angles. It also means they are **complementary angles**, following the definition in Section 2.1.

For consistency, when we are labeling the parts of the right triangle, we will use the letters  $A$  and  $B$  to denote the acute angles and  $C$  to denote the right angle. The letters  $a$ ,  $b$ , and  $c$  will denote the sides opposite these angles, respectively. Thus, side  $c$  is the hypotenuse of the right triangle. See Fig. 4.25.

In solving right triangles, we will find it convenient to express the trigonometric functions of the acute angles in terms of the sides. By placing the vertex of angle  $A$  at the origin and the vertex of right angle  $C$  on the positive  $x$ -axis, as shown in Fig. 4.26, we have the following ratios for angle  $A$  in terms of the sides of the triangle.

$$\begin{aligned} \sin A &= \frac{a}{c} & \cos A &= \frac{b}{c} & \tan A &= \frac{a}{b} \\ \cot A &= \frac{b}{a} & \sec A &= \frac{c}{b} & \csc A &= \frac{c}{a} \end{aligned} \tag{4.3}$$

If we should place the vertex of  $B$  at the origin, instead of the vertex of angle  $A$ , we would obtain the following ratios for the functions of angle  $B$  (see Fig. 4.27):

$$\begin{aligned} \sin B &= \frac{b}{c} & \cos B &= \frac{a}{c} & \tan B &= \frac{b}{a} \\ \cot B &= \frac{a}{b} & \sec B &= \frac{c}{a} & \csc B &= \frac{c}{b} \end{aligned} \tag{4.4}$$

Eqs. (4.3) and (4.4) show that we may generalize our definitions of the trigonometric functions of an acute angle of a right triangle (we have chosen  $\angle A$  in Fig. 4.28) to be as follows:

$$\begin{aligned} \sin A &= \frac{\text{side opposite } A}{\text{hypotenuse}} & \csc A &= \frac{\text{hypotenuse}}{\text{side opposite } A} \\ \cos A &= \frac{\text{side adjacent } A}{\text{hypotenuse}} & \sec A &= \frac{\text{hypotenuse}}{\text{side adjacent } A} \\ \tan A &= \frac{\text{side opposite } A}{\text{side adjacent } A} & \cot A &= \frac{\text{side adjacent } A}{\text{side opposite } A} \end{aligned} \tag{4.5}$$

Using the definitions in this form, we can solve right triangles without placing the angle in standard position. The angle need only be a part of any right triangle.

In general, regardless of its orientation, any triangle having a right angle and a specified angle of interest  $\theta$  can have its sides and angles related using the trigonometric function definitions. In Fig. 4.29, if we let the side adjacent to angle  $\theta$  be  $a$ , the side opposite the angle  $\theta$  be  $o$ , and the hypotenuse of the right triangle be  $h$ , then we have the following general trigonometric functions:

$$\begin{aligned} \sin \theta &= o/h & \csc \theta &= h/o \\ \cos \theta &= a/h & \sec \theta &= h/a \\ \tan \theta &= o/a & \cot \theta &= a/o \end{aligned} \tag{4.6}$$



**LEARNING TIP**

The first three basic trigonometric relations on the left-hand side of Eq. (4.6) are often remembered with a useful mnemonic, **soh cah toa**, where each leading letter in the triplet corresponds to the trigonometric function name (sine, cosine, and tangent, respectively), and the other two letters in each triplet correspond to the names of the sides in the ratio that defines each function, i.e., **soh** means  $\sin \theta = o/h$ .

We note from the above discussion of Fig. 4.28 that  $\sin A = \cos B$ ,  $\tan A = \cot B$ , and  $\sec A = \csc B$ . From this, we conclude that *cofunctions of acute complementary angles are equal*. The sine function and cosine function are cofunctions, the tangent function and cotangent function are cofunctions, and the secant function and cosecant function are cofunctions.

**EXAMPLE 2 Cofunctions of complementary angles**

Given  $a = 4$ ,  $b = 7$ , and  $c = \sqrt{65}$  (see Fig. 4.30), find  $\sin A$ ,  $\cos A$ ,  $\tan A$ ,  $\sin B$ ,  $\cos B$ , and  $\tan B$  in exact form and in approximate decimal form (to three significant digits).

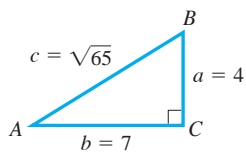


Fig. 4.30

$$\begin{aligned} \sin A &= \frac{\text{side opposite angle } A}{\text{hypotenuse}} = \frac{4}{\sqrt{65}} = 0.496 & \sin B &= \frac{\text{side opposite angle } B}{\text{hypotenuse}} = \frac{7}{\sqrt{65}} = 0.868 \\ \cos A &= \frac{\text{side adjacent angle } A}{\text{hypotenuse}} = \frac{7}{\sqrt{65}} = 0.868 & \cos B &= \frac{\text{side adjacent angle } B}{\text{hypotenuse}} = \frac{4}{\sqrt{65}} = 0.496 \\ \tan A &= \frac{\text{side opposite angle } A}{\text{side adjacent angle } A} = \frac{4}{7} = 0.571 & \tan B &= \frac{\text{side opposite angle } B}{\text{side adjacent angle } B} = \frac{7}{4} = 1.75 \end{aligned}$$

We see that  $A$  and  $B$  are complementary angles. Comparing values of the functions of angles  $A$  and  $B$ , we see that  $\sin A = \cos B$  and  $\cos A = \sin B$ . ■

We are now ready to solve right triangles. See how the following procedure is used in the examples that follow.

**Procedure for Solving a Right Triangle**

1. Sketch a right triangle and label the known and unknown sides and angles.
2. Express each of the three unknown parts in terms of the known parts and solve for the unknown parts.
3. Check the results. The sum of the angles should be  $180^\circ$ . If only one side is given, check the computed side with the Pythagorean theorem. If two sides are given, check the angles and computed side by using appropriate trigonometric functions.

**EXAMPLE 3 Given angle and side—find other parts**

Solve the right triangle with  $A = 50.0^\circ$  and  $b = 6.70$ .

We first sketch the right triangle shown in Fig. 4.31. (In making the sketch, we should be careful to follow proper labelling of the triangle as outlined on page 127.) We then express unknown side  $a$  in terms of known side  $b$  and known angle  $A$  and solve for  $a$ . We will then do the same for unknown side  $c$  and unknown angle  $B$ .

Finding side  $a$ , we know that  $\tan A = \frac{a}{b}$ , which means that  $a = b \tan A$ . Thus,

$$a = 6.70 \tan 50.0^\circ = 7.98$$

Next, solving for side  $c$ , we have  $\cos A = \frac{b}{c}$ , which means  $c = \frac{b}{\cos A}$ .

$$c = \frac{6.70}{\cos 50.0^\circ} = 10.4$$

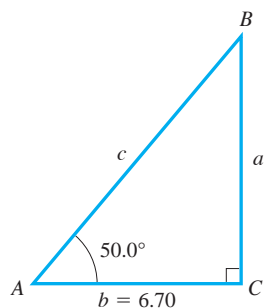


Fig. 4.31

**Practice Exercise**

1. In a right triangle, find  $a$  if  $B = 20.0^\circ$  and  $c = 8.50$ .

**LEARNING TIP**

In finding the unknown parts, we first expressed them in terms of the known parts. We do this because *it is best to use given values in calculations*. If we use one computed value to find another computed value, any error in the first would be carried to the value of the second. For instance, in Example 3, if we were to find the value of  $c$  by using the value of  $a$ , any error in  $a$  would cause  $c$  to be in error as well.

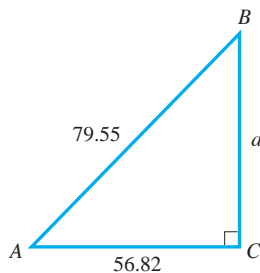


Fig. 4.32

**Practice Exercise**

2. In a right triangle, find  $B$  if  $a = 20.0$  and  $b = 28.0$ .

Now, solving for  $B$ , we know that  $A + B = 90^\circ$ , or

$$B = 90^\circ - A$$

$$B = 90^\circ - 50.0^\circ = 40.0^\circ$$

Therefore,  $a = 7.98$ ,  $c = 10.4$ , and  $B = 40.0^\circ$ .

Checking the angles:  $A + B + C = 50.0^\circ + 40.0^\circ + 90^\circ = 180^\circ$

Checking the sides:  $c^2 = a^2 + b^2$

$$10.4^2 = 7.98^2 + 6.70^2$$

$$108.16 \approx 108.57$$

Since the computed values were rounded off, the values 108.16 and 108.57 show that the values for sides  $a$  and  $c$  check. As we calculate the values of the unknown parts, if we store each in the calculator memory, we can get a better check of the solution. ■

We should also point out that, by inspection, we can make a rough check on the sides and angles of any triangle.

*The longest side is always opposite the largest angle, and the shortest side is always opposite the smallest angle.*

In a right triangle, *the hypotenuse is always the longest side*. We see that this is true for the sides and angles for the triangle in Example 3, where  $c$  is the longest side (opposite the  $90^\circ$  angle) and  $b$  is the shortest side and is opposite the angle of  $40^\circ$ .

**EXAMPLE 4 Given two sides, find other parts**

Solve the right triangle with  $b = 56.82$  and  $c = 79.55$ .

We sketch the right triangle as shown in Fig. 4.32. Since two sides are given, we will use the Pythagorean theorem to find the third side  $a$ . Also, we will use the cosine to find  $\angle A$ .

Since  $c^2 = a^2 + b^2$ ,

$$a^2 = c^2 - b^2$$

$$a = \sqrt{c^2 - b^2}$$

$$a = \sqrt{79.55^2 - 56.82^2}$$

$$a = 55.67$$

Since  $\cos A = \frac{b}{c}$ , we have

$$\cos A = \frac{56.82}{79.55}$$

$$A = \cos^{-1}\left(\frac{56.82}{79.55}\right)$$

$$A = 44.42^\circ$$

It is not necessary to actually calculate the ratio  $56.82/79.55$ . In the same way, we find the value of angle  $B$ :

$$\sin B = \frac{56.82}{79.55}$$

$$B = \sin^{-1}\left(\frac{56.82}{79.55}\right)$$

$$B = 45.58^\circ$$

Although we used a different function, we did use exactly the same ratio to find  $B$  as we used to find  $A$ . Therefore, many texts would find  $B$  from the fact that  $A + B = 90^\circ$ , or  $B = 90^\circ - A = 90^\circ - 44.42^\circ = 45.58^\circ$ . This is also acceptable since any possible error should be discovered when the solution is checked.

We have now found that

$$a = 55.67 \quad A = 44.42^\circ \quad B = 45.58^\circ$$

Checking the sides and angles, we first note that side  $a$  is the shortest side and is opposite the smallest angle,  $\angle A$ . Also, the hypotenuse is the longest side. Next, using the sine function (we could use the cosine or tangent) to check the sides, we have

$$\begin{aligned} \sin 44.42^\circ &= \frac{55.67}{79.55}, \text{ or} & \sin 45.58^\circ &= \frac{56.82}{79.55}, \text{ or} \\ 0.6999 &\approx 0.6998 & 0.7142 &\approx 0.7143 \end{aligned}$$

This shows that the values check. As we noted at the end of Example 3, we would get a more accurate check if we save the calculator values as they are found and use them for the check. ■

■ As we noted earlier, the symbol  $\approx$  means “equals approximately.”

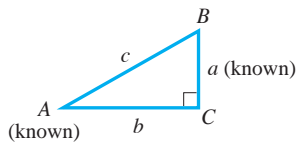


Fig. 4.33

**EXAMPLE 5** Unknown parts in terms of known parts

If  $A$  and  $a$  are known, express the unknown parts of a right triangle in terms of  $A$  and  $a$ . We sketch a right triangle as in Fig. 4.33, and then set up the required expressions.

Since  $\frac{a}{b} = \tan A$ , we have  $a = b \tan A$ , or  $b = \frac{a}{\tan A}$ .

Since  $\frac{a}{c} = \sin A$ , we have  $a = c \sin A$ , or  $c = \frac{a}{\sin A}$ .

Since  $A$  is known,  $B = 90^\circ - A$ . ■

**EXERCISES 4.4**

In Exercises 1–4, make the given changes in the indicated examples of this section and then find the indicated values.

1. In Example 2, interchange the values for  $a$  and  $b$  and then find the values.
2. In Example 3, change 6.70 to 7.60 and then solve the triangle.
3. In Example 4, change 56.82 to 65.82 and then solve the triangle.
4. In Example 5, in line 1, change  $a$  to  $b$  and then find the expressions for the unknown parts.

In Exercises 5–8, draw appropriate figures and verify through observation that only one triangle may contain the given parts (that is, any others which may be drawn will be congruent).

5. A  $60^\circ$  angle included between sides of 3 cm and 6 cm
6. A side of 4 cm included between angles of  $40^\circ$  and  $50^\circ$
7. A right triangle with a hypotenuse of 5 cm and a leg of 3 cm
8. A right triangle with a  $70^\circ$  angle between the hypotenuse and a leg of 5 cm

In Exercises 9–32, solve the right triangles with the given parts. Round off results. Refer to Fig. 4.34.

9.  $A = 77.8^\circ, a = 6700$
10.  $A = 18.4^\circ, c = 0.0897$
11.  $a = 153, c = 345$
12.  $a = 932, c = 1240$
13.  $B = 32.1^\circ, c = 23.8$
14.  $B = 64.3^\circ, b = 0.652$

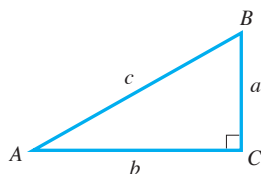


Fig. 4.34

- |                                   |                                      |
|-----------------------------------|--------------------------------------|
| 15. $b = 82, c = 88$              | 16. $a = 5920, b = 4110$             |
| 17. $A = 32.10^\circ, c = 56.85$  | 18. $B = 12.60^\circ, c = 18.42$     |
| 19. $a = 56.73, b = 44.09$        | 20. $a = 9.908, c = 12.63$           |
| 21. $B = 37.5^\circ, a = 0.862$   | 22. $A = 87.25^\circ, b = 8.450$     |
| 23. $B = 74.18^\circ, b = 1.849$  | 24. $A = 51.36^\circ, a = 3692$      |
| 25. $a = 591.87, b = 264.93$      | 26. $b = 2.9507, c = 5.0864$         |
| 27. $A = 2.975^\circ, b = 14.592$ | 28. $B = 84.942^\circ, a = 7413.5$   |
| 29. $B = 9.56^\circ, c = 0.0973$  | 30. $a = 1.28, b = 16.3$             |
| 31. $a = 35.0, C = 90.0^\circ$    | 32. $A = 25.7^\circ, B = 64.3^\circ$ |

In Exercises 33–36, find the part of the triangle labelled either  $x$  or  $A$  in the indicated figure.

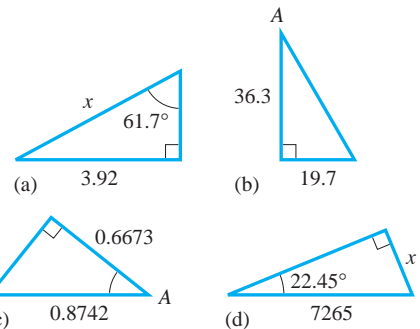


Fig. 4.35

- |                  |                  |
|------------------|------------------|
| 33. Fig. 4.35(a) | 34. Fig. 4.35(b) |
| 35. Fig. 4.35(c) | 36. Fig. 4.35(d) |

In Exercises 37–40, find the indicated part of the right triangle that has the given parts.

37. One leg is 25.6, and the hypotenuse is 37.5. Find the smaller acute angle.  
 38. One leg is 8.50, and the angle opposite this leg is  $52.3^\circ$ . Find the other leg.  
 39. The hypotenuse is 827, and one angle is  $17.6^\circ$ . Find the longer leg.  
 40. The legs are 0.596 and 0.842. Find the larger acute angle.

In Exercises 41–44, refer to Fig. 4.34. In Exercises 41–43, the listed parts are assumed known. Express the other parts in terms of the known parts.

41.  $A, c$                       42.  $a, b$                       43.  $B, a$   
 44. In Fig. 4.34, is there any combination of two given parts (not including  $C$ ) that does not give a unique solution of the triangle? Explain.

#### Answers to Practice Exercises

1.  $a = 7.99$     2.  $54.5^\circ$

## 4.5 Applications of Right Triangles

### Angles of Elevation and Depression • Indirect Measurements of Distances and Angles

■ See the chapter introduction.

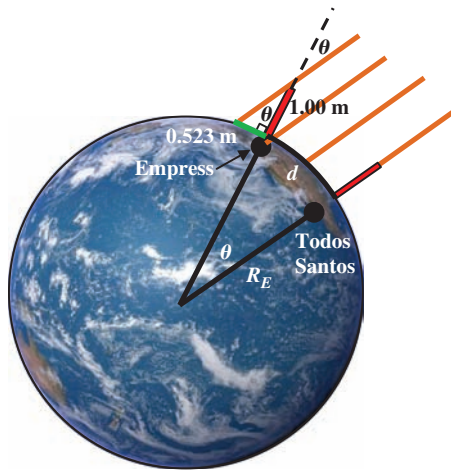


Fig. 4.36

Many problems in science, technology, and everyday life can be solved by finding the missing parts of a right triangle. In this section, we illustrate a number of these in the examples and exercises.

#### EXAMPLE 1 Determining the circumference and radius of earth

During the summer solstice at noon, the sun is directly straight up in the sky at locations on earth that are at the Tropic of Cancer (latitude approximately  $23^\circ 26' N$ ). Therefore, any vertical stick will not cast a shadow. Assuming a spherical earth, at any location north of the Tropic of Cancer, the sun in the sky at noon will be at a position closer to the horizon, thereby causing the stick to cast a shadow (see Fig. 4.36). Since the sun's rays striking earth at any location are essentially parallel, the length of the shadow of a vertical stick can be used to calculate the circumference and radius of earth. Todos Santos, Mexico, lies on the tropic of Cancer, and 3060 km directly north is the village of Empress, Alberta. A person in Empress, Alberta, takes a 1.00 m long ruler and orients it vertically at noon on the summer solstice. The shadow that it casts is 52.3 cm long. Determine the circumference and radius of earth.

The sun's rays are parallel and lie along the same direction as the vertical stick at Todos Santos. At Empress, a vertical stick makes an angle  $\theta$  with respect to the sun's rays as shown. The angle  $\theta$  is a corresponding angle to the angle subtended by the two towns with vertex at the centre of earth. From the shadow that is cast,

$$\begin{aligned}\tan \theta &= \frac{0.523 \text{ m}}{1.00 \text{ m}} \\ \theta &= \tan^{-1}(0.523) && \text{tangent of required angle} = \frac{\text{given opposite side}}{\text{given adjacent side}} \\ \theta &= 27.6^\circ \times \frac{\pi \text{ rad}}{180^\circ} \\ \theta &= 0.482 \text{ rad}\end{aligned}$$

To find earth's radius  $R_E$ , we use Eq. (2.12), which states that any distance formed along a circular arc if the angle is in radians is

$$\begin{aligned}s &= \theta r \\ d &= \theta R_E \\ R_E &= d/\theta \\ R_E &= 3060 \text{ km}/0.482 \\ R_E &= 6350 \text{ km}\end{aligned}$$

To find earth's circumference  $c_E$ , we use Eq. (2.9),

$$\begin{aligned}c &= 2\pi r \\ c_E &= 2\pi R_E\end{aligned}$$

$$c_E = 2\pi(6350 \text{ km})$$

$$c_E = 39\,900 \text{ km}$$

Both values are close to the average radius of earth, 6370 km, and circumference, 40 000 km. ■

■ See the chapter introduction.

### EXAMPLE 2 Angle of elevation

Horseshoe Falls on the Canadian side of Niagara Falls can be seen from a small boat 760 m downstream. The **angle of elevation** (the angle between the horizontal and the line of sight, when the object is above the horizontal) from the observer to the top of Horseshoe Falls is  $4.0^\circ$ . How high are the falls?

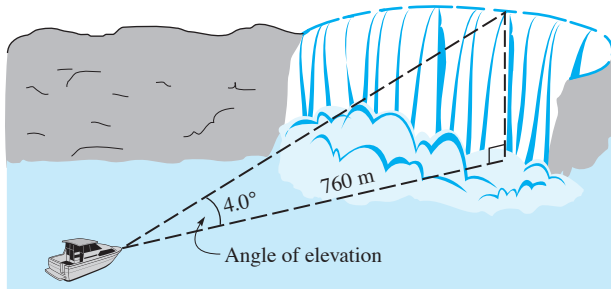


Fig. 4.37

By drawing an appropriate figure, as shown in Fig. 4.37, we show the given information and what we are to find. Here, we let  $h$  be the height of the falls and label the horizontal distance from the boat to the base of the falls as 760 m. From the figure, we see that

$$\frac{h}{760} = \tan 4.0^\circ \quad \frac{\text{required opposite side}}{\text{given adjacent side}} = \text{tangent of given angle}$$

$$h = 760 \tan 4.0^\circ$$

$$h = 53 \text{ m}$$

We have rounded off the result since the data are good only to two significant digits. (Niagara Falls is on the border between the United States and Canada and is divided into the American Falls and the Horseshoe Falls. About 500 000 tonnes of water flow over the falls each minute.) ■

### EXAMPLE 3 Angle of depression

A blimp is 565 m above the ground and south of Olympic Stadium in Montreal during a Grey Cup game. The **angle of depression** (the angle between the horizontal and the line of sight, when the object is below the horizontal) of the north goal line from the blimp is  $58.5^\circ$ . How far is the observer in the blimp from the goal line?

Again, we sketch a figure as shown in Fig. 4.38. Here, we let  $d$  be the distance between the blimp and the north goal line. From the figure, we see that

$$\frac{565}{d} = \sin 58.5^\circ \quad \frac{\text{given opposite side}}{\text{required hypotenuse}} = \text{sine of known angle}$$

$$d = \frac{565}{\sin 58.5^\circ}$$

$$d = 663 \text{ m}$$

In this case, we have rounded off the result to three significant digits, which is the accuracy of the given information. (The Grey Cup games in 1977, 1979, 1981, 1985, 2001, and 2008 were played in the Olympic Stadium.) ■

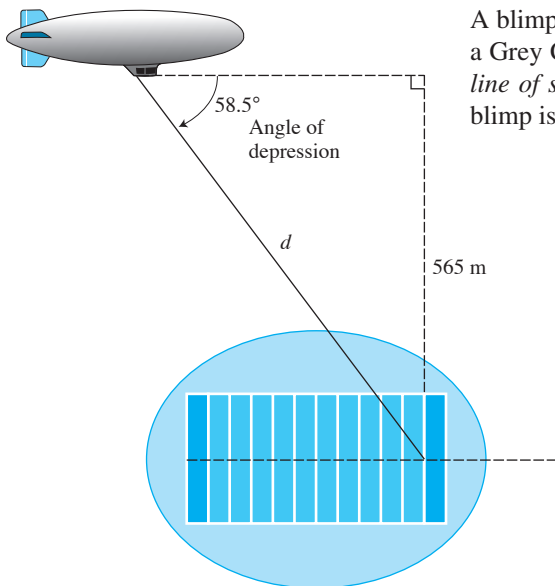


Fig. 4.38

Carefully note the difference between the angle of elevation and the angle of depression. The angle of elevation is the angle through which an object is observed by *elevating* the line of sight above the horizontal. The angle of depression is the angle through which the object is observed by *depressing* (lowering) the line of sight below the horizontal.

### EXAMPLE 4 Height of a missile

A missile is launched at an angle of  $26.55^\circ$  with respect to the horizontal. If it travels in a straight line over level terrain for 2.000 min and its average speed is 6355 km/h, what is its altitude at this time?

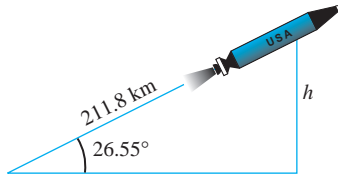


Fig. 4.39

In Fig. 4.39, we let  $h$  represent the altitude of the missile after 2.000 min (altitude is measured on a perpendicular line). Also, we determine that in this time the missile has flown 211.8 km in a direct line from the launching site. This is found from the fact that it travels at 6355 km/h for 2.000 min  $\left(\frac{1 \text{ h}}{60 \text{ min}}\right)$  and distance = speed  $\times$  time. We

therefore have  $\left(6355 \frac{\text{km}}{\text{h}}\right)\left(\frac{1}{30.00} \text{ h}\right) = 211.8 \text{ km}$ . This means

$$\frac{h}{211.8} = \sin 26.55^\circ \quad \frac{\text{required opposite side}}{\text{known hypotenuse}} = \text{sine of given angle}$$

$$h = 211.8 (\sin 26.55^\circ)$$

$$h = 94.67 \text{ km}$$

### EXAMPLE 5 Measurement of an angle

A driver coming to an intersection sees the word STOP in the roadway. From the measurements shown in Fig. 4.40, find the angle  $\theta$  that the letters make at the driver's eye.

From the figure, we know sides  $BS$  and  $BE$  in triangle  $BES$  and sides  $BT$  and  $BE$  in triangle  $BET$ . This means we can find  $\angle TEB$  and  $\angle SEB$  by use of the tangent. We then find  $\theta$  from the fact that  $\theta = \angle TEB - \angle SEB$ .

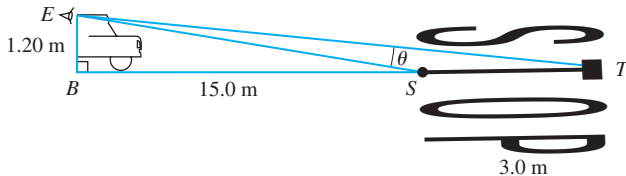


Fig. 4.40

$$\tan \angle TEB = \frac{18.0}{1.20}, \quad \angle TEB = 86.2^\circ$$

$$\tan \angle SEB = \frac{15.0}{1.20}, \quad \angle SEB = 85.4^\circ$$

$$\theta = 86.2^\circ - 85.4^\circ = 0.8^\circ$$

### EXAMPLE 6 Surveyor—indirect measurement

Using lasers, a surveyor makes the measurements shown in Fig. 4.41, where points  $B$  and  $C$  are in a marsh. Find the distance between  $B$  and  $C$ .

Since the distance  $BC = AC - AB$ ,  $BC$  is found by finding  $AC$  and  $AB$  and subtracting:

$$\frac{AB}{265.74} = \tan 21.66^\circ$$

$$AB = 265.74 \tan 21.66^\circ$$

$$\frac{AC}{265.74} = \tan (21.66^\circ + 8.85^\circ)$$

$$AC = 265.74 \tan 30.51^\circ$$

$$BC = AC - AB = 265.74 \tan 30.51^\circ - 265.74 \tan 21.66^\circ$$

$$BC = 51.06 \text{ m}$$

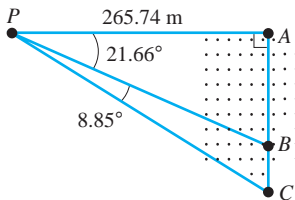


Fig. 4.41

Lasers were first produced in the late 1950s.

The indirect measurement of distances, such as the one illustrated in Examples 1 and 6, has been one of the most useful applications of trigonometry. Methods of indirect measurement have also been used in many fields, such as surveying and navigation.

## EXERCISES 4.5

In Exercises 1 and 2, make the given changes in the indicated examples of this section and then find the indicated values.

- In Example 3, in line 4, change  $58.5^\circ$  to  $62.1^\circ$  and then find the distance.
- In Example 4, in line 2, change 2.000 min to 3.000 min and then find the altitude.

In Exercises 3–40, solve the given problems. Sketch an appropriate figure, unless the figure is given.

- A straight 122-m culvert is built down a hillside that makes an angle of  $54.0^\circ$  with the horizontal. Find the height of the hill.



4. In 2000, about 70 tonnes of soil were removed from under the Leaning Tower of Pisa, and the angle the tower made with the ground was increased by about  $0.5^\circ$ . Before that, a point near the top of the tower was 50.5 m from a point at the base (measured along the tower), and this top point was directly above a point on the ground 4.25 m from the same base point. See Fig. 4.42. How much did the point on the ground move toward the base point?

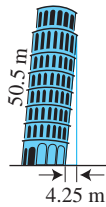


Fig. 4.42

5. A tree has a shadow 22.8 m long when the angle of elevation of the sun is  $62.6^\circ$ . How tall is the tree?
6. The straight arm of a robot is 1.25 m long and makes an angle of  $13.0^\circ$  above a horizontal conveyor belt. How high above the belt is the end of the arm? See Fig. 4.43.

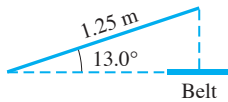


Fig. 4.43

7. The headlights of an automobile are set such that the beam drops 5.10 cm for each 7.50 m in front of the car. What is the angle between the beam and the road?
8. A bullet was fired such that it just grazed the top of a table. It entered a wall, which is 3.84 m from the graze point in the table, at a point 1.41 m above the tabletop. At what angle was the bullet fired above the horizontal? See Fig. 4.44.

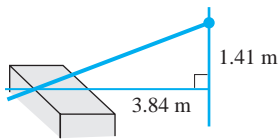


Fig. 4.44

9. A robot is on the surface of Mars. The angle of depression from a camera in the robot to a rock on the surface of Mars is  $13.33^\circ$ . The camera is 196.0 cm above the surface. How far from the camera is the rock?
10. The CN Tower in Toronto can be seen from a point on the ground known to be 1600 m away from the base of the tower. The angle of elevation from the observer to the top of the tower is  $19^\circ$ . How high is the CN Tower? See Fig. 4.45.

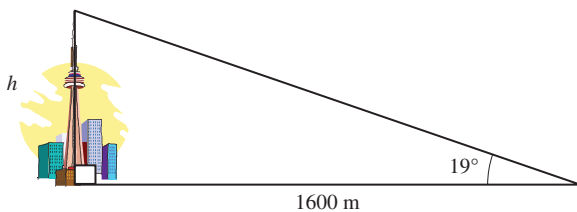


Fig. 4.45

11. In the design of a new building, a doorway is 795 mm above the ground. A ramp for the disabled, at an angle of  $6.00^\circ$  with the ground, is to be built to the doorway. How long will the ramp be?
12. On a test flight, during the landing of the space shuttle, the ship was 105 m above the end of the landing strip. It then came in on a constant angle of  $7.50^\circ$  with the landing strip. How far from the end of the landing strip did it first touch ground?

13. From the southernmost point in Sleeping Giant Provincial Park in Thunder Bay, Ontario, it is found that the angle of elevation of the highest point on the Sleeping Giant is  $3.9^\circ$ . If the horizontal distance between the points is 5.5 km, how much higher is the point on the summit?
14. What is the steepest angle between the surface of a board 3.50 cm thick and a nail 5.00 cm long if the nail is hammered into the board such that it does not go through?
15. A rectangular piece of plywood 1200 mm by 2400 mm is cut from one corner to an opposite corner. What are the angles between edges of the resulting pieces?
16. A guardrail is to be constructed around the top of a circular observation tower. The diameter of the observation area is 12.3 m. If the railing is constructed with 30 equal straight sections, what should be the length of each section?
17. The angle of inclination of a road is often expressed as *percent grade*, which is the vertical rise divided by the horizontal run (expressed as a percent). See Fig. 4.46. A 6.0% grade corresponds to a road that rises 6.0 m for every 100 m along the horizontal. Find the angle of inclination that corresponds to a 6.0% grade.



Fig. 4.46

18. A tabletop is in the shape of a regular octagon (eight sides). What is the greatest distance across the table if one side of the octagon is 0.750 m?
19. To get a good view of a person in front of a teller's window, it is determined that a surveillance camera at a bank should be directed at a point 5.17 m to the right and 2.25 m below the camera. See Fig. 4.47. At what angle of depression should the camera be directed?

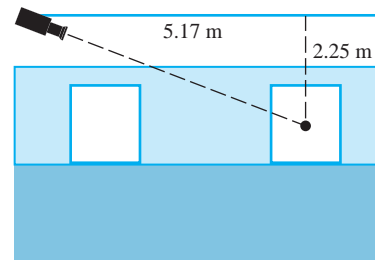


Fig. 4.47

20. A street light is designed as shown in Fig. 4.48. How high above the street is the light?

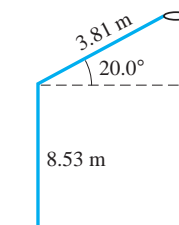


Fig. 4.48

21. A straight driveway is 85.0 m long, and the top is 12.0 m above the bottom. What angle does it make with the horizontal?
22. Part of the Tower Bridge in London is a drawbridge. This part of the bridge is 76.0 m long. When each half is raised, the distance between them is 8.0 m. What angle does each half make with the horizontal? See Fig. 4.49.

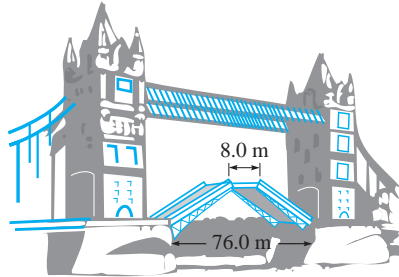


Fig. 4.49

23. A square wire loop is rotating in the magnetic field between two poles of a magnet in order to induce an electric current. The axis of rotation passes through the centre of the loop and is midway between the poles, as shown in the side view in Fig. 4.50. How far is the edge of the loop from either pole if the side of the square is 7.30 cm and the poles are 7.66 cm apart when the angle between the loop and the vertical is  $78.0^\circ$ ?

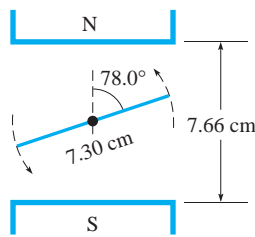


Fig. 4.50

24. From a space probe circling Io, one of Jupiter's moons, at an altitude of 552 km, it was observed that the angle of depression of the horizon was  $39.7^\circ$ . What is the radius of Io?
25. A manufacturing plant is designed to be in the shape of a regular pentagon with 92.5 m on each side. A security fence surrounds the building to form a circle, and each corner of the building is to be 25.0 m from the closest point on the fence. How much fencing is required?
26. A traveller on a boat along the Saguenay River in Quebec wishes to estimate the height of the cliff at Cape Trinity. Fig. 4.51 shows the angle measurements she made at points 250 m apart. How high is the cliff? (In the figure, the triangle containing the height  $h$  is vertical and perpendicular to the path followed by the boat. The height of the boat can be ignored.)

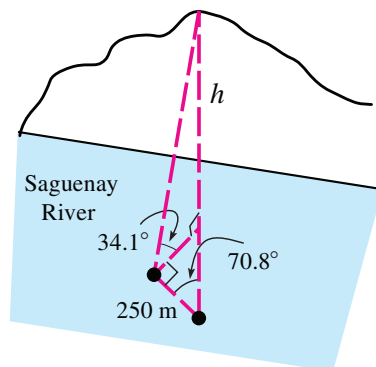


Fig. 4.51

27. Find the angle  $\theta$  in the taper shown in Fig. 4.52. (The front face is an isosceles trapezoid.)
28. What is the circumference of the Arctic Circle (latitude  $66^\circ 32' N$ )? The radius of the earth is 6370 km.
29. A draftsman sets the legs of a pair of dividers so that the angle between them is  $32.0^\circ$ . If each leg is 11.4 cm long, what is the distance between their ends where they touch the paper?

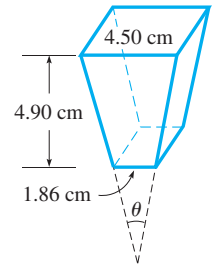


Fig. 4.52

30. The ratio of the width to the height of an HDTV screen is 16 to 9. What is the angle between the width and a diagonal of the screen to the nearest  $0.1^\circ$ ?
31. A stairway 1.0 m wide goes from the bottom of a cylindrical storage tank to the top at a point halfway around the tank. The hand-rail on the outside of the stairway makes an angle of  $31.8^\circ$  with the horizontal, and the radius of the tank is 11.8 m. Find the length of the handrail. See Fig. 4.53.

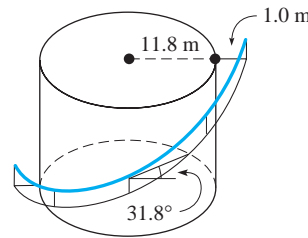


Fig. 4.53

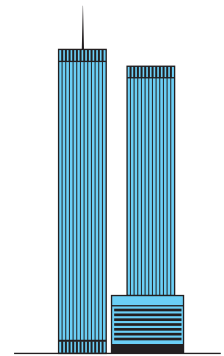


Fig. 4.54

32. An antenna was on the top of the World Trade Center (before it was destroyed in 2001). From a point on the river 2400 m from the Center, the angles of elevation of the top and bottom of the antenna were  $12.1^\circ$  and  $9.9^\circ$ , respectively. How tall was the antenna? (Disregard the small part of the antenna near the base that could not be seen.) The former World Trade Center is shown in Fig. 4.54. (This problem is included in memory of those who suffered and died as a result of the terrorist attack of September 11, 2001.)
33. Some of the streets of Montreal, Quebec, are shown in Fig. 4.55. (a) How far is it between intersections  $B$  and  $C$ ? (b) How far is it between intersections  $C$  and  $D$ ?

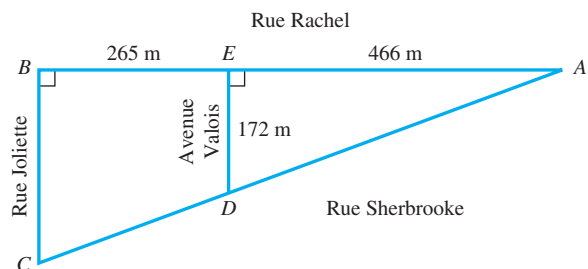


Fig. 4.55

34. A supporting girder structure is shown in Fig. 4.56. Find the length  $x$ .

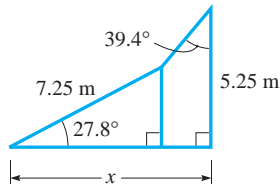


Fig. 4.56

35. The diameter  $d$  of a pipe can be determined by noting the distance  $x$  on the V-gauge shown in Fig. 4.57. Points  $A$  and  $B$  indicate where the pipe touches the gauge, and  $x$  equals either  $AV$  or  $VB$ . Find a formula for  $d$  in terms of  $x$  and  $\theta$ .

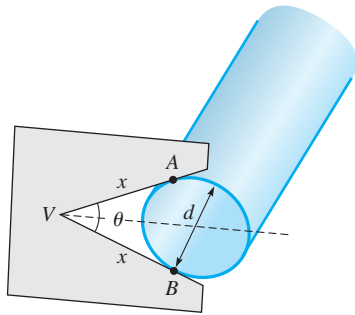


Fig. 4.57

36. The political banner shown in Fig. 4.58 is in the shape of a parallelogram. Find its area.

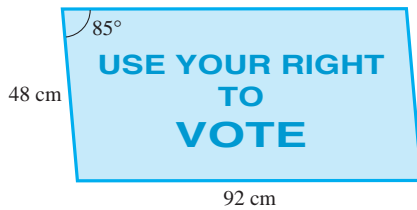


Fig. 4.58

37. Find a formula for the area of the trapezoidal aqueduct cross section shown in Fig. 4.59.

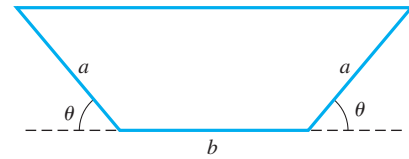


Fig. 4.59

38. A communications satellite is in orbit 35 300 km directly above the earth's equator. What is the greatest latitude  $\phi$  from which a signal can travel from the earth's surface to the satellite in a straight line? The radius of the earth is 6370 km.
39. What is the angle between the base of a cubical glass paper-weight and a diagonal of the cube (from one corner to the opposite corner) to 3 significant digits?
40. Find the angle of view  $\theta$  of the camera lens (see Fig. 4.60), given the measurements shown in the figure.

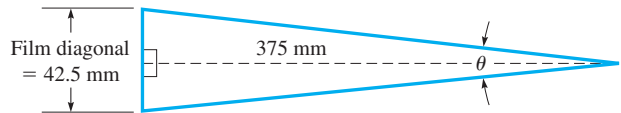
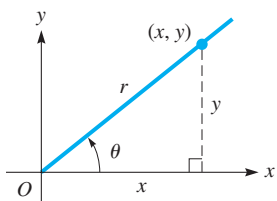


Fig. 4.60

Answer to Practice Exercise

1.  $\theta = 0.5^\circ$

CHAPTER 4 EQUATIONS



sine of  $\theta$ :  $\sin \theta = \frac{y}{r}$

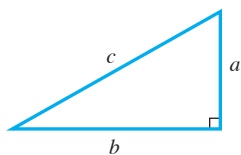
cosecant of  $\theta$ :  $\csc \theta = \frac{1}{\sin \theta} = \frac{r}{y}$

cosine of  $\theta$ :  $\cos \theta = \frac{x}{r}$

secant of  $\theta$ :  $\sec \theta = \frac{1}{\cos \theta} = \frac{r}{x}$  (4.1)

tangent of  $\theta$ :  $\tan \theta = \frac{y}{x}$

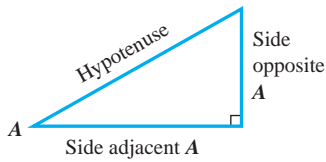
cotangent of  $\theta$ :  $\cot \theta = \frac{1}{\tan \theta} = \frac{x}{y}$



Pythagorean theorem

$c^2 = a^2 + b^2$

(4.2)



$$\sin A = \frac{\text{side opposite } A}{\text{hypotenuse}}$$

$$\csc A = \frac{\text{hypotenuse}}{\text{side opposite } A}$$

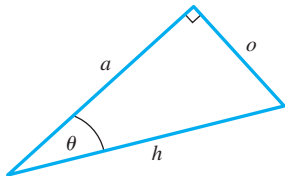
$$\cos A = \frac{\text{side adjacent } A}{\text{hypotenuse}}$$

$$\sec A = \frac{\text{hypotenuse}}{\text{side adjacent } A}$$

$$\tan A = \frac{\text{side opposite } A}{\text{side adjacent } A}$$

$$\cot A = \frac{\text{side adjacent } A}{\text{side opposite } A}$$

(4.5)



$$\sin \theta = o/h \quad \csc \theta = h/o$$

$$\cos \theta = a/h \quad \sec \theta = h/a$$

$$\tan \theta = o/a \quad \cot \theta = a/o$$

(4.6)

## CHAPTER 4 REVIEW EXERCISES

In Exercises 1–4, find the smallest positive angle and the smallest negative angle (numerically) coterminal with but not equal to the given angle.

1.  $17.0^\circ$     2.  $248.3^\circ$     3.  $-217.5^\circ$     4.  $-7.6^\circ$

In Exercises 5–8, express the given angles in decimal form (to the nearest  $0.01^\circ$ ).

5.  $31^\circ 54'$     6.  $574^\circ 45'$     7.  $-38^\circ 6'$     8.  $321^\circ 27'$

In Exercises 9–12, express the given angles to the nearest minute.

9.  $17.5^\circ$     10.  $-65.4^\circ$     11.  $749.7^\circ$     12.  $126.25^\circ$

In Exercises 13–16, determine the trigonometric functions of the angles (in standard position) whose terminal side passes through the given points. Give answers in exact form.

13.  $(24, 7)$     14.  $(5, 4)$     15.  $(48, 48)$     16.  $(1.2, 0.5)$

In Exercises 17–20, find the indicated trigonometric functions. Give answers in decimal form, rounded off to three significant digits.

17. Given  $\sin \theta = \frac{5}{13}$ , find  $\cos \theta$  and  $\cot \theta$ .  
 18. Given  $\cos \theta = \frac{3}{8}$ , find  $\sin \theta$  and  $\tan \theta$ .  
 19. Given  $\tan \theta = 2$ , find  $\cos \theta$  and  $\csc \theta$ .  
 20. Given  $\cot \theta = 40$ , find  $\sin \theta$  and  $\sec \theta$ .

In Exercises 21–28, find the values of the trigonometric functions. Round off results.

21.  $\sin 72.1^\circ$     22.  $\cos 40.3^\circ$   
 23.  $\tan 88.64^\circ$     24.  $\sin 0.91^\circ$   
 25.  $\sec 18.4^\circ$     26.  $\csc 82.4^\circ$   
 27.  $(\cot 7.06^\circ)(\sin 7.06^\circ) - \cos 7.06^\circ$   
 28.  $(\sec 79.36^\circ)(\sin 79.36^\circ) - \tan 79.36^\circ$

In Exercises 29–40, find  $\theta$  for each of the given trigonometric functions. Round off results.

29.  $\cos \theta = 0.950$     30.  $\sin \theta = 0.63052$   
 31.  $\tan \theta = 1.574$     32.  $\cos \theta = 0.0135$

33.  $\csc \theta = 4.713$

34.  $\cot \theta = 0.7561$

35.  $\sec \theta = 25.4$

36.  $\csc \theta = 1.92$

37.  $\cot \theta = 7.117$

38.  $\sec \theta = 1.006$

39.  $\sin \theta = 0.9998$

40.  $\cos \theta = 1.402$

In Exercises 41–52, solve the right triangles with the given parts. Refer to Fig. 4.61.

41.  $A = 17.0^\circ, b = 6.00$

42.  $B = 68.1^\circ, a = 1080$

43.  $a = 81.0, b = 64.5$

44.  $a = 106, c = 382$

45.  $A = 37.5^\circ, a = 12.0$

46.  $B = 15.7^\circ, c = 126$

47.  $b = 6.508, c = 7.642$

48.  $a = 0.721, b = 0.144$

49.  $A = 49.67^\circ, c = 0.8253$

50.  $B = 4.38^\circ, b = 5682$

51.  $a = 11.652, c = 15.483$

52.  $a = 724.39, b = 852.44$

In Exercises 53–94, solve the given problems.

53. Find the value of  $x$  for the triangle shown in Fig. 4.62.

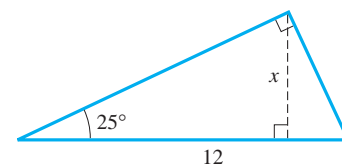


Fig. 4.62

54. Explain three ways in which the value of  $x$  can be found for the triangle shown in Fig. 4.63. Which of these methods is the easiest?

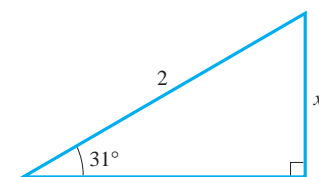


Fig. 4.63

55. Find the perimeter of a regular octagon (eight equal sides with equal interior angles) that is inscribed in a circle (all vertices of the octagon touch the circle) of radius 10.
56. Explain why values of  $\sin \theta$  increase as  $\theta$  increases from  $0^\circ$  to  $90^\circ$ .
57. What is  $x$  if  $(3, 2)$  and  $(x, 7)$  are on the same terminal side of an acute angle?
58. Two legs of a right triangle are 2.607 and 4.517. What is the smaller acute angle?
59. Show that the side  $c$  of any triangle  $ABC$  is related to the perpendicular  $h$  from  $C$  to side  $AB$  by the equation

$$c = h \cot A + h \cot B.$$

60. For the isosceles triangle shown in Fig. 4.64, show that  $c = 2a \sin \frac{A}{2}$ .

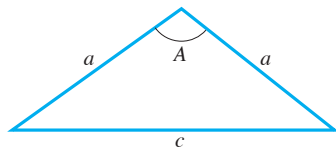


Fig. 4.64

61. If  $x = \tan \theta$ , what is  $\csc \theta$  in terms of  $x$ ?
62. Find the angle between the line passing through the origin and  $(3, 2)$ , and the line passing through the origin and  $(2, 3)$ .
63. A sloped cathedral ceiling is between walls that are 2.50 m high and 4.00 m high. If the walls are 5.00 m apart, at what angle does the ceiling rise?
64. A pendulum 1.25 m long swings through an angle of  $5.60^\circ$ . What is the distance between the extreme positions of the pendulum?
65. The voltage  $E$  at any instant in a coil of wire that is turning in a magnetic field is given by  $E = E_M \cos \alpha$ , where  $E_M$  is the maximum voltage and  $\alpha$  is the angle the coil makes with the field. Find the acute angle  $\alpha$  if  $E = 56.9$  V and  $E_M = 339$  V.
66. A formula for the area of a quadrilateral is  $A = \frac{1}{2}d_1d_2 \sin \theta$ , where  $d_1$  and  $d_2$  are the lengths of the diagonals and  $\theta$  is the angle between them. Find the area of a four-sided carpet remnant with diagonals 1050 mm and 1330 mm and  $\theta = 72.0^\circ$ .
67. For a car rounding a curve, the road should be banked at an angle  $\theta$  according to the equation  $\tan \theta = \frac{v^2}{gr}$ . Here,  $v$  is the speed of the car and  $r$  is the radius of the curve in the road. See Fig. 4.65. Find  $\theta$  for  $v = 24.2$  m/s,  $g = 9.80$  m/s<sup>2</sup>, and  $r = 282$  m.

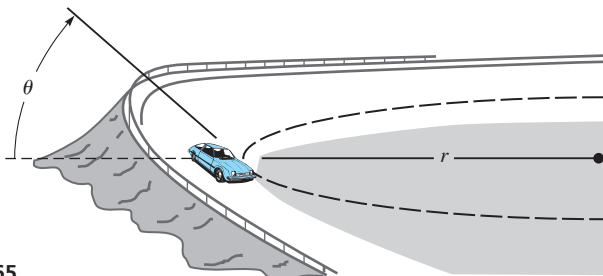


Fig. 4.65

68. The apparent power  $S$  in an electric circuit in which the power is  $P$  and the impedance phase angle is  $\theta$  is given by  $S = P \sec \theta$ . Given  $P = 12.0$  V  $\cdot$  A and  $\theta = 29.4^\circ$ , find  $S$ .

69. A surveyor measures two sides and the included angle of a triangular tract of land to be  $a = 31.96$  m,  $b = 47.25$  m, and  $C = 64.09^\circ$ . (a) Show that a formula for the area  $A$  of the tract is  $A = \frac{1}{2}ab \sin C$ . (b) Find the area of the tract.
70. A water channel has the cross section of an isosceles trapezoid. See Fig. 4.66. (a) Show that a formula for the area of the cross section is  $A = bh + h^2 \cot \theta$ . (b) Find  $A$  if  $b = 12.6$  m,  $h = 4.75$  m, and  $\theta = 37.2^\circ$ .

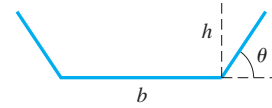


Fig. 4.66

71. In tracking an airplane on radar, it is found that the plane is 27.5 km on a direct line from the control tower, with an angle of elevation of  $10.3^\circ$ . What is the altitude of the plane?
72. A straight emergency chute for an airplane is 5.5 m long. In being tested, the top of the chute is 2.9 m above the ground. What angle does the chute make with the ground?
73. The windshield on an automobile is inclined  $42.5^\circ$  with respect to the horizontal. Assuming that the windshield is flat and rectangular, what is its area if it is 1.50 m wide and the bottom is 0.480 m in front of the top?
74. A water slide at an amusement park is 25.0 m long and is inclined at an angle of  $52.0^\circ$  with the horizontal. How high is the top of the slide above the water level?
75. Find the area of the patio shown in Fig. 4.67.

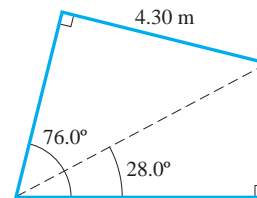


Fig. 4.67

76. The cross section (a regular trapezoid) of a levee to be built along a river is shown in Fig. 4.68. What is the volume of rock and soil that will be needed for a 1-km length of the levee?

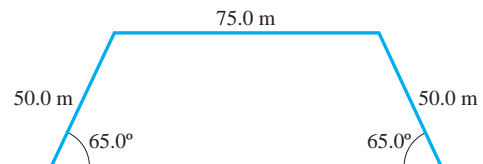


Fig. 4.68

77. The vertical cross section of an attic room in a house is shown in Fig. 4.69. Find the distance  $d$  across the floor.

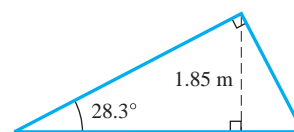


Fig. 4.69

78. The impedance  $Z$  and resistance  $R$  in an ac circuit may be represented by letting the impedance be the hypotenuse of a right triangle and the resistance be the side adjacent to the phase angle  $\theta$ . If  $R = 1750 \Omega$  and  $\theta = 17.38^\circ$ , find  $Z$ .

79. A typical aqueduct built by the Romans dropped on average at an angle of about  $0.030^\circ$  to allow gravity to move the water from the source to the city. For such an aqueduct of 65 km in length, how much higher was the source than the city?
80. The distance from the ground level to the underside of a cloud is called the *ceiling*. See Fig. 4.70. A ground observer 950 m from a searchlight aimed vertically notes that the angle of elevation of the spot of light on a cloud is  $76^\circ$ . What is the ceiling?

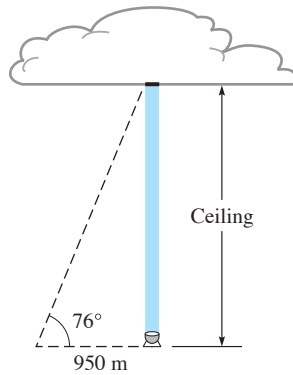


Fig. 4.70

81. The window of a house is shaded as shown in Fig. 4.71. What percent of the window is shaded when the angle of elevation  $\theta$  of the sun is  $65^\circ$ ?

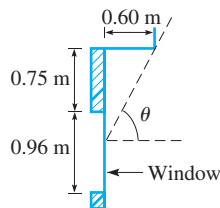


Fig. 4.71

82. A person standing on a level plain hears the sound of a plane, looks in the direction of the sound, but the plane is not there (familiar?). When the sound was heard, it was coming from a point at an angle of elevation of  $25^\circ$ , and the plane was travelling at 720 km/h (201 m/s) at a constant altitude of 850 m along a straight line. If the plane later passes directly over the person, at what angle of elevation should the person have looked directly to see the plane when the sound was heard? (The speed of sound is 340 m/s.) See Fig. 4.72.

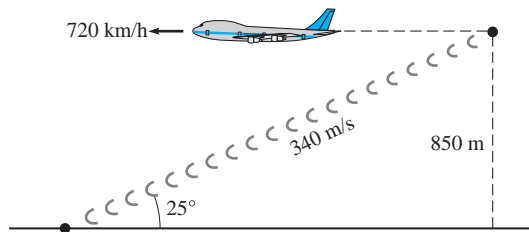


Fig. 4.72

83. In the structural support shown in Fig. 4.73, find  $x$ .

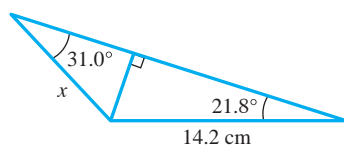


Fig. 4.73

84. A Coast Guard boat 2.75 km from a straight beach can travel at 37.5 km/h. By travelling along a line that is at  $69.0^\circ$  with the beach, how long will it take it to reach the beach? See Fig. 4.74.

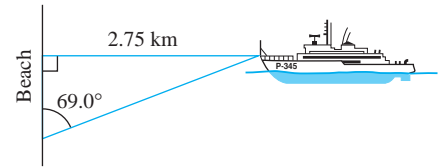


Fig. 4.74

85. One span of the multi-span Thousand Island International Bridge connecting Canada and the United States over the St. Lawrence River is 230 m long (see Fig. 4.75). The angle subtended by the span at the eye of an observer in a helicopter is  $2.2^\circ$ . Show that the distance calculated from the helicopter to the span is about the same if the line of sight is perpendicular to the end or to the middle of the span.

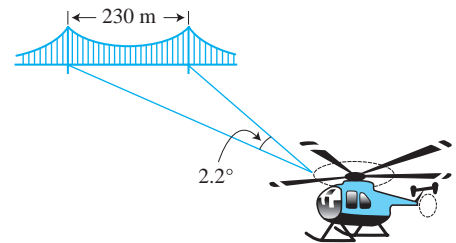


Fig. 4.75

86. Each side piece of the trellis shown in Fig. 4.76 makes an angle of  $80.0^\circ$  with the ground. Find the length of each side piece and the area covered by the trellis.

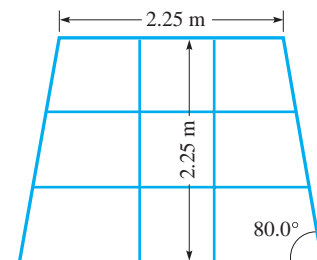


Fig. 4.76

87. A laser beam is transmitted with a "width" of  $0.00200^\circ$ . What is the diameter of a spot of the beam on an object 52 500 km distant? See Fig. 4.77.

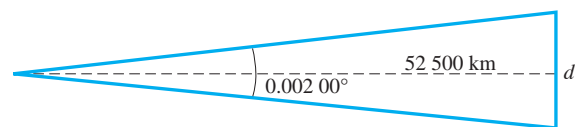


Fig. 4.77

88. Find the gear angle  $\theta$  in Fig. 4.78 if  $t = 0.180$  cm.

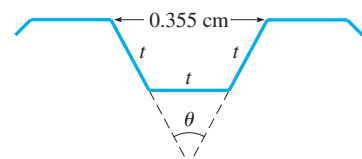


Fig. 4.78



89. The surface of a soccer ball consists of 20 regular hexagons (six sides) interlocked around 12 regular pentagons (five sides). See Fig. 4.79. (a) If the side of each hexagon and pentagon is 45.0 mm, what is the surface area of the soccer ball? (b) Find the surface area, given that the diameter of the ball is 222 mm, (c) Assuming that the given values are accurate, account for the difference in the values found in parts (a) and (b).



Fig. 4.79

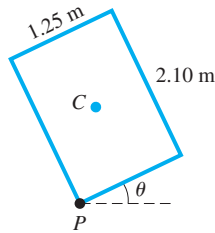


Fig. 4.80

90. Through what angle  $\theta$  must the crate shown in Fig. 4.80 be tipped in order that its centre of gravity  $C$  is directly above the pivot point  $P$ ?
91. A hang glider is directly above the shore of a lake. An observer on a hill is 375 m along a straight line from the shore. From the observer, the angle of elevation of the hang glider is  $42.0^\circ$ , and the angle of depression of the shore is  $25.0^\circ$ . How far above the shore is the hang glider?
92. A ground observer sights a weather balloon to the east at an angle of elevation of  $15.0^\circ$ . A second observer 2.35 km to the east of the first also sights the balloon to the east at an angle of elevation of  $24.0^\circ$ . How high is the balloon? See Fig. 4.81.

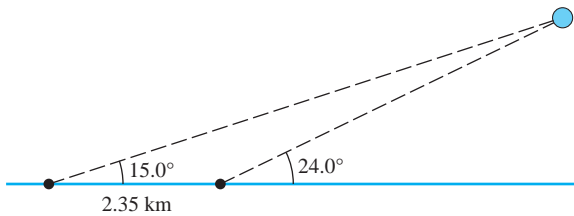


Fig. 4.81

93. A uniform strip of wood 5.0 cm wide frames a trapezoidal window, as shown in Fig. 4.82. Find the left dimension  $l$  of the outside of the frame.

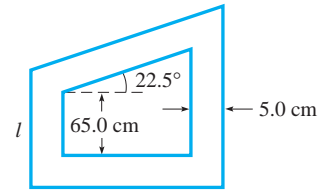


Fig. 4.82

94. A crop-dusting plane flies over a level field at a height of 8.0 m. If the dust leaves the plane through a  $30^\circ$  angle and hits the ground after the plane travels 25 m, how wide a strip is dusted? See Fig. 4.83.

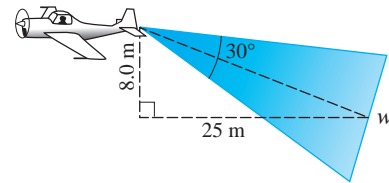


Fig. 4.83

### Writing Exercise

95. A patio is designed in the shape of an isosceles trapezoid with bases 5.0 m and 7.0 m. The other sides are 6.0 m each. Write one or two paragraphs explaining how to use (a) the sine and (b) the cosine to find the internal angles of the patio, and (c) the tangent in finding the area of the patio.

## CHAPTER 4 PRACTICE TEST

- Express  $37^\circ 39'$  in decimal form.
- Find the value of  $\tan 73.8^\circ$ .
- Find  $\theta$  if  $\cos \theta = 0.3726$ .
- A ship's captain, desiring to travel due south, discovers due to an improperly functioning instrument, the ship has gone 22.62 km in a direction  $4.05^\circ$  east of south. How far from its course (to the east) is the ship?
- Find  $\tan \theta$  in fractional form if  $\sin \theta = \frac{2}{3}$ .
- Find  $\csc \theta$  if  $\tan \theta = 1.294$ .
- Solve the right triangle in Fig. 4.84 if  $A = 37.4^\circ$  and  $b = 52.8$ .

- Solve the right triangle in Fig. 4.84 if  $a = 2.49$  and  $c = 3.88$ .

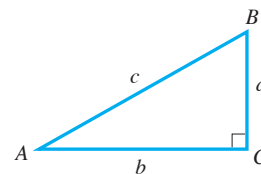


Fig. 4.84

- The equal sides of an isosceles triangle are each 12.0, and each base angle is  $42.0^\circ$ . What is the length of the third side?
- If  $\tan \theta = 9/40$ , find values of  $\sin \theta$  and  $\cos \theta$ . Then evaluate  $\sin \theta / \cos \theta$ .

11. In finding the wavelength  $\lambda$  (the Greek letter lambda) of light, the equation  $\lambda = d \sin \theta$  is used. Find  $\lambda$  if  $d = 30.05 \mu\text{m}$  and  $\theta = 1.167^\circ$ . ( $\mu$  is the prefix for  $10^{-6}$ .)
12. Determine the trigonometric functions of an angle in standard position if its terminal side passes through  $(5, 2)$ . Give answers in exact and decimal forms.
13. The loading ramp at the back of a truck is 3.2 m long. What angle does it make with the ground if the top of the ramp is 1.0 m above the ground?
14. A surveyor sights two points directly ahead. Both are at an elevation 18.525 m lower than the observation point. How far apart

are the points if the angles of depression are  $13.500^\circ$  and  $21.375^\circ$ , respectively? See Fig. 4.85.

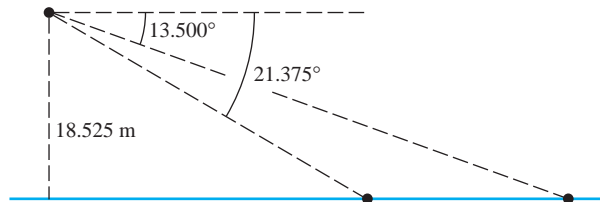


Fig. 4.85

# 5

## LEARNING OUTCOMES

After completion of this chapter, the student should be able to:

- Identify and solve a linear equation of one variable
- Identify a system of equations in two or three variables
- Calculate the slope of a linear function
- Graph a linear function
- Use the slope-intercept form of a line
- Solve a system of linear equations graphically
- Identify an inconsistent system of equations, and a dependent system of equations
- Solve a system of two or three linear equations algebraically
- Solve a system of two or three linear equations using determinants (Cramer's rule)
- Solve application problems involving systems of linear equations

► Kirchhoff's circuit laws deal with the concepts of conservation of electric charge and energy in electric circuits. The systems of equations generated from these laws for specific circuits will be analysed in this chapter.

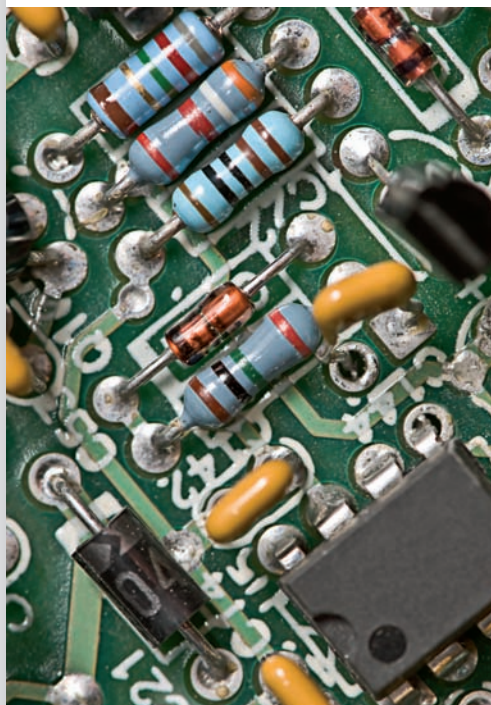
# Systems of Linear Equations; Determinants

As knowledge about electric circuits was first developing, in 1848 the German physicist Gustav Kirchhoff formulated what are now known as *Kirchhoff's current law* and *Kirchhoff's voltage law*, using principles of conservation of charge and energy. These laws are still widely used today, and when they are applied, more than one equation is usually set up. To find the needed information about the circuit, it is necessary to find solutions that satisfy all equations at the same time. The solution of some circuits using Kirchhoff's laws will be shown in this chapter. Although best known for these laws of electric circuits, Kirchhoff is also credited in the study of optics as a founder of the modern chemical identification method known as *spectrum analysis*.

Mathematical methods of solving such *systems* of equations were well known to Kirchhoff, and this allowed the study of electricity to progress rapidly. In fact, 100 years earlier a book by the English mathematician Colin Maclaurin was published (2 years after his death) in which many well organized topics in algebra were covered, including a general method of solving systems of equations. This method is now called *Cramer's rule* (named for the Swiss mathematician Gabriel Cramer, who popularized it in a book he wrote in 1750.) Some of the methods of solution, including Cramer's rule, are included in this chapter.

Two or more equations that relate variables are found in many fields of science and technology. These include aeronautics, business, transportation, the analysis of forces on a structure, medical doses, and robotics, as well as electric circuits. These applications often require solutions that satisfy all equations simultaneously.

In this chapter, our attention is restricted to *linear* equations (variables occur only to the first power). We will consider systems of two equations with two unknowns and systems of three equations with three unknowns. Systems with other kinds of equations and systems with more unknowns are covered in later chapters.



## 5.1 Linear Equations

Linear Equation in One Unknown • Linear Equation in Two Unknowns • System of Linear Equations

Many of the equations we have encountered are examples of a very important type of equation, the *linear equation*. In general, an equation is termed **linear** in a given set of variables if each term contains only one variable, to the first power, or is a constant.

### EXAMPLE 1 Meaning of linear

The equation  $5x - t + 6 = 0$  is linear in  $x$  and  $t$ , but  $5x^2 - t + 6 = 0$  is not linear, due to the presence of  $x^2$ .

The equation  $4x + y = 8$  is linear in  $x$  and  $y$ , but  $4xy + y = 8$  is not, due to the presence of  $xy$ .

The equation  $x - 6y + z - 4w = 7$  is linear in  $x, y, z,$  and  $w$ , but the equation  $x - \frac{6}{y} + z - 4w = 7$  is not, due to the presence of  $\frac{6}{y}$ , where  $y$  appears in the denominator (i.e.,  $y$  has a power of  $-1$ ). ■

■ Recall that  $\frac{1}{y} = y^{-1}$ . See Eq. (1.8).

An equation that can be written in the form

$$ax + b = 0 \quad (5.1)$$

is known as a **linear equation in one unknown**. Here,  $a$  and  $b$  are constants. We discussed this type of equation in Section 1.10. The **solution**, or **root**, of the equation is  $x = -b/a$ . We can also see that the solution is the same as the *zero* of the **linear function**  $f(x) = ax + b$ .

### EXAMPLE 2 Linear equation in one unknown

The equation  $2x + 7 = 0$  is a linear equation of the form of Eq. (5.1), with  $a = 2$  and  $b = 7$ .

The solution to the linear equation  $2x + 7 = 0$ , or the zero of the linear function  $f(x) = 2x + 7$ , is  $-7/2$ . From Chapter 3, recall that the *zero* of a function  $f(x)$  is the value of  $x$  for which  $f(x) = 0$ . ■

In the next example, two specific illustrations are given of applied problems that involve more than one unknown.

### EXAMPLE 3 Applications of linear equations

(a) A basic law of direct-current electricity, known as *Kirchhoff's current law*, may be stated as "The algebraic sum of the currents entering any junction in a circuit is zero." If three wires are joined at a junction as in Fig. 5.1, this law leads to the linear equation

$$i_1 + i_2 + i_3 = 0$$

where  $i_1, i_2,$  and  $i_3$  are the currents in each of the wires. (Either one or two of these currents must have a negative sign, showing that it is actually leaving the junction.)

(b) Two forces,  $F_1$  and  $F_2$ , acting on a beam might be related by the equation

$$2F_1 + 4F_2 = 200 \quad \blacksquare$$

An equation that can be written in the form

$$ax + by = c \quad (5.2)$$

#### Practice Exercise

- Find the zero of the function  $f(x) = 3x + 9$ .

■ See the chapter introduction.

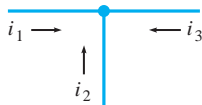


Fig. 5.1

**LEARNING TIP**

A **solution** is any set of numbers, one for each variable, that satisfies the equation.

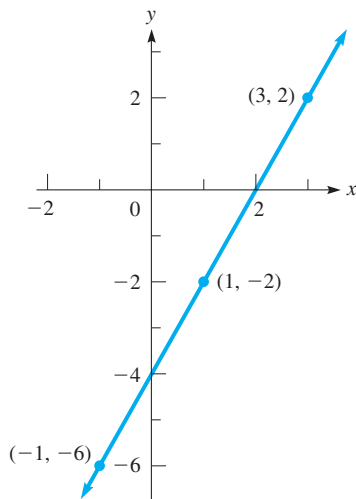


Fig. 5.2

is a **linear equation in two unknowns**. For such equations, we know that for each value of  $x$  there is a corresponding value of  $y$ . Each of these pairs of numbers is a *solution* to the equation.

**EXAMPLE 4** Linear equation in two unknowns

The equation  $2x - y - 4 = 0$  is a linear equation in two unknowns,  $x$  and  $y$ , since we can write it in the form of Eq. (5.2) as  $2x - y = 4$ . To graph this equation, we can write it as  $y = 2x - 4$ . From Fig. 5.2, we see that the graph is a straight line.

The coordinates of any point on the line give us a solution of this equation. For example, the point  $(1, -2)$  is on the line. This means that  $x = 1, y = -2$  is a solution of the equation. We can show that these values are a solution by substituting in the equation  $2x - y - 4 = 0$ . This gives us

$$\begin{aligned} 2(1) - (-2) - 4 &= 0 \\ 2 + 2 - 4 &= 0 \\ 0 &= 0 \end{aligned}$$

Since we have equality,  $x = 1, y = -2$  is a solution. In the same way, we can show that  $x = 3, y = 2$  is a solution.

If we are given the value of one variable, we find the value of the other variable, which gives a solution by substitution. For example, for the equation  $2x - y - 4 = 0$ , if  $x = -1$ , we have

$$\begin{aligned} 2(-1) - y - 4 &= 0 \\ -2 - y - 4 &= 0 \\ y &= -6 \end{aligned}$$

Therefore,  $x = -1, y = -6$  is a solution, and the point  $(-1, -6)$  is on the graph. ■

*Two linear equations, each containing the same two unknowns,  $x$  and  $y$ ,*

**Practice Exercise**

2. Is  $x = -3, y = 2$  a solution to the equation of Example 4?

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned} \tag{5.3}$$

*are said to form a system of simultaneous linear equations*, where  $a_1, a_2, b_1, b_2, c_1,$  and  $c_2$  are all constants. A **solution of the system** is any pair of values  $(x, y)$  that satisfies both equations. Methods of finding the solutions to such systems are the principal concern of this chapter.

**EXAMPLE 5** System of equations – perimeter application

The perimeter of the page on which this problem is printed (also see Fig. 5.3) is 98.2 cm. The length is 6.1 cm more than the width.

From the two statements, we can set up two equations in the two unknown quantities, the length and the width. Letting  $l =$  the length and  $w =$  the width, we have

$$\begin{aligned} 2l + 2w &= 98.2 \\ l - w &= 6.1 \end{aligned}$$

as a system of simultaneous linear equations. The solution of this system is  $l = 27.6$  cm and  $w = 21.5$  cm. These values satisfy both equations since

$$2(27.6) + 2(21.5) = 98.2 \quad \text{and} \quad 27.6 - 21.5 = 6.1$$

This is the only pair of values that satisfies *both* equations. Methods for finding such solutions are presented in later sections of this chapter. ■

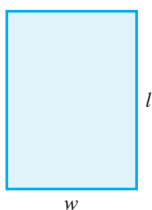


Fig. 5.3

## EXERCISES 5.1

In Exercises 1–3, answer the given questions about the indicated examples of this section.

- In the second line of the third paragraph of Example 1, if  $\frac{6}{5}$  is replaced by its reciprocal, is there any change in the conclusion drawn?
- In the first line of Example 2, if + is changed to -, what changes result in the rest of the example?
- In Example 4, what is the solution if  $x = 4$ ?

In Exercises 4–7, determine whether or not the given pairs of values are solutions of the given linear equations in two unknowns.

- $2x + 3y = 9$ ;  $(3, 1)$ ,  $(5, \frac{1}{3})$
- $5x + 2y = 1$ ;  $(0.2, -1)$ ;  $(1, -2)$
- $-3x + 5y = 13$ ;  $(-1, 2)$ ,  $(4, 5)$
- $x - 4y = 10$ ;  $(2, -2)$ ,  $(2, 2)$

In Exercises 8–13, for each given value of  $x$ , determine the value of  $y$  that gives a solution to the given linear equation in two unknowns.

- $3x - 2y = 12$ ;  $x = 2, x = -3$
- $-5x + 6y = 60$ ;  $x = -10, x = 8$
- $x - 4y = 2$ ;  $x = 3, x = -0.4$
- $3x - 2y = 9$ ;  $x = \frac{2}{3}, x = -3$
- $24x - 9y = 16$ ;  $x = 2/3, x = -1/2$
- $2.4y - 4.5x = -3.0$ ;  $x = -0.4, x = 2.0$

In Exercises 14–24, determine whether or not the given pair of values is a solution of the given system of simultaneous linear equations.

- $x - y = 5$   $x = 4, y = -1$   
 $2x + y = 7$
- $2x + y = 8$   $x = -1, y = 10$   
 $3x - y = -13$
- $A + 5B = -7$   $A = -2, B = 1$   
 $3A - 4B = -4$
- $-30x + 5y = 1$   $x = \frac{1}{3}, y = 2$   
 $6x - 3y = -4$
- $2x - 5y = 0$   $x = \frac{1}{2}, y = -\frac{1}{5}$   
 $4x + 10y = 4$
- $6i_1 + i_2 = 5$   $i_1 = 1, i_2 = -1$   
 $3i_1 - 4i_2 = -1$
- $3x - 2y = 2.2$   $x = 0.6, y = -0.2$   
 $5x + y = 2.8$
- $s - 7t = -3.2$   $s = -1.1, t = 0.3$   
 $2s + t = 2.5$
- $l - w = 3.7$   $l = 26.4, w = 22.7$   
 $2l + 2w = 98.2$
- $30x - 21y = -675$   $x = -12, y = 15$   
 $42x + 15y = -279$

$$24. \quad 3.8a + 7.5b = -26.5 \quad a = 2.5, b = -4.8$$

$$2.6a - 12.6b = -51.1$$

In Exercises 25–30, solve the given problems.

- If  $x = -2$  is a root of the equation  $3x + b = 0$ , find  $b$ .
- If  $x = 2, y = -3$  is a solution of the system of equations  
 $ax + y = 1$   
 $x - ay = c$   
find  $a$  and  $c$ .
- In planning a search pattern from an aircraft carrier, a pilot plans to fly at  $p$  km/h relative to a wind that is blowing at  $w$  km/h. Travelling with the wind, the ground speed would be 300 km/h, and against the wind the ground speed would be 220 km/h. This leads to two equations:  
 $p + w = 300$   
 $p - w = 220$   
Are the speeds 260 km/h and 40 km/h?
- The electric resistance  $R$  of a certain resistor is a function of the temperature  $T$  given by the equation  $R = aT + b$ , where  $a$  and  $b$  are constants. If  $R = 1200 \Omega$  when  $T = 10.0^\circ\text{C}$  and  $R = 1280 \Omega$  when  $T = 50.0^\circ\text{C}$ , we can find the constants  $a$  and  $b$  by substituting and obtaining the equations  
 $1200 = 10.0a + b$   
 $1280 = 50.0a + b$   
Are the constants  $a = 4.00 \Omega/^\circ\text{C}$  and  $b = 1160 \Omega$ ?
- The forces acting on part of a structure are shown in Fig. 5.4. An analysis of the forces leads to the equations  
 $0.80F_1 + 0.50F_2 = 50$   
 $0.60F_1 - 0.87F_2 = 12$   
Are the forces 45 N and 28 N?
- A student earned \$4000 during the summer and decided to put half into a Registered Retirement Savings Plan (RRSP). If the RRSP was invested in two accounts earning 4.0% and 5.0%, the total income for the first year is \$92. The equations to determine the amounts of  $x$  and  $y$  are  
 $x + y = 2000$   
 $0.040x + 0.050y = 92$   
Are the amounts  $x = \$1200$  and  $y = \$800$ ?

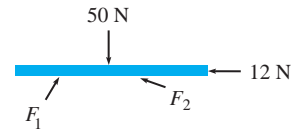


Fig. 5.4

## Answers to Practice Exercises

- $x = -3$
- No



## 5.2 Graphs of Linear Functions

### Slope of a Line • Slope-Intercept Form • Sketching Lines by Intercepts

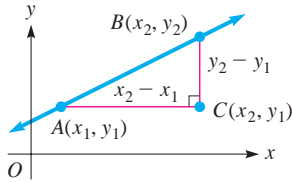


Fig. 5.5

#### Slope of a Line

$$m = \frac{y_2 - y_1}{x_2 - x_1} \quad (5.4)$$

The first method of solving a system of equations that will be studied is a graphical method. Before presenting this method in the next section, additional ways of graphing a linear equation will be developed. We will then be able to analyse and check the graph of a linear equation quickly, often by inspection.

Consider the line that passes through points  $A$ , with coordinates  $(x_1, y_1)$ , and  $B$ , with coordinates  $(x_2, y_2)$ , in Fig. 5.5. Point  $C$  is horizontal from  $A$  and vertical from  $B$ . Thus,  $C$  has coordinates  $(x_2, y_1)$ , and there is a right angle at  $C$ . One way of measuring the steepness of this line is to find the ratio of the vertical distance to the horizontal distance between two points. Therefore, we define the **slope** of the line through two points as the difference in the  $y$ -coordinates divided by the difference in the  $x$ -coordinates. For points  $A$  and  $B$ , the slope  $m$  is

The slope is often referred to as the *rise* (vertical change) over the *run* (horizontal change). Note that the slope of a vertical line, for which  $x_2 = x_1$ , is undefined (for  $x_2 = x_1$ , the denominator of Eq. 5.4 is zero).

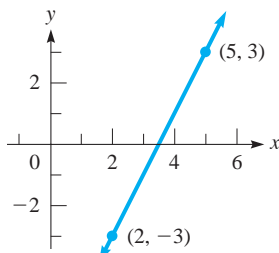


Fig. 5.6

#### EXAMPLE 1 Slope of a line through two points

Find the slope of the line through the points  $(2, -3)$  and  $(5, 3)$ .

In Fig. 5.6, we draw the line through the two given points. By taking  $(5, 3)$  as  $(x_2, y_2)$ , then  $(x_1, y_1)$  is  $(2, -3)$ . We may choose either point as  $(x_2, y_2)$ , but *once the choice is made the order must be maintained*. Using Eq. (5.4), the slope is

$$\begin{aligned} m &= \frac{3 - (-3)}{5 - 2} \\ m &= \frac{6}{3} = 2 \end{aligned}$$

The rise is 2 units for each unit (of run) in going from left to right. ■

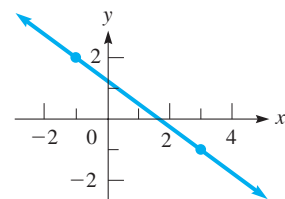


Fig. 5.7

#### EXAMPLE 2 Slope of a line through two points

Find the slope of the line through  $(-1, 2)$  and  $(3, -1)$ .

In Fig. 5.7, we draw the line through these two points. By taking  $(x_2, y_2)$  as  $(3, -1)$  and  $(x_1, y_1)$  as  $(-1, 2)$ , the slope is

$$\begin{aligned} m &= \frac{-1 - 2}{3 - (-1)} \\ m &= \frac{-3}{3 + 1} = -\frac{3}{4} \end{aligned}$$

The negative sign means the line *falls* 3 units for each 4 units in going from left to right. ■

#### Practice Exercise

- Find the slope of the line through  $(-2, 4)$  and  $(-5, -6)$ .

#### LEARNING TIP

Note in Example 1 that as  $x$  increases,  $y$  increases and that slope is **positive**. In Example 2, as  $x$  increases,  $y$  decreases and that slope is **negative**. Also, the larger the absolute value of the slope, the steeper is the line. The closer the slope is to zero, the more the line approaches a horizontal line.

#### EXAMPLE 3 Slope and steepness of a line

For each of the following lines shown in Fig. 5.8, we show the difference in the  $y$ -coordinates and in the  $x$ -coordinates between two points.

In Fig. 5.8(a), a line with a slope of 5 is shown. It rises sharply.

In Fig. 5.8(b), a line with a slope of  $\frac{1}{2}$  is shown. It rises slowly.

In Fig. 5.8(c), a line with a slope of  $-5$  is shown. It falls sharply.  
 In Fig. 5.8(d), a line with a slope of  $-\frac{1}{2}$  is shown. It falls slowly.

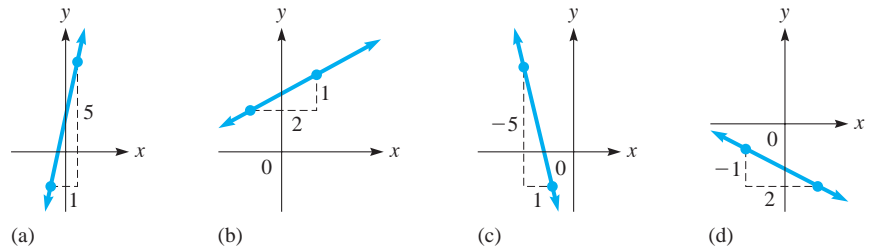


Fig. 5.8

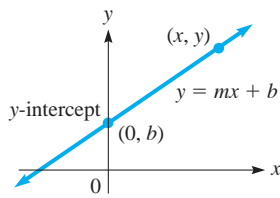


Fig. 5.9

**SLOPE-INTERCEPT FORM OF THE EQUATION OF A STRAIGHT LINE**

We now show how the slope is related to the equation of a straight line. In Fig. 5.9, if we have two points,  $(0, b)$  and a general point  $(x, y)$ , the slope is

$$m = \frac{y - b}{x - 0}$$

Simplifying this, we have  $mx = y - b$ , or

$$y = mx + b \tag{5.5}$$

**LEARNING TIP**

$y = mx + b$  is the **slope-intercept form** of the equation of a straight line. The coefficient of  $x$  is the slope, and the added constant is the **ordinate of the y-intercept**. The point  $(0, b)$  and simply  $b$  are both referred to as the **y-intercept**.

In Eq. (5.5),  $m$  is the slope, and  $b$  is the  $y$ -coordinate of the point where it crosses the  $y$ -axis. This point is the **y-intercept** of the line, and its coordinates are  $(0, b)$ .

**EXAMPLE 4 Slope-intercept form of an equation**

Find the slope and the  $y$ -intercept of the line  $y = \frac{3}{2}x - 3$ . Since the coefficient of  $x$  is  $\frac{3}{2}$ , this means that the slope is  $\frac{3}{2}$ . Also, because we can write

$$y = \frac{3}{2}x + (-3)$$

↙ slope
↙ y-intercept ordinate

we see that the constant is  $-3$ , which means the  $y$ -intercept is the point  $(0, -3)$ . The line is shown in Fig. 5.10.

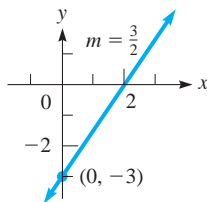


Fig. 5.10

**EXAMPLE 5 Slope-intercept form of an equation**

Find the slope and the  $y$ -intercept of the line  $2x + 3y = 4$ .

We must first write the equation in slope-intercept form. Solving for  $y$ , we have

$$3y = -2x + 4$$

$$y = -\frac{2}{3}x + \frac{4}{3}$$

↙ slope
↙ y-intercept ordinate

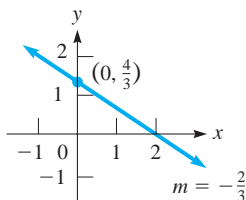


Fig. 5.11

Therefore, the slope is  $-\frac{2}{3}$ , and the  $y$ -intercept is the point  $(0, \frac{4}{3})$ . See Fig. 5.11.

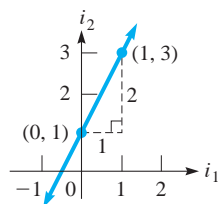


Fig. 5.12

**EXAMPLE 6** Application of the slope-intercept form

In analysing an electric current, two of the currents,  $i_1$  and  $i_2$ , were related by  $i_2 = 2i_1 + 1$ . Graph this equation using the slope-intercept form.

Since the equation is solved for  $i_2$ , we treat  $i_1$  as the independent variable and  $i_2$  as the dependent variable. This means that the slope of the line is 2. Also, since the  $b$ -term is 1, we can see that the intercept is  $(0, 1)$ .

We can use this information to sketch the line, as shown in Fig. 5.12. Since the slope is 2, we know that  $i_2$  increases 2 units for each unit increase of  $i_1$ . Thus, starting at the  $i_2$ -intercept  $(0, 1)$ , if  $i_1$  increases by 1,  $i_2$  increases by 2, and we are at the point  $(1, 3)$ . The line must pass through  $(1, 3)$ , as well as  $(0, 1)$ . Therefore, we draw the line through these points. (Negative values may be used for electric currents since the sign shows the direction of flow.) ■

**SKETCHING LINES BY INTERCEPTS**

Another way of sketching the graph of a straight line is to find two points on the line and then draw the line through these points. Two points that are easily determined are those where the line crosses the  $y$ -axis and the  $x$ -axis. We already know that the point where it crosses the  $y$ -axis is the  $y$ -intercept. In the same way, *the point where it crosses the  $x$ -axis is called the  $x$ -intercept*, and the coordinates of the  $x$ -intercept are  $(a, 0)$ .

The intercepts are easily found because one of the coordinates is zero. By setting  $x = 0$  and  $y = 0$ , in turn, and finding the value of the other variable, we get the coordinates of the intercepts. This method works except when both intercepts are at the origin, and we must find one other point, or use the slope-intercept method. In using the intercept method, a third point may be found as a check. The next example shows how a line is sketched by finding its intercepts.

**EXAMPLE 7** Sketching a line by its intercepts

Sketch the graph of the line  $2x - 3y = 6$  by finding its intercepts and one check point. See Fig. 5.13.

First, we let  $x = 0$ . This gives us  $-3y = 6$ , or  $y = -2$ . This gives us the  $y$ -intercept, which is the point  $(0, -2)$ . Next, we let  $y = 0$ , which gives us  $2x = 6$ , or  $x = 3$ . This means the  $x$ -intercept is the point  $(3, 0)$ .

The intercepts are enough to sketch the line as shown in Fig. 5.13. To find a check point, we can use any value for  $x$  other than 3 or any value of  $y$  other than  $-2$ . Choosing  $x = 1$ , we find that  $y = -\frac{4}{3}$ . This means that the point  $(1, -\frac{4}{3})$  should be on the line. In Fig. 5.13, we can see that it is on the line.

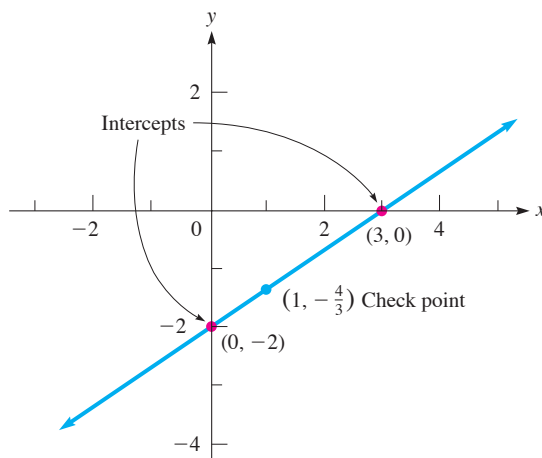


Fig. 5.13

**Practice Exercise**

2. Write the equation  $3x - 5y - 15 = 0$  in slope-intercept form.

**Practice Exercise**

3. Find the intercepts of the line  $3x - 5y - 15 = 0$ .

■ Further details and discussion of slope and the graphs of linear equations are found in Chapter 21.

## EXERCISES 5.2

In Exercises 1–4, answer the given questions about the indicated examples of this section.

- In Example 2, if the first  $y$ -coordinate is changed to  $-2$ , what changes result in the example?
- In Example 4, if  $\frac{3}{2}$  is changed to  $-\frac{3}{4}$ , what changes result in the example?
- In the first line of Example 5, if  $+$  is changed to  $-$ , what changes result in the example?
- In the first line of Example 7, if  $-$  is changed to  $+$ , what changes occur in the graph?

In Exercises 5–12, find the slope of the line that passes through the given points.

- |                               |                               |
|-------------------------------|-------------------------------|
| 5. $(1, 0), (3, 8)$           | 6. $(3, 1), (2, -7)$          |
| 7. $(-1, 2), (-4, 17)$        | 8. $(-1, -2), (2, 10)$        |
| 9. $(5, -3), (-2, -5)$        | 10. $(-3, 4), (-7, -4)$       |
| 11. $(0.4, 0.5), (-0.2, 0.2)$ | 12. $(-2.8, 3.4), (1.2, 4.2)$ |

In Exercises 13–20, sketch the line with the given slope and  $y$ -intercept.

- |                               |                                |
|-------------------------------|--------------------------------|
| 13. $m = 2, (0, -1)$          | 14. $m = 3, (0, 1)$            |
| 15. $m = 0, (0, 2)$           | 16. $m = -4, (0, -2)$          |
| 17. $m = \frac{1}{2}, (0, 0)$ | 18. $m = \frac{2}{3}, (0, -1)$ |
| 19. $m = -9, (0, 20)$         | 20. $m = -0.3, (0, -1.4)$      |

In Exercises 21–28, find the slope and the  $y$ -intercept of the line with the given equation and sketch the graph using the slope and the  $y$ -intercept.

- |                      |                            |
|----------------------|----------------------------|
| 21. $y = -2x + 1$    | 22. $y = -4x$              |
| 23. $y = x - 4$      | 24. $y = \frac{4}{5}x + 2$ |
| 25. $5x - 2y = 40$   | 26. $-2y = 7$              |
| 27. $24x + 40y = 15$ | 28. $1.5x - 2.4y = 3.0$    |

In Exercises 29–36, find the  $x$ -intercept and the  $y$ -intercept of the line with the given equation. Sketch the line using the intercepts.

- |                     |                       |
|---------------------|-----------------------|
| 29. $x + 2y = 4$    | 30. $3x + y = 3$      |
| 31. $4x - 3y = 12$  | 32. $5y - x = 5$      |
| 33. $y = 3x + 6$    | 34. $y = -2x - 4$     |
| 35. $y = -12x + 30$ | 36. $y = 0.25x + 4.5$ |

In Exercises 37 and 38, solve for  $k$ .

- Find the value of  $k$  such that the line  $kx - 2y = 9$  has a slope of 3.
- Find the value of  $k$  such that the line through  $(0, k)$  and  $(-k, -1)$  has a slope of  $1/2$ .

In Exercises 39–42, sketch the indicated lines.

- The diameter of the large end,  $d$  (in cm), of a certain type of machine tool can be found from the equation  $d = 0.2l + 1.2$ , where  $l$  is the length of the tool. Sketch  $d$  as a function of  $l$ , for values of  $l$  to 10 cm. See Fig. 5.14.

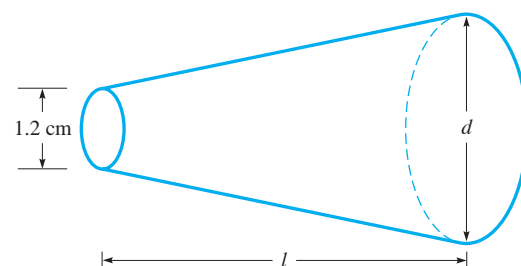


Fig. 5.14

- In mixing 85-octane gasoline and 93-octane gasoline to produce 91-octane gasoline, the equation  $0.85x + 0.93y = 910$  is used. Sketch the graph.
- Two electric currents,  $I_1$  and  $I_2$  (in mA), in part of a circuit in a computer are related by the equation  $4I_1 - 5I_2 = 2$ . Sketch  $I_2$  as a function of  $I_1$ . These currents can be negative.
- In 2010, there were  $2.93 \times 10^9$  email accounts worldwide, and in 2012, there were  $3.38 \times 10^9$  email accounts worldwide. Assuming that the growth in the number of email accounts is linear, set up a function relating the number  $N$  of email accounts and the time  $t$  (in years). Let  $t = 0$  be the year 2010. Sketch the graph.

## Answers to Practice Exercises

1.  $10/3$    2.  $y = \frac{3}{5}x - 3$    3.  $(5, 0)$  and  $(0, -3)$

## 5.3 Solving Systems of Two Linear Equations in Two Unknowns Graphically

Determine the Point of Intersection •  
Using a Graphing Calculator •  
Inconsistent and Dependent Systems

Since a solution of a system of simultaneous linear equations in two unknowns is any pair of values  $(x, y)$  that satisfies *both* equations, graphically *the solution would be the coordinates of the point of intersection of the two lines*. This must be the case, for the coordinates of this point constitute the only pair of values to satisfy *both* equations. (In some special cases, there may be no solution; in others, there may be many solutions. See Examples 5 and 6.)

Therefore, when we solve two simultaneous linear equations in two unknowns graphically, *we must graph each line and determine the point of intersection*. This may, of course, lead to *approximate results* if the lines cross at points not used to determine the graph.

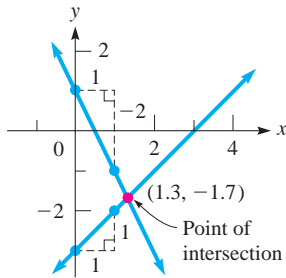


Fig. 5.15

**EXAMPLE 1** Determine the point of intersection

Solve the system of equations

$$y = x - 3$$

$$y = -2x + 1$$

Since each of the equations is in slope-intercept form, note that  $m = 1$  and  $b = -3$  for the first line and that  $m = -2$  and  $b = 1$  for the second line. Using these values, we sketch the lines, as shown in Fig. 5.15.

From the figure, it can be seen that *the lines cross at about the point*  $(1.3, -1.7)$ . This means that the solution is approximately

$$x = 1.3 \quad y = -1.7$$

(The exact solution is  $x = \frac{4}{3}, y = -\frac{5}{3}$ .)

**COMMON ERROR**

**When checking the solution, ensure that you substitute the values into *both* equations.**

It is a common error to substitute the values into only one equation, which, even if it checks, may not illustrate a valid solution for both equations simultaneously.

Substituting these solutions gives

$$\begin{aligned} -1.7 &\stackrel{?}{=} 1.3 - 3 & \text{and} & & -1.7 &\stackrel{?}{=} -2(1.3) + 1 \\ -1.7 &= -1.7 & & & -1.7 &\approx -1.6 \end{aligned}$$

These values show that the solution checks. (The point  $(1.3, -1.7)$  is *on* the first line and *almost on* the second line. The difference in values when checking the values for the second line is due to the fact that the solution is *approximate*.) ■

**EXAMPLE 2** Determine the point of intersection

Solve the system of equations

$$2x + 5y = 10$$

$$3x - y = 6$$

We could write each equation in slope-intercept form in order to sketch the lines. Also, we could use the form in which they are written to find the intercepts. Choosing to find the intercepts and draw lines through them, let  $y = 0$ ; then  $x = 0$ . Therefore, the intercepts of the first line are the points  $(5, 0)$  and  $(0, 2)$ . A third point is  $(-1, \frac{12}{5})$ . The intercepts of the second line are  $(2, 0)$  and  $(0, -6)$ . A third point is  $(1, -3)$ . Plotting these points and drawing the proper straight lines, we see that the lines cross at about  $(2.3, 1.1)$ . [The exact values are  $(\frac{40}{17}, \frac{18}{17})$ .] The solution of the system of equations is approximately  $x = 2.3, y = 1.1$  (see Fig. 5.16).

Checking, we have

$$\begin{aligned} 2(2.3) + 5(1.1) &\stackrel{?}{=} 10 & \text{and} & & 3(2.3) - 1.1 &\stackrel{?}{=} 6 \\ 10.1 &\approx 10 & & & 5.8 &\approx 6 \end{aligned}$$

This shows the solution is correct to the accuracy we can get from the graph. ■

**SOLVING SYSTEMS OF EQUATIONS USING A GRAPHING CALCULATOR**

A graphing calculator can be used to find the point of intersection with much greater accuracy than is possible by hand-sketching the lines. Once we locate the point of intersection, the features of the calculator allow us to get the accuracy we need.

**EXAMPLE 3** Using a graphing calculator to solve a system of equations

Using a graphing calculator, we can solve the systems of equations in Examples 1 and 2.

Solving the system for Example 1, let  $y_1 = x - 3$  and  $y_2 = -2x + 1$ . After displaying the graphs in the proper *window* settings, we use the *intersect* feature to find (rounded to the nearest 0.001) that the solution is  $x = 1.333, y = -1.667$ . ■

**Practice Exercise**

- In Example 2, change the 6 to 3, and then solve. (Can be done using Fig. 5.16.)

■ We could use the *trace* and *zoom* features, but the *intersect* feature is meant for this type of problem, and is more accurate.

**WORD PROBLEMS INVOLVING TWO LINEAR EQUATIONS**

Linear equations in two unknowns are often useful in solving word problems. Just as in Section 1.12, we must read the statement carefully in order to identify the unknowns and the information for setting up the equations.

**EXAMPLE 4 Application of solving a system**

A driver travelled for 1.5 h at a constant speed along a highway. Then, through a construction zone, the driver reduced the car’s speed by 40 km/h for 30 min. If 200 km were covered in the 2.0 h, what were the two speeds?

First, let  $v_h$  = the highway speed and  $v_c$  = the speed in the construction zone. Two equations are found by using

1. distance = rate  $\times$  time [for units, km = ( $\frac{\text{km}}{\text{h}}$ )h], and
2. the fact that “the driver reduced the car’s speed by 40 km/h for 30 min.”

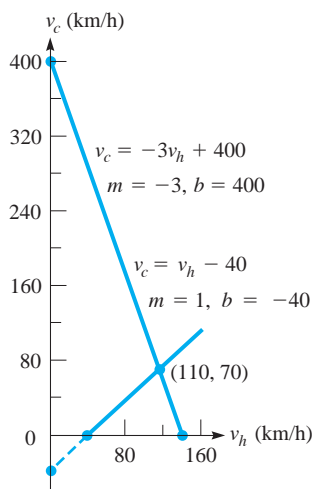
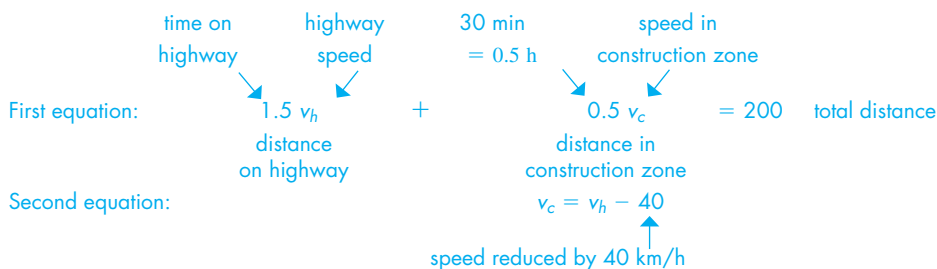


Fig. 5.17



Using  $v_h$  as the independent variable and  $v_c$  as the dependent variable, the sketch is shown in Fig. 5.17.

We see that the point of intersection is (110, 70), which means that the solution is  $v_h = 110$  km/h and  $v_c = 70$  km/h. Checking in the statement of the problem, we have  $(1.5 \text{ h})(110 \text{ km/h}) + (0.5 \text{ h})(70 \text{ km/h}) = 200 \text{ km}$  and  $70 \text{ km/h} = 110 \text{ km/h} - 40 \text{ km/h}$ . ■

**INCONSISTENT AND DEPENDENT SYSTEMS**

The lines of each system in the previous examples intersect in a single point, and each system has one solution. Such systems are called consistent and independent. Most systems that we will encounter have just one solution. However, as we now show, not all systems have just one such solution. The following examples illustrate a system that has no solution and a system that has an infinite number of solutions.

**EXAMPLE 5 Inconsistent system**

Solve the system of equations

$$x = 2y + 6$$

$$6y = 3x - 6$$

Writing each of these equations in slope-intercept form (Eq. 5.5), we have for the first equation

$$y = \frac{1}{2}x - 3$$

For the second equation, we have

$$y = \frac{1}{2}x - 1$$

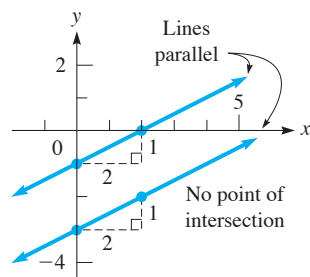


Fig. 5.18

From these, we see that each line has a slope of  $\frac{1}{2}$  and that the y-intercepts are (0, -3) and (0, -1). Therefore, we know that the y-intercepts are different, but the slopes are the same. Since the slope indicates that each line rises  $\frac{1}{2}$  unit for y for each unit x increases, the lines are parallel and do not intersect, as shown in Fig. 5.18. This means that there are no solutions for this system of equations. Such a system is called inconsistent. ■



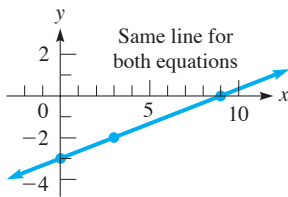


Fig. 5.19

**EXAMPLE 6** Dependent system

Solve the system of equations

$$\begin{aligned}x - 3y &= 9 \\ -2x + 6y &= -18\end{aligned}$$

We find that the intercepts and a third point for the first line are  $(9, 0)$ ,  $(0, -3)$ , and  $(3, -2)$ . For the second line, we then find that the intercepts are the same as for the first line. We also find that the check point  $(3, -2)$  also satisfies the equation of the second line. This means that the two lines are really the same line.

Another check is to write each equation in slope-intercept form. This gives us the equation  $y = \frac{1}{3}x - 3$  for each line. See Fig. 5.19.

Since the lines are the same, *the coordinates of any point on this common line constitute a solution of the system—that is to say, there are an infinite number of solutions.* Since *no unique solution can be determined*, the system is called **dependent**. ■

Later in the chapter, we will show algebraic ways of finding out whether a given system is consistent, inconsistent, or dependent.

**EXERCISES 5.3**

In Exercises 1 and 2, answer the given questions about the indicated examples of this section.

- In the second equation of Example 2, if  $-$  is replaced with  $+$ , what is the solution?
- In Example 6, by changing what one number in the first equation does the system become (a) inconsistent? (b) consistent?

In Exercises 3–20, solve each system of equations by sketching the graphs. Use the slope and the y-intercept or both intercepts. Estimate each result to the nearest 0.1 if necessary.

- |   |   |
|---|---|
| 3. $y = -x + 4$<br>$y = x - 2$              | 4. $y = \frac{1}{2}x - 1$<br>$y = -x + 8$ |
| 5. $y = 2x - 6$<br>$y = -\frac{1}{3}x + 1$  | 6. $y = \frac{1}{2}x - 4$<br>$y = 2x + 2$ |
| 7. $3x + 2y = 6$<br>$x - 3y = 3$            | 8. $4R - 3V = -8$<br>$6R + V = 6$         |
| 9. $2x - 5y = 10$<br>$3x + 4y = -12$        | 10. $-5x + 3y = 15$<br>$2x + 7y = 14$     |
| 11. $s - 4t = 8$<br>$2s = t + 4$            | 12. $y = 4x - 6$<br>$y = 2x + 4$          |
| 13. $y = -x + 3$<br>$y = -2x + 3$           | 14. $p - 6 = 6v$<br>$v = 3 - 3p$          |
| 15. $x - 4y = 6$<br>$2y = x + 4$            | 16. $x + y = 3$<br>$3x - 2y = 14$         |
| 17. $-2r_1 + 2r_2 = 7$<br>$4r_1 - 2r_2 = 1$ | 18. $2x - 3y = -5$<br>$3x + 2y = 12$      |
| 19. $15y = 20x - 48$<br>$32x + 21y = 60$    | 20. $7A = 8B + 17$<br>$5B = 12A - 22$     |

In Exercises 21–32, solve each system of equations graphically to the nearest 0.001 for each variable.

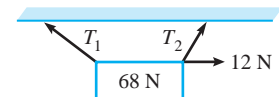
- |  |  |
|--|--|
| 21. $x = 4y + 2$<br>$3y = 2x + 3$              | 22. $1.2x - 2.4y = 4.8$<br>$3.0x = -2.0y + 7.2$  |
| 23. $4.0x - 3.5y = 1.5$<br>$0.7y + 0.1x = 0.7$ | 24. $5F - 2T = 7$<br>$3F + 4T = 8$               |
| 25. $x - 5y = 10$<br>$2x - 10y = 20$           | 26. $18x - 3y = 7$<br>$2y = 1 + 12x$             |
| 27. $1.9v = 3.2t$<br>$1.2t - 2.6v = 6$         | 28. $3y = 14x - 9$<br>$12x + 23y = 0$            |
| 29. $5x = y + 3$<br>$4x = 2y - 3$              | 30. $0.75u + 0.67v = 5.9$<br>$2.1u - 3.9v = 4.8$ |
| 31. $7R = 18V + 13$<br>$-1.4R + 3.6V = 2.6$    | 32. $y = 6x + 2$<br>$12x - 2y = -4$              |

In Exercises 33–38, graphically solve the given problems.

33. Chains support a crate, as shown in Fig. 5.20. The equations relating tensions  $T_1$  and  $T_2$  are given below. Determine the tensions to the nearest 1 N from the graph.

$$\begin{aligned}0.8T_1 - 0.6T_2 &= 12 \\ 0.6T_1 + 0.8T_2 &= 68\end{aligned}$$

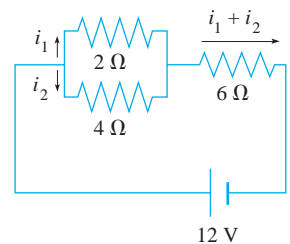
Fig. 5.20



34. The equations relating the currents  $i_1$  and  $i_2$  shown in Fig. 5.21 are given below. Find the currents to the nearest 0.1 A.

$$\begin{aligned}2i_1 + 6(i_1 + i_2) &= 12 \\ 4i_2 + 6(i_1 + i_2) &= 12\end{aligned}$$

Fig. 5.21



35. An architect designing a parking lot has a row 62.0 m wide to divide into spaces for compact cars and full-size cars. The architect determines that 16 compact car spaces and 6 full-size car spaces use the width, or that 12 compact car spaces and 9 full-size car spaces use all but 0.2 m of the width. What are the widths (to the nearest 0.1 m) of the spaces being planned? See Fig. 5.22.

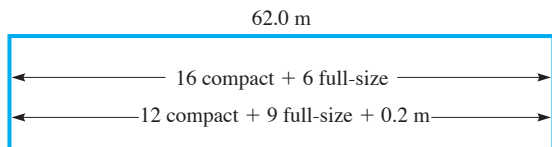


Fig. 5.22

36. An airplane flies into a headwind with an effective ground speed of 140 km/h. On the return trip it flies with the tailwind and has

an effective ground speed of 240 km/h. Find the speed  $p$  of the plane in still air, and the speed  $w$  of the wind (to the nearest 1 km/h).

37. One alloy is 70% lead and 30% zinc, and another alloy is 40% lead and 60% zinc. Find the amount of each alloy needed to make 120 kg of an alloy that is 50% lead and 50% zinc (to the nearest kg).
38. A certain car uses 7.2 L/100 km in city driving and 5.4 L/100 km in highway driving. If 36 L of gas are used in travelling 598 km, how many kilometres were driven in the city, and how many were driven on the highway (assuming that only the given rates of usage were actually used, to the nearest km)?

#### Answer to Practice Exercise

1.  $x = 1.5, y = 1.4$

## 5.4 Solving Systems of Two Linear Equations in Two Unknowns Algebraically

Solution of a System of Equations by Substitution • Solution by Elimination

The graphical method of solving two linear equations in two unknowns is good for getting a “picture” of the solution. One problem is that graphical methods usually give *approximate* results, although the accuracy obtained when using a graphing calculator is excellent. If *exact* solutions are required, we turn to other methods. In this section, we present two algebraic methods of solution.

### SOLUTION OF A SYSTEM OF EQUATIONS BY SUBSTITUTION

The first method involves the **substitution of one equation into the other**. The basic idea of this method is to *solve one of the equations for one of the unknowns and substitute this into the other equation*. The result is an equation with only one unknown, and this equation can then be solved for the unknown. Following is the basic procedure to be used.

#### LEARNING TIP

In following the substitution technique, you must first choose an unknown for which to solve. Often it makes little difference, but if it is easier to solve a particular equation for one of the unknowns, that is the one to use.

#### Solution of Two Linear Equations by Substitution

1. Solve one equation for one of the unknowns.
2. **Substitute** this solution into the **other** equation and simplify. At this point, we have a linear equation in one unknown.
3. Solve the resulting equation for the value of the unknown it contains.
4. Substitute this value into the equation of step 1 and solve for the other unknown.
5. Check the values in **both original equations**.

#### EXAMPLE 1 Solution by substitution

Solve the following system of equations by substitution.

$$x - 3y = 6$$

$$2x + 3y = 3$$

Here, it is easiest to solve the first equation for  $x$ :

$$\begin{array}{ll}
 \text{step 1} & x = 3y + 6 \quad (\text{A1}) \\
 & \swarrow \text{in second equation, } x \text{ replaced by } 3y + 6 \\
 \text{step 2} & 2(3y + 6) + 3y = 3 \quad \text{substituting} \\
 \text{step 3} & 6y + 12 + 3y = 3 \quad \text{solving for } y \\
 & 9y = -9 \\
 & y = -1
 \end{array}$$

Now, substitute the value  $y = -1$  into the first of the original equations. Since this equation is already solved for  $x$  in terms of  $y$ , Eq. (A1), we obtain

$$\text{step 4} \quad x = 3(-1) + 6 = 3 \quad \text{solving for } x$$

Therefore, the solution of the system is  $x = 3, y = -1$ . As a check, substitute these values into each of the original equations. This gives us  $3 - 3(-1) = 6$  and  $2(3) + 3(-1) = 3$ , which verifies the solution. ■

### EXAMPLE 2 Solution by substitution

Solve the following system of equations by substitution.

$$\begin{array}{l}
 -5x + 2y = -4 \\
 10x + 6y = 3
 \end{array}$$

It makes little difference which equation or which unknown is chosen. Therefore,

$$\begin{array}{ll}
 2y = 5x - 4 & \text{solving first equation for } y \\
 y = \frac{5x - 4}{2} & (\text{B1}) \\
 \swarrow \text{in second equation, } y \text{ replaced by } \frac{5x - 4}{2} \\
 10x + 6\left(\frac{5x - 4}{2}\right) = 3 & \text{substituting}
 \end{array}$$

#### Practice Exercise

1. Solve the system in Example 2 by first solving for  $x$  and then substituting.

$$\begin{array}{ll}
 10x + 3(5x - 4) = 3 & \text{solving for } x \\
 10x + 15x - 12 = 3 \\
 25x = 15 \\
 x = \frac{3}{5}
 \end{array}$$

Substituting this value into the expression for  $y$ , Eq. (B1), we obtain

$$y = \frac{5(3/5) - 4}{2} = \frac{3 - 4}{2} = -\frac{1}{2} \quad \text{solving for } y$$

Therefore, the solution of this system is  $x = \frac{3}{5}, y = -\frac{1}{2}$ . Substituting these values in both original equations shows that the solution checks. ■

### SOLUTION BY ELIMINATION

The method of substitution is useful if one of the equations can easily be solved for one of the unknowns. However, often a fraction results (such as in Example 2), and this leads to additional algebraic steps to find the solution. Even worse, if the coefficients are themselves decimals or fractions, the algebraic steps are even more involved.

Therefore, we now present another algebraic method of solving a system of equations, **elimination of a variable by addition or subtraction** (also called **the addition or subtraction method**). Following is the basic procedure to be used.

■ For reference, Eq. (5.3) is

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}$$

■ Some prefer always to use addition of the terms of the resulting equations. This method is appropriate because it avoids possible errors that may be caused when subtracting. However, it may require that all signs of one equation be changed.

### LEARNING TIP

The addition or subtraction of any two equations produces another statement of equality. For example,  $1 + 4 = 5$  and  $6 - 4 = 2$  are two equations.

If we add them

$$\begin{array}{r} 1 + 4 = 5 \\ + \quad 6 - 4 = 2 \\ \hline 1 + 4 + 6 - 4 = 5 + 2 \\ 7 = 7 \end{array}$$

If we subtract them

$$\begin{array}{r} 1 + 4 = 5 \\ - \quad 6 - 4 = 2 \\ \hline 1 + 4 - (6 - 4) = 5 - 2 \\ 3 = 3 \end{array}$$

■ Some calculators have a specific feature for solving simultaneous linear equations. It is necessary only to enter the coefficients and constants to get the solution.

### Practice Exercise

2. Solve the system in Example 3 by first multiplying the terms of the first equation by 2, and then subtracting.

### Solution of Two Linear Equations by Elimination

1. Write the equations in the form of Eq. (5.3), if they are not already in this form.
2. If necessary, multiply all terms of each equation by a constant chosen so that the coefficients of one unknown will be numerically the same in both equations. (They can have the same or different signs.)
3. (a) If the numerically equal coefficients have **different** signs, **add** the terms on each side of the resulting equations.  
(b) If the numerically equal coefficients have the **same** sign, **subtract** the terms on each side of one equation from the terms of the other equation.
4. Solve the resulting linear equation in the other unknown.
5. Substitute this value into one of the **original** equations to find the value of the other unknown.
6. Check by substituting both values into both original equations.

### EXAMPLE 3 Solution by elimination, using addition

Use the method of elimination by addition or subtraction to solve the system of equations:

$$\begin{aligned} x - 3y &= 6 && \text{already in the form of Eq. (5.3)} \\ 2x + 3y &= 3 \end{aligned}$$

We look at the coefficients to determine the best way to eliminate one of the unknowns. Since the coefficients of the  $y$ -terms are *numerically the same and opposite in sign*, we may immediately *add* terms of the two equations together to eliminate  $y$ . Adding the terms of the left sides and adding terms of the right sides, we obtain

$$\begin{aligned} (x - 3y) + (2x + 3y) &= 6 + 3 && \text{the terms with } y \text{ cancel (they are eliminated)} \\ 3x &= 9 \\ x &= 3 \end{aligned}$$

Substituting this value into the first equation, we obtain

$$\begin{aligned} 3 - 3y &= 6 \\ -3y &= 3 \\ y &= -1 \end{aligned}$$

The solution  $x = 3, y = -1$  agrees with the results obtained for the same problem illustrated in Example 1. ■

### EXAMPLE 4 Solution by elimination, using subtraction

Use the method of addition or subtraction to solve the following system of equations:

$$\begin{aligned} 3x - 2y &= 4 \\ x + 3y &= 2 \end{aligned}$$

Looking at the coefficients of  $x$  and  $y$ , we see that we must multiply the second equation by 3 to make the coefficients of  $x$  the same. To make the coefficients of  $y$  numerically the same, we must multiply the first equation by 3 and the second equation by 2. Thus, the best method is to multiply the second equation by 3 and eliminate  $x$ . Doing this, the coefficients of  $x$  have the *same sign*. Therefore, we *subtract* terms of the second equation from those of the first equation:

**COMMON ERROR**

When multiplying any equation by a constant, ensure that you multiply the constant with the terms on both sides of the equation. It is a common error to forget to multiply the value on the right.

$$\begin{array}{r}
 3x - 2y = 4 \\
 \underline{3x + 9y = 6} \quad \text{each term of second equation multiplied by 3} \\
 3x - 3x = 0 \quad \text{subtract} \\
 \phantom{3x} - 11y = -2 \\
 \phantom{3x} \phantom{-11y} = \frac{2}{11} \quad 4 - 6 = -2 \\
 -2y - (+9y) = -11y \quad y = \frac{2}{11}
 \end{array}$$

In order to find the value of  $x$ , substitute  $y = \frac{2}{11}$  into one of the original equations. Choosing the second equation (its form is somewhat simpler), we have

$$\begin{array}{r}
 x + 3\left(\frac{2}{11}\right) = 2 \\
 11x + 6 = 22 \quad \text{multiply each term by 11} \\
 x = \frac{16}{11}
 \end{array}$$

Therefore, the solution is  $x = \frac{16}{11}$ ,  $y = \frac{2}{11}$  (see Fig. 5.23). Substituting these values into both of the original equations shows that the solution checks. ■

**EXAMPLE 5 System of Example 4 solved by elimination, using addition**

As noted in Example 4, we can solve that system of equations by first multiplying the terms of the first equation by 3 and those of the second equation by 2, and thereby eliminating  $y$ . Doing this, we have

$$\begin{array}{r}
 9x - 6y = 12 \quad \text{each term of first equation multiplied by 3} \\
 \underline{2x + 6y = 4} \quad \text{each term of second equation multiplied by 2} \\
 11x \phantom{-6y} = 16 \quad \text{add} \\
 9x + 2x = 11x \quad \phantom{-6y} \\
 -6y + 6y = 0 \quad x = \frac{16}{11} \quad 12 + 4 = 16
 \end{array}$$

$$\begin{array}{r}
 3\left(\frac{16}{11}\right) - 2y = 4 \quad \text{substituting } x = 16/11 \text{ in the first original equation} \\
 48 - 22y = 44 \quad \text{multiply each term by 11} \\
 -22y = -4 \\
 y = \frac{2}{11}
 \end{array}$$

Therefore, the solution is  $x = \frac{16}{11}$ ,  $y = \frac{2}{11}$ , as found in Example 4 (see Fig. 5.23). ■

**EXAMPLE 6 Application – solution by elimination, using subtraction**

By weight, one alloy is 70% copper and 30% zinc. Another alloy is 40% copper and 60% zinc. How many grams of each are required to make 300 g of an alloy that is 60% copper and 40% zinc?

Let  $A$  = the required number of grams of the first alloy and  $B$  = the required number of grams of the second alloy. Our equations are determined from:

1. The total weight of the final alloy is 300 g:  $A + B = 300$ .
2. The final alloy will have 180 g of copper (60% of 300 g), and this comes from 70% of  $A$  ( $0.70A$ ) and 40% of  $B$  ( $0.40B$ ):  $0.70A + 0.40B = 180$ .

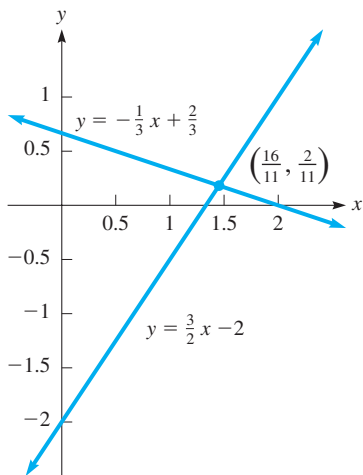


Fig. 5.23

■ The second equation is based on the amount of copper. We could have used the amount of zinc, which would have led to the equation  $0.30A + 0.60B = 0.40(300)$ . Since we need only two equations, we may use any two of these three equations to find the solution. In fact, if the system has a unique solution, this third equation is not independent of the other two. It could be obtained through adding or subtracting multiples of the two equations used.

These two equations can now be solved simultaneously:

$$\begin{array}{rcl}
 A + B = 300 & \text{sum of weights is 300 g} \\
 \text{copper} \rightarrow 0.70A + 0.40B = 180 & \leftarrow \text{60\% of 300 g} \\
 \begin{array}{c} \nearrow 70\% \text{ weight} \\ \uparrow \text{of first} \\ \uparrow \text{alloy} \end{array} & & \begin{array}{c} \nearrow 40\% \text{ weight} \\ \uparrow \text{of second} \\ \uparrow \text{alloy} \end{array} \\
 4A + 4B = 1200 & \text{multiply each term of first equation by 4} \\
 7A + 4B = 1800 & \text{multiply each term of second equation by 10} \\
 \hline
 3A = 600 & \text{subtract first equation from second equation} \\
 A = 200 \text{ g} & \\
 B = 100 \text{ g} & \text{by substituting into first equation}
 \end{array}$$

Checking with the statement of the problem, using the percentages of zinc, we have  $0.30(200) + 0.60(100) = 0.40(300)$ , or  $60 \text{ g} + 60 \text{ g} = 120 \text{ g}$ . We use zinc here since we used the percentages of copper for the equation. ■

### EXAMPLE 7 Inconsistent system with elimination

In solving the system of equations

$$\begin{array}{r}
 4x = 2y + 3 \\
 -y + 2x - 2 = 0
 \end{array}$$

first note that the equations are not in the correct form. Therefore, writing them in the form of Eq. (5.3), we have

$$\begin{array}{r}
 4x - 2y = 3 \\
 2x - y = 2
 \end{array}$$

Now, multiply the second equation by 2 and subtract to get

$$\begin{array}{r}
 4x - 2y = 3 \\
 4x - 2y = 4 \\
 \hline
 4x - 4x = 0 \rightarrow 0 = -1 \leftarrow 3 - 4 = -1 \\
 -2y - (-2y) = 0
 \end{array}$$

Since  $0$  does not equal  $-1$ , we conclude that there is no solution. See Fig. 5.24, which shows that the lines representing the equations are parallel and do not intersect.

#### LEARNING TIP

- When the result is  $0 = a$  ( $a \neq 0$ ), the system is *inconsistent*. From Section 5.3, we know this means the lines representing the equations are parallel.
- If the result of solving a system is  $0 = 0$ , the system is *dependent*. As shown in Section 5.3, this means there is an unlimited number of solutions, and the lines that represent the equations are really the same line.

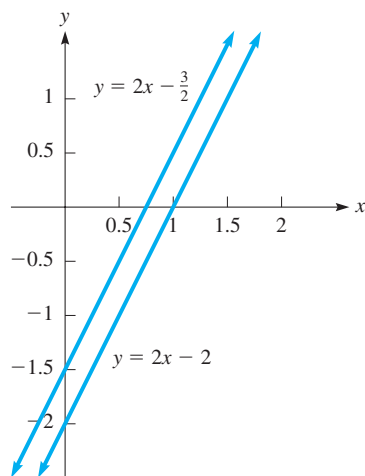


Fig. 5.24



## EXERCISES 5.4

In Exercises 1–4, make the given changes in the indicated examples of this section and then solve the resulting problems.

- In Example 1, change the + to – in the second equation and then solve the system of equations.
- In Example 3, change the + to – in the second equation and then solve the system of equations.
- In Example 4, change  $x$  to  $2x$  in the second equation and then solve the system of equations.
- In Example 7, change 3 to 4 in the first equation and then find if there is any change in the conclusion that is drawn.

In Exercises 5–14, solve the given systems of equations by the method of substitution.

- |                     |                    |
|---------------------|--------------------|
| 5. $x = y + 3$      | 6. $x = 2y + 1$    |
| $x - 2y = 5$        | $2x - 3y = 4$      |
| 7. $p = V - 4$      | 8. $y = 2x + 10$   |
| $V + p = 10$        | $2x + y = -2$      |
| 9. $x + y = -5$     | 10. $3x + y = 1$   |
| $2x - y = 2$        | $3x - 2y = 16$     |
| 11. $2x + 3y = 7$   | 12. $2s + 2t = 1$  |
| $6x - y = 1$        | $4s - 2t = 17$     |
| 13. $33x + 2y = 34$ | 14. $3A + 3B = -1$ |
| $40y = 9x + 11$     | $5A = -6B - 1$     |

In Exercises 15–24, solve the given systems of equations by the method of elimination by addition or subtraction.

- |                       |                        |
|-----------------------|------------------------|
| 15. $x + 2y = 5$      | 16. $x + 3y = 7$       |
| $x - 2y = 1$          | $2x + 3y = 5$          |
| 17. $2x - 3y = 4$     | 18. $R - 4r = 17$      |
| $2x + y = -4$         | $3R + 4r = 3$          |
| 19. $12t + 9y = 14$   | 20. $3x - y = 3$       |
| $6t = 7y - 16$        | $4x = 3y + 14$         |
| 21. $v + 2t = 7$      | 22. $3x - y = 5$       |
| $2v + 4t = 9$         | $-9x + 3y = -15$       |
| 23. $2x - 3y - 4 = 0$ | 24. $3i_1 + 5 = -4i_2$ |
| $3x + 2 = 2y$         | $3i_2 = 5i_1 - 2$      |

In Exercises 25–36, solve the given systems of equations by either method of this section.

- |                        |                       |
|------------------------|-----------------------|
| 25. $2x - y = 5$       | 26. $3x + 2y = 4$     |
| $6x + 2y = -5$         | $6x - 6y = 13$        |
| 27. $6x + 3y + 4 = 0$  | 28. $1 + 6q = 5p$     |
| $5y = -9x - 6$         | $3p - 4q = 7$         |
| 29. $15x + 10y = 11$   | 30. $2x + 6y = -3$    |
| $20x - 25y = 7$        | $-6x - 18y = 5$       |
| 31. $12V + 108 = -84C$ | 32. $66x + 66y = -77$ |
| $36C + 48V + 132 = 0$  | $33x - 132y = 143$    |

$$33. \begin{aligned} 44A &= 1 - 15B \\ 5B &= 22 + 7A \end{aligned}$$

$$35. \begin{aligned} 2b &= 6a - 16 \\ 33a &= 4b + 39 \end{aligned}$$

$$34. \begin{aligned} 3.0x - 2.0y &= 4.0 \\ 2.9x - 2.0y &= 8.0 \end{aligned}$$

$$36. \begin{aligned} 30P &= 55 - Q \\ 19P + 14Q + 32 &= 0 \end{aligned}$$

In Exercises 37–40, in order to make the coefficients easier to work with, first multiply each term of the equation or divide each term of the equation by a number selected by inspection. Then proceed with the solution of the system by an appropriate algebraic method.

$$37. \begin{aligned} 0.3x - 0.7y &= 0.4 \\ 0.2x + 0.5y &= 0.7 \end{aligned}$$

$$38. \begin{aligned} 250R + 225Z &= 400 \\ 375R - 675Z &= 325 \end{aligned}$$

$$39. \begin{aligned} 40s - 30t &= 60 \\ 20s - 40t &= -50 \end{aligned}$$

$$40. \begin{aligned} 0.060x + 0.048y &= -0.084 \\ 0.013x - 0.065y &= -0.078 \end{aligned}$$

In Exercises 41–44, solve the given systems of equations by an appropriate algebraic method.

41. Find the voltages  $V_1$  and  $V_2$  of the batteries shown in Fig. 5.25. The terminals are aligned in the same direction in Fig. 5.25(a) and in the opposite directions in Fig. 5.25(b).

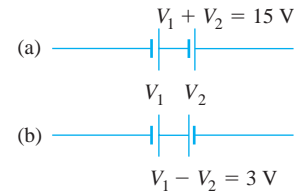


Fig. 5.25

42. A spring of length  $L$  is stretched  $x$  cm for each newton of weight hung from it. Weights of 3 N and then 5 N are hung from the spring, leading to the equations
- $$\begin{aligned} L + 3x &= 18 \\ L + 5x &= 22 \end{aligned}$$
- Solve for  $L$  and  $x$ .
43. Two grades of gasoline are mixed to make a blend with 1.50% of a special additive. Combining  $x$  litres of a grade with 1.80% of the additive to  $y$  litres of a grade with 1.00% of the additive gives 10 000 L of the blend. The equations relating  $x$  and  $y$  are
- $$\begin{aligned} x + y &= 10\,000 \\ 0.0180x + 0.0100y &= 0.0150(10\,000) \end{aligned}$$
- Find  $x$  and  $y$  (to three significant digits).
44. A 6.0% solution and a 15.0% solution of a drug are added to 200 mL of a 20.0% solution to make 1200 mL of a 12.0% solution for a proper dosage. The equations relating the number of millilitres of the added solutions are
- $$\begin{aligned} x + y + 200 &= 1200 \\ 0.060x + 0.150y + 0.200(200) &= 0.120(1200) \end{aligned}$$
- Find  $x$  and  $y$  (to three significant digits).

In Exercises 45–56, set up appropriate systems of two linear equations and solve the systems algebraically. All data are accurate to at least two significant digits.

- 45. A person's email for a week contained a total of 79 messages. The number of spam messages was four more than twice the number of other messages. How many were spam?
- 46. A 150-m cable is cut into two pieces such that one piece is four times as long as the other. How long is each piece?
- 47. The weight  $W_f$  supported by the front wheels of a certain car and the weight  $W_r$  supported by the rear wheels together equal the weight of the car, 17 700 N. See Fig. 5.26. Also, the ratio of  $W_r$  to  $W_f$  is 0.847. What are the weights supported by each set of wheels (to 3 significant digits)?

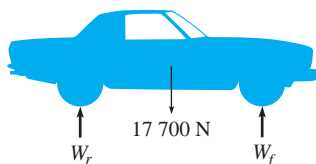


Fig. 5.26

- 48. A sprinkler system is used to water two areas. If the total water flow is 980 L/h and the flow through one sprinkler is 65% as much as the other, what is the flow in each?
- 49. In a test of a heat-seeking rocket, a first rocket is launched at 600 m/s, and the heat-seeking rocket is launched along the same flight path 12 s later at a speed of 960 m/s. Find the times  $t_1$  and  $t_2$  of flight of the rockets until the heat-seeking rocket destroys the first rocket.
- 50. The *moment* of a force (also called *torque*) is the product of the force and the perpendicular distance from a specified axis of rotation. If a lever is supported at only one point (called the *fulcrum*) and is in balance, the sum of the torques (about the support) of forces acting on one side of the support must equal the sum of the torques of the forces acting on the other side. Find the forces  $F_1$  and  $F_2$  that are in the positions shown in Fig. 5.27(a) and then move to the positions in Fig. 5.27(b). The lever weighs 20 N and is in balance in each case.

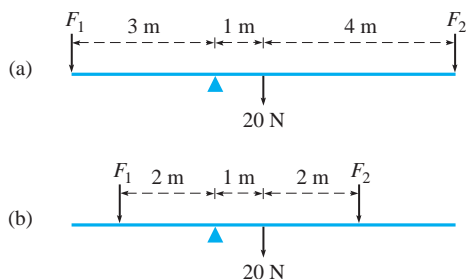


Fig. 5.27

- 51. There are two types of offices in an office building, and a total of 54 offices. One type rents for \$900/month and the other type rents for \$1250/month. If all offices are occupied and the total rental income is \$55 600/month, how many of each type are there?

- 52. An underwater (but near the surface) explosion is detected by sonar on a ship 30 s before it is heard on the deck. If sound travels at 1500 m/s in water and 330 m/s in air, how far is the ship from the explosion (to 3 significant digits)? See Fig. 5.28.

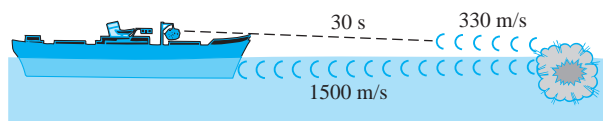


Fig. 5.28

- 53. A small isolated farm uses a windmill and a gas generator for power. During a 10-day period, they produced 3010 kW · h of energy with the windmill operating at 45.0% of capacity and the generator at capacity. During the following 10-day period, they produced 2900 kW · h with the windmill at 72.0% of capacity and the generator down 60 h for repairs (at capacity otherwise). What is the power capacity (in kW) of each?
- 54. In mixing a weed-killing chemical, a 40% solution of the chemical is mixed with an 85% solution to get 20 L of a 60% solution. How much of each solution is needed?
- 55. What conclusion can you draw from a sales report that states that “sales this month were \$8000 more than last month, which means that total sales for both months are \$4000 more than twice the sales last month”?
- 56. For an electric circuit, a report stated that current  $i_1$  is twice current  $i_2$  and that twice the sum of the two currents less 6 times  $i_2$  is 6 mA. Explain your conclusion about the values of the currents found from this report.

In Exercises 57–60, answer the given questions.

- 57. What condition(s) must be placed on the constants of the system of equations
 
$$ax + y = c$$

$$bx + y = d$$
 such that there is a unique solution for  $x$  and  $y$ ?
- 58. What conditions must be placed on the constants of the system of equations in Exercise 57 such that the system is (a) inconsistent? (b) dependent?
- 59. For the dependent system of Example 6 of Section 5.3, both equations can be written as  $y = \frac{1}{3}x - 3$ . The solution for the system can then be shown in terms of a general point on the graph as  $(x, \frac{1}{3}x - 3)$ . This is referred to as the solution with arbitrary  $x$ . Find the solutions in this form for this system for  $x = -3$ , and  $x = 9$ .
- 60. For the dependent system of Example 6 of Section 5.3, write the form for the solution with arbitrary  $y$ . See Exercise 59.

Answers to Practice Exercises

- 1.  $x = 3/5, y = -1/2$     2.  $x = 3, y = -1$

## 5.5 Solving Systems of Two Linear Equations in Two Unknowns by Determinants

Determinant of the Second Order •  
Cramer's Rule • Solving Systems  
of Equations by Determinants

Consider two linear equations in two unknowns, as given in Eq. (5.3):

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned} \quad (5.3)$$

Note that  $a_1$  and  $a_2$  are constants and represent the **coefficients of the first variable**  $x$  in the two equations.  $b_1$  and  $b_2$  are constants and represent the **coefficients of the second variable**  $y$  in the two equations.  $c_1$  and  $c_2$  are also constants in the system of equations. In Section 16.4, these sets of coefficients and constants will be used again as systems of equations are solved using matrices.

If we multiply the first of the equations in Eq. (5.3) by  $b_2$  and the second by  $b_1$ , we obtain

$$\begin{aligned} a_1b_2x + b_1b_2y &= c_1b_2 \\ a_2b_1x + b_2b_1y &= c_2b_1 \end{aligned} \quad (5.6)$$

We see that the coefficients of  $y$  are the same. Thus, subtracting the second equation from the first, we can solve for  $x$ . The solution can be shown to be

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1} \quad (5.7)$$

In the same manner, we may show that

$$y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1} \quad (5.8)$$

■ Determinants were invented by the German mathematician Gottfried Wilhelm Leibniz (1646–1716).

The expression  $a_1b_2 - a_2b_1$ , which appears in each of the denominators of Eqs. (5.7) and (5.8), is an example of a special kind of expression called a *determinant of the second order*. The determinant  $a_1b_2 - a_2b_1$  is denoted by

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

**Second-Order Determinant** Therefore, by definition, a **determinant of the second order** is

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1 \quad (5.9)$$

The numbers  $a_1$  and  $b_1$  are called the **elements** of the first **row** of the determinant. The numbers  $a_1$  and  $a_2$  are the elements of the first **column** of the determinant. In the same manner, the numbers  $a_2$  and  $b_2$  are the elements of the second row, and the numbers  $b_1$  and  $b_2$  are the elements of the second column. The numbers  $a_1$  and  $b_2$  are the elements

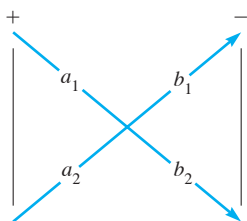


Fig. 5.29

of the **principal diagonal**, and the numbers  $a_2$  and  $b_1$  are the elements of the **secondary diagonal**. Thus, one way of stating the definition indicated in Eq. (5.9) is that the value of a determinant of the second order is found by taking the product of the elements of the principal diagonal and subtracting the product of the elements of the secondary diagonal.

A diagram that is often helpful for remembering the expansion of a second-order determinant is shown in Fig. 5.29. The following examples illustrate how we carry out the evaluation of determinants.

### EXAMPLE 1 Evaluating a second-order determinant

$$\begin{vmatrix} -5 & 8 \\ 3 & 7 \end{vmatrix} = (-5)(7) - 3(8) = -35 - 24 = -59$$

### EXAMPLE 2 Evaluating second-order determinants

$$(a) \begin{vmatrix} 4 & 6 \\ 3 & 17 \end{vmatrix} = 4(17) - (3)(6) = 68 - 18 = 50$$

$$(b) \begin{vmatrix} 4 & 6 \\ -3 & 17 \end{vmatrix} = 4(17) - (-3)(6) = 68 + 18 = 86$$

$$(c) \begin{vmatrix} 3.6 & 6.1 \\ -3.2 & -17.2 \end{vmatrix} = 3.6(-17.2) - (-3.2)(6.1) = -42.4$$

#### Practice Exercise

1. Evaluate the determinant:  $\begin{vmatrix} 2 & -4 \\ 3 & 5 \end{vmatrix}$

Note the signs of the terms being combined. ■

We note that the numerators and denominators of Eqs. (5.7) and (5.8) may be written as determinants. The numerators of the equations are

$$\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

Therefore, the solutions for  $x$  and  $y$  of the system of equations

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned} \tag{5.3}$$

may be written directly in terms of determinants, without algebraic operations, as

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \quad \text{and} \quad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \tag{5.10}$$

Note Carefully the Location of  $c_1$  and  $c_2$

For this reason, determinants provide a quick and easy method of solution of systems of equations. Again, the denominator of each of the equations in Eq. (5.10) is the same.

The determinant of the denominator is made up of the coefficients of  $x$  and  $y$ . Also, the determinant of the numerator of the solution for  $x$  is obtained from the determinant of the denominator by **replacing the column of  $a$ 's by the column of  $c$ 's**. The determinant of the numerator of the solution for  $y$  is obtained from the determinant of the denominator by **replacing the column of  $b$ 's by the column of  $c$ 's**.

■ Cramer's rule is named for the Swiss mathematician Gabriel Cramer (1704–1752).

This result is referred to as **Cramer's rule**. In using Cramer's rule, we must be sure that the equations are written in the form of Eq. (5.3) before setting up the determinants.

The following examples illustrate the method of solving systems of equations by determinants.

### EXAMPLE 3 Solving a system using Cramer's rule

Solve the following system of equations by determinants:

$$\begin{aligned} 2x + y &= 1 \\ 5x - 2y &= -11 \end{aligned}$$

First, note that the equations are in the proper form of Eq. (5.3) for solution by determinants. Next, set up the determinant for the denominator, which consists of the four coefficients in the system, written as shown. It is

$$\begin{array}{c} \left. \begin{array}{|c|} \hline 2 \\ 5 \\ \hline \end{array} \right\} \text{x-coefficients} \quad \left| \begin{array}{cc} 2 & 1 \\ 5 & -2 \end{array} \right| \quad \left. \begin{array}{|c|} \hline 1 \\ -2 \\ \hline \end{array} \right\} \text{y-coefficients} \end{array}$$

For finding  $x$ , the determinant in the numerator is obtained from this determinant by replacing the first column by the constants that appear on the right sides of the equations. Thus, *the numerator for the solution for  $x$*  is

$$\left. \begin{array}{|c|} \hline 1 \\ -11 \\ \hline \end{array} \right\} \text{replace x-coefficients with the constants} \quad \left| \begin{array}{cc} 1 & 1 \\ -11 & -2 \end{array} \right|$$

For finding  $y$ , the determinant in the numerator is obtained from the determinant of the denominator by replacing the second column by the constants that appear on the right sides of the equations. Thus, *the numerator for the solution for  $y$*  is

$$\left. \begin{array}{|c|} \hline 2 \\ 5 \\ \hline \end{array} \right\} \text{replace y-coefficients with the constants} \quad \left| \begin{array}{cc} 2 & 1 \\ 5 & -11 \end{array} \right|$$

Now, set up the solutions for  $x$  and  $y$  using the determinants above:

$$\begin{aligned} x &= \frac{\left| \begin{array}{cc} 1 & 1 \\ -11 & -2 \end{array} \right|}{\left| \begin{array}{cc} 2 & 1 \\ 5 & -2 \end{array} \right|} = \frac{1(-2) - (-11)(1)}{2(-2) - (5)(1)} = \frac{-2 + 11}{-4 - 5} = \frac{9}{-9} = -1 \\ y &= \frac{\left| \begin{array}{cc} 2 & 1 \\ 5 & -11 \end{array} \right|}{\left| \begin{array}{cc} 2 & 1 \\ 5 & -2 \end{array} \right|} = \frac{2(-11) - (5)(1)}{-9} = \frac{-22 - 5}{-9} = 3 \end{aligned}$$

■ Determinants can be evaluated on a graphing calculator. This is described on page 171.

#### LEARNING TIP

Since the same determinant appears in each denominator, *it needs to be evaluated only once*. This means that three determinants are to be evaluated in order to solve the system.

Therefore, the solution to the system of equations is  $x = -1, y = 3$ .

Substituting these values into the equations to check, we have

$$\begin{aligned} 2(-1) + 3 &\stackrel{?}{=} 1 & \text{and} & \quad 5(-1) - 2(3) \stackrel{?}{=} -11 \\ 1 &= 1 & & \quad -11 = -11 \end{aligned}$$

which shows the solutions are correct. ■

**EXAMPLE 4** Solving a system using Cramer's rule

Solve the following system of equations by determinants. Numbers are approximate.

$$5.3x + 7.2y = 4.5$$

$$3.2x - 6.9y = 5.7$$

$$x = \frac{\begin{array}{c} \text{constants} \rightarrow \\ \begin{vmatrix} 4.5 & 7.2 \\ 5.7 & -6.9 \end{vmatrix} \\ \leftarrow \text{coefficients} \end{array}}{\begin{array}{c} \begin{vmatrix} 5.3 & 7.2 \\ 3.2 & -6.9 \end{vmatrix} \\ \leftarrow \text{coefficients} \end{array}} = \frac{4.5(-6.9) - 5.7(7.2)}{5.3(-6.9) - 3.2(7.2)} = \frac{-72.09}{-59.61} = 1.2$$

$$y = \frac{\begin{array}{c} \begin{vmatrix} 5.3 & 4.5 \\ 3.2 & 5.7 \end{vmatrix} \\ \leftarrow \text{coefficients} \end{array}}{\begin{array}{c} \begin{vmatrix} 5.3 & 7.2 \\ 3.2 & -6.9 \end{vmatrix} \\ \leftarrow \text{coefficients} \end{array}} = \frac{5.3(5.7) - 3.2(4.5)}{-59.61} = \frac{15.81}{-59.61} = -0.27$$

**LEARNING TIP**

Determinants are often easier to use than other algebraic methods when the coefficients for the variables in the equations are decimals.

The calculations can all be done on a calculator, using the following procedure: (1) Evaluate the denominator and store this value; (2) divide the value of each numerator by the value of the denominator, storing the values of  $x$  and  $y$ ; (3) use the stored values of  $x$  and  $y$  to check the solution; (4) round off results (if the numbers are approximate). For this system, we store the value of the denominator,  $D = -59.61$ , in order to calculate  $x$  and  $y$ , stored as  $X$  and  $Y$ . ■

**EXAMPLE 5** Solving a system using Cramer's rule—investment application

Two investments totalling \$18 000 yield an annual income of \$700. If the first investment has an interest rate of 5.5% and the second a rate of 3.0%, what is the value of each?

Let  $x$  = the value of the first investment and  $y$  = the value of the second investment. We know that the total of the two investments is \$18 000. This leads to the equation  $x + y = 18\,000$ . The first investment yields  $0.055x$  dollars annually, and the second yields  $0.030y$  dollars annually. This leads to the equation  $0.055x + 0.030y = 700$ . These two equations are then solved simultaneously:

$$\begin{array}{rcl} x + & y = & 18\,000 \quad \text{sum of investments} \\ 0.055x + 0.030y = & 700 & \leftarrow \text{income} \\ \begin{array}{c} \uparrow \quad \uparrow \\ \text{5.5\% value} \quad \text{3.0\% value} \end{array} & & \end{array}$$

$$x = \frac{\begin{vmatrix} 18\,000 & 1 \\ 700 & 0.030 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 0.055 & 0.030 \end{vmatrix}} = \frac{540 - 700}{0.030 - 0.055} = \frac{-160}{-0.025} = 6400$$

**LEARNING TIP**

The equations must be in the form of Eq. (5.3) before the determinants are set up. The specific positions of the values in the determinants are based on that form of writing the system. If either unknown is missing from an equation, a zero must be placed in the proper position.

The value of  $y$  can be found most easily by substituting this value of  $x$  into the first equation,  $y = 18\,000 - x = 18\,000 - 6400 = 11\,600$ .

Therefore, the values invested are \$6400 and \$11 600. Checking, the total income is  $\$6400(0.055) + \$11\,600(0.030) = \$700$ , which agrees with the statement of the problem. ■

**INCONSISTENT OR DEPENDENT SOLUTIONS WITH DETERMINANTS**

If the determinant of the denominator is zero, we do not have a unique solution since this would require division by zero. If the determinant of the denominator is zero and that of the numerator is not zero, the system is *inconsistent*. If the determinants of both numerator and denominator are zero, the system is *dependent*.



**EXAMPLE 6** Inconsistent and dependent systems using determinants

(a) Solve the following system of equations using determinants.

$$\begin{aligned}x - 4y &= 6 \\ -3x + 12y &= -6\end{aligned}$$

$$x = \frac{\begin{vmatrix} 6 & -4 \\ -6 & 12 \end{vmatrix}}{\begin{vmatrix} 1 & -4 \\ -3 & 12 \end{vmatrix}} = \frac{72 - 24}{12 - 12} = \frac{48}{0}$$

$$y = \frac{\begin{vmatrix} 1 & 6 \\ -3 & -6 \end{vmatrix}}{\begin{vmatrix} 1 & -4 \\ -3 & 12 \end{vmatrix}} = \frac{-6 - (-18)}{12 - 12} = \frac{12}{0}$$

Both solutions yield a division by zero error, so the system is inconsistent. (There are no solutions.)

(b) Solve the following system of equations using determinants.

$$\begin{aligned}5x + 2y &= -3 \\ 15x + 6y &= -9\end{aligned}$$

$$x = \frac{\begin{vmatrix} -3 & 2 \\ -9 & 6 \end{vmatrix}}{\begin{vmatrix} 5 & 2 \\ 15 & 6 \end{vmatrix}} = \frac{-18 - (-18)}{30 - 30} = \frac{0}{0}$$

$$y = \frac{\begin{vmatrix} 5 & -3 \\ 15 & -9 \end{vmatrix}}{\begin{vmatrix} 1 & -4 \\ -3 & 12 \end{vmatrix}} = \frac{-45 - (-45)}{30 - 30} = \frac{0}{0}$$

Both solutions yield an indeterminate  $0/0$  division, so the system is dependent. (There are an infinite number of intersections since the two lines are the same line.)

**EXERCISES 5.5**

In Exercises 1–4, make the given changes in the indicated examples of this section and then solve the resulting problems.

- In Example 2(a), change the 6 to  $-6$  and then evaluate.
- In Example 2(a), change the 4 to  $-4$  and the 6 to  $-6$  and then evaluate.
- In Example 3, change the  $+$  to  $-$  in the first equation and then solve the system of equations.
- In Example 5, change \$700 to \$830 and then solve for the values of the investments.

In Exercises 5–16, evaluate the given determinants.

$$\begin{array}{lll} 5. \begin{vmatrix} 2 & 4 \\ 3 & 1 \end{vmatrix} & 6. \begin{vmatrix} -1 & 3 \\ 2 & 6 \end{vmatrix} & 7. \begin{vmatrix} 3 & -5 \\ 7 & -2 \end{vmatrix} \\ 8. \begin{vmatrix} -4 & 7 \\ 1 & -3 \end{vmatrix} & 9. \begin{vmatrix} 8 & -10 \\ 0 & 4 \end{vmatrix} & 10. \begin{vmatrix} -4 & -3 \\ -8 & -6 \end{vmatrix} \end{array}$$

$$11. \begin{vmatrix} -20 & 110 \\ -70 & -80 \end{vmatrix} \quad 12. \begin{vmatrix} -6.5 & 12.2 \\ -15.5 & 34.6 \end{vmatrix} \quad 13. \begin{vmatrix} 0.75 & -1.32 \\ 0.15 & 1.18 \end{vmatrix}$$

$$14. \begin{vmatrix} 0.20 & -0.05 \\ 0.28 & 0.09 \end{vmatrix} \quad 15. \begin{vmatrix} 16 & -8 \\ 42 & -15 \end{vmatrix} \quad 16. \begin{vmatrix} 43 & -7 \\ -81 & 16 \end{vmatrix}$$

In Exercises 17–26, solve the given systems of equations by determinants. (These are the same as those for Exercises 15–24 of Section 5.4.)

$$\begin{array}{ll} 17. \begin{cases} x + 2y = 5 \\ x - 2y = 1 \end{cases} & 18. \begin{cases} x + 3y = 7 \\ 2x + 3y = 5 \end{cases} \\ 19. \begin{cases} 2x - 3y = 4 \\ 2x + y = -4 \end{cases} & 20. \begin{cases} R - 4r = 17 \\ 3R + 4r = 3 \end{cases} \\ 21. \begin{cases} 12t + 9y = 14 \\ 6t = 7y - 16 \end{cases} & 22. \begin{cases} 3x - y = 3 \\ 4x = 3y + 14 \end{cases} \end{array}$$

23.  $v + 2t = 7$   
 $2v + 4t = 9$
24.  $3x - y = 5$   
 $-9x + 3y = -15$
25.  $2x - 3y - 4 = 0$   
 $3x + 2 = 2y$
26.  $3i_1 + 5 = -4i_2$   
 $3i_2 = 5i_1 - 2$

In Exercises 27–34, solve the given systems of equations by determinants. All numbers are approximate. (Exercises 27–30 are the same as Exercises 37–40 of Section 5.4.)

27.  $0.3x - 0.7y = 0.4$   
 $0.2x + 0.5y = 0.7$
28.  $250R + 225Z = 400$   
 $375R - 675Z = 325$
29.  $40s - 30t = 60$   
 $20s - 40t = -50$
30.  $0.060x + 0.048y = -0.084$   
 $0.013x - 0.065y = -0.078$
31.  $301x - 529y = 1520$   
 $385x - 741y = 2540$
32.  $0.25d + 0.63n = -0.37$   
 $-0.61d - 1.80n = 0.55$
33.  $1.2y + 10.8 = -8.4x$   
 $3.5x + 4.8y + 12.9 = 0$
34.  $6541x + 4397y = -7732$   
 $3309x - 8755y = 7622$

In Exercises 35–38, answer the given questions about the determinant to the right.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

35. What is the value of the determinant if  $c = d = 0$ ?
36. What change in value occurs if rows and columns are interchanged ( $b$  and  $c$  are interchanged)?
37. What is the value of the determinant if  $a = kb$  and  $c = kd$ ?
38. How does the value change if  $a$  and  $c$  are doubled?

In Exercises 39–42, solve the given systems of equations by determinants. All numbers are accurate to at least two significant digits.

39. The forces acting on a link of an industrial robot are shown in Fig. 5.30. The equations for finding forces  $F_1$  and  $F_2$  are
- $$F_1 + F_2 = 21$$
- $$2F_1 = 5F_2$$
- Find  $F_1$  and  $F_2$ .

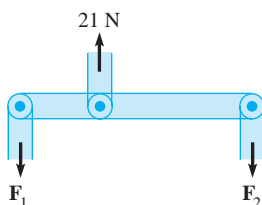


Fig. 5.30

$$A = \frac{1}{2} \left( \begin{vmatrix} x_0 & x_1 \\ y_0 & y_1 \end{vmatrix} + \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} + \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} + \begin{vmatrix} x_3 & x_0 \\ y_3 & y_0 \end{vmatrix} \right)$$

where  $(x_0, y_0)$ ,  $(x_1, y_1)$ ,  $(x_2, y_2)$ , and  $(x_3, y_3)$  are the rectangular coordinates of the vertices of the quadrilateral, listed counterclockwise. (This *surveyor's formula* can be generalized to find the area of any polygon.)

A surveyor records the locations of the vertices of a quadrilateral building lot on a rectangular coordinate system as  $(12.79, 0.00)$ ,  $(67.21, 12.30)$ ,  $(53.05, 47.12)$ , and  $(10.09, 53.11)$ , where distances are in metres. Find the area of the lot.

41. An airplane begins a flight with a total of 144 L of fuel stored in two separate wing tanks. During the flight, 25.0% of the fuel in one tank is used, and in the other tank 37.5% of the fuel is used. If the total fuel used is 44.8 L, the amounts  $x$  and  $y$  used from each tank can be found by solving the system of equations
- $$x + y = 144$$
- $$0.250x + 0.375y = 44.8$$
- Find  $x$  and  $y$ .

42. In applying Kirchhoff's laws (see the chapter introduction; for the equations, see, for example, Beiser, *Modern Technical Physics*, 6th ed., p. 550) to the electric circuit shown in Fig. 5.31, the following equations are found. Find the indicated currents  $I_1$  and  $I_2$  (in A).
- $$52I_1 - 27I_2 = -420$$
- $$-27I_1 + 76I_2 = 210$$

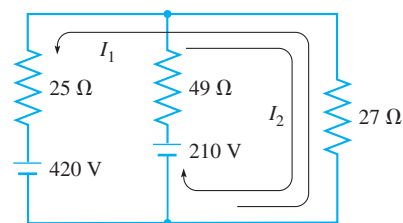


Fig. 5.31

In Exercises 43–52, set up appropriate systems of two linear equations in two unknowns and then solve the systems by determinants. All numbers are accurate to at least two significant digits.

43. A developer built twice as many three-bedroom homes as four-bedroom homes in a new development. The profit from each three-bedroom home was \$25 000, and the profit from each four-bedroom home was \$35 000. If the total profit was \$6 800 000, how many of each were built?
44. Two joggers are 2.0 km apart. If they jog toward each other, they will meet in 12 min. If they jog in the same direction, the faster one will overtake the slower one in 2.0 h. At what rate does each jog?
45. A shipment of 320 cell phones and radar detectors was destroyed due to a truck accident. On the insurance claim, the shipper stated that each phone was worth \$110, each detector was worth \$160, and their total value was \$40 700. How many of each were in the shipment?
46. Two types of electromechanical carburetors are being assembled and tested. Each of the first type requires 15 min of assembly time and 2 min of testing time. Each of the second type requires 12 min of assembly time and 3 min of testing time. If 222 min of assembly time and 45 min of testing time are available, how many of each type can be assembled and tested, if all the time is used?
47. A machinery sales representative receives a fixed salary plus a sales commission each month. If \$6200 is earned on sales of \$70 000 in one month and \$4700 is earned on sales of \$45 000 in the following month, what are the fixed salary and the commission percent?
48. A moving walkway at an airport is 65.0 m long. A child running at a constant speed takes 20.0 s to run along the walkway in the direction it is moving, and then 52.0 s to run all the way back. What are the speed of the walkway and the speed of the child?
49. A boat carrying illegal drugs leaves a port and travels at 63 km/h. A Coast Guard cutter leaves the port 24 min later and travels at 75 km/h in pursuit of the boat. Find the times each has travelled when the cutter overtakes the boat with drugs. See Fig. 5.32.

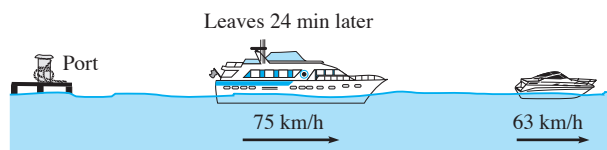


Fig. 5.32

50. Sterling silver is 92.5% silver and 7.5% copper. One silver-copper alloy is 94.0% silver, and a second silver-copper alloy is 85.0% silver. How much of each should be used in order to make 100 g of sterling silver?
51. In an experiment, a variable voltage  $V$  is in a circuit with a fixed voltage  $V_0$  and a resistance  $R$ . The voltage  $V$  is related to the current  $i$  in the circuit by  $V = Ri - V_0$ . If  $V = 5.8$  V for  $i = 2.0$  A and  $V = 24.7$  V for  $i = 6.2$  A, find  $V$  as a function of  $i$ .
52. The velocity of sound in steel is 4850 m/s faster than the velocity of sound in air. One end of a long steel bar is struck, and an

instrument at the other end measures the time it takes for the sound to reach it. The sound in the bar takes 0.0120 s, and the sound in the air takes 0.180 s. What are the velocities of sound in air and in steel?

### Answers to Practice Exercises

1. 22    2.  $x = -2, y = 3$

## 5.6 Solving Systems of Three Linear Equations in Three Unknowns Algebraically

### Algebraic Method Using Addition or Subtraction

Many technical problems involve systems of linear equations with more than two unknowns. In this section, we solve systems with three unknowns, and later we will show how systems with even more unknowns are solved.

Solving such systems is very similar to solving systems in two unknowns. In this section, we will show the algebraic method, and in the next section we will show how determinants are used. Graphical solutions are not typically used since a linear equation in three unknowns represents a plane in space, and drawing the intersection of three such planes is difficult to visualize. Nevertheless, graphical interpretations of systems of three linear equations will be briefly shown at the end of this section.

A system of three linear equations in three unknowns written in the form

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned} \quad (5.11)$$

has as its solution the set of values  $x$ ,  $y$ , and  $z$  that satisfy all three equations simultaneously. The method of solution involves multiplying *two* of the equations by the proper numbers to eliminate *one* of the unknowns between these equations. We then repeat this process, using a *different pair* of the original equations, being sure that we eliminate the same unknown as we did between the first pair of equations. At this point we have two linear equations in two unknowns that can be solved by any of the methods previously discussed.

### EXAMPLE 1 Algebraically solving a system

Solve the following system of equations:

$$\begin{array}{rcl} (1) & 4x + y + 3z & = 1 \\ (2) & 2x - 2y + 6z & = 11 \\ (3) & -6x + 3y + 12z & = -4 \\ \hline (4) & 8x + 2y + 6z & = 2 & (1) \text{ multiplied by } 2 \\ & 2x - 2y + 6z & = 11 & (2) \\ \hline (5) & 10x & + 12z & = 13 & \text{adding} \\ \hline (6) & 12x + 3y + 9z & = 3 & (1) \text{ multiplied by } 3 \\ & -6x + 3y + 12z & = -4 & (3) \\ \hline (7) & 18x & - 3z & = 7 & \text{subtracting} \end{array}$$

■ We can choose to first eliminate any one of the three unknowns. We have chosen  $y$ .

■ At this point we could have used Eqs. (1) and (3) or Eqs. (2) and (3), but we must set them up to eliminate  $y$ .

**Practice Exercise**

1. Solve the system in Example 1 by first eliminating  $x$ .

■ Some graphing calculators have a specific feature for solving simultaneous linear equations. It is necessary only to enter the coefficients and constants to get the solution.

$$\begin{array}{rcl}
 (8) & 10x + 12z & = 13 & (5) \\
 (9) & 72x - 12z & = 28 & (7) \text{ multiplied by } 4 \\
 (10) & 82x & = 41 & \text{adding} \\
 (11) & x & = \frac{1}{2} & \\
 (12) & 18\left(\frac{1}{2}\right) - 3z & = 7 & \text{substituting (10) in (7)} \\
 (13) & -3z & = -2 & \\
 (14) & z & = \frac{2}{3} & \\
 (15) & 4\left(\frac{1}{2}\right) + y + 3\left(\frac{2}{3}\right) & = 1 & \text{substituting (13) and (10) in (1)} \\
 (16) & 2 + y + 2 & = 1 & \\
 & y & = -3 & 
 \end{array}$$

Thus, the solution is  $x = \frac{1}{2}, y = -3, z = \frac{2}{3}$ . Substituting in the equations, we have

$$\begin{array}{rcl}
 4\left(\frac{1}{2}\right) + (-3) + 3\left(\frac{2}{3}\right) & \stackrel{?}{=} & 1 \\
 2\left(\frac{1}{2}\right) - 2(-3) + 6\left(\frac{2}{3}\right) & \stackrel{?}{=} & 11 \\
 -6\left(\frac{1}{2}\right) + 3(-3) + 12\left(\frac{2}{3}\right) & \stackrel{?}{=} & -4 \\
 1 = 1 & & 11 = 11 & & -4 = -4
 \end{array}$$

We see that the solution checks. If we had first eliminated  $x$  or  $z$ , we would have completed the solution in a similar manner. ■

**EXAMPLE 2 Solving a system—application**

An industrial robot and the forces acting on its main link are shown in Fig. 5.33. An analysis of the forces leads to the following equations relating the forces. Determine the forces.

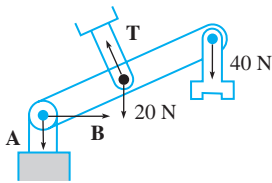


Fig. 5.33

$$\begin{array}{rcl}
 (1) & A + 60 & = 0.8T \\
 (2) & B & = 0.6T \\
 (3) & 8A + 6B + 80 & = 5T & \begin{array}{l} \text{choose 5 to} \\ \text{clear decimals} \end{array} \\
 (4) & 5A & - 4T = -300 & (1) \text{ multiplied by 5 and rewritten} \\
 (5) & 5B - 3T & = 0 & (2) \text{ multiplied by 5 and rewritten} \\
 (6) & 8A + 6B - 5T & = -80 & (3) \text{ rewritten} \\
 (7) & 40A & - 32T = -2400 & (4) \text{ multiplied by 8} \\
 (8) & 40A + 30B - 25T & = -400 & (6) \text{ multiplied by 5} \\
 (9) & -30B - 7T & = -2000 & \text{subtracting} \\
 (10) & 30B - 18T & = 0 & (5) \text{ multiplied by 6} \\
 (11) & -25T & = -2000 & \text{adding} \\
 (12) & T & = 80 \text{ N} & \\
 (13) & A + 60 & = 64 & (12) \text{ substituted in (1)} \\
 (14) & A & = 4 \text{ N} & \\
 (15) & B & = 48 \text{ N} & (12) \text{ substituted in (2)}
 \end{array}$$

Therefore, the forces are  $A = 4 \text{ N}, B = 48 \text{ N},$  and  $T = 80 \text{ N}.$  The solution checks when substituted into the original equations. Note that we could have started the solution by substituting Eq. (2) into Eq. (3), thereby first eliminating  $B.$  ■

**EXAMPLE 3 Setting up and solving a system—voltages in series application**

Three voltages,  $V_1, V_2,$  and  $V_3,$  where  $V_3$  is three times  $V_1,$  are in series with the same polarity (see Fig. 5.34(a)) and have a total voltage of 85 mV. If  $V_2$  is reversed in polarity (see Fig. 5.34(b)), the voltage is 35 mV. Find the voltages.

Since the voltages are in series with the same polarity,  $V_1 + V_2 + V_3 = 85.$  Then, since  $V_3$  is three times  $V_1,$  we have  $V_3 = 3V_1.$  Then, with the reversed polarity of  $V_2,$  we have  $V_1 - V_2 + V_3 = 35.$  Writing these equations in standard form, we have the following solution:

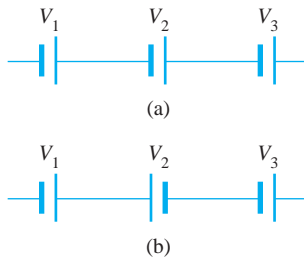


Fig. 5.34

(1)	$V_1 + V_2 + V_3 = 85$	
(2)	$3V_1 - V_3 = 0$	rewriting second equation
(3)	$V_1 - V_2 + V_3 = 35$	
(4)	$2V_1 + 2V_3 = 120$	adding (1) and (3)
(5)	$6V_1 - 2V_3 = 0$	(2) multiplied by 2
(6)	$8V_1 = 120$	adding
(7)	$V_1 = 15 \text{ mV}$	
(8)	$3(15) - V_3 = 0$	substituting (7) in (2)
(9)	$V_3 = 45 \text{ mV}$	
(10)	$15 + V_2 + 45 = 85$	substituting (7) and (9) in (1)
(11)	$V_2 = 25 \text{ mV}$	

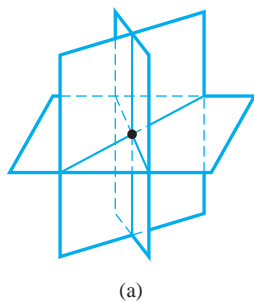
Therefore, the three voltages are 15 mV, 25 mV, and 45 mV.

Checking the solution, the sum of the three voltages is 85 mV,  $V_3$  is three times  $V_1$ , and the sum of  $V_1$  and  $V_3$  less  $V_2$  is 35 mV. ■

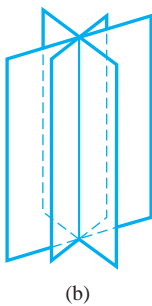
Systems with four or more unknowns are solved in a manner similar to that used for three unknowns. With four unknowns, one is eliminated between three different pairs of equations, and the resulting three equations are then solved.

Linear systems with more than two unknowns may have an unlimited number of solutions or be inconsistent. After eliminating unknowns, if we have  $0 = 0$ , there is an unlimited number of solutions. If we have  $0 = a$  ( $a \neq 0$ ), the system is inconsistent, and there is no solution. (See Exercises 29–32.)

As noted earlier, a linear equation in three unknowns represents a plane in space. For three linear equations in three unknowns, if the planes intersect at a point, there is a unique solution (Fig. 5.35(a)); if they intersect in a line, there is an unlimited number of solutions (Fig. 5.35(b)). If the planes do not have a common intersection, the system is inconsistent. The planes can be parallel (Fig. 5.36(a)), two can be parallel (Fig. 5.36(b)), or they can intersect in three parallel lines (Fig. 5.36(c)). If one plane is coincident with another plane, the system has an unlimited number of solutions if they intersect, or it is inconsistent.

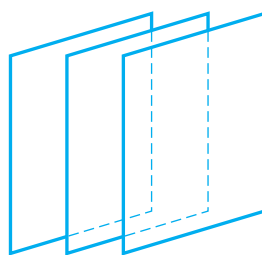


(a)

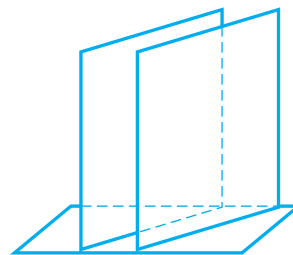


(b)

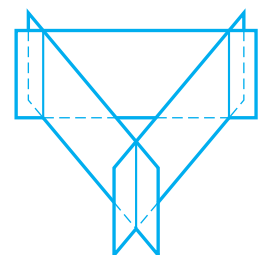
Fig. 5.35



(a)



(b)



(c)

Fig. 5.36

## EXERCISES 5.6

In Exercises 1 and 2, make the given changes in Example 1 of this section and then solve the resulting system of equations.

1. In the second equation, change the constant to the right of the = sign from 11 to 12, and in the third equation, change the constant to the right of the = sign from  $-4$  to  $-14$ .
2. Change the second equation to  $8x + 9z = 10$  (no  $y$ -term).

In Exercises 3–16, solve the given systems of equations.

- |   |   |
|---|---|
| <ol style="list-style-type: none"> <li>3. <math>x + y + z = 2</math></li> <li style="padding-left: 20px;"><math>x - z = 1</math></li> <li style="padding-left: 20px;"><math>x + y = 1</math></li> <li>5. <math>2x + 3y + z = 2</math></li> <li style="padding-left: 20px;"><math>-x + 2y + 3z = -1</math></li> <li style="padding-left: 20px;"><math>-3x - 3y + z = 0</math></li> </ol> | <ol style="list-style-type: none"> <li>4. <math>x + y - z = -3</math></li> <li style="padding-left: 20px;"><math>x + z = 2</math></li> <li style="padding-left: 20px;"><math>2x - y + 2z = 3</math></li> <li>6. <math>2x + y - z = 4</math></li> <li style="padding-left: 20px;"><math>4x - 3y - 2z = -2</math></li> <li style="padding-left: 20px;"><math>8x - 2y - 3z = 3</math></li> </ol> |
|---|---|

7.  $5l + 6w - 3h = 6$   
 $4l - 7w - 2h = -3$   
 $3l + w - 7h = 1$
9.  $2x - 2y + 3z = 5$   
 $2x + y - 2z = -1$   
 $4x - y - 3z = 0$
11.  $3x - 7y + 3z = 6$   
 $3x + 3y + 6z = 1$   
 $5x - 5y + 2z = 5$
13.  $10x + 15y - 25z = 35$   
 $40x - 30y - 20z = 10$   
 $16x - 2y + 8z = 6$
15.  $r - s - 3t - u = 1$   
 $2r + 4s - 2u = 2$   
 $3r + 4s - 2t = 0$   
 $r + 2t - 3u = 3$
8.  $3r + s - t = 2$   
 $r - 2s + t = 0$   
 $4r - s + t = 3$
10.  $2u + 2v + 3w = 0$   
 $3u + v + 4w = 21$   
 $-u - 3v + 7w = 15$
12.  $18x + 24y + 4z = 46$   
 $63x + 6y - 15z = -75$   
 $-90x + 30y - 20z = -55$
14.  $2i_1 - 4i_2 - 4i_3 = 3$   
 $3i_1 + 8i_2 + 2i_3 = -11$   
 $4i_1 + 6i_2 - i_3 = -8$
16.  $3x + 2y - 4z + 2t = 3$   
 $5x - 3y - 5z + 6t = 8$   
 $2x - y + 3z - 2t = 1$   
 $-2x + 3y + 2z - 3t = -2$

In Exercises 17 and 18, three solutions of the equation  $Ax + By + Cz = D$  are given, along with the value of  $D$ . Find the constants  $A$ ,  $B$ , and  $C$ , and write the equation.

17.  $x = 2, y = 4, z = 4$   
 $x = 3, y = -2, z = 8$   
 $x = -1, y = 8, z = 6$   
 $D = 12$
18.  $x = 4, y = 1, z = 1$   
 $x = -1, y = 2, z = 3$   
 $x = 2, y = -1, z = 0$   
 $D = 5$

In Exercises 19–28, solve the systems of equations. In Exercises 23–28, it is necessary to set up the appropriate equations. All numbers are accurate to at least three significant digits.

19. A medical supply company has 1150 worker-hours for production, maintenance, and inspection. Using this and other factors, the number of hours used for each operation,  $P$ ,  $M$ , and  $I$ , respectively, is found by solving the following system of equations:
- $$P + M + I = 1150$$
- $$P = 4I - 100$$
- $$P = 6M + 50$$
20. Three oil pumps fill three different tanks. The pumping rates of the pumps (in L/h) are  $r_1$ ,  $r_2$ , and  $r_3$ , respectively. Because of malfunctions, they do not operate at capacity each time. Their rates can be found by solving the following system of equations:
- $$r_1 + r_2 + r_3 = 14\,000$$
- $$r_1 + 2r_2 = 13\,000$$
- $$3r_1 + 3r_2 + 2r_3 = 36\,000$$
21. The forces acting on a certain girder, as shown in Fig. 5.37, can be found by solving the following system of equations:
- $$0.707F_1 - 0.800F_2 = 0$$
- $$0.707F_1 + 0.600F_2 - F_3 = 10.0$$
- $$3.00F_2 - 3.00F_3 = 20.0$$

Find the forces, in newtons.

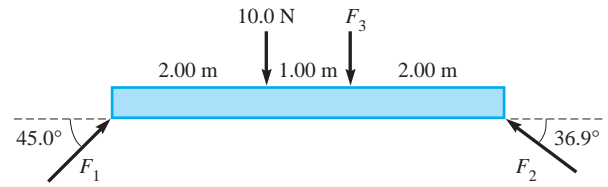


Fig. 5.37

22. Using Kirchhoff's laws (see the chapter introduction; for the equations, see, for example, Beiser, *Modern Technical Physics*, 6th ed., p. 550) with the electric circuit shown in Fig. 5.38, the following equations are found. Find the indicated currents (in A)  $I_1$ ,  $I_2$ , and  $I_3$ .

$$1.0I_1 + 3.0(I_1 - I_3) = 12$$

$$2.0I_2 + 4.0(I_2 + I_3) = 12$$

$$1.0I_1 - 2.0I_2 + 3.0I_3 = 0$$

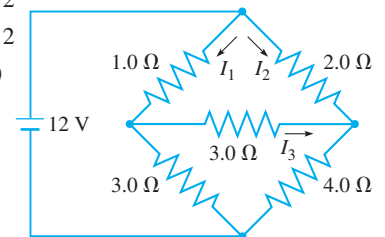


Fig. 5.38

23. Find angles  $A$ ,  $B$ , and  $C$  in the roof truss shown in Fig. 5.39.

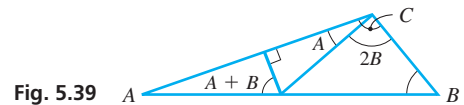


Fig. 5.39

24. Under certain conditions, the cost per kilometre  $C$  of operating a car is a function of the speed  $v$  (in km/h) of the car, given by  $C = av^2 + bv + c$ . If  $C = 28$  ¢/km for  $v = 10$  km/h,  $C = 22$  ¢/km for  $v = 50$  km/h, and  $C = 24$  ¢/km for  $v = 80$  km/h, find  $C$  as a function of  $v$ .
25. The angle  $\theta$  (in degrees) between two links of a robot arm is given by  $\theta = at^3 + bt^2 + ct$ , where  $t$  is the time during an 11.8-s cycle. If  $\theta = 19.0^\circ$  for  $t = 1.00$  s,  $\theta = 30.9^\circ$  for  $t = 3.00$  s, and  $\theta = 19.8^\circ$  for  $t = 5.00$  s, find the equation  $\theta = f(t)$ . See Fig. 5.40.

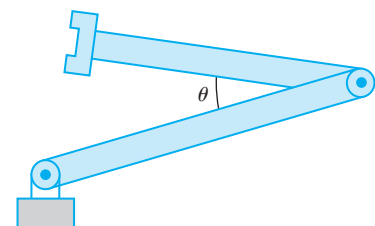


Fig. 5.40

26. The computer systems at three weather bureaus have a combined hard-disk memory capacity of 8.0 TB (terabytes). The memory capacity of systems A and C have 0.2 TB more memory than twice that of system B, and twice the sum of the memory capacities of systems A and B is three times that of system C. What are the memory capacities of each of these computer systems?



27. By weight, one fertilizer is 20% potassium, 30% nitrogen, and 50% phosphorus. A second fertilizer has percents of 10, 20, and 70, respectively, and a third fertilizer has percents of 0, 30, and 70, respectively. How much of each must be mixed to get 200 kg of fertilizer with percents of 12, 25, and 63, respectively?
28. The average traffic flow (number of vehicles) from noon until 1 P.M. in a certain section of one-way streets in a city is shown in Fig. 5.41. Explain why an analysis of the flow through these intersections is not sufficient to obtain unique values for  $x$ ,  $y$ , and  $z$ .

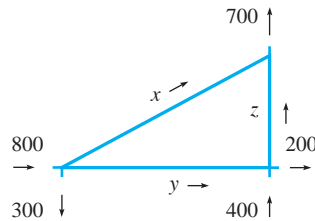


Fig. 5.41

In Exercises 29–32, show that the given systems of equations have either an unlimited number of solutions or no solution. If there is an unlimited number of solutions, find one of them.

29.  $x - 2y - 3z = 2$   
 $x - 4y - 13z = 14$   
 $-3x + 5y + 4z = 0$
30.  $x - 2y - 3z = 2$   
 $x - 4y - 13z = 14$   
 $-3x + 5y + 4z = 2$
31.  $3x + 3y - 2z = 2$   
 $2x - y + z = 1$   
 $x - 5y + 4z = -3$
32.  $3x + y - z = -3$   
 $x + y - 3z = -5$   
 $-5x - 2y + 3z = -7$

Answer to Practice Exercise

1.  $x = 1/2, y = -3, z = 2/3$

## 5.7 Solving Systems of Three Linear Equations in Three Unknowns by Determinants

Determinant of the Third Order • Cramer's Rule • Solving Systems of Equations by Determinants • Determinants on the Calculator

Just as systems of two linear equations in two unknowns can be solved by determinants, so can systems of three linear equations in three unknowns. The system

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned} \tag{5.11}$$

can be solved in general terms by the method of elimination by addition or subtraction. This leads to the following solutions for  $x$ ,  $y$ , and  $z$ .

$$\begin{aligned} x &= \frac{d_1b_2c_2 + d_3b_1c_2 + d_2b_3c_1 - d_3b_2c_1 - d_1b_3c_2 - d_2b_1c_3}{a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3} \\ y &= \frac{a_1d_2c_3 + a_3d_1c_2 + a_2d_3c_1 - a_3d_2c_1 - a_1d_3c_2 - a_2d_1c_3}{a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3} \\ z &= \frac{a_1b_2d_3 + a_3b_1d_2 + a_2b_3d_1 - a_3b_2d_1 - a_1b_3d_2 - a_2b_1d_3}{a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3} \end{aligned} \tag{5.12}$$

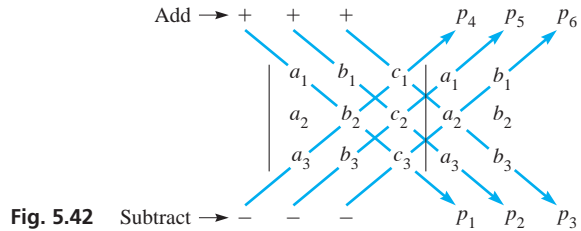
The expressions that appear in the numerators and denominators of Eq. (5.12) are examples of a **determinant of the third order**. This determinant is defined by

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3 \tag{5.13}$$

The elements, rows, columns, and diagonals of a third-order determinant are defined just as are those of a second-order determinant. For example, the principal diagonal is made up of the elements  $a_1$ ,  $b_2$ , and  $c_3$ .

■ This rewriting method is used only for third-order determinants. It does not work for determinants of order higher than three.

Probably the easiest way of remembering the method of finding the value of a third-order determinant is as follows: Rewrite the first two columns to the right of the determinant. The products of the elements of the principal diagonal and the two parallel diagonals to the right of it are then added. The products of the elements of the secondary diagonal and the two parallel diagonals to the right of it are subtracted from the first sum. The algebraic sum of these six products gives the value of the determinant. These products are indicated in Fig. 5.42.



Examples 1 and 2 illustrate this method of evaluating third-order determinants.

**Practice Exercise**

1. Evaluate the determinant

$$\begin{vmatrix} 2 & -2 & 0 \\ -1 & 5 & 1 \\ 3 & 4 & 5 \end{vmatrix}$$

**EXAMPLE 1** Evaluating a third-order determinant

$$\begin{vmatrix} 1 & 5 & 4 \\ -2 & 3 & -1 \\ 2 & -1 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 5 & p_1 & p_2 & p_3 \\ -2 & 3 & & & \\ 2 & -1 & & & \end{vmatrix} = 15 + (-10) + (+8) - (24) - (1) - (-50) = 38$$

**EXAMPLE 2** Evaluating a determinant on a calculator

$$\begin{vmatrix} 3 & -2 & 8 \\ -5 & 5 & 0 \\ 4 & 9 & -6 \end{vmatrix} = \begin{vmatrix} 3 & -2 & & & \\ -5 & 5 & & & \\ 4 & 9 & & & \end{vmatrix} = (-90) + 0 + (-360) - 160 - 0 - (-60) = -550$$

■ Determinants can be evaluated on a calculator using the *matrix* feature. The numbers are entered into a *matrix* (an array of numbers; see Section 16.1), and the determinant is calculated using the *det* instruction.

Inspection of Eq. (5.12) reveals that the numerators of these solutions may also be written in terms of determinants. Thus, we may write the general solution to a system of three equations in three unknowns as

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} \quad y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} \quad z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} \quad (5.14)$$

As in Section 5.5, if the determinant of the denominator is not zero, there is a unique solution to the system of equations. If all determinants are zero, there is an unlimited number of solutions. If the determinant of the denominator is zero and any of the determinants of the numerators is not zero, the system is *inconsistent*, and there is no solution.

An analysis of Eq. (5.14) shows that the situation is precisely the same as it was when we were using determinants to solve systems of two linear equations. That is, the determinants in the denominators in the expressions for  $x$ ,  $y$ , and  $z$  are the same. They consist of elements that are the coefficients of the unknowns. The determinant of the numerator of the solution for  $x$  is the same as that of the denominator, except that the column of  $d$ 's replaces the column of  $a$ 's. The determinant in the numerator of the solution for  $y$  is the same as that of the denominator, except that the column of  $d$ 's replaces the column of  $b$ 's. The determinant of the numerator of the solution for  $z$  is the same as the determinant of the denominator, except that the column of  $d$ 's replaces the column of  $c$ 's. To summarize:

### Cramer's Rule for Three Equations and Three Unknowns

*The determinant in the denominator of each is made up of the coefficients of  $x$ ,  $y$ , and  $z$ . The determinants in the numerators are the same as that in the denominator, except that the column of  $d$ 's replaces the column of coefficients of the unknown for which we are solving.*

Remember, the equations must be written in the standard form shown in Eq. (5.11) before the determinants are formed.

#### EXAMPLE 3 Solving a system using Cramer's rule

Solve the following system by determinants:

$$3x + 2y - 5z = -1$$

$$2x - 3y - z = 11$$

$$5x - 2y + 7z = 9$$

constants  $\rightarrow$

$$x = \frac{\begin{vmatrix} -1 & 2 & -5 & -1 & 2 \\ 11 & -3 & -1 & 11 & -3 \\ 9 & -2 & 7 & 9 & -2 \end{vmatrix}}{\begin{vmatrix} 3 & 2 & -5 & 3 & 2 \\ 2 & -3 & -1 & 2 & -3 \\ 5 & -2 & 7 & 5 & -2 \end{vmatrix}}$$

evaluate the denominator first; if it is zero, there is no need to evaluate the numerator

coefficients  $\rightarrow$

$$x = \frac{21 - 18 + 110 - 135 + 2 - 154}{-63 - 10 + 20 - 75 - 6 - 28} = \frac{-174}{-162} = \frac{29}{27}$$

constants  $\rightarrow$

$$y = \frac{\begin{vmatrix} 3 & -1 & -5 & 3 & -1 \\ 2 & 11 & -1 & 2 & 11 \\ 5 & 9 & 7 & 5 & 9 \end{vmatrix}}{-162} = \frac{231 + 5 - 90 + 275 + 27 + 14}{-162}$$

denominator = -162 from solution for  $x$

$$y = \frac{462}{-162} = -\frac{77}{27}$$

constants  $\rightarrow$

$$z = \frac{\begin{vmatrix} 3 & 2 & -1 & 3 & 2 \\ 2 & -3 & 11 & 2 & -3 \\ 5 & -2 & 9 & 5 & -2 \end{vmatrix}}{-162} = \frac{-81 + 110 + 4 - 15 + 66 - 36}{-162}$$

denominator = -162 from solution for  $x$

$$z = \frac{48}{-162} = -\frac{8}{27}$$

Substituting in each of the original equations shows that the solution checks:

$$3\left(\frac{29}{27}\right) + 2\left(-\frac{77}{27}\right) - 5\left(-\frac{8}{27}\right) = \frac{87 - 154 + 40}{27} = \frac{-27}{27} = -1$$

$$2\left(\frac{29}{27}\right) - 3\left(-\frac{77}{27}\right) - \left(-\frac{8}{27}\right) = \frac{58 + 231 + 8}{27} = \frac{297}{27} = 11$$

$$5\left(\frac{29}{27}\right) - 2\left(-\frac{77}{27}\right) + 7\left(-\frac{8}{27}\right) = \frac{145 + 154 - 56}{27} = \frac{243}{27} = 9$$

After the values of  $x$  and  $y$  were determined, we could also have evaluated  $z$  by substituting the values of  $x$  and  $y$  into one of the original equations. ■

#### EXAMPLE 4 Setting up and solving a system of three linear equations using determinants

An 8.0% solution, an 11% solution, and an 18% solution of nitric acid are to be mixed to get 150 mL of a 12% solution. If the volume of acid from the 8.0% solution equals half the volume of acid from the other two solutions, how much of each is needed?

Let  $x$  = volume of 8.0% solution needed,  $y$  = volume of 11% solution needed, and  $z$  = volume of 18% solution needed.

The fact that the sum of the volumes of the three solutions is 150 mL leads to the equation  $x + y + z = 150$ . Since there are  $0.080x$  mL of pure acid from the first solution,  $0.11y$  mL from the second solution,  $0.18z$  mL from the third solution, and  $0.12(150)$  mL in the final solution, we are led to the equation  $0.080x + 0.11y + 0.18z = 18$ . Finally, using the last stated condition, we have the equation  $0.080x = 0.5(0.11y + 0.18z)$ . These equations are then written in the form of Eq. (5.11) and solved.

	$x + y + z = 150$	sum of volumes
	$0.080x + 0.11y + 0.18z = 18$	volumes of pure acid
acid in 8.0% solution	$0.080x = 0.055y + 0.090z$	one-half of acid in others
	$x + y + z = 150$	standard form of
	$0.080x + 0.11y + 0.18z = 18$	Eq. (5.11) by rewriting
	$0.080x - 0.055y - 0.090z = 0$	third equation

$$x = \frac{\begin{vmatrix} 150 & 1 & 1 \\ 18 & 0.11 & 0.18 \\ 0 & -0.055 & -0.090 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ 0.080 & 0.11 & 0.18 \\ 0.080 & -0.055 & -0.090 \end{vmatrix}} = \frac{150(1)(-0.090) - 1(1)(-0.090) + 1(1)(0.11)}{-0.0099 + 0.0144 - 0.0044 - 0.0088 + 0.0099 + 0.0072} = \frac{0.630}{0.0084} = 75$$

$$y = \frac{\begin{vmatrix} 1 & 150 & 1 \\ 0.080 & 18 & 0.18 \\ 0.080 & 0 & -0.090 \end{vmatrix}}{0.0084} = \frac{-1.620 + 2.160 + 0 - 1.440 - 0 + 1.080}{0.0084} = \frac{0.180}{0.0084} = 21$$

$$z = \frac{\begin{vmatrix} 1 & 1 & 150 \\ 0.080 & 0.11 & 18 \\ 0.080 & -0.055 & 0 \end{vmatrix} \begin{vmatrix} 1 & 1 \\ 0.080 & 0.11 \\ 0.080 & -0.055 \end{vmatrix}}{0.0084}$$

the value of  $z$  can also be found by substituting  $x = 75$  and  $y = 21$  into the first equation

$$z = \frac{0 + 1.440 - 0.660 - 1.320 + 0.990 - 0}{0.0084} = \frac{0.450}{0.0084} = 54$$

Therefore, 75 mL of the 8.0% solution, 21 mL of the 11% solution, and 54 mL of the 18% solution are required to make the 12% solution. Results have been rounded off to two significant digits, the accuracy of the data. Checking with the statement of the problem, we see that these volumes total 150 mL. ■

As we previously noted, the calculations can be done completely on the calculator using either the *matrix* feature (as illustrated with Examples 2 and 3) or using the method outlined after Example 4 in Section 5.5. Briefly, using this method for three equations is: (1) Evaluate and store the value of the denominator; (2) divide the value of each numerator by the value of the denominator, storing the values of  $x$ ,  $y$ , and  $z$ ; (3) use the stored values of  $x$ ,  $y$ , and  $z$  to check the solution.

## EXERCISES 5.7

In Exercises 1 and 2, make the given change in the indicated examples of this section and then solve the resulting problems.

- In Example 1, interchange the first and second rows of the determinant and then evaluate it.
- In Example 3, change the constant to the right of the = sign in the first equation from  $-1$  to  $-3$ , change the constant to the right of the = sign in the third equation from  $9$  to  $11$ , and then solve the resulting system of equations.

In Exercises 3–14, evaluate the given third-order determinants.

<p>3. <math>\begin{vmatrix} 5 &amp; 4 &amp; -1 \\ -2 &amp; -6 &amp; 8 \\ 7 &amp; 1 &amp; 1 \end{vmatrix}</math></p>	<p>4. <math>\begin{vmatrix} -7 &amp; 0 &amp; 0 \\ 2 &amp; 4 &amp; 5 \\ 1 &amp; 4 &amp; 2 \end{vmatrix}</math></p>
<p>5. <math>\begin{vmatrix} 8 &amp; 9 &amp; -6 \\ -3 &amp; 7 &amp; 2 \\ 4 &amp; -2 &amp; 5 \end{vmatrix}</math></p>	<p>6. <math>\begin{vmatrix} -2 &amp; 4 &amp; -1 \\ 5 &amp; -10 &amp; 4 \\ 4 &amp; -8 &amp; 2 \end{vmatrix}</math></p>
<p>7. <math>\begin{vmatrix} -3 &amp; -4 &amp; -8 \\ 5 &amp; -1 &amp; 0 \\ 2 &amp; 10 &amp; -1 \end{vmatrix}</math></p>	<p>8. <math>\begin{vmatrix} 10 &amp; 2 &amp; -7 \\ -2 &amp; -3 &amp; 6 \\ 6 &amp; 5 &amp; -2 \end{vmatrix}</math></p>
<p>9. <math>\begin{vmatrix} 4 &amp; -3 &amp; -11 \\ -9 &amp; 2 &amp; -2 \\ 0 &amp; 1 &amp; -5 \end{vmatrix}</math></p>	<p>10. <math>\begin{vmatrix} 9 &amp; -2 &amp; 0 \\ -1 &amp; 3 &amp; -6 \\ -4 &amp; -6 &amp; -2 \end{vmatrix}</math></p>
<p>11. <math>\begin{vmatrix} 25 &amp; 18 &amp; -50 \\ -30 &amp; 40 &amp; -12 \\ -20 &amp; 55 &amp; -22 \end{vmatrix}</math></p>	<p>12. <math>\begin{vmatrix} 20 &amp; 0 &amp; -15 \\ -4 &amp; 30 &amp; 1 \\ 6 &amp; -1 &amp; 40 \end{vmatrix}</math></p>
<p>13. <math>\begin{vmatrix} 0.1 &amp; -0.2 &amp; 0 \\ -0.5 &amp; 1 &amp; 0.4 \\ -2 &amp; 0.8 &amp; 2 \end{vmatrix}</math></p>	<p>14. <math>\begin{vmatrix} 0.25 &amp; -0.54 &amp; -0.42 \\ 1.20 &amp; 0.35 &amp; 0.28 \\ -0.50 &amp; 0.12 &amp; -0.44 \end{vmatrix}</math></p>

In Exercises 15–30, solve the given systems of equations by use of determinants. (Exercises 17–26 are the same as Exercises 3–12 of Section 5.6.) Assume all numbers are accurate to 3 significant digits.

<p>15. <math>2x + 3y + z = 4</math>  <math>3x - z = -3</math>  <math>x - 2y + 2z = -5</math></p>	<p>16. <math>4x + y + z = 2</math>  <math>2x - y - z = 4</math>  <math>3y + z = 2</math></p>
<p>17. <math>x + y + z = 2</math>  <math>x - z = 1</math>  <math>x + y = 1</math></p>	<p>18. <math>x + y - z = -3</math>  <math>x + z = 2</math>  <math>2x - y + 2z = 3</math></p>
<p>19. <math>2x + 3y + z = 2</math>  <math>-x + 2y + 3z = -1</math>  <math>-3x - 3y + z = 0</math></p>	<p>20. <math>2x + y - z = 4</math>  <math>4x - 3y - 2z = -2</math>  <math>8x - 2y - 3z = 3</math></p>
<p>21. <math>5l + 6w - 3h = 6</math>  <math>4l - 7w - 2h = -3</math>  <math>3l + w - 7h = 1</math></p>	<p>22. <math>3r + s - t = 2</math>  <math>r - 2s + t = 0</math>  <math>4r - s + t = 3</math></p>
<p>23. <math>2x - 2y + 3z = 5</math>  <math>2x + y - 2z = -1</math>  <math>4x - y - 3z = 0</math></p>	<p>24. <math>2u + 2v + 3w = 0</math>  <math>3u + v + 4w = 21</math>  <math>-u - 3v + 7w = 15</math></p>
<p>25. <math>3x - 7y + 3z = 6</math>  <math>3x + 3y + 6z = 1</math>  <math>5x - 5y + 2z = 5</math></p>	<p>26. <math>18x + 24y + 4z = 46</math>  <math>63x + 6y - 15z = -75</math>  <math>-90x + 30y - 20z = -55</math></p>
<p>27. <math>p + 2q + 2r = 0</math>  <math>2p + 6q - 3r = -1</math>  <math>4p - 3q + 6r = -8</math></p>	<p>28. <math>9x + 12y + 2z = 23</math>  <math>21x + 2y - 5z = -25</math>  <math>-18x + 6y - 4z = -11</math></p>
<p>29. <math>3.0x + 4.5y - 7.5z = 10.5</math>  <math>4.8x - 3.6y - 2.4z = 1.2</math>  <math>4.0x - 0.5y + 2.0z = 1.5</math></p>	<p>30. <math>26L - 52M - 52N = 39</math>  <math>45L + 96M + 40N = -80</math>  <math>55L + 62M - 11N = -48</math></p>

In Exercises 31–34, use the determinant at the right. Answer the questions about the determinant for the changes given in each exercise.

$$\begin{vmatrix} 4 & 2 & 1 \\ 3 & 6 & 5 \\ 7 & 9 & 8 \end{vmatrix} = 19$$

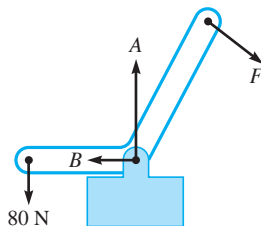
31. How does the value change if the first two rows are interchanged?
32. What is the value if the second row is replaced with the first row (the first row remains unchanged—the first and second rows are the same)?
33. How does the value change if the elements of the first row are added to the corresponding elements of the second row (the first row remains unchanged)?
34. How does value change if each element of the first row is multiplied by 2?

In Exercises 35–45, solve the given problems by determinants. In Exercises 38–45, set up appropriate systems of equations. All numbers are accurate to at least two significant digits.

35. In analysing the forces on the bell-crank mechanism shown in Fig. 5.43, the following equations are found. Find the forces.

$$\begin{aligned} A - 0.60F &= 80 \\ B - 0.80F &= 0 \\ 6.0A - 10F &= 0 \end{aligned}$$

Fig. 5.43



36. Using Kirchoff's laws (see the chapter introduction; for the equations, see, for example, Beiser, *Modern Technical Physics*, 6th ed., p. 550) with the electric circuit shown in Fig. 5.44, the following equations are found. Find the indicated currents (in A)  $I_1$ ,  $I_2$ , and  $I_3$ .

$$\begin{aligned} 19I_1 - 12I_2 &= 60 \\ 12I_1 - 18I_2 + 6.0I_3 &= 0 \\ 6.0I_2 - 18I_3 &= 0 \end{aligned}$$

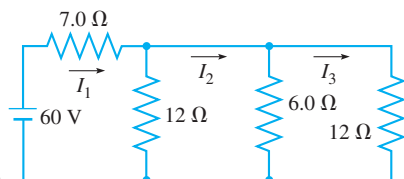


Fig. 5.44

37. In a laboratory experiment to measure the acceleration of an object, the distances travelled by the object were recorded for three different time intervals. These data led to the following equations:

$$\begin{aligned} s_0 + 2v_0 + 2a &= 20 \\ s_0 + 4v_0 + 8a &= 54 \\ s_0 + 6v_0 + 18a &= 104 \end{aligned}$$

Here,  $s_0$  is the initial displacement (in m),  $v_0$  is the initial velocity (in m/s), and  $a$  is the acceleration (in  $m/s^2$ ). Find  $s_0$ ,  $v_0$ , and  $a$ .

38. A certain 18-hole golf course has par-3, par-4, and par-5 holes, and there are twice as many par-4 holes as par-5 holes. How many holes of each type are there if a golfer has par on every hole for a score of 70?

39. The voltage  $V$  across part of an electric circuit is a function of the temperature  $T$  (in  $^{\circ}C$ ) given by  $V = a + bT + cT^2$ , where  $a$ ,  $b$ , and  $c$  are constants. By evaluating  $a$ ,  $b$ , and  $c$ , find  $V = f(T)$  if  $V = 6.4$  V for  $T = 2.0^{\circ}C$ ,  $V = 8.6$  V for  $T = 4.0^{\circ}C$ , and  $V = 11.6$  V for  $T = 6.0^{\circ}C$ .

40. A mail-order company charges \$4 for shipping orders of less than \$50, \$6 for orders from \$50 to \$200, and \$8 for orders over \$200. One day the total shipping charges were \$2160 for 384 orders. Find the number of orders shipped at each rate if the number of orders under \$50 was 12 more than twice the number of orders over \$200.

41. An alloy used in electrical transformers contains nickel (Ni), iron (Fe), and molybdenum (Mo). The percent of Ni is 1% less than five times the percent of Fe. The percent of Fe is 1% more than three times the percent of Mo. Find the percent of each in the alloy.

42. A person invests \$20 000, partly at 5.00%, partly at 6.00%, and the remainder at 6.50%. The total annual interest is \$1170. Three times the amount invested at 6.00% equals the amount invested at 5.00% and 6.50% combined. How much is invested at each rate?

43. A person spent 1.10 h in a car going to an airport, 1.95 h flying in a jet, and 0.520 h in a taxi to reach the final destination. The jet's speed averaged 12.0 times that of the car, which averaged 15.0 km/h more than the taxi. What was the average speed of each if the trip covered 1140 km?

44. An intravenous aqueous solution is made from three mixtures to get 500 mL with 6.0% of one medication, 8.0% of a second medication, and 86% water. The percents in the mixtures are, respectively, 5.0, 20, 75 (first), 0, 5.0, 95 (second), and 10, 5.0, 85 (third). How much of each is used?

45. Three unknown forces are exerted at the pin joint shown in a static structure (see Fig. 5.45). It is known that the sum of the magnitudes of the three forces is 125 N. Analysing the force components in the horizontal and vertical directions yields two more equations:

$$\begin{aligned} -0.174F_1 - 0.985F_2 + 0.927F_3 &= 0 \\ 0.985F_1 - 0.174F_2 - 0.375F_3 &= 0 \end{aligned}$$

Solve for the forces.

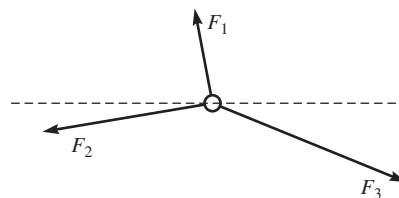


Fig. 5.45

Answer to Practice Exercise

1. 26



## CHAPTER 5 EQUATIONS

$$\text{Linear equation in one unknown} \quad ax + b = 0 \quad (5.1)$$

$$\text{Linear equation in two unknowns} \quad ax + by = c \quad (5.2)$$

$$\begin{aligned} \text{System of two linear equations} \quad a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned} \quad (5.3)$$

$$\text{Definition of slope} \quad m = \frac{y_2 - y_1}{x_2 - x_1} \quad (5.4)$$

$$\text{Slope-intercept form} \quad y = mx + b \quad (5.5)$$

$$\text{Second-order determinant} \quad \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1 \quad (5.9)$$

$$\text{Cramer's rule} \quad x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \quad \text{and} \quad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \quad (5.10)$$

$$\begin{aligned} \text{System of three linear equations} \quad a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned} \quad (5.11)$$

$$\text{Third-order determinant} \quad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3 \quad (5.13)$$

$$\text{Cramer's rule} \quad x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} \quad y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} \quad z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} \quad (5.14)$$

## CHAPTER 5 REVIEW EXERCISES

In Exercises 1–4, evaluate the given determinants.

$$1. \begin{vmatrix} -2 & 5 \\ 3 & 1 \end{vmatrix}$$

$$2. \begin{vmatrix} 40 & 10 \\ -20 & -60 \end{vmatrix}$$

$$3. \begin{vmatrix} -18 & -33 \\ -21 & 44 \end{vmatrix}$$

$$4. \begin{vmatrix} 0.91 & -1.2 \\ 0.73 & -5.0 \end{vmatrix}$$

In Exercises 5–8, find the slopes of the lines that pass through the given points.

$$5. (2, 0), (4, -8)$$

$$6. (-1, -5), (-4, 4)$$

$$7. (40, -20), (-30, -40)$$

$$8. \left(-6, \frac{1}{2}\right), \left(1, -\frac{7}{2}\right)$$

In Exercises 9–12, find the slopes and the y-intercepts of the lines with the given equations. Sketch the graphs.

$$9. y = -2x + 4$$

$$10. 2y = \frac{2}{3}x - 3$$

$$11. 8x - 2y = 5$$

$$12. 3x = 8 + 3y$$

In Exercises 13–20, solve the given systems of equations graphically.

$$13. y = 2x - 4$$

$$14. y = -3x + 3$$

$$y = -\frac{3}{2}x + 3$$

$$y = 2x - 6$$

$$15. 4A - B = 6$$

$$16. 2x - 5y = 10$$

$$3A + 2B = 12$$

$$3x + y = 6$$

$$\begin{array}{ll} 17. 7x = 2y + 14 & 18. 5v = 15 - 3t \\ y = -4x + 4 & t = 6v - 12 \\ 19. 3M + 4N = 6 & 20. 5x + 2y = 5 \\ 2M - 3N = 2 & 2x - 4y = 3 \end{array}$$

In Exercises 21–30, solve the given systems of equations by using an appropriate algebraic method.

$$\begin{array}{ll} 21. x + 2y = 5 & 22. 2x - y = 7 \\ x + 3y = 7 & x + y = 2 \\ 23. 4x + 3y = -4 & 24. r = -3s - 2 \\ y = 2x - 3 & -2r - 9s = 2 \\ 25. 10i - 27v = 29 & 26. 3x - 6y = 5 \\ 40i + 33v = 69 & 7x + 2y = 4 \\ 27. 7x = 2y - 6 & 28. 3R = 8 - 5I \\ 7y = 12 - 4x & 6I = 8R + 11 \\ 29. 90x - 110y = 40 & 30. 0.42x - 0.56y = 1.26 \\ 30x - 15y = 25 & 0.98x - 1.40y = -0.28 \end{array}$$

In Exercises 31–40, solve the given systems of equations by determinants. (These are the same as for Exercises 21–30.)

$$\begin{array}{ll} 31. x + 2y = 5 & 32. 2x - y = 7 \\ x + 3y = 7 & x + y = 2 \\ 33. 4x + 3y = -4 & 34. r = -3s - 2 \\ y = 2x - 3 & -2r - 9s = 2 \\ 35. 10i - 27v = 29 & 36. 3x - 6y = 5 \\ 40i + 33v = 69 & 7x + 2y = 4 \\ 37. 7x = 2y - 6 & 38. 3R = 8 - 5I \\ 7y = 12 - 4x & 6I = 8R + 11 \\ 39. 90x - 110y = 40 & 40. 0.42x - 0.56y = 1.26 \\ 30x - 15y = 25 & 0.98x - 1.40y = -0.28 \end{array}$$

In Exercises 41–43, choose an exercise from among Exercises 31–40 that you think is most easily solved by the indicated method. In Exercises 41–44, explain your answers.

41. Substitution    42. Addition or subtraction    43. Determinants  
44. Considering the slopes  $m$  and the  $y$ -intercepts  $b$ , for a system of two equations, with two unknowns, explain how to determine if there is (a) a unique solution, (b) an inconsistent solution, or (c) a dependent solution.

In Exercises 45–48, evaluate the given determinants.

$$\begin{array}{ll} 45. \begin{vmatrix} 4 & -1 & 8 \\ -1 & 6 & -2 \\ 2 & 1 & -1 \end{vmatrix} & 46. \begin{vmatrix} -500 & 0 & -500 \\ 250 & 300 & -100 \\ -300 & 200 & 200 \end{vmatrix} \\ 47. \begin{vmatrix} -2.2 & -4.1 & 7.0 \\ 1.2 & 6.4 & -3.5 \\ -7.2 & 2.4 & -1.0 \end{vmatrix} & 48. \begin{vmatrix} 30 & 22 & -12 \\ 0 & -34 & 44 \\ 35 & -41 & -27 \end{vmatrix} \end{array}$$

In Exercises 49–54, solve the given systems of equations algebraically. In Exercises 53 and 54, the numbers are approximate, and are accurate to 3 significant digits.

$$\begin{array}{ll} 49. 2x + y + z = 4 & 50. x + 2y + z = 2 \\ x - 2y - z = 3 & 3x - 6y + 2z = 2 \\ 3x + 3y - 2z = 1 & 2x - z = 8 \\ 51. 2r + s + 2t = 8 & 52. 4u + 4v - 2w = -4 \\ 3r - 2s - 4t = 5 & 20u - 15v + 10w = -10 \\ -2r + 3s + 4t = -3 & 24u - 12v - 9w = 39 \\ 53. 3.6x + 5.2y - z = -2.2 & 54. 32t + 24u + 63v = 32 \\ 3.2x - 4.8y + 3.9z = 8.1 & 42t - 31u + 19v = 132 \\ 6.4x + 4.1y + 2.3z = 5.1 & 48t + 12u + 11v = 0 \end{array}$$

In Exercises 55–60, solve the given systems of equations by determinants. In Exercises 59 and 60, the numbers are approximate. (These systems are the same as for Exercises 49–54.)

$$\begin{array}{ll} 55. 2x + y + z = 4 & 56. x + 2y + z = 2 \\ x - 2y - z = 3 & 3x - 6y + 2z = 2 \\ 3x + 3y - 2z = 1 & 2x - z = 8 \\ 57. 2r + s + 2t = 8 & 58. 4u + 4v - 2w = -4 \\ 3r - 2s - 4t = 5 & 20u - 15v + 10w = -10 \\ -2r + 3s + 4t = -3 & 24u - 12v - 9w = 39 \\ 59. 3.6x + 5.2y - z = -2.2 & 60. 32t + 24u + 63v = 32 \\ 3.2x - 4.8y + 3.9z = 8.1 & 42t - 31u + 19v = 132 \\ 6.4x + 4.1y + 2.3z = 5.1 & 48t + 12u + 11v = 0 \end{array}$$

In Exercises 61–64, solve for  $x$ . Here, we see that we can solve an equation in which the unknown is an element of a determinant.

$$\begin{array}{ll} 61. \begin{vmatrix} 2 & 5 \\ 1 & x \end{vmatrix} = 3 & 62. \begin{vmatrix} -1 & x \\ 3 & 4 \end{vmatrix} = 7 \\ 63. \begin{vmatrix} x & 1 & 2 \\ 0 & -1 & 3 \\ -2 & 2 & 1 \end{vmatrix} = 5 & 64. \begin{vmatrix} 1 & 2 & -1 \\ -2 & 3 & x \\ -1 & 2 & -2 \end{vmatrix} = -3 \end{array}$$

In Exercises 65–68, let  $1/x = u$  and  $1/y = v$ . Solve for  $u$  and  $v$  and then solve for  $x$  and  $y$ . In this way, we see how to solve systems of equations involving reciprocals.

$$\begin{array}{ll} 65. \frac{1}{x} - \frac{1}{y} = \frac{1}{2} & 66. \frac{1}{x} + \frac{1}{y} = 3 \\ \frac{1}{x} + \frac{1}{y} = \frac{1}{4} & \frac{2}{x} + \frac{1}{y} = 1 \\ 67. \frac{2}{x} + \frac{3}{y} = 3 & 68. \frac{3}{x} - \frac{2}{y} = 4 \\ \frac{5}{x} - \frac{6}{y} = 3 & \frac{2}{x} + \frac{4}{y} = 1 \end{array}$$

In Exercises 69 and 70, determine the value of  $k$  that makes the system dependent. In Exercises 71 and 72, determine the value of  $k$  that makes the system inconsistent.

$$\begin{array}{ll} 69. 3x - ky = 6 & 70. 5x + 20y = 15 \\ x + 2y = 2 & 2x + ky = 6 \\ 71. kx - 2y = 5 & 72. 2x - 5y = 7 \\ 4x + 6y = 1 & kx + 10y = 2 \end{array}$$

In Exercises 73 and 74, solve the given systems of equations by any appropriate method. All numbers in 73 are accurate to two significant digits, and in 74 they are accurate to 3 significant digits.

73. A 20-m crane arm with a supporting cable and with a 9000-N box suspended from its end has forces acting on it, as shown in Fig. 5.46. Find the forces (in N) from the following equations.

$$F_1 + 2.0F_2 = 26\,000$$

$$0.87F_1 - F_3 = 0$$

$$3.0F_1 - 4.0F_2 = 54\,000$$

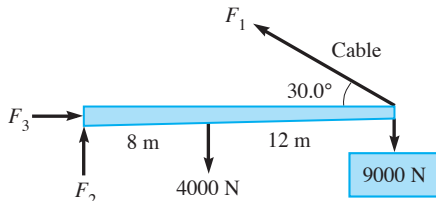


Fig. 5.46

74. In applying Kirchhoff's laws (see Exercise 22 of Section 5.6) to the electric circuit shown in Fig. 5.47, the following equations result. Find the indicated currents (in A).

$$i_1 + i_2 + i_3 = 0$$

$$5.20i_1 - 3.25i_2 = 8.33 - 6.45$$

$$3.25i_2 - 2.62i_3 = 6.45 - 9.80$$

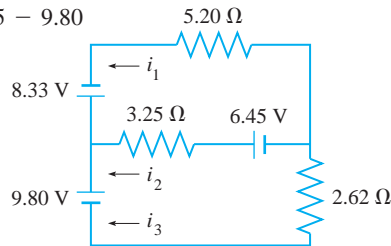


Fig. 5.47

In Exercises 75–94, set up systems of equations and solve by any appropriate method. All numbers are accurate to at least 3 significant digits.

75. A study found that the fuel consumption for transportation contributes a percent  $p_1$  that is 16% less than the percent  $p_2$  of all other sources combined. Find  $p_1$  and  $p_2$ .
76. A certain amount of a fuel contains 150 MJ of potential heat. Part is burned at 80% efficiency, and the rest is burned at 70% efficiency, such that the total amount of heat actually delivered is 114 MJ. Find the amounts burned at each efficiency.
77. The sales representatives of a company have a choice of being paid 10% of their sales in a month, or \$2400 plus 4% of their sales in the month. For what monthly sales is the income the same, and what is that income?
78. A person invested a total of \$20 900 into two bonds, one with an annual interest rate of 6.00% and the other with an annual interest rate of 5.00% per year. If the total annual interest from the bonds is \$1170, how much is invested in each bond?
79. A total of 42 tonnes of two types of ore is to be loaded into a smelter. The first type contains 6.0% copper, and the second contains 2.4% copper. Find the necessary amounts of each ore to produce 2 tonnes of copper.
80. As a 40-m pulley belt makes one revolution, one of the two pulleys makes one more revolution than the other. Another pulley

of half the radius replaces the smaller pulley and makes six more revolutions than the larger pulley for one revolution of the belt. Find the circumferences of the pulleys. See Fig. 5.48.

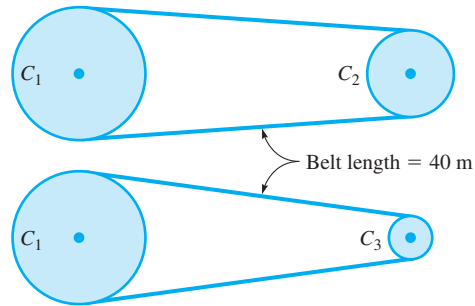


Fig. 5.48

81. A computer analysis showed that the temperature  $T$  of the ocean water within 1000 m of a nuclear-plant discharge pipe was given by  $T = \frac{a}{x + 100} + b$ , where  $x$  is the distance from the pipe and  $a$  and  $b$  are constants. If  $T = 14^\circ\text{C}$  for  $x = 0$  and  $T = 10^\circ\text{C}$  for  $x = 900$  m, find  $a$  and  $b$ .
82. In measuring the angles of a triangular parcel, a surveyor noted that one of the angles equalled the sum of the other two angles. What conclusion can be drawn?
83. A satellite is to be launched from a space shuttle. It is calculated that the satellite's speed will be 24 200 km/h if launched directly ahead of the shuttle or 21 400 km/h if launched directly to the rear of the shuttle. What is the speed of the shuttle and the launching speed of the satellite relative to the shuttle? See Fig. 5.49.

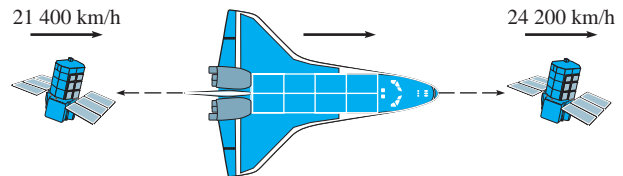


Fig. 5.49

84. The velocity  $v$  of sound is a function of the temperature  $T$  according to the function  $v = aT + b$ , where  $a$  and  $b$  are constants. If  $v = 337.5$  m/s for  $T = 10.0^\circ\text{C}$  and  $v = 346.6$  m/s for  $T = 25.0^\circ\text{C}$ , find  $v$  as a function of  $T$ .
85. The power  $P$  (in W) dissipated in an electric resistance  $R$  (in Ω) equals the resistance times the square of the current  $I$  (in A). If 1.0 A flows through resistance  $R_1$  and 3.0 A flows through resistance  $R_2$ , the total power dissipated is 14.0 W. If 3.0 A flows through  $R_1$  and 1.0 A flows through  $R_2$ , the total power dissipated is 6.0 W. Find  $R_1$  and  $R_2$ .
86. Twelve equal rectangular ceiling panels are placed as shown in Fig. 5.50. If each panel is 150 mm longer than it is wide and a

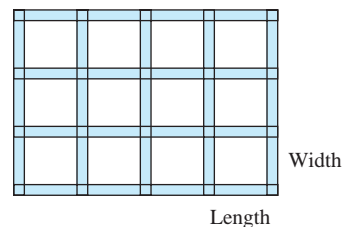


Fig. 5.50

total of 40.0 m of edge and middle strips is used, what are the dimensions of the room?

87. The weight of a lever may be considered to be at its centre. A 10-m lever of weight  $w$  is balanced on a fulcrum 4 m from one end by a load  $L$  at that end. If a load of  $4L$  is placed at that end, it requires a 20-N weight at the other end to balance the lever. What are the initial load  $L$  and the weight  $w$  of the lever? (See Exercise 50 of Section 5.4.) See Fig. 5.51.

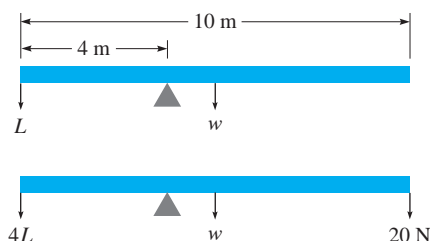


Fig. 5.51

88. Two fuel mixtures, one of 2.0% oil and 98.0% gasoline and another of 8.0% oil and 92.0% gasoline, are to be used to make 10.0 L of a fuel that is 4.0% oil and 96.0% gasoline for use in a chain saw. How much of each mixture is needed?
89. For the triangular truss shown in Fig. 5.52,  $\angle A$  is  $55^\circ$  less than twice  $\angle B$ , and  $\angle C$  is  $25^\circ$  less than  $\angle B$ . Find the measures of the three angles.

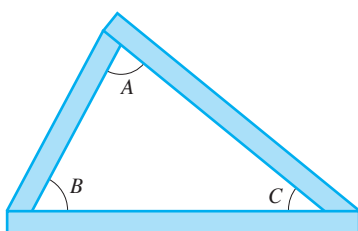


Fig. 5.52

90. A manufacturer produces three models of DVD players in a year. Four times as many of model A are produced as model C, and 7000 more of model B than model C. If the total production for the year is 97 000 units, how many of each are produced?
91. Three computer programs  $A$ ,  $B$ , and  $C$ , require a total of 140 MB (megabytes) of hard-disk memory. If three other programs, two requiring the same memory as  $B$  and one the same as  $C$ , are added to a disk with  $A$ ,  $B$ , and  $C$ , a total of 236 MB are required. If three other programs, one requiring the same memory as  $A$

and two the same memory as  $C$ , are added to a disk with  $A$ ,  $B$ , and  $C$ , a total of 304 MB are required. How much memory is required for each of  $A$ ,  $B$ , and  $C$ ?

92. One ampere of electric current is passed through a solution of sulfuric acid, silver nitrate, and cupric sulfate, releasing hydrogen gas, silver, and copper. A total mass of 1.750 g is released. The mass of silver deposited is 3.40 times the mass of copper deposited, and the mass of copper and 70.0 times the mass of hydrogen combined equals the mass of silver deposited less 0.037 g. How much of each is released?
93. Gold loses about 5.3%, and silver about 10%, of its weight when immersed in water. If a gold-silver alloy weighs 6.0 N in air and 5.6 N in water, find the weight in air of the gold and the silver in the alloy.
94. Three unknown forces are exerted at the pin joint shown in a static structure (see Fig. 5.53). It is known that the sum of the magnitudes of forces  $F_2$  and  $F_3$  is 71.0 kN. Analysing the force components in the horizontal and vertical directions yields two more equations:
- $$0.766F_1 - 0.985F_2 - 0.309F_3 = 0$$
- $$0.643F_1 - 0.174F_2 - 0.951F_3 = 0$$
- Solve for the forces.

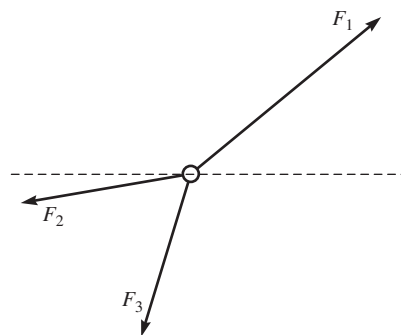


Fig. 5.53

**Writing Exercise**

95. Write one or two paragraphs giving reasons for choosing a particular method of solving the following problem. If a first pump is used for 2.2 h and a second pump is used for 2.7 h, 32 m<sup>3</sup> can be removed from a wastewater-holding tank. If the first pump is used for 1.4 h and the second for 2.5 h, 24 m<sup>3</sup> can be removed. How much can each pump remove in 1.0 h? (What is the result to 3 significant digits?)

**CHAPTER 5 PRACTICE TEST**

1. Find the slope of the line through  $(2, -5)$  and  $(-1, 4)$ .
2. Is  $x = -2, y = 3$  a solution to the system of equations
- $$\begin{cases} 2x + 5y = 11 \\ y - 5x = 12 \end{cases}$$
3. Evaluate:
- $$\begin{vmatrix} -1 & 3 & -2 \\ 4 & -3 & 0 \\ 5 & -4 & 2 \end{vmatrix}$$
- Explain.

4. Solve by substitution:
- $$\begin{cases} x + 2y = 5 \\ 4y = 3 - 2x \end{cases}$$
5. Solve by determinants:
- $$\begin{cases} 3x - 2y = 4 \\ 2x + 5y = -1 \end{cases}$$
6. By finding the slope and y-intercept, sketch the graph of  $2x + y = 4$ .

7. The perimeter of a rectangular ranch is 24 km, and the length is 6.0 km more than the width. Set up equations relating the length  $l$  and the width  $w$  and then solve for  $l$  and  $w$ .
8. Solve by addition or subtraction:  
 $6N - 2P = 13$   
 $4N + 3P = -13$
9. In testing an anticholesterol drug, it was found that each milligram of drug administered reduced a person's blood cholesterol level by 2 units. Set up the function relating the cholesterol level  $C$  as a function of the dosage  $d$  for a person whose cholesterol level is 310 before taking the drug. Sketch the graph.
10. Solve the following system of equations graphically. Determine the values of  $x$  and  $y$  to the nearest 0.1.  
 $2x - 3y = 6$   
 $4x + y = 4$
11. By volume, one alloy is 60% copper, 30% zinc, and 10% nickel. A second alloy has percents of 50, 30, and 20, respectively. A third alloy is 30% copper and 70% nickel. How much of each alloy is needed to make  $100 \text{ cm}^3$  of a resulting alloy with percents of 40, 15, and 45, respectively?
12. Solve for  $y$  by determinants:  
 $3x + 2y - z = 4$   
 $2x - y + 3z = -2$   
 $x + \quad \quad 4z = 5$

# Factoring and Fractions

In this chapter, we further develop operations with products, quotients, and fractions and show how they are used in solving equations. These additional algebraic methods are needed in later chapters, and they will be useful in solving many application problems.

The development of the symbols now used in algebra in itself led to advances in mathematics and science. Until about 1500, most problems and their solutions were stated in words, which made them very lengthy and often difficult to follow. For example, to solve the equation  $3x + 7 = 8(2x - 5)$ , the problem would be stated something like “find a number such that seven added to three times the number is equal to the product of 8 and the quantity of five subtracted from twice the number.” Then the solution would be written out, word by word.

As time went on, writers abbreviated some of the words in a problem, but they still essentially wrote out the solution in words. Some symbols did start to come into use. For example, the  $+$  and  $-$  signs first appeared in a published book in the late 1400s, the square root symbol  $\sqrt{\quad}$  was first used in 1525, and the  $=$  sign was introduced in 1557. Then, in the late 1500s, the French lawyer François Viète wrote several articles in which he used symbols, including letters to represent numbers. He so improved the symbolism of algebra that he is often called “the father of algebra.” By the mid-1600s, the notation being used was reasonably similar to that we use today.

The use of symbols in algebra made it more useful in all fields of mathematics. It was important in the development of calculus in the 1600s and 1700s. In turn, this gave scientists a powerful tool that allowed for much greater advancement in all areas of science and technology.

Although the primary purpose of this chapter is to develop additional algebraic methods, many technical applications of these operations will be shown. The important applications in optics are noted by the picture of the Hubble telescope below. Other areas of application include electronics, mechanical design, thermodynamics, and fluid mechanics.



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## 6

### LEARNING OUTCOMES

After completion of this chapter, the student should be able to:

- Identify and perform various algebraic special products
- Factor algebraic expressions using common factors, difference of squares, sum and difference of cubes, and other techniques
- Simplify expressions involving algebraic fractions
- Solve equations involving algebraic fractions

◀ Great advances in our knowledge of the universe have been made through the use of the Hubble space telescope (shown at the left during release from the cargo bay of Space Shuttle *Atlantis*) since the mid-1990s. In Section 6.8, we illustrate the use of algebraic operations in the study of planetary motion and in the design of lenses and reflectors in telescopes.



## 6.1 Special Products

### Special Algebraic Products • Special Products Involving Cubes

Certain types of algebraic products are used so often that we should be very familiar with them. These are shown in the following equations:

#### SPECIAL ALGEBRAIC PRODUCTS

$$a(x + y) = ax + ay \quad (6.1)$$

$$(x + y)(x - y) = x^2 - y^2 \quad (6.2)$$

$$(x + y)^2 = x^2 + 2xy + y^2 \quad (6.3)$$

$$(x - y)^2 = x^2 - 2xy + y^2 \quad (6.4)$$

$$(x + a)(x + b) = x^2 + (a + b)x + ab \quad (6.5)$$

$$(ax + b)(cx + d) = acx^2 + (ad + bc)x + bd \quad (6.6)$$

We see that Eq. (6.1) is the very important *distributive law*. Eqs. (6.2) to (6.6) are found by using the distributive law along with other basic operations that were developed in Chapter 1.

Eqs. (6.1) to (6.6) are important **special products** and should be known **thoroughly**. By carefully studying the products, you will **see how they are formed**, and this allows them to be used quickly. Eq. (6.6) looks complicated, but when you see how each of the terms on the right side is formed, you will not have to actually memorize it. In using these equations, we must realize that any of the literal numbers may represent any symbol or expression that represents a number.

■ Eqs. (6.1) to (6.6) are identities. (See page 42.) Each is valid for all sets of values of the literal numbers.

#### EXAMPLE 1 Using special products (distributive law and difference of squares)

(a) Using Eq. (6.1) in the following product, we have

$$\begin{array}{c} \text{the 6 factor is distributed to both terms by multiplication} \\ \downarrow \quad \downarrow \\ 6(3r + 2s) = 6(3r) + 6(2s) = 18r + 12s \end{array}$$

(b) Using Eq. (6.2), we have

Difference of Squares

$$\begin{array}{c} (3r + 2s)(3r - 2s) = (3r)^2 - (2s)^2 = 9r^2 - 4s^2 \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \text{sum of} & \text{difference} & \text{difference} \\ \text{3r and 2s} & \text{of 3r} & \text{of squares} \\ & \text{and 2s} & \end{array} \end{array}$$

Using Eq. (6.1) in (a), we have  $a = 6$ . In (a) and (b),  $3r = x$  and  $2s = y$ . ■

#### EXAMPLE 2 Using special products involving the square of a binomial

Using Eqs. (6.3) and (6.4) in the following products, we have

$$\begin{array}{c} \text{■ } (5a + 2)^2 = (5a + 2)(5a + 2) \\ \text{(a) } (5a + 2)^2 = \overbrace{(5a)^2}^{\text{square}} + \underbrace{2(5a)(2)}_{\text{twice product}} + \overbrace{2^2}^{\text{square}} = 25a^2 + 20a + 4 \quad \text{Eq. (6.3)} \end{array}$$

$$\text{(b) } (5a - 2)^2 = (5a)^2 - \underbrace{2(5a)(2)}_{\text{twice product}} + 2^2 = 25a^2 - 20a + 4 \quad \text{Eq. (6.4)}$$

In these illustrations, we let  $x = 5a$  and  $y = 2$ . ■

#### COMMON ERROR

It should be emphasized that  $(5a + 2)^2$  is **not**  $(5a)^2 + 2^2$ , or  $25a^2 + 4$ . Ensure that you use the forms of Eqs. (6.3) and (6.4) very carefully by **including the middle term, 20a**.

**LEARNING TIP**

When expanding the square of a binomial, three terms result: the square of the first term, the square of the last term, and a middle term of the products of the two terms multiplied by 2:

$$(3x - 2y)^2 = (3x)^2 + 2(3x)(-2y) + (-2y)^2$$

First term squared
Twice product
Last term squared

$$(3x - 2y)^2 = 9x^2 - 12xy + 4y^2$$

**Practice Exercises**

Find the indicated products

1.  $(3x + 4)(3x - 4)$
2.  $(3x + 4)^2$
3.  $(3x + 4)(x - 2)$

**EXAMPLE 3 Using special products involving the distributive property**

Using Eqs. (6.5) and (6.6) in the following products, we have

(a)  $(x + 5)(x - 3) = x^2 + [5 + (-3)]x + (5)(-3) = x^2 + 2x - 15$  Eq. (6.5)  
 each factor is distributed to both terms in the other factor

(b)  $(4x + 5)(2x - 3) = (4x)(2x) + [(4)(-3) + (5)(2)]x + (5)(-3)$  Eq. (6.6)  
 $= 8x^2 - 2x - 15$  ■

Generally, when we use these special products, we find the middle term mentally and write down the result directly, as shown in the next example.

**EXAMPLE 4 Middle term often found mentally**

(a)  $(y - 5)(y + 5) = y^2 - 25$  no middle term—Eq. (6.2)

(b)  $(3x - 2)^2 = 9x^2 - 12x + 4$  middle term =  $2(3x)(-2)$ —Eq. (6.4)

(c)  $(x^2 - 4)(x^2 + 7) = x^4 + 3x^2 - 28$  middle term =  $(7 - 4)x^2$ —Eq. (6.5)  
 $(x^2)^2 = x^4$  ■

At times, it is necessary to use more than one of the special products to simplify an algebraic expression. When this happens, it may be necessary to write down an intermediate step. This is illustrated in the following examples.

**EXAMPLE 5 Applications of special products**

- (a) When analysing the forces on a certain type of beam, the expression  $Fa(L - a)(L + a)$  occurs. In expanding this expression, we first multiply  $L - a$  by  $L + a$  by use of Eq. (6.2). The expansion is completed by using Eq. (6.1), the distributive law.

$$Fa(L - a)(L + a) = Fa(L^2 - a^2) \quad \text{Eq. (6.2)}$$

$$= FaL^2 - Fa^3 \quad \text{Eq. (6.1)}$$

- (b) The electrical power delivered to the resistor  $R$  in Fig. (6.1) is  $R(i_1 + i_2)^2$ . Here,  $i_1$  and  $i_2$  are electric currents. To expand this expression, we first perform the square by use of Eq. (6.3) and then complete the expansion by use of Eq. (6.1).

$$R(i_1 + i_2)^2 = R(i_1^2 + 2i_1i_2 + i_2^2) \quad \text{Eq. (6.3)}$$

$$= Ri_1^2 + 2Ri_1i_2 + Ri_2^2 \quad \text{Eq. (6.1)} \quad \blacksquare$$

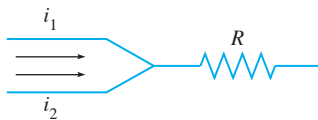


Fig. 6.1

**EXAMPLE 6 Group terms then use special products**

In determining the product  $(x + y - 2)^2$ , we may group the quantity  $(x + y)$  in an intermediate step. This leads to

$$(x + y - 2)^2 = [(x + y) - 2]^2$$

$$= (x + y)^2 - 2(x + y)(2) + 2^2$$

$$= x^2 + 2xy + y^2 - 4x - 4y + 4$$

In this example, we used Eqs. (6.3) and (6.4). ■

**SPECIAL PRODUCTS INVOLVING CUBES**

Four other special products occur less frequently. However, they are sufficiently important that they should be readily recognized. They are shown in Eqs. (6.7) to (6.10).

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3 \quad (6.7)$$

$$(x - y)^3 = x^3 - 3x^2y + 3xy^2 - y^3 \quad (6.8)$$

$$(x + y)(x^2 - xy + y^2) = x^3 + y^3 \quad (6.9)$$

$$(x - y)(x^2 + xy + y^2) = x^3 - y^3 \quad (6.10)$$

**EXAMPLE 7** Using special products involving the cube of a binomial

$$\begin{aligned} \text{(a)} \quad (x + 4)^3 &= x^3 + 3(x^2)(4) + 3(x)(4^2) + 4^3 \quad \text{Eq. (6.7)} \\ &= x^3 + 12x^2 + 48x + 64 \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad (2x - 5)^3 &= (2x)^3 - 3(2x)^2(5) + 3(2x)(5^2) - 5^3 \quad \text{Eq. (6.8)} \\ &= 8x^3 - 60x^2 + 150x - 125 \end{aligned}$$

**EXAMPLE 8** Using special products involving the difference or sum of cubes

**Sum of Cubes**

$$\begin{aligned} \text{(a)} \quad (x + 3)(x^2 - 3x + 9) &= x^3 + 3^3 \quad \text{Eq. (6.9)} \\ &= x^3 + 27 \end{aligned}$$

**Difference of Cubes**

$$\begin{aligned} \text{(b)} \quad (x - 2)(x^2 + 2x + 4) &= x^3 - 2^3 \quad \text{Eq. (6.10)} \\ &= x^3 - 8 \end{aligned}$$

**EXERCISES 6.1**

In Exercises 1–4, make the given changes in the indicated examples of this section and then solve the resulting problems.

1. In Example 1(b), interchange the + and - signs and then find the product.
2. In Example 3(b), interchange the + and - signs and then find the product.
3. In Example 6, group  $y - 2$  instead of  $x + y$  and then find the product.
4. In Example 7(a), change + to - in the original expression and then find the product.

In Exercises 5–36, find the indicated products directly by inspection. It should not be necessary to write down intermediate steps [except possibly when using Eq. (6.6)].

- |                            |                            |
|----------------------------|----------------------------|
| 5. $40(x - y)$             | 6. $2x(a - 3)$             |
| 7. $2x^2(x - 4)$           | 8. $3a^2(2a + 7)$          |
| 9. $(T + 6)(T - 6)$        | 10. $(s + 2t)(s - 2t)$     |
| 11. $(3v - 2)(3v + 2)$     | 12. $(ab - c)(ab + c)$     |
| 13. $(4x - 5y)(4x + 5y)$   | 14. $(7s + 2t)(7s - 2t)$   |
| 15. $(12 + 5ab)(12 - 5ab)$ | 16. $(2xy - 11)(2xy + 11)$ |
| 17. $(5f + 4)^2$           | 18. $(i_1 + 3)^2$          |
| 19. $(2x + 17)^2$          | 20. $(9a + 8b)^2$          |
| 21. $(L^2 - 1)^2$          | 22. $(b^2 - 6)^2$          |

- |                            |                           |
|----------------------------|---------------------------|
| 23. $(4a + 7xy)^2$         | 24. $(3A + 10z)^2$        |
| 25. $(0.6s - t)^2$         | 26. $(0.3p - 0.4q)^2$     |
| 27. $(x + 1)(x + 5)$       | 28. $(y - 8)(y + 5)$      |
| 29. $(3 + C^2)(6 + C^2)$   | 30. $(1 - e^3)(7 - e^3)$  |
| 31. $(4x - 5)(5x + 1)$     | 32. $(2y - 1)(3y - 1)$    |
| 33. $(10v - 3)(4v + 15)$   | 34. $(7s + 12)(8s + 5)$   |
| 35. $(30x + 7y)(20x - 9y)$ | 36. $(24x - y)(15x + 4y)$ |

Use the special products of this section to determine the products of Exercises 37–62. You may need to write down one or two intermediate steps.

- |                                |                            |
|--------------------------------|----------------------------|
| 37. $2(x - 2)(x + 2)$          | 38. $25(V - 5)(V + 5)$     |
| 39. $2a(2a - 1)(2a + 1)$       | 40. $4c(2c - 3)(2c + 3)$   |
| 41. $6a(x + 2b)^2$             | 42. $7r(5r + 2b)^2$        |
| 43. $5n^2(2n + 5)^2$           | 44. $8T(T - 7)^2$          |
| 45. $[(2R + 3r)(2R - 3r)]^2$   | 46. $[(6t - y)(6t + y)]^2$ |
| 47. $(x + y + 1)^2$            | 48. $(x + 2 + 3y)^2$       |
| 49. $(3 - x - y)^2$            | 50. $2(x - y + 1)^2$       |
| 51. $(5 - t)^3$                | 52. $(2s + 3)^3$           |
| 53. $(3L + 7R)^3$              | 54. $(2A - 5B)^3$          |
| 55. $(w + h - 1)(w + h + 1)$   |                            |
| 56. $(2a - c + 2)(2a - c - 2)$ |                            |

57.  $(x + 2)(x^2 - 2x + 4)$   
 58.  $(s - 3)(s^2 + 3s + 9)$   
 59.  $(4 - 3x)(16 + 12x + 9x^2)$   
 60.  $(2x + 3a)(4x^2 - 6ax + 9a^2)$   
 61.  $(x + y)^2(x - y)^2$   
 62.  $(x - y)(x + y)(x^2 + y^2)$

In Exercises 63–72, use the special products of this section to determine the products. Each comes from the technical area indicated.

63.  $P_1(P_0c + G)$  (computers)  
 64.  $akT(t_2 - t_1)$  (heat conduction)  
 65.  $4(p + DA)^2$  (photography)  
 66.  $(2J + 3)(2J - 1)$  (lasers)  
 67.  $\frac{1}{2}\pi(R + r)(R - r)$  (architecture)  
 68.  $k(T - T_0)(T + T_0)(T^2 + T_0^2)$  (radiation)  
 69.  $\frac{L}{6}(x - a)^3$  (mechanics: beams)  
 70.  $(1 - z)^2(1 + z)$  (motion: gyroscope)  
 71.  $L_0[1 + a(T - T_0)]$  (thermal expansion)  
 72.  $(s + 1 + j)(s + 1 - j)$  (electricity)

In Exercises 73–80, solve the given problems.

73. Multiply 49 by 51 by writing  $(49)(51) = (50 - 1)(50 + 1)$  and using one of the special products.  
 74. Multiply 82 by 78 by writing  $(82)(78) = (80 + 2)(80 - 2)$  and using one of the special products.  
 75. The length of a piece of rectangular floor tile is 3 cm more than twice the side  $x$  of a second square piece of tile. The width of the rectangular piece is 3 cm less than twice the side of the square piece. Find the area of the rectangular piece in terms of  $x$  (in expanded form).

76. The radius of a circular oil spill is  $r$ . It then increases in radius by 40 m before being contained. Find the area of the oil spill at the time it is contained in terms of  $r$  (in expanded form). See Fig. 6.2.

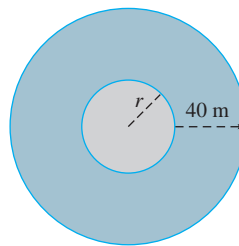


Fig. 6.2

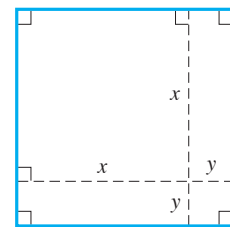


Fig. 6.3

77. Referring to Fig. 6.3, find (a) an expression for the area of the large square (the entire figure) as the power of a binomial; (b) the sum of the areas of the four figures within the large square as a trinomial. Noting that the results of (a) and (b) should be equal, what special product is shown geometrically?  
 78. If  $(x + k)(x - 1) = x^2 + k(x + 2) - (x - 3)$ , find the value of  $k$ .  
 79. Find the product  $(2x - y - 2)(2x + y + 2)$  by first grouping terms similar to Example 6, by grouping the last two terms of each factor.  
 80. Verify Eqs. (6.8) and (6.9) by multiplication. Then explain why we may refer to Eqs. (6.1) to (6.10) as *identities*.

#### Answers to Practice Exercises

1.  $9x^2 - 16$    2.  $9x^2 + 24x + 16$    3.  $3x^2 - 2x - 8$

## 6.2 Factoring: Common Factor and Difference of Squares

Factoring • Prime Factors • Common Monomial Factors • Factoring Difference of Two Squares • Complete Factoring • Factoring by Grouping

At times, we want to determine which expressions can be multiplied together to equal a given algebraic expression. We know from Section 1.7 that when an algebraic expression is the product of two or more quantities, each of these quantities is a **factor** of the expression. Therefore, *determining these factors, which is essentially reversing the process of finding a product, is called factoring*. Factoring is **very** useful when you are determining the zeros of a function (that is, solving for values of the variable that produce 0 as the value of a function).

In our work on factoring, we will consider only the factoring of polynomials (see Section 1.7) that have integers as coefficients for all terms. Also, all factors will have integral coefficients. A *polynomial or a factor is called prime if it contains no factors other than +1 or -1 and plus or minus itself*. Also, we say that an *expression is factored completely if it is expressed as a product of its prime factors*.

**EXAMPLE 1** Factoring completely

When we factor the expression  $12x + 6x^2$  as

$$12x + 6x^2 = 2(6x + 3x^2)$$

we see that it has not been factored completely. The factor  $6x + 3x^2$  is not prime, because it may be factored as

$$6x + 3x^2 = 3x(2 + x)$$

Therefore, the expression  $12x + 6x^2$  is factored completely as

$$12x + 6x^2 = 6x(2 + x)$$

The factors  $x$  and  $(2 + x)$  are prime. We can factor the numerical coefficient, 6, as  $2(3)$ , but it is standard not to write numerical coefficients in factored form. ■

To factor expressions easily, we must know how to do algebraic multiplication and really know the special products of the previous section. The special products also give us a way of checking answers and deciding whether a given factor is prime.

**LEARNING TIP**

*The ability to factor algebraic expressions depends heavily on the proper recognition of the special products.*

■ For reference, Eq. (6.1) is  $a(x + y) = ax + ay$ .

**COMMON MONOMIAL FACTORS**

Often an expression contains a monomial that is common to each term of the expression. Therefore, *the first step in factoring any expression should be to factor out any **common monomial factor** that may exist.* To do this, we note the common factor by inspection and then use the reverse of the distributive law, Eq. (6.1), to show the factored form. The following examples illustrate factoring a common monomial factor out of an expression.

**EXAMPLE 2** Common monomial factor

In factoring  $6x - 2y$ , we note each term contains a factor of 2:

$$6x - 2y = 2(3x) - 2y = 2(3x - y)$$

Here, 2 is the common monomial factor, and  $2(3x - y)$  is the required factored form of  $6x - 2y$ . Once the common factor has been identified, it is not actually necessary to write a term like  $6x$  as  $2(3x)$ . The result can be written directly.

In this case,

$$2(3x - y) = 6x - 2y$$

Since the result of the multiplication gives the original expression, the factored form is correct. ■

The next example illustrates the case where the common factor is the same as one of the terms, and special care must be taken to complete the factoring correctly.

**EXAMPLE 3** Common factor same as term

Factor  $4ax^2 + 2ax$ .

The numerical factor 2 and the literal factors  $a$  and  $x$  are common to each term. Therefore, the common monomial factor of  $4ax^2 + 2ax$  is  $2ax$ . This means that

$$4ax^2 + 2ax = 2ax(2x) + 2ax(1) = 2ax(2x + 1)$$

Note the presence of the 1 in the factored form. When we divide  $4ax^2 + 2ax$  by  $2ax$ , we get

$$\begin{aligned} \frac{4ax^2 + 2ax}{2ax} &= \frac{4ax^2}{2ax} + \frac{2ax}{2ax} \\ &= 2x + 1 \end{aligned}$$

**LEARNING TIP**

Usually, the factorization of common monomial factors is done by inspection. However, once the common factor is found, the other factor can also be determined by dividing the original expression by the common factor. For example,  $2abc + 4ab$  has a common factor of  $2ab$  in each term, so

$$2abc + 4ab = 2ab\left(\frac{2abc}{2ab} + \frac{4ab}{2ab}\right)$$

$$2abc + 4ab = 2ab(c + 2)$$

**LEARNING TIP**

You can check your factoring result by multiplication.

**Practice Exercises**

Factor: 1.  $3cx^3 - 9cx$  2.  $9cx^3 - 3cx$

noting very carefully that  $2ax$  divided by  $2ax$  is 1 and *does not simply cancel out leaving nothing*. Without the 1, when the factored form is multiplied out, we would not obtain the original expression. ■

**COMMON ERROR**

When the common factor is the same as one of the terms in the expression, remember that the term divides to 1, so do not omit the 1 and cancel the term out to nothing!

$$5rt + 5r = 5r(t + 1)$$

**Not**

$$5rt + 5r \neq 5r(t + 0)$$

**EXAMPLE 4** Common factor by inspection

Factor  $6a^5x^2 - 9a^3x^3 + 3a^3x^2$ .

After inspecting each term, we determine that each contains a factor of 3,  $a^3$ , and  $x^2$ . Thus, the common monomial factor is  $3a^3x^2$ . This means that

$$6a^5x^2 - 9a^3x^3 + 3a^3x^2 = 3a^3x^2(2a^2 - 3x + 1)$$
 ■

In these examples, note that factoring an expression does not actually change the expression, although it does change the *form* of the expression. In equating the expression to its factored form, we write an *identity*.

It is often necessary to use factoring when solving an equation. This is illustrated in the following example.

**EXAMPLE 5** Using factoring in solving an equation

An equation used in the analysis of FM reception is  $R_F = \alpha(2R_A + R_F)$ . Solve for  $R_F$ . The steps in the solution are as follows:

$$R_F = \alpha(2R_A + R_F) \quad \text{original equation}$$

$$R_F = 2\alpha R_A + \alpha R_F \quad \text{use distributive law}$$

$$R_F - \alpha R_F = 2\alpha R_A \quad \text{subtract } \alpha R_F \text{ from both sides}$$

$$R_F(1 - \alpha) = 2\alpha R_A \quad \text{factor out } R_F \text{ on left}$$

$$R_F = \frac{2\alpha R_A}{1 - \alpha} \quad \text{divide both sides by } 1 - \alpha$$

■ FM radio was developed in the early 1930s.

We see that we collected both terms containing  $R_F$  on the left so that we could factor and thereby solve for  $R_F$ . ■

**FACTORING THE DIFFERENCE OF TWO SQUARES**

In Eq. (6.2), we see that the product of the sum and the difference of two numbers results in the difference between the squares of two numbers. Therefore, *factoring the difference of two squares gives factors that are the sum and the difference of the numbers*.

■ For reference, Eq. (6.2) is  $(x + y)(x - y) = x^2 - y^2$ .

**LEARNING TIP**

Usually, in factoring an expression of this type, where it is very clear what numbers are squared, we do not actually write out the middle step as shown. However, if in doubt, write it out.

**EXAMPLE 6** Factoring the difference of two squares

In factoring  $x^2 - 16$ , note that  $x^2$  is the square of  $x$  and 16 is the square of 4. Therefore,

$$x^2 - 16 = x^2 - 4^2 = (x + 4)(x - 4)$$

↑ squares  
↑ difference     ↑ sum     ↑ difference



**EXAMPLE 7** Factoring the difference of two squares where a factor is itself a composite factor

(a) Since  $4x^2$  is the square of  $2x$  and 9 is the square of 3, we may factor  $4x^2 - 9$  as

$$4x^2 - 9 = (2x)^2 - 3^2 = (2x + 3)(2x - 3)$$

(b) In the same way,

$$\begin{aligned}(y - 3)^2 - 16x^4 &= [(y - 3) + 4x^2][(y - 3) - 4x^2] \\ &= (y - 3 + 4x^2)(y - 3 - 4x^2)\end{aligned}$$

where we note that  $16x^4 = (4x^2)^2$ . ■

**COMPLETE FACTORING**

As noted before, *a common monomial factor should be factored out first*. However, we must be careful to *see if the other factor can itself be factored*. Be sure to include only prime factors in the final result.

**COMMON ERROR**

Always factor your equations completely. It is a common error to stop before the expression involves only prime factors.

For example,  $4a^2 - a^2x^2$  can be factored to  $a^2(4 - x^2)$ , and it is a common error to think you are now finished. However, the second factor is a difference of squares. The complete factorization is

$$\begin{aligned}4a^2 - a^2x^2 &= a^2(4 - x^2) \\ &= a^2(2 + x)(2 - x)\end{aligned}$$

**EXAMPLE 8** Complete factoring

(a) In factoring  $20x^2 - 45$ , note a common factor of 5 in each term. Therefore,  $20x^2 - 45 = 5(4x^2 - 9)$ . However, the factor  $4x^2 - 9$  itself is the difference of squares. Therefore,  $20x^2 - 45$  is completely factored as

$$20x^2 - 45 = 5(4x^2 - 9) = 5(2x + 3)(2x - 3)$$

(b) In factoring  $x^4 - y^4$ , note that we have the difference of two squares. Therefore,  $x^4 - y^4 = (x^2 + y^2)(x^2 - y^2)$ . However, the factor  $x^2 - y^2$  is also the difference of squares. This means that

$$x^4 - y^4 = (x^2 + y^2)(x^2 - y^2) = (x^2 + y^2)(x + y)(x - y) \quad \blacksquare$$

**COMMON ERROR**

*The factor  $x^2 + y^2$  is prime.* It is not equal to  $(x + y)^2$ . (See Example 2 of Section 6.1.)

**Practice Exercises**

Factor: 3.  $9c^2 - 64$  4.  $18c^2 - 128$

**FACTORING BY GROUPING**

The terms in a polynomial can sometimes be grouped such that the polynomial can then be factored by the methods of this section. The following example illustrates this method of *factoring by grouping*.

**EXAMPLE 9** Factoring by grouping

Factor  $2x - 2y + ax - ay$ .

We see that there is no common factor to all four terms, but that each of the first two terms contains a factor of 2, and each of the third and fourth terms contains a factor of  $a$ . Grouping terms this way and then factoring each group, we have

$$\begin{aligned} 2x - 2y + ax - ay &= (2x - 2y) + (ax - ay) \\ &= 2(x - y) + a(x - y) \quad \text{now note the common factor of } (x - y) \\ &= (x - y)(2 + a) \end{aligned}$$

The general method of factoring by grouping can be used with several types of groupings. We will discuss another type in the following section.

**EXERCISES 6.2**

In Exercises 1–4, make the given changes in the indicated examples of this section and then solve the indicated problems.

- In Example 3, change the  $+$  sign to  $-$  and then factor.
- In Example 3, set the given expression equal to  $B$  and then solve for  $a$ .
- In Example 8(a), change the coefficient of the first term from 20 to 5 and then factor.
- In Example 9, change both  $-$  signs to  $+$  and then factor.

In Exercises 5–44, factor the given expressions completely.

- |                                 |                              |                         |
|---------------------------------|------------------------------|-------------------------|
| 5. $6x + 6y$                    | 6. $3a - 3b$                 | 7. $5a - 5$             |
| 8. $2x^2 + 2$                   | 9. $3x^2 - 9x$               | 10. $4s^2 + 20s$        |
| 11. $7b^2h - 28b$               | 12. $5a^2 - 20ax$            | 13. $288n^2 + 24n$      |
| 14. $90p^3 - 15p^2$             | 15. $2x + 4y - 8z$           | 16. $23a - 46b + 69c$   |
| 17. $3ab^2 - 6ab + 12ab^3$      | 18. $4pq - 14q^2 - 16pq^2$   |                         |
| 19. $12pq^2 - 8pq - 28pq^3$     | 20. $27a^2b - 24ab - 9a$     |                         |
| 21. $2a^2 - 2b^2 + 4c^2 - 6d^2$ | 22. $5a + 10ax - 5ay - 20az$ |                         |
| 23. $x^2 - 4$                   | 24. $r^2 - 25$               | 25. $100 - 9A^2$        |
| 26. $49 - Z^4$                  | 27. $36a^4 + 1$              | 28. $324z^2 - 4$        |
| 29. $162s^2 - 50t^2$            | 30. $36s^2 - 121t^2$         | 31. $144n^2 - 169p^4$   |
| 32. $36a^2b^2 + 169c^2$         | 33. $(x + y)^2 - 9$          | 34. $(a - b)^2 - 1$     |
| 35. $2x^2 - 8$                  | 36. $5a^2 - 125$             | 37. $300x^2 - 2700z^2$  |
| 38. $28x^2 - 700y^2$            | 39. $2(I - 3)^2 - 8$         | 40. $a(x + 2)^2 - ay^2$ |
| 41. $x^4 - 16$                  | 42. $y^4 - 81$               | 43. $x^8 - 1$           |
| 44. $2x^4 - 8y^4$               |                              |                         |

In Exercises 45–50, solve for the indicated letter.

- |   |                                      |
|---|--------------------------------------|
| 45. $2a - b = ab + 3$ , for $a$                 | 46. $n(x + 1) = 5 - x$ , for $x$     |
| 47. $3 - 2s = 2(3 - st)$ , for $s$              | 48. $k(2 - y) = y(2k - 1)$ , for $y$ |
| 49. $(x + 2k)(x - 2) = x^2 + 3x - 4k$ , for $k$ |                                      |
| 50. $(2x - 3k)(x + 1) = 2x^2 - x - 3$ , for $k$ |                                      |

In Exercises 51–58, factor the given expressions by grouping as illustrated in Example 9.

- |                            |                               |
|----------------------------|-------------------------------|
| 51. $3x - 3y + bx - by$    | 52. $am + an + cn + cm$       |
| 53. $a^2 + ax - ab - bx$   | 54. $2y - y^2 - 6y^4 + 12y^3$ |
| 55. $x^3 + 3x^2 - 4x - 12$ | 56. $S^3 - 5S^2 - S + 5$      |
| 57. $x^2 - y^2 + x - y$    | 58. $4p^2 - q^2 + 2p + q$     |

In Exercises 59 and 60, evaluate the given expressions by using factoring. The results may be checked with a calculator.

$$59. \frac{8^9 - 8^8}{7} \qquad 60. \frac{5^9 - 5^7}{7^2 - 5^2}$$

In Exercises 61 and 62, give the required explanations.

- Factor  $n^2 + n$ , and then explain why it represents a positive even integer if  $n$  is a positive integer.
- Factor  $n^3 - n$ , and then explain why it represents a multiple of 6 if  $n$  is an integer greater than 1.

In Exercises 63–70, factor the expressions completely. In Exercises 69 and 70, it is necessary to set up the proper expression. Each expression comes from the technical area indicated.

- $2\pi rh + 2\pi r^2$  (container surface area)
- $4d^2D^2 - 4d^3D - d^4$  (machine design)
- $Rv + Rv^2 + Rv^3$  (business)
- $PbL^2 - Pb^3$  (architecture)
- $rR^2 - r^3$  (pipeline flow)
- $p_1R^2 - p_1r^2 - p_2R^2 + p_2r^2$  (fluid flow)
- As large as possible a square is cut from a circular metal plate of radius  $r$ . Express in factored form the area of the metal pieces that are left.
- A pipe of outside diameter  $d$  is inserted into a pipe of inside radius  $r$ . Express in factored form the cross-sectional area within the larger pipe that is outside the smaller pipe.

In Exercises 71–76, solve for the indicated letter. Each equation comes from the technical area indicated.

- $i_1R_1 = (i_2 - i_1)R_2$ , for  $i_1$  (electricity: ammeter)
- $nV + n_1v = n_1V$ , for  $n_1$  (acoustics)
- $3BY + 5Y = 9BS$ , for  $B$  (physics: elasticity)
- $Sq + Sp = Spq + p$ , for  $q$  (computer design)
- $ER = AtT_0 - AtT_1$ , for  $t$  (energy conservation)
- $R = kT_2^4 - kT_1^4$ , for  $k$  (factor resulting denominator) (radiation)

**Answers to Practice Exercises**

- $3cx(x^2 - 3)$
- $3cx(3x^2 - 1)$
- $(3c - 8)(3c + 8)$
- $2(3c - 8)(3c + 8)$

## 6.3 Factoring Trinomials

When Coefficient of Square Term is 1 •  
Importance of the Middle Term •  
Factoring General Trinomials •  
Factoring by Grouping

For reference, Eq. (6.5) is  
 $(x + a)(x + b) = x^2 + (a + b)x + ab$ .

### LEARNING TIP

When factoring trinomials of the form  $x^2 + bx + c$ , we may be able to find the two numbers that multiply to  $c$ , and add to  $b$ , giving the form:

$$x^2 + bx + c = (x + \text{number}_1) \times (x + \text{number}_2)$$

In the previous section, we considered factoring based on the special products of Eqs. (6.1) and (6.2). The trinomials formed from Eqs. (6.3) to (6.6) are important expressions to be factored, and this section is devoted to them.

### FACTORIZING TRINOMIALS FOR WHICH COEFFICIENT OF $x^2$ IS 1

When factoring is based on Eq. (6.5), we start with the expression on the right and then find the factors on the left. By writing Eq. (6.5) with sides reversed, we have

$$\text{coefficient} = 1 \rightarrow x^2 + (a + b)x + ab = (x + a)(x + b)$$

We find integers  $a$  and  $b$  by noting that

1. the coefficient of  $x^2$  is 1,
2. the final constant is the product of the constants  $a$  and  $b$  in the factors, and
3. the coefficient of  $x$  is the sum of  $a$  and  $b$ .

As in Section 6.2, we consider only factors in which all terms have integral coefficients.

### EXAMPLE 1 Factoring the trinomial $x^2 + (a + b)x + ab$

In factoring  $x^2 + 3x + 2$ , we set it up as

$$x^2 + 3x + 2 = (x(\quad))(x(\quad))$$

The constant 2 tells us that the product of the required integers is 2. Thus, the only possibilities are 2 and 1 (or 1 and 2). The + sign before the 2 indicates that the sign before the 1 and 2 in the factors must be the same. The + sign before the 3, the sum of the integers, tells us that both signs are positive. Therefore,

$$x^2 + 3x + 2 = (x + 2)(x + 1)$$

In factoring  $x^2 - 3x + 2$ , the analysis is the same until we note that the middle term is negative. This tells us that both signs are negative in this case. Therefore,

$$x^2 - 3x + 2 = (x - 2)(x - 1)$$

For a trinomial with first term  $x^2$  and constant  $+2$  to be factorable, the middle term must be  $+3x$  or  $-3x$ . No other middle terms are possible. This means, for example, the expressions  $x^2 + 4x + 2$  and  $x^2 - x + 2$  cannot be factored. ■

### EXAMPLE 2 Factoring trinomials with $x^2$ coefficient of 1

(a) In order to factor  $x^2 + 7x - 8$ , we must find two integers whose product is  $-8$  and whose sum is  $+7$ . The possible factors of  $-8$  are

$$-8 \text{ and } +1 \quad +8 \text{ and } -1 \quad -4 \text{ and } +2 \quad +4 \text{ and } -2$$

Inspecting these, we see that only  $+8$  and  $-1$  have the sum of  $+7$ . Therefore,

$$x^2 + 7x - 8 = (x + 8)(x - 1)$$

### LEARNING TIP

Always multiply the factors together to check to see that you get the correct middle term.

In choosing the correct values for the integers, it is usually fairly easy to find a pair for which the product is the final term. However, choosing the pair of integers that correctly fits the middle term is the step that often is not done properly. *Special attention must be given to choosing the integers so that the expansion of the resulting factors has the correct middle term of the original expression.*

(b) In the same way, we have

$$x^2 - x - 12 = (x - 4)(x + 3)$$

since  $-4$  and  $+3$  is the only pair of integers whose product is  $-12$  and whose sum is  $-1$ .

### COMMON ERROR

Ensure that you have the **proper signs** on the two numbers chosen to make the product and the sum leading to the trinomial coefficients. To factor  $x^2 - x - 6$ , we require two numbers that multiply to  $-6$ . The pairs  $+2$  and  $-3$  and  $-2$  and  $+3$  both work. However, only one of those pairs adds to the coefficient of the middle term ( $+2$  and  $-3$ ). Therefore,

$$x^2 - x - 6 = (x + 2)(x - 3)$$

$$x^2 - x - 6 \neq (x - 2)(x + 3)$$

(c) Also,

$$x^2 - 5xy + 6y^2 = (x - 3y)(x - 2y)$$

since  $-3$  and  $-2$  is the only pair of integers whose product is  $+6$  and whose sum is  $-5$ . Here, we find second terms of each factor with a product of  $6y^2$  and sum of  $-5xy$ , which means that each second term must have a factor of  $y$ , as we have shown above. ■

■ For reference, Eqs. (6.3) and (6.4) are  $(x + y)^2 = x^2 + 2xy + y^2$  and  $(x - y)^2 = x^2 - 2xy + y^2$ .

In factoring a trinomial in which the first and third terms are perfect squares, we may find that the expression fits the form of Eq. (6.3) or Eq. (6.4), as well as the form of Eq. (6.5). The following example illustrates this case.

### EXAMPLE 3 Factoring perfect square trinomials

(a) To factor  $x^2 + 10x + 25$ , we must find two integers whose product is  $+25$  and whose sum is  $+10$ . Since  $5^2 = 25$ , we note that this expression may fit the form of Eq. (6.3). This can be the case only if the first and third terms are perfect squares. Since the sum of  $+5$  and  $+5$  is  $+10$ , we have

$$x^2 + 10x + 25 = (x + 5)(x + 5)$$

or

$$x^2 + 10x + 25 = (x + 5)^2$$

(b) To factor  $A^4 - 20A^2 + 100$ , we note that  $A^4 = (A^2)^2$  and  $100 = 10^2$ . This means that this expression may fit the form of Eq. (6.4). Since the sum of  $-10$  and  $-10$  is  $-20$ , we have

$$\begin{aligned} A^4 - 20A^2 + 100 &= (A^2 - 10)(A^2 - 10) \\ &= (A^2 - 10)^2 \end{aligned}$$

(c) Just because the first and third terms are perfect squares, we must realize that the expression may be factored but does *not* fit the form of either Eq. (6.3) or (6.4). One example, using  $x^2$  and  $100$  for the first and third terms, is

$$x^2 + 29x + 100 = (x + 25)(x + 4) \quad \blacksquare$$

### Practice Exercises

Factor: 1.  $x^2 - x - 2$  2.  $x^2 - 4x + 4$

### FACTORING GENERAL TRINOMIALS

Factoring a trinomial for which the coefficient of  $x^2$  is any positive integer involves a slightly more challenging method. Factoring based on the special product of Eq. (6.6) often requires some trial and error. However, the amount of trial and error can be kept

■ For reference, Eq. (6.6) is  
 $(ax + b)(cx + d) =$   
 $acx^2 + (ad + bc)x + bd.$

to a minimum by carefully noting the coefficients of  $x^2$  and the constant. Rewriting Eq. (6.6) with sides reversed, we have

$$\begin{array}{c}
 \text{product} \leftarrow \text{coefficients} \\
 \downarrow \qquad \qquad \qquad \uparrow \\
 acx^2 + (ad + bc)x + bd = (ax + b)(cx + d) \\
 \qquad \qquad \qquad \qquad \qquad \uparrow \qquad \qquad \qquad \downarrow \\
 \qquad \qquad \qquad \text{product} \leftarrow \text{integers} \\
 \\
 \text{outer product} \\
 \downarrow \qquad \qquad \qquad \uparrow \\
 acx^2 + (ad + bc)x + bd = (ax + b)(cx + d) \\
 \qquad \qquad \qquad \qquad \qquad \uparrow \qquad \qquad \qquad \downarrow \\
 \qquad \qquad \qquad \text{inner product}
 \end{array}$$

This diagram shows us that

1. the coefficient of  $x^2$  is the product of the coefficients  $a$  and  $c$  in the factors,
2. the final constant is the product of the constants  $b$  and  $d$  in the factors, and
3. the coefficient of  $x$  is the sum of the inner and outer products.

In finding the factors, we must try possible combinations of  $a$ ,  $b$ ,  $c$ , and  $d$  that give

*the proper inner and outer products for the middle term*

of the given expression.

#### EXAMPLE 4 Factoring a general trinomial

To factor  $2x^2 + 11x + 5$ , we take the factors of 2 to be +2 and +1 (we use only positive coefficients  $a$  and  $c$  when the coefficient of  $x^2$  is positive). We set up the factoring as

$$2x^2 + 11x + 5 = (2x \quad)(x \quad)$$

Since the product of the integers to be found is +5, only integers of the same sign need to be considered. Also since *the sum of the outer and inner products is +11*, the integers are positive. The factors of +5 are +1 and +5, and -1 and -5, which means that +1 and +5 is the only possible pair. Now, trying the factors

$$\begin{array}{c}
 (2x + 5)(x + 1) \\
 \begin{array}{ccc}
 \downarrow & \downarrow & \\
 & +5x & \\
 & +2x & \\
 \hline
 & +7x &
 \end{array}
 \end{array}
 \quad +2x + 5x = +7x$$

we see that  $7x$  is *not* the correct middle term. Therefore, we now try

$$\begin{array}{c}
 (2x + 1)(x + 5) \\
 \begin{array}{ccc}
 \downarrow & \downarrow & \\
 & +x & \\
 & +10x & \\
 \hline
 & +11x &
 \end{array}
 \end{array}
 \quad +x + 10x = +11x$$

and we have the correct sum of +11x. Therefore,

$$2x^2 + 11x + 5 = (2x + 1)(x + 5)$$

For a trinomial with a first term  $2x^2$  and a constant +5 to be factorable, we can now see that the middle term must be either +11x or +7x. This means that  $2x^2 + 7x + 5 = (2x + 5)(x + 1)$ , but a trinomial such as  $2x^2 + 8x + 5$  is not factorable. ■

#### EXAMPLE 5 Factoring a general trinomial with multiple leading coefficient factors

In factoring  $4x^2 + 4x - 3$ , the coefficient 4 in  $4x^2$  shows that the possible coefficients of  $x$  in the factors are 4 and 1, or 2 and 2. The 3 shows that the only possible constants in the factors are 1 and 3, and *the minus sign with the 3 tells us that these integers have*

*different signs.* This gives us the following possible combinations of factors, along with the resulting sum of the outer and inner products:

$$\begin{aligned}(4x + 3)(x - 1): & -4x + 3x = -x \\(4x + 1)(x - 3): & -12x + x = -11x \\(4x - 3)(x + 1): & 4x - 3x = +x \\(4x - 1)(x + 3): & 12x - x = +11x \\(2x + 3)(2x - 1): & -2x + 6x = +4x \\(2x - 3)(2x + 1): & 2x - 6x = -4x\end{aligned}$$

### LEARNING TIP

Another hint given by the coefficients of the original expression is that *the plus sign on the  $x$ -term tells us that the larger of the outer and inner product factors must be positive.*

We see that the factors that have the correct middle term of  $+4x$  are  $(2x + 3)(2x - 1)$ . This means that

$$4x^2 + 4x - 3 = (2x + 3)(2x - 1)$$

Expressing the result with the factors reversed is an equally correct answer. ■

### EXAMPLE 6 Factoring a general trinomial with multiple factor combinations on the $x^2$ -term and the last term

$$6s^2 + 19st - 20t^2 = (6s - 5t)(s + 4t)$$

There are numerous possibilities for the combinations of 6 and  $-20$ . However, we must remember to *check carefully that the middle term of the expression is the proper result of the factors we have chosen.* ■

### EXAMPLE 7 Check the middle term carefully

- (a) In factoring  $9x^2 - 6x + 1$ , note that  $9x^2$  is the square of  $3x$  and 1 is the square of 1. Therefore, we recognize that this expression might fit the perfect square form of Eq. (6.4). This leads us to factor it tentatively as

$$9x^2 - 6x + 1 = (3x - 1)^2 \quad \text{check middle term: } 2(3x)(-1) = -6x$$

However, before we can be certain that this is correct, we must check to see if the middle term of the expansion of  $(3x - 1)^2$  is  $-6x$ , which is what it must be to fit the form of Eq. (6.4). When we expand  $(3x - 1)^2$ , we find that the middle term is  $-6x$  and therefore that the factorization is correct.

- (b) In the same way, we have

$$36x^2 + 84xy + 49y^2 = (6x + 7y)^2 \quad \text{check middle term: } 2(6x)(7y) = 84xy$$

since the middle term of the expansion of  $(6x + 7y)^2$  is  $84xy$ . *We must be careful to include the factors of  $y$  in the second terms of the factors.* ■

### Practice Exercise

Factor: 3.  $4x^2 + x - 5$

### COMMON ERROR

When the last term in a trinomial is not just a constant but also has a variable dependence, it is common to forget this variable when stating the factors:

$$2x^2 + 5xy - 3y^2 = (2x - y)(x + 3y)$$

$$2x^2 + 5xy - 3y^2 \neq (2x - 1)(x + 3)$$



As noted before, we must be careful to factor an expression completely. We first look for common monomial factors and then check each resulting factor to see if it can be factored when we complete each step.

### EXAMPLE 8 General trinomial with common monomial factor

When factoring  $2x^2 + 6x - 8$ , first note the common monomial factor of 2. This leads to

$$2x^2 + 6x - 8 = 2(x^2 + 3x - 4)$$

Now, notice that  $x^2 + 3x - 4$  is also factorable. Therefore,

$$2x^2 + 6x - 8 = 2(x + 4)(x - 1)$$

Having noted the common factor of 2 prevents our having to check factors of 2 and 8. If we had not noted the common factor, we might have arrived at

$$(2x + 8)(x - 1) \quad \text{or} \quad (2x - 2)(x + 4)$$

(possibly after a number of trials). Each is correct as far as it goes, but also each of these is *not complete*. Since  $2x + 8 = 2(x + 4)$ , or  $2x - 2 = 2(x - 1)$ , we can arrive at the proper result shown above. In this way, we would have

$$\begin{aligned} 2x^2 + 6x - 8 &= (2x + 8)(x - 1) \\ &= 2(x + 4)(x - 1) \end{aligned}$$

$$\begin{aligned} \text{or} \quad 2x^2 + 6x - 8 &= (2x - 2)(x + 4) \\ &= 2(x - 1)(x + 4) \end{aligned}$$

Although these factorizations are correct, *it is more efficient to factor out the common factor first*. By doing so, the number of possible factoring combinations is greatly reduced. ■

### EXAMPLE 9 Factoring a trinomial—application

A study of the path of a certain rocket leads to the expression  $16t^2 + 240t - 1600$ , where  $t$  is the time of flight. Factor this expression.

An inspection shows that there is a common factor of 16. (This might be found by noting successive factors of 2 or 4.) Factoring out 16 leads to

$$\begin{aligned} 16t^2 + 240t - 1600 &= 16(t^2 + 15t - 100) && \text{numbers that multiply} \\ &= 16(t + 20)(t - 5) && \text{to } -100 \text{ and sum to } 15 \\ &&& \text{are } 20 \text{ and } -5 \end{aligned}$$

Here, factors of 100 need to be checked for sums equal to 15. This might take some time, but it is much simpler than looking for factors of 16 and 1600 with sums of 240. ■

## FACTORIZING BY GROUPING

Just as in Example 9 of Section 6.2, we can sometimes factor trinomials by grouping terms in the expression together and finding common factors between them. To factor the trinomial  $ax^2 + bx + c$  by grouping, we use the following procedure.

### Method for Factoring a Trinomial by Grouping

1. Find two numbers whose product is  $ac$  and whose sum is  $b$ .
2. Write the trinomial with two  $x$ -terms having these numbers as coefficients.
3. Complete the factorization by grouping.

#### Practice Exercise

Factor: 4.  $6x^2 + 9x - 6$

■ Liquid-fuel rockets were designed in the United States in the 1920s but were developed by German engineers. They were first used in the 1940s during World War II.

**EXAMPLE 10** Factoring by grouping

Factor the trinomial  $6x^2 + 7x - 20$  by grouping.

We begin by finding two numbers with a product of  $-120$  and a sum of  $7$ . Trying products of numbers *with different signs* and a sum of  $7$ , we find the numbers that work are  $-8$  and  $15$ . We then complete steps 2 and 3 of the above procedure as follows:

$$\begin{aligned} 6x^2 + 7x - 20 &= 6x^2 - 8x + 15x - 20 && 7x = -8x + 15x \\ &= (6x^2 - 8x) + (15x - 20) && \text{group first two terms and last two terms} \\ &= 2x(3x - 4) + 5(3x - 4) && \text{find common factor of each group and} \\ &= (3x - 4)(2x + 5) && \text{note common factor of } 3x - 4 \end{aligned}$$

Multiplication verifies that these are the correct factors. ■

**EXAMPLE 11** Factoring by grouping

Factor  $x^2 - 4xy + 4y^2 - 9$ .

The first three terms represent  $(x - 2y)^2$ . Thus, grouping these terms, we have

$$\begin{aligned} x^2 - 4xy + 4y^2 - 9 &= (x^2 - 4xy + 4y^2) - 9 && \text{group terms} \\ &= (x - 2y)^2 - 9 && \text{factor grouping (note difference of squares)} \\ &= [(x - 2y) + 3][(x - 2y) - 3] && \text{factor difference of squares} \\ &= (x - 2y + 3)(x - 2y - 3) \end{aligned}$$

**Not all groupings work.** Noting the factorable combination  $4y^2 - 9$ , and then grouping the first two terms and the last two terms, would not have led to the factorization. ■

**EXERCISES 6.3**

In Exercises 1–6, make the given changes in the indicated examples of this section and then factor.

- In Example 1, change the 3 to 4 and the 2 to 3.
- In Example 2(a), change the + before  $7x$  to  $-$ .
- In Example 4, change the + before  $11x$  to  $-$ .
- In Example 6, change the coefficient 19 to 7.
- In Example 8, change the 8 to 36.
- In Example 10, change the + before  $7x$  to  $-$ .

In Exercises 7–56, factor the given expressions completely.

- |                         |                          |
|-------------------------|--------------------------|
| 7. $x^2 + 5x + 4$       | 8. $x^2 - 5x - 6$        |
| 9. $s^2 - s - 42$       | 10. $a^2 + 14a - 32$     |
| 11. $t^2 + 5t - 24$     | 12. $r^3 - 11r^2 + 18r$  |
| 13. $x^2 + 2x + 1$      | 14. $D^2 + 8D + 16$      |
| 15. $L^2 - 4LK + 4K^2$  | 16. $b^2 - 12bc + 36c^2$ |
| 17. $3x^2 - 5x - 2$     | 18. $2n^2 - 13n - 7$     |
| 19. $12y^2 - 32y - 12$  | 20. $25x^2 + 45x - 10$   |
| 21. $2s^2 + 13s + 11$   | 22. $7y^2 - 12y + 5$     |
| 23. $3f^4 - 16f^2 + 5$  | 24. $5R^4 - 3R^2 - 2$    |
| 25. $2t^2 + 7t - 15$    | 26. $3n^2 - 20n + 20$    |
| 27. $3t^2 - 7tu + 4u^2$ | 28. $3x^2 + xy - 14y^2$  |
| 29. $4x^2 - 3x - 7$     | 30. $2z^2 + 13z - 5$     |
| 31. $9x^2 + 7xy - 2y^2$ | 32. $4r^2 + 11rs - 3s^2$ |

- |                                  |                                 |
|----------------------------------|---------------------------------|
| 33. $4m^2 + 20m + 25$            | 34. $16q^2 + 24q + 9$           |
| 35. $8x^2 - 24x + 18$            | 36. $3a^2c^2 - 6ac + 3$         |
| 37. $9t^2 - 15t + 4$             | 38. $6t^4 + t^2 - 12$           |
| 39. $8b^6 + 31b^3 - 4$           | 40. $12n^4 + 8n^2 - 15$         |
| 41. $4p^2 - 25pq + 6q^2$         | 42. $12x^2 + 4xy - 5y^2$        |
| 43. $12x^2 + 47xy - 4y^2$        | 44. $8r^2 - 14rs - 9s^2$        |
| 45. $2x^2 - 14x + 12$            | 46. $6y^2 - 33y - 18$           |
| 47. $4x^5 + 14x^3 - 8x$          | 48. $12B^2 + 22BH - 4H^2$       |
| 49. $ax^3 + 4a^2x^2 - 12a^3x$    | 50. $6x^4 - 13x^3 + 5x^2$       |
| 51. $a^2 + 2ab + b^2 - 4$        | 52. $x^2 - 6xy + 9y^2 - 4z^2$   |
| 53. $25a^2 - 25x^2 - 10xy - y^2$ | 54. $r^2 - s^2 + 2st - t^2$     |
| 55. $4x^{2n} + 13x^n - 12$       | 56. $12B^{2n} + 19B^nH - 10H^2$ |

In Exercises 57–68, factor the given expressions completely. Each is from the technical area indicated.

- $16t^2 - 80t + 64$  (projectile motion)
- $9x^2 - 33Lx + 30L^2$  (civil engineering)
- $4s^2 + 16s + 12$  (electricity)
- $3\eta^2 + 18\eta - 1560$  (fuel efficiency)
- $200n^2 - 2100n - 3600$  (biology)
- $bT^2 - 40bT + 400b$  (thermodynamics)
- $V^2 - 2nBV + n^2B^2$  (chemistry)
- $a^4 + 8a^2\pi^2f^2 + 16\pi^4f^4$  (periodic motion: energy)

- 65.  $wx^4 - 5wLx^3 + 6wL^2x^2$  (beam design)
- 66.  $1 - 2r^2 + r^4$  (lasers)
- 67.  $3Adu^2 - 4Aduv + Adv^2$  (water power)
- 68.  $k^2A^2 + 2k\lambda A + \lambda^2 - \alpha^2$  (robotics)

In Exercises 69–75, solve the given problems.

- 69. Find the integral value of  $k$  that makes  $4x^2 + 4x - k$  a perfect square trinomial, and express the result in factored form.
- 70. Find the integral value of  $k$  that makes  $49x^2 - 70x + k$  a perfect square trinomial, and express the result in factored form.
- 71. Often, the sum of squares cannot be factored, but  $x^4 + 4$  can be factored. Add and subtract  $4x^2$  and then use factoring by grouping.

- 72. Using the method of Exercise 71, factor  $x^4 + x^2 + 1$ . (First add and subtract  $x^2$ .)
- 73. If sum of squares  $36x^2 + 9$  can be factored, factor it.
- 74. Explain why most students would find  $24x^2 - 23x - 12$  more difficult to factor than  $23x^2 - 18x - 5$ .
- 75. Factor the statements in Exercises 40–43 using grouping.

**Answers to Practice Exercises**

- 1.  $(x - 2)(x + 1)$     2.  $(x - 2)^2$
- 3.  $(4x + 5)(x - 1)$     4.  $3(2x - 1)(x + 2)$

## 6.4 The Sum and Difference of Cubes

**Factoring the Sum of Cubes • Factoring the Difference of Cubes • Summary of Methods of Factoring**

■ Note carefully the positions of the + and - signs.

We have seen that the difference of squares can be factored, but that the sum of squares often cannot be factored. We now turn our attention to the sum and difference of cubes, both of which can be factored. Writing Eqs. (6.9) and (6.10) with sides reversed,

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2) \tag{6.9}$$

$$x^3 - y^3 = (x - y)(x^2 + xy + y^2) \tag{6.10}$$

In these equations, the second factors are prime, assuming  $x$  and  $y$  are themselves prime factors.

**EXAMPLE 1 Factoring the sum and difference of cubes**

**Table of Cubes**

$1^3 = 1$
$2^3 = 8$
$3^3 = 27$
$4^3 = 64$
$5^3 = 125$
$6^3 = 216$

(a)  $x^3 + 8 = x^3 + 2^3$

$$= (x + 2)[(x)^2 - 2x + 2^2]$$

$$= (x + 2)(x^2 - 2x + 4)$$

(b)  $x^3 - 1 = x^3 - 1^3$

$$= (x - 1)[(x)^2 + (1)(x) + 1^2]$$

$$= (x - 1)(x^2 + x + 1)$$

(c)  $8 - 27x^3 = 2^3 - (3x)^3$

$$= (2 - 3x)[2^2 + 2(3x) + (3x)^2]$$

$$= (2 - 3x)(4 + 6x + 9x^2)$$

**EXAMPLE 2 Factoring a difference of cubes with an initial common factor**

In factoring  $ax^5 - ax^2$ , we first note that each term has a common factor of  $ax^2$ . This is factored out to get  $ax^2(x^3 - 1)$ . However, the expression is not completely factored since  $1 = 1^3$ , which means that  $x^3 - 1$  is the difference of cubes. We complete the factoring by the use of Eq. (6.10). Therefore,

$$ax^5 - ax^2 = ax^2(x^3 - 1) = ax^2(x - 1)(x^2 + x + 1)$$

**EXAMPLE 3 Factoring a difference of cubes—application**

The volume of material used to make a steel bearing with a hollow core is given by  $\frac{4}{3}\pi R^3 - \frac{4}{3}\pi r^3$ . Factor this expression.

$$\begin{aligned} \frac{4}{3}\pi R^3 - \frac{4}{3}\pi r^3 &= \frac{4}{3}\pi(R^3 - r^3) && \text{common factor of } \frac{4}{3}\pi \\ &= \frac{4}{3}\pi(R - r)(R^2 + Rr + r^2) && \text{using Eq. (6.10)} \end{aligned}$$

**Practice Exercises**

Factor: 1.  $x^3 + 216$     2.  $x^3 - 216$

### Summary of Methods for Factoring

The following is a step-by-step guide to factoring algebraic expressions.

1. *Common algebraic factor.* **Always check for this first.** Remember that this can be a common numeric factor, a common monomial algebraic factor, or even a common factor consisting of identical multiplied algebraic expressions.
2. *Difference of squares*
3. *Factorable trinomial*
4. *Sum or difference of cubes*
5. *Factoring by grouping*
6. *Check.* Be sure to check that each factor is factored completely.

## EXERCISES 6.4

In Exercises 1 and 2, make the given changes in the indicated examples of this section and then factor.

1. In Example 1(a), change the + before the 8 to -.
2. In Example 2, change  $-ax^2$  to  $+a^4x^2$ .

In Exercises 3–26, factor the given expressions completely.

- |                           |                           |                       |               |
|---------------------------|---------------------------|-----------------------|---------------|
| 3. $x^3 + 1$              | 4. $R^3 + 27$             | 5. $8 - t^3$          | 6. $8r^3 - 1$ |
| 7. $27x^3 - 8a^6$         | 8. $64x^4 + 125x$         | 9. $4x^3 + 32$        |               |
| 10. $3y^3 - 81$           | 11. $6A^6 - 6A^3$         | 12. $8s^9 - 64$       |               |
| 13. $162x^3y - 6x^3y^4$   | 14. $12a^3 + 96a^3b^3$    | 15. $x^6y^3 + x^3y^6$ |               |
| 16. $16r^3 - 432$         | 17. $3a^6 - 3a^3$         | 18. $x^6 - 27y^3$     |               |
| 19. $0.001R^3 - 0.064r^3$ | 20. $0.027x^3 + 0.125$    |                       |               |
| 21. $27L^6 + 216L^3$      | 22. $a^3s^5 - 8000a^3s^2$ |                       |               |
| 23. $(a + b)^3 + 64$      | 24. $125 + (2x + y)^3$    |                       |               |
| 25. $64 - x^6$            | 26. $a^6 - 27b^6$         |                       |               |

In Exercises 27–32, factor the given expressions completely. Each is from the technical area indicated.

27.  $2x^3 + 250$  (computer image)
28.  $kT^3 - kT_0^3$  (thermodynamics)
29.  $D^4 - d^3D$  (machine design)
30.  $(h + 2t)^3 - h^3$  (container design)

31.  $QH^4 + Q^4H$  (thermodynamics)

32.  $\left(\frac{s}{r}\right)^{12} - \left(\frac{s}{r}\right)^6$  (molecular interaction)

In Exercises 33 and 34, perform the indicated operations.

33. Perform the division  $(x^5 - y^5) \div (x - y)$ . Noting the result, determine the quotient  $(x^7 - y^7) \div (x - y)$ , without dividing. From these results, factor  $x^5 - y^5$  and  $x^7 - y^7$ .
34. Perform the division  $(x^5 + y^5) \div (x + y)$ . Noting the result, determine the quotient  $(x^7 + y^7) \div (x + y)$ , without dividing. From these results, factor  $x^5 + y^5$  and  $x^7 + y^7$ .

In Exercises 35–38, solve the given problems.

35. Factor  $x^6 - y^6$  as the difference of cubes.
36. Factor  $x^6 - y^6$  as the difference of squares. Then explain how the result of Exercise 35 can be shown to be the same as the result in this exercise. (*Hint:* See Exercise 72 on page 196.)
37. By factoring  $(n^3 + 1)$ , explain why this expression represents a number that is not prime if  $n$  is an integer greater than one.
38. Explain how Eq. (6.9) can be used to factor the *difference* of cubes.

### Answers to Practice Exercises

1.  $(x + 6)(x^2 - 6x + 36)$
2.  $(x - 6)(x^2 + 6x + 36)$

## 6.5 Equivalent Fractions

**Fundamental Principle of Fractions • Equivalent Fractions • Simplest Form, or Lowest Terms, of a Fraction • Factors That Differ Only in Sign**

When we deal with algebraic expressions, we must be able to work effectively with fractions. Since algebraic expressions are representations of numbers, the basic operations on fractions from arithmetic form the basis of our algebraic operations. In this section, we demonstrate a very important property of fractions, and in the following two sections, we establish the basic algebraic operations with fractions.

This important property of fractions is often referred to as the **fundamental principle of fractions**.

**Fundamental Principle of Fractions**

The value of a fraction is unchanged if both the numerator and the denominator are multiplied or divided by the same number, provided that this number is not zero.

Two fractions are said to be **equivalent** if one can be obtained from the other by use of the fundamental principle.

**EXAMPLE 1** Equivalent arithmetic fractions

If we multiply the numerator and the denominator of the fraction  $\frac{6}{8}$  by 2, we obtain the equivalent fraction  $\frac{12}{16}$ . If we divide the numerator and the denominator of  $\frac{6}{8}$  by 2, we obtain the equivalent fraction  $\frac{3}{4}$ . This means that the fractions  $\frac{6}{8}$ ,  $\frac{3}{4}$ , and  $\frac{12}{16}$  are equivalent. Obviously, there is an unlimited number of other fractions that are equivalent to these fractions. ■

**EXAMPLE 2** Equivalent algebraic fractions

We may write

$$\frac{ax}{2} = \frac{3a^2x}{6a}$$

since we get the fraction on the right by multiplying the numerator and the denominator of the fraction on the left by  $3a$ . This means that the fractions are equivalent. ■

**SIMPLEST FORM, OR LOWEST TERMS, OF A FRACTION**

One of the most important operations with a fraction is that of **reducing** it to its **simplest form**, or **lowest terms**.

A fraction is said to be in its simplest form if the numerator and the denominator have no common factors other than  $+1$  or  $-1$ .

In *reducing* a fraction to its simplest form, use the fundamental principle of fractions by *dividing* both the numerator and the denominator by **all factors** that are common to each. *If any of the literal symbols were to be evaluated, numerical values must be such that none of the denominators would be equal to zero.* Therefore, all operations will be done properly since the undefined operation of division by zero is avoided. This can also be indicated by stating explicitly which factors cannot equal zero.

**EXAMPLE 3** Reducing a fraction to its lowest terms

In order to reduce the fraction

$$\frac{16ab^3c^2}{24ab^2c^5}$$

to its lowest terms, note that both the numerator and the denominator contain the factor  $8ab^2c^2$ . Therefore,

$$\frac{16ab^3c^2}{24ab^2c^5} = \frac{2b(8ab^2c^2)}{3c^3(8ab^2c^2)} = \frac{2b}{3c^3} \quad \text{— common factor}$$

Here, we divided out the common factor of  $8ab^2c^2$ . The resulting fraction is in its lowest terms, since there are no common factors in the numerator and the denominator other than  $+1$  or  $-1$ . In performing this division, we also assume that neither  $a$ ,  $b$ , nor  $c$  is equal to zero so as to avoid the possibility of dividing by zero, that is,  $a, b, c \neq 0$ . ■

**Practice Exercise**

1. Reduce to lowest terms:  $\frac{9xy^5}{15x^3y^2}$

In simplifying a fraction, be very careful in performing the basic step of reducing the fraction to its simplest form, through **cancellation**.

**Cancellation** is the process of *dividing* both the numerator and denominator *by the common factor*.

### COMMON ERROR

You cannot remove just any expression that appears in both the numerator and denominator. If a **term** is the same in both numerator and denominator, it *cannot* be cancelled. Only **factors** that are in both the numerator and denominator can be cancelled.

In the following example, we illustrate this very common error in the simplification of fractions. Follow it very carefully.

### EXAMPLE 4 Cancel factors, not terms

When simplifying the expression

$$\frac{x^2(x-2)}{x^2-4}$$

a term, but not a factor, of the denominator

many students would “cancel” the  $x^2$  from the numerator and the denominator. This is incorrect, since  $x^2$  is a **term only** of the denominator.

In order to simplify the above fraction properly, we should factor the denominator. We get

$$\frac{x^2(\cancel{x-2})}{(\cancel{x-2})(x+2)} = \frac{x^2}{x+2}, \text{ where } x \neq 2$$

Here, the common **factor**  $(x-2)$  has been divided out. ■

The following examples illustrate the proper simplification of fractions.

### EXAMPLE 5 Distinguishing between term and factor

(a)  $\frac{2a}{2ax} = \frac{1}{x}$  —  $2a$  is a factor of the numerator and the denominator,  $a \neq 0$

We divide out the common factor of  $2a$ .

(b)  $\frac{2a}{2a+x}$  —  $2a$  is a term, but not a factor, of the denominator

**This cannot be reduced** since there are no common **factors** in the numerator and the denominator. ■

### EXAMPLE 6 Remaining factor of 1 in the denominator

$$\begin{aligned} \frac{2x^2+8x}{x+4} &= \frac{2x(x+4)}{(x+4)} = \frac{2x}{1}, x \neq -4 \\ &= 2x \end{aligned}$$

The numerator and the denominator were each divided by  $x+4$  after factoring the numerator. The only remaining factor in the denominator is 1, and it is generally not written in the final result. Another way of writing the denominator is  $1(x+4)$ , which shows the **factor** of 1 more clearly. ■

### Practice Exercise

2. Reduce to lowest terms:  $\frac{4a}{4a-2x}$



**EXAMPLE 7** Cancel factors only

$$\frac{x^2 - 4x + 4}{x^2 - 4} = \frac{(x-2)\overset{1}{\cancel{(x-2)}}}{(x+2)\overset{1}{\cancel{(x-2)}}}, x \neq 2$$

$$= \frac{x-2}{x+2}$$

—  $x$  is a term but not a factor

**Practice Exercise**

3. Reduce to lowest terms:  $\frac{x^2 - x - 2}{x^2 + 3x + 2}$

Here, the numerator and the denominator have each been *factored first and then the common factor  $x - 2$  has been divided out*. In the final form, neither the  $x$ 's nor the 2's may be cancelled, since they are not common *factors*. ■

**EXAMPLE 8** Reducing a fraction—application

In the mathematical analysis of the vibrations in a certain mechanical system, the following expression and simplification are used:

$$\frac{8s + 12}{4s^2 + 26s + 30} = \frac{4(2s + 3)}{2(2s^2 + 13s + 15)} = \frac{\overset{2}{\cancel{4}}\overset{1}{\cancel{(2s+3)}}}{\underset{1}{\cancel{2}}\overset{1}{\cancel{(2s+3)}}(s+5)}, s \neq -\frac{3}{2}$$

$$= \frac{2}{s+5}$$

In the third fraction, note that the factors common to both the numerator and the denominator are 2 and  $(2s + 3)$ . ■

**FACTORS THAT DIFFER ONLY IN SIGN**

In simplifying fractions, we must be able to distinguish between factors that differ only in *sign*. Since  $-(y - x) = -y + x = x - y$ , we have

$$x - y = -(y - x) \quad (6.11)$$

Here, the *factors  $x - y$  and  $y - x$  differ only in sign*. The following examples illustrate the simplification of fractions where a change of signs is necessary.

**EXAMPLE 9** Factors that differ only in sign

$$\frac{x^2 - 1}{1 - x} = \frac{\overset{1}{\cancel{(x-1)}}(x+1)}{\underset{1}{\cancel{-(x-1)}}} = \frac{x+1}{-1} = -(x+1), x \neq 1$$

In the second fraction, *we replaced  $1 - x$  with the equal expression  $-(x - 1)$* . In the third fraction, the common factor  $x - 1$  was divided out. Finally, we expressed the result in the more convenient form by dividing  $x + 1$  by  $-1$ . Replacing  $1 - x$  with  $-(x - 1)$  is the same as factoring  $-1$  from the terms of  $1 - x$ .

It should be clear that the factors  $1 - x$  and  $x - 1$  differ only in sign. ■

**COMMON ERROR**

When simplifying fractions, it is important to look for factors that differ only in sign. They can provide a cancellation that often goes unnoticed, yielding a fraction which is **not in its lowest terms**.

**EXAMPLE 10** Factors that differ only in sign

$$\begin{aligned} \frac{2x^4 - 128x}{20 + 7x - 3x^2} &= \frac{2x(x^3 - 64)}{(4 - x)(5 + 3x)} = \frac{2x(\cancel{x-4})(x^2 + 4x + 16)}{-(\cancel{x-4})(3x + 5)}, x \neq 4 \\ &= \frac{2x(x^2 + 4x + 16)}{3x + 5} \end{aligned}$$

Again, the factor  $4 - x$  has been replaced by the equal expression  $-(x - 4)$ . This allows us to recognize the common factor of  $x - 4$ .

Also, note that the order of the terms of the factor  $5 + 3x$  was changed in writing the third fraction. This was done only to write the terms in the more standard form with the  $x$ -term first. However, since both terms are *positive*, it is simply an application of the commutative law of addition, and the factor itself is not actually changed. ■

**EXERCISES 6.5**

In Exercises 1–4, make the given changes in the indicated examples of this section and then solve the resulting problems.

- In Example 3, change the numerator to  $18abc^6$  and then reduce the fraction to lowest terms.
- In Example 4, change the  $-$  sign in the numerator to  $+$  and then simplify.
- In Example 7, change the  $-$  sign in the numerator to  $+$  and then simplify.
- In Example 10, change the numerator to  $2x^4 - 32x^2$  and then simplify.

In Exercises 5–12, multiply the numerator and the denominator of each fraction by the given factor and obtain an equivalent fraction.

- $\frac{2}{3}$  (by 7)
- $\frac{7}{5}$  (by 9)
- $\frac{ax}{y}$  (by  $2x$ )
- $\frac{2x^2y}{3n}$  (by  $2xn^2$ )
- $\frac{2}{x+3}$  (by  $x-2$ )
- $\frac{7}{a-1}$  (by  $a+2$ )
- $\frac{a(x-y)}{x-2y}$  (by  $x+y$ )
- $\frac{B-1}{B+1}$  (by  $B-1$ )

In Exercises 13–20, divide the numerator and the denominator of each fraction by the given factor and obtain an equivalent fraction.

- $\frac{28}{44}$  (by 4)
- $\frac{25}{65}$  (by 5)
- $\frac{4x^2y}{8xy^2}$  (by  $2x$ )
- $\frac{6a^3b^2}{9a^5b^4}$  (by  $3a^2b^2$ )
- $\frac{2(R-1)}{(R-1)(R+1)}$  (by  $R-1$ )
- $\frac{(x+5)(x-3)}{3(x+5)}$  (by  $x+5$ )

- $\frac{s^2 - 3s - 10}{2s^2 + 3s - 2}$  (by  $s + 2$ )
- $\frac{6x^2 + 13x - 5}{6x^3 - 2x^2}$  (by  $3x - 1$ )

In Exercises 21–28, replace the  $A$  with the proper expression such that the fractions are equivalent.

- $\frac{3x}{2y} = \frac{A}{6y^2}$
- $\frac{2R}{R+T} = \frac{2R^2T}{A}$
- $\frac{7}{a+5} = \frac{7a-35}{A}$
- $\frac{a+1}{5a^2c} = \frac{A}{5a^3c - 5a^2c}$
- $\frac{2x^3 + 2x}{x^4 - 1} = \frac{A}{x^2 - 1}$
- $\frac{n^2 - 1}{n^3 + 1} = \frac{A}{n^2 - n + 1}$
- $\frac{x^2 + 3bx - 4b^2}{x - b} = \frac{x + 4b}{A}$
- $\frac{4y^2 - 1}{4y^2 + 6y - 4} = \frac{A}{2y + 4}$

In Exercises 29–64, reduce each fraction to simplest form.

- $\frac{2a}{8a}$
- $\frac{6x}{15x}$
- $\frac{18x^2y}{24xy}$
- $\frac{2a^2xy}{6axyz^2}$
- $\frac{a+b}{5a^2+5ab}$
- $\frac{t-a}{t^2-a^2}$
- $\frac{4a-4b}{4a-2b}$
- $\frac{20s-5r}{10r-5s}$
- $\frac{4x^2+1}{4x^2-1}$
- $\frac{x^2-y^2}{x^2+y^2}$
- $\frac{3x^2-6x}{x-2}$
- $\frac{10T^2+15T}{2T+3}$
- $\frac{3+2y}{4y^3+6y^2}$
- $\frac{6-3t}{4t^3-8t^2}$
- $\frac{x^2-8x+16}{x^2-16}$
- $\frac{4a^2+12ab+9b^2}{4a^2+6ab}$

45.  $\frac{2w^4 + 5w^2 - 3}{w^4 + 11w^2 + 24}$

47.  $\frac{5x^2 - 6x - 8}{x^3 + x^2 - 6x}$

49.  $\frac{N^4 - 16}{8N - 16}$

51.  $\frac{t + 4}{(2t + 9)t + 4}$

53.  $\frac{(x - 1)(3 + x)}{(3 - x)(1 - x)}$

55.  $\frac{y^2 - x^2}{2x - 2y}$

57.  $\frac{x^3 + x^2 - x - 1}{x^3 - x^2 - x + 1}$

59.  $\frac{(x + 5)(x - 2)(x + 2)(3 - x)}{(2 - x)(5 - x)(3 + x)(2 + x)}$

60.  $\frac{(2x - 3)(3 - x)(x - 7)(3x + 1)}{(3x + 2)(3 - 2x)(x - 3)(7 + x)}$

61.  $\frac{x^3 + y^3}{2x + 2y}$

63.  $\frac{6x^2 + 2x}{27x^3 + 1}$

46.  $\frac{3y^3 + 7y^2 + 4y}{y^2 + 5y + 4}$

48.  $\frac{5s^2 + 8rs - 4s^2}{6r^2 - 17rs + 5s^2}$

50.  $\frac{3 + x(4 + x)}{3 + x}$

52.  $\frac{2A^3 + 8A^4 + 8A^5}{4A + 2}$

54.  $\frac{(2x - 1)(x + 6)}{(x - 3)(1 - 2x)}$

56.  $\frac{x^2 - y^2 - 4x + 4y}{x^2 - y^2 + 4x - 4y}$

58.  $\frac{3a^2 - 13a - 10}{5 + 4a - a^2}$

62.  $\frac{w^3 - 8}{w^2 + 2w + 4}$

64.  $\frac{24 - 3a^3}{a^2 - 4a + 4}$

66. (a)  $\frac{2x + 3}{2x + 6}$  (b)  $\frac{2(x + 6)}{2x + 6}$

67. (a)  $\frac{x^2 - x - 2}{x^2 - x}$  (b)  $\frac{x^2 - x - 2}{x^2 + x}$

68. (a)  $\frac{x^3 - x}{1 - x}$  (b)  $\frac{2x^2 + 4x}{2x^2 + 4}$

In Exercises 69–72, reduce each fraction to simplest form. Each is from the indicated area of application.

69.  $\frac{mu^2 - mv^2}{mu - mv}$  (nuclear energy)

70.  $\frac{16(t^2 - 2tt_0 + t_0^2)(t - t_0 - 3)}{3t - 3t_0}$  (rocket motion)

71.  $\frac{E^2R^2 - E^2r^2}{(R^2 + 2Rr + r^2)^2}$  (electricity)

72.  $\frac{r_0^3 - r_i^3}{r_0^2 - r_i^2}$  (machine design)

### Answers to Practice Exercises

1.  $\frac{3y^3}{5x^2}$  2.  $\frac{2a}{2a - x}$  3.  $\frac{x - 2}{x + 2}, x \neq -1$  4.  $-\frac{1}{y + 2}, y \neq 2$

In Exercises 65–68, after finding the simplest form of each fraction, explain why it cannot be simplified more.

65. (a)  $\frac{x^2(x + 2)}{x^2 + 4}$  (b)  $\frac{x^4 + 4x^2}{x^4 - 16}$

## 6.6 Multiplication and Division of Fractions

### Multiplication of Fractions • Division of Fractions

From arithmetic, recall that *the product of two fractions is a fraction whose numerator is the product of the numerators and whose denominator is the product of the denominators of the given fractions*. Also, *the quotient of two fractions is found by inverting the denominator and proceeding as in multiplication*. Symbolically, these operations are shown as

#### Multiplication of Fractions

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$$

#### Division of Fractions

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$$

The rule for division may be verified by use of the fundamental principle of fractions. By multiplying the numerator and the denominator of the fraction

$$\frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc} = \frac{ad}{1 \cdot bc} = \frac{ad}{bc}$$

by  $\frac{d}{c}$  we obtain

The following three examples illustrate the multiplication of fractions.

### EXAMPLE 1 Multiplying basic fractions

$$(a) \frac{3}{5} \times \frac{2}{7} = \frac{(3)(2)}{(5)(7)} = \frac{6}{35}$$

← multiply numerators  
← multiply denominators

$$(b) \frac{3a}{5b} \times \frac{15b^2}{a} = \frac{(3a)(15b^2)}{(5b)(a)} = \frac{45ab^2}{5ab} = \frac{9b}{1} = 9b, \text{ where } a, b \neq 0$$

$$(c) (6x) \left( \frac{2y}{3x^2} \right) = \left( \frac{6x}{1} \right) \left( \frac{2y}{3x^2} \right) = \frac{12xy}{3x^2} = \frac{4y}{x}, \text{ where } x \neq 0$$

#### LEARNING TIP

When multiplying fractions, since all the factors in the numerators and denominators are to be multiplied, it is best to **factor** the numerator and denominator and **cancel any common factors before multiplying**.

This technique will simplify the fraction before the multiplication expansion, and make it easier to see how to factor and simplify the result.

In (b), we divided out the common factor  $5ab$  to reduce the resulting fraction to its lowest terms. In (c), note that we treated  $6x$  as  $\frac{6x}{1}$ .

When multiplying fractions, we usually want to express the final result in simplest form, which is generally its most useful form. If we were to multiply out the numerator and the denominator before factoring, it is very possible that we would not see how to factor the result and therefore would not be able to simplify it. The following example illustrates this point.

### EXAMPLE 2 Factor and simplify first

In performing the multiplication

$$\frac{3(x-y)}{(x-y)^2} \times \frac{(x^2-y^2)}{6x+9y}$$

if we multiplied out the numerators and the denominators before performing any factoring, we would have to simplify the fraction

$$\frac{3x^3 - 3x^2y - 3xy^2 + 3y^3}{6x^3 - 3x^2y - 12xy^2 + 9y^3}$$

It is possible to factor the resulting numerator and denominator, but finding any common factors this way is very difficult. However, as stated above, we should **first factor** the numerator and denominator, then **divide out (cancel)** any common factors, and consequently the solution is much easier. Doing this, we have

$$\begin{aligned} \frac{3(x-y)}{(x-y)^2} \times \frac{(x^2-y^2)}{6x+9y} &= \frac{3(x-y)(x^2-y^2)}{(x-y)^2(6x+9y)} \\ &= \frac{3(x-y)(x+y)(x-y)}{(x-y)^2(3)(2x+3y)} \\ &= \frac{\cancel{3}(x-y)^2(x+y)}{\cancel{3}(x-y)^2(2x+3y)}, \text{ where } x \neq y \\ &= \frac{x+y}{2x+3y} \end{aligned}$$

The common factor  $3(x-y)^2$  is readily recognized using this procedure.

#### Practice Exercise

1. Multiply:  $\frac{2x}{4x+8} \times \frac{x+2}{xy-x}$

■ It is possible to factor and indicate the product of the factors, showing only a single step, as we have done here.

**LEARNING TIP**

It is permissible to multiply out the final form of the numerator and denominator, but it is preferable to leave the numerator and denominator in factored form.

The factored form makes it easier to identify important features of a function, such as zeros or vertical asymptotes. These ideas will become important in later chapters.

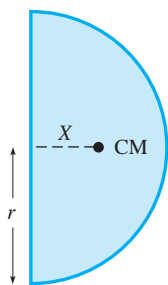


Fig. 6.4

**EXAMPLE 3** Multiplying algebraic fractions

$$\left(\frac{2x-4}{4x+12}\right)\left(\frac{2x^2+x-15}{3x-1}\right) = \frac{2(x-2)(2x-5)(x+3)}{2(x+3)(3x-1)} \leftarrow \begin{array}{l} \text{multiplications} \\ \text{indicated} \end{array}$$

$$= \frac{(x-2)(2x-5)}{2(3x-1)}, \text{ where } x \neq -3$$

Here, the common factor is  $2(x+3)$ .

The following examples illustrate the division of fractions.

**EXAMPLE 4** Dividing basic algebraic fractions

(a)  $\frac{6x}{7} \div \frac{5}{3} = \frac{6x}{7} \times \frac{3}{5} = \frac{18x}{35}$

(b)  $\frac{3a^2}{5c} \div \frac{2c^2}{a} = \frac{3a^2}{5c} \times \frac{a}{2c^2} = \frac{3a^3}{10c^3}$

**EXAMPLE 5** Division by a fraction—application

When finding the centre of mass (CM) of a uniform flat semicircular metal plate, the equation  $X_{CM} = \frac{4\pi r^3}{3} \div \left(\frac{\pi r^2}{2} \times 2\pi\right)$  is derived. Simplify the right side of this equation to find  $X_{CM}$  as a function of  $r$  in simplest form. See Fig. 6.4.

The parentheses indicate that we should perform the multiplication first:

$$X_{CM} = \frac{4\pi r^3}{3} \div \left(\frac{\pi r^2}{2} \times 2\pi\right) = \frac{4\pi r^3}{3} \div \left(\frac{2\pi^2 r^2}{2}\right) \quad 2\pi = \frac{2\pi}{1}$$

$$X_{CM} = \frac{4\pi r^3}{3} \div (\pi^2 r^2) = \frac{4\pi r^3}{3} \times \frac{1}{\pi^2 r^2}$$

$$X_{CM} = \frac{4\pi r^3}{3\pi^2 r^2} = \frac{4r}{3\pi} \quad \text{divide out the common factor of } \pi r^2$$

This is the exact solution. Approximately,  $X_{CM} = 0.424r$ .

**EXAMPLE 6** Dividing algebraic fractions

$$(x+y) \div \frac{2x+2y}{6x+15y} = \frac{x+y}{1} \times \frac{6x+15y}{2x+2y} = \frac{(x+y)(3)(2x+5y)}{2(x+y)} \quad \begin{array}{l} \text{factor and} \\ \text{cancel} \end{array}$$

$$= \frac{3(2x+5y)}{2} \quad \begin{array}{l} \text{restriction} \\ \text{where } x \neq -y \end{array}$$

$$= \frac{3(2x+5y)}{2} \quad \text{simplify}$$

**Practice Exercise**

2. Divide:  $\frac{3x}{a+1} \div \frac{x^2+2x}{a^2+a}$

**EXAMPLE 7** Dividing compound algebraic fractions

$$\frac{4-x^2}{x^2-3x+2} \div \frac{x+2}{x^2-9} = \frac{4-x^2}{x^2-3x+2} \times \frac{x^2-9}{x+2} \quad \text{invert}$$

$$\begin{aligned}
 &= \frac{(2-x)(2+x)(x-3)(x+3)}{(x-2)(x-1)(x+2)} && \text{factor and cancel} \\
 &= \frac{-\cancel{(x-2)}\cancel{(x+2)}(x-3)(x+3)}{\cancel{(x-2)}(x-1)\cancel{(x+2)}} && \begin{array}{l} \text{replace } (2-x) \text{ with } -(x-2) \\ \text{and } (2+x) \text{ with } (x+2) \end{array} \\
 &= \frac{(x-3)(x+3)}{x-1} \quad \text{or} \quad \frac{(x-3)(x+3)}{1-x} && \text{restriction where } x \neq \pm 2 \\
 & && \text{simplify}
 \end{aligned}$$

Note the use of Eq. (6.11) when the factor  $(2-x)$  was replaced by  $-(x-2)$  to get the first form of the answer. As shown, it can also be used to get the alternate form of the answer, although it is not necessary to give this form. ■

## EXERCISES 6.6

In Exercises 1–4, make the given changes in the indicated examples of this section and then solve the resulting problems.

- In Example 2, change the first numerator to  $4x + 6y$  and do the multiplication.
- In Example 3, change the first denominator to  $4x - 10$  and do the multiplication.
- In Example 6, change the denominator  $6x + 15y$  to  $6x + 12y$  and then do the division.
- In Example 7, change the numerator  $x + 2$  of the divisor to  $x + 3$  and then simplify.

In Exercises 5–44, simplify the given expressions involving the indicated multiplications and divisions.

- $\frac{3}{8} \times \frac{2}{7}$
- $11 \times \frac{13}{33}$
- $\frac{4x}{3y} \times \frac{9y^2}{2}$
- $\frac{18sy^3}{ax^2} \times \frac{(ax)^2}{3s}$
- $\frac{2}{9} \div \frac{4}{7}$
- $\frac{5}{16} \div \frac{25}{13}$
- $\frac{xy}{az} \div \frac{bz}{ay}$
- $\frac{sr^2}{2t} \div \frac{st}{4}$
- $\frac{4x+12}{5} \times \frac{15t}{3x+9}$
- $\frac{2y^2+6y}{6z} \times \frac{z^3}{y^2-9}$
- $\frac{u^2-v^2}{u+2v} (3u+6v)$
- $(x-y) \frac{x+2y}{x^2-y^2}$
- $\frac{2a+8}{15} \div \frac{a^2+8a+16}{125}$
- $\frac{a^2-a}{3a+9} \div \frac{a^2-2a+1}{18-2a^2}$
- $\frac{x^4-9}{x^2} \div (x^2+3)^2$
- $\frac{9B^2-16}{B+1} \div (4-3B)$
- $\frac{3ax^2-9ax}{10x^2+5x} \times \frac{2x^2+x}{a^2x-3a^2}$
- $\frac{4R^2-36}{R^3-25R} \times \frac{7R-35}{3R^2+9R}$
- $\left(\frac{x^4-1}{8x+16}\right) \left(\frac{2x^2-8x}{x^3+x}\right)$
- $\left(\frac{2x^2-4x-6}{x^2-3x}\right) \left(\frac{x^3-4x^2}{4x^2-4x-8}\right)$

- $\frac{x^2+ax}{2b-cx} \div \frac{a^2+2ax+x^2}{2bx-cx^2}$
- $\frac{x^4-11x^2+28}{4-x^2} \div \frac{2x^2+6}{2x^2+3}$
- $\frac{35a+25}{12a+33} \div \frac{28a+20}{36a+99}$
- $\frac{2a^3+a^2}{2b^3+b^2} \div \frac{2ab+a}{2ab+b}$
- $\frac{x^2-6x+5}{4x^2-17x-15} \times \frac{6x+21}{2x^2+5x-7}$
- $\frac{n^2+5n}{3n^2+8n+4} \times \frac{2n^2-8}{n^3+3n^2-10n}$
- $\frac{6T^2-NT-N^2}{2V^2-9V-35} \div \frac{8L^3-9L}{2L^3-3L^2}$
- $\frac{7x^2}{3a} \div \left(\frac{a}{x} \times \frac{a^2x}{x^2}\right)$
- $\left(\frac{4t^2-1}{t-5} \div \frac{2t+1}{2t}\right) \times \frac{2t^2-50}{4t^2+4t+1}$
- $\frac{2x^2-5x-3}{x-4} \div \left(\frac{x-3}{x^2-16} \times \frac{1}{3-x}\right)$
- $\frac{x^3-y^3}{2x^2-2y^2} \times \frac{y^2+2xy+x^2}{x^2+xy+y^2}$
- $\frac{2M^2+4M+2}{6M-6} \div \frac{5M+5}{M^2-1}$
- $\left(\frac{ax+bx+ay+by}{p-q}\right) \left(\frac{3p^2+4pq-7q^2}{a+b}\right)$
- $\frac{x^4+x^5-1-x}{x-1} \div \frac{x-1}{x}$
- $\frac{x}{2x+4} \times \frac{x^2-4}{3x^2}$
- $\frac{4x^2-25}{4x^2} \div \frac{4x+10}{8}$

43.  $\frac{2x^2 + 3x - 2}{2 + 3x - 2x^2} \div \frac{5x + 10}{4x + 2}$

44.  $\frac{16x^2 - 8x + 1}{9x} \times \frac{12x + 3}{1 - 16x^2}$

In Exercises 45–48, simplify the given expressions. The technical application of each is indicated.

45.  $\frac{d}{2} \div \frac{v_1d + v_2d}{4v_1v_2}$  (average velocity)

46.  $\frac{c\lambda^2 - c\lambda_0^2}{\lambda_0^2} \div \frac{\lambda^2 + \lambda_0^2}{\lambda_0^2}$  (cosmology)

47.  $\frac{2\pi}{\lambda} \left( \frac{a+b}{2ab} \right) \left( \frac{ab\lambda}{2a+2b} \right)$  (optics)

48.  $(p_1 - p_2) \div \left( \frac{\pi a^4 p_1 - \pi a^4 p_2}{81u} \right)$  (hydrodynamics)

#### Answers to Practice Exercises

1.  $\frac{1}{2(y-1)}, x \neq 0, -2$     2.  $\frac{3a}{x+2}, x \neq 0, a \neq -1$

## 6.7 Addition and Subtraction of Fractions

### Lowest Common Denominator • Addition and Subtraction of Fractions • Complex Fractions

From arithmetic, recall that *the sum of a set of fractions that all have the same denominator is the sum of the numerators divided by the common denominator*. Since algebraic expressions represent numbers, this fact is also true in algebra. Addition and subtraction of such fractions are illustrated in the following example.

#### EXAMPLE 1 Combining basic fractions

$$\begin{aligned} \text{(a)} \quad \frac{5}{9} + \frac{2}{9} - \frac{4}{9} &= \frac{5+2-4}{9} \quad \leftarrow \text{sum of numerators} \\ &\quad \leftarrow \text{same denominators} \\ &= \frac{3}{9} = \frac{1}{3} \quad \leftarrow \text{final result in lowest terms} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \frac{b}{ax} + \frac{1}{ax} - \frac{2b-1}{ax} &= \frac{b+1-(2b-1)}{ax} \quad \leftarrow \text{use parentheses to show subtraction of both terms} \\ &= \frac{b+1-2b+1}{ax} \\ &= \frac{2-b}{ax} \end{aligned}$$

#### COMMON ERROR

When subtracting a fraction with multiple terms in the numerator, ensure that the negative sign is applied to *all* terms. The signs of all the terms must be changed before they can be combined with other terms. It is a common error to forget to apply the subtraction to both terms.

For example,

$$\frac{7}{5} - \frac{x+4}{5} = \frac{7-(x+4)}{5} = \frac{7-x-4}{5} = \frac{3-x}{5}$$

is correct.

$$\frac{7}{5} - \frac{x+4}{5} \neq \frac{7-x+4}{5} = \frac{11-x}{5}$$

is not correct.

#### LOWEST COMMON DENOMINATOR

If the fractions to be combined do not all have the same denominator, we must first change each to an equivalent fraction so that the resulting fractions do have the same denominator. Normally, the denominator that is most convenient and useful is the **lowest common denominator** (abbreviated as **LCD**).



**LEARNING TIP**

The LCD is the product of all the prime factors that appear in the denominators, with each factor raised to the highest power to which it appears in any one of the denominators. This means that the lowest common denominator is the simplest algebraic expression into which all given denominators will divide exactly.

Following is the procedure for finding the lowest common denominator of a set of fractions.

**Procedure for Finding the Lowest Common Denominator**

1. Factor each denominator into its prime factors.
2. For each different prime factor that appears, note the highest power to which it is raised in any one of the denominators.
3. Form the product of all the different prime factors, each raised to the power found in step 2. This product is the lowest common denominator.

The two examples that follow illustrate the method of finding the LCD.

**EXAMPLE 2** Lowest common denominator (LCD) with algebraic fractions

Find the LCD of the fractions

$$\frac{3}{4a^2b} \quad \frac{5}{6ab^3} \quad \frac{1}{4ab^2}$$

We now express each denominator in terms of powers of its prime factors:

$$4a^2b = 2^2a^2b \quad 6ab^3 = 2 \times 3 \times ab^3 \quad 4ab^2 = 2^2ab^2$$

highest powers
already seen to be highest power of 2

The prime factors to be considered are 2, 3,  $a$ , and  $b$ . The largest exponent of 2 that appears is 2. Therefore,  $2^2$  is a factor of the LCD.

*What matters is that the highest power of 2 that appears is 2, not the fact that 2 appears in all three denominators with a total of five factors.*

The largest exponent of 3 that appears is 1 (understood in the second denominator). Therefore, 3 is a factor of the LCD. The largest exponent of  $a$  that appears is 2, and the largest exponent of  $b$  that appears is 3. Thus,  $a^2$  and  $b^3$  are factors of the LCD. Therefore, the LCD of the fractions is

$$2^2 \times 3 \times a^2b^3 = 12a^2b^3$$

This is the simplest expression into which *each* of the denominators above will divide exactly. ■

Finding the LCD of a set of fractions can be a source of difficulty. Remember, it is necessary to find *all of the prime factors* that may be in any of the denominators, and then find *the highest power to which each is raised in any of the denominators*. The following example illustrates this when factoring the denominators is the first step.

**EXAMPLE 3** Lowest common denominator with factored denominators

Find the LCD of the following fractions:

$$\frac{x-4}{x^2-2x+1} \quad \frac{1}{x^2-1} \quad \frac{x+3}{x^2-x}$$

Factoring each of the denominators, we find that the fractions are

$$\frac{x-4}{(x-1)^2} \quad \frac{1}{(x-1)(x+1)} \quad \frac{x+3}{x(x-1)}$$

The factor  $(x-1)$  appears in all the denominators. It is squared in the first fraction and appears only to the first power in the other two fractions. Thus, we must have

**Practice Exercise**

1. Find the LCD of the following fractions:

$$\frac{x}{2x^2 + 2x}, \quad \frac{1}{x^2 + 2x + 1}, \quad \frac{2x + 4}{4x^2}$$

$(x - 1)^2$  as a factor in the LCD. We do not need a higher power of  $(x - 1)$  since, as far as this factor is concerned, each denominator will divide into it evenly. Next, the second denominator has a factor of  $(x + 1)$ . Therefore, the LCD must also have a factor of  $(x + 1)$ ; otherwise, the second denominator would not divide into it exactly. Finally, the third denominator shows that a factor of  $x$  is also needed. The LCD is therefore  $x(x + 1)(x - 1)^2$ . All three denominators will divide exactly into this expression, and there is no simpler expression for which this is true. ■

**ADDITION AND SUBTRACTION OF FRACTIONS**

Once we have found the LCD for the fractions, we multiply the numerator and the denominator of each fraction by the proper quantity to make the resulting denominator in each case the lowest common denominator. After this step, it is necessary only to add the numerators, place this result over the common denominator, and simplify.

**EXAMPLE 4 Adding fractions using LCD**

Combine:  $\frac{2}{3r^2} + \frac{4}{rs^3} - \frac{5}{3s}$ .

By looking at the denominators, notice that the factors necessary in the LCD are 3,  $r$ , and  $s$ . The 3 appears only to the first power, the largest exponent of  $r$  is 2, and the largest exponent of  $s$  is 3. Therefore, the LCD is  $3r^2s^3$ . Now, write each fraction with this quantity as the denominator.

Since the denominator of the first fraction already contains factors of 3 and  $r^2$ , **it is necessary to introduce the factor of  $s^3$** . In other words, we must multiply the numerator and the denominator of this fraction by  $s^3$ . For similar reasons, we must multiply the numerators and the denominators of the second and third fractions by  $3r$  and  $r^2s^2$ , respectively. This leads to

$$\begin{aligned} \frac{2}{3r^2} + \frac{4}{rs^3} - \frac{5}{3s} &= \frac{2(s^3)}{(3r^2)(s^3)} + \frac{4(3r)}{(rs^3)(3r)} - \frac{5(r^2s^2)}{(3s)(r^2s^2)} && \text{change to equivalent} \\ & && \text{fractions with LCD} \\ & && \text{factors needed in each} \\ &= \frac{2s^3}{3r^2s^3} + \frac{12r}{3r^2s^3} - \frac{5r^2s^2}{3r^2s^3} \\ &= \frac{2s^3 + 12r - 5r^2s^2}{3r^2s^3} && \text{combine numerators over LCD} \end{aligned}$$

Note that the minus sign for the third term is used in the numerator. ■

**EXAMPLE 5 Adding fractions using LCD, binomial factors**

$$\begin{aligned} \frac{a}{x-1} + \frac{a}{x+1} &= \frac{a(x+1)}{(x-1)(x+1)} + \frac{a(x-1)}{(x+1)(x-1)} && \text{change to equivalent} \\ & && \text{fractions with LCD} \\ & && \text{factors needed} \\ &= \frac{ax + a + ax - a}{(x+1)(x-1)} && \text{combine numerators} \\ & && \text{over LCD} \\ &= \frac{2ax}{(x+1)(x-1)} && \text{simplify} \end{aligned}$$

When we multiply each fraction by the quantity required to obtain the proper denominator, we do not actually have to write the common denominator under each numerator. Placing all the products that appear in the numerators over the common

denominator is sufficient. Hence, the illustration in this example would appear as

$$\begin{aligned}\frac{a}{x-1} + \frac{a}{x+1} &= \frac{a(x+1) + a(x-1)}{(x-1)(x+1)} = \frac{ax + a + ax - a}{(x-1)(x+1)} \\ &= \frac{2ax}{(x-1)(x+1)}\end{aligned}$$

### EXAMPLE 6 Adding fractions—application

The following expression is found in the analysis of the dynamics of missile firing. The indicated addition is performed as shown.

$$\begin{aligned}\frac{1}{s} - \frac{1}{s+4} + \frac{8}{s^2 + 8s + 16} &= \frac{1}{s} - \frac{1}{s+4} + \frac{8}{(s+4)^2} && \text{factor third denominator} \\ &= \frac{1(s+4)^2 - 1(s)(s+4) + 8s}{s(s+4)^2} && \text{LCD has one factor of } s \text{ and two factors of } (s+4) \\ &= \frac{s^2 + 8s + 16 - s^2 - 4s + 8s}{s(s+4)^2} && \text{expand terms of the numerator} \\ &= \frac{12s + 16}{s(s+4)^2} = \frac{4(3s+4)}{s(s+4)^2} && \text{simplify/factor}\end{aligned}$$

#### Practice Exercise

2. Combine:

$$\frac{3}{2x+2} - \frac{2}{x^2+2x+1} + \frac{1}{2x^3+2x^2}$$

We factored the numerator in the final result to see whether or not there were any factors common to the numerator and the denominator. Since there are none, either form of the result is acceptable. ■

### EXAMPLE 7 Combining fractions, factored denominators with binomial factors

$$\begin{aligned}\frac{3x}{x^2 - x - 12} - \frac{x-1}{x^2 - 8x + 16} - \frac{6-x}{2x-8} &= \frac{3x}{(x-4)(x+3)} - \frac{x-1}{(x-4)^2} - \frac{6-x}{2(x-4)} && \text{factor denominators} \\ &= \frac{3x(2)(x-4) - (x-1)(2)(x+3) - (6-x)(x-4)(x+3)}{2(x-4)^2(x+3)} && \text{change to equivalent fraction with LCD} \\ &= \frac{[6x^2 - 24x] - [2x^2 + 4x - 6] - [-x^3 + 7x^2 + 6x - 72]}{2(x-4)^2(x+3)} && \text{expand in numerator} \\ &= \frac{6x^2 - 24x - 2x^2 - 4x + 6 + x^3 - 7x^2 - 6x + 72}{2(x-4)^2(x+3)} = \frac{x^3 - 3x^2 - 34x + 78}{2(x-4)^2(x+3)} && \text{simplify}\end{aligned}$$

## COMPLEX FRACTIONS

A **complex fraction** is one in which the numerator, the denominator, or both the numerator and the denominator contain fractions. The following examples illustrate the simplification of complex fractions. Like in Section 6.6, when dividing algebraic fractions, you can multiply by the reciprocal of the divisor. ■

**EXAMPLE 8** Complex fraction

■ The original complex fraction can be written as a division as follows:

$$\frac{2}{x} \div \left(1 - \frac{4}{x}\right)$$

$$\frac{\frac{2}{x}}{1 - \frac{4}{x}} = \frac{\frac{2}{x}}{\frac{x-4}{x}}$$

first, perform subtraction in denominator

$$= \frac{2}{x} \times \frac{x}{x-4} = \frac{2x}{x(x-4)}$$

invert divisor and multiply

$$= \frac{2}{x-4}, \text{ where } x \neq 0$$

simplify

**EXAMPLE 9** Complex fraction

$$\frac{1 - \frac{2}{x}}{\frac{1}{x} + \frac{2}{x^2 + 4x}} = \frac{1 - \frac{2}{x}}{\frac{1}{x} + \frac{2}{x(x+4)}} = \frac{\frac{x-2}{x}}{\frac{x+4+2}{x(x+4)}}$$

perform subtraction and addition in numerator and denominator separately

$$= \frac{x-2}{x} \times \frac{x(x+4)}{x+6}$$

invert divisor and multiply

$$= \frac{(x-2)(\cancel{x})(x+4)}{\cancel{x}(x+6)}$$

show common factors

$$= \frac{(x-2)(x+4)}{x+6}, \text{ where } x \neq 0$$

simplify

**EXERCISES 6.7**

In Exercises 1–4, make the given changes in the indicated examples of this section and then solve the resulting problems.

- In Example 2, change the denominator of the third fraction to  $4a^2b^2$  and then find the LCD.
- In Example 5, add the fraction  $\frac{2}{x^2 - 1}$  to those being added and then find the result.
- In Example 7, change the denominator of the third fraction to  $2x + 6$  and then find the result.
- In Example 8, change the fraction in the numerator to  $2/x^2$  and then simplify.

In Exercises 5–44, perform the indicated operations and simplify.

- $\frac{3}{5} + \frac{6}{5}$
- $\frac{2}{13} + \frac{6}{13}$
- $\frac{1}{x} + \frac{7}{x}$
- $\frac{2}{a} + \frac{3}{a}$
- $\frac{1}{2} + \frac{3}{4}$
- $\frac{5}{9} - \frac{1}{3}$
- $\frac{3}{4x} + \frac{7a}{4} + 2$
- $\frac{t-3}{a} - \frac{t}{2a}$
- $\frac{a}{x} - \frac{b}{x^2}$
- $\frac{3}{2s^2} + \frac{5}{4s}$
- $\frac{6}{5x^3} + \frac{a}{25x}$
- $\frac{a}{6y} - \frac{2b}{3y^4}$

$$17. \frac{2}{5a} + \frac{1}{a} - \frac{a}{10}$$

$$19. \frac{x+1}{2x} - \frac{y-3}{4y} - \frac{2-x}{xy}$$

$$21. \frac{3}{2x-1} + \frac{1}{4x-2}$$

$$23. \frac{4}{x(x+1)} - \frac{3}{2x}$$

$$25. \frac{s}{2s-6} + \frac{1}{4} - \frac{3s}{4s-12}$$

$$27. \frac{3R}{R^2-9} - \frac{2}{3R+9}$$

$$29. \frac{3}{x^2-8x+16} - \frac{2}{4-x}$$

$$31. \frac{v+4}{v^2+5v+4} - \frac{v-2}{v^2-5v+6}$$

$$33. \frac{x-1}{3x^2-13x+4} - \frac{3x+1}{4-x}$$

$$34. \frac{x}{4x^2-12x+5} + \frac{2x-1}{4x^2-4x-15}$$

$$35. \frac{t}{t^2-t-6} - \frac{2t}{t^2+6t+9} + \frac{t}{9-t^2}$$

$$18. \frac{1}{2A} - \frac{6}{B} - \frac{9}{4C}$$

$$20. 5 + \frac{1-x}{2y} - \frac{3+x}{4y}$$

$$22. \frac{5}{6y+3} - \frac{a}{8y+4}$$

$$24. \frac{3}{ax+ay} - \frac{1}{a^2}$$

$$26. \frac{2}{x+2} - \frac{3-x}{x^2+2x} + \frac{1}{x}$$

$$28. \frac{2}{n^2+4n+4} - \frac{3}{4+2n}$$

$$30. \frac{2a-b}{c-3d} - \frac{b-2a}{3d-c}$$

$$32. \frac{N-1}{2N^3-4N^2} - \frac{5}{2-N}$$

$$36. \frac{5}{2x^3 - 3x^2 + x} - \frac{x}{x^4 - x^2} + \frac{2-x}{2x^2 + x - 1}$$

$$37. \frac{1}{w^3 + 1} + \frac{1}{w + 1} - 2$$

$$39. \frac{\frac{1}{x}}{1 - \frac{1}{x}}$$

$$41. \frac{\frac{x}{y} - \frac{y}{x}}{1 + \frac{y}{x}}$$

$$43. \frac{\frac{3}{x} + \frac{1}{x^2 + x}}{\frac{1}{x+1} - \frac{1}{x-1}}$$

$$38. \frac{2}{8 - x^3} + \frac{1}{x^2 - x - 2}$$

$$40. \frac{x - \frac{1}{x}}{1 - \frac{1}{x}}$$

$$42. \frac{\frac{V^2 - 9}{V}}{\frac{1}{V} - \frac{1}{3}}$$

$$44. \frac{\frac{1}{u-v} + \frac{1}{2u+2v}}{\frac{2u}{2u^2 - 3uv + v^2} + \frac{2}{2u-v}}$$

The expression  $f(x+h) - f(x)$  is frequently used in the study of calculus. (If necessary, refer to Section 3.1 for a review of functional notation.) In Exercises 45–48, determine and then simplify this expression for the given functions.

$$45. f(x) = \frac{x}{x+1} \qquad 46. f(x) = \frac{3}{2x-1}$$

$$47. f(x) = \frac{1}{x^2} \qquad 48. f(x) = \frac{2}{x^2+4}$$

In Exercises 49–57, simplify the given expressions. In Exercise 58, answer the given question.

49. Using the definitions of the trigonometric functions given in Section 4.2, find an expression that is equivalent to  $(\tan \theta)(\cot \theta) + (\sin \theta)^2 - \cos \theta$ , in terms of  $x$ ,  $y$ , and  $r$ .
50. Using the definitions of the trigonometric functions given in Section 4.2, find an expression that is equivalent to  $\sec \theta - (\cot \theta)^2 + \csc \theta$ , in terms of  $x$ ,  $y$ , and  $r$ .
51. If  $f(x) = 2x - x^2$ , find  $f\left(\frac{1}{a}\right)$ .
52. If  $f(x) = x^2 + x$ , find  $f\left(a + \frac{1}{a}\right)$ .
53. If  $f(x) = x - \frac{2}{x}$ , find  $f(a+1)$ .
54. If  $f(x) = 2x - 3$ , find  $f[1/f(x)]$ .
55. The sum of two numbers  $a$  and  $b$  is divided by the sum of their reciprocals. Simplify the expression for this quotient.
56. Find  $A$  and  $B$  if  $\frac{2x-9}{x^2-x-6} = \frac{A}{x-3} + \frac{B}{x+2}$ .

$$57. \text{ If } x = \frac{mn}{m+n} \text{ and } y = \frac{mn}{m-n}, \text{ show that } \frac{y^2 - x^2}{y^2 + x^2} = \frac{2mn}{m^2 + n^2}.$$

58. When adding fractions, explain why it is better to find the lowest common denominator rather than any denominator that is common to the fractions.

In Exercises 59–68, perform the indicated operations. Each expression occurs in the indicated area of application.

$$59. \frac{3}{4\pi} - \frac{3H_0}{4\pi H} \quad (\text{transistor theory})$$

$$60. 1 + \frac{9}{128T} - \frac{27p}{64T^3} \quad (\text{thermodynamics})$$

$$61. \frac{2n^2 - n - 4}{2n^2 + 2n - 4} + \frac{1}{n-1} \quad (\text{optics})$$

$$62. \frac{b}{x^2 + y^2} - \frac{2bx^2}{x^4 + 2x^2y^2 + y^4} \quad (\text{magnetic field})$$

$$63. \left(\frac{3Px}{2L^2}\right)^2 + \left(\frac{P}{2L}\right)^2 \quad (\text{force of a weld})$$

$$64. \frac{a}{b^2h} + \frac{c}{bh^2} - \frac{1}{6bh} \quad (\text{strength of materials})$$

$$65. \frac{\frac{L}{C} + \frac{R}{sC}}{sL + R + \frac{1}{sC}} \quad (\text{electricity})$$

$$66. \frac{\frac{m}{c}}{1 - \frac{p^2}{c^2}} \quad (\text{airfoil design})$$

$$67. \frac{1}{R^2} + \left(\omega c - \frac{1}{\omega L}\right)^2 \quad (\text{electricity})$$

$$68. \frac{\frac{x}{h_1} + \frac{x-L}{h_2}}{1 + \frac{x(L-x)}{h_1h_2}} \quad (\text{optics})$$

#### Answers to Practice Exercises

$$1. 4x^2(x+1)^2 \quad 2. \frac{3x^3 - x^2 + x + 1}{2x^2(x+1)^2}$$

## 6.8 Equations Involving Fractions

**Multiply Each Term by the LCD • Solving Equations • Applications • Extraneous Solutions**

### LEARNING TIP

To solve equations dealing with fractions, **multiply each term of the equation by the LCD**. The resulting equation can be solved by using methods discussed previously and will not involve fractions.

■ Note carefully that we are multiplying both sides of the equation by the LCD and not combining terms over a common denominator as when adding fractions.

Many important equations in science, engineering, and technology have fractions in them. Although the solution of these equations will still involve the use of the basic operations stated in Section 1.10, an additional procedure can be used to eliminate the fractions and thereby help lead to the solution. The following examples illustrate how to solve equations involving fractions.

### EXAMPLE 1 Multiply each term by the LCD

$$\text{Solve for } x: \frac{x}{12} - \frac{1}{8} = \frac{x+2}{6}.$$

First, note that the LCD of the terms of the equation is 24. Therefore, multiply each term by 24. This gives

$$\frac{24(x)}{12} - \frac{24(1)}{8} = \frac{24(x+2)}{6} \quad \text{each term multiplied by LCD}$$

Reduce each term to its lowest terms and solve the resulting equation:

$$\begin{aligned} 2x - 3 &= 4(x + 2) && \text{each term reduced} \\ 2x - 3 &= 4x + 8 \\ -2x &= 11 \\ x &= -\frac{11}{2} \end{aligned}$$

When we check this solution in the original equation, we obtain  $-7/12$  on each side of the equal sign. Therefore, the solution is correct. ■

### EXAMPLE 2 Solving a literal equation

$$\text{Solve for } x: \frac{x}{2} - \frac{1}{b^2} = \frac{x}{2b}.$$

First, determine that the LCD of the terms of the equation is  $2b^2$ . Then multiply each term by  $2b^2$  and continue with the solution:

$$\begin{aligned} \frac{2b^2(x)}{2} - \frac{2b^2(1)}{b^2} &= \frac{2b^2(x)}{2b} && \text{each term multiplied by LCD} \\ b^2x - 2 &= bx && \text{each term reduced} \\ b^2x - bx &= 2 \\ x(b^2 - b) &= 2 && \text{factor} \\ x &= \frac{2}{b^2 - b} = \frac{2}{b(b - 1)} \end{aligned}$$

Note the use of factoring in arriving at the final result. Checking shows that each side of the original equation is equal to  $\frac{1}{b^2(b-1)}$ . ■

### Expanded Procedure for Solving Equations

The procedure for solving equations covered in Section 1.10 can now be expanded as:

1. If the equation has any fractions, remove the fractions by multiplying by the LCD.
2. Remove grouping symbols (distributive law).
3. Combine like terms on each side.
4. Perform the same operations on both sides to simplify and factor, if necessary, with the goal of isolating the variable  $x$ .

**EXAMPLE 3** Solving an equation—optics application

An equation relating the focal length  $f$  of a lens with the object distance  $p$  and the image distance  $q$  is given below. See Fig. 6.5. Solve for  $q$ .

$$f = \frac{pq}{p + q} \quad \text{given equation}$$

Since the only denominator is  $p + q$ , the LCD is also  $p + q$ . By first multiplying each term by  $p + q$ , the solution is completed as follows:

$$f(p + q) = \frac{pq(p + q)}{p + q} \quad \text{each term multiplied by LCD}$$

$$fp + fq = pq \quad \text{reduce term on right}$$

$$fq - pq = -fp$$

$$q(f - p) = -fp \quad \text{factor}$$

$$q = \frac{-fp}{f - p} \quad \text{divide by } f - p$$

$$q = -\frac{fp}{-(p - f)} \quad \text{use Eq. (6.11)}$$

$$q = \frac{fp}{p - f}$$

The last form is preferred since there is no minus sign before the fraction. However, either form of the result is correct. ■

**EXAMPLE 4** Solving an equation—astrophysics application

In astronomy, when developing the equations that describe the motion of the planets, the equation

$$\frac{1}{2}v^2 - \frac{GM}{r} = -\frac{GM}{2a}$$

is found. Solve for  $M$ .

First, determine that the LCD of the terms of the equation is  $2ar$ . Multiplying each term by  $2ar$  and proceeding, we have

$$\frac{2ar(v^2)}{2} - \frac{2ar(GM)}{r} = -\frac{2ar(GM)}{2a} \quad \text{each term multiplied by LCD}$$

$$arv^2 - 2aGM = -rGM \quad \text{each term reduced}$$

$$rGM - 2aGM = -arv^2$$

$$M(rG - 2aG) = -arv^2 \quad \text{factor}$$

$$M = -\frac{arv^2}{rG - 2aG}$$

$$M = \frac{arv^2}{2aG - rG}$$

$$M = \frac{arv^2}{G(2a - r)}$$

The second form of the result is obtained by using Eq. (6.11). Again, note the use of factoring to arrive at the final result. ■

■ See the chapter introduction.

■ The first telescope was invented by Lippershay, a Dutch lens maker, in about 1608.

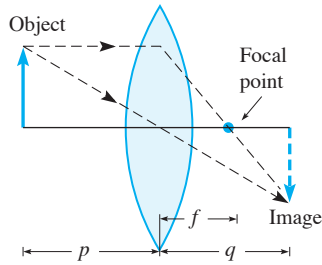


Fig. 6.5

■ For reference, Eq. (6.11) is  $x - y = -(y - x)$ .

■ See the chapter introduction.

■ In 1609, the Italian scientist Galileo (1564–1642) learned of the invention of the telescope and developed it for astronomical observations. Among his first discoveries were the four largest moons of the planet Jupiter.

**Practice Exercise**

1. Solve for  $x$ :  $\frac{x - 1}{3a} = \frac{5}{6} + \frac{x}{a^2}$



**EXAMPLE 5** No solution

Solve for  $x$ :  $\frac{2}{x+1} - \frac{1}{x} = -\frac{2}{x^2+x}$ .

Note first that  $x = 0$  and  $x = -1$  make a denominator equal to zero, so these are not possible values of the solution. The solution proceeds as follows:

$$\begin{aligned} \frac{2(x)(x+1)}{x+1} - \frac{x(x+1)}{x} &= -\frac{2x(x+1)}{x(x+1)} && \text{multiplying each side by the LCD} \\ &&& \text{of } x(x+1), x \neq 0, -1 \\ 2x - (x+1) &= -2 && \text{simplifying each fraction} \\ 2x - x - 1 &= -2 && \text{completing the solution} \\ x &= -1 \end{aligned}$$

As we said earlier,  $x = -1$  cannot be a solution. Therefore, *there is no solution to this equation.* ■

**COMMON ERROR**

It is a common error to give a value that makes the denominator zero as a solution to an equation that involves fractions. Such an error can be avoided by stating explicitly from the beginning the values that are not valid solutions (because they make a denominator equal to zero), as we did in Example 5. Checking your solution in the original equation also helps identify values that cannot be solutions.

**LEARNING TIP**

Whenever we multiply each term by a common denominator that **contains the unknown**, it is possible to obtain a value that is not a solution of the original equation. Such a value is termed an **extraneous solution**. Only certain equations will lead to extraneous solutions, but we must be careful to identify them when they occur.

■ The first large-scale electronic computer was the ENIAC. It was constructed at the Univ. of Pennsylvania in the mid-1940s and used until 1955. It had 18 000 vacuum tubes and occupied 1400 m<sup>2</sup> of floor space.

■ A military programmable computer called *Colossus* was used to break the German codes in World War II. It was developed, in secret, before ENIAC.

A number of stated problems give rise to equations involving fractions. The following examples illustrate the solution of such problems.

**EXAMPLE 6** Solving a word problem with fractions

An industrial firm uses a computer system that processes and prints out its data for an average day in 20 h. To process the data more rapidly and to handle increased future computer needs, the firm plans to add new components to the system. One set of new components can process the data in 12 h, without the present system. How long would it take the new system, a combination of the present system and the new components, to process the data?

First, let  $x =$  the number of hours for the new system to process the data. Next, we know that it takes the present system 20 h to do it. This means that it processes  $\frac{1}{20}$  of the data in 1 h, or  $\frac{1}{20}x$  of the data in  $x$  h. In the same way, the new components can process  $\frac{1}{12}x$  of the data in  $x$  h. When  $x$  h have passed, the new system will have processed all of the data. Therefore,

$$\begin{aligned} \frac{x}{20} + \frac{x}{12} &= 1 && \begin{array}{l} \text{part of data processed} \\ \text{by present system} \end{array} \quad \begin{array}{l} \text{part of data processed} \\ \text{by new components} \end{array} \quad \begin{array}{l} \text{one complete processing} \\ \text{(all of data)} \end{array} \\ \frac{60x}{20} + \frac{60x}{12} &= 60(1) && \text{each term multiplied by LCD of 60} \\ 3x + 5x &= 60 && \text{each term reduced} \\ 8x &= 60 \\ x &= \frac{60}{8} = 7.5 \text{ h} \end{aligned}$$

Therefore, the new system should take about 7.5 h to process the data. ■

**EXAMPLE 7** Solving a word problem with fractions

A bus averaging 80.0 km/h takes 1.50 h longer to travel from city A to city B than a train that averages 96.0 km/h. How far apart are the cities?

Let  $d$  = the distance from city A to city B. Since distance = rate  $\times$  time, then distance  $\div$  rate = time. Therefore, the time the train takes is  $d/96.0$  h, and the time the bus takes is  $d/80.0$  h. This means

$$\begin{aligned} t_{\text{bus}} - t_{\text{train}} &= 1.50 \text{ h} & 80 &= 16(5) = (2^4)(5) \\ \frac{d}{80.0} - \frac{d}{96.0} &= 1.50 & 96 &= 32(3) = (2^5)(3) \\ \frac{480d}{80.0} - \frac{480d}{96.0} &= 480(1.50) & \text{LCD} &= (2^5)(3)(5) = 480 \\ 6.00d - 5.00d &= 720 \\ d &= 720 \text{ km} \end{aligned}$$

**EXERCISES 6.8**

In Exercises 1–4, make the given changes in the indicated examples of this section and then solve for the indicated variable.

- In Example 2, change the second denominator from  $b^2$  to  $b$  and then solve for  $x$ .
- In Example 3, solve for  $p$ .
- In Example 4, solve for  $G$ .
- In Example 5, change the numerator on the right to 1 and then solve for  $x$ .

In Exercises 5–32, solve the given equations and check the results.

- $\frac{x}{2} + 6 = 2x$
- $\frac{x}{5} + 2 = \frac{15 + x}{10}$
- $\frac{x}{6} - \frac{1}{2} = \frac{x}{3}$
- $\frac{3N}{8} - \frac{3}{4} = \frac{N - 4}{2}$
- $1 - \frac{t - 5}{6} = \frac{3}{4}$
- $\frac{2x - 7}{3} + 5 = \frac{1}{5}$
- $\frac{3x}{7} - \frac{5}{21} = \frac{2 - x}{14}$
- $\frac{F - 3}{12} - \frac{2}{3} = \frac{1 - 3F}{2}$
- $\frac{3}{T} + 2 = \frac{5}{3}$
- $\frac{1}{2y} - \frac{1}{2} = 4$
- $3 - \frac{x - 2}{5x} = \frac{1}{5}$
- $\frac{1}{2R} - \frac{1}{3} = \frac{2}{3R}$
- $\frac{2y}{y - 1} = 5$
- $\frac{x}{2x - 3} = 4$
- $\frac{2}{s} = \frac{3}{s - 1}$
- $\frac{5}{n + 2} = \frac{3}{2n}$
- $\frac{5}{2x + 4} + \frac{3}{6x + 12} = 2$
- $\frac{3}{4x - 6} + \frac{1}{4} = \frac{5}{3 - 2x}$
- $\frac{2}{Z - 5} - \frac{3}{10 - 2Z} = 3$
- $\frac{4}{4 - x} + 2 - \frac{2}{12 - 3x} = \frac{1}{3}$
- $\frac{1}{4x} + \frac{3}{2x} = \frac{2}{x + 1}$
- $\frac{3}{t + 3} - \frac{1}{t} = \frac{5}{6 + 2t}$
- $\frac{7}{y} = \frac{3}{y - 4} + \frac{7}{2y^2 - 8y}$
- $\frac{1}{2x + 3} = \frac{5}{2x} - \frac{4}{2x^2 + 3x}$

- $\frac{1}{x^2 - x} - \frac{1}{x} = \frac{1}{x - 1}$
- $\frac{2}{x^2 - 1} - \frac{2}{x + 1} = \frac{1}{x - 1}$
- $\frac{2}{B^2 - 4} - \frac{1}{B - 2} = \frac{1}{2B + 4}$
- $\frac{2}{2x^2 + 5x - 3} - \frac{1}{4x - 2} + \frac{3}{2x + 6} = 0$

In Exercises 33–48, solve for the indicated letter. In Exercises 37–48, each of the given formulas arises in the technical or scientific area of study listed.

- $2 - \frac{1}{b} + \frac{3}{c} = 0$ , for  $c$
- $\frac{2}{3} - \frac{h}{x} = \frac{1}{6x}$ , for  $x$
- $\frac{t - 3}{b} - \frac{t}{2b - 1} = \frac{1}{2}$ , for  $t$
- $\frac{1}{a^2 + 2a} - \frac{y}{2a} = \frac{2y}{a + 2}$ , for  $y$
- $\frac{s - s_0}{t} = \frac{v + v_0}{2}$ , for  $v$  (velocity of object)
- $S = \frac{P}{A} + \frac{Mc}{I}$ , for  $A$  (machine design)
- $V = 1.2 \left( 5.0 + \frac{8.0R}{8.0 + R} \right)$ , for  $R$  (electric resistance in Fig. 6.6)

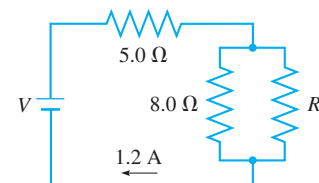


Fig. 6.6

- $C = \frac{7p}{100 - p}$ , for  $p$  (environmental pollution)

$$41. z = \frac{1}{g_m} - \frac{jX}{g_m R}, \text{ for } R \text{ (FM transmission)}$$

$$42. A = \frac{1}{2} wp - \frac{1}{2} w^2 - \frac{\pi}{8} w^2, \text{ for } p \text{ (architecture)}$$

$$43. p = \frac{RT}{V - b} - \frac{a}{V^2}, \text{ for } T \text{ (thermodynamics)}$$

$$44. \frac{1}{x} + \frac{1}{nx} = \frac{1}{f}, \text{ for } f \text{ (photography)}$$

$$45. \frac{1}{C} = \frac{1}{C_2} + \frac{1}{C_1 + C_3}, \text{ for } C_1 \text{ (electricity: capacitors)}$$

$$46. D = \frac{wx^4}{24EI} - \frac{wLx^3}{6EI} + \frac{wL^2x^2}{4EI}, \text{ for } w \text{ (beam design)}$$

$$47. f(n - 1) = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}}, \text{ for } R_1 \text{ (optics)}$$

$$48. P = \frac{\frac{1}{1+i}}{1 - \frac{1}{1+i}}, \text{ for } i \text{ (business)}$$

In Exercises 49–58, set up appropriate equations and solve the given stated problems. All numbers are accurate to at least 3 significant digits.

49. One pump can empty an oil tanker in 5.00 h, and a second pump can empty the tanker in 8.00 h. How long would it take the two pumps working together to empty the tanker?
50. One company determines that it will take its crew 450 h to clean up a chemical dump site, and a second company determines that it will take its crew 600 h to clean up the site. How long will it take the two crews working together?
51. One automatic packaging machine can package 100 boxes of machine parts in 12.0 min, and a second machine can do it in 10.0 min. A newer model machine can do it in 8.00 min. How long will it take the three machines working together?
52. A painting crew can paint a structure in 12.0 h, or the crew can paint it in 7.20 h when working with a second crew. How long would it take the second crew to do the job if working alone?
53. An elevator travelled from the first floor to the top floor of a building at an average speed of 2.00 m/s and returned to the first floor at 2.20 m/s. If it was on the top floor for 90.0 s and the total elapsed time was 5.00 min, how far above the first floor is the top floor?

54. A commuter rapid transit train travels 24.0 km farther between stops A and B than between stops B and C. If it averages 60.0 km/h from A to B and 30.0 km/h between B and C, and an express averages 50.0 km/h between A and C (not stopping at B), how far apart are stops A and C?
55. A jet takes the same time to travel 2580 km with the wind as it does to travel 1800 km against the wind. If its speed relative to the air is 450 km/h, what is the speed of the wind?
56. An engineer travels from St. John's, Newfoundland, to the Hibernia oil field in the North Atlantic Ocean on a ship that averages 28.0 km/h. After spending 6.00 h at the field, the engineer returns to St. John's in a helicopter that averages 140 km/h. If the total trip takes 19.5 h, how far is the Hibernia oil field from St. John's?
57. The current through each of the resistances  $R_1$  and  $R_2$  in Fig. 6.7 equals the voltage  $V$  divided by the resistance. The sum of the currents equals the current  $i$  in the rest of the circuit. Find the voltage if  $i = 1.20$  A,  $R_1 = 2.70 \Omega$ , and  $R_2 = 6.00 \Omega$ .

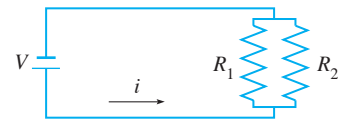


Fig. 6.7

58. A fox, pursued by a greyhound, has a start of 60 leaps. He makes 9 leaps while the greyhound makes but 6; but, 3 leaps of the greyhound are equivalent to 7 of the fox. How many leaps must the greyhound make to overcome the fox? (Copied from Davies, Charles, *Elementary Algebra*, New York: A. S. Barnes & Burr, 1852.) (*Hint*: Let the unit of distance be one fox leap.)

In Exercises 59 and 60, find constants  $A$  and  $B$  such that the equation is true.

$$59. \frac{x - 12}{x^2 + x - 6} = \frac{A}{x + 3} + \frac{B}{x - 2}$$

$$60. \frac{23 - x}{2x^2 + 7x - 4} = \frac{A}{2x - 1} - \frac{B}{x + 4}$$

#### Answer to Practice Exercise

$$1. x = \frac{5a^2 + 2a}{2a - 6}$$

## CHAPTER 6 EQUATIONS

$$a(x + y) = ax + ay \quad (6.1)$$

$$(x + y)(x - y) = x^2 - y^2 \quad (6.2)$$

$$(x + y)^2 = x^2 + 2xy + y^2 \quad (6.3)$$

$$(x - y)^2 = x^2 - 2xy + y^2 \quad (6.4)$$

$$(x + a)(x + b) = x^2 + (a + b)x + ab \quad (6.5)$$

$$(ax + b)(cx + d) = acx^2 + (ad + bc)x + bd \quad (6.6)$$

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3 \quad (6.7)$$

$$(x - y)^3 = x^3 - 3x^2y + 3xy^2 - y^3 \quad (6.8)$$

$$(x + y)(x^2 - xy + y^2) = x^3 + y^3 \quad (6.9)$$

$$(x - y)(x^2 + xy + y^2) = x^3 - y^3 \quad (6.10)$$

$$x - y = -(y - x) \quad (6.11)$$

## CHAPTER 6 REVIEW EXERCISES

In Exercises 1–12, find the products by inspection. No intermediate steps should be necessary.

- |                            |                              |
|----------------------------|------------------------------|
| 1. $3a(4x + 5a)$           | 2. $-7xy(4x^2 - 7y)$         |
| 3. $(2a + 7b)(2a - 7b)$    | 4. $(x - 4z)(x + 4z)$        |
| 5. $(2a + 1)^2$            | 6. $(5C - 2D)^2$             |
| 7. $(b - 4)(b + 7)$        | 8. $(y - 5)(y - 7)$          |
| 9. $(2x + 5)(x - 9)$       | 10. $(4ax - 3)(5ax + 7)$     |
| 11. $(2c^a + d)(2c^a - d)$ | 12. $(3s^n - 2t)(3s^n + 2t)$ |

In Exercises 13–44, factor the given expressions completely.

- |                           |                                |
|---------------------------|--------------------------------|
| 13. $3s + 9t$             | 14. $7x - 28y$                 |
| 15. $a^2x^2 + a^2$        | 16. $3ax - 6ax^4 - 9a$         |
| 17. $W^2b^{x+2} - 144b^x$ | 18. $900t^a - t^{a+4}$         |
| 19. $16(x + 2)^2 - t^4$   | 20. $25s^4 - 36t^2$            |
| 21. $36t^2 - 24t + 4$     | 22. $4x^2 - 12x + 9$           |
| 23. $25t^2 + 10t + 1$     | 24. $4c^2 + 36cd + 81d^2$      |
| 25. $x^2 + x - 56$        | 26. $x^2 - 4x - 45$            |
| 27. $t^4 - 5t^2 - 36$     | 28. $3N^4 - 33N^2 + 30$        |
| 29. $2k^2 - k - 36$       | 30. $5x^2 + 2x - 3$            |
| 31. $8x^2 - 8x - 70$      | 32. $27F^3 + 21F^2 - 48F$      |
| 33. $10b^2 + 23b - 5$     | 34. $12x^2 - 7xy - 12y^2$      |
| 35. $4x^2 - 64y^2$        | 36. $4a^2x^2 + 26a^2x + 36a^2$ |
| 37. $250 - 16y^6$         | 38. $8a^4 + 64a$               |
| 39. $8x^3 + 27$           | 40. $R^3 - 125r^3$             |
| 41. $ab^2 - 3b^2 + a - 3$ | 42. $axy - ay + ax - a$        |
| 43. $nx + 5n - x^2 + 25$  | 44. $ty - 4t + y^2 - 16$       |

In Exercises 45–68, perform the indicated operations and express results in simplest form.

- |   |  |
|---|--|
| 45. $\frac{48ax^3y^6}{9a^3xy^6}$                                      | 46. $\frac{-39r^2s^4t^8}{52rs^5t}$   |
| 47. $\frac{6x^2 - 7x - 3}{4x^2 - 8x + 3}$                             | 48. $\frac{p^4 - 4p^2 - 4}{p^4 - p^2 - 12}$                                  |
| 49. $\frac{4x + 4y}{35x^2} \times \frac{28x}{x^2 - y^2}$              | 50. $\left(\frac{6x - 3}{x^2}\right)\left(\frac{4x^2 - 12x}{6 - 12x}\right)$ |
| 51. $\frac{18 - 6L}{L^2 - 6L + 9} \div \frac{L^2 - 2L - 15}{L^2 - 9}$ |  |

$$52. \frac{6x^2 - xy - y^2}{2x^2 + xy - y^2} \div \frac{16y^2 - 4x^2}{2x^2 + 6xy + 4y^2}$$

$$53. \frac{\frac{3x}{7x^2 + 13x - 2}}{6x^2} \div \frac{x^2 + 4x + 4}{x^2 + 4x + 4}$$

$$55. \frac{x + \frac{1}{x} + 1}{x^2 - \frac{1}{x}}$$

$$57. \frac{4}{9x} - \frac{5}{12x^2}$$

$$59. \frac{6}{x} - \frac{7}{2x} + \frac{3}{xy}$$

$$61. \frac{a + 1}{a + 2} - \frac{a + 3}{a}$$

$$63. \frac{2x}{x^2 + 2x - 3} - \frac{1}{6x + 2x^2}$$

$$65. \frac{3x}{2x^2 - 2} - \frac{2}{4x^2 - 5x + 1}$$

$$67. \frac{3x}{x^2 + 2x - 3} - \frac{2}{x^2 + 3x} + \frac{x}{x - 1}$$

$$68. \frac{3}{y^4 - 2y^3 - 8y^2} + \frac{y - 1}{y^2 + 2y} - \frac{y - 3}{y^2 - 4y}$$

$$54. \frac{\frac{3x - 3y}{2x^2 + 3xy - 2y^2}}{3x^2 - 3y^2} \div \frac{x^2 + 4xy + 4y^2}{x^2 + 4xy + 4y^2}$$

$$56. \frac{\frac{4}{y} - 4y}{2 - \frac{2}{y}}$$

$$58. \frac{3}{10a^2} + \frac{1}{4a^3}$$

$$60. \frac{T}{T^2 + 2} - \frac{1}{2T + T^3}$$

$$62. \frac{y}{y + 2} - \frac{1}{y^2 + 2y}$$

$$64. \frac{x}{4x^2 + 4x - 3} - \frac{3}{4x^2 - 9}$$

$$66. \frac{2n - 1}{4 - n} + \frac{n + 2}{5n - 20}$$

In Exercises 69–76, factor the given expressions. In Exercises 69–72, disregard the restriction that coefficients and exponents must be integers. The expressions in Exercises 73–76 can be factored if they are first rewritten in a different form.

- |   |               |
|---|---------------|
| 69. $x^2 - 5$                                   | 70. $x - y$   |
| 71. $x - 1$                                     | 72. $7 - x^2$ |
| 73. $x + y$ (one of two factors is to be $x$ )  |               |
| 74. $x + y$ (one of two factors is to be $a$ )  |               |
| 75. $x - 2y$ (one of two factors is to be 2)    |               |
| 76. $x - y$ (one of two factors is to be $-y$ ) |               |

In Exercises 77–84, solve the given equations.

$$77. \frac{x}{2} - 3 = \frac{x - 10}{4} \qquad 78. \frac{2x}{c} - \frac{1}{2c} = \frac{3}{c} - x, \text{ for } x$$

$$79. \frac{2}{t} - \frac{1}{at} = 2 + \frac{a}{t}, \text{ for } t \qquad 80. \frac{3}{a^2y} - \frac{1}{ay} = \frac{9}{a}, \text{ for } y$$

$$81. \frac{2x}{2x^2 - 5x} - \frac{3}{x} = \frac{1}{4x - 10} \qquad 82. \frac{3}{x^2 + 3x} - \frac{1}{x} = \frac{1}{3 + x}$$

$$83. \text{ Given } f(x) = \frac{1}{x + 2}, \text{ solve for } x \text{ if } f(x + 2) = 2f(x).$$

$$84. \text{ Given } f(x) = \frac{x}{x + 1}, \text{ solve for } x \text{ if } 3f(x) + f\left(\frac{1}{x}\right) = 2.$$

In Exercises 85–102, solve the given problems. Where indicated, the expression is found in the stated technical area.

85. In an algebraic fraction, what effect on the result is made if the sign is changed of (a) an odd number of factors? (b) an even number of factors?

86. Algebraically show that the reciprocal of the reciprocal of the number  $x$  is  $x$ .

$$87. \text{ Show that } xy = \frac{1}{4} [(x + y)^2 - (x - y)^2].$$

$$88. \text{ Show that } x^2 + y^2 = \frac{1}{2} [(x + y)^2 + (x - y)^2].$$

89. Multiply:  $2zS(S + 1)$  (solid-state physics)

90. Expand:  $kr(R - r)$  (blood flow)

91. Expand:  $[2b + (n - 1)\lambda]^2$  (optics)

92. Factor:  $\pi r_1^2 l - \pi r_2^2 l$  (jet plane fuel supply)

93. Factor:  $cT_2 - cT_1 + RT_2 - RT_1$  (pipeline flow)

94. Factor:  $9600t + 8400t^2 - 1200t^3$  (solar energy)

95. Simplify and express in factored form:  $(2R - r)^2 - (r^2 + R^2)$  (aircraft radar)

96. Express in factored form:  $2R(R + r) - (R + r)^2$  (electricity: power)

97. Expand and simplify:  $(n + 1)^3(2n + 1)^3$  (fluid flow in pipes)

98. Expand and simplify:  $2(e_1 - e_2)^2 + 2(e_2 - e_3)^2$  (mechanical design)

99. Expand and simplify:  $10a(T - t) + a(T - t)^2$  (instrumentation)

100. Expand the third term and then factor by grouping:  $pa^2 + (1 - p)b^2 - [pa + (1 - p)b]^2$  (nuclear physics)

101. A metal cube of edge  $x$  is heated and each edge increases by 4 mm. Express the increase in volume in factored form.

102. Express the difference in volumes of two ball bearings of radii  $r$  mm and 3 mm in factored form. ( $r > 3$  mm)

In Exercises 103–114, perform the indicated operations and simplify the given expressions. Each expression is from the indicated technical area of application.

$$103. \left(\frac{2wtv^2}{Dg}\right)\left(\frac{b\pi^2D^2}{n^2}\right)\left(\frac{6}{bt^2}\right) \text{ (machine design)}$$

$$104. \frac{m}{c} \div \left[1 - \left(\frac{p}{c}\right)^2\right] \text{ (airfoil design)}$$

$$105. \frac{\frac{\pi ka}{2}(R^4 - r^4)}{\pi ka(R^2 - r^2)} \text{ (flywheel rotation)}$$

$$106. \frac{V}{kp} - \frac{RT}{k^2p^2} \text{ (electric motors)}$$

$$107. 1 - \frac{d^2}{2} + \frac{d^4}{24} - \frac{d^6}{120} \text{ (aircraft emergency locator transmitter)}$$

$$108. \frac{wx^2}{2T_0} + \frac{kx^4}{12T_0} \text{ (bridge design)}$$

$$109. \frac{4k - 1}{4k - 4} + \frac{1}{2k} \text{ (spring stress)}$$

$$110. \frac{Am}{k} - \frac{g}{2}\left(\frac{m}{k}\right)^2 + \frac{AML}{k} \text{ (rocket fuel)}$$

$$111. 1 - \frac{3a}{4r} - \frac{a^3}{4r^3} \text{ (hydrodynamics)}$$

$$112. \frac{1}{F} + \frac{1}{f} - \frac{d}{fF} \text{ (optics)}$$

$$113. \frac{\frac{u^2}{2g} - x}{\frac{1}{2gc^2} - \frac{u^2}{2g} + x} \text{ (mechanism design)}$$

$$114. \frac{V}{\frac{1}{2R} + \frac{1}{2R + 2}} \text{ (electricity)}$$

In Exercises 115–124, solve for the indicated letter. Each equation is from the indicated technical area of application.

115.  $W = mgh_2 - mgh_1$ , for  $m$  (work done on object)

116.  $R_1(V_2 - V_1) + R_2V_2 = 3R_1R_2$ , for  $V_2$  (electricity)

117.  $R = \frac{wL}{H(w + L)}$ , for  $L$  (architecture)

118.  $\frac{J}{T} = \frac{t}{\omega_1 - \omega_2}$ , for  $\omega_2$  (rotational torque)

119.  $E = V_0 + \frac{(m + M)V^2}{2} + \frac{p^2}{2I}$ , for  $I$  (nuclear physics)

120.  $\frac{q_2 - q_1}{d} = \frac{f + q_1}{D}$ , for  $q_1$  (photography)

121.  $s^2 + \frac{cs}{m} + \frac{kL^2}{mb^2} = 0$ , for  $c$  (mechanical vibrations)

122.  $I = \frac{A}{x^2} + \frac{B}{(10 - x)^2}$ , for  $A$  (optics)

123.  $V = \frac{i}{sC} + \frac{V_0}{s}$ , for  $C$  (electricity)

124.  $C = \frac{k}{n(1 - k)}$ , for  $k$  (reinforced concrete design)

In Exercises 125–132, set up appropriate equations and solve the given stated problems. All numbers are accurate to at least three significant digits.

125. If a certain car's lights are left on, the battery will be dead in 4.00 h. If only the radio is left on, the battery will be dead in 24.0 h. How long will the battery last if both the lights and the radio are left on?
126. Two pumps are being used to fight a fire. One pumps 5000 L in 20 min, and the other pumps 5000 L in 25.0 min. How long will it take the two pumps together to pump 5000 L?
127. A number  $m$  is the *harmonic mean* of number  $x$  and  $y$  if  $1/m$  equals the average of  $1/x$  and  $1/y$  (divide  $1/x + 1/y$  by 2). Find the harmonic mean of the musical notes that have frequencies of 400 Hz and 1200 Hz.
128. An auto mechanic can do a certain motor job in 3.00 h, and with an assistant he can do it in 2.10 h. How long would it take the assistant to do the job alone?
129. The *relative density*, also known as specific gravity  $sg$ , of an object may be defined as its weight in air  $w_a$ , divided by the difference of its weight in air and its weight when submerged in water,  $w_w$ . For a lead weight,  $w_a = 1.097 w_w$ . Find the specific gravity of lead.

130. A car travels halfway to its destination at 80.0 km/h and the remainder of the distance at 60.0 km/h. What is the average speed of the car for the trip?
131. For electric resistors in parallel, the reciprocal of the combined resistance equals the sum of the reciprocals of the individual resistances. For three resistors of  $12.0 \Omega$ ,  $R$  ohms, and  $2R$  ohms, in parallel, the combined resistance is  $6.00 \Omega$ . Find  $R$ .
132. An ambulance averaged 36.0 km/h going to an accident and 48.0 km/h on its return to the hospital. If the total time for the round-trip was 40.0 min, including 5.00 min at the accident scene, how far from the hospital was the accident?

### Writing Exercise

133. In analysing a certain electric circuit, the expression

$$\frac{\left(1 + \frac{1}{s}\right)\left(1 + \frac{1}{s/2}\right)}{3 + \frac{1}{s} + \frac{1}{s/2}}$$

is used. Simplify this expression, and write a paragraph describing your procedure. When you “cancel,” explain what basic operation is being performed.

## CHAPTER 6 PRACTICE TEST

- Find the product:  $2x(2x - 3)^2$ .
- The following equation is used in electricity.  
Solve for  $R_1$ :  $\frac{1}{R} = \frac{1}{R_1 + r} + \frac{1}{R_2}$
- Reduce to simplest form:  $\frac{2x^2 + 5x - 3}{2x^2 + 12x + 18}$
- Factor:  $4x^2 - 16y^2$
- Factor:  $pb^3 + 8a^3p$  (business application)
- Factor:  $2a - 4T - ba + 2bT$
- Factor:  $36x^2 + 14x - 16$

In Problems 8–10, perform the indicated operations and simplify.

- $\frac{3}{4x^2} - \frac{2}{x^2 - x} - \frac{x}{2x - 2}$
- $\frac{x^2 + x}{2 - x} \div \frac{x^2}{x^2 - 4x + 4}$
- $1 - \frac{3}{2x + 2}$
- $\frac{x}{5} - \frac{1}{2}$
- If one riveter can do a job in 12 days, and a second riveter can do it in 16 days, how long would it take for them to do it together?
- Solve for  $x$ :  $\frac{3}{2x^2 - 3x} + \frac{1}{x} = \frac{3}{2x - 3}$



# 7

## LEARNING OUTCOMES

After completion of this chapter, the student should be able to:

- Identify quadratic functions
- Solve quadratic functions by factoring, by completing the square, and by using the quadratic formula
- Solve engineering and technology problems involving quadratic equations
- Graph a quadratic function and identify important points on the curve

# Quadratic Equations

We showed earlier how to solve some basic equations, and in this chapter we develop methods of solving the important *quadratic equation*. In the first three sections, we present algebraic methods of solution, and in Section 7.4 we discuss graphical solutions.

The graph of the *quadratic function* is a *parabola*, which has many modern applications in science and technology (Examples 2 and 3 on page 99 illustrate the graph of a parabola). We will discuss the parabola more in this chapter and in detail in Chapter 21.

Satellite communications with television microwave dishes mounted on homes is an application of the parabola. The signal is intercepted by the entire diameter of the dish and is reflected to the detector located at the *focus* of the dish. The fact that parabolic surfaces have this property of reflection was first shown by the Greek mathematician Diocles in about 200 B.C.E. He wanted to find a mirror to reflect the rays of the sun to a point and cause burning. He proved that this was true for a mirror with a parabolic surface.

Another important application of a parabola (and thereby of quadratic functions) is that of a projectile, examples of which are a baseball, an artillery shell, and a rocket. When Galileo showed that the time an object falls does not depend on its mass, he discovered that the distance fallen depends on the square of the time of fall. This, in turn, was shown to mean that the path of a projectile is parabolic (not considering air resistance). To find the location of a projectile for any given time of flight requires the solution of a quadratic equation.

The Babylonians developed methods of solving quadratic equations nearly 4000 years ago. Today, in addition to those mentioned above, quadratic equations have applications in architecture, electric circuits, mechanical systems, forces on structures, product design, fluid mechanics, and astrophysics.

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► In Section 7.4 we see how a quadratic function arises when analysing the bending moment of certain beams.



## 7.1 Quadratic Equations; Solution by Factoring

General Quadratic Equation • Solutions of a Quadratic Equation • Procedure for Solving by Factoring

Given that  $a$ ,  $b$ , and  $c$  are constants ( $a \neq 0$ ), the equation

$$ax^2 + bx + c = 0 \quad (7.1)$$

is called the **general quadratic equation in  $x$** . The left side of Eq. (7.1) is a polynomial function of degree 2. This function,  $f(x) = ax^2 + bx + c$ , is known as the **quadratic function**. Any equation that can be simplified and then written in the form of Eq. (7.1) is a quadratic equation in one unknown.

Among the applications of quadratic equations and functions are the following examples: In finding the time  $t$  of flight of a projectile, we have the equation  $s = s_0 + v_0t + \frac{1}{2}at^2$ ; in analysing the electric current  $i$  in a circuit, the function  $f(i) = Ei - Ri^2$  is found; and in determining the forces at a distance  $x$  along a beam, the function  $f(x) = ax^2 + bLx + cL^2$  is used.

Since it is the term with  $x^2$  in Eq. (7.1) that distinguishes the quadratic equation from other types of equations, the equation is not quadratic if  $a = 0$ . However, either  $b$  or  $c$  (or both) may be zero, and the equation is still quadratic. No power of  $x$  higher than two may be present in a quadratic equation. Also, we should be able to identify a quadratic equation properly, even when it does not initially appear in the form of Eq. (7.1). The following two examples illustrate how we may recognize quadratic equations.

### EXAMPLE 1 Examples of quadratic equations

The following are quadratic equations.

$$\begin{array}{ccc} x^2 & - & 4x & - & 5 & = & 0 \\ \uparrow & & \uparrow & & \uparrow & & \\ a = 1 & & b = -4 & & c = -5 & & \end{array}$$

To show this equation in the form of Eq. (7.1), it can be written as  $1x^2 + (-4)x + (-5) = 0$ .

$$\begin{array}{ccc} 3x^2 & - & 6 & = & 0 \\ \uparrow & & \uparrow & & \\ a = 3 & & c = -6 & & \end{array}$$

Since there is no term with  $x$ ,  $b = 0$ .

$$\begin{array}{ccc} 2x^2 & + & 7x & = & 0 \\ \uparrow & & \uparrow & & \\ a = 2 & & b = 7 & & \end{array}$$

Since no constant appears,  $c = 0$ .

$$(m - 3)x^2 - mx + 7 = 0$$

The constants in Eq. (7.1) may include literal expressions. In this case,  $m - 3$  takes the place of  $a$ ,  $-m$  takes the place of  $b$ , and  $c = 7$ .

$$4x^2 - 2x = x^2$$

After all nonzero terms have been collected on the left side, the equation becomes  $3x^2 - 2x = 0$ .

$$(x + 1)^2 = 4$$

Expanding the left side and collecting all nonzero terms on the left, we have  $x^2 + 2x - 3 = 0$ . ■

■ If either (or both) the first-power term or the constant is missing, the quadratic is called *incomplete*.

### EXAMPLE 2 Examples of equations that are not quadratic

The following are not quadratic equations.

$$bx - 6 = 0 \quad \text{There is no term with } x^2.$$

$$x^3 - x^2 - 5 = 0 \quad \text{There should be no term of degree higher than 2. Thus, there can be no term with } x^3 \text{ in a quadratic equation.}$$

$$x^2 + x - 7 = x^2 \quad \text{When terms are collected, there will be no term with } x^2. \quad \blacksquare$$

### SOLUTIONS OF A QUADRATIC EQUATION

Recall that the **solution** of an equation consists of all numbers (**roots**) which, when substituted in the equation, give equality. There are two roots for a quadratic equation. At times, these roots are equal (see Example 3), so only one number is actually a solution. Also, the roots can be imaginary, and if this happens, all we wish to do at this point is to recognize that they are imaginary.

#### EXAMPLE 3 Solutions (roots) of a quadratic equation

- (a) The quadratic equation  $3x^2 - 7x + 2 = 0$  has roots  $x = 1/3$  and  $x = 2$ . This is seen by substituting these numbers in the equation.

$$3\left(\frac{1}{3}\right)^2 - 7\left(\frac{1}{3}\right) + 2 = 3\left(\frac{1}{9}\right) - \frac{7}{3} + 2 = \frac{1}{3} - \frac{7}{3} + 2 = \frac{0}{3} = 0$$

$$3(2)^2 - 7(2) + 2 = 3(4) - 14 + 2 = 14 - 14 = 0$$

- (b) The quadratic equation  $4x^2 - 4x + 1 = 0$  has a **double root** (both roots are the same) of  $x = 1/2$ . Showing that this number is a solution, we have

$$4\left(\frac{1}{2}\right)^2 - 4\left(\frac{1}{2}\right) + 1 = 4\left(\frac{1}{4}\right) - 2 + 1 = 1 - 2 + 1 = 0$$

- (c) The quadratic equation  $x^2 + 9 = 0$  has the imaginary roots  $x = 3j$  and  $x = -3j$ , which means  $x = 3\sqrt{-1}$  and  $x = -3\sqrt{-1}$ . ■

■ Until the 1600s, most mathematicians did not accept negative, irrational, or imaginary roots of an equation. It was also generally accepted that an equation had only one root.

#### LEARNING TIP

The **multiplicity** of a root refers to how many times a particular root forms a solution of an equation. For double roots of quadratics, they are said to have a **multiplicity of 2**.

This section deals only with quadratic equations whose quadratic expression is factorable. Therefore, all roots will be rational. Using the fact that

*a product is zero if any of its factors is zero*

we have the following steps in solving a quadratic equation.

#### Procedure for Solving A Quadratic Equation by Factoring

1. Collect all terms on the left and simplify (to the form of Eq. (7.1)).
2. Factor the quadratic expression.
3. Set each factor equal to zero.
4. Solve the resulting linear equations. These numbers are the roots of the quadratic equation.
5. Check the solutions in the original equation.

#### LEARNING TIP

Quadratics are often easier to graph and factor if  $a > 0$  and the coefficients of the equation are integers. This can be accomplished by multiplying the entire equation by  $-1$  if necessary to ensure  $a > 0$  and/or by multiplying the entire equation by the LCD if there are fractional coefficients.

#### EXAMPLE 4 Solving a quadratic equation by factoring

$$\begin{array}{rcl} x^2 - x - 12 = 0 & & \\ (x - 4)(x + 3) = 0 & \text{factor} & \\ \begin{array}{l} \swarrow \quad \searrow \\ x - 4 = 0 \quad x + 3 = 0 \\ x = 4 \quad x = -3 \end{array} & \text{set each factor equal to zero} & \\ & \text{solve} & \end{array}$$

The roots are  $x = 4$  and  $x = -3$ . We can check them in the original equation by substitution. Therefore,

$$\begin{array}{rcl} (4)^2 - (4) - 12 \stackrel{?}{=} 0 & (-3)^2 - (-3) - 12 \stackrel{?}{=} 0 & \\ 0 = 0 & 0 = 0 & \end{array}$$

Both roots satisfy the original equation. ■

#### Practice Exercise

1. Solve for  $x$ :  $x^2 + 4x - 21 = 0$

**EXAMPLE 5** Solving a quadratic equation by factoring with  $a \neq 1$ 

$$\begin{aligned}
 2N^2 + 7N - 4 &= 0 \\
 (2N - 1)(N + 4) &= 0 && \text{factor} \\
 2N - 1 = 0, \quad N &= \frac{1}{2} && \text{set each factor to zero and solve} \\
 N + 4 = 0, \quad N &= -4
 \end{aligned}$$

Therefore, the roots are  $N = \frac{1}{2}$  and  $N = -4$ . These roots can be checked by the same procedure used in Example 4. ■

**EXAMPLE 6** Rearranging an equation into general quadratic form

$$\begin{aligned}
 x^2 + 4 &= 4x && \text{equation not in form of Eq. (7.1)} \\
 x^2 - 4x + 4 &= 0 && \text{subtract } 4x \text{ from both sides} \\
 (x - 2)^2 &= 0 && \text{factor} \\
 x - 2 = 0, \quad x &= 2 && \text{solve}
 \end{aligned}$$

Since  $(x - 2)^2 = (x - 2)(x - 2)$ , both factors are the same. This means there is a double root of  $x = 2$ . Substitution shows that  $x = 2$  satisfies the original equation. ■

A number of equations involving fractions lead to quadratic equations after the fractions are eliminated. The following two examples, the second being a stated problem, illustrate the process of solving such equations with fractions.

**EXAMPLE 7** Fractional equation solved as a quadratic

Solve for  $x$ :

$$\frac{1}{x} + 3 = \frac{2}{x + 2}$$

Note first that  $x \neq 0, -2$ . We rearrange the fractions so there is a common denominator. This can be achieved by placing every term in the equation over  $x(x + 2)$ , the LCD.

$$\frac{1(x + 2) + 3x(x + 2)}{x(x + 2)} = \frac{2x}{x(x + 2)}$$

Since the denominators are equal, the numerators must also be equal

$$(x + 2) + 3x(x + 2) = 2x$$

Alternatively, we could have started by multiplying each term of the original equation by the LCD.

$$\begin{aligned}
 \frac{x(x + 2)}{x} + 3x(x + 2) &= \frac{2x(x + 2)}{x + 2} && \text{multiply each term by the LCD, } x(x + 2) \\
 x + 2 + 3x^2 + 6x &= 2x && \text{reduce each term} \\
 3x^2 + 5x + 2 &= 0 && \text{collect terms on left} \\
 (3x + 2)(x + 1) &= 0 && \text{factor} \\
 3x + 2 = 0, \quad x &= -\frac{2}{3} && \text{set each factor equal to zero and solve} \\
 x + 1 = 0, \quad x &= -1
 \end{aligned}$$

**Practice Exercise**

2. Solve for  $x$ :  $9x^2 + 1 = 6x$

**LEARNING TIP**

When solving a quadratic equation, it is important to write the equation in the general form as a first step, setting the quadratic equation to be equal to zero. If the product is equal to zero, then setting each factor equal to zero will lead to the roots. If the product is not zero, then it is likely that neither factor will give a correct root.

Checking in the original equation, we have

$$\begin{aligned} \frac{1}{-\frac{2}{3}} + 3 &\stackrel{?}{=} \frac{2}{-\frac{2}{3} + 2} & \frac{1}{-1} + 3 &\stackrel{?}{=} \frac{2}{-1 + 2} \\ -\frac{3}{2} + 3 &\stackrel{?}{=} \frac{2}{\frac{4}{3}} & -1 + 3 &\stackrel{?}{=} \frac{2}{1} \\ \frac{3}{2} &= \frac{6}{4} & 2 &= 2 \\ \frac{3}{2} &= \frac{3}{2} \end{aligned}$$

We see that the roots check. Remember, if either value gives division by zero, the root is extraneous, and must be excluded from the solution. ■

**EXAMPLE 8 Solving a word problem with quadratics**

A lumber truck travels 60.0 km from a sawmill to a lumber camp and then back in 7.00 h travel time. If the truck averages 5.00 km/h less on the return trip than on the trip to the camp, find its average speed to the camp. See Fig. 7.1.

Let  $v =$  the average speed (in km/h) of the truck going to the camp. This means that the average speed of the return trip was  $(v - 5)$  km/h.

We also know that  $d = vt$  (distance equals speed times time), which tells us that  $t = d/v$ . Thus, the time for each part of the trip is the distance divided by the speed.

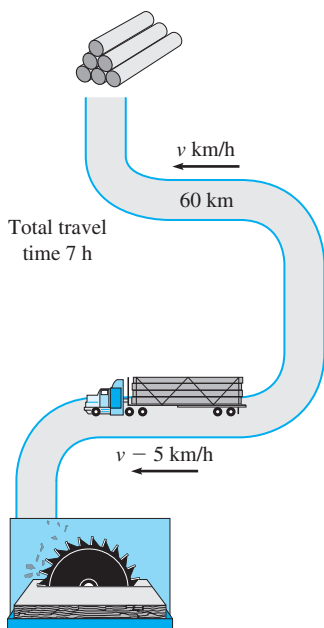


Fig. 7.1

time to camp	time from camp	total time	
$\frac{60}{v}$	$\frac{60}{v - 5}$	$= 7$	
$60(v - 5) + 60v = 7v(v - 5)$			multiply each term by $v(v - 5)$
$7v^2 - 155v + 300 = 0$			collect terms on the left
$(7v - 15)(v - 20) = 0$			factor
$7v - 15 = 0, \quad v = \frac{15}{7}$			set each factor equal to zero and solve
$v - 20 = 0$			
$v = 20$			

The value  $v = 15/7$  km/h cannot be a solution since the return speed of 5.00 km/h less would be negative. Thus, the solution is  $v = 20.0$  km/h, which means the return speed was 15.0 km/h. The trip to the camp took 3.00 h, and the return took 4.00 h, which shows the solution checks. ■

**EXERCISES 7.1**

In Exercises 1 and 2, make the given changes in the indicated examples of this section and then solve the resulting quadratic equations.

1. In Example 5, change the + sign before  $7N$  to  $-$  and then solve.
2. In Example 7, change the numerator of the first term to 2 and the numerator of the term on the right to 1 and then solve.

In Exercises 3–8, determine whether or not the given equations are quadratic. If the resulting form is quadratic, identify  $a$ ,  $b$ , and  $c$ , with  $a > 0$ . Otherwise, explain why the resulting form is not quadratic.

- |                           |                             |
|---------------------------|-----------------------------|
| 3. $x(x - 2) = 4$         | 4. $(3x - 2)^2 = 2$         |
| 5. $x^2 = (x + 2)^2$      | 6. $x(2x^2 + 5) = 7 + 2x^2$ |
| 7. $n(n^2 + n - 1) = n^3$ | 8. $(T - 7)^2 = (2T + 3)^2$ |

In Exercises 9–38, solve the given quadratic equations by factoring.

- |                          |                            |
|--------------------------|----------------------------|
| 9. $x^2 - 4 = 0$         | 10. $B^2 - 400 = 0$        |
| 11. $4x^2 = 9$           | 12. $x^2 = 0.16$           |
| 13. $x^2 - 8x - 9 = 0$   | 14. $x^2 + x - 6 = 0$      |
| 15. $R^2 + 12 = 7R$      | 16. $x^2 + 30 = 11$        |
| 17. $40x - 16x^2 = 0$    | 18. $15L = 20L^2$          |
| 19. $27m^2 = 3$          | 20. $a^2x^2 = 9$           |
| 21. $3x^2 - 13x + 4 = 0$ | 22. $A^2 + 8A + 16 = 0$    |
| 23. $7x^2 + 3x = 4$      | 24. $4x^2 + 25 = 20x$      |
| 25. $6x^2 = 13x - 6$     | 26. $6z^2 = 6 + 5x$        |
| 27. $4x(x + 1) = 3$      | 28. $9t^2 = 9 - t(43 + t)$ |

29.  $6y^2 + by = 2b^2$       30.  $2x^2 - 7ax + 4a^2 = a^2$   
 31.  $8s^2 + 16s = 90$       32.  $18t^2 = 48t - 32$   
 33.  $(x + 2)^3 = x^3 + 8$       34.  $V(V^2 - 4) = V^2(V - 1)$   
 35.  $(x + a)^2 - b^2 = 0$       36.  $2x^2 = 2b^2 - 3xb$   
 37.  $x^2 + 2ax = b^2 - a^2$   
 38.  $x^2(a^2 + 2ab + b^2) = x(a + b)$   
 39. In Eq. (7.1), for  $a = 2$ ,  $b = -7$ , and  $c = 3$ , show that the sum of the roots is  $-b/a$ .  
 40. For the equation of Exercise 39, show that the product of the roots is  $c/a$ .  
 41. The voltage  $V$  across a semiconductor in a computer is given by  $V = \alpha I + \beta I^2$ , where  $I$  is the current (in A). If a 6.00-V battery is conducted across the semiconductor, find the current if  $\alpha = 2.00 \Omega$  and  $\beta = 0.500 \Omega/\text{A}$ .  
 42. The mass  $m$  (in Mg) of the fuel supply in the first-stage booster of a rocket is  $m = 135 - 6t - t^2$ , where  $t$  is the time (in s) after launch. When does the booster run out of fuel? (Round to 3 significant digits.)  
 43. The power  $P$  (in MW) produced between midnight and noon by a nuclear power plant is  $P = 4h^2 - 48h + 744$ , where  $h$  is the hour of the day. At what time is the power 664 MW?  
 44. In determining the speed  $v$  (in km/h) of a car while studying its fuel economy, the equation  $v^2 - 16v = 3072$  is used. Find  $v$  to the nearest km/h.

In Exercises 45 and 46, although the equations are not quadratic, factoring will lead to one quadratic factor and the solution can be completed by factoring as with a quadratic equation. Find the three roots of each equation.

45.  $x^3 - x = 0$       46.  $x^3 - 4x^2 - x + 4 = 0$

In Exercises 47–50, solve the given equations involving fractions.

47.  $\frac{1}{x-3} + \frac{4}{x} = 2$       48.  $2 - \frac{1}{x} = \frac{3}{x+2}$   
 49.  $\frac{1}{2x} - \frac{3}{4} = \frac{1}{2x+3}$       50.  $\frac{x}{2} + \frac{1}{x-3} = 3$

## 7.2 Completing the Square

### Solving a Quadratic Equation by Completing the Square

■ The method of completing the square is used later in Section 7.4 and also in Chapters 21, 28, and 31.

Most quadratic equations that arise in applications cannot be solved by factoring. Therefore, we now develop a method, called **completing the square**, that can be used to solve any quadratic equation. In the next section, this method is used to develop a general formula that can be used to solve any quadratic equation.

In the first example, we show the solution of a type of quadratic equation that arises while using the method of completing the square. In the examples that follow it, the method itself is used and described.

#### EXAMPLE 1 Quadratic equation solutions

In solving  $x^2 = 16$ , we may write  $x^2 - 16 = 0$ , and complete the solution by factoring as a difference of squares. This yields  $(x + 4)(x - 4) = 0$ , giving solutions  $x = 4$  and  $x = -4$ .

In Exercises 51–54, set up the appropriate quadratic equations and solve.

51. The spring constant  $k$  is the force  $F$  divided by the amount  $x$  the spring stretches ( $k = F/x$ ). See Fig. 7.2(a). For two springs in series (see Fig. 7.2(b)), the reciprocal of the spring constant  $k_c$  for the combination equals the sum of the reciprocals of the individual spring constants. Find the spring constants for each of two springs in series if  $k_c = 2 \text{ N/cm}$  and one spring constant is  $3 \text{ N/cm}$  more than the other.

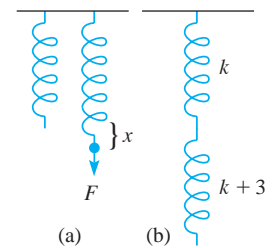


Fig. 7.2

52. The reciprocal of the combined resistance  $R$  of two resistances  $R_1$  and  $R_2$  connected in parallel (see Fig. 7.3(a)) is equal to the sum of the reciprocals of the individual resistances. If the two resistances are connected in series (see Fig. 7.3(b)), their combined resistance is the sum of their individual resistances. If two resistances connected in parallel have a combined resistance of  $3.0 \Omega$  and the same two resistances have a combined resistance of  $16 \Omega$  when connected in series, what are the resistances?

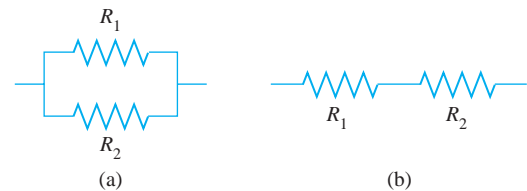


Fig. 7.3

53. A hydrofoil made the round-trip of 120 km between two islands in 3.5 h of travel time. If the average speed going was 10 km/h less than the average speed returning, find these speeds.  
 54. A rectangular solar panel is 20 cm by 30 cm. By adding the same amount to each dimension, the area is doubled. How much is added?

#### Answers to Practice Exercises

1.  $x = -7, x = 3$       2.  $x = 1/3, x = 1/3$

We can also solve the original equation by taking the square root of each side of the expression. Since  $x^2 = 16$ , we have  $x = \pm\sqrt{16}$ , and thus  $x = \pm 4$ . Therefore the principal root of 16 and its negative both satisfy the original equation.

We may solve  $(x - 3)^2 = 16$  in a similar fashion:

$$\begin{aligned}(x - 3)^2 &= 16 \\ x - 3 &= \pm 4 \\ x &= -1 \quad \text{or} \quad x = 7\end{aligned}$$

Even if the number on the right side is not a perfect square, we can still use the same method for equations of the same form. For example, we solve  $(x - 3)^2 = 17$  as

$$\begin{aligned}(x - 3)^2 &= 17 \\ x - 3 &= \pm\sqrt{17} \\ x &= 3 + \sqrt{17} \quad \text{or} \quad x = 3 - \sqrt{17}\end{aligned}$$

Decimal approximations of these roots are 7.12 and  $-1.12$ . ■

### EXAMPLE 2 Method of completing the square

To find the roots of the quadratic equation

$$x^2 - 6x - 8 = 0$$

first note that the left side is not factorable. However,

$$x^2 - 6x \text{ is part of the special product } (x - 3)^2 = x^2 - 6x + 9$$

and this special product is a *perfect square*. By adding 9 to  $x^2 - 6x$ , we have  $(x - 3)^2$ . Therefore, we rewrite the original equation as

$$\begin{aligned}x^2 - 6x &= 8 \\ x^2 - 6x + 9 &= 17 && \text{add 9 to both sides} \\ (x - 3)^2 &= 17 && \text{rewriting the left side} \\ x - 3 &= \pm\sqrt{17} && \text{as in third illustration of Example 1}\end{aligned}$$

The  $\pm$  sign means that  $x - 3 = \pm\sqrt{17}$  or  $x - 3 = -\sqrt{17}$ .

By adding 3 to each side, we obtain

$$x = 3 \pm \sqrt{17}$$

which means  $x = 3 + \sqrt{17}$  and  $x = 3 - \sqrt{17}$  are the two roots of the equation.

Therefore, by creating an expression that is a perfect square and then using the principal square root and its negative, we were finally able to solve the equation as two linear equations. ■

How we determine the number to be added to complete the square is based on the special products in Eqs. (6.3) and (6.4). We rewrite these as

$$(x + a)^2 = x^2 + 2ax + a^2 \quad (7.2)$$

$$(x - a)^2 = x^2 - 2ax + a^2 \quad (7.3)$$

We must be certain that the coefficient of the  $x^2$ -term is 1 before we start to complete the square. The coefficient of  $x$  in each case is numerically  $2a$ , and the number added to complete the square is  $a^2$ . **Thus, if we take half the coefficient of the  $x$ -term and square this result, we have the number that completes the square.** In our example, the numerical coefficient of the  $x$ -term was 6, and 9 was added to complete

#### Practice Exercise

- Solve by completing the square:  
 $x^2 + 4x - 12 = 0$

the square. Therefore, the procedure for solving a quadratic equation by completing the square is as follows:

### Solving a Quadratic Equation by Completing the Square

1. Divide each side by  $a$  (the coefficient of  $x^2$ ).
2. Rewrite the equation with the constant on the right side.
3. Complete the square: Add the square of one-half of the coefficient of  $x$  to both sides.
4. Write the left side as a square and simplify the right side.
5. Equate the square root of the left side to the principal square root of the right side and to its negative.
6. Solve the two resulting linear equations.

### EXAMPLE 3 Method of completing the square

Solve  $2x^2 + 16x - 9 = 0$  by completing the square.

$$x^2 + 8x - \frac{9}{2} = 0$$

1. Divide each term by 2 to make coefficient of  $x^2$  equal to 1.

$$x^2 + 8x = \frac{9}{2}$$

2. Put constant on right by adding  $9/2$  to each side.

$$\frac{1}{2}(8) = 4; \quad 4^2 = 16$$

3. Divide coefficient 8 of  $x$  by 2, square the 4, and add to both sides.

$$x^2 + 8x + 16 = \frac{9}{2} + 16 = \frac{41}{2}$$

$$(x + 4)^2 = \frac{41}{2}$$

4. Write left side as  $(x + 4)^2$ .

$$x + 4 = \pm \sqrt{\frac{41}{2}}$$

5. Take the square root of each side and equate the  $x + 4$  to the principal square root of  $41/2$  and its negative.

$$x = -4 \pm \sqrt{\frac{41}{2}}$$

6. Solve for  $x$ .

■ Quadratic equations can be solved using the *Solver* feature of some graphing calculators.

Therefore, the roots are  $-4 + \sqrt{\frac{41}{2}}$  and  $-4 - \sqrt{\frac{41}{2}}$ . ■

The method of completing the square is useful in situations other than the solutions of quadratic equations. For example, writing an expression as the sum of two squares is an important algebraic step when computing some integrals (Section 28.6) and when finding inverse Laplace transforms (Section 31.11). Example 4 illustrates how the square is completed for such cases.

### EXAMPLE 4 Writing an expression as the sum of two squares

Write the expression  $x^2 + 4x + 13$  as the sum of two squares.

The square of one-half of the coefficient of  $x$  is 4, so 4 is the number that completes the square. We rewrite 13 as the sum  $13 = 4 + 9$ , so that

$$\begin{aligned} x^2 + 4x + 13 &= x^2 + 4x + 4 + 9 \\ &= (x + 2)^2 + 9 \\ &= (x + 2)^2 + 3^2 \end{aligned}$$

■



## EXERCISES 7.2

In Exercises 1 and 2, make the given changes in the indicated examples of this section and then solve the resulting quadratic equations by completing the square.

- In Example 2, change the  $-$  sign before  $6x$  to  $+$ .
- In Example 3, change the coefficient of the second term from 16 to 12.

In Exercises 3–10, solve the given quadratic equations by finding appropriate square roots as in Example 1.

- $x^2 = 25$
- $x^2 = 100$
- $x^2 = 7$
- $x^2 = 15$
- $(x - 2)^2 = 25$
- $(x + 2)^2 = 10$
- $(x + 3)^2 = 7$
- $(x - \frac{5}{2})^2 = 100$

In Exercises 11–30, solve the given quadratic equations by completing the square. Exercises 11–14 and 17–20 may be checked by factoring.

- $x^2 + 2x - 8 = 0$
- $x^2 - 8x - 20 = 0$
- $D^2 + 3D + 2 = 0$
- $t^2 + 5t - 6 = 0$
- $n^2 = 4n - 2$
- $(R + 9)(R + 1) = 13$
- $v(v + 2) = 15$
- $Z^2 + 12 = 8Z$
- $2s^2 + 5s = 3$
- $4x^2 + x = 3$
- $3y^2 = 3y + 2$
- $3x^2 = 3 - 4x$
- $2y^2 - y - 2 = 0$
- $9v^2 - 6v - 2 = 0$

- $5T^2 - 10T + 4 = 0$
- $4V^2 + 9 = 12V$
- $9x^2 + 6x + 1 = 0$
- $2x^2 - 3x + 2a = 0$
- $x^2 + 2bx + c = 0$
- $px^2 + qx + r = 0$

In Exercises 31 and 32, use completing the square to write the given expression as the sum of two squares.

- $x^2 + 6x + 13$
- $x^2 - 8x + 17$

In Exercises 33–36, use completing the square to solve the given problems.

- The voltage  $V$  across a certain electronic device is related to the temperature  $T$  (in  $^{\circ}\text{C}$ ) by  $V = 4.00T - 0.200T^2$ . For what temperature(s) is  $V = 15.0$  V?
- A flare is shot vertically into the air such that its distance  $s$  (in m) above the ground is given by  $s = 20.0t - 5.00t^2$ , where  $t$  is the time (in s) after it was fired. Find  $t$  for  $s = 15.0$  m.
- A surveillance camera is 12.0 m on a direct line from an ATM. The camera is 5.00 m more to the right of the ATM than it is above the ATM. How far above the ATM is the camera?
- A rectangular storage area is 8.00 m longer than it is wide. If the area is  $28.0$  m<sup>2</sup>, what are its dimensions?

## Answer to Practice Exercise

- $x = -6, x = 2$

## 7.3 The Quadratic Formula

## The Quadratic Formula • Character of the Roots of a Quadratic Equation

We now use the method of completing the square to derive a general formula that may be used for the solution of any quadratic equation.

Consider Eq. (7.1), the general quadratic equation:

$$ax^2 + bx + c = 0 \quad (a \neq 0)$$

When we divide through by  $a$ , we obtain

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Subtracting  $c/a$  from each side, we have

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Half of  $b/a$  is  $b/2a$ , which squared is  $b^2/4a^2$ . Adding  $b^2/4a^2$  to each side gives us

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2}$$

Writing the left side as a perfect square and combining fractions on the right side, we have

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

Equating  $x + \frac{b}{2a}$  to the principal square root of the right side and its negative,

$$x + \frac{b}{2a} = \frac{\pm \sqrt{b^2 - 4ac}}{2a}$$

■ The quadratic formula, with the  $\pm$  sign, means that the solutions to the quadratic equation

$$ax^2 + bx + c = 0$$

are

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

and

$$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

When we subtract  $b/2a$  from each side and simplify the resulting expression, we obtain the **quadratic formula**:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad (7.4)$$

The quadratic formula gives us a quick general way of solving any quadratic equation. We need only write the equation in the standard form of Eq. (7.1), substitute these numbers into the formula, and simplify.

### EXAMPLE 1 Quadratic formula—rational roots

$$\text{Solve: } x^2 - 5x + 6 = 0.$$

$\uparrow$   $\uparrow$   $\uparrow$   
 $a = 1$   $b = -5$   $c = 6$

Here, using the indicated values of  $a$ ,  $b$ , and  $c$  in the quadratic formula, we have

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(6)}}{2(1)} = \frac{5 \pm \sqrt{25 - 24}}{2} = \frac{5 \pm 1}{2}$$

$$x = \frac{5 + 1}{2} = 3 \quad \text{or} \quad x = \frac{5 - 1}{2} = 2$$

The roots  $x = 3$  and  $x = 2$  check when substituted in the original equation. ■

### COMMON ERROR

It must be emphasized that, in using the quadratic formula, the entire expression  $-b \pm \sqrt{b^2 - 4ac}$  is divided by  $2a$ . It is a relatively common error to divide only the radical  $\sqrt{b^2 - 4ac}$  by  $2a$ .

### EXAMPLE 2 Quadratic formula—irrational roots

$$\text{Solve: } 2x^2 - 7x - 5 = 0.$$

$\uparrow$   $\uparrow$   $\uparrow$   
 $a = 2$   $b = -7$   $c = -5$

Substituting the values for  $a$ ,  $b$ , and  $c$  in the quadratic formula, we have

$$x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(2)(-5)}}{2(2)} = \frac{7 \pm \sqrt{49 + 40}}{4} = \frac{7 \pm \sqrt{89}}{4}$$

$$x = \frac{7 + \sqrt{89}}{4} = 4.108 \quad \text{or} \quad x = \frac{7 - \sqrt{89}}{4} = -0.6085$$

The exact roots are  $x = \frac{7 \pm \sqrt{89}}{4}$  (this form is often used when the roots are irrational).

Approximate decimal values are  $x = 4.11$  and  $x = -0.608$ . ■

### Practice Exercise

1. Solve using the quadratic formula:  
 $3x^2 + x - 5 = 0$

**EXAMPLE 3** Quadratic formula—double rootSolve:  $9x^2 + 24x + 16 = 0$ .In this example,  $a = 9$ ,  $b = 24$ , and  $c = 16$ . Thus,

$$x = \frac{-24 \pm \sqrt{24^2 - 4(9)(16)}}{2(9)} = \frac{-24 \pm \sqrt{576 - 576}}{18} = \frac{-24 \pm 0}{18} = -\frac{4}{3}$$

Here, both roots are  $-\frac{4}{3}$ , and we write the result as  $x = -\frac{4}{3}$  and  $x = -\frac{4}{3}$ . We will get a double root when  $b^2 = 4ac$ , as in this case. ■**EXAMPLE 4** Quadratic formula—imaginary rootsSolve:  $3x^2 - 5x + 4 = 0$ .In this example,  $a = 3$ ,  $b = -5$ , and  $c = 4$ . Therefore,

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(3)(4)}}{2(3)} = \frac{5 \pm \sqrt{25 - 48}}{6} = \frac{5 \pm \sqrt{-23}}{6}$$

These roots contain imaginary numbers. This happens if  $b^2 < 4ac$ . ■Examples 1–4 illustrate the *character of the roots* of a quadratic equation. If  $a$ ,  $b$ , and  $c$  are rational numbers (see Section 1.1), by noting the value of  $b^2 - 4ac$  (called the **discriminant**), we have the following:**LEARNING TIP**

If  $b^2 - 4ac$  is positive and a perfect square,  $ax^2 + bx + c$  is factorable. We can use the value of  $b^2 - 4ac$  to help in checking the roots or in finding the character of the roots without having to solve the equation completely.

**Character of the Roots of a Quadratic Equation**

1. If  $b^2 - 4ac$  is positive and a perfect square (see Section 1.6), the roots are real, rational, and unequal. (See Example 1, where  $b^2 - 4ac = 1$ .)
2. If  $b^2 - 4ac$  is positive but not a perfect square, the roots are real, irrational, and unequal. (See Example 2, where  $b^2 - 4ac = 89$ .)
3. If  $b^2 - 4ac = 0$ , the roots are real, rational, and equal. (See Example 3, where  $b^2 - 4ac = 0$ .)
4. If  $b^2 - 4ac < 0$ , the roots contain imaginary numbers and are unequal. (See Example 4, where  $b^2 - 4ac = -23$ .)

**EXAMPLE 5** Quadratic formula—literal numbersThe equation  $s = s_0 + v_0t - \frac{1}{2}gt^2$  is used in the analysis of projectile motion (see Fig. 7.4). Solve for  $t$ .

$$gt^2 - 2v_0t - 2(s_0 - s) = 0 \quad \text{multiply by } -2, \text{ put in form of Eq. (7.1)}$$

In this form, we see that  $a = g$ ,  $b = -2v_0$ , and  $c = -2(s_0 - s)$ :

$$t = \frac{-(-2v_0) \pm \sqrt{(-2v_0)^2 - 4g(-2)(s_0 - s)}}{2g} = \frac{2v_0 \pm \sqrt{4(v_0^2 + 2gs_0 - 2gs)}}{2g}$$

$$t = \frac{2v_0 \pm 2\sqrt{v_0^2 + 2gs_0 - 2gs}}{2g} = \frac{v_0 \pm \sqrt{v_0^2 + 2gs_0 - 2gs}}{g}$$

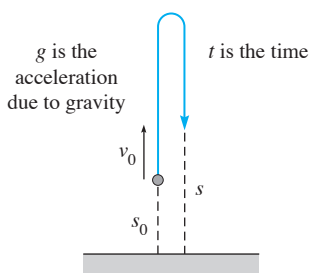


Fig. 7.4

**EXAMPLE 6** Quadratic formula—word problem

A rectangular area 17.0 m long and 12.0 m wide is to be used for a patio with a rectangular pool. One end and one side of the patio area around the pool (the chairs, sunning,

etc.) are to be the same width. The other end with the diving board is to be twice as wide, and the other side is to be three times as wide as the narrow side. The pool area is to be  $96.5 \text{ m}^2$ . What are the widths of the patio ends and sides, and the dimensions of the pool? See Fig. 7.5.

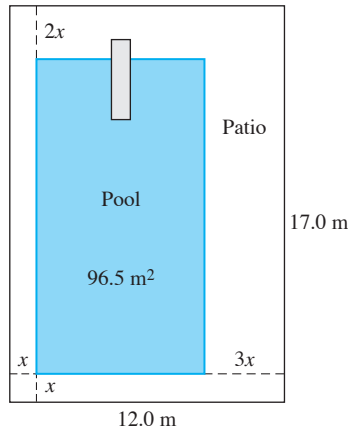


Fig. 7.5

First, let  $x$  = the width of the narrow end and side of the patio. The other end is then  $2x$  in width, and the other side is  $3x$  in width. Since the pool area is  $96.5 \text{ m}^2$ , we have

$$\begin{aligned} & \text{pool length} \quad \text{pool width} \quad \text{pool area} \\ & (17.0 - 3x)(12.0 - 4x) = 96.5 \\ & 204 - 68.0x - 36.0x + 12x^2 = 96.5 \\ & 12x^2 - 104.0x + 107.5 = 0 \\ & x = \frac{-(-104.0) \pm \sqrt{(-104.0)^2 - 4(12)(107.5)}}{2(12)} = \frac{104.0 \pm \sqrt{5656}}{24} \end{aligned}$$

Evaluating, we get  $x = 7.47 \text{ m}$  and  $x = 1.20 \text{ m}$ . The value  $7.47 \text{ m}$  cannot be the required result since the width of the patio would be greater than the width of the entire area. For  $x = 1.20 \text{ m}$ , the pool would have a length  $13.4 \text{ m}$  and width  $7.20 \text{ m}$ . These give an area of  $96.5 \text{ m}^2$ , which checks. The widths of the patio area are then  $1.20 \text{ m}$ ,  $1.20 \text{ m}$ ,  $2.40 \text{ m}$ , and  $3.60 \text{ m}$ . ■

## EXERCISES 7.3

In Exercises 1–4, make the given changes in the indicated examples of this section and then solve the resulting equations by the quadratic formula.

- In Example 1, change the  $-$  sign before  $5x$  to  $+$ .
- In Example 2, change the coefficient of  $x^2$  from 2 to 3.
- In Example 3, change the  $+$  sign before  $24x$  to  $-$ .
- In Example 4, change 4 to 3.

In Exercises 5–36, solve the given quadratic equations, using the quadratic formula. Exercises 5–8 are the same as Exercises 11–14 of Section 7.2.

- $x^2 + 2x - 8 = 0$
- $D^2 + 3D + 2 = 0$
- $x^2 - 4x + 2 = 0$
- $v^2 = 15 - 2v$
- $2s^2 + 5s = 3$
- $3y^2 = 3y + 2$
- $y + 2 = 2y^2$
- $30y^2 + 23y - 40 = 0$
- $8t^2 + 61t = -120$
- $s^2 = 9 + s(1 - 2s)$
- $25y^2 = 121$
- $15 + 4z = 32z^2$
- $x^2 - 0.200x - 0.400 = 0$
- $0.290Z^2 - 0.180 = 0.630Z$
- $x^2 - 8x - 20 = 0$
- $t^2 + 5t - 6 = 0$
- $x^2 + 10x - 4 = 0$
- $8V - 12 = V^2$
- $4x^2 + x = 3$
- $3x^2 = 3 - 4x$
- $2 + 6v = 9v^2$
- $40x^2 - 62x - 63 = 0$
- $2d(d - 2) = -7$
- $20r^2 = 20r + 1$
- $37T = T^2$
- $4x^2 - 12x = 7$
- $3.20x^2 = 2.50x + 7.60$
- $12.5x^2 + 13.2x = 15.5$

- $x^2 + 2cx - 1 = 0$
- $b^2x^2 + 1 - a = (b + 1)x$
- $x^2 - 7x + (6 + a) = 0$
- $c^2x^2 - x - 1 = x^2$

In Exercises 37–40, without solving the given equations, determine the character of the roots.

- $2x^2 - 7x = -8$
- $3.6t^2 + 2.1 = 7.7t$
- $3x^2 + 19x = 14$
- $0.45s^2 + 0.33 = 0.12s$

In Exercises 41–58, solve the given problems. All numbers are accurate to at least 3 significant digits.

- Find  $k$  if the equation  $x^2 + 4x + k = 0$  has a real double root.
- Find the smallest positive integral value of  $k$  if the equation  $x^2 + 3x + k = 0$  has roots with imaginary numbers.
- Solve the equation  $x^4 - 5x^2 + 4 = 0$  for  $x$ . (Hint: The equation can be written as  $(x^2)^2 - 5(x^2) + 4 = 0$ . First solve for  $x^2$ .)
- Without drawing the graph or completely solving the equation, explain how to find the number of  $x$ -intercepts of a quadratic function.
- In machine design, in finding the outside diameter  $D_0$  of a hollow shaft, the equation  $D_0^2 - DD_0 - 0.250D^2 = 0$  is used. Solve for  $D_0$  if  $D = 3.625 \text{ cm}$ .
- A missile is fired vertically into the air. The distance  $s$  (in m) above the ground as a function of time  $t$  (in s) is given by  $s = 100 + 500t - 4.9t^2$ . (a) When will the missile hit the ground? (b) When will the missile be 1000 m above the ground?
- For a rectangle, if the ratio of the length to the width equals the ratio of the length plus the width to the length, the ratio is called

the *golden ratio*. Find the value of the golden ratio, which the ancient Greeks thought had the most pleasing properties to look at.

48. When focusing a camera, the distance  $r$  the lens must move from the infinity setting is given by  $r = f^2 / (p - f)$ , where  $p$  is the distance from the object to the lens, and  $f$  is the focal length of the lens. Solve for  $f$ .
49. In calculating the current in an electric circuit with an inductance  $L$ , a resistance  $R$ , and a capacitance  $C$ , it is necessary to solve the equation  $Lm^2 + Rm + 1/C = 0$ . Solve for  $m$  in terms of  $L$ ,  $R$ , and  $C$ . See Fig. 7.6.



Fig. 7.6

50. In finding the radius  $r$  of a circular arch of height  $h$  and span  $b$ , an architect used the following formula. Solve for  $h$ .
- $$r = \frac{b^2 + 4h^2}{8h}$$
51. A flat-screen computer monitor is 37.0 cm wide and 31.0 cm high with a uniform edge around the viewing screen. If the edge covers 20.0% of the monitor front, what is the width of the edge?
52. An investment of \$2000 is deposited at a certain annual interest rate. One year later, \$3000 is deposited in another account at the same rate. At the end of the second year, the accounts have a total value of \$5319.05. What is the interest rate?
53. The length of a tennis court is 12.8 m more than its width. If the area of the tennis court is 262 m<sup>2</sup>, what are its dimensions? See Fig. 7.7.

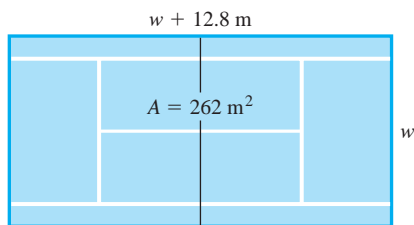


Fig. 7.7

54. Two circular oil spills are tangent to each other. If the distance between centres is 800 m and they cover a combined area of  $1.02 \times 10^6 \text{ m}^2$ , what is the radius of each? See Fig. 7.8.

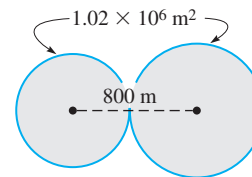


Fig. 7.8

55. In remodeling a house, an architect finds that by adding the same amount to each dimension of a 3.80-m by 5.00-m rectangular room, the area would be increased by 11.0 m<sup>2</sup>. How much must be added to each dimension?
56. Two pipes together drain a wastewater-holding tank in 6.00 h. If used alone to empty the tank, one takes 2.00 h longer than the other. How long does each take to empty the tank if used alone?
57. On some highways, a car can legally travel 20.0 km/h faster than a truck. Travelling at maximum legal speeds, a car can travel 120 km in 18.0 min less than a truck. What are the maximum legal speeds for cars and for trucks?
58. For electric capacitors connected in series, the sum of the reciprocals of the capacitances equals the reciprocal of the combined capacitance. If one capacitor has 5.00  $\mu\text{F}$  more capacitance than another capacitor and they are connected in series, what are their capacitances if their combined capacitance is 4.00  $\mu\text{F}$ ?

Answer to Practice Exercise

$$1. x = \frac{-1 \pm \sqrt{61}}{6}$$

## 7.4 The Graph of the Quadratic Function

### The Parabola • Vertex and y-intercept • Solving Quadratic Equations Graphically

$x$	$y$
-4	5
-3	0
-2	-3
-1	-4
0	-3
1	0
2	5

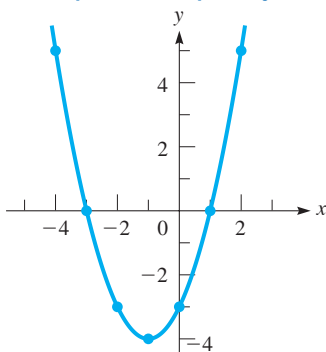


Fig. 7.9

In this section, we discuss the graph of the quadratic function  $ax^2 + bx + c$  and show the graphical solution of a quadratic equation. By letting  $y = ax^2 + bx + c$ , we can graph this function, as in Chapter 3. The next example shows the graph of the quadratic function.

#### EXAMPLE 1 Graphing a quadratic function

Graph the function  $f(x) = x^2 + 2x - 3$ .

First, let  $y = x^2 + 2x - 3$ . We can now set up a table of values and graph the function as shown in Fig. 7.9.

The graph of the quadratic function shown in Fig. 7.9 is that of a **parabola**, and the graph of any quadratic function  $y = ax^2 + bx + c$  will have the same basic shape. (In Section 3.4, we briefly noted that graphs in Examples 2 and 3 were parabolas.) A

parabola can open upward (as in Example 1) or downward. The location of the parabola and how it opens depends on the values of  $a$ ,  $b$ , and  $c$ , as we will show after Example 2.

In Example 1, the parabola has a minimum point at  $(-1, -4)$ , and the curve opens upward. All parabolas have an **extreme point** of this type. For  $y = ax^2 + bx + c$ , if  $a > 0$ , the parabola has a **minimum point** and opens **upward**. If  $a < 0$ , the parabola has a **maximum point** and opens **downward**. The extreme point of the parabola is known as its **vertex**.

### EXAMPLE 2 Parabola—extreme points

The graph of  $y = 2x^2 - 8x + 6$  is shown in Fig. 7.10(a). For this parabola,  $a = 2$  ( $a > 0$ ) and it opens upward. The vertex (a minimum point) is  $(2, -2)$ .

The graph of  $y = -2x^2 + 8x - 6$  is shown in Fig. 7.10(b). For this parabola,  $a = -2$  ( $a < 0$ ) and it opens downward. The vertex (a maximum point) is  $(2, 2)$ . ■

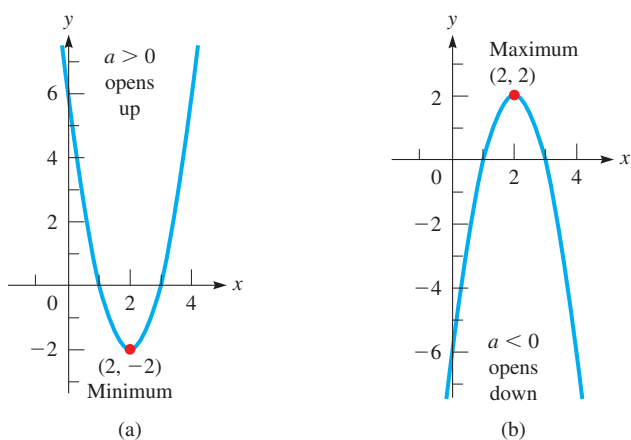


Fig. 7.10

#### LEARNING TIP

Since  $x = -\frac{b}{2a}$  is the  $x$ -coordinate of the vertex, if we substitute this value into Equation 7.5, it gives

$$y = a\left(-\frac{b}{2a} + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} = c - \frac{b^2}{4a}$$

Therefore, the coordinates of the vertex point of any parabola are:

$$\text{Vertex } (x, y) = \left(-\frac{b}{2a}, c - \frac{b^2}{4a}\right)$$

We can sketch a graph of parabolas like these by using its basic shape and knowing two or three points, including the vertex.

In order to find the coordinates of the extreme point, start with the quadratic function

$$y = ax^2 + bx + c$$

Then factor  $a$  from the two terms containing  $x$ , obtaining

$$y = a\left(x^2 + \frac{b}{a}x\right) + c$$

Now, completing the square of the terms within parentheses, we have

$$y = a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + c - \frac{b^2}{4a}$$

$$y = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} \quad (7.5)$$

Now, look at the factor  $\left(x + \frac{b}{2a}\right)^2$ . If  $x = -b/2a$ , the term is zero. If  $x$  is any other value,  $\left(x + \frac{b}{2a}\right)^2$  is positive. Thus, if  $a > 0$ , the value of  $y$  increases from the minimum value, where  $x = -b/2a$ , and if  $a < 0$ , the value of  $y$  decreases from the maximum value, where  $x = -b/2a$ . This means that  $x = -b/2a$  is the  $x$ -coordinate of the vertex (extreme point). The  $y$ -coordinate is found by substituting this  $x$ -value in the function.

Another easily found point is the  $y$ -intercept. As with a linear equation, we find the  $y$ -intercept where  $x = 0$ . For  $y = ax^2 + bx + c$ , if  $x = 0$ , then  $y = c$ . This means that the point  $(0, c)$  is the  $y$ -intercept.

### EXAMPLE 3 Graphing a parabola—vertex— $y$ -intercept

For the graph of the function  $y = 2x^2 - 8x + 6$ , find the vertex and  $y$ -intercept and sketch the graph. (This function is also used in Example 2.)

First,  $a = 2$  and  $b = -8$ . This means that the  $x$ -coordinate of the vertex is

$$\frac{-b}{2a} = \frac{-(-8)}{2(2)} = \frac{8}{4} = 2$$

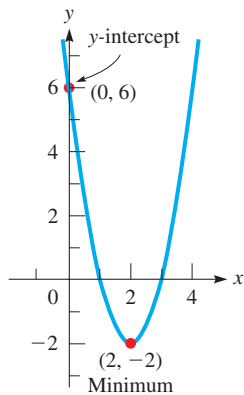


Fig. 7.11

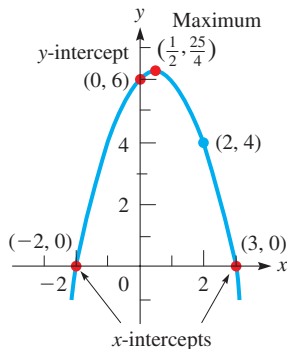


Fig. 7.12

**Practice Exercise**

1. (a) Find the vertex and y-intercept for the graph of the function  $y = 3x^2 + 12x - 4$ .
- (b) Is the vertex a minimum or a maximum point?

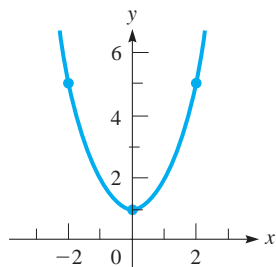


Fig. 7.13

and the y-coordinate is

$$y = 2(2^2) - 8(2) + 6 = -2$$

Thus, the vertex is  $(2, -2)$ . Since  $a > 0$ , it is a minimum point.

Since  $c = 6$ , the y-intercept is  $(0, 6)$ .

We can use the minimum point  $(2, -2)$  and the y-intercept  $(0, 6)$ , along with the fact that the graph is a parabola, to get an approximate sketch of the graph. Noting that a parabola increases (or decreases) away from the vertex in the same way on each side of it (it is *symmetric* to a vertical line through the vertex), we sketch the graph in Fig. 7.11. It is the same graph as that shown in Fig. 7.10(a). ■

If we are not using a graphing calculator, we may need one or two additional points to get a good sketch of a parabola. This would be true if the y-intercept is close to the vertex. Two points we can find are the x-intercepts, if the parabola crosses the x-axis (one point if the vertex is on the x-axis). They are found by setting  $y = 0$  and solving the quadratic equation  $ax^2 + bx + c = 0$ . Also, we may simply find one or two points other than the vertex and the y-intercept. Sketching a parabola in this way is shown in the following two examples.

**EXAMPLE 4** Graphing a parabola—using vertex, y-intercept, and x-intercepts

Sketch the graph of  $y = -x^2 + x + 6$ .

We first note that  $a = -1$  and  $b = 1$ . Therefore, the x-coordinate of the maximum point ( $a < 0$ ) is  $-\frac{1}{2(-1)} = \frac{1}{2}$ . The y-coordinate is  $-\left(\frac{1}{2}\right)^2 + \frac{1}{2} + 6 = \frac{25}{4}$ . This means that the maximum point is  $\left(\frac{1}{2}, \frac{25}{4}\right)$ .

The y-intercept is  $(0, 6)$ .

Using these points in Fig. 7.12, since they are close together, they do not give a good idea of how wide the parabola opens. Therefore, setting  $y = 0$ , we solve the equation

$$-x^2 + x + 6 = 0 \quad \text{multiply each term by } -1$$

or

$$x^2 - x - 6 = 0$$

This equation is factorable. Thus,

$$(x - 3)(x + 2) = 0$$

$$x = 3, -2$$

This means that the x-intercepts are  $(3, 0)$  and  $(-2, 0)$ , as shown in Fig. 7.12.

Also, rather than finding the x-intercepts, we can let  $x = 2$  (or some value to the right of the vertex) and then use the point  $(2, 4)$ . ■

**EXAMPLE 5** Graphing a parabola—no x-term

Sketch the graph of  $y = x^2 + 1$ .

Since there is no x-term,  $b = 0$ . This means that the x-coordinate of the minimum point ( $a > 0$ ) is 0 and that the minimum point and the y-intercept are both  $(0, 1)$ . We know that the graph opens upward, since  $a > 0$ , which in turn means that it does not cross the x-axis. Now, letting  $x = 2$  and  $x = -2$ , find the points  $(2, 5)$  and  $(-2, 5)$  on the graph, which is shown in Fig. 7.13. ■

We can see that the domain of the quadratic function is all  $x$ . Knowing that the graph has either a maximum point or a minimum point, the range of the quadratic function must be restricted to certain values of  $y$ . In Example 3, the range is  $f(x) \geq -2$ ; in Example 4, the range is  $f(x) \leq \frac{25}{4}$ ; and in Example 5, the range is  $f(x) \geq 1$ .



See the chapter introduction

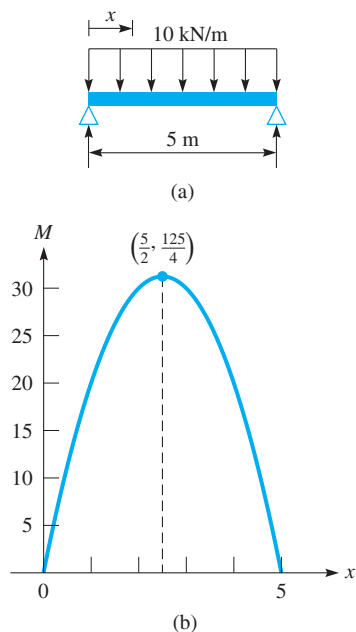


Fig. 7.14

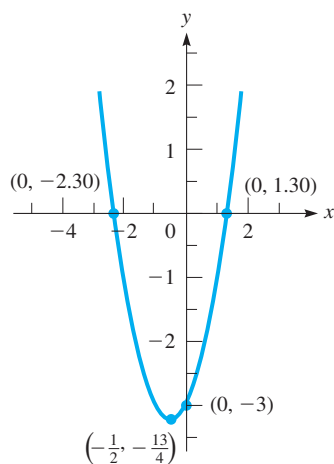


Fig. 7.15

### EXAMPLE 6 Graphing a parabola—application

The bending moment  $M$  (in  $\text{kN} \cdot \text{m}$ ) at any point along a beam is the algebraic sum of the moments of all the forces acting on the portion of the beam to the right or to the left of that point. Consider a simply supported 5 m beam that supports a uniformly distributed load of 10  $\text{kN/m}$ , as shown in Fig. 7.14(a). For this beam, the bending moment  $x$  metres from one end of the beam is given by  $M = 25x - 5x^2$ , where  $0 \leq x \leq 5$ . Graph this function and determine where the bending moment is zero and where the maximum bending moment occurs.

Since  $c = 0$ , the  $M$ -intercept is at the origin. We also have that  $a = -5$  and  $b = 25$ . Therefore, the parabola opens downward and the  $x$  coordinate of the maximum point is  $-\frac{b}{2a} = -\frac{25}{2(-5)} = \frac{5}{2}$ . Substituting into the function, the  $M$ -coordinate of the maximum is  $25\left(\frac{5}{2}\right) - 5\left(\frac{5}{2}\right)^2 = \frac{125}{4}$ .

Because of symmetry, this is enough information to obtain the graph, as shown in Fig. 7.14(b). From this graph we see that the bending moment is zero for  $x = 0$  and  $x = 5$  (at both ends of the beam), and that the maximum bending moment occurs at the centre of the beam. ■

### SOLVING QUADRATIC EQUATIONS GRAPHICALLY

In Chapter 3, we showed how an equation can be solved graphically. Following that method, to solve the equation  $ax^2 + bx + c = 0$ , let  $y = ax^2 + bx + c$  and graph the function. The roots of the equation are the  $x$ -coordinates of the points for which  $y = 0$  (the  $x$ -intercepts). The following example illustrates solving quadratic equations graphically.

### EXAMPLE 7 Solving a quadratic equation graphically

Solve the equation  $3x = x(2 - x) + 3$  graphically.

First, collect all terms on the left of the equal sign. This gives  $x^2 + x - 3 = 0$ . We then let  $y = x^2 + x - 3$  and graph this function. The minimum point is  $(-\frac{1}{2}, -\frac{13}{4})$  and the  $y$ -intercept is  $(0, -3)$ . Using the *zero* feature (or the *trace* and *zoom* features) of a graphing calculator, we find the roots to be

$$x = -2.30 \quad \text{and} \quad x = 1.30$$

The complete graph is shown in Fig. 7.15. ■

## EXERCISES 7.4

In Exercises 1 and 2, make the given changes in the indicated examples of this section and then solve the resulting problems.

- In Example 3, change the  $-$  sign before  $8x$  to  $+$  and then sketch the graph.
- In Example 7, change the coefficient of  $x$  on the left from 3 to 5 and then find the solution graphically.

In Exercises 3–8, sketch the graph of each parabola by using only the vertex and the  $y$ -intercept.

- $y = x^2 - 6x + 5$
- $y = -x^2 - 4x - 3$
- $y = -3x^2 + 10x - 4$
- $s = 2t^2 + 8t - 5$
- $R = v^2 - 4v$
- $y = -2x^2 - 5x$

In Exercises 9–12, sketch the graph of each parabola by using the vertex, the  $y$ -intercept, and the  $x$ -intercepts.

- $y = x^2 - 4$
- $y = x^2 + 3x$
- $y = -2x^2 - 6x + 8$
- $u = -3v^2 + 12v - 5$

In Exercises 13–16, sketch the graph of each parabola by using the vertex, the  $y$ -intercept, and two other points, not including the  $x$ -intercepts.

- $y = 2x^2 + 3$
- $y = x^2 + 2x + 2$
- $y = -2x^2 - 2x - 6$
- $y = -3x^2 - x$

In Exercises 17–24, solve the given equations graphically. If there are no real roots, state this as the answer.

17.  $2x^2 - 3 = 0$                       18.  $5 - x^2 = 0$   
 19.  $-3x^2 + 11x - 5 = 0$             20.  $2t^2 = 7t + 4$   
 21.  $x(2x - 1) = -3$                 22.  $2w - 5 = w^2$   
 23.  $6R^2 = 18 - 7R$                 24.  $3x^2 - 25 = 20x$

In Exercises 25–30, graph all three parabolas on the same coordinate system. In Exercises 25–28, describe (a) the shifts (see page 106) of  $y = x^2$  that occur and (b) how each parabola opens. In Exercises 29 and 30, describe (a) the shifts and (b) how the shape of each parabola changes.

25. (a)  $y = x^2$  (b)  $y = x^2 + 3$  (c)  $y = x^2 - 3$   
 26. (a)  $y = x^2$  (b)  $y = (x - 3)^2$  (c)  $y = (x + 3)^2$   
 27. (a)  $y = x^2$  (b)  $y = (x - 2)^2 + 3$  (c)  $y = (x + 2)^2 - 3$   
 28. (a)  $y = x^2$  (b)  $y = -x^2$  (c)  $y = -(x - 2)^2$   
 29. (a)  $y = x^2$  (b)  $y = 3x^2$  (c)  $y = \frac{1}{3}x^2$   
 30. (a)  $y = x^2$  (b)  $y = -3(x - 2)^2$  (c)  $y = \frac{1}{3}(x + 2)^2$

In Exercises 31–46, solve the given applied problem.

31. A quadratic equation  $f(x) = 0$  has a solution  $x = -1$ . Its graph has its vertex at (3, 4). What is the other solution?
32. Find  $c$  such that  $y = x^2 - 12x + c$  has exactly one real root.
33. Find the smallest integral value of  $c$  such that  $y = 2x^2 - 4x - c$  has two real roots.
34. Find the smallest integral value of  $c$  such that  $y = 3x^2 - 12x + c$  has no real roots.
35. The vertical distance  $d$  (in cm) of the end of a robot arm above a conveyor belt in its 8-s cycle is given by  $d = 2t^2 - 16t + 47$ . Sketch the graph of  $d = f(t)$ .
36. When mineral deposits form a uniform coating 1 mm thick on the inside of a pipe of radius  $r$  (in mm), the cross-sectional area  $A$  through which water can flow is  $A = \pi(r^2 - 2r + 1)$ . Sketch  $A = f(r)$ .
37. An equipment company determines that the area  $A$  (in  $\text{m}^2$ ) covered by a rectangular tarpaulin is given by  $A = w(8 - w)$ , where  $w$  is the width of the tarpaulin and its perimeter is 16 m. Sketch the graph of  $A$  as a function of  $w$ .
38. Under specified conditions, the pressure loss  $h_L$  (in  $\kappa\text{Pa}$  per 100 m), in the flow of water through a water line in which the flow is  $Q$  L/min, is given by  $h_L = 0.0001Q^2 + 0.005Q$ . Sketch the graph of  $h_L$  as a function of  $Q$ , for  $Q < 400$  L/min.
39. When analysing the power  $P$  (in W) dissipated in an electric circuit, the equation  $P = 50i - 3i^2$  results. Here,  $i$  is the current (in A). Sketch the graph of  $P = f(i)$ .
40. Tests show that the power  $P$  (in kW) of an automobile engine as a function of  $r$  (in r/min) is given by  $P = -5.0 \times 10^{-6}r^2 + 0.050r - 45$  ( $1500 < r < 6000$  r/min). Sketch the graph of  $P$  vs.  $r$  and find the maximum power that is produced.
41. A missile is fired vertically upward such that its distance  $s$  (in m) above the ground is given by  $s = 50 + 90t - 4.9t^2$ , where  $t$  is the time (in s). Sketch the graph and then determine from the graph (a) when the missile will hit the ground (to 0.1 s), (b) how high it will go (to three significant digits), and (c) how long it will take to reach a height of 250 m (to 0.1 s).
42. In a certain electric circuit, the resistance  $R$  (in  $\Omega$ ) that gives resonance is found by solving the equation  $25R = 3(R^2 + 4)$ . Solve this equation graphically (to 0.1  $\Omega$ ).
43. If the radius of a circular solar cell is increased by 1.00 cm, its area is  $96.0 \text{ cm}^2$ . What was the original radius?
44. The diagonal of a rectangular floor is 1.00 m less than twice the length of one of the sides. If the other side is 5.00 m long, what is the area of the floor?
45. A security fence is to be built around a rectangular parking area of  $2000 \text{ m}^2$ . If the front side of the fence costs  $\$60/\text{m}$  and the other three sides cost  $\$30/\text{m}$ , solve graphically for the dimensions (to 1 m) of the parking area if the fence is to cost  $\$7500$ . See Fig. 7.16.

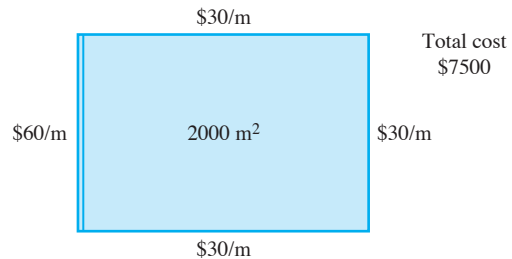


Fig. 7.16

46. An airplane pilot could decrease the time  $t$  (in h) needed to travel the 5400 km from Ottawa to London by 60 min if the plane's speed  $v$  is increased by 40 km/h. Set up the appropriate equation and solve graphically for  $v$  (to 2 significant digits).

#### Answers to Practice Exercise

1. (a)  $(-2, -16)$ ,  $(0, -4)$       (b) minimum

## CHAPTER 7 EQUATIONS

### Quadratic equation

$$ax^2 + bx + c = 0 \quad (7.1)$$

### Quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad (7.4)$$

## CHAPTER 7 REVIEW EXERCISES

In Exercises 1–12, solve the given quadratic equations by factoring.

1.  $x^2 + 3x - 4 = 0$
2.  $x^2 + 3x - 10 = 0$
3.  $x^2 - 10x + 16 = 0$
4.  $P^2 - 27 = 6P$
5.  $3x^2 + 11x = 4$
6.  $6y^2 = 11y - 3$
7.  $6t^2 = 13t - 5$
8.  $3x^2 + 5x + 2 = 0$
9.  $6s^2 = 25s$
10.  $23n + 35 = 6n^2$
11.  $4B^2 = 8B + 21$
12.  $6x^2 = 8 - 47x$

In Exercises 13–24, solve the given quadratic equations by using the quadratic formula.

13.  $x^2 - x - 110 = 0$
14.  $x^2 + 3x - 18 = 0$
15.  $m^2 + 2m = 6$
16.  $1 + 7D = D^2$
17.  $2x^2 - x = 36$
18.  $3x^2 = 14 - x$
19.  $3s + 2 = 4s^2$
20.  $5x^2 + 7x - 2 = 0$
21.  $2.1x^2 + 2.3x + 5.5 = 0$
22.  $0.30R^2 - 0.42R = 0.15$
23.  $6x^2 = 9 - 4x$
24.  $24x^2 = 25x + 20$

In Exercises 25–36, solve the given quadratic equations by any appropriate algebraic method.

25.  $x^2 + 4x - 4 = 0$
26.  $x^2 + 3x + 1 = 0$
27.  $3x^2 + 8x + 2 = 0$
28.  $3p^2 = 28 - 5p$
29.  $4v^2 = v + 5$
30.  $n - 2 = 6n^2$
31.  $7 + 3C = -2C^2$
32.  $4y^2 - 5y = 8$
33.  $a^2x^2 + 2ax + 2 = 0$
34.  $16r^2 = 8r - 1$
35.  $ay^2 = a - 3y$
36.  $2bx = x^2 - 3b$

In Exercises 37–40, solve the given quadratic equations by completing the square.

37.  $x^2 - x - 30 = 0$
38.  $x^2 = 2x + 5$
39.  $2t^2 = t + 4$
40.  $4x^2 - 8x = 3$

In Exercises 41–44, solve the given equations.

41.  $\frac{x-4}{x-1} = \frac{2}{x}$
42.  $\frac{V-1}{3} = \frac{5}{V} + 1$
43.  $\frac{x^2-3x}{x-3} = \frac{x^2}{x+2}$
44.  $\frac{x-2}{x-5} = \frac{15}{x^2-5x}$

In Exercises 45–48, sketch the graphs of the given functions by using the vertex, the y-intercept, and one or two other points.

45.  $y = 2x^2 - x - 1$
46.  $y = -4x^2 - 1$
47.  $y = x - 3x^2$
48.  $y = 2x^2 + 8x - 10$

In Exercises 49–52, solve the given equations graphically. If there are no real roots, state this as the answer.

49.  $2x^2 + x - 4 = 0$
50.  $-4x^2 - x - 1 = 0$
51.  $3x^2 = -x - 2$
52.  $x(15x - 12) = 8$

In Exercises 53 and 54, solve the given problems.

53. A quadratic equation  $f(x) = 0$  has a solution  $x = 2$ . Its graph has its vertex at  $(-1, 6)$ . What is the other solution?
54. Find  $c$  such that  $y = 2x^2 + 16x + c$  has exactly one real root.

In Exercises 55–68, solve the given quadratic equations by any appropriate method. All numbers are accurate to at least 3 significant digits.

55. The bending moment  $M$  of a simply supported beam of length  $L$  with a uniform load of  $w$  kg/m at a distance  $x$  from one end is  $M = 0.5wLx - 0.5wx^2$ . For what values of  $x$  is  $M = 0$ ?
56. A nuclear power plant supplies a fixed power level at a constant voltage. The current  $I$  (in A) is found by solving the equation  $I^2 - 17I - 12 = 0$ . Solve for  $I > 0$ .
57. A computer analysis shows that the cost  $C$  (in dollars) for a company to make  $x$  units of a certain product is given by  $C = 0.1x^2 + 0.8x + 7$ . How many units can be made for \$50?
58. For laminar flow of fluids, the coefficient  $K$  used to calculate energy loss due to sudden enlargements (where pipe size suddenly increases) is given by  $K = 1.00 - 2.67R + R^2$ , where  $R$  is the ratio of cross-sectional areas. If  $K = 0.500$ , what is the value of  $R$ ?
59. At an altitude  $h$  (in m) above sea level, the boiling point of water is lower by  $T^\circ\text{C}$  ( $T > 0$ ) than the boiling point at sea level, which is  $100.0^\circ\text{C}$ . The difference can be approximated by solving the equation  $T^2 + 244T - h = 0$ . What is the boiling point in Calgary, Alberta (altitude 1050 m)?
60. In a natural gas pipeline, the velocity  $v$  (in m/s) of the gas as a function of the distance  $x$  (in cm) from the wall of the pipe is given by  $v = 5.20x - x^2$ . Determine  $x$  for  $v = 4.80$  m/s.
61. The height  $h$  of an object shot at an angle  $\theta$  moving with velocity  $v$  is given by  $h = vt \sin \theta - 4.9t^2$ , where  $t$  is the time of flight. Find  $t$  (to the nearest 0.01 s) if  $v = 15.0$  m/s,  $\theta = 65.0^\circ$ , and  $h = 6.00$  m.
62. In studying the emission of light, in order to determine the angle at which the intensity is a given value, the equation  $\sin^2 A - 4 \sin A + 1 = 0$  must be solved. Find angle  $A$  (to  $0.1^\circ$ ). ( $\sin^2 A = (\sin A)^2$ )
63. A computer analysis shows that the number  $n$  of electronic components a company should produce for supply to equal demand is found by solving  $\frac{n^2}{500\,000} = 144 - \frac{n}{500}$ . Find  $n$ .
64. To determine the resistances of two resistors that are to be in parallel in an electric circuit, it is necessary to solve the equation  $\frac{20}{R} + \frac{20}{R+10} = \frac{1}{5}$ . Find  $R$  (to nearest  $1 \Omega$ ).
65. In designing a cylindrical container, the formula  $A = 2\pi r^2 + 2\pi rh$  (Eq. 2.19) is used. Solve for  $r$ .
66. In determining the number of bytes  $b$  that can be stored on a hard disk, the equation  $b = kr(R - r)$  is used. Solve for  $r$ .

67. In the study of population growth, the equation  $p_2 = p_1 + rp_1(1 - p_1)$  occurs. Solve for  $p_1$ .
68. In the study of the velocities of deep-water waves, the equation  $v^2 = k^2\left(\frac{L}{C} + \frac{C}{L}\right)$  occurs. Solve for  $L$ .

In Exercises 69–82, set up the necessary equation where appropriate and solve the given problems. All numbers are accurate to at least 2 significant digits.

69. In testing the effects of a drug, the percent of the drug in the blood was given by  $p = 0.090t - 0.015t^2$ , where  $t$  is the time (in h) after the drug was administered. Sketch the graph of  $p = f(t)$ .
70. In an electric circuit, the voltage  $V$  as a function of the time  $t$  (in min) is given by  $V = 9.8 - 9.2t + 2.3t^2$ . Sketch the graph of  $V = f(t)$ , for  $t \leq 5$  min.
71. By adding the same amount to its length and its width, a developer increased the area of a rectangular lot by  $3000 \text{ m}^2$  to make it 80 m by 100 m. What were the original dimensions of the lot? See Fig. 7.17.

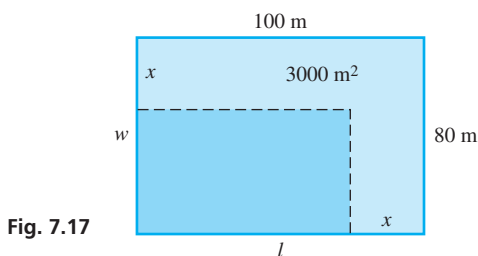


Fig. 7.17

72. A thick-walled, cylindrical pipe of mass  $m$ , outer radius  $r$ , and thickness  $t$  is to be rotated along its axis of symmetry. The pipe possesses a resistance to a change in its rotation called moment of inertia  $I$ , given by  $I = mr^2\left(1 - \frac{t}{r} + \frac{t^2}{2r^2}\right)$ . A particular pipe is measured to have a mass of 8.00 kg, an outer radius of 0.250 m, and a moment of inertia  $0.410 \text{ kg} \cdot \text{m}^2$ . Determine the thickness of the wall of the pipe.
73. Concrete contracts as it dries. If the volume of a cubical concrete block is  $29.0 \text{ cm}^3$  less and each edge is 0.100 cm less after drying, what was the original length of an edge of the block?
74. A jet flew 1250 km with a tailwind of 50.0 km/h. The tailwind then changed to 20.0 km/h for the remaining 575 km of the flight. If the total time of the flight was 3.00 h, find the speed of the jet relative to the air.
75. The length of one rectangular field is 400 m more than the side of a square field. The width is 100 m more than the side of the square field. If the rectangular field has twice the area of the square field, what are the dimensions of each field?
76. A military jet flies directly over and at right angles to the straight course of a commercial jet. The military jet is flying at 200 km/h faster than four times the speed of the commercial jet. How fast is each going if they are 2050 km apart (on a direct line) after 1.00 h?

77. The width of a rectangular TV screen is 14.5 cm more than the height. If the diagonal is 68.6 cm, find the dimensions of the screen. See Fig. 7.18.

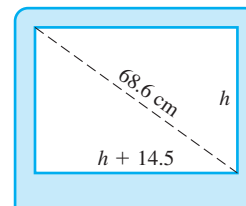


Fig. 7.18

78. Find the exact value of  $x$  that is defined in terms of the *continuing fraction* at the right (the pattern continues endlessly). Explain your method of solution.
- $$x = 2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \dots}}}$$
79. An electric utility company is placing utility poles along a road. It is determined that five fewer poles per kilometre would be necessary if the distance between poles were increased by 10 m. How many poles are being placed each kilometre?
80. An architect is designing a Norman window (a semicircular part over a rectangular part) as shown in Fig. 7.19. If the area of the window is to be  $1.75 \text{ m}^2$  and the height of the rectangular part is 1.35 m, find the radius of the circular part.

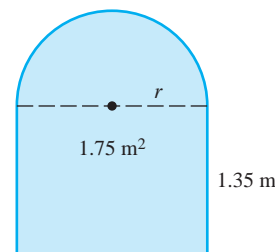


Fig. 7.19

81. A testing station found  $p$  parts per million (ppm) of sulfur dioxide in the air as a function of the hour  $h$  of the day to be  $p = 0.00174(10 + 24h - h^2)$ . Sketch the graph of  $p = f(h)$  and, from the graph, find the time when  $p = 0.205$  ppm.
82. A compact disc (CD) is made such that it is 53.0 mm from the edge of the centre hole to the edge of the disc. Find the radius of the hole if 1.36% of the disc is removed in making the hole. See Fig. 7.20.

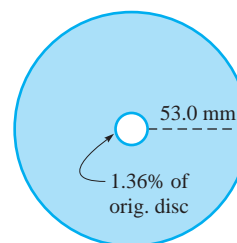


Fig. 7.20

### Writing Exercise

83. An electronics student is asked to solve the equation  $\frac{1}{2} = \frac{1}{R} + \frac{1}{R+1}$  for  $R$ . Write one or two paragraphs explaining your procedure for the solution, including a discussion of what methods of the chapter may be used in completing the solution, assuming  $R > 0$ .

## CHAPTER 7 PRACTICE TEST

1. Solve by factoring:  $2x^2 + 5x = 12$
2. Solve by using the quadratic formula:  $x^2 = 3x + 5$
3. Solve graphically using a calculator:  $4x^2 - 5x - 3 = 0$
4. Solve algebraically:  $2x^2 - x = 6 - 2x(3 - x)$
5. Solve algebraically:  $\frac{3}{x} - \frac{2}{x+2} = 1$
6. Sketch the graph of  $y = 2x^2 + 8x + 5$  using the extreme point and the  $y$ -intercept.
7. In electricity, the formula  $P = EI - RI^2$  is used. Solve for  $I$  in terms of  $E$ ,  $P$ , and  $R$ .
8. Solve by completing the square:  $x^2 - 6x - 9 = 0$ .
9. The perimeter of a rectangular window is 8.4 m, and its area is  $3.8 \text{ m}^2$ . Find its dimensions.
10. If  $y = x^2 - 8x + 8$ , sketch the graph using the vertex and any other useful points and find graphically the values of  $x$  for which  $y = 0$ .



# 8

## LEARNING OUTCOMES

After completion of this chapter, the student should be able to:

- Determine the magnitude and sign of any trigonometric function of any angle
- Identify reference angles and use them to solve for trigonometric functions of angles in any quadrant
- Express an angle in degrees or radians and convert between the two measurements
- Given the value of a trigonometric function of an angle, find the angle
- Use radian measure of angle to find arc length
- Solve problems involving area of a sector of a circle
- Solve problems involving angular velocity
- Solve application problems involving trigonometric functions of any angle

► Gears and pulleys are common elements in the design of mechanical systems. In Section 8.4, we will see how radian measure of an angle is used to analyse the rotational properties of gears and pulleys.

# Trigonometric Functions of Any Angle

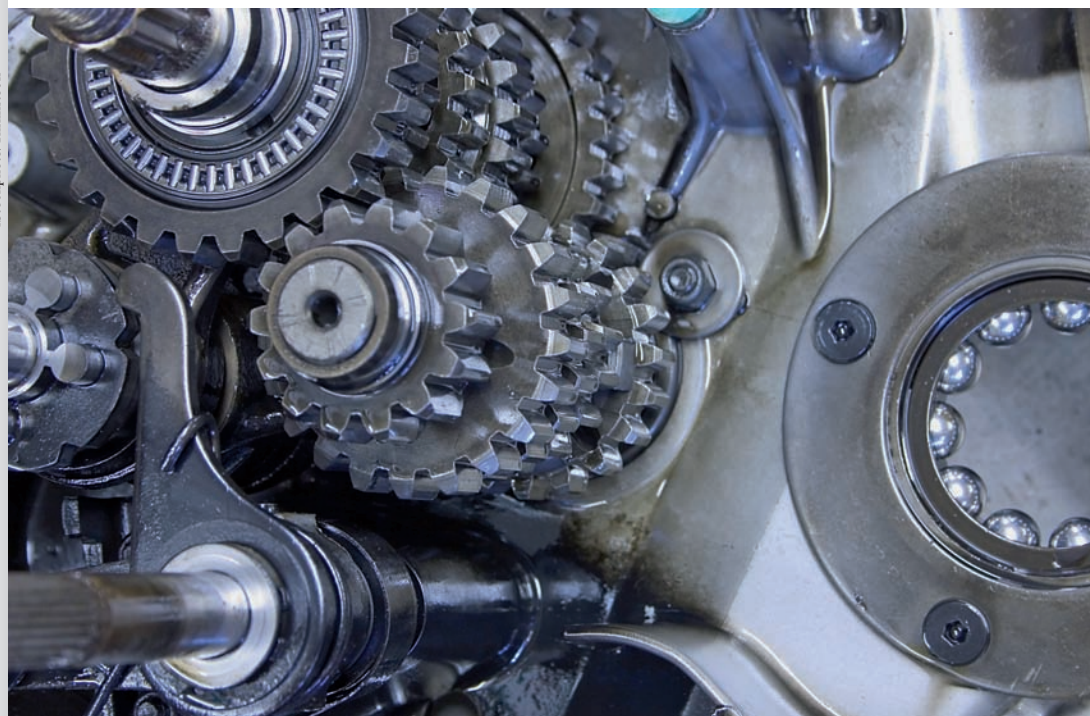
**T**rigonometric functions were introduced in Chapter 4 as ratios of sides in a right triangle, and although defined generally, the values of the functions were computed only for acute angles. In this chapter, we show how these functions are used with angles of any size and with angles measured in radians.

By the mid-1700s, the trigonometric functions had been used for many years as ratios. It was also known that they were useful in describing the behaviour of periodic functions (functions for which values repeat at specific intervals) without reference to actual triangles. In about 1750, this led Swiss mathematician Leonhard Euler to include, for the first time in a textbook, the trigonometric functions of numbers (not angles). As will be shown in this chapter, this is equivalent to using these functions with arguments of angles of any size measured in radians.

Euler was one of the most prolific mathematicians in history. Although he was blind for the last 17 years of his life, this had little effect on his productivity. Euler wrote more than 70 volumes in mathematics and related fields, such as astronomy, classical mechanics, music, optics, and fluids. Euler was the first to use the notation  $f(x)$  to signify a function  $f$  being applied to an argument  $x$  (see Chapter 3). He also introduced the modern notation for the trigonometric functions (see Chapter 4), was a pioneer of the methods of calculus (see Chapters 23–28), and is well known for his development of power series (see Chapters 19 and 30) and for his contributions to the study of complex numbers (see Chapter 12). Noted as one of the greatest mathematicians of all time, Euler's interest in applied subjects often led him to study and develop topics in mathematics that were used in these applications.

Today, trigonometric functions of numbers are important to many applications, such as electric circuits, mechanical vibrations, and rotational motion. These functions developed by Euler and others were critical to the development of electronics in the 1900s. Some electronics applications will be discussed in this chapter and in Chapter 10.

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## 8.1 Signs of the Trigonometric Functions

### Signs of the Trigonometric Functions in Each of the Four Quadrants • Evaluating Functions Knowing Point on Terminal Side

Recall the definitions of the trigonometric functions that were given in Section 4.2. Here, the point  $(x, y)$  is a point on the terminal side of angle  $\theta$ , and  $r$  is the radius vector. See Fig. 8.1.

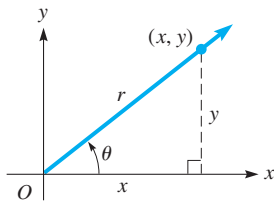


Fig. 8.1

#### Trigonometric Functions

$$\begin{aligned} \sin \theta &= \frac{y}{r} & \csc \theta &= \frac{1}{\sin \theta} = \frac{r}{y} \\ \cos \theta &= \frac{x}{r} & \sec \theta &= \frac{1}{\cos \theta} = \frac{r}{x} \\ \tan \theta &= \frac{y}{x} & \cot \theta &= \frac{1}{\tan \theta} = \frac{x}{y} \end{aligned} \quad (8.1)$$

As noted before, these definitions are valid for a standard-position angle of any size. In this section, we determine the *sign* of each function of angles with terminal sides in each of the four quadrants.

We can find the values of the functions if we know the coordinates  $(x, y)$  on the terminal side and the radius vector  $r$ . Since  $r$  is always taken to be positive, the functions will vary in sign, depending on the values of  $x$  and  $y$ . If either  $x$  or  $y$  is zero in the denominator, the function is undefined. We will consider this in the next section.

Since  $\sin \theta = y/r$ , the sign of  $\sin \theta$  depends on the sign of  $y$ . Since  $y > 0$  in the first and second quadrants and  $y < 0$  in the third and fourth quadrants,  $\sin \theta$  is positive if the terminal side is in the first or second quadrant, and  $\sin \theta$  is negative if the terminal side is in the third or fourth quadrant. See Fig. 8.2.

	I	II	III	IV
$\sin \theta$	+	+	-	-

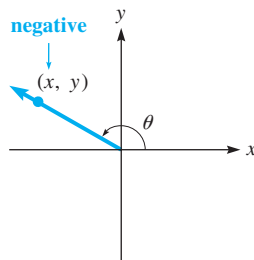
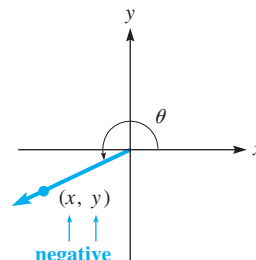
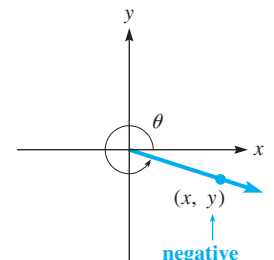


Fig. 8.2

(a)



(b)



(c)

#### EXAMPLE 1 Sign of $\sin \theta$ in each quadrant

The value of  $\sin 20^\circ$  is positive, because the terminal side of  $20^\circ$  is in the first quadrant, where  $y > 0$ . The value of  $\sin 160^\circ$  is positive, because the terminal side of  $160^\circ$  is in the second quadrant, where  $y > 0$ . The values of  $\sin 200^\circ$  and  $\sin 340^\circ$  are negative, because the terminal sides are in the third and fourth quadrants, respectively, where  $y < 0$ . ■

	I	II	III	IV
$\tan \theta$	+	-	+	-

Since  $\tan \theta = y/x$ , the sign of  $\tan \theta$  depends on the ratio of  $y$  to  $x$ . Since  $x$  and  $y$  are both positive in the first quadrant, both negative in the third quadrant, and have different signs in the second and fourth quadrants,  $\tan \theta$  is positive if the terminal side is in the first or third quadrant, and  $\tan \theta$  is negative if the terminal side is in the second or fourth quadrant. See Fig. 8.2.

#### EXAMPLE 2 Sign of $\tan \theta$ in each quadrant

The values of  $\tan 20^\circ$  and  $\tan 200^\circ$  are positive, because the terminal sides of these angles are in the first and third quadrants, respectively. In these quadrants, the ratio of  $y$  to  $x$  is always positive (the ratio of two positive numbers in Quadrant I, or the ratio of two negative numbers in Quadrant III). The values of  $\tan 160^\circ$  and  $\tan 340^\circ$  are negative, because the terminal sides of these angles are in the second and fourth quadrants, respectively. In these quadrants, the ratio of  $y$  to  $x$  is always negative (the ratio of one positive to one negative number in both Quadrant II and Quadrant IV). ■



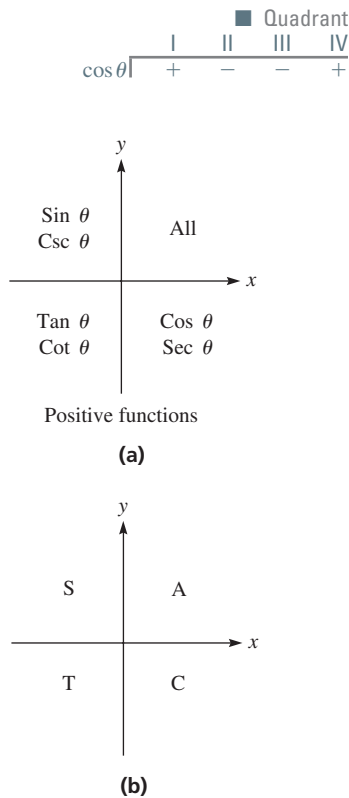


Fig. 8.3

Since  $\cos \theta = x/r$ , the sign of  $\cos \theta$  depends on the sign of  $x$ . Since  $x > 0$  in the first and fourth quadrants and  $x < 0$  in the second and third quadrants,  $\cos \theta$  is positive if the terminal side is in the first or fourth quadrant, and  $\cos \theta$  is negative if the terminal side is in the second or third quadrant. See Fig. 8.2.

**EXAMPLE 3** Sign of  $\cos \theta$  in each quadrant

The values of  $\cos 20^\circ$  and  $\cos 340^\circ$  are positive, because the terminal sides of these angles are in the first and fourth quadrants, respectively, where  $x > 0$ . The values of  $\cos 160^\circ$  and  $\cos 200^\circ$  are negative, because the terminal sides of these angles are in the second and third quadrants, respectively, where  $x < 0$ .

Since  $\csc \theta$  is defined in terms of  $y$  and  $r$ , as in  $\sin \theta$ ,  $\csc \theta$  has the same sign as  $\sin \theta$ . For similar reasons,  $\cot \theta$  has the same sign as  $\tan \theta$ , and  $\sec \theta$  has the same sign as  $\cos \theta$ . Therefore, as shown in Fig. 8.3(a),

**All functions of first-quadrant angles are positive. Sin  $\theta$  and csc  $\theta$  are positive for second-quadrant angles. Tan  $\theta$  and cot  $\theta$  are positive for third-quadrant angles. Cos  $\theta$  and sec  $\theta$  are positive for fourth-quadrant angles. All others are negative.**

A useful mnemonic to remember the signs of the trigonometric functions is the word CAST, spelled counterclockwise from Quadrant IV. Each letter represents the basic trigonometric function that is positive in that quadrant. C = Cosine positive (and its reciprocal), A = All positive (and their reciprocals), S = Sine positive (and its reciprocal), and T = Tangent positive (and its reciprocal). This is shown in Fig. 8.3(b).

If you don't like the idea of starting in Quadrant IV, start in Quadrant I and make your own sentence. A common one is "All Students Take Calculus." Use whichever method works for you.

This discussion does not include the *quadrantal angles*, those angles with terminal sides on one of the axes. They will be discussed in the next section.

**Practice Exercise**

- Determine the sign of the given functions: (a)  $\sin 140^\circ$  (b)  $\tan 255^\circ$  (c)  $\sec 175^\circ$

**EXAMPLE 4** Positive and negative functions

- (a) The following are *positive*:

$$\sin 150^\circ \quad \cos 290^\circ \quad \tan 190^\circ \quad \cot 260^\circ \quad \sec 350^\circ \quad \csc 100^\circ$$

- (b) The following are *negative*:

$$\sin 300^\circ \quad \cos 150^\circ \quad \tan 100^\circ \quad \cot 300^\circ \quad \sec 200^\circ \quad \csc 250^\circ$$

**EXAMPLE 5** Sign of function related to terminal side

- If  $\sin \theta > 0$ , then the terminal side of  $\theta$  is in either the first or second quadrant.
- If  $\sec \theta < 0$ , then the terminal side of  $\theta$  is in either the second or third quadrant.
- If  $\cos \theta > 0$ , and  $\tan \theta < 0$ , then the terminal side of  $\theta$  is in the fourth quadrant. Only in the fourth quadrant are both signs correct.

**EXAMPLE 6** Evaluating functions

Determine the trigonometric functions of  $\theta$  to 3 significant digits if the terminal side of  $\theta$  passes through  $(-1, \sqrt{3})$ . See Fig. 8.4.

We know that  $x = -1$ ,  $y = \sqrt{3}$ , and from the Pythagorean theorem, we find that  $r = 2$ . Therefore, the trigonometric functions of  $\theta$  are

$$\begin{aligned} \sin \theta &= \frac{\sqrt{3}}{2} = 0.866 & \cos \theta &= -\frac{1}{2} = -0.500 & \tan \theta &= -\sqrt{3} = -1.73 \\ \cot \theta &= -\frac{1}{\sqrt{3}} = -0.577 & \sec \theta &= -2 = -2.00 & \csc \theta &= \frac{2}{\sqrt{3}} = 1.16 \end{aligned}$$

The point  $(-1, \sqrt{3})$  is in the second quadrant, and the signs of the functions of  $\theta$  are those of a second-quadrant angle.

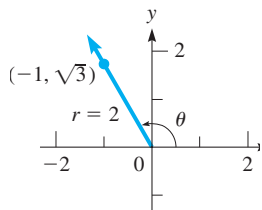


Fig. 8.4

**Practice Exercise**

- Determine the value of  $\cos \theta$  if the terminal side of  $\theta$  passes through  $(-1, 4)$ .

When dealing with negative angles, or angles greater than  $360^\circ$ , the sign of the function is still determined by the location of the terminal side of the angle. For example,  $\sin(-120^\circ) < 0$  since the terminal side of  $-120^\circ$  is in the third quadrant. Also, for the same reason,  $\sin 600^\circ < 0$  (same terminal side as  $-120^\circ$  or  $240^\circ$ ).

## EXERCISES 8.1

In Exercises 1 and 2, answer the given questions about the indicated examples of this section.

- In Example 4, if  $90^\circ$  is added to each angle, what is the sign of each resulting function?
- In Example 6, if the point  $(-1, \sqrt{3})$  is replaced with the point  $(1, -\sqrt{3})$ , what are the resulting values?

In Exercises 3–14, determine the sign of the given functions.

- $\sin 36^\circ$ ,  $\cos 120^\circ$
- $\tan 320^\circ$ ,  $\sec 185^\circ$
- $\csc 98^\circ$ ,  $\cot 82^\circ$
- $\cos 260^\circ$ ,  $\csc 290^\circ$
- $\sec 150^\circ$ ,  $\tan 220^\circ$
- $\sin 335^\circ$ ,  $\cot 265^\circ$
- $\cos 348^\circ$ ,  $\csc 238^\circ$
- $\cot 110^\circ$ ,  $\sec 309^\circ$
- $\tan 460^\circ$ ,  $\sin(-110^\circ)$
- $\csc(-200^\circ)$ ,  $\cos 550^\circ$
- $\cot(-2^\circ)$ ,  $\cos 710^\circ$
- $\sin 539^\circ$ ,  $\tan(-480^\circ)$

In Exercises 15–22, find the trigonometric functions of  $\theta$  if the terminal side of  $\theta$  passes through the given point.

- $(2, 1)$
- $(-1, 1)$
- $(-2, -3)$
- $(4, -3)$
- $(-0.5, 1.2)$
- $(-9, -12)$
- $(20, -8)$
- $(0.9, 4)$

In Exercises 23–28, for the given values, determine the quadrant(s) in which the terminal side of the angle lies.

- $\sin \theta = 0.500$
- $\cos \theta = 0.866$
- $\tan \theta = 1.50$
- $\sin \theta = -0.866$
- $\cos \theta = -0.500$
- $\tan \theta = -0.750$

In Exercises 29–38, determine the quadrant in which the terminal side of  $\theta$  lies, subject to both given conditions.

- $\sin \theta > 0$ ,  $\cos \theta < 0$
- $\tan \theta > 0$ ,  $\cos \theta < 0$
- $\sec \theta < 0$ ,  $\cot \theta < 0$
- $\cos \theta > 0$ ,  $\csc \theta < 0$
- $\csc \theta < 0$ ,  $\tan \theta < 0$
- $\tan \theta < 0$ ,  $\cos \theta > 0$
- $\sin \theta < 0$ ,  $\tan \theta > 0$
- $\sec \theta > 0$ ,  $\csc \theta < 0$
- $\sin \theta > 0$ ,  $\cot \theta < 0$
- $\tan \theta > 0$ ,  $\csc \theta < 0$

In Exercises 39–42, with  $(x, y)$  in the given quadrant, determine whether the given ratio is positive or negative.

- III,  $\frac{x}{r}$
- II,  $\frac{y}{r}$
- IV,  $\frac{y}{x}$
- III,  $\frac{x}{y}$

### Answers to Practice Exercises

- (a) + (b) + (c) -
- $-1/\sqrt{17}$

## 8.2 Trigonometric Functions of Any Angle

### Reference Angle • Evaluating Functions • Quadrantal Angles • Negative Angles

Any negative angle or angle greater than  $360^\circ$  is coterminal with a positive angle less than  $360^\circ$ . Since the terminal sides of coterminal angles are the same, the trigonometric functions of coterminal angles are equal. Therefore, we can evaluate the functions of an angle of any size if we can find the trigonometric functions of  $0^\circ$  and positive angles less than  $360^\circ$ . Such values will be needed for vectors, oblique triangles, and graphs in Chapters 9 and 10.

### EXAMPLE 1 Coterminal angles

The following pairs of angles are coterminal:

$$390^\circ \text{ and } 30^\circ \quad 900^\circ \text{ and } 180^\circ \quad -150^\circ \text{ and } 210^\circ \text{ (see Fig. 8.5)}$$

The trigonometric functions of both angles in each pair are equal. For example,

$$\sin 390^\circ = \sin 30^\circ \quad \text{and} \quad \tan(-150^\circ) = \tan 210^\circ$$

The values of the functions depend on the values of  $x$ ,  $y$ , and  $r$ . Therefore, the absolute value of a function of a second-quadrant angle equals the value of the same function of a first-quadrant angle. For example, in Fig. 8.6,  $\tan \theta_2 = -4/3$ , or  $|\tan \theta_2| = 4/3$ , and  $\tan \theta_1 = 4/3$ . This means  $|\tan \theta_2| = \tan \theta_1$ . The triangles with  $\theta_1$  and  $\theta_{\text{ref}}$  are congruent, which means  $\theta_1 = \theta_{\text{ref}}$ . If  $F$  represents any of the trigonometric functions, knowing that the absolute value of a function of  $\theta_2$  equals the value of the same function of  $\theta_1$  means that

$$|F(\theta_2)| = |F(\theta_1)| = |F(\theta_{\text{ref}})| \quad (8.2)$$

The angle labelled  $\theta_{\text{ref}}$  is called the reference angle. The reference angle of a given angle is the acute angle formed by the terminal side of the angle and the  $x$ -axis.

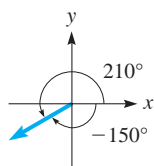


Fig. 8.5

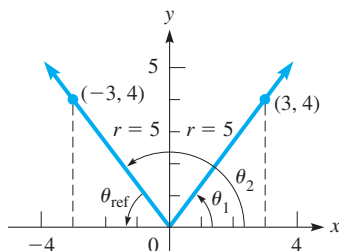


Fig. 8.6

**LEARNING TIP**

The sign used (+ or -) depends on the sign the function has in the quadrant for the original angle. Since all trigonometric functions are positive in the first quadrant, by finding an acute reference angle the trigonometric function of the original angle will be the same magnitude as the trigonometric function evaluated at the reference angle, but with the sign appropriate to the function in that quadrant.

Using Eq. (8.2) and the fact that  $\theta_{\text{ref}} = 180^\circ - \theta_2$ , we may conclude that the value of any trigonometric function of any second-quadrant angle is found from

$$F(\theta_2) = \pm F(180^\circ - \theta_2) = \pm F(\theta_{\text{ref}}) \tag{8.3}$$

**EXAMPLE 2** Quadrant II reference angles

In Quadrant II,  $\sin \theta$  and  $\csc \theta$  are (+), but  $\cos \theta$ ,  $\sec \theta$ ,  $\tan \theta$ , and  $\cot \theta$  are (-). Therefore, the trigonometric functions of  $\theta_2$  in Fig. 8.6 are as follows (in exact form and to 3 significant digits):

$$\begin{aligned} \sin \theta_2 &= \sin(180 - \theta_2) = \sin \theta_{\text{ref}} = \sin \theta_1 = \frac{4}{5} = 0.800 \\ \cos \theta_2 &= -\cos(180 - \theta_2) = -\cos \theta_{\text{ref}} = -\cos \theta_1 = -\frac{3}{5} = -0.600 \\ \tan \theta_2 &= -\tan(180 - \theta_2) = -\tan \theta_{\text{ref}} = -\tan \theta_1 = -\frac{4}{3} = -1.33 \\ \csc \theta_2 &= \csc(180 - \theta_2) = \csc \theta_{\text{ref}} = \csc \theta_1 = \frac{5}{4} = 1.25 \\ \sec \theta_2 &= -\sec(180 - \theta_2) = -\sec \theta_{\text{ref}} = -\sec \theta_1 = -\frac{5}{3} = -1.67 \\ \cot \theta_2 &= -\cot(180 - \theta_2) = -\cot \theta_{\text{ref}} = -\cot \theta_1 = -\frac{3}{4} = -0.750 \end{aligned}$$

In the same way, we derive the formulas for trigonometric functions of any third- or fourth-quadrant angle. In Fig. 8.7, the reference angle  $\theta_{\text{ref}}$  is found by subtracting  $180^\circ$  from  $\theta_3$ , and the functions of  $\theta_{\text{ref}}$  and  $\theta_1$  are numerically equal. In Fig. 8.8, the reference angle  $\theta_{\text{ref}}$  is found by subtracting  $\theta_4$  from  $360^\circ$ .

$$F(\theta_3) = \pm F(\theta_3 - 180^\circ) = \pm F(\theta_{\text{ref}}) \tag{8.4}$$

$$F(\theta_4) = \pm F(360^\circ - \theta_4) = \pm F(\theta_{\text{ref}}) \tag{8.5}$$

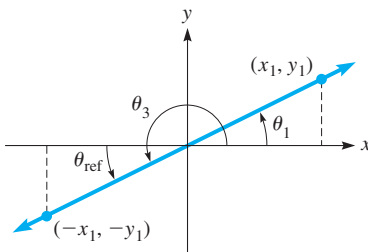


Fig. 8.7

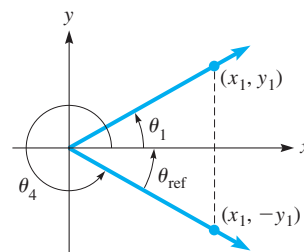


Fig. 8.8

**EXAMPLE 3** Quadrant III reference angles

The trigonometric functions of  $\theta_3 = 210^\circ$  can be expressed in terms of the reference angle of  $30^\circ$  (see Fig. 8.9), and then evaluated, as follows (to 3 significant digits):

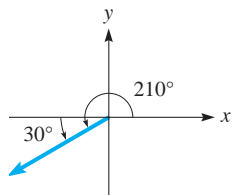


Fig. 8.9

$$\begin{aligned} \sin 210^\circ &= -\sin(210^\circ - 180^\circ) = -\sin 30^\circ = -\frac{1}{2} = -0.500 \\ \csc 210^\circ &= -\csc 30^\circ = -2.00 \\ \cos 210^\circ &= -\cos 30^\circ = -0.866 \\ \sec 210^\circ &= -\sec 30^\circ = -1.16 \\ \tan 210^\circ &= +\tan 30^\circ = +0.577 \\ \cot 210^\circ &= +\cot 30^\circ = +1.73 \end{aligned}$$

Annotations: "same function" and "reference angle" point to the top two equations. "quadrant III" points to the signs in the first two equations. "sin theta, csc theta are (-) in Quadrant III" points to the negative signs. "cos theta, sec theta are (-) in Quadrant III" points to the negative signs in the third and fourth equations. "tan theta, cot theta are (+) in Quadrant III" points to the positive signs in the last two equations.

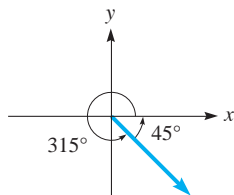


Fig. 8.10

**EXAMPLE 4** Quadrant IV reference angles

If  $\theta_4 = 315^\circ$ , the functions of  $\theta_4$  are found to 3 significant digits by using Eq. (8.5) as follows. See Fig. 8.10.

$$\begin{array}{l} \text{reference angle} \\ \sin 315^\circ = -\sin(360^\circ - 315^\circ) = -\sin 45^\circ = -0.707 \\ \text{quadrant IV} \quad \uparrow \quad \text{sin } \theta, \text{ csc } \theta \text{ are } (-) \text{ in Quadrant IV} \\ \text{csc } 315^\circ = -\text{csc } 45^\circ = -1.41 \\ \cos 315^\circ = +\cos 45^\circ = +0.707 \quad \text{sec } 315^\circ = +\sec 45^\circ = +1.41 \\ \text{cos } \theta, \text{ sec } \theta \text{ are } (+) \text{ in Quadrant IV} \\ \tan 315^\circ = -\tan 45^\circ = -1.00 \quad \cot 315^\circ = -\cot 45^\circ = -1.00 \\ \text{tan } \theta, \text{ cot } \theta \text{ are } (-) \text{ in Quadrant IV} \end{array}$$

**Practice Exercise**

- Express each of the functions of  $225^\circ$  in terms of the reference angle.

**LEARNING TIP**

In most examples, we will round off trigonometric function values to three significant digits (as in Examples 3, 4, and 5). However, *if the angle is approximate* (meaning it is a **measured value**), then we round the trigonometric function values to the same number of significant digits as are in the angle.

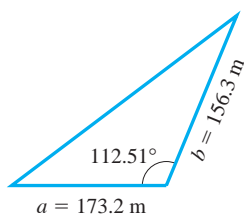


Fig. 8.11

**EXAMPLE 5** Evaluating trigonometric functions using reference angles

$$\begin{array}{l} \text{same function} \quad \text{reference angle} \\ \sin 160^\circ = +\sin(180^\circ - 160^\circ) = \sin 20^\circ = 0.342 \\ \tan 110^\circ = -\tan(180^\circ - 110^\circ) = -\tan 70^\circ = -2.75 \\ \cos 225^\circ = -\cos(225^\circ - 180^\circ) = -\cos 45^\circ = -0.707 \\ \cot 260^\circ = +\cot(260^\circ - 180^\circ) = \cot 80^\circ = 0.176 \\ \sec 304^\circ = +\sec(360^\circ - 304^\circ) = \sec 56^\circ = 1.79 \\ \sin 357^\circ = -\sin(360^\circ - 357^\circ) = -\sin 3^\circ = -0.0523 \\ \text{determines quadrant} \quad \text{proper sign for function in quadrant} \end{array}$$

A calculator will give values, with the proper signs, of functions like those in Examples 3, 4, and 5. The reciprocal key is used for  $\cot \theta$ ,  $\sec \theta$ , and  $\csc \theta$ , as shown in Section 4.2.

**EXAMPLE 6** Quadrant II angle—application

A formula for finding the area of a triangle, knowing sides  $a$  and  $b$  and the included  $\angle C$ , is  $A = \frac{1}{2}ab \sin C$ . A surveyor uses this formula to find the area of a triangular tract of land for which  $a = 173.2$  m,  $b = 156.3$  m, and  $C = 112.51^\circ$ . See Fig. 8.11.

$$\begin{aligned} A &= \frac{1}{2} (173.2) (156.3) \sin 112.51^\circ \\ &= 12\,500 \text{ m}^2 \quad \text{rounded to four significant digits} \end{aligned}$$

The calculator automatically uses a positive value for  $\sin 112.51^\circ$ .

**COMMON ERROR**

Knowing how to use the reference angle is important when using a calculator to solve for an angle, because *for a given value of a trigonometric function, there are multiple solutions for the angle*. There will always be two solutions between  $0^\circ$  and  $360^\circ$ , but an infinite number of solutions are possible for any integer number of full rotations from those two angles (giving identical terminal arms from each new full rotation).

**The calculator will not necessarily give the required angle.**

It will give an angle we can use, but whether or not it is the required angle for the problem will depend on the problem being solved, and upon the quadrant in which the desired angle lies.

■ The reason that the calculator displays these particular angles is shown in Chapter 20, when the inverse trigonometric functions are discussed in detail.

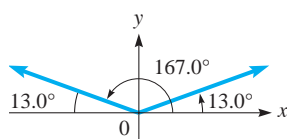


Fig. 8.12

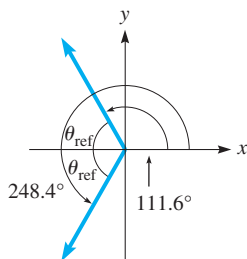


Fig. 8.13

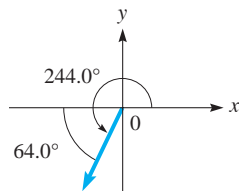


Fig. 8.14

#### Practice Exercise

2. If  $\cos \theta = 0.5736$ , find  $\theta$  for  $0^\circ \leq \theta < 360^\circ$ .

#### LEARNING TIP

The calculator gives the reference angle (disregarding any minus signs) in all cases except when  $\cos \theta$  is negative. To avoid confusion from the angle displayed by the calculator, a good procedure is to find the reference angle first. Then it can be used to determine the angle required by the problem.

When a value of a trigonometric function is entered into a calculator, it is programmed to solve the angle as follows:

For values of  $\sin \theta$ , the calculator displays angles from  $-90^\circ$  to  $90^\circ$ :

$$[\sin(-90^\circ) = -1, \sin 0^\circ = 0, \sin 90^\circ = 1], \theta \text{ from } [-90^\circ, 90^\circ]$$

For values of  $\cos \theta$ , the calculator displays angles from  $0^\circ$  to  $180^\circ$ :

$$[\cos 0^\circ = 1, \cos 90^\circ = 0, \cos 180^\circ = -1], \theta \text{ from } [0^\circ, 180^\circ]$$

For values of  $\tan \theta$ , the calculator displays angles **between**  $-90^\circ$  and  $90^\circ$ :

$$[\tan 0^\circ = 0, \text{ if } \tan \theta < 0, \theta \text{ is between } -90^\circ \text{ and } 0^\circ], \theta \text{ from } (-90^\circ, 90^\circ)$$

This means that if  $\sin \theta$  or  $\tan \theta$  is negative, the displayed angle is negative. If  $\cos \theta$  is negative, the displayed angle is greater than  $90^\circ$  and no greater than  $180^\circ$ .

#### EXAMPLE 7 Finding angles given $\sin \theta$

If  $\sin \theta = 0.225$ , we have  $\theta = \sin^{-1}(0.225) = 13.0^\circ$  (rounded off).

This result is correct, but remember that

$$\begin{aligned} \sin(180^\circ - 13.0^\circ) &= \sin 167.0^\circ \\ &= 0.225 \end{aligned}$$

also. If we need only an acute angle,  $\theta = 13.0^\circ$  is correct. However, if a second-quadrant angle is required, we see that  $\theta = 167.0^\circ$  is the angle (see Fig. 8.12). These can be checked by finding the values of  $\sin 13.0^\circ$  and  $\sin 167.0^\circ$ . ■

#### EXAMPLE 8 Finding angles given $\sec \theta$

For  $\sec \theta = -2.722$  and  $0^\circ \leq \theta < 360^\circ$  (this means  $\theta$  may equal  $0^\circ$  or be between  $0^\circ$  and  $360^\circ$ ), we have  $\theta = \cos^{-1}\left(\frac{-1}{2.722}\right) = 111.6^\circ$  (rounded off).

The angle  $111.6^\circ$  is the second-quadrant angle, but  $\sec \theta < 0$  in the third quadrant as well. The reference angle is  $\theta_{\text{ref}} = 180^\circ - 111.6^\circ = 68.4^\circ$ , and the third quadrant angle is  $180^\circ + 68.4^\circ = 248.4^\circ$ . Therefore, the two angles between  $0^\circ$  and  $360^\circ$  for which  $\sec \theta = -2.722$  are  $111.6^\circ$  and  $248.4^\circ$  (see Fig. 8.13). These angles can be checked by finding  $\sec 111.6^\circ$  and  $\sec 248.4^\circ$ . ■

#### EXAMPLE 9 Finding angles given $\tan \theta$

Given that  $\tan \theta = 2.05$  and  $\cos \theta < 0$ , find  $\theta$  for  $0^\circ \leq \theta < 360^\circ$ .

Since  $\tan \theta$  is positive and  $\cos \theta$  is negative,  $\theta$  must be a third-quadrant angle. A calculator will display an angle of  $64.0^\circ$  (rounded off) for  $\tan \theta = 2.05$ . However, since we need a third-quadrant angle, we must add  $64.0^\circ$  to  $180^\circ$ . Thus, the required angle is  $244.0^\circ$  (see Fig. 8.14). Check by finding  $\tan 244.0^\circ$ .

If  $\tan \theta = -2.05$  and  $\cos \theta < 0$ , the calculator will display an angle of  $-64.0^\circ$  for  $\tan \theta = -2.05$ . We would then have to recognize that the reference angle is  $64.0^\circ$  and subtract it from  $180^\circ$  to get  $116.0^\circ$ , the required second-quadrant angle. This can be checked by finding  $\tan 116.0^\circ$ . ■

We can find the reference angle by entering the absolute value of the function. The displayed angle will be the reference angle. The required angle  $\theta$  is found by using the reference angle as described earlier and as shown in Eq. (8.6).

$$\begin{aligned} \theta &= \theta_{\text{ref}} && \text{(first quadrant)} \\ \theta &= 180^\circ - \theta_{\text{ref}} && \text{(second quadrant)} \\ \theta &= 180^\circ + \theta_{\text{ref}} && \text{(third quadrant)} \\ \theta &= 360^\circ - \theta_{\text{ref}} && \text{(fourth quadrant)} \end{aligned} \tag{8.6}$$

**EXAMPLE 10** Using the reference angle

Given that  $\cos \theta = -0.1298$ , find  $\theta$  for  $0^\circ \leq \theta < 360^\circ$ .

Since  $\cos \theta$  is negative,  $\theta$  is either a second-quadrant angle or a third-quadrant angle. Using 0.1298, the calculator tells us that the reference angle is  $82.54^\circ$ .

For the second-quadrant angle, we subtract  $82.54^\circ$  from  $180^\circ$  to get  $97.46^\circ$ . For the third-quadrant angle, we add  $82.54^\circ$  to  $180^\circ$  to get  $262.54^\circ$ .

Using  $-0.1298$ , the calculator displays the second-quadrant angle of  $97.46^\circ$ . For the third-quadrant angle, we must subtract  $97.46^\circ$  from  $180^\circ$  to get the reference angle  $82.54^\circ$ , which is then added to  $180^\circ$  to get  $262.54^\circ$ . ■

**Quadrantal Angles**

We use Eqs. (8.3), (8.4), and (8.5) when the terminal side is in one of the quadrants. A **quadrantal angle** has its terminal side along one of the axes. Using the definitions of the functions (recalling that  $r > 0$ ), we get the following values for quadrantal angles. These values may be verified by referring to Fig. 8.15.

$\theta$	$\sin \theta$	$\cos \theta$	$\tan \theta$	$\cot \theta$	$\sec \theta$	$\csc \theta$
$0^\circ$	0.000	1.000	0.000	undef.	1.000	undef.
$90^\circ$	1.000	0.000	undef.	0.000	undef.	1.000
$180^\circ$	0.000	-1.000	0.000	undef.	-1.000	undef.
$270^\circ$	-1.000	0.000	undef.	0.000	undef.	-1.000
$360^\circ$	Same as the functions of $0^\circ$ (same terminal side)					

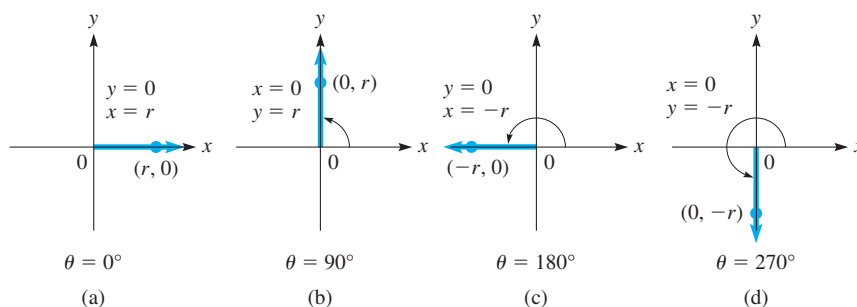


Fig. 8.15

**EXAMPLE 11** Quadrantal angles

Since  $\sin \theta = y/r$ , by looking at Fig. 8.15(a), we can see that  $\sin 0^\circ = 0/r = 0$ .

Since  $\tan \theta = y/x$ , from Fig. 8.15(b), we see that  $\tan 90^\circ = r/0$ , which is undefined due to the division by zero. Using a calculator to find  $\tan 90^\circ$ , the display would indicate an error (due to division by zero).

Since  $\cos \theta = x/r$ , from Fig. 8.15(c), we see that  $\cos 180^\circ = -r/r = -1$ .

Since  $\cot \theta = x/y$ , from Fig. 8.15(d), we see that  $\cot 270^\circ = 0/-r = 0$ . ■

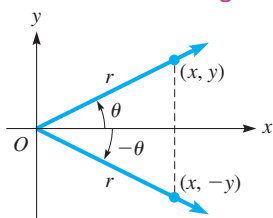
**Negative Angles**

Fig. 8.16

To evaluate functions of negative angles, we can use functions of corresponding positive angles, if we use the correct **sign**. In Fig. 8.16, note that  $\sin \theta = y/r$ , and  $\sin(-\theta) = -y/r$ , which means  $\sin(-\theta) = -\sin \theta$ . In the same way, we can get all of the relations between functions of  $-\theta$  and the functions of  $\theta$ . Therefore, using the definitions, we have

$$\begin{aligned} \sin(-\theta) &= -\sin \theta & \cos(-\theta) &= \cos \theta & \tan(-\theta) &= -\tan \theta \\ \csc(-\theta) &= -\csc \theta & \sec(-\theta) &= \sec \theta & \cot(-\theta) &= -\cot \theta \end{aligned} \quad (8.7)$$



**EXAMPLE 12** Negative angles

Solve the following trigonometric functions of negative angles, to 3 significant digits:

$$\begin{aligned}\sin(-60^\circ) &= -\sin 60^\circ = -0.866 & \cos(-60^\circ) &= \cos 60^\circ = 0.500 \\ \tan(-60^\circ) &= -\tan 60^\circ = -1.73 & \cot(-60^\circ) &= -\cot 60^\circ = -0.577 \\ \sec(-60^\circ) &= \sec 60^\circ = 2.00 & \csc(-60^\circ) &= -\csc 60^\circ = -1.16\end{aligned}$$

**EXERCISES 8.2**

In Exercises 1–4, make the given changes in the indicated examples of this section and then solve the resulting problems.

- In Example 5, add  $40^\circ$  to each angle, express in terms of the same function of a positive acute angle, and then evaluate.
- In Example 7, make the value of the function negative and then find  $\theta$  between  $0^\circ$  and  $360^\circ$ .
- In Example 9, change 2.05 to  $-2.05$  and  $<$  to  $>$ , then find  $\theta$ .
- In Example 10, change the  $-$  to  $+$  and then find  $\theta$ .

In Exercises 5–10, express the given trigonometric function in terms of the same function of a positive acute angle.

- $\sin 160^\circ$ ,  $\cos 220^\circ$
- $\tan 91^\circ$ ,  $\sec 345^\circ$
- $\tan 105^\circ$ ,  $\csc 302^\circ$
- $\cos 190^\circ$ ,  $\cot 290^\circ$
- $\cos 400^\circ$ ,  $\tan(-400^\circ)$
- $\tan 920^\circ$ ,  $\csc(-550^\circ)$

In Exercises 11–44, the given angles are approximate. In Exercises 11–18, find the values of the given trigonometric functions by finding the reference angle and attaching the proper sign.

- $\sin 195^\circ$
- $\tan 311^\circ$
- $\cos 106.3^\circ$
- $\sin 103.4^\circ$
- $\sec 328.33^\circ$
- $\cot 516.53^\circ$
- $\tan(-31.5^\circ)$
- $\csc(-108.4^\circ)$

In Exercises 19–26, find the values of the given trigonometric functions directly from a calculator.

- $\tan 152.4^\circ$
- $\cos 341.4^\circ$
- $\sin 310.36^\circ$
- $\tan 242.68^\circ$
- $\csc 194.82^\circ$
- $\sec 441.08^\circ$
- $\cos(-72.61^\circ)$
- $\sin(-215.5^\circ)$

In Exercises 27–40, find  $\theta$  for  $0^\circ \leq \theta < 360^\circ$ .

- $\sin \theta = -0.8480$
- $\tan \theta = -1.830$
- $\cos \theta = 0.4003$
- $\sin \theta = 0.6374$
- $\cot \theta = -0.0122$
- $\csc \theta = -8.09$
- $\sin \theta = 0.870$ ,  $\cos \theta < 0$
- $\tan \theta = 0.932$ ,  $\sin \theta < 0$
- $\cos \theta = -0.12$ ,  $\tan \theta > 0$
- $\sin \theta = -0.192$ ,  $\tan \theta < 0$
- $\tan \theta = -1.366$ ,  $\cos \theta > 0$
- $\cos \theta = 0.0726$ ,  $\sin \theta < 0$
- $\sec \theta = 2.047$ ,  $\cot \theta < 0$
- $\cot \theta = -0.3256$ ,  $\csc \theta > 0$

In Exercises 41–44, determine the function that satisfies the given conditions.

- Find  $\tan \theta$  when  $\sin \theta = -0.5736$  and  $\cos \theta > 0$ .
- Find  $\sin \theta$  when  $\cos \theta = 0.422$  and  $\tan \theta < 0$ .
- Find  $\cos \theta$  when  $\tan \theta = -0.809$  and  $\csc \theta > 0$ .
- Find  $\cot \theta$  when  $\sec \theta = 6.122$  and  $\sin \theta < 0$ .

In Exercises 45–48, insert the proper sign,  $>$  or  $<$  or  $=$ , between the given expressions. Explain your answers.

- $\sin 90^\circ$   $2 \sin 45^\circ$
- $\cos 360^\circ$   $2 \cos 180^\circ$
- $\tan 180^\circ$   $\tan 0^\circ$
- $\sin 270^\circ$   $3 \sin 90^\circ$

In Exercises 49–52, solve the given problems. In Exercises 51 and 52, assume  $0^\circ < \theta < 90^\circ$ . (Hint: Review cofunctions on page 128.)

- Using the fact that  $\sin 75^\circ = 0.9659$ , evaluate  $\cos 195^\circ$ .
- Using the fact that  $\cot 20^\circ = 2.747$ , evaluate  $\tan 290^\circ$ .
- Express  $\tan(270^\circ - \theta)$  in terms of  $\cot \theta$ .
- Express  $\cos(90^\circ + \theta)$  in terms of  $\sin \theta$ .

In Exercises 53–56, evaluate the given expressions.

- The current  $i$  in an alternating-current circuit is given by  $i = i_m \sin \theta$ , where  $i_m$  is the maximum current in the circuit. Find  $i$  if  $i_m = 0.0259$  A and  $\theta = 495.2^\circ$ .
- The force  $F$  that a rope exerts on a crate is related to force  $F_x$  directed along the  $x$ -axis by  $F = F_x \sec \theta$ , where  $\theta$  is the standard-position angle for  $F$ . See Fig. 8.17. Find  $F$  if  $F_x = -365$  N and  $\theta = 127.0^\circ$ .

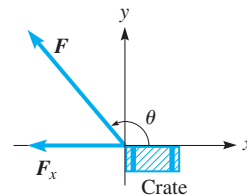


Fig. 8.17

- For the slider mechanism shown in Fig. 8.18,  $y \sin \alpha = x \sin \beta$ . Find  $y$  if  $x = 6.78$  cm,  $\alpha = 31.3^\circ$ , and  $\beta = 104.7^\circ$ .

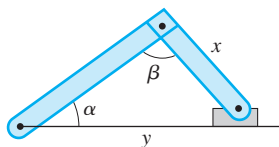


Fig. 8.18

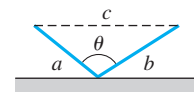


Fig. 8.19

- A laser follows the path shown in Fig. 8.19. The angle  $\theta$  is related to the distances  $a$ ,  $b$ , and  $c$  by  $2ab \cos \theta = a^2 + b^2 - c^2$ . Find  $\theta$  if  $a = 12.9$  cm,  $b = 15.3$  cm, and  $c = 24.6$  cm.

**Answers to Practice Exercises**

- $\sin 225^\circ = -\sin 45^\circ$     $\cos 225^\circ = -\cos 45^\circ$   
 $\tan 225^\circ = \tan 45^\circ$     $\cot 225^\circ = \cot 45^\circ$   
 $\sec 225^\circ = -\sec 45^\circ$     $\csc 225^\circ = -\csc 45^\circ$
- $55.00^\circ, 305.00^\circ$



## 8.3 Radians

### Definition of Radian • Procedure for Converting Angle Measurements • Evaluations on Calculator

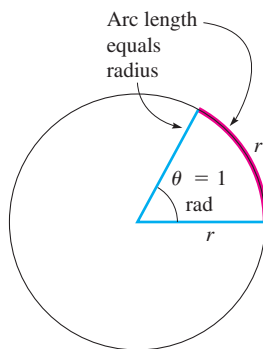


Fig. 8.20

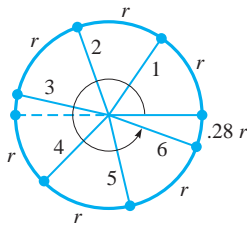


Fig. 8.21

For many problems in which trigonometric functions are used, particularly those involving the solution of triangles, degree measurements of angles are convenient and quite sufficient. However, division of a circle into 360 equal parts is by definition, and it is arbitrary and artificial (see the margin comment on page 56).

In numerous other types of applications and in more theoretical discussions, the *radian* is a more meaningful measure of an angle. We defined the radian in Chapter 2 and reviewed it briefly in Chapter 4. In this section, we discuss the radian in detail and start by reviewing its definition.

**A radian** is the measure of an angle with its vertex at the centre of a circle and with an intercepted arc on the circle equal in length to the radius of the circle. See Fig. 8.20.

Since the circumference of any circle in terms of its radius is given by  $c = 2\pi r$ , the ratio of the circumference to the radius is  $2\pi$ . This means that the radius may be laid out  $2\pi$  (about 6.28) times along the circumference, regardless of the length of the radius. Therefore, note that radian measure is independent of the radius of the circle. The definition of a radian is based on an important property of a circle and is therefore a more natural measure of an angle. In Fig. 8.21, the numbers on each of the radii indicate the number of radians in the angle measured in standard position. The circular arrow shows an angle of 6 radians.

Since the radius may be laid out  $2\pi$  times along the circumference, it follows that there are  $2\pi$  radians in one complete rotation. Also, there are  $360^\circ$  in one complete rotation. Therefore,  $360^\circ$  is equivalent to  $2\pi$  radians. It then follows that the relation between degrees and radians is  $2\pi \text{ rad} = 360^\circ$ , or

$$\pi \text{ rad} = 180^\circ \quad (8.8)$$

Converting Angles

$$1^\circ = \frac{\pi}{180} \text{ rad} = 0.01745 \text{ rad} \quad (8.9)$$

Degrees to Radians

$$1 \text{ rad} = \frac{180^\circ}{\pi} = 57.30^\circ \quad (8.10)$$

Radians to Degrees

From Eqs. (8.8), (8.9), and (8.10), note that we convert angle measurements from degrees to radians or radians to degrees using a procedure identical to that of all unit conversions (see Section 1.3).

### Procedure for Converting Angle Measurements

1. To convert an angle measured in degrees to the same angle measured in radians, **multiply the number of degrees by**  $\left(\frac{\pi \text{ rad}}{180^\circ}\right)$ . Note the degrees in the multiplied factor are in the denominator, which will cancel with the degrees in the original angle.
2. To convert an angle measured in radians to the same angle measured in degrees, **multiply the number of radians by**  $\left(\frac{180^\circ}{\pi \text{ rad}}\right)$ . Note the radians in the multiplied factor are in the denominator, which will cancel with the radians in the original angle.

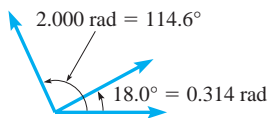


Fig. 8.22

**EXAMPLE 1** Converting to and from radians

(a)  $18.0^\circ = 18.0^\circ \left( \frac{\pi \text{ rad}}{180^\circ} \right) = \frac{\pi}{10.0} \text{ rad} = 0.314 \text{ rad}$

← converting degrees to radians  
↑ degrees cancel (See Fig. 8.22.)

(b)  $2.000 \text{ rad} = 2.000 \text{ rad} \left( \frac{180^\circ}{\pi \text{ rad}} \right) = \frac{360.0^\circ}{\pi} = 114.6^\circ$  (See Fig. 8.22.)

← converting radians to degrees

Multiplying by  $\pi/180^\circ$  or  $180^\circ/\pi$  is actually multiplying by 1, because  $\pi \text{ rad} = 180^\circ$ . The unit of measurement is different, but *the angle is the same*. ■

Because of the definition of the radian, it is common to express radians in terms of  $\pi$ , particularly for angles whose degree measure is a fraction of  $180^\circ$ .

**EXAMPLE 2** Radians in terms of  $\pi$

(a) Converting  $30^\circ$  to radian measure, we have

$$30^\circ = 30^\circ \left( \frac{\pi \text{ rad}}{180^\circ} \right) = \frac{\pi}{6} \text{ rad} \quad (\text{See Fig. 8.23.})$$

(b) Converting  $3\pi/4 \text{ rad}$  to degrees, we have

$$\frac{3\pi}{4} \text{ rad} = \frac{3\pi}{4} \text{ rad} \left( \frac{180^\circ}{\pi \text{ rad}} \right) = 135^\circ \quad (\text{See Fig. 8.23.}) \quad \blacksquare$$

We wish now to make a very important point. Since  $\pi$  is a number (a little greater than 3) that is the ratio of the circumference of a circle to its diameter, it is the ratio of one length to another. This means *radians have no units*, and *radian measure amounts to measuring angles in terms of real numbers*. It is this property that makes radians useful in many applications. If the symbol rad is used, it is only to emphasize that the angle is in radians.

**Practice Exercises**

1. Convert  $36^\circ$  to radians in terms of  $\pi$ .
2. Convert  $7\pi/9$  to degrees.

**LEARNING TIP**

When the angle is measured in radians, it is customary that no units are shown. The radian is understood to be the unit of measurement.

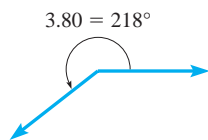


Fig. 8.24

**EXAMPLE 3** No angle units indicates radians

(a)  $60.0^\circ = 60.0^\circ \left( \frac{\pi}{180^\circ} \right) = \frac{\pi}{3.00} = 1.05$       $1.05 = 1.05 \text{ rad}$

(b)  $3.80 = 3.80 \left( \frac{180^\circ}{\pi} \right) = 218^\circ$

← no units indicates radian measure

Because no units are shown for 1.05 and 3.80, they are known to be measured in radians. See Fig. 8.24. ■

■ Using the *angle* feature, a calculator can be used directly to change an angle expressed in degrees to an angle expressed in radians, or from an angle expressed in radians to an angle expressed in degrees (as pointed out in Section 4.1).

We can use a calculator to find the value of a function of an angle in radians. If the calculator is in radian mode, it then uses values in radians directly and will *consider any angle entered to be in radians*. The mode can be changed as needed, but *always be careful to have your calculator in the proper mode*.

**COMMON ERROR**

Check the setting in the *mode* feature on the calculator. When computing the values of trigonometric functions, or when solving for angles using inverse trigonometric functions, **you must know what mode your calculator is in** to ensure that you are using or obtaining the angle with the correct unit. It is a common error to use the values given by a calculator without ensuring the proper mode is set (radians or degrees).

**EXAMPLE 4** Calculator evaluations

- (a) To find the value of  $\sin 0.7538$ , put the calculator in radian mode (note that no units are shown with 0.7538), then evaluate

no units  
indicates  
radian  
measure

$$\sin 0.7538 = 0.6844$$

- (b) Similarly,  $\tan 0.9977$  is found as

$$\tan 0.9977 = 1.550$$

**Practice Exercise**

3. Find the value of  $\sin 3.56$ .

In the following application, the resulting angle is a unitless number, and it is therefore in radian measure.

**EXAMPLE 5** Application of radian measure

The velocity  $v$  of an object undergoing simple harmonic motion at the end of a spring is given by

$$v = A \sqrt{\frac{k}{m}} \cos \left( \sqrt{\frac{k}{m}}(t) \right) \quad \text{the angle is } \left( \sqrt{\frac{k}{m}}(t) \right)$$

Here,  $m$  is the mass of the object (in g),  $k$  is a constant depending on the spring,  $A$  is the maximum distance the object moves, and  $t$  is the time (in s). Find the velocity (in cm/s) after 0.100 s of a 36.0-g object at the end of a spring for which  $k = 400 \text{ g/s}^2$ , if  $A = 5.00 \text{ cm}$ .

Substituting, we have

$$v = (5.00 \text{ cm}) \sqrt{\frac{400 \text{ g/s}^2}{36.0 \text{ g}}} \cos \left( \sqrt{\frac{400 \text{ g/s}^2}{36.0 \text{ g}}}(0.100 \text{ s}) \right)$$

Using calculator memory for  $\sqrt{\frac{400}{36.0}}$ , and with the calculator in radian mode, we have

$$v = 15.7 \text{ cm/s}$$

If we need a reference angle in radians, recall that  $\pi/2 = 90^\circ$ ,  $\pi = 180^\circ$ ,  $3\pi/2 = 270^\circ$ , and  $2\pi = 360^\circ$  (see Fig. 8.25). These and their decimal approximations are shown in Table 8.1.

Table 8.1 Quadrantal Angles

Degrees	Radians	Radians (decimal)
$90^\circ$	$\frac{1}{2} \pi$	1.571
$180^\circ$	$\pi$	3.142
$270^\circ$	$\frac{3}{2} \pi$	4.712
$360^\circ$	$2\pi$	6.283

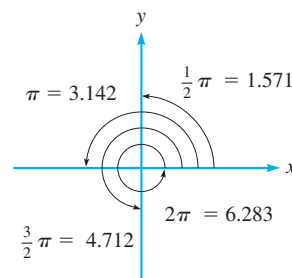


Fig. 8.25

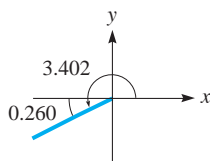


Fig. 8.26

**EXAMPLE 6** Angles of any size in radians and reference angles in radians

An angle of 3.402 is greater than 3.142 but less than 4.712. Thus, it is a third-quadrant angle, and the reference angle is  $3.402 - \pi = 0.260$ . The calculator ( $\pi$ ) key can be used. See Fig. 8.26.

An angle of 5.210 is between 4.712 and 6.283. Therefore, it is in the fourth quadrant, and the reference angle is  $2\pi - 5.210 = 1.073$ .

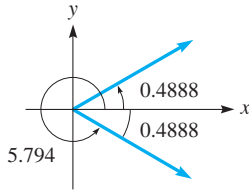


Fig. 8.27

**EXAMPLE 7** Find an angle in radians for a given function

Express  $\theta$  in radians, such that  $\cos \theta = 0.8829$  and  $0 \leq \theta < 2\pi$ .

Since  $\cos \theta$  is positive and  $\theta$  is between 0 and  $2\pi$ , we want a first-quadrant angle and a fourth-quadrant angle. With the calculator in radian mode,

$$\begin{aligned}\cos^{-1} 0.8829 &= 0.4888 && \text{first-quadrant angle} \\ 2\pi - 0.4888 &= 5.794 && \text{fourth-quadrant angle} \\ \theta &= 0.4888 \quad \text{or} \quad \theta = 5.794 && \text{see Fig. 8.27} \quad \blacksquare\end{aligned}$$

**COMMON ERROR**

When one first encounters radian measure, expressions such as  $\sin 1$  and  $\sin \theta = 1$  are often confused. The first is equivalent to  $\sin 57.30^\circ$  (since 1 radian =  $57.30^\circ$ ). The second means  $\theta$  is the angle for which the sine is 1. Since  $\sin 90^\circ = 1$ , we can say that  $\theta = 90^\circ$  or  $\theta = \pi/2$ .

**Practice Exercise**

4. If  $\sin \theta = 0.4235$ , find the smallest positive  $\theta$  (in radians).

**EXAMPLE 8** Angle in radians and value of a trigonometric function relationship

- (a)  $\sin \pi/3 = \sqrt{3}/2$       (b)  $\cos \theta = 0.5$ ,  $\theta = 60^\circ = \pi/3$  (smallest positive  $\theta$ )  
(c)  $\tan 2 = \tan 114.6^\circ$       (d)  $\tan \theta = 2$ ,  $\theta = 1.107$  (smallest positive  $\theta$ )       $\blacksquare$

**EXERCISES 8.3**

In Exercises 1–4, make the given changes in the indicated examples of this section and then solve the resulting problems.

- In Example 3(b), change 3.80 to 2.80.
- In Example 4(a), change  $\sin$  to  $\cos$ .
- In Example 7, change  $\cos$  to  $\sin$ .
- In Example 8(a), change  $\frac{\pi}{3}$  to  $\frac{\pi}{4}$ .

In Exercises 5–12, express the given angle measurements in radian measure in terms of  $\pi$ .

5.  $15^\circ$ ,  $150^\circ$     6.  $12^\circ$ ,  $225^\circ$     7.  $75^\circ$ ,  $330^\circ$     8.  $36^\circ$ ,  $315^\circ$   
9.  $210^\circ$ ,  $27^\circ$     10.  $5^\circ$ ,  $300^\circ$     11.  $720^\circ$ ,  $-9^\circ$     12.  $66^\circ$ ,  $540^\circ$

In Exercises 13–20, the given numbers express angle measure. Express the measure of each angle in terms of degrees.

13.  $\frac{2\pi}{5}$ ,  $\frac{3\pi}{2}$     14.  $\frac{3\pi}{10}$ ,  $\frac{5\pi}{6}$     15.  $\frac{\pi}{18}$ ,  $\frac{7\pi}{4}$     16.  $\frac{8\pi}{15}$ ,  $\frac{4\pi}{3}$   
17.  $\frac{7\pi}{18}$ ,  $\frac{5\pi}{3}$     18.  $\frac{\pi}{40}$ ,  $\frac{5\pi}{4}$     19.  $-\frac{\pi}{9}$ ,  $\frac{3\pi}{20}$     20.  $\frac{9\pi}{2}$ ,  $\frac{4\pi}{15}$

In Exercises 21–28, express the given angles in radian measure. Round off results to the number of significant digits in the given angle.

21.  $23.0^\circ$     22.  $54.3^\circ$     23.  $252^\circ$     24.  $104^\circ$   
25.  $333.5^\circ$     26.  $168.7^\circ$     27.  $478.5^\circ$     28.  $-86.1^\circ$

In Exercises 29–36, the given numbers express the angle measure. Express the measure of each angle in terms of degrees, with the same accuracy as the given value.

29. 0.750    30. 0.240    31. 3.407    32. 1.703  
33. 12.6    34. 34.4    35. -16.42    36. 100.00

In Exercises 37–44, evaluate the given trigonometric functions by first changing the radian measure to degree measure. Round off results to four significant digits.

37.  $\sin \frac{\pi}{4}$     38.  $\cos \frac{\pi}{6}$     39.  $\tan \frac{5\pi}{12}$   
40.  $\sin \frac{7\pi}{36}$     41.  $\cos \frac{5\pi}{6}$     42.  $\tan \frac{7\pi}{3}$   
43.  $\sec 4.5920$     44.  $\cot 3.2732$

In Exercises 45–52, evaluate the given trigonometric functions directly, without first changing to degree measure.

45.  $\tan 0.7359$     46.  $\cos 0.9308$     47.  $\sin 4.24$   
48.  $\tan 3.47$     49.  $\sec 2.07$     50.  $\sin(-2.34)$   
51.  $\cot(-4.86)$     52.  $\csc 6.19$

In Exercises 53–60, find  $\theta$  to four significant digits for  $0 \leq \theta < 2\pi$ .

53.  $\sin \theta = 0.3090$     54.  $\cos \theta = -0.9135$   
55.  $\tan \theta = -0.2126$     56.  $\sin \theta = -0.0436$   
57.  $\cos \theta = 0.6742$     58.  $\cot \theta = 1.860$   
59.  $\sec \theta = -1.307$     60.  $\csc \theta = 3.940$

In Exercises 61–64, solve the given problems. (Hint: Review cofunctions on page 128.)

61. Using the fact that  $\sin \frac{\pi}{8} = 0.3827$ , find the value of  $\cos \frac{5\pi}{8}$ . (A calculator should be used only to check the result.)  
62. Using the fact that  $\tan \frac{\pi}{6} = 0.5774$ , find the value of  $\cot \frac{5\pi}{3}$ . (A calculator should be used only to check the result.)  
63. Express  $\tan\left(\frac{\pi}{2} + \theta\right)$  in terms of  $\cot \theta$ . ( $0 < \theta < \frac{\pi}{2}$ )  
64. Express  $\cos\left(\frac{3\pi}{2} + \theta\right)$  in terms of  $\sin \theta$ . ( $0 < \theta < \frac{\pi}{2}$ )

In Exercises 65–72, evaluate the given problems.

- 65. A unit of angle measurement used in artillery is the *mil*, which is defined as a central angle of a circle that intercepts an arc equal in length to  $1/6400$  of the circumference. How many mils are in a central angle of  $34.4^\circ$ ?
- 66. Through how many radians does the minute hand of a clock move in 25 min?
- 67. After the brake was applied, a bicycle wheel went through 1.75 rotations. Through how many radians did a spoke rotate?
- 68. Through how many radians does a Ferris wheel with 18 seats move when loading passengers on the first 12 seats, assuming the loading process started at seat 1?
- 69. A flat plate of weight  $W$  oscillates as shown in Fig. 8.28. Its potential energy  $V$  is given by  $V = \frac{1}{2}Wb\theta^2$ , where  $\theta$  is measured in radians. Find  $V$  if  $W = 8.75$  N,  $b = 0.75$  m, and  $\theta = 5.5^\circ$ .

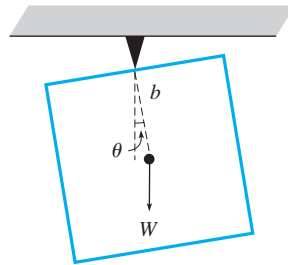


Fig. 8.28

- 70. The charge  $q$  (in C) on a capacitor as a function of time is  $q = A \sin \omega t$ . If  $t$  is measured in seconds, in what units is  $\omega$  measured? Explain.
- 71. The height  $h$  of a rocket launched 1200 m from an observer is found to be  $h = 1200 \tan \frac{5t}{3t + 10}$  for  $t < 10$  s, where  $t$  is the time after launch. Find  $h$  for  $t = 8.0$  s.
- 72. The electric intensity  $I$  (in  $\text{W}/\text{m}^2$ ) from the two radio antennas in Fig. 8.29 is a function of  $\theta$  given by  $I = 0.023 \cos^2(\pi \sin \theta)$ . Find  $I$  for  $\theta = 40.0^\circ$ . ( $\cos^2 \alpha = (\cos \alpha)^2$ .)

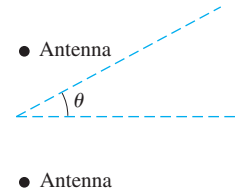


Fig. 8.29

Answers to Practice Exercises

1.  $\pi/5$     2.  $140^\circ$     3.  $-0.406$     4.  $0.4373$  rad

## 8.4 Applications of Radian Measure

Arc Length • Area of a Sector of a Circle • Angular Velocity

Radian measure has numerous applications in mathematics and technology, some of which were illustrated in the last eight exercises of the previous section. In this section, several more applications are shown.

### ARC LENGTH

From geometry (see Section 2.4 and Eqs. (2.9) and (2.12)), we know that *the length of an arc on a circle is proportional to the central angle* formed by the radii that intercept the arc. The length of arc of a complete circle is the circumference. Letting  $s$  represent the length of arc, we may state that  $s = 2\pi r$  for a complete circle. Since  $2\pi$  is the central angle (in radians) of the complete circle, *we have for the length of arc*

$$s = \theta r \quad (\theta \text{ in radians}) \tag{8.11}$$

for any circular arc with central angle  $\theta$ . Therefore, if we know the central angle  $\theta$  in radians and the radius of a circle, we can find the length of a circular arc directly by using Eq. (8.11). See Fig. 8.30.

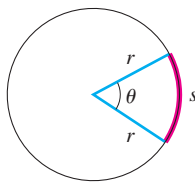


Fig. 8.30

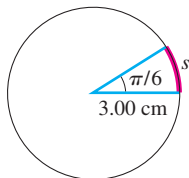


Fig. 8.31

#### EXAMPLE 1 Arc length

Find the length of arc on a circle of radius  $r = 3.00$  cm, for which the central angle  $\theta = \pi/6$ . See Fig. 8.31.

$$\begin{aligned}
 & \theta \text{ in radians} \\
 & \downarrow \\
 s &= \theta r \\
 s &= \left(\frac{\pi}{6}\right)(3.00 \text{ cm}) = \frac{\pi}{2.00} \text{ cm} \\
 s &= 1.57 \text{ cm}
 \end{aligned}$$

Therefore, the length of arc  $s$  is 1.57 cm. ■

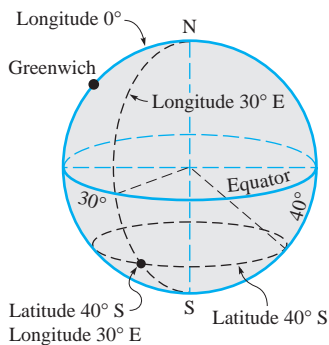


Fig. 8.32

Among the important applications of arc length are distances on the earth’s surface. For most purposes, the earth may be regarded as a sphere (the diameter at the equator is slightly greater than the distance between the poles). A *great circle* of the earth (or any other sphere) is the circle of intersection of the surface of the sphere and a plane that passes through the centre.

The equator is a great circle and is designated as  $0^\circ$  latitude. Other *parallels of latitude* are parallel to the equator with diameters decreasing to zero at the poles, which are  $90^\circ$  N and  $90^\circ$  S. See Fig. 8.32.

*Meridians of longitude* are half great circles between the poles. The *prime meridian* through Greenwich, England, is designated as  $0^\circ$ , with meridians to  $180^\circ$  measured east and west from Greenwich. Positions on the surface of the earth are designated by longitude and latitude coordinates.

**Practice Exercise**

- Find  $\theta$  in degrees if  $s = 2.50$  m and  $r = 1.75$  m.

**EXAMPLE 2 Arc length—nautical mile**

The traditional definition of a *nautical mile* is the length of arc along a great circle of the earth for a central angle of  $1'$ . The modern international definition is a distance of 1852 m. What measurement of the earth’s radius does this definition use?

Here,  $\theta = 1' = (1/60)^\circ$ , and  $s = 1852$  m. Solving for  $r$ , we have

$$r = \frac{s}{\theta} = \frac{1852 \text{ m}}{\left(\frac{1}{60}\right)^\circ \left(\frac{\pi}{180^\circ}\right)} = 6.367 \times 10^6 \text{ m}$$

$$r = 6367 \text{ km}$$

Historically, the fact that the earth is not a perfect sphere has led to many variations in the distance used for a nautical mile. ■

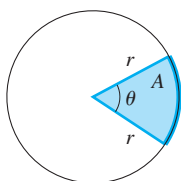


Fig. 8.33

**AREA OF A SECTOR OF A CIRCLE**

Another application of radians is finding the area of a sector of a circle (see Fig. 8.33). Recall from geometry that areas of sectors of circles are proportional to their central angles. From Eq. (2.10), the area of a circle is  $A = \pi r^2$ , which can be written as  $A = \frac{1}{2}(2\pi)r^2$ . Since the angle for a complete circle is  $2\pi$ , the area of any sector of a circle in terms of the radius and central angle (in radians) is

$$A = \frac{1}{2}\theta r^2 \quad (\theta \text{ in radians}) \tag{8.12}$$

**EXAMPLE 3 Area of a sector of a circle**

- (a) The area of a sector of a circle with central angle  $218^\circ$  and a radius of 5.25 cm (see Fig. 8.34(a)) is

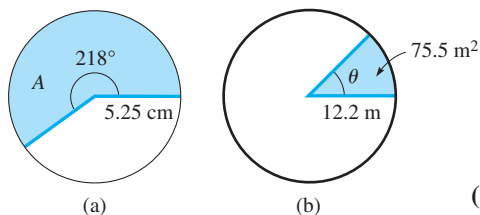


Fig. 8.34

$$A = \frac{1}{2}(218^\circ) \left(\frac{\pi}{180^\circ}\right) (5.25 \text{ cm})^2 = 52.4 \text{ cm}^2$$

- (b) Given that the area of a sector is  $75.5 \text{ m}^2$  and the radius is 12.2 m (see Fig. 8.34(b)), we find the central angle by solving for  $\theta$  and then substituting:

$$\theta = \frac{2A}{r^2} = \frac{2(75.5 \text{ m}^2)}{(12.2 \text{ m})^2} = 1.01$$

no units indicates radian measure

**Practice Exercise**

- Find  $A$  if  $r = 17.5$  cm and  $\theta = 125^\circ$ .

This means that the central angle is 1.01 rad, or  $57.9^\circ$ . ■

**ANGULAR VELOCITY**

The average velocity of a moving object is defined by  $v = s/t$ , where  $v$  is the average velocity,  $s$  is the distance travelled, and  $t$  is the elapsed time. For an object moving in a circular path with constant speed, the distance travelled is the length of arc through which it moves. Therefore, if we divide both sides of Eq. (8.11) by  $t$ , we obtain

$$\frac{s}{t} = \frac{\theta r}{t} = \frac{\theta}{t} r$$

where

$$\omega = \theta/t \tag{8.13}$$

is defined to be the angular velocity. Therefore,

$$v = \omega r \tag{8.14}$$

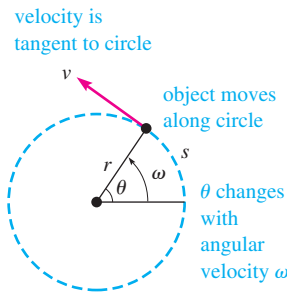


Fig. 8.35

Some of the typical units used for angular velocity are

- rad/s    rad/min    rad/h
- °/s    °/min
- r/s    r/min

( $r$  represents revolutions.)

( $r/\text{min}$  is the same as rpm. This text does not use rpm.)

Eq. (8.14) expresses the relationship between the **linear velocity  $v$**  and the **angular velocity  $\omega$**  of an object moving around a circle of radius  $r$ . See Fig. 8.35. In the figure,  $v$  is shown directed tangent to the circle, for that is its direction for the position shown. The direction of  $v$  changes constantly.

The standard SI units for  $\omega$  are radians per second (rad/s). In this way, the formula can be used directly. However, in practice,  $\omega$  is often given in revolutions per minute or in some similar unit. In these cases, it is necessary to convert the units of  $\omega$  to radians per unit of time before substituting in Eq. (8.14).

**EXAMPLE 4 Angular velocity—application**

A person on a hang glider is moving in a horizontal circular arc of radius 90.0 m with an angular velocity of 0.125 rad/s. The person’s linear velocity is

$$v = (0.125 \text{ rad/s}) (90.0 \text{ m}) = 11.3 \text{ m/s}$$

(Remember that radians are numbers and are not included in the final set of units.) This means that the person is moving along the circumference of the arc at 11.3 m/s (40.7 km/h).

**EXAMPLE 5 Angular velocity—geosynchronous satellites**

A communications satellite remains at an altitude of 35 920 km above a point on the equator. If the radius of the earth is 6370 km, what is the velocity of the satellite?

In order for the satellite to remain over a point on the equator, it must rotate exactly once each day around the centre of the earth (and it must remain at an altitude of 35 920 km). Since there are  $2\pi$  radians in each revolution, the angular velocity is

$$\omega = \frac{1 \text{ r}}{1 \text{ day}} \left( \frac{1 \text{ day}}{24 \text{ h}} \right) \left( \frac{2\pi \text{ rad}}{1 \text{ r}} \right) = 0.2618 \text{ rad/h}$$

The radius of the circle through which the satellite moves is its altitude plus the radius of the earth, or  $35\,920 + 6370 = 42\,290$  km. Thus, the velocity is

$$\begin{aligned} v &= \omega r \\ v &= (0.2618 \text{ rad/h}) (42\,290 \text{ km}) \\ v &= 11\,070 \text{ km/h} \\ v &= 11\,070 \text{ km/h} \left( \frac{1000 \text{ m}}{1 \text{ km}} \right) \left( \frac{1 \text{ h}}{3600 \text{ s}} \right) \\ v &= 3075 \text{ m/s} \end{aligned}$$

Mechanical elements that are connected on their edges (like pulleys with a belt, or meshed gears) have the same linear velocity  $v$  on their edges. Mechanical elements that are

The first U.S. communications satellite was launched in July 1962.

**Practice Exercise**

3. Find  $r$  if  $v = 25.0$  m/s and the angular velocity is  $6.0^\circ/\text{min}$ .



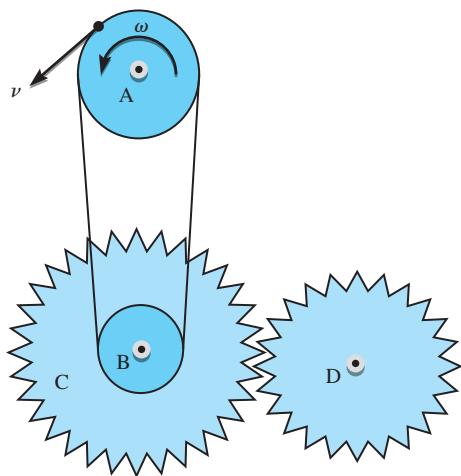


Fig. 8.36

■ See the chapter introduction.

mounted on the same shaft have the same rotational properties (same amount of rotation, same rate of rotation).

### EXAMPLE 6 Angular velocity—application, gears and pulleys

A pulley belt 2.00 m long takes 0.800 s to make one complete revolution around pulleys A and B. Pulley B and Gear C are mounted on the same shaft, and the teeth of gear D mesh with those of gear C. The radius of pulley A is 0.150 m, the radius of pulley B is 0.100 m, the radius of gear C is 0.300 m, and the radius of gear D is 0.200 m. What is the resulting angular velocity (in rad/s and in r/min) of gear D? See Fig. 8.36.

Since the linear velocity of a point on the edge of pulleys A and B is the same as the linear velocity of the belt,

$$\begin{aligned}v &= s/t \\v &= 2.00 \text{ m}/0.800 \text{ s} \\v &= 2.50 \text{ m/s}\end{aligned}$$

The angular velocity  $\omega$  for pulley B is then

$$\begin{aligned}\omega_B &= v_B/r_B \\ \omega_B &= 2.50 \text{ m/s}/0.100 \text{ m} \\ \omega_B &= 25.0 \text{ rad/s}\end{aligned}$$

Since gear C and pulley B are on the same shaft, they have the same angular velocity,  $\omega_C = \omega_B = 25.0 \text{ rad/s}$ , so the linear velocity of the edge of gear C can be found.

$$\begin{aligned}v_C &= \omega_C r_C \\ v_C &= 25.0 \text{ rad/s} (0.300 \text{ m}) \\ v_C &= 7.50 \text{ m/s}\end{aligned}$$

Since gear C and gear D are meshed, the linear velocities of their edges (their teeth) must be the same, so  $v_D = v_C = 7.50 \text{ m/s}$ . The angular velocity of gear D can then be found.

$$\begin{aligned}\omega_D &= v_D/r_D \\ \omega_D &= 7.50 \text{ m/s}/0.200 \text{ m} \\ \omega_D &= 37.5 \text{ rad/s} \\ \omega_D &= 37.5 \text{ rad/s} \left( \frac{1 \text{ r}}{2\pi \text{ rad}} \right) \left( \frac{60 \text{ s}}{1 \text{ min}} \right) \\ \omega_D &= 358 \text{ r/min}\end{aligned}$$

Since A, B, and C all spin counterclockwise, at the teeth connection point between gears C and D, the teeth will be moving vertically upward. Thus, the rotation of gear D must be clockwise. ■

### ANGULAR ACCELERATION

The average acceleration of a moving object is defined by  $a = \frac{v_f - v_i}{t}$ , where  $a$  is the average acceleration,  $v_f$  and  $v_i$  are the final and initial velocities respectively, and  $t$  is the elapsed time. For an object moving in a circular path with a changing angular velocity, if we divide the acceleration equation by the radius  $r$ , we obtain

$$\frac{a}{r} = \frac{\frac{v_f}{r} - \frac{v_i}{r}}{t}$$

However, from Eq. (8.14) we see that the ratio of linear velocity to radius is angular velocity  $\omega$ . We thus define

$$\alpha = \frac{\omega_f - \omega_i}{t} \quad (8.15)$$

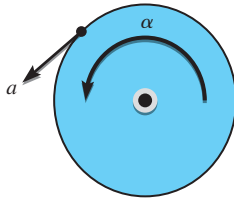


Fig. 8.37

where  $\alpha$  is the angular acceleration. Comparing the above two equations gives

$$a = \alpha r \quad (8.16)$$

Eq. (8.16) represents the relationship between the linear acceleration  $a$  and the angular acceleration  $\alpha$  of an object moving in a circle with a changing magnitude for its velocity. See Fig. 8.37. The standard SI units for  $\alpha$  are  $\text{rad/s}^2$ .

### EXAMPLE 7 Angular acceleration—application

A flywheel has a radius 40.0 cm, and its edge is accelerating at the rate of  $4.25 \text{ m/s}^2$ . Determine its angular acceleration  $\alpha$ .

The wheel radius expressed in metres is  $r = 40.0 \text{ cm} = 0.400 \text{ m}$ . Eq. (8.16) gives

$$a = \alpha r$$

$$\alpha = \frac{a}{r}$$

$$\alpha = \frac{4.25 \text{ m/s}^2}{0.400 \text{ m}}$$

$$\alpha = 10.6 \text{ rad/s}^2$$

### COMMON ERROR

When dealing with arc length, area of a sector of a circle, angular velocity, or angular acceleration, the equations require that the angle  $\theta$  be expressed in radians. A common error is to use  $\theta$  in degrees.

### EXAMPLE 8 Application to electric current

The current at any time in a certain alternating-current electric circuit is given by  $i = I \sin 120\pi t$ , where  $I$  is the maximum current and  $t$  is the time in seconds. Given that  $I = 0.0685 \text{ A}$ , find  $i$  for  $t = 0.00500 \text{ s}$ .

Substituting, with the calculator in radian mode, we get

$$i = (0.0685 \text{ A}) \sin [(120\pi)(0.00500)]$$

$$i = 0.0651 \text{ A}$$

$(120\pi)(0.00500)$  is a pure number, and therefore is an angle in radians.

## EXERCISES 8.4

In Exercises 1–4, make the given changes in the indicated examples of this section, and then solve the resulting problems.

- In Example 1, change  $\pi/6$  to  $\pi/4$ .
- In Example 3(a), change  $218^\circ$  to  $258^\circ$ .
- In Example 4, change 90.0 m to 115 m.
- In Example 6, change 0.800 s to 1.20 s.

In Exercises 5–16, for an arc length  $s$ , area of sector  $A$ , and central angle  $\theta$  of a circle of radius  $r$ , find the indicated quantity for the given values.

- $r = 3.30 \text{ cm}$ ,  $\theta = \pi/3$ ,  $s = ?$
- $r = 21.2 \text{ cm}$ ,  $\theta = 2.65$ ,  $s = ?$
- $s = 1010 \text{ mm}$ ,  $\theta = 136.0^\circ$ ,  $r = ?$
- $s = 0.3456 \text{ m}$ ,  $\theta = 73.61^\circ$ ,  $A = ?$
- $s = 0.3913 \text{ km}$ ,  $r = 0.9449 \text{ km}$ ,  $A = ?$
- $s = 3.19 \text{ m}$ ,  $r = 2.29 \text{ m}$ ,  $\theta = ?$

- $r = 4.9 \text{ cm}$ ,  $\theta = 5.6$ ,  $A = ?$
- $r = 46.3 \text{ dm}$ ,  $\theta = 2\pi/5$ ,  $A = ?$
- $A = 0.0119 \text{ m}^2$ ,  $\theta = 326.0^\circ$ ,  $r = ?$
- $A = 1200 \text{ mm}^2$ ,  $\theta = 17^\circ$ ,  $s = ?$
- $A = 16.5 \text{ m}^2$ ,  $r = 4.02 \text{ m}$ ,  $s = ?$
- $A = 67.8 \text{ km}^2$ ,  $r = 67.8 \text{ km}$ ,  $\theta = ?$

In Exercises 17–56, solve the given problems.

- In travelling three-fourths of the way around a traffic circle a car travels 0.203 km. What is the radius of the traffic circle? See Fig. 8.38.

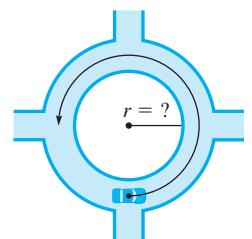


Fig. 8.38

18. The latitude of Manila, Philippines, is  $15^\circ$  N, and the latitude of Shanghai, China, is  $31^\circ$  N. Both are at a longitude of  $121^\circ$  W. What is the distance between Manila and Shanghai? Explain how the angle used in the solution is found. The radius of the earth is 6370 km.
19. Of the estimated natural gas reserves in North America,  $4.59 \times 10^9$  m<sup>3</sup> are in the United States,  $2.66 \times 10^9$  m<sup>3</sup> are in Canada, and  $1.99 \times 10^9$  m<sup>3</sup> are in Mexico. In making a *circle graph* (circular sectors represent percentages of the whole—a *pie chart*) with a radius of 4.00 cm for these data, what are the central angle and area of the sector that represents Canada's reserves?
20. A section of sidewalk is a circular sector of radius 1.25 m and central angle  $50.6^\circ$ . What is the area of this section of sidewalk?
21. When between 12:00 noon and 1:00 P.M. are the minute and hour hands of a clock  $180^\circ$  apart?
22. A cam is in the shape of a circular sector, as shown in Fig. 8.39. What is the perimeter of the cam?

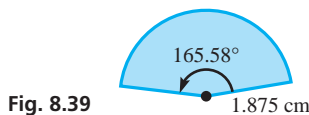


Fig. 8.39

23. A lawn sprinkler can water up to a distance of 25.0 m. It turns through an angle of  $115.0^\circ$ . What area can it water?
24. A spotlight beam sweeps through a horizontal angle of  $75.0^\circ$ . If the range of the spotlight is 115 m, what area can it cover?
25. If a car makes a U-turn in 6.0 s, what is its average angular velocity in the turn?
26. The roller on a computer printer makes 2200 r/min. What is its angular velocity (in rad/s)?
27. What is the floor area of the hallway shown in Fig. 8.40? The outside and inside of the hallway are circular arcs.

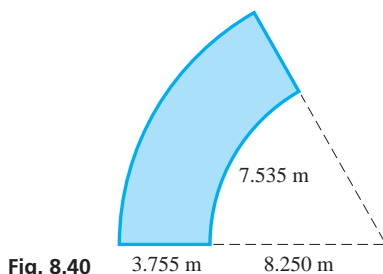


Fig. 8.40

28. The arm of a car windshield wiper is 32.4 cm long and is attached at the middle of a 38.1-cm blade. (Assume that the arm and blade are in line.) What area of the windshield is cleaned by the wiper if it swings through  $110.0^\circ$  arcs?
29. Part of a railroad track follows a circular arc with a central angle of  $28.0^\circ$ . If the radius of the arc of the inner rail is 28.55 m and the rails are 1.44 m apart, how much longer is the outer rail than the inner rail?
30. A wrecking ball is dropped as shown in Fig. 8.41. Its velocity at the bottom of its swing is  $v = \sqrt{2gh}$ , where  $g$  is the acceleration due to gravity. What is its angular velocity at the bottom if  $g = 9.80$  m/s<sup>2</sup> and  $h = 4.80$  m?

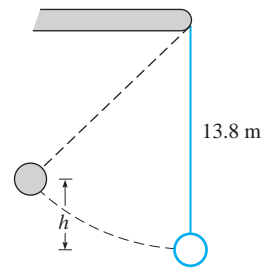


Fig. 8.41

31. Part of a security fence is built 2.50 m from a cylindrical storage tank 11.2 m in diameter. What is the area between the tank and this part of the fence if the central angle of the fence is  $75.5^\circ$ ? See Fig. 8.42.

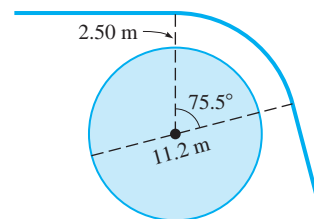


Fig. 8.42

32. Through what angle does the drum in Fig. 8.43 turn in order to lower the crate 10.3 m? If the crate accelerates downward at  $3.75$  m/s<sup>2</sup>, what is the angular acceleration of the drum?

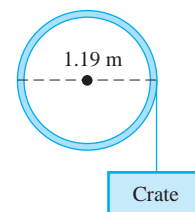


Fig. 8.43

33. A section of road follows a circular arc with a central angle of  $15.6^\circ$ . The radius of the inside of the curve is 285.0 m, and the road is 15.2 m wide. What is the volume of the concrete in the road if it is 0.305 m thick?
34. The propeller of the motor on a motorboat is rotating at 133 rad/s. What is the linear velocity of a point on the tip of a blade if it is 22.5 cm long?
35. A storm causes a pilot to follow a circular-arc route, with a central angle of  $12.8^\circ$ , from city A to city B rather than the straight-line route of 185.0 km. How much farther does the plane fly due to the storm?
36. A freeway interchange exit is a circular arc 335 m long with a central angle of  $79.4^\circ$ . What is the radius of curvature of the exit?
37. The paddles of a riverboat have a radius of 2.59 m and revolve at 20.0 r/min. What is the speed of a tip of one of the paddles (in m/s)?
38. The sweep second hand of a watch is 15.0 mm long. What is the linear velocity of the tip?
39. A DVD has a diameter of 12.1 cm and rotates at 360.0 r/min. What is the linear velocity of a point on the outer edge? At startup the DVD accelerates from rest to 360.0 r/min in 0.200 s. Find the linear acceleration of a point on its edge.

40. A Ferris wheel 18.0 m in diameter makes one revolution in 3.00 min. Find the speed of a seat on the rim (in m/s).
41. In 2005 a planet was found revolving about a distant sunlike star in 2.88 days. The radius of its orbit is 5 600 000 km. What is its linear velocity (in km/h) about its star?
42. Assume that earth rotates around the sun in a circular orbit of radius 150 000 000 km (which is approximately correct). What is earth's linear velocity (in km/h)? (Use 1 yr = 365.25 d.)
43. The sprocket assembly for a 70.0-cm bike is shown in Fig. 8.44. How fast (in r/min) does the rider have to pedal in order to go 25.0 km/h on level ground?

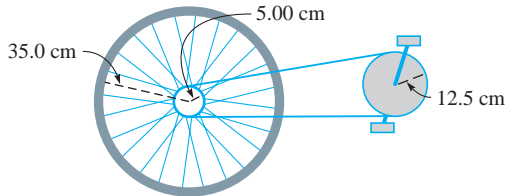


Fig. 8.44

44. The flywheel of a car engine is 0.36 m in diameter. If it is revolving at 750 r/min, through what distance does a point on the rim move in 2.00 s?
45. Two streets meet at an angle of  $82.0^\circ$ . What is the length of the piece of curved curbing at the intersection if it is constructed along the arc of a circle 5.50 m in radius? See Fig. 8.45.

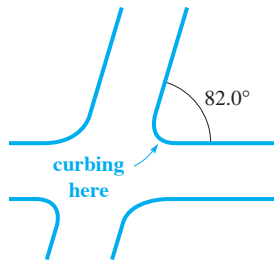


Fig. 8.45

46. An ammeter needle is deflected  $52.00^\circ$  by a current of 0.2500 A. The needle is 3.750 cm long, and a circular scale is used. How long is the scale for a maximum current of 1.500 A?
47. A drill bit 9.53 mm in diameter rotates at 1260 r/min. What is the linear velocity of a point on its circumference (in mm/s)?
48. A helicopter blade is 2.75 m long and from rest accelerates for 5.00 s until it is rotating at 420 r/min. What is the average linear acceleration of the tip of the blade (in  $\text{m/s}^2$ )?
49. A waterwheel used to generate electricity has paddles 3.75 m long. The speed of the end of a paddle is one-fourth that of the water. If the water is flowing at the rate of 6.50 m/s, what is the angular velocity of the waterwheel?
50. A jet is travelling westward with the sun directly overhead (the jet is on a line between the sun and the centre of the earth). How fast must the jet fly in order to keep the sun directly overhead? (Assume that the earth's radius is 6370 km, the altitude of the jet is 11 km, and the earth rotates about its axis once in 24.0 h.)
51. What is the linear velocity of a point in Winnipeg, Manitoba, which is at a latitude of  $49^\circ 53' \text{ N}$ ? The radius of the earth is 6370 km.

52. Through what total angle does the drive shaft of a car rotate in 1.0 s when the tachometer reads 2400 r/min? If it starts from rest, what is the drive shaft's average angular acceleration if it takes 3.25 s to reach 2400 r/min?
53. A baseball field is designed such that the outfield fence is along the arc of a circle with its centre at second base. If the radius of the circle is 85.0 m, what is the playing area of the field? See Fig. 8.46.

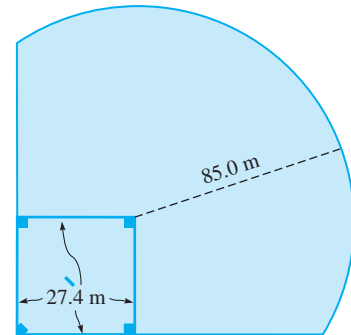


Fig. 8.46

54. A patio is in the shape of a circular sector with a central angle of  $160.0^\circ$ . It is enclosed by a railing of which the circular part is 11.6 m long. What is the area of the patio?
55. An oil storage tank 4.25 m long has a flat bottom as shown in Fig. 8.47. The radius of the circular part is 1.10 m. What volume of oil does the tank hold?

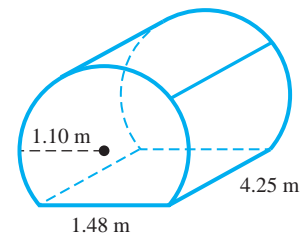


Fig. 8.47

56. Two equal beams of light illuminate the area shown in Fig. 8.48. What area is lit by both beams?

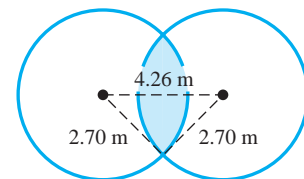


Fig. 8.48

In Exercises 57–60, another use of radians is illustrated.

57. Use a calculator (in radian mode) to evaluate the ratios  $(\sin \theta)/\theta$  and  $(\tan \theta)/\theta$  for  $\theta = 0.1, 0.01, 0.001, \text{ and } 0.0001$ . From these values, explain why it is possible to say that
- $$\sin \theta = \tan \theta = \theta \quad (8.17)$$
- approximately for very small angles.
58. Using Eq. (8.17), evaluate  $\tan 0.001^\circ$ . Compare with a calculator value.

59. An astronomer observes that a star 12.5 light-years away moves through an angle of  $0.2''$  in 1 year. Assuming it moved in a straight line perpendicular to the initial line of observation, how many kilometres did the star move? (1 light-year =  $9.46 \times 10^{12}$  km.) Use Eq. (8.17) and express the answer to 3 significant digits.
60. In calculating a back line of a lot, a surveyor discovers an error of  $0.05^\circ$  in an angle measurement. If the lot is 136.0 m deep, by how much is the back-line calculation in error? See Fig. 8.49. Use Eq. (8.17) and express the answer to 3 significant digits.

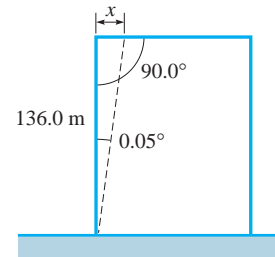
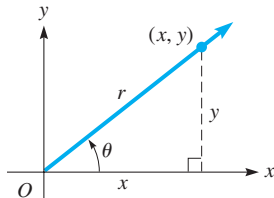


Fig. 8.49

Answers to Practice Exercises

1.  $81.9^\circ$     2.  $334 \text{ cm}^2$     3. 14 300 m

CHAPTER 8 EQUATIONS



$$\begin{aligned} \sin \theta &= \frac{y}{r} & \csc \theta &= \frac{1}{\sin \theta} = \frac{r}{y} \\ \cos \theta &= \frac{x}{r} & \sec \theta &= \frac{1}{\cos \theta} = \frac{r}{x} \\ \tan \theta &= \frac{y}{x} & \cot \theta &= \frac{1}{\tan \theta} = \frac{x}{y} \end{aligned} \tag{8.1}$$

$$F(\theta_2) = \pm F(180^\circ - \theta_2) = \pm F(\theta_{\text{ref}}) \quad (\theta_{\text{ref}} \text{ is reference angle}) \tag{8.3}$$

$$F(\theta_3) = \pm F(\theta_3 - 180^\circ) = \pm F(\theta_{\text{ref}}) \tag{8.4}$$

$$F(\theta_4) = \pm F(360^\circ - \theta_4) = \pm F(\theta_{\text{ref}}) \tag{8.5}$$

Using reference angles

$$\begin{aligned} \theta &= \theta_{\text{ref}} && \text{(first quadrant)} \\ \theta &= 180^\circ - \theta_{\text{ref}} && \text{(second quadrant)} \\ \theta &= 180^\circ + \theta_{\text{ref}} && \text{(third quadrant)} \\ \theta &= 360^\circ - \theta_{\text{ref}} && \text{(fourth quadrant)} \end{aligned} \tag{8.6}$$

Negative angles

$$\begin{aligned} \sin(-\theta) &= -\sin \theta & \cos(-\theta) &= \cos \theta & \tan(-\theta) &= -\tan \theta \\ \csc(-\theta) &= -\csc \theta & \sec(-\theta) &= \sec \theta & \cot(-\theta) &= -\cot \theta \end{aligned} \tag{8.7}$$

Radian-degree conversions

$$\pi \text{ rad} = 180^\circ \tag{8.8}$$

$$1^\circ = \frac{\pi}{180} \text{ rad} = 0.01745 \text{ rad} \tag{8.9}$$

$$1 \text{ rad} = \frac{180^\circ}{\pi} = 57.30^\circ \tag{8.10}$$

Circular arc length

$$s = \theta r \quad (\theta \text{ in radians}) \tag{8.11}$$

Circular sector area

$$A = \frac{1}{2} \theta r^2 \quad (\theta \text{ in radians}) \tag{8.12}$$

Angular velocity

$$\omega = \frac{\theta}{t} \tag{8.13}$$

Linear and angular velocity

$$v = \omega r \tag{8.14}$$

Angular acceleration

$$\alpha = \frac{\omega_f - \omega_i}{t} \tag{8.15}$$

Linear and angular acceleration

$$a = \alpha r \tag{8.16}$$

Small angle approximation

$$\sin \theta = \tan \theta = \theta \quad (\text{for small } \theta \text{ in radians}) \tag{8.17}$$

## CHAPTER 8 REVIEW EXERCISES

In Exercises 1–4, find the trigonometric functions of  $\theta$ . The terminal side of  $\theta$  passes through the given point.

1. (6, 8)
2. (-12, 5)
3. (42, -12)
4. (-2, -3)

In Exercises 5–8, express the given trigonometric functions in terms of the same function of a positive acute angle.

5.  $\cos 132^\circ$ ,  $\tan 194^\circ$
6.  $\sin 243^\circ$ ,  $\cot 318^\circ$
7.  $\sin 289^\circ$ ,  $\sec(-15^\circ)$
8.  $\cos 463^\circ$ ,  $\csc(-100^\circ)$

In Exercises 9–12, express the given angle measurements in terms of  $\pi$ .

9.  $40^\circ$ ,  $153^\circ$
10.  $22.5^\circ$ ,  $324^\circ$
11.  $408^\circ$ ,  $202.5^\circ$
12.  $12^\circ$ ,  $-162^\circ$

In Exercises 13–20, the given numbers represent angle measure. Express the measure of each angle in degrees.

13.  $\frac{7\pi}{5}$ ,  $\frac{13\pi}{18}$
14.  $\frac{3\pi}{8}$ ,  $\frac{7\pi}{20}$
15.  $\frac{\pi}{15}$ ,  $\frac{11\pi}{6}$
16.  $\frac{27\pi}{10}$ ,  $\frac{5\pi}{4}$
17. 0.560
18. -1.354
19. 36.07
20. 14.5

In Exercises 21–28, express the given angles in radians (not in terms of  $\pi$ ).

21.  $102^\circ$
22.  $305^\circ$
23.  $20.25^\circ$
24.  $148.38^\circ$
25.  $262.05^\circ$
26.  $-18.72^\circ$
27.  $-636.2^\circ$
28.  $385.4^\circ$

In Exercises 29–48, determine the values of the given trigonometric functions directly on a calculator. The angles are approximate. Express answers to Exercises 41–44 to four significant digits.

29.  $\cos 245.5^\circ$
30.  $\sin 141.3^\circ$
31.  $\cot 295^\circ$
32.  $\tan 184^\circ$
33.  $\csc 247.82^\circ$
34.  $\sec 96.17^\circ$
35.  $\sin 565.24^\circ$
36.  $\cos 326.72^\circ$
37.  $\tan 301.4^\circ$
38.  $\sin 703.9^\circ$
39.  $\tan 436.42^\circ$
40.  $\cos(-162.32^\circ)$
41.  $\sin \frac{9\pi}{5}$
42.  $\sec \frac{5\pi}{8}$
43.  $\cos \frac{7\pi}{6}$
44.  $\tan \frac{23\pi}{12}$
45.  $\sin 0.5906$
46.  $\tan 0.8035$
47.  $\csc 2.153$
48.  $\cos(-7.190)$

In Exercises 49–52, find  $\theta$  in degrees for  $0^\circ \leq \theta < 360^\circ$ .

49.  $\tan \theta = 0.1817$
50.  $\sin \theta = -0.9323$
51.  $\cos \theta = -0.4730$
52.  $\cot \theta = 1.196$

In Exercises 53–56, find  $\theta$  in radians for  $0 \leq \theta < 2\pi$ .

53.  $\cos \theta = 0.8387$
54.  $\sin \theta = 0.1045$
55.  $\sin \theta = -0.8650$
56.  $\tan \theta = 8.480$

In Exercises 57–60, find  $\theta$  in degrees for  $0^\circ \leq \theta < 360^\circ$ .

57.  $\cos \theta = -0.7222$ ,  $\sin \theta < 0$
58.  $\tan \theta = -1.683$ ,  $\cos \theta < 0$
59.  $\cot \theta = 0.4291$ ,  $\cos \theta < 0$
60.  $\sin \theta = 0.2626$ ,  $\tan \theta < 0$

In Exercises 61–68, for an arc of length  $s$ , area of sector  $A$ , and central angle  $\theta$  of circle of radius  $r$ , find the indicated quantity for the given values.

61.  $s = 20.3$  cm,  $\theta = 107.5^\circ$ ,  $r = ?$
62.  $s = 5840$  m,  $r = 1060$  m,  $\theta = ?$
63.  $A = 265$  mm<sup>2</sup>,  $r = 12.8$  mm,  $\theta = ?$
64.  $A = 0.908$  km<sup>2</sup>,  $\theta = 234.5^\circ$ ,  $r = ?$
65.  $r = 4.62$  m,  $A = 32.8$  m<sup>2</sup>,  $s = ?$
66.  $\theta = 98.5^\circ$ ,  $A = 0.493$  dm<sup>2</sup>,  $s = ?$
67.  $\theta = 0.85^\circ$ ,  $s = 7.94$  cm,  $A = ?$
68.  $r = 254$  cm,  $s = 76.1$  cm,  $A = ?$

In Exercises 69–92, solve the given problems.

69. Show that the area  $A$  of a minor segment of a circle of radius  $r$  intercepted by a central angle  $\theta$  (in radians) (see Fig. 8.50) is

$$A = \frac{1}{2}r^2(\theta - \sin \theta).$$

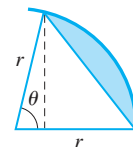


Fig. 8.50

70. The cross section of a tunnel is the major segment of a circle of radius 12.0 m wide. The base of the tunnel is 20.0 m wide. What is the area of the cross section? See Exercise 69.
71. Find the area of the parcel of land shown in Fig. 8.51. It is a right triangle attached to a circle sector.

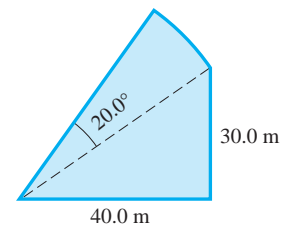


Fig. 8.51

72. The speedometer of a car is designed to be accurate with tires that are 35.6 cm in radius. If the tires are changed to 38.1 cm in radius, and the speedometer shows 88.0 km/h, how fast is the car actually going?



73. The instantaneous power  $P$  (in W) input to a resistor in an alternating-current circuit is  $P = P_m \sin^2 377t$ , where  $P_m$  is the maximum power input and  $t$  is the time (in s). Find  $P$  for  $P_m = 0.120$  W and  $t = 2.00$  ms. ( $\sin^2 \theta = (\sin \theta)^2$ .)
74. The horizontal distance  $x$  through which a pendulum moves is given by  $x = a(\theta + \sin \theta)$ , where  $a$  is a constant and  $\theta$  is the angle between the vertical and the pendulum. Find  $x$  for  $a = 45.0$  cm and  $\theta = 0.175$ .
75. A sector gear with a pitch radius of 8.25 cm and a 6.60-cm arc of contact is shown in Fig. 8.52. What is the sector angle  $\theta$  (in  $^\circ$ )?

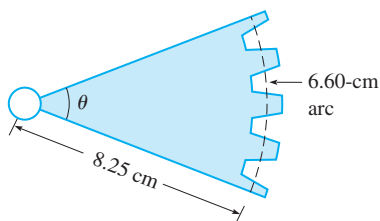


Fig. 8.52

76. Two pulleys have radii of 10.0 cm and 6.00 cm, and their centres are 40.0 cm apart. If the pulley belt is uncrossed, what must be the length of the belt?
77. A special vehicle for travelling on glacial ice in Jasper National Park, Canada, has tires that are 1.36 m in diameter. If the vehicle travels at 5.60 km/h, what is the angular velocity (in r/min) of the tire?
78. A rotating circular restaurant at the top of a hotel normally completes one revolution in 24.0 min. What is the angular acceleration required to accelerate the restaurant from rest to normal speed in 5.00 min.
79. Find the velocity (in km/h) of the moon as it revolves about the earth. Assume it takes 28 days for one revolution at a distance of 390 000 km from the earth.
80. The stopboard of a shot-put circle is a circular arc 1.22 m in length. The radius of the circle is 1.06 m. What is the central angle (in degrees)?
81. The longitude of Whitehorse, Yukon, is 135 W, and the longitude of St. Petersburg, Russia, is 30 $^\circ$  E. Both cities are at a latitude of 60 $^\circ$  N. (a) Find the great circle distance (see page 254) from Whitehorse to St. Petersburg over the north pole. (b) Find the distance between them along the 60 $^\circ$  N latitude arc. The radius of the earth is 6370 km. What do the results show?
82. A piece of circular filter paper 15.0 cm in diameter is folded such that its effective filtering area is the same as that of a sector with central angle of 220 $^\circ$ . What is the filtering area?
83. To produce an electric current, a circular loop of wire of diameter 25.0 cm is rotating about its diameter at 60.0 r/s in a magnetic field. What is the greatest linear velocity of any point on the loop (in m/s)?
84. Find the area of the decorative glass panel shown in Fig. 8.53. The panel is made of two equal circular sectors and an isosceles triangle.

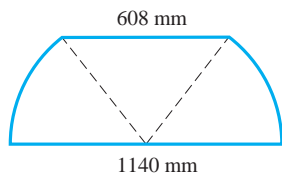


Fig. 8.53

85. A circular hood is to be used over a piece of machinery. It is to be made from a circular piece of sheet metal 1.08 m in radius. A hole 0.25 m in radius and a sector of central angle 80.0 $^\circ$  are to be removed to make the hood. What is the area of the top of the hood?
86. The chain on a chain saw is driven by a sprocket 7.50 cm in diameter. If the chain is 108 cm long and makes one revolution in 0.250 s, what is the angular velocity (in r/s) of the sprocket?
87. An *ultracentrifuge*, used to observe the sedimentation of particles such as proteins, may rotate as fast as 80 000 r/min. If it rotates at this rate and is 7.20 cm in diameter, what is the linear velocity of a particle at the outer edge (in m/s to 3 significant digits)?
88. A computer is programmed to shade in a sector of a pie chart 2.44 cm in radius. If the perimeter of the shaded sector is 7.32 cm, what is the central angle (in degrees) of the sector? See Fig. 8.54.

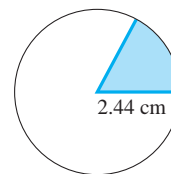


Fig. 8.54

89. A Gothic arch, commonly used in medieval European structures, is formed by two circular arcs. In one type, each arc is one-sixth of a circle, with the centre of each at the base on the end of the other arc. See Fig. 8.55. Therefore, the width of the arch equals the radius of each arc. For such an arch, find the area of the opening if the width is 15.0 m.

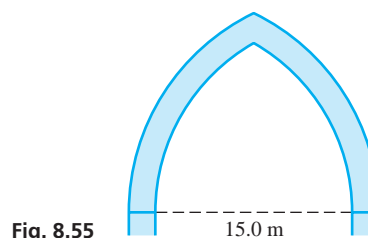


Fig. 8.55

90. The Trans-Alaska Pipeline was assembled in sections 12.2 m long and 1.22 m in diameter. If the depth of the oil in one horizontal section is 0.305 m, what is the volume of oil in this section?
91. A laser beam is transmitted with a “width” of 0.0008 $^\circ$  and makes a circular spot of radius 2.50 km on a distant object. How far is the object from the source of the laser beam? Use Eq. (8.17).
92. The planet Venus subtends an angle of 15" to an observer on earth. If the distance between Venus and earth is 167 Gm, what is the diameter of Venus? Use Eq. (8.17).

### Writing Exercise

93. Write a paragraph explaining how you determine the units for the result of the following problem: An astronaut in a spacecraft circles the moon once each 1.95 h. If the altitude of the spacecraft is constant at 113 km, what is its velocity? The radius of the moon is 1740 km. (What is the answer?)



## CHAPTER 8 PRACTICE TEST

1. Change  $150^\circ$  to radians in terms of  $\pi$ .
2. Determine the sign of (a)  $\sec 285^\circ$ , (b)  $\sin 245^\circ$ , (c)  $\cot 265^\circ$ .
3. Express  $\sin 205^\circ$  in terms of the sine of a positive acute angle. Do not evaluate.
4. Find  $\sin \theta$  and  $\sec \theta$  if  $\theta$  is in standard position and the terminal side passes through  $(-9, 12)$ .
5. An airplane propeller blade is 1.40 m long and rotates at 2200 r/min. What is the linear velocity of a point on the tip of the blade?
6. Given that 3.572 is the measure of an angle, express the angle in degrees.
7. If  $\tan \theta = 0.2396$ , find  $\theta$ , in degrees, for  $0^\circ \leq \theta < 360^\circ$ .
8. If  $\cos \theta = -0.8244$  and  $\csc \theta < 0$ , find  $\theta$  in radians for  $0 \leq \theta < 2\pi$ .
9. The floor of a sunroom is in the shape of a circular sector of arc length 16.0 m and radius 4.25 m. What is the area of the floor?
10. A circular sector has an area of  $38.5 \text{ cm}^2$  and a diameter of 12.2 cm. What is the arc length of the sector?

# 9

## LEARNING OUTCOMES

After completion of this chapter, the student should be able to:

- Distinguish between a scalar and a vector
- Add vectors by the polygon method
- Add vectors by the parallelogram method
- Resolve a vector into its  $x$ - and  $y$ -components
- Add vectors by components
- Solve application problems involving vectors
- Solve oblique triangles using the law of sines and/or the law of cosines
- Solve application problems involving oblique triangles

# Vectors and Oblique Triangles

In many applications, we often deal with such things as forces and velocities. To study them, both their *magnitudes* and *directions* must be known. In general, a quantity for which we must specify *both magnitude and direction* is called a *vector*.

In basic applications, a vector is usually represented by an arrow showing its magnitude and direction, although this was not common before the 1800s. In 1743, the French mathematician d'Alembert published a paper on dynamics in which he used some diagrams, but most of the text was algebraic. In 1788, the French mathematician Lagrange wrote a classic work on *Analytical Mechanics*, but the text was all algebraic and included no diagrams.

In the 1800s, calculus was used to advance the use of vectors, and in turn, these advancements became very important in further developments in scientific fields such as electromagnetic theory. Also in the 1800s, mathematicians defined a vector more generally and opened up study in new areas of advanced mathematics. In this text, we deal only with the basic meaning of a vector. However, a vector is an excellent example of a mathematical concept that came from a basic physics concept.

After studying vectors, we then develop methods of solving *oblique* triangles (triangles that are not right triangles). Although obviously not known in their modern forms, one of these methods, the *law of cosines*, was known to the Greek astronomer Ptolemy (90–168), and the other method, the *law of sines*, was known to Islamic mathematicians of the 1100s. In solving oblique triangles, we often use the trigonometric functions of obtuse angles.

Vectors are of great importance in many fields of science and technology, including physics, engineering, structural design, and navigation. One modern application is illustrated in the picture below. As with right triangles, the applications of oblique triangles are found in many fields of science and technology.

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► The wind must be considered to find the proper heading for an aircraft. In Section 9.5, we use vectors and oblique triangles to show how this may be done.

## 9.1 Introduction to Vectors

Scalars and Vectors • Addition of Vectors • Scalar Multiple of a Vector • Subtraction of Vectors

We deal with many quantities that may be described by a number that shows only the magnitude. These include lengths, areas, time intervals, monetary amounts, and temperatures. *Quantities such as these, described only by the **magnitude**, are known as scalars.*

As we said on the previous page, *many other quantities, called **vectors**, are fully described only when both the **magnitude** and **direction** are specified.* The following example shows the difference between scalars and vectors.

### EXAMPLE 1 Scalars and vectors—application

A jet is travelling at 600 km/h. From this statement alone, we know only the *speed* of the jet. *Speed is a scalar quantity*, and it tells us only the *magnitude* of the rate. Knowing only the speed of the jet, we know the rate at which it is moving, but we do not know where it is headed.

If the phrase “in a direction  $10^\circ$  south of west” is added to the sentence about the jet, we specify the direction of travel as well as the speed. We then know the *velocity* of the jet; that is, the *direction* of travel as well as the rate at which it is moving. *Velocity is a vector quantity.* ■

### EXAMPLE 2 Action of two vectors—application

Consider a boat moving in a river. We will assume that the boat is driven by a motor that can move it at 8 km/h in still water and that the river’s current is going 6 km/h downstream, as shown in Fig. 9.1. We quickly see that the movement of the boat depends on the direction in which it is headed. If it heads downstream, it moves at 14 km/h, for the water is moving at 6 km/h and the boat moves at 8 km/h with respect to the water. If it heads upstream, however, it moves only at 2 km/h, since the river is acting directly against the motor. If the boat heads directly across the river, the point it reaches on the other side is not directly opposite the point from which it started. This is so because the river is moving the boat downstream *at the same time* the boat moves across the river.

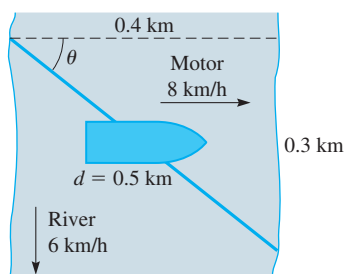


Fig. 9.1

Checking this last case further, assume that the river is 0.4 km wide where the boat is crossing. It takes 0.05 h ( $0.4 \text{ km} \div 8 \text{ km/h} = 0.05 \text{ h}$ ) to cross. In 0.05 h, the river will carry the boat 0.3 km ( $0.05 \text{ h} \times 6 \text{ km/h} = 0.3 \text{ km}$ ) downstream. This means the boat went 0.3 km downstream as it went 0.4 km across the river. From the Pythagorean theorem, we see that it went 0.5 km from its starting point to its finishing point:

$$d^2 = 0.4^2 + 0.3^2 = 0.25$$

$$d = 0.5 \text{ km}$$

Since the 0.5 km was travelled in 0.05 h, the magnitude of the velocity (the *speed*) of the boat was actually

$$v = \frac{d}{t} = \frac{0.5 \text{ km}}{0.05 \text{ h}} = 10 \text{ km/h}$$

Also, note that the direction of this velocity can be represented along a line that makes an angle  $\theta$  with the line directed directly across the river, as shown in Fig. 9.1. We can find this angle by noting that

$$\tan \theta = \frac{0.3 \text{ km}}{0.4 \text{ km}} = 0.75$$

$$\theta = \tan^{-1} 0.75 = 37^\circ$$

Therefore, when headed directly across the river, the boat’s velocity is 10 km/h directed at an angle of  $37^\circ$  downstream from a line directly across the river. ■

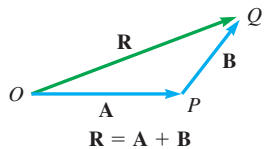


Fig. 9.2

**LEARNING TIP**

Since a vector is determined by its magnitude and direction, two vectors in different locations are considered the same if they have the same magnitude and direction.

**ADDITION OF VECTORS**

We have just seen two velocity vectors being *added*. Note that these vectors are not added the way numbers are added. We must take into account their directions as well as their magnitudes. Reasoning along these lines, let us now define the sum of two vectors.

We will represent a vector quantity by a letter printed in **boldface** type. The same letter in *italic* (lightface) type represents the magnitude only. Thus, **A** is a vector of magnitude *A*. In handwriting, one usually places an arrow over the letter to represent a vector, such as  $\vec{A}$ .

Let **A** and **B** represent vectors directed from *O* to *P* and *P* to *Q*, respectively (see Fig. 9.2). *The vector sum  $\mathbf{A} + \mathbf{B}$  is the vector **R**, from the initial point *O* to the terminal point *Q*. Here, vector **R** is called the **resultant**. In general, a resultant is a single vector that is the vector sum of any number of other vectors.*

There are two common methods of adding vectors by means of a diagram: the polygon method and the parallelogram method. For both methods, vectors must be drawn with reasonable accuracy and moved to specific locations without changing their magnitude and direction.

The polygon method is illustrated in Fig. 9.3. It can be described as follows.

**Polygon Method of Adding Vectors**

To add two vectors **A** and **B**, move **B** so that its tail touches the head of **A**. The vector sum  $\mathbf{A} + \mathbf{B}$  is the resultant vector **R**, which is drawn from the tail of **A** to the head of **B**.

Similarly, to add three or more vectors, the initial point of the second vector is placed at the terminal point of the first vector, the initial point of the third vector is placed at the terminal point of the second vector, and so on. The resultant is the vector from the initial point of the first vector to the terminal point of the last vector.

Note that the order in which vectors are added does not matter.

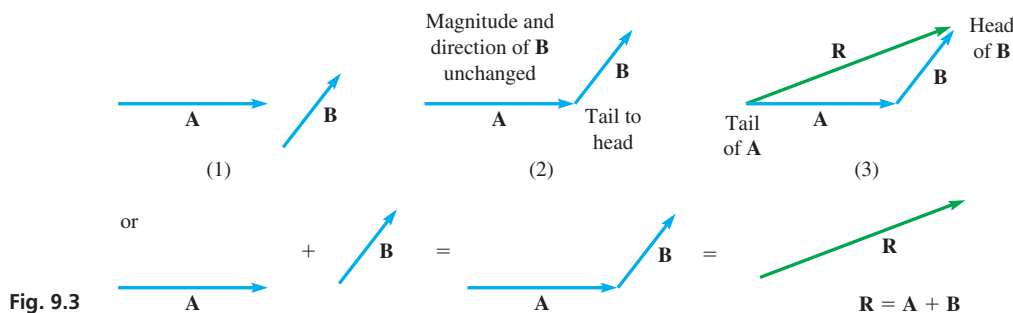


Fig. 9.3

**EXAMPLE 3 Adding vectors—polygon method**

The addition of vectors **A**, **B**, and **C** is shown in Fig. 9.4.

**Practice Exercise**

- For the vectors in Example 3, show that  $\mathbf{R} = \mathbf{B} + \mathbf{C} + \mathbf{A}$ .

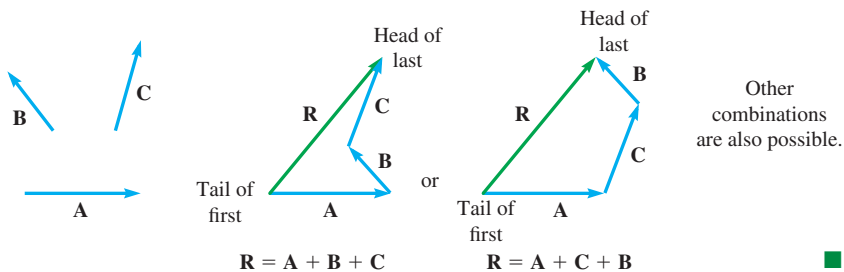


Fig. 9.4

Other combinations are also possible.

The parallelogram method is illustrated in Example 4. It can be described as follows.

**Parallelogram Method of Adding Vectors**

To add two vectors **A** and **B**, move both vectors tail to tail, so that they have a common initial point. Let the vectors be the sides of a parallelogram. The vector sum  $\mathbf{A} + \mathbf{B}$  is the resultant vector **R**, which corresponds to the *diagonal* of the parallelogram.

**EXAMPLE 4 Adding vectors—parallelogram method**

The addition of vectors **A** and **B** is shown in Fig. 9.5.

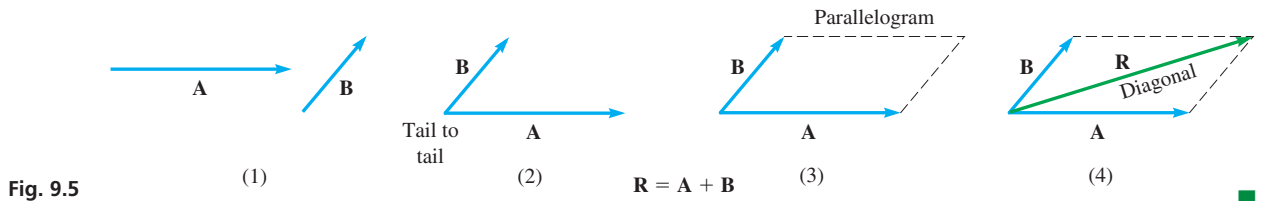


Fig. 9.5

**Scalar Multiple of Vector**

If vector **C** is in the same direction as vector **A** and **C** has a magnitude  $n$  times that of **A**, then  $\mathbf{C} = n\mathbf{A}$ , where *the vector  $n\mathbf{A}$  is called the scalar multiple of vector **A***. This means that  $2\mathbf{A}$  is a vector that is twice as long as **A** but is *in the same direction*. Note carefully that only the magnitudes of **A** and  $2\mathbf{A}$  are different, and their directions are the same. The addition of scalar multiples of vectors is illustrated in the following example.

**EXAMPLE 5 Scalar multiple of a vector**

For vectors **A** and **B** in Fig. 9.6, find vector  $3\mathbf{A} + 2\mathbf{B}$ .

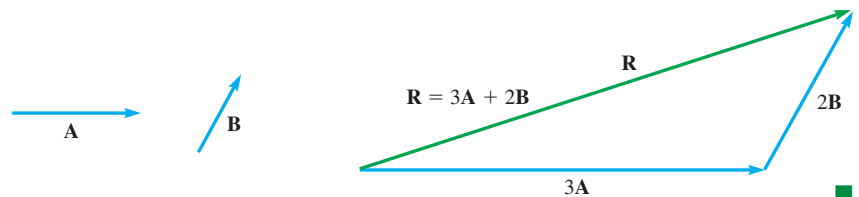


Fig. 9.6

**Subtraction of Vectors**

Vector **B** is subtracted from vector **A** by reversing the direction of **B** and proceeding as in vector addition. Thus,  $\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B})$ , where *the minus sign indicates that vector  $-\mathbf{B}$  has the opposite direction of vector **B***. Vector subtraction is illustrated in the following example.

**EXAMPLE 6 Subtracting vectors**

For vectors **A** and **B** in Fig. 9.7, find vector  $2\mathbf{A} - \mathbf{B}$ .

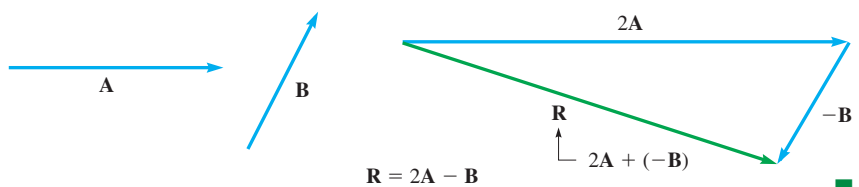


Fig. 9.7

**Practice Exercise**

- For the vectors in Example 6, find vector  $\mathbf{B} - 3\mathbf{A}$ .

Among the most important applications of vectors is that of the forces acting on a structure or on an object. The next example shows the addition of forces by using the parallelogram method.

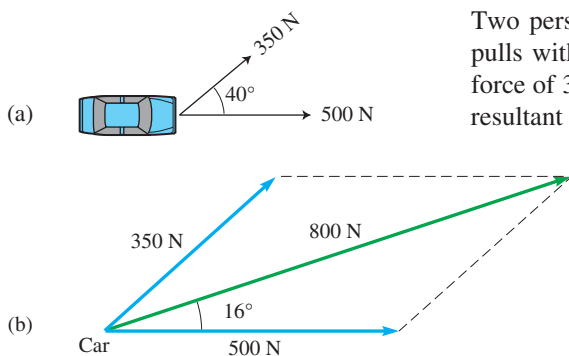


Fig. 9.8

**EXAMPLE 7 Adding vectors—application**

Two persons pull horizontally on ropes attached to a car mired in mud. One person pulls with a force of 500 N directly to the right, while the other person pulls with a force of 350 N at an angle of 40° from the first force, as shown in Fig. 9.8(a). Find the resultant force on the car.

We make a scale drawing of the forces as shown in Fig. 9.8(b), measuring the magnitudes of the forces with a ruler and the angles with a protractor. (The scale drawing of the forces is made larger and with a different scale than that in Fig. 9.8(a) in order to get better accuracy.) We then complete the parallelogram and draw in the diagonal that represents the resultant force. Finally, we find that the resultant force is about 800 N and that it acts at an angle of about 16° from the first force. ■

Two other important vector quantities are *velocity* and *displacement*. Velocity as a vector is illustrated in Examples 1 and 2. The **displacement** of an object is the change in its position. Displacement is given by the distance from a reference point and the angle from a reference direction. The following example illustrates the difference between *distance* and *displacement*.

**EXAMPLE 8 Adding vectors—displacement**

To avoid a storm, a jet travels at 60° north of east from Detroit for 310 km and then turns to a direction of 10° north of east for 450 km to Ottawa. Find the displacement of Ottawa from Detroit.

We make a scale drawing in Fig. 9.9 to show the route taken by the jet. Measuring distances with a ruler and angles with a protractor, we find that Ottawa is about 680 km from Detroit, at an angle of about 30° north of east. By giving both the magnitude and the **direction**, we have given the displacement.

If the jet returned directly from Ottawa to Detroit, its *displacement* from Detroit would be *zero*, although it travelled a *distance* of about 1440 km. ■

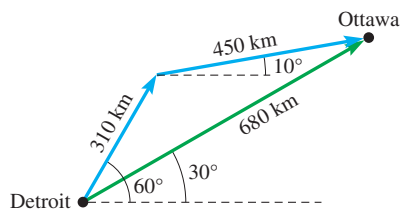


Fig. 9.9

**EXERCISES 9.1**

In Exercises 1–4, find the resultant vectors if the given changes are made in the indicated examples of this section.

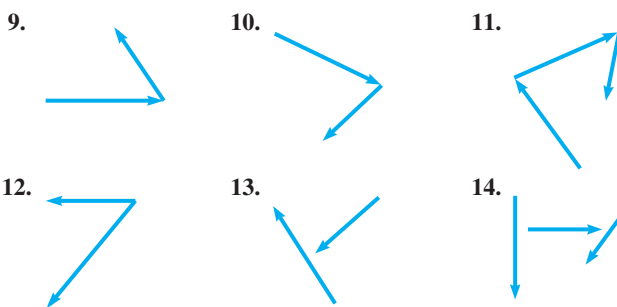
1. In Example 3, what is the resultant of the three vectors if the direction of vector **A** is reversed?
2. In Example 5, for vectors **A** and **B**, what is vector  $2\mathbf{A} + 3\mathbf{B}$ ?
3. In Example 6, for vectors **A** and **B**, what is vector  $2\mathbf{B} - \mathbf{A}$ ?
4. In Example 7, if 20° replaces 40°, what is the resultant force?

In Exercises 5–8, determine whether a scalar or a vector is described in (a) and (b). Explain your answers.

5. (a) A soccer player runs 15 m from the centre of the field.  
(b) A soccer player runs 15 m from the centre of the field toward the opponents' goal.
6. (a) A small-craft warning reports 35 km/h winds.  
(b) A small-craft warning reports 35 km/h winds from the north.
7. (a) An arm of an industrial robot pushes with a 10-N force downward on a part.  
(b) A part is being pushed with a 10-N force by an arm of an industrial robot.

8. (a) A ballistics test shows that a bullet hit a wall at a speed of 100 m/s.  
(b) A ballistics test shows that a bullet hit a wall at a speed of 100 m/s perpendicular to the wall.

In Exercises 9–14, add the given vectors by drawing the appropriate resultant.

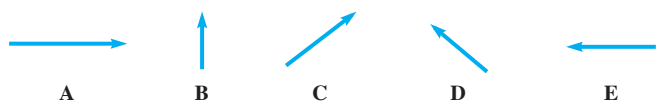




In Exercises 15–18, draw the given vectors and find their sum graphically. The magnitude is shown first, followed by the direction as an angle in standard position.

15. 3.6 cm,  $0^\circ$ ; 4.3 cm,  $90^\circ$       16. 2.3 cm,  $45^\circ$ ; 5.2 cm,  $120^\circ$   
 17. 6.0 cm,  $150^\circ$ ; 1.8 cm,  $315^\circ$       18. 7.5 cm,  $240^\circ$ ; 2.3 cm,  $30^\circ$

In Exercises 19–40, find the indicated vector sums and differences with the given vectors by means of diagrams. (You might find graph paper to be helpful.)



19.  $A + B$       20.  $B + C$       21.  $C + D$       22.  $D + E$   
 23.  $A + C + E$       24.  $B + D + A$       25.  $2A + D + E$   
 26.  $B + E + A$       27.  $B + 3E$       28.  $A + 2C$   
 29.  $3C + E$       30.  $2D + A$       31.  $A - B$   
 32.  $C - D$       33.  $E - B$       34.  $C - 2A$   
 35.  $3B + \frac{1}{2}A$       36.  $2B - \frac{3}{2}E$       37.  $B + 2C - E$   
 38.  $A + 2D - 3B$       39.  $C - B - \frac{3}{4}A$       40.  $D - 2C - \frac{1}{2}E$

In Exercises 41–48, solve the given problems. Use a ruler and protractor as in Examples 7 and 8.

41. Two forces that act on an airplane wing are called the *lift* and the *drag*. Find the resultant of these forces acting on the airplane wing in Fig. 9.10.

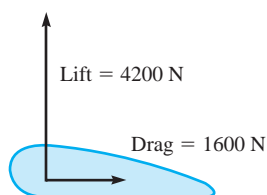


Fig. 9.10

42. Two electric charges create an electric field intensity, a vector quantity, at a given point. The field intensity is 30 kN/C to the right and 60 kN/C at an angle of  $45^\circ$  above the horizontal to the right. Find the resultant electric field intensity at this point.
43. A ski tow is moving skiers vertically upward at 24 m/min and horizontally at 44 m/min. What is the velocity of the tow?
44. A small plane travels at 180 km/h in still air. It is headed due south in a wind of 50 km/h from the northeast. What is the resultant velocity of the plane?
45. A driver takes the wrong road at an intersection and travels 4 km north, then 6 km east, and finally 10 km to the southeast to reach the home of a friend. What is the displacement of the friend's home from the intersection?
46. A ship travels 20 km in a direction of  $30^\circ$  south of east and then turns due south for another 40 km. What is the ship's displacement from its initial position?
47. Three ropes hold a helium-filled balloon in place, but two of the ropes break. The remaining rope holds the balloon with a tension of 510 N at an angle of  $80^\circ$  with the ground due to a wind. The weight (a vertical force) of the balloon and contents is 400 N, and the upward buoyant force is 900 N. The wind creates a horizontal force of 90 N on the balloon. What is the resultant force on the balloon?
48. While unloading a crate weighing 610 N, the chain from a crane supports it with a force of 650 N at an angle of  $20^\circ$  from the vertical. What force must a horizontal rope exert on the crate so that the total force (including its weight) on the crate is zero?

### Answers to Practice Exercises

1. Same as  $R$  in Fig. 9.4.      2. (Half scale)

## 9.2 Components of Vectors

Resolving a Vector into Components • Finding  $x$ - and  $y$ -Components • Meaning of Sign of Component

### Resolving a Vector into Components

Using diagrams is useful in developing an understanding of vectors. However, unless the diagrams are drawn with great care, the results we get are not too accurate. Therefore, other methods are needed in order to get more accurate results.

In this section, we show how a given vector can be written as the sum of two other vectors, with any required degree of accuracy. In the next section, we show how this lets us add vectors to get their sum with the required accuracy in the result.

Two vectors that, when added together, have a resultant equal to the original vector are called **components** of the original vector. In the illustration of the boat in Section 9.1, the velocities of 8 km/h across the river and 6 km/h downstream are components of the 10 km/h vector directed at the angle  $\theta$ .

Certain components of a vector are of particular importance. If the initial point of a vector is placed at the origin of a rectangular coordinate system and its direction is given by an angle in standard position, we may find its  **$x$ - and  $y$ -components**. These components are vectors directed along the axes that, when added together, equal the given vector. The initial points of these components are at the origin, and the terminal points are at the points where perpendicular lines from the terminal point of the given vector cross the axes. Finding these component vectors is called **resolving** the vector into its components.



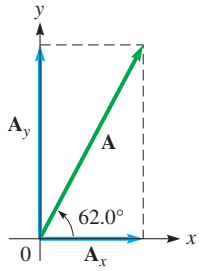


Fig. 9.11

**EXAMPLE 1** Components of a first-quadrant vector

Find the  $x$ - and  $y$ -components of the vector  $\mathbf{A}$  shown in Fig. 9.11. The magnitude of  $\mathbf{A}$  is 7.25.

From the figure, we see that  $A_x$ , the magnitude of the  $x$ -component of  $\mathbf{A}_x$ , is related to  $\mathbf{A}$  by

$$\frac{A_x}{A} = \cos 62.0^\circ$$

$$A_x = A \cos 62.0^\circ$$

In the same way,  $A_y$ , the magnitude of the  $y$ -component of  $\mathbf{A}_y$ , is related to  $\mathbf{A}$  ( $\mathbf{A}_y$  could be placed along the vertical dashed line) by

$$\frac{A_y}{A} = \sin 62.0^\circ$$

$$A_y = A \sin 62.0^\circ$$

From these relations, knowing that  $A = 7.25$ , we have

$$A_x = 7.25 \cos 62.0^\circ = 3.40$$

$$A_y = 7.25 \sin 62.0^\circ = 6.40$$

This means that the  $x$ -component is directed along the  $x$ -axis to the right and has a magnitude of 3.40. Also, the  $y$ -component is directed along the  $y$ -axis upward and its magnitude is 6.40. These two component vectors can replace vector  $\mathbf{A}$ , since the effect they have is the same as  $\mathbf{A}$ . ■

**Practice Exercise**

- For the vector in Example 1, change the angle to  $25.0^\circ$  and find the  $x$ - and  $y$ -components.

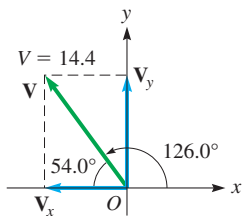


Fig. 9.12

**EXAMPLE 2** Components of a second-quadrant vector

Resolve a vector 14.4 units long and directed at an angle of  $126.0^\circ$  into its  $x$ - and  $y$ -components. See Fig. 9.12.

Placing the initial point of the vector at the origin and putting the angle in standard position, note that the vector directed along the  $x$ -axis,  $\mathbf{V}_x$ , is related to the vector  $\mathbf{V}$  of magnitude  $V$  by

$$V_x = \underbrace{V}_{\text{magnitude of vector}} \cos \underbrace{126.0^\circ}_{\text{standard-position angle}}$$

Since the vector directed along the  $y$ -axis,  $\mathbf{V}_y$ , could also be placed along the vertical dashed line, it is related to the vector  $\mathbf{V}$  by

$$V_y = V \sin 126.0^\circ$$

Thus, the vectors  $\mathbf{V}_x$  and  $\mathbf{V}_y$  have the magnitudes

$$V_x = 14.4 \cos 126.0^\circ = -8.46 \quad V_y = 14.4 \sin 126.0^\circ = 11.6$$

Therefore, we have resolved the given vector into two components: one, directed along the negative  $x$ -axis, of magnitude 8.46, and the other, directed along the positive  $y$ -axis, of magnitude 11.6.

It is also possible to use the reference angle, as long as *the proper sign is attached to each component*. In this case, the reference angle is  $54.0^\circ$ , and therefore

$$V_x = \underbrace{-14.4}_{\text{directed along negative } x\text{-axis}} \cos 54.0^\circ = -8.46 \quad V_y = \underbrace{+14.4}_{\text{directed along positive } y\text{-axis}} \sin 54.0^\circ = 11.6$$

The minus sign shows that the  $x$ -component is directed to the left. ■

**Practice Exercise**

- For the vector in Example 2, change the angle to  $306.0^\circ$  and find the  $x$ - and  $y$ -components.

From Examples 1 and 2, we can see that the steps used in finding the  $x$ - and  $y$ -components of a vector are as follows:

### Steps Used in Finding the $x$ - and $y$ -Components of a Vector

1. Place vector  $\mathbf{A}$  such that  $\theta$  is in standard position.
2. Calculate  $A_x$  and  $A_y$  from  $A_x = A \cos \theta$  and  $A_y = A \sin \theta$ . We may use the reference angle if we note the direction of the component.
3. Check the components to see if each is in the correct direction and has a magnitude that is proper for the reference angle.

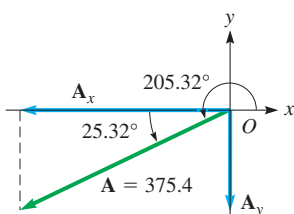


Fig. 9.13

### EXAMPLE 3 Components of a third-quadrant vector

Resolve vector  $\mathbf{A}$ , of magnitude 375.4 and direction  $\theta = 205.32^\circ$ , into its  $x$ - and  $y$ -components. See Fig. 9.13.

By placing  $\mathbf{A}$  such that  $\theta$  is in standard position, we see that

$$A_x = A \cos 205.32^\circ = 375.4 \cos 205.32^\circ = -339.3$$

and

$$A_y = A \sin 205.32^\circ = 375.4 \sin 205.32^\circ = -160.5$$

↑ directed along  
negative axis

The angle  $205.32^\circ$  places the vector in the third quadrant, and each of the components is directed along the negative axis. This must be the case for a third-quadrant angle. Also, the reference angle is  $25.32^\circ$ , and we see that the magnitude of  $A_x$  is greater than the magnitude of  $A_y$ , which must be true for a reference angle that is less than  $45^\circ$ . ■

### EXAMPLE 4 Vector components—application to tension

The tension  $\mathbf{T}$  in a cable supporting the sign shown in Fig. 9.14(a) is 85.0 N. If the cable makes an angle of  $53.5^\circ$  with the horizontal, find the horizontal and vertical components of the tension.

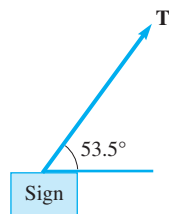
The tension is the force the cable exerts on the sign. Showing the tension in Fig. 9.14(b), we see that

$$\begin{aligned} T_y &= T \sin 53.5^\circ = 85.0 \sin 53.5^\circ \\ &= 68.3 \text{ N} \end{aligned}$$

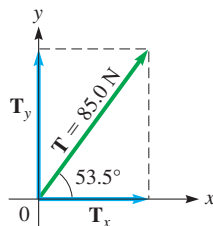
$$\begin{aligned} T_x &= T \cos 53.5^\circ = 85.0 \cos 53.5^\circ \\ &= 50.6 \text{ N} \end{aligned}$$

Note that  $T_y > T_x$ , which should be the case for an acute angle greater than  $45^\circ$ . ■

Fig. 9.14 (a)



(b)



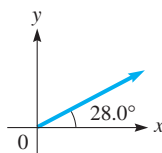
## EXERCISES 9.2

In Exercises 1–4, find the component vectors if the given changes are made in the indicated examples of this section.

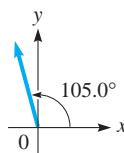
1. In Example 2, change  $126.0^\circ$  to  $216.0^\circ$ .
2. In Example 3, change  $205.32^\circ$  to  $295.32^\circ$ .
3. In Example 3, change  $205.32^\circ$  to  $270.00^\circ$ .
4. In Example 4, change  $53.5^\circ$  to  $45.0^\circ$ .

In Exercises 5–10, find the horizontal and vertical components of the vectors shown in the given figures. In each, the magnitude of the vector is 750.

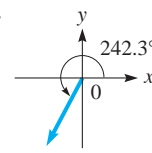
5.

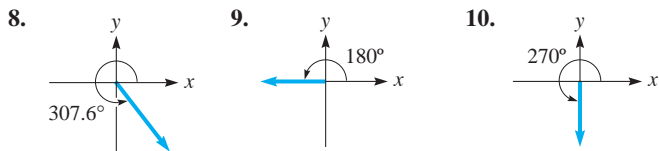


6.



7.





In Exercises 11–20, find the  $x$ - and  $y$ -components of the given vectors by use of the trigonometric functions. The magnitude is shown first, followed by the direction as an angle in standard position.

- |   |   |
|---|---|
| 11. 8.60 N, $\theta = 68.0^\circ$                   | 12. 9750 N, $\theta = 243.0^\circ$                  |
| 13. 76.8 m/s, $\theta = 145.0^\circ$                | 14. 0.0998 dm/s, $\theta = 296.0^\circ$             |
| 15. 9040 mm/s <sup>2</sup> , $\theta = 270.0^\circ$ | 16. 16.4 cm/s <sup>2</sup> , $\theta = 156.5^\circ$ |
| 17. 2.65 mN, $\theta = 197.3^\circ$                 | 18. 678 N, $\theta = 22.5^\circ$                    |
| 19. 0.8734 dm, $\theta = 157.83^\circ$              | 20. 509.4 m, $\theta = 360.0^\circ$                 |

In Exercises 21–34, find the required horizontal and vertical components of the given vectors.

21. A nuclear submarine approaches the surface of the ocean at 25.0 km/h and at an angle of  $17.3^\circ$  with the surface. What are the components of its velocity? See Fig. 9.15.

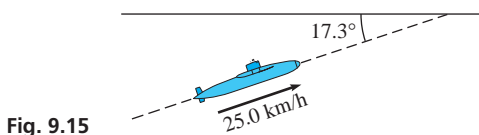


Fig. 9.15

22. Water is flowing downhill at 8.0 m/s through a pipe that is at an angle of  $66.4^\circ$  with the horizontal. What are the components of its velocity?
23. A car is being unloaded from a ship. It is supported by a cable from a crane and guided into position by a horizontal rope. If the tension in the cable is 12 400 N and the cable makes an angle of  $3.5^\circ$  with the vertical, what are the weight  $W$  of the car and the tension  $T$  in the rope? (The weight of the cable is negligible to that of the car.) See Fig. 9.16.

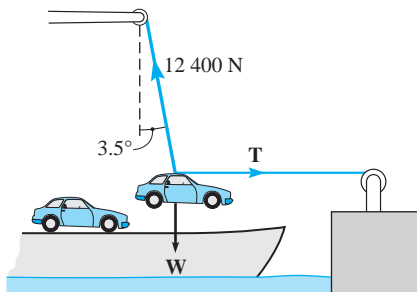


Fig. 9.16

24. A car travelling at 25 km/h hits a stone wall at an angle of  $65^\circ$  with the wall. What is the component of the car's velocity that is perpendicular to the wall?
25. With the sun directly overhead, a plane is taking off at 125 km/h at an angle of  $22.0^\circ$  above the horizontal. How fast is the plane's shadow moving along the runway?
26. A jet is 145 km at a position  $37.5^\circ$  north of east of Vancouver. What are the components of the jet's displacement from Vancouver?

27. The end of a robot arm is 1.20 m on a line  $78.6^\circ$  above the horizontal from the point where it does a weld. What are the components of the displacement from the end of the robot arm to the welding point?
28. The tension in a rope attached to a boat is 55.0 N. The rope is attached to the boat 4.00 m below the level at which it is being drawn in. At the point where there is 12.0 m of rope out, what force tends to bring the boat toward the wharf, and what force tends to raise the boat? See Fig. 9.17.



Fig. 9.17

29. A person applies a force of 210 N perpendicular to a jack handle that is at an angle of  $25^\circ$  above the horizontal. What are the horizontal and vertical components of the force?
30. A water skier is pulled up the ramp shown in Fig. 9.18 at 8.5 m/s. How fast is the skier rising when leaving the ramp?

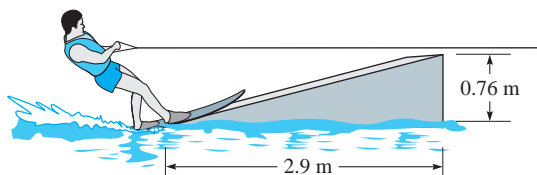


Fig. 9.18

31. A wheelbarrow is being pushed with a force of 310 N directed along the handles, which are at an angle of  $20^\circ$  above the horizontal. What is the effective force for moving the wheelbarrow forward?
32. Two upward forces are acting on a bolt. One force of 60.5 N acts at an angle of  $82.4^\circ$  above the horizontal, and the other force of 37.2 N acts at an angle of  $50.5^\circ$  below the first force. What is the total upward force on the bolt? See Fig. 9.19.

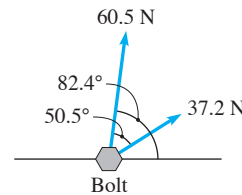


Fig. 9.19

33. Vertical wind shear in the lowest 100 m above the ground is of great importance to aircraft when taking off or landing. It is defined as the rate at which the wind velocity changes per metre above ground. If the vertical wind shear at 50 m above the ground is 0.75 (km/h)/m directed at angle of  $40^\circ$  above the ground, what are its vertical and horizontal components?
34. At one point, the *Pioneer* space probe was entering the gravitational field of Jupiter at an angle of  $2.55^\circ$  below the horizontal with a velocity of 29 870 km/h. What were the components of its velocity?

Answers to Practice Exercises

1.  $A_x = 6.57$ ,  $A_y = 3.06$     2.  $V_x = 8.46$ ,  $V_y = -11.6$

## 9.3 Vector Addition by Components

Adding Vectors at Right Angles •  
 Adding  $x$ -Components and  
 Adding  $y$ -Components •  
 Standard-Position Angle of Resultant

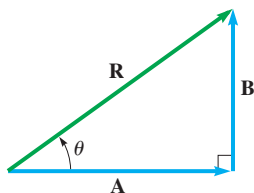


Fig. 9.20

Now that we have developed the meaning of the components of a vector, we are able to add vectors to any degree of required accuracy. To do this, we use the components of the vector, the Pythagorean theorem, and the tangent of the standard-position angle of the resultant. In the following example, two vectors at right angles are added.

### EXAMPLE 1 Adding vectors at right angles

Add vectors  $\mathbf{A}$  and  $\mathbf{B}$ , with  $A = 14.5$  and  $B = 9.10$ . The vectors are at right angles, as shown in Fig. 9.20.

We can find the magnitude  $R$  of the resultant vector  $\mathbf{R}$  by use of the Pythagorean theorem. This leads to

$$\begin{aligned} R &= \sqrt{A^2 + B^2} = \sqrt{(14.5)^2 + (9.10)^2} \\ &= 17.1 \end{aligned}$$

We now determine the direction of the resultant vector  $\mathbf{R}$  by specifying its direction as the angle  $\theta$  in Fig. 9.20, that is, the angle that  $\mathbf{R}$  makes with vector  $\mathbf{A}$ . Therefore,

$$\begin{aligned} \tan \theta &= \frac{B}{A} = \frac{9.10}{14.5} \\ \theta &= 32.1^\circ \end{aligned}$$

Therefore, we see that  $\mathbf{R}$  is a vector of magnitude  $R = 17.1$  and in a direction  $32.1^\circ$  from vector  $\mathbf{A}$ .

Note that Fig. 9.20 shows vectors  $\mathbf{A}$  and  $\mathbf{B}$  as horizontal and vertical, respectively. This means they are the horizontal and vertical components of the resultant vector  $\mathbf{R}$ . However, we would find the resultant of any two vectors *at right angles* in the same way. ■

### COMMON ERROR

*Remember, a vector is not completely specified unless both its magnitude and its direction are specified. A common error is to determine the magnitude, but not to find the angle  $\theta$  that is used to define its direction.*

If the vectors being added are not at right angles, the resultant is found by first finding the components of the given vectors and then combining components. The general procedure will be illustrated in the following examples and can be summarized as follows.

### Procedure for Adding Vectors by Components

1. Place each vector with its tail at the origin.
2. Resolve each vector into its  $x$ - and  $y$ -components.
3. Add the magnitudes of the  $x$ -components of the given vectors to obtain  $R_x$ , the magnitude of the  $x$ -component of the resultant.
4. Add the magnitudes of the  $y$ -components of the given vectors to obtain  $R_y$ , the magnitude of the  $y$ -component of the resultant.
5. Find the magnitude of the resultant  $\mathbf{R}$  as

$$R = \sqrt{R_x^2 + R_y^2}$$

6. Find the standard-position angle  $\theta$  for the resultant  $\mathbf{R}$ . If the angle is not in the first quadrant, find the reference angle  $\theta_{\text{ref}}$  first as

$$\theta_{\text{ref}} = \tan^{-1} \left| \frac{R_y}{R_x} \right|$$

Depending on the quadrant, the required angle  $\theta$  is then found through Eq. (8.6).

**EXAMPLE 2** Adding by first combining components

Find the resultant of two vectors  $\mathbf{A}$  and  $\mathbf{B}$  such that  $A = 1200$ ,  $\theta_A = 270.0^\circ$ ,  $B = 1750$ , and  $\theta_B = 115.0^\circ$ .

First, place the vectors on a coordinate system with the tail of each at the origin as shown in Fig. 9.21(a). Then resolve each vector into its  $x$ - and  $y$ -components, as shown in Fig. 9.21(b) and as calculated below. (Note that  $\mathbf{A}$  is vertical and has no horizontal component.) Next, the components are combined, as in Fig. 9.21(c) and as calculated. Finally, the magnitude of the resultant and the angle  $\theta$  (to determine the direction), as shown in Fig. 9.21(d), are calculated.

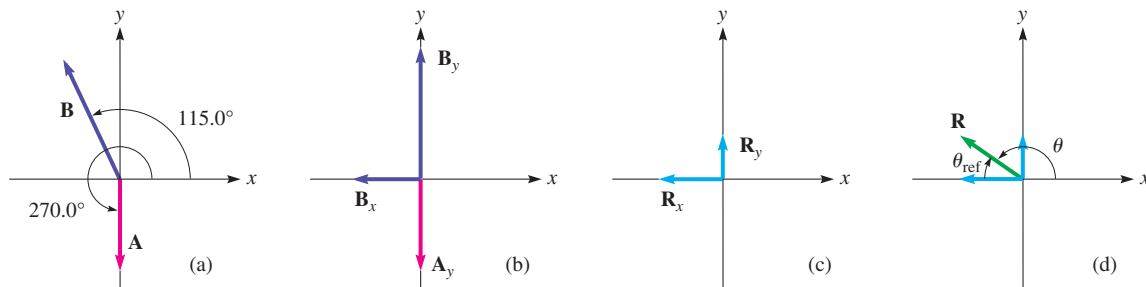


Fig. 9.21

$$A_x = A \cos 270.0^\circ = 1200 \cos 270.0^\circ = 0$$

$$B_x = B \cos 115.0^\circ = 1750 \cos 115.0^\circ = -739.6$$

$$A_y = A \sin 270.0^\circ = 1200 \sin 270.0^\circ = -1200$$

$$B_y = B \sin 115.0^\circ = 1750 \sin 115.0^\circ = 1586$$

Fig. 9.21 (b)

$$R_x = A_x + B_x = 0 - 739.6 = -739.6$$

$$R_y = A_y + B_y = -1200 + 1586 = 386$$

Fig. 9.21 (c)

$$R = \sqrt{R_x^2 + R_y^2} = \sqrt{(-739.6)^2 + 386^2} = 834$$

$$\tan \theta_{\text{ref}} = \left| \frac{R_y}{R_x} \right| = \frac{386}{739.6}$$

$$\theta_{\text{ref}} = 27.6^\circ$$

$$\theta = 152.4^\circ \leftarrow 180^\circ - 27.6^\circ$$

Fig. 9.21 (d)

**Practice Exercise**

- In Example 2, change  $270^\circ$  to  $90^\circ$  and then add the vectors.

In the last line of the display, we subtracted  $\theta_{\text{ref}}$  from  $180^\circ$  because  $R_x$  is negative and  $R_y$  is positive, placing  $\theta$  in the second quadrant. Thus, the resultant has a magnitude of 834 and is directed at a standard-position angle of  $152.4^\circ$ .

Note that if we obtained the tangent of  $\theta$  directly from  $R_y/R_x$ , the calculator would give us an angle of  $-27.6^\circ$ , where we would have to recognize  $27.6^\circ$  as the reference angle. Instead, we obtain the reference angle right away by disregarding the signs of  $R_x$  and  $R_y$ .

The values in this example are shown rounded off. However, to reduce the accumulation of rounding errors when using the calculator,  $R_x$  and  $R_y$  are each calculated in one step and stored for the calculation of  $R$  and  $\theta$ .

When we found  $R_y$ , we saw that we could do a vector addition as a scalar addition. Also, since the magnitude of the components and the resultant are much smaller than either of the original vectors, this vector addition would have been difficult to do accurately by means of a diagram. ■

**EXAMPLE 3** Adding by first combining components

Find the resultant  $\mathbf{R}$  of the two vectors shown in Fig. 9.22(a),  $\mathbf{A}$  of magnitude 8.075 and standard-position angle of  $57.26^\circ$  and  $\mathbf{B}$  of magnitude 5.437 and standard-position angle of  $322.15^\circ$ .

In Fig. 9.22(b), we show the components of vectors  $\mathbf{A}$  and  $\mathbf{B}$ , and then in Fig. 9.22(c), we show the resultant and its components.

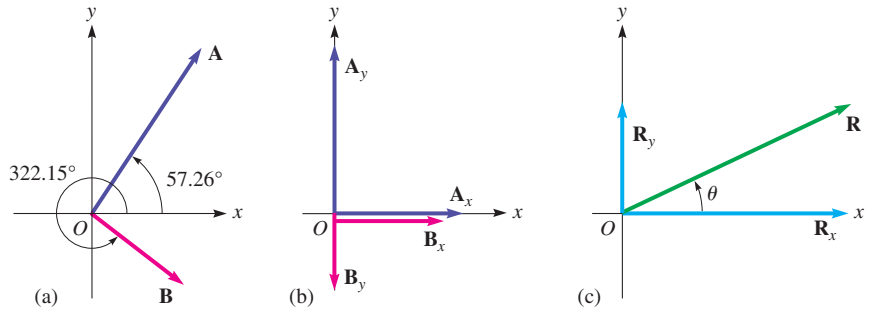


Fig. 9.22

Vector	Magnitude	Angle	Magnitude of x-Component	Magnitude of y-Component
A	8.075	57.26°	$A_x = 8.075 \cos 57.26^\circ = 4.367$	$A_y = 8.075 \sin 57.26^\circ = 6.792$
B	5.437	322.15°	$B_x = 5.437 \cos 322.15^\circ = 4.293$	$B_y = 5.437 \sin 322.15^\circ = -3.336$
R			$R_x = A_x + B_x = 8.660$	$R_y = A_y + B_y = 3.456$

$$R = \sqrt{R_x^2 + R_y^2} = \sqrt{(8.660)^2 + (3.456)^2} = 9.324$$

$$\theta = \tan^{-1} \frac{R_y}{R_x} = \tan^{-1} \left( \frac{3.456}{8.660} \right) = 21.76^\circ \leftarrow \text{don't forget the direction}$$

The resultant vector is 9.324 units long and is directed at a standard-position angle of 21.76°, as shown in Fig. 9.22(c). We know that the resultant is in the first quadrant since both  $R_x$  and  $R_y$  are positive. Once again, the values are shown rounded, but values have been stored for use in the calculation of  $R$  and  $\theta$ . ■

Some general formulas can be derived from the previous examples. For a given vector **A**, directed at an angle  $\theta$ , of magnitude  $A$ , and with components of magnitude  $A_x$  and  $A_y$ , we have the following relations:

$$A_x = A \cos \theta \quad A_y = A \sin \theta \tag{9.1}$$

$$A = \sqrt{A_x^2 + A_y^2} \tag{9.2}$$

$$\theta_{\text{ref}} = \tan^{-1} \frac{|A_y|}{|A_x|} \tag{9.3}$$

The value of  $\theta$  is found by using the reference angle from Eq. (9.3) and the quadrant in which the resultant lies.

**EXAMPLE 4** Addition of three vectors

Find the resultant of the three given vectors in Fig. 9.23. The magnitudes of these vectors are  $T = 422$ ,  $U = 405$ , and  $V = 210$ .

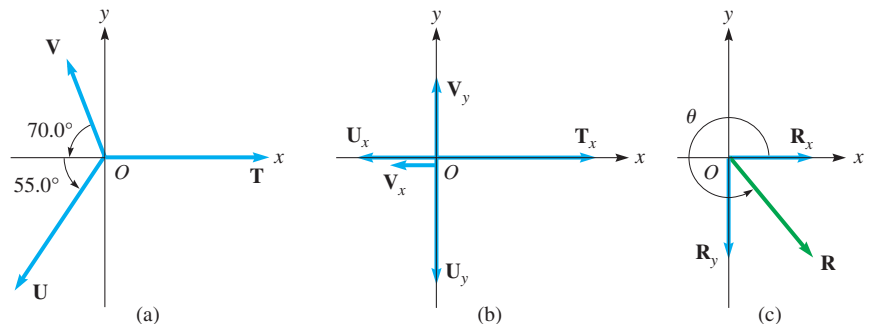


Fig. 9.23

**Practice Exercise**

- In Example 4, find the resultant of vectors **U** and **V**. (Do not include vector **T**.)

We could change the given angles to standard-position angles. However, we will use the given angles, *being careful to give the proper sign to each component*. In the following table, we show the  $x$ - and  $y$ -components of the given vectors, and the sums of these components give us the components of  $\mathbf{R}$ .

Vector	Magnitude	Ref. Angle	Magnitude of $x$ -Component	Magnitude of $y$ -Component
$\mathbf{T}$	422	$0^\circ$	$422 \cos 0^\circ = 422.0$	$422 \sin 0^\circ = 0.0$
$\mathbf{U}$	405	$55.0^\circ$	$-405 \cos 55.0^\circ = -232.3$	$-405 \sin 55.0^\circ = -331.8$
$\mathbf{V}$	210	$70.0^\circ$	$-210 \cos 70.0^\circ = -71.8$	$+210 \sin 70.0^\circ = 197.3$
$\mathbf{R}$			117.9	-134.5

From these components, we find  $R$  and  $\theta_{\text{ref}}$ :

$$\mathbf{R} = \sqrt{R_x^2 + R_y^2} = \sqrt{(117.9)^2 + (-134.5)^2} = 179$$

$$\theta_{\text{ref}} = \tan^{-1} \frac{|-134.5|}{|117.9|} = 48.8^\circ$$

Since  $R_x$  is positive and  $R_y$  is negative, we know that  $\theta$  is a fourth-quadrant angle. Therefore, to find  $\theta$ , we subtract  $\theta_{\text{ref}}$  from  $360^\circ$ . This means

$$\theta = 360^\circ - 48.8^\circ = 311.2^\circ$$

### EXERCISES 9.3

In Exercises 1 and 2, find the resultant vectors if the given changes are made in the indicated examples of this section.

- In Example 2, find the resultant if  $\theta_A$  is changed to  $0^\circ$ .
- In Example 3, find the resultant if  $\theta_B$  is changed to  $232.15^\circ$ .

In Exercises 3–6, vectors  $\mathbf{A}$  and  $\mathbf{B}$  are at right angles. Find the magnitude and direction (the angle from vector  $\mathbf{A}$ ) of the resultant.

- |                               |                             |
|-------------------------------|-----------------------------|
| 3. $A = 14.7$<br>$B = 19.2$   | 4. $A = 592$<br>$B = 195$   |
| 5. $A = 3.086$<br>$B = 7.143$ | 6. $A = 1734$<br>$B = 3297$ |

In Exercises 7–14, with the given sets of components, find  $R$  and  $\theta$ .

- |                               |                                 |
|-------------------------------|---------------------------------|
| 7. $R_x = 5.18, R_y = 8.56$   | 8. $R_x = 89.6, R_y = -52.0$    |
| 9. $R_x = -0.982, R_y = 2.56$ | 10. $R_x = -729, R_y = -209$    |
| 11. $R_x = -646, R_y = 2030$  | 12. $R_x = -31.2, R_y = -41.2$  |
| 13. $R_x = 6941, R_y = -1246$ | 14. $R_x = 7.627, R_y = -6.353$ |

In Exercises 15–30, add the given vectors by components.

- |   |  |
|---|--|
| 15. $A = 18.0, \theta_A = 0.00^\circ$<br>$B = 12.0, \theta_B = 27.0^\circ$  | 16. $F = 154, \theta_F = 90.0^\circ$<br>$T = 128, \theta_T = 43.0^\circ$                                       |
| 17. $C = 5650, \theta_C = 76.0^\circ$<br>$D = 1280, \theta_D = 160.0^\circ$                                       | 18. $A = 6.89, \theta_A = 123.0^\circ$<br>$B = 29.0, \theta_B = 260.0^\circ$                                   |
| 19. $A = 9.821, \theta_A = 34.27^\circ$<br>$B = 17.45, \theta_B = 752.50^\circ$                                   | 20. $E = 1653, \theta_E = 36.37^\circ$<br>$F = 9807, \theta_F = 253.06^\circ$                                  |
| 21. $A = 21.9, \theta_A = 236.2^\circ$<br>$B = 96.7, \theta_B = 11.5^\circ$<br>$C = 62.9, \theta_C = 143.4^\circ$ | 22. $R = 630, \theta_R = 189.6^\circ$<br>$F = 176, \theta_F = 320.1^\circ$<br>$T = 324, \theta_T = 75.4^\circ$ |

23.  $U = 0.364, \theta_U = 175.7^\circ$   
 $V = 0.596, \theta_V = 319.5^\circ$   
 $W = 0.129, \theta_W = 100.6^\circ$

25. The vectors in Fig. 9.24.

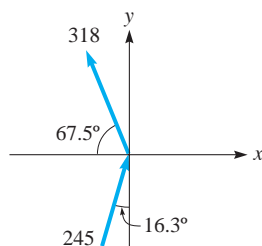


Fig. 9.24

24.  $A = 6.4, \theta_A = 126^\circ$   
 $B = 5.9, \theta_B = 238^\circ$   
 $C = 3.2, \theta_C = 72^\circ$

26. The vectors in Fig. 9.25.

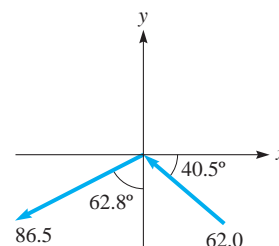


Fig. 9.25

27. The vectors in Fig. 9.26.

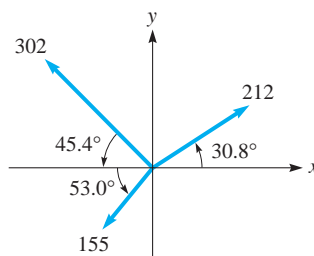


Fig. 9.26

28. The vectors in Fig. 9.27.

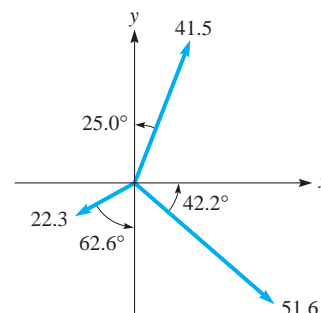


Fig. 9.27



29. In order to move an ocean liner into the channel, three tugboats exert the forces shown in Fig. 9.28. What is the resultant of these forces?

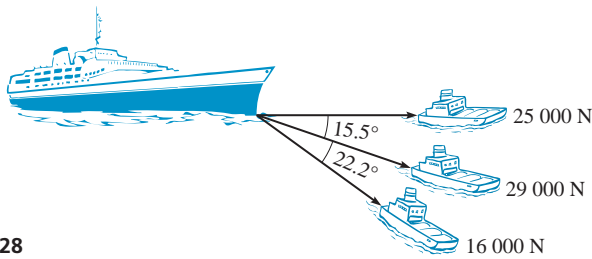


Fig. 9.28

30. A naval cruiser on maneuvers travels 54.0 km at 18.7° west of north, then turns and travels 64.5 km at 15.6° south of east, and finally turns to travel 72.4 km at 38.1° east of south. Find its displacement from its original position. See Fig. 9.29.

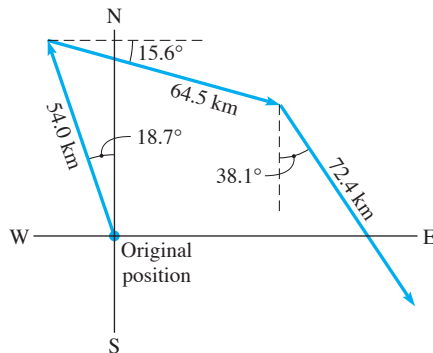


Fig. 9.29

In Exercises 31–34, solve the given problems.

- 31. The angle between two forces of 17 N and 25 N is 37°. What is the magnitude of the resultant force?
- 32. What is the magnitude of a velocity vector that makes an angle of 18.0° with its horizontal component of 420 km/h?
- 33. The angle between two equal-momentum vectors of 15.0 kg · m/s in magnitude is 72.0°. What is the magnitude of the resultant?
- 34. The resultant **R** of displacements **A** and **B** is perpendicular to **A**. If  $A = 25$  m,  $B = 35$  m, and  $R = 25$  m, find the angle between **A** and **B**.

Answers to Practice Exercises

- 1.  $R = 2880$ ,  $\theta = 104.9^\circ$
- 2.  $R = 332$ ,  $\theta = 203.8^\circ$

## 9.4 Applications of Vectors

Forces • Displacement • Velocity • Other Applications

In Section 9.1, we introduced the important vector quantities of force, velocity, and displacement, and we found vector sums by use of diagrams. Now, we can use the method of Section 9.3 to find sums of these kinds of vectors and others and to use them in various types of applications.

### EXAMPLE 1 Forces at right angles

In centering a figurine on a table, two persons apply forces on it. These forces are at right angles and have magnitudes of 6.00 N and 8.00 N. The angle between their lines of action is 90.0°. What is the resultant of these forces on the figurine?

By means of an appropriate diagram (Fig. 9.30), we may better visualize the actual situation. Note that a good choice of axes (unless specified, it is often convenient to choose the  $x$ - and  $y$ -axes to fit the problem) is to have the  $x$ -axis in the direction of the 6.00-N force and the  $y$ -axis in the direction of the 8.00-N force. (This is possible since the angle between them is 90°.) With this choice, note that the two given forces will be the  $x$ - and  $y$ -components of the resultant. Therefore, we arrive at the following results:

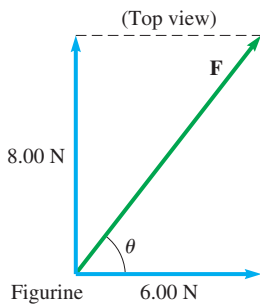


Fig. 9.30

$$\begin{aligned}
 F_x &= 6.00 \text{ N}, F_y = 8.00 \text{ N} \\
 F &= \sqrt{(6.00)^2 + (8.00)^2} = 10.0 \text{ N} \\
 \theta &= \tan^{-1} \frac{F_y}{F_x} = \tan^{-1} \frac{8.00}{6.00} \\
 &= 53.1^\circ
 \end{aligned}$$

The resultant has a magnitude of 10.0 N and acts at an angle of 53.1° from the 6.00-N force.

■ The metric unit of force, the newton (N), is named for the great English mathematician and physicist Sir Isaac Newton (1642–1727). His name will appear on other pages of this text, as some of his many accomplishments are noted.

**EXAMPLE 2** Ship displacement

A ship sails 32.50 km due east and then turns 41.25° north of east. After sailing another 16.18 km, where is it with reference to the starting point?

In this problem, we are to find the resultant displacement of the ship from the two given displacements. The problem is diagrammed in Fig. 9.31, where the first displacement is labelled vector **A** and the second as vector **B**.

Since east corresponds to the positive  $x$ -direction, note that the  $x$ -component of the resultant is  $A + B_x$  and the  $y$ -component of the resultant is  $B_y$ . Therefore, we have the following results:

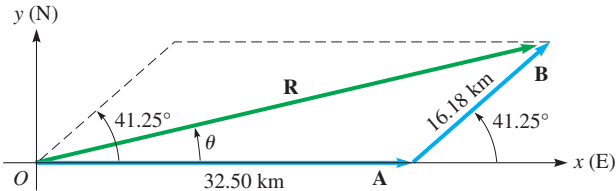


Fig. 9.31

$$\begin{aligned} R_x &= A + B_x = 32.50 + 16.18 \cos 41.25^\circ \\ &= 32.50 + 12.16 \\ &= 44.66 \text{ km} \\ R_y &= 16.18 \sin 41.25^\circ = 10.67 \text{ km} \\ R &= \sqrt{(44.66)^2 + (10.67)^2} = 45.92 \text{ km} \\ \theta &= \tan^{-1} \frac{10.67}{44.66} \\ &= 13.44^\circ \end{aligned}$$

**Practice Exercise**

- In Example 2, change “41.25° north of east” to “in the direction of 41.25° north of west,” and then find the resultant displacement of the ship.

Therefore, the ship is 45.92 km from the starting point, in a direction 13.44° north of east. ■

■ An aircraft’s *heading* is the direction in which it is pointed. Its *air speed* is the speed at which it travels through the air surrounding it. Due to the wind, the heading and air speed, and its actual direction and speed relative to the ground, will differ.

**EXAMPLE 3** Airplane velocity

An airplane headed due east is in a wind blowing from the southeast. What is the resultant velocity of the plane with respect to the surface of the earth if the velocity of the plane with respect to the air is 600 km/h and that of the wind is 100 km/h? See Fig. 9.32.

Let  $v_{px}$  be the velocity of the plane in the  $x$ -direction (east),  $v_{py}$  the velocity of the plane in the  $y$ -direction,  $v_{wx}$  the  $x$ -component of the velocity of the wind,  $v_{wy}$  the  $y$ -component of the velocity of the wind, and  $v_{pa}$  the velocity of the plane with respect to the air. Therefore,

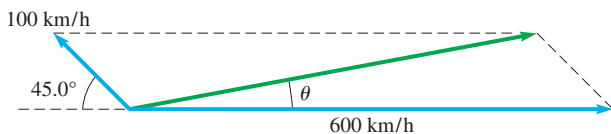


Fig. 9.32

$$\begin{aligned} v_{px} &= v_{pa} + v_{wx} = 600 - 100(\cos 45.0^\circ) = 529 \text{ km/h} \\ v_{py} &= v_{wy} = 100(\sin 45.0^\circ) = 70.7 \text{ km/h} \\ v &= \sqrt{(529)^2 + (70.7)^2} = 534 \text{ km/h} \\ \theta &= \tan^{-1} \frac{v_{py}}{v_{px}} = \tan^{-1} \frac{70.7}{529} \\ &= 8^\circ \end{aligned}$$

The plane is travelling at 534 km/h and is flying in a direction 8° north of east. From this, we observe that a plane does not necessarily head in the direction of its destination. ■

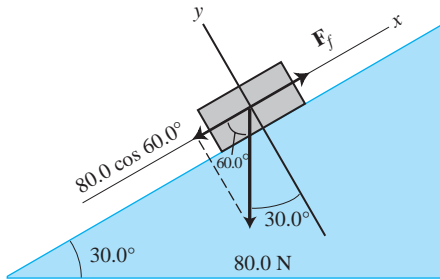
**Equilibrium of Forces**

As we have seen, an important vector quantity is the force acting on an object. One of the most important applications of vectors involves forces that are in **equilibrium**. For an object to be in equilibrium, the net force acting on it in any direction must be zero. This condition is satisfied if the sum of the  $x$ -components of the force is zero and the sum of the  $y$ -components of the force is also zero. The following two examples illustrate forces in equilibrium.

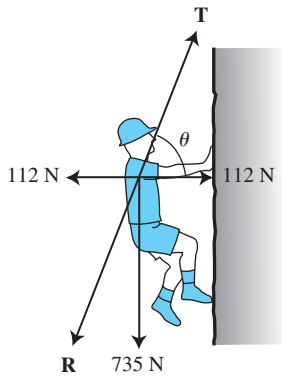
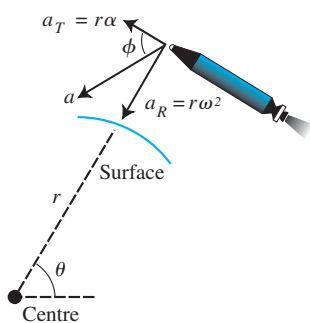
**EXAMPLE 4** Forces on a block on an inclined plane

A cement block is resting on a straight inclined plank that makes an angle of 30.0° with the horizontal. If the block weighs 80.0 N, what is the force of friction between the block and the plank?

The weight of the cement block is the force exerted on the block due to gravity. Therefore, the weight is directed vertically downward. The frictional force tends to oppose the motion


**Fig. 9.33**

■ **Newton's third law of motion** states that when an object exerts a force on another object, the second object exerts on the first object a force of the same magnitude but in the opposite direction. The force exerted by the plank on the block is an illustration of this law. (Sir Isaac Newton, again. See page 277.)


**Fig. 9.34**

**Fig. 9.35**

of the block and is directed upward along the plank. The frictional force must be sufficient to counterbalance that component of the weight of the block that is directed down the plank for the block to be at rest (not moving). The plank itself “holds up” that component of the weight that is perpendicular to the plank. A convenient set of coordinates (see Fig. 9.33) is one with the origin at the centre of the block and with the  $x$ -axis directed up the plank and the  $y$ -axis perpendicular to the plank. The magnitude of the frictional force  $F_f$  is given by

$$\begin{aligned} F_f &= 80.0 \cos 60.0^\circ && \text{component of weight down plank} \\ &= 40.0 \text{ N} && \text{equals frictional force} \end{aligned}$$

We have used the  $60.0^\circ$  angle since it is the reference angle. We could have expressed the frictional force as  $F_f = 80.0 \sin 30.0^\circ$ .

Here, it is assumed that the block is small enough that we may calculate all forces as though they act at the centre of the block (although we know that the frictional force acts at the surface between the block and the plank). ■

### EXAMPLE 5 Forces for equilibrium

A 735-N mountain climber suspended by a rope pushes on the side of a cliff with a horizontal force of 112 N. What is the tension  $T$  in the rope if the climber is in equilibrium? See Fig. 9.34.

For equilibrium, the tension in the rope must be equal and opposite to the resultant of the climber's weight and force against the cliff. This means the magnitude of the  $x$ -component of the tension is 112 N (the reaction force of the cliff—another illustration of Newton's third law—see Example 4) and the magnitude of the  $y$ -component is 735 N (see Fig. 9.34).

$$\begin{aligned} T &= \sqrt{112^2 + 735^2} \\ &= 743 \text{ N} \\ \theta &= \tan^{-1} \frac{735}{112} = 81.3^\circ \end{aligned}$$

As with any vector, we must find the direction in which the tension acts (in this case, along the rope). ■

### EXAMPLE 6 Acceleration of a spacecraft

For a spacecraft moving in a circular path around the earth, the tangential component  $a_T$  and the centripetal component  $a_R$  of its acceleration are given by the expressions shown in Fig. 9.35. The radius of the circle through which it is moving (from the centre of the earth to the spacecraft) is  $r$ , its angular velocity is  $\omega$ , and its angular acceleration is  $\alpha$  (the rate at which  $\omega$  is changing).

While going into orbit, at one point a spacecraft is moving in a circular path 230 km above the surface of the earth. At this point,  $r = 6.60 \times 10^6$  m,  $\omega = 1.10 \times 10^{-3}$  rad/s, and  $\alpha = 0.420 \times 10^{-6}$  rad/s<sup>2</sup>. Calculate the magnitude of the resultant acceleration and the angle it makes with the tangential component.

$$\begin{aligned} a_R &= r\omega^2 = (6.60 \times 10^6) (1.10 \times 10^{-3})^2 \\ &= 7.99 \text{ m/s}^2 \\ a_T &= r\alpha = (6.60 \times 10^6) (0.420 \times 10^{-6}) \\ &= 2.77 \text{ m/s}^2 \end{aligned}$$

Since a tangent line to a circle is perpendicular to the radius at the point of tangency,  $a_T$  is perpendicular to  $a_R$ . Thus,

$$\begin{aligned} a &= \sqrt{a_T^2 + a_R^2} = \sqrt{7.99^2 + 2.77^2} \\ &= 8.45 \text{ m/s}^2 \\ \phi &= \tan^{-1} \frac{a_R}{a_T} = \tan^{-1} \frac{7.99}{2.77} \\ &= 70.9^\circ \end{aligned}$$

## EXERCISES 9.4

In Exercises 1 and 2, find the necessary quantities if the given changes are made in the indicated examples of this section.

- In Example 2, find the resultant location if the  $41.25^\circ$  angle is changed to  $31.25^\circ$ .
- In Example 3, find the resultant velocity if the wind is from the southwest, rather than the southeast.

In Exercises 3–36, solve the given problems.

- Two hockey players strike the puck at the same time, hitting it with horizontal forces of 34.5 N and 19.5 N that are perpendicular to each other. Find the resultant of these forces.
- To straighten a small tree, two horizontal ropes perpendicular to each other are attached to the tree. If the tensions in the ropes are 82.3 N and 102 N, what is the resultant force on the tree?
- In lifting a heavy piece of equipment from the mud, a cable from a crane exerts a vertical force of 6500 N, and a cable from a truck exerts a force of 8300 N at  $10.0^\circ$  above the horizontal. Find the resultant of these forces.
- At a point in the plane, two electric charges create an electric field (a vector quantity) of 25.9 kN/C at  $10.8^\circ$  above the horizontal to the right and 12.6 kN/C at  $83.4^\circ$  below the horizontal to the right. Find the resultant electric field.
- A motorboat leaves a dock and travels 1580 m due west, then turns  $35.0^\circ$  to the south and travels another 1640 m to a second dock. What is the displacement of the second dock from the first dock?
- Regina is 530 km at  $6.9^\circ$  north of west from Winnipeg. Saskatoon is 230 km at  $53^\circ$  north of west from Regina. What is the displacement of Saskatoon from Winnipeg?
- From a fixed point, a surveyor locates a pole at 215.6 m due east and a building corner at 358.2 m at  $37.72^\circ$  north of east. What is the displacement of the building from the pole?
- A rocket is launched with a vertical component of velocity of 2840 km/h and a horizontal component of velocity of 1520 km/h. What is its resultant velocity?
- In testing the behaviour of a tire on ice, a force of 2080 N is exerted to the side, and a force of 3120 N is exerted to the front. What is the resultant force on the tire?
- To raise a crate, two ropes are attached to its top. If the force in one rope is 960 N at  $25^\circ$  from the vertical, what must be the force in the second rope at  $35^\circ$  from the vertical in order to lift the crate straight upward?
- A storm front is moving east at 22.0 km/h and south at 12.5 km/h. Find the resultant velocity of the front.
- To move forward, a helicopter pilot tilts the helicopter forward. If the rotor generates a force of 14 000 N, with a horizontal component (thrust) of 1900 N, what is the vertical component (lift)?
- The acceleration (a vector quantity) of gravity on a sky diver is  $9.8 \text{ m/s}^2$ . If the force of the wind also causes an acceleration of  $1.2 \text{ m/s}^2$  at an angle of  $15^\circ$  above the horizontal, what is the resultant acceleration of the sky diver?
- In an accident, a truck with momentum (a vector quantity) of  $22\,100 \text{ kg} \cdot \text{m/s}$  strikes a car with momentum of  $17\,800 \text{ kg} \cdot \text{m/s}$

from the rear. The angle between their directions of motion is  $25.0^\circ$ . What is the resultant momentum?

- In an automobile safety test, a shoulder and seat belt exerts a force of 425 N directly backward and a force of 368 N backward at an angle of  $20.0^\circ$  below the horizontal on a dummy. If the belt holds the dummy from moving farther forward, what force did the dummy exert on the belt? See Fig. 9.36.

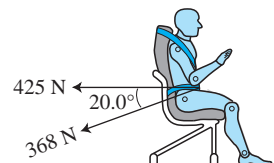


Fig. 9.36

- Two perpendicular forces act on a ring at the end of a chain that passes over a pulley and holds an automobile engine. If the forces have the values shown in Fig. 9.37, what is the weight of the engine?

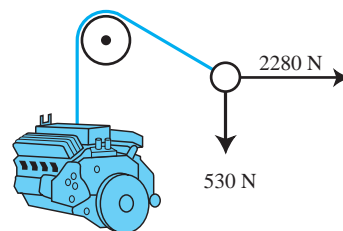


Fig. 9.37

- A plane flies at 550 km/h into a head wind of 60 km/h at  $78^\circ$  with the direction of the plane. Find the resultant velocity of the plane with respect to the ground. See Fig. 9.38.

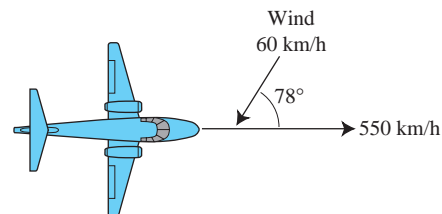


Fig. 9.38

- A ship's navigator determines that the ship is moving through the water at 17.5 km/h with a heading of  $26.3^\circ$  north of east, but the ship is actually moving at 19.3 km/h in a direction of  $33.7^\circ$  north of east. What is the velocity of the current?
- A space shuttle is moving in orbit at 29 370 km/h. A satellite is launched to the rear at 190 km/h at an angle of  $5.20^\circ$  from the direction of the shuttle. Find the velocity of the satellite.
- A block of ice slides down a (frictionless) ramp with an acceleration of  $5.3 \text{ m/s}^2$ . If the ramp makes an angle of  $32.7^\circ$  with the horizontal, find  $g$ , the acceleration due to gravity. See Fig. 9.39.

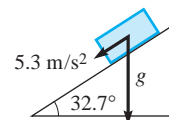


Fig. 9.39

23. In navigation, one method of expressing *bearing* is to give a single angle measured clockwise from due north. A radar station notes the bearing of two planes, one 15.5 km away on a bearing of  $22.0^\circ$ , and the other 12.0 km away on a bearing of  $180.0^\circ$ . Find the displacement of the second plane from the first. See Fig. 9.40.

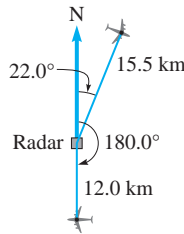


Fig. 9.40

24. As a satellite travels around earth at an altitude of 35 940 km, the magnitude of its linear velocity (see Example 5, page 255) remains unchanged, but its *direction* changes continually, which means there is a change in velocity.

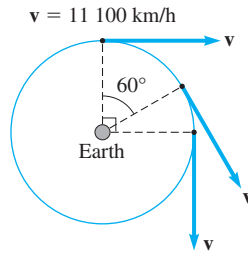


Fig. 9.41

25. A passenger on a cruise ship travelling due east at a speed of 32 km/h notes that the smoke from the ship's funnels makes an angle of  $15^\circ$  with the ship's wake. If the wind is from the south-west, find the speed of the wind.
26. A mine shaft goes due west 75 m (measured horizontally) from the opening at an angle of  $25^\circ$  below the horizontal surface. It then becomes horizontal and turns  $30^\circ$  north of west and continues for another 45 m. What is the displacement of the end of the tunnel from the opening?
27. A crowbar 1.5 m long is supported at a fulcrum 1.2 m from the end, with the other end under a boulder. If the crowbar is at an angle of  $18^\circ$  with the horizontal and a person pushes down, perpendicular to the crowbar, with a force of 240 N, what vertical force is applied to the boulder? See Fig. 9.42. (The forces are related by  $1.2\mathbf{F}_1 = 0.3\mathbf{F}_2$ . See Exercise 50 on page 159.)

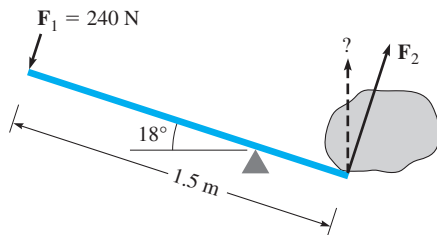


Fig. 9.42

28. A scuba diver's body is directed downstream at  $75^\circ$  to the bank of a river. If the diver swims at 25 m/min, and the water is moving at 5.0 m/min, what is the diver's velocity?
29. While starting up, a circular saw blade 8.20 cm in diameter is rotating at 212 rad/min and has an angular acceleration of 318 rad/min<sup>2</sup>. What is the acceleration of the tip of one of the teeth? (See Example 6.)
30. A boat travels across a river, reaching the opposite bank at a point directly opposite that from which it left. If the boat travels 6.00 km/h in still water and the current of the river flows at 3.00 km/h, what was the velocity of the boat in the water?

31. In searching for a boat lost at sea, a Coast Guard cutter leaves St. John's and travels 75.0 km due east. It then turns  $65^\circ$  north of east and travels another 75.0 km, and finally turns another  $65.0^\circ$  toward the west and travels another 75.0 km. What is its displacement from the port?
32. A car is held stationary on a ramp by two forces. One is the force of 2140 N by the brakes, which hold it from rolling down the ramp. The other is a reaction force by the ramp of 9850 N, perpendicular to the ramp. This force keeps the car from going through the ramp. See Fig. 9.43. What is the weight of the car, and at what angle with the horizontal is the ramp inclined?

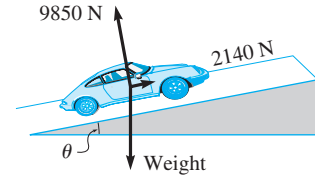


Fig. 9.43

33. A plane is moving at 75.0 m/s, and a package with weather instruments is ejected horizontally from the plane at 15.0 m/s, perpendicular to the direction of the plane. If the vertical velocity  $v_v$  (in m/s), as a function of time  $t$  (in s) of fall, is given by  $v_v = 9.80t$ , what is the velocity of the package after 2.00 s (before its parachute opens)?
34. A flat rectangular barge, 48.0 m long and 20.0 m wide, is headed directly across a stream at 4.5 km/h. The stream flows at 3.8 km/h. What is the velocity, relative to the riverbed, of a person walking diagonally across the barge at 5.0 km/h while facing the opposite upstream bank?
35. In Fig. 9.44, a long, straight conductor perpendicular to the plane of the paper carries an electric current  $i$ . A bar magnet having poles of strength  $m$  lies in the plane of the paper. The vectors  $\mathbf{H}_i$ ,  $\mathbf{H}_N$ , and  $\mathbf{H}_S$  represent the components of the magnetic intensity  $\mathbf{H}$  due to the current and to the N and S poles of the magnet, respectively. The magnitudes of the components of  $\mathbf{H}$  are given by

$$H_i = \frac{1}{2\pi} \frac{i}{a} \quad H_N = \frac{1}{4\pi} \frac{m}{b^2} \quad H_S = \frac{1}{4\pi} \frac{m}{c^2}$$

Given that  $a = 0.300$  m,  $b = 0.400$  m,  $c = 0.300$  m, the length of the magnet is 0.500 m,  $i = 4.00$  A, and  $m = 2.00$  A · m, calculate the resultant magnetic intensity  $\mathbf{H}$ . The component  $\mathbf{H}_i$  is parallel to the magnet.

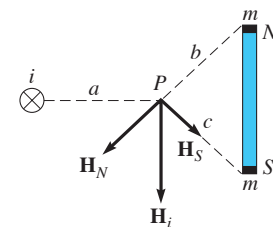


Fig. 9.44

36. Solve the problem of Exercise 35 if  $\mathbf{H}_i$  is directed away from the magnet, making an angle of  $10.0^\circ$  with the direction of the magnet.

Answers to Practice Exercise

1.  $R = 22.96$  km,  $\theta = 27.68^\circ$

## 9.5 Oblique Triangles, the Law of Sines

Oblique Triangles • Law of Sines •  
 Case 1: Two Angles and One Side •  
 Case 2: Two Sides and the Angle Opposite  
 One of Them • Ambiguous Case

To this point, we have limited our study of triangle solution to right triangles. However, *many triangles that require solution do not contain a right angle. Such triangles are called oblique triangles.* We now discuss solutions of oblique triangles.

In Section 4.4, we stated that we need to know three parts, at least one of them a side, to solve a triangle. There are four possible such combinations of parts, and these combinations are as follows:

**Case 1.** Two angles and one side

**Case 2.** Two sides and the angle opposite one of them

**Case 3.** Two sides and the included angle

**Case 4.** Three sides

There are several ways in which oblique triangles may be solved, but we restrict our attention to the two most useful methods, the **law of sines** and the **law of cosines**. In this section, we discuss the law of sines and show that it may be used to solve Case 1 and Case 2.

Let  $ABC$  be an oblique triangle with sides  $a$ ,  $b$ , and  $c$  opposite angles  $A$ ,  $B$ , and  $C$ , respectively. By drawing a perpendicular  $h$  from  $B$  to side  $b$ , as shown in Fig. 9.45(a), note that  $h/c = \sin A$  and  $h/a = \sin C$ , and therefore

$$h = c \sin A \quad \text{or} \quad h = a \sin C \quad (9.4)$$

This relationship also holds if one of the angles is obtuse. By drawing a perpendicular  $h$  to the extension of side  $b$ , as shown in Fig. 9.45(b), we see that  $h/c = \sin A$ .

Also, in Fig. 9.45(b),  $h/a = \sin(180^\circ - C)$ , where  $180^\circ - C$  is also the reference angle for the obtuse angle  $C$ . For a second-quadrant reference angle  $180^\circ - C$ , we have  $\sin[180^\circ - (180^\circ - C)] = \sin C$ . Putting this all together,  $\sin(180^\circ - C) = \sin C$ . This means that

$$h = c \sin A \quad \text{or} \quad h = a \sin(180^\circ - C) = a \sin C \quad (9.5)$$

The results are precisely the same in Eqs. (9.4) and (9.5). Setting the results for  $h$  equal to each other, we have

$$c \sin A = a \sin C$$

Dividing each side by  $\sin A$  multiplied by  $\sin C$  and reversing sides gives us

$$\frac{a}{\sin A} = \frac{c}{\sin C} \quad (9.6)$$

By dropping a perpendicular from  $A$  to  $a$ , we also derive the result

$$c \sin B = b \sin C$$

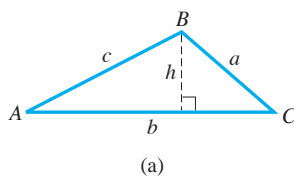
which, when each side is divided by  $\sin B$  multiplied by  $\sin C$ , and reversing sides, gives us

$$\frac{b}{\sin B} = \frac{c}{\sin C} \quad (9.7)$$

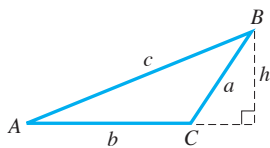
If we had dropped a perpendicular to either side  $a$  or side  $c$ , equations similar to Eqs. (9.6) and (9.7) would have resulted.

Combining Eqs. (9.6) and (9.7), for any triangle with sides  $a$ ,  $b$ , and  $c$ , opposite angles  $A$ ,  $B$ , and  $C$ , respectively, such as the one shown in Fig. 9.46, we have the **law of sines**:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \quad (9.8)$$



(a)



(b)

Fig. 9.45

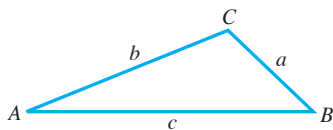


Fig. 9.46

Law of Sines



**LEARNING TIP**

Note that *there are actually three equations combined in Eq. (9.8)*. Of these, we use the one with three known parts of the triangle, and we find the fourth part. In finding the complete solution of a triangle, it may be necessary to use two of the three equations.

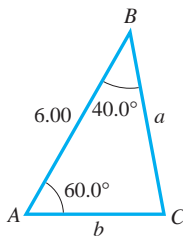


Fig. 9.47

**Practice Exercise**

- In Example 1, change the value of  $B$  to  $65.0^\circ$ , and then find  $a$ .

**LEARNING TIP**

The solution of a triangle can be checked approximately by noting that *the smallest angle is opposite the shortest side, and the largest angle is opposite the longest side*. Note that this is the case in Example 1, where  $b$  (shortest side) is opposite  $B$  (smallest angle), and  $c$  (longest side) is opposite  $C$  (largest angle).

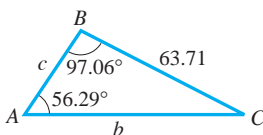


Fig. 9.48

Another form of the law of sines can be obtained by equating the reciprocals of each of the fractions in Eq. (9.8). The law of sines is a statement of proportionality between the sides of a triangle and the sines of the angles opposite them.

**CASE 1: TWO ANGLES AND ONE SIDE**

Now, we see how the law of sines is used in the solution of a triangle in which two angles and one side are known. If two angles are known, the third may be found from the fact that the sum of the angles in a triangle is  $180^\circ$ . At this point, we must be able to find the ratio between the given side and the sine of the angle opposite it. Then, by use of the law of sines, we may find the other two sides.

**EXAMPLE 1 Case 1: Two angles and one side**

Given  $c = 6.00$ ,  $A = 60.0^\circ$ , and  $B = 40.0^\circ$ , find  $a$ ,  $b$ , and  $C$ .

First, we can see that

$$C = 180.0^\circ - (60.0^\circ + 40.0^\circ) = 80.0^\circ$$

We now know side  $c$  and angle  $C$ , which allows us to use Eq. (9.8). Therefore, using the equation relating  $a$ ,  $A$ ,  $c$ , and  $C$ , we have

$$\frac{a}{\sin 60.0^\circ} = \frac{6.00}{\sin 80.0^\circ} \quad \text{or} \quad a = \frac{6.00 \sin 60.0^\circ}{\sin 80.0^\circ} = 5.28$$

Now, using the equation relating  $b$ ,  $B$ ,  $c$ , and  $C$ , we have

$$\frac{b}{\sin 40.0^\circ} = \frac{6.00}{\sin 80.0^\circ} \quad \text{or} \quad b = \frac{6.00 \sin 40.0^\circ}{\sin 80.0^\circ} = 3.92$$

Thus,  $a = 5.28$ ,  $b = 3.92$ , and  $C = 80.0^\circ$ . See Fig. 9.47. We could also have used the form of Eq. (9.8) relating  $a$ ,  $A$ ,  $b$ , and  $B$  in order to find  $b$ , but any error in calculating  $a$  would make  $b$  in error as well. Of course, any error in calculating  $C$  would make both  $a$  and  $b$  in error. ■

**EXAMPLE 2 Case 1: Two angles and one side**

Solve the triangle with the following given parts:  $a = 63.71$ ,  $A = 56.29^\circ$ , and  $B = 97.06^\circ$ . See Fig. 9.48.

From the figure, we see that we are to find angle  $C$  and sides  $b$  and  $c$ . We first determine angle  $C$ :

$$\begin{aligned} C &= 180^\circ - (A + B) = 180^\circ - (56.29^\circ + 97.06^\circ) \\ &= 26.65^\circ \end{aligned}$$

Noting the three angles, we know that  $c$  is the shortest side ( $C$  is the smallest angle) and  $b$  is the longest side ( $B$  is the largest angle). This means that the length of  $a$  is between  $c$  and  $b$ , or  $c < 63.71$  and  $b > 63.71$ . Now using the ratio  $a/\sin A$  of Eq. (9.8) (the law of sines) to find sides  $b$  and  $c$ , we have

$$\begin{aligned} \frac{b}{\sin 97.06^\circ} &= \frac{63.71}{\sin 56.29^\circ} \quad \text{or} \quad b = \frac{63.71 \sin 97.06^\circ}{\sin 56.29^\circ} = 76.01 \\ \frac{c}{\sin 26.65^\circ} &= \frac{63.71}{\sin 56.29^\circ} \quad \text{or} \quad c = \frac{63.71 \sin 26.65^\circ}{\sin 56.29^\circ} = 34.35 \end{aligned}$$



Thus,  $b = 76.01$ ,  $c = 34.35$ , and  $C = 26.65^\circ$ . Note that  $c < a$  and  $b > a$ , as expected. ■

If the given information is appropriate, the law of sines may be used to solve applied problems. The following example illustrates the use of the law of sines in such a problem.

**EXAMPLE 3 Case 1: Application**

Two observers  $A$  and  $B$  sight a helicopter due east. The observers are 1540 m apart, and the angles of elevation they each measure to the helicopter are  $32.0^\circ$  and  $44.0^\circ$ , respectively. How far is observer  $A$  from the helicopter? See Fig. 9.49.

Letting  $H$  represent the position of the helicopter, we see that angle  $B$  within the triangle  $ABH$  is  $180^\circ - 44.0^\circ = 136.0^\circ$ . This means that the angle at  $H$  within the triangle is

$$H = 180^\circ - (32.0^\circ + 136.0^\circ) = 12.0^\circ$$

Now, using the law of sines to find required side  $b$ , we have

$$\begin{array}{c} \text{required side} \rightarrow \\ \text{opposite} \\ \text{known angle} \end{array} \frac{b}{\sin 136.0^\circ} = \frac{1540}{\sin 12.0^\circ} \begin{array}{c} \leftarrow \text{known side} \\ \text{opposite} \\ \text{known angle} \end{array}$$

or

$$b = \frac{1540 \sin 136.0^\circ}{\sin 12.0^\circ} = 5150 \text{ m}$$

Thus, observer  $A$  is about 5150 m from the helicopter. ■

■ The first successful helicopter was made in the United States by Igor Sikorsky in 1939.

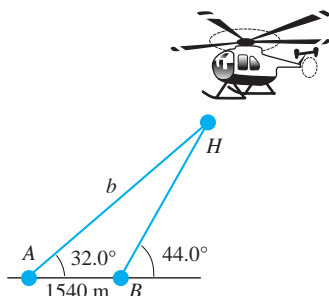


Fig. 9.49

**CASE 2: TWO SIDES AND THE ANGLE OPPOSITE ONE OF THEM**

For a triangle in which we know two sides and the angle opposite one of the given sides, the solution will be either *one triangle*, or *two triangles*, or even possibly *no triangle*. The following examples illustrate how each of these results is possible.

**EXAMPLE 4 Case 2: Two sides and angle opposite**

Solve the triangle with the following given parts:  $a = 40.0$ ,  $b = 60.0$ , and  $A = 30.0^\circ$ .

First, make a good scale drawing (Fig. 9.50(a)) by drawing angle  $A$  and measuring off 60 for  $b$ . This will more clearly show that side  $a = 40.0$  will intersect side  $c$  at either position  $B$  or  $B'$ . This means there are two triangles that satisfy the given values. Using the law of sines, we solve the case for which  $B$  is an acute angle:

$$\frac{60.0}{\sin B} = \frac{40.0}{\sin 30.0^\circ} \quad \text{or} \quad \sin B = \frac{60.0 \sin 30.0^\circ}{40.0}$$

$$B = \sin^{-1}\left(\frac{60.0 \sin 30.0^\circ}{40.0}\right) = 48.6^\circ$$

$$C = 180^\circ - (30.0^\circ + 48.6^\circ) = 101.4^\circ$$

Therefore,  $B = 48.6^\circ$  and  $C = 101.4^\circ$ . Using the law of sines again to find  $c$ , we have

$$\begin{aligned} \frac{c}{\sin 101.4^\circ} &= \frac{40.0}{\sin 30.0^\circ} \\ c &= \frac{40.0 \sin 101.4^\circ}{\sin 30.0^\circ} \\ &= 78.4 \end{aligned}$$

Thus,  $B = 48.6^\circ$ ,  $C = 101.4^\circ$ , and  $c = 78.4$ . See Fig. 9.50(b).

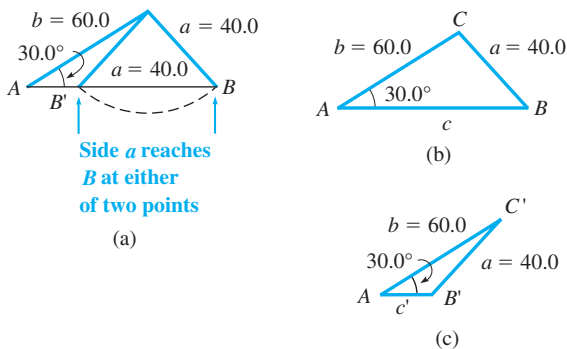


Fig. 9.50

The other solution is the case in which  $B'$ , opposite side  $b$ , is an obtuse angle. Therefore,

$$\begin{aligned} B' &= 180^\circ - B = 180^\circ - 48.6^\circ \\ &= 131.4^\circ \\ C' &= 180^\circ - (30.0^\circ + 131.4^\circ) \\ &= 18.6^\circ \end{aligned}$$

Using the law of sines to find  $c'$ , we have

$$\begin{aligned} \frac{c'}{\sin 18.6^\circ} &= \frac{40.0}{\sin 30.0^\circ} \\ c' &= \frac{40.0 \sin 18.6^\circ}{\sin 30.0^\circ} \\ &= 25.5 \end{aligned}$$

This means that the second solution is  $B' = 131.4^\circ$ ,  $C' = 18.6^\circ$ , and  $c' = 25.5$ . See Fig. 9.50(c).

**EXAMPLE 5 Case 2: Possible solutions**

In Example 4, if  $a$  were 60.0, only one solution would result. In this case, side  $a$  would intercept side  $c$  at  $B$ . It also intercepts the extension of side  $c$ , but this would require that angle  $A$  not be included in the triangle (see Fig. 9.51). Thus, only one solution may result if  $a > b$ .

In Example 4, there would be *no solution* if side  $a$  were not at least 30.0. If this were the case, side  $a$  would not be long enough to even touch side  $c$ . It can be seen that  $a$  must at least equal  $b \sin A$ . If it is just equal to  $b \sin A$ , there is *one solution*, a right triangle. See Fig. 9.52.

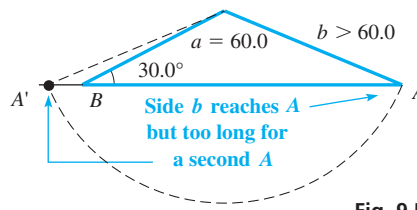


Fig. 9.51

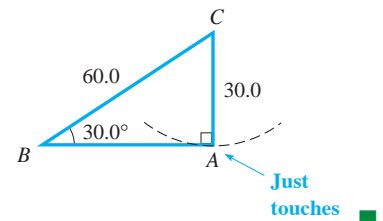


Fig. 9.52

**Ambiguous Case**

Summarizing the results for Case 2 as illustrated in Examples 4 and 5, we make the following conclusions. Given sides  $a$  and  $b$  and angle  $A$  (assuming here that  $a$  and  $A$  ( $A < 90^\circ$ ) are corresponding parts), we have the following summary of solutions for Case 2.

**Practice Exercise**

- Determine which of the four possible solution types occurs if  $a = 28$ ,  $b = 48$ , and  $A = 30^\circ$ .

**Summary of Solutions:**

**Two Sides and the Angle Opposite One of Them**

- No solution** if  $a < b \sin A$ . See Fig. 9.53(a).
- A right triangle solution** if  $a = b \sin A$ . See Fig. 9.53(b).
- Two solutions** if  $b \sin A < a < b$ . See Fig. 9.53(c).
- One solution** if  $a > b$ . See Fig. 9.53(d).

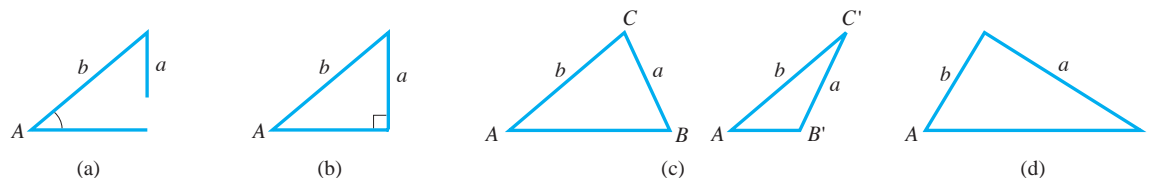


Fig. 9.53

**LEARNING TIP**

Note that *in order to have two solutions, we must know two sides and the angle opposite one of the sides, and the shorter side must be opposite the known angle.*

If there is *no solution*, the calculator will indicate an *error*. If the solution is a *right triangle*, the calculator will show an angle of *exactly 90°* (no extra decimal digits will be displayed).

■ See the chapter introduction

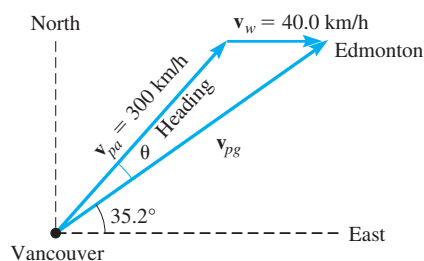


Fig. 9.54

For the reason that two solutions may result from it, Case 2 is called the **ambiguous case**. The following example illustrates Case 2 in an applied problem.

**EXAMPLE 6 Case 2: Application**

Edmonton is  $35.2^\circ$  north of east of Vancouver. What should be the heading of a plane from Vancouver to Edmonton if the wind is from the west at  $40.0$  km/h and the plane's speed with respect to the air is  $300$  km/h?

The heading should be set so that the resultant of the plane's velocity with respect to the air  $\mathbf{v}_{pa}$  and the velocity of the wind  $\mathbf{v}_w$  will be in the direction from Vancouver to Edmonton. This means that the resultant velocity  $\mathbf{v}_{pg}$  of the plane with respect to the ground must be at an angle of  $35.2^\circ$  north of east of Vancouver.

Using the given information, we draw the vector triangle shown in Fig. 9.54. In the triangle, we know that the angle at Edmonton is  $35.2^\circ$  by noting the alternate-interior angles (see page 58). By finding  $\theta$ , the required heading can be found. There can be only one solution, since  $v_{pa} > v_w$ . Using the law of sines, we have

$$\begin{array}{c} \text{known side} \rightarrow 40.0 \\ \text{opposite} \\ \text{required angle} \end{array} \rightarrow \frac{40.0}{\sin \theta} = \frac{300}{\sin 35.2^\circ} \leftarrow \begin{array}{c} \text{known side} \\ \text{opposite} \\ \text{known angle} \end{array}$$

$$\sin \theta = \frac{40.0 \sin 35.2^\circ}{300}, \quad \theta = 4.4^\circ$$

Therefore, the heading should be  $35.2^\circ + 4.4^\circ = 39.6^\circ$  north of east. Compare this example with Example 3 on page 278. ■

If we try to use the law of sines for Case 3 or Case 4, we find that we do not have enough information to complete any of the ratios. These cases can, however, be solved by the law of cosines as shown in the next section.

**EXAMPLE 7 Cases 3 and 4 not solvable by the law of sines**

Given (Case 3) two sides and the included angle of a triangle  $a = 2$ ,  $b = 3$ ,  $C = 45^\circ$ , and (Case 4) the three sides  $a = 5$ ,  $b = 6$ ,  $c = 7$ , we set up the ratios

$$\text{(Case 3)} \frac{2}{\sin A} = \frac{3}{\sin B} = \frac{c}{\sin 45^\circ}, \quad \text{and} \quad \text{(Case 4)} \frac{5}{\sin A} = \frac{6}{\sin B} = \frac{7}{\sin C}$$

The solution cannot be found since each of the three possible equations in either Case 3 or Case 4 contains two unknowns. ■

**EXERCISES 9.5**

In Exercises 1 and 2, solve the resulting triangles if the given changes are made in the indicated examples of this section.

- In Example 2, solve the triangle if the value of  $B$  is changed to  $82.94^\circ$ .
- In Example 4, solve the triangle if the value of  $b$  is changed to  $70.0$ .

In Exercises 3–22, solve the triangles with the given parts.

- $a = 45.7$ ,  $A = 65.0^\circ$ ,  $B = 49.0^\circ$
- $b = 3.07$ ,  $A = 26.0^\circ$ ,  $C = 120.0^\circ$
- $c = 4380$ ,  $A = 37.4^\circ$ ,  $B = 34.6^\circ$
- $a = 93.2$ ,  $B = 0.9^\circ$ ,  $C = 82.6^\circ$

- $a = 4.601$ ,  $b = 3.107$ ,  $A = 18.23^\circ$
- $b = 362.2$ ,  $c = 294.6$ ,  $B = 69.37^\circ$
- $b = 7751$ ,  $c = 3642$ ,  $B = 20.73^\circ$
- $a = 150.4$ ,  $c = 250.9$ ,  $C = 76.43^\circ$
- $b = 0.0742$ ,  $B = 51.0^\circ$ ,  $C = 3.40^\circ$
- $c = 729$ ,  $B = 121.0^\circ$ ,  $C = 44.2^\circ$
- $a = 63.8$ ,  $B = 58.4^\circ$ ,  $C = 22.2^\circ$
- $a = 0.130$ ,  $A = 55.2^\circ$ ,  $B = 117.5^\circ$
- $b = 4384$ ,  $B = 47.43^\circ$ ,  $C = 64.56^\circ$
- $b = 283.2$ ,  $B = 13.79^\circ$ ,  $C = 76.38^\circ$
- $a = 5.240$ ,  $b = 4.446$ ,  $B = 48.13^\circ$

- 18.  $a = 89.45, c = 37.36, C = 15.62^\circ$
- 19.  $b = 2880, c = 3650, B = 31.4^\circ$
- 20.  $a = 0.841, b = 0.965, A = 57.1^\circ$
- 21.  $a = 450, b = 1260, A = 64.8^\circ$
- 22.  $a = 20, c = 10, C = 30^\circ$

In Exercises 23–41, use the law of sines to solve the given problems.

- 23. Two angles of a triangle measure  $29.0^\circ$  and  $57.0^\circ$ . The longest side is 52.0 cm long. What is the length of the shortest side?
- 24. Two angles of a triangle measure  $45.0^\circ$  and  $55.0^\circ$ . The side opposite the  $55.0^\circ$  angle is 8.75 cm long. What is the length of the longest side?
- 25. The loading ramp at a package delivery service is 3.2 m long and makes a  $22.5^\circ$  angle with the level ground. If this ramp is replaced with one that is 7.5 m long, what angle does the new ramp make with the ground?
- 26. The longest side of a triangular parcel of land is 172 m long, and the shortest side is 105 m long. If the largest angle is  $82.0^\circ$ , what is the length of the third side?
- 27. The floor of the Bastion in Nanaimo, British Columbia, is a regular octagon, 2.3 m on each side. Find the greatest distance across the floor (that is, find the length of the longest diagonal).
- 28. Two ropes hold a 175-N crate as shown in Fig. 9.55. Find the tensions  $T_1$  and  $T_2$  in the ropes. (*Hint:* Move the vectors so that they are tail to head to form a triangle. The vector sum  $T_1 + T_2$  must equal 175 N for equilibrium. See page 278.)

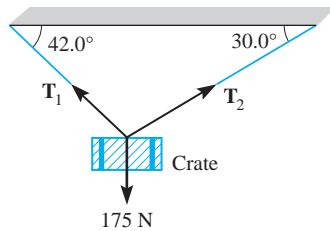


Fig. 9.55

- 29. Find the tension  $T$  in the left guy wire attached to the top of the tower shown in Fig. 9.56. (*Hint:* The horizontal components of the tensions must be equal and opposite for equilibrium. See page 278. Thus, move the tension vectors tail to head to form a triangle with a vertical resultant. This resultant equals the upward force at the top of the tower for equilibrium. This last force is not shown and does not have to be calculated.)

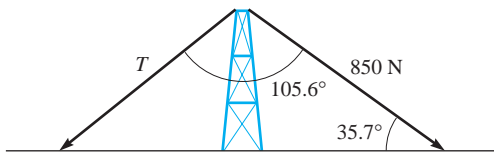


Fig. 9.56

- 30. Find the distance from Golden to Radium, from Fig. 9.57. The route that joins these landmarks across three national parks is known as the Golden Triangle.

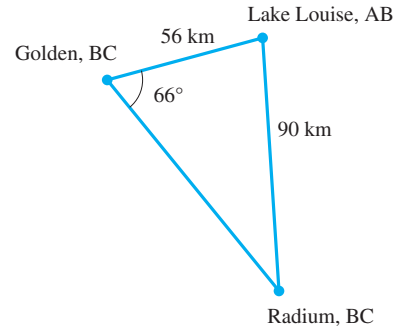


Fig. 9.57

- 31. Find the length of Cordelia St. in Brisbane, Queensland, Australia, from Fig. 9.58.

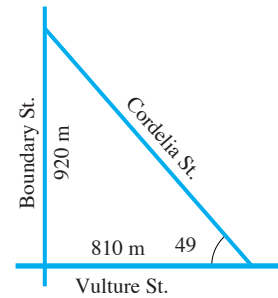


Fig. 9.58

- 32. When an airplane is landing on a 2510-m runway, the angles of depression to the ends of the runway are  $10.0^\circ$  and  $13.5^\circ$ . How far is the plane from the near end of the runway?
- 33. Find the total length of the path of the laser beam that is shown in Fig. 9.59.

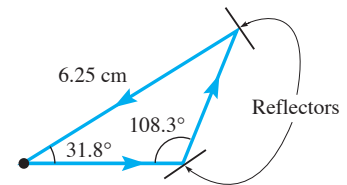


Fig. 9.59

- 34. In widening a highway, it is necessary for a construction crew to cut into the bank along the highway. The present angle of elevation of the straight slope of the bank is  $23.0^\circ$ , and the new angle is to be  $38.5^\circ$ , leaving the top of the slope at its present position. If the slope of the present bank is 66.0 m long, how far horizontally into the bank at its base must they dig?
- 35. A communications satellite is directly above the extension of a line between receiving towers  $A$  and  $B$ . It is determined from radio signals that the angle of elevation of the satellite from tower  $A$  is  $89.2^\circ$ , and the angle of elevation from tower  $B$  is  $86.5^\circ$ . See Fig. 9.60. If  $A$  and  $B$  are 1290 km apart, how far is the satellite from  $A$ ? (Ignore the curvature of the earth.)

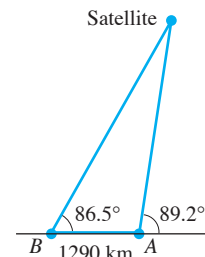


Fig. 9.60

36. An astronaut on the moon drives a lunar rover 16 km in the direction  $60.0^\circ$  north of east from the base. (a) Through what angle must the rover then be turned so that by driving 12 km farther the astronaut can turn again to return to base along a north-south line? (b) How long is the last leg of the trip? (c) Can the astronaut make it back to base if the maximum range of the rover is 40 km?
37. A boat owner wishes to cross a river 2.60 km wide and go directly to a point on the opposite side 1.75 km downstream. The boat goes 8.00 km/h in still water, and the stream flows at 3.50 km/h. What should the boat's heading be?
38. A motorist travelling along a level highway at 75 km/h directly toward a mountain notes that the angle of elevation of the mountain top changes from about  $20^\circ$  to about  $30^\circ$  in a 20-min period. How much closer on a direct line did the mountain top become?
39. A hillside is inclined at  $23^\circ$  with the horizontal. From a given point on the slope, it has been found that a vein of gold is 55 m directly below. At what point downhill and at what angle below the hillside slope must a straight 65-m shaft be dug to reach the vein?
40. Point  $P$  on the mechanism shown in Fig. 9.61 is driven back and forth horizontally. If the minimum value of angle  $\theta$  is  $32.0^\circ$ , what is the distance between extreme positions of  $P$ ? What is the maximum possible value of angle  $\theta$ ?

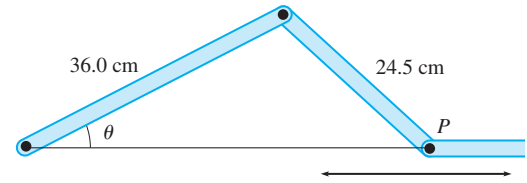


Fig. 9.61

41. The floor of the Winnipeg Art Gallery is in the shape of a triangle. Find the length of the side facing Memorial Boulevard, from Fig. 9.62.

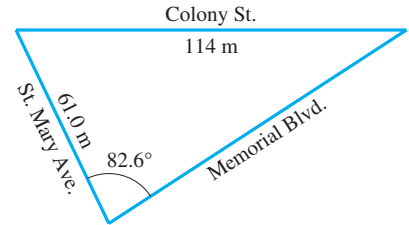


Fig. 9.62

Answers to Practice Exercises

1.  $a = 6.34$     2. Two solutions

## 9.6 The Law of Cosines

Law of Cosines • Case 3: Two Sides and Included Angle • Case 4: Three Sides • Summary of Solving Oblique Triangles

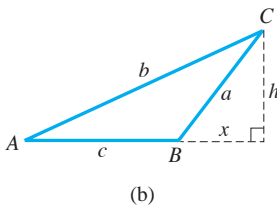
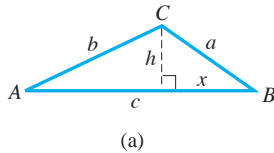


Fig. 9.63

As noted in the last section, the law of sines cannot be used for Case 3 (two sides and the included angle) and Case 4 (three sides). In this section, we develop the *law of cosines*, which can be used for Cases 3 and 4. After finding another part of the triangle using the law of cosines, we will often find it easier to complete the solution using the law of sines.

Consider any oblique triangle—for example, either triangle shown in Fig. 9.63. For each triangle,  $h/b = \sin A$ , or  $h = b \sin A$ . Also, using the Pythagorean theorem, we obtain  $a^2 = h^2 + x^2$  for each triangle. Therefore (with  $(\sin A)^2 = \sin^2 A$ ),

$$a^2 = b^2 \sin^2 A + x^2 \tag{9.9}$$

In Fig. 9.63(a), note that  $(c - x)/b = \cos A$ , or  $c - x = b \cos A$ . Solving for  $x$ , we have  $x = c - b \cos A$ . In Fig. 9.63(b),  $c + x = b \cos A$ , and solving for  $x$ , we have  $x = b \cos A - c$ . Substituting these relations into Eq. (9.9), we obtain

$$a^2 = b^2 \sin^2 A + (c - b \cos A)^2 \tag{9.10}$$

and

$$a^2 = b^2 \sin^2 A + (b \cos A - c)^2$$

respectively. When expanded, these both give

$$\begin{aligned} a^2 &= b^2 \sin^2 A + b^2 \cos^2 A + c^2 - 2bc \cos A \\ &= b^2 (\sin^2 A + \cos^2 A) + c^2 - 2bc \cos A \end{aligned} \tag{9.11}$$

Recalling the definitions of the trigonometric functions, we know that  $\sin \theta = y/r$  and  $\cos \theta = x/r$ . Thus,  $\sin^2 \theta + \cos^2 \theta = (y^2 + x^2)/r^2$ . However,  $x^2 + y^2 = r^2$ , which means

$$\sin^2 \theta + \cos^2 \theta = 1 \tag{9.12}$$

This equation is valid for any angle  $\theta$ , since we have made no assumptions as to the properties of  $\theta$ . Thus, by substituting Eq. (9.12) into Eq. (9.11), we arrive at the **law of cosines**:

$$a^2 = b^2 + c^2 - 2bc \cos A \tag{9.13}$$

Using the method above, we may also show that

$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$

**CASE 3: TWO SIDES AND THE INCLUDED ANGLE**

If two sides and the included angle of a triangle are known, the forms of the law of cosines show that we may directly solve for the side opposite the given angle. Then, as noted earlier, the solution may be completed using the law of sines.

**EXAMPLE 1 Case 3: Two sides and included angle**

Solve the triangle with  $a = 45.0$ ,  $b = 67.0$ , and  $C = 35.0^\circ$ . See Fig. 9.64.

Since angle  $C$  is known, first solve for side  $c$ , using the law of cosines in the form  $c^2 = a^2 + b^2 - 2ab \cos C$ . Substituting, we have

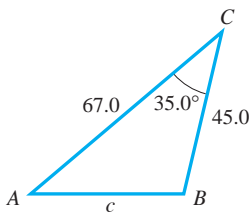


Fig. 9.64

$$c^2 = 45.0^2 + 67.0^2 - 2(45.0)(67.0) \cos 35.0^\circ$$

← unknown side opposite known angle →  
↑ known sides ↑

$$c = \sqrt{45.0^2 + 67.0^2 - 2(45.0)(67.0) \cos 35.0^\circ} = 39.7$$

From the law of sines, we now have

$$\frac{45.0}{\sin A} = \frac{67.0}{\sin B} = \frac{39.7}{\sin 35.0^\circ}$$

← sides opposite angles

$$\sin A = \frac{45.0 \sin 35.0^\circ}{39.7}, \quad A = 40.6^\circ$$

**Practice Exercise**

- In Example 1, change the value of  $a$  to 95.0, and find  $A$  and  $B$ .

Finally, rather than use the law of sines again, we obtain angle  $B$  by subtraction. We find that  $B = 180^\circ - (35.0^\circ + 40.6^\circ) = 104.4^\circ$ . Therefore,  $c = 39.7$ ,  $A = 40.6^\circ$ , and  $B = 104.4^\circ$ . ■

**COMMON ERROR**

If after finding side  $c$  we had solved for angle  $B$  rather than angle  $A$ , the calculator would have shown  $75.6^\circ$ . This would have given us the value of the reference angle for angle  $B$ , and not the correct value for the obtuse angle  $B$  ( $B = 180^\circ - 75.6^\circ = 104.4^\circ$ ). Since only the largest angle of a triangle can be greater than  $90^\circ$ , we can avoid this possible source of error if we **first solve for the smaller unknown angle** (the angle opposite the shorter known side). The larger unknown angle can then be found by subtraction.

**EXAMPLE 2 Case 3: Application**

Two forces are acting on a bolt. One is a 78.0-N force acting horizontally to the right, and the other is a force of 45.0 N acting upward to the right,  $15.0^\circ$  from the vertical. Find the resultant force  $F$ . See Fig. 9.65.

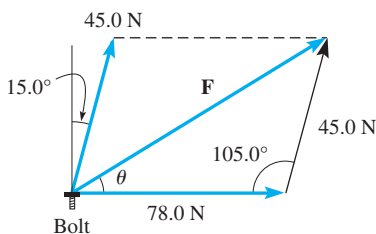


Fig. 9.65

Moving the 45.0-N vector to the right and using the lower triangle with the  $105.0^\circ$  angle, the magnitude of  $F$  is

$$F = \sqrt{78.0^2 + 45.0^2 - 2(78.0)(45.0) \cos 105.0^\circ} = 99.6 \text{ N}$$

To find  $\theta$ , use the law of sines:

$$\frac{45.0}{\sin \theta} = \frac{99.6}{\sin 105.0^\circ}, \quad \sin \theta = \frac{45.0 \sin 105.0^\circ}{99.6}$$

This gives us  $\theta = 25.9^\circ$ .

We can also solve this problem using vector components. ■

**LEARNING TIP**

The best procedure for Case 4 is to find the largest angle first. This avoids the ambiguous case if we switch to the law of sines and there is an obtuse angle. The largest angle is opposite the longest side. Another procedure is to use the law of cosines to find two angles.

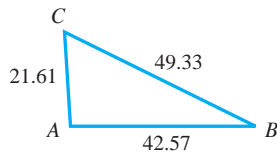


Fig. 9.66

**CASE 4: THREE SIDES**

Given the three sides of a triangle, we may solve for the angle opposite any side using the law of cosines.

**EXAMPLE 3 Case 4: Three sides**

Solve the triangle with sides  $a = 49.33$ ,  $b = 21.61$ , and  $c = 42.57$ . See Fig. 9.66.

Since the longest side is  $a = 49.33$ , first solve for angle  $A$ :

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{21.61^2 + 42.57^2 - 49.33^2}{2(21.61)(42.57)}$$

$$A = 94.81^\circ$$

From the law of sines, we now have

$$\frac{49.33}{\sin A} = \frac{21.61}{\sin B} = \frac{42.57}{\sin C}$$

Therefore,

$$B = \sin^{-1} \frac{21.61 \sin 94.81^\circ}{49.33} = 25.88^\circ$$

$$\text{and } C = 180^\circ - (94.81^\circ + 25.88^\circ) = 59.31^\circ. \quad \blacksquare$$

**EXAMPLE 4 Case 4: Application**

A vertical radio antenna is to be built on a hillside with a constant slope. A guy wire is to be attached at a point 29.0 m up the antenna, and at a point 22.0 m from the base of the antenna up the hillside. If the guy wire is 34.0 m long, what angle does the antenna make with the hillside?

From Fig. 9.67, we can set up the equation necessary for the solution.

$$34.0^2 = 22.0^2 + 29.0^2 - 2(22.0)(29.0)\cos \theta$$

$$\theta = \cos^{-1} \frac{22.0^2 + 29.0^2 - 34.0^2}{2(22.0)(29.0)} = 82.4^\circ \quad \blacksquare$$

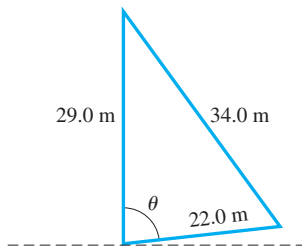


Fig. 9.67

■ Summary of solving an oblique triangle using the law of sines or the law of cosines

**Practice Exercise**

2. Using Fig. 9.67, and only the data given in Example 4, find the angle between the guy wire and the hillside.

**SOLVING OBLIQUE TRIANGLES****Case 1: Two Angles and One Side**

Find the unknown angle by subtracting the sum of the known angles from  $180^\circ$ . Use the **law of sines** to find the unknown sides.

**Case 2: Two Sides and the Angle Opposite One of Them**

Use the known side and the known angle opposite it to find the angle opposite the other known side. Find the third angle from the fact that the sum of the angles is  $180^\circ$ . Use the **law of sines** to find the third side.

**CAUTION:** There may be two solutions. See page 285 for a summary of Case 2 and the *ambiguous case*.

**Case 3: Two Sides and the Included Angle**

Find the third side by using the **law of cosines**. Find the *smaller* unknown angle (opposite the shorter side) by using the **law of sines**. Complete the solution using the fact that the sum of the angles is  $180^\circ$ .

**Case 4: Three Sides**

Find the *largest angle* (opposite the longest side) by using the **law of cosines**. Find a second angle by using the **law of sines**. Complete the solution by using the fact that the sum of the angles is  $180^\circ$ .



Other variations in finding the solutions can be used. For example, after finding the third side in Case 3 or finding the largest angle in Case 4, the solution can be completed by using the law of sines. All angles in Case 4 can be found by using the law of cosines. The methods shown above are those normally used.

## EXERCISES 9.6

In Exercises 1 and 2, solve the resulting triangles if the given changes are made in the indicated examples of this section.

- In Example 1, solve the triangle if the value of  $C$  is changed to  $145^\circ$ .
- In Example 3, solve the triangle if the value of  $a$  is changed to 29.33.

In Exercises 3–22, solve the triangles with the given parts.

- $a = 6.00, b = 7.56, C = 54.0^\circ$
- $b = 87.3, c = 34.0, A = 130.0^\circ$
- $a = 4530, b = 924, C = 98.0^\circ$
- $a = 0.0845, c = 0.116, B = 85.0^\circ$
- $a = 39.53, b = 45.22, c = 67.15$
- $a = 2.331, b = 2.726, c = 2.917$
- $a = 385.4, b = 467.7, c = 800.9$
- $a = 0.2433, b = 0.2635, c = 0.1538$
- $a = 320, b = 847, C = 158.0^\circ$
- $b = 18.3, c = 27.1, A = 8.7^\circ$
- $a = 2140, c = 428, B = 86.3^\circ$
- $a = 1.13, b = 0.510, C = 77.6^\circ$
- $b = 103.7, c = 159.1, C = 104.67^\circ$
- $a = 49.32, b = 54.55, B = 114.36^\circ$
- $a = 0.4937, b = 0.5956, c = 0.6398$
- $a = 69.72, b = 49.30, c = 22.29$
- $a = 723, b = 598, c = 158$
- $a = 1.78, b = 6.04, c = 4.80$
- $a = 1500, A = 15^\circ, B = 140^\circ$
- $a = 17, b = 24, c = 42$ . Explain your answer.

In Exercises 23–40, use the law of cosines to solve the given problems.

- For a triangle with sides  $a, b$ , and  $c$  opposite angles  $A, B$ , and  $C$ , respectively, show that  $1 + \cos A = \frac{(b + c + a)(b + c - a)}{2bc}$ .
- Set up equations (do not solve) to solve the triangle in Fig. 9.68 by the law of cosines. Why is the law of sines easier to use?

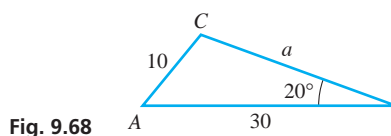


Fig. 9.68

- On page 285, we saw that it was possible to be given two sides and the angle opposite one of them, and find that there is no triangle that can have those parts. Is it possible in Case 3 or in Case 4 that there is no solution for a set of given parts?
- Three circles of radii 24 cm, 32 cm, and 42 cm are externally tangent to each other (each is tangent to the other two). Find the largest angle of the triangle formed by joining their centres.
- A nuclear submarine leaves its base and travels at 23.5 km/h. For 2.00 h, it travels along a course of  $32.1^\circ$  north of west. It then turns an additional  $21.5^\circ$  north of west and travels for another 1.00 h. How far from its base is it?
- The robot arm shown in Fig. 9.69 places packages on a conveyor belt. What is the distance  $x$ ?

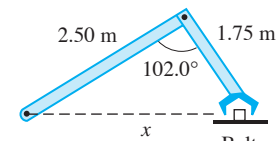


Fig. 9.69

- Find the angle between the front legs and the back legs of the folding chair shown in Fig. 9.70.

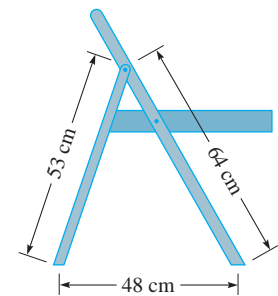


Fig. 9.70

- In a baseball field, the four bases are at the vertices of a square 90.0 ft on a side. The pitching rubber is 60.5 ft from home plate. See Fig. 9.71. How far is it from the pitching rubber to first base? (1 ft = 0.3048 m)

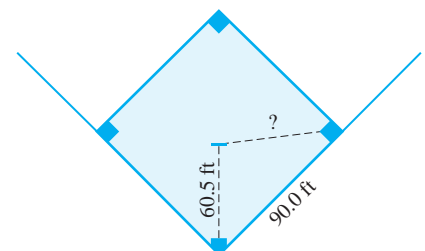


Fig. 9.71

31. A plane leaves an airport and travels 624 km due east. It then turns toward the north and travels another 326 km. It then turns again less than  $180^\circ$  and travels another 846 km directly back to the airport. Through what angles did it turn?
32. The apparent depth of an object submerged in water is less than its actual depth. A coin is actually 5.00 cm from an observer's eye just above the surface, but it appears to be only 4.25 cm. The real light ray from the coin makes an angle with the surface that is  $8.1^\circ$  greater than the angle the apparent ray makes. How much deeper is the coin than it appears to be? See Fig. 9.72.

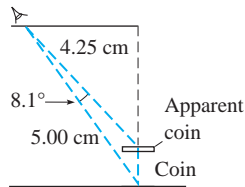


Fig. 9.72

33. A nut is in the shape of a regular hexagon (six sides). If each side is 9.53 mm, what opening on a wrench is necessary to tighten the nut? See Fig. 9.73.

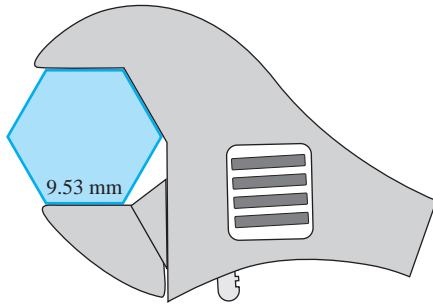


Fig. 9.73

34. Two ropes support a 78.3-N crate from above. The tensions in the ropes are 50.6 N and 37.5 N. What is the angle between the ropes? (See Exercise 28 of Section 9.5.)
35. A ferryboat travels at 11.5 km/h with respect to the water. Because of the river current, it is travelling at 12.7 km/h with

respect to the land in the direction of its destination. If the ferryboat's heading is  $23.6^\circ$  from the direction of its destination, what is the velocity of the current?

36. Find the distance across Georgian Bay from Nottawasaga Island to Christian Island, from Fig. 9.74.

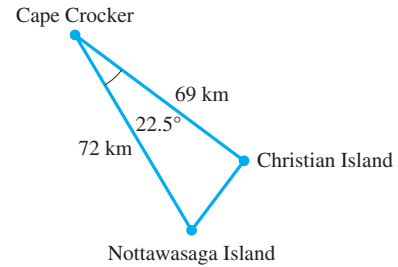


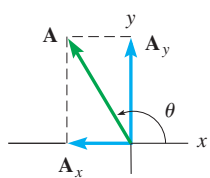
Fig. 9.74

37. An air traffic controller sights two planes that are due east from the control tower and headed toward each other. One is 15.8 km from the tower at an angle of elevation of  $26.4^\circ$ , and the other is 32.7 km from the tower at an angle of elevation of  $12.4^\circ$ . How far apart are the planes?
38. A ship's captain notes that a second ship is 14.5 km away at a bearing (see Exercise 23 of Section 9.4) of  $46.3^\circ$ , and that a third ship was at a distance of 21.7 km at a bearing of  $201.0^\circ$ . How far apart are the second and third ships?
39. A park is in the shape of a parallelogram with sides of 1.25 km and 1.90 km that meet in a  $78.0^\circ$  angle. The park has two diagonal paths. What is the length of each path?
40. A triangular machine part has sides of 5 cm and 8 cm. Explain why the law of sines, or the law of cosines, is used to start the solution of the triangle if the third known part is (a) the third side, (b) the angle opposite the 8-cm side, or (c) the angle between the 5-cm and 8-cm sides.

Answers to Practice Exercises

1.  $B = 43.8^\circ$ ,  $A = 101.2^\circ$     2.  $57.7^\circ$

CHAPTER 9 EQUATIONS



Vector components

$$A_x = A \cos \theta \quad A_y = A \sin \theta \quad (9.1)$$

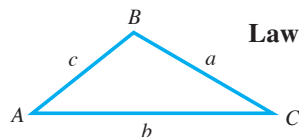
$$A = \sqrt{A_x^2 + A_y^2} \quad (9.2)$$

$$\theta_{\text{ref}} = \tan^{-1} \frac{|A_y|}{|A_x|} \quad (9.3)$$

Law of sines

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \quad (9.8)$$

Law of cosines



$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc \cos A \\ b^2 &= a^2 + c^2 - 2ac \cos B \\ c^2 &= a^2 + b^2 - 2ab \cos C \end{aligned} \quad (9.13)$$

## CHAPTER 9 REVIEW EXERCISES

In Exercises 1–4, find the  $x$ - and  $y$ -components of the given vectors by use of the trigonometric functions.

- $A = 65.0, \theta_A = 28.0^\circ$
- $A = 8050, \theta_A = 149.0^\circ$
- $A = 0.9204, \theta_A = 215.59^\circ$
- $A = 657.1, \theta_A = 343.74^\circ$

In Exercises 5–8, vectors  $\mathbf{A}$  and  $\mathbf{B}$  are at right angles. Find the magnitude and direction of the resultant.

- $A = 327$   
 $B = 505$
- $A = 6.8$   
 $B = 2.9$
- $A = 4964$   
 $B = 3298$
- $A = 26.52$   
 $B = 89.86$

In Exercises 9–16, add the given vectors by components.

- $A = 780, \theta_A = 28.0^\circ$   
 $B = 346, \theta_B = 320.0^\circ$
- $J = 0.0120, \theta_J = 370.5^\circ$   
 $K = 0.00781, \theta_K = 260.0^\circ$
- $A = 22.51, \theta_A = 130.16^\circ$   
 $B = 7.604, \theta_B = 200.09^\circ$
- $A = 18\,760, \theta_A = 110.43^\circ$   
 $B = 4835, \theta_B = 350.20^\circ$
- $Y = 51.33, \theta_Y = 12.25^\circ$   
 $Z = 42.61, \theta_Z = 291.77^\circ$
- $A = 703.1, \theta_A = 122.54^\circ$   
 $B = 302.9, \theta_B = 214.82^\circ$
- $A = 0.750, \theta_A = 15.0^\circ$   
 $B = 0.265, \theta_B = 192.4^\circ$   
 $C = 0.548, \theta_C = 344.7^\circ$
- $S = 8120, \theta_S = 141.9^\circ$   
 $T = 1540, \theta_T = 165.2^\circ$   
 $U = 3470, \theta_U = 296.0^\circ$

In Exercises 17–36, solve the triangles with the given parts.

- $A = 48.0^\circ, B = 68.0^\circ, a = 145$
- $A = 132.0^\circ, b = 7.50, C = 32.0^\circ$
- $a = 22.8, B = 33.5^\circ, C = 125.3^\circ$
- $A = 71.0^\circ, B = 48.5^\circ, c = 8.42$
- $A = 17.85^\circ, B = 154.16^\circ, c = 7863$
- $a = 1.985, b = 4.189, c = 3.652$
- $b = 7607, c = 4053, B = 110.09^\circ$
- $A = 77.06^\circ, a = 12.07, c = 5.104$
- $b = 14.5, c = 13.0, C = 56.6^\circ$
- $B = 40.6^\circ, b = 7.00, c = 18.0$
- $a = 186, B = 130.0^\circ, c = 106$
- $b = 750, c = 1100, A = 56^\circ$
- $a = 7.86, b = 2.45, C = 2.5^\circ$
- $a = 0.208, c = 0.697, B = 165.4^\circ$
- $A = 67.16^\circ, B = 96.84^\circ, c = 532.9$
- $A = 43.12^\circ, a = 7.893, b = 4.113$
- $a = 17, b = 12, c = 25$
- $a = 9064, b = 9953, c = 1106$
- $a = 0.530, b = 0.875, c = 1.25$
- $a = 47.4, b = 40.0, c = 45.5$

In Exercises 37–71, solve the given problems.

37. For any triangle  $ABC$  show that

$$\frac{a^2 + b^2 + c^2}{2abc} = \frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c}$$

38. In solving a triangle for Case 3 (two sides and the included angle), explain what type of solution is obtained if the included angle is a right angle.

39. Two angles of a triangle measure  $22^\circ$  and  $112^\circ$ . The shortest side is 54 cm. What is the longest side?
40. Two of the angles of a triangle measure  $42.0^\circ$  and  $59.5^\circ$ . The longest side is 5.00 cm longer than the shortest side. What is the perimeter of the triangle?

41. An architect determines the two acute angles and one of the legs of a right triangular wall panel. Show that the area  $A_t$  is

$$A_t = \frac{a^2 \sin B}{2 \sin A}$$

42. A surveyor determines the three angles and one side of a triangular tract of land. (a) Show that the area  $A_t$  can be found from  $A_t = \frac{a^2 \sin B \sin C}{2 \sin A}$ . (b) For a right triangle, show that this agrees with the formula in Exercise 41.

43. Find the horizontal and vertical components of the force shown in Fig. 9.75.

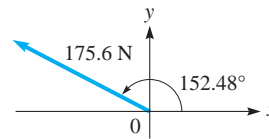


Fig. 9.75

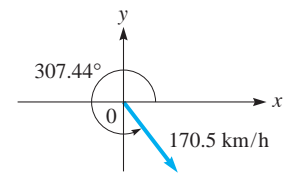


Fig. 9.76

44. Find the horizontal and vertical components of the velocity shown in Fig. 9.76.
45. In a ballistics test, a bullet was fired into a block of wood with a velocity of 670 m/s and at an angle of  $71.3^\circ$  with the surface of the block. What was the component of the velocity perpendicular to the surface?
46. A storm cloud is moving at 15 km/h from the northwest. A television tower is  $60^\circ$  south of east of the cloud. What is the component of the cloud's velocity toward the tower?
47. During a 3.00-min period after taking off, the supersonic jet Concorde travelled at 480 km/h at an angle of  $24.0^\circ$  above the horizontal. What was its gain in altitude during this period?
48. A rocket is launched at an angle of  $42.0^\circ$  with the horizontal and with a speed of 760 m/s. What are its horizontal and vertical components of velocity?
49. Three forces of 3200 N, 1300 N, and 2100 N act on a bolt as shown in Fig. 9.77. Find the resultant force.

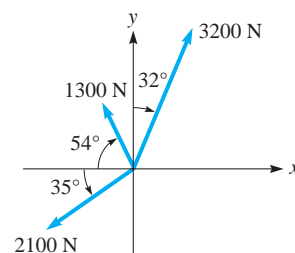


Fig. 9.77

50. In Fig. 9.78, force  $F$  represents the total surface tension force around the circumference on the liquid in the capillary tube. The vertical component of  $F$  holds up the liquid in the tube above the liquid surface outside the tube. What is the vertical component of  $F$ ?

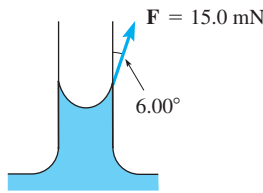


Fig. 9.78

51. A helium-filled balloon rises vertically at 3.5 m/s as the wind carries it horizontally at 5.0 m/s. What is the resultant velocity of the balloon?
52. A shearing pin is designed to break and disengage gears before damage is done to a machine. In a test, a vertically upward force of 8250 N and a force of 7520 N at  $35.0^\circ$  below the horizontal are applied to a shearing pin. What is the resultant force?
53. A magnetic force of 0.15 N is applied at an angle of  $22.5^\circ$  above the horizontal on an iron bar. A second magnetic force of 0.20 N is applied from the opposite side at an angle of  $15.0^\circ$  above the horizontal. What is the upward force on the bar?
54. At a certain point, the angle of elevation of the cliff at Head-Smashed-In Buffalo Jump, Alberta (a UNESCO World Heritage Site) is measured as  $55^\circ$ . Measured a distance 11.0 m farther away from the cliff, the angle of elevation is  $29^\circ$ . Find the height of the cliff. See Fig. 9.79.

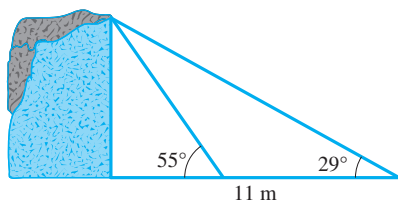


Fig. 9.79

55. A water molecule ( $H_2O$ ) consists of two hydrogen atoms and one oxygen atom. The distance from the nucleus of each hydrogen atom to the nucleus of the oxygen atom is 0.96 pm, and the bond angle (see Fig. 9.80) is  $105^\circ$ . How far is one hydrogen nucleus from the other?

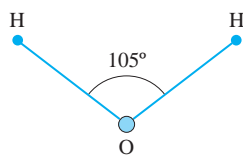


Fig. 9.80

56. A crater on the moon is 150 km in diameter. If the distance to the moon (to each side of the crater) from the earth is 390 000 km, what angle is subtended by the crater at an observer's position on the earth?

57. In Fig. 9.81, a damper mechanism in an air-conditioning system is shown. If  $\theta = 27.5^\circ$  when the spring is at its shortest and longest lengths, what are these lengths?

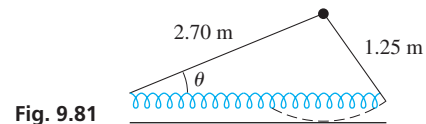


Fig. 9.81

58. A bullet is fired from the ground of a level field at an angle of  $39.0^\circ$  above the horizontal. It travels in a straight line at 670 m/s for 0.20 s when it strikes a target. The sound of the strike is recorded 0.32 s later on the ground. If sound travels at 350 m/s, where is the recording device located?
59. In order to get around an obstruction, an oil pipeline is constructed in two straight sections, one 3.756 km long and the other 4.675 km long, with an angle of  $168.85^\circ$  between the sections where they are joined. How much more pipeline was necessary due to the obstruction?
60. Three pipes of radii 2.50 cm, 3.25 cm, and 4.25 cm are welded together lengthwise. See Fig. 9.82. Find the angles between the centre-to-centre lines.

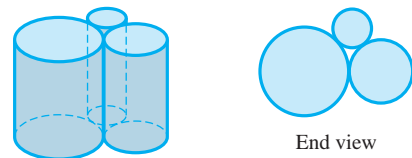


Fig. 9.82

61. Two satellites are being observed at the same observing station. One is 36 200 km from the station, and the other is 30 100 km away. The angle between their lines of observation is  $105.4^\circ$ . How far apart are the satellites?
62. Find the side  $x$  in the truss in Fig. 9.83.

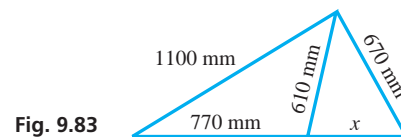
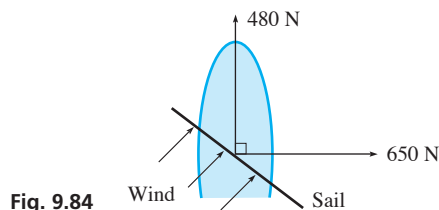


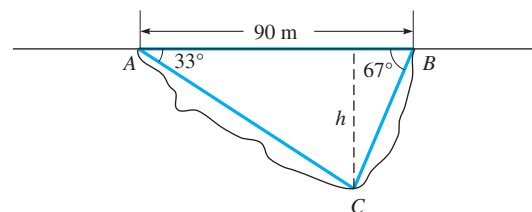
Fig. 9.83

63. The angle of depression of a fire noticed west of a fire tower is  $6.2^\circ$ . The angle of depression of a pond, also west of the tower, is  $13.5^\circ$ . If the fire and pond are at the same altitude, and the tower is 2.25 km from the pond on a direct line, how far is the fire from the pond?
64. A surveyor wishes to find the distance between two points between which there is a security-restricted area. The surveyor measures the distance from each of these points to a third point and finds them to be 226.73 m and 185.12 m. If the angle between the lines of sight from the third point to the other points is  $126.724^\circ$ , how far apart are the two points?
65. In Australia, Adelaide is 805 km and  $69.0^\circ$  south of east of Alice Springs. The pilot of an airplane due north of Adelaide radios Alice Springs and finds the plane is on a line  $10.5^\circ$  south of east from Alice Springs. How far is the plane from Alice Springs?
66. In going around a storm, a plane flies 125 km south, then 140 km at  $30.0^\circ$  south of west, and finally 225 km at  $15.0^\circ$  north of west. What is the displacement of the plane from its original position?

67. One end of a 725-m bridge is sighted from a distance of 1630 m. The angle between the lines of sight of the ends of the bridge is  $25.2^\circ$ . From these data, how far is the observer from the other end of the bridge?
68. A plane is travelling horizontally at 400 m/s. A missile is fired horizontally from it  $30.0^\circ$  from the direction in which the plane is travelling. If the missile leaves the plane at 650 m/s, what is its velocity 10.0 s later if the vertical component is given by  $v_V = -9.80t$  (in m/s)?
69. A sailboat is headed due north, and its sail is set perpendicular to the wind, which is from the south of west. The component of the force of the wind in the direction of the heading is 480 N, and the component perpendicular to the heading (the *drift* component) is 650 N. What is the force exerted by the wind, and what is the direction of the wind? See Fig. 9.84.



70. Boston is 650 km and  $21.0^\circ$  south of west from Halifax, Nova Scotia. Radio signals locate a ship  $10.5^\circ$  east of south from Halifax and  $5.6^\circ$  north of east from Boston. How far is the ship from each city?
71. A bridge 90 m long spans a valley, as in Fig. 9.85. From one end, the angle of depression of a certain point at the bottom of the valley is  $33^\circ$ . From the other end, the angle of depression of the same point is  $67^\circ$ . Find the height of the bridge.



### Writing Exercise

72. The resultant of three horizontal forces, 45 N, 35 N, and 25 N, that act on a bolt is zero. Write a paragraph or two explaining how to find the angles between the forces.

## CHAPTER 9 PRACTICE TEST

In all triangle solutions, sides  $a$ ,  $b$ , and  $c$  are opposite angles  $A$ ,  $B$ , and  $C$ , respectively.

1. By use of a diagram, find the vector sum  $2\mathbf{A} + \mathbf{B}$  for the given



vectors.

2. For the triangle in which  $a = 22.5$ ,  $B = 78.6^\circ$ , and  $c = 30.9$ , find  $b$ .
3. A surveyor locates a tree 36.50 m to the northeast of a set position. The tree is 21.38 m north of a utility pole. What is the displacement of the utility pole from the set position?
4. For the triangle in which  $A = 18.9^\circ$ ,  $B = 104.2^\circ$ , and  $a = 426$ ,

find  $c$ .

5. Solve the triangle in which  $a = 9.84$ ,  $b = 3.29$ , and  $c = 8.44$ .
6. For vector  $\mathbf{R}$ , find  $R$  and standard position  $\theta$  if  $R_x = -235$  and  $R_y = 152$ .
7. Find the horizontal and vertical components of a vector of magnitude 871 that is directed at a standard-position angle of  $284.3^\circ$ .
8. A ship leaves a port and travels due west. At a certain point it turns  $31.5^\circ$  north of west and travels an additional 42.0 km to a point 63.0 km on a direct line from the port. How far from the port is the point where the ship turned?
9. Find the sum of the vectors for which  $A = 449$ ,  $\theta_A = 74.2^\circ$ ,  $B = 285$ , and  $\theta_B = 208.9^\circ$  by components.



# 10

## LEARNING OUTCOMES

After completion of this chapter, the student should be able to:

- Graph the trigonometric functions  $y = \sin x$  and  $y = \cos x$  and identify their important values for sketching
- Define amplitude, period, and displacement
- Graph the functions  $y = a \sin x$  and  $y = a \cos x$
- Graph the functions  $y = a \sin bx$  and  $y = a \cos bx$
- Graph the functions  $y = a \sin(bx + c)$  and  $y = a \cos(bx + c)$
- Graph the functions  $y = \tan x$ ,  $y = \cot x$ ,  $y = \sec x$ , and  $y = \csc x$
- Solve application problems involving graphs of trigonometric functions
- Graph composite trigonometric curves by addition of coordinates
- Graph Lissajous figures

# Graphs of the Trigonometric Functions

The electronics era is thought by many to have started in the 1880s, with the discovery of the vacuum tube by the American inventor Thomas Edison and the discovery of radio waves by the German physicist Heinrich Hertz. Then, in the 1890s, the cathode-ray tube was developed and, as an oscilloscope, has been used since that time to analyse various types of wave forms, such as sound waves and radio waves. Since the mid-1900s, devices similar to a cathode-ray tube have been used in TV picture tubes and computer displays.

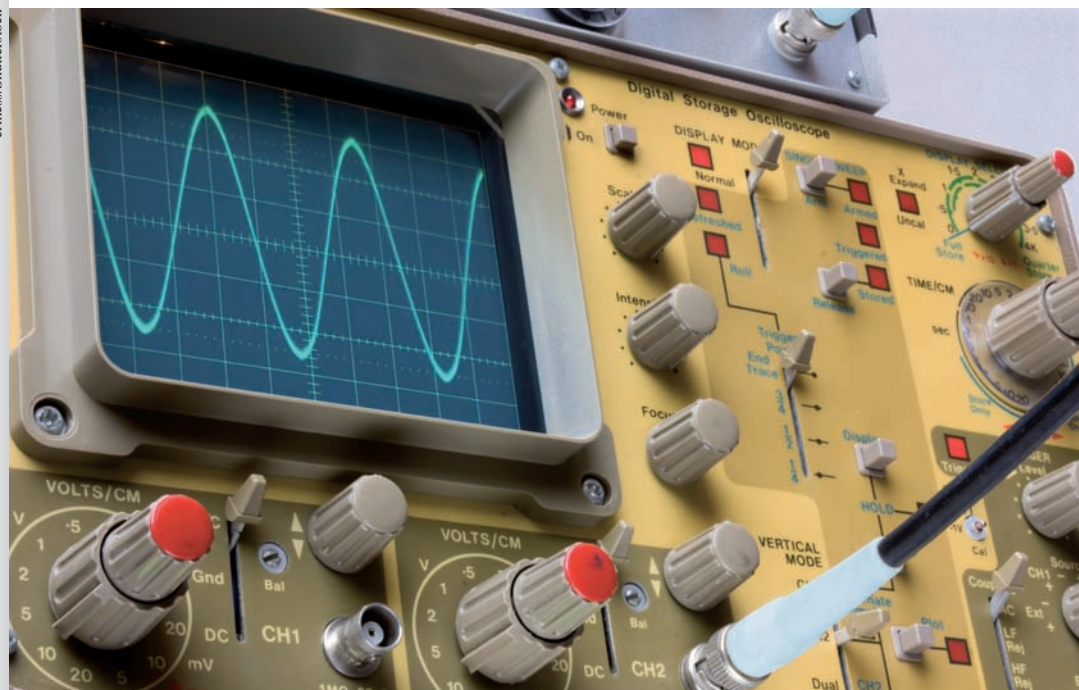
What is seen on the screen of an oscilloscope are electric signals that are represented by graphs of trigonometric functions. As noted earlier, the basic method of graphing was developed in the mid-1600s, and using trigonometric functions of numbers has been common since the mid-1700s. Therefore, the graphs of the trigonometric functions were well known in the late 1800s and became very useful in the development of electronics.

The graphs of the trigonometric functions are useful in many areas of application, particularly those that involve wave motion and periodic values. Filtering electronic signals in communications, mixing musical sounds on a tape in a recording studio, studying the seasonal temperatures of an area, and analysing ocean waves and tides are some of the many applications of periodic motion.

Aside from their use in applications, the graphs of the trigonometric functions give us one of the clearest ways of showing the properties of the various functions. Therefore, in this chapter, we show graphs of these functions, with emphasis on the sine and cosine functions.

► The graphs of electrical signals (including sinusoidal signals) can be displayed on an oscilloscope, and properties of the signal, such as period, frequency, phase, and amplitude, can be measured. In Section 10.6, we show the patterns that result when two signals are combined and displayed.

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## 10.1 Graphs of $y = a \sin x$ and $y = a \cos x$

Graphs of  $y = \sin x$  and  $y = \cos x$  • Amplitude • Graphs of  $y = a \sin x$  and  $y = a \cos x$

Recall from Section 8.3 that when angles are expressed in radian measure, both the independent variable and the dependent variable of trigonometric functions are real numbers. For this reason, *we only consider angles in radians* when plotting and sketching the graphs of trigonometric functions. If necessary, review Section 8.3 on radian measure.

In this section, we discuss the graphs of the sine and cosine functions. We begin by making a table of values of  $x$  and  $y$  for the function  $y = \sin x$ , where  $x$  and  $y$  are used in the standard way as the *independent variable* and *dependent variable*. We plot the points to obtain the graph in Fig. 10.1.

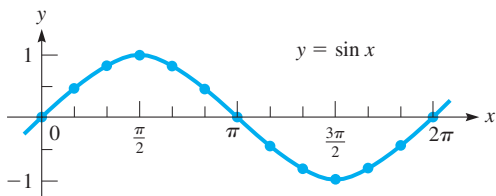


Fig. 10.1

$x$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	$\pi$
$y$	0	0.5	0.87	1	0.87	0.5	0

$x$	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	$2\pi$
$y$	-0.5	-0.87	-1	-0.87	-0.5	0

The graph of  $y = \cos x$  may be drawn in the same way. The next table gives values for plotting the graph of  $y = \cos x$ , and the graph is shown in Fig. 10.2.

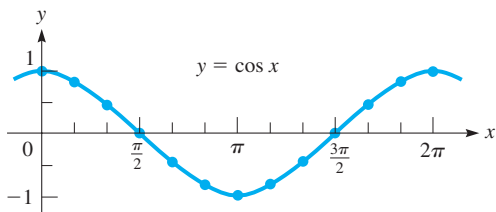


Fig. 10.2

$x$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	$\pi$
$y$	1	0.87	0.5	0	-0.5	-0.87	-1

$x$	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	$2\pi$
$y$	-0.87	-0.5	0	0.5	0.87	1

The graphs are continued beyond the values shown in the tables to indicate that *they continue indefinitely in each direction*. To show this more clearly, in Figs. 10.3 and 10.4, note the graphs of  $y = \sin x$  and  $y = \cos x$  from  $x = -10$  to  $x = 10$ .

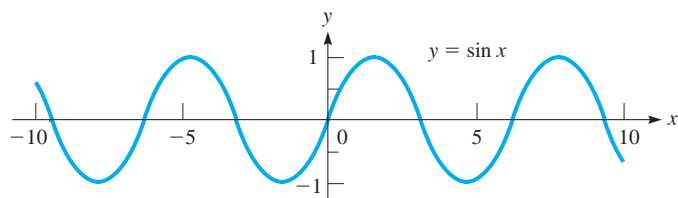


Fig. 10.3

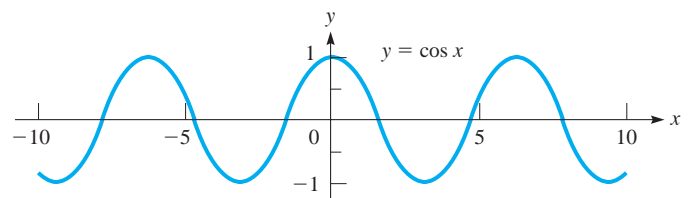


Fig. 10.4

From these tables and graphs, we can see that the graphs of  $y = \sin x$  and  $y = \cos x$  have the basic features listed in the following table. These features (illustrated in Fig. 10.5) will be especially valuable in *sketching* similar curves. Since some of these features remain the same, it will not be necessary to plot numerous points every time we wish to sketch such a curve.



Table 10.1

Key values	$y = \sin x$	$y = \cos x$
0	0	1
$\frac{\pi}{2}$	1	0
$\pi$	0	-1
$\frac{3\pi}{2}$	-1	0
$2\pi$	0	1

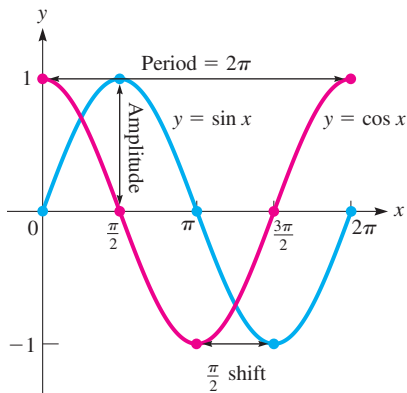


Fig. 10.5

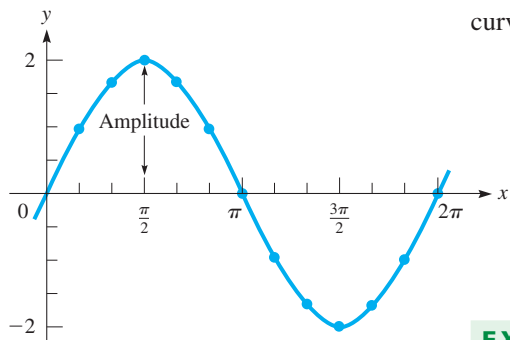


Fig. 10.6

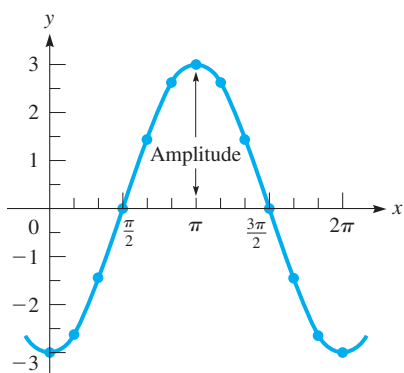


Fig. 10.7

### Basic Features of the Graphs of $y = \sin x$ and $y = \cos x$

1. The domain is all values of  $x$ .
2. The range is  $-1 \leq y \leq 1$ , or  $[-1, 1]$ .
3. The **amplitude** (half the distance between the maximum value and the minimum value) is 1.
4. Both graphs are exactly the same shape (called **sinusoidal**).
5. The graph of the cosine curve is shifted  $\pi/2$  units to the left of the sine curve.
6. For both graphs, the values of  $y$  repeat every  $2\pi$  units of  $x$ . We therefore say that the functions are **periodic** with **period**  $2\pi$ .
7. The functions have zeros, maximum values, and minimum values when  $x$  is a multiple of  $\pi/2$ . The behaviour from 0 to  $2\pi$  at these key values is summarized in Table 10.1.

To obtain the graph of  $y = a \sin x$ , note that all the  $y$ -values obtained for the graph of  $y = \sin x$  are to be multiplied by the number  $a$ . In this case, the greatest value of the sine function is  $|a|$ , and the curve will have no value less than  $-|a|$ . Therefore, the range is  $[-|a|, |a|]$ , and the amplitude of the curve is  $|a|$ . This is also true for  $y = a \cos x$ .

#### EXAMPLE 1 Plotting the graph of $y = a \sin x$

Plot the graph of  $y = 2 \sin x$ .

Since  $a = 2$ , the amplitude of this curve is  $|2| = 2$ . This means that the maximum value of  $y$  is 2 and the minimum value is  $y = -2$ . The table of values follows, and the curve is shown in Fig. 10.6.

$x$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	$\pi$
$y$	0	1	1.73	2	1.73	1	0

$x$	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	$2\pi$
$y$	-1	-1.73	-2	-1.73	-1	0

#### EXAMPLE 2 Plotting the graph of $y = a \cos x$

Plot the graph of  $y = -3 \cos x$ .

In this case,  $a = -3$ , and this means that the amplitude is  $|-3| = 3$ . Therefore, the maximum value of  $y$  is 3, and the minimum value of  $y$  is  $-3$ . The table of values follows, and the curve is shown in Fig. 10.7.

$x$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	$\pi$
$y$	-3	-2.6	-1.5	0	1.5	2.6	3

$x$	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	$2\pi$
$y$	2.6	1.5	0	-1.5	-2.6	-3

Note that the effect of the negative sign with the number  $a$  is to **invert** the curve about the  $x$ -axis.

Apart from the fact that the range of the functions  $y = a \sin x$  and  $y = a \cos x$  is  $[-|a|, |a|]$ , we can see from the previous examples that the number  $a$  has no other effect on the basic features of these functions as compared to those of the functions



3.  $y = \sin x$                       4.  $y = \cos x$   
 5.  $y = 3 \cos x$                     6.  $y = -4 \sin x$

In Exercises 7–22, sketch the graphs of the given functions.

7.  $y = 3 \sin x$                       8.  $y = 5 \sin x$   
 9.  $y = \frac{5}{2} \sin x$                     10.  $y = 35 \sin x$   
 11.  $y = 200 \cos x$                   12.  $y = 0.25 \cos x$   
 13.  $y = 0.8 \cos x$                   14.  $y = \frac{3}{2} \cos x$   
 15.  $y = -\sin x$                       16.  $y = -300 \sin x$   
 17.  $y = -1500 \sin x$               18.  $y = -0.2 \sin x$   
 19.  $y = -\cos x$                     20.  $y = -8 \cos x$   
 21.  $y = -50 \cos x$                 22.  $y = -0.4 \cos x$

Although units of  $\pi$  are convenient, we must remember that  $\pi$  is only a number. Numbers that are not multiples of  $\pi$  may be used. In Exercises 23–26, plot the indicated graphs by finding the values of  $y$  that correspond to values of  $x$  of 0, 1, 2, 3, 4, 5, 6, and 7 on a calculator. (Remember, the numbers 0, 1, 2, and so on represent radian measure.)

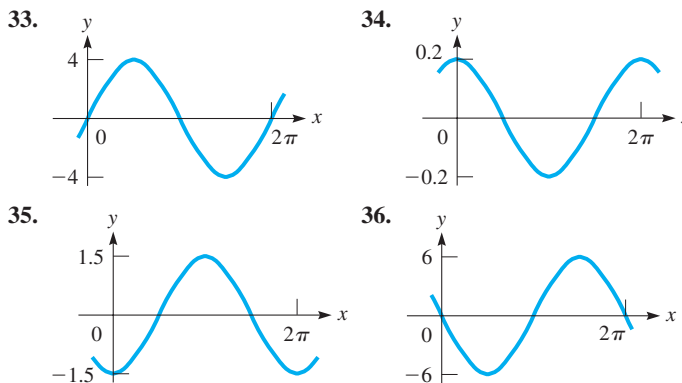
23.  $y = \sin x$                           24.  $y = -30 \sin x$   
 25.  $y = 12 \cos x$                     26.  $y = 2 \cos x$

In Exercises 27–32, solve the given problems.

27. Find the function and graph it for a function of the form  $y = a \sin x$  that passes through  $(\pi/2, -2)$ .  
 28. Find the function and graph it for a function of the form  $y = a \sin x$  that passes through  $(3\pi/2, -2)$ .  
 29. Find the function and graph it for a function of the form  $y = a \cos x$  that passes through  $(\pi, 2)$ .  
 30. Find the function and graph it for a function of the form  $y = a \cos x$  that passes through  $(2\pi, -2)$ .

31. The graph displayed on an oscilloscope can be represented by  $y = -0.05 \sin x$ . Display this curve on a graphing calculator.  
 32. The displacement  $y$  (in cm) of the end of a robot arm for welding is  $y = 4.75 \cos t$ , where  $t$  is the time (in s). Display this curve on a graphing calculator.

In Exercises 33–36, the graph of a function of the form  $y = a \sin x$  or  $y = a \cos x$  is shown. Determine the specific function of each.



In Exercises 37–40, find the value of  $a$  for either  $y = a \sin x$  or  $y = a \cos x$ , whichever is correct, such that the given point is on the graph. The amplitude of each function is 2.50. Thereby determine the function. (All points are located such that the  $x$  value is between  $-\pi$  and  $\pi$ .)

37.  $(0.67, -1.55)$                     38.  $(-1.20, 0.90)$   
 39.  $(2.07, 1.20)$                     40.  $(-2.47, -1.55)$

Answers to Practice Exercise

1.

$x$	0	$\frac{\pi}{2}$	$\pi$	$\frac{3\pi}{2}$	$2\pi$
$y$	0	-6	0	6	0
		min.		max.	

## 10.2 Graphs of $y = a \sin bx$ and $y = a \cos bx$

Period of a Function • Graphs of  $y = a \sin bx$  and  $y = a \cos bx$  • Important Values for Sketching • Cycle

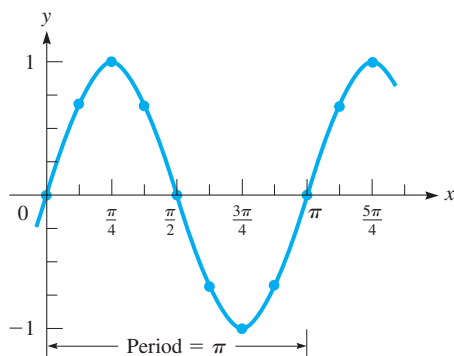


Fig. 10.10

We learned in Section 10.1 that the values of the functions  $y = a \sin x$  and  $y = a \cos x$  repeat every  $2\pi$  units of  $x$ , making them periodic with period  $2\pi$ . More generally, we say that a function  $F$  has period  $P$  if  $F(x) = F(x + P)$  for all  $x$  in the domain of  $F$ , and  $P$  is the smallest such number. In other words, the period is the  $x$ -distance between a point and the next corresponding point after which the values of  $y$  repeat.

Let us now plot the curve  $y = \sin 2x$ . This means that we choose a value of  $x$ , multiply this value by 2, and find the sine of the result. This leads to the following table of values for this function:

$x$	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$	$\frac{5\pi}{8}$	$\frac{3\pi}{4}$	$\frac{7\pi}{8}$	$\pi$	$\frac{9\pi}{8}$	$\frac{5\pi}{4}$
$2x$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	$\pi$	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	$2\pi$	$\frac{9\pi}{4}$	$\frac{5\pi}{2}$
$y$	0	0.7	1	0.7	0	-0.7	-1	-0.7	0	0.7	1

Plotting these points, we have the curve shown in Fig. 10.10.

From the table and Fig. 10.10, note that  $y = \sin 2x$  repeats after  $\pi$  units of  $x$ . The effect of the 2 is that the period of  $y = \sin 2x$  is half the period of the curve of  $y = \sin x$ .

**LEARNING TIP**

If the period of a function  $F(x)$  is  $P$ , then the period of  $F(bx)$  is  $P/b$ . Since each of the functions  $\sin x$  and  $\cos x$  has a period of  $2\pi$ , each of the functions  $\sin bx$  and  $\cos bx$  has a period of  $2\pi/b$ .

**Practice Exercise**

Find the period of each function.

1.  $y = \sin \pi x$     2.  $y = \cos \frac{1}{3}x$

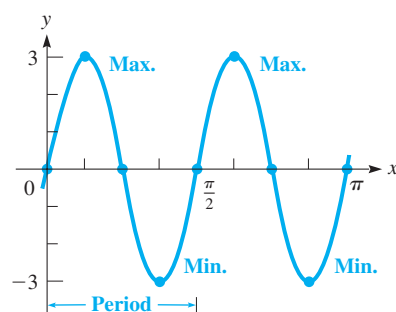


Fig. 10.11

**LEARNING TIP**

By finding one-fourth of the period, we can easily find the important values for sketching the curve.

**EXAMPLE 1 Finding the period of a function**

- (a) The period of  $\cos 4x$  is  $\frac{2\pi}{4} = \frac{\pi}{2}$ .      (b) The period of  $\sin 3\pi x$  is  $\frac{2\pi}{3\pi} = \frac{2}{3}$ .  
 (c) The period of  $\sin \frac{1}{2}x$  is  $\frac{2\pi}{\frac{1}{2}} = 4\pi$ .      (d) The period of  $\cos \frac{\pi}{4}x$  is  $\frac{2\pi}{\frac{\pi}{4}} = 8$ .

In (a), the period tells us that the curve of  $y = \cos 4x$  will repeat every  $\pi/2$  (approximately 1.57) units of  $x$ . In (b), we see that the curve of  $y = \sin 3\pi x$  will repeat every  $2/3$  of a unit. In (c) and (d), the periods are longer than those of  $y = \sin x$  and  $y = \cos x$ . ■

Combining the value of the period with the value of the amplitude from Section 10.1, we conclude that *the functions  $y = a \sin bx$  and  $y = a \cos bx$  have an amplitude of  $|a|$  and a period of  $2\pi/b$* . These properties are very useful in sketching these functions.

**EXAMPLE 2 Sketching the graph of  $y = a \sin bx$** 

Sketch the graph of  $y = 3 \sin 4x$  for  $0 \leq x \leq \pi$ .

Since  $a = 3$ , the amplitude is 3. The  $4x$  tells us that the period is  $2\pi/4 = \pi/2$ . This means that  $y = 0$  for  $x = 0$  and for  $x = \pi/2$ . Since this sine function is zero halfway between  $x = 0$  and  $x = \pi/2$ , we find that  $y = 0$  for  $x = \pi/4$ . Also, the fact that the graph of the sine function reaches its maximum and minimum values halfway between zeros means that  $y = 3$  for  $x = \pi/8$ , and  $y = -3$  for  $x = 3\pi/8$ . Note that the values of  $x$  in the following table are those for which  $4x = 0, \pi/2, \pi, 3\pi/2, 2\pi$ , and so on, which correspond to the key values listed in Tables 10.1 and 10.2.

$x$	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$	$\frac{5\pi}{8}$	$\frac{3\pi}{4}$	$\frac{7\pi}{8}$	$\pi$
$y$	0	3	0	-3	0	3	0	-3	0

Using the values from the table and the fact that the curve is sinusoidal in form, we sketch the graph of this function in Fig. 10.11. We see that the key values where the function has zeros, maxima, and minima occur when  $x$  is a multiple of  $\pi/8$ , which is exactly one-fourth of the period. ■

Note from Example 2 that *an important distance in sketching a sine curve or a cosine curve is one-fourth of the period*. For  $y = a \sin bx$ , it is one-fourth of the period from the origin to the first value of  $x$  where  $y$  is at its maximum (or minimum) value. Then we proceed another one-fourth period to a zero, another one-fourth period to the next minimum (or maximum) value, another to the next zero (this is where the period is completed), and so on.

Similarly, one-fourth of the period is useful in sketching the graph of  $y = \cos bx$ . For this function, the maximum (or minimum) value occurs for  $x = 0$ . At the following one-fourth-period values, there is a zero, a minimum (or maximum), a zero, and a maximum (or minimum) at the start of the next period.

We now summarize the important values for sketching the graphs of  $y = a \sin bx$  and  $y = a \cos bx$ .

**Important Values for Sketching  $y = a \sin bx$  and  $y = a \cos bx$** 

1. The amplitude:  $|a|$
2. The period:  $2\pi/b$
3. Values of the function for each one-fourth period

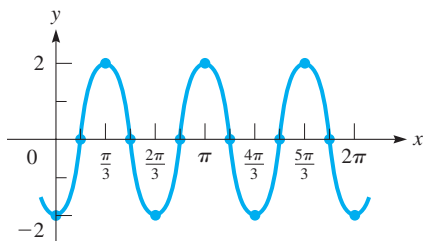


Fig. 10.12

**EXAMPLE 3** Using important values to sketch a graph

Sketch the graph of  $y = -2 \cos 3x$  for  $0 \leq x \leq 2\pi$ .

Note that the amplitude is 2 and the period is  $\frac{2\pi}{3}$ . This means that one-fourth of the period is  $\frac{1}{4} \times \frac{2\pi}{3} = \frac{\pi}{6}$ . Since the cosine curve is at a maximum or minimum for  $x = 0$ , we find that  $y = -2$  for  $x = 0$  (the negative value is due to the minus sign before the function), which means it is a minimum point. The curve then has a zero at  $x = \frac{\pi}{6}$ , a maximum value of 2 at  $x = 2(\frac{\pi}{6}) = \frac{\pi}{3}$ , a zero at  $x = 3(\frac{\pi}{6}) = \frac{\pi}{2}$ , and its next value of  $-2$  at  $x = 4(\frac{\pi}{6}) = \frac{2\pi}{3}$ , and so on. Therefore, we have the following table:

$x$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	$\pi$	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	$2\pi$
$y$	-2	0	2	0	-2	0	2	0	-2	0	2	0	-2

Using this table and the sinusoidal shape of the cosine curve, we sketch the function in Fig. 10.12. ■

For a periodic function, a **cycle** is any section of the graph that includes exactly one period. Fig. 10.11 shows two cycles, whereas Fig. 10.12 shows three.

**EXAMPLE 4** Graph of  $y = a \cos bx$ —application

A generator produces a voltage  $V = 200 \cos 50\pi t$ , where  $t$  is the time in seconds ( $50\pi$  is angular velocity, so it has units of rad/s; thus,  $50\pi t$  is an angle in radians). Graph  $V$  as a function of  $t$  for  $0 \leq t \leq 0.06$  s.

The amplitude is 200 V and the period is  $2\pi / (50\pi) = 0.04$  s. Since one-fourth of the period is 0.01 s, the function has zeros, maxima, and minima when  $x$  is a multiple of 0.01. Thus, we have the following table of key values:

$t$ (s)	0	0.01	0.02	0.03	0.04	0.05	0.06
$V$ (V)	200	0	-200	0	200	0	-200

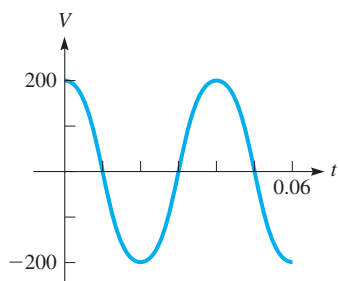


Fig. 10.13

The graph is shown in Fig. 10.13. Note that between 0 and 0.06 s, the function completes 1.5 cycles. We do not consider negative values of  $t$ , for they have no real meaning in this problem. ■

**EXERCISES 10.2**

In Exercises 1 and 2, graph the function if the given changes are made in the indicated examples of this section.

- In Example 2, if the coefficient of  $x$  is changed from 4 to 6, sketch the graph of the resulting function.
- In Example 3, if the coefficient of  $x$  is changed from 3 to 4, sketch the graph of the resulting function.

In Exercises 3–22, find the period of each function.

- |                                |                                 |
|--------------------------------|---------------------------------|
| 3. $y = 2 \sin 6x$             | 4. $y = 4 \sin 2x$              |
| 5. $y = 3 \cos 8x$             | 6. $y = 28 \cos 10x$            |
| 7. $y = -2 \sin 12x$           | 8. $y = -\frac{1}{3} \sin 5x$   |
| 9. $y = -\cos 16x$             | 10. $y = -4 \cos 2x$            |
| 11. $y = 520 \sin 2\pi x$      | 12. $y = 2 \sin 3\pi x$         |
| 13. $y = 3 \cos 4\pi x$        | 14. $y = 4 \cos 10\pi x$        |
| 15. $y = 15 \sin \frac{1}{3}x$ | 16. $y = -25 \sin \frac{2}{5}x$ |

- |  |   |
|--|---|
| 17. $y = -\frac{1}{2} \cos \frac{2}{3}x$ | 18. $y = \frac{1}{3} \cos \frac{1}{4}x$ |
| 19. $y = 0.4 \sin \frac{2\pi x}{3}$      | 20. $y = 1.5 \cos \frac{\pi x}{10}$     |
| 21. $y = 3.3 \cos \pi^2 x$               | 22. $y = -12.5 \sin \frac{2x}{\pi}$     |

In Exercises 23–42, sketch the graphs of the given functions. (These are the same functions as in Exercises 3–22.)

- |                                |                                 |
|--------------------------------|---------------------------------|
| 23. $y = 2 \sin 6x$            | 24. $y = 4 \sin 2x$             |
| 25. $y = 3 \cos 8x$            | 26. $y = 28 \cos 10x$           |
| 27. $y = -2 \sin 12x$          | 28. $y = -\frac{1}{3} \sin 5x$  |
| 29. $y = -\cos 16x$            | 30. $y = -4 \cos 2x$            |
| 31. $y = 520 \sin 2\pi x$      | 32. $y = 2 \sin 3\pi x$         |
| 33. $y = 3 \cos 4\pi x$        | 34. $y = 4 \cos 10\pi x$        |
| 35. $y = 15 \sin \frac{1}{3}x$ | 36. $y = -25 \sin \frac{2}{5}x$ |

37.  $y = -\frac{1}{2} \cos \frac{2}{3}x$       38.  $y = \frac{1}{3} \cos \frac{1}{4}x$   
 39.  $y = 0.4 \sin \frac{2\pi x}{3}$       40.  $y = 1.5 \cos \frac{\pi x}{10}$   
 41.  $y = 3.3 \cos \pi^2 x$       42.  $y = -12.5 \sin \frac{2x}{\pi}$

In Exercises 43–46, the period is given for a function of the form  $y = \sin bx$ . Write the function corresponding to the given period.

43.  $\frac{\pi}{3}$       44.  $\frac{2\pi}{5}$       45.  $\frac{1}{3}$       46. 6

In Exercises 47–50, graph the given functions. In Exercises 47 and 48, first rewrite the function with a positive angle, and then graph the resulting function. (Refer to Eq. 8.7 for trigonometric functions of negative angles.)

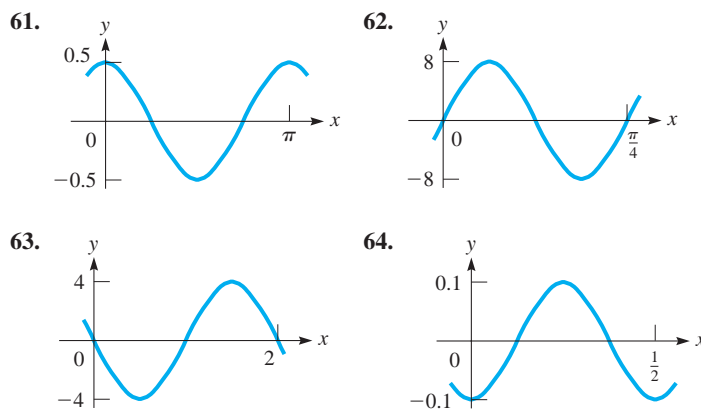
47.  $y = 3 \sin(-2x)$       48.  $y = -5 \cos(-4\pi x)$   
 49.  $y = 8 \left| \cos \left( \frac{\pi}{2}x \right) \right|$       50.  $y = 0.4 \left| \sin 6x \right|$

In Exercises 51–60, solve the given problems.

51. By noting the periods of  $\sin 2x$  and  $\sin 3x$ , find the minimum period of the function  $y = \sin 2x + \sin 3x$ .  
 52. By noting the period of  $\cos \frac{1}{2}x$  and  $\cos \frac{1}{3}x$ , find the minimum period of the function  $y = \cos \frac{1}{2}x + \cos \frac{1}{3}x$ .  
 53. Find the function and graph it for a function of the form  $y = -2 \sin bx$  that passes through  $(\pi/4, -2)$  and for which  $b$  has the smallest possible positive value.  
 54. Find the function and graph it for a function of the form  $y = 2 \sin bx$  that passes through  $(\pi/6, 2)$  and for which  $b$  has the smallest possible positive value.  
 55. Find the function and graph it for a function of the form  $y = 2 \cos bx$  that passes through  $(\pi, 0)$  and for which  $b$  has the smallest possible positive value.

56. Find the function and graph it for a function of the form  $y = -2 \cos bx$  that passes through  $(\pi/2, 2)$  and for which  $b$  has the smallest possible positive value.  
 57. The standard electric voltage in a 60-Hz alternating-current circuit is given by  $V = 170 \sin 120\pi t$ , where  $t$  is the time in seconds. Sketch the graph of  $V$  as a function of  $t$  for  $0 \leq t \leq 0.05$  s.  
 58. To tune the instruments of an orchestra before a concert, an A note is struck on a piano. The piano wire vibrates with a displacement  $y$  (in mm) given by  $y = 3.2 \cos 880\pi t$ , where  $t$  is in seconds. Sketch the graph of  $y$  vs.  $t$  for  $0 \leq t \leq 0.01$  s.  
 59. The velocity  $v$  (in cm/s) of a piston is  $v = 450 \cos 3600t$ , where  $t$  is in seconds. Sketch the graph of  $v$  vs.  $t$  for  $0 \leq t \leq 0.006$  s.  
 60. On a certain day in St. John, New Brunswick, the difference between high tide and low tide was 6.4 m. The period was about 12.4 h. Find a cosine function that describes these tides if high tide was at midnight.

In Exercises 61–64, the graph of a function of the form  $y = a \sin bx$  or  $y = a \cos bx$  is shown. Determine the specific function of each.



Answers to Practice Exercises

1. 2      2.  $6\pi$

### 10.3 Graphs of $y = a \sin(bx + c)$ and $y = a \cos(bx + c)$

Phase Angle • Displacement •  
 Graphs of  $y = a \sin(bx + c)$  and  
 $y = a \cos(bx + c)$

In the function  $y = a \sin(bx + c)$ ,  $c$  represents the **phase angle**. It is another very important quantity in graphing the sine and cosine functions. Its meaning is illustrated in the following example.

**EXAMPLE 1** Sketch of a function with phase angle

Sketch the graph of  $y = \sin\left(2x + \frac{\pi}{4}\right)$ .

Here,  $c = \pi/4$ . Therefore, in order to obtain values for the table, we assume a value for  $x$ , multiply it by 2, add  $\pi/4$  to this value, and then find the sine of the result. The values shown are those for which  $2x + \pi/4 = 0, \pi/4, \pi, 2, 3\pi/4, \pi$ , and so on, which are the important values for  $y = \sin 2x$ .

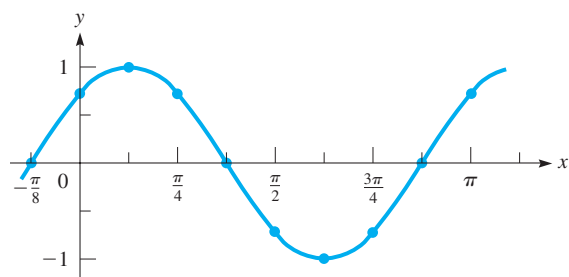


Fig. 10.14

$x$	$-\frac{\pi}{8}$	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$	$\frac{5\pi}{8}$	$\frac{3\pi}{4}$	$\frac{7\pi}{8}$	$\pi$
$y$	0	0.7	1	0.7	0	-0.7	-1	-0.7	0	0.7

Solving  $2x + \pi/4 = 0$ , we get  $x = -\pi/8$ , which gives  $y = \sin 0 = 0$ . The other values for  $y$  are found in the same way. See Fig. 10.14. ■



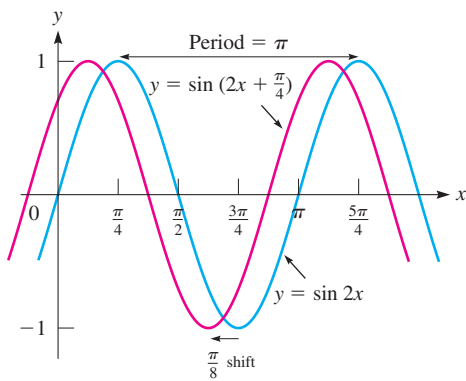


Fig. 10.15

Note from Example 1 that the graph of  $y = \sin(2x + \frac{\pi}{4})$  is precisely the same as the graph of  $y = \sin 2x$ , except that it is **shifted**  $\frac{\pi}{8}$  units to the left. Fig. 10.15 shows the graphs of  $y = \sin 2x$  and  $y = \sin(2x + \frac{\pi}{4})$ . We see that the shapes are the same and that the graph of  $y = \sin(2x + \frac{\pi}{4})$  is about 0.4 unit ( $\frac{\pi}{8} \approx 0.39$ ) to the left of the graph of  $y = \sin 2x$ .

In general, the effect of  $c$  in the equation  $y = \sin(bx + c)$  can be understood when we write  $bx + c = b(x + \frac{c}{b})$ . This means that the function  $y = \sin(bx + c)$  is obtained as a result of adding the constant  $c/b$  to  $x$  in the function  $y = \sin bx$ . As we discussed in Section 3.5, by adding a constant to  $x$ , the graph of  $y = a \sin bx$  is shifted to the left or to the right. In this case, the graph is shifted to the left if  $c > 0$  and to the right if  $c < 0$ . The direction and magnitude of the shift is called the **displacement** (or **phase shift**), and it is given by  $-c/b$ . Note that the displacement can be obtained by solving for  $x$  in the equation  $bx + c = 0$ . In Example 1, the displacement is  $-\pi/8$ .

We can verify that the displacement is  $-c/b$  by noting corresponding points on the graphs of  $y = \sin bx$  and  $y = \sin(bx + c)$ . For  $y = \sin bx$ , when  $x = 0$ , then  $y = 0$ . For  $y = \sin(bx + c)$ , when  $x = -c/b$ , then  $y = 0$ . The point  $(-c/b, 0)$  on the graph of  $y = \sin(bx + c)$  is  $-c/b$  units to the left of the point  $(0, 0)$  on the graph of  $y = \sin bx$ .

We can use the displacement combined with the amplitude and the period along with the other information from the previous sections to sketch curves of the functions  $y = a \sin(bx + c)$  and  $y = a \cos(bx + c)$ , where  $b > 0$ .

**LEARNING TIP**

Carefully note the difference between  $y = \sin(bx + c)$  and  $y = \sin bx + c$ . Writing  $\sin(bx + c)$  means to find the sine of the quantity  $bx + c$ , whereas  $\sin bx + c$  means to find the sine of  $bx$  and then add the value  $c$ .

**LEARNING TIP**

Note that the constant  $c$  and the displacement  $-c/b$  differ in sign:

- If  $c > 0$ , the graph is shifted to the **left**, and displacement is **negative**.
- If  $c < 0$ , the graph is shifted to the **right**, and displacement is **positive**.

**Important Quantities for Sketching Graphs of  $y = a \sin(bx + c)$  and  $y = a \cos(bx + c)$**

Amplitude =  $|a|$

Period =  $\frac{2\pi}{b}$  (10.1)

Displacement =  $-\frac{c}{b}$

These quantities allow us to evaluate the function at key values each one-fourth period. Table 10.3 summarizes these key values for the cycle that starts at  $x = -c/b$  and is completed at  $x = -c/b + \text{period}$ . A general illustration of the graph of  $y = a \sin(bx + c)$  is shown in Fig. 10.16.

Table 10.3

Key values (one cycle)	$y = a \sin(bx + c)$	$y = a \cos(bx + c)$
$-\frac{c}{b}$	0	$a$
$-\frac{c}{b} + \frac{\text{period}}{4}$	$a$	0
$-\frac{c}{b} + \frac{\text{period}}{2}$	0	$-a$
$-\frac{c}{b} + \frac{3 \cdot \text{period}}{4}$	$-a$	0
$-\frac{c}{b} + \text{period}$	0	$a$

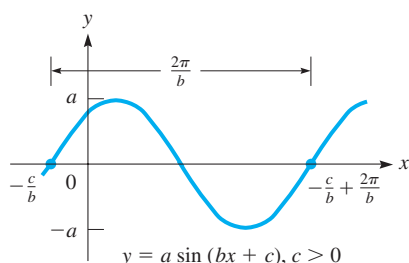
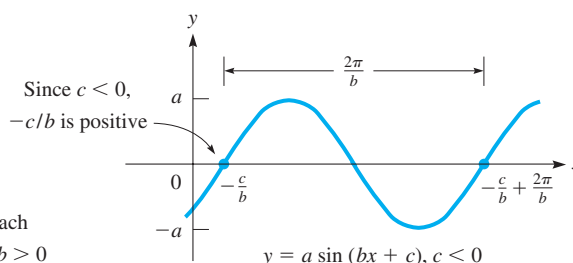


Fig. 10.16

(a)



For each  $a > 0, b > 0$

(b)



**EXAMPLE 2** Sketching the graph of  $y = a \sin(bx + c)$ 

Sketch the graph of  $y = 2 \sin(3x - \pi)$ .

First, note that  $a = 2$ ,  $b = 3$ , and  $c = -\pi$ . Therefore, the amplitude is 2, the period is  $2\pi/3$ , and the displacement is  $-\pi/3 = \pi/3$ . (We can also get the displacement from  $3x - \pi = 0$ ,  $x = \pi/3$ .)

One-fourth of the period is  $\frac{1}{4}(\frac{2\pi}{3}) = \frac{\pi}{6}$ , so key values for one full cycle start at  $\pi/3$ , end at  $\pi$ , and are found every  $\pi/6$  units in between. This cycle is represented by the magenta portion of the graph in Fig. 10.17, obtained from the following table of important values.

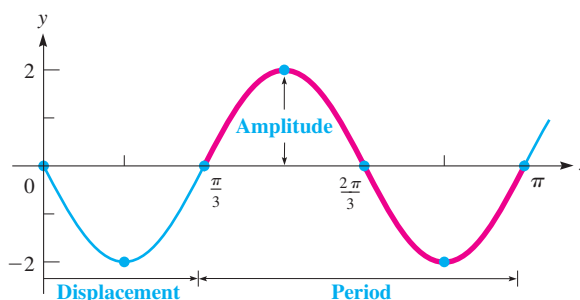


Fig. 10.17

$x$	$0$	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	$\pi$
$y$	$0$	$-2$	$0$	$2$	$0$	$-2$	$0$

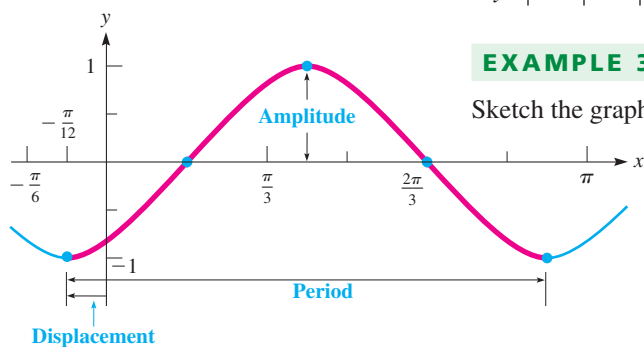


Fig. 10.18

**EXAMPLE 3** Sketching the graph of  $y = a \cos(bx + c)$ 

Sketch the graph of the function  $y = -\cos(2x + \frac{\pi}{6})$ .

First, we determine that the amplitude is 1, the period is  $\frac{2\pi}{2} = \pi$ , and the displacement is  $-\frac{\pi}{6} \div 2 = -\frac{\pi}{12}$ . Moreover, one-fourth of the period is  $\pi/4$ , so key values for one cycle are found every  $\pi/4$  units beginning at  $-\frac{\pi}{12}$  and ending at  $\frac{11\pi}{12}$ . This cycle is represented by the magenta portion of the graph in Fig. 10.18, sketched from the following table:

$x$	$-\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{5\pi}{12}$	$\frac{2\pi}{3}$	$\frac{11\pi}{12}$
$y$	$-1$	$0$	$1$	$0$	$-1$

**Practice Exercise**

1. For the graph of  $y = 8 \sin(2x - \pi/3)$ , determine amplitude, period, and displacement.

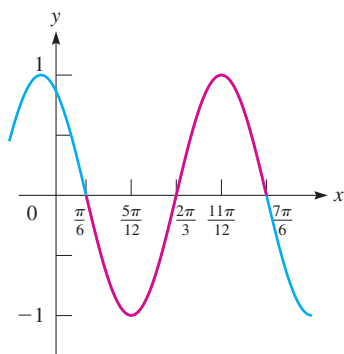


Fig. 10.19

**EXAMPLE 4** Graph with  $b < 0$ 

Sketch the graph of the function  $y = \sin(-2x + \frac{\pi}{3})$ .

We first rewrite the function with a positive angle. We use the fact that  $\sin(-x) = -\sin x$  (see Eq. 8.7) to write

$$y = \sin(-2x + \frac{\pi}{3}) = \sin(-(2x - \frac{\pi}{3})) = -\sin(2x - \frac{\pi}{3})$$

Therefore, we have  $a = -1$ ,  $b = 2$ ,  $c = -\frac{\pi}{3}$ . We determine that the amplitude is 1, the period is  $2\pi \div 2 = \pi$ , and the displacement is  $-(-\frac{\pi}{3}) \div 2 = \frac{\pi}{6}$ . Also, one-fourth of the period is  $\frac{\pi}{4}$ , so key values are  $\frac{\pi}{4}$  units apart, starting at  $\frac{\pi}{6}$  and ending at  $\frac{7\pi}{6}$ . We now make a table of important values:

$x$	$\frac{\pi}{6}$	$\frac{5\pi}{12}$	$\frac{2\pi}{3}$	$\frac{11\pi}{12}$	$\frac{7\pi}{6}$
$y$	$0$	$-1$	$0$	$1$	$0$

From this table, we sketch the graph in Fig. 10.19.

**EXAMPLE 5** Using a trigonometric function to model daylight hours

At 48°N latitude, the number of daylight hours during the longest day of the year (June 21) is 16.1 hours. During the shortest day of the year (December 21), the number of daylight hours is 8.31 hours. Approximate the number of hours  $h$  of daylight each day during a year with a function of the form  $h = a \sin(bt + c) + d$ , where time is measured in days. Cities at 48°N latitude include Paris, Vienna, and Victoria, British Columbia.

We determine the values of  $a$ ,  $b$ ,  $c$ , and  $d$  as follows:

- The amplitude is half the distance between the maximum and the minimum daylight hours, so  $a = \frac{16.1 - 8.31}{2} = 3.90$ .
- The period is 365 days. Therefore,  $\frac{2\pi}{b} = 365$  and  $b = \frac{2\pi}{365}$ .
- The maximum of the sine function occurs at one fourth of the period, which is  $\frac{365}{4} = 91.25$ . Since the maximum for the data at hand occurs at  $t = 172$  (June 21), we must shift the sine function to the right  $172 - 91.25 = 80.75$  days. This gives  $-\frac{c}{b} = 80.75$ , so that  $c = -80.75 \times \frac{2\pi}{365} = -1.39$ .
- Finally, the range is  $[8.31, 16.1]$  instead of  $[-3.90, 3.90]$ , so we must shift the sine function  $16.1 - 3.90 = 12.2$  hours vertically by setting  $d = 12.2$ .

Our sine model is

$$h = 3.90 \sin\left(\frac{2\pi}{365}t - 1.39\right) + 12.2$$

The graph is shown in Fig. 10.20. ■

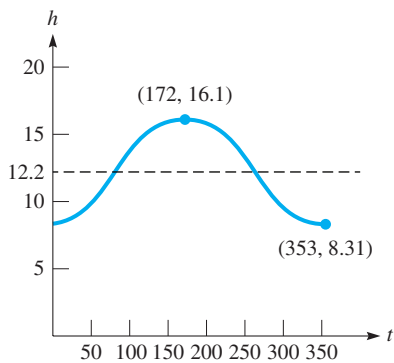


Fig. 10.20

**EXERCISES 10.3**

In Exercises 1 and 2, graph the function if the given changes are made in the indicated examples of this section.

- In Example 3, if the sign before  $\pi/6$  is changed, sketch the graph of the resulting function.
- In Example 4, if the sign before  $\pi/6$  is changed, sketch the graph of the resulting function.

In Exercises 3–26, determine the amplitude, period, and displacement for each function. Then sketch the graphs of the functions.

- |   |   |
|---|---|
| 3. $y = \sin\left(x - \frac{\pi}{6}\right)$                         | 4. $y = 3 \sin\left(x + \frac{\pi}{4}\right)$                       |
| 5. $y = \cos\left(x + \frac{\pi}{6}\right)$                         | 6. $y = 2 \cos\left(x - \frac{\pi}{8}\right)$                       |
| 7. $y = 0.2 \sin\left(2x + \frac{\pi}{2}\right)$                    | 8. $y = -\sin\left(3x - \frac{\pi}{2}\right)$                       |
| 9. $y = -\cos(2x - \pi)$  | 10. $y = 0.4 \cos\left(3x + \frac{\pi}{3}\right)$                   |
| 11. $y = \frac{1}{2} \sin\left(\frac{1}{2}x - \frac{\pi}{4}\right)$ | 12. $y = 2 \sin\left(\frac{1}{4}x + \frac{\pi}{2}\right)$           |
| 13. $y = 30 \cos\left(\frac{1}{3}x + \frac{\pi}{3}\right)$          | 14. $y = \frac{1}{3} \cos\left(\frac{1}{2}x - \frac{\pi}{8}\right)$ |
| 15. $y = \sin\left(\pi x + \frac{\pi}{8}\right)$                    | 16. $y = -2 \sin(2\pi x - \pi)$                                     |
| 17. $y = 0.08 \cos\left(4\pi x - \frac{\pi}{5}\right)$              | 18. $y = 25 \cos\left(3\pi x + \frac{\pi}{2}\right)$                |

- |   |   |
|---|---|
| 19. $y = -0.6 \sin(2\pi x - 1)$                                 | 20. $y = 1.8 \sin\left(\pi x + \frac{1}{3}\right)$          |
| 21. $y = 40 \cos(3\pi x + 2)$                                   | 22. $y = 360 \cos(6\pi x - 1)$                              |
| 23. $y = \sin(\pi^2 x - \pi)$                                   | 24. $y = -\frac{1}{2} \sin\left(2x - \frac{1}{\pi}\right)$  |
| 25. $y = -\frac{3}{2} \cos\left(\pi x + \frac{\pi^2}{6}\right)$ | 26. $y = \pi \cos\left(\frac{1}{\pi}x + \frac{1}{3}\right)$ |

In Exercises 27–30, write the equation for the given function with the given amplitude, period, and displacement, respectively ( $a > 0$ ).

- |                                |                                   |
|--------------------------------|-----------------------------------|
| 27. sine, 4, $3\pi$ , $-\pi/4$ | 28. cosine, 8, $2\pi/3$ , $\pi/3$ |
| 29. cosine, 12, $1/2$ , $1/8$  | 30. sine, 18, 4, $-1$             |

● In Exercises 31–34, explain why the given equations are correct. The method using a graphing calculator is indicated in Exercise 31.

- By viewing the graphs of  $y_1 = \sin x$  and  $y_2 = \cos(x - \pi/2)$ , show that  $\cos(x - \pi/2) = \sin x$ .
- Show that  $\cos(2x - 3\pi/8) = \cos(3\pi/8 - 2x)$ .
- Show that  $\sin(x/2 - 3\pi/4) = -\sin(3\pi/4 - x/2)$ .
- Show that  $2|\sin(x - \pi/3)| = \sin(\pi/3 - x) - \sin(x - \pi/3)$  for  $-2\pi/3 < x < \pi/3$ .

In Exercises 35–40, solve the given problems. In Exercises 39 and 40, use a graphing calculator to view the indicated curves.

- Find the function and graph it for a function of the form  $y = 2 \sin(2x + c)$  that passes through  $(-\pi/8, 0)$  and for which  $c$  has the smallest possible positive value.

36. Find the function and graph it for a function of the form  $y = 2 \cos(2x - c)$  that passes through  $(\pi/6, 2)$  and for which  $c$  has the smallest possible positive value.
37. A wave travelling in a string may be represented by the equation  $y = A \sin 2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right)$ . Here,  $A$  is the amplitude,  $t$  is the time the wave has travelled,  $x$  is the distance from the origin, and  $T$  is the time required for the wave to travel one wavelength  $\lambda$  (the Greek letter lambda). Sketch three cycles of the wave for which  $A = 2.00$  cm,  $T = 0.100$  s,  $\lambda = 20.0$  cm, and  $x = 5.00$  cm.
38. The electric current  $i$  (in  $\mu\text{A}$ ) in a certain circuit is given by  $i = 3.8 \cos 2\pi(t + 0.20)$ , where  $t$  is the time in seconds. Sketch three cycles of this function.
39. A certain satellite circles the earth such that its distance  $y$ , in kilometres north or south (altitude is not considered) from the equator, is  $y = 7200 \cos(0.025t - 0.25)$ , where  $t$  is the time (in min) after launch. View two cycles of the graph.
40. In performing a test on a patient, a medical technician used an ultrasonic signal given by the equation  $I = A \sin(\omega t + \theta)$ . View two cycles of the graph of  $I$  vs.  $t$  if  $A = 5$  nW/m<sup>2</sup>,  $\omega = 2 \times 10^5$  rad/s, and  $\theta = 0.4$ .

In Exercises 41–44, give the specific form of the equation by evaluating  $a$ ,  $b$ , and  $c$  through an inspection of the given curve. Explain how  $a$ ,  $b$ , and  $c$  are found.

41.  $y = a \sin(bx + c)$   
Fig. 10.21

42.  $y = a \cos(bx + c)$   
Fig. 10.21

43.  $y = a \cos(bx + c)$   
Fig. 10.22

44.  $y = a \sin(bx + c)$   
Fig. 10.22

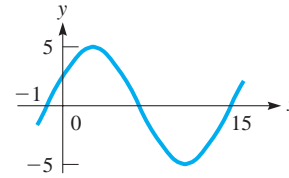


Fig. 10.21

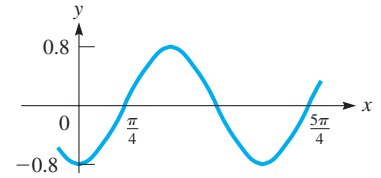


Fig. 10.22

45. In Sydney, Australia, the number of daylight hours during the shortest day of the year (June 21) is 9.90 hours. During the longest day of the year (December 21), the number of daylight hours is 14.4 hours. Approximate the number of hours  $h$  of daylight each day during a year with a function of the form  $h = a \sin(bt + c) + d$ , where time is measured in days.

Answers to Practice Exercise

1. amp. = 8, per. =  $\pi$ , disp. =  $\pi/6$

## 10.4 Graphs of $y = \tan x$ , $y = \cot x$ , $y = \sec x$ , $y = \csc x$

Graph of  $y = \tan x$  • Reciprocal Functions • Asymptotes • Graphs of  $y = \cot x$ ,  $y = \sec x$ ,  $y = \csc x$

The graph of  $y = \tan x$  is shown in Fig. 10.23. Then, knowing that  $\csc x = 1/\sin x$ ,  $\sec x = 1/\cos x$ , and  $\cot x = 1/\tan x$  (see Eq. 4.1 in Section 4.2), we are able to find the values of  $y = \csc x$ ,  $y = \sec x$ , and  $y = \cot x$  from the indicated corresponding reciprocal functions. Using these values, we can graph these functions, and in Figs. 10.24–10.26 we show the resulting graphs.

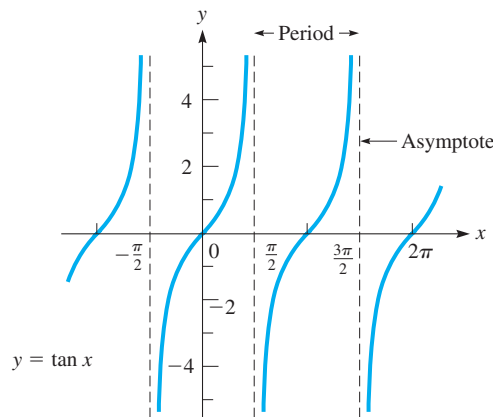


Fig. 10.23

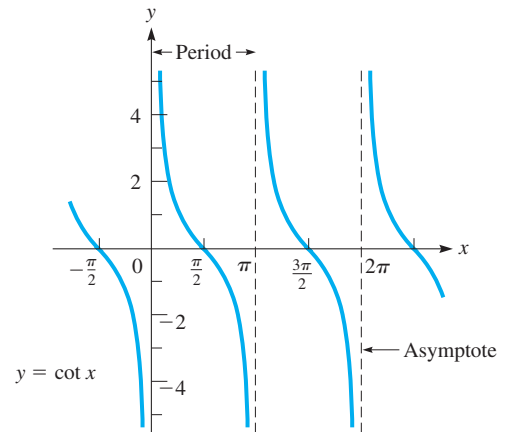


Fig. 10.24

■ Repeating from above the reciprocal relationships that are the basis for graphing  $y = \cot x$ ,  $y = \sec x$ , and  $y = \csc x$ , we have

$$\begin{aligned} \csc x &= \frac{1}{\sin x} & \sec x &= \frac{1}{\cos x} \\ \cot x &= \frac{1}{\tan x} \end{aligned} \quad (10.2)$$

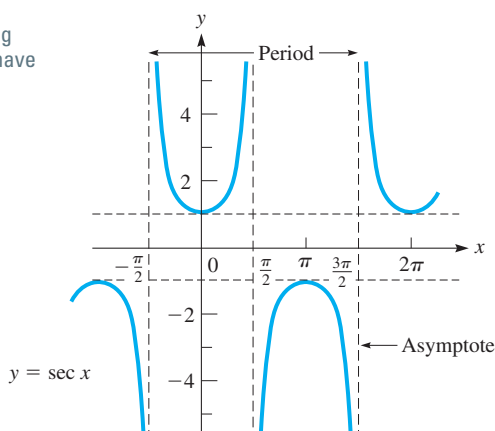


Fig. 10.25

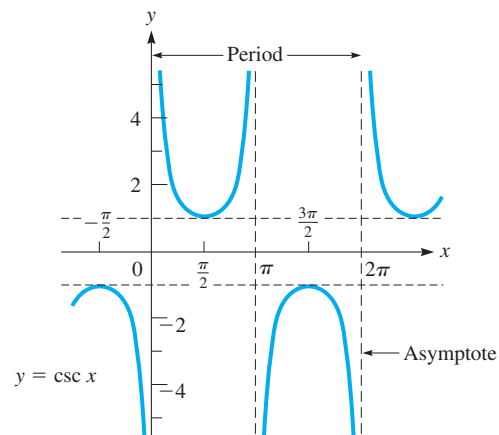


Fig. 10.26

From these graphs, note that  $y = \tan x$  and  $y = \cot x$  have period  $\pi$  and have all real numbers as their range. The functions  $y = \sec x$  and  $y = \csc x$  have period  $2\pi$ , but their ranges do not include the real numbers between  $-1$  and  $1$ .

The vertical dashed lines on the graphs are **vertical asymptotes** (see Section 3.4). The curves *approach* these lines but never actually reach them. The values of  $x$  for which the curve has an asymptote are not included in the domain of the function.

As we can see from the graphs, the functions have asymptotes, zeros, and maximum or minimum values when  $x$  is a multiple of  $\pi/2$ , that is to say, at the same key values which allowed us to sketch the graphs of the sine and cosine functions easily. The characteristics of the trigonometric functions at the key values from  $0$  to  $2\pi$  are compared in Table 10.4. For example, note that when the sine is zero, its reciprocal (the cosecant) has an asymptote.

Table 10.4

Key values	$y = \sin x$	$y = \cos x$	$y = \tan x$
$0$	$0$	$1$	$0$
$\frac{\pi}{2}$	$1$	$0$	asymptote
$\pi$	$0$	$-1$	$0$
$\frac{3\pi}{2}$	$-1$	$0$	asymptote
$2\pi$	$0$	$1$	$0$
Key values	$y = \csc x$	$y = \sec x$	$y = \cot x$
$0$	asymptote	$1$	asymptote
$\frac{\pi}{2}$	$1$	asymptote	$0$
$\pi$	asymptote	$-1$	asymptote
$\frac{3\pi}{2}$	$-1$	asymptote	$0$
$2\pi$	asymptote	$1$	asymptote

To sketch functions such as  $y = a \sec x$ , first sketch  $y = \sec x$  and then multiply the  $y$ -values by  $a$ . Here,  $a$  is not an amplitude, since the ranges of these functions are not limited in the same way they are for the sine and cosine functions.

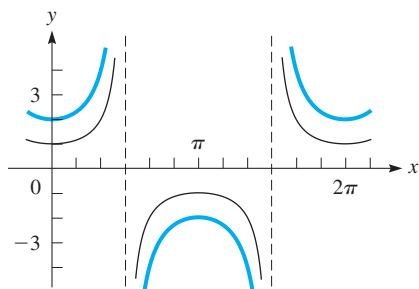


Fig. 10.27

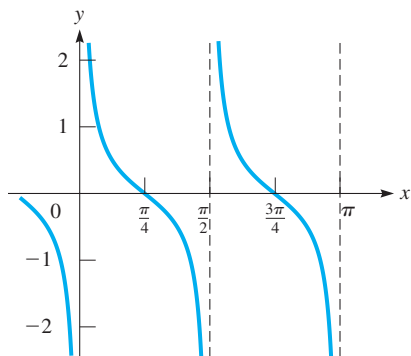


Fig. 10.28

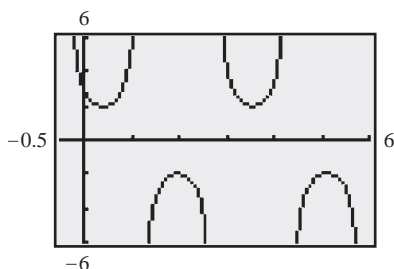


Fig. 10.29

**EXAMPLE 1** Sketching the graph of  $y = a \sec x$ 

Sketch the graph of  $y = 2 \sec x$ .

First, we sketch in  $y = \sec x$ , shown as the light curve in Fig. 10.27. Then we multiply the  $y$ -values of this secant function by 2. Although we can only estimate these values and do this approximately, a reasonable graph can be sketched this way. The desired curve is shown in Fig. 10.27. ■

**EXAMPLE 2** Graph of  $y = a \cot bx$ 

Sketch the graph of  $y = 0.5 \cot 2x$ .

Since the period of  $y = \cot x$  is  $\pi$ , the period of  $y = \cot 2x$  is  $\pi/2$ . Therefore, we have the following table of key values:

$x$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	$\pi$
$y$	asymptote	0	asymptote	0	asymptote

Since  $a = 0.5$ , the function increases more slowly than  $y = \cot x$ . We sketch the graph as shown in Fig. 10.28. ■

Using a graphing calculator, we can display the graphs of these functions more easily and more accurately than by sketching them. By knowing the general shape and period of the function, the values for the *window* settings can be determined.

**EXAMPLE 3** Calculator graph of  $y = a \csc (bx + c)$ 

View at least two periods of the graph of  $y = 2 \csc (2x + \pi/4)$  on a graphing calculator.

Since the period of  $\csc x$  is  $2\pi$ , the period of  $\csc (2x + \pi/4)$  is  $2\pi/2 = \pi$ . Recalling that  $\csc x = (\sin x)^{-1}$ , the curve will have the same displacement as  $y = \sin (2x + \pi/4)$ . Therefore, displacement is  $-\frac{\pi/4}{2} = -\frac{\pi}{8}$ . There is some flexibility in choosing the *window* settings, and as an example, we choose the following settings:

$$X_{\min} = -0.5 \quad (\text{the displacement is } -\pi/8 = -0.4)$$

$$X_{\max} = 6 \quad (\text{displacement} = -\pi/8; \text{period} = \pi, -\pi/8 + 2\pi = 15\pi/8 = 5.9)$$

$$Y_{\min} = -6, Y_{\max} = 6 \quad (\text{there is no curve between } y = -2 \text{ and } y = 2)$$

With  $y_1 = 2(\sin(2x + \pi/4))^{-1}$ , and with the calculator in radian mode, Fig. 10.29 shows the calculator view. ■

**EXERCISES 10.4**

In Exercises 1 and 2, view the graphs on a graphing calculator if the given changes are made in the indicated examples of this section.

- In Example 2, change 0.5 to 5.
- In Example 3, change the sign before  $\pi/4$ .

In Exercises 3–6, fill in the following table for each function and plot the graph from these points.

$x$	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	$\pi$
$y$													

3.  $y = \tan x$

4.  $y = \cot x$

5.  $y = \sec x$

6.  $y = \csc x$

In Exercises 7–14, sketch the graphs of the given functions by use of the basic curve forms (Figs. 10.23, 10.24, 10.25, and 10.26). See Example 1.

7.  $y = 2 \tan x$

8.  $y = 3 \cot x$

9.  $y = \frac{1}{2} \sec x$

10.  $y = \frac{3}{2} \csc x$

11.  $y = -8 \cot x$

12.  $y = -0.1 \tan x$

13.  $y = -3 \csc x$

14.  $y = -60 \sec x$

In Exercises 15–24, view at least two cycles of the graphs of the given functions on a graphing calculator.

15.  $y = \tan 2x$

16.  $y = 2 \cot 3x$

17.  $y = \frac{1}{2} \sec 3x$

18.  $y = 0.4 \csc 3x$

19.  $y = 2 \cot\left(2x + \frac{\pi}{6}\right)$

20.  $y = \tan\left(3x - \frac{\pi}{2}\right)$

21.  $y = 18 \csc\left(3x - \frac{\pi}{3}\right)$       22.  $y = 12 \sec\left(2x + \frac{\pi}{4}\right)$   
 23.  $y = 75 \tan\left(0.5x - \frac{\pi}{8}\right)$       24.  $y = 0.5 \sec\left(0.2x + \frac{\pi}{25}\right)$

In Exercises 25 and 26, solve the given problems. In Exercises 27–30, sketch the appropriate graphs.

25. Write the equation of a secant function with zero displacement, a period of  $4\pi$ , and that passes through  $(0, -3)$ .  
 26. Use a graphing calculator to show that  $\sin x < \tan x$  for  $0 < x < \pi/2$ , although  $\sin x$  and  $\tan x$  are nearly equal for the values near zero.  
 27. Near Antarctica, an iceberg with a vertical face 200 m high is seen from a small boat. At a distance  $x$  from the iceberg, the angle of elevation  $\theta$  of the top of the iceberg can be found from the equation  $x = 200 \cot \theta$ . Sketch  $x$  as a function of  $\theta$ .  
 28. In a laser experiment, two mirrors move horizontally in equal and opposite distances from point  $A$ . The laser path from and to point  $B$  is shown in Fig. 10.30. From the figure, we see that  $x = a \tan \theta$ . Sketch the graph of  $x = f(\theta)$  for  $a = 5.00$  cm.

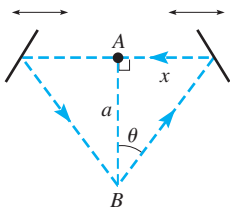


Fig. 10.30

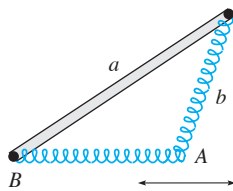


Fig. 10.31

29. A mechanism with two springs is shown in Fig. 10.31, where point  $A$  is restricted to move horizontally. From the law of sines, we see that  $b = (a \sin B) \csc A$ . Sketch the graph of  $b$  as a function of  $A$  for  $a = 4.00$  cm and  $B = \pi/4$ .  
 30. A cantilever column of length  $L$  will buckle if too large a downward force  $P$  is applied  $d$  units off centre. The horizontal deflection  $x$  (see Fig. 10.32) is  $x = d(\sec(kL) - 1)$ , where  $k$  is a constant depending on  $P$ , and  $0 < kL < \pi/2$ . For a constant  $d$ , sketch the graph of  $x$  as a function of  $kL$ .

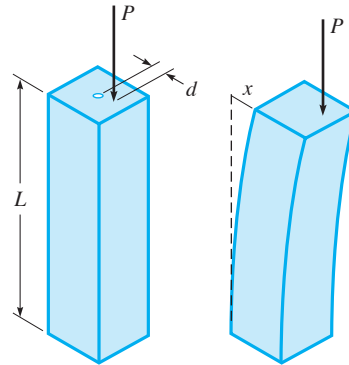


Fig. 10.32

## 10.5 Applications of the Trigonometric Graphs

### Simple Harmonic Motion • Alternating Electric Current • Frequency

When an object moves in a circular path with constant velocity (see Section 8.4), its *projection* on a diameter moves with **simple harmonic motion**. For example, the shadow of a ball at the end of a string and moving at a constant rate moves with simple harmonic motion. We now consider this physical concept and some of its applications.

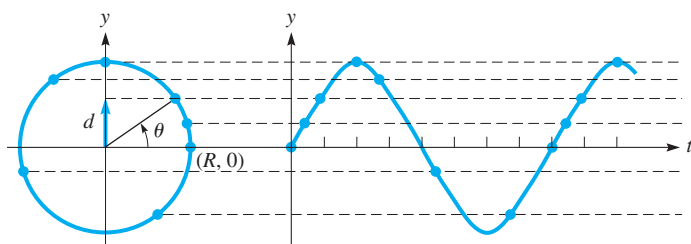


Fig. 10.33

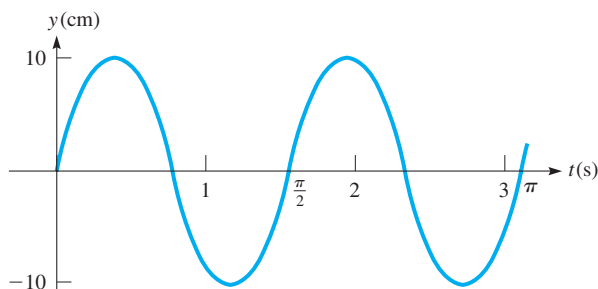


Fig. 10.34

### EXAMPLE 1 Simple harmonic motion

In Fig. 10.33, assume that a particle starts at the end of the radius at  $(R, 0)$  and moves counterclockwise around the circle with constant angular velocity  $\omega$ . The displacement of the projection on the  $y$ -axis is  $d$  and is given by  $d = R \sin \theta$ . The displacement is shown for a few different positions of the end of the radius.

Since  $\theta/t = \omega$ , or  $\theta = \omega t$ , we have

$$d = R \sin \omega t \quad (10.3)$$

as the equation for the displacement of this projection, with time  $t$  as the independent variable.

For the case where  $R = 10.0$  cm and  $\omega = 4.00$  rad/s, we have

$$d = 10.0 \sin 4.00t$$

By sketching or viewing the graph of this function, we can find the displacement  $d$  of the projection for a given time  $t$ . The graph is shown in Fig. 10.34. ■

In Example 1, note that *time is the independent variable*. This is motion for which the object (the end of the projection) remains at the same horizontal position ( $x = 0$ ) and moves only vertically according to a sinusoidal function. In the previous sections, we dealt with functions in which  $y$  is a sinusoidal function of the horizontal displacement  $x$ . Think of a water wave. At *one point* of the wave, the motion is only vertical and sinusoidal with time. At *one given time*, a picture would indicate a sinusoidal movement from one horizontal position to the next.

### EXAMPLE 2 Simple harmonic motion—application

A windmill is used to pump water. The radius of the blade is 2.5 m, and it is moving with constant angular velocity. If the vertical displacement of the end of the blade is timed from the point it is at an angle of  $45^\circ$  ( $\pi/4$  rad) from the horizontal (see Fig. 10.35(a)), the displacement  $d$  is given by

$$d = 2.5 \sin\left(\omega t + \frac{\pi}{4}\right)$$

If the blade makes an angle of  $90^\circ$  ( $\pi/2$  rad) when  $t = 0$  (see Fig. 10.35(b)), the displacement  $d$  is given by

$$d = 2.5 \sin\left(\omega t + \frac{\pi}{2}\right) \quad \text{or} \quad d = 2.5 \cos \omega t$$

If timing started at the first maximum for the displacement, the resulting curve for the displacement would be that of the cosine function. ■

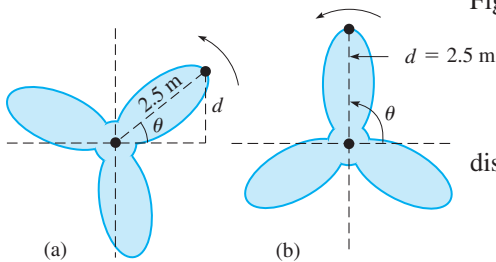


Fig. 10.35

#### Practice Exercise

1. If the windmill blade in Example 2 starts at an angle of  $135^\circ$ , what equation gives its displacement in terms of the cosine function?

Other examples of simple harmonic motion are (1) the movement of a pendulum bob through its arc (a very close approximation to simple harmonic motion), (2) the motion of an object “bobbing” in water, (3) the movement of the end of a vibrating rod (which we hear as sound), and (4) the displacement of a weight moving up and down on a spring. Other phenomena that give rise to equations like those for simple harmonic motion are found in the fields of optics, sound, and electricity. The equations for such phenomena have the same mathematical form because they result from vibratory movement or motion in a circle.

### EXAMPLE 3 Alternating current

A very important use of the trigonometric curves arises in the study of alternating current, which is caused by the motion of a wire passing through a magnetic field. If the wire is moving in a circular path, with angular velocity  $\omega$ , the current  $i$  in the wire at time  $t$  is given by an equation of the form

$$i = I_m \sin(\omega t + \alpha)$$

where  $I_m$  is the maximum current attainable and  $\alpha$  is the phase angle.

The current may be represented by a sinusoidal wave. Given that  $I_m = 6.00$  A,  $\omega = 120\pi$  rad/s, and  $\alpha = \pi/6$ , we have the equation

$$i = 6.00 \sin\left(120\pi t + \frac{\pi}{6}\right)$$

From this equation, note that the amplitude is 6.00 A, the period is  $\frac{1}{60}$  s, and the displacement is  $-\frac{1}{720}$  s. From these values, we draw the graph as shown in Fig. 10.36. Since the current takes on both positive and negative values, we conclude that it moves alternately in one direction and then the other. ■

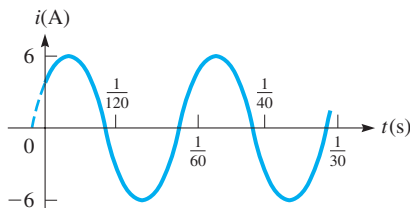


Fig. 10.36

It is a common practice to express the rate of rotation in terms of *the frequency*  $f$ , *the number of cycles per second*, rather than directly in terms of the angular velocity  $\omega$ , the number of radians per second. *The unit for frequency is the hertz* (Hz), and  $1 \text{ Hz} = 1 \text{ cycle/s}$ . Since there are  $2\pi$  rad in one cycle, we have

■ The unit *Hertz* is named for the German physicist Heinrich Hertz (1857–1894).

$$\omega = 2\pi f$$

(10.4)



It is the frequency  $f$  that is referred to in electric current, on radio stations, for musical tones, and so on.

#### EXAMPLE 4 Frequency—hertz

For the electric current in Example 3,  $\omega = 120\pi$  rad/s. The corresponding frequency  $f$  is

$$f = \frac{120\pi}{2\pi} = 60 \text{ Hz}$$

This means that  $120\pi$  rad/s corresponds to 60 cycles/s. This is the standard frequency used for alternating current. ■

## EXERCISES 10.5

In Exercises 1 and 2, answer the given questions about the indicated examples of this section.

- In Example 1, what is the equation relating  $d$  and  $t$  if the end of the radius starts at  $(0, R)$ ?
- In Example 2, if the blade starts at an angle of  $-45^\circ$ , what is the equation relating  $d$  and  $t$  as (a) a sine function? (b) A cosine function?

A graphing calculator may be used in the following exercises.

In Exercises 3 and 4, sketch two cycles of the curve of the projection of Example 1 as a function of time for the given values.

3.  $R = 2.40$  cm,  $\omega = 2.00$  rad/s    4.  $R = 1.80$  m,  $f = 0.250$  Hz

In Exercises 5 and 6, a point on a cam is 8.30 cm from the centre of rotation. The cam is rotating with a constant angular velocity, and the vertical displacement  $d = 8.30$  cm for  $t = 0$  s. See Fig. 10.37. Sketch two cycles of  $d$  as a function of  $t$  for the given values.

- $f = 3.20$  Hz
- $\omega = 3.20$  rad/s

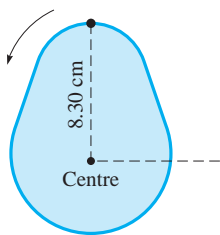


Fig. 10.37

In Exercises 7 and 8, a satellite is orbiting the earth such that its displacement  $D$  north of the equator (or south if  $D < 0$ ) is given by  $D = A \sin(\omega t + \alpha)$ . Sketch two cycles of  $D$  as a function of  $t$  for the given values.

- $A = 500$  km,  $\omega = 3.60$  rad/h,  $\alpha = 0$
- $A = 850$  km,  $f = 1.6 \times 10^{-4}$  Hz,  $\alpha = \pi/3$

In Exercises 9 and 10, for an alternating-current circuit in which the voltage  $V$  is given by  $V = E \cos(\omega t + \alpha)$ , sketch two cycles of the voltage as a function of time for the given values.

- $E = 170$  V,  $f = 60.0$  Hz,  $\alpha = -\pi/3$
- $E = 80$  V,  $\omega = 377$  rad/s,  $\alpha = \pi/2$

In Exercises 11 and 12, refer to the wave in the string described in Exercise 37 of Section 10.3. For a point on the string, the displacement  $y$  is given by  $y = A \sin 2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right)$ . We see that each point on the string moves with simple harmonic motion. Sketch two cycles of  $y$  as a function of  $t$  for the given values.

- $A = 3.20$  cm,  $T = 0.050$  s,  $\lambda = 40.0$  cm,  $x = 5.00$  cm
- $A = 0.750$  cm,  $T = 0.250$  s,  $\lambda = 24.0$  cm,  $x = 20.0$  cm

In Exercises 13 and 14, the air pressure within a plastic container changes above and below the external atmospheric pressure by  $p = p_0 \sin 2\pi ft$ . Sketch two cycles of  $p$  as a function of  $t$  for the given values.

- $p_0 = 280$  kPa,  $f = 2.30$  Hz
- $p_0 = 45.0$  kPa,  $f = 0.450$  Hz

In Exercises 15–22, sketch the required curves.

- The angular displacement  $\theta$  of a certain pendulum bob in terms of its initial displacement  $\theta_0$  is  $\theta = \theta_0 \cos \omega t$ . If  $\omega = 2.00$  rad/s, and  $\theta_0 = \pi/30$  rad, draw two cycles for the resulting equation.
- A study found that, when breathing normally, the increase in volume  $V$  (in L) of air in a person's lungs as a function of the time  $t$  (in s) is  $V = 0.30 \sin 0.50\pi t$ . Sketch two cycles.
- Sketch two cycles of the radio signal  $V = 0.014 \cos(2\pi ft + \pi/4)$  ( $V$  in volts,  $f$  in hertz, and  $t$  in seconds) for a station broadcasting with  $f = 950$  kHz ("95" on the AM radio dial).
- Sketch two cycles of the acoustical intensity  $I$  of the sound wave for which  $I = A \cos(2\pi ft - \alpha)$ , given that  $t$  is in seconds,  $A = 0.027$  W/cm<sup>2</sup>,  $f = 240$  Hz, and  $\alpha = 0.80$ .
- The rotating beacon of a parked police car is 12 m from a straight wall. (a) Sketch the graph of the length  $L$  of the light beam, where  $L = 12 \sec \pi t$ , for  $0 \leq t \leq 2.0$  s. (b) Which part(s) of the graph show meaningful values? Explain.
- The motion of a piston of a car engine approximates simple harmonic motion. Given that the stroke (twice the amplitude) is 0.100 m, the engine runs at 2800 r/min, and the piston starts at the middle of its stroke, find the equation for the displacement  $d$  as a function of  $t$ . Sketch two cycles.

21. The paddle wheel of the *S. S. Beaver*, Canada's first steamship, had a radius of 1.98 m and rotated at 30 r/min when moving at top speed. Find the equation of motion of the vertical displacement (from the centre of the wheel)  $y$  of the end of a paddle as a function of the time  $t$  (in s) if the paddle was initially horizontal. Sketch two cycles.
22. The sinusoidal electromagnetic wave emitted by an antenna in a cellular phone system has a frequency of  $7.5 \times 10^9$  Hz and an amplitude of 0.045 V/m. Find the equation representing the wave if it starts at the origin. Sketch two cycles.

**Answer to Practice Exercise**

1.  $d = 2.5 \cos(\omega t + \pi/4)$

## 10.6 Composite Trigonometric Curves

Addition of Ordinates • Parametric Equations • Lissajous Figures

Many applications involve functions that in themselves are a combination of two or more simpler functions. In this section, we discuss methods by which the curve of such a function can be found by combining values from the simpler functions.

### EXAMPLE 1 Function that is a sum of simpler functions

Sketch the graph of  $y = 2 + \sin 2x$ .

This function is the sum of the simpler functions  $y_1 = 2$  and  $y_2 = \sin 2x$ . We may find values for  $y$  by adding 2 to each important value of  $y_2 = \sin 2x$ .

For  $y_2 = \sin 2x$ , the amplitude is 1, and the period is  $2\pi/2 = \pi$ . Therefore, we obtain the values in the following table and sketch the graph in Fig. 10.38.

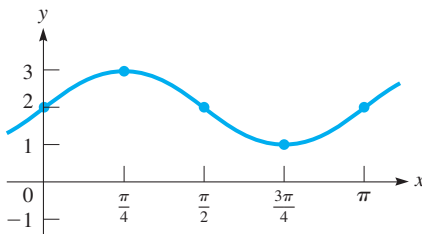


Fig. 10.38

$x$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	$\pi$
$\sin 2x$	0	1	0	-1	0
$2 + \sin 2x$	2	3	2	1	2

Note that this is a vertical shift of 2 units of the graph of  $y = \sin 2x$ , in the same way as discussed on page 106. ■

### ADDITION OF ORDINATES

Another way to sketch the resulting graph is to *first sketch the two simpler curves and then add the  $y$ -values graphically*. This method is called **addition of ordinates** and is illustrated in the following example.

### EXAMPLE 2 Addition of ordinates

Sketch the graph of  $y = 2 \cos x + \sin 2x$ .

On the same set of coordinate axes, we sketch the curves  $y = 2 \cos x$  and  $y = \sin 2x$ . These are shown as dashed and solid light curves in Fig. 10.39. For various values of  $x$ , we determine the distance above or below the  $x$ -axis of each curve and add these distances, noting that those above the axis are positive and those below the axis are negative. We thereby *graphically add the  $y$ -values* of these curves to get points on the resulting curve, shown in colour in Fig. 10.39.

At  $A$ , add the two lengths (shown side-by-side for clarity) to get the length for  $y$ . At  $B$ , both lengths are negative, and the value for  $y$  is the sum of these negative values. At  $C$ , one is positive and the other negative, and we must subtract the lower length from the upper one to get the length for  $y$ .

We combine these lengths for enough  $x$ -values to get a good curve. Some points are easily found. Where one curve crosses the  $x$ -axis, its value is zero, and the resulting curve has its point on the other curve. Here, where  $\sin 2x$  is zero, the points for the resulting curve lie on the curve of  $2 \cos x$ .

We should also add values where each curve is at its maximum or its minimum. *Extra care should be taken for those values of  $x$  for which one curve is **positive** and the other is **negative**.* ■

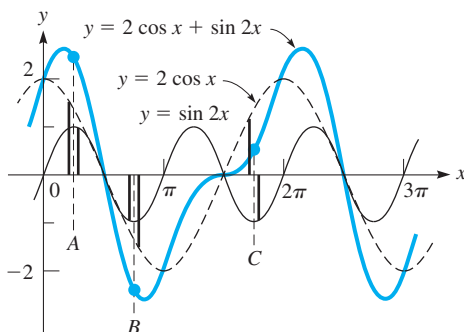


Fig. 10.39

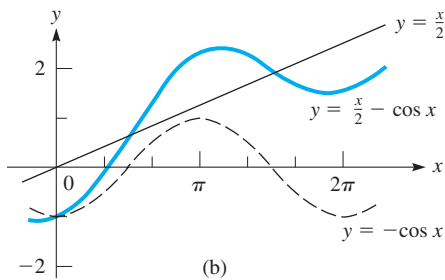
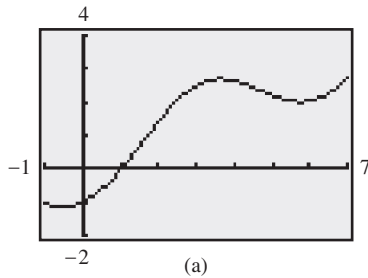


Fig. 10.40

We have seen how a fairly complex curve can be sketched graphically. It is expected that a graphing calculator (or computer *grapher*) will generally be used to view most graphs, particularly ones that are difficult to sketch. A graphing calculator can display such curves much more easily, and with much greater accuracy. Information about the amplitude, period, and displacement is useful in choosing *window settings*. For these graphs, it is important that the calculator is in *radian mode*.

### EXAMPLE 3 Graphing calculator addition of ordinates

Use a graphing calculator to display the graph of  $y = \frac{x}{2} - \cos x$ .

Here, we note that the curve is a combination of the straight line  $y = x/2$  and the trigonometric curve  $y = \cos x$ . There are several good choices for the *window settings*, depending on how much of the curve is to be viewed. To see a little more than one period of  $\cos x$ , we can make the following choices:

$$X_{\min} = -1 \text{ (to start to left of } y\text{-axis)}$$

$$X_{\max} = 7 \text{ (period of } \cos x \text{ is } 2\pi = 6.3)$$

$$Y_{\min} = -2 \text{ (line passes through } (0, 0)\text{; amplitude of } y = \cos x \text{ is } 1)$$

$$Y_{\max} = 4 \text{ (slope of line is } 1/2)$$

See Fig. 10.40(a). The graphs of  $y = x/2 - \cos x$ ,  $y = x/2$ , and  $y = -\cos x$  are shown in Fig. 10.40(b).

We show  $y = -\cos x$  rather than  $y = \cos x$  because for *addition of ordinates*, it is easier to add graphic values than to subtract them. Here we can **add**  $y_1 = x/2$  and  $y_2 = -\cos x$  in order to get  $y = x/2 - \cos x$ . ■

### EXAMPLE 4 Graphing calculator addition of ordinates

View the graph of  $y = \cos \pi x - 2 \sin 2x$  on a graphing calculator.

The combination of  $y = \cos \pi x$  and  $y = 2 \sin 2x$  leads to the following choices for the *window settings*:

$$X_{\min} = -1 \text{ (to start to the left of the } y\text{-axis)}$$

$$X_{\max} = 7 \text{ (the periods are } 2 \text{ and } \pi\text{; this shows at least two periods of each)}$$

$$Y_{\min} = -3, Y_{\max} = 3 \text{ (the sum of the amplitudes is } 3)$$

There are many possible choices for  $X_{\min}$  and  $X_{\max}$  to get a good view of the graph on a calculator. However, since the sum of the amplitudes is 3, note that the curve cannot be below  $y = -3$  or above  $y = 3$ .

The graphing calculator view is shown in Fig. 10.41.

This graph can be constructed by using addition of ordinates, although it is difficult to do so very accurately. ■

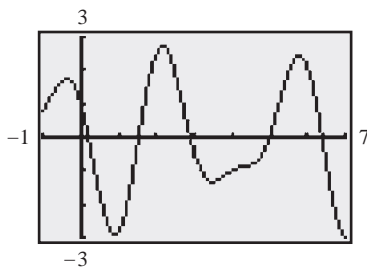


Fig. 10.41

## LISSAJOUS FIGURES

An important application of trigonometric curves is made when they are added at *right angles*. The methods for doing this are shown in the following examples.

### EXAMPLE 5 Graphing parametric equations

Plot the graph for which the values of  $x$  and  $y$  are given by the equations  $y = \sin 2\pi t$  and  $x = 2 \cos \pi t$ . Equations given in this form,  $x$  and  $y$  in terms of a third variable, are called **parametric equations**.

Since both  $x$  and  $y$  are in terms of  $t$ , by assuming values of  $t$ , we find corresponding values of  $x$  and  $y$  and use these values to plot the graph. Since the periods of  $\sin 2\pi t$  and  $2 \cos \pi t$  are  $t = 1$  and  $t = 2$ , respectively, we will use values of  $t = 0, 1/4, 1/2, 3/4, 1$ , and so on. These give us convenient values of  $0, \pi/4, \pi/2, 3\pi/4, \pi$ , and so on to use in the table. We plot the points in Fig. 10.42.

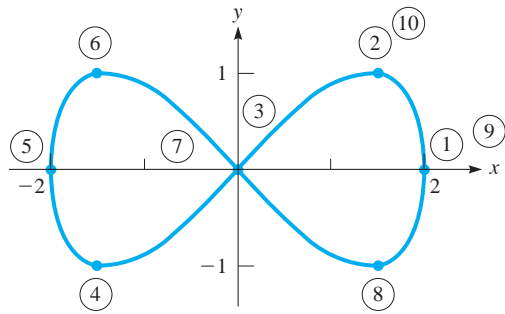


Fig. 10.42

■ Lissajous figures are named for the French physicist Jules Lissajous (1822–1880).

■ See the chapter introduction.

$t$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	$\frac{5}{4}$	$\frac{3}{2}$	$\frac{7}{4}$	2	$\frac{9}{4}$
$x$	2	1.4	0	-1.4	-2	-1.4	0	1.4	2	1.4
$y$	0	1	0	-1	0	1	0	-1	0	1
Point number	1	2	3	4	5	6	7	8	9	10

Since  $x$  and  $y$  are trigonometric functions of a third variable  $t$ , and since the  $x$ -axis is at right angles to the  $y$ -axis, values of  $x$  and  $y$  obtained in this way result in a combination of two trigonometric curves at right angles. Figures obtained in this way are called **Lissajous figures**. Note that the Lissajous figure in Fig. 10.42 is not a function since there are two values of  $y$  for each value of  $x$  (except  $x = -2, 0, 2$ ) in the domain.

In practice, Lissajous figures can be displayed on an *oscilloscope* by applying an electric signal between a pair of horizontal plates and another signal between a pair of vertical plates. These signals are then seen on the screen of the oscilloscope. This type of screen (a cathode-ray tube) is similar to that used on most older TV sets.

**EXAMPLE 6 Graphing Lissajous figures**

If a circle is placed on the  $x$ -axis and another on the  $y$ -axis, we may represent the coordinates  $(x, y)$  for the curve of Example 5 by the lengths of the projections (see Example 1 of Section 10.5) of a point moving around each circle. A careful study of Fig. 10.43 will clarify this. We note that the radius of the circle giving the  $x$ -values is 2 and that the radius of the circle giving the  $y$ -values is 1. This is due to the way in which  $x$  and  $y$  are defined. Also, due to these definitions, the point revolves around the  $y$ -circle twice as fast as the corresponding point around the  $x$ -circle.

■ See the chapter introduction.

■ On an oscilloscope, the curve would result when two electric signals are plotted against each other, one in  $x$  and the other in  $y$ . The first would have twice the amplitude and half the frequency of the other.

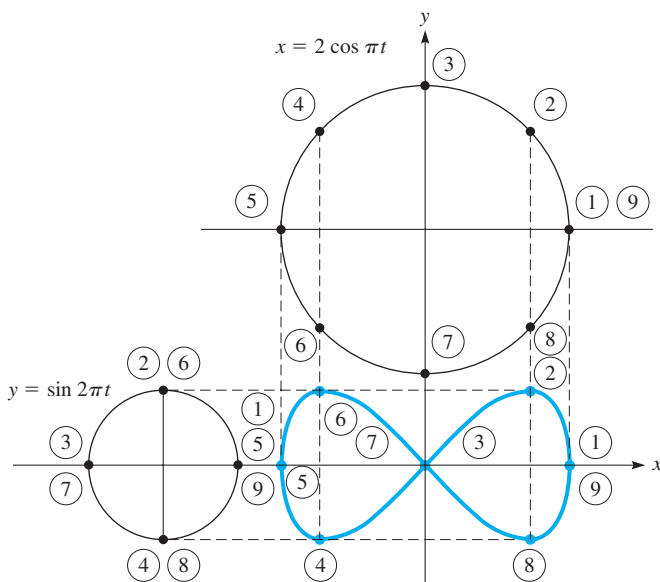


Fig. 10.43

Most graphing calculators can be used to display a curve defined by parametric equations. It is necessary to use the *mode* feature and select *parametric equations*. Use the manual for the calculator, as there are some differences as to how this is done on various calculators.

**EXAMPLE 7 Graphing calculator Lissajous figures**

Use a graphing calculator to display the graph defined by the parametric equations  $x = 2 \cos \pi t$  and  $y = \sin 2\pi t$ . These are the same equations as those used in Examples 5 and 6.

First, select the parametric equation option from the *mode* feature and enter the parametric equations  $x_{1T} = 2 \cos \pi t$  and  $y_{1T} = \sin 2\pi t$ . Then make the following *window* settings:

- Tmin = 0 (standard default settings, and the usual choice)
- Tmax = 2 (the periods are 2 and 1; the longer period is 2)
- Tstep = .1047 (standard default setting; curve is smoother with 0.01)
- Xmin = -2, Xmax = 2 (smallest and largest possible values of  $x$ ), Xscl = 1
- Ymin = -1, Ymax = 1 (smallest and largest possible values of  $y$ ), Yscl = 0.5

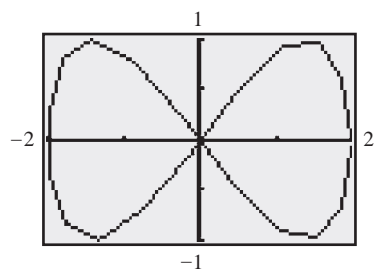


Fig. 10.44

The calculator graph is shown in Fig. 10.44.

## EXERCISES 10.6

In Exercises 1–8, sketch the curves of the given functions by addition of ordinates.

1.  $y = 1 + \sin x$
2.  $y = 3 - 2 \cos x$
3.  $y = \frac{1}{3}x + \sin 2x$
4.  $y = x - \sin x$
5.  $y = \frac{1}{10}x^2 - \sin \pi x$
6.  $y = \frac{1}{4}x^2 + \cos 3x$
7.  $y = \sin x + \cos x$
8.  $y = \sin x + \sin 2x$

In Exercises 9–20, display the graphs of the given functions on a graphing calculator.

9.  $y = x^3 + 10 \sin 2x$
10.  $y = \frac{1}{x^2 + 1} - \cos \pi x$
11.  $y = \sin x - 1.5 \sin 2x$
12.  $y = \cos 3x - 3 \sin x$
13.  $y = 20 \cos 2x + 30 \sin x$
14.  $y = \frac{1}{2} \sin 4x + \cos 2x$
15.  $y = 2 \sin x - \cos 1.5x$
16.  $y = 8 \sin 0.5x - 12 \sin x$
17.  $y = \sin \pi x - \cos 2x$
18.  $y = 2 \cos 4x - \cos\left(x - \frac{\pi}{4}\right)$
19.  $y = 2 \sin\left(2x - \frac{\pi}{6}\right) + \cos\left(2x + \frac{\pi}{3}\right)$
20.  $y = 3 \cos 2\pi x + \sin \frac{\pi}{2}x$

In Exercises 21–24, plot the Lissajous figures.

21.  $x = 3 \sin t, y = 2 \sin t$
22.  $x = 2 \cos t, y = \cos(t + 4)$
23.  $x = 2 \cos 2\pi t, y = \sin \pi t$
24.  $x = \cos\left(t + \frac{\pi}{4}\right), y = \sin 2t$

In Exercises 25–32, use a graphing calculator to display the Lissajous figures.

25.  $x = \cos \pi\left(t + \frac{1}{6}\right), y = 2 \sin \pi t$
26.  $x = \sin^2 \pi t, y = \cos \pi t$
27.  $x = 2 \cos 3t, y = \cos 2t$
28.  $x = 2 \sin \pi t, y = 3 \sin 3\pi t$
29.  $x = \sin(t + 1), y = \sin 5t$
30.  $x = 5 \cos t, y = 3 \sin 5t$
31.  $x = 2 \cos \pi t, y = 3 \sin\left(2\pi t - \frac{\pi}{4}\right)$
32.  $x = 1.5 \cos 3\pi t, y = 0.5 \cos 5\pi t$

In Exercises 33–44, sketch the appropriate curves. A graphing calculator may be used.

33. An object oscillating on a spring has a displacement (in m) given by  $y = 0.4 \sin 4t + 0.3 \cos 4t$ , where  $t$  is the time (in s). Sketch the graph.
34. The voltage  $V$  in a certain electric circuit is given by  $V = 50 \sin 50\pi t + 80 \sin 60\pi t$ , where  $t$  is the time (in s). Sketch the graph.
35. An analysis of the temperature records for Montreal indicates that the average daily temperature  $T$  (in °C) during the year is approximately  $T = 6 - 15 \cos\left[\frac{\pi}{6}(x - 0.5)\right]$ , where  $x$  is measured in months ( $x = 0.5$  is Jan. 15, etc.). Sketch the graph of  $T$  vs.  $x$  for one year.
36. An analysis of data shows that the mean density  $d$  (in  $\text{mg}/\text{cm}^3$ ) of a calcium compound in the bones of women is given by  $d = 139.3 + 48.6 \sin(0.0674x - 0.210)$ , where  $x$  represents the ages of women ( $20 \leq x \leq 80$  years). (A woman is considered to be osteoporotic if  $d < 115 \text{ mg}/\text{cm}^3$ .) Sketch the graph.

37. A normal person with a pulse rate of 60 beats/min has a blood pressure of “120 over 80.” This means the pressure is oscillating between a high (systolic) of 120 mm of mercury (shown as mmHg) and a low (diastolic) of 80 mmHg. Assuming a sinusoidal type of function, find the pressure  $p$  as a function of the time  $t$  if the initial pressure is 120 mmHg. Sketch the graph for the first 5 s.
38. The world’s highest tides occur in the Bay of Fundy on the Atlantic coast of Canada. One day in August 2012, the first low tide at Minas Basin, Nova Scotia, occurred at 6:21 A.M. The water level at low tide was 1.8 m; later, at high tide, it was 14.9 m. The next low tide occurred at 6:45 P.M. Assuming a sinusoidal type of function, find the height of the water  $h$  (in m) as a function of time  $t$  (in hours since midnight). Sketch the graph for one day.
39. The electric current  $i$  (in mA) in a certain circuit is given by  $i = 0.32 + 0.50 \sin t - 0.20 \cos 2t$ , where  $t$  is in milliseconds. Sketch two cycles of  $i$  as a function of  $t$ .
40. The available solar energy depends on the amount of sunlight, and the available time in a day for sunlight depends on the time of the year. An approximate correction factor (in min) to standard time is  $C = 10 \sin \frac{1}{29}(n - 80) - 7.5 \cos \frac{1}{58}(n - 80)$ , where  $n$  is the number of the day of the year. Sketch  $C$  as a function of  $n$ .
41. Two signals are seen on an oscilloscope as being at right angles. The equations for the displacements of these signals are  $x = 4 \cos \pi t$  and  $y = 2 \sin 3\pi t$ . Sketch the figure that appears on the oscilloscope.
42. In the study of optics, light is said to be *elliptically polarized* if certain optic vibrations are out of phase. These may be represented by Lissajous figures. Determine the Lissajous figure for two light waves given by  $w_1 = \sin \omega t$  and  $w_2 = \sin\left(\omega t + \frac{\pi}{4}\right)$ .
43. In checking electric circuit elements, a *square wave* such as that shown in Fig. 10.45 may be displayed on an oscilloscope. Display the graph of  $y = 1 + \frac{4}{\pi} \sin\left(\frac{\pi x}{4}\right) + \frac{4}{3\pi} \sin\left(\frac{3\pi x}{4}\right)$  on a graphing calculator, and compare it with Fig. 10.45. This equation gives the first three terms of a *Fourier series*. As more terms of the series are added, the approximation to a square wave is better.

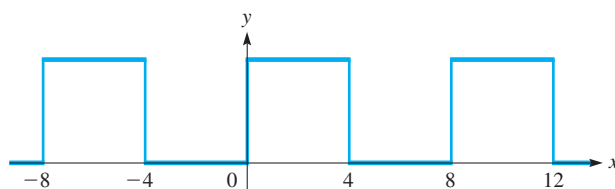


Fig. 10.45

44. Another type of display on an oscilloscope may be a *sawtooth wave* such as that shown in Fig. 10.46. Display the graph of  $y = 1 - \frac{8}{\pi^2}\left(\cos \frac{\pi x}{2} + \frac{1}{9} \cos \frac{3\pi x}{2}\right)$  on a graphing calculator and compare it with Fig. 10.46. See Exercise 43.

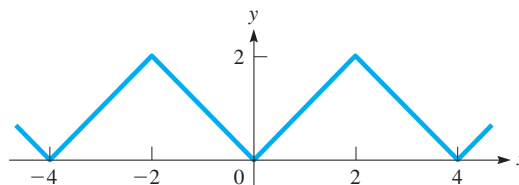
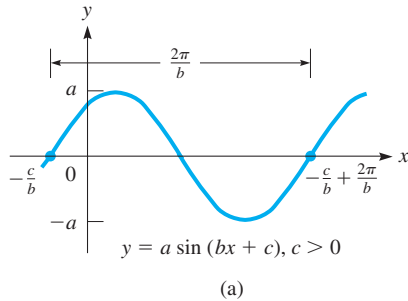


Fig. 10.46

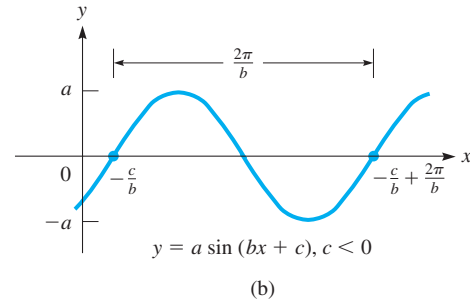
## CHAPTER 10 EQUATIONS

For the graphs of  $y = a \sin(bx + c)$   
and  $y = a \cos(bx + c)$

$$\text{Amplitude} = |a| \quad \text{Period} = \frac{2\pi}{b} \quad \text{Displacement} = -\frac{c}{b} \quad (10.1)$$



For each  
 $a > 0, b > 0$



Reciprocal relationships

$$\csc x = \frac{1}{\sin x} \quad \sec x = \frac{1}{\cos x} \quad \cot x = \frac{1}{\tan x} \quad (10.2)$$

Simple harmonic motion

$$d = R \sin \omega t \quad (10.3)$$

Angular velocity and frequency

$$\omega = 2\pi f \quad (10.4)$$

## CHAPTER 10 REVIEW EXERCISES

In Exercises 1–28, sketch the curves of the given trigonometric functions.

1.  $y = \frac{2}{3} \sin x$
2.  $y = -4 \sin x$
3.  $y = -2 \cos x$
4.  $y = 2.3 \cos(-x)$
5.  $y = 2 \sin 3x$
6.  $y = 4.5 \sin 12x$
7.  $y = 0.4 \cos 4x$
8.  $y = 24 \cos 6x$
9.  $y = 3 \cos \frac{1}{3}x$
10.  $y = 3 \sin(-0.5x)$
11.  $y = \sin \pi x$
12.  $y = 36 \sin 4\pi x$
13.  $y = 5 \cos\left(\frac{\pi x}{2}\right)$
14.  $y = -\cos 6\pi x$
15.  $y = -0.5 \sin\left(-\frac{\pi x}{6}\right)$
16.  $y = 8 \sin \frac{\pi}{4}x$
17.  $y = 2 \sin\left(3x - \frac{\pi}{2}\right)$
18.  $y = 3 \sin\left(\frac{x}{2} + \frac{\pi}{2}\right)$
19.  $y = -2 \cos(4x + \pi)$
20.  $y = 0.8 \cos\left(\frac{x}{6} - \frac{\pi}{2}\right)$
21.  $y = -\sin\left(\pi x + \frac{\pi}{6}\right)$
22.  $y = 250 \sin(3\pi x - \pi)$
23.  $y = 8 \cos\left(4\pi x - \frac{\pi}{2}\right)$
24.  $y = 3 \cos(2\pi x + \pi)$
25.  $y = 0.3 \tan 0.5x$
26.  $y = \frac{1}{4} \sec x$
27.  $y = -\frac{1}{3} \csc x$
28.  $y = -5 \cot \pi x$

In Exercises 29–32, sketch the curves of the given functions by addition of ordinates.

29.  $y = 2 + \frac{1}{2} \sin 2x$
30.  $y = \frac{1}{2}x - \cos \frac{1}{3}x$
31.  $y = \sin 2x + 3 \cos x$
32.  $y = \sin 3x + 2 \cos 2x$

In Exercises 33–40, display the curves of the given functions on a graphing calculator.

33.  $y = 2 \sin x - \cos 2x$
34.  $y = 10 \sin 3x - 20 \cos x$
35.  $y = \cos\left(x + \frac{\pi}{4}\right) - 0.4 \sin 2x$
36.  $y = 2 \cos \pi x + \cos(2\pi x - \pi)$
37.  $y = \frac{\sin x}{x}$
38.  $y = \sqrt{x} \sin 0.5x$
39.  $y = \sin^2 x + \cos^2 x$  ( $\sin^2 x = (\sin x)^2$ )  
What conclusion can be drawn from the graph?
40.  $y = \sin\left(x + \frac{\pi}{4}\right) - \cos\left(x - \frac{\pi}{4}\right) + 1$   
What conclusion can be drawn from the graph?



In Exercises 41–44, give the specific form of the indicated equation by evaluating  $a$ ,  $b$ , and  $c$  through an inspection of the given curve.

41.  $y = a \sin(bx + c)$   
(Fig. 10.47)

42.  $y = a \cos(bx + c)$   
(Fig. 10.47)

43.  $y = a \cos(bx + c)$   
(Fig. 10.48)

44.  $y = a \sin(bx + c)$   
(Fig. 10.48)

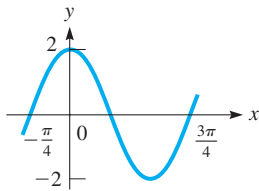


Fig. 10.47

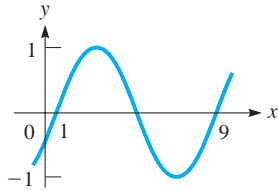


Fig. 10.48

In Exercises 45–48, display the Lissajous figures on a graphing calculator.

45.  $x = -\cos 2\pi t, y = 2 \sin \pi t$

46.  $x = \sin\left(t + \frac{\pi}{6}\right), y = \sin t$

47.  $x = 2 \cos\left(2\pi t + \frac{\pi}{4}\right), y = \cos \pi t$

48.  $x = \cos\left(t - \frac{\pi}{6}\right), y = \cos\left(2t + \frac{\pi}{3}\right)$

In Exercises 49–60, solve the given problems.

49. Display the function  $y = 2|\sin 0.2\pi x| - |\cos 0.4\pi x|$  on a graphing calculator.
50. Display the function  $y = 0.2|\tan 2x|$  on a graphing calculator.
51. Show that  $\cos\left(x + \frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4} - x\right)$  on a graphing calculator.
52. Show that  $\tan\left(x - \frac{\pi}{3}\right) = -\tan\left(\frac{\pi}{3} - x\right)$  on a graphing calculator.
53. What is the period of the function  $y = 2 \cos 0.5x + \sin 3x$ ?
54. What is the period of the function  $y = \sin \pi x + 3 \sin 0.25\pi x$ ?
55. Find the function and graph it if it is of the form  $y = a \sin x$  and passes through  $(5\pi/2, 3)$ .
56. Find the function and graph it if it is of the form  $y = a \cos x$  and passes through  $(4\pi, -3)$ .
57. Find the function and graph it if it is of the form  $y = 3 \cos bx$  and passes through  $(\pi/3, -3)$  and  $b$  has the smallest possible positive value.
58. Find the function and graph it if it is of the form  $y = 3 \sin bx$  and passes through  $(\pi/3, 0)$  and  $b$  has the smallest possible positive value.
59. Find the function and graph it for a function of the form  $y = 3 \sin(\pi x + c)$  that passes through  $(-0.25, 0)$  and for which  $c$  has the smallest possible positive value.
60. Write the equation of the cosecant function with zero displacement, a period of 2, and that passes through  $(0.5, 4)$ .

In Exercises 61–82, sketch the appropriate curves. A graphing calculator may be used.

61. The range  $R$  of a rocket is given by  $R = \frac{v_0^2 \sin 2\theta}{g}$ . Sketch  $R$  as a function of  $\theta$  for  $v_0 = 1000$  m/s and  $g = 9.8$  m/s<sup>2</sup>. See Fig. 10.49.

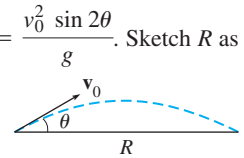


Fig. 10.49

62. The blade of a saber saw moves vertically up and down at 18 strokes per second. The vertical displacement  $y$  (in cm) is given by  $y = 1.2 \sin 36\pi t$ , where  $t$  is in seconds. Sketch at least two cycles of the graph of  $y$  vs.  $t$ .
63. The velocity  $v$  (in cm/s) of a piston in a certain engine is given by  $v = \omega D \cos \omega t$ , where  $\omega$  is the angular velocity of the crankshaft in radians per second and  $t$  is the time in seconds. Sketch the graph of  $v$  vs.  $t$  if the engine is at 3000 r/min and  $D = 3.6$  cm.
64. A light wave for the colour yellow can be represented by the equation  $y = A \sin 3.4 \times 10^{15} t$ . With  $A$  as a constant, sketch two cycles of  $y$  as a function of  $t$  (in s).
65. The electric current  $i$  (in A) in a circuit in which there is a full-wave rectifier is  $i = 10 |\sin 120\pi t|$ . Sketch the graph of  $i = f(t)$  for  $0 \leq t \leq 0.05$  s. What is the period of the current?
66. A circular disc suspended by a thin wire attached to the centre of one of its flat faces is twisted through an angle  $\theta$ . Torsion in the wire tends to turn the disc back in the opposite direction (thus, the name *torsion pendulum* is given to this device). The angular displacement  $\theta$  (in rad) as a function of time  $t$  (in s) is  $\theta = \theta_0 \cos(\omega t + \alpha)$ , where  $\theta_0$  is the maximum angular displacement,  $\omega$  is a constant that depends on the properties of the disc and wire, and  $\alpha$  is the phase angle. Sketch the graph of  $\theta$  vs.  $t$  if  $\theta_0 = 0.100$  rad,  $\omega = 2.50$  rad/s, and  $\alpha = \pi/4$ . See Fig. 10.50.

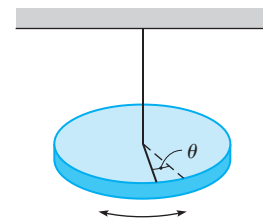


Fig. 10.50

67. The vertical displacement  $y$  of a point at the end of a propeller blade of a small boat is  $y = 14.0 \sin 40.0\pi t$ . Sketch two cycles of  $y$  (in cm) as a function of  $t$  (in s).
68. In optics, two waves are said to interfere destructively if, when they both pass through a medium, the amplitude of the resulting wave is zero. Sketch the graph of  $y = \sin x + \cos(x + \pi/2)$  and explain whether or not it would represent destructive interference of two waves.
69. The vertical displacement  $y$  (in dm) of a buoy floating in water is given by  $y = 3.0 \cos 0.2t + 1.0 \sin 0.4t$ , where  $t$  is in seconds. Sketch the graph of  $y$  as a function of  $t$  for the first 40 s.



70. The vertical motion of a rubber raft on a lake approximates simple harmonic motion due to the waves. If the amplitude of the motion is 0.250 m and the period is 3.00 s, find an equation for the vertical displacement  $y$  as a function of the time  $t$ . Sketch two cycles.
71. A drafting student draws a circle through the three vertices of a right triangle. The hypotenuse of the triangle is the diameter  $d$  of the circle, and from Fig. 10.51, we see that  $d = a \sec \theta$ . Sketch the graph of  $d$  as a function of  $\theta$  for  $a = 3.00$  cm.

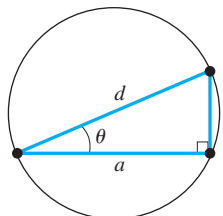


Fig. 10.51

72. The height  $h$  (in m) of a certain rocket ascending vertically is given by  $h = 800 \tan \theta$ , where  $\theta$  is the angle of elevation from an observer 800 m from the launch pad. Sketch  $h$  as a function of  $\theta$ .
73. The number of hours  $h$  of daylight each day during the year in the city of Toronto, Ontario, can be approximated by  $h = 12.2 + 3.3 \sin\left(\frac{2\pi}{365}t - \frac{160\pi}{365}\right)$ , where  $t$  is measured in days ( $t = 15$  is January 15, etc.). Sketch the graph of  $h$  vs.  $t$  for one year.
74. The equation in Exercise 73 can be used to approximate the number of hours of daylight in Christchurch, New Zealand—at approximately the same latitude as Toronto, but in the southern hemisphere. Explain what change is necessary and determine the proper equation. Sketch the graph for one year.
75. If the upper end of a spring is not fixed and is being moved with a sinusoidal motion, the motion of the bob at the end of the spring is affected. Sketch the curve if the motion of the upper end of a spring is being moved by an external force and the bob moves according to the equation  $y = 4 \sin 2t - 2 \cos 2t$ .
76. The loudness  $L$  (in decibels) of a fire siren as a function of the time  $t$  (in s) is approximately  $L = 40 - 35 \cos 2t + 60 \sin t$ . Sketch this function for  $0 \leq t \leq 10$  s.
77. The path of a roller mechanism used in an assembly-line process is given by  $x = \theta - \sin \theta$  and  $y = 1 - \cos \theta$ . Sketch the path for  $0 \leq \theta \leq 2\pi$ .
78. The equations for two voltage signals that give a resulting curve on an oscilloscope are  $x = 6 \sin \pi t$  and  $y = 4 \cos 4\pi t$ . Sketch the graph of the curve displayed on the oscilloscope.
79. The impedance  $Z$  (in  $\Omega$ ) and resistance  $R$  (in  $\Omega$ ) for an alternating-current circuit are related by  $Z = R \sec \theta$ , where  $\theta$  is called the phase angle. Sketch the graph for  $Z$  as a function of  $\theta$  for  $-\pi/2 < \theta < \pi/2$ .
80. For an object sliding down an inclined plane at constant speed, the coefficient of friction  $\mu$  between the object and the plane is given by  $\mu = \tan \theta$ , where  $\theta$  is the angle between the plane and the horizontal. Sketch the graph of  $\mu$  vs.  $\theta$ .
81. The charge  $q$  (in C) on a certain capacitor as a function of the time  $t$  (in s) is given by  $q = 0.0003(3 - 2 \sin 100t \cos 100t)$ . Sketch two cycles of  $q$  vs.  $t$ .
82. The instantaneous power  $P$  (in W) in an electric circuit is defined as the product of the instantaneous voltage  $V$  and the instantaneous current  $i$  (in A). If we have  $V = 100 \cos 200t$  and  $i = 2 \cos\left(200t + \frac{\pi}{4}\right)$ , plot the graph  $V$  vs.  $t$  and the graph of  $i$  vs.  $t$  on the same coordinate system. Then sketch the graph of  $P$  vs.  $t$  by multiplying appropriate values of  $V$  and  $i$ .

### Writing Exercise

83. A wave passing through a string can be described at any instant by the equation  $y = a \sin(bx + c)$ . Write one or two paragraphs explaining the change in the wave (a) if  $a$  is doubled, (b) if  $b$  is doubled, and (c) if  $c$  is doubled.

## CHAPTER 10 PRACTICE TEST

1. Determine the amplitude, period, and displacement of the function  $y = -3 \sin(4\pi x - \pi/3)$ .

In Problems 2–5, sketch the graphs of the given functions.

2.  $y = 0.5 \cos \frac{\pi}{2}x$
3.  $y = 2 + 3 \sin x$
4.  $y = 3 \sec x$
5.  $y = 2 \sin\left(2x - \frac{\pi}{3}\right)$
6. A wave is travelling in a string. The displacement  $y$  (in cm) as a function of the time  $t$  (in s) from its equilibrium position is given by  $y = A \cos(2\pi/T)t$ .  $T$  is the period (in s) of the motion. If  $A = 0.200$  cm and  $T = 0.100$  s, sketch two cycles of  $y$  vs.  $t$ .

7. Sketch the graph of  $y = 2 \sin x + \cos 2x$  by addition of ordinates.
8. Use a graphing calculator to display the Lissajous figure for which  $x = \sin \pi t$  and  $y = 2 \cos 2\pi t$ .
9. Sketch two cycles of the curve of a projection of the end of a radius on the  $y$ -axis. The radius is of length  $R$  and it is rotating counterclockwise about the origin at 2.00 rad/s. It starts at an angle of  $\pi/6$  with the positive  $x$ -axis.
10. Find the function of the form  $y = 2 \sin bx$  if its graph passes through  $(\pi/3, 2)$  and  $b$  is the smallest possible positive value. Then graph the function.

# 11

## LEARNING OUTCOMES

After completion of this chapter, the student should be able to:

- Use the laws of exponents to simplify expressions
- Convert radicals to fractional exponents and vice versa
- Perform mathematical operations with exponents and radicals
- Simplify expressions containing radicals
- Rationalize a denominator containing radicals

# Exponents and Radicals

Exponents and radicals are useful in many technical applications. To set the foundation to study these applications, further development of the connection between fractional exponents and radicals must be undertaken. For example, the concept  $\sqrt{x} = x^{1/2}$  will be illustrated. The use of fractional exponents can be more convenient than the use of radicals in advanced mathematical operations.

As we noted in Chapter 6, the use of symbols led to advances in mathematics and science. As letters began to be used as algebraic symbols for numbers in the 1600s, it was common to write, for example,  $x^3$  as  $xxx$ . For larger powers, this is obviously inconvenient, and the modern use of exponents came into use. The first to use exponents consistently was the French mathematician René Descartes in the 1630s.

The meaning of negative and fractional exponents was first defined by the English mathematician Wallis in the 1650s, although he did not write them as we do today. In the 1670s, it was the great English mathematician and physicist Isaac Newton who first used all exponents (positive, negative, and fractional) with modern notation. This improvement in notation made the development of many areas of mathematics, particularly calculus, easier. In this way, it also helped to make many advances in the applications of mathematics.

As we develop the various operations with exponents and radicals, we will show their uses in some technical areas of application. They are used in a number of formulas in areas such as electronics, hydrodynamics, optics, solar energy, and machine design.

- ▼ In finding the rate at which solar radiation changes at a solar-energy collector, the following expression is found:

$$\frac{(t^4 + 100)^{1/2} - 2t^3(t + 6)(t^4 + 100)^{-1/2}}{[(t^4 + 100)^{1/2}]^2}$$

In Section 11.2, we show that this can be written in a much simpler form.

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## 11.1 Simplifying Expressions with Integral Exponents

Laws of Exponents • Simplifying Expressions • Care with Zero and Negative Exponents

The laws of exponents were given in Section 1.4. For reference, they are

$$a^m \times a^n = a^{m+n} \quad (11.1)$$

$$\frac{a^m}{a^n} = a^{m-n} \quad \text{or} \quad \frac{a^m}{a^n} = \frac{1}{a^{n-m}} \quad (a \neq 0) \quad (11.2)$$

$$(a^m)^n = a^{mn} \quad (11.3)$$

$$(ab)^n = a^n b^n, \quad \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n} \quad (b \neq 0) \quad (11.4)$$

$$a^0 = 1 \quad (a \neq 0) \quad (11.5)$$

$$a^{-n} = \frac{1}{a^n} \quad (a \neq 0) \quad (11.6)$$

Although Eqs. (11.1) to (11.4) were originally defined for positive integers as exponents, we showed in Section 1.4 that, with the definitions in Eqs. (11.5) and (11.6), they are valid for all integral exponents. Later in the chapter, we will show how fractions are used as exponents. These equations are very important to the developments of this chapter and should be reviewed and learned thoroughly.

### EXAMPLE 1 Simplifying basic expressions

(a) Applying Eq. (11.1) and then Eq. (11.6), we have

$$a^3 \times a^{-5} = a^{3-5} = a^{-2} = \frac{1}{a^2}$$

*Negative exponents are generally not used in the expression of the final result, unless specified otherwise. However, they are often used in intermediate steps.*

(b) Applying Eq. (11.1), then (11.6), and then (11.4), we have

$$(10^3 \times 10^{-4})^2 = (10^{3-4})^2 = (10^{-1})^2 = \left(\frac{1}{10}\right)^2 = \frac{1}{10^2} = \frac{1}{100}$$

The result is in proper form as either  $1/10^2$  or  $1/100$ . If the exponent is large, it is common to leave the exponent in the answer. ■

### LEARNING TIP

Although negative exponents are not often used in the final result, if *scientific notation is being used, then the form with the negative exponent is indicated*. For example,

$$\begin{aligned} 0.00625 \text{ m} &= 6.25 \times \frac{1}{1000} \text{ m} = \\ &6.25 \times \frac{1}{10^3} \text{ m} = 6.25 \times 10^{-3} \text{ m} \end{aligned}$$

### EXAMPLE 2 Simplifying basic expressions with multiple variables

(a) Applying Eqs. (11.2) and (11.5), we have

$$\frac{a^2 b^3 c^0}{ab^7} = \frac{a^{2-1} (1)}{b^{7-3}} = \frac{a}{b^4}$$

(b) Applying Eqs. (11.4) and (11.3), and then (11.6), we have

$$(x^{-2}y)^3 = (x^{-2})^3 (y^3) = x^{-6}y^3 = \frac{y^3}{x^6} \quad \blacksquare$$

Often, several different combinations of Eqs. (11.1) to (11.6) can be used to simplify an expression. This is illustrated in the next example.

**EXAMPLE 3** Simplification can be done in different ways

$$(a) (x^2y)^2\left(\frac{2}{x}\right)^{-2} = \frac{(x^4y^2)}{\left(\frac{2}{x}\right)^2} = \frac{x^4y^2}{\frac{4}{x^2}} = \frac{x^4y^2}{1} \times \frac{x^2}{4} = \frac{x^6y^2}{4}$$

$$(b) (x^2y)^2\left(\frac{2}{x}\right)^{-2} = (x^4y^2)\left(\frac{2^{-2}}{x^{-2}}\right) = (x^4y^2)\left(\frac{x^2}{2^2}\right) = \frac{x^6y^2}{4}$$

In (a), we first used Eq. (11.6) and then (11.4). The simplification was completed by changing the division of a fraction to multiplication and using Eq. (11.1). In (b), we first used Eq. (11.3), then (11.6), and finally (11.1). ■

**EXAMPLE 4** Exponents and units of measurement

As in Chapters 1 through 3, when writing a denominate number, units of measurement can appear in the denominator. They can also be written using negative exponents. For example, the metric unit for pressure is the *pascal*, where  $1 \text{ Pa} = 1 \text{ N/m}^2$ . This can be written as

$$1 \text{ Pa} = 1 \text{ N/m}^2 = 1 \text{ N} \cdot \text{m}^{-2}$$

where  $1/\text{m}^2 = \text{m}^{-2}$ .

The metric unit for energy is the *joule*, where  $1 \text{ J} = 1 \text{ kg} \cdot (\text{m} \cdot \text{s}^{-1})^2$ , or

$$1 \text{ J} = 1 \text{ kg} \cdot \text{m}^2 \cdot \text{s}^{-2} = 1 \text{ kg} \cdot \text{m}^2/\text{s}^2$$

Care must be taken to apply the laws of exponents properly. Certain common problems are pointed out in the following examples.

**EXAMPLE 5** Managing zero exponents

The expression  $(-5x)^0$  equals 1, whereas the expression  $-5x^0$  equals  $-5$ . For  $(-5x)^0$ , the parentheses show that the expression  $-5x$  is raised to the zero power, whereas for  $-5x^0$ , using proper order of operations, only  $x$  is raised to the zero power, and we have

$$-5x^0 = -5(1) = -5$$

Similarly,  $(-2)^2 = 4$  and  $-2^2 = -4$ .

**Practice Exercise**

1. Simplify:  $x\left(\frac{3}{x}\right)^{-3}$

■ The unit *pascal* is named for the French mathematician and scientist Blaise Pascal (1623–1662).

■ The unit *joule* is named for the English physicist James Prescott Joule (1818–1899).

**Practice Exercise**

2. Simplify:  $3/x^0$

**COMMON ERROR**

$$(-5)^0 = 1 \text{ but } -5^0 = -(5^0) = -1$$

Order of operations requires exponents to be performed before multiplication. A common error is to apply the exponent to all multiplied factors instead of just the one to which the exponent is applied.

$$2x^{-1} = \frac{2}{x} \text{ not } 2x^{-1} \neq \frac{1}{2x}$$

All factors must be raised to the exponent for it to be applied to all:

$$(2x)^{-1} = \frac{1}{2x}$$

**LEARNING TIP**

The special case  $0^0$  requires its own discussion. For many purposes,  $0^0$  can be *defined* to have a definite value. For example, some series notations and the binomial theorem (discussed in Chapter 19) require that  $0^0 = 1$ . For the power rule of differential calculus (discussed in Chapter 23), it is required that  $0^0 = 0$ . But in evaluating some limits (limits are also discussed in Chapter 23),  $0^0$  can be treated as indeterminate (it has no definite value). It depends on the mathematical context.

**EXAMPLE 6** Simplification done in different ways on a binomial with exponents

$$\begin{aligned} \text{(a)} \quad (2a + b^{-1})^{-2} &= \frac{1}{(2a + b^{-1})^2} = \frac{1}{\left(2a + \frac{1}{b}\right)^2} = \frac{1}{\left(\frac{2ab + 1}{b}\right)^2} \\ &= \frac{1}{\frac{(2ab + 1)^2}{b^2}} = \frac{b^2}{(2ab + 1)^2} \quad \text{not necessary to expand the denominator} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad (2a + b^{-1})^{-2} &= \left(2a + \frac{1}{b}\right)^{-2} = \left(\frac{2ab + 1}{b}\right)^{-2} \\ &= \frac{(2ab + 1)^{-2}}{b^{-2}} = \frac{b^2}{(2ab + 1)^2} \quad \text{positive exponents used in the final result} \end{aligned}$$

**COMMON ERROR**

A common error occurs when simplifying an expression in which multiple terms are raised to an exponent. Remember that

$$(2a + b^{-1})^{-2} \text{ is not equal to } (2a)^{-2} + (b^{-1})^{-2} \text{ or } \frac{1}{4a^2} + b^2$$

As in Section 6.1, *when raising any multinomial to a power, we cannot simply raise each term to the power applied to obtain the result.*

■ For reference, Eq. (11.4) is  $(ab)^n = a^n b^n$ .

Remember, when raising a product of factors to a power, we use Eq. (11.4). Thus,

$$(2ab^{-1})^{-2} = (2a)^{-2}(b^{-1})^{-2} = \frac{b^2}{(2a)^2} = \frac{b^2}{4a^2}$$

We see that we must be careful to distinguish between the power of a sum of terms and the power of a product of factors.

**LEARNING TIP**

When a factor is moved from the denominator to the numerator of a fraction (or vice versa), the sign of the exponent is changed. We should carefully note the word *factor*; this rule does not apply to moving *terms* in the numerator or the denominator.

**EXAMPLE 7** Factor moved from numerator to denominator

$$\begin{aligned} \text{(a)} \quad 3L^{-1} - (2L)^{-2} &= \frac{3}{L} - \frac{1}{(2L)^2} = \frac{3}{L} - \frac{1}{4L^2} \\ &= \frac{12L - 1}{4L^2} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad 3^{-1} \left( \frac{4^{-2}}{3 - 3^{-1}} \right) &= \frac{1}{3} \left( \frac{1}{4^2} \right) \left( \frac{1}{3 - \frac{1}{3}} \right) = \frac{1}{3 \times 4^2} \left( \frac{1}{\frac{9-1}{3}} \right) \\ &= \frac{1}{3 \times 4^2} \left( \frac{3}{8} \right) = \frac{1}{128} \end{aligned}$$

**Practice Exercise**

3. Simplify:  $(3a)^{-1} - 3a^{-2}$

**EXAMPLE 8** Simplifying complex expressions

$$\begin{aligned}
 \frac{1}{x^{-1}} \left( \frac{x^{-1} - y^{-1}}{x^2 - y^2} \right) &= \frac{x \left( \frac{\frac{1}{x} - \frac{1}{y}}{x^2 - y^2} \right)}{1} = x \left( \frac{\frac{y - x}{xy}}{x^2 - y^2} \right) \\
 &= \frac{x(y - x)}{xy} \\
 &= \frac{x(y - x)}{(x - y)(x + y)} \\
 &= \frac{x(y - x)}{xy} \times \frac{1}{(x - y)(x + y)} \\
 &= \frac{x(y - x)}{xy(x - y)(x + y)} = \frac{-(x - y)}{y(x - y)(x + y)} \\
 &= -\frac{1}{y(x + y)}
 \end{aligned}$$

Note that in this example, *the  $x^{-1}$  and  $y^{-1}$  in the numerator could not be moved directly to the denominator with positive exponents* because they are only *terms* of the original numerator. ■

**EXAMPLE 9** Simplifying an expression from calculus

$$\begin{aligned}
 3(x + 4)^2(x - 3)^{-2} - 2(x - 3)^{-3}(x + 4)^3 &= \frac{3(x + 4)^2}{(x - 3)^2} - \frac{2(x + 4)^3}{(x - 3)^3} \\
 &= \frac{3(x - 3)(x + 4)^2 - 2(x + 4)^3}{(x - 3)^3} = \frac{(x + 4)^2[3(x - 3) - 2(x + 4)]}{(x - 3)^3} \\
 &= \frac{(x + 4)^2(x - 17)}{(x - 3)^3}
 \end{aligned}$$

Expressions like the one in this example are often found in problems in calculus. ■

**EXERCISES 11.1**

In Exercises 1–4, solve the resulting problems if the given changes are made in the indicated examples of this section.

- In Example 3, change the factor  $x^2$  to  $x^{-2}$  and then find the result.
- In Example 6, change the term  $2a$  to  $2a^{-1}$  and then find the result.
- In Example 7(b), change the  $3^{-1}$  in the denominator to  $3^{-2}$  and then find the result.
- In Example 8, change the sign in the numerator from  $-$  to  $+$  and then find the result.

In Exercises 5–52, express each of the given expressions in simplest form with only positive exponents.

- |                          |                        |                             |   |   |   |
|--------------------------|------------------------|-----------------------------|---|---|---|
| 5. $x^7x^{-4}$           | 6. $y^9y^{-2}$         | 7. $2a^2a^{-6}$             | 20. $(3x)^{-2}$   | 21. $(7a^{-1}x)^{-3}$   | 22. $7a^{-1}x^{-3}$                               |
| 8. $5ss^{-5}$            | 9. $5^0 \times 5^{-3}$ | 10. $(3^2 \times 4^{-3})^3$ | 23. $\left(\frac{2}{n^3}\right)^{-3}$   | 24. $\left(\frac{3}{x^3}\right)^{-2}$   | 25. $3\left(\frac{a}{b^{-2}}\right)^{-3}$         |
| 11. $(2\pi x^{-1})^2$    | 12. $(3xy^{-2})^3$     | 13. $2(5an^{-2})^{-1}$      | 26. $5\left(\frac{2n^{-2}}{D^{-1}}\right)^{-2}$                               | 27. $(a + b)^{-1}$  | 28. $a^{-1} + b^{-1}$                             |
| 14. $4(6s^2t^{-1})^{-2}$ | 15. $(-4)^0$           | 16. $-4^0$                  | 29. $3x^{-2} + 2y^{-2}$   | 30. $(3x + 2y)^{-2}$  | 31. $(2a^{-n})^2\left(\frac{3}{2a^n}\right)^{-1}$ |
| 17. $-7x^0$              | 18. $(-7x)^0$          | 19. $3x^{-2}$               | 32. $(7 \times 3^{-a})\left(\frac{3^a}{7}\right)^2$                           | 33. $\left(\frac{3a^2}{4b}\right)^{-3}\left(\frac{4}{a}\right)^{-5}$          | 34. $(2np^{-2})^{-2}(4^{-1}p^2)^{-1}$             |
|                          |                        |                             | 35. $\left(\frac{V^{-1}}{2t}\right)^{-2}\left(\frac{t^2}{V^{-2}}\right)^{-3}$ | 36. $ab\left(\frac{a^{-2}}{b^2}\right)^{-3}\left(\frac{a^{-3}}{b^5}\right)^2$ | 37. $2a^{-2} + (2a^{-2})^4$                       |
|                          |                        |                             | 39. $2 \times 3^{-1} + 4 \times 3^{-2}$                                       | 38. $3(a^{-1}z^2)^{-3} + c^{-2}z^{-1}$  | 40. $5 \times 2^{-2} - 3^{-1} \times 2^3$         |



41.  $(R_1^{-1} + R_2^{-1})^{-1}$     42.  $2(2a - b^{-2})^{-1}$     43.  $(n^{-2} - 2n^{-1})^2$   
 44.  $2(2^{-3} - 4^{-1})^{-2}$     45.  $\frac{6^{-1}}{4^{-2} + 2}$     46.  $\frac{x - y^{-1}}{x^{-1} - y}$   
 47.  $\frac{x^{-2} - y^{-2}}{x^{-1} - y^{-1}}$     48.  $\frac{ax^{-2} + a^{-2}x}{a^{-1} + x^{-1}}$     49.  $2t^{-2} + t^{-1}(t + 1)$   
 50.  $3x^{-1} - x^{-3}(y + 2)$     51.  $(D - 1)^{-1} + (D + 1)^{-1}$   
 52.  $4(2x - 1)(x + 2)^{-1} - (2x - 1)^2(x + 2)^{-2}$
- In Exercises 53–70, solve the given problems.
53. If  $x < 0$ , is it ever true that  $x^{-2} < x^{-1}$ ?  
 54. Is it true that  $(a + b)^0 = 1$  for all values of  $a$  and  $b$ ?  
 55. Express  $4^2 \times 64$  (a) as a power of 4 and (b) as a power of 2.  
 56. Express  $1/81$  (a) as a power of 9 and (b) as a power of 3.  
 57. (a) By use of Eqs. (11.4) and (11.6), show that  

$$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$$
 (b) Verify the equation in part (a) by evaluating each side with  $a = 3.576$ ,  $b = 8.091$ , and  $n = 7$ .
58. For what integral values of  $n$  is  $(-3)^{-n} = -3^{-n}$ ? Explain.  
 59. For what integral value(s) of  $n$  is  $n^\pi > \pi^n$ ? (Hint: Graph  $y = x^\pi - \pi^x$  on a graphing calculator.)  
 60. Evaluate  $(8^{19})^{12} / (8^{16})^{14}$ . What happens when you try to evaluate this on a calculator?  
 61. Solve for  $x$ :  $2^{5x} = 2^7(2^{2x})^2$ .  
 62. In analysing the tuning of an electronic circuit, the expression  $[\omega\omega_0^{-1} - \omega_0\omega^{-1}]^2$  is used. Expand and simplify this expression.  
 63. The metric unit of energy, the *joule* (J), can be expressed as  $\text{kg} \cdot \text{s}^{-2} \cdot \text{m}^2$ . Simplify these units and include *newtons* (see Section 1.3) and only positive exponents in the final result.
64. The units for the electric quantity called *permittivity* are  $\text{C}^2 \cdot \text{N}^{-1} \cdot \text{m}^{-2}$ . Given that  $1 \text{ F} = 1 \text{ C}^2 \cdot \text{J}^{-1}$ , show that the units of permittivity are F/m. See Section 1.3.  
 65. When studying a solar energy system, the units encountered are  $\text{kg} \cdot \text{s}^{-1} (\text{m} \cdot \text{s}^{-2})^2$ . Simplify these units and include *joules* (see Example 4) and only positive exponents in the final result.  
 66. The metric units for the velocity of an object are  $\text{m} \cdot \text{s}^{-1}$ , and the units for the acceleration  $a$  of the object are  $\text{m} \cdot \text{s}^{-2}$ . What are the units for  $v/a$ ?  
 67. Given that  $v = a^p t^r$ , where  $v$  is the velocity of an object,  $a$  is its acceleration, and  $t$  is the time, use the metric units given in Exercise 66 to show that  $p = r = 1$ .  
 68. An idealized model of the thermodynamic process in a gasoline engine is the *Otto cycle*. The efficiency  $\eta$  of the process is  

$$\eta = \frac{\frac{T_1 r^\gamma}{r} - \frac{T_2 r^\gamma}{r} - T_1 + T_2}{\frac{T_1 r^\gamma}{r} - \frac{T_2 r^\gamma}{r}}.$$
 Show that  $\eta = 1 - \frac{1}{r^{\gamma-1}}$ .  
 69. An expression encountered in finance is  

$$\frac{p(1+i)^{-1}[(1+i)^{-n} - 1]}{(1+i)^{-1} - 1}$$
 where  $n$  is an integer. Simplify this expression.  
 70. In optics, the combined focal length  $F$  of two lenses is given by  $F = [f_1^{-1} + f_2^{-1} + d(f_1 f_2)^{-1}]^{-1}$ , where  $f_1$  and  $f_2$  are the focal lengths of the lenses and  $d$  is the distance between them. Simplify the right side of this equation.

### Answers to Practice Exercises

1.  $\frac{x^4}{27}$     2. 3    3.  $\frac{a-9}{3a^2}$

## 11.2 Fractional Exponents

Meaning and Interpretation of  
 Fractional Exponents • Evaluations •  
 Graphs • Simplifying Expressions

In Section 11.1, we reviewed the use of integral exponents, including exponents that are negative integers and zero. We now show how rational numbers may be used as exponents. With the appropriate definitions, all the laws of exponents are valid for all rational numbers as exponents.

Eq. (11.3) states that  $(a^m)^n = a^{mn}$ . If we were to let  $m = \frac{1}{2}$  and  $n = 2$ , we would have  $(a^{1/2})^2 = a^1$ . However, we already have a way of writing a quantity that, when squared, equals  $a$ . This is written as  $\sqrt{a}$ . To be consistent with previous definitions and to allow the laws of exponents to hold, we define

$$a^{1/n} = \sqrt[n]{a} \quad (11.7)$$

In order that Eqs. (11.3) and (11.7) may hold at the same time, we define

$$a^{m/n} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m \quad (11.8)$$

These definitions are valid for all the laws of exponents. We must note that Eqs. (11.7) and (11.8) are valid as long as  $\sqrt[n]{a}$  does not involve the even root of a negative number. Such numbers are imaginary and are considered in Chapter 12.



**COMMON ERROR**

The radical  $\sqrt[n]{a}$  is the  $n$ th root of the number  $a$ . Do not confuse it with  $n\sqrt{a}$ , which is  $n$  multiplied by the root of the number  $a$ .

**LEARNING TIP**

We may interpret  $a^{m/n}$  as the  $m$ th power of the  $n$ th root of  $a$ , as well as the  $n$ th root of the  $m$ th power of  $a$ , or

$$a^{m/n} = (a^{1/n})^m = (\sqrt[n]{a})^m$$

or

$$a^{m/n} = (a^m)^{1/n} = \sqrt[n]{a^m}$$

■ For reference, Eq. (11.1) is  $a^m \times a^n = a^{m+n}$ .

■ Eq. (11.3) is  $(a^m)^n = a^{mn}$ .

**EXAMPLE 1** Meaning of a fractional exponent

We now verify that Eq. (11.1) holds for the above definitions:

$$a^{1/4}a^{1/4}a^{1/4}a^{1/4} = a^{(1/4)+(1/4)+(1/4)+(1/4)} = a^1$$

Now,  $a^{1/4} = \sqrt[4]{a}$  by definition. Also, by definition,  $\sqrt[4]{a} \sqrt[4]{a} \sqrt[4]{a} \sqrt[4]{a} = a$ . Eq. (11.1) is thereby verified for  $n = 4$  in Eq. (11.7).

Eq. (11.3) is verified by the following:

$$(a^{1/4})(a^{1/4})(a^{1/4})(a^{1/4}) = (a^{1/4})^4 = a^1 = (\sqrt[4]{a})^4 \quad \blacksquare$$

**EXAMPLE 2** Interpretation of a fractional exponent

$$8^{2/3} = (\sqrt[3]{8})^2 = (2)^2 = 4 \quad \text{or} \quad 8^{2/3} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4 \quad \blacksquare$$

Although both interpretations of Eq. (11.8) are possible, as indicated in Example 2, in evaluating numerical expressions involving fractional exponents without a calculator, it is almost always best to *find the root first, as indicated by the denominator* of the fractional exponent. This allows us to find the root of the smaller number, which is normally easier to find.

**EXAMPLE 3** Order of operations for fractional exponents

To evaluate  $(64)^{5/2}$ , we should proceed as follows:

$$(64)^{5/2} = [(64)^{1/2}]^5 = 8^5 = 32\,768$$

If we raised 64 to the fifth power first, we would have

$$(64)^{5/2} = (64^5)^{1/2} = (1\,073\,741\,824)^{1/2}$$

We would now have to evaluate the indicated square root. This demonstrates why it is preferable to find the indicated root first.  $\blacksquare$

**EXAMPLE 4** Evaluating fractional exponents

$$\text{(a)} \quad (16)^{3/4} = (16^{1/4})^3 = 2^3 = 8$$

$$\text{(b)} \quad 4^{-1/2} = \frac{1}{4^{1/2}} = \frac{1}{2} \quad \text{(c)} \quad 9^{3/2} = (9^{1/2})^3 = 3^3 = 27$$

We note in (b) that Eq. (11.6) must also hold for negative rational exponents. In writing  $4^{-1/2}$  as  $\frac{1}{4^{1/2}}$ , the *sign* of the exponent is changed.  $\blacksquare$

Fractional exponents allow us to find roots of numbers on a calculator. By use of the appropriate key (on most calculators  $x^y$  or  $\wedge$ ), we may raise any positive number to any power. For roots, we use the equivalent fractional exponent. Powers that are fractions or decimal in form are entered directly.

**EXAMPLE 5** Fractional exponents—application

The thermodynamic temperature  $T$  (in kelvins (K)) is related to the pressure  $p$  (in kPa) of a gas by the equation  $T = 80.5p^{2/7}$ . Find the value of  $T$  for  $p = 750$  kPa.

Substituting, we have

$$T = 80.5(750)^{2/7} = 534 \text{ K} \quad \blacksquare$$

**Practice Exercises**

Evaluate: 1.  $81^{5/4}$     2.  $27^{-4/3}$

■ The unit *kelvin* is named for the Scottish physicist Lord Kelvin (1824–1907).

When finding powers of negative numbers, some calculators will show an error. If this is the case, enter the positive value of the number and then enter a negative sign for the result when appropriate. From Section 1.6, we recall that *an even root of a negative number is imaginary and an odd root of a negative number is negative*. If it is an integral power, the basic laws of signs are used.

### EXAMPLE 6 Fractional exponents and graphs

Plot the graph of the function  $y = 2x^{1/3}$ .

In obtaining points for the graph, we use  $(1/3)$  as the power when using the calculator. If your calculator does not evaluate powers of negative numbers, enter positive values for the negative values of  $x$  and then make the results negative. Since  $x^{1/3} = \sqrt[3]{x}$ , we know that  $x^{1/3}$  is negative for negative values of  $x$  because we have an *odd* root of a negative number. We get the following table of values:

$x$	-3	-2	-1	0	1	2	3
$y$	-2.9	-2.5	-2.0	0	2.0	2.5	2.9

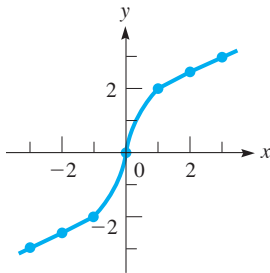


Fig. 11.1

The graph is shown in Fig. 11.1. Of course, this curve can easily be shown on a graphing calculator. ■

Fractional exponents are often easier to use in more complex expressions involving roots. This is true in algebra and in topics from more advanced mathematics. Any expression with radicals can also be expressed with fractional exponents and then simplified.

### EXAMPLE 7 Simplifying expressions with fractional exponents

$$\begin{aligned} \text{(a)} \quad (8a^2b^4)^{1/3} &= [(8^{1/3})(a^2)^{1/3}(b^4)^{1/3}] && \text{using Eq. (11.4)} \\ &= 2a^{2/3}b^{4/3} && \text{using Eqs. (11.7) and (11.3)} \end{aligned}$$

$$\text{(b)} \quad a^{3/4}a^{4/5} = a^{(3/4)+(4/5)} = a^{31/20} \quad \text{using Eq. (11.1)}$$

$$\text{(c)} \quad \left( \frac{4^{-3/2}x^{2/3}y^{-7/4}}{2^{3/2}x^{-1/3}y^{3/4}} \right)^{2/3} = \left( \frac{x^{2/3}x^{1/3}}{2^{3/2}4^{3/2}y^{3/4}y^{7/4}} \right)^{2/3} \quad \text{using Eq. (11.6)}$$

$$= \left( \frac{x^{(2/3)+(1/3)}}{2^{3/2}4^{3/2}y^{(3/4)+(7/4)}} \right)^{2/3} \quad \text{using Eq. (11.1)}$$

$$= \frac{x^{(1)(2/3)}}{2^{(3/2)(2/3)}4^{(3/2)(2/3)}y^{(10/4)(2/3)}} \quad \text{using Eq. (11.4)}$$

$$= \frac{x^{2/3}}{8y^{5/3}}$$

$$\text{(d)} \quad (4x^4)^{-1/2} - 3x^{-3} = \frac{1}{(4x^4)^{1/2}} - \frac{3}{x^3} \quad \text{using Eq. (11.6)}$$

$$= \frac{1}{2x^2} - \frac{3}{x^3} = \frac{x-6}{2x^3} \quad \text{using Eq. (11.7); common denominator}$$

#### Practice Exercise

- Simplify: 3.  $(64a^2c^3)^{2/3}$   
4.  $2x^{-4} - (8x^3)^{-1/3}$

### EXAMPLE 8 Simplifying an expression—application

During a day, the rate  $R$  at which the radiation changes at a solar-radiation collector is

$$R = \frac{(t^4 + 100)^{1/2} - 2t^3(t+6)(t^4 + 100)^{-1/2}}{[(t^4 + 100)^{1/2}]^2}$$

Here,  $R$  is measured in  $\text{kW}/(\text{m}^2 \cdot \text{h})$ ,  $t$  is the number of hours from noon, and  $-6 \text{ h} \leq t \leq 8 \text{ h}$ . Express the right side of this equation in simpler form and find  $R$  for  $t = 0$  (noon) and for  $t = 4 \text{ h}$  (4 P.M.).

■ See the chapter introduction.

■ Solar collectors supply the power for most space satellites.

■ For reference, basic forms of Eq. (11.1) to (11.7) are as follows:

$$a^m \times a^n = a^{m+n} \quad (11.1)$$

$$\frac{a^m}{a^n} = a^{m-n} \quad (11.2)$$

$$(a^m)^n = a^{mn} \quad (11.3)$$

$$(ab)^n = a^n b^n \quad (11.4)$$

$$a^0 = 1 \quad (11.5)$$

$$a^{-n} = \frac{1}{a^n} \quad (11.6)$$

$$a^{1/n} = \sqrt[n]{a} \quad (11.7)$$

Performing the simplification, we have the following steps:

$$R = \frac{(t^4 + 100)^{1/2} - \frac{2t^3(t+6)}{(t^4 + 100)^{1/2}}}{(t^4 + 100)}$$

← using Eq. (11.6)  
← using Eq. (11.3)

$$R = \frac{(t^4 + 100)^{1/2}(t^4 + 100)^{1/2} - 2t^3(t+6)}{(t^4 + 100)^{1/2}}$$

← common denominator

$$R = \frac{(t^4 + 100) - 2t^3(t+6)}{(t^4 + 100)^{1/2}} \times \frac{1}{t^4 + 100}$$

← invert divisor and multiply

$$R = \frac{100 - 12t^3 - t^4}{(t^4 + 100)^{1/2}(t^4 + 100)} = \frac{100 - 12t^3 - t^4}{(t^4 + 100)^{3/2}}$$

← using Eq. (11.1)

For  $t = 0$ :  $R = \frac{100 - 12(0^3) - 0^4}{(0^4 + 100)^{3/2}} = 0.100 \text{ kW}/(\text{m}^2 \cdot \text{h})$

For  $t = 4 \text{ h}$ :  $R = \frac{100 - 12(4^3) - 4^4}{(4^4 + 100)^{3/2}} = -0.138 \text{ kW}/(\text{m}^2 \cdot \text{h})$

This shows that the radiation is increasing at noon and decreasing at 4 P.M. ■

## EXERCISES 11.2

In Exercises 1–4, solve the resulting problems if the given changes are made in the indicated examples of this section.

- In Example 2, change the exponent to  $4/3$ .
- In Example 4(b), change the exponent to  $-3/2$ .
- In Example 6, change the exponent to  $-1/3$ .
- In Example 7(d), change the exponent  $-1/2$  to  $-3/2$ .

In Exercises 5–28, evaluate the given expressions.

- |                                      |                                    |   |   |
|--------------------------------------|------------------------------------|---|---|
| 5. $25^{1/2}$                        | 6. $27^{1/3}$                      | 7. $81^{1/4}$   | 8. $125^{2/3}$  |
| 9. $100^{25/2}$                      | 10. $-16^{5/4}$                    | 11. $8^{-1/3}$  | 12. $16^{-1/4}$   |
| 13. $64^{-2/3}$                      | 14. $-32^{-4/5}$                   | 15. $5^{1/2}5^{3/2}$                                  | 16. $(4^4)^{3/2}$   |
| 17. $(3^6)^{2/3}$                    | 18. $\frac{121^{-1/2}}{100^{1/2}}$ | 19. $\frac{1000^{1/3}}{-400^{-1/2}}$                  | 20. $\frac{-7^{-1/2}}{6^{-1}7^{1/2}}$                     |
| 21. $\frac{15^{2/3}}{5^2 15^{-1/3}}$ | 22. $\frac{(-27)^{1/3}}{6}$        | 23. $\frac{(-8)^{2/3}}{-2}$                           | 24. $\frac{-4^{-1/2}}{(-64)^{-2/3}}$                      |
| 25. $125^{-2/3} - 100^{-3/2}$        | 26. $32^{0.4} + 25^{-0.5}$         | 27. $\frac{16^{-0.25}}{5} + \frac{2^{-0.6}}{2^{0.4}}$ | 28. $\frac{4^{-1}}{36^{-1/2}} - \frac{5^{-1/2}}{5^{1/2}}$ |

In Exercises 29–32, use a calculator to evaluate each expression.

- |                     |                       |
|---------------------|-----------------------|
| 29. $17.98^{1/4}$   | 30. $(-750.81)^{2/3}$ |
| 31. $4.0187^{-4/9}$ | 32. $0.1863^{-1/6}$   |

In Exercises 33–56, simplify the given expressions. Express all answers with positive exponents.

- |                      |                       |                                 |
|----------------------|-----------------------|---------------------------------|
| 33. $B^{2/3}B^{1/2}$ | 34. $x^{5/6}x^{-1/3}$ | 35. $\frac{y^{-1/2}}{-y^{2/5}}$ |
|----------------------|-----------------------|---------------------------------|

- |   |  |   |
|---|--|---|
| 36. $\frac{s^{1/4}s^{2/3}}{s^{-1}}$                                   | 37. $\frac{x^{3/10}}{x^{-1/5}x^2}$   | 38. $\frac{R^{-2/5}R^2}{R^{-3/10}}$                           |
| 39. $(8a^3b^6)^{1/3}$   | 40. $(8b^{-4}c^2)^{2/3}$   | 41. $(16a^4b^3)^{-3/4}$                                       |
| 42. $(32C^5D^4)^{-2/5}$   | 43. $\left(\frac{a^{5/7}}{a^{2/3}}\right)^{7/4}$                             | 44. $\left(\frac{4a^{5/6}b^{-1/5}}{a^{2/3}b^2}\right)^{-1/2}$ |
| 45. $\frac{1}{2}(4x^2 + 1)^{-1/2}(8x)$                                | 46. $\frac{(x + x^{1/2})(x - x^{1/2})}{x}$                                   |   |
| 47. $\frac{y^{3/8}(y^{5/8} - y^{13/8})}{y^{1/2}(y^{1/2} - y^{-1/2})}$ | 48. $\frac{3^{-1}a^{1/2}}{4^{-1/2}b} \div \frac{9^{1/2}a^{-1/3}}{2b^{-1/4}}$ |   |
| 49. $(T^{-1} + 2T^{-2})^{-1/2}$                                       | 50. $(a^{-2} - a^{-4})^{-1/4}$   |   |
| 51. $(a^3)^{-4/3} + a^{-2}$   | 52. $(4N^6)^{-1/2} - 2N^{-1}$  |   |
| 53. $[(a^{1/2} - a^{-1/2})^2 + 4]^{1/2}$                              | 54. $4x^{1/2} + \frac{1}{2}x^{-1/2}(4x + 1)$                                 |   |
| 55. $x^2(2x - 1)^{-1/2} + 2x(2x - 1)^{1/2}$                           | 56. $(3n - 1)^{-2/3}(1 - n) - (3n - 1)^{1/3}$                                |   |

In Exercises 57–60, graph the given functions.

- |                       |                       |
|-----------------------|-----------------------|
| 57. $f(x) = 3x^{1/2}$ | 58. $f(x) = 2x^{2/3}$ |
| 59. $f(t) = t^{-4/5}$ | 60. $f(V) = 4V^{3/2}$ |

In Exercises 61–68, perform the indicated operations.

61. Simplify  $(x^{n-1} \div x^{n-3})^{1/3}$  and express the result as a radical.
62. (a) Simplify  $(x^2 - 4x + 4)^{1/2}$ . (b) For what values of  $x$  is your answer in part (a) valid? Explain.

63. A factor used in determining the performance of a solar-energy storage system is  $(A/S)^{-1/4}$ , where  $A$  is the actual storage capacity and  $S$  is a standard storage capacity. If this factor is 0.5, explain how to find the ratio  $A/S$ .
64. A factor used in measuring the loudness sensed by the human ear is  $(I/I_0)^{0.3}$ , where  $I$  is the intensity of the sound and  $I_0$  is a reference intensity. Evaluate this factor for  $I = 3.20 \times 10^{-6} \text{ W/m}^2$  (ordinary conversation) and  $I_0 = 10^{-12} \text{ W/m}^2$ .
65. The period  $T$  of a satellite circling earth is given by  $T^2 = kR^3 \left(1 + \frac{d}{R}\right)^3$ , where  $R$  is the radius of earth,  $d$  is the distance of the satellite above earth, and  $k$  is a constant. Solve for  $R$ , using fractional exponents in the result.
66. The withdrawal resistance  $R$  of a nail of diameter  $d$  indicates its holding power. One formula for  $R$  is  $R = k(sg)^{5/2}dh$ , where  $k$  is a constant,  $sg$  is the specific gravity of the wood, and  $h$  is the depth of the nail in the wood. Solve for  $sg$  using fractional exponents in the result.
67. The electric current  $i$  (in A) in a circuit with a battery of voltage  $E$ , a resistance  $R$ , and an inductance  $L$  is  $i = \frac{E}{R} (1 - e^{-Rt/L})$ , where  $t$

is the time after the circuit is closed. See Fig. 11.2. Find  $i$  for  $E = 6.20 \text{ V}$ ,  $R = 1.20 \Omega$ ,  $L = 3.24 \text{ H}$ , and  $t = 0.00100 \text{ s}$ . (The number  $e$  is an irrational constant and can be found from the calculator.)

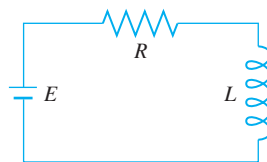


Fig. 11.2

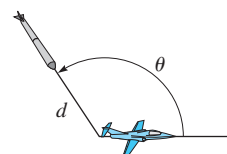


Fig. 11.3

68. For a heat-seeking rocket in pursuit of an aircraft, the distance  $d$  (in km) from the rocket to the aircraft is  $d = \frac{500(\sin \theta)^{1/2}}{(1 - \cos \theta)^{3/2}}$  where  $\theta$  is shown in Fig. 11.3. Find  $d$  for  $\theta = 125.0^\circ$ .

#### Answers to Practice Exercises

1. 243    2.  $\frac{1}{81}$     3.  $16a^{4/3}c^2$     4.  $\frac{4 - x^3}{2x^4}$

## 11.3 Simplest Radical Form

Operations with Radicals • Removing Perfect  $n$ th-power Factors • Reducing the Order of a Radical • Rationalizing the Denominator

As we have said, any expression with radicals can also be expressed with fractional exponents. For adding or subtracting radicals, there is little advantage to changing form, but with multiplication or division of radicals, fractional exponents have some advantages. Therefore, we now define operations with radicals so that they are consistent with the laws of exponents. This will let us use the form that is more convenient for the operation being performed.

$$\sqrt[n]{a^n} = (\sqrt[n]{a})^n = a \quad (11.9)$$

$$\sqrt[n]{a}\sqrt[n]{b} = \sqrt[n]{ab} \quad (11.10)$$

$$\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a} \quad (11.11)$$

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}} \quad (b \neq 0) \quad (11.12)$$

The number under the radical is called the **radicand**, and the number indicating the root being taken is called the **order** (or **index**) of the radical. To avoid difficulties with imaginary numbers (which are considered in the next chapter), we will assume that all letters represent positive numbers.

### EXAMPLE 1 Operation with radicals

Following are illustrations using Eqs. (11.9) to (11.12).

(a)  $\sqrt[5]{4^5} = (\sqrt[5]{4})^5 = 4$  using Eq. (11.9)

(b)  $\sqrt[3]{2}\sqrt[3]{3} = \sqrt[3]{2 \times 3} = \sqrt[3]{6}$  using Eq. (11.10)

(c)  $\sqrt[3]{\sqrt{5}} = \sqrt[3 \times 2]{5} = \sqrt[6]{5}$  using Eq. (11.11)

(d)  $\frac{\sqrt{7}}{\sqrt{3}} = \sqrt{\frac{7}{3}}$  using Eq. (11.12)

**EXAMPLE 2** Root of a sum and root of a product

In Example 5 of Section 1.6, we saw that

$$\sqrt{16 + 9} \text{ is not equal to } \sqrt{16} + \sqrt{9}$$

However, using Eq. (11.10),

$$\begin{aligned}\sqrt{16 \times 9} &= \sqrt{16} \times \sqrt{9} \\ &= 4 \times 3 = 12\end{aligned}$$

**COMMON ERROR**

Because multiplication of factors inside radicals allows separation into two factors in separate radicals  $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$ , it is a common error to apply this procedure to addition or subtraction terms inside radicals. Remember that  $\sqrt[n]{a + b} \neq \sqrt[n]{a} + \sqrt[n]{b}$ . **Be careful to distinguish between the root of a sum of terms and the root of a product of factors.**

**LEARNING TIP**

Always remove all perfect  $n$ th-power factors from the radicand of a radical of order  $n$ .

**EXAMPLE 3** Removing perfect square factors

To simplify  $\sqrt{75}$ , we know that  $75 = (25)(3)$  and that  $\sqrt{25} = 5$ . As in Section 1.6 and now using Eq. (11.10), we write

$$\sqrt{75} = \sqrt{(25)(3)} = \sqrt{25}\sqrt{3} = 5\sqrt{3}$$

↑ perfect square

This illustrates one step that should always be carried out in simplifying radicals.

**EXAMPLE 4** Removing perfect  $n$ th-power factors

$$(a) \sqrt{72} = \sqrt{(36)(2)} = \sqrt{36}\sqrt{2} = 6\sqrt{2}$$

↑ perfect square

$$(b) \sqrt{a^3b^2} = \sqrt{(a^2)(a)(b^2)} = \sqrt{a^2}\sqrt{a}\sqrt{b^2} = ab\sqrt{a}$$

↑ perfect squares

$$(c) \text{ cube root } \rightarrow \sqrt[3]{40} = \sqrt[3]{(8)(5)} = \sqrt[3]{8}\sqrt[3]{5} = 2\sqrt[3]{5}$$

↑ perfect cube

$$\begin{aligned}(d) \text{ fifth root } \rightarrow \sqrt[5]{64x^8y^{12}} &= \sqrt[5]{(32)(2)(x^5)(x^3)(y^{10})(y^2)} \\ &= \sqrt[5]{(32)(x^5)(y^{10})}\sqrt[5]{2x^3y^2} \\ &= 2xy^2\sqrt[5]{2x^3y^2}\end{aligned}$$

↑ perfect fifth powers

The next example illustrates another procedure used to simplify radicals. It is to *reduce the order of the radical*, when it is possible to do so.

**EXAMPLE 5** Reducing the order of a radical using fractional exponents

$$(a) \sqrt[6]{8} = \sqrt[6]{2^3} = 2^{3/6} = 2^{1/2} = \sqrt{2}$$

Here, we started with a sixth root and ended with a square root, thereby reducing the order of the radical. Fractional exponents are often helpful for this.

$$(b) \sqrt[8]{16} = \sqrt[8]{2^4} = 2^{4/8} = 2^{1/2} = \sqrt{2}$$

$$(c) \frac{\sqrt[4]{9}}{\sqrt{3}} = \frac{\sqrt[4]{3^2}}{\sqrt{3}} = \frac{3^{2/4}}{3^{1/2}} = 1$$

$$(d) \frac{\sqrt[6]{8}}{\sqrt{7}} = \frac{\sqrt[6]{2^3}}{\sqrt{7}} = \frac{2^{1/2}}{7^{1/2}} = \sqrt{\frac{2}{7}}$$

$$\begin{aligned}(e) \sqrt[9]{27x^6y^{12}} &= \sqrt[9]{3^3x^6y^9y^3} = 3^{3/9}x^{6/9}y^{9/9}y^{3/9} = 3^{1/3}x^{2/3}yy^{1/3} \\ &= y\sqrt[3]{3x^2y}\end{aligned}$$

**Practice Exercises**

Simplify: 1.  $\sqrt{8x^5}$       2.  $\sqrt[3]{16a^7b^2}$

If a radical is to be written in its *simplest form*, the two operations illustrated in the last four examples must be performed. Therefore, we have the following:

### Steps to Reduce a Radical to Simplest Form

1. Remove all perfect  $n$ th-power factors from a radical of order  $n$ .
2. If possible, reduce the order of the radical using fractional exponents where necessary.

### RATIONALIZING THE DENOMINATOR

When working with fractions, it has traditionally been the practice to *write a fraction with radicals in a form in which the denominator contains no radicals*. Such a fraction was not considered to be in simplest form unless this was done. This step of simplification was performed primarily for ease of calculation, but with a calculator it does not matter to any extent that there is a radical in the denominator. However, the procedure of writing a radical in this form, called **rationalizing the denominator**, is at times useful for other purposes. Therefore, the following examples show how the process of rationalizing the denominator is carried out.

#### EXAMPLE 6 Rationalizing a denominator

To write  $\sqrt{\frac{2}{5}}$  in an equivalent form in which the denominator is not included under the radical sign, we *create a perfect square in the denominator* by multiplying the expression by the fraction  $\frac{\sqrt{5}}{\sqrt{5}}$ . Since this fraction is equivalent to 1, it does not change the expression. When multiplying the radicals, this gives us  $\sqrt{\frac{10}{25}}$ , which may be written as  $\frac{1}{5}\sqrt{10}$  or  $\frac{\sqrt{10}}{5}$ . These steps are written as follows:

$$\sqrt{\frac{2}{5}} = \frac{\sqrt{2}}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} = \frac{\sqrt{2 \times 5}}{\sqrt{5 \times 5}} = \sqrt{\frac{10}{25}} = \frac{\sqrt{10}}{\sqrt{25}} = \frac{\sqrt{10}}{5}$$

↑ perfect square

#### EXAMPLE 7 Rationalizing a denominator

$$(a) \frac{5}{\sqrt{18}} = \frac{5}{3\sqrt{2}} = \frac{5}{3\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{5\sqrt{2}}{3\sqrt{4}} = \frac{5\sqrt{2}}{6}$$

$$(b) \text{ cube root } \rightarrow \sqrt[3]{\frac{2}{3}} = \sqrt[3]{\frac{2}{3}} \times \frac{\sqrt[3]{3^2}}{\sqrt[3]{3^2}} = \sqrt[3]{\frac{18}{27}} = \frac{\sqrt[3]{18}}{\sqrt[3]{27}} = \frac{\sqrt[3]{18}}{3}$$

↑ perfect cube

#### Practice Exercise

Rationalize the denominator: 3.  $\sqrt{\frac{a}{3b}}$

In (a), a perfect square was made by multiplying by  $\sqrt{2}/\sqrt{2}$ . We can indicate this as we have shown or by multiplying the numerator by  $\sqrt{2}$  and multiplying the 2 under the radical in the denominator by 2, in which case the denominator would be  $3\sqrt{2} \times 2$ . In (b), we want a perfect cube, since a cube root is being found.

#### EXAMPLE 8 Rationalizing an expression—application

The period  $T$  (in s) for one cycle of a simple pendulum is given by  $T = 2\pi\sqrt{L/g}$ , where  $L$  is the length of the pendulum and  $g$  is the acceleration due to gravity. Rationalize the denominator on the right side of this equation.

Substituting and then rationalizing, we have

$$\begin{aligned} T &= 2\pi\sqrt{\frac{L}{g}} \\ T &= 2\pi\sqrt{\frac{L}{g}} \times \frac{\sqrt{g}}{\sqrt{g}} \\ T &= \frac{2\pi}{g}\sqrt{gL} \end{aligned}$$

### EXAMPLE 9 Simplifying an expression with multiple terms in a radical

Simplify  $\sqrt{\frac{1}{2a^2} + 2b^{-2}}$  and rationalize the denominator.

We will write the expression using only positive exponents, and then perform the required operations.

$$\begin{aligned} \text{first combine fractions over lowest common denominator} \rightarrow \sqrt{\frac{1}{2a^2} + \frac{2}{b^2}} &= \sqrt{\frac{b^2 + 4a^2}{2a^2b^2}} = \frac{\sqrt{b^2 + 4a^2}}{ab\sqrt{2}} \quad \leftarrow \text{sum of squares—radical cannot be simplified} \\ &= \frac{\sqrt{b^2 + 4a^2}}{ab\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} \\ &= \frac{\sqrt{2(b^2 + 4a^2)}}{2ab} \end{aligned}$$

## EXERCISES 11.3

In Exercises 1–4, simplify the resulting expressions if the given changes are made in the indicated examples of this section.

- In Example 4(b), change the exponent of  $b$  to 4 and then find the resulting expression.
- In Example 5(d), change the 8 to 27 and then find the resulting expression.
- In Example 7(b), change the root to a fourth root and then find the resulting expression.
- In Example 9, replace  $b^{-2}$  with  $b^{-4}$  and then find the resulting expression.

In Exercises 5–64, write each expression in simplest radical form. If a radical appears in the denominator, rationalize the denominator.

- |                                      |                               |                                 |  |                                |  |
|--------------------------------------|-------------------------------|---------------------------------|--|--------------------------------|--|
| 5. $\sqrt{24}$                       | 6. $\sqrt{150}$               | 7. $\sqrt{108}$                 | 35. $\sqrt[6]{64}$                       | 36. $\sqrt[9]{27}$             | 37. $\sqrt{4 \times 10^4}$             |
| 8. $\sqrt{98}$                       | 9. $\sqrt{x^2y^5}$            | 10. $\sqrt{pq^2r^7}$            | 38. $\sqrt{4 \times 10^5}$               | 39. $\sqrt{4 \times 10^6}$     | 40. $\sqrt[3]{16 \times 10^5}$         |
| 11. $\sqrt{x^2y^4z^3}$               | 12. $\sqrt{12ab^2}$           | 13. $\sqrt{18R^5TV^4}$          | 41. $\sqrt[4]{4a^2}$                     | 42. $\sqrt[6]{b^2c^4}$         | 43. $\sqrt[4]{\frac{1}{4}}$            |
| 14. $\sqrt{132M^2N^3}$               | 15. $\sqrt[3]{16}$            | 16. $\sqrt[4]{48}$              | 44. $\frac{\sqrt[4]{320}}{\sqrt[4]{5}}$  | 45. $\sqrt[4]{\sqrt[3]{16}}$   | 46. $\sqrt[5]{\sqrt[4]{9}}$            |
| 17. $\sqrt[5]{96}$                   | 18. $\sqrt[3]{-512}$          | 19. $\sqrt[3]{8a^2}$            | 47. $\sqrt{\sqrt{\sqrt{n}}}$             | 48. $\sqrt{b^4\sqrt{a}}$       | 49. $\sqrt{28u^3v^{-5}}$               |
| 20. $\sqrt[3]{5a^4b^2}$              | 21. $\sqrt[4]{64r^3s^4t^5}$   | 22. $\sqrt[5]{16x^5y^3z^{11}}$  | 50. $\sqrt{98x^6y^{-7}}$                 | 51. $\sqrt{64 + 144}$          | 52. $\sqrt{9 + 81}$                    |
| 23. $\sqrt[5]{8}\sqrt[5]{4}$         | 24. $\sqrt[4]{4}\sqrt[4]{64}$ | 25. $\sqrt[3]{P}\sqrt[3]{P^2V}$ | 53. $\sqrt{\frac{2x}{3c^4}}$             | 54. $\sqrt{\frac{n}{m^3}}$     | 55. $\sqrt{\frac{5}{4} - \frac{1}{8}}$ |
| 26. $\sqrt[6]{3m^5n^8}\sqrt[6]{9mn}$ | 27. $\sqrt{\frac{3}{2}}$      | 28. $\sqrt{\frac{11}{12}}$      | 56. $\sqrt{\frac{1}{a^2} + \frac{1}{b}}$ | 57. $\sqrt{xy^{-1} + x^{-1}y}$ | 58. $\sqrt{x^2 + 4^{-1}}$              |
| 29. $\sqrt[3]{\frac{3}{4}}$          | 30. $\sqrt[4]{\frac{2}{25}}$  | 31. $\sqrt[5]{\frac{1}{9}}$     | 59. $\sqrt{\frac{C-2}{C+2}}$             | 60. $\sqrt{a^2 + 2ab + b^2}$   | 61. $\sqrt{a^2 + b^2}$                 |
| 32. $\sqrt[6]{\frac{5}{4}}$          | 33. $\sqrt[4]{400}$           | 34. $\sqrt[8]{81}$              | 62. $\sqrt{4x^2 - 1}$                    | 63. $\sqrt{9x^2 - 6x + 1}$     | 64. $\sqrt{\frac{1}{2} + 2r + 2r^2}$   |

In Exercises 65–72, perform the required operation.

- Change to radicals of the same order:  $\sqrt{a}$ ;  $\sqrt[3]{b}$ ;  $\sqrt[6]{c}$ .
- Change to radicals of the same order:  $3x^{2/3}$ ;  $2y^{1/2}$ ;  $(5z)^{1/4}$ .
- Display the graphs of  $y_1 = \sqrt{x+2}$  and  $y_2 = \sqrt{x} + \sqrt{2}$  on a calculator to show that  $\sqrt{x+2}$  is not equal to  $\sqrt{x} + \sqrt{2}$ .
- An approximate equation for the efficiency  $\eta$  (in percent) of an engine is  $\eta = 100(1 - 1/\sqrt[5]{R^2})$ , where  $R$  is the compression ratio. Explain how this equation can be written with fractional exponents and then find  $\eta$  for  $R = 7.35$ .



69. When analysing the velocity of an object falling through a great distance, the expression  $a\sqrt{2g/a}$  arises. Show by rationalizing the denominator that this expression takes on a simpler form.
70. According to the Doppler effect, the observed frequency of light  $f_o$  when the observer is moving away with a velocity  $v$  from the light source emitting a signal of frequency  $f$  is given by  $f_o = \frac{fc}{c-v} \sqrt{1 - \left(\frac{v}{c}\right)^2}$ , where  $c$  is the speed of light. Simplify the right side of the equation, leaving a radical in the denominator.
71. In analysing an electronic filter circuit, the expression  $\frac{8A}{\pi^2\sqrt{1 + (f_0/f)^2}}$  is used. Rationalize the denominator, expressing the answer without the fraction  $f_0/f$ .
72. The expression  $\frac{1}{2L}\sqrt{R^2 - \frac{4L}{C}}$  occurs in the study of electric circuits. Simplify this expression by combining terms under the radical and rationalizing the denominator.

#### Answers to Practice Exercises

1.  $2x^2\sqrt{2x}$     2.  $2a^2\sqrt[3]{2ab^2}$     3.  $\frac{\sqrt{3ab}}{3b}$

## 11.4 Addition and Subtraction of Radicals

### Similar Radicals • Adding and Subtracting Radicals • Calculator Verification of Result

When we add or subtract algebraic expressions, we combine similar terms, those that differ only in numerical coefficients. This is true when adding or subtracting radicals. *The radicals must be similar to perform the addition*, rather than simply indicate the addition. *Radicals are similar if they differ only in their numerical coefficients*. This means they must have the same order and have the same radicand.

#### Procedure for Adding or Subtracting Radicals

- Express each radical in simplest form.
- Rationalize any denominators.
- Combine any radicals that are similar. If there are radicals that are not similar, just indicate the addition or subtraction.
- Factor the expression, if there are common factors.

#### EXAMPLE 1 Adding similar radicals

$$(a) 2\sqrt{7} - 5\sqrt{7} + \sqrt{7} = -2\sqrt{7} \quad \text{all similar radicals}$$

This result follows the distributive law, as it should. We can write

$$2\sqrt{7} - 5\sqrt{7} + \sqrt{7} = (2 - 5 + 1)\sqrt{7} = -2\sqrt{7}$$

We can also see that the terms combine just as

$$2x - 5x + x = -2x$$

$$(b) \sqrt[5]{6} + 4\sqrt[5]{6} - 2\sqrt[5]{6} = 3\sqrt[5]{6} \quad \text{all similar radicals}$$

$$(c) \sqrt{5} + 2\sqrt{3} - 5\sqrt{5} = 2\sqrt{3} - 4\sqrt{5} \quad \text{answer contains two terms}$$

similar radicals

We note in (c) that we are only able to indicate the final subtraction since the radicals are not similar. ■

#### EXAMPLE 2 Adding radicals requiring reduction

$$(a) \sqrt{2} + \sqrt{8} = \sqrt{2} + \sqrt{4 \times 2} = \sqrt{2} + \sqrt{4}\sqrt{2} \\ = \sqrt{2} + 2\sqrt{2} = 3\sqrt{2}$$

$$(b) \sqrt[3]{24} + \sqrt[3]{81} = \sqrt[3]{8 \times 3} + \sqrt[3]{27 \times 3} = \sqrt[3]{8}\sqrt[3]{3} + \sqrt[3]{27}\sqrt[3]{3} \\ = 2\sqrt[3]{3} + 3\sqrt[3]{3} = 5\sqrt[3]{3}$$

Notice that  $\sqrt{8}$ ,  $\sqrt[3]{24}$ , and  $\sqrt[3]{81}$  were simplified before performing the additions. We also note that  $\sqrt{2} + \sqrt{8}$  is not equal to  $\sqrt{2} + 8$ . ■

We note in the illustrations of Example 2 that the radicals do not initially appear to be similar. However, after each is simplified, we are able to recognize the similar radicals.

### EXAMPLE 3 Simplifying each and combining similar radicals

$$\begin{aligned} \text{(a)} \quad 6\sqrt{7} - \sqrt{28} + 3\sqrt{63} &= 6\sqrt{7} - \sqrt{4 \times 7} + 3\sqrt{9 \times 7} \\ &= 6\sqrt{7} - 2\sqrt{7} + 3(3\sqrt{7}) \\ &= 6\sqrt{7} - 2\sqrt{7} + 9\sqrt{7} \quad \leftarrow \\ &= 13\sqrt{7} \quad \text{all similar radicals} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad 3\sqrt{125} - \sqrt{20} + \sqrt{27} &= 3\sqrt{25 \times 5} - \sqrt{4 \times 5} + \sqrt{9 \times 3} \\ &= 3(5\sqrt{5}) - 2\sqrt{5} + 3\sqrt{3} \\ &= 13\sqrt{5} + 3\sqrt{3} \quad \leftarrow \text{not similar to others} \end{aligned}$$

#### Practice Exercise

Combine: 1.  $5\sqrt{44} - \sqrt{99}$

### EXAMPLE 4 Simplifying each, rationalizing denominators, and combining similar radicals

$$\begin{aligned} \sqrt{24} + \sqrt{\frac{3}{2}} &= \sqrt{4 \times 6} + \sqrt{\frac{3 \times 2}{2 \times 2}} = \sqrt{4}\sqrt{6} + \sqrt{\frac{6}{4}} \\ &= 2\sqrt{6} + \frac{\sqrt{6}}{2} = \frac{4\sqrt{6} + \sqrt{6}}{2} = \frac{5}{2}\sqrt{6} \end{aligned}$$

One radical was simplified by removing the perfect square factor, and in the other, we rationalized the denominator. Note that we would not be able to combine the radicals if we did not rationalize the denominator of the second radical.

Next is an example of adding radical expressions that contain literal numbers.

### EXAMPLE 5 Radicals with literal numbers

$$\begin{aligned} \sqrt{\frac{2}{3a}} - 2\sqrt{\frac{3}{2a}} &= \frac{\sqrt{2}}{\sqrt{3a}} \times \frac{\sqrt{3a}}{\sqrt{3a}} - 2 \frac{\sqrt{3}}{\sqrt{2a}} \times \frac{\sqrt{2a}}{\sqrt{2a}} \\ &= \frac{\sqrt{6a}}{3a} - \frac{2\sqrt{6a}}{2a} \\ &= \frac{\sqrt{6a}}{3a} - \frac{\sqrt{6a}}{a} \\ &= \frac{\sqrt{6a} - 3\sqrt{6a}}{3a} \\ &= \frac{-2\sqrt{6a}}{3a} \end{aligned}$$

#### Practice Exercise

Combine: 2.  $\sqrt{\frac{8x}{5y}} - \sqrt{10xy}$

## EXERCISES 11.4

In Exercises 1 and 2, simplify the resulting expressions if the given changes are made in the indicated examples of this section.

- In Example 3(b), change  $\sqrt{27}$  to  $\sqrt{45}$  and then find the resulting simplified expression.
- In Example 4, change  $\sqrt{24}$  to  $\sqrt{54}$  and then find the resulting simplified expression.

In Exercises 3–38, express each radical in simplest form, rationalize denominators, and perform the indicated operations.

- $2\sqrt{3} + 5\sqrt{3}$
- $8\sqrt{11} - 3\sqrt{11}$
- $\sqrt{28} + \sqrt{5} - 3\sqrt{7}$
- $8\sqrt{6} - \sqrt{12} - 5\sqrt{6}$
- $\sqrt{5} + \sqrt{16} + 4$
- $\sqrt{7} + \sqrt{36} + 27$
- $2\sqrt{3t^2} - 3\sqrt{12t^2}$
- $4\sqrt{2n^2} - \sqrt{50n^2}$

11.  $\sqrt{8a} - \sqrt{32a}$
12.  $\sqrt{27x} + 2\sqrt{18x}$
13.  $2\sqrt{28} + 3\sqrt{175}$
14.  $\sqrt{100 + 25} - 7\sqrt{80}$
15.  $2\sqrt{200} - \sqrt{1250} - \sqrt{450}$
16.  $2\sqrt{44} - \sqrt{99} + \sqrt{2}\sqrt{88}$
17.  $3\sqrt{75R} + 2\sqrt{48R} - 2\sqrt{18R}$
18.  $2\sqrt{28} - \sqrt{108} - 2\sqrt{175}$
19.  $\sqrt{60} + \sqrt{\frac{5}{3}}$
20.  $3\sqrt{84} - \sqrt{\frac{3}{7}}$
21.  $\sqrt{\frac{1}{2}} + \sqrt{\frac{25}{2}} - 4\sqrt{18}$
22.  $\sqrt{6} - \sqrt{\frac{3}{5}} - \sqrt{18}$
23.  $\sqrt[3]{81} + \sqrt[3]{3000}$
24.  $\sqrt[3]{-16} + \sqrt[3]{54}$
25.  $\sqrt[4]{32} - \sqrt[8]{4}$
26.  $\sqrt[6]{\sqrt{2}} - \sqrt[12]{2^{13}}$
27.  $5\sqrt{a^3b} - \sqrt{4ab^5}$
28.  $\sqrt{2R^2I} + \sqrt{8\sqrt{I^3}}$
29.  $\sqrt{6}\sqrt{5}\sqrt{3} - \sqrt{40a^2}$
30.  $3\sqrt{60b^2n} - b\sqrt{135n}$
31.  $\sqrt[3]{24a^2b^4} - \sqrt[3]{3a^5b}$
32.  $\sqrt[5]{32a^6b^4} + 3a\sqrt[5]{243ab^9}$
33.  $\sqrt{\frac{a}{c^5}} - \sqrt{\frac{c}{a^3}}$
34.  $\sqrt{\frac{2x}{3y}} + \sqrt{\frac{27y}{8x}}$
35.  $\sqrt[3]{ab^{-1}} - \sqrt[3]{8a^{-2}b^2}$
36.  $\sqrt{\frac{2xy^{-1}}{3}} + \sqrt{\frac{27x^{-1}y}{8}}$
37.  $\sqrt{\frac{T-V}{T+V}} - \sqrt{\frac{T+V}{T-V}}$
38.  $\sqrt{\frac{16}{x} + 8 + x} - \sqrt{1 - \frac{1}{x}}$

In Exercises 39–44, express each radical in simplest form, rationalize denominators, and perform the indicated operations. Then use a calculator to verify the result.

39.  $3\sqrt{45} + 3\sqrt{75} - 2\sqrt{500}$
40.  $2\sqrt{40} + 3\sqrt{90} - 5\sqrt{250}$
41.  $2\sqrt{\frac{2}{3}} + \sqrt{24} - 5\sqrt{\frac{3}{2}}$
42.  $\sqrt{\frac{2}{7}} - 2\sqrt{\frac{7}{2}} + 5\sqrt{56}$
43.  $2\sqrt[3]{16} - \sqrt[3]{\frac{1}{4}}$
44.  $5\sqrt[4]{810} - \sqrt[4]{\frac{5}{8}}$

In Exercises 45–52, solve the given problems.

45. Find the exact sum of the positive roots of  $x^2 - 2x - 2 = 0$  and  $x^2 + 2x - 11 = 0$ .
46. For the quadratic equation  $ax^2 + bx + c = 0$ , if  $a$ ,  $b$ , and  $c$  are integers, the sum of the roots is a rational number. Explain.

47. Without calculating the actual value, determine whether  $10\sqrt{11} - \sqrt{1000}$  is positive or negative. Explain.
48. The adjacent sides of a parallelogram are  $\sqrt{12}$  and  $\sqrt{27}$  units long. What is the perimeter of the parallelogram?
49. The two legs of a right triangle are  $2\sqrt{2}$  and  $2\sqrt{6}$  units long. What is the perimeter of the triangle?

50. The current  $I$  (in A) passing through a resistor  $R$  (in  $\Omega$ ) in which  $P$  watts of power are dissipated is  $I = \sqrt{P/R}$ . If the power dissipated in the resistors shown in Fig. 11.4 is  $W$  watts, what is the sum of the currents in radical form?

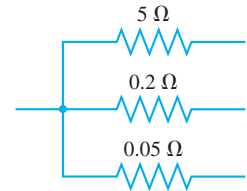


Fig. 11.4

51. A rectangular piece of plywood 1200 mm by 2400 mm has corners cut from it, as shown in Fig. 11.5. Find the perimeter of the remaining piece in exact form and in decimal form (to 3 significant digits).

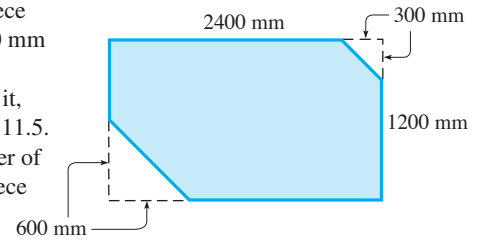


Fig. 11.5

52. Three squares with areas of  $150 \text{ cm}^2$ ,  $54 \text{ cm}^2$ , and  $24 \text{ cm}^2$  are displayed on a computer monitor. What is the sum (in radical form) of the perimeters of these squares?

Answers to Practice Exercises

1.  $7\sqrt{11}$
2.  $\frac{(2 - 5y)\sqrt{10xy}}{5y}$

## 11.5 Multiplication and Division of Radicals

Multiplication of Radicals • Power of a Radical • Division of Radicals • Rationalizing the Denominator

When multiplying expressions containing radicals, we use Eq. (11.10), along with the normal procedures of algebraic multiplication. Note that the orders of the radicals being multiplied in Eq. (11.10) are the same. The following examples illustrate the method.

### EXAMPLE 1 Multiplying monomial radicals

$$\begin{aligned} \text{(a)} \quad & \sqrt{5}\sqrt{2} = \sqrt{5 \times 2} = \sqrt{10} \\ \text{(b)} \quad & \sqrt{33}\sqrt{3} = \sqrt{33 \times 3} = \sqrt{99} = \sqrt{9 \times 11} = \sqrt{9}\sqrt{11} \\ & = 3\sqrt{11} \end{aligned}$$

↑ perfect square

$$\begin{aligned} \text{or} \quad & \sqrt{33}\sqrt{3} = \sqrt{33 \times 3} = \sqrt{11 \times 3 \times 3} \\ & = 3\sqrt{11} \end{aligned}$$

↑ perfect square

Note that we express the resulting radical in simplest form.

$$\begin{aligned} \text{(c)} \quad & \sqrt[3]{6}\sqrt[3]{4} = \sqrt[3]{6(4)} = \sqrt[3]{24} = \sqrt[3]{8}\sqrt[3]{3} \\ & = 2\sqrt[3]{3} \end{aligned}$$

↑ perfect cube

$$\begin{aligned} \text{(d)} \quad & \sqrt[5]{8a^3b^4}\sqrt[5]{8a^2b^3} = \sqrt[5]{(8a^3b^4)(8a^2b^3)} = \sqrt[5]{64a^5b^7} = \sqrt[5]{32a^5b^5}\sqrt[5]{2b^2} \\ & = 2ab\sqrt[5]{2b^2} \end{aligned}$$

↑ perfect fifth power

■ For reference, Eq. (11.10) is  $\sqrt[n]{a}\sqrt[n]{b} = \sqrt[n]{ab}$ .

**EXAMPLE 2** Multiplying binomial radicals

$$\begin{aligned} \text{(a)} \quad \sqrt{2}(3\sqrt{5} - 4\sqrt{2}) &= 3\sqrt{2}\sqrt{5} - 4\sqrt{2}\sqrt{2} = 3\sqrt{10} - 4\sqrt{4} \\ &= 3\sqrt{10} - 4(2) = 3\sqrt{10} - 8 \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad (5\sqrt{7} - 2\sqrt{3})(4\sqrt{7} + 3\sqrt{3}) &= (5\sqrt{7})(4\sqrt{7}) + (5\sqrt{7})(3\sqrt{3}) - (2\sqrt{3})(4\sqrt{7}) - (2\sqrt{3})(3\sqrt{3}) \\ &= (5)(4)\sqrt{7}\sqrt{7} + (5)(3)\sqrt{7}\sqrt{3} - (2)(4)\sqrt{3}\sqrt{7} - (2)(3)\sqrt{3}\sqrt{3} \\ &= 20(7) + 15\sqrt{21} - 8\sqrt{21} - 6(3) \\ &= 140 + 7\sqrt{21} - 18 \\ &= 122 + 7\sqrt{21} \end{aligned}$$

■ For reference, Eqs. (6.3) and (6.4) are  
 $(x + y)^2 = x^2 + 2xy + y^2$   
 $(x - y)^2 = x^2 - 2xy + y^2$

**Practice Exercises**

- Multiply: 1.  $2\sqrt{5}(\sqrt{15} - 3\sqrt{3})$   
 2.  $(2\sqrt{6} - \sqrt{3})(3\sqrt{6} + 2\sqrt{3})$

**LEARNING TIP**

Remember, to multiply radicals and combine them under one radical sign, it is necessary that the order of the radicals be the same. If necessary, we can make the order of each radical the same by appropriate operations on each radical separately. Fractional exponents are frequently useful for this purpose.

When raising a single-term radical expression to a power, we use the basic meaning of the power. When raising a binomial to a power, we proceed as with any binomial. We use Eqs. (6.3) and (6.4) along with Eq. (11.10) if the binomial is squared. These are illustrated in the next two examples.

**EXAMPLE 3** Power of a radical

$$\text{(a)} \quad (2\sqrt{7})^2 = 2^2(\sqrt{7})^2 = 4(7) = 28$$

$$\begin{aligned} \text{(b)} \quad (2\sqrt{7})^3 &= 2^3(\sqrt{7})^3 = 8(\sqrt{7})^2(\sqrt{7}) = 8(7)\sqrt{7} \\ &= 56\sqrt{7} \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad (3 + \sqrt{5})^2 &= 3^2 + 2(3)\sqrt{5} + (\sqrt{5})^2 = 9 + 6\sqrt{5} + 5 \\ &= 14 + 6\sqrt{5} \end{aligned}$$

$$\begin{aligned} \text{(d)} \quad (\sqrt{a} - \sqrt{b})^2 &= (\sqrt{a})^2 - 2\sqrt{a}\sqrt{b} + (\sqrt{b})^2 \\ &= a + b - 2\sqrt{ab} \end{aligned}$$

**EXAMPLE 4** Use of fractional exponents

$$\text{(a)} \quad \sqrt[3]{2\sqrt{5}} = 2^{1/3}5^{1/2} = 2^{2/6}5^{3/6} = (2^25^3)^{1/6} = \sqrt[6]{500}$$

$$\begin{aligned} \text{(b)} \quad \sqrt[3]{4a^2b}\sqrt[4]{8a^3b^2} &= (2^2a^2b)^{1/3}(2^3a^3b^2)^{1/4} = (2^2a^2b)^{4/12}(2^3a^3b^2)^{3/12} \\ &= (2^8a^8b^4)^{1/12}(2^9a^9b^6)^{1/12} = (2^{17}a^{17}b^{10})^{1/12} \\ &= 2a(2^5a^5b^{10})^{1/12} \\ &= 2a\sqrt[12]{32a^5b^{10}} \end{aligned}$$

**DIVISION OF RADICALS**

If a fraction involving a radical is to be changed in form, *rationalizing the denominator* or *rationalizing the numerator* is the principal step. Although calculators have made the rationalization of denominators unnecessary for calculation, this process often makes the form of the fraction simpler. Also, rationalizing numerators is useful at times in more advanced mathematics. We now consider rationalizing when the denominator (or numerator) to be rationalized has more than one term.

**LEARNING TIP**

If the denominator is the sum (or difference) of two terms, at least one of which is a square root, the denominator is rationalized by multiplying both the numerator and denominator by the difference (or sum) of the same two terms. This **forces the denominator to become a difference of squares, thus eliminating the square root(s)**. The binomial that must be multiplied in order to make a difference of squares is called the **conjugate** of the original denominator, and **it consists of changing the sign on one term** in the original denominator.

This process works whether one or both terms are radicals. The result is a denominator free of radicals.

**EXAMPLE 5** Rationalizing a denominator with conjugate multiplication

The fraction  $\frac{1}{\sqrt{3} - \sqrt{2}}$  can be rationalized by multiplying the numerator and the denominator by  $\sqrt{3} + \sqrt{2}$ . In this way, the radicals will be removed from the denominator.

$$\frac{1}{\sqrt{3} - \sqrt{2}} \times \frac{\sqrt{3} + \sqrt{2}}{\sqrt{3} + \sqrt{2}} = \frac{\sqrt{3} + \sqrt{2}}{(\sqrt{3})^2 - (\sqrt{2})^2} = \frac{\sqrt{3} + \sqrt{2}}{3 - 2} = \sqrt{3} + \sqrt{2}$$

change sign

We note that, after rationalizing the denominator, the result can have a simpler form than the original expression.

**EXAMPLE 6** Rationalizing a denominator with literal numbers

Rationalize the denominator of  $\frac{4\sqrt{x}}{\sqrt{7} + \sqrt{x}}$  and simplify the result.

$$\frac{4\sqrt{x}}{\sqrt{7} + \sqrt{x}} \times \frac{\sqrt{7} - \sqrt{x}}{\sqrt{7} - \sqrt{x}} = \frac{4\sqrt{7x} - 4x}{7 - \sqrt{7x} + \sqrt{7x} - x}$$

$$= \frac{4(\sqrt{7x} - x)}{7 - x}$$

change sign

**Practice Exercise**

Rationalize the denominator and simplify:

3.  $\frac{3 - 2\sqrt{3}}{2 + 7\sqrt{3}}$

As we noted earlier, in certain types of algebraic operations, it may be necessary to rationalize the numerator of an expression. This procedure is illustrated in the following example.

**EXAMPLE 7** Rationalizing—application

In studying the properties of a semiconductor, the expression

$$\frac{C_1 + C_2\sqrt{1 + 2V}}{\sqrt{1 + 2V}}$$

is used. Here,  $C_1$  and  $C_2$  are constants and  $V$  is the voltage across a junction of the semiconductor. Rationalize the numerator of this expression.

Multiplying numerator and denominator by the conjugate  $C_1 - C_2\sqrt{1 + 2V}$ , we have

$$\frac{C_1 + C_2\sqrt{1 + 2V}}{\sqrt{1 + 2V}} = \frac{(C_1 + C_2\sqrt{1 + 2V})(C_1 - C_2\sqrt{1 + 2V})}{\sqrt{1 + 2V}(C_1 - C_2\sqrt{1 + 2V})}$$

$$= \frac{C_1^2 - C_2^2(\sqrt{1 + 2V})^2}{C_1\sqrt{1 + 2V} - C_2\sqrt{1 + 2V}\sqrt{1 + 2V}}$$

$$= \frac{C_1^2 - C_2^2(1 + 2V)}{C_1\sqrt{1 + 2V} - C_2(1 + 2V)}$$

## EXERCISES 11.5

In Exercises 1–4, perform the indicated operations on the resulting expressions if the given changes are made in the indicated examples of this section.

- In Example 2(a), change  $4\sqrt{2}$  to  $4\sqrt{8}$  and then perform the multiplication.
- In Example 4(a), change  $\sqrt{5}$  to  $\sqrt[6]{5}$  and then perform the multiplication.
- In Example 5, change the sign in the denominator from  $-$  to  $+$  and then rationalize the denominator.
- In Example 6, rationalize the numerator of the given expression.

In Exercises 5–46, perform the indicated operations, expressing answers in simplest form with rationalized denominators.

- $\sqrt{3}\sqrt{10}$
- $\sqrt{2}\sqrt{51}$
- $\sqrt{6}\sqrt{2}$
- $\sqrt{7}\sqrt{14}$
- $\sqrt[3]{4}\sqrt[3]{2}$
- $\sqrt[5]{4}\sqrt[5]{16}$
- $(5\sqrt{2})^2$
- $2(3\sqrt{5})^3$
- $\sqrt{8}\sqrt{\frac{5}{2}}$
- $\sqrt[6]{7}\sqrt{\frac{2}{3}}$
- $\sqrt{3}(\sqrt{2} - \sqrt{5})$
- $3\sqrt{5}(\sqrt{15} - 2\sqrt{5})$
- $(2 - \sqrt{5})(2 + \sqrt{5})$
- $4(2 - \sqrt{7})^2$
- $(3\sqrt{30} - 2\sqrt{3})(6\sqrt{30} + 7\sqrt{3})$
- $(3\sqrt{7a} - \sqrt{8})(\sqrt{7a} + \sqrt{2})$
- $(3\sqrt{11} - \sqrt{x})(2\sqrt{11} + 5\sqrt{x})$
- $(2\sqrt{10} + 3\sqrt{15})(\sqrt{10} - 7\sqrt{15})$
- $\sqrt{a}(\sqrt{ab} + \sqrt{c^3})$
- $\sqrt{3x}(^3\sqrt{3x} - \sqrt{xy})$
- $\frac{\sqrt{6} - 3}{\sqrt{6}}$
- $\frac{5 - \sqrt{10}}{\sqrt[4]{10}}$
- $(\sqrt{2a} - \sqrt{b})(\sqrt{2a} + 3\sqrt{b})$
- $(2\sqrt{mn} + 3\sqrt{n})^2$
- $\sqrt{2}\sqrt[3]{3}$
- $\sqrt[5]{16}\sqrt[3]{8}$
- $\frac{1}{\sqrt{7} + \sqrt{3}}$
- $\frac{6\sqrt[4]{25}}{5 - 2\sqrt{5}}$
- $\frac{\sqrt{2} - 1}{\sqrt{7} - 3\sqrt{2}}$
- $\frac{2\sqrt{15} - 3}{\sqrt{15} + 4}$
- $\frac{2\sqrt{3} - 5\sqrt{5}}{\sqrt{3} + 2\sqrt{5}}$
- $\frac{\sqrt{15} - 3\sqrt{5}}{2\sqrt{15} - \sqrt{5}}$
- $\frac{2\sqrt{x}}{\sqrt{x} - \sqrt{5}}$
- $\frac{\sqrt{2c} + 3d}{\sqrt{2c} - d}$
- $(\sqrt[5]{\sqrt{6} - \sqrt{5}})(\sqrt[5]{\sqrt{6} + \sqrt{5}})$
- $\sqrt[3]{5 - \sqrt{17}}\sqrt[3]{5 + \sqrt{17}}$
- $\left(\sqrt{\frac{2}{R}} + \sqrt{\frac{R}{2}}\right)\left(\sqrt{\frac{2}{R}} - 2\sqrt{\frac{R}{2}}\right)$
- $(3 + \sqrt{6 - 2a})(2 - \sqrt{6 - 2a})$
- $\frac{\sqrt{x+y}}{\sqrt{x-y} - \sqrt{x}}$
- $\frac{\sqrt{1+a}}{a - \sqrt{1-a}}$
- $\frac{\sqrt{a} + \sqrt{a-2}}{\sqrt{a} - \sqrt{a-2}}$
- $\frac{\sqrt{T^4 - V^4}}{\sqrt{V^{-2} - T^{-2}}}$

In Exercises 47–50, perform the indicated operations, expressing answers in simplest form with rationalized denominators. Then verify the result with a calculator.

- $(\sqrt{11} + \sqrt{6})(\sqrt{11} - 2\sqrt{6})$
- $(2\sqrt{5} - \sqrt{7})(3\sqrt{5} + \sqrt{7})$

$$49. \frac{2\sqrt{6} - \sqrt{5}}{3\sqrt{6} - 4\sqrt{5}} \qquad 50. \frac{\sqrt{7} - 4\sqrt{2}}{5\sqrt{7} - 4\sqrt{2}}$$

In Exercises 51–54, combine the terms into a single fraction, but do not rationalize the denominators.

$$51. 2\sqrt{x} + \frac{1}{\sqrt{x}} \qquad 52. \frac{3}{2\sqrt{3x-4}} - \sqrt{3x-4}$$

$$53. \frac{x^2}{\sqrt{2x+1}} + 2x\sqrt{2x+1} \qquad 54. 4\sqrt{x^2+1} - \frac{4x}{\sqrt{x^2+1}}$$

In Exercises 55–58, rationalize the numerator of each fraction.

$$55. \frac{\sqrt{5} + 2\sqrt{2}}{3\sqrt{10}} \qquad 56. \frac{\sqrt{19} - 3}{5}$$

$$57. \frac{\sqrt{x+h} - \sqrt{x}}{h} \qquad 58. \frac{\sqrt{3x+4} + \sqrt{3x}}{8}$$

In Exercises 59–72, solve the given problems.

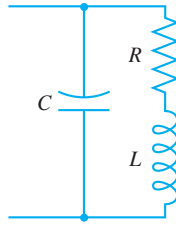
- By substitution, show that  $x = 1 - \sqrt{2}$  is a solution of the equation  $x^2 - 2x - 1 = 0$ .
- For the quadratic equation  $ax^2 + bx + c = 0$ , if  $a$ ,  $b$ , and  $c$  are integers, the product of the roots is a rational number. Explain.
- Determine the relationship between  $a$  and  $c$  in  $ax^2 + bx + c = 0$  if the roots of the equation are reciprocals.
- Evaluate  $r^2 - s^2$  if  $r = -b + \sqrt{b^2 - 4ac}$  and  $s = -b - \sqrt{b^2 - 4ac}$ .
- Rationalize the denominator of  $\frac{1}{\sqrt[3]{x^2} + \sqrt[3]{x} + 1}$ . (Hint: See Eq. 6.10.)
- One leg of a right triangle is  $2\sqrt{7}$ , and the hypotenuse is 6. What is the area of the triangle?
- For an object oscillating at the end of a spring and on which there is a force that retards the motion, the equation  $m^2 + bm + k^2 = 0$  must be solved. Here,  $b$  is a constant related to the retarding force, and  $k$  is the spring constant. By substitution, show that  $m = \frac{1}{2}(\sqrt{b^2 - 4k^2} - b)$  is a solution.
- Among the products of a specialty furniture company are tables with tops in the shape of a regular octagon (eight sides). Express the area  $A$  of a table top as a function of the side  $s$  of the octagon. (Hint: Draw a square around the octagon.)
- An expression used in determining the characteristics of a spur gear is  $\frac{50}{50 + \sqrt{V}}$ . Rationalize the denominator.
- When studying the orbits of earth satellites, the expression  $\left(\frac{GM}{4\pi^2}\right)^{1/3} T^{2/3}$  arises. Express it in simplest rationalized radical form.
- In analysing a tuned amplifier circuit, the expression  $\frac{2Q}{\sqrt{2} - 1}$  is used. Rationalize the denominator.
- When analysing the ratio of resultant forces when forces with magnitudes  $F$  and  $T$  act on a structure, the expression  $\sqrt{F^2 + T^2}/\sqrt{F^{-2} + T^{-2}}$  arises. Simplify this expression.

71. The resonant frequency  $\omega$  of a capacitance  $C$  in parallel with a resistance  $R$  and inductance  $L$  (see Fig. 11.6) is

$$\omega = \frac{1}{\sqrt{LC}} \sqrt{1 - \frac{R^2 C}{L}}$$

Combine terms under the radical, rationalize the denominator, and simplify.

Fig. 11.6



72. In fluid dynamics, the expression  $\frac{a}{1 + m/\sqrt{r}}$  arises. Combine terms in the denominator, rationalize the denominator, and simplify.

Answers to Practice Exercises

1.  $2(5\sqrt{3} - 3\sqrt{15})$     2.  $3(10 + \sqrt{2})$     3.  $\frac{-48 + 25\sqrt{3}}{143}$

CHAPTER 11 EQUATIONS

Exponents

$$a^m \times a^n = a^{m+n}$$

$$\frac{a^m}{a^n} = a^{m-n} \quad \text{or} \quad \frac{a^m}{a^{-n}} = \frac{1}{a^{n-m}} \quad (a \neq 0)$$

$$(a^m)^n = a^{mn}$$

$$(ab)^n = a^n b^n, \quad \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n} \quad (b \neq 0)$$

$$a^0 = 1 \quad (a \neq 0)$$

$$a^{-n} = \frac{1}{a^n} \quad (a \neq 0)$$

(11.1) Fractional  $a^{1/n} = \sqrt[n]{a}$  (11.7)

(11.2) exponents

(11.3)  $a^m b^n = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$  (11.8)

(11.4) Radicals

(11.9)  $\sqrt[n]{a^n} = (\sqrt[n]{a})^n = a$

(11.10)  $\sqrt[n]{a} \sqrt[n]{b} = \sqrt[n]{ab}$

(11.5)

(11.11)  $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$

(11.6)

(11.12)  $\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}} \quad (b \neq 0)$

CHAPTER 11 REVIEW EXERCISES

In Exercises 1–28, express each expression in simplest form with only positive exponents.

1.  $2a^{-2}b^0$     2.  $(2c)^{-1}z^{-2}$     3.  $\frac{9c^{-1}}{a^{-3}}$     4.  $\frac{-5x^0}{3y^{-1}}$   
 5.  $3(25)^{3/2}$     6.  $32^{2/5}$     7.  $400^{-3/2}$     8.  $7(1000)^{-2/3}$   
 9.  $\left(\frac{3}{t^2}\right)^{-2}$     10.  $\left(\frac{2x^3}{3}\right)^{-3}$     11.  $\frac{-8^{2/3}}{49^{-1/2}}$     12.  $\frac{4^{-45}}{2^{-90}}$   
 13.  $(2a^{1/3}b^{5/6})^6$     14.  $(ax^{-1/2}y^{1/4})^8$     15.  $(-32m^{15}n^{10})^{3/5}$   
 16.  $(27x^{-6}y^9)^{2/3}$     17.  $2L^2 - 4C^{-1}$     18.  $C^4C^{-1} + (C^4)^{-1}$   
 19.  $\frac{2x^{-1}}{2x^{-1} + y^{-1}}$     20.  $\frac{9a}{(2a)^{-1} - a}$     21.  $(a - 3b^{-1})^{-1}$   
 22.  $(2s^{-2} + t)^{-2}$     23.  $(x^3 - y^{-3})^{1/3}$   
 24.  $(8a^3)^{2/3}(a^{-2} + 1)^{1/2}$     25.  $(W^2 + 2WH + H^2)^{-1/2}$   
 26.  $\left[\frac{(9a)^0(4x^2)^{1/3}(3b^{1/2})}{(2b^0)^2}\right]^{-6}$   
 27.  $2x(x - 1)^{-2} - 2(x^2 + 1)(x - 1)^{-3}$   
 28.  $4(1 - T^2)^{1/2} - (1 - T^2)^{-1/2}$

In Exercises 29–78, perform the indicated operations and express the result in simplest radical form with rationalized denominators.

29.  $\sqrt{68}$     30.  $4\sqrt{96}$     31.  $\sqrt{ab^5c^2}$     32.  $\sqrt{x^3y^4z^6}$   
 33.  $\sqrt{9a^3b^4}$     34.  $\sqrt{8x^5y^2}$     35.  $\sqrt{84st^3u^{-2}}$     36.  $\sqrt{52L^2C^{-5}}$   
 37.  $\frac{5}{\sqrt{2s}}$     38.  $\frac{3a}{\sqrt{5x}}$     39.  $\sqrt{\frac{11}{27}}$     40.  $\sqrt{\frac{7}{8V}}$

41.  $\sqrt[4]{8m^6n^9}$     42.  $\sqrt[3]{9a^7b^{-3}}$     43.  $\sqrt[4]{\sqrt[3]{64}}$   
 44.  $\sqrt{a^{-3}\sqrt[5]{b^{12}}}$     45.  $\sqrt{36 + 4} - 2\sqrt{10}$   
 46.  $2\sqrt{68x} - \sqrt{153x}$     47.  $\sqrt{63} - 2\sqrt{112} - \sqrt{28}$   
 48.  $7\sqrt{20} - \sqrt{80} - 2\sqrt{125}$     49.  $a\sqrt{2x^3} + \sqrt{8a^2x^3}$   
 50.  $2\sqrt{m^2n^3} - \sqrt{n^5}$     51.  $\sqrt[3]{8a^4} + b\sqrt[3]{a}$   
 52.  $\sqrt[4]{2xy^5} - \sqrt[3]{32xy}$     53.  $5\sqrt{5}(6\sqrt{5} - \sqrt{35})$   
 54.  $2\sqrt{8}(5\sqrt{2} - \sqrt{6})$     55.  $2\sqrt{2}(\sqrt{6} - \sqrt{10})$   
 56.  $3\sqrt{5}(\sqrt{15m^2} + 2\sqrt{35m^2})$   
 57.  $(2 - 3\sqrt{17B})(3 + \sqrt{17B})$   
 58.  $(5\sqrt{6} - 4)(3\sqrt{6} + 5)$   
 59.  $(2\sqrt{7} - 3\sqrt{a})(3\sqrt{7} + \sqrt{a})$   
 60.  $(3\sqrt{2} - \sqrt{13})(5\sqrt{2} + 3\sqrt{13})$   
 61.  $\frac{\sqrt{3x}}{2\sqrt{3x} - \sqrt{y}}$     62.  $\frac{5\sqrt{a}}{2\sqrt{a} - c}$     63.  $\frac{\sqrt{2}}{\sqrt{3} - 4\sqrt{2}}$   
 64.  $\frac{4}{3 - 2\sqrt{7}}$     65.  $\frac{\sqrt{7} - \sqrt{5}}{\sqrt{5} + 3\sqrt{7}}$     66.  $\frac{7 - 2\sqrt{6}}{3 + 2\sqrt{6}}$   
 67.  $\frac{2\sqrt{x} - a}{3\sqrt{x} + 5a}$     68.  $\frac{2\sqrt{3y} - \sqrt{2}}{9\sqrt{3y} - 5\sqrt{2}}$     69.  $\sqrt{4b^2 + 1}$   
 70.  $x^{3/8}(x^{5/8} - 8x^{13/8})$   
 71.  $(3x^{1/2} - 2x)(3x^{1/2} + 2x)$   
 72.  $(x^{3n-1}/x^{n-1})^{1/n}$   
 73.  $(1 + 6^{1/2})(3^{1/2} + 2^{1/2})(3^{1/2} - 2^{1/2})$   
 74.  $\sqrt{a^{-2} + \frac{1}{b^2}}$





# Complex Numbers

Among the important areas of research in the mid-nineteenth century was the study of light, which was determined to be a form of electromagnetic radiation. Today, extensive use of electromagnetic waves is made in a great variety of situations. These include the transmission of signals for radio, television, and cell phones. Also, radar, microwave ovens, and X-ray machines are among the many other applications of electromagnetic waves.

In the 1860s, James Clerk Maxwell, a Scottish physicist, developed a set of very important equations for electromagnetic radiation. In doing so, he actually predicted *mathematically* the existence of electromagnetic waves, such as radio waves, which move at the speed of light.

It was not until 1887 that Heinrich Hertz, a German physicist, produced and observed radio waves in the laboratory. We see that mathematics was used in predicting the existence of one of the most important phenomena in use today—over 20 years before it was actually observed.

Important in the study of electromagnetic waves and many areas of electricity and electronics are *complex numbers*, which we study in this chapter. Complex numbers include imaginary numbers as well as real numbers. Other than briefly noting imaginary numbers in Chapters 1 and 7, we have purposely avoided discussing them until now.

Despite their names, complex numbers and imaginary numbers have very real and useful applications. Besides electricity and electronics, they are used in the study of mechanical vibrations, optics, acoustics, and fluid mechanics, as well as to generate beautiful geometric images called fractals.



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# 12

## LEARNING OUTCOMES

After completion of this chapter, the student should be able to:

- Express complex numbers in rectangular form, polar form, exponential form, and graphically and convert from one form to another
- Perform mathematical operations on complex numbers in rectangular, polar, and exponential forms
- Add complex numbers graphically
- Solve algebraic equations involving complex numbers
- Find  $n$ th roots of a complex number using DeMoivre's theorem
- Solve application problems involving complex numbers, including the analysis of alternating-current circuits

◀ In Section 12.7, we see that tuning in a radio station involves a basic application of complex numbers to electricity.

## 12.1 Basic Definitions

Imaginary Unit • Rectangular Form of a Complex Number • Equality of Complex Numbers • Conjugate

Since the square of a positive number or a negative number is positive, it is not possible to square any real number and have a negative result. Therefore, we must define a number system to include square roots of negative numbers. We will find that such numbers can be used to great advantage in certain applications.

If the radicand in a square root is negative, we can express the indicated root as the product of  $\sqrt{-1}$  and the square root of a positive number. *The symbol  $\sqrt{-1}$  is defined as the imaginary unit and is denoted by the symbol  $j$ .* Therefore, we have

$$j = \sqrt{-1} \quad \text{and} \quad j^2 = -1 \tag{12.1}$$

Generally, mathematicians use the symbol  $i$  for  $\sqrt{-1}$ , and therefore most nontechnical textbooks use  $i$ . However, one of the major technical applications of complex numbers is in electronics, where  $i$  represents electric current. Therefore, we will use  $j$  for  $\sqrt{-1}$ , which is also the standard symbol in electronics textbooks.

### EXAMPLE 1 Square roots in terms of $j$

Express the following square roots in terms of  $j$ .

- (a)  $\sqrt{-9} = \sqrt{(9)(-1)} = \sqrt{9}\sqrt{-1} = 3j$
- (b)  $\sqrt{-0.25} = \sqrt{0.25}\sqrt{-1} = 0.5j$
- (c)  $\sqrt{-5} = \sqrt{5}\sqrt{-1} = \sqrt{5}j = j\sqrt{5}$  the form  $j\sqrt{5}$  is better than  $\sqrt{5}j$  ■

When performing operations involving square roots of negative numbers, these must always be expressed in terms of  $j$ , as in the previous example.

### EXAMPLE 2 Expressing numbers in terms of $j$

- (a)  $\sqrt{-6} = \sqrt{(6)(-1)} = \sqrt{6}\sqrt{-1} = j\sqrt{6}$   
this step is correct if only one factor is negative, as in this case
- (b)  $-\sqrt{-75} = -\sqrt{(25)(3)(-1)} = -\sqrt{(25)(3)}\sqrt{-1} = -5j\sqrt{3}$  ■

**LEARNING TIP**

For any positive real number  $a$  we have  $\sqrt{-a} = \sqrt{(-1)(a)} = j\sqrt{a}$ , or simply

$$\sqrt{-a} = j\sqrt{a} \quad (a > 0) \tag{12.2}$$

Note that we write the  $j$  before the radical. This clearly shows that the  $j$  is not *under* the radical.

**Practice Exercises**  
 Simplify: 1.  $\sqrt{-5}\sqrt{-20}$  2.  $-\sqrt{-48}$

**COMMON ERROR** Note that  $-\sqrt{-75}$  is not  $\sqrt{75}$ .

We can avoid this common error if we keep in mind the rules for order of operations. The square root is performed *before* multiplying by  $-1$ , so writing the expression in terms of  $j$  is done before changing sign.

### EXAMPLE 3 Be careful when simplifying

$$(\sqrt{-4})^2 = (\sqrt{4}\sqrt{-1})^2 = (j\sqrt{4})^2 = 4j^2 = -4$$

The simplification of this expression does not follow Eq. (11.10), which states that  $\sqrt{ab} = \sqrt{a}\sqrt{b}$  for square roots. This is the reason it was noted as being valid only if  $a$  and  $b$  are not negative. Therefore, we see that *Eq. (11.10) does not always hold for negative values of  $a$  and  $b$ .* ■

**COMMON ERROR**

In simplifying  $(\sqrt{-4})^2$ , we can write it as  $(\sqrt{-4})(\sqrt{-4})$ . However, we **cannot now write this product as  $\sqrt{(-4)(-4)}$** , for this leads to an incorrect result of +4.

Once again, we can avoid this common error if we keep in mind the rules for order of operations. The square root is the first operation applied on  $-4$ , so we immediately write  $\sqrt{-4}$  as  $j\sqrt{4}$ . After that, the product (or the square) can be performed, as in Example 2.

We cannot overemphasize that

$$(\sqrt{-4})^2 \text{ is not equal to } \sqrt{(-4)(-4)}$$

**EXAMPLE 4** First, express number in terms of  $j$ 

To illustrate the method of handling square roots of negative numbers further, consider the difference between  $\sqrt{-3}\sqrt{-12}$  and  $\sqrt{(-3)(-12)}$ . For these expressions, we have

$$\begin{aligned}\sqrt{-3}\sqrt{-12} &= (j\sqrt{3})(j\sqrt{12}) = (\sqrt{3}\sqrt{12})j^2 = (\sqrt{36})j^2 \\ &= 6(-1) = -6 \\ \sqrt{(-3)(-12)} &= \sqrt{36} = 6\end{aligned}$$

For  $\sqrt{-3}\sqrt{-12}$ , we have the product of square roots of negative numbers, so the square root is performed first, and we use Eq. (12.2). For  $\sqrt{(-3)(-12)}$ , we have the product of negative numbers under the radical, so the multiplication is performed first. We must be careful to note the difference. ■

At times, we need to raise imaginary numbers to some power. Using the definitions of exponents and of  $j$ , we have the following results:

$$\begin{array}{ll}j = j & j^5 = j^4j = j \\ j^2 = -1 & j^6 = j^4j^2 = (1)(-1) = -1 \\ j^3 = j^2j = -j & j^7 = j^4j^3 = (1)(-j) = -j \\ j^4 = j^2j^2 = (-1)(-1) = 1 & j^8 = j^4j^4 = (1)(1) = 1\end{array}$$

The powers of  $j$  go through the cycle,  $j, -1, -j, 1, j, -1, -j, 1$ , and so forth. Noting this—and the fact that  $j$  raised to a power that is a multiple of 4 equals 1—allows us to raise  $j$  to any integral power easily.

**EXAMPLE 5** Powers of  $j$ 

$$(a) j^{10} = j^8j^2 = (1)(-1) = -1$$

$$(b) j^{45} = j^{44}j = (1)(j) = j$$

$$(c) j^{531} = j^{528}j^3 = (1)(-j) = -j$$

↑ exponents 8, 44, 528 are multiples of 4

**Practice Exercise**

Simplify: 3.  $-j^{25}$

■ Even negative numbers were not widely accepted by mathematicians until late in the sixteenth century.

■ Imaginary numbers were so named because the French mathematician René Descartes (1596–1650) referred to them as “imaginaires.” Most of the mathematical development of them occurred in the eighteenth century.

■ Complex numbers were named by the German mathematician Karl Friedrich Gauss (1777–1855).

**RECTANGULAR FORM OF A COMPLEX NUMBER**

Using real numbers and the imaginary unit  $j$ , we define a new kind of number. A **complex number** is any number that can be written in the form  $a + bj$ , where  $a$  and  $b$  are real numbers. If  $a = 0$  and  $b \neq 0$ , we have a number of the form  $bj$ , which is a **pure imaginary number**. If  $b = 0$ , then  $a + bj$  is a real number. The form  $a + bj$  is known as the **rectangular form** of a complex number, where  $a$  is known as the **real part** and  $b$  is known as the **imaginary part**. We see that complex numbers include all real numbers and all pure imaginary numbers.

A comment here about the words *imaginary* and *complex* is in order. The choice of the names of these numbers is historical in nature, and unfortunately it leads to some misconceptions about the numbers. The use of *imaginary* does not imply that the numbers do not exist. Imaginary numbers do in fact exist, as they are defined above. In the same way, the use of *complex* does not imply that the numbers are complicated and therefore difficult to understand. With the proper definitions and operations, we can work with complex numbers just as with any other type of number.

Since a complex number is the sum of a real number and an imaginary number, it is not positive or negative in the usual sense, and it is not equal in the usual way to another complex number. However, the real part is positive or negative (or zero), and the same is true of the imaginary part. Thus, *two complex numbers are equal if and only if their real parts are equal and their imaginary parts are equal*. That is,

*two complex numbers,  $a + bj$  and  $x + yj$ , are equal if  $a = x$  and  $b = y$ .*

**EXAMPLE 6** Equality of complex numbers

(a)  $a + bj = 3 + 4j$  if  $a = 3$  and  $b = 4$   
 real part  $\rightarrow$   $\leftarrow$  imaginary part

(b)  $x + 3j = yj - 5$   
 $x - yj = -5 - 3j$  if  $x = -5$  and  $y = 3$

**EXAMPLE 7** Equality of complex numbers

What values of  $x$  and  $y$  satisfy the equation  $x + 3(xj + y) = 5 - j - yj$ ?

Rewriting each side in the form  $a + bj$ , and equating real parts and equating imaginary parts, we have

$$(x + 3y) + 3xj = 5 + (-1 - y)j$$

For equality,  $x + 3y = 5$  and  $3x = -1 - y$ . The solution of this system of equations is  $x = -1$  and  $y = 2$ .

**Practice Exercise**

Evaluate  $x$  and  $y$ :

4.  $4 - 6j - x = j + yj$

**Conjugate**

The **conjugate** of the complex number  $a + bj$  is the complex number  $a - bj$ . We see that the sign of the imaginary part is changed to obtain the conjugate.

**EXAMPLE 8** Conjugates

(a)  $3 - 2j$  is the conjugate of  $3 + 2j$ . We may also say that  $3 + 2j$  is the conjugate of  $3 - 2j$ . Thus, each is the conjugate of the other.

(b)  $-2 - 5j$  and  $-2 + 5j$  are conjugates.

(c)  $6j$  and  $-6j$  are conjugates.

(d)  $3$  is the conjugate of  $3$  (imaginary part is zero).

**EXERCISES 12.1**

In Exercises 1–4, perform the indicated operations on the resulting expressions if the given changes are made in the indicated examples of this section.

1. In Example 2(a), put a  $j$  in front of the radical and then simplify.
2. In Example 4, put a  $-$  sign before the first radical of the first illustration and then simplify.
3. In Example 5(a), add 40 to the exponent and then evaluate.
4. In Example 7, change the 5 on the right side of the equation to  $-5$  and then solve.

In Exercises 5–16, express each number in terms of  $j$ .

5.  $\sqrt{-81}$
6.  $\sqrt{-121}$
7.  $-\sqrt{-4}$
8.  $-\sqrt{-49}$
9.  $\sqrt{-0.36}$
10.  $-\sqrt{-0.01}$
11.  $4\sqrt{-8}$
12.  $3\sqrt{-48}$
13.  $\sqrt{-\frac{7}{4}}$
14.  $-\sqrt{-\frac{5}{9}}$
15.  $-\sqrt{-4e^2}$
16.  $\sqrt{-\pi^4}$

In Exercises 17–32, simplify each of the given expressions.

17. (a)  $(\sqrt{-7})^2$
18. (a)  $\sqrt{(-15)^2}$
19. (a)  $\sqrt{(-2)(-8)}$
20. (a)  $\sqrt{-9}\sqrt{-16}$
21.  $\sqrt{-\frac{1}{15}}\sqrt{-\frac{27}{5}}$
22.  $-\sqrt{(-\frac{4}{7})(-\frac{49}{16})}$
23.  $-\sqrt{-5}\sqrt{-2}\sqrt{-10}$
24.  $-\sqrt{(-3)(-7)}\sqrt{-21}$
25. (a)  $-j^6$
26. (a)  $-j^{21}$
27.  $j^2 - j^6$
28.  $2j^5 - 1/j^{-2}$
29.  $j^{15} - j^{13}$
30.  $3j^{48} + j^{200}$
31.  $-\sqrt{(-j)^2}$
32.  $-\sqrt{-j^2}$

In Exercises 33–44, perform the indicated operations and simplify each complex number to its rectangular form.

33.  $2 + \sqrt{-9}$
34.  $-26 + \sqrt{-64}$
35.  $3j - \sqrt{-100}$

36.  $-\sqrt{1} - \sqrt{-400}$  37.  $\sqrt{-4j^2} + \sqrt{-4}$  38.  $5 - 2\sqrt{25j^2}$   
 39.  $2j^2 + 3j$  40.  $j^3 - 6$  41.  $\sqrt{18} - \sqrt{-8}$   
 42.  $\sqrt{-27} + \sqrt{12}$  43.  $(\sqrt{-2})^2 + j^4$  44.  $(2\sqrt{2})^2 - j^6$

In Exercises 45–48, find the conjugate of each complex number.

45. (a)  $6 - 7j$  (b)  $8 + j$  45. (a)  $-3 + 2j$  (b)  $-9 - j$   
 47. (a)  $2j$  (b)  $-4$  48. (a)  $6$  (b)  $-5j$

In Exercises 49–54, find the values of  $x$  and  $y$  that satisfy the given equations.

49.  $7x - 2yj = 14 + 4j$  50.  $2x + 3yj = -6 + 12j$   
 51.  $6j - 7 = 3 - x - yj$  52.  $9 - j = xj + 1 - y$   
 53.  $x - 2j^2 + 7j = yj + 2xj^3$   
 54.  $2x - 6xj^3 - 3j^2 = yj - y + 7j^5$

In Exercises 55–64, answer the given questions.

55. How is a number changed if it is multiplied by (a)  $j^4$ ; (b)  $j^2$ ?  
 56. Evaluate: (a)  $j^{-8}$ ; (b)  $j^{-6}$ .

57. Are  $8j$  and  $-8j$  the solutions to the equation  $x^2 + 64 = 0$ ?

58. Are  $2j$  and  $-2j$  solutions to the equation  $x^4 + 16 = 0$ ?

59. Evaluate  $j + j^2 + j^3 + j^4 + j^5 + j^6 + j^7 + j^8$ .

60. What is the smallest positive value of  $n$  for which  $j^{-1} = j^n$ ?

61. Is it possible that a given complex number and its conjugate are equal? Explain.

62. Is it possible that a given complex number and the negative of its conjugate are equal? Explain.

63. Show that the real part of  $x + yj$  equals the imaginary part of  $j(x + yj)$ .

64. Explain why a real number is a complex number, but a complex number may not be a real number.

#### Answers to Practice Exercises

1.  $-10$  2.  $-4j\sqrt{3}$  3.  $-j$  4.  $x = 4, y = -7$

## 12.2 Basic Operations with Complex Numbers

### Addition, Subtraction, Multiplication, and Division of Complex Numbers

The basic operations of addition, subtraction, multiplication, and division of complex numbers are based on the operations for binomials with real coefficients (see Chapters 1 and 6). In performing these operations, we treat  $j$  as we would any other literal number, although we must properly handle any powers of  $j$  that might occur. However, **we must be careful to express all complex numbers in terms of  $j$  before performing these operations.** We have the following definitions.

#### Basic Operations on Complex Numbers

Addition:

$$(a + bj) + (c + dj) = (a + c) + (b + d)j \quad (12.3)$$

Subtraction:

$$(a + bj) - (c + dj) = (a - c) + (b - d)j \quad (12.4)$$

Multiplication:

$$(a + bj)(c + dj) = (ac - bd) + (ad + bc)j \quad (12.5)$$

Division:

$$\frac{a + bj}{c + dj} = \frac{(a + bj)(c - dj)}{(c + dj)(c - dj)} = \frac{(ac + bd) + (bc - ad)j}{c^2 + d^2} \quad (12.6)$$

■ Compare Eq. (12.5) with Eq. (6.6).

Eqs. (12.3) and (12.4) show that we *add and subtract complex numbers by combining the real parts and combining the imaginary parts.*

#### EXAMPLE 1 Adding and subtracting complex numbers

$$\begin{aligned} \text{(a)} \quad (3 - 2j) + (-5 + 7j) &= (3 - 5) + (-2 + 7)j \\ &= -2 + 5j \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad (7 + 9j) - (6 - 4j) &= (7 - 6) + (9 - (-4))j \\ &= 1 + 13j \end{aligned}$$



When complex numbers are multiplied, Eq. (12.5) indicates that *we express numbers in terms of  $j$ , proceed as with any algebraic multiplication, and note that  $j^2 = -1$ .*

### EXAMPLE 2 Multiplying complex numbers

$$\begin{aligned}
 \text{(a)} \quad (6 - \sqrt{-4})(\sqrt{-9}) &= (6 - 2j)(3j) \quad \text{write in terms of } j \\
 &= 18j - 6j^2 = 18j - 6(-1) \\
 &= 6 + 18j \\
 \text{(b)} \quad (-9.4 - 6.2j)(2.5 + 1.5j) &= (-9.4)(2.5) + (-9.4)(1.5j) \\
 &\quad + (-6.2j)(2.5) + (-6.2j)(1.5j) \\
 &= -23.5 - 14.1j - 15.5j - 9.3j^2 \\
 &= -23.5 - 29.6j - 9.3(-1) \\
 &= -14.2 - 29.6j
 \end{aligned}$$

#### Practice Exercise

Multiply: 1.  $(3 - 7j)(9 + 2j)$

### LEARNING TIP

To divide by a complex number, multiply the numerator and the denominator by the conjugate of the denominator.

Most calculators are programmed for complex numbers. They use  $i$  instead of  $j$ .

The procedure shown in Eq. (12.6) for dividing by a complex number is the same as that used for rationalizing the denominator of a fraction. The result is in the proper form of a complex number.

### EXAMPLE 3 Dividing complex numbers

$$\begin{aligned}
 \text{(a)} \quad \frac{7 - 2j}{3 + 4j} &= \frac{(7 - 2j)(3 - 4j)}{(3 + 4j)(3 - 4j)} \quad \leftarrow \text{multiply by conjugate of denominator} \\
 &= \frac{21 - 28j - 6j + 8j^2}{9 - 16j^2} = \frac{21 - 34j + 8(-1)}{9 - 16(-1)} \\
 &= \frac{13 - 34j}{25}
 \end{aligned}$$

This could be written in the form  $a + bj$  as  $\frac{13}{25} - \frac{34}{25}j$ , but this type of result is generally left as a single fraction. In decimal form, the result would be expressed as  $0.52 - 1.36j$ .

$$\begin{aligned}
 \text{(b)} \quad \frac{1}{j} + \frac{2}{3 + j} &= \frac{(3 + j) + 2j}{j(3 + j)} = \frac{3 + 3j}{3j - 1} = \frac{3 + 3j}{-1 + 3j} \cdot \frac{-1 - 3j}{-1 - 3j} \\
 &= \frac{-3 - 12j - 9j^2}{1 - 9j^2} = \frac{6 - 12j}{10} \\
 &= \frac{3 - 6j}{5}
 \end{aligned}$$

#### Practice Exercise

Divide: 2.  $\frac{5 - 3j}{2 + 7j}$

### EXAMPLE 4 Dividing complex numbers—application

In an alternating-current circuit, the voltage  $V$  is given by  $V = IZ$ , where  $I$  is the current (in A) and  $Z$  is the impedance (in  $\Omega$ ). Each of these can be represented by complex numbers. Find the complex number representation for  $I$  if  $V = 4.20 - 3.00j$  volts and  $Z = 5.30 + 2.65j$  ohms. (This type of circuit is discussed in Section 12.7.)

Since  $I = V/Z$ , we have

$$\begin{aligned}
 I &= \frac{4.20 - 3.00j}{5.30 + 2.65j} = \frac{(4.20 - 3.00j)(5.30 - 2.65j)}{(5.30 + 2.65j)(5.30 - 2.65j)} \quad \leftarrow \text{multiply by conjugate of denominator} \\
 &= \frac{22.26 - 11.13j - 15.90j + 7.95j^2}{5.30^2 - 2.65^2j^2} = \frac{22.26 - 27.03j}{5.30^2 + 2.65^2} \\
 &= \frac{14.31 - 27.03j}{35.11} = 0.408 - 0.770j \text{ amperes}
 \end{aligned}$$



## EXERCISES 12.2

In Exercises 1–4, perform the indicated operations on the resulting expressions if the given changes are made in the indicated examples of this section.

- In Example 1(b), change the sign in the first parentheses from + to – and then perform the addition.
- In Example 2(b), change the sign before  $6.2j$  from – to + and then perform the multiplication.
- In Example 3(a), change the sign in the denominator from + to – and then simplify.
- In Example 3(b), change the sign in the second denominator from + to – and then simplify.

In Exercises 5–42, perform the indicated operations, expressing all answers in the form  $a + bj$ .

- $(3 - 7j) + (2 - j)$
- $(7j - 6) - (19 - 3j)$
- $0.23 - (0.46 - 0.19j) + 0.67j$
- $(7 - j) - (4 - 4j) + (6 - j)$
- $(12j - 21) - (15 - 18j) - 9j$
- $(0.062j - 0.073) - 0.030j - (0.121 - 0.051j)$
- $(7 - j)(7j)$
- $(-4 - j) + (-7 - 4j)$
- $(4 - j)(5 + 2j)$
- $(\sqrt{-18}\sqrt{-4})(3j)$
- $7j^3 - 7\sqrt{-9}$
- $j\sqrt{-7} - j^6\sqrt{112} + 3j$
- $(3 - 7j)^2$
- $(1 - j)^3$
- $\frac{6j}{2 - 5j}$
- $\frac{1 - j}{3j}$
- $\frac{j\sqrt{2} - 5}{j\sqrt{2} + 3}$
- $\frac{j^2 - j}{2j - j^8}$
- $\frac{4j}{1 - j} - \frac{8 + j}{2 + 3j}$
- $(4j^5 - 5j^4 + 2j^3 - 3j^2)^2$
- $(3j^2 - 5j^4)(4j^3 - 6j)$
- $\frac{(2 - j^3)^4}{(j^8 - j^6)^3} + j$
- $(-2.2j)(1.5j - 4.0)$
- $(8j - 5)(7 + 4j)$
- $\sqrt{-6}\sqrt{-12}\sqrt{30}$
- $6j - 5j^2\sqrt{-63}$
- $j^2\sqrt{-7} - \sqrt{-28} + 8$
- $(8j + 20)^2$
- $(1 + j)(1 - j)^2$
- $\frac{0.25}{3.0 - \sqrt{-1.0}}$
- $\frac{6 + 5j}{3 - 4j}$
- $\frac{j^5 - j^3}{3 + j}$
- $\frac{3}{2j} + \frac{5}{6 - j}$
- $\frac{(6j + 5)(2 - 4j)}{(5 - j)(4j + 1)}$
- $(2j^2 - 3j^3 + 2j^4 - 2j^5)^6$
- $(5j - 4j^2 + 3j^7)(2j^{12} - j^{13})$
- $(1 + j)^{-3}(2 - j)^{-2}$

In Exercises 43–54, answer the given problems.

- Show that  $-1 - j$  is a solution to the equation  $x^2 + 2x + 2 = 0$ .
- Show that  $1 - j\sqrt{3}$  is a solution to the equation  $x^2 + 4 = 2x$ .
- Multiply  $-3 + j$  by its conjugate.
- Divide  $2 - 3j$  by its conjugate.
- Write the reciprocal of  $3 - j$  in rectangular form.
- Write the reciprocal of  $2 + 5j$  in rectangular form.
- Write  $j^{-2} + j^{-3}$  in rectangular form.
- Solve for  $x$ :  $(x + 2j)^2 = 5 + 12j$
- Solve for  $x$ :  $(x + 3j)^2 = 7 - 24j$
- For  $\frac{3}{5} + \frac{4}{5}j$ , find: (a) the conjugate; (b) the reciprocal.

- If  $f(x) = x + \frac{1}{x}$ , find  $f(1 + 3j)$ .

- When finding the current in a certain electric circuit, the expression  $(s + 1 + 4j)(s + 1 - 4j)$  occurs. Simplify this expression.

In Exercises 55–58, solve the given problems. Refer to Example 4.

- If  $I = 0.835 - 0.427j$  amperes and  $Z = 250 + 170j$  ohms, find the complex-number representation for  $V$ .
- If  $V = 5.70 - 3.65j$  volts and  $I = 0.360 - 0.525j$  amperes, find the complex-number representation for  $Z$ .
- If  $V = 85 + 74j$  volts and  $Z = 2500 - 1200j$  ohms, find the complex-number representation for  $I$ .
- In an alternating-current circuit, two impedances  $Z_1$  and  $Z_2$  have a total impedance  $Z_T$  of  $Z_T = \frac{Z_1 Z_2}{Z_1 + Z_2}$ . Find  $Z_T$  for  $Z_1 = 3200 + 4800j$  ohms and  $Z_2 = 4800 - 6400j$  ohms.

In Exercises 59–62, answer or explain as indicated.

- What type of number is the result of (a) adding a complex number to its conjugate and (b) subtracting a complex number from its conjugate?
- If the reciprocal of  $a + bj$  equals  $a - bj$ , what condition must  $a$  and  $b$  satisfy?
- Explain why the product of a complex number and its conjugate is real and nonnegative.
- Explain how to show that the reciprocal of the imaginary unit is the negative of the imaginary unit.

## Answers to Practice Exercises

- $41 - 57j$
- $\frac{-11 - 41j}{53}$

## 12.3 Graphical Representation of Complex Numbers

### Complex Plane • Complex Number as a Point • Adding and Subtracting Complex Numbers Graphically

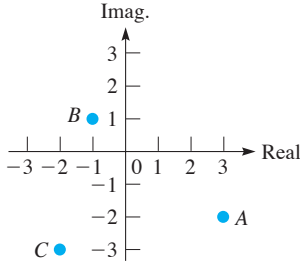


Fig. 12.1

Graphically, we represent a complex number as a *point*, designated as  $a + bj$ , in the rectangular coordinate system. The *real part* is the *x-value* of the point, and the *imaginary part* is the *y-value* of the point. Used in this way, the coordinate system is called the **complex plane**, the horizontal axis is the **real axis**, and the vertical axis is the **imaginary axis**.

#### EXAMPLE 1 Complex number in complex plane

In Fig. 12.1, *A* represents the complex number  $3 - 2j$ , *B* represents  $-1 + j$ , and *C* represents  $-2 - 3j$ . These complex numbers are represented by the points  $(3, -2)$ ,  $(-1, 1)$ , and  $(-2, -3)$ , respectively, of the standard rectangular coordinate system. ■

In the complex plane, consider two complex numbers—for example,  $1 + 2j$  and  $3 + j$ —and their sum  $4 + 3j$ . Drawing lines from the origin to these points (see Fig. 12.2), note that if we think of the complex numbers as being vectors, their sum is the vector sum. Because complex numbers can be used to represent vectors, the numbers are particularly important. *Any complex number can be thought of as representing a vector from the origin to its point in the complex plane.* This leads to the method used to add complex numbers graphically.

#### LEARNING TIP

Keep in mind that the meaning given to the points representing complex numbers in the complex plane is different from the meaning given to the points in the standard rectangular coordinate system. A point in the complex plane represents a **single complex number**, whereas a point in the rectangular coordinate system represents a **pair of real numbers**.

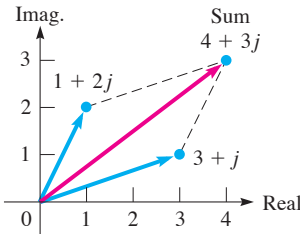


Fig. 12.2

#### Steps to Add Complex Numbers Graphically

1. Find the point corresponding to one of the numbers and draw a line from the origin to this point.
2. Repeat step 1 for the second number.
3. Complete a parallelogram with the lines drawn as adjacent sides. The resulting fourth vertex is the point representing the sum.

Note that this is equivalent to adding vectors by graphical means (see Section 9.1).

#### EXAMPLE 2 Adding complex numbers graphically

(a) Add the complex numbers  $5 - 2j$  and  $-2 - j$  graphically.

The solution is shown in Fig. 12.3. We see that the fourth vertex of the parallelogram is at  $3 - 3j$ . This is, of course, the algebraic sum of these two complex numbers.

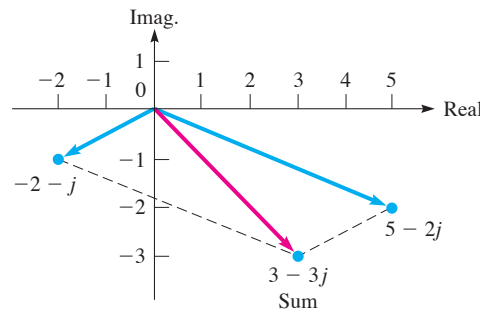


Fig. 12.3

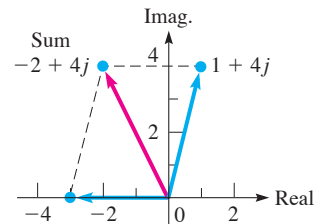


Fig. 12.4

(b) Add the complex numbers  $-3$  and  $1 + 4j$ .

First, note that  $-3 = -3 + 0j$ , which means that the point representing  $-3$  is on the negative real axis. In Fig. 12.4, we show the numbers  $-3$  and  $1 + 4j$  on the graph and complete the parallelogram. From the graph, we see that the sum is  $-2 + 4j$ . ■

**EXAMPLE 3** Subtracting complex numbers graphically

Subtract  $4 - 2j$  from  $2 - 3j$  graphically.

Subtracting  $4 - 2j$  is equivalent to adding  $-4 + 2j$ . Therefore, we complete the solution by adding  $-4 + 2j$  and  $2 - 3j$ , as shown in Fig. 12.5. The result is  $-2 - j$ . ■

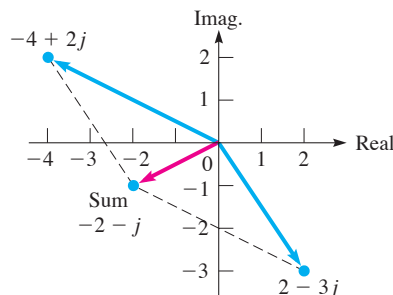


Fig. 12.5

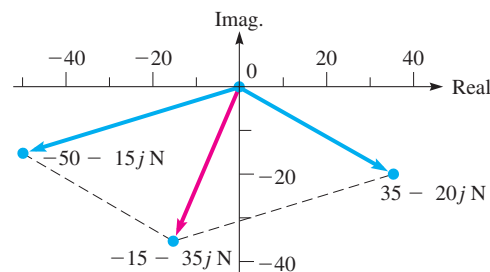


Fig. 12.6

**EXAMPLE 4** Adding complex numbers—application

Two forces acting on an overhead bolt can be represented by  $35 - 20j$  N and  $-50 - 15j$  N. Find the resultant force graphically.

The forces are shown in Fig. 12.6. From the graph, we can see that the sum of the forces, which is the resultant force, is  $-15 - 35j$  N. ■

**EXERCISES 12.3**

In Exercises 1 and 2, perform the indicated operations for the resulting complex numbers if the given changes are made in the indicated examples of this section.

- In Example 2(a), change the sign of the imaginary part of the second complex number and then add the numbers graphically.
- In Example 3, change the sign of the imaginary part of the second complex number and do the subtraction graphically.

In Exercises 3–8, locate the given numbers in the complex plane.

- |             |             |              |
|-------------|-------------|--------------|
| 3. $2 + 6j$ | 4. $-5 + j$ | 5. $-4 - 3j$ |
| 6. 10       | 7. $-3j$    | 8. $3 - 4j$  |

In Exercises 9–28, perform the indicated operations graphically. Check them algebraically.

- |                                   |                                   |
|-----------------------------------|-----------------------------------|
| 9. $2 + (3 + 4j)$                 | 10. $2j + (-2 + 3j)$              |
| 11. $(5 - j) + (3 + 2j)$          | 12. $(3 - 2j) + (-1 - j)$         |
| 13. $5j - (1 - 4j)$               | 14. $(0.2 - 0.1j) - 0.1$          |
| 15. $(2 - 4j) + (-2 + j)$         | 16. $(-1 - 2j) + (6 - j)$         |
| 17. $(3 - 2j) - (4 - 6j)$         | 18. $(-25 - 40j) - (20 - 55j)$    |
| 19. $(80 + 300j) - (260 + 150j)$  | 20. $(-j - 2) - (-1 - 3j)$        |
| 21. $(1.5 - 0.5j) + (3.0 + 2.5j)$ | 22. $(3.5 + 2.0j) - (1.5j - 4.0)$ |
| 23. $(3 - 6j) - (-1 - 8j)$        | 24. $(-6 - 3j) + (2 - 7j)$        |
| 25. $(2j + 1) - 3j - (j + 1)$     | 26. $(6 - j) - 9 - (2j - 3)$      |
| 27. $(j - 6) - j + (j - 7)$       | 28. $j - (1 - j) + (3 + 2j)$      |

In Exercises 29–32, show the given number, its negative, and its conjugate on the same coordinate system.

29.  $3 + 2j$     30.  $-2 + 4j$     31.  $-3 - 5j$     32.  $5 - j$

In Exercises 33 and 34, show the numbers  $a + bj$ ,  $3(a + bj)$ , and  $-3(a + bj)$  on the same coordinate system. The multiplication of a complex number by a real number is called **scalar multiplication** of the complex number.

33.  $3 - j$     34.  $-10 - 30j$

In Exercises 35–38, perform the indicated vector operations graphically on the complex number  $2 + 4j$ .

- Graph the complex number and its conjugate. Describe the relative positions.
- Add the number and its conjugate. Describe the result.
- Subtract the conjugate from the number. Describe the result.
- Graph the number, the number multiplied by  $j$ , the number multiplied by  $j^2$ , and the number multiplied by  $j^3$  on the same graph. Describe the result of multiplying a complex number by  $j$ .

In Exercises 39 and 40, perform the indicated vector additions graphically. Check them algebraically.

- Two ropes hold a boat at a dock. The tensions in the ropes can be represented by  $40 + 10j$  N and  $50 - 25j$  N. Find the resultant force.
- Relative to the air, a plane heads north of west with a velocity that can be represented by  $-480 + 210j$  km/h. The wind is blowing from south of west with a velocity that can be represented by  $60 + 210j$  km/h. Find the resultant velocity of the plane.

## 12.4 Polar Form of a Complex Number

### Polar Form $r(\cos \theta + j \sin \theta)$ • Expressing Numbers in Polar Form

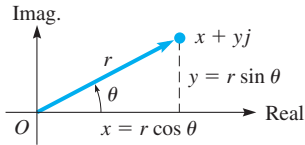


Fig. 12.7

■ If  $\theta_{\text{ref}}$  is a reference angle, according to Eq. (8.6),

- $\theta = \theta_{\text{ref}}$  (first quadrant)
- $\theta = 180^\circ - \theta_{\text{ref}}$  (second quadrant)
- $\theta = 180^\circ + \theta_{\text{ref}}$  (third quadrant)
- $\theta = 360^\circ - \theta_{\text{ref}}$  (fourth quadrant)

In this section, we use the fact that a complex number can be represented by a vector to write complex numbers in another form. This form has advantages when multiplying and dividing complex numbers, and we will discuss these operations later in this chapter.

In the complex plane, by drawing a vector from the origin to the point that represents the number  $x + jy$ , an angle  $\theta$  in standard position is formed. The point  $x + jy$  is  $r$  units from the origin. In fact, we can find any point in the complex plane by knowing the angle  $\theta$  and the value of  $r$ . The equations relating  $x$ ,  $y$ ,  $r$ , and  $\theta$  are similar to those developed for vectors (Eqs. 9.1 to 9.3). Referring to Fig. 12.7, we see that

$$x = r \cos \theta \quad y = r \sin \theta \tag{12.7}$$

$$r^2 = x^2 + y^2 \quad \tan \theta = \frac{y}{x} \tag{12.8}$$

Note that, except for the case of applications to circuits, we will find angles that are not in the first quadrant by disregarding the signs of  $x$  and  $y$  and using the tangent to find a reference angle. Depending on the quadrant, the required angle  $\theta$  is then found through Eq. (8.6).

Substituting Eq. (12.7) into the rectangular form  $x + jy$  of a complex number, we have  $x + jy = r \cos \theta + j(r \sin \theta)$ , or

$$x + jy = r(\cos \theta + j \sin \theta) \tag{12.9}$$

The right side of Eq. (12.9) is called the **polar form** (sometimes the **trigonometric form**). The length  $r$  is the **absolute value**, or the **modulus**, and  $\theta$  is the **argument** of the complex number. Eq. (12.9), along with Eq. (12.8), defines the polar form of a complex number.

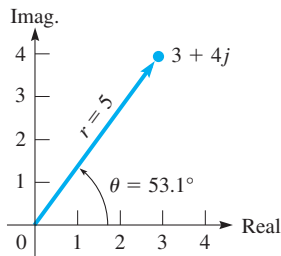


Fig. 12.8

#### EXAMPLE 1 Representing a number in polar form

Represent the complex number  $3 + 4j$  graphically and give its polar form.

From the rectangular form  $3 + 4j$ , we see that  $x = 3$  and  $y = 4$ . Using Eq. (12.8), we have

$$r = \sqrt{3^2 + 4^2} = 5 \quad \theta = \tan^{-1} \frac{4}{3} = 53.1^\circ$$

Thus, the polar form is  $5(\cos 53.1^\circ + j \sin 53.1^\circ)$ . See Fig. 12.8. ■

In Example 1, in expressing the polar form as  $5(\cos 53.1^\circ + j \sin 53.1^\circ)$ , we rounded the angle to the nearest  $0.1^\circ$ , as it is not possible to express the result *exactly* in degrees. In dealing with nonexact numbers, we will express angle to the nearest  $0.1^\circ$ . Other results that cannot be written exactly will be expressed to three significant digits, unless a different accuracy is given in the problem. Of course, in applied situations, most numbers are approximate, as they are derived through measurement.

Another convenient and widely used notation for the polar form is  $r/\theta$ . We must remember in using this form that it represents a complex number and is simply a shorthand way of writing  $r(\cos \theta + j \sin \theta)$ . Therefore,

$$r/\theta = r(\cos \theta + j \sin \theta) \tag{12.10}$$

#### EXAMPLE 2 Polar form $r/\theta$

- (a)  $3(\cos 40^\circ + j \sin 40^\circ) = 3/40^\circ$
- (b)  $6.26(\cos 217.3^\circ + j \sin 217.3^\circ) = 6.26/217.3^\circ$
- (c)  $5/120^\circ = 5(\cos 120^\circ + j \sin 120^\circ)$
- (d)  $14.5/306.2^\circ = 14.5(\cos 306.2^\circ + j \sin 306.2^\circ)$  ■

#### Practice Exercise

Write in polar form: 1.  $15 - 8j$

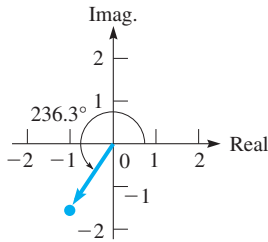


Fig. 12.9

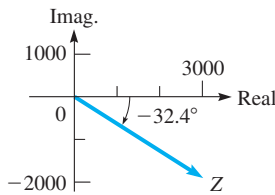


Fig. 12.10

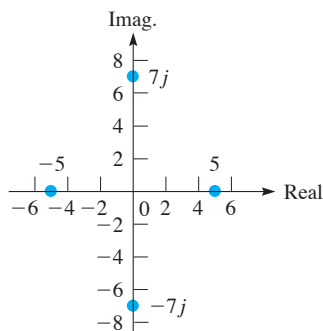
**Practice Exercise**Write in rectangular form: 2.  $2.50 \angle 120^\circ$ 

Fig. 12.11

**EXAMPLE 3 Rectangular form to polar form**

Represent the complex number  $-1.04 - 1.56j$  graphically and give its polar form.

The graphical representation is shown in Fig. 12.9. From Eq. (12.8), we have

$$r = \sqrt{(-1.04)^2 + (-1.56)^2} = 1.87$$

$$\theta_{\text{ref}} = \tan^{-1} \frac{1.56}{1.04} = 56.3^\circ \quad \theta = 180^\circ + 56.3^\circ = 236.3^\circ$$

Since both the real and imaginary parts are negative,  $\theta$  is a third-quadrant angle. Therefore, we found the reference angle before finding  $\theta$ . This means the polar forms are

$$1.87 (\cos 236.3^\circ + j \sin 236.3^\circ) = 1.87 \angle 236.3^\circ$$

**EXAMPLE 4 Polar form to rectangular form**

The impedance  $Z$  (in  $\Omega$ ) in an alternating-current circuit is given by  $Z = 3560 \angle -32.4^\circ$ . Express this in rectangular form.

From the polar form, we have  $r = 3560 \Omega$  and  $\theta = -32.4^\circ$  (it is common to use negative angles in this type of application). This means that we can also write

$$Z = 3560 (\cos(-32.4^\circ) + j \sin(-32.4^\circ))$$

See Fig. 12.10. This means that

$$x = 3560 \cos(-32.4^\circ) = 3010$$

$$y = 3560 \sin(-32.4^\circ) = -1910$$

Therefore, the rectangular form is  $Z = 3010 - 1910j \Omega$ .

**EXAMPLE 5 Polar form—real and imaginary numbers**

Represent the numbers 5,  $-5$ ,  $7j$ , and  $-7j$  in polar form.

Since a positive real number is on the positive real axis, its polar form is

$$a = a (\cos 0^\circ + j \sin 0^\circ) = a \angle 0^\circ$$

Negative real numbers, being on the negative real axis, are written as

$$a = |a| (\cos 180^\circ + j \sin 180^\circ) = |a| \angle 180^\circ$$

Thus,

$$5 = 5 (\cos 0^\circ + j \sin 0^\circ) = 5 \angle 0^\circ$$

$$-5 = 5 (\cos 180^\circ + j \sin 180^\circ) = 5 \angle 180^\circ$$

Positive pure imaginary numbers lie on the positive imaginary axis and are expressed in polar form by

$$bj = b (\cos 90^\circ + j \sin 90^\circ) = b \angle 90^\circ$$

Similarly, negative pure imaginary numbers, being on the negative imaginary axis, are written as

$$bj = |b| (\cos 270^\circ + j \sin 270^\circ) = |b| \angle 270^\circ$$

Thus,

$$7j = 7 (\cos 90^\circ + j \sin 90^\circ) = 7 \angle 90^\circ$$

$$-7j = 7 (\cos 270^\circ + j \sin 270^\circ) = 7 \angle 270^\circ$$

The complex numbers 5,  $-5$ ,  $7j$ , and  $-7j$  are shown graphically in Fig. 12.11.

## EXERCISES 12.4

In Exercises 1 and 2, change the sign of the real part of the complex number in the indicated example of this section and then perform the indicated operations for the resulting complex number.

1. Example 1

2. Example 3

In Exercises 3–18, represent each complex number graphically and give the polar form of each.

- |                            |                      |                      |
|----------------------------|----------------------|----------------------|
| 3. $8 + 6j$                | 4. $-8 - 15j$        | 5. $30 - 40j$        |
| 6. $-5 + 12j$              | 7. $-2.00 + 3.00j$   | 8. $7.00 - 5.00j$    |
| 9. $-0.55 - 0.24j$         | 10. $460 - 460j$     | 11. $1 + j\sqrt{3}$  |
| 12. $\sqrt{2} - j\sqrt{2}$ | 13. $3.514 - 7.256j$ | 14. $62.31 + 95.27j$ |
| 15. $-3$                   | 16. $60$             | 17. $9j$             |
|                            |                      | 18. $-2j$            |

In Exercises 19–36, represent each complex number graphically and give the rectangular form of each.

- |  |   |
|--|---|
| 19. $5.00(\cos 54.0^\circ + j \sin 54.0^\circ)$      | 20. $6(\cos 180^\circ + j \sin 180^\circ)$  |
| 21. $160(\cos 150.0^\circ + j \sin 150.0^\circ)$     |   |
| 22. $2.50(\cos 315.0^\circ + j \sin 315.0^\circ)$    |   |
| 23. $3.00(\cos 232.0^\circ + j \sin 232.0^\circ)$    |   |
| 24. $220.8(\cos 155.13^\circ + j \sin 155.13^\circ)$ |   |
| 25. $0.08(\cos 360^\circ + j \sin 360^\circ)$        | 26. $15(\cos 0^\circ + j \sin 0^\circ)$     |
| 27. $120(\cos 270^\circ + j \sin 270^\circ)$         | 28. $\cos 600.0^\circ + j \sin 600.0^\circ$ |

- |                           |                           |
|---------------------------|---------------------------|
| 29. $4.75/172.8^\circ$    | 30. $1.50/62.3^\circ$     |
| 31. $0.9326/229.54^\circ$ | 32. $277.8/-342.63^\circ$ |
| 33. $7.32/-270^\circ$     | 34. $18.3/540.0^\circ$    |
| 35. $86.42/94.62^\circ$   | 36. $4629/182.44^\circ$   |

In Exercises 37–44, solve the given problems.

- What is the argument for any negative real number?
- For  $x + yj$ , what is the argument if  $x = y < 0$ ?
- Show that the conjugate of  $r/\theta$  is  $r/-\theta$ .
- Find  $r$  and  $\tan(\theta_{\text{ref}})$  for the complex number  $a^b(a + bj)$ . ( $a > 0$ )
- The voltage of a certain generator is represented by  $2.84 - 1.06j$  kV. Write this voltage in polar form.
- Find the magnitude and direction of a force on a bolt that is represented by  $40.5 + 24.5j$  newtons.
- The electric field intensity of a light wave can be described by  $12.4/78.3^\circ$  V/m. Write this in rectangular form.
- The current in a certain microprocessor circuit is represented by  $3.75/15.0^\circ$   $\mu$ A. Write this in rectangular form.

## Answers to Practice Exercises

- $17(\cos 331.9^\circ + j \sin 331.9^\circ)$
- $-1.25 + 2.17j$

## 12.5 Exponential Form of a Complex Number

### Exponential Form $re^{j\theta}$ • Expressing Numbers in Exponential Form • Summary of Important Forms

■ The number  $e$  is named for the Swiss mathematician Leonhard Euler (1707–1783). His works in mathematics and other fields filled about 70 volumes.

Another important form of a complex number is the *exponential form*. It is commonly used in electronics, engineering, and physics applications. As we will see in the next section, it is also convenient for multiplication and division of complex numbers, as the rectangular form is for addition and subtraction.

The **exponential form** of a complex number is written as  $re^{j\theta}$ , where  $r$  and  $\theta$  have the same meanings as given in the previous section, although  $\theta$  is expressed in radians. The number  $e$  is a special irrational number and has an approximate value:

$$e = 2.718\ 281\ 828\ 459\ 045\ 2$$

This number  $e$  is very important in mathematics, and we will see it again in the next chapter. For now, it is necessary to accept the value for  $e$ , although in calculus its meaning is shown along with the reason it has the above value. We can find its value on a calculator by using the  $e^x$  key, with  $x = 1$ , or by using the  $e$  key.

In advanced mathematics, it is shown that

$$re^{j\theta} = r(\cos \theta + j \sin \theta) \quad (12.11)$$

## LEARNING TIP

We will always express  $\theta$  in radians when using the exponential form.

By expressing  $\theta$  in radians, the expression  $j\theta$  is an exponent that can be shown to obey all the laws of exponents as discussed in Chapter 11.

**EXAMPLE 1** Polar form to exponential form

Express the number  $8.50/136.3^\circ$  in exponential form.

Since this complex number is in polar form, we note that  $r = 8.50$  and that we must express  $136.3^\circ$  in radians. Changing  $136.3^\circ$  to radians, we have

$$136.3^\circ = 136.3^\circ \left( \frac{\pi \text{ rad}}{180^\circ} \right) = 2.38 \text{ rad}$$

Therefore, the required exponential form is  $8.50e^{2.38j}$ . This means that

$$8.50/136.3^\circ = 8.50e^{2.38j}$$

degrees to radians  
value of  $r$

We see that the principal step in changing from polar form to exponential form is to change  $\theta$  from degrees to radians. ■

**EXAMPLE 2** Rectangular form to exponential form

Express the number  $3.07 - 7.43j$  in exponential form.

From the rectangular form, we have  $x = 3.07$  and  $y = -7.43$ . Therefore, using Eq. (12.8), we get

$$r = \sqrt{(3.07)^2 + (-7.43)^2} = 8.04$$

$$\theta_{\text{ref}} = \tan^{-1} \frac{7.43}{3.07} = 67.6^\circ \quad \theta = 360^\circ - 67.6^\circ = 292.4^\circ \quad \theta \text{ is in the fourth quadrant}$$

Since  $292.4^\circ = 5.10 \text{ rad}$ , the exponential form is  $8.04e^{5.10j}$ . This means that

$$3.07 - 7.43j = 8.04e^{5.10j}$$

Note that we could have used radian mode directly to obtain  $\theta_{\text{ref}}$  in radians and then find  $\theta$  by subtracting from  $2\pi$ —that is,  $\theta = 2\pi - \tan^{-1}(7.43/3.07) = 5.10$ . ■

**EXAMPLE 3** Exponential form to other forms

Express the complex number  $2.00e^{4.80j}$  in polar and rectangular forms.

We first express  $4.80 \text{ rad}$  as  $275.0^\circ$ . From the exponential form, we know that  $r = 2.00$ . Thus, the polar form is

$$2.00(\cos 275.0^\circ + j \sin 275.0^\circ)$$

Using the distributive law, we rewrite the polar form and then evaluate. Thus,

$$\begin{aligned} 2.00e^{4.80j} &= 2.00(\cos 275.0^\circ + j \sin 275.0^\circ) \\ &= 2.00 \cos 275.0^\circ + (2.00 \sin 275.0^\circ)j \\ &= 0.174 - 1.99j \end{aligned}$$

**EXAMPLE 4** Power of a number in exponential form

Express  $(1.846e^{1.229j})^2$  in polar and rectangular forms.

First, we note that  $(1.846e^{1.229j})^2 = 1.846^2 e^{2.458j}$ . Since  $2.458 \text{ rad} = 140.83^\circ$ , the polar form is  $1.846^2/140.83^\circ = 3.408/140.83^\circ$ .

For the rectangular form, using radian mode, we have

$$\begin{aligned} (1.846e^{1.229j})^2 &= 1.846^2 e^{2.458j} = 1.846^2 (\cos 2.458 + j \sin 2.458) \\ &= -2.642 + 2.152j \end{aligned}$$

As we noted earlier, an important application of the use of complex numbers is in alternating-current analysis. When an alternating current flows through a given circuit, usually the current and voltage have different phases. That is, they do not reach their

**Practice Exercise**

- Express  $25.0(\cos 127.0^\circ + j \sin 127.0^\circ)$  in exponential form.

**Practice Exercise**

- Express  $-20.5 - 16.8j$  in exponential form.

■ Calculators that operate with complex numbers treat the exponential and polar forms as being the same, since each uses  $r$  and  $\theta$ .



peak values at the same time. Therefore, one way of accounting for the magnitude as well as the phase of an electric current or voltage is to write it as a complex number. Here, the modulus is the actual magnitude of the current or voltage, and the argument  $\theta$  is a measure of the phase.

**EXAMPLE 5 Exponential form—application**

A current of  $2.00 - 4.00j$  amperes flows in a given circuit. Express this current in exponential form and find the magnitude of the current.

From the rectangular form, we have  $x = 2.00$  and  $y = -4.00$ . Therefore,

$$r = \sqrt{(2.00)^2 + (-4.00)^2} = 4.47 \text{ A}$$

$$\theta = \tan^{-1} \frac{-4.00}{2.00} = -63.4^\circ$$

It is normal to express the phase in terms of negative angles when the imaginary part is negative. Therefore,  $63.4^\circ = 1.11 \text{ rad}$ , which means the exponential form is  $4.47e^{-1.11j}$ . The modulus of 4.47 means the magnitude of the current is 4.47 A. ■

We now summarize the three important forms of a complex number. See Fig. 12.12.

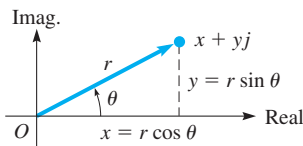


Fig. 12.12

Rectangular:	$x + yj$
Polar:	$r(\cos \theta + j \sin \theta) = r \angle \theta$
Exponential:	$re^{j\theta}$

It follows that

$$x + yj = r(\cos \theta + j \sin \theta) = r \angle \theta = re^{j\theta} \tag{12.12}$$

where

$$r^2 = x^2 + y^2 \quad \tan \theta = \frac{y}{x} \tag{12.8}$$

In Eq. (12.12), the argument  $\theta$  is the same for exponential and polar forms. It is usually expressed in radians in exponential form and in degrees in polar form.

**EXERCISES 12.5**

In Exercises 1–4, perform the indicated operations for the resulting complex numbers if the given changes are made in the indicated examples of this section.

- In Example 1, change  $136.3^\circ$  to  $226.3^\circ$  and then find the exponential form.
- In Example 2, change the sign before  $7.43j$  from  $-$  to  $+$  and then find the exponential form.
- In Example 3, change the exponent to  $3.80j$  and then find the polar and rectangular forms.
- In Example 4, change the exponent 2 to 3 and then find the polar and rectangular forms.

In Exercises 5–24, express the given numbers in exponential form.

- $3.00(\cos 60.0^\circ + j \sin 60.0^\circ)$
- $575(\cos 135^\circ + j \sin 135^\circ)$
- $0.450(\cos 282.3^\circ + j \sin 282.3^\circ)$

- $2.10(\cos 588.7^\circ + j \sin 588.7^\circ)$
- $375.5[\cos(-95.46^\circ) + j \sin(-95.46^\circ)]$
- $16.72[\cos(-7.14^\circ) + j \sin(-7.14^\circ)]$
- $0.515 \angle 198.3^\circ$
- $4650 \angle 326.5^\circ$
- $4.06 \angle -61.4^\circ$
- $0.0192 \angle 76.7^\circ$
- $9245 \angle 296.3^\circ$
- $82.76 \angle 470.1^\circ$
- $3 - 4j$
- $-1 - 5j$
- $-30 + 20j$
- $600 + 100j$
- $5.90 + 2.40j$
- $47.3 - 10.9j$
- $-634.6 - 528.2j$
- $-8573 + 5477j$

In Exercises 25–32, express the given complex numbers in polar and rectangular forms.

- $3.00e^{0.500j}$
- $20.0e^{1.00j}$
- $464e^{1.85j}$
- $2.50e^{3.84j}$
- $3.20e^{-5.41j}$
- $0.800e^{3.00j}$
- $0.1724e^{2.391j}$
- $820.7e^{-3.492j}$



**EXAMPLE 2** Multiplication in  $r/\theta$  form

When we use the  $r/\theta$  polar form to multiply the two complex numbers in Example 1, we have

$$\begin{aligned}
 r_1 &= 3.61 & \theta_1 &= 56.3^\circ \\
 r_2 &= 1.41 & \theta_2 &= 315.0^\circ \\
 (3.61/56.3^\circ) (1.41/315.0^\circ) &= (3.61)(1.41)/56.3^\circ + 315.0^\circ \\
 &= 5.09/371.3^\circ \\
 &= 5.09/11.3^\circ
 \end{aligned}$$

**Practice Exercise**

1. Find the polar form product:  
 $(3/50^\circ)(5/65^\circ)$

Again, note that the angle in the final result is between  $0^\circ$  and  $360^\circ$ . ■

If we wish to *divide* one complex number in exponential form by another, we arrive at the following result:

$$r_1 e^{j\theta_1} \div r_2 e^{j\theta_2} = \frac{r_1}{r_2} e^{j(\theta_1 - \theta_2)} \tag{12.14}$$

Therefore, we have Eq. (12.15) for dividing in polar form.

**LEARNING TIP**

The quotient of two complex numbers in polar form is

$$\begin{aligned}
 \frac{r_1(\cos \theta_1 + j \sin \theta_1)}{r_2(\cos \theta_2 + j \sin \theta_2)} &= \frac{r_1}{r_2} [\cos(\theta_1 - \theta_2) + j \sin(\theta_1 - \theta_2)] \\
 \frac{r_1/\theta_1}{r_2/\theta_2} &= \frac{r_1}{r_2} / \theta_1 - \theta_2
 \end{aligned} \tag{12.15}$$

That is, the magnitudes are divided, and the angles are subtracted.

**EXAMPLE 3** Division in polar form

Divide the first complex number of Example 1 by the second.

Using the polar forms, we have

$$\begin{aligned}
 \frac{3.61(\cos 56.3^\circ + j \sin 56.3^\circ)}{1.41(\cos 315.0^\circ + j \sin 315.0^\circ)} &= \frac{3.61}{1.41} [\cos(56.3^\circ - 315.0^\circ) + j \sin(56.3^\circ - 315.0^\circ)] \\
 &= 2.56 [\cos(-258.7^\circ) + j \sin(-258.7^\circ)] \\
 &= 2.56(\cos 101.3^\circ + j \sin 101.3^\circ)
 \end{aligned}$$

**Practice Exercise**

2. Find the polar form quotient:  
 $\frac{3(\cos 50^\circ + j \sin 50^\circ)}{5(\cos 65^\circ + j \sin 65^\circ)}$

**EXAMPLE 4** Division in  $r/\theta$  form

Repeating Example 3 using  $r/\theta$  polar form, we have

$$\begin{aligned}
 \frac{3.61/56.3^\circ}{1.41/315.0^\circ} &= \frac{3.61}{1.41} / 56.3^\circ - 315.0^\circ = 2.56 / -258.7^\circ \\
 &= 2.56 / 101.3^\circ
 \end{aligned}$$

We have just seen that multiplying and dividing numbers in polar form can be easily performed. However, if we are to add or subtract numbers in polar form, *we must do the addition or subtraction by using rectangular form.*

**EXAMPLE 5** Addition in polar form

Perform the addition  $1.563/37.56^\circ + 3.827/146.23^\circ$ .

In order to do this addition, we must change each number to rectangular form:

$$\begin{aligned} & 1.563/37.56^\circ + 3.827/146.23^\circ \\ &= 1.563(\cos 37.56^\circ + j \sin 37.56^\circ) + 3.827(\cos 146.23^\circ + j \sin 146.23^\circ) \\ &= 1.2390 + 0.9528j - 3.1813 + 2.1273j \\ &= -1.9423 + 3.0801j \end{aligned}$$

Now, we change this to polar form:

$$\begin{aligned} r &= \sqrt{(-1.9423)^2 + (3.0801)^2} = 3.641 \\ \tan \theta &= \frac{3.0801}{-1.9423} \quad \theta = 122.24^\circ \end{aligned}$$

Therefore,

$$1.563/37.56^\circ + 3.827/146.23^\circ = 3.641/122.24^\circ \quad \blacksquare$$

**DEMOIVRE'S THEOREM**

To raise a complex number to a power, we may use the exponential form of the number in Eq. (11.3). This leads to

$$(re^{j\theta})^n = r^n e^{jn\theta} \quad (12.16)$$

We extend this to polar form as follows.

**LEARNING TIP**

The power of a complex number in polar form is

$$\begin{aligned} [r(\cos \theta + j \sin \theta)]^n &= r^n (\cos n\theta + j \sin n\theta) \\ (r/\theta)^n &= r^n/n\theta \end{aligned} \quad (12.17)$$

That is, the magnitude is raised to the power, and the angle is multiplied by the power.

■ For reference, Eq. (11.3) is  $(a^m)^n = a^{mn}$ .

■ DeMoivre's theorem is named for the mathematician Abraham DeMoivre (1667–1754).

Equation (12.17) is known as **DeMoivre's theorem** and is valid for all real values of  $n$ . It is also used for finding roots of complex numbers if  $n$  is a fractional exponent.

**EXAMPLE 6** Power by DeMoivre's theorem

Using DeMoivre's theorem, find  $(2 + 3j)^3$ .

From Example 1, we know  $r = 3.61$  and  $\theta = 56.3^\circ$ . Therefore,

$$\begin{aligned} [3.61(\cos 56.3^\circ + j \sin 56.3^\circ)]^3 &= (3.61)^3 [\cos(3 \times 56.3^\circ) + j \sin(3 \times 56.3^\circ)] \\ &= 47.0(\cos 168.9^\circ + j \sin 168.9^\circ) \\ &= 47.0/168.9^\circ \end{aligned}$$

Expressing  $\theta$  in radians, we have  $\theta = 56.3^\circ = 0.983$  rad. Therefore,

$$\begin{aligned} (3.61e^{0.983j})^3 &= (3.61)^3 e^{3 \times 0.983j} = 47.0e^{2.95j} \\ (2 + 3j)^3 &= 47.0(\cos 168.9^\circ + j \sin 168.9^\circ) \\ &= 47.0/168.9^\circ \\ &= 47.0e^{2.95j} = -46 + 9j \end{aligned} \quad \blacksquare$$

**Practice Exercise**

3. Find the polar form power:  $(3 \cos 50^\circ)^8$

**EXAMPLE 7** Cube roots by DeMoivre's theorem

Find the cube root of  $-1$ .

Since we know that  $-1$  is a real number, we can find its cube root by means of the definition. That is,  $(-1)^3 = -1$ . We check this by DeMoivre's theorem. Writing  $-1$  in polar form, we have

$$-1 = 1(\cos 180^\circ + j \sin 180^\circ)$$

Applying DeMoivre's theorem, with  $n = \frac{1}{3}$ , we obtain

$$\begin{aligned} (-1)^{1/3} &= 1^{1/3}(\cos \frac{1}{3}180^\circ + j \sin \frac{1}{3}180^\circ) = \cos 60^\circ + j \sin 60^\circ \\ &= \frac{1}{2} + j \frac{\sqrt{3}}{2} && \text{exact answer} \\ &= 0.5000 + 0.8660j && \text{decimal approximation} \end{aligned}$$

Observe that we did not obtain  $-1$  as the answer. If we check the answer, in the form  $\frac{1}{2} + j\frac{\sqrt{3}}{2}$ , by actually cubing it, we obtain  $-1$ ! Therefore, it is a correct answer.

We should note that it is possible to take  $\frac{1}{3}$  of any angle up to  $1080^\circ$  and still have an angle less than  $360^\circ$ . Since  $180^\circ$  and  $540^\circ$  have the same terminal side, let us try writing  $-1$  as  $1(\cos 540^\circ + j \sin 540^\circ)$ . Using DeMoivre's theorem, we have

$$(-1)^{1/3} = 1^{1/3}(\cos \frac{1}{3}540^\circ + j \sin \frac{1}{3}540^\circ) = \cos 180^\circ + j \sin 180^\circ = -1$$

We have found the answer we originally anticipated.

Angles of  $180^\circ$  and  $90^\circ$  also have the same terminal side, so we try

$$\begin{aligned} (-1)^{1/3} &= 1^{1/3}(\cos \frac{1}{3}900^\circ + j \sin \frac{1}{3}900^\circ) = \cos 300^\circ + j \sin 300^\circ \\ &= \frac{1}{2} - j \frac{\sqrt{3}}{2} && \text{exact answer} \\ &= 0.5000 - 0.8660j && \text{decimal approximation} \end{aligned}$$

Checking this, we find that it is also a correct root. We may try  $1260^\circ$ , but  $\frac{1}{3}(1260^\circ) = 420^\circ$ , which has the same functional values as  $60^\circ$ , and would give us the answer  $0.5000 + 0.8660j$  again.

We have found, therefore, *three cube roots* of  $-1$ . They are

$$-1, \quad \frac{1}{2} + j \frac{\sqrt{3}}{2}, \quad \frac{1}{2} - j \frac{\sqrt{3}}{2}$$

These roots are graphed in Fig. 12.13. Note that they are equally spaced on the circumference of a circle of radius 1. ■

When the results of Example 7 are generalized, it can be proved that *there are  $n$   $n$ th roots of a complex number*. When graphed, these roots are on a circle of radius  $r^{1/n}$  and are equally spaced  $360^\circ/n$  apart. Following is the method for finding these  $n$  roots.

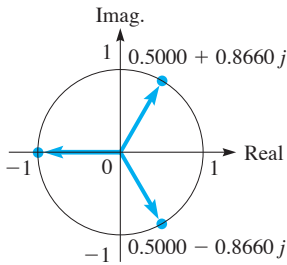


Fig. 12.13

### Using DeMoivre's Theorem to Find the $n$ th Roots of a Complex Number

1. Express the number in polar form.
2. Express the root as a fractional exponent.
3. Use Eq. (12.17) with  $\theta$  to find one root.
4. Use Eq. (12.17) and add  $360^\circ/n$  to  $\theta$ ,  $n - 1$  times, to find the other roots.

**EXAMPLE 8 Square roots by DeMoivre's theorem**

Find the two square roots of  $2j$ .

First, we write  $2j$  in polar form as  $2j = 2(\cos 90^\circ + j \sin 90^\circ)$ . To find square roots, we use the exponent  $1/2$ . The first square root is

$$(2j)^{1/2} = 2^{1/2} \left( \cos \frac{90^\circ}{2} + j \sin \frac{90^\circ}{2} \right) = \sqrt{2} (\cos 45^\circ + j \sin 45^\circ) = 1 + j$$

To find the other square root, we add  $360^\circ$  to  $90^\circ$ . This gives us

$$(2j)^{1/2} = 2^{1/2} \left( \cos \frac{450^\circ}{2} + j \sin \frac{450^\circ}{2} \right) = \sqrt{2} (\cos 225^\circ + j \sin 225^\circ) = -1 - j$$

Therefore, the two square roots of  $2j$  are  $1 + j$  and  $-1 - j$ . We see in Fig. 12.14 that they are on a circle of radius  $\sqrt{2}$  and  $180^\circ$  apart.

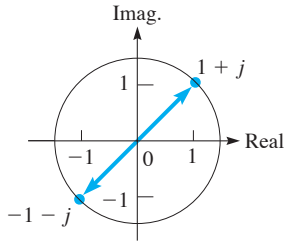


Fig. 12.14

**EXAMPLE 9 Sixth roots by DeMoivre's theorem**

Find all the roots of the equation  $x^6 - 64 = 0$ .

Solving for  $x$ , we have  $x^6 = 64$ , or  $x = \sqrt[6]{64}$ . Therefore, we have to find the six sixth roots of 64. Writing 64 in polar form, we have  $64 = 64(\cos 0^\circ + j \sin 0^\circ)$ . Using the exponent  $1/6$  for the sixth root, we have the following solutions:

First root:  $64^{1/6} = 64^{1/6} \left( \cos \frac{0^\circ}{6} + j \sin \frac{0^\circ}{6} \right) = 2(\cos 0^\circ + j \sin 0^\circ) = 2$

Second root:  $64^{1/6} = 64^{1/6} \left( \cos \frac{0^\circ + 360^\circ}{6} + j \sin \frac{0^\circ + 360^\circ}{6} \right)$   
 $= 2(\cos 60^\circ + j \sin 60^\circ) = 1 + j\sqrt{3}$

Third root:  $64^{1/6} = 64^{1/6} \left( \cos \frac{0^\circ + 720^\circ}{6} + j \sin \frac{0^\circ + 720^\circ}{6} \right)$   
 $= 2(\cos 120^\circ + j \sin 120^\circ) = -1 + j\sqrt{3}$

Fourth root:  $64^{1/6} = 64^{1/6} \left( \cos \frac{0^\circ + 1080^\circ}{6} + j \sin \frac{0^\circ + 1080^\circ}{6} \right)$   
 $= 2(\cos 180^\circ + j \sin 180^\circ) = -2$

Fifth root:  $64^{1/6} = 64^{1/6} \left( \cos \frac{0^\circ + 1440^\circ}{6} + j \sin \frac{0^\circ + 1440^\circ}{6} \right)$   
 $= 2(\cos 240^\circ + j \sin 240^\circ) = -1 - j\sqrt{3}$

Sixth root:  $64^{1/6} = 64^{1/6} \left( \cos \frac{0^\circ + 1800^\circ}{6} + j \sin \frac{0^\circ + 1800^\circ}{6} \right)$   
 $= 2(\cos 300^\circ + j \sin 300^\circ) = 1 - j\sqrt{3}$

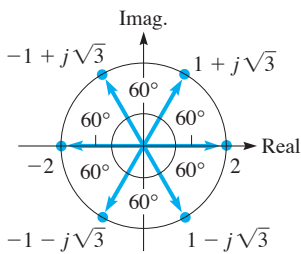


Fig. 12.15

These roots are graphed in Fig. 12.15. Note that they are equally spaced  $60^\circ$  apart on the circumference of a circle of radius 2.

From the text and examples of this and previous sections, we see the advantages for the various forms of writing complex numbers. The rectangular form can be used for all of the basic operations but lends itself best to addition and subtraction. The polar form is used for multiplication, division, raising to powers, and finding roots. The exponential form can be used for multiplication, division, and powers, and also for theoretical purposes (for example, deriving DeMoivre's theorem).

## EXERCISES 12.6

In Exercises 1–4, perform the indicated operations for the resulting complex numbers if the given changes are made in the indicated examples of this section.

- In Example 1, change the sign of the complex part of the second complex number and then perform the multiplication.
- In Example 3, divide the first number of Exercise 1 by the second.
- In Example 6, change the exponent to 5 and then find the result.
- In Example 8, replace  $2j$  with  $-2j$  and then find the roots.

In Exercises 5–20, perform the indicated operations. Leave the result in polar form.

- $[4(\cos 60^\circ + j \sin 60^\circ)][2(\cos 20^\circ + j \sin 20^\circ)]$
- $[3(\cos 120^\circ + j \sin 120^\circ)][5(\cos 45^\circ + j \sin 45^\circ)]$
- $(0.5/140^\circ)(6/110^\circ)$
- $(0.4/320^\circ)(5.5/-150^\circ)$
- $\frac{8(\cos 100^\circ + j \sin 100^\circ)}{4(\cos 65^\circ + j \sin 65^\circ)}$
- $\frac{9(\cos 230^\circ + j \sin 230^\circ)}{45(\cos 80^\circ + j \sin 80^\circ)}$
- $\frac{12/320^\circ}{5/-210^\circ}$
- $\frac{2/90^\circ}{4/75^\circ}$
- $[0.2(\cos 35^\circ + j \sin 35^\circ)]^3$
- $[3(\cos 120^\circ + j \sin 120^\circ)]^4$
- $(2/135^\circ)^8$
- $(1/142^\circ)^{10}$
- $\frac{(50/236^\circ)(2/84^\circ)}{125/47^\circ}$
- $\frac{36/274^\circ}{(2/141^\circ)(6/195^\circ)}$
- $\frac{(4/24^\circ)(10/326^\circ)}{(1/186^\circ)(8/77^\circ)}$
- $\frac{(25/194^\circ)(6/239^\circ)}{(30/17^\circ)(10/29^\circ)}$

In Exercises 21–24, perform the indicated operations. Express results in polar form. See Example 5.

- $2.78/56.8^\circ + 1.37/207.3^\circ$
- $15.9/142.6^\circ - 18.5/71.4^\circ$
- $7085/115.62^\circ - 4667/296.34^\circ$
- $307.5/326.54^\circ + 726.3/96.41^\circ$

In Exercises 25–36, change each number to polar form and then perform the indicated operations. Express the result in rectangular and polar forms. Check by performing the same operation in rectangular form.

- $(3 + 4j)(5 - 12j)$
- $(7 - 3j)(8 + j)$
- $\frac{7}{1 - 3j}$
- $\frac{30 + 40j}{5 - 12j}$
- $(3 + 4j)^4$
- $(2 + 3j)^5$
- $(-2 + 5j)(-1 - j)$
- $(1 + 5j)(4 + 2j)$
- $\frac{40j}{7 + 2j}$
- $\frac{-2 + 5j}{-1 - j}$
- $(-1 - j)^8$
- $(1 - 2j)^6$

In Exercises 37–42, use DeMoivre's theorem to find all the indicated roots. Be sure to find all roots.

- The two square roots of  $4(\cos 60^\circ + j \sin 60^\circ)$
- The three cube roots of  $27(\cos 120^\circ + j \sin 120^\circ)$
- The three cube roots of  $3 - 4j$
- The two square roots of  $-5 + 12j$
- The square roots of  $1 + j$
- The cube roots of  $\sqrt{3} + j$

In Exercises 43–48, find all of the roots of the given equations.

- $x^4 - 1 = 0$
- $x^3 - 8 = 0$
- $x^3 + 27j = 0$
- $x^4 - j = 0$
- $x^5 + 32 = 0$
- $x^6 + 8 = 0$

In Exercises 49–58, perform the indicated operations.

- Using the results of Example 7, find the cube roots of  $-125$ .
- Using the results of Example 8, find the square roots of  $32j$ .
- In Example 7, we showed that one cube root of  $-1$  is  $\frac{1}{2} - \frac{1}{2}j\sqrt{3}$ . Cube this number in rectangular form and show that the result is  $-1$ .
- Explain why the two square roots of a complex number are negatives of each other.



53. The cube roots of  $-1$  can be found by solving the equation  $x^3 + 1 = 0$ . Find these roots by factoring  $x^3 + 1$  as the sum of cubes and compare with Example 7.
54. The cube roots of  $8$  can be found by solving the equation  $x^3 - 8 = 0$ . Find these roots by factoring  $x^3 - 8$  as the difference of cubes and compare with Exercise 44.
55. The electric power  $P$  (in W) supplied to an element in a circuit is the product of the voltage  $V$  (in V) and the current  $I$  (in A). Find the expression for the power supplied if  $V = 6.80/\underline{56.3^\circ}$  volts and  $I = 0.0705/\underline{-15.8^\circ}$  amperes.
56. The displacement  $d$  (in cm) of a weight suspended on a system of two springs is  $d = 6.03/\underline{22.5^\circ} + 3.26/\underline{76.0^\circ}$  cm. Perform the addition and express the answer in polar form.
57. The voltage across a certain inductor is  $V = (8.66/\underline{90.0^\circ}) (50.0/\underline{135.0^\circ}) / (10.0/\underline{60.0^\circ})$  volts. Simplify this expression and find the magnitude of the voltage.
58. In a microprocessor circuit, the current is  $I = 3.75/\underline{15.0^\circ} \mu\text{A}$  and the impedance is  $Z = 2500/\underline{-35.0^\circ}$  ohms. Find the voltage  $V$  in rectangular form. (See Example 4 of Section 12.2.)

#### Answers to Practice Exercises

1.  $15/\underline{115^\circ}$    2.  $0.6/\underline{345^\circ}$    3.  $6561/\underline{40^\circ}$

## 12.7 An Application to Alternating-Current (ac) Circuits

**Basic Circuit with Resistance, Inductance, and Capacitance • Impedance • Phase Angle • Phasor • Resonance**

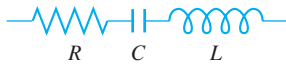


Fig. 12.16

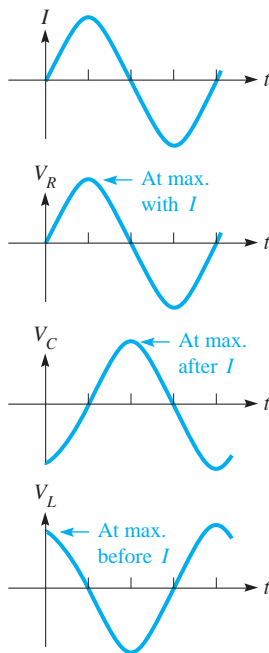


Fig. 12.17

We now show an application of complex numbers in a basic type of alternating-current circuit. We show how the voltage is measured between any two points in a circuit containing a resistance, a capacitance, and an inductance. This circuit is similar to one noted in earlier examples and exercises in this chapter.

A *resistance* is any part of a circuit that tends to obstruct the flow of electric current through the circuit. It is denoted by  $R$  (units in ohms,  $\Omega$ ) and in diagrams by  $\text{---}\text{---}\text{---}$ , as shown in Fig. 12.16. A *capacitance* is two nonconnected plates in a circuit; no current actually flows across the gap between them. In an ac circuit, an electric charge is continually going to and from each plate and, therefore, the current in the circuit is not effectively stopped. It is denoted by  $C$  (units in farads, F) and in diagrams by  $\text{---}\text{---}$  (see Fig. 12.16). An *inductance* is basically a coil of wire in which current is induced because the current is continually changing in the circuit. It is denoted by  $L$  (units in henrys, H) and in diagrams by  $\text{---}\text{---}\text{---}$  (see Fig. 12.16). All these elements affect the voltage in an alternating-current circuit. We state here the relation each has to the voltage and current in the circuit.

In Chapter 10, we noted that the current and voltage in an ac circuit could be represented by a sine or a cosine curve. Therefore, each reaches peak values periodically. *If they reach their respective peak values at the same time, they are in phase. If the voltage reaches its peak before the current, the voltage leads the current. If the voltage reaches its peak after the current, the voltage lags the current.*

In the study of electricity, it is shown that the voltage across a resistance is in phase with the current. The voltage across a capacitor lags the current by  $90^\circ$ , and the voltage across an inductance leads the current by  $90^\circ$ . This is shown in Fig. 12.17, where, in a given circuit,  $I$  represents the current,  $V_R$  is the voltage across a resistor,  $V_C$  is the voltage across a capacitor,  $V_L$  is the voltage across an inductor, and  $t$  represents time.

Each element in an ac circuit tends to offer a type of resistance to the flow of current. *The effective resistance of any part of the circuit is called the reactance*, and it is denoted by  $X$ . The voltage across any part of the circuit whose reactance is  $X$  is given by  $V = IX$ , where  $I$  is the current (in amperes) and  $V$  is the voltage (in volts). Therefore,

*the voltage  $V_R$  across a resistor with resistance  $R$ ,*

*the voltage  $V_C$  across a capacitor with reactance  $X_C$ , and*

*the voltage  $V_L$  across an inductor with reactance  $X_L$*

are, respectively,

$$V_R = IR \quad V_C = IX_C \quad V_L = IX_L \quad (12.18)$$

■ Eq. (12.18) is based on Ohm's law, which states that the current is proportional to the voltage for a constant resistance. It is named for the German physicist Georg Ohm (1787–1854). The ohm ( $\Omega$ ) is also named for him.

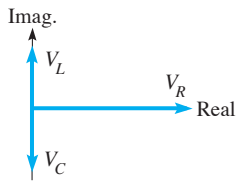


Fig. 12.18

To determine the voltage across a combination of these elements of a circuit, we must account for the reactance, as well as the phase of the voltage across the individual elements. Since the voltage across a resistor is in phase with the current, we represent  $V_R$  along the positive real axis as a real number. Since the voltage across an inductance leads the current by  $90^\circ$ , we represent this voltage as a positive, pure imaginary number. In the same way, by representing the voltage across a capacitor as a negative, pure imaginary number, we show that the voltage *lags* the current by  $90^\circ$ . These representations are meaningful since the positive imaginary axis is  $+90^\circ$  from the positive real axis, and the negative imaginary axis is  $-90^\circ$  from the positive real axis. See Fig. 12.18.

The circuit elements shown in Fig. 12.16 are in *series*, and all circuits we consider (except Exercises 22 and 23) are series circuits. The total voltage across a series of all three elements is given by  $V_R + V_L + V_C$ , which we represent by  $V_{RLC}$ . Therefore,

$$V_{RLC} = IR + IX_Lj - IX_Cj = I[R + j(X_L - X_C)]$$

This expression is also written as

$$V_{RLC} = IZ \tag{12.19}$$

■ In the 1880s, it was decided that alternating current (favored by George Westinghouse) would be used to distribute electric power. Thomas Edison had argued for the use of direct current.

where the symbol  $Z$  is called the **impedance** of the circuit. It is the total effective resistance to the flow of current by a combination of the elements in the circuit, taking into account the phase of the voltage in each element. From its definition, we see that  $Z$  is a complex number.

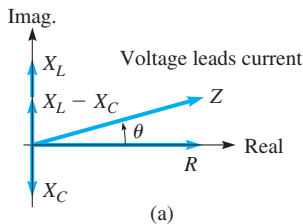
$$Z = R + j(X_L - X_C) \tag{12.20}$$

with magnitude

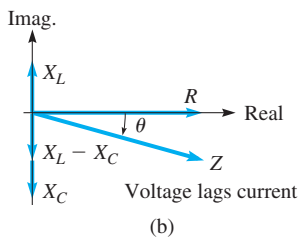
$$|Z| = \sqrt{R^2 + (X_L - X_C)^2} \tag{12.21}$$

Also, as a complex number, it makes an angle  $\theta$  with the  $x$ -axis, given by

$$\theta = \tan^{-1} \frac{X_L - X_C}{R} \tag{12.22}$$



(a)



(b)

Fig. 12.19

All these equations are based on phase relations of voltages with respect to the current. Therefore, *the angle  $\theta$  represents the phase angle between the current and the voltage*. The standard way of expressing  $\theta$  is to use a positive angle if the voltage leads the current and use a negative angle if the voltage lags the current. Using Eq. (12.18), a calculator will give the correct angle even when  $\tan \theta < 0$ .

If the voltage leads the current, then  $X_L > X_C$  as shown in Fig. 12.19(a). If the voltage lags the current, then  $X_L < X_C$  as shown in Fig. 12.19(b).

In the examples and exercises of this section, the commonly used units and symbols for them are used. For a summary of these units and symbols, including prefixes, see Section 1.3.

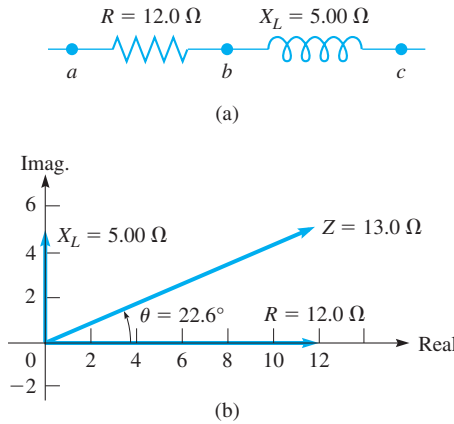


Fig. 12.20

**EXAMPLE 1 Finding the impedance and the voltage**

In the series circuit shown in Fig. 12.20(a),  $R = 12.0 \Omega$  and  $X_L = 5.00 \Omega$ . A current of 2.00 A is in the circuit. Find the voltage across each element, the impedance, the voltage across the combination, and the phase angle between the current and the voltage.

The voltage across the resistor (between points  $a$  and  $b$ ) is the product of the current and the resistance ( $V = IR$ ). This means  $V_R = (2.00)(12.0) = 24.0$  V. The voltage across the inductor (between points  $b$  and  $c$ ) is the product of the current and the reactance, or  $V_L = (2.00)(5.00) = 10.0$  V.

To find the voltage across the combination, between points  $a$  and  $c$ , we must first find the magnitude of the impedance. Note that **the voltage is not the arithmetic sum of  $V_R$  and  $V_L$** , as we must account for the phase.

By Eq. (12.20), the impedance is (there is no capacitor)

$$Z = 12.0 + 5.00j$$

with magnitude

$$|Z| = \sqrt{R^2 + X_L^2} = \sqrt{(12.0)^2 + (5.00)^2} = 13.0 \Omega$$

Thus, the magnitude of the voltage across the combination of the resistor and the inductance is

$$|V_{RL}| = (2.00)(13.0) = 26.0 \text{ V}$$

The phase angle between the voltage and the current is found by Eq. (12.22). This gives

$$\theta = \tan^{-1} \frac{5.00}{12.0} = 22.6^\circ$$

The voltage *leads* the current by  $22.6^\circ$ , and this is shown in Fig. 12.20(b). ■

**EXAMPLE 2 Finding the impedance and the phase angle**

For a circuit in which  $R = 8.00 \Omega$ ,  $X_L = 7.00 \Omega$ , and  $X_C = 13.0 \Omega$ , find the impedance and the phase angle between the current and the voltage.

By the definition of impedance, Eq. (12.20), we have

$$Z = 8.00 + (7.00 - 13.0)j = 8.00 - 6.00j$$

where the magnitude of the impedance is

$$|Z| = \sqrt{(8.00)^2 + (-6.00)^2} = 10.0 \Omega$$

The phase angle is found by

$$\theta = \tan^{-1} \frac{-6.00}{8.00} = -36.9^\circ$$

The angle  $\theta = -36.9^\circ$  is given directly by the calculator, and it is the angle we want. As we noted after Eq. (12.22), we express  $\theta$  as a negative angle if the voltage lags the current, as it does in this example. See Fig. 12.21.

From the values above, we write the impedance in polar form as

$$Z = 10.0 \angle -36.9^\circ \text{ ohms.} \quad \blacksquare$$

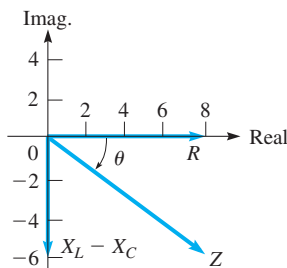


Fig. 12.21

Note that the resistance is represented in the same way as a vector along the positive  $x$ -axis. Actually, resistance is not a vector quantity but is represented in this manner in order to assign an angle as the phase of the current. The important concept in this analysis is that **the phase difference between the current and voltage is constant**, and therefore any direction may be chosen arbitrarily for one of them. Once this choice is made, other phase angles are measured with respect to this direction. A common choice, as above, is to make the phase angle of the current zero. If an arbitrary angle is chosen, it is necessary to treat the current, voltage, and impedance as complex numbers.

**EXAMPLE 3** Finding the voltage

In a particular circuit, the current ( $I$ ) is  $2.00 - 3.00j$  A and the impedance ( $Z$ ) is  $6.00 + 2.00j$  ohms. The voltage across this part of the circuit is

$$\begin{aligned} V &= IZ \\ V &= (2.00 - 3.00j)(6.00 + 2.00j) \\ &= 12.0 - 14.0j - 6.00j^2 \\ &= 12.0 - 14.0j + 6.00 \\ &= 18.0 - 14.0j \text{ volts} \end{aligned}$$

The magnitude of the voltage is

$$|V| = \sqrt{(18.0)^2 + (-14.0)^2} = 22.8 \text{ V}$$

- The ampere (A) is named for the French physicist André Ampère (1775–1836).
- The volt (V) is named for the Italian physicist Alessandro Volta (1745–1827).
- The farad (F) is named for the British physicist Michael Faraday (1791–1867).
- The henry (H) is named for the U.S. physicist Joseph Henry (1797–1878).
- The coulomb (C) is named for the French physicist Charles Coulomb (1736–1806).

Since the voltage across a resistor is in phase with the current, this voltage can be represented as having a phase difference of zero with respect to the current. Therefore, the resistance is indicated as an arrow in the positive real direction, denoting the fact that the current and the voltage are in phase. *Such a representation is called a phasor.* The arrow denoted by  $R$ , as in Fig. 12.20, is actually the phasor representing the voltage across the resistor. Remember, the positive real axis is arbitrarily chosen as the direction of the phase of the current.

To show properly that the voltage across an inductance leads the current by  $90^\circ$ , its reactance (effective resistance) is multiplied by  $j$ . We know that there is a positive  $90^\circ$  angle between a positive real number and a positive imaginary number. In the same way, by multiplying the capacitive reactance by  $-j$ , we show the  $90^\circ$  difference in phase between the voltage and the current in a capacitor, with the current leading. Therefore,  $jX_L$  represents the phasor for the voltage across an inductor and  $-jX_C$  is the phasor for the voltage across the capacitor. The phasor for the voltage across the combination of the resistance, inductance, and capacitance is  $Z$ , where the phase difference between the voltage and the current for the combination is the angle  $\theta$ .

From this, we see that *multiplying a phasor by  $j$  means to perform the operation of rotating it through  $90^\circ$ .* For this reason,  $j$  is also called the  *$j$ -operator*.

**EXAMPLE 4** Multiplication by  $j$

Multiplying a positive real number  $A$  by  $j$ , we have  $A \times j = Aj$ , which is a positive imaginary number. In the complex plane,  $Aj$  is  $90^\circ$  from  $A$ , which means that by multiplying  $A$  by  $j$  we rotated  $A$  by  $90^\circ$ . Similarly, we see that  $Aj \times j = Aj^2 = -A$ , which is a negative real number, rotated  $90^\circ$  from  $Aj$ . Therefore, successive multiplications of  $A$  by  $j$  give us

$A \times j = Aj$	positive imaginary number
$Aj \times j = Aj^2 = -A$	negative real number
$-A \times j = -Aj$	negative imaginary number
$-Aj \times j = -Aj^2 = A$	positive real number

See Fig. 12.22. (See Exercise 38 on page 349.)

An alternating current is produced by a coil of wire rotating through a magnetic field. If the angular velocity of the wire is  $\omega$ , the capacitive and inductive reactances are given by

$$X_C = \frac{1}{\omega C} \quad \text{and} \quad X_L = \omega L \tag{12.23}$$

Therefore, if  $\omega$ ,  $C$ , and  $L$  are known, the reactance of the circuit can be found.

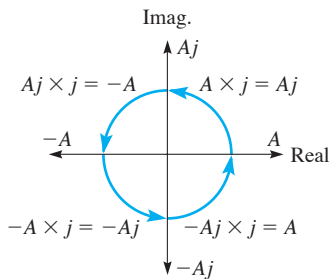


Fig. 12.22

**EXAMPLE 5** Finding the voltage—current phase difference

If  $R = 12.0 \Omega$ ,  $L = 0.300 \text{ H}$ ,  $C = 250 \mu\text{F}$ , and  $\omega = 80.0 \text{ rad/s}$ , find the impedance and the phase difference between the current and the voltage.

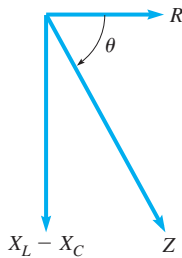


Fig. 12.23

$$X_C = \frac{1}{(80.0)(250 \times 10^{-6})} = 50.0 \Omega$$

$$X_L = (0.300)(80.0) = 24.0 \Omega$$

$$Z = 12.0 + (24.0 - 50.0)j = 12.0 - 26.0j$$

$$|Z| = \sqrt{(12.0)^2 + (-26.0)^2} = 28.6 \Omega$$

$$\theta = \tan^{-1} \frac{-26.0}{12.0} = -65.2^\circ$$

$$Z = 28.6 \angle -65.2^\circ \Omega$$

The voltage *lags* the current (see Fig. 12.23). ■

Recall from Section 10.5 that the angular velocity  $\omega$  is related to the frequency  $f$  by the relation  $\omega = 2\pi f$ . It is very common to use frequency when discussing alternating current.

An important concept in the application of this theory is **resonance**. *For resonance, the impedance of any circuit is a minimum, or the total impedance is  $R$ .* Thus,  $X_L - X_C = 0$ . Also, it can be seen that the current and the voltage are in phase under these conditions. Resonance is required for the tuning of radio and television receivers.

■ See the chapter introduction.

**EXAMPLE 6** Resonance

In the antenna circuit of a radio, the inductance is  $4.20 \text{ mH}$ , and the capacitance is variable. What range of values of capacitance is necessary for the radio to receive the AM band of radio stations, with frequencies from  $530 \text{ kHz}$  to  $1600 \text{ kHz}$ ?

For proper tuning, the circuit should be in resonance, or  $X_L = X_C$ . This means that

$$2\pi fL = \frac{1}{2\pi fC} \quad \text{or} \quad C = \frac{1}{(2\pi f)^2 L}$$

For  $f_1 = 530 \text{ kHz} = 5.30 \times 10^5 \text{ Hz}$  and  $L = 4.20 \text{ mH} = 4.20 \times 10^{-3} \text{ H}$ ,

$$C_1 = \frac{1}{(2\pi)^2 (5.30 \times 10^5)^2 (4.20 \times 10^{-3})} = 2.15 \times 10^{-11} \text{ F} = 21.5 \text{ pF}$$

and for  $f_2 = 1600 \text{ kHz} = 1.60 \times 10^6 \text{ Hz}$  and  $L = 4.20 \times 10^{-3} \text{ H}$ , we have

$$C_2 = \frac{1}{(2\pi)^2 (1.60 \times 10^6)^2 (4.20 \times 10^{-3})} = 2.36 \times 10^{-12} \text{ F} = 2.36 \text{ pF}$$

The capacitance should be capable of varying from  $2.36 \text{ pF}$  to  $21.6 \text{ pF}$ . ■

■ From Section 1.3, the following prefixes are defined as follows:

Prefix	Factor	Symbol
pico	$10^{-12}$	p
milli	$10^{-3}$	m
kilo	$10^3$	k

**EXERCISES 12.7**

In Exercises 1 and 2, perform the indicated operations if the given changes are made in the indicated examples of this section.

- In Example 1, change the value of  $X_L$  to  $16.0 \Omega$  and then solve the given problem.
- In Example 5, double the values of  $L$  and  $C$  and then solve the given problem.

In Exercises 3–6, use the circuit shown in Fig. 12.24. The current in the circuit is  $5.75 \text{ mA}$ . Determine the indicated quantities.

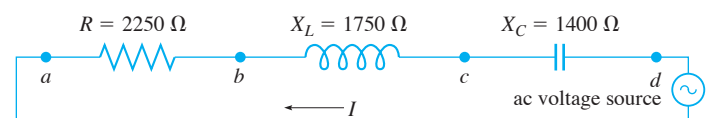


Fig. 12.24

3. The voltage across the resistor (between points  $a$  and  $b$ ).
4. The voltage across the inductor (between points  $b$  and  $c$ ).
5. (a) The magnitude of the impedance across the resistor and the inductor (between points  $a$  and  $c$ ).
- (b) The phase angle between the current and the voltage for this combination.
- (c) The voltage across this combination.
6. (a) The magnitude of the impedance across the resistor, inductor, and capacitor (between points  $a$  and  $d$ ).
- (b) The phase angle between the current and the voltage for this combination.
- (c) The voltage across this combination.

In Exercises 7–10, an ac circuit contains the given combination of circuit elements from among a resistor ( $R = 45.0 \Omega$ ), a capacitor ( $C = 86.2 \mu\text{F}$ ), and an inductor ( $L = 42.9 \text{ mH}$ ). If the frequency in the circuit is  $f = 60.0 \text{ Hz}$ , find (a) the magnitude of the impedance and (b) the phase angle between the current and the voltage.

7. The circuit has the inductor and the capacitor (an  $LC$  circuit).
8. The circuit has the resistor and the capacitor (an  $RC$  circuit).
9. The circuit has the resistor and the inductor (an  $RL$  circuit).
10. The circuit has the resistor, the inductor, and the capacitor (an  $RLC$  circuit).

In Exercises 11–24, solve the given problems.

11. Given that the current in a given circuit is  $3.90 - 6.04j \text{ mA}$  and the impedance is  $5.16 + 1.14j \text{ k}\Omega$ , find the magnitude of the voltage.
12. Given that the voltage in a given circuit is  $8.375 - 3.140j \text{ V}$  and the impedance is  $2.146 - 1.114j \Omega$ , find the magnitude of the current.
13. A resistance ( $R = 25.3 \Omega$ ) and a capacitance ( $C = 2.75 \text{ nF}$ ) are in an AM radio circuit. If  $f = 1200 \text{ kHz}$ , find the impedance across the resistor and the capacitor.
14. A resistance ( $R = 64.5 \Omega$ ) and an inductance ( $L = 1.08 \text{ mH}$ ) are in a telephone circuit. If  $f = 8.53 \text{ kHz}$ , find the impedance across the resistor and inductor.
15. The reactance of an inductor is  $1200 \Omega$  for  $f = 280 \text{ Hz}$ . What is the inductance?
16. A resistor, an inductor, and a capacitor are connected in series across an ac voltage source. A voltmeter measures  $12.0 \text{ V}$ ,  $15.5 \text{ V}$ , and  $10.5 \text{ V}$ , respectively, when placed across each element separately. What is the voltage of the source?
17. An inductance of  $12.5 \mu\text{H}$  and a capacitance of  $47.0 \text{ nF}$  are in series in an amplifier circuit. Find the frequency for resonance.
18. A capacitance ( $C = 95.2 \text{ nF}$ ) and an inductance are in series in the circuit of a receiver for navigation signals. Find the inductance if the frequency for resonance is  $50.0 \text{ kHz}$ .
19. In Example 6, what should be the capacitance in order to receive a  $680\text{-kHz}$  radio signal?
20. A  $220\text{-V}$  source with  $f = 60.0 \text{ Hz}$  is connected in series to an inductance ( $L = 2.05 \text{ H}$ ) and a resistance  $R$  in an electric-motor circuit. Find  $R$  if the current is  $0.250 \text{ A}$ .
21. The power  $P$  (in  $\text{W}$ ) supplied to a series combination of elements in an ac circuit is  $P = VI \cos \theta$ , where  $V$  is the effective voltage,  $I$  is the effective current, and  $\theta$  is the phase angle between the current and voltage. If  $V = 225 \text{ mV}$  across the resistor, capacitor, and inductor combination in Exercise 10, determine the power supplied to these elements.
22. For two impedances  $Z_1$  and  $Z_2$  in parallel, the reciprocal of the combined impedance  $Z_C$  is the sum of the reciprocals of  $Z_1$  and  $Z_2$ . Find the combined impedance for the parallel circuit elements in Fig. 12.25 if the current in the circuit has a frequency of  $60.0 \text{ Hz}$ .

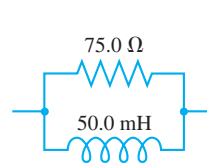


Fig. 12.25

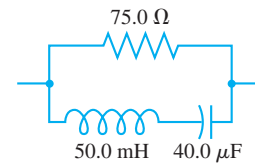


Fig. 12.26

23. Find the combined impedance of the circuit elements in Fig. 12.26. The frequency of the current in the circuit is  $60.0 \text{ Hz}$ . See Exercise 22.
24. (a) If the complex number  $j$ , in polar form, is multiplied by itself, what is the resulting number in polar and rectangular forms? (b) In the complex plane, where is the resulting complex number in relation to  $j$ ?

## CHAPTER 12 EQUATIONS

### Chapter Equations for Complex Numbers

#### Imaginary unit

$$j = \sqrt{-1} \quad \text{and} \quad j^2 = -1 \quad (12.1)$$

$$\sqrt{-a} = j\sqrt{a} \quad (a > 0) \quad (12.2)$$

#### Basic operations

$$(a + bj) + (c + dj) = (a + c) + (b + d)j \quad (12.3)$$

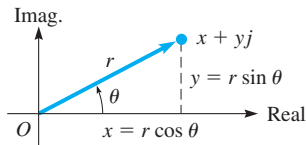
$$(a + bj) - (c + dj) = (a - c) + (b - d)j \quad (12.4)$$

$$(a + bj)(c + dj) = (ac - bd) + (ad + bc)j \quad (12.5)$$

$$\frac{a + bj}{c + dj} = \frac{(a + bj)(c - dj)}{(c + dj)(c - dj)} = \frac{(ac + bd) + (bc - ad)j}{c^2 + d^2} \quad (12.6)$$



## Complex number forms

Rectangular:  $x + yj$ Polar:  $r(\cos \theta + j \sin \theta) = r/\theta$ Exponential:  $re^{j\theta}$ 

$$x = r \cos \theta \quad y = r \sin \theta \quad (12.7)$$

$$r^2 = x^2 + y^2 \quad \tan \theta = \frac{y}{x} \quad (12.8)$$

$$x + yj = r(\cos \theta + j \sin \theta) = r/\theta = re^{j\theta} \quad (12.12)$$

## Product in polar form

$$r_1(\cos \theta_1 + j \sin \theta_1)r_2(\cos \theta_2 + j \sin \theta_2) = r_1r_2[\cos(\theta_1 + \theta_2) + j \sin(\theta_1 + \theta_2)]$$

$$(r_1/\theta_1)(r_2/\theta_2) = r_1r_2/\theta_1 + \theta_2 \quad (12.13)$$

## Quotient in polar form

$$\frac{r_1(\cos \theta_1 + j \sin \theta_1)}{r_2(\cos \theta_2 + j \sin \theta_2)} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + j \sin(\theta_1 - \theta_2)] \quad (12.15)$$

$$\frac{r_1/\theta_1}{r_2/\theta_2} = \frac{r_1/\theta_1 - \theta_2}{r_2}$$

## DeMoivre's theorem

$$[r(\cos \theta + j \sin \theta)]^n = r^n(\cos n\theta + j \sin n\theta) \quad (12.17)$$

$$(r/\theta)^n = r^n/n\theta$$

## Chapter Equations for Alternating-Current Circuits

## Voltage, current, reactance

$$V_R = IR \quad V_C = IX_C \quad V_L = IX_L \quad (12.18)$$

## Impedance

$$V_{RLC} = IZ \quad (12.19)$$

$$Z = R + j(X_L - X_C) \quad (12.20)$$

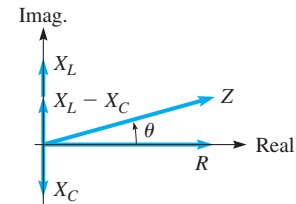
$$|Z| = \sqrt{R^2 + (X_L - X_C)^2} \quad (12.21)$$

## Phase angle

$$\theta = \tan^{-1} \frac{X_L - X_C}{R} \quad (12.22)$$

## Capacitive reactance and inductive reactance

$$X_C = \frac{1}{\omega C} \quad \text{and} \quad X_L = \omega L \quad (12.23)$$



## CHAPTER 12 REVIEW EXERCISES

In Exercises 1–16, perform the indicated operations, expressing all answers in simplest rectangular form.

1.  $(6 - 2j) + (4 + j)$

2.  $(12 + 7j) + (-8 + 6j)$

3.  $(18 - 3j) - (12 - 5j)$

4.  $(-4 - 2j) - \sqrt{-49}$

5.  $(2 + j)(4 - j)$

6.  $(-5 + 3j)(8 - 4j)$

7.  $(2j^5)(6 - 3j)(4 + 3j)$

8.  $j(3 - 2j) - (j^3)(5 + j)$

9.  $\frac{3}{7 - 6j}$

10.  $\frac{24j}{2 + 9j}$

11.  $\frac{6 - \sqrt{-16}}{\sqrt{-4}}$

12.  $\frac{3 + \sqrt{-4}}{4 - j}$

13.  $\frac{5j - (3 - j)}{4 - 2j}$

14.  $\frac{2 + (j - 6)}{1 - 2j}$

15.  $\frac{j(7 - 3j)}{2 - j^7}$

16.  $\frac{(2 - j)(3 + 2j)}{4 + 3j^3}$

In Exercises 17–20, find the values of  $x$  and  $y$  for which the equations are valid.

17.  $3x - 2j = yj - 9$

18.  $2xj - 2y = (y + 3)j - 3$

19.  $(3 - 2j)(x + yj) = 4 + j^9$

20.  $(x + yj)(7j - 4) = j(x - 5)$

In Exercises 21–24, perform the indicated operations graphically. Check them algebraically.

21.  $(-1 + 5j) + (4 + 6j)$

22.  $(7 - 2j) + (-5 + 4j)$

23.  $(9 + 2j) - (5 - 6j)$

24.  $(8 + 4j) - (11 - 3j)$

In Exercises 25–32, give the polar and exponential forms of each of the complex numbers.

25.  $1 - j$

26.  $4 + 3j$

27.  $-22 - 77j$

28.  $60 - 20j$



29.  $1.07 + 4.55j$                       30.  $-327 + 158j$   
 31. 5000                                  32.  $-4j^5$

In Exercises 33–44, give the rectangular form of each number.

33.  $2(\cos 225^\circ + j \sin 225^\circ)$       34.  $48(\cos 60^\circ + j \sin 60^\circ)$   
 35.  $5.011(\cos 123.82^\circ + j \sin 123.82^\circ)$   
 36.  $2.417(\cos 656.26^\circ + j \sin 656.26^\circ)$   
 37.  $0.62/-72^\circ$                       38.  $20/160^\circ$   
 39.  $27.08/346.27^\circ$                       40.  $1.689/194.36^\circ$   
 41.  $2.00e^{0.25j}$                           42.  $e^{-3.62j}$   
 43.  $(35.37e^{1.096j})^2$                       44.  $(13.6e^{2.158j})(3.27e^{3.888j})$

In Exercises 45–60, perform the indicated operations. Leave the result in polar form.

45.  $[3(\cos 32^\circ + j \sin 32^\circ)][5(\cos 52^\circ + j \sin 52^\circ)]$   
 46.  $[2.5(\cos 162^\circ + j \sin 162^\circ)][8(\cos 115^\circ + j \sin 115^\circ)]$   
 47.  $(40/18^\circ)(0.5/245^\circ)$   
 48.  $(0.1254/172.38^\circ)(27.17/204.34^\circ)$   
 49.  $\frac{24(\cos 165^\circ + j \sin 165^\circ)}{3(\cos 106^\circ + j \sin 106^\circ)}$       50.  $\frac{18(\cos 403^\circ + j \sin 403^\circ)}{4(\cos 192^\circ + j \sin 192^\circ)}$   
 51.  $\frac{245.6/326.44^\circ}{17.19/192.83^\circ}$                       52.  $\frac{4/206^\circ}{100/-320^\circ}$   
 53.  $0.983/47.2^\circ + 0.366/95.1^\circ$   
 54.  $17.8/110.4^\circ - 14.9/226.3^\circ$   
 55.  $7644/294.36^\circ - 6871/17.86^\circ$   
 56.  $4.944/327.49^\circ + 8.009/7.37^\circ$   
 57.  $[2(\cos 16^\circ + j \sin 16^\circ)]^{10}$       58.  $[3(\cos 36^\circ + j \sin 36^\circ)]^6$   
 59.  $(7/110.5^\circ)^3$                           60.  $(536/220.3^\circ)^4$

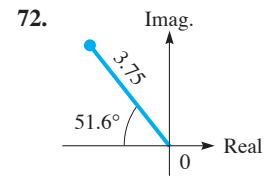
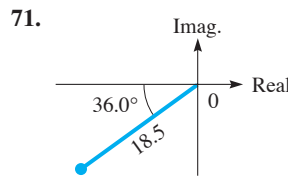
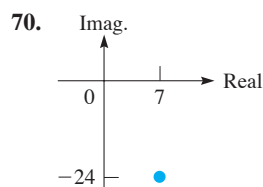
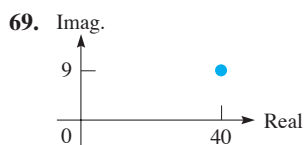
In Exercises 61–64, change each number to polar form and then perform the indicated operations. Express the final result in rectangular and polar forms. Check by performing the same operation in rectangular form.

61.  $(1 - j)^{10}$                               62.  $(\sqrt{3} + j)^8(1 + j)^5$   
 63.  $\frac{(5 + 5j)^4}{(-1 - j)^6}$                                   64.  $(\sqrt{3} - j)^{-8}$

In Exercises 65–68, find all the roots of the given equations.

65.  $x^3 + 8 = 0$                               66.  $x^3 - 1 = 0$   
 67.  $x^4 + j = 0$                               68.  $x^5 - 32j = 0$

In Exercises 69–72, determine the rectangular form and the polar form of the complex number for which the graphical representation is shown in the given figure.



In Exercises 73–84, solve the given problems.

73. Evaluate  $x^2 - 2x + 4$  for  $x = 5 - 2j$ .  
 74. Evaluate  $2x^2 + 5x - 7$  for  $x = -8 + 7j$ .  
 75. Using the quadratic formula, solve for  $x$ :  $x^2 + 3jx - 2 = 0$ .  
 76. Using the quadratic formula, solve for  $x$ :  $jx^2 - 2x - 3j = 0$ .  
 77. Find a quadratic equation with roots  $2 + j$  and  $2 - j$ .  
 78. Find a quadratic equation with roots  $-3 + 4j$  and  $-3 - 4j$ .  
 79. Are  $1 - j$  and  $-1 - j$  solutions to the equation  $x^2 - 2x + 2 = 0$ ?  
 80. Show that  $\frac{1}{2}(1 + j\sqrt{3})$  is the reciprocal of its conjugate.  
 81. Solve for  $x$ :  $(1 + jx)^2 = 1 + j - x^2$   
 82. What is the argument for any negative imaginary number?  
 83. If  $f(x) = 2x - (x - 1)^{-1}$ , find  $f(1 + 2j)$ .  
 84. If  $f(x) = x^{-2} + 3x^{-1}$ , find  $f(4 + j)$ .

In Exercises 85–96, find the required quantities.

85. A 60-V ac voltage source is connected in series across a resistor, an inductor, and a capacitor. The voltage across the inductor is 60 V, and the voltage across the capacitor is 60 V. What is the voltage across the resistor?  
 86. In a series ac circuit with a resistor, an inductor, and a capacitor,  $R = 6.50 \Omega$ ,  $X_C = 3.74 \Omega$ , and  $Z = 7.50 \Omega$ . Find  $X_L$ .  
 87. In a series ac circuit with a resistor, an inductor, and a capacitor,  $R = 6250 \Omega$ ,  $Z = 6720 \Omega$ , and  $X_L = 1320 \Omega$ . Find the phase angle  $\theta$ .  
 88. A coil of wire rotates at 120.0 r/s. If the coil generates a current in a circuit containing a resistance of  $12.07 \Omega$ , an inductance of  $0.1405 \text{ H}$ , and an impedance of  $22.35 \Omega$ , what must be the value of a capacitor (in F) in the circuit?  
 89. What is the frequency  $f$  for resonance in a circuit for which  $L = 2.65 \text{ H}$  and  $C = 18.3 \mu\text{F}$ ?  
 90. The displacement of an electromagnetic wave is given by  $d = A(\cos \omega t + j \sin \omega t) + B(\cos \omega t - j \sin \omega t)$ . Find the expressions for the magnitude and phase angle of  $d$ .  
 91. Two cables lift a crate. The tensions in the cables can be represented by  $2100 - 1200j \text{ N}$  and  $1200 + 5600j \text{ N}$ . Express the resultant tension in polar form.  
 92. A boat is headed across a river with a velocity (relative to the water) that can be represented as  $6.5 + 1.7j \text{ km/h}$ . The velocity of the river current can be represented as  $-1.1 - 4.3j \text{ km/h}$ . Express the resultant velocity of the boat in polar form.

93. In the study of shearing effects in the spinal column, the expression  $\frac{1}{\mu + j\omega n}$  is found. Express this in rectangular form.

94. In the theory of light reflection on metals, the expression

$$\frac{\mu(1 - kj) - 1}{\mu(1 - kj) + 1}$$

is encountered. Simplify this expression.

95. Show that  $e^{j\pi} = -1$ .

96. Show that  $(e^{j\pi})^{1/2} = j$ .

### Writing Exercise

97. A computer programmer is writing a program to determine the  $n$ th roots of a real number. Part of the program is to show the number of real roots and the number of pure imaginary roots. Write one or two paragraphs explaining how these numbers of roots can be determined without actually finding the roots.

## CHAPTER 12 PRACTICE TEST

1. Add, expressing the resultant in rectangular form:  
 $(3 - \sqrt{-4}) + (5\sqrt{-9} - 1)$ .

2. Multiply, expressing the resultant in polar form:  
 $(2\angle 130^\circ)(3\angle 45^\circ)$ .

3. Express  $2 - 7j$  in polar form.

4. Express in terms of  $j$ : (a)  $-\sqrt{-64}$  (b)  $-j^{15}$ .

5. Add graphically:  $(4 - 3j) + (-1 + 4j)$ .

6. Simplify, expressing the result in rectangular form:  $\frac{2 - 4j}{5 + 3j}$ .

7. Express  $2.56(\cos 125.2^\circ + j \sin 125.2^\circ)$  in exponential form.

8. For an ac circuit in which  $R = 3.50 \Omega$ ,  $X_L = 6.20 \Omega$ , and  $X_C = 7.35 \Omega$ , find the impedance and the phase angle between the current and the voltage.

9. Express  $3.47 - 2.81j$  in exponential form.

10. Find the values of  $x$  and  $y$ :  $x + 2j - y = yj - 3xj$ .

11. What is the capacitance of the circuit in a radio that has an inductance of 8.75 mH if it is to receive a station with frequency 600 kHz?

12. Find the cube roots of  $j$ .

# 13

## LEARNING OUTCOMES

After completion of this chapter, the student should be able to:

- Evaluate and graph an exponential function
- Recognize the connection between properties of logarithms and laws of exponents
- Change equations from exponential form to logarithmic form and vice versa
- Evaluate and graph a logarithmic function
- Identify the exponential and logarithmic functions as inverse functions
- Solve logarithmic and exponential equations
- Change a logarithm in one base to a logarithm in another base
- Solve application problems involving logarithmic and exponential functions
- Graph functions on logarithmic or semilogarithmic paper

► Because the human ear is sensitive to a broad range of sound intensities, a logarithmic intensity scale is used. In Section 13.6 we show how logarithms are used to compare the sound intensity of a jackhammer to the sound intensity of a city street.

# Exponential and Logarithmic Functions

By the early 1600s, astronomy had progressed to the point of finding accurate information about the motion of the heavenly bodies. Also, navigation had led to a more systematic exploration of earth. In making the accurate measurements needed in astronomy and navigation, many lengthy calculations involving large numbers had to be performed, and all such calculations had to be done by hand.

Noting that astronomers' calculations usually involved sines of angles, John Napier (1550–1617), a Scottish mathematician, constructed a table of values that allowed multiplication of these sines by addition of values from the table. These tables of *logarithms* first appeared in 1614. Therefore, logarithms were essentially invented to turn the more difficult operation of multiplication into the easier process of addition.

Napier's logarithms did not really make use of a number base. It was the English mathematician Henry Briggs (1561–1630) who, recognizing the usefulness of logarithms, suggested to Napier that logarithms should use the base 10 in order to make the calculations even easier. Napier agreed, and Briggs spent many years developing the first extensive table of base 10 logarithms, published in 1624.

Mathematicians and scientists enthusiastically received logarithms as a long-needed tool for lengthy calculations. The great French mathematician Pierre Laplace (1749–1827) was hardly exaggerating when he said that logarithms “by shortening the labors, doubled the life of the astronomer.” Logarithms were commonly used for calculations until the 1970s, when the scientific calculator came into use.

As we will learn in this chapter, logarithms are still used in some calculations and in some graphical displays, but they are of great importance in many scientific and technical applications and in advanced mathematics. The units used to measure the intensity of sound, the intensity of earthquakes, the distinction of a base from an acid, and the power gains and losses in electrical transmission lines are all measured in logarithmic units.

In this chapter we will also study the closely related exponential functions, which are crucial in many areas of science, mathematics, and statistics. The applications we discuss here include compound interest and population growth, as well as applications in electronics, mechanical systems, thermodynamics, and nuclear physics.



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## 13.1 Exponential Functions

The Exponential Function • Graphing Exponential Functions • Features of Exponential Functions

In Chapter 11, we showed that any rational number can be used as an exponent. Now letting *the exponent be a variable*, we define the **exponential function** as

$$y = b^x \quad (13.1)$$

where  $b > 0$ ,  $b \neq 1$ , and  $x$  is any real number. *The number  $b$  is called the base.* More generally, for  $b > 0$ ,  $b \neq 1$ , and  $a \neq 0$ , we also refer to a function of the form  $y = ab^x$  as an **exponential function**.

We noted in Chapter 3 that the functions we consider must always have real values. That is why we impose the restriction  $b > 0$ . If  $b$  were negative and  $x$  were a fractional exponent with an even-number denominator,  $y$  would be imaginary (as in  $(-3)^{1/2}$ ). Also, we restrict  $b \neq 1$  in order to exclude the constant function  $y = 1^x$  (since 1 to any real power is 1).

### EXAMPLE 1 Exponential functions

From the definition,  $y = 3^x$  is an exponential function, but  $y = (-3)^x$  is not because the base is negative. However,  $y = -3^x$  is an exponential function, because it is  $-1$  times  $3^x$ , and any real-number multiple of an exponential function is also an exponential function.

Also,  $y = (\sqrt{3})^x$  is an exponential function since it can be written as  $y = 3^{x/2}$ . As long as  $x$  is a real number, so is  $x/2$ . Therefore, the exponent of 3 is real.

The function  $y = 3^{-x}$  is an exponential function. If  $x$  is real, so is  $-x$ .

Other exponential functions are  $y = -2(8^{-0.55x})$  and  $y = 35(1.0001)^x$ . ■

### EXAMPLE 2 Evaluating an exponential function

Evaluate the function  $y = -2(4^x)$  for the given values of  $x$ .

- (a) If  $x = 2$ ,  $y = -2(4^2) = -2(16) = -32$ .  
 (b) If  $x = -2$ ,  $y = -2(4^{-2}) = -2/16 = -1/8$ .  
 (c) If  $x = 3/2$ ,  $y = -2(4^{3/2}) = -2(8) = -16$ .  
 (d) If  $x = \sqrt{2}$ ,  $y = -2(4^{\sqrt{2}}) = -14.206$ . ■

### GRAPHING EXPONENTIAL FUNCTIONS

We now show some representative graphs of the exponential function.

### EXAMPLE 3 Graphing an exponential function

Plot the graph of  $y = 2^x$ .

For this function, we have the values in the following table:

$x$	-3	-2	-1	0	1	2	3
$y$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

$\uparrow$   $2^{-3} = \frac{1}{8}$        $\uparrow$   $2^0 = 1$        $\uparrow$   $2^3 = 8$

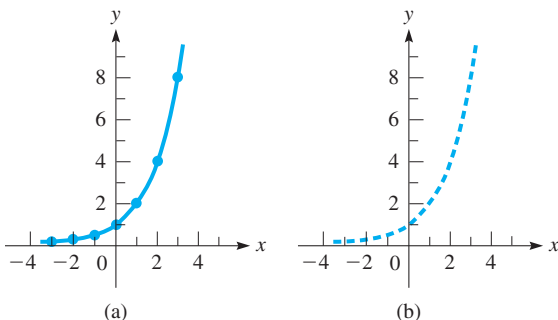


Fig. 13.1

The curve is shown in Fig. 13.1(a). The values from the table can be seen as enlarged points on the plot, all corresponding to integer exponents. Using rational exponents (introduced in Chapter 11) would fill in many points between those for integers, but all the points for irrational exponents would be missing and the curve would be dotted (see Fig. 13.1(b)). Using all real numbers for exponents, including the irrational numbers, results in the smooth curve shown in Fig. 13.1(a).

## Practice Exercises

Evaluate  $y = 16^x$  for:

- $x = 3/2$
- $x = -0.5$

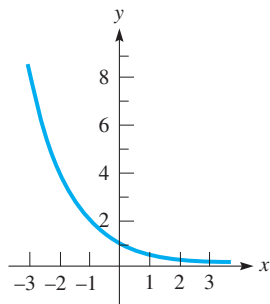


Fig. 13.2

We see that the  $x$ -axis is an *asymptote* of the curve. As noted on page 100, an asymptote is a line that the curve gets closer and closer to as values of  $x$  increase (or decrease) without bound, although the curve never actually touches the asymptote. We can also see that as we move from left to right, the graph rises, which means that the function increases as  $x$  increases. ■

## EXAMPLE 4 Graphing an exponential function

Plot the graph of  $y = \left(\frac{1}{2}\right)^x$ .For this function,  $(1/2)^x = 1/2^x = 2^{-x}$ . We have the values in the following table:

$x$	-3	-2	-1	0	1	2	3
$y$	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

The graph is shown in Fig. 13.2. Note that  $y$  becomes very large for negative values of  $x$ . Moreover, notice that as we move from left to right, the graph falls, which means the function decreases as  $x$  increases. ■

Any exponential curve where  $b > 1$  will be similar in shape to that shown in Fig. 13.1, and if  $b < 1$ , it will be similar to the curve in Fig. 13.2. From these examples, we can see that exponential functions have the following basic features.

Basic Features of Exponential Functions of the Form  $y = b^x$ 

- The domain is all values of  $x$ ; the range is  $y > 0$ .
- The  $x$ -axis is an asymptote of the graph.
- As  $x$  increases, the function increases if  $b > 1$  and decreases if  $b < 1$ .

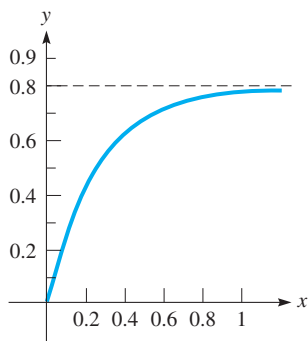


Fig. 13.3

Exponential functions are important in many applications. We now illustrate an application in the next example, and others are shown in the exercises. ■

## EXAMPLE 5 Exponential function—application

In an electric circuit in which there is a battery, an inductor, and a resistor, the current  $i$  (in A) as a function of the time  $t$  (in s) is  $i = 0.8(1 - e^{-4t})$ . Graph this function. Here,  $e$  is the same number we introduced on page 352 and is equal to approximately 2.718.

Here,  $e^{-4t} = 1$  for  $t = 0$ , and this means  $i = 0$  for  $t = 0$ . Also,  $e^{-4t}$  becomes very small in a very short time, and this means  $i$  cannot be greater than 0.8 A, and that  $i = 0.8$  A is an asymptote. The curve is shown in Fig. 13.3. ■

## EXERCISES 13.1

In Exercises 1 and 2, perform the indicated operations if the given changes are made in the indicated examples of this section.

- In Example 2(c), change the sign of  $x$  and then evaluate.
- In Example 3, change the sign of the exponent and then plot the graph.

In Exercises 3–6, determine if the given functions are exponential functions.

- (a)  $y = 5^x$  (b)  $y = 5^{-x}$
- (a)  $y = -7^x$  (b)  $y = (-7)^{-x}$
- (a)  $y = -7(-5)^{-x}$  (b)  $y = -7(5^{-x})$
- (a)  $y = (\sqrt{5})^{-x}$  (b)  $y = -(\sqrt{-5})^x$

In Exercises 7–12, evaluate the exponential function  $y = 9^x$  for the given values of  $x$ .

7.  $x = 0.5$       8.  $x = 4$       9.  $x = -2$   
 10.  $x = -0.5$       11.  $x = -3/2$       12.  $x = 5/2$

In Exercises 13–18, plot the graphs of the given functions.

13.  $y = 4^x$       14.  $y = 0.25^x$       15.  $y = 0.2(10^{-x})$   
 16.  $y = -5(1.6^{-x})$       17.  $y = 0.5\pi^x$       18.  $y = 2e^x$

In Exercises 19–24, display the graphs of the given functions on a graphing calculator.

19.  $y = 0.3(2.55)^x$       20.  $y = -1.5(4.15)^x$   
 21.  $y = 0.1(0.25^{2x})$       22.  $y = 0.4(0.95)^x$   
 23.  $i = 1.2(2 + 6^{-t})$       24.  $y = 0.5e^{-x}$

In Exercises 25–38, solve the given problems. Where necessary, round results to 3 significant digits.

25. Find the base  $b$  of the function  $y = b^x$  if its graph passes through the point  $(3, 64)$ .  
 26. Find the base  $b$  of the function  $y = b^x$  if its graph passes through the point  $(-2, 64)$ .  
 27. For the function  $f(x) = b^x$ , show that  $f(c + d) = f(c) \cdot f(d)$ .  
 28. For the function  $f(x) = b^x$ , show that  $f(c - d) = f(c)/f(d)$ .  
 29. Use a graphing calculator to graph the function  $y = 2^{|x|}$ .  
 30. To show the *damping effect* of an exponential function, use a graphing calculator to display the graph of  $y = (x^3)(2^{-x})$ . Be sure to use appropriate window settings.  
 31. Use a graphing calculator to find the value(s) for which  $x^2 = 2^x$ .  
 32. Use a graphing calculator to find the integral values of  $x$  for which  $x^3 > 3^x$ .  
 33. The value  $V$  of a bank account in which \$250 is invested at 5.00% interest, compounded annually, is  $V = 250(1.0500)^t$ , where  $t$  is the time in years. Find the value of the account after 4 years.

34. The intensity  $I$  of an earthquake is given by  $I = I_0(10)^R$ , where  $I_0$  is a minimum intensity reference value, and  $R$  is the Richter scale magnitude of the earthquake. Evaluate  $I$  in terms of  $I_0$  if  $R = 5.5$ .

35. The electric current  $i$  (in mA) in the circuit shown in Fig. 13.4 is  $i = 2.5(1 - e^{-0.10t})$ , where  $t$  is the time (in s). Evaluate  $i$  for  $t = 5.0$  ms.

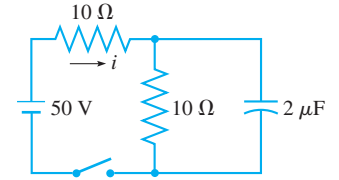


Fig. 13.4

36. A projection of the annual growth rate  $p$  (in %) of the number of users of the Internet is  $p = 8.5(1.2^{-t} + 1)$ , where  $t$  is the number of years after 2005. Display the graph of this function on a graphing calculator from 2005 to 2015.  
 37. The flash unit on a camera operates by releasing the stored charge on a capacitor. For a particular unit, the charge  $q$  (in  $\mu\text{C}$ ) as a function of the time  $t$  (in s) is  $q = 100e^{-10t}$ . Display the graph on a graphing calculator.  
 38. The height  $y$  (in m) of the Gateway Arch in St. Louis (see Fig. 13.5) is given by  $y = 230.9 - 19.5(e^{x/38.9} + e^{-x/38.9})$ , where  $x$  is the distance (in m) from the point on the ground level directly below the top. Display the graph on a graphing calculator.



Fig. 13.5

Answers to Practice Exercises

1. 64      2. 1/4

## 13.2 Logarithmic Functions

Exponential Form • Logarithmic Form • The Logarithmic Function • Graphing Logarithmic Functions • Features of Logarithmic Functions • Inverse Functions

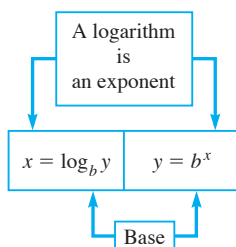


Fig. 13.6

For many uses in mathematics and for many applications, it is necessary to express the exponent  $x$  in the exponential function  $y = b^x$  in terms of  $y$  and the base  $b$ . This is done by defining a *logarithm*. Therefore, if  $y = b^x$ , the exponent  $x$  is the **logarithm** of the number  $y$  to the base  $b$ . We write this as

$$\text{If } y = b^x, \text{ then } x = \log_b y. \tag{13.2}$$

This means that  $x$  is the power to which the base  $b$  must be raised in order to equal the number  $y$ . That is,  $x$  is a logarithm, and **a logarithm is an exponent**. As with the exponential function, for the equation  $x = \log_b y$ ,  $x$  may be any real number,  $b$  is a positive number other than 1, and  $y$  is a positive real number. In Eq. (13.2),

$$y = b^x \text{ is the exponential form, and } x = \log_b y \text{ is the logarithmic form.}$$

See Fig. 13.6.



**LEARNING TIP**

Remember, *the exponent may be negative*. The base must be positive.

**Practice Exercises**

- Change  $125^{2/3} = 25$  to logarithmic form.
- Change  $\log_3(1/3) = -1$  to exponential form.

**LEARNING TIP**

In order to change a function of the form  $y = ab^x$  into logarithmic form, we must first write it as  $y/a = b^x$ . **The coefficient of  $b^x$  must be equal to 1**, which is the form of Eq. (13.1). In the same way, the coefficient of  $\log_b y$  must be 1 in order to change it into exponential form.

**EXAMPLE 1 Exponential form and logarithmic form**

The equation  $y = 2^x$  is written as  $x = \log_2 y$  when written in logarithmic form. When we choose values of  $y$  to find corresponding values of  $x$  from this equation, we ask ourselves “2 raised to what power  $x$  gives  $y$ ?”

This means that if  $y = 8$ , we ask “what power of 2 gives us 8?” Then knowing that  $2^3 = 8$ , we know that  $x = 3$ . Therefore,  $3 = \log_2 8$ . ■

**EXAMPLE 2 Changing to logarithmic form**

- (a)  $3^2 = 9$  in logarithmic form is  $2 = \log_3 9$ .  
 (b)  $4^{-1} = 1/4$  in logarithmic form is  $-1 = \log_4(1/4)$ . ■

**EXAMPLE 3 Changing between forms**

- (a)  $(64)^{1/3} = 4$  in logarithmic form is  $\frac{1}{3} = \log_{64} 4$ .  
 (b)  $(32)^{3/5} = 8$  in logarithmic form is  $\frac{3}{5} = \log_{32} 8$ .  
 (c)  $\log_2 32 = 5$  in exponential form is  $32 = 2^5$ .  
 (d)  $\log_6\left(\frac{1}{36}\right) = -2$  in exponential form is  $\frac{1}{36} = 6^{-2}$ . ■

**EXAMPLE 4 Solving for unknowns by changing form**

- (a) Find  $b$ , given that  $-4 = \log_b\left(\frac{1}{81}\right)$ .

Writing this in exponential form, we have  $\frac{1}{81} = b^{-4}$ . Thus,  $\frac{1}{81} = \frac{1}{b^4}$  or  $\frac{1}{3^4} = \frac{1}{b^4}$ . Therefore,  $b = 3$ .

- (b) Given  $\log_4 y = 1/2$ , in exponential form it becomes  $y = 4^{1/2}$ , or  $y = 2$ . ■

We see that the exponential form is very useful for determining values written in logarithmic form. For this reason, it is important that you learn to transform readily from one form to the other.

**EXAMPLE 5 Solving for unknowns—application**

The power supply  $P$  (in W) of a certain satellite is given by  $P = 75e^{-0.005t}$ , where  $t$  is the time (in days) after launch. By writing this equation in logarithmic form, solve for  $t$ .

In order to have the equation in the exponential form of Eq. (13.1), we must have only  $e^{-0.005t}$  on the right. Therefore, by dividing by 75, we have

$$\frac{P}{75} = e^{-0.005t}$$

Writing this in logarithmic form, we have  $\log_e(P/75) = -0.005t$ , or

$$t = \frac{\log_e\left(\frac{P}{75}\right)}{-0.005} = -200 \log_e\left(\frac{P}{75}\right) \quad \frac{1}{-0.005} = -200$$

We recall from Section 12.5 that  $e$  is the special irrational number equal to about 2.718. It is important as a base of logarithms, and this will be discussed in Section 13.5. ■

**The Logarithmic Function**

When we are working with functions, we must keep in mind that a function is defined by the operation being performed on the independent variable, and not by the letter chosen to represent it. However, for consistency, it is standard practice to let  $y$



**LEARNING TIP**

Eqs. (13.2) and (13.3) do not represent different *functions*, due to the difference in location of the variables, since they represent the *same operation* on the independent variable that appears in each. However, Eq. (13.3) expresses the function with the standard dependent and independent variables.

**Practice Exercise**

- For the function in Example 6, evaluate  $y$  for  $x = 8$ .

represent the dependent variable and  $x$  represent the independent variable. Therefore, *the logarithmic function is*

$$y = \log_b x \tag{13.3}$$

As with the exponential function,  $b > 0$  and  $b \neq 1$ .

**EXAMPLE 6 Evaluating a logarithmic function**

For the logarithmic function  $y = \log_2 x$ , we have the standard independent variable  $x$  and the standard dependent variable  $y$ .

If  $x = 16$ ,  $y = \log_2 16$ , which means that  $y = 4$ , since  $2^4 = 16$ .

If  $x = \frac{1}{16}$ ,  $y = \log_2 \left(\frac{1}{16}\right)$ , which means that  $y = -4$ , since  $2^{-4} = \frac{1}{16}$ . ■

**GRAPHING LOGARITHMIC FUNCTIONS**

We now show some representative graphs for the logarithmic function. From these graphs, we can see the basic properties of the logarithmic function.

**EXAMPLE 7 Graphing a logarithmic function**

Plot the graph of  $y = \log_2 x$ .

We can find the points for this graph more easily if we first put the equation in exponential form:  $x = 2^y$ . By assuming values for  $y$ , we can find the corresponding values for  $x$ .

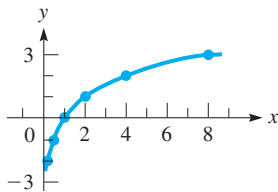


Fig. 13.7

	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
$x$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
$y$	-2	-1	0	1	2	3

Using these values, we construct the graph seen in Fig. 13.7. ■

From the graph in Example 7, we can see that logarithmic functions have the following features. We consider only the features for  $b > 1$ , for these are the bases of importance.

**Basic Features of Logarithmic Functions of the Form  $y = \log_b x$  ( $b > 1$ )**

- The domain is  $x > 0$ ; the range is all values of  $y$ .
- The negative  $y$ -axis is an asymptote of the graph.
- If  $0 < x < 1$ ,  $\log_b x < 0$ ; if  $x = 1$ ,  $\log_b x = 0$ ; if  $x > 1$ ,  $\log_b x > 0$ .
- If  $x > 1$ ,  $\log_b x$  increases more slowly than  $x$ .

Table 1

$x$	1	4	16	64
$\log_2 x$	0	2	4	6
$2^x$	2	16	65 536	$1.8 \times 10^{19}$

We just noted that if  $b > 1$  and  $x > 1$ ,  $\log_b x$  increases more slowly than  $x$ . It is also true that  $b^x$  increases more rapidly than  $x$ . Actually, as  $x$  becomes larger,  $\log_b x$  increases very slowly, and  $b^x$  increases very rapidly. Using  $\log_2 x$  and  $2^x$  and a calculator, we have the table of values shown to the left. This shows that we must choose values of  $x$  carefully when graphing these functions.

**INVERSE FUNCTIONS**

As we have seen, the exponential function  $y = b^x$  and the logarithmic function  $y = \log_b x$  are closely related functions. In fact, the exponential and the logarithmic functions are **inverses** of each other. We first encountered inverse functions in Chapter 4, when we discussed inverse trigonometric functions. Recall that, in general, the inverse of a

function undoes the process that the function performs. In this case, the exponential function operates on a base to raise it to a certain exponent and return a value, while the logarithmic function operates on a value and returns the exponent to which the base has to be raised to produce that number.

### Procedure for Finding the Inverse of a Function

1. Solve for the independent variable as a function of the dependent variable.
2. Interchange the variables.

### LEARNING TIP

Note that the  $x$  and  $y$  coordinates of inverse functions are interchanged. As a result, the graphs of inverse functions are mirror images of each other across the line  $y = x$ .

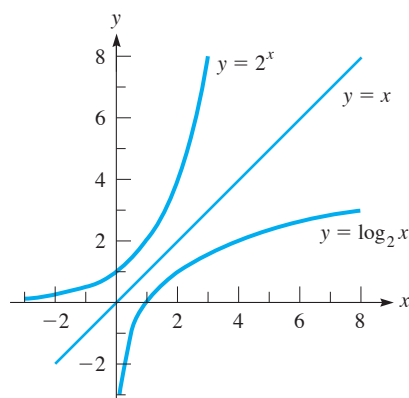


Fig. 13.8

For a function, there is exactly one value of  $y$  in the range for each value of  $x$  in the domain. This must also hold for the inverse function. Thus, for a function to have an inverse, there must be only one  $x$  for each  $y$ . This is true for  $y = b^x$  and  $y = \log_b x$ , as we have seen earlier in this section.

In the following example we use the procedure for finding an inverse to verify that an exponential function and a logarithmic function are inverses of each other. Other examples are found in the exercises.

### EXAMPLE 8 Inverse functions

The functions  $y = 2^x$  and  $y = \log_2 x$  are inverse functions. We show this by solving  $y = 2^x$  for  $x$  and then interchanging  $x$  and  $y$ .

Writing  $y = 2^x$  in logarithmic form gives us  $x = \log_2 y$ . Then interchanging  $x$  and  $y$ , we have  $y = \log_2 x$ , which is the inverse function.

Making a table of values for each function, we have

$x$	-3	-2	-1	0	1	2	3
$y$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

$x$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
$y$	-3	-2	-1	0	1	2	3

We see that the coordinates are interchanged. In Fig. 13.8, note that the graphs of these two functions reflect each other across the line  $y = x$ . ■

## EXERCISES 13.2

In Exercises 1–4, perform the indicated operations if the given changes are made in the indicated examples of this section.

1. In Example 3(b), change the exponent to  $4/5$  and then make any other necessary changes.
2. In Example 4(b), change the  $1/2$  to  $5/2$  and then make any other necessary changes.
3. In Example 6, change the logarithm base to 4 and then make any other necessary changes.
4. In Example 7, change the logarithm base to 4 and then plot the graph.

In Exercises 5–16, express the given equations in logarithmic form.

5.  $3^3 = 27$
6.  $5^2 = 25$
7.  $4^4 = 256$
8.  $2^7 = 128$
9.  $7^{-2} = \frac{1}{49}$
10.  $3^{-2} = \frac{1}{9}$

11.  $2^{-6} = \frac{1}{64}$
12.  $(12)^0 = 1$
13.  $8^{1/3} = 2$
14.  $(81)^{3/4} = 27$
15.  $(\frac{1}{4})^2 = \frac{1}{16}$
16.  $(\frac{1}{2})^{-2} = 4$

In Exercises 17–28, express the given equations in exponential form.

17.  $\log_3 81 = 4$
18.  $\log_{11} 121 = 2$
19.  $\log_9 9 = 1$
20.  $\log_{15} 1 = 0$
21.  $\log_{25} 5 = \frac{1}{2}$
22.  $\log_8 16 = \frac{4}{3}$
23.  $\log_{243} 3 = 0.2$
24.  $\log_{32}(\frac{1}{8}) = -0.6$
25.  $\log_{10} 0.1 = -1$
26.  $\log_7(\frac{1}{49}) = -2$
27.  $\log_{0.5} 16 = -4$
28.  $\log_{1/3} 3 = -1$

In Exercises 29–44, determine the value of the unknown.

29.  $\log_4 16 = x$
30.  $\log_5 125 = x$

31.  $\log_{10} 0.01 = x$       32.  $\log_{16} \left(\frac{1}{4}\right) = x$   
 33.  $\log_7 y = 3$       34.  $\log_8 (N + 1) = 3$   
 35.  $\log_8 (A - 2) = -\frac{2}{3}$       36.  $\log_7 y = -2$   
 37.  $\log_b 5 = 2$       38.  $\log_b 625 = 4$   
 39.  $\log_b 4 = -\frac{1}{3}$       40.  $\log_b 4 = \frac{2}{3}$   
 41.  $\log_{10} 10^{0.2} = x$       42.  $\log_5 5^{2.3} = R + 1$   
 43.  $\log_3 27^{-1} = x + 1$       44.  $\log_b \left(\frac{1}{4}\right) = -0.5$

In Exercises 45–50, plot the graphs of the given functions.

45.  $y = \log_3 x$       46.  $y = \log_4 x$   
 47.  $y = \log_{0.5} x$       48.  $y = 3 \log_2 x$   
 49.  $N = 0.2 \log_4 v$       50.  $A = 2.4 \log_{10} (2r)$

In Exercises 51–54, display the graphs of the given functions on a graphing calculator.

51.  $y = 3 \log_e x$       52.  $y = 5 \log_{10} |x|$   
 53.  $y = -\log_{10} (-x)$       54.  $y = -\log_e (-2x)$

In Exercises 55–72, perform the indicated operations.

55. Evaluate  $\log_4 x$  for (a)  $x = 1/64$  and (b)  $x = -1/2$ .  
 56. Evaluate: (a)  $\log_b b$  (b)  $\log_b 1$   
 57. If  $f(x) = \log_5 x$ , find: (a)  $f(\sqrt{5})$  (b)  $f(0)$   
 58. If  $f(x) = \log_b x$  and  $f(3) = 2$ , find  $f(9)$ .  
 59. On a graphing calculator, display the graphs of: (a)  $y_1 = \log_{10} x$   
 (b)  $y_2 = \log_{10}(x + 2)$  (c)  $y_3 = \log_{10}(x - 2)$  (See page 107.)  
 60. On a graphing calculator, display the graphs of  $y_1 = 2 \log_{10} x$   
 and  $y_2 = \log_{10} x^2$ . Describe any similarities or differences.  
 61. Using a graphing calculator, solve the equation  $\log_{10} x = x - 2$ .  
 62. Use a graphing calculator to display the graphs (on the same  
 screen) of  $y = \log_e (x^2 + c)$ , with  $c = -4, 0, 4$ . Describe the  
 results.  
 63. The magnitudes (visual brightness),  $m_1$  and  $m_2$ , of two stars are  
 related to their (actual) brightnesses,  $b_1$  and  $b_2$ , by the equation  
 $m_1 - m_2 = 2.5 \log_{10} (b_2/b_1)$ . Solve for  $b_2$ .  
 64. The velocity  $v$  of a rocket when its fuel is completely burned is  
 given by  $v = u \log_e (w_0/w)$ , where  $u$  is the exhaust velocity,  $w_0$   
 is the liftoff weight, and  $w$  is the burnout weight. Solve for  $w$ .

65. An equation relating the number  $N$  of atoms of radium at any  
 time  $t$  in terms of the number  $N_0$  of atoms at  $t = 0$  is  
 $\log_e (N/N_0) = -kt$ , where  $k$  is a constant. Solve for  $N$ .  
 66. The capacitance  $C$  of a cylindrical capacitor is given by  
 $C = k/\log_e (R_2/R_1)$ , where  $R_1$  and  $R_2$  are the inner and outer  
 radii. Solve for  $R_1$ .  
 67. The work  $W$  done by a sample of nitrogen gas during an isother-  
 mal (constant temperature) change from volume  $V_1$  to volume  $V_2$   
 is given by  $W = k \log_e (V_2/V_1)$ . Solve for  $V_1$ .  
 68. An equation used in measuring the flow of water in a channel is  
 $C = -a \log_{10} (b/R)$ . Solve for  $R$ .  
 69. The time  $t$  (in ps) required for  $N$  calculations by a certain com-  
 puter design is  $t = N + \log_2 N$ . Sketch the graph of this  
 function.  
 70. When a tractor-trailer turns a right angle corner, its rear wheels  
 follow a curve called a *tractrix*, the equation for which is  
 $y = \log_e \left( \frac{1 + \sqrt{1 - x^2}}{x} \right) - \sqrt{1 - x^2}$ . Display the curve on a  
 graphing calculator.  
 71. Graph the functions  $y = (1/3)^x$  and  $y = 3^{-x}$  and then compare  
 them. Explain what your comparison shows.  
 72. In Exercise 47, the graph of  $y = \log_{0.5} x$  is plotted. By inspecting  
 the graph and noting the properties of  $\log_{0.5} x$ , describe some of  
 the differences between logarithms to a base less than 1 and those  
 to a base greater than 1.

In Exercises 73–76, show that the given functions are inverse  
 functions of each other. Then display the graphs of each function and  
 the line  $y = x$  on a graphing calculator and verify that each is the  
 mirror image of the other across  $y = x$ .

73.  $y = 10^{x/2}$  and  $y = 2 \log_{10} x$   
 74.  $y = e^x$  and  $y = \log_e x$   
 75.  $y = 3x$  and  $y = x/3$   
 76.  $y = 2x + 4$  and  $y = 0.5x - 2$

#### Answers to Practice Exercises

1.  $2/3 = \log_{125} 25$     2.  $1/3 = 3^{-1}$     3. 3

## 13.3 Properties of Logarithms

**Logarithm of a Product • Logarithm of a  
 Quotient • Logarithm of a  
 Power • Logarithms of 1 and b**

Since a logarithm is an exponent, it must follow the laws of exponents. The laws used in this section to derive the very useful properties of logarithms are listed here for reference.

$$b^u b^v = b^{u+v} \quad (13.4)$$

$$\frac{b^u}{b^v} = b^{u-v} \quad (13.5)$$

$$(b^u)^n = b^{un} \quad (13.6)$$

■ A logarithm is an exponent. However, a historical curiosity is that logarithms were developed before exponents were used.

The next example shows the reasoning used in deriving the properties of logarithms.

### EXAMPLE 1 Logarithm of a product

We know that  $8 \times 16 = 128$ . Writing these numbers as powers of 2, we have

$$8 = 2^3 \quad 16 = 2^4 \quad 128 = 2^7 = 2^{3+4}$$

The logarithmic forms can be written as

$$3 = \log_2 8 \quad 4 = \log_2 16 \quad 3 + 4 = \log_2 128$$

This means that

$$\log_2 8 + \log_2 16 = \log_2 128$$

where

$$8 \times 16 = 128$$

The *sum of the logarithms* of 8 and 16 equals the logarithm of 128, where the *product* of 8 and 16 equals 128. ■

■ For reference, Eqs. (13.4) to (13.6) are

$$\begin{aligned} b^u b^v &= b^{u+v} \\ \frac{b^u}{b^v} &= b^{u-v} \\ (b^u)^n &= b^{nu} \end{aligned}$$

Following Example 1, if we let  $u = \log_b x$  and  $v = \log_b y$  and write these equations in exponential form, we have  $x = b^u$  and  $y = b^v$ . Therefore, forming the product of  $x$  and  $y$ , we obtain

$$xy = b^u b^v = b^{u+v} \quad \text{or} \quad xy = b^{u+v}$$

Writing this last equation in logarithmic form yields

$$u + v = \log_b xy$$

or

#### Logarithm of a Product

$$\log_b xy = \log_b x + \log_b y \quad (13.7)$$

Eq. (13.7) states the property that *the logarithm of the product of two numbers is equal to the sum of the logarithms of the numbers*.

#### COMMON ERROR

The logarithm of a product is the sum of the logarithms. However, this property **does not** give us a way to rewrite the logarithm of a sum. For instance,  $\log_2(3x)$  can be rewritten as  $\log_2(3x) = \log_2 3 + \log_2 x$ , but  $\log_2(3 + x)$  **cannot** be rewritten in terms of  $\log_2 3$  and  $\log_2 x$ .

Using the same definitions of  $u$  and  $v$  to form the quotient of  $x$  and  $y$ , we then have

$$\frac{x}{y} = \frac{b^u}{b^v} = b^{u-v} \quad \text{or} \quad \frac{x}{y} = b^{u-v}$$

Writing this last equation in logarithmic form, we have

$$u - v = \log_b \left( \frac{x}{y} \right)$$

or

#### Logarithm of a Quotient

$$\log_b \left( \frac{x}{y} \right) = \log_b x - \log_b y \quad (13.8)$$

Eq. (13.8) states the property that *the logarithm of the quotient of two numbers is equal to the logarithm of the numerator minus the logarithm of the denominator.*

**COMMON ERROR**

The logarithm of a quotient is the difference of the logarithms. However, this property **does not** give us a way to rewrite the logarithm of a difference. For instance,  $\log_3\left(\frac{5}{x}\right)$  can be rewritten as  $\log_3\left(\frac{5}{x}\right) = \log_3 5 - \log_3 x$ , but  $\log_3(5 - x)$  **cannot** be rewritten in terms of  $\log_3 5$  and  $\log_3 x$ .

If we again let  $u = \log_b x$  and write this in exponential form, we have  $x = b^u$ . To find the  $n$ th power of  $x$ , we write

$$x^n = (b^u)^n = b^{nu}$$

Expressing this equation in logarithmic form yields

$$nu = \log_b(x^n)$$

or

**Logarithm of a Power**

$$\log_b(x^n) = n \log_b x \quad (13.9)$$

■ In advanced mathematics, the logarithms of negative and imaginary numbers are defined.

**LEARNING TIP**

In Section 13.2, we showed that the base  $b$  of logarithms must be a positive number. Since  $x = b^u$  and  $y = b^v$ , this means that  $x$  and  $y$  are also positive numbers. Therefore, the *properties of logarithms that have just been derived are valid only for positive values of  $x$  and  $y$ .*

Eq. (13.9) states that *the logarithm of the  $n$ th power of a number is equal to  $n$  times the logarithm of the number.* The exponent  $n$  may be any real number, which, of course, includes all rational and irrational numbers.

**EXAMPLE 2 Using the properties of logarithms**

(a) Using Eq. (13.7), we may express  $\log_4 15$  as a sum of logarithms:

$$\log_4 15 = \log_4(3 \times 5) = \log_4 3 + \log_4 5 \quad \begin{array}{l} \text{logarithm of product} \\ \text{is sum of logarithms} \end{array}$$

(b) Using Eq. (13.8), we may express  $\log_4\left(\frac{5}{3}\right)$  as the difference of logarithms:

$$\log_4\left(\frac{5}{3}\right) = \log_4 5 - \log_4 3 \quad \begin{array}{l} \text{logarithm of quotient is} \\ \text{difference of logarithms} \end{array}$$

(c) Using Eq. (13.9), we may express  $\log_4(t^2)$  as twice  $\log_4 t$ :

$$\log_4(t^2) = 2 \log_4 t \quad \begin{array}{l} \text{logarithm of power is} \\ \text{multiple of logarithm} \end{array}$$

(d) Using Eq. (13.8) and then Eq. (13.7), we have

$$\begin{aligned} \log_4\left(\frac{xy}{z}\right) &= \log_4(xy) - \log_4 z \\ &= \log_4 x + \log_4 y - \log_4 z \end{aligned}$$

**Practice Exercises**

Express as a sum or difference of logarithms.

1.  $\log_3 10$    2.  $\log_3(2a/5)$

**EXAMPLE 3 Sum of logarithms as a single quantity**

We may also express a sum or difference of logarithms as the logarithm of a single quantity.

(a)  $\log_4 3 + \log_4 x = \log_4(3 \times x) = \log_4 3x$    using Eq. (13.7)

(b)  $\log_4 3 - \log_4 x = \log_4\left(\frac{3}{x}\right)$    using Eq. (13.8)

(c)  $\log_4 3 + 2 \log_4 x = \log_4 3 + \log_4(x^2) = \log_4 3x^2$    using Eqs. (13.7) and (13.9)

(d)  $\log_4 3 + 2 \log_4 x - \log_4 y = \log_4\left(\frac{3x^2}{y}\right)$    using Eqs. (13.7), (13.8), and (13.9) ■

In Section 13.2, we noted that  $\log_b 1 = 0$ . Also, since  $b = b^1$  in logarithmic form is  $\log_b b = 1$ , we have  $\log_b(b^n) = n \log_b b = n(1) = n$ .

Summarizing these properties, we have

$$\log_b 1 = 0 \quad \log_b b = 1 \quad (13.10)$$

$$\log_b(b^n) = n \quad (13.11)$$

These equations may be used to find exact values of certain logarithms.

#### EXAMPLE 4 Exact values for certain logarithms

(a) We may evaluate  $\log_3 9$  using Eq. (13.11):

$$\log_3 9 = \log_3(3^2) = 2$$

We can establish the exact value since the base of the logarithm and the number being raised to the power are the same. Of course, this could have been evaluated directly from the definition of a logarithm.

(b) Using Eq. (13.11), we can write  $\log_3(3^{0.4}) = 0.4$ . Although we did not evaluate  $3^{0.4}$ , we can evaluate  $\log_3(3^{0.4})$ . ■

#### Practice Exercise

3. Find the exact value of  $2 \log_2 8$ .

#### EXAMPLE 5 Using the properties of logarithms

(a)  $\log_2 6 = \log_2(2 \times 3) = \log_2 2 + \log_2 3 = 1 + \log_2 3$

(b)  $\log_5 \frac{1}{5} = \log_5 1 - \log_5 5 = 0 - 1 = -1$

(c)  $\log_7 \sqrt{7} = \log_7(7^{1/2}) = \frac{1}{2} \log_7 7 = \frac{1}{2}$  ■

#### EXAMPLE 6 Evaluation of logarithms in two ways

The following illustration shows the evaluation of a logarithm in two different ways. Either method is appropriate.

(a)  $\log_5\left(\frac{1}{25}\right) = \log_5 1 - \log_5 25 = 0 - \log_5(5^2) = -2$

(b)  $\log_5\left(\frac{1}{25}\right) = \log_5(5^{-2}) = -2$  ■

#### EXAMPLE 7 Solving an equation with logarithms

Use the basic properties of logarithms to solve the following equation for  $y$  in terms of  $x$ :  $\log_b y = 2 \log_b x + \log_b a$ .

Using Eq. (13.9) and then Eq. (13.7), we have

$$\log_b y = \log_b(x^2) + \log_b a = \log_b(ax^2)$$

Since we have the logarithm to the base  $b$  of different expressions on each side of the resulting equation, the expressions must be equal. Therefore,

$$y = ax^2 \quad \blacksquare$$

#### EXAMPLE 8 Solving an equation with logarithms—application

An equation for the current  $i$  and the time  $t$  in an electric circuit containing a resistance  $R$  and a capacitance  $C$  is  $\log_e i - \log_e I = -t/RC$  (where  $R$  and  $C$  are both factors in the denominator). Here,  $I$  is the current for  $t = 0$ . Solve for  $i$  as a function of  $t$ .

Using Eq. (13.8), we rewrite the left side of this equation, obtaining

$$\log_e\left(\frac{i}{I}\right) = -\frac{t}{RC}$$

Rewriting this in exponential form, we have

$$\frac{i}{I} = e^{-t/RC} \quad \text{or} \quad i = Ie^{-t/RC}$$

### EXERCISES 13.3

In Exercises 1–8, perform the indicated operations on the resulting expressions if the given changes are made in original expressions of the indicated examples of this section.

- In Example 2(a), change the 15 to 21.
- In Example 2(d), change the  $x$  to 2 and the  $z$  to 3.
- In Example 3(b), change the 3 to 5.
- In Example 3(c), change the 2 to 3.
- In Example 4(a), change the 9 to 27.
- In Example 5(a), change the 6 to 10.
- In Example 5(c), change the 7's to 5's.
- In Example 7, change the 2 to 3.

In Exercises 9–20, express each as a sum, difference, or multiple of logarithms. See Example 2.

- |   |  |
|---|--|
| 9. $\log_5 33$                                | 10. $\log_3 14$                                |
| 11. $\log_7\left(\frac{5}{3}\right)$          | 12. $\log_3\left(\frac{2}{11}\right)$          |
| 13. $\log_2(a^3)$                             | 14. $2 \log_8(n^5)$                            |
| 15. $\log_6 abc$                              | 16. $\log_2\left(\frac{xy}{z^2}\right)$        |
| 17. $8 \log_5 \sqrt[4]{y}$                    | 18. $\log_4 \sqrt[7]{x}$                       |
| 19. $\log_2\left(\frac{\sqrt{x}}{a^2}\right)$ | 20. $\log_3\left(\frac{\sqrt[3]{y}}{7}\right)$ |

In Exercises 21–28, express each as the logarithm of a single quantity. See Example 3.

- |                                     |   |
|-------------------------------------|---|
| 21. $\log_b a + \log_b c$           | 22. $\log_2 3 + \log_2 x$               |
| 23. $\log_5 9 - \log_5 3$           | 24. $-\log_8 R + \log_8 V$              |
| 25. $-\log_b \sqrt{x} + \log_b x^2$ | 26. $\log_4 3^3 + \log_4 9$             |
| 27. $2 \log_e 2 + 3 \log_e \pi$     | 28. $\frac{1}{2} \log_b a - 2 \log_b 5$ |

In Exercises 29–36, determine the exact value of each of the given logarithms.

- |                                       |                                       |
|---------------------------------------|---------------------------------------|
| 29. $\log_2\left(\frac{1}{32}\right)$ | 30. $\log_3\left(\frac{1}{81}\right)$ |
| 31. $\log_2(2^{2.5})$                 | 32. $\log_5(5^{0.1})$                 |
| 33. $6 \log_7 \sqrt{7}$               | 34. $\pi \log_6 \sqrt[3]{6}$          |
| 35. $\log_3 \sqrt[4]{27}$             | 36. $\log_5 \sqrt[3]{25}$             |

In Exercises 37–44, express each as a sum, difference, or multiple of logarithms. In each case, part of the logarithm may be determined exactly.

- |                                      |                           |
|--------------------------------------|---------------------------|
| 37. $\log_3 18$                      | 38. $\log_5 75$           |
| 39. $\log_2\left(\frac{1}{6}\right)$ | 40. $\log_{10}(0.05)$     |
| 41. $\log_3 \sqrt{6}$                | 42. $\log_2 \sqrt[3]{24}$ |
| 43. $\log_{10} 3000$                 | 44. $\log_{10}(40^2)$     |

In Exercises 45–56, solve for  $y$  in terms of  $x$ .

- |   |                                      |
|---|--------------------------------------|
| 45. $\log_b y = \log_b 2 + \log_b x$                | 46. $\log_b y = \log_b 6 - \log_b x$ |
| 47. $\log_4 y = \log_4 x - \log_4 5 + \log_4 3$     |                                      |
| 48. $\log_3 y = -2 \log_3(x + 1) + \log_3 7$        |                                      |
| 49. $\log_{10} y = 2 \log_{10} 7 - 3 \log_{10} x$   |                                      |
| 50. $\log_b y = 3 \log_b \sqrt{x} + 2 \log_b 10$    |                                      |
| 51. $5 \log_2 y - \log_2 x = 3 \log_2 4 + \log_2 a$ |                                      |
| 52. $4 \log_2 x - 3 \log_2 y = \log_2 27$           |                                      |
| 53. $\log_2 x + \log_2 y = 1$                       | 54. $\pi \log_4 x + \log_4 y = 1$    |
| 55. $\frac{2 \log_5 x}{\log_5 3} - \log_5 y = 2$    | 56. $\log_8 x = 2 \log_8 y + 4$      |

In Exercises 57–68, solve the given problems.

- Explain why  $\log_{10}(x + 3)$  is not equal to  $\log_{10} x + \log_{10} 3$ .
- Evaluate  $2 \log_2(2x) - \log_2 x^2$ . For what values of  $x$  is the value of this expression valid? Explain.
- Display the graphs of  $y = \log_e(e^2 x)$  and  $y = 2 + \log_e x$  on a graphing calculator and explain why they are the same.
- If  $x = \log_b 2$  and  $y = \log_b 3$ , express  $\log_b 12$  in terms of  $x$  and  $y$ .
- If  $\log_b x = 2$  and  $\log_b y = 3$ , find  $\log_b \sqrt{x^2 y^4}$ .
- Is it true that  $\log_b(ab)^x = x \log_b a + x$ ?
- On the same screen of a graphing calculator, display the graphs of  $y_1 = \log_{10} x - \log_{10}(x^2 + 1)$  and  $y_2 = \log_{10} \frac{x}{x^2 + 1}$ . What conclusion can be drawn from the display?
- The use of the insecticide DDT was banned in Canada in 1970. A computer analysis shows that an expression relating the amount  $A$  still present in an area, the original amount  $A_0$ , and the time  $t$  (in years) since 1970 is  $\log_{10} A = \log_{10} A_0 + 0.1t \log_{10} 0.8$ . Solve for  $A$  as a function of  $t$ .



65. A study of urban density shows that the population density  $D$  (in persons/km<sup>2</sup>) is related to the distance  $r$  (in km) from the city centre by  $\log_e D = \log_e a - br + cr^2$ , where  $a$ ,  $b$ , and  $c$  are positive constants. Solve for  $D$  as a function of  $r$ .
66. When a person ingests a medication capsule, it is found that the rate  $R$  (in mg/min) at which it enters the bloodstream at time  $t$  (in min) is given by  $\log_{10} R - \log_{10} 5 = t \log_{10} 0.95$ . Solve for  $R$  as a function of  $t$ .
67. A container of water is heated to 90°C and then placed in a room at 0°C. The temperature  $T$  of the water is related to the time  $t$  (in min) by  $\log_e T = \log_e 90.0 - 0.23 t$ . Find  $T$  as a function of  $t$ .
68. In analysing the power gain in an electric circuit, the equation  $N = 10(2 \log_{10} I_1 - 2 \log_{10} I_2 + \log_{10} R_1 - \log_{10} R_2)$  is used. Express this with a single logarithm on the right side.

#### Answers to Practice Exercises

1.  $\log_3 2 + \log_3 5$    2.  $\log_3 2 + \log_3 a - \log_3 5$    3. 6

## 13.4 Logarithms to the Base 10

Common Logarithms • Antilogarithm • Calculations Using Base 10 Logarithms

### LEARNING TIP

1. When no base is shown, the logarithm is assumed to be to the base 10.
2. The *decimal* part of a logarithm is expressed with the same number of significant digits as there are in the number whose logarithm is being found.

In Section 13.2, we stated that a base of logarithms must be a positive number, not equal to 1. In the examples and exercises of the previous sections, we used a number of different bases. There are, however, only two bases that are generally used. They are 10 and  $e$ , where  $e$  is the irrational number approximately equal to 2.718 that we introduced in Section 12.5 and have used in the previous sections of this chapter. We discuss logarithms to the base 10 in this section and those to the base  $e$  in the next section.

*Logarithms to the base 10 are called **common logarithms**.* They may be found directly by use of a calculator, and the  $\boxed{\log}$  key is used for this purpose.

### EXAMPLE 1 Base 10 logarithms

Using a calculator to find  $\log 426$ , we find that

$$\log 426 = 2.629$$

↑ no base shown means base is 10

when the result is rounded off. Because 426 has 3 significant digits, we have retained 3 digits in the decimal part of the logarithm.

Since  $10^2 = 100$  and  $10^3 = 1000$ , and in this case

$$10^{2.629} = 426$$

we see that the 2.629 power of 10 gives a number between 100 and 1000. ■

### EXAMPLE 2 Negative base 10 logarithm

Finding  $\log 0.036\ 54$ , we see that

$$\log 0.036\ 54 = -1.4372$$

We note that the logarithm here is negative. Raising 10 to a negative power gives us a number between 0 and 1, and here we have

$$10^{-1.4372} = 0.036\ 54$$

We may also use a calculator to find a number  $N$  if we know  $\log N$ . *In this case, we refer to  $N$  as the **antilogarithm** of  $\log N$ .* On the calculator, we use the  $\boxed{10^x}$  key. We note that it shows the basic definition of a logarithm. (On many scientific calculators, the key sequence  $\boxed{\text{inv}}\boxed{\log}$  is used. Note that this sequence makes it clear that the exponential and logarithmic functions are inverse functions.)

**EXAMPLE 3 Antilogarithm—inverse logarithm**

Given  $\log N = 1.1854$ , find  $N$ .

Since  $1 < \log N < 2$ , we know that  $N$  must be a number between  $10^1 = 10$  and  $10^2 = 100$ . We find the exact number as an antilogarithm. We have

$$N = 10^{1.1854} = 15.32$$

where the result has been rounded off to 4 significant digits because the decimal part of the logarithm had 4 significant digits. ■

The following example illustrates an application in which a measurement requires the direct use of the value of a logarithm.

**EXAMPLE 4 Application—decibels**

The power gain  $G$  (in decibels (dB)) of an electronic device is given by  $G = 10 \log(P_0/P_i)$ , where  $P_0$  is the output power (in W) and  $P_i$  is the input power. Determine the power gain for an amplifier for which  $P_0 = 15.8$  W and  $P_i = 0.625$  W.

Substituting the given values, we have

$$G = 10 \log \frac{15.8}{0.625} = 14.0 \text{ dB} \quad \blacksquare$$

■ The unit of sound intensity level (used for power gain), the bel (B), is named for the inventor Alexander Graham Bell (1847–1922). The decibel is the commonly used unit.

As noted in the chapter introduction, logarithms were developed for calculational purposes. They were first used in the seventeenth century for making tedious and complicated calculations that arose in astronomy and navigation. These complicated calculations were greatly simplified, since logarithms allowed them to be performed by means of basic additions, subtractions, multiplications, and divisions. Performing calculations in this way provides an opportunity to understand better the meaning and properties of logarithms. Also, certain calculations cannot be done directly on a calculator but can be done by logarithms.

**Practice Exercises**

Evaluate  $x$  using a calculator.

1.  $x = \log 0.5392$     2.  $\log x = 2.2901$

**EXAMPLE 5 Calculation using logarithms**

A certain computer design has 64 different sequences of ten binary digits so that the total number of possible states is  $(2^{10})^{64} = 1024^{64}$ . Evaluate  $1024^{64}$  using logarithms. (It is very possible that your calculator cannot do this calculation directly.)

Since  $\log x^n = n \log x$ , we know that  $\log 1024^{64} = 64 \log 1024$ . Although most calculators will not directly evaluate  $1024^{64}$ , we can use one to find the value of  $64 \log 1024$ . Since  $1024^{64}$  is *exact*, we will show ten calculator digits until we round off the result to 3 significant digits. We therefore evaluate  $1024^{64}$  as follows:

$$\text{Let } N = 1024^{64}.$$

$$\log N = \log 1024^{64} = 64 \log 1024 \quad \text{using Eq. (13.9): } \log_b x^n = n \log_b x$$

$$= 192.6591972$$

$$N = 10^{192.6591972}$$

meaning of logarithm

$$= 10^{192} \times 10^{0.6591972}$$

using Eq. (13.4):  $b^u b^v = b^{u+v}$

$$= (10^{192}) \times (4.56)$$

antilogarithm of 0.6591972 is 4.56  
(rounded off)

$$= 4.56 \times 10^{192}$$

By using Eq. (13.4),  $10^{0.6591972}$  represents a number between 1 and 10 ( $10^0 = 1$  and  $10^1 = 10$ ), and we can write the result immediately in scientific notation.

Note that although we have used a calculator to find a logarithm and an antilogarithm (in the past, this would have been done by consulting a table of logarithms), the only actual calculation we have done is a multiplication by 64. The use of logarithms has allowed us to change the calculation of a power into the simpler process of multiplication. ■

Example 5 shows that calculations using logarithms are based on Eqs. (13.7), (13.8), and (13.9). Multiplication is performed by addition of logarithms, division is performed by the subtraction of logarithms, and a power is found by a multiple of a logarithm. A root of a number is found by using the fractional exponent form of the power.

## EXERCISES 13.4

In Exercises 1 and 2, find the indicated values if the given changes are made in the indicated examples of this section.

- In Example 2, change 0.036 54 to 0.3654 and then find the required value.
- In Example 3, change 1.1854 to 2.1854 and then find the required value.

In Exercises 3–10, find the common logarithm of each of the given numbers by using a calculator.

- |                       |                           |
|-----------------------|---------------------------|
| 3. 567                | 4. 0.0640                 |
| 5. $9.24 \times 10^6$ | 6. $3.19^3$               |
| 7. $1.174^{-4}$       | 8. $8.043 \times 10^{-8}$ |
| 9. $\sqrt{274}$       | 10. $\log_2 16$           |

In Exercises 11–18, find the antilogarithm of each of the given logarithms by using a calculator.

- |              |              |               |             |
|--------------|--------------|---------------|-------------|
| 11. 4.437    | 12. 0.929    | 13. -1.3045   | 14. -6.9788 |
| 15. 3.301 12 | 16. 8.824 36 | 17. -2.237 46 | 18. -10.336 |

In Exercises 19–22, use logarithms to evaluate the given expressions. All exponents are exact.

- |                                    |  |
|------------------------------------|--|
| 19. $(5.98)(14.3)$                 | 20. $\frac{895}{73.4^{86}}$            |
| 21. $(\sqrt[10]{7.32})(2470)^{30}$ | 22. $\frac{126\,000^{20}}{2.63^{2.5}}$ |

In Exercises 23–26, use a calculator to verify the given values.

- |                                   |                                    |
|-----------------------------------|------------------------------------|
| 23. $\log 14 + \log 0.5 = \log 7$ | 24. $\log 500 - \log 20 = \log 25$ |
| 25. $\log 81 = 4 \log 3$          | 26. $\log 6 = 0.5 \log 36$         |

In Exercises 27–31, find the logarithms of the given numbers.

- The signal used by some cell phones has a frequency of  $9.00 \times 10^8$  Hz.
- A large sunspot may be  $3.5 \times 10^4$  km in diameter.
- About  $1.3 \times 10^{-14}\%$  of carbon atoms are carbon-14, the radioactive isotope used in determining the age of samples from ancient sites.
- In an air sample taken in an urban area,  $5/10^6$  of the air was carbon monoxide.
- The mass of an electron is  $9.11 \times 10^{-31}$  kg.

In Exercises 32–35, find the indicated values.

- Find  $T$  (in K) if  $\log T = 8$ , where  $T$  is the temperature sufficient for nuclear fission.

- Find  $v$  (in m/s) if  $\log v = 7.423$ , where  $v$  is the speed of an electron in a TV picture tube.
- Find  $\eta$  if  $\log \eta = -0.35$ , where  $\eta$  is the efficiency of a certain gasoline engine.
- Find  $E$  (in J) if  $\log E = -18.49$ , where  $E$  is the energy of a photon of visible light.

In Exercises 36–42, solve the given problems by evaluating the appropriate logarithms or antilogarithms.

- Simplify:  $\frac{\log_b x^2}{\log 100}$ .
- Evaluate:  $\log(\log 10^{100})$ . (See Exercise 52 on page 29.)
- Evaluate:  $2(10^{\log 0.1}) + 3(10^{\log 0.01})$ .
- A stereo amplifier has an input power of 0.750 W and an output power of 25.0 W. What is the power gain? (See Example 4.)
- The percent transmittance (% $T$ ) of a substance is related to its absorbance ( $A$ ) by  $A = -\log(\%T/100)$ . Find % $T$  if  $A = 0.27$ .
- Measured on the Richter scale, the magnitude of an earthquake of intensity  $I$  is defined as  $R = \log(I/I_0)$ , where  $I_0$  is a minimum level for comparison. What is the Richter scale reading for the 2007 Peruvian earthquake for which  $I = 79\,000\,000I_0$ ?
- The moment magnitude scale  $M_W$  was devised as a successor to the Richter scale to measure the size of very large earthquakes adequately. In terms of the seismic moment of the earthquake  $M_0$  (in N·m), it is given by  $M_W = (\log M_0 + 7)/1.5 - 10.7$ . What is the moment magnitude scale reading for the 2010 Haiti earthquake for which  $M_0 = 3.54 \times 10^{19}$  N·m?

In Exercises 43 and 44, use logarithms to perform the indicated calculations.

- A certain type of optical switch in a fibre-optic system allows a light signal to continue in either of two fibres. How many possible paths could a light signal follow if it passes through 400 such switches? Express your answer in scientific notation.
- The peak current  $I_m$  (in A) in an alternating-current circuit is given by  $I_m = \sqrt{\frac{2P}{Z \cos \theta}}$ , where  $P$  is the power developed,  $Z$  is the magnitude of the impedance, and  $\theta$  is the phase angle between the current and voltage. Evaluate  $I_m$  for  $P = 5.25$  W,  $Z = 320 \Omega$ , and  $\theta = 35.4^\circ$ .

### Answers to Practice Exercises

- 0.2683
- 195.0

## 13.5 Natural Logarithms

Natural Logarithm (ln  $x$ ) • Change of Base • Calculator Values and Graphs

As we have noted, another number important as a base of logarithms is the number  $e$ . *Logarithms to the base  $e$  are called natural logarithms.* Since  $e$  is an irrational number equal to about 2.718, it may appear to be a very unnatural choice as a base of logarithms. However, in calculus, the reason for its choice and the fact that it is a very natural number for a base of logarithms are shown.

Just as  $\log x$  refers to logarithms to the base 10, the notation  $\ln x$  is used to denote logarithms to the base  $e$ . We briefly noted this in Section 13.2 in discussing the graphing calculator. Due to the extensive use of natural logarithms, the notation  $\ln x$  is more convenient than  $\log_e x$ , although they mean the same thing.

Since more than one base is important, at times it is useful to change a logarithm from one base to another. If  $u = \log_b x$ , then  $b^u = x$ . Taking logarithms of both sides of this last expression to the base  $a$ , we have

$$\begin{aligned}\log_a b^u &= \log_a x \\ u \log_a b &= \log_a x \\ u &= \frac{\log_a x}{\log_a b}\end{aligned}$$

However,  $u = \log_b x$ , which means that

$$\log_b x = \frac{\log_a x}{\log_a b} \quad (13.12)$$

Eq. (13.12) allows us to change a logarithm in one base to a logarithm in another base. The following examples illustrate the method of performing this operation.

### EXAMPLE 1 Change of base to find natural log

Find  $\ln 20$  using only logarithms to the base 10.

Using Eq. (13.12) with  $a = 10$ ,  $b = e$ , and  $x = 20$ , we have

$$\begin{aligned}\log_e 20 &= \frac{\log_{10} 20}{\log_{10} e} \\ \text{or} \quad \ln 20 &= \frac{\log 20}{\log e} = 2.996\end{aligned}$$

This means that  $e^{2.996} = 20$ . ■

### COMMON ERROR

A common error is to confuse Eq. (13.12), which uses a quotient of logarithms to change the base, with Eq. (13.8), which expresses the logarithm of a quotient as the difference of logarithms. Make sure that you understand the difference between these expressions:

$$\begin{aligned}\log_3 x &= \frac{\log x}{\log 3} \text{ evaluates a logarithm to the base 3 using logarithms to the base 10} \\ \log\left(\frac{x}{3}\right) &= \log x - \log 3 \text{ expresses the logarithm of a quotient as the difference of logarithms}\end{aligned}$$

### EXAMPLE 2 Change of base to find log to base 5

Find  $\log_5 560$ .

In Eq. (13.12), if we let  $a = 10$ ,  $b = 5$ , and  $x = 560$ , we have

$$\log_5 560 = \frac{\log 560}{\log 5} = 3.932$$

### Practice Exercise

1. Find  $\log_3 23$ .

From the definition of a logarithm, this means that

$$5^{3.932} = 560 \quad \blacksquare$$

Since natural logarithms are used extensively, it is often convenient to have Eq. (13.12) written specifically for use with logarithms to the base 10 and natural logarithms. First, using  $a = 10$  and  $b = e$ , we have

$$\ln x = \frac{\log x}{\log e} \quad (13.13)$$

Then with  $a = e$  and  $b = 10$ , we have

$$\log x = \frac{\ln x}{\ln 10} \quad (13.14)$$

Note that we really found  $\ln 20$  in Example 1 by using Eq. (13.13).

Values of natural logarithms can be found directly on a calculator. The  $\boxed{\ln}$  key is used for this purpose. In order to find the antilogarithm of a natural logarithm, we use the  $\boxed{e^x}$  key. The following example illustrates finding a natural logarithm and an antilogarithm on a graphing calculator.

### EXAMPLE 3 Evaluating natural logarithms

(a) We evaluate  $\ln 236.5$  directly on the calculator. We find that

$$\ln 236.5 = 5.4659$$

which means that  $e^{5.4659} = 236.5$ .

(b) Given that  $\ln N = -0.8729$ , we determine  $N$  by finding  $e^{-0.8729}$  on the calculator. This gives us

$$N = 0.4177 \quad \blacksquare$$

Using Eq. (13.12), we can display the graph of a logarithmic function with any base on a graphing calculator, as we show in the next example.

### EXAMPLE 4 Graphing a logarithm to the base 2

Graph the function  $y = 3 \log_2 x$ .

The function  $y = \log_2 x$  cannot be entered directly into a graphing calculator, so we must use the fact that

$$\log_2 x = \frac{\log x}{\log 2} \quad \text{or} \quad \log x = \frac{\ln x}{\ln 2}$$

Therefore, we graph the function

$$y = \frac{3 \log x}{\log 2} \quad \left( \text{or } y = \frac{3 \ln x}{\ln 2} \right)$$

The curve is shown in Fig. 13.9. \blacksquare

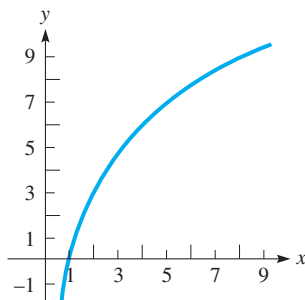


Fig. 13.9

#### Practice Exercise

2. Use a calculator to evaluate  $\ln 57.2$ .

Applications of natural logarithms are found in many fields of technology. One such application is shown in the next example, and others are found in the exercises.

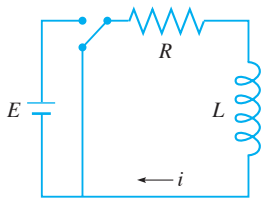


Fig. 13.10

**EXAMPLE 5** Natural log—application

The electric current  $i$  in a circuit containing a resistance and an inductance (see Fig. 13.10) is given by  $\ln(i/I) = -Rt/L$ , where  $I$  is the current at  $t = 0$ ,  $R$  is the resistance,  $t$  is the time, and  $L$  is the inductance. Calculate how long (in s) it takes  $i$  to reach 0.430 A, if  $I = 0.750$  A,  $R = 7.50$   $\Omega$ , and  $L = 1.25$  H.

Solving for  $t$  and then evaluating, we have

$$t = \frac{L \ln(i/I)}{R} = \frac{L(\ln i - \ln I)}{R} \quad \text{either form can be used}$$

$$= \frac{1.25(\ln 0.430 - \ln 0.750)}{7.50} = 0.0927 \text{ s} \quad \text{evaluating}$$

Therefore, the current changes from 0.750 A to 0.430 A in 0.0927 s. ■

**EXERCISES 13.5**

In Exercises 1 and 2, find the indicated values if the given changes are made in the indicated examples of this section.

- In Example 1, change 20 to 200 and then evaluate.
- In Example 2, change the base to 4 and then evaluate.

In Exercises 3–8, use logarithms to the base 10 to find the natural logarithms of the given numbers.

- 26.0
- 631
- 1.562
- 0.5017
- 0.007 326 7
- 0.000 443 48

In Exercises 9–14, use logarithms to the base 10 to find the indicated logarithms.

- $\log_7 42$
- $\log_2 86$
- $\log_\pi 245$
- $\log_{12} 122$
- $\log_{40} 750$
- $\log_{100} 3720$

In Exercises 15–22, find the natural logarithms of the given numbers.

- 51.4
- 293
- 1.394
- 6552
- 0.9917
- 0.002 086
- $(0.012 937)^4$
- $\sqrt{0.000 060 808}$

In Exercises 23–26, use Eq. (13.14) to find the common logarithms of the given numbers.

- 45.17
- 8765
- 0.685 28
- 0.001 429 8

In Exercises 27–34, find the natural antilogarithms of the given logarithms.

- 2.190
- 5.420
- 0.008 421 0
- 0.632
- 0.7429
- 2.942 18
- 23.504
- 0.008 04

In Exercises 35–38, use a graphing calculator to display the indicated graphs.

- The graph of  $y = \log_5 x$
- The graph of  $y = 2 \log_8 x$
- Verify that  $y = 2^x$  and  $y = \log_2 x$  are inverse functions by displaying the graphs of each function and the line  $y = x$  on a graphing calculator. (See Example 8 of Section 13.2.)

- Explain what is meant by the expression  $\ln(\ln x)$ . Display the graph of  $y = \ln(\ln x)$  on a calculator.

In Exercises 39–42, use a calculator to verify the given values.

- $\ln 5 + \ln 8 = \ln 40$
- $2 \ln 6 - \ln 3 = \ln 12$
- $4 \ln 3 = \ln 81$
- $\ln 5 - 0.5 \ln 25 = \ln 1$

In Exercises 43–56, solve the given problems.

- Evaluate:  $\sqrt{\ln e^9}$ .
- Solve for  $y$  in terms of  $x$ :  $\ln y + 2 \ln x = 1 + \ln 5$ .
- Solve for  $x$ :  $\ln(\log x) = 0$ .
- If  $\ln x = 3$  and  $\ln y = 4$ , find  $\sqrt{x^2 y}$ .
- If  $x = \ln 4$  and  $y = \ln 5$ , express  $\ln 80$  in terms of  $x$  and  $y$ .
- On a graphing calculator, compare the graphs of  $y_1 = \ln(x^2 + 1)$  and  $y_2 = \ln(x^2 + 1)^{-1}$ .
- Find  $f$  (in Hz) if  $\ln f = 21.619$ , where  $f$  is the frequency of the microwaves in a microwave oven.
- Find  $k$  (in 1/Pa) if  $\ln k = -21.504$ , where  $k$  is the compressibility of water.
- If interest is compounded continuously (daily compounded interest closely approximates this), with an interest rate  $i$ , a bank account will double in  $t$  years according to  $i = (\ln 2)/t$ . Find  $i$  if the account is to double in 8.5 years. (2 is exact.)
- One approximate formula for world population growth is  $T = 50.0 \ln 2$ , where  $T$  is the number of years for the population to double. According to this formula, how long does it take for the population to double? (2 is exact.)
- For the electric circuit of Example 5, find how long it takes the current to reach 10% of the initial value of 0.750 A.
- The velocity  $v$  (in m/s) of a rocket increases as fuel is consumed and ejected. Considering fuel as part of the mass  $m$  of a rocket in flight and  $m_0$  as its original mass, the velocity is given by  $v = 2500(\ln m_0 - \ln m)$ . If 0.75 of the original mass is fuel, find  $v$  when all the fuel is used.

55. The distance  $x$  travelled by a motorboat  $t$  seconds after the engine is cut off is given by  $x = k^{-1} \ln(kv_0t + 1)$ , where  $v_0$  is the velocity of the boat at the time the engine is cut and  $k$  is a constant. Find how long it takes a boat to go 150 m if  $v_0 = 12.0$  m/s and  $k = 6.80 \times 10^{-3}$ /m.

56. The electric current  $i$  (in A) in a circuit that has a 1-H inductor, a 10- $\Omega$  resistor, and a 6-V battery, and the time  $t$  (in s) are related by the equation  $10t = -\ln(1 - i/0.6)$ . Solve for  $i$ .

Answers to Practice Exercises

1. 2.854    2. 4.047

## 13.6 Exponential and Logarithmic Equations

Exponential Equations • Logarithmic Equations

### EXPONENTIAL EQUATIONS

An equation in which the variable occurs in an exponent is called an **exponential equation**. Although some may be solved by changing to logarithmic form, they are generally solved by taking the logarithm of each side and then using the properties of logarithms.

#### EXAMPLE 1 Two ways to solve an exponential equation

Solve the equation  $2^x = 8$ .

(a) We can solve the exponential equation  $2^x = 8$  by writing it in logarithmic form. This gives

$$x = \log_2 8 = 3 \quad 2^3 = 8$$

This method is good if we can directly evaluate the resulting logarithm.

(b) Since  $2^x$  and 8 are equal, the logarithms of  $2^x$  and 8 are also equal. Therefore, we can also solve  $2^x = 8$  in a more general way by taking logarithms (to any proper base) of both sides and equating these logarithms. This gives us

$$\begin{aligned} \log 2^x &= \log 8 & \text{or} & \quad \ln 2^x = \ln 8 \\ x \log 2 &= \log 8 & & \quad x \ln 2 = \ln 8 & \text{using Eq. (13.9)} \\ x &= \frac{\log 8}{\log 2} = 3 & & \quad x = \frac{\ln 8}{\ln 2} = 3 & \text{using a calculator} \end{aligned}$$

#### EXAMPLE 2 Solving an exponential equation

Solve the equation  $3^{x-2} = 5$ .

Taking logarithms of each side and equating them, we have

$$\begin{aligned} \log 3^{x-2} &= \log 5 \\ (x - 2) \log 3 &= \log 5 & \text{using Eq. (13.9)} \\ x &= 2 + \frac{\log 5}{\log 3} = 3.465 \end{aligned}$$

This solution means that

$$3^{3.465-2} = 3^{1.465} = 5$$

which can be checked by a calculator.

■ For reference, Eqs. (13.7), (13.8), and (13.9) are

$$\begin{aligned} \log_b xy &= \log_b x + \log_b y \\ \log_b \left(\frac{x}{y}\right) &= \log_b x - \log_b y \\ \log_b (x^n) &= n \log_b x \end{aligned}$$

Practice Exercise

1. Solve for  $x$ :  $2^{x+1} = 7$



**EXAMPLE 3** Solving an exponential equation

Solve the equation  $2(4^{x-1}) = 17^x$ .

By taking logarithms of each side, we have the following:

$$\begin{aligned} \log 2 + (x - 1)\log 4 &= x \log 17 && \text{using Eqs. (13.7) and (13.9)} \\ x \log 4 - x \log 17 &= \log 4 - \log 2 \\ x(\log 4 - \log 17) &= \log 4 - \log 2 \\ x &= \frac{\log 4 - \log 2}{\log 4 - \log 17} = \frac{\log(4/2)}{\log 4 - \log 17} && \text{using Eq. (13.8)} \\ &= \frac{\log 2}{\log 4 - \log 17} = -0.479 \end{aligned}$$

**EXAMPLE 4** Exponential equation—application to atmospheric pressure

At constant temperature, the atmospheric pressure  $p$  (in Pa) at an altitude  $h$  (in m) is given by  $p = p_0 e^{kh}$ , where  $p_0$  is the pressure where  $h = 0$  (usually taken as sea level). Given that  $p_0 = 101.3$  kPa (atmospheric pressure at sea level) and  $p = 68.9$  kPa for  $h = 3050$  m, find the value of  $k$ .

Since the equation is defined in terms of  $e$ , we can solve it most easily by taking natural logarithms of *each side*. By doing this, we have the following solution:

$$\begin{aligned} \ln p &= \ln(p_0 e^{kh}) = \ln p_0 + \ln e^{kh} && \text{using Eq. (13.7)} \\ &= \ln p_0 + kh \ln e = \ln p_0 + kh && \text{using Eq. (13.9), } \ln e = 1 \\ \ln p - \ln p_0 &= kh \\ k &= \frac{\ln p - \ln p_0}{h} \end{aligned}$$

Substituting the given values, we have

$$k = \frac{\ln(68.9 \times 10^3) - \ln(101.3 \times 10^3)}{3050} = -0.000126/\text{m}$$

**LOGARITHMIC EQUATIONS**

Some of the important measurements in scientific and technical work are defined in terms of logarithms. Using these formulas can lead to solving a **logarithmic equation**, which is an equation with the logarithm of an expression involving the variable. In solving logarithmic equations, we use the basic properties of logarithms to help change them into a usable form. There is, however, no general algebraic method for solving such equations, and we consider only some special cases.

**EXAMPLE 5** Logarithmic equation—application to sound intensity

The human ear responds to sound over a broad range of sound intensities. Therefore, a logarithmic intensity scale is usually used, where the intensity of sound is compared to a reference value. In particular, the sound intensity level  $SL$  (measured in dB) is defined by the equation  $SL = 10 \log\left(\frac{I}{I_0}\right)$ , where  $I$  is the intensity of the sound and  $I_0$  is the minimum detectable intensity.

A jackhammer has a sound level of 100 dB, and a busy street has a sound level of 70 dB. To find how many times greater the intensity  $I_j$  of the sound of a jackhammer is than the intensity  $I_c$  of the sound of the city street, we begin by substituting the given sound levels and their corresponding intensities separately into the above equation. This gives

$$70 = 10 \log\left(\frac{I_c}{I_0}\right) \quad \text{and} \quad 100 = 10 \log\left(\frac{I_j}{I_0}\right)$$

■ See the chapter introduction.

To solve for  $I_c$  and  $I_j$ , we divide each side by 10 and then use exponential form:

$$\begin{aligned} 7.0 &= \log\left(\frac{I_c}{I_0}\right) & \text{and} & & 10 &= \log\left(\frac{I_j}{I_0}\right) \\ \frac{I_c}{I_0} &= 10^{7.0} & & & \frac{I_j}{I_0} &= 10^{10} \\ I_c &= I_0(10^{7.0}) & & & I_j &= I_0(10^{10}) \end{aligned}$$

Since we want the number of times  $I_j$  is greater than  $I_c$ , we divide  $I_j$  by  $I_c$ :

$$\frac{I_j}{I_c} = \frac{I_0(10^{10})}{I_0(10^{7.0})} = \frac{10^{10}}{10^{7.0}} = 10^{3.0} \quad \text{or} \quad I_j = 10^{3.0}I_c = 1000I_c$$

Thus, the sound of a jackhammer is 1000 times as intense as the sound of the city street. ■

■ This demonstrates that sound intensities change much more than sound levels. (See Exercise 50.)

■ For reference,

$$\log_b xy = \log_b x + \log_b y \quad (13.7)$$

$$\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y \quad (13.8)$$

$$\log_b(x^n) = n\log_b x \quad (13.9)$$

### EXAMPLE 6 Logarithmic equation—application to population growth

Under a medium-growth projection scenario, an analysis of the population of Canada from 1981 to 2008 led to the equation  $\log_2 P = \log_2 33.7 + 0.378$ , where  $P$  is the projected population (in millions) in 2036. Determine this population.

We solve the logarithmic equation as follows:

$$\begin{aligned} \log_2 P &= \log_2 33.7 + 0.378 \\ \log_2(P/33.7) &= 0.378 && \text{using Eq. (13.8)} \\ P/33.7 &= 2^{0.378} && \text{changing to exponential form} \\ P &= 33.7(2^{0.378}) \\ P &= 43.8 \text{ million} && \text{projected 2036 population} \end{aligned}$$

### EXAMPLE 7 Solving a logarithmic equation

Solve the logarithmic equation  $2 \ln 2 + \ln x = \ln 3$ .

Using the properties of logarithms, we have the following solution:

$$\begin{aligned} 2 \ln 2 + \ln x &= \ln 3 \\ \ln 2^2 + \ln x - \ln 3 &= 0 && \text{using Eq. (13.9)} \\ \ln \frac{4x}{3} &= 0 && \text{using Eqs. (13.7) and (13.8)} \\ \frac{4x}{3} &= e^0 = 1 && \text{exponential form} \\ 4x &= 3 \\ x &= 3/4 \end{aligned}$$

Since  $\ln(3/4) = \ln 3 - \ln 4$ , this solution checks in the *original equation*. ■

### EXAMPLE 8 Solving a logarithmic equation

Solve the logarithmic equation  $2 \log x - 1 = \log(1 - 2x)$ .

$$\begin{aligned} \log x^2 - \log(1 - 2x) &= 1 && \text{using Eq. (13.9)} \\ \log \frac{x^2}{1 - 2x} &= 1 && \text{using Eq. (13.8)} \\ \frac{x^2}{1 - 2x} &= 10^1 && \text{exponential form} \\ x^2 &= 10 - 20x \\ x^2 + 20x - 10 &= 0 \\ x &= \frac{-20 \pm \sqrt{400 + 40}}{2} = -10 \pm \sqrt{110} && \text{using the quadratic formula} \\ &&& \text{Eq. (7.4)} \end{aligned}$$

#### LEARNING TIP

Many exponential and logarithmic equations cannot be solved algebraically, as we have done in the examples in this section. However, they can be solved graphically using a graphing calculator as described in Section 3.5.

For instance, to solve the equation  $2 \ln 2 + \ln x = \ln 3$  from Example 7 graphically, we can find the *zero* of the function  $y_1 = 2 \ln 2 + \ln x - \ln 3$ , or the *intersection* of the functions  $y_1 = 2 \ln 2 + \ln x$  and  $y_2 = \ln 3$ .

Solutions can also be approximated numerically.

## Practice Exercise

2. Solve for  $x$ :  $3 \log 2 - \log(x + 1) = 1$  and cannot be used in the first term of the original equation, we have that the unique solution is

$$x = -10 + \sqrt{110} = 0.488$$

## EXERCISES 13.6

In Exercises 1 and 2, find the indicated values if the given changes are made in the indicated examples of this section.

- In Example 2, change the sign in the exponent from  $-$  to  $+$  and then solve the equation.
- In Example 7, change  $\ln 3$  to  $\ln 6$  and then solve the equation.

In Exercises 3–30, solve the given equations. When necessary, round expressions involving logarithms of integers to 3 significant digits in their decimal part.

- $2^x = 16$
- $\pi^x = 15$
- $6^{x+1} = 78$
- $0.8^x = 0.4$
- $3 \log_8 x = -2$
- $\log x^2 = (\log x)^2$
- $\log_2 x + \log_2 7 = \log_2 21$
- $2 \log(3 - x) = 1$
- $\log 12x^2 - \log 3x = 3$
- $3 \ln 2 + \ln(x - 1) = \ln 24$
- $\frac{1}{2} \log(x + 2) + \log 5 = 1$
- $\log(2x - 1) + \log(x + 4) = 1$
- $\ln(2x - 1) - 2 \ln 4 = 3 \ln 2$
- $3^x = 1/81$
- $3^{-x} = 0.525$
- $5^{x-1} = 0.07$
- $0.6^x = 2^{x^2}$
- $5^x = 0.3$
- $e^{-x} = 17.54$
- $3(14^x) = 400$
- $15.6^{x+2} = 23^x$
- $5 \log_{32} x = -3$
- $x^{\log x} = 1000x^2$
- $2 \log_2 3 - \log_2 x = \log_2 45$
- $3 \log(2x - 1) = 1$
- $\ln x - \ln(1/3) = 1$
- $\log_2 x + \log_2(x + 2) = 3$
- $2 \log_x 2 + \log_2 x = 3$

In Exercises 31–38, use a graphing calculator to solve the given equations. Round your answers to 3 significant digits.

- $15^{-x} = 1.326$
- $4(3^x) = 5$
- $3 \ln 2x = 2$
- $2 \ln 2 - \ln x = -1$
- $e^{2x} = 3.625$
- $5^{x+2} = e^{2x}$
- $\log 4x + \log x = 2$
- $\log(x - 3) + \log x = \log 4$

In Exercises 39–56, find the indicated quantities.

- Solve for  $x$ :  $e^x + e^{-x} = 3$ . (Hint: Multiply each term by  $e^x$  and then it can be treated as a quadratic equation in  $e^x$ .)
- Solve for  $x$ :  $3^x + 3^{-x} = 4$ . See Exercise 39.
- If  $y = 1.5e^{-0.90x}$ , find  $y$  when  $x = 7.1$ .
- What values of  $x$  cannot be solutions of the equation  $y = \log(2x - 5) + \log(x^2 + 1)$ ?
- Use logarithms to find the  $x$ -intercept of the graph of  $y = 3 - 4^{x+2}$ .
- Use a graphing calculator to find the point of intersection of the curves of  $3x + 5y = 6$  and  $y = 1.5 \ln(x + 3)$ .
- In computer design, the number  $N$  of bits of memory is often expressed as a power of 2, or  $N = 2^x$ . Find  $x$  if  $N = 2.68 \times 10^8$  bits.
- Referring to Exercise 64 on page 381, in what year will the amount of DDT be 25% of the original amount?
- Forensic scientists determine the temperature  $T$  (in  $^\circ\text{C}$ ) of a body  $t$  hours after death from the equation  $T = T_0 + (37 - T_0)0.97^t$ , where  $T_0$  is the air temperature. If a body is discovered at midnight with a body temperature of  $27^\circ\text{C}$  in a room at  $22^\circ\text{C}$ , at what time did death occur?
- When a camera flash goes off, the batteries recharge the flash's capacitor to a charge  $Q$  according to  $Q = Q_0(1 - e^{-kt})$ , where  $Q_0$  is the maximum charge and  $t$  is in seconds. How long does it take to recharge the capacitor to 90% of capacity if  $k = 0.5$ ?
- In chemistry, the pH value of a solution is a measure of its acidity. The pH value is defined by  $\text{pH} = -\log(\text{H}^+)$ , where  $\text{H}^+$  is the hydrogen-ion concentration (in mol/L). If the pH of a sample of rainwater is 4.764, find the hydrogen-ion concentration. (If  $\text{pH} < 7$ , the solution is acid. If  $\text{pH} > 7$ , the solution is basic.) Acid rain has a pH between 4 and 5, and normal rain is slightly acidic with a pH of about 5.6.
- Referring to Example 5, show that if the difference in sound level of two sounds is  $d$  decibels, the louder sound is  $10^{d/10}$  more intense than the quieter sound.
- The moment magnitude scale  $M_W$  was devised as a successor to the Richter scale to measure the size of very large earthquakes adequately. In terms of the seismic moment of the earthquake  $M_0$  (in N·m), it is given by  $M_W = (\log M_0 + 7)/1.5 - 10.7$ . What was the seismic moment of the 2011 earthquake off the coast of Japan whose moment magnitude was 9.0?
- How many times stronger (in terms of seismic moment) was the 2004 Indian Ocean earthquake off the coast of Sumatra ( $M_W = 9.1$ ) than the 2012 Haida Gwaii earthquake off the coast of British Columbia ( $M_W = 7.7$ )? (See Exercise 51.)
- Studies have shown that the concentration  $c$  (in  $\text{mg}/\text{cm}^3$  of blood) of aspirin in a typical person is related to the time  $t$  (in h) after the aspirin reaches maximum concentration by the equation  $\ln c = \ln 15 - 0.20t$ . Solve for  $c$  as a function of  $t$ .
- In an electric circuit containing a resistor and a capacitor with an initial charge  $q_0$ , the charge  $q$  on the capacitor at any time  $t$  after closing the switch can be found by solving the equation  $\ln q = -\frac{t}{RC} + \ln q_0$ . Here,  $R$  is the resistance and  $C$  is the capacitance. Solve for  $q$  as a function of  $t$ .
- An earth satellite loses 0.1% of its remaining power each week. An equation relating the power  $P$ , the initial power  $P_0$ , and the time  $t$  (in weeks) is  $\ln P = t \ln 0.999 + \ln P_0$ . Solve for  $P$  as a function of  $t$ .

56. In finding the path of a certain plane, the equation  $\ln r = \ln a - \ln \cos \theta - (v/w) \ln(\sec \theta + \tan \theta)$  is used. Solve for  $r$ . (The conditions of the plane's flight are similar to those described in Exercise 96, p. 340.)

In Exercises 57–60, solve the given equations graphically. Round your answers to 3 significant digits.

57.  $2^x + 3^x = 50$

58.  $4^x + x^2 = 25$

59. The curve in which a uniform wire or rope hangs under its own weight is called a *catenary*. An example of a catenary that we see every day is a wire strung between utility poles, as shown in Fig. 13.11. For a particular wire, the equation of the catenary it forms is  $y = 2(e^{x/4} + e^{-x/4})$ , where  $(x, y)$  is a point on the curve. Find  $x$  for  $y = 5.8$  m.

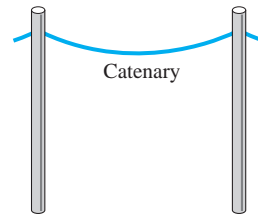


Fig. 13.11

60. In finding the current  $i$  in a certain electric circuit, the equation  $i = 2 \ln(t + 2) - t$  relates the current and the time  $t$  (in s). Find  $t$  for  $i = 0.05$  A.

Answers to Practice Exercises

1. 1.8    2.  $-1/5$

## 13.7 Graphs on Logarithmic and Semilogarithmic Paper

Logarithmic Scale • Semilogarithmic (Semilog) Paper • Logarithmic (Log-Log) Paper

When constructing the graphs of some functions, one of the variables changes much more rapidly than the other. We saw this in graphing the exponential and logarithmic functions in Sections 13.1 and 13.2. The following example illustrates this point.

### EXAMPLE 1 Graph of an exponential function

Plot the graph of  $y = 4(3^x)$ .

Constructing the following table of values,

$x$	-1	0	1	2	3	4	5
$y$	1.3	4	12	36	108	324	972

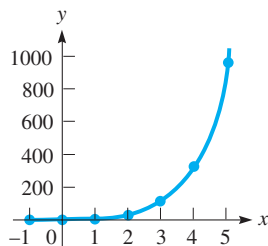


Fig. 13.12

we then plot these values as shown in Fig. 13.12.

We see that as  $x$  changes from  $-1$  to  $5$ ,  $y$  changes much more rapidly, from about 1 to nearly 1000. Also, because of the scale that must be used, we see that it is not possible to show accurately the differences in the  $y$ -values on the graph. ■

It is possible to graph a function with a large change in values, for one or both variables, more accurately than can be done on the standard rectangular coordinate system. This is done by using a scale marked off in distances proportional to the logarithms of the values being represented. Such a scale is called a **logarithmic scale**. For example,  $\log 1 = 0$ ,  $\log 2 = 0.301$ , and  $\log 10 = 1$ . Thus, on a logarithmic scale, the 2 is placed 0.301 unit of distance from the 1 to the 10. Fig. 13.13 shows a logarithmic scale with the numbers represented and the distance used for each.

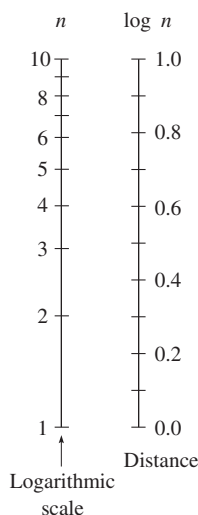


Fig. 13.13

On a logarithmic scale, the distances between the integers are not equal, but this scale does allow for a much greater range of values and much greater accuracy for many of the values. There is another advantage to using logarithmic scales. Many equations that would have more complex curves when graphed on the standard rectangular coordinate system will have simpler curves, often straight lines, when graphed using logarithmic scales. In many cases, this makes the analysis of the curve much easier.

Zero and negative numbers do not appear on the logarithmic scale. In fact, all numbers used on the logarithmic scale must be positive, since the domain of the logarithmic function includes only positive real numbers. Thus, the logarithmic scale must start at some number greater than zero. This number must be a power of 10 and can be very small, say,  $10^{-6} = 0.000\,001$ , but still positive.

If we wish to use a large range of values for only one of the variables, we use what is known as **semilogarithmic**, or **semilog**, graph paper. On this graph paper, only one

axis (usually the y-axis) uses a logarithmic scale. If we wish to use a large range of values for both variables, we use **logarithmic**, or **log-log**, graph paper. Both axes are marked with logarithmic scales.

The following examples illustrate the use of semilog and log-log graph paper.

**EXAMPLE 2 Graph on semilogarithmic paper**

Construct the graph of  $y = 4(3^x)$  on semilogarithmic graph paper.

This is the same function as in Example 1, and we repeat the table of values:

$x$	-1	0	1	2	3	4	5
$y$	1.3	4	12	36	108	324	972

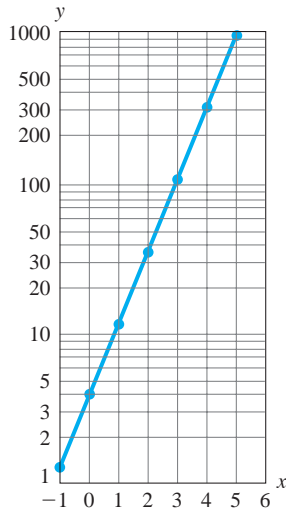


Fig. 13.14

Again, we see that the range of y-values is large. When we plotted this curve on the rectangular coordinate system in Example 1, we had to use large units along the y-axis. This made the values of 1.3, 4, 12, and 36 appear at practically the same level. However, when we use semilog graph paper, we can label each axis such that all y-values are as accurately plotted as the x-values.

The logarithmic scale is shown in **cycles**, and we must label the base line of the first cycle as 1 times a power of 10 (0.01, 0.1, 1, 10, 100, and so on) with the following cycle labelled with the next power of 10. The lines between are labelled with 2, 3, 4, and so on, times the proper power of 10. See the vertical scale in Fig. 13.14. The logarithmic scale in Fig. 13.14 has *three cycles*, since all values of three powers of 10 are represented.

We now plot the points in the table on the graph. The resulting graph is a straight line, as we see in Fig. 13.14. Taking logarithms of each side of the equation, we have

$$\begin{aligned} \log y &= \log[4(3^x)] = \log 4 + \log 3^x && \text{using Eq. (13.7)} \\ &= \log 4 + x \log 3 && \text{using Eq. (13.9)} \end{aligned}$$

However, since  $\log y$  was plotted automatically (because we used semilogarithmic paper), the graph really represents

$$u = \log 4 + x \log 3$$

where  $u = \log y$ ;  $\log 3$  and  $\log 4$  are constants, and therefore this equation is of the form  $u = mx + b$ , which is a straight line (see Section 5.2). ■

**EXAMPLE 3 Graph on logarithmic paper**

Construct the graph of  $x^4y^2 = 1$  on logarithmic paper.

First, we solve for y and make a table of values. Considering positive values of x and y, we have

$$y = \sqrt{\frac{1}{x^4}} = \frac{1}{x^2}$$

$x$	0.5	1	2	8	20
$y$	4	1	0.25	0.0156	0.0025

We plot these values on log-log paper on which both scales are logarithmic, as shown in Fig. 13.15. We again see that we have a straight line. Taking logarithms of both sides of the equation, we have

$$\begin{aligned} \log(x^4y^2) &= \log 1 \\ \log x^4 + \log y^2 &= 0 && \text{using Eq. (13.7)} \\ 4 \log x + 2 \log y &= 0 && \text{using Eq. (13.9)} \end{aligned}$$

If we let  $u = \log y$  and  $v = \log x$ , we then have

$$4v + 2u = 0 \quad \text{or} \quad u = -2v$$

which is the equation of a straight line, as shown in Fig. 13.15. Note, however, that not all graphs on logarithmic paper are straight lines. ■

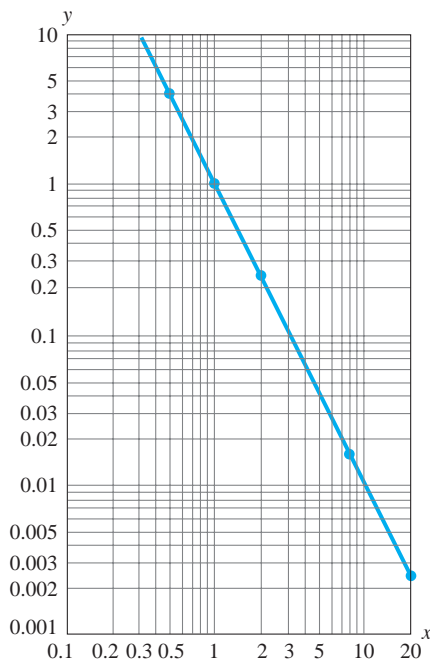


Fig. 13.15

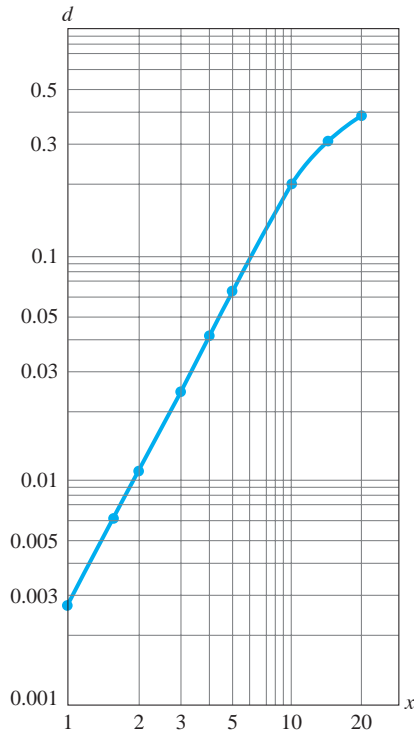


Fig. 13.16

**EXAMPLE 4** Deflection of a beam graphed on log-log paper

The deflection (in m) of a certain cantilever beam as a function of the distance  $x$  (in m) from one end is

$$d = 0.0001(30x^2 - x^3)$$

If the beam is 20.0 m long, plot a graph of  $d$  as a function of  $x$  on log-log paper.

Constructing a table of values, we have

$x$ (m)	1.00	1.50	2.00	3.00	4.00
$d$ (m)	0.002 90	0.006 41	0.0112	0.0243	0.0416

$x$ (m)	5.00	10.0	15.0	20.0
$d$ (m)	0.0625	0.200	0.338	0.400

Since the beam is 20.0 m long, there is no meaning to values of  $x$  greater than 20.0 m. The graph is shown in Fig. 13.16. ■

Logarithmic and semilogarithmic paper may be useful for plotting data derived from experimentation. Often, the data cover too large a range of values to be plotted on ordinary graph paper. The next example illustrates the use of semilogarithmic paper to plot data.

**EXAMPLE 5** Vapour pressure graphed on semilog paper

The vapour pressure of water depends on the temperature. The following table gives the vapour pressure (in kPa) for the corresponding values of temperature (in °C):

$T$ (°C)	10	20	40	60	80	100	120	140	160
$p$ (kPa)	1.19	2.33	7.34	19.9	47.3	101	199	361	617

These data are then plotted on semilogarithmic paper, as shown in Fig. 13.17. Intermediate values of temperature and pressure can then be read directly from the graph. ■

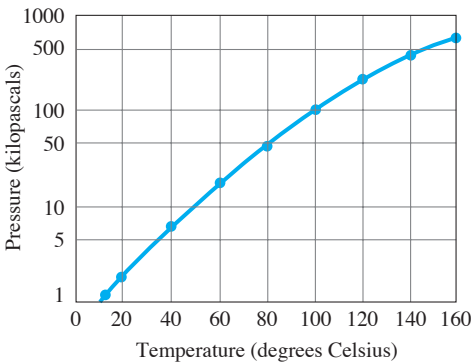


Fig. 13.17

**EXERCISES 13.7**

In Exercises 1 and 2, make the given changes in the indicated examples, and then draw the graphs.

- In Example 2, change the 4 to 2 and then make the graph.
- In Example 3, change the 1 to 4 and then make the graph.

In Exercises 3–10, plot the graphs of the given functions on semilogarithmic paper.

- $y = 2^x$
- $y = 6^{-x}$
- $y = 2x^3 + 6x$
- $y = 5^x$
- $y = 5(4^x)$
- $y = x^3$
- $y = 2x^4$
- $y = 4x^3 + 2x^2$

In Exercises 11–18, plot the graphs of the given functions on log-log paper.

- $y = 0.01x^4$
- $xy = 40$
- $x^2y^2 = 25$
- $y = \sqrt{x}$
- $xy = 40$
- $x^2y^3 = 16$
- $x^3y = 8$

In Exercises 19–26, determine the type of graph paper on which the graph of the given function is a straight line. Using the appropriate paper, sketch the graph.

- $y = 3^{-x}$
- $y = 0.2x^3$
- $y = 3x^6$



22.  $y = 5(10^{-x})$     23.  $y = 4^{x/2}$     24.  $xy^3 = 10$   
 25.  $x\sqrt{y} = 4$     26.  $y(2^x) = 3$

In Exercises 27–38, plot the indicated graphs.

27. On the moon, the distance  $s$  (in m) a rock will fall due to gravity is  $s = 0.81t^2$ , where  $t$  is the time (in s) of fall. Plot the graph of  $s$  as a function of  $t$  for  $0 \leq t \leq 10$  s on (a) a regular rectangular coordinate system and (b) a logarithmic coordinate system.
28. By pumping, the air pressure in a tank is reduced by 18% each second. Thus, the pressure  $p$  (in kPa) in the tank is given by  $p = 101(0.82)^t$ , where  $t$  is the time (in s). Plot the graph of  $p$  as a function of  $t$  for  $0 \leq t \leq 30$  s on (a) a regular rectangular coordinate system and (b) a semilogarithmic coordinate system.
29. Strontium-90 decays according to the equation  $N = N_0e^{-0.028t}$ , where  $N$  is the amount present after  $t$  years and  $N_0$  is the original amount. Plot  $N$  as a function of  $t$  on semilog paper if  $N_0 = 1000$  g.
30. The electric power  $P$  (in W) in a certain battery as a function of the resistance  $R$  (in  $\Omega$ ) in the circuit is given by  $P = \frac{100R}{(0.50 + R)^2}$ . Plot  $P$  as a function of  $R$  on semilog paper, using the logarithmic scale for  $R$  and values of  $R$  from  $0.01 \Omega$  to  $10 \Omega$ . Compare the graph with that in Fig. 3.19 on page 101.
31. The acceleration  $g$  (in  $m/s^2$ ) produced by the gravitational force of the earth on a spacecraft is given by  $g = 3.99 \times 10^{14}/r^2$ , where  $r$  is the distance from the centre of the earth to the spacecraft. On log-log paper, graph  $g$  as a function of  $r$  from  $r = 6.37 \times 10^6$  m (the earth's surface) to  $r = 3.91 \times 10^8$  m (the distance to the moon).
32. In undergoing an adiabatic (no *heat* gained or lost) expansion of a gas, the relation between the pressure  $p$  (in kPa) and the volume  $v$  (in  $m^3$ ) is  $p^2v^3 = 850$ . On log-log paper, graph  $p$  as a function of  $v$  from  $v = 0.10 \text{ m}^3$  to  $v = 10 \text{ m}^3$ .
33. The number of cell phone subscribers in Canada from 1985 to 2011 is shown in the following table. Plot  $N$  as a function of the year on semilog paper.

Year	1985	1990	1995	2000	2005	2010	2011
$N (\times 10^6)$	0.006	0.584	2.59	8.73	17.0	25.8	27.4

34. The period  $T$  (in years) and mean distance  $d$  (given as a ratio of that of earth) from the sun to the planets (Mercury, Venus, Earth, Mars, Jupiter, Saturn, Uranus, Neptune) are given below. Plot  $T$  as a function of  $d$  on log-log paper.

Planet	M	V	E	M	J	S	U	N
$d$	0.39	0.72	1.00	1.52	5.20	9.54	19.2	30.1
$T$	0.24	0.62	1.00	1.88	11.9	29.5	84.0	165

35. The intensity level  $B$  (in dB) and the frequency  $f$  (in Hz) for a sound of constant loudness were measured as shown in the table that follows. Plot the data for  $B$  as a function of  $f$  on semilog paper, using the log scale for  $f$ .

$f$ (Hz)	100	200	500	1000	2000	5000	10 000
$B$ (dB)	40	30	22	20	18	24	30

36. The atmospheric pressure  $p$  (in kPa) at a given altitude  $h$  (in km) is given in the following table. On semilog paper, plot  $p$  as a function of  $h$ .

$h$ (km)	0	10	20	30	40
$p$ (kPa)	101	25	6.3	2.0	0.53

37. One end of a very hot steel bar is sprayed with a stream of cool water. The rate of cooling  $R$  (in  $^\circ\text{C/s}$ ) as a function of the distance  $d$  (in cm) from one end of the bar is then measured, with the results shown in the following table. On log-log paper, plot  $R$  as a function of  $d$ . Such experiments are made to determine the hardness of steel.

$d$ (cm)	0.063	0.13	0.19	0.25
$R$ ( $^\circ\text{C/s}$ )	600	190	100	72

$d$ (cm)	0.38	0.50	0.75	1.0	1.5
$R$ ( $^\circ\text{C/s}$ )	46	29	17	10	6.0

38. The magnetic intensity  $H$  (in A/m) and flux density  $B$  (in teslas) of annealed iron are given in the following table. Plot  $H$  as a function of  $B$  on log-log paper.

$B$ (T)	0.0042	0.043	0.67	1.01
$H$ (A/m)	10	50	100	150

$B$ (T)	1.18	1.44	1.58	1.72
$H$ (A/m)	200	500	1000	10 000

In Exercises 39 and 40, plot the indicated semilogarithmic graphs for the following application.

In a particular electric circuit, called a low-pass filter, the input voltage  $V_i$  is across a resistor and a capacitor, and the output voltage  $V_0$  is across the capacitor (see Fig. 13.18). The voltage gain  $G$  (in dB) is given by

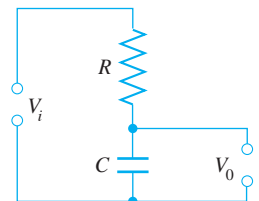


Fig. 13.18

$$G = 20 \log \frac{1}{\sqrt{1 + (\omega T)^2}}$$

where  $\tan \phi = -\omega T$ .

Here,  $\phi$  is the phase angle of  $V_0/V_i$ . For values of  $\omega T$  of 0.01, 0.10, 0.30, 1.0, 3.0, 10.0, 30.0, and 100, plot the indicated graphs. These graphs are called a Bode diagram for the circuit.

39. Calculate values of  $G$  for the given values of  $\omega T$  and plot a semilogarithmic graph of  $G$  vs.  $\omega T$ .
40. Calculate values of  $\phi$  (as negative angles) for the given values of  $\omega T$  and plot a semilogarithmic graph of  $\phi$  vs.  $\omega T$ .



## CHAPTER 13 EQUATIONS

<b>Exponential function</b>	$y = b^x$	(13.1)
<b>Logarithmic form</b>	$x = \log_b y$	(13.2)
<b>Logarithmic function</b>	$y = \log_b x$	(13.3)
<b>Laws of exponents</b>	$b^u b^v = b^{u+v}$	(13.4)
	$\frac{b^u}{b^v} = b^{u-v}$	(13.5)
	$(b^u)^n = b^{nu}$	(13.6)
<b>Properties of logarithms</b>	$\log_b xy = \log_b x + \log_b y$	(13.7)
	$\log_b \left(\frac{x}{y}\right) = \log_b x - \log_b y$	(13.8)
	$\log_b (x^n) = n \log_b x$	(13.9)
	$\log_b 1 = 0 \quad \log_b b = 1$	(13.10)
	$\log_b (b^n) = n$	(13.11)
<b>Changing base of logarithms</b>	$\log_b x = \frac{\log_a x}{\log_a b}$	(13.12)
	$\ln x = \frac{\log x}{\log e}$	(13.13)
	$\log x = \frac{\ln x}{\ln 10}$	(13.14)

## CHAPTER 13 REVIEW EXERCISES

In Exercises 1–12, determine the value of  $x$ .

- $\log_{10} x = 4$
- $\log_9 x = 3$
- $\log_5 x = -1$
- $\log_4 (\sin x) = -0.5$
- $2 \log_2 8 = x$
- $\log_{12} 144 = x - 3$
- $\log_8 32 = x$
- $\log_9 27 = x$
- $\log_x 36 = 2$
- $\log_x (1/243) = 5$
- $\log_x 10 = \frac{1}{2}$
- $\log_x 8 = 0$

In Exercises 13–24, express each as a sum, difference, or multiple of logarithms. Wherever possible, evaluate logarithms of the result.

- $\log_3 2x$
- $\log_5 \left(\frac{7}{a}\right)$
- $\log_3 (t^2)$
- $\log_6 \sqrt{5}$
- $\log_2 28$
- $\log_7 98$
- $\log_4 \sqrt{48}$
- $\log_2 \sqrt[4]{32y}$
- $\log_3 \left(\frac{9}{x}\right)$
- $\log_6 \left(\frac{5}{36}\right)$
- $\log_{10} (1000x^4)$
- $\log_3 (9^2 \times 6^3)$

In Exercises 25–36, solve for  $y$  in terms of  $x$ .

- $\log_6 y = \log_6 4 - \log_6 x$
- $2 \ln y = \ln e^2 - 3 \ln x$
- $3 \ln y = 2 + 3 \ln x$
- $2(\log_9 y + 2 \log_9 x) = 1$
- $\log_3 y = \frac{1}{2} \log_3 7 + \frac{1}{2} \log_3 x$
- $(\log_2 3)(\log_2 y - \log_2 x) = 3$
- $\log_5 x + \log_5 y = \log_5 3 + 1$
- $\log_7 y = 2 \log_7 5 + \log_7 x + 2$
- $2(\log_4 y - 3 \log_4 x) = 3$
- $\frac{\log_7 x}{\log_7 4} - \log_7 y = 1$
- $2^y = e^x$
- $10^y = 3^{x+1}$

In Exercises 37–44, display the graphs of the given functions on a graphing calculator.

- $y = 0.5(5^x)$
- $y = 3(2^{-x})$
- $R = 0.2 \log_4 r$
- $y = 10 \log_{16} x$
- $y = \log_{3.15} x$
- $y = 0.1 \log_{4.05} x$
- $y = 1 - e^{-|x|}$
- $s = 2(1 - e^{-0.2t})$

In Exercises 45–48, use logarithms to the base 10 to find the natural logarithms of the given numbers.

45. 8.86                                      46. 0.303  
47.  $\sin 2.07$                                 48.  $\sqrt{0.542}$

In Exercises 49–52, use natural logarithms to find logarithms to the base 10 of the given numbers.

49. 65.89                                      50. 0.0781  
51.  $0.1197^3$                                 52.  $\log 1000$

In Exercises 53–60, solve the given equations. Round all results to 3 significant digits.

53.  $e^{2x} = 5$                                     54.  $2(5^x) = 15$   
55.  $3^{x+2} = 5^x$                                 56.  $2^x/3^{1-x} = 12^{x+1}$   
57.  $\log_4 z + \log_4 6 = \log_4 12$             58.  $\log_8(x+2) = 2 - \log_8 2$   
59.  $2 \log_3 2 - \log_3(x+1) = \log_3 5$   
60.  $\log(n+2) + \log n = 0.4771$

In Exercises 61 and 62, plot the graphs of the given functions on semilogarithmic paper. In Exercises 63 and 64, plot the graphs of the given functions on log-log paper.

61.  $y = 8^x$                                       62.  $y = 5x^3$   
63.  $y = \sqrt[3]{x}$                                     64.  $xy^4 = 16$

If  $x$  is eliminated between Eqs. (13.1) and (13.2), we have

$$y = b^{\log_b y} \quad (13.15)$$

In Exercises 65–68, evaluate the given expressions using Eq. (13.15).

65.  $10^{\log 4}$                                     66.  $2e^{\ln 7.5}$   
67.  $3e^{2 \ln 2}$                                     68.  $5(10^{2 \log 3})$

In Exercises 69–104, solve the given problems. When necessary, round results to 3 significant digits.

69. Use a calculator to verify that  $2 \log 3 - \log 6 = \log 1.5$ .  
70. Use a calculator to verify that  $3 \ln 2 + 0.5 \ln 64 = 3 \ln 4$ .  
71. Evaluate  $\sqrt[3]{\ln e^8} - \sqrt{\log 10^4}$ .  
72. Solve for  $x$ :  $2^x + 32(2^{-x}) = 12$ .  
73. If  $f(x) = 2 \log_b x$  and  $f(8) = 3$ , find  $f(2)$ .  
74. Evaluate:  $\log(2 \log 1000)$ .  
75. Evaluate:  $7(10^{\log 0.1}) + 6000(100^{\log 0.001})$ .  
76. If  $x = \log_b 7$  and  $y = \log_b 2$ , express  $\log_b 42$  in terms of  $x$  and  $y$ .  
77. Use a graphing calculator to solve the equation  $\log_5 x = 2x - 7$ .  
78. Use logarithms to find the  $x$ -intercept of the graph of  $y = 2 - 5^{2x-3}$ .  
79. If an amount of  $P$  dollars is invested at an annual interest rate  $r$  (expressed as a decimal), the value  $V$  of the investment after  $t$  years is  $V = P(1 + r/n)^{nt}$ , if interest is compounded  $n$  times a year. If \$1000 is invested at an annual interest rate of 6%, compounded semiannually, express  $V$  as a function of  $t$  and solve for  $t$ .  
80. The current  $i$  (in A) in a certain electric circuit is given by  $i = 16(1 - e^{-250t})$ , where  $t$  is the time (in s). Solve for  $t$ .  
81. The formula  $\ln(I/I_0) = -\beta h$  is used in estimating the thickness of the ozone layer. Here,  $I_0$  is the intensity of a wavelength

of sunlight before reaching the earth's atmosphere,  $I$  is the intensity of the light after passing through  $h$  cm of the ozone layer, and  $\beta$  is a constant. Solve for  $I$ .

82. The amount  $A$  of cesium-137 (a dangerous radioactive element) remaining after  $t$  years is given by  $A = A_0(0.5^{t/30.3})$ , where  $A_0$  is the initial amount. In what year will the cesium-137 be 10% of the amount released at the Fukushima Daiichi nuclear power plant in Japan after the March 2011 earthquake and tsunami damaged its cooling systems?  
83. The bending moment  $M$  (in  $\text{N} \cdot \text{m}$ ) of a particular concrete column is given by  $\log M = 6.663$ . What is the value of  $M$ ?  
84. A state lottery pays \$500 for a \$1 ticket if a person picks the correct three-digit number determined by the random draw of three numbered balls. The probability  $p$  of winning this lottery at least once in  $x$  attempts is given by  $p = 1 - 0.999^x$ . How many attempts are necessary for a person to have a 50% chance ( $p = 0.50$ ) of winning this lottery at least once?  
85. An approximate formula for the population (in millions) of India since 1995 is  $P = 937e^{0.0137t}$ , where  $t$  is the number of years since 1995. Sketch the graph of  $P$  vs.  $t$  for 1995 to 2020.  
86. A computer analysis of the luminous efficiency  $E$  (in lumens/W) of a tungsten lamp as a function of its input power  $P$  (in W) is given by  $E = 22.0(1 - 0.65e^{-0.008P})$ . Sketch the graph of  $E$  as a function of  $P$  for  $0 \leq P \leq 1000$  W.  
87. An original amount of 100 mg of radium radioactively decomposes such that  $N$  mg remain after  $t$  years. The function relating  $t$  and  $N$  is  $t = 2350(\ln 100 - \ln N)$ . Sketch the graph.  
88. The time  $t$  (in s) to chemically change 5 kg of a certain substance into another is given by  $t = -5 \log\left(\frac{5-x}{5}\right)$ , where  $x$  is the number of kilograms that have been changed at any time. Sketch the graph.  
89. An equation that may be used for the angular velocity  $\omega$  of the slider mechanism in Fig. 13.19 is  $2 \ln \omega = \ln 3g + \ln \sin \theta - \ln l$ , where  $g$  is the acceleration due to gravity. Solve for  $\sin \theta$ .

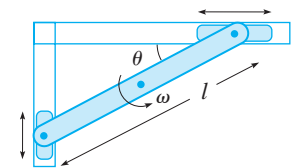


Fig. 13.19

90. Taking into account the weight loss of fuel, the maximum velocity  $v_m$  of a rocket is  $v_m = u(\ln m_0 - \ln m_s) - gt_f$ , where  $m_0$  is the initial mass of the rocket and fuel,  $m_s$  is the mass of the rocket shell,  $t_f$  is the time during which fuel is expended,  $u$  is the velocity of the expelled fuel, and  $g$  is the acceleration due to gravity. Solve for  $m_0$ .  
91. An equation used to calculate the capacity  $C$  (in bits/s) of a telephone channel is  $C = B \log_2(1 + R)$ , where  $B$  is the bandwidth of the channel (in Hz) and  $R$  is the signal-to-noise ratio. Solve for  $R$ .  
92. An equation used in studying the action of a protein molecule is  $\ln A = \ln \theta - \ln(1 - \theta)$ . Solve for  $\theta$ .

93. The magnitudes (visual brightnesses),  $m_1$  and  $m_2$ , of two stars are related to their (actual) brightnesses,  $b_1$  and  $b_2$ , by the equation  $m_1 - m_2 = 2.5 \log(b_2/b_1)$ . As a result of this definition, magnitudes may be negative, and *magnitudes decrease as brightnesses increase*. The magnitude of the brightest star, Sirius, is  $-1.4$ , and the magnitudes of the faintest stars observable with the naked eye are about  $6.0$ . How much brighter is Sirius than these faintest stars?
94. The power gain of an electronic device such as an amplifier is defined as  $n = 10 \log(P_0/P_i)$ , where  $n$  is measured in decibels,  $P_0$  (in W) is the power output, and  $P_i$  (in W) is the power input. If  $P_0 = 10.0$  W and  $P_i = 0.125$  W, calculate the power gain. (See Example 4 on page 383.)
95. In studying the frictional effects on a flywheel, the revolutions per minute  $R$  that it makes as a function of the time  $t$  (in min) are given by  $R = 4520(0.750)^{2.50t}$ . Find  $t$  for  $R = 1950$  r/min.
96. The efficiency  $\eta$  of a gasoline engine as a function of its compression ratio  $r$  is given by  $\eta = 1 - r^{1-\gamma}$ , where  $\gamma$  is a constant. Find  $\gamma$  for  $\eta = 0.55$  and  $r = 7.5$ .
97. The intensity  $I$  of light decreases from its value  $I_0$  as it passes a distance  $x$  through a medium. Given that  $x = k(\ln I_0 - \ln I)$ , where  $k$  is a constant depending on the medium, find  $x$  for  $I = 0.850I_0$  and  $k = 5.00$  cm.
98. According to Benford's law, in lists of numbers from many real-life sources of data (such as tables of physical constants, accounting data, or scientific calculations), the leading digit 1 occurs much more often than the others. In particular, the probability that the first digit of a randomly selected number from such a list is  $n$  is given by

$$P(n) = \log\left(\frac{n+1}{n}\right)$$

How many times more likely is a 1 than a 9 as the leading digit of a number in a data set that satisfies Benford's law?

99. Under certain conditions, the temperature  $T$  and pressure  $p$  are related by the following equation, where  $T_0$  is the temperature at pressure  $p_0$ :  $\frac{T}{T_0} = \left(\frac{p}{p_0}\right)^{\frac{k-1}{k}}$ . Solve for  $k$ .

100. The temperature  $T$  of an object with an initial temperature  $T_1$  in water at temperature  $T_0$  as a function of the time  $t$  is given by  $T = T_1 + (T_0 - T_1)e^{-kt}$ . Solve for  $t$ .
101. Pure water is running into a brine solution, and the same amount of solution is running out. The number  $n$  of kilograms of salt in the solution after  $t$  min is found by solving the equation  $\ln n = -0.04t + \ln 20$ . Solve for  $n$  as a function of  $t$ .
102. For the circuit in Fig. 13.20, the current  $i$  (in mA) is given by  $i = 1.6e^{-100t}$ . Plot the graph of  $i$  as a function of  $t$  for the first 0.05 s on semilog paper.

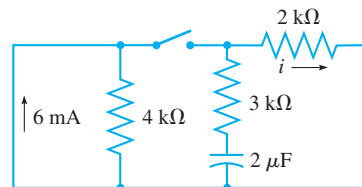


Fig. 13.20

103. For a particular solar-energy system, the collector area  $A$  required to supply a fraction  $F$  of the total energy is given by  $A = 480F^{2.2}$ . Plot  $A$  (in  $m^2$ ) as a function of  $F$ , from  $F = 0.1$  to  $F = 0.9$ , on semilog paper.
104. The current  $I$  (in  $\mu A$ ) and resistance  $R$  (in  $\Omega$ ) were measured as follows in a certain microcomputer circuit:

$R$ ( $\Omega$ )	100	200	500	1000	2000	5000	10 000
$I$ ( $\mu A$ )	81	41	16	8.2	4.0	1.6	0.8

Plot  $I$  as a function of  $R$  on log-log paper.

### Writing Exercise

105. While checking logarithmic curves on a calculator, a machine-design student noted that a certain robotic arm was shaped like part of the graph of  $y = \ln(2x^{2/3})$ . As a check, the student re-wrote the equation as  $y = (2 \ln x)/3 + \ln 4 - \ln(\ln e^2)$ . Write a paragraph explaining (a) whether the second equation is equivalent to the first, and (b) if the graphs of the two equations are identical.

## CHAPTER 13 PRACTICE TEST

In Problems 1–4, determine the value of  $x$ .

- $\log_9 x = -\frac{1}{2}$
- $\log_3 x - \log_3 2 = 2$
- $\log_x 64 = 3$
- $3^{3x+1} = 8$
- Graph the function  $y = 2 \log_4 x$ .
- Graph the function  $y = 2(3^x)$  on semilog paper.
- Express  $\log_5\left(\frac{4a^3}{7}\right)$  as a combination of a sum, difference, and multiple of logarithms, including  $\log_5 2$ .
- Solve for  $y$  in terms of  $x$ :  $3 \log_7 x - \log_7 y = 2$ .

- An equation used for a certain electric circuit is  $\ln i - \ln I = -t/RC$ . Solve for  $i$ .
- Evaluate:  $\frac{2 \ln 0.9523}{\log 6066}$ .
- Evaluate:  $\log_5 7.32$ .
- If  $A_0$  dollars are invested at 8%, compounded continuously for  $t$  years, the value  $A$  of the investment is given by  $A = A_0 e^{0.08t}$ . Determine how long it takes for the investment to double in value.

# Additional Types of Equations and Systems of Equations

In this chapter, we discuss graphical and algebraic solutions of systems of equations of types different from those of earlier chapters. We also consider solutions of two special types of equations.

One of the methods involves the use of graphs for the solutions, similar to that we used in Chapter 3. The intersection of curves was very much part of the basic method of solution of equations used in the mid-1600s by René Descartes, who developed the coordinate system. Since that time, graphical solutions of equations have been very common and very useful in science and technology.

In the study of optics in the 1800s, it was found that light travelled much faster in free space than in other mediums, such as glass and water. Scientists then defined  $n$ , the *index of refraction*, to be the ratio of the speed of light in free space to the speed of light in a particular substance. In 1836, the French mathematician Cauchy developed the equation  $n = A + B\lambda^{-2} + C\lambda^{-4}$  that related the index of refraction with the wavelength of light  $\lambda$  in a medium. Once the constants  $A$ ,  $B$ , and  $C$  were found for a particular medium (by solving three simultaneous equations, as in Chapter 5), this equation can be solved for  $\lambda$  for a particular value of  $n$ , using a method known at the time that is used in this chapter (see Exercise 36 on page 410). Again, we see that an earlier mathematical method was useful in dealing with a new scientific discovery.

Applications of the types of equations and systems of equations of this chapter are found in many fields of science and technology. These include physics, electricity, business, and structural design.

# 14

## LEARNING OUTCOMES

After completion of this chapter, the student should be able to:

- Graph conic sections using a calculator
- Solve systems involving conics and other nonlinear equations graphically
- Solve systems involving nonlinear equations algebraically by the method of substitution or by the method of addition and subtraction
- Solve application problems involving systems of nonlinear equations
- Solve equations in quadratic form
- Solve equations with radicals
- Solve application problems involving equations in quadratic form or equations with radicals



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◀ A three-dimensional holographic image is formed without the use of a lens. In Section 14.4, the dimensions of a holograph are determined.

## 14.1 Graphical Solution of Systems of Equations

Conic Sections • Parabola, Circle, Ellipse,  
Hyperbola • Solving Systems of Equations

In this section, we first discuss the graphs of the *circle*, *parabola*, *ellipse*, and *hyperbola*, which are known as the **conic sections**. We discussed the parabola earlier when discussing quadratic functions in Chapter 7. Then we will find graphical solutions of systems of equations involving these and other nonlinear equations.

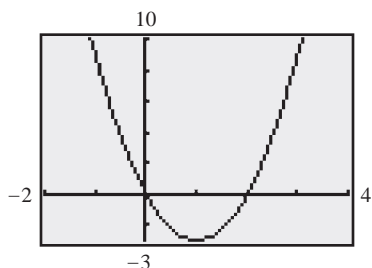


Fig. 14.1

### EXAMPLE 1 Calculator graph—parabola

Graph the equation  $y = 3x^2 - 6x$ .

We graphed equations of this form in Section 7.4. Since the general quadratic function is  $y = ax^2 + bx + c$ , for  $y = 3x^2 - 6x$ , we have  $a = 3$ ,  $b = -6$ , and  $c = 0$ . Therefore,  $-b/(2a) = 1$ , which means the  $x$ -coordinate of the vertex is  $-(-6)/6 = 1$ . Since  $y = -3$  for  $x = 1$ , the vertex is  $(1, -3)$ . It is a minimum point since  $a > 0$ .

Knowing the vertex and the fact that the graph goes through the origin ( $c = 0$ ), we choose appropriate *window* settings and have the display shown in Fig. 14.1.

As we showed in Section 7.4, the curve is a **parabola**, and a parabola always results if the equation is of the form of the quadratic function  $y = ax^2 + bx + c$ . ■

### EXAMPLE 2 Plotted graph—circle

Plot the graph of the equation  $x^2 + y^2 = 25$ .

We first solve this equation for  $y$ , and we obtain  $y = \sqrt{25 - x^2}$ , or  $y = -\sqrt{25 - x^2}$ , which we write as  $y = \pm\sqrt{25 - x^2}$ . We now assume values for  $x$  and find the corresponding values for  $y$ .

$x$	0	$\pm 1$	$\pm 2$	$\pm 3$	$\pm 4$	$\pm 5$
$y$	$\pm 5$	$\pm 4.9$	$\pm 4.6$	$\pm 4$	$\pm 3$	0

If  $x > 5$ , values of  $y$  are imaginary. We cannot plot these because  $x$  and  $y$  must both be real. When we show  $y = \pm 3$  for  $x = \pm 4$ , this is a short way of representing four points. These points are  $(4, 3)$ ,  $(4, -3)$ ,  $(-4, 3)$ , and  $(-4, -3)$ .

In Fig. 14.2, *the resulting curve is a circle*. A circle with its centre at the origin results from an equation of the form  $x^2 + y^2 = r^2$ , where  $r$  is the radius. ■

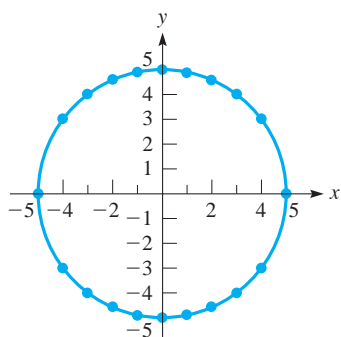


Fig. 14.2

■ See the *vertical-line test* on page 102.

From the graph of the circle in Fig. 14.2, we see that *the equation of a circle does not represent a function*. There are *two* values of  $y$  for most of the values of  $x$  in the domain. We must take this into account when displaying the graph of such an equation on a graphing calculator. This is illustrated in the next example.

### EXAMPLE 3 Calculator graph—ellipse

Display the graph of the equation  $2x^2 + 5y^2 = 10$  on a graphing calculator.

First, solving for  $y$ , we get  $y = \pm\sqrt{\frac{10 - 2x^2}{5}}$ . To display the graph of this equation on a calculator, *we must enter both functions*, one as  $y_1 = \sqrt{(10 - 2x^2)/5}$ , and the other as  $y_2 = -\sqrt{(10 - 2x^2)/5}$ .

Trying some *window* settings (or noting that the domain is from  $-\sqrt{5}$  to  $\sqrt{5}$  and the range is from  $-\sqrt{2}$  to  $\sqrt{2}$ ), we get the graphing calculator display that is shown in Fig. 14.3.

*The curve is an ellipse*. An ellipse will be the resulting curve if the equation is of the form  $ax^2 + by^2 = c$ , where the constants  $a$ ,  $b$ , and  $c$  have the same sign, and for which  $a \neq b$ . ■

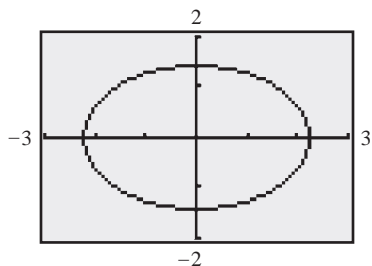


Fig. 14.3



■ A more complete discussion of the conic sections is found in Chapter 21.

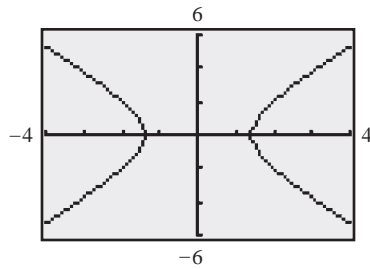


Fig. 14.4

When these types of curves are displayed on the graphing calculator, it is possible that there are small gaps in the curve. The appearance of such gaps depends on the pixel width and the functional values used by the calculator.

**EXAMPLE 4** Calculator graph—hyperbola

Display the graph of  $2x^2 - y^2 = 4$  on a graphing calculator.

Solving for  $y$ , we get

$$y = \pm \sqrt{2x^2 - 4}$$

As in Example 3, we enter two functions in the calculator, one with the plus sign and the other with the minus sign, and we have the display shown in Fig. 14.4. The *window* settings are chosen by noting that the values  $-\sqrt{2} < x < \sqrt{2}$  are not in the domain of either  $y_1 = \sqrt{2x^2 - 4}$  or  $y_2 = -\sqrt{2x^2 - 4}$ . These values of  $x$  would lead to imaginary values of  $y$ .

The curve is a **hyperbola**, which results when we have an equation of the form  $ax^2 + by^2 = c$ , if  $a$  and  $b$  have *different* signs. ■

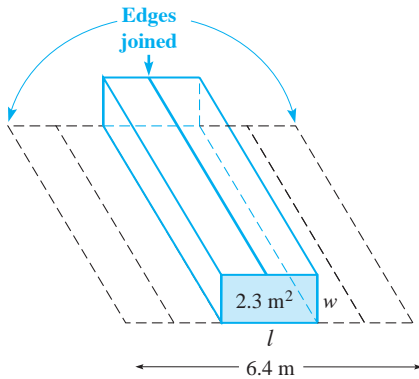


Fig. 14.5

**SOLVING SYSTEMS OF EQUATIONS**

As in solving systems of linear equations, we solve any system by finding the values of  $x$  and  $y$  that satisfy both equations at the same time. To solve a system graphically, we graph the equations and find the coordinates of all points of intersection. If the curves do not intersect, the system has no real solutions.

**EXAMPLE 5** Calculator—solving a system

For proper ventilation, the vent for a hot-air heating system is to have a rectangular cross-sectional area of  $2.3 \text{ m}^2$  and is to be made from sheet metal  $6.4 \text{ m}$  wide. Find the dimensions of this cross-sectional area of the vent.

In Fig. 14.5, we have let  $l$  = the length and  $w$  = the width of the area. Since the area is  $2.3 \text{ m}^2$ , we have  $lw = 2.3$ . Also, since the sheet metal is  $6.4 \text{ m}$  wide, this is the perimeter of the area. This gives us  $2l + 2w = 6.4$ , or  $l + w = 3.2$ . This means that the system of equations to be solved is

$$\begin{aligned} lw &= 2.3 \\ l + w &= 3.2 \end{aligned}$$

Solving each equation for  $l$ , we have

$$l = 2.3/w \quad \text{and} \quad l = 3.2 - w$$

We now display the graphs of these two equations on a graphing calculator, using  $x$  for  $w$  and  $y$  for  $l$ . Since negative values of  $l$  and  $w$  have no meaning to the solution and since the straight line  $l = 3.2 - w$  has intercepts of  $(3.2, 0)$  and  $(0, 3.2)$ , we choose the *window* settings as shown in Fig. 14.6.

Using the *intersect* feature (or the *trace* and *zoom* features) with the graph in Fig. 14.6, we find that the solutions are approximately  $(1.1, 2.1)$  and  $(2.1, 1.1)$ . Using the length as the longer dimension, we have the solution of

$$l = 2.1 \text{ m} \quad \text{and} \quad w = 1.1 \text{ m}$$

We see that this checks with the statement of the problem. ■

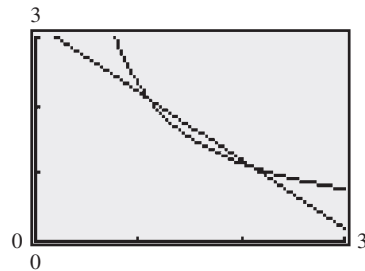


Fig. 14.6

In Example 5, we graphed the equation  $xy = 2.3$  (having used  $x$  for  $w$  and  $y$  for  $l$ ). The graph of this equation is also a *hyperbola*, another form of which is  $xy = c$ . We now show the solutions of two more systems of equations.

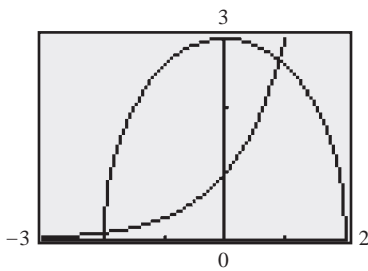


Fig. 14.7

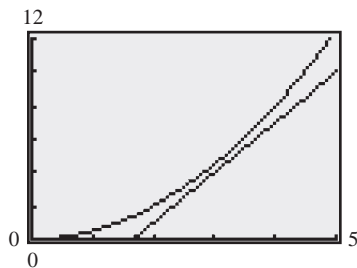


Fig. 14.8

**EXAMPLE 6** Calculator—solving a system

Graphically solve the system of equations

$$\begin{aligned} 9x^2 + 4y^2 &= 36 \\ y &= 3^x \end{aligned}$$

Solving the first equation for  $y$ , we have  $y = \frac{1}{2}\sqrt{36 - 9x^2}$ . Since this is an ellipse, as shown in Example 3, its domain extends from  $x = -2$  to  $x = 2$  ( $9x^2$  cannot be greater than 36). The exponential curve cannot be negative and increases rapidly, as discussed in Chapter 13. This means we need only graph the upper part of the ellipse (using the + sign). Therefore, we choose the *window* settings as shown in the calculator display in Fig. 14.7.

Using the *intersection* feature, we find the approximate solutions of  $x = -2.00$ ,  $y = 0.11$  and  $x = 0.90$ ,  $y = 2.68$ . ■

**EXAMPLE 7** System of equations—no real solutions

Graphically solve the system of equations

$$\begin{aligned} x^2 &= 2y \\ 3x - y &= 5 \end{aligned}$$

Solving the first equation for  $y$ , we get  $y = \frac{1}{2}x^2$ . This parabola has its vertex at the origin, and it opens upward since  $a > 0$ . This means there cannot be a point of intersection for  $y < 0$ .

The straight line has intercepts of  $(0, -5)$  and  $(5/3, 0)$ . This means that any possible point of intersection must be in the first quadrant. Therefore, we have the *window* settings as shown in the calculator display in Fig. 14.8.

From the figure, we see that the two curves do not intersect. This means that there are *no real solutions* to this system of equations. The two complex solutions of the system can be found algebraically using the methods of the next section. ■

**EXERCISES 14.1**

In Exercises 1–4, make the given changes in the indicated examples of this section, and then perform the indicated operations.

- In Example 1, change the  $-$  sign before  $6x$  to  $+$ .
- In Example 2, change the  $+$  sign before  $y^2$  to  $-$ .
- In Example 6, change the coefficient of  $x^2$  to 25.
- In Example 7, change the coefficient of  $y$  in the first equation to 3.

In Exercises 5–30, solve the given systems of equations graphically by using a graphing calculator. Find all values to the nearest 0.01.

- $y = 2x$   
 $x^2 + y^2 = 16$
- $x^2 + 2y^2 = 8$   
 $x - 2y = 4$
- $y = x^2 - 2$   
 $4y = 12x - 17$
- $8y = 11x^2$   
 $xy = 30$
- $y = -x^2 + 4$   
 $x^2 + y^2 = 9$
- $x^2 - 4y^2 = 16$   
 $x^2 + y^2 = 1$
- $3x - y = 4$   
 $y = 6 - 2x^2$
- $y = 3x - 6$   
 $xy = 6$
- $4x^2 + 25y^2 = 21$   
 $10y = 31 - 9x$
- $y = -2x^2$   
 $y = x^2 - 6$
- $y = x^3$   
 $x^2 + 2y^2 = 16$
- $y = 2x^2 - 4x$   
 $x^2y = -4$

$$17. \begin{aligned} 2x^2 + 3y^2 &= 19 \\ x^2 + y^2 &= 9 \end{aligned}$$

$$19. \begin{aligned} x^2 + y^2 &= 7 \\ y(x + 2) &= 3 \end{aligned}$$

$$21. \begin{aligned} y &= x^2 \\ y &= \sin x \end{aligned}$$

$$23. \begin{aligned} y &= e^{-x} \\ x + y &= 2 \end{aligned}$$

$$25. \begin{aligned} x^2 - y^2 &= 7 \\ y &= 4 \log_2 x \end{aligned}$$

$$27. \begin{aligned} y &= \ln(x - 1) \\ y &= \sin \frac{1}{2}x \end{aligned}$$

$$29. \begin{aligned} 10^{x+y} &= 150 \\ y &= x^2 \end{aligned}$$

$$18. \begin{aligned} x^2 - y^2 &= 4 \\ 2x^2 + y^2 &= 16 \end{aligned}$$

$$20. \begin{aligned} x^2 + y^2 &= 0.25 \\ x^2 - y^2 &= 0.07 \end{aligned}$$

$$22. \begin{aligned} y &= 3 + 2x - x^2 \\ y &= 2 \cos 2x \end{aligned}$$

$$24. \begin{aligned} y &= 2^{x+1} \\ x^2 + y^2 &= 4 \end{aligned}$$

$$26. \begin{aligned} x^2 + 4y^2 &= 16 \\ y &= 2 \ln x \end{aligned}$$

$$28. \begin{aligned} y &= \cos x \\ y &= \log_3 x \end{aligned}$$

$$30. \begin{aligned} e^{x^2+y^2} &= 20 \\ xy &= 4 \end{aligned}$$

In Exercise 31, draw the appropriate figures. In Exercises 32–38, set up systems of equations and solve them graphically.

- By drawing rough sketches, show that a parabola and an ellipse can have 0, 1, 2, 3, or 4 possible points of intersection.
- A rectangular security area is to be enclosed by fencing and divided in two equal parts of 1600 m<sup>2</sup> each by a fence parallel to the shorter sides. Find the dimensions of the security area if the total amount of fencing is 280 m.



33. A helicopter is located 5.2 km north of east of a radio tower such that it is three times as far north as it is east from the tower. Find the northern and eastern components of the displacement from the tower.
34. A 4.60-m insulating strip is placed completely around a rectangular solar panel with an area of  $1.20 \text{ m}^2$ . What are the dimensions of the panel?
35. The power developed in an electric resistor is  $i^2R$ , where  $i$  is the current. If a first current passes through a  $2.0\text{-}\Omega$  resistor and a second current passes through a  $3.0\text{-}\Omega$  resistor, the total power produced is 12 W. If the resistors are reversed, the total power produced is 16 W. Find the currents (in A) if  $i > 0$ .
36. A circular hot tub is located on the square deck of a home. The side of the deck is 7.3 m more than the radius of the hot tub, and there are  $72.5 \text{ m}^2$  of deck around the tub. Find the radius of the hot tub and the length of the side of the deck. Explain your answer.
37. Assume earth is a sphere, with  $x^2 + y^2 = 41$  as the equation of a circumference (distance in thousands of km). If a meteorite approaching earth has a path described as  $y^2 = 20x + 140$ , will the meteorite strike earth? If so, where?
38. Two people meet at the intersection of two perpendicular roads. Each then leaves along a different road with one walking  $1.0 \text{ km/h}$  faster than the other. If they are  $7.0 \text{ km}$  apart (on a direct line) after  $1.0 \text{ h}$ , how fast is each walking?

## 14.2 Algebraic Solution of Systems of Equations

**Solution by Substitution • Solution by Addition or Subtraction**

Often, using the graphical method is the easiest way to solve a system of equations. With a graphing calculator, it is possible to find the result with good accuracy. However, the graphical method does not usually give the *exact* answer. Using algebraic methods to find exact solutions for some systems of equations is either not possible or quite involved. There are systems, however, for which there are relatively simple algebraic solutions. In this section, we consider two useful methods, both of which we discussed before when we were studying systems of linear equations.

### SOLUTION BY SUBSTITUTION

The first method is *substitution*. If we can solve one of the equations for one of its variables, we can substitute this solution into the other equation. We then have only one unknown in the resulting equation, and we can then solve this equation by methods discussed in earlier chapters.

#### EXAMPLE 1 Solution by substitution

By substitution, solve the system of equations

$$\begin{aligned} 2x - y &= 4 \\ x^2 - y^2 &= 4 \end{aligned}$$

We solve the first equation for  $y$ , obtaining  $y = 2x - 4$ . We now substitute  $2x - 4$  for  $y$  in the second equation, getting

$$x^2 - (2x - 4)^2 = 4 \quad \text{in second equation, } y \text{ replaced by } 2x - 4$$

When simplified, this gives a quadratic equation:

$$\begin{aligned} x^2 - (4x^2 - 16x + 16) &= 4 \\ -3x^2 + 16x - 20 &= 0 \\ x &= \frac{-16 \pm \sqrt{256 - 4(-3)(-20)}}{-6} = \frac{-16 \pm \sqrt{16}}{-6} = \frac{-16 \pm 4}{-6} = \frac{10}{3}, 2 \end{aligned}$$

We now find the corresponding values of  $y$  by substituting into  $y = 2x - 4$ . Thus, we have the solutions  $x = \frac{10}{3}$ ,  $y = \frac{8}{3}$ , and  $x = 2$ ,  $y = 0$ . As a check, we find that these values also satisfy the equation  $x^2 - y^2 = 4$ . We can check the solutions graphically from Fig. 14.9.

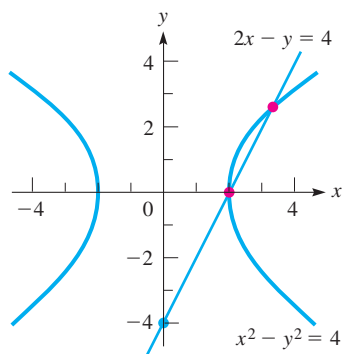


Fig. 14.9

**EXAMPLE 2** Solution by substitution

By substitution, solve the system of equations

$$\begin{aligned} xy &= -2 \\ 2x + y &= 2 \end{aligned}$$

From the first equation, we have  $y = -2/x$ . Substituting this into the second equation, we have

$$\begin{aligned} 2x + \left(-\frac{2}{x}\right) &= 2 && \text{in second equation, } y \text{ replaced by } -\frac{2}{x} \\ 2x^2 - 2 &= 2x \\ x^2 - x - 1 &= 0 \\ x &= \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2} \end{aligned}$$

By substituting these values for  $x$  into either of the original equations, we find the corresponding values of  $y$ , and we have the solutions

$$x = \frac{1 + \sqrt{5}}{2}, y = 1 - \sqrt{5} \quad \text{and} \quad x = \frac{1 - \sqrt{5}}{2}, y = 1 + \sqrt{5}$$

These can be checked by substituting in the original equations. In decimal form, they are

$$x \approx 1.618, y \approx -1.236 \quad \text{and} \quad x \approx -0.618, y \approx 3.236$$

The graphical solutions are shown in Fig. 14.10.

The solutions can also be found by first solving the second equation for  $y$ , or either equation for  $x$ , and then substituting in the other equation. ■

**SOLUTION BY ADDITION OR SUBTRACTION**

The other algebraic method is that of elimination by *addition or subtraction*. This method is most useful if both equations have only squared terms and constants.

**EXAMPLE 3** Solution by addition

By addition or subtraction, solve the system of equations

$$\begin{aligned} 2x^2 + y^2 &= 9 \\ x^2 - y^2 &= 3 \end{aligned}$$

We note that if we add the corresponding sides of each equation,  $y^2$  is eliminated. This leads to the solution:

$$\begin{aligned} 2x^2 + y^2 &= 9 \\ x^2 - y^2 &= 3 \\ \hline 3x^2 &= 12 && \text{add} \\ x^2 &= 4 \\ x &= \pm 2 \end{aligned}$$

For  $x = 2$ , we have two corresponding  $y$ -values,  $y = \pm 1$ . Also, for  $x = -2$ , we have two corresponding  $y$ -values,  $y = \pm 1$ . Thus, we have four solutions:

$$x = 2, y = 1 \quad x = 2, y = -1 \quad x = -2, y = 1 \quad x = -2, y = -1$$

Each solution checks in the original equations. The graphical solutions are shown in Fig. 14.11. ■

**Practice Exercise**

- Solve by substitution:  $2x^2 + y^2 = 3$   
 $x - y = 2$

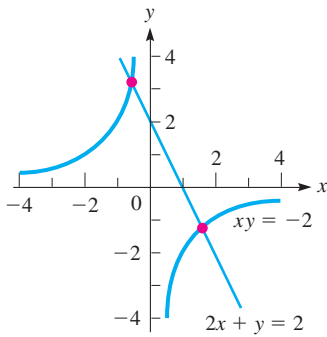


Fig. 14.10

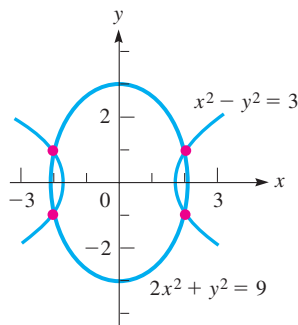


Fig. 14.11

**Practice Exercise**

2. Solve by addition or subtraction:

$$\begin{aligned} x^2 + y^2 &= 6 \\ 2x^2 - y^2 &= 6 \end{aligned}$$

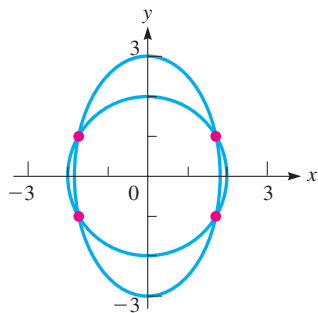


Fig. 14.12

**EXAMPLE 4 Solution by subtraction**

By addition or subtraction, solve the system of equations

$$\begin{aligned} 5x^2 + 2y^2 &= 17 \\ x^2 + y^2 &= 4 \end{aligned}$$

Multiplying the second equation by 2, and subtracting the resulting equations, we have

$$\begin{aligned} 5x^2 + 2y^2 &= 17 \\ 2x^2 + 2y^2 &= 8 && \text{each term of second equation multiplied by 2} \\ \hline 3x^2 &= 9 && \text{subtract} \\ x^2 &= 3, && x = \pm\sqrt{3} \end{aligned}$$

The corresponding values of  $y$  for each value of  $x$  are  $\pm 1$ . Again, we have four solutions:

$$\begin{aligned} x = \sqrt{3}, y = 1 & & x = \sqrt{3}, y = -1 \\ x = -\sqrt{3}, y = 1 & & x = -\sqrt{3}, y = -1 \end{aligned}$$

Each solution checks when substituted in the original equations. The graphical solutions are shown in Fig. 14.12. ■

**EXAMPLE 5 Algebraic solution—application**

A certain number of machine parts cost \$1000. If they cost \$5 less per part, 10 additional parts could be purchased for the same amount of money. What is the cost of each part?

Let  $c$  = the cost per part, and  $n$  = the number of parts. From the first statement, we see that  $cn = 1000$ . From the second statement,  $(c - 5)(n + 10) = 1000$ . Rewriting these equations, we have

$$\begin{aligned} n &= 1000/c \\ cn + 10c - 5n - 50 &= 1000 \\ c\left(\frac{1000}{c}\right) + 10c - 5\left(\frac{1000}{c}\right) - 50 &= 1000 && \text{substituting first equation in second} \\ 1000 + 10c - \frac{5000}{c} - 50 &= 1000 && \text{equation} \\ 10c - \frac{5000}{c} - 50 &= 0 \\ c^2 - 5c - 500 &= 0 \\ (c + 20)(c - 25) &= 0 \\ c &= -20, 25 \end{aligned}$$

Since a negative answer has no significance in this particular situation, we see that the solution is  $c = \$25$  per part. Checking with the original statement of the problem, we see that this is correct. ■

**EXERCISES 14.2**

In Exercises 1–4, make the given changes in the indicated examples of this section and then solve the resulting systems of equations.

- In Example 1, change the sign before  $y$  in the first equation from  $-$  to  $+$  and then solve the system.
- In Example 2, change the right side of the second equation from 2 to 3 and then solve the system.
- In Example 3, change the coefficient of  $x^2$  in the first equation from 2 to 1 and then solve the system.
- In Example 4, change the left side of the first equation from  $5x^2 + 2y^2$  to  $2x^2 + 5y^2$  and then solve the system.

In Exercises 5–28, solve the given systems of equations algebraically.

5.  $y = x + 1$   
 $y = x^2 + 1$
7.  $x + 2y = 3$   
 $x^2 + y^2 = 26$
9.  $x + y = 1$   
 $x^2 - y^2 = 1$
11.  $2x - y = 2$   
 $2x^2 + 3y^2 = 4$
13.  $wh = 1$   
 $w + h = 2$
15.  $xy = 4$   
 $4x - 3y = 2$
17.  $y = x^2$   
 $y = 3x^2 - 50$
19.  $x^2 - y = -1$   
 $x^2 + y^2 = 5$
21.  $D^2 - 1 = R$   
 $D^2 - 2R^2 = 1$
23.  $x^2 + y^2 = 25$   
 $x^2 - 2y^2 = 7$
25.  $x^2 + 3y^2 = 37$   
 $2x^2 - 9y^2 = 14$
27.  $x^2 + y^2 + 4x = 1$   
 $x^2 + y^2 - 2y = 9$
6.  $y = 2x - 1$   
 $y = 2x^2 + 2x - 3$
8.  $p^2 + 4h^2 = 4$   
 $h = p + 1$
10.  $x + y = 2$   
 $2x^2 - y^2 = 1$
12.  $6y - x = 6$   
 $x^2 + 3y^2 = 36$
14.  $xy = 100$   
 $x + y = 20$
16.  $xy = -4$   
 $2x + y = -2$
18.  $M = L^2 - 1$   
 $2L^2 - M^2 = 2$
20.  $s^2 + t^2 = 8$   
 $3s - t = 2$
22.  $2y^2 - 4x = 7$   
 $y^2 + 2x^2 = 3$
24.  $3x^2 - y^2 = 4$   
 $x^2 + 4y^2 = 10$
26.  $5x^2 - 4y^2 = 15$   
 $3y^2 + 4x^2 = 12$
28.  $x^2 + y^2 - 4x - 2y + 4 = 0$   
 $x^2 + y^2 - 2x - 4y + 4 = 0$

(Hint for Exercises 27 and 28: First, subtract one equation from the other to get an equation relating  $x$  and  $y$ . Then substitute this equation in either given equation.)

In Exercises 29–44, solve the indicated systems of equations algebraically. In Exercises 33–44, it is necessary to set up the systems of equations.

29. Solve for  $x$  and  $y$ :  $x^2 - y^2 = a^2 - b^2$ ;  $x - y = a - b$ .
30. For what value of  $b$  are the two solutions of the system  $x^2 - 2y = 5$ ;  $y = 3x + b$  equal to each other? For this value of  $b$ , what is true of the graphs of the two functions?
31. A rocket is fired from behind a ship and follows the path given by  $h = 3x - 0.05x^2$ , where  $h$  is its altitude (in km) and  $x$  is the horizontal distance travelled (in km). A missile fired from the ship follows the path given by  $h = 0.8x - 15$ . For  $h > 0$  and  $x > 0$ , find where the paths of the rocket and missile cross.
32. A 2-kg block collides with an 8-kg block. Using the physical laws of conservation of energy and conservation of momentum, along with given conditions, the following equations involving the velocities are established:  
 $v_1^2 + 4v_2^2 = 41$   
 $2v_1 + 8v_2 = 12$

Find these velocities (in m/s) if  $v_2 > 0$ .

33. One face of a washer has an area of  $37.7 \text{ cm}^2$ . The inner radius is 2.00 cm less than the outer radius. What are the radii?
34. A right triangular sail has a perimeter of 9.30 m and a hypotenuse of 4.17 m. Find the lengths of the sides of the sail.
35. A rectangular computer chip has a surface area of  $2.1 \text{ cm}^2$  and a perimeter of 5.8 cm. Find the length and the width of the chip.
36. On a map, the path of an underground stream can be approximated by the parabola  $y = x^2 - 20x + 84$ . Wells need to be dug at two locations on the line  $y = 2x + 12$ . Find the coordinates of the points where the wells should be dug.
37. A roof truss is in the shape of a right triangle. If there are 4.60 m of lumber in the truss and the longest side is 2.20 m long, what are the lengths of the other two sides of the truss?
38. In a certain roller mechanism, the radius of one steel ball is 2.00 cm greater than the radius of a second steel ball. If the difference in their masses is 7100 g, find the radii of the balls. The density of steel is  $7.70 \text{ g/cm}^3$ .
39. A set of equal electrical resistors in series has a total resistance (the sum of the resistances) of  $78.0 \Omega$ . Another set of two fewer equal resistors in series also has a total resistance of  $78.0 \Omega$ . If the resistance of each resistor in the second set is  $1.3 \Omega$  greater than that of each resistor in the first, how many resistors are in each set?
40. Security fencing encloses a rectangular storage area of  $1600 \text{ m}^2$  that is divided into two sections by additional fencing parallel to the shorter sides. Find the dimensions of the storage area if 220 m of fencing are used.
41. An open liner for a box is to be made from a rectangular sheet of cardboard of area  $216 \text{ cm}^2$  by cutting equal 2.00-cm squares from each corner and bending up the sides. If the volume within the liner is  $224 \text{ cm}^3$ , what are dimensions of the cardboard sheet?
42. Two guy wires, one 140 m long and the other 120 m long, are attached at the same point of a TV tower with the longer one secured in the (level) ground 30 m farther from the base of the tower than the shorter one. How high up on the tower are they attached?
43. A jet travels at 990 km/h relative to the air. It takes the jet 1.4 h longer to travel the 5200 km from London to Montreal, Quebec, against the wind than it takes from Montreal to London with the wind. Find the velocity of the wind.
44. In a marketing survey, a company found that the total gross income for selling  $t$  tables at a price of  $p$  dollars each was \$35 000. It then increased the price of each table by \$100 and found that the total income was only \$27 000 because 40 fewer tables were sold. Find  $p$  and  $t$ .

#### Answers to Practice Exercises

1.  $x = 1/3, y = -5/3; x = 1, y = -1$
2.  $x = 2, y = \sqrt{2}; x = 2, y = -\sqrt{2};$   
 $x = -2, y = \sqrt{2}; x = -2, y = -\sqrt{2}$

## 14.3 Equations in Quadratic Form

Substituting to Fit Quadratic Form •  
Solving Equations in Quadratic Form •  
Extraneous Roots

Often, we encounter equations that can be solved by methods applicable to quadratic equations, even though these equations are not actually quadratic. They do have the property, however, that *with a proper substitution they may be written in the form of a quadratic equation*. All that is necessary is that the equation have terms including some variable quantity, its square, and perhaps a constant term. The following example illustrates these types of equations.

### EXAMPLE 1 Identifying a quadratic form

(a) The equation  $x - 2\sqrt{x} - 5 = 0$  is an equation in quadratic form because if we let  $y = \sqrt{x}$ , we have  $x = (\sqrt{x})^2 = y^2$ , and the resulting equation is  $y^2 - 2y - 5 = 0$ .

(b)  $t^{-4} - 5t^{-2} + 3 = 0$

By letting  $y = t^{-2}$ , we have  $y^2 - 5y + 3 = 0$ .

(c)  $t^3 - 3t^{3/2} - 7 = 0$

By letting  $y = t^{3/2}$ , we have  $y^2 - 3y - 7 = 0$ .

(d)  $(x + 1)^4 - (x + 1)^2 - 1 = 0$

By letting  $y = (x + 1)^2$ , we have  $y^2 - y - 1 = 0$ .

(e)  $x^{10} - 2x^5 + 1 = 0$

By letting  $y = x^5$ , we have  $y^2 - 2y + 1 = 0$ . ■

A method for solving equations in quadratic form is summarized as follows.

#### Procedure for Solving an Equation in Quadratic Form

1. Identify the equation as quadratic: find a quantity involving the unknown variable whose square is included in the equation. (Some algebraic manipulation may be necessary.) Let this quantity be  $y$ .
2. Substitute  $y$  into the original equation to rewrite it in the form  $ay^2 + by + c = 0$ .
3. Solve the quadratic equation for  $y$ .
4. Substitute each of the solutions for  $y$  into the equation relating  $y$  and the original variable, and solve for the original variable.
5. **Check all answers in the original equation.** The substitution could have added **extraneous roots** (that is, roots of a subsequent equation that are not roots of the original equation). Extraneous roots must be identified and discarded.

**EXAMPLE 2** Solving an equation in quadratic form

Solve the equation  $2x^4 + 7x^2 = 4$ .

We first let  $y = x^2$  to write the equation in quadratic form. We will then solve the resulting quadratic equation for  $y$ . However, solutions for  $x$  are required, so we again let  $y = x^2$  to solve for  $x$ .

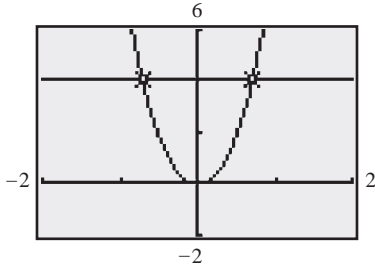


Fig. 14.13

$$\begin{aligned}
 2y^2 + 7y - 4 &= 0 && \text{let } y = x^2 \\
 (2y - 1)(y + 4) &= 0 && \text{factor and solve for } y \\
 y = \frac{1}{2} && \text{or } && y = -4 \\
 x^2 = \frac{1}{2} && \text{or } && x^2 = -4 \quad y = x^2 \text{ (to solve for } x) \\
 x = \pm \frac{1}{\sqrt{2}} && \text{or } && x = \pm 2j
 \end{aligned}$$

We can let  $y_1 = 2x^4 + 7x^2$  and  $y_2 = 4$  to solve the system on a graphing calculator. The display is shown in Fig. 14.13, and we see that there are two points of intersection, one for  $x = 0.7071$  and the other for  $x = -0.7071$ . Since  $1/\sqrt{2} = 0.7071$ , this verifies the real solutions. The imaginary solutions cannot be found graphically. Substitution of each value *in the original equation* shows each value to be a solution. ■

Two of the solutions in Example 2 are complex numbers. We were able to find these solutions directly from the definition of the square root of a negative number. In some cases (see Exercise 22 of this section), it is necessary to use the method of Section 12.6 to find such complex-number solutions.

**EXAMPLE 3** Solving an equation containing a square root

Solve the equation  $x - \sqrt{x} - 2 = 0$ .

By letting  $y = \sqrt{x}$ , we have

$$\begin{aligned}
 y^2 - y - 2 &= 0 \\
 (y - 2)(y + 1) &= 0 \\
 y = 2 &\quad \text{or } \quad y = -1
 \end{aligned}$$

Since  $y = \sqrt{x}$ , we note that  $y$  cannot be negative, and this means  $y = -1$  cannot lead to a solution. For  $y = 2$ , we have  $x = 4$ . Checking, we find that  $x = 4$  satisfies the original equation. Therefore, the only solution is  $x = 4$ .

The graph of  $y = x - \sqrt{x} - 2$  is shown in Fig. 14.14. Note that the only zero is at  $x = 4$ , confirming that  $y = -1$  is an extraneous root and should be discarded. ■

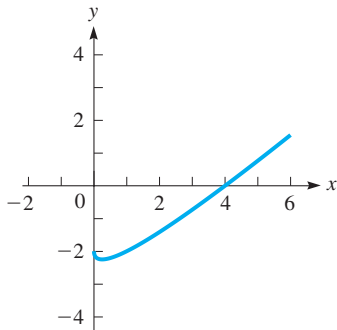


Fig. 14.14

**COMMON ERROR**

It is a common error to give an extraneous root as the solution to an equation. As we said earlier, it is important to check all answers in the original equation so that extraneous roots can be identified and discarded.

**EXAMPLE 4** Solving an equation containing negative exponents

Solve the equation  $x^{-2} + 3x^{-1} + 1 = 0$ .

By substituting  $y = x^{-1}$ , we have  $y^2 + 3y + 1 = 0$ . To solve this equation, we use the quadratic formula:

$$y = \frac{-3 \pm \sqrt{9 - 4}}{2} = \frac{-3 \pm \sqrt{5}}{2}$$

Since  $x = 1/y$ , we have

$$x = \frac{2}{-3 + \sqrt{5}} \quad \text{or} \quad x = \frac{2}{-3 - \sqrt{5}}$$

**Practice Exercise**

- Solve for  $x$ :  $x^{-4} - 8x^{-2} + 16 = 0$

**LEARNING TIP**

Note that when checking decimal answers in the original equation, it is more accurate to store values in the memory of the calculator and use them without rounding than to use rounded values.

These answers, to the nearest 0.001, are

$$x \approx -2.618 \quad \text{or} \quad x \approx -0.382$$

These results check when substituted in the original equation. ■

**EXAMPLE 5 Solving an equation containing grouped terms**

Solve the equation  $(x^2 - x)^2 - 8(x^2 - x) + 12 = 0$ .

By substituting  $y = x^2 - x$ , we have

$$\begin{aligned} y^2 - 8y + 12 &= 0 \\ (y - 2)(y - 6) &= 0 \\ y = 2 \quad \text{or} \quad y &= 6 \\ x^2 - x = 2 \quad \text{or} \quad x^2 - x &= 6 \quad y = x^2 - x \end{aligned}$$

Solving each of these equations, we have

$$\begin{aligned} x^2 - x - 2 &= 0 & x^2 - x - 6 &= 0 \\ (x - 2)(x + 1) &= 0 & (x - 3)(x + 2) &= 0 \\ x = 2 \quad \text{or} \quad x &= -1 & x = 3 \quad \text{or} \quad x &= -2 \end{aligned}$$

Each value checks when substituted in the original equation. ■

**EXAMPLE 6 Quadratic form—application**

A rectangular photograph has an area of  $120 \text{ cm}^2$ . The diagonal of the photograph is 17 cm. Find its length and width. See Fig. 14.15.

Let  $l$  = the length of the photograph and let  $w$  = its width. Since the area is  $120 \text{ cm}^2$ ,  $lw = 120$ . Also, using the Pythagorean theorem and the fact that the diagonal is 17 cm, we have the equation  $l^2 + w^2 = 17^2 = 289$ . Therefore, we are to solve the system of equations

$$lw = 120 \quad l^2 + w^2 = 289$$

Solving the first equation for  $l$ , we have  $l = 120/w$ . Substituting this into the second equation, we have

$$\begin{aligned} \left(\frac{120}{w}\right)^2 + w^2 &= 289 \\ \frac{14\,400}{w^2} + w^2 &= 289 \\ 14\,400 + w^4 &= 289w^2 \end{aligned}$$

Let  $y = w^2$ .

$$\begin{aligned} y^2 - 289y + 14\,400 &= 0 \\ y &= \frac{-(-289) \pm \sqrt{(-289)^2 - 4(1)(14\,400)}}{2(1)} \\ y &= 225 \quad \text{or} \quad y = 64 \end{aligned}$$

Therefore,  $w^2 = 225$  or  $w^2 = 64$ .

Solving for  $w$ , we get  $w = \pm 15$  or  $w = \pm 8.0$ . Only the positive values are meaningful in this problem, which means if  $w = 8.0 \text{ cm}$ , then  $l = 15 \text{ cm}$ , and if  $w = 15 \text{ cm}$ ,  $l = 8.0 \text{ cm}$ . Therefore, the picture is 15 cm by 8.0 cm, and the solution satisfies the statement of the problem. ■

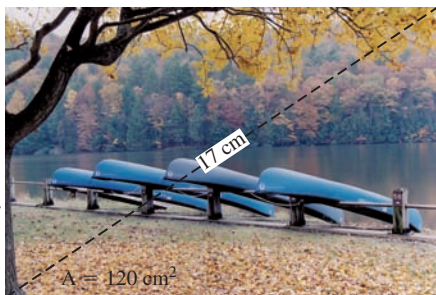


Fig. 14.15

■ The solution to the problem of finding the length and width of a rectangle given the diagonal and the area was known to Old Babylonian mathematicians around 1770 B.C.E.

■ The first permanent photograph was taken in 1816 by the French inventor Joseph Niépce (1765–1833).



## EXERCISES 14.3

In Exercises 1 and 2, make the given changes in the indicated examples of this section and then solve the resulting equations.

- In Example 2, change the + before the  $7x^2$  to - and then solve the equation.
- In Example 3, change the 2 to 6 and then solve the equation.

In Exercises 3–26, solve the given equations algebraically.

- $x^4 - 13x^2 + 36 = 0$
- $x^{-2} - 2x^{-1} - 8 = 0$
- $x^{-4} + 2x^{-2} = 24$
- $2x - 7\sqrt{x} + 5 = 0$
- $3\sqrt[3]{x} - 5\sqrt[6]{x} + 2 = 0$
- $x^{2/3} - 2x^{1/3} - 15 = 0$
- $2n^{1/2} - 5n^{1/4} = 3$
- $(x - 1) - \sqrt{x - 1} = 20$
- $(C + 1)^{-2/3} + 5(C + 1)^{-1/3} - 6 = 0$
- $(x^2 - 2x)^2 - 11(x^2 - 2x) + 24 = 0$
- $(x^2 - 1)^2 + (x^2 - 1)^{-2} = 2$
- $x - 3\sqrt{x - 2} = 6$  (Let  $y = \sqrt{x - 2}$ .)
- $x^6 + 7x^3 - 8 = 0$
- $\frac{1}{s^2 + 1} + \frac{2}{s^2 + 3} = 1$
- $(x + \frac{2}{x})^2 - 6x - \frac{12}{x} = -9$
- $e^{2x} - e^x = 0$
- $10^{2x} - 2(10^x) = 0$

In Exercises 27–32, solve the given equations algebraically and check the solutions with a graphing calculator.

- $x^4 - 20x^2 + 64 = 0$
- $x^{-2} - x^{-1} - 42 = 0$
- $x + 2 = 3\sqrt{x}$
- $x^{2/3} - 4x^{1/3} = 12$
- $(\log x)^2 - 3 \log x + 2 = 0$
- $2^x + 32(2^{-x}) = 12$

In Exercises 33–40, solve the given problems algebraically.

- Solve for  $x$ :  $\log(x^4 + 4) - \log(5x^2) = 0$ .
- If  $f(x) = x^2 + 1$ , find  $x$  if  $f(x^2 - 3) = 5$ .
- The equivalent resistance  $R_T$  of two resistors  $R_1$  and  $R_2$  in parallel is given by  $R_T^{-1} = R_1^{-1} + R_2^{-1}$ . If  $R_T = 1.00 \Omega$  and  $R_2 = \sqrt{R_1}$ , find  $R_1$  and  $R_2$ .
- An equation used in the study of the dispersion of light is  $\mu = A + B\lambda^{-2} + C\lambda^{-4}$ . Solve for  $\lambda$ . (See the chapter introduction.)
- In the theory dealing with optical interferometers, the equation  $\sqrt{F} = 2\sqrt{p}/(1 - p)$  is used. Solve for  $p$  if  $F = 16$ .
- A special washer is made from a circular disc 3.50 cm in radius by removing a rectangular area of  $12.0 \text{ cm}^2$  from the centre. If each corner of the rectangular area is 0.50 cm from the outer edge of the washer, what are the dimensions of the area that is removed?
- A rectangular TV screen has an area of  $2240 \text{ cm}^2$  and a diagonal of 68.6 cm. Find the dimensions of the screen.
- The impedance  $Z$  in an alternating-current circuit is  $2.00 \Omega$ . If the resistance  $R$  is numerically equal to the square of the reactance  $X$ , find  $R$  and  $X$ . See Section 12.7.

## Answer to Practice Exercise

- $1/2, 1/2, -1/2, -1/2$

## 14.4 Equations with Radicals

Solve by Squaring Both Sides • Solve by Isolating a Radical • Solve by Using Quadratic Form

Equations with radicals can usually be solved using the following method.

## Procedure for Solving an Equation with Radicals

- Isolate the radical (or one of the radicals if more than one radical is present) by rewriting the equation with the radical on one side and all other terms on the other side.
- Raise both sides of the equation to the order of the radical (for instance, square both sides when the radical is a square root). **Be careful to raise the complete expression on each side, not just the terms separately.**
- Repeat steps 1 and 2 until the equation does not contain radicals.
- Solve the resulting equation.
- Check all solutions in the original equation.** Identify and discard extraneous roots.

**EXAMPLE 1** Solve by squaring both sidesSolve the equation  $\sqrt{x-4} = 2$ .

By squaring both sides of the equation, we have

$$\begin{aligned}(\sqrt{x-4})^2 &= 2^2 \\ x-4 &= 4 \\ x &= 8\end{aligned}$$

This solution checks when put into the original equation. ■

**EXAMPLE 2** Solve by squaring both sidesSolve the equation  $2\sqrt{3x-1} = 3x$ .

Squaring both sides of the equation gives us

$$\begin{aligned}(2\sqrt{3x-1})^2 &= (3x)^2 \quad \leftarrow \text{don't forget to square the 2} \\ 4(3x-1) &= 9x^2 \\ 12x-4 &= 9x^2 \\ 9x^2-12x+4 &= 0 \\ (3x-2)^2 &= 0 \\ x &= \frac{2}{3} \quad \text{(double root)}\end{aligned}$$

Checking this solution in the original equation, we have

$$2\sqrt{3\left(\frac{2}{3}\right)-1} \stackrel{?}{=} 3\left(\frac{2}{3}\right), \quad 2\sqrt{2-1} \stackrel{?}{=} 2, \quad 2 = 2$$

Therefore, the solution  $x = \frac{2}{3}$  checks.

We can check this solution graphically by letting  $y_1 = 2\sqrt{3x-1}$  and  $y_2 = 3x$ . The calculator display is shown in Fig. 14.16. The *intersection* feature shows that the only  $x$ -value that the curves have in common is  $x = 0.6667$ , which agrees with the solution of  $x = 2/3$ . This also means the line  $y_2$  is tangent to the curve of  $y_1$ . ■

**Practice Exercise**

1. Solve for
- $x$
- :
- $\sqrt{2x+3} = x$

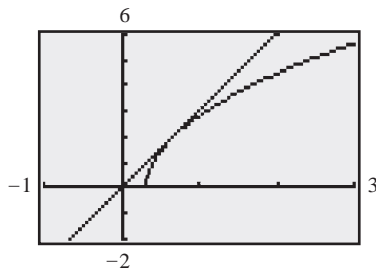


Fig. 14.16

**EXAMPLE 3** Solve by cubing both sidesSolve the equation  $\sqrt[3]{x-8} = 2$ .

Cubing both sides of the equation, we have

$$\begin{aligned}x-8 &= 8 \\ x &= 16\end{aligned}$$

Checking this solution in the original equation, we get

$$\sqrt[3]{16-8} \stackrel{?}{=} 2, \quad 2 = 2$$

Therefore, the solution checks. ■

**EXAMPLE 4** Solve by isolating the radicalSolve the equation  $\sqrt{x-1} + 3 = x$ .

We first isolate the radical by subtracting 3 from each side. This gives us

$$\sqrt{x-1} = x-3$$

We now square both sides and proceed with the solution:

$$\begin{aligned}(\sqrt{x-1})^2 &= (x-3)^2 \\ x-1 &= x^2-6x+9 \quad \leftarrow \text{square the expression on each side,} \\ x^2-7x+10 &= 0 \quad \leftarrow \text{not just the terms separately} \\ (x-5)(x-2) &= 0 \\ x &= 5 \quad \text{or} \quad x = 2\end{aligned}$$

The solution  $x = 5$  checks, but the solution  $x = 2$  gives  $4 = 2$ . Thus, the solution is  $x = 5$ . The value  $x = 2$  is an extraneous root. ■

**Practice Exercise**

2. Solve for
- $x$
- :
- $\sqrt{x+4} + 2 = x$

**EXAMPLE 5** Solve by isolating the radical

Solve the equation  $\sqrt{x+1} + \sqrt{x-4} = 5$ .

We start by isolating the first radical and then squaring both sides of the resulting equation:

$$\begin{aligned}\sqrt{x+1} &= 5 - \sqrt{x-4} \quad \leftarrow \text{two terms} \\ (\sqrt{x+1})^2 &= (5 - \sqrt{x-4})^2 \\ x+1 &= 25 - 10\sqrt{x-4} + (\sqrt{x-4})^2 \quad \leftarrow \text{be careful!} \\ &= 25 - 10\sqrt{x-4} + x - 4\end{aligned}$$

Now, isolating the radical on one side of the equation and squaring again, we have

$$\begin{aligned}10\sqrt{x-4} &= 20 \\ \sqrt{x-4} &= 2 \quad \text{divide by 10} \\ x-4 &= 4 \quad \text{square both sides} \\ x &= 8\end{aligned}$$

This solution checks. ■

**COMMON ERROR**

It should be emphasized that when squaring both sides of an equation, the complete expression is squared, and not just the terms separately.

$(5 - \sqrt{x-4})^2$  is **not**  $25 + (x-4)$ . You are calculating the square of a binomial, so you must have three terms, including the middle term  $-10\sqrt{x-4}$  (which still contains a radical).

As we see in the following example, equations with radicals can sometimes be solved as equations in quadratic form.

**EXAMPLE 6** Solve by using a quadratic form

Solve the equation  $\sqrt{x} - \sqrt[4]{x} = 2$ .

By letting  $y = \sqrt[4]{x}$  ( $y \geq 0$ ), we have

$$\begin{aligned}y^2 - y - 2 &= 0 \\ (y-2)(y+1) &= 0 \\ y = 2 \quad \text{or} \quad y &= -1\end{aligned}$$

Since  $y \geq 0$ , we discard the negative root. For the other value,  $y = 2$ , we have

$$\sqrt[4]{x} = 2, \quad x = 16$$

This checks, because  $\sqrt{16} - \sqrt[4]{16} = 4 - 2 = 2$ . Therefore, the only solution of the original equation is  $x = 16$ . ■

**EXAMPLE 7** Equation with radicals—application

Each cross section of a holographic image is in the shape of a right triangle. The perimeter of the cross section is 60 cm, and its area is  $120 \text{ cm}^2$ . Find the length of each of the three sides.

If we let the two legs of the triangle be  $x$  and  $y$ , as shown in Fig. 14.17, from the formulas for the perimeter  $p$  and the area  $A$  of a triangle, we have

$$p = x + y + \sqrt{x^2 + y^2} \quad \text{and} \quad A = \frac{1}{2}xy$$

where the hypotenuse was found by use of the Pythagorean theorem. Using the information given in the statement of the problem, we arrive at the equations

$$x + y + \sqrt{x^2 + y^2} = 60 \quad \text{and} \quad xy = 240$$

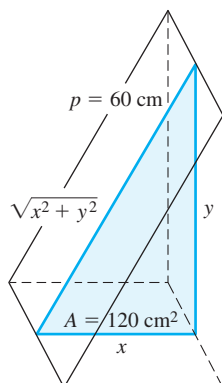


Fig. 14.17

Isolating the radical in the first equation and then squaring both sides, we have

$$\begin{aligned}\sqrt{x^2 + y^2} &= 60 - x - y \\ x^2 + y^2 &= 3600 - 120x - 120y + x^2 + 2xy + y^2 \\ 0 &= 3600 - 120x - 120y + 2xy\end{aligned}$$

Solving the second of the original equations for  $y$ , we have  $y = 240/x$ . Substituting, we have

$$0 = 3600 - 120x - 120\left(\frac{240}{x}\right) + 2x\left(\frac{240}{x}\right)$$

$$0 = 3600x - 120x^2 - 120(240) + 480x \quad \text{multiply by } x$$

$$0 = 30x - x^2 - 240 + 4x \quad \text{divide by } 120$$

$$x^2 - 34x + 240 = 0 \quad \text{collect terms on left}$$

$$(x - 10)(x - 24) = 0$$

$$x = 10 \text{ cm} \quad \text{or} \quad x = 24 \text{ cm}$$

If  $x = 10$  cm, then  $y = 24$  cm, or if  $x = 24$  cm, then  $y = 10$  cm. Therefore, the legs of the holographic cross section are 10 cm and 24 cm, and the hypotenuse is 26 cm. For these sides,  $p = 60$  cm and  $A = 120$  cm<sup>2</sup>. We see that these values check with the statement of the problem. ■

■ See the chapter introduction.

■ Holography is a method of producing a three-dimensional image without the use of a lens. The theory of holography was developed in the late 1940s by the British engineer Dennis Gabor (1900–1979). After the invention of lasers, the first holograms were produced in the early 1960s.

## EXERCISES 14.4

In Exercises 1–4, make the given changes in the indicated examples of this section, and then solve the resulting equations.

- In Example 2, change the  $3x$  on the right to 3.
- In Example 3, change the 8 under the radical to 19.
- In Example 4, change the 3 on the left to 7.
- In Example 5, change the 4 under the second radical to 14.

In Exercises 5–34, solve the given equations. In Exercises 19 and 20, explain how the extraneous root is introduced.

- $\sqrt{x-8} = 2$
- $\sqrt{8-2x} = x$
- $\sqrt{3x+2} = 3x$
- $2\sqrt{3-x} - x = 5$
- $\sqrt[3]{y-5} = 3$
- $5\sqrt{s} - 6 = s$
- $\sqrt{x^2-9} = 4$
- $\sqrt{x+4} + 8 = x$
- $\sqrt{5+\sqrt{x}} = \sqrt{x}-1$
- $3\sqrt{1-2t} + 1 = 2t$
- $2\sqrt{x+2} - \sqrt{3x+4} = 1$
- $\sqrt{5x+1} - 1 = 3\sqrt{x}$
- $\sqrt{6x-5} - \sqrt{x+4} = 2$
- $\sqrt{x-9} = \frac{36}{\sqrt{x-9}} - \sqrt{x}$
- $\sqrt{x-2} = \sqrt[4]{x-2} + 12$
- $\sqrt{x+4} = 3$
- $2\sqrt{2P+5} = P$
- $\sqrt{5x-1} + 3 = x$
- $x - 3\sqrt{2x+1} = -5$
- $\sqrt[4]{5-x} = 2$
- $3\sqrt{x} + 2 = 2x$
- $t^2 = 3 - \sqrt{2t^2 - 3}$
- $\sqrt{x+15} + 5 = x$
- $\sqrt{13+\sqrt{x}} = \sqrt{x} + 1$
- $1 - 2\sqrt{y+4} = y$
- $\sqrt{x-1} + \sqrt{x+2} = 3$
- $\sqrt{x-7} = \sqrt{x}-7$
- $\sqrt{5x-4} - \sqrt{x} = 2$
- $\sqrt[4]{x+10} = \sqrt{x-2}$
- $\sqrt{3x+\sqrt{3x+4}} = 4$

In Exercises 35–38, solve the given equations algebraically and check the solutions graphically with a graphing calculator.

- $\sqrt{3x+4} = x$
- $\sqrt{x-2} + 3 = x$
- $\sqrt{2x+1} + 3\sqrt{x} = 9$
- $\sqrt{2x+1} - \sqrt{x+4} = 1$

In Exercises 39–52, solve the given problems.

- If  $f(x) = \sqrt{x+3}$ , find  $x$  if  $f(x+6) = 5$ .
- Solve  $\sqrt{x} - \sqrt{2x} = 2$  algebraically, and check the solution from the graph on a calculator.
- Solve  $\sqrt{x-1} + x = 3$  algebraically. Then compare the solution with that of Example 4. Noting that the algebraic steps after isolating the radical are identical, why is the solution different?
- The resonant frequency  $f$  in an electric circuit with an inductance  $l$  and a capacitance  $C$  is given by  $f = \frac{1}{2\pi\sqrt{LC}}$ . Solve for  $L$ .
- A formula used in calculating the range  $r$  for radio communication is  $R = \sqrt{2rh + h^2}$ . Solve for  $h$ .
- An equation used in analysing a certain type of concrete beam is  $k = \sqrt{2np + (np)^2} - np$ . Solve for  $p$ .
- In the study of spur gears in contact, the equation  $kC = \sqrt{R_1^2 - R_2^2} + \sqrt{r_1^2 - r_2^2} - A$  is used. Solve for  $r_1^2$ .
- The focal length  $f$  of a lens, in terms of the image distance  $q$  and the object distance  $p$ , is given by  $\frac{1}{f} = \frac{1}{p} + \frac{1}{q}$ . Find  $p$  and  $q$  if  $f = 4$  cm and  $p = \sqrt{q}$ .
- One section of a roof is in the shape of an isosceles triangle. Its perimeter is three times the height to its base of 4.0 m. What is the area of this section of the roof?

48. A smaller of two cubical boxes is centred on the larger box, and they are taped together with a wide adhesive that just goes around both boxes (see Fig. 14.18). If the edge of the larger box is 1.00 cm greater than that of the smaller box, what are the lengths of the edges of the boxes if 100.0 cm of tape is used?

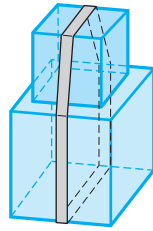


Fig. 14.18

49. A freighter is 5.2 km farther from a Coast Guard station on a straight coast than from the closest point  $A$  on the coast. If the station is 8.3 km from  $A$ , how far is it from the freighter?
50. The velocity  $v$  of an object that falls through a distance  $h$  is given by  $v = \sqrt{2gh}$ , where  $g$  is the acceleration due to gravity. Two objects are dropped from heights that differ by 10.0 m such that the sum of their velocities when they strike the ground is 20.0 m/s. Find the heights from which they are dropped if  $g = 9.80 \text{ m/s}^2$ .
51. A point  $T$  on Thorah Island in Lake Simcoe, Ontario, is 3.8 km from Beaverton ( $B$ ). A person in a motorboat travels straight from

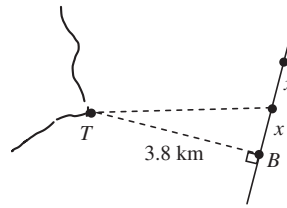


Fig. 14.19

$T$  to a point on Alsop's Beach,  $x$  km from  $B$ , and then travels  $x$  km farther along the beach away from  $B$ . (Assume that the coast is straight, which is nearly the case.) Find  $x$  if the person travelled a total of 5.4 km. See Fig. 14.19.

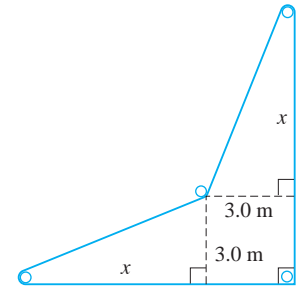


Fig. 14.20

52. The length of the roller belt in Fig. 14.20 is 28.0 m. Find  $x$ .

Answers to Practice Exercises

1. 3    2. 5

CHAPTER 14 REVIEW EXERCISES

In Exercises 1–10, solve the given systems of equations by use of a graphing calculator. Find all values to the nearest 0.01.

- |                                       |   |   |
|---------------------------------------|---|---|
| 1. $x + 2y = 6$<br>$y = 4x^2$         | 2. $x + y = 3$<br>$x^2 + y^2 = 25$      | 3. $3x + 2y = 6$<br>$x^2 + 4y^2 = 4$            |
| 4. $x^2 - 2y = 0$<br>$y = 3x - 5$     | 5. $y = x^2 + 1$<br>$4x^2 + 16y^2 = 29$ | 6. $\frac{x^2}{4} + y^2 = 1$<br>$x^2 - y^2 = 1$ |
| 7. $y = 11 - x^2$<br>$y = 2x^2 - 1$   | 8. $x^2y = 63$<br>$y = 25 - 2x^2$       |   |
| 9. $y = x^2 - 2x$<br>$y = 1 - e^{-x}$ | 10. $y = \ln x$<br>$y = \sin x$         |   |

In Exercises 11–20, solve each of the given systems of equations algebraically.

- |   |   |   |
|---|---|---|
| 11. $y = 4x^2$<br>$y = 8x$                  | 12. $x + y = 12$<br>$xy = 20$   | 13. $2R = L^2$<br>$R^2 + L^2 = 3$           |
| 14. $y = x^2$<br>$2x^2 - y^2 = 1$           | 15. $4u^2 + v = 3$<br>$2u + 3v = 1$   | 16. $x^2 + 7y^2 = 56$<br>$2x^2 - 8y^2 = 90$ |
| 17. $4x^2 - 7y^2 = 21$<br>$x^2 + 2y^2 = 99$ | 18. $s - t = 6$<br>$\sqrt{s} - \sqrt{t} = 1$                                  |   |
| 19. $4x^2 + 3xy = 4$<br>$x + 3y = 4$        | 20. $\frac{6}{x} + \frac{3}{y} = 4$<br>$\frac{36}{x^2} + \frac{36}{y^2} = 13$ |   |

In Exercises 21–38, solve the given equations.

- |                                  |                                   |
|----------------------------------|-----------------------------------|
| 21. $x^4 - 20x^2 + 64 = 0$       | 22. $t^6 - 26t^3 - 27 = 0$        |
| 23. $x^{3/2} - 9x^{3/4} + 8 = 0$ | 24. $x^{1/2} + 3x^{1/4} - 28 = 0$ |
| 25. $D^{-2} + 4D^{-1} - 21 = 0$  | 26. $2x - 3\sqrt{x} - 5 = 0$      |
| 27. $4(\ln x)^2 - \ln x^2 = 0$   | 28. $e^x + e^{-x} = 2$            |

29.  $\frac{4}{r^2 + 1} + \frac{7}{2r^2 + 1} = 2$
30.  $(x^2 + 5x)^2 - 5(x^2 + 5x) = 6$
31.  $3\sqrt{2Z + 4} = 2Z$
32.  $\sqrt[3]{x - 2} = 3$
33.  $\sqrt{5x + 9} + 1 = x$
34.  $2\sqrt{5x - 3} - 1 = 2x$
35.  $\sqrt{x + 1} + \sqrt{x} = 2$
36.  $\sqrt{3x^2 - 2} - \sqrt{x^2 + 7} = 1$
37.  $\sqrt{n + 4} + 2\sqrt{n + 2} = 3$
38.  $\sqrt{3x - 2} - \sqrt{x + 7} = 1$

In Exercises 39 and 40, find the value of the constant exactly. Explain your method. (In each, the expression on the right is called a **continued radical**. Also,  $\dots$  means that the pattern continues indefinitely.) Hint: Square both sides.

39.  $x = \sqrt{2 + \sqrt{2 + \sqrt{2 + \dots}}}$

40.  $\tau = \sqrt{1 + \sqrt{1 + \sqrt{1 + \dots}}}$

The constant  $\tau$  is known as the *golden ratio*. It has many applications in mathematics, architecture, biology, music, and art.

In Exercises 41–46, solve the given equations algebraically and check the solutions graphically with a graphing calculator.

- |   |  |
|---|--|
| 41. $x^3 - 2x^{3/2} - 48 = 0$           | 42. $(x + 1)^4 - 54 = 3(x + 1)^2$            |
| 43. $2\sqrt{3x + 1} - \sqrt{x - 1} = 6$ | 44. $3\sqrt{x} + \sqrt{x - 9} = 11$          |
| 45. $\sqrt[3]{x^3 - 7} = x - 1$         | 46. $\sqrt{x^2 + 7} + \sqrt[4]{x^2 + 7} = 6$ |

In Exercises 47–58, solve the given problems.

47. Solve for  $x$  and  $y$ :  $x^2 - y^2 = 2a + 1$ ;  $x - y = 1$
48. Solve for  $x$ :  $\log(\sqrt{x} + 38) - \log x = 1$
49. Solve  $\sqrt{\sqrt{x} - 1} = 2$  for  $x$ . Check using the graph on a calculator.
50. If  $f(x) = \sqrt{8 - 2x}$  and  $f(x + 1) = 2$ , find  $x$ .

51. Use a graphing calculator to solve the following system of three equations:  $x^2 + y^2 = 13$ ,  $y = x - 1$ ,  $xy = 6$ .
52. Algebraically solve the following system of three equations:  $y = -x^2$ ,  $y = x - 1$ ,  $xy = 1$ . Explain the results.
53. In the study of atomic structure, the equation  $L = \frac{h}{2\pi} \sqrt{l(l+1)}$  is used. Solve for  $l$  ( $l > 0$ ).
54. The frequency  $\omega$  of a certain  $RLC$  circuit is given by  $\omega = \frac{\sqrt{R^2 + 4(L/C) + R}}{2L}$ . Solve for  $C$ .
55. In the theory dealing with a suspended cable, the equation  $y = \sqrt{s^2 - m^2} - m$  is used. Solve for  $m$ .
56. The equation  $V = e^2cr^{-2} - e^2Zr^{-1}$  is used in spectroscopy. Solve for  $r$ .
57. In an experiment, an object is allowed to fall, stops, and then falls for twice the initial time. The total distance the object falls is 392 cm. The equations relating the times  $t_1$  and  $t_2$  (in s) of fall are  $490t_1^2 + 490t_2^2 = 392$  and  $t_2 = 2t_1$ . Find the times of fall.
58. If two objects collide and the kinetic energy remains constant, the collision is termed perfectly elastic. Under these conditions, if an object of mass  $m_1$  and initial velocity  $u_1$  strikes a second object (initially at rest) of mass  $m_2$ , such that the velocities after collision are  $v_1$  and  $v_2$ , the following equations are found:
- $$m_1u_1 = m_1v_1 + m_2v_2$$
- $$\frac{1}{2}m_1u_1^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$$
- Solve these equations for  $m_2$  in terms of  $u_1$ ,  $v_1$ , and  $m_1$ .

In Exercises 59–70, set up the appropriate equations and solve them.

59. A wrench is dropped by a worker at a construction site. Four seconds later the worker hears it hit the ground below. How high is the worker above the ground? (The velocity of sound is 331 m/s, and the distance the wrench falls as a function of time is  $s = 4.9t^2$ .)
60. The rectangular flat screen for a computer monitor has a perimeter of 122.6 cm, and a diagonal of 43.6 cm. Find the dimensions of the screen.
61. The perimeter of a banner in the shape of a tall isosceles triangle (that is, with height longer than its base) is 72 dm, and its area is  $240 \text{ dm}^2$ . Using a calculator, graphically find the lengths of the sides of the banner.
62. A rectangular field is enclosed by fencing and a wall along one long side and half of an adjacent side. See Fig. 14.21. If the area of the field is  $9000 \text{ m}^2$  and 240 m of fencing are used, what are the dimensions of the field?

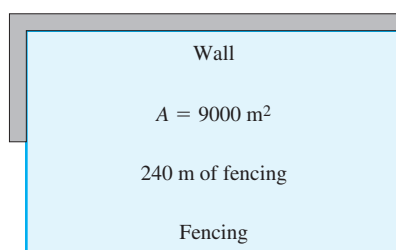


Fig. 14.21

63. A circuit on a computer chip is designed to be within the area shown in Fig. 14.22. If this part of the chip has an area of  $9.0 \text{ mm}^2$  and a perimeter of 16 mm, find  $x$  and  $y$ .

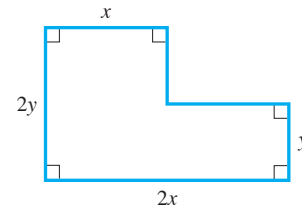


Fig. 14.22

64. For the plywood piece shown in Fig. 14.23, find  $x$  and  $y$ .

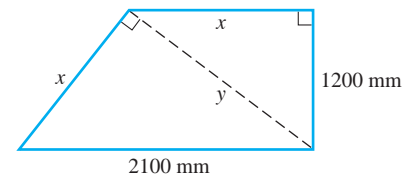


Fig. 14.23

65. The viewing window on a graphing calculator has an area of  $1770 \text{ mm}^2$  and a diagonal of 62 mm. What are the length and width of the rectangle?
66. A trough is made from a piece of sheet metal 12.0 cm wide. The cross section of the trough is shown in Fig. 14.24. Find  $x$ .

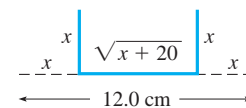


Fig. 14.24

67. The circular solar cell and square solar cell shown in Fig. 14.25 have a combined surface area of  $40.0 \text{ cm}^2$ . Find the radius of the circular cell and the side of the square cell.

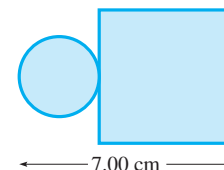


Fig. 14.25

68. A plastic band 19.0 cm long is bent into the shape of a triangle with sides  $\sqrt{x-1}$ ,  $\sqrt{5x-1}$ , and 9. Find  $x$ .
69. A ferry travels from Digby, Nova Scotia, to St. John, New Brunswick, and later it returns to Digby at an average speed that is 3.2 km/h slower. If Digby is 72 km from St. John, and the total travel time is 5 hours and 42 minutes, find the average speed of the ferry in each direction.
70. Two trains are approaching the same crossing on tracks that are at right angles to each other. Each is travelling at 60.0 km/h. If one is 6.00 km from the crossing when the other is 3.00 km from it, how much later will they be 4.00 km apart (on a direct line and before reaching the crossing)?

### Writing Exercise

71. Using a computer, an engineer designs a triangular support structure with sides (in m) of  $x$ ,  $\sqrt{x-1}$ , and 4.00 m. If the perimeter is to be 9.00 m, the equation to be solved is  $x + \sqrt{x-1} + 4.00 = 9.00$ . Write one or two paragraphs explaining how to solve this equation by two different methods discussed in this chapter.

## CHAPTER 14 PRACTICE TEST

1. Solve for  $x$ :  $x^{1/2} - 2x^{1/4} = 3$ .
2. Solve for  $x$ :  $3\sqrt{x-2} - \sqrt{x+1} = 1$ .
3. Solve for  $x$ :  $x^4 - 17x^2 + 16 = 0$ .
4. Solve for  $x$  and  $y$  algebraically:  
$$x^2 - 2y = 5$$
$$2x + 6y = 1$$
5. Solve for  $x$ :  $\sqrt[3]{2x+5} = 5$ .
6. The velocity  $v$  of an object falling under the influence of gravity in terms of its initial velocity  $v_0$ , the acceleration due to gravity  $g$ , and the height  $h$  fallen is given by  $v = \sqrt{v_0^2 + 2gh}$ . Solve for  $h$ .
7. Solve for  $x$  and  $y$  graphically:  
$$x^2 - y^2 = 4$$
$$xy = 2$$
8. A rectangular desktop has a perimeter of 14.0 m and an area of  $10.0 \text{ m}^2$ . Find the length and the width of the desktop.



# Equations of Higher Degree

The desire to eliminate errors in the computation of mathematical tables led the British mathematician Charles Babbage (1792–1871) to design a machine for solving polynomial equations. Nevertheless, despite years of government financing, his *difference engine* was not realized in its entirety during his lifetime. (The first working difference engine was built in 1991, faithful to one of Babbage’s designs.) Babbage also designed an *analytical engine*, which he hoped would perform many kinds of calculations. Although never built, it did have the important features of a modern computer: input, storage, control unit, and output. Because of this design, Babbage is often considered the father of the computer.

We see that polynomials played an important role in the development of computers. Today, among many other things, computers are used to solve equations, including polynomial equations of higher degree, the topic of this chapter.

Algebraic methods for solving third- and fourth-degree polynomial equations were developed in the 1500s, but the search for an algebraic method for solving equations of degree five continued for a few hundred years. Although known prior to 1650, the *fundamental theorem of algebra* (which we state in this chapter) was proved in 1799 by Karl Friedrich Gauss, considered by many as the greatest mathematician of all time. The theorem guaranteed the existence of solutions to all polynomial equations, although it said nothing about the existence of exact formulas for solving them. In the early nineteenth century, it was finally established that no general algebraic formula exists for solving polynomials of degree higher than four.

In this chapter we study some methods for solving higher-degree polynomial equations. The solutions we find include all possible roots, including complex-number roots. Applications of higher-degree equations arise in a number of technical areas. Among them we find the calculation of resistances in an electric circuit, of the dimensions of a container or structure, and of various business production costs.



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# 15

## LEARNING OUTCOMES

After completion of this chapter, the student should be able to:

- Use the remainder theorem to evaluate polynomials and to find remainders
- Use the factor theorem to identify factors and zeros of polynomials
- Perform synthetic division
- Use the fundamental theorem of algebra to determine the number of roots of an equation
- Solve equations given at least one root
- Determine the possible rational roots of an equation
- Determine the maximum possible number of positive and negative roots by using Descartes’ rule of signs
- Solve polynomial equations of degree three and higher
- Solve application problems involving polynomial equations

◀ In Section 15.3, we see how the design of a box to hold a product involves the solution of a higher-degree equation.

## 15.1 The Remainder and Factor Theorems; Synthetic Division

Polynomial Function • Remainder Theorem • Factor Theorem • Synthetic Division

In this section, we present two theorems and a simplified method for algebraic division. These will help us in solving polynomial equations later in the chapter.

Any function of the form

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n \quad (15.1)$$

where  $a_0 \neq 0$  and  $n$  is a positive integer or zero is called a **polynomial function**. We will be considering only polynomials in which the coefficients  $a_0, a_1, \dots, a_n$  are real numbers.

If we divide a polynomial by  $x - r$ , we can rewrite it in the form

$$f(x) = (x - r)q(x) + R \quad (15.2)$$

where  $q(x)$  is the quotient and  $R$  is the remainder.

### EXAMPLE 1 Division with remainder

Divide  $f(x) = 3x^2 + 5x - 8$  by  $x - 2$ .

The division is shown at the left, and it shows that

$$3x^2 + 5x - 8 = (x - 2)(3x + 11) + 14$$

where, for this function  $f(x)$  with  $r = 2$ , we identify  $q(x)$  and  $R$  as

$$q(x) = 3x + 11 \quad R = 14$$

If we now set  $x = r$  in Eq. (15.2), we have  $f(r) = q(r)(r - r) + R = q(r)(0) + R$ , which leads us to the following theorem.

### The Remainder Theorem

If a polynomial  $f(x)$  is divided by  $(x - r)$ , then the remainder  $R$  is a constant given by  $f(r)$ . That is,

$$f(r) = R \quad (15.3)$$

In other words, without having to perform the division, we can find the remainder by evaluating the function at  $x = r$ . Similarly, without having to evaluate the function, we can find  $f(r)$  by performing the division and noting the remainder.

### EXAMPLE 2 Verifying the remainder theorem

In Example 1,  $f(x) = 3x^2 + 5x - 8$ ,  $R = 14$ , and  $r = 2$ .

We find that

$$\begin{aligned} f(2) &= 3(2^2) + 5(2) - 8 \\ &= 12 + 10 - 8 \\ &= 14 \end{aligned}$$

Therefore,  $f(2) = 14$  verifies that  $f(r) = R$  for this example.

### EXAMPLE 3 Using the remainder theorem

By using the remainder theorem, determine the remainder when  $3x^3 - x^2 - 20x + 5$  is divided by  $x + 4$ .

We start by identifying the value of  $r$ . We write  $x + 4 = x - (-4)$ , which means that  $r = -4$ . We therefore evaluate the function  $f(x) = 3x^3 - x^2 - 20x + 5$  for

$$\begin{array}{r} \phantom{x - 2} \overline{3x + 11} \\ x - 2 \overline{) 3x^2 + 5x - 8} \\ \underline{3x^2 - 6x} \phantom{- 8} \\ 11x - 8 \\ \underline{11x - 22} \\ 14 \end{array}$$

$x = -4$ , or find  $f(-4)$ :

$$\begin{aligned} f(-4) &= 3(-4)^3 - (-4)^2 - 20(-4) + 5 \\ &= -192 - 16 + 80 + 5 \\ &= -123 \end{aligned}$$

The remainder is  $-123$  when  $3x^3 - x^2 - 20x + 5$  is divided by  $x + 4$ . ■

The remainder theorem allows us to establish an important connection between zeros of polynomials and factors of polynomials. We see in Eq. (15.2) that if the remainder  $R = 0$ , then  $f(x) = (x - r)q(x)$ , and this shows that  $x - r$  is a factor of  $f(x)$ . Therefore, we have the following theorem.

**The Factor Theorem**

If  $f(x)$  is a polynomial and  $f(r) = 0$ , then  $x - r$  is a factor of  $f(x)$ .

Consequently, if  $f(r) = 0$ , then

- $x = r$  is a **zero** of  $f(x)$
- $x - r$  is a **factor** of  $f(x)$
- $x = r$  is a **root** of the equation  $f(x) = 0$

**EXAMPLE 4 Using the factor theorem**

(a) We determine that  $t + 1$  is a factor of  $f(t) = t^3 + 2t^2 - 5t - 6$  because  $f(-1) = 0$ , as we now show:

$$f(-1) = (-1)^3 + 2(-1)^2 - 5(-1) - 6 = -1 + 2 + 5 - 6 = 0$$

(b) However,  $t + 2$  is not a factor of  $f(t)$  because  $f(-2)$  is not zero, as we now show:

$$f(-2) = (-2)^3 + 2(-2)^2 - 5(-2) - 6 = -8 + 8 + 10 - 6 = 4$$
 ■

**Practice Exercise**

1. Use the factor theorem to determine whether  $x - 1/2$  is a factor of  $f(x) = 2x^3 + 9x^2 - 11x + 3$ .

**SYNTHETIC DIVISION**

In the sections that follow, we will find that division of a polynomial by the factor  $x - r$  is also useful in solving polynomial equations. Therefore, we now develop a simplified form of long division, known as **synthetic division**. It allows us to easily find the coefficients of the quotient and the remainder. If the degree of the equation is high, it is easier to use synthetic division than to calculate  $f(r)$ . The method for synthetic division is developed in the following example.

**EXAMPLE 5 Developing synthetic division**

Divide  $x^4 + 4x^3 - x^2 - 16x - 14$  by  $x - 2$ .

We first perform this division in the usual manner:

■

$$\begin{array}{r} -2 \overline{) 1 \ 4 \ -1 \ -16 \ -14} \\ \underline{-2} \phantom{0} \\ 6 \\ \underline{-12} \\ 11 \\ \underline{-22} \\ 6 \\ \underline{-12} \\ -2 \end{array}$$

$$\begin{array}{r} x^3 + 6x^2 + 11x + 6 \\ x - 2 \overline{) x^4 + 4x^3 - x^2 - 16x - 14} \\ \underline{x^4 - 2x^3} \phantom{0} \\ 6x^3 - x^2 \phantom{0} \\ \underline{6x^3 - 12x^2} \phantom{0} \\ 11x^2 - 16x \phantom{0} \\ \underline{11x^2 - 22x} \phantom{0} \\ 6x - 14 \\ \underline{6x - 12} \\ -2 \end{array}$$

In doing the division, notice that we repeat many terms and that the only important numbers are the coefficients. This means there is no need to write in the powers of  $x$ . To the left of the division, we write it without  $x$ 's and without identical terms.

$$\begin{array}{r|rrrrr} -2 & 1 & 4 & -1 & -16 & -14 \\ & & -2 & -12 & -22 & -12 \\ \hline & 1 & 6 & 11 & 6 & -2 \end{array}$$

$$\begin{array}{r|rrrrr} & 1 & 4 & -1 & -16 & -14 \\ & & -2 & -12 & -22 & -12 \\ \hline & 1 & 6 & 11 & 6 & -2 \end{array} \quad \begin{array}{l} \\ \\ \\ \\ \end{array}$$

$$\begin{array}{r|rrrrr} & 1 & 4 & -1 & -16 & -14 \\ & & 2 & 12 & 22 & 12 \\ \hline & 1 & 6 & 11 & 6 & -2 \end{array}$$

All numbers below the dividend may be written in two lines. Then all coefficients of the quotient, except the first, appear in the bottom line. Therefore, the line above the dividend is omitted, and we have the table at the left.

Now, write the first coefficient (in this case, 1) in the bottom line. Also, change the  $-2$  to 2, which is the actual value of  $r$ . Then in the table at the left, write the 2 on the right. *In this table, the 1, 6, 11, and 6 are the coefficients of the  $x^3, x^2, x,$  and constant term of the quotient. The  $-2$  is the remainder.*

Finally, it is easier to use addition rather than subtraction in the process, so we change the signs of the numbers in the middle row. Remember that originally the bottom line was found by subtraction. Therefore, we have the last table on the left.

In the last table at the left, we have  $1$  (of the bottom row)  $\times 2 (= r) = 2$ , the first number of the middle row. In the second column,  $4 + 2 = 6$ , the second number in the bottom row. Then,  $6 \times 2 (= r) = 12$ , the second number of the second row;  $-1 + 12 = 11$ ;  $11 \times 2 = 22$ ;  $22 + (-16) = 6$ ;  $6 \times 2 = 12$ ; and  $12 + (-14) = -2$ .

We read the bottom line of the last table, the one we use in *synthetic division*, as

$$1x^3 + 6x^2 + 11x + 6 \text{ with a remainder of } -2$$

The method of synthetic division shown in the last table is outlined below. ■

### Procedure for Synthetic Division

Let  $f(x)$  be a polynomial of degree  $n$ . To divide  $f(x)$  by  $x - r$ :

1. Write the coefficients of  $f(x)$ . Be certain that the powers are in descending order and that zeros are inserted for missing powers.
2. Carry down the left coefficient, then multiply it by  $r$ , and place this product under the second coefficient of the top line.
3. Add the two numbers in the second column and place the result below. Multiply this sum by  $r$  and place the product under the third coefficient of the top line.
4. Continue this process until the bottom row has as many numbers as the top row.
5. Read the coefficients of  $x^{n-1}, \dots, x,$  the constant term, and the remainder from the last line of the table.

### EXAMPLE 6 Using synthetic division

Divide  $x^5 + 2x^4 - 4x^2 + 3x - 4$  by  $x + 3$  using synthetic division.

Since the powers of  $x$  are in descending order, write down the coefficients of  $f(x)$ . In doing so, we must be certain to include a zero for the missing  $x^3$  term. Next, note that the divisor is  $x + 3$ , which means that  $r = -3$ . The  $-3$  is placed to the right. This gives us a top line of

$$\text{coefficients} \rightarrow 1 \quad 2 \quad 0 \quad -4 \quad 3 \quad -4 \quad \boxed{-3} \leftarrow r$$

Next, we carry the left coefficient, 1, to the bottom line and multiply it by  $r, -3$ , placing the product,  $-3$ , in the middle line under the second coefficient, 2. We then add the 2 and the  $-3$  and place the result,  $-1$ , below. This gives

$$\begin{array}{r|rrrrrr} & 1 & 2 & 0 & -4 & 3 & -4 \\ & & -3 & & 12 & -9 & & \\ \hline & 1 & -1 & & 8 & -6 & -4 \end{array}$$

add

$1 \times -3 = -3$

Now, multiply the  $-1$  by  $-3$  ( $= r$ ) and place the result,  $3$ , in the middle line under the zero. Now, add and continue the process, obtaining the following table:

$$\begin{array}{r|rrrrrr}
 1 & 2 & 0 & -4 & 3 & -4 & \\
 & -3 & 3 & -9 & 39 & -126 & \\
 \hline
 1 & -1 & 3 & -13 & 42 & -130 & \\
 \end{array}$$

↑   ↑   ↑   ↑   ↑
↑
coefficients and constant  
of quotient
←
remainder

Since the degree of the dividend is 5, the degree of the quotient is 4. This means that the quotient is  $x^4 - x^3 + 3x^2 - 13x + 42$  and the remainder is  $-130$ . In turn, this means that for  $f(x) = x^5 + 2x^4 - 4x^2 + 3x - 4$ , we have  $f(-3) = -130$ . ■

### EXAMPLE 7 Checking a factor with synthetic division

By synthetic division, determine whether or not  $t + 4$  is a factor of  $t^4 + 2t^3 - 15t^2 - 32t - 16$ .

$$\begin{array}{r|rrrrr}
 1 & 2 & -15 & -32 & -16 & \\
 & -4 & 8 & 28 & 16 & \\
 \hline
 1 & -2 & -7 & -4 & 0 & \\
 \end{array}$$

Since the remainder is zero,  $t + 4$  is a factor. We may also conclude that

$$f(t) = (t + 4)(t^3 - 2t^2 - 7t - 4) \quad \blacksquare$$

### EXAMPLE 8 Checking a rational factor

By using synthetic division, determine whether  $2x - 3$  is a factor of  $2x^3 - 3x^2 + 8x - 12$ .

We first note that the coefficient of  $x$  in the divisor is not 1. Thus we write  $2x - 3 = 2\left(x - \frac{3}{2}\right)$ , and identify  $r = \frac{3}{2}$  from the factorization. We have

$$\begin{array}{r|rrrr}
 2 & -3 & 8 & -12 & \\
 & 3 & 0 & 12 & \\
 \hline
 2 & 0 & 8 & 0 & \\
 \end{array}$$

Since the remainder is zero,  $x - \frac{3}{2}$  is a factor. The result  $2x^2 + 8$  can now be divided by 2 (the coefficient of  $x$  in the divisor) to complete the division. We obtain the quotient  $x^2 + 4$ , with remainder 0. Thus,  $2\left(x - \frac{3}{2}\right) = 2x - 3$  is indeed a factor of the function, and we have that

$$2x^3 - 3x^2 + 8x - 12 = (2x - 3)(x^2 + 4) \quad \blacksquare$$

### EXAMPLE 9 Checking a zero

Determine whether or not  $-12.5$  is a zero of the function

$$f(x) = 6x^3 + 61x^2 - 171x + 100.$$

We use synthetic division to determine whether  $x - (-12.5)$  is a factor of  $f(x)$ . We have

$$\begin{array}{r|rrrr}
 6 & 61 & -171 & 100 & \\
 & -75 & 175 & -50 & \\
 \hline
 6 & -14 & 4 & 50 & \\
 \end{array}$$

Since the remainder is not zero,  $x - (-12.5)$  is not a factor of  $f(x)$ , and therefore  $-12.5$  is not a zero of  $f(x)$ . ■

#### LEARNING TIP

The procedure for synthetic division requires that the divisor be of the form  $x - r$ . If the coefficient of  $x$  in the divisor is not 1, the coefficient needs to be factored out and the value of  $r$  identified from the factorization. The resulting quotient is then divided by the value of the factored coefficient to complete the division.

For example, if the divisor is  $3x + 1$ , we rewrite it as  $3\left(x - \left(-\frac{1}{3}\right)\right)$ , and divide by  $\left(x - \left(-\frac{1}{3}\right)\right)$ . The result is then divided by 3 to complete the division.

#### Practice Exercise

2. Use synthetic division to determine whether  $x + 2$  is a factor of  $2x^3 + x^2 - 12x - 8$ .

## EXERCISES 15.1

In Exercises 1–4, make the given changes in the indicated examples of this section, and then perform the indicated operations.

- In Example 3, change the  $x + 4$  to  $x + 3$  and then find the remainder.
- In Example 4(a), change the  $t + 1$  to  $t - 1$  and then determine if  $t - 1$  is a factor.
- In Example 6, change the  $x + 3$  to  $x + 2$  and then perform the synthetic division.
- In Example 8, change the  $2x - 3$  to  $2x + 3$  and then determine whether  $2x + 3$  is a factor.

In Exercises 5–10, find the remainder by long division.

- $(x^3 + 2x + 3) \div (x + 1)$
- $(x^4 - 4x^3 - x^2 + x - 100) \div (x + 3)$
- $(2x^5 - x^2 + 8x + 44) \div (x + 2)$
- $(4s^3 - 9s^2 - 24s - 17) \div (s - 5)$
- $(2x^4 - 3x^3 - 2x^2 - 15x - 16) \div (2x - 3)$
- $(2x^4 - 10x^2 + 30x - 60) \div (x + 4)$

In Exercises 11–16, find the remainder using the remainder theorem. Do not use synthetic division.

- $(R^4 + R^3 - 9R^2 + 3) \div (R + 4)$
- $(4x^4 - x^2 + 5x - 7) \div (x - 3)$
- $(2x^4 - 7x^3 - x^2 + 8) \div (x - 3)$
- $(3n^4 - 13n^2 + 10n - 10) \div (n + 4)$
- $(x^5 - 3x^3 + 5x^2 - 10x + 6) \div (x - 2)$
- $(3x^4 - 12x^3 - 60x + 4) \div (x - 0.5)$

In Exercises 17–22, use the factor theorem to determine whether or not the second expression is a factor of the first expression. Do not use synthetic division.

- $4x^3 + x^2 - 16x - 4, x - 2$
- $3x^3 + 14x^2 + 7x - 4, x + 4$
- $3V^4 - 7V^3 + V + 8, V - 2$
- $x^5 - 2x^4 + 3x^3 - 6x^2 - 4x + 8, x - 2$
- $x^{61} - 1, x + 1$
- $x^7 - 128^{-1}, x + 2^{-1}$

In Exercises 23–32, perform the indicated divisions by synthetic division.

- $(x^3 + 2x^2 - x - 2) \div (x - 1)$
- $(x^3 - 3x^2 - x + 2) \div (x - 2)$
- $(x^3 + 2x^2 - 3x + 4) \div (x + 1)$
- $(2x^3 - 4x^2 + x - 1) \div (x + 2)$
- $(p^6 - 6p^3 - 2p^2 - 6) \div (p - 2)$
- $(x^5 + 4x^4 - 8) \div (x + 1)$
- $(x^7 - 128) \div (x - 2)$
- $(20x^4 + 11x^3 - 89x^2 + 60x - 77) \div (x + 2.75)$
- $(2x^4 + x^3 + 3x^2 - 1) \div (2x - 1)$
- $(6t^4 + 5t^3 - 10t + 4) \div (3t - 2)$

In Exercises 33–40, use the factor theorem and synthetic division to determine whether or not the second expression is a factor of the first.

- $2x^5 - x^3 + 3x^2 - 4; x + 1$
- $t^5 - 3t^4 - t^2 - 6; t - 3$

- $4x^3 - 6x^2 + 2x - 2; x - \frac{1}{2}$
- $3x^3 - 5x^2 + x + 1; x + \frac{1}{3}$
- $2Z^4 - Z^3 - 4Z^2 + 1; 2Z - 1$
- $6x^4 + 5x^3 - x^2 + 6x - 2; 3x - 1$
- $4x^4 + 2x^3 - 8x^2 + 3x + 12; 2x + 3$
- $3x^4 - 2x^3 + x^2 + 15x + 4; 3x + 4$

In Exercises 41–44, use synthetic division to determine whether or not the given numbers are zeros of the given functions.

- $x^4 - 5x^3 - 15x^2 + 5x + 14; 7$
- $r^4 + 5r^3 - 18r - 8; -4$
- $85x^3 + 348x^2 - 263x + 120; -4.8$
- $2x^3 + 13x^2 + 10x - 4; \frac{1}{2}$

In Exercises 45–58, solve the given problems.

- If  $f(x) = 2x^3 + 3x^2 - 19x - 4$ , and  $f(x) = (x + 4)g(x)$ , find  $g(x)$ .
- Using synthetic division, divide  $ax^2 + bx + c$  by  $x + 1$ .
- By division, show that  $2x - 1$  is a factor of  $f(x) = 4x^3 + 8x^2 - x - 2$ . May we therefore conclude that  $f(1) = 0$ ? Explain.
- By division, show that  $x^2 + 2$  is a factor of  $f(x) = 3x^3 - x^2 + 6x - 2$ . May we therefore conclude that  $f(-2) = 0$ ? Explain.
- For what value of  $k$  is  $x - 2$  a factor of  $f(x) = 2x^3 + kx^2 - x + 14$ ?
- For what value of  $k$  is  $x + 1$  a factor of  $f(x) = 3x^4 + 3x^3 + 2x^2 + kx - 4$ ?
- Use synthetic division:  $(x^3 - 3x^2 + x - 3) \div (x + j)$ .
- Use synthetic division:  $(2x^3 - 7x^2 + 10x - 6) \div [x - (1 + j)]$ .
- If  $f(x) = -g(x)$ , do the functions have the same zeros? Explain.
- Do the functions  $f(x)$  and  $f(-x)$  have the same zeros? Explain.
- In finding the electric current in a certain circuit, it is necessary to factor the denominator of  $\frac{2s}{s^3 + 5s^2 + 4s + 20}$ . Is (a)  $(s - 2)$  or (b)  $(s + 5)$  a factor?
- In the theory of the motion of a sphere moving through a fluid, the function  $f(r) = 4r^3 - 3ar^2 - a^3$  is used. Is (a)  $r = a$  or (b)  $r = 2a$  a zero of  $f(r)$ ?
- In finding the volume  $V$  (in  $\text{cm}^3$ ) of a certain gas in equilibrium with a liquid, it is necessary to solve the equation  $V^3 - 6V^2 + 12V = 8$ . Use synthetic division to determine if  $V = 2 \text{ cm}^3$ .
- An architect is designing a window in the shape of a segment of a circle. An approximate formula for the area is  $A = \frac{h^3}{2w} + \frac{2wh}{3}$ , where  $A$  is the area,  $w$  is the width, and  $h$  is the height of the segment. If the width is 1.500 m and the area is  $0.5417 \text{ m}^2$ , use synthetic division to show that  $h = 0.500 \text{ m}$ .

## Answers to Practice Exercises

- Yes ( $R = 0$ )
- No ( $R = 4$ )



## 15.2 The Roots of an Equation

The Fundamental Theorem of Algebra • Linear Factors • Complex Roots • Remaining Quadratic Factor

■ The fundamental theorem of algebra was first proved in 1799 by the German mathematician Karl Gauss (1777–1855) for his doctoral thesis. During his lifetime, Gauss gave three other different proofs of the theorem. See the chapter introduction.

In this section, we present certain theorems that are useful in determining the number of roots in the equation  $f(x) = 0$  and the nature of some of these roots. In dealing with polynomial equations of higher degree, it is helpful to have as much of this kind of information as we can find before actually solving for all of the roots, including any possible complex roots. A graph does not show the complex roots an equation may have, but we can verify the real roots, as we show in some of the examples that follow.

The first of these theorems is so important that it is called the **fundamental theorem of algebra**. As we will see, the other two theorems are a direct consequence of the fundamental theorem.

### The Fundamental Theorem of Algebra and Related Theorems

1. (**The fundamental theorem of algebra**) Every polynomial equation has at least one real or complex root.
2. A polynomial of degree  $n$  can be factored into  $n$  linear factors.
3. A polynomial equation of degree  $n$  has exactly  $n$  roots.

The proof of the fundamental theorem is of an advanced nature, and therefore we accept its validity at this time. However, using the fundamental theorem, we can show the validity of the other two statements.

Let us assume that we have a polynomial equation  $f(x) = 0$  and that we are looking for its roots. By the fundamental theorem, we know that it has at least one root. Assuming that we can find this root by some means (the factor theorem, for example), we call this root  $r_1$ . Thus,

$$f(x) = (x - r_1)f_1(x)$$

where  $f_1(x)$  is the polynomial quotient found by dividing  $f(x)$  by  $(x - r_1)$ . However, since the fundamental theorem states that any polynomial equation has at least one root, this must apply to  $f_1(x) = 0$  as well. Let us assume that  $f_1(x) = 0$  has the root  $r_2$ . Therefore, this means that  $f(x) = (x - r_1)(x - r_2)f_2(x)$ . Continuing this process until one of the quotients is a constant  $a$ , we have

$$f(x) = a(x - r_1)(x - r_2) \cdots (x - r_n)$$

Note that one linear factor appears each time a root is found and that the degree of the quotient is one less each time. Therefore, if  $f(x)$  is of degree  $n$ , there are  $n$  linear factors, with one root associated with each of them.

### EXAMPLE 1 Illustrating the fundamental theorem

For the equation  $f(x) = 2x^4 - 3x^3 - 12x^2 + 7x + 6 = 0$ , we are given the factors that we show. In the next section, we will see how to find these factors.

For the function  $f(x)$ , we have

$$\begin{aligned} 2x^4 - 3x^3 - 12x^2 + 7x + 6 &= (x - 3)(2x^3 + 3x^2 - 3x - 2) \\ 2x^3 + 3x^2 - 3x - 2 &= (x + 2)(2x^2 - x - 1) \\ 2x^2 - x - 1 &= (x - 1)(2x + 1) \\ 2x + 1 &= 2(x + \frac{1}{2}) \end{aligned}$$

Therefore,

$$2x^4 - 3x^3 - 12x^2 + 7x + 6 = 2(x - 3)(x + 2)(x - 1)(x + \frac{1}{2}) = 0$$

The degree of  $f(x)$  is 4. There are four linear factors:  $(x - 3)$ ,  $(x + 2)$ ,  $(x - 1)$ , and  $(x + \frac{1}{2})$ . There are four roots of the equation: 3,  $-2$ , 1, and  $-\frac{1}{2}$ . Thus, we have verified each of the theorems above for this example. ■



It is not necessary for each root of an equation to be different from the other roots. For example, the equation  $(x - 1)^2 = 0$  has two roots, both of which are 1. Such roots are referred to as *multiple* (or *repeated*) roots.

When we solve the equation  $x^2 + 1 = 0$ , the roots are  $j$  and  $-j$ . In fact, for any quadratic equation (with real coefficients) that has a root of the form  $a + bj$  ( $b \neq 0$ ), there is also a root of the form  $a - bj$ . This is so because we can find the solutions of an equation of the form  $ax^2 + bx + c = 0$  from the quadratic formula as

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

and the only difference between these roots is the sign before the radical. The result is also true for polynomial equations of any degree.

**LEARNING TIP**

If the coefficients of the equation  $f(x) = 0$  are real and  $a + bj$  ( $b \neq 0$ ) is a complex root, then its conjugate,  $a - bj$ , is also a root.

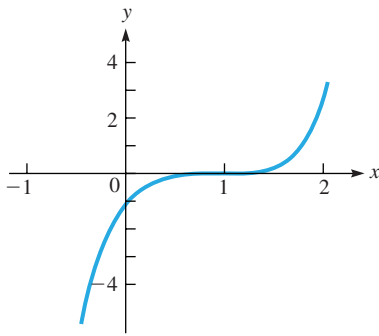


Fig. 15.1

**EXAMPLE 2 Illustrating complex roots**

Consider the equation  $f(x) = (x - 1)^3(x^2 + x + 1) = 0$ .

The factor  $(x - 1)^3$  shows that there is a triple root of 1, and there is a total of five roots, since the highest-power term would be  $x^5$  if we were to multiply out the function. To find the other two roots, we use the quadratic formula on the factor  $(x^2 + x + 1)$ . This is permissible, since we are finding the values of  $x$  for

$$x^2 + x + 1 = 0$$

For this, we have

$$x = \frac{-1 \pm \sqrt{1 - 4}}{2}$$

Thus,

$$x = \frac{-1 + j\sqrt{3}}{2} \quad \text{and} \quad x = \frac{-1 - j\sqrt{3}}{2}$$

Therefore, the roots of  $f(x) = 0$  are 1, 1, 1,  $\frac{-1 + j\sqrt{3}}{2}$ , and  $\frac{-1 - j\sqrt{3}}{2}$ . As shown in Fig. 15.1, the graph of  $f(x)$  intersects the  $x$ -axis at  $x = 1$ , so that  $x = 1$  is verified as a root of  $f(x) = 0$ . However, it is not possible to tell from the graph that this is a triple root or that there are complex roots. ■

**LEARNING TIP**

From Example 2, we can see that whenever enough roots are known so that the remaining factor is quadratic, it is possible to find the remaining roots from the quadratic formula. This is true for finding real or complex roots.

$$\begin{array}{r|rrrr} 3 & 10 & -16 & -32 & -\frac{4}{3} \\ & -4 & -8 & 32 & \\ \hline & 3 & 6 & -24 & 0 \end{array}$$

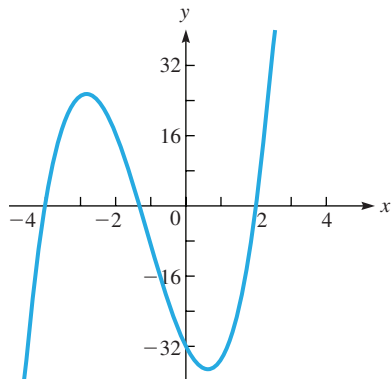


Fig. 15.2

**EXAMPLE 3 Solving an equation given one root**

Solve the equation  $3x^3 + 10x^2 - 16x - 32 = 0$ ;  $-\frac{4}{3}$  is a root.

Using synthetic division and the given root, we have the table shown at the left. From this, we see that

$$3x^3 + 10x^2 - 16x - 32 = (x + \frac{4}{3})(3x^2 + 6x - 24)$$

We know that  $x + \frac{4}{3}$  is a factor from the given root and that  $3x^2 + 6x - 24$  is a factor found from synthetic division. This second factor can be factored as

$$3x^2 + 6x - 24 = 3(x^2 + 2x - 8) = 3(x + 4)(x - 2)$$

Therefore, we have

$$3x^3 + 10x^2 - 16x - 32 = 3(x + \frac{4}{3})(x + 4)(x - 2)$$

This means the roots are  $-\frac{4}{3}$ ,  $-4$ , and  $2$ . The three real roots are the three intersections of  $f(x)$  with the  $x$ -axis, as shown in Fig. 15.2. ■

$$\begin{array}{r|rrrrr} 1 & 3 & -4 & -10 & -4 & \\ & -1 & -2 & 6 & 4 & \\ \hline 1 & 2 & -6 & -4 & 0 & \end{array} \quad | -1$$

$$\begin{array}{r|rrrr} 1 & 2 & -6 & -4 & \\ & 2 & 8 & 4 & \\ \hline 1 & 4 & 2 & 0 & \end{array} \quad | 2$$

**Practice Exercise**

1. Solve  $x^4 - x^3 - 2x^2 - 4x - 24 = 0$ , given that  $-2$  and  $3$  are roots.

$$\begin{array}{r|rrrrr} 3 & -26 & 63 & -36 & -20 & \\ & 6 & -40 & 46 & 20 & \\ \hline 3 & -20 & 23 & 10 & 0 & \end{array} \quad | 2$$

$$\begin{array}{r|rrrr} 3 & -20 & 23 & 10 & \\ & 6 & -28 & -10 & \\ \hline 3 & -14 & -5 & 0 & \end{array} \quad | 2$$

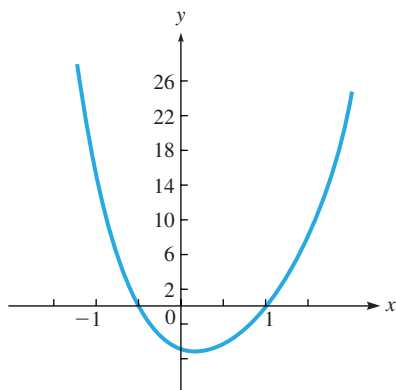


Fig. 15.3

**EXAMPLE 4 Solving an equation given two roots**

Solve  $x^4 + 3x^3 - 4x^2 - 10x - 4 = 0$ ;  $-1$  and  $2$  are roots.

Using synthetic division and the root  $-1$ , the first table at the left shows that

$$x^4 + 3x^3 - 4x^2 - 10x - 4 = (x + 1)(x^3 + 2x^2 - 6x - 4)$$

We now know that  $x - 2$  must be a factor of  $x^3 + 2x^2 - 6x - 4$ , since it is a factor of the original function. Again, using synthetic division and this time the root  $2$ , we have the second table at the left. Thus,

$$x^4 + 3x^3 - 4x^2 - 10x - 4 = (x + 1)(x - 2)(x^2 + 4x + 2)$$

Since the original equation can now be written as

$$(x + 1)(x - 2)(x^2 + 4x + 2) = 0$$

the remaining two roots are found by solving

$$x^2 + 4x + 2 = 0$$

by the quadratic formula. This gives us

$$x = \frac{-4 \pm \sqrt{16 - 8}}{2} = \frac{-4 \pm 2\sqrt{2}}{2} = -2 \pm \sqrt{2}$$

Therefore, the roots are  $-1, 2, -2 + \sqrt{2}$ , and  $-2 - \sqrt{2}$ .

**EXAMPLE 5 Solving an equation given a double root**

Solve the equation  $3x^4 - 26x^3 + 63x^2 - 36x - 20 = 0$ , given that  $2$  is a double root.

Using synthetic division, we have the first table at the left. It tells us that

$$3x^4 - 26x^3 + 63x^2 - 36x - 20 = (x - 2)(3x^3 - 20x^2 + 23x + 10)$$

Also, since  $2$  is a double root, it must be a root of  $3x^3 - 20x^2 + 23x + 10 = 0$ . Using synthetic division again, we have the second table at the left. This second quotient  $3x^2 - 14x - 5$  factors into  $(3x + 1)(x - 5)$ . The roots are  $2, 2, -\frac{1}{3}$ , and  $5$ .

Since the quotient of the first division is the dividend for the second division, both divisions can be done without rewriting the first quotient as follows:

$$\begin{array}{l} \text{first division} \rightarrow \left[ \begin{array}{r|rrrrr} 3 & -26 & 63 & -36 & -20 & \\ & 6 & -40 & 46 & 20 & \\ \hline 3 & -20 & 23 & 10 & 0 & \end{array} \right. \\ \text{second division} \rightarrow \left[ \begin{array}{r|rrrr} 3 & -20 & 23 & 10 & \\ & 6 & -28 & -10 & \\ \hline 3 & -14 & -5 & 0 & \end{array} \right. \end{array} \quad | 2$$

**EXAMPLE 6 Solving an equation given a complex root**

Solve the equation  $2x^4 - 5x^3 + 11x^2 - 3x - 5 = 0$ , given that  $1 + 2j$  is a root.

Since  $1 + 2j$  is a root, we know that  $1 - 2j$  is also a root. Using synthetic division twice, we can then reduce the remaining factor to a quadratic function.

$$\begin{array}{r|rrrrr} 2 & -5 & 11 & -3 & -5 & \\ & 2 + 4j & -11 - 2j & 4 - 2j & 5 & \\ \hline 2 & -3 + 4j & -2j & 1 - 2j & 0 & \\ & 2 - 4j & -1 + 2j & -1 + 2j & & \\ \hline 2 & -1 & -1 & 0 & & \end{array} \quad \begin{array}{l} | 1 + 2j \\ | 1 - 2j \end{array}$$

The quadratic factor  $2x^2 - x - 1$  factors into  $(2x + 1)(x - 1)$ . Therefore, the roots of the equation are  $1 + 2j, 1 - 2j, 1$ , and  $-\frac{1}{2}$ . As we can see, the graph in Fig. 15.3 shows only the real roots.

**Graphical Features Indicating Number and Types of Roots**

If  $f(x)$  is of degree  $n$  and the roots of  $f(x) = 0$  are real and different, the graph of  $f(x)$  crosses the  $x$ -axis  $n$  times and  $f(x)$  changes signs as it crosses. See Fig. 15.4, where  $n = 4$ , the equation has four changes of sign, and the graph crosses four times.

In Fig. 15.4, we see that for two points on the graph, *the graph must cross the  $x$ -axis an odd number of times between points on the different sides of the  $x$ -axis and an even number of times between points on the same side of the  $x$ -axis, if at all.*

For each pair of complex roots, the number of times the graph crosses the  $x$ -axis is reduced by two (Fig. 15.5). For multiple roots, the graph crosses the  $x$ -axis once if the multiple is odd, or is tangent to the  $x$ -axis if the multiple is even (Fig. 15.6).

The graph must cross the  $x$ -axis at least once if the degree of  $f(x)$  is odd, because the range includes all real numbers (Figs. 15.5 and 15.6). The curve may not cross the  $x$ -axis if the degree of  $f(x)$  is even, because the range is bounded at a minimum point or a maximum point (Fig. 15.7), as we have seen for the quadratic function.

$$\begin{aligned} y = f(x) \\ &= x^4 - 5x^3 + 5x^2 + 5x - 6 \\ &= (x + 1)(x - 1)(x - 2)(x - 3) \end{aligned}$$

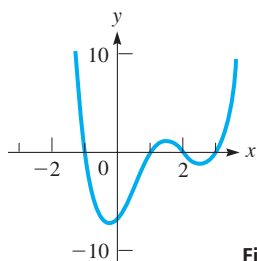


Fig. 15.4

$$\begin{aligned} y = f(x) \\ &= x^3 - 2x^2 + 3 \\ &= (x + 1)(x^2 - 3x + 3) \end{aligned}$$

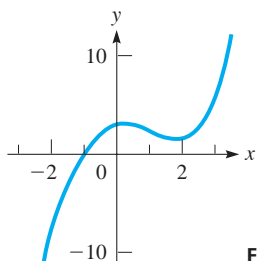


Fig. 15.5

$$\begin{aligned} y = f(x) \\ &= x^5 + x^4 - 2x^3 - 2x^2 + x + 1 \\ &= (x + 1)^3(x - 1)^2 \end{aligned}$$

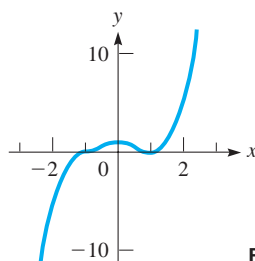


Fig. 15.6

$$\begin{aligned} y = f(x) \\ &= x^4 + 4 \\ &= (x^2 - 2x + 2)(x^2 + 2x + 2) \end{aligned}$$

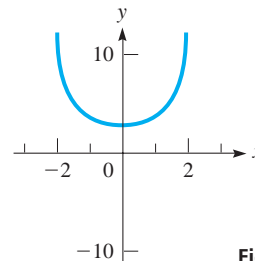


Fig. 15.7

**EXERCISES 15.2**

In Exercises 1 and 2, make the given changes in the indicated examples of this section and then solve the resulting equation.

- In Example 2, change the middle term of the second factor to  $2x$ .
- In Example 6, change the middle three terms to the left of the  $=$  sign to  $-3x^3 + 7x^2 + 7x$ , given the same root.

In Exercises 3–6, find the roots of the given equations by inspection.

- $(x - 2)(x^2 - 9) = 0$
- $x(2x + 5)^2(x^2 - 64) = 0$
- $(x^2 + 6x + 9)(x^2 + 4) = 0$
- $(4x^2 + 9)(4x^2 + 4x + 1) = 0$

In Exercises 7–26, solve the given equations using synthetic division, given the roots indicated.

- $x^3 + x^2 - 8x - 12 = 0$  ( $r_1 = -2$ )
- $R^3 + 1 = 0$  ( $r_1 = -1$ )
- $2x^3 + 11x^2 + 20x + 12 = 0$  ( $r_1 = -\frac{3}{2}$ )
- $4x^3 - 20x^2 - x + 5 = 0$  ( $r_1 = \frac{1}{2}$ )
- $3x^3 + 2x^2 + 3x + 2 = 0$  ( $r_1 = j$ )
- $x^3 + 5x^2 + 9x + 5 = 0$  ( $r_1 = -2 + j$ )
- $t^4 + t^3 - 2t^2 + 4t - 24 = 0$  ( $r_1 = 2, r_2 = -3$ )
- $x^4 - 2x^3 - 20x^2 - 8x - 96 = 0$  ( $r_1 = 6, r_2 = -4$ )
- $2x^4 - 19x^3 + 39x^2 + 35x - 25 = 0$  (5 is a double root)
- $4n^4 + 28n^3 + 61n^2 + 42n + 9 = 0$  ( $-3$  is a double root)
- $6x^4 + 5x^3 - 15x^2 + 4 = 0$  ( $r_1 = -\frac{1}{2}, r_2 = \frac{2}{3}$ )

- $6x^4 - 5x^3 - 14x^2 + 14x - 3 = 0$  ( $r_1 = \frac{1}{3}, r_2 = \frac{3}{2}$ )
- $2x^4 - x^3 - 4x^2 + 10x - 4 = 0$  ( $r_1 = 1 + j$ )
- $s^4 - 8s^3 - 72s - 81 = 0$  ( $r_1 = 3j$ )
- $x^5 - 3x^4 + 4x^3 - 4x^2 + 3x - 1 = 0$  (1 is a triple root)
- $12x^5 - 7x^4 + 41x^3 - 26x^2 - 28x + 8 = 0$   
( $r_1 = 1, r_2 = \frac{1}{4}, r_3 = -\frac{2}{3}$ )
- $P^5 - 3P^4 - P + 3 = 0$  ( $r_1 = 3, r_2 = j$ )
- $4x^5 + x^3 - 4x^2 - 1 = 0$  ( $r_1 = 1, r_2 = \frac{1}{2}j$ )
- $(x^6 + 2x^5 - 4x^4 - 10x^3 - 41x^2 - 72x - 36 = 0$   
( $-1$  is a double root;  $2j$  is a root)
- $x^6 - x^5 - 2x^3 - 3x^2 - x - 2 = 0$  ( $j$  is a double root)

In Exercises 27 and 28, answer the given questions.

- Why cannot a third-degree polynomial function with real coefficients have zeros of 1, 2, and  $j$ ?
- How can the graph of a fourth-degree polynomial equation have its only  $x$ -intercepts as 0, 1, and 2?

In Exercises 29 and 30, form the indicated equations.

- Form a polynomial equation of degree 3 and with integral coefficients, having a root of  $1 + j$ , and for which  $f(2) = 4$ .
- Form a polynomial equation of the smallest possible degree and with integral coefficients, having a double root of 3 and a root of  $j$ .

**Answer to Practice Exercise**

- $-2, 3, 2j, -2j$

## 15.3 Rational and Irrational Roots

Possible Rational Roots •  
 Descartes' Rule of Signs •  
 Roots of a Polynomial Equation

### LEARNING TIP

In a polynomial equation  $f(x) = 0$  with integral coefficients, if the coefficient of the highest power is 1, then any integral roots are factors of the constant term of  $f(x)$ .

The product  $(x + 2)(x - 4)(x + 3)$  equals  $x^3 + x^2 - 14x - 24$ . Note how the constant 24 is determined only by the 2, 4, and 3. These numbers represent the roots of the equation if the given function is set equal to zero. In fact, if we find all the integral roots of an equation with integral coefficients and represent the equation in the form

$$f(x) = (x - r_1)(x - r_2) \cdots (x - r_k)f_{k+1}(x) = 0$$

where all the roots indicated are integers, the constant term of  $f(x)$  must have factors of  $r_1, r_2, \dots, r_k$ . This leads us to the result in the Learning Tip to the left.

### EXAMPLE 1 Possible integral roots if $a_0 = 1$

The equation  $x^5 - 4x^4 - 7x^3 + 14x^2 - 44x + 120 = 0$  can be written as

$$(x - 5)(x + 3)(x - 2)(x^2 + 4) = 0$$

We now note that  $5(3)(2)(4) = 120$ . Thus, the roots 5,  $-3$ , and 2 are numerical factors of  $|120|$ . The theorem states nothing about the signs involved. ■

If the coefficient  $a_0$  of the highest-power term of  $f(x)$  is an integer not equal to 1, the polynomial equation  $f(x) = 0$  may have rational roots that are not integers. We can factor  $a_0$  from every term of  $f(x)$ . Thus, any polynomial equation  $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$  with integral coefficients can be written in the form

$$f(x) = a_0 \left( x^n + \frac{a_1}{a_0}x^{n-1} + \cdots + \frac{a_n}{a_0} \right) = 0$$

Since  $a_n$  and  $a_0$  are integers,  $a_n/a_0$  is a rational number. Using the same reasoning as with integral roots applied to the polynomial within the parentheses, we see that any rational roots are factors of  $a_n/a_0$ . This leads to the following theorem:

### Rational Roots of a Polynomial Equation

Any rational root  $r_r$  of a polynomial equation (with integral coefficients)

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$$

is an integral factor of  $a_n$  divided by an integral factor of  $a_0$ . Therefore,

$$r_r = \frac{\text{integral factor of } a_n}{\text{integral factor of } a_0} \quad (15.4)$$

### EXAMPLE 2 Possible rational roots

If  $f(x) = 4x^3 - 3x^2 - 25x - 6$ , any possible rational roots must be integral factors of 6 divided by integral factors of 4. These factors of 6 are 1, 2, 3, and 6, and these factors of 4 are 1, 2, and 4. Forming all possible positive and negative quotients, any possible rational roots that exist will be found in the following list:  $\pm 1, \pm \frac{1}{2}, \pm \frac{1}{4}, \pm 2, \pm 3, \pm \frac{3}{2}, \pm \frac{3}{4}, \pm 6$ .

The roots of this equation are  $-2, 3$ , and  $-\frac{1}{4}$ . ■

There are 16 different possible rational roots in Example 2, but we cannot tell which of these are the actual roots. Therefore we now present a rule, known as *Descartes' rule of signs*, which will help us find these roots.

■ Descartes' rule of signs is named for the French mathematician René Descartes (1596–1650). (See page 95.)

**Descartes' Rule of Signs**

1. The number of positive roots of a polynomial equation  $f(x) = 0$  cannot exceed the number of changes in sign in  $f(x)$  in going from one term to the next in  $f(x)$ .
2. The number of negative roots cannot exceed the number of sign changes in  $f(-x)$ .

If  $f(x)$  has all positive terms, then any positive number substituted in  $f(x)$  must give a positive value for  $f(x)$ . This indicates that the number substituted in the function is not a root. Thus, there must be at least one negative and one positive term in the function for any positive number to be a root.

When the number of positive roots is less than the number of sign changes of  $f(x)$ , it is always less by an even number (because complex roots come in pairs). Therefore, for a single sign change we have the conclusion in the Learning Tip to the left.

**LEARNING TIP**

When there is just one change of sign in  $f(x)$ , there is one positive root.  
When there is just one change of sign in  $f(-x)$ , there is one negative root.

**EXAMPLE 3 Using Descartes' rule of signs**

By Descartes' rule of signs, determine the maximum possible number of positive and negative roots of  $3x^3 - x^2 - x + 4$ .

Here,  $f(x) = 3x^3 - x^2 - x + 4$ . The first term is positive, and the second is negative, which indicates a change of sign. The third term is also negative; there is no change of sign from the second to the third term. The fourth term is positive, thus giving us a second change of sign, from the third to the fourth term. Hence, there are two changes in sign, which we can show as follows:

$$f(x) = 3x^3 - x^2 - x + 4$$

Since there are *two* changes of sign in  $f(x)$ , there are *no more than two* positive roots of  $f(x) = 0$ .

To find the maximum possible number of negative roots, we must find the number of sign changes in  $f(-x)$ . Thus,

$$\begin{aligned} f(-x) &= 3(-x)^3 - (-x)^2 - (-x) + 4 \\ &= -3x^3 - x^2 + x + 4 \end{aligned}$$

There is only one change of sign in  $f(-x)$ ; therefore, there is one negative root. ■

**EXAMPLE 4 Using Descartes' rule of signs**

For the equation  $4x^5 - x^4 - 4x^3 + x^2 - 5x - 6 = 0$ , we write

$$\begin{aligned} f(x) &= 4x^5 - x^4 - 4x^3 + x^2 - 5x - 6 \\ f(-x) &= -4x^5 - x^4 + 4x^3 + x^2 + 5x - 6 \end{aligned}$$

Thus, there are no more than three positive and two negative roots. ■

We now summarize all the information that can be used to find the roots of a polynomial equation  $f(x) = 0$  of degree  $n$  and with real coefficients algebraically. We include two rules we have not seen yet, but which will be illustrated in the examples that follow the summary, together with the complete method for solving polynomial equations.

**Practice Exercise**

1. Determine the maximum possible number of positive and negative roots of  $9x^4 - 12x^3 - 5x^2 + 12x - 4 = 0$ .

■ Some calculators have a specific feature for solving higher-degree equations.

**Practice Exercise**

2. Find the roots of the equation  $9x^4 - 12x^3 - 5x^2 + 12x - 4 = 0$ .

$$\begin{array}{r} 2 \ 1 \ 5 \ -3 \ \underline{1} \\ 2 \ 3 \ 8 \ 5 \end{array}$$

$$\begin{array}{r} 2 \ 1 \ 5 \ -3 \ \underline{\frac{1}{2}} \\ 1 \ 1 \ 3 \\ 2 \ 2 \ 6 \ 0 \end{array}$$

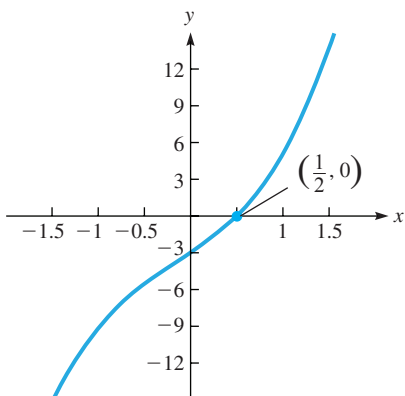


Fig. 15.8

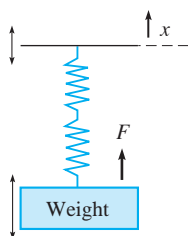


Fig. 15.9

**Roots of a Polynomial Equation of Degree  $n$**

1. There are  $n$  real or complex roots.
2. Complex roots appear in conjugate pairs.
3. Any rational roots must be factors of the constant term divided by factors of the coefficient of the highest-power term.
4. The maximum number of positive roots is the number of sign changes in  $f(x)$ , and the maximum number of negative roots is the number of sign changes in  $f(-x)$ .
5. Use synthetic division to try possible roots. When a root is found, continue working with the quotient, which is one degree less than the degree of the dividend. When  $n - 2$  roots have been found, the remaining two roots are found using the quadratic formula.
6. When trying a positive root, if the bottom row of the synthetic division table contains all positive numbers, then there are no roots larger than the value tried.
7. When trying a negative root, if the bottom row of the synthetic division table contains alternating signs, then there are no roots less than the value tried.

**EXAMPLE 5 Finding all the roots of an equation**

Find the roots of the equation  $2x^3 + x^2 + 5x - 3 = 0$ .

Since  $n = 3$ , there are three roots. We have

$$f(x) = 2x^3 + x^2 + 5x - 3 \quad \text{and} \quad f(-x) = -2x^3 + x^2 - 5x - 3$$

which shows there is one positive root and no more than two negative roots, which may or may not be rational. The possible rational roots are  $\pm 1, \pm \frac{1}{2}, \pm \frac{3}{2}, \pm 3$ .

First, trying the root 1 (always a possibility if there are positive roots), we have the synthetic division shown at the left. The remainder of 5 tells us that 1 is not a root. Moreover, the bottom row of the table contains all positive numbers, so that there is no positive root larger than 1. Indeed, if we tried any positive number larger than 1, the results in the last row would be larger positive numbers than the ones we now have. Therefore, there is no reason to try  $+3/2$  and  $+3$  as roots.

Now, let us try  $\frac{1}{2}$ , as shown at the left. The zero remainder tells us that  $\frac{1}{2}$  is a root, and the remaining factor is  $2x^2 + 2x + 6$ , which itself factors to  $2(x^2 + x + 3)$ . By the quadratic formula, we find the remaining roots by solving the equation  $x^2 + x + 3 = 0$ . This gives us

$$x = \frac{-1 \pm \sqrt{1 - 12}}{2} = \frac{-1 \pm j\sqrt{11}}{2}$$

The three roots are  $\frac{1}{2}, \frac{-1 + j\sqrt{11}}{2}$ , and  $\frac{-1 - j\sqrt{11}}{2}$ . The graph in Fig. 15.8 verifies the one real root at  $x = 1/2$ . ■

**EXAMPLE 6 Solving an equation—application**

During a cycle of the movement of the weight on the double spring in Fig. 15.9, the force  $F$  (in N) on the weight by the spring is

$$F = x^4 - 7x^3 + 12x^2 + 4x$$

where  $x$  is the displacement (in cm) of the top of the double spring. For what values of  $x$  is  $F = 16$  N?

Substituting 16 for  $F$ , we see that we are to solve the equation

$$x^4 - 7x^3 + 12x^2 + 4x - 16 = 0$$

To solve this equation, we write

$$\begin{aligned}
 f(x) &= x^4 - 7x^3 + 12x^2 + 4x - 16 \\
 f(-x) &= x^4 + 7x^3 + 12x^2 - 4x - 16
 \end{aligned}$$

We see that there are four roots; there are no more than three positive roots, and there is one negative root. Since the coefficient of  $x^4$  is 1, any possible rational roots must be integers. These possible rational roots are  $\pm 1, \pm 2, \pm 4, \pm 8,$  and  $\pm 16$ . Since there is only one negative root, we look for this one first. Trying  $-2$ , we have

$$\begin{array}{r}
 1 \quad -7 \quad +12 \quad +4 \quad -16 \quad \underline{-2} \\
 \quad -2 \quad +18 \quad -60 \quad +112 \\
 \hline
 1 \quad -9 \quad +30 \quad -56 \quad +96
 \end{array}$$

The remainder is not zero, so  $-2$  is not a root. Moreover, the signs of the bottom row of the table alternate in sign, so if we tried any number less than  $-2$ , the numbers would still alternate from term to term in the quotient. Hence  $-4, -8,$  and  $-16$  cannot be roots.

Next, we try  $-1$ , as shown at the left. The remainder of zero tells us that  $-1$  is the negative root.

Now that we have found the one negative root, we look for the positive roots. Trying 1, we have the table at the left. The remainder of  $-3$  tells us that 1 is not a root. Next, we try 2, as shown at the left. We see that 2 is a root.

It is not necessary to find any more roots by trial and error. We may now use the quadratic formula or factoring on the equation  $x^2 - 6x + 8 = 0$ . The remaining roots are 2 and 4. Thus, the roots are  $-1, 2, 2,$  and 4. (Note that 2 is a double root.)

These roots now indicate that  $F = 16$  N for displacements of  $-1$  cm, 2 cm, and 4 cm. ■

By the methods we have presented, we can look for *all roots* of a polynomial equation. These include any possible *complex roots* and *exact values of the rational and irrational roots*, if they exist. These methods allow us to solve a great many polynomial equations for these roots, but there are numerous other equations for which these methods are not sufficient.

When a polynomial equation has more than two irrational roots, we cannot generally find these roots by the methods we have developed. Approximate values can be found on a calculator either graphically or by evaluating the function. These methods are illustrated in the following two examples. Another approach for approximating roots is discussed in Section 24.2.

**EXAMPLE 7** Finding roots with a calculator

Find the roots of the equation  $x^3 - 2x^2 - 3x + 2 = 0$ , using a graphing calculator.

Since the degree of the equation is 3, we know there are three roots, and since the degree is odd, there is at least one real root. Using Descartes' rule of signs, we have

$$\begin{aligned}
 f(x) &= x^3 - 2x^2 - 3x + 2 && \text{two changes of sign} \\
 f(-x) &= -x^3 - 2x^2 + 3x + 2 && \text{one change of sign}
 \end{aligned}$$

This means there is one negative root and no more than two positive roots, if any. Therefore, setting

$$y = x^3 - 2x^2 - 3x + 2$$

and using the *window settings* (after two or three trials) shown in the calculator display in Fig. 15.10, the view shows one negative and two positive roots. Using the *zero* feature (or the *trace* and *zoom* features), these roots are

$$-1.34, 0.53, 2.81 \quad \text{to the nearest } 0.01$$

$$\begin{array}{r}
 \blacksquare \quad 1 \quad -7 \quad +12 \quad +4 \quad -16 \quad \underline{-1} \\
 \quad -1 \quad 8 \quad -20 \quad 16 \\
 \hline
 1 \quad -8 \quad 20 \quad -16 \quad 0
 \end{array}$$

$$\begin{array}{r}
 \blacksquare \quad 1 \quad -8 \quad 20 \quad -16 \quad \underline{1} \\
 \quad 1 \quad -7 \quad -13 \\
 \hline
 1 \quad -7 \quad 13 \quad -3
 \end{array}$$

$$\begin{array}{r}
 \blacksquare \quad 1 \quad -8 \quad 20 \quad -16 \quad \underline{2} \\
 \quad 2 \quad -12 \quad 16 \\
 \hline
 1 \quad -6 \quad 8 \quad 0
 \end{array}$$

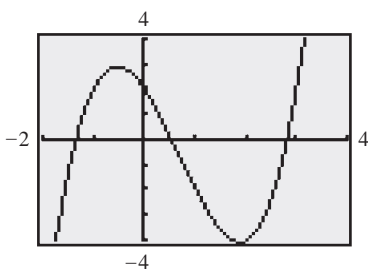


Fig. 15.10



The roots of a polynomial equation can be approximated by successive evaluation of the function. The method can be summarized as follows.

### Approximating Roots by Successive Evaluation

1. Find an interval  $(a, b)$  such that  $f(a)$  and  $f(b)$  have opposite signs.
2. Choose a point  $c$  in  $(a, b)$  and evaluate  $f(c)$ .
3. If  $f(c) = 0$ , then  $c$  is a root. Otherwise, if  $f(a)$  and  $f(c)$  have opposite signs, then the root is in  $(a, c)$ . If  $f(c)$  and  $f(b)$  have opposite signs, then the root is in  $(c, b)$ .
4. Repeat steps 2 and 3 until the interval is sufficiently small.

### EXAMPLE 8 Solving an applied problem by successive evaluation

The bottom part of a box to hold a jigsaw puzzle is to be made from a rectangular piece of cardboard 37.0 cm by 31.0 cm by cutting out equal squares from the corners, bending up the sides, and taping the corners. See Fig. 15.11. If the volume of the bottom part is to be  $2770 \text{ cm}^3$ , find the side of the square that is to be cut out.

Let  $x =$  the side of the square to be cut out. This means

$$\begin{aligned} 2770 &= x(37.0 - 2x)(31.0 - 2x) && \text{volume} = 2770 \text{ cm}^3 \\ &= 4x^3 - 136x^2 + 1147x \end{aligned}$$

$$4x^3 - 136x^2 + 1147x - 2770 = 0 \quad \text{simplify with terms on the left}$$

We evaluate  $f(x) = 4x^3 - 136x^2 + 1147x - 2770$  for integral values of  $x$  and find that  $f(4) = -102$  and  $f(5) = 65$ . Since  $f(4)$  and  $f(5)$  have opposite signs, this means that there is a root in  $(4, 5)$ . We pick  $c = 4.5$  and evaluate  $f(4.5) = 2$ . Now  $f(4)$  and  $f(4.5)$  have opposite signs, so there is a root in  $(4, 4.5)$ . As we continue the evaluations, we obtain the following table:

Interval	$c$	$f(c)$	New interval
$(4, 5)$	4.5	2	$(4, 4.5)$
$(4, 4.5)$	4.4	-15.42	$(4.4, 4.5)$
$(4.4, 4.5)$	4.48	-1.353	$(4.48, 4.5)$
$(4.48, 4.5)$	4.49	0.3318	$(4.48, 4.49)$

From these values,  $x = 4.49$  cm, to three significant digits. This value gives a volume

$$V = 4.49(37.0 - 2(4.49))(31.0 - 2(4.49)) = 2770 \text{ cm}^3$$

which means that the solution checks.

Checking values greater than  $x = 5$ , we find that there is another root between 6 and 7. Using the same procedure as above, we find that  $x = 6.79$  cm is also a solution. (A third root  $x = 22.7$  cm is obviously too large.) Therefore, there are two possible squares that can be cut from each corner to get a bottom part with a volume of  $2770 \text{ cm}^3$ .

■ See the chapter introduction.

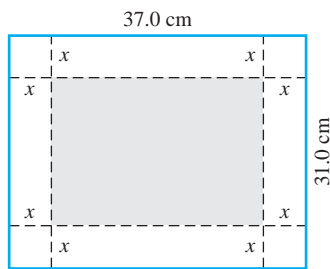


Fig. 15.11

■ Methods of evaluating functions on a graphing calculator are described on page 88.

## EXERCISES 15.3

In Exercises 1 and 2, make the given changes in the indicated examples of this section and then perform the indicated operation.

- In Example 4, change all signs between terms.
- In Example 5, change the + sign before the  $5x$  to  $-$ .

In Exercises 3–20, solve the given equations.

- $x^3 + 2x^2 - x - 2 = 0$
- $x^3 + x^2 - 5x + 3 = 0$
- $x^3 + 2x^2 - 5x - 6 = 0$
- $t^3 - 12t - 16 = 0$
- $3x^3 - x - 2 = 0$
- $3t^3 + 8t^2 - 1 = 0$
- $3x^3 + 11x^2 + 5x - 3 = 0$
- $4x^3 - 5x^2 - 23x + 6 = 0$
- $x^4 - 11x^2 - 12x + 4 = 0$
- $8x^4 - 32x^3 - x + 4 = 0$
- $5n^4 - 2n^3 + 40n - 16 = 0$
- $4n^4 - 17n^2 + 14n - 3 = 0$
- $12x^4 + 44x^3 + 21x^2 - 11x = 6$
- $9x^4 - 3x^3 + 34x^2 - 12x = 8$
- $D^5 + D^4 - 9D^3 - 5D^2 + 16D + 12 = 0$
- $x^6 - x^4 - 14x^2 + 24 = 0$
- $4x^5 - 24x^4 + 49x^3 - 38x^2 + 12x - 8 = 0$
- $2x^5 + 5x^4 - 4x^3 - 19x^2 - 16x = 4$

In Exercises 21–24, use a graphing calculator to solve the given equations to the nearest 0.01.

- $x^3 - 2x^2 - 5x + 4 = 0$
- $2x^4 - 15x^2 - 7x + 3 = 0$
- $8x^4 + 36x^3 + 35x^2 - 4x - 4 = 0$
- $2x^5 - 3x^4 + 8x^3 - 4x^2 - 4x + 2 = 0$

In Exercises 25–28, find the irrational root (to the nearest 0.01) that lies between the given values by successive evaluation. (See Example 8.)

- $x^3 - 6x^2 + 10x - 4 = 0$  (0 and 1)
- $r^4 - r^3 - 3r^2 - r - 4 = 0$  (2 and 3)
- $3x^3 + 13x^2 + 3x - 4 = 0$  (-1 and 0)
- $3x^4 - 3x^3 - 11x^2 - x - 4 = 0$  (-2 and -1)

In Exercises 29–46, solve the given problems. Use a graphing calculator if necessary.

- Solve the following system algebraically:  
 $y = x^4 - 11x^2$ ;  $y = 12x - 4$
- Find rational values of  $a$  such that  $(x - a)$  will divide into  $x^3 + x^2 - 4x - 4$  with a remainder of zero.
- Where does the graph of the function  $f(x) = 4x^3 + 3x^2 - 20x - 15$  cross the  $x$ -axis?
- Where does the graph of the function  $f(s) = 2s^4 - s^3 - 5s^2 + 7s - 6$  cross the  $s$ -axis?

- The angular acceleration  $\alpha$  (in  $\text{rad/s}^2$ ) of the wheel of a car is given by  $\alpha = -0.2t^3 + t^2$ , where  $t$  is the time (in s). For what values of  $t$  is  $\alpha = 2.0 \text{ rad/s}^2$ ?
- In finding one of the dimensions  $d$  (in cm) of the support columns of a building, the equation  $3d^3 + 5d^2 - 400d - 18\,000 = 0$  is found. What is this dimension?
- The deflection  $y$  of a beam at a horizontal distance  $x$  from one end is given by  $y = k(x^4 - 2Lx^3 - L^3x)$ , where  $L$  is the length of the beam and  $k$  is a constant. For what values of  $x$  is the deflection zero?
- The specific gravity  $s$  of a sphere of radius  $r$  that sinks to a depth  $h$  in water is given by  $s = \frac{3rh^2 - h^3}{4r^3}$ . Find the depth to which a spherical buoy of radius 4.0 cm sinks if  $s = 0.50$ .
- A variable electric voltage in a circuit is given by  $V = 0.1t^4 - 1.0t^3 + 3.5t^2 - 5.0t + 2.3$ , where  $t$  is the time (in s). If the voltage is on for 5.0 s, when is  $V = 0$ ?
- The pressure difference  $p$  (in kPa) at a distance  $x$  (in km) from one end of an oil pipeline is given by  $p = x^5 - 3x^4 - x^2 + 7x$ . If the pipeline is 4 km long, where is  $p = 0$ ?
- A rectangular tray is made from a square piece of sheet metal 10.0 cm on a side by cutting equal squares from each corner, bending up the sides, and then welding them together. How long is the side of the square that must be cut out if the volume of the tray is  $70.0 \text{ cm}^3$ ?
- The angle  $\theta$  of a robot arm with the horizontal as a function of time  $t$  (in s) is given by  $\theta = 15 + 20t^2 - 4t^3$  for  $0 \leq t \leq 5$  s. Find  $t$  for  $\theta = 40^\circ$ .
- The radii of four different-sized ball bearings differ by 1.00 mm in radius from one size to the next. If the volume of the largest equals the volumes of the other three combined, find the radii.
- A rectangular safe is to be made of steel of uniform thickness, including the door. The inside dimensions are 1.20 m, 1.20 m, and 2.00 m. If the volume of steel is  $1.25 \text{ m}^3$ , find its thickness.
- For electrical resistors connected in parallel, the reciprocal of the combined resistance equals the sum of the reciprocals of the individual resistances. If three resistors are connected in parallel such that the second resistance is  $1400 \Omega$  more than the first and the third is  $5600 \Omega$  more than the first, find the resistances for a combined resistance of  $1400 \Omega$ .
- Each of three revolving doors has a perimeter of 6.60 m and revolves through a volume of  $9.50 \text{ m}^3$  in one revolution about their common vertical side. What are the doors' dimensions?
- If  $a$ ,  $b$ , and  $c$  are positive integers, find the combinations of the possible positive, negative, and complex roots if  $f(x) = ax^3 - bx^2 + c = 0$ .
- An equation  $f(x) = 0$  involves only odd powers of  $x$  with positive coefficients. Explain why this equation has no real root except  $x = 0$ .

## Answers to Practice Exercises

- 3 positive, 1 negative
- 1,  $2/3$ ,  $2/3$ , 1

## CHAPTER 15 EQUATIONS

## Polynomial function

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n \quad (15.1)$$

## Remainder theorem

$$f(x) = (x - r)q(x) + R \quad (15.2)$$

$$f(r) = R \quad (15.3)$$

## Rational roots

$$r_r = \frac{\text{integral factor of } a_n}{\text{integral factor of } a_0} \quad (15.4)$$

## CHAPTER 15 REVIEW EXERCISES

In Exercises 1–4, find the remainder of the indicated division by the remainder theorem.

- $(2x^3 - 4x^2 - x + 4) \div (x - 1)$
- $(x^3 - 2x^2 + 9) \div (x + 2)$
- $(4n^3 + n + 4) \div (n + 3)$
- $(x^4 - 5x^3 + 8x^2 + 15x - 2) \div (x - 3)$

In Exercises 5–8, use the factor theorem to determine whether or not the second expression is a factor of the first.

- $x^4 + x^3 + x^2 - 2x - 3$ ;  $x + 1$
- $2s^3 - 6s - 4$ ;  $s - 2$
- $2t^4 - 10t^3 - t^2 - 3t + 10$ ;  $t + 5$
- $9v^3 + 6v^2 + 4v + 2$ ;  $3v + 1$

In Exercises 9–16, use synthetic division to perform the indicated divisions.

- $(x^3 + 3x^2 + 6x + 1) \div (x - 1)$
- $(3x^3 - 2x^2 + 7) \div (x - 3)$
- $(2x^3 - 3x^2 - 4x + 3) \div (x + 2)$
- $(3D^3 + 8D^2 - 16) \div (D + 4)$
- $(x^4 + 3x^3 - 20x^2 - 2x + 56) \div (x + 6)$
- $(x^4 - 6x^3 + x - 8) \div (x - 3)$
- $(2m^5 - 46m^3 + m^2 - 9) \div (m - 5)$
- $(x^6 + 63x^3 + 5x^2 - 9x - 8) \div (x + 4)$

In Exercises 17–20, use synthetic division to determine whether or not the given numbers are zeros of the given functions.

- $y^3 + 5y^2 - 6$ ;  $-3$
- $8y^4 - 32y^3 - y + 4$ ;  $4$
- $2x^4 - x^3 + 2x^2 + x - 1$ ;  $\frac{1}{2}$
- $6W^4 - 7W^3 + 2W^2 - 9W - 6$ ;  $-\frac{2}{3}$

In Exercises 21–32, find all the roots of the given equations, using synthetic division and the given roots.

- $x^3 - 4x^2 - 7x + 10 = 0$  ( $r_1 = 5$ )
- $3B^3 - B^2 - 24B + 28 = 0$  ( $r_1 = 2$ )

- $x^4 - 10x^3 + 35x^2 - 50x + 24 = 0$  ( $r_1 = 1, r_2 = 2$ )
- $x^4 - x^3 - 5x^2 - x - 6 = 0$  ( $r_1 = 3, r_2 = -2$ )
- $4p^4 - p^2 - 18p + 9 = 0$  ( $r_1 = \frac{1}{2}, r_2 = \frac{3}{2}$ )
- $15x^4 + 4x^3 + 56x^2 + 16x - 16 = 0$   
( $r_1 = 2/5, r_2 = -2/3$ )
- $4x^4 + 4x^3 + x^2 + 4x - 3 = 0$  ( $r_1 = j$ )
- $x^4 + 2x^3 - 4x - 4 = 0$  ( $r_1 = -1 + j$ )
- $s^5 + 3s^4 - s^3 - 11s^2 - 12s - 4 = 0$  ( $-1$  is a triple root)
- $24x^5 + 10x^4 + 7x^2 - 6x + 1 = 0$   
( $r_1 = -1, r_2 = \frac{1}{4}, r_3 = \frac{1}{3}$ )
- $V^5 + 4V^4 + 5V^3 - V^2 - 4V - 5 = 0$   
( $r_1 = 1, r_2 = -2 + j$ )
- $2x^5 - x^4 + 8x - 4 = 0$  ( $r_1 = \frac{1}{2}, r_2 = 1 + j$ )

In Exercises 33–40, solve the given equations.

- $x^3 + x^2 - 10x + 8 = 0$
- $x^3 - 8x^2 + 20x = 16$
- $2r^3 - 3r^2 + 1 = 0$
- $2x^3 - 3x^2 - 11x + 6 = 0$
- $6x^3 - x^2 - 12x = 5$
- $6y^3 + 19y^2 + 2y = 3$
- $4t^4 - 17t^2 + 14t - 3 = 0$
- $2x^4 + 5x^3 - 14x^2 - 23x + 30 = 0$

In Exercises 41–62, solve the given problems. Where appropriate, set up the required equations.

- What are the possible numbers of real zeros (double roots count as two, etc.) for a polynomial with real coefficients and of degree 5?
- What are the possible combinations of real and complex zeros (double roots count as two, etc.) of a fourth-degree polynomial?
- If a calculator shows a real root, how many complex roots are possible for a sixth-degree polynomial equation  $f(x) = 0$ ?
- Find rational values of  $a$  such that  $(x - a)$  will divide into  $x^3 - 13x + 3$  with a remainder of  $-9$ .
- Explain how to find  $k$  if  $x + 2$  is a factor of  $f(x) = 3x^3 + kx^2 - 8x - 8$ . What is  $k$ ?

46. Explain how to find  $k$  if  $x - 3$  is a factor of  $f(x) = kx^4 - 15x^2 - 5x - 12$ . What is  $k$ ?
47. Form a polynomial equation of degree 3 with integral coefficients and having roots of  $j$  and 5.
48. If  $f(x) = 3x^4 - 18x^3 - 2x^2 + 13x - 6$ , and  $f(x) = g(x)(x - 6)$ , find  $g(x)$ .
49. Solve the following system algebraically:  $x^2 = y + 3$ ;  $xy = 2$ .
50. Where does the graph of the function  $f(x) = 2x^4 - 7x^3 + 11x^2 - 28x + 12$  cross the  $x$ -axis?
51. Find the irrational root of the equation  $3x^3 - x^2 - 8x - 2 = 0$  that lies between 1 and 2.
52. A propane tank is in the shape of a cylinder with hemispheres on each end. The cylinder and the hemispheres have the same radius. If the total length of the tank is 4 m and the volume is  $7.0 \text{ m}^3$ , what is the radius?
53. The bending moment  $M$  (in  $\text{kN} \cdot \text{m}$ ) along a beam with a non-uniformly distributed load is given by  $M = -\frac{25}{6}x^3 + 600x$ , where  $x$  is the horizontal distance from one end. For what values of  $x$  is the bending moment  $850 \text{ kN} \cdot \text{m}$ ?
54. A company determined that the number  $s$  (in thousands) of computer chips that it could supply at a price  $p$  of less than \$5 is given by  $s = 4p^2 - 25$ , whereas the demand  $d$  (in thousands) for the chips is given by  $d = p^3 - 22p + 50$ . For what price is the supply equal to the demand?
55. In order to find the diameter  $d$  (in cm) of a helical spring subject to given forces, it is necessary to solve the equation  $64d^3 - 144d^2 + 108d - 27 = 0$ . Solve for  $d$ .
56. A cubical tablet for purifying water is wrapped in a sheet of foil 0.500 mm thick. The total volume of tablet and foil is 33.1% greater than the volume of the tablet alone. Find the length of the edge of the tablet.

57. For the mirror shown in Fig. 15.12, the reciprocal of the focal distance  $f$  equals the sum of the reciprocals of the object distance  $p$  and image distance  $q$  (in cm). Find  $p$ , if  $q = p + 4$  and  $f = (p + 1)/p$ .

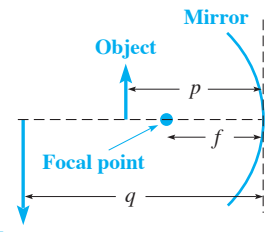


Fig. 15.12

58. Three electric capacitors are connected in series. The capacitance of the second is  $1 \mu\text{F}$  more than that of the first, and the third is  $2 \mu\text{F}$  more than the second. The capacitance of the combination is  $1.33 \mu\text{F}$ . The equation used to determine  $C$ , the capacitance of the first capacitor, is
- $$\frac{1}{C} + \frac{1}{C + 1} + \frac{1}{C + 3} = \frac{3}{4}$$
- Find the values of the capacitances.
59. The height of a cylindrical oil tank is 3.2 m more than the radius. If the volume of the tank is  $680 \text{ m}^3$ , what are the radius and the height of the tank?
60. A grain storage bin has a square base, each side of which is 5.5 m longer than the height of the bin. If the bin holds  $160 \text{ m}^3$  of grain, find its dimensions.
61. A rectangular door has a diagonal brace that is 0.300 m longer than the height of the door. If the area of the door is  $2.7 \text{ m}^2$ , find its dimensions.
62. The radius of one ball bearing is 1.0 mm greater than the radius of a second ball bearing. If the sum of their volumes is  $100 \text{ mm}^3$ , find the radius of each.

### Writing Exercise

63. A computer science student is to write a computer program that will print out the values of  $n$  for which  $x + r$  is a factor of  $x^n + r^n$ . Write a paragraph that states the values of  $n$  and explains how they are found.

## CHAPTER 15 PRACTICE TEST

- Is  $-3$  a zero for the function  $2x^3 + 3x^2 + 7x - 6$ ? Explain.
- Find the remaining roots of the equation  $x^4 - 2x^3 - 7x^2 + 20x - 12 = 0$ ; 2 is a double root.
- Use synthetic division to perform the division  $(x^3 - 5x^2 + 4x - 9) \div (x - 3)$ .
- Use the factor theorem and synthetic division to determine whether or not  $2x + 1$  is a factor of  $2x^4 + 15x^3 + 23x^2 - 16$ .
- Use the remainder theorem to find the remainder of the division  $(x^3 + 4x^2 + 7x - 9) \div (x + 4)$ .
- Solve for  $x$ :  $2x^4 - x^3 + 5x^2 - 4x - 12 = 0$ .
- The ends of a 10-m beam are supported at different levels. The deflection  $y$  of the beam is given by  $y = kx^2(x^3 + 436x - 4000)$ , where  $x$  is the horizontal distance from one end and  $k$  is a constant. Find the values of  $x$  for which the deflection is zero.
- A cubical metal block is heated such that its edge increases by 1.0 mm and its volume is doubled. Find the edge of the cube to tenths.

# Matrices; Systems of Linear Equations

# 16

While working with systems of linear transformations in the 1850s, the English mathematician Arthur Cayley developed the use of a *matrix*. (As we will see, a *matrix* is simply a rectangular array of numbers.) Cayley's interests, as well as those of the many other mathematicians who contributed to the theory of matrices during the nineteenth century, were on the purely mathematical aspects of the theory, with little or no reference to possible applications.

It was not until the 1920s that one of the first major applications of matrices was made, when physicists used them in developing theories about the elementary particles within the atom. Their work is still very important in atomic and nuclear physics.

In the 1930s, many authors started using matrix notation to describe Gaussian elimination, an important method for solving systems of linear equations. (We will discuss Gaussian elimination in Section 16.5.) The matrix interpretation of Gaussian elimination became very important after World War II, with the development of the first electronic computers. The method was used to study the accuracy of calculations made by the machines.

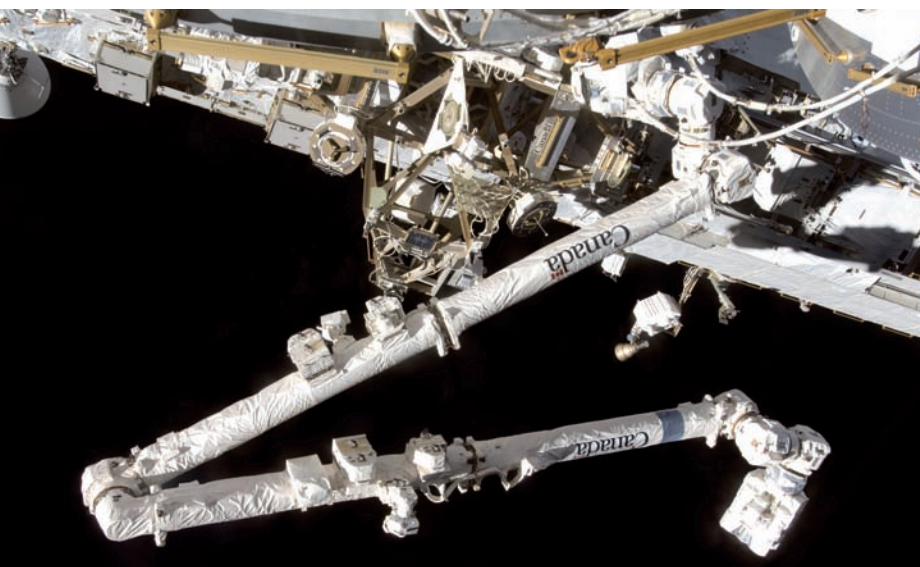
Although Gaussian elimination is named in honour of German mathematician Karl Friedrich Gauss, the method first appeared in a third-century B.C.E. Chinese text. Isaac Newton is responsible for an independent development of the method in seventeenth-century Europe. Thanks to his influence, the method was a standard lesson in algebra textbooks by the turn of the nineteenth century. Around 1800, Gauss developed his systematic version of Gaussian elimination in order to solve astronomical problems by the method of least squares (see Section 22.6).

Since about 1950, matrices have become a very important and useful tool in many other areas of application, such as social science and economics. They are now used extensively in business and industry in making appropriate decisions in research, development, and production.

## LEARNING OUTCOMES

After completion of this chapter, the student should be able to:

- Add and subtract matrices
- Multiply a matrix by a scalar
- Solve a matrix equation
- Perform matrix multiplication
- Find the inverse of a matrix
- Solve a system of linear equations by using the inverse of the matrix of coefficients
- Solve a system of linear equations by using Gaussian elimination
- Identify a system of linear equations as having a unique solution, no solution, or an unlimited number of solutions
- Evaluate determinants of any order
- Solve a system of linear equations by determinants



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◀ The robotic arm Canadarm2 was used to assemble the International Space Station while in space. It is routinely used to move supplies, equipment, and even astronauts. In Section 16.2 we show how the translational and rotational displacements of robotic arms are obtained by multiplication of matrices.

## 16.1 Matrices: Definitions and Basic Operations

Elements of a Matrix • Square Matrix • Equality of Matrices • Matrix Addition and Subtraction • Scalar Multiplication

In this section, we introduce the definitions and some basic operations with matrices. In the three sections that follow, we develop additional operations and show how they are used in solving systems of linear equations. This is only an introduction to the use of matrices, and additional uses and operations with matrices are shown in other sources.

A **matrix** is an ordered rectangular array of numbers (called the **elements** of the matrix). If the array has  $n$  rows and  $m$  columns, it is an  $n \times m$  matrix. When  $n = m$ , the matrix is called a **square matrix**. It is convenient to designate a given matrix by a capital letter (for example, matrix  $A$ ).

### EXAMPLE 1 Illustrations of matrices

Some examples of matrices are shown here:



$$A = \begin{bmatrix} 5 & 0 & -1 \\ 1 & 2 & 6 \\ 0 & -4 & -5 \end{bmatrix} \quad B = \begin{bmatrix} 9 \\ 8 \\ 1 \\ 5 \end{bmatrix} \quad C = [-1 \quad 6 \quad 8 \quad 9] \quad O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Matrix  $A$  is an example of a square  $3 \times 3$  matrix, matrix  $B$  is a  $4 \times 1$  matrix, matrix  $C$  is a  $1 \times 4$  matrix, and matrix  $O$  is a  $2 \times 2$  matrix. Since all of the elements of  $O$  are zero, it is called a **zero matrix**. ■

■ We first introduced the term *matrix* on page 171 when we described the evaluation of a determinant on a calculator. A determinant is a specific number associated with a square matrix. It is not a matrix.

To be able to refer to specific elements of a matrix and to give a general representation, a double-subscript notation is usually employed. That is,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

row  column 

We see that the first subscript refers to the row in which the element lies and the second subscript refers to the column in which the element lies.

*Two matrices are said to be equal if and only if they are identical.* That is, they must have the same number of columns, the same number of rows, and the elements must respectively be equal.

### EXAMPLE 2 Equality of matrices

(a)  $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} 1 & -5 & 0 \\ 4 & 6 & -3 \end{bmatrix}$

if and only if  $a_{11} = 1$ ,  $a_{12} = -5$ ,  $a_{13} = 0$ ,  $a_{21} = 4$ ,  $a_{22} = 6$ , and  $a_{23} = -3$ .

(b) The matrices

$$\begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & -5 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 2 & -5 \\ -1 & -2 & 3 \end{bmatrix}$$

are not equal, since the elements in the third column are reversed.

(c) The matrices

$$\begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 & 3 & 0 \\ -1 & 5 & 0 \end{bmatrix}$$

are not equal, since the number of columns is different. This is true despite the fact that both elements of the third column are zeros. ■



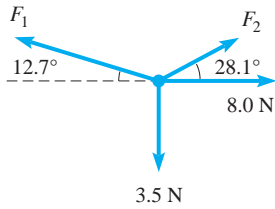


Fig. 16.1

**EXAMPLE 3** Matrix equation

The forces acting on a bolt are in equilibrium, as shown in Fig. 16.1. Analysing the horizontal and vertical components as in Section 9.4, we find the following matrix equation. Find forces  $F_1$  and  $F_2$ .

$$\begin{bmatrix} 0.98F_1 - 0.88F_2 \\ 0.22F_1 + 0.47F_2 \end{bmatrix} = \begin{bmatrix} 8.0 \\ 3.5 \end{bmatrix}$$

From the equality of matrices, we know that  $0.98F_1 - 0.88F_2 = 8.0$  and  $0.22F_1 + 0.47F_2 = 3.5$ . Therefore, to find the forces  $F_1$  and  $F_2$ , we must solve the system of equations

$$0.98F_1 - 0.88F_2 = 8.0$$

$$0.22F_1 + 0.47F_2 = 3.5$$

Using determinants, we have

$$F_1 = \frac{\begin{vmatrix} 8.0 & -0.88 \\ 3.5 & 0.47 \end{vmatrix}}{\begin{vmatrix} 0.98 & -0.88 \\ 0.22 & 0.47 \end{vmatrix}} = \frac{8.0(0.47) - 3.5(-0.88)}{0.98(0.47) - 0.22(-0.88)} = 10.5 \text{ N}$$

Using determinants again, or by substituting this value into either equation, we find that  $F_2 = 2.6 \text{ N}$ . These values check when substituted into the original matrix equation. ■

**MATRIX ADDITION AND SUBTRACTION**

If two matrices have the same number of rows and the same number of columns, their **sum** is defined as the matrix consisting of the sums of the corresponding elements. If the number of rows or the number of columns of the two matrices is not equal, they cannot be added.

**EXAMPLE 4** Adding matrices

(a)

$$\begin{bmatrix} 8 & 1 & -5 & 9 \\ 0 & -2 & 3 & 7 \end{bmatrix} + \begin{bmatrix} -3 & 4 & 6 & 0 \\ 6 & -2 & 6 & 5 \end{bmatrix} = \begin{bmatrix} 8 + (-3) & 1 + 4 & -5 + 6 & 9 + 0 \\ 0 + 6 & -2 + (-2) & 3 + 6 & 7 + 5 \end{bmatrix} \\ = \begin{bmatrix} 5 & 5 & 1 & 9 \\ 6 & -4 & 9 & 12 \end{bmatrix}$$

**Practice Exercise**

1. For matrices  $A$  and  $B$ , find  $A + B$ .

$$A = \begin{bmatrix} 2 & -4 \\ -7 & 5 \end{bmatrix} \quad B = \begin{bmatrix} -9 & -6 \\ 7 & 3 \end{bmatrix}$$

(b) The matrices

$$\begin{bmatrix} 3 & -5 & 8 \\ 2 & 9 & 0 \\ 4 & -2 & 3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 3 & -5 & 8 & 0 \\ 2 & 9 & 0 & 0 \\ 4 & -2 & 3 & 0 \end{bmatrix}$$

cannot be added since the second matrix has one more column than the first matrix. This is true even though the extra column contains only zeros. ■

The product of a number and a matrix (known as **scalar multiplication** of a matrix) is defined as the matrix whose elements are obtained by multiplying each element of the given matrix by the given number. Thus, we obtain matrix  $kA$  by multiplying the elements of matrix  $A$  by  $k$ . In this way,  $A + A$  and  $2A$  are the same matrix.



**EXAMPLE 5** Scalar multiplicationFor the given matrix  $A$ , find  $2A$ :

$$A = \begin{bmatrix} -5 & 7 \\ 3 & 0 \end{bmatrix} \quad 2A = \begin{bmatrix} 2(-5) & 2(7) \\ 2(3) & 2(0) \end{bmatrix} = \begin{bmatrix} -10 & 14 \\ 6 & 0 \end{bmatrix}$$

By combining the definitions for the addition of matrices and for the scalar multiplication of a matrix, we can define the subtraction of matrices. That is, the **difference** of matrices  $A$  and  $B$  is given by  $A - B = A + (-B)$ . Therefore, we change the sign of each element of  $B$ , and proceed as in addition.

**EXAMPLE 6** Subtracting matrices

$$\begin{bmatrix} 7 & -4 \\ -9 & 3 \end{bmatrix} - \begin{bmatrix} -2 & 6 \\ -8 & 5 \end{bmatrix} = \begin{bmatrix} 7 & -4 \\ -9 & 3 \end{bmatrix} + \begin{bmatrix} 2 & -6 \\ 8 & -5 \end{bmatrix} = \begin{bmatrix} 9 & -10 \\ -1 & -2 \end{bmatrix}$$

The operations of addition, subtraction, and multiplication of a matrix by a number are like those for real numbers. For these operations, the algebra of matrices is like the algebra of real numbers. We see that the following laws hold for matrices:

**Practice Exercise**

2. For matrices  $A$  and  $B$  in Practice Exercise 1, find  $A - 2B$ .

$$A + B = B + A \quad (\text{commutative law}) \quad (16.1)$$

$$A + (B + C) = (A + B) + C \quad (\text{associative law}) \quad (16.2)$$

$$k(A + B) = kA + kB \quad (16.3)$$

$$A + O = A \quad (16.4)$$

Here, we have let  $O$  represent the zero matrix. We will find in the next section that not all laws for matrix operations are like those for real numbers.

**EXERCISES 16.1**

In Exercises 1 and 2, make the given changes in the indicated examples of this section and then perform the indicated operations.

- In Example 4(a), interchange the second and third columns of the second matrix and then add the matrices.
- In Example 5, find the matrix  $-2A$ .

In Exercises 3–10, determine the value of the literal numbers in each of the given matrix equalities.

- $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 4 & 7 \end{bmatrix}$
- $\begin{bmatrix} x \\ x + y \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$
- $\begin{bmatrix} x & 2y & z \\ r/4 & -s & -5t \end{bmatrix} = \begin{bmatrix} -2 & 10 & -9 \\ 12 & -4 & 5 \end{bmatrix}$
- $[a + bj \quad 2c - dj \quad 3e + fj] = [5j \quad a + 6 \quad 3b + c]$   
( $j = \sqrt{-1}$ )
- $\begin{bmatrix} C + D \\ 2C - D \\ D - 2E \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 6 \end{bmatrix}$
- $\begin{bmatrix} 2x - 3y \\ x + 4y \end{bmatrix} = \begin{bmatrix} 13 \\ 1 \end{bmatrix}$
- $\begin{bmatrix} x - 3 & x + y \\ x - z & y + z \\ x + t & y - t \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 4 & -1 \end{bmatrix}$
- $\begin{bmatrix} x \\ x + y \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 4 & -3 \end{bmatrix}$

In Exercises 11–14, find the indicated sums of matrices.

- $\begin{bmatrix} 2 & 3 \\ -5 & 4 \end{bmatrix} + \begin{bmatrix} -1 & 7 \\ 5 & -2 \end{bmatrix}$
- $\begin{bmatrix} 1 & 0 & 9 \\ 3 & -5 & -2 \end{bmatrix} + \begin{bmatrix} 4 & -1 & 7 \\ 2 & 0 & -3 \end{bmatrix}$
- $\begin{bmatrix} 50 & -82 \\ -34 & 57 \\ -15 & 62 \end{bmatrix} + \begin{bmatrix} -55 & 82 \\ 45 & 14 \\ 26 & -67 \end{bmatrix}$
- $\begin{bmatrix} 4.7 & 2.1 & -9.6 \\ -6.8 & 4.8 & 7.4 \\ -1.9 & 0.7 & 5.9 \end{bmatrix} + \begin{bmatrix} -4.9 & -9.6 & -2.1 \\ 3.4 & 0.7 & 0.0 \\ 5.6 & 10.1 & -1.6 \end{bmatrix}$

In Exercises 15–34, use matrices  $A$ ,  $B$ , and  $C$  to find the indicated matrices.

$$A = \begin{bmatrix} -1 & 4 & -7 \\ 2 & -6 & 11 \end{bmatrix} \quad B = \begin{bmatrix} 7 & 9 & -6 \\ 4 & -1 & -8 \end{bmatrix} \quad C = \begin{bmatrix} 3 & 19 \\ 7 & -5 \end{bmatrix}$$

- $A + B$
- $A - B$
- $A + C$
- $B + C$
- $2A + B$
- $2B + A$
- $A - 2B$
- $3A - B$
- $-4A$
- $-3B$
- $-C - A$
- $-\frac{1}{2}A + B$

27.  $3A$       28.  $-2C$       29.  $A + 3B$       30.  $B - 0.5A$   
 31.  $A - C$       32.  $-4A - 3B$       33.  $-6B + A$       34.  $A - B + 2C$

In Exercise 35–38, use matrices  $A$  and  $B$  to show that the indicated laws hold for these matrices. In Exercise 35, explain the meaning of the result.

$$A = \begin{bmatrix} -1 & 2 & 3 & 7 \\ 0 & -3 & -1 & 4 \\ 9 & -1 & 0 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 4 & -1 & -3 & 0 \\ 5 & 0 & -1 & 1 \\ 1 & 11 & 8 & 2 \end{bmatrix}$$

35.  $A + B = B + A$       36.  $A + O = A$   
 37.  $-(A - B) = B - A$       38.  $3(A + B) = 3A + 3B$

In Exercises 39 and 40, find the unknown quantities in the given matrix equations.

39. An airplane is flying in a direction  $21.0^\circ$  north of east at 235 km/h but is headed  $14.5^\circ$  north of east. The wind is from the southeast. Find the speed of the wind  $v_w$  and the speed of the plane  $v_p$  relative to the wind from the given matrix equation. See Fig. 16.2.

$$\begin{bmatrix} v_p \cos 14.5^\circ - v_w \cos 45.0^\circ \\ v_p \sin 14.5^\circ + v_w \sin 45.0^\circ \end{bmatrix} = \begin{bmatrix} 235 \cos 21.0^\circ \\ 235 \sin 21.0^\circ \end{bmatrix}$$

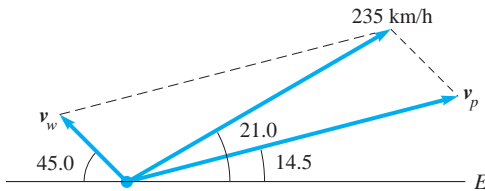


Fig. 16.2

40. Find the electric currents shown in Fig. 16.3 by solving the following matrix equation:

$$\begin{bmatrix} I_1 + I_2 + I_3 \\ -2I_1 + 3I_2 \\ -3I_2 + 6I_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 24 \\ 0 \end{bmatrix}$$

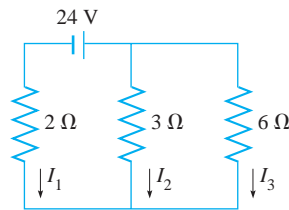


Fig. 16.3

In Exercises 41–44, perform the indicated matrix operations.

41. The contractor of a housing development constructs four different types of houses, with either a carport, a one-car garage, or a

two-car garage. The following matrix shows the number of houses of each type and the type of garage.

	Type A	Type B	Type C	Type D
Carport	96	75	0	0
1-car garage	62	44	24	0
2-car garage	0	35	68	78

If the contractor builds two additional identical developments, find the matrix showing the total number of each house-garage type built in the three developments.

42. The inventory of a drug supply company shows that the following numbers of cases of bottles of vitamins C and B<sub>3</sub> (niacin) are in stock: vitamin C—25 cases of 100-mg bottles, 10 cases of 250-mg bottles, and 32 cases of 500-mg bottles; vitamin B<sub>3</sub>—30 cases of 100-mg bottles, 18 cases of 250-mg bottles, and 40 cases of 500-mg bottles. This is represented by matrix  $A$  below. After two shipments are sent out, each of which can be represented by matrix  $B$  below, find the matrix that represents the remaining inventory.

$$A = \begin{bmatrix} 25 & 10 & 32 \\ 30 & 18 & 40 \end{bmatrix} \quad B = \begin{bmatrix} 10 & 5 & 6 \\ 12 & 4 & 8 \end{bmatrix}$$

43. One serving of brand K of breakfast cereal provides the given percentages of the given vitamins and minerals: vitamin A, 15%; vitamin C, 25%; calcium, 10%; iron, 25%. One serving of brand G provides: vitamin A, 10%; vitamin C, 10%; calcium, 10%; iron, 45%. One serving of tomato juice provides: vitamin A, 15%; vitamin C, 30%; calcium, 3%; iron, 3%. One serving of orange-pineapple juice provides vitamin A, 0%; vitamin C, 100%; calcium, 2%; iron, 2%. Set up a two-row, four-column matrix  $B$  to represent the data for the cereals and a similar matrix  $J$  for the juices.

44. Referring to Exercise 43, find the matrix  $B + J$  and explain the meaning of its elements.

Answers to Practice Exercises

1.  $\begin{bmatrix} -7 & -10 \\ 0 & 8 \end{bmatrix}$       2.  $\begin{bmatrix} 20 & 8 \\ -21 & -1 \end{bmatrix}$

## 16.2 Multiplication of Matrices

Multiplication of Matrices • Identity Matrix • Inverse of a Matrix

The definition for the multiplication of matrices does not have an intuitive basis. However, through the solution of a system of linear equations we can, at least in part, show why multiplication is defined as it is. Consider Example 1.

### EXAMPLE 1 Reasoning for the definition of multiplication

If we solve the system of equations

$$\begin{aligned} 2x + y &= 1 \\ 7x + 3y &= 5 \end{aligned}$$

we get  $x = 2, y = -3$ . Checking this solution in each of the equations, we get

$$\begin{aligned} 2(2) + 1(-3) &= 1 \\ 7(2) + 3(-3) &= 5 \end{aligned}$$

Let us represent the coefficients of the equations by the matrix  $\begin{bmatrix} 2 & 1 \\ 7 & 3 \end{bmatrix}$  and the solutions by the matrix  $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$ . If we now indicate the multiplications of these matrices and perform it as shown

$$\begin{bmatrix} 2 & 1 \\ 7 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 2(2) + 1(-3) \\ 7(2) + 3(-3) \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

we note that we obtain a matrix that properly represents the right-side values of the equations. (Note the products and sums in the resulting matrix.) ■

Following reasons along the lines indicated in Example 1, we now define the **multiplication of matrices**. If the number of columns in a first matrix equals the number of rows in a second matrix, the product of these matrices is formed as follows: *The element in a specified row and a specified column of the product matrix is the sum of the products formed by multiplying each element in the specified row of the first matrix by the corresponding element in the specific column of the second matrix.* The product matrix will have the same number of rows as the first matrix and the same number of columns as the second matrix. We summarize this as follows.

### Multiplication of Matrices

- If  $A$  is an  $n \times k$  matrix, and  $B$  is a  $k \times m$  matrix, then their product  $AB$  is an  $n \times m$  matrix.
- The  $ij^{\text{th}}$  element of  $AB$  is formed by multiplying each element of row  $i$  in  $A$  by the corresponding element of column  $j$  in  $B$  and then adding these products.
- The product  $BA$  requires that  $n = m$ , so just because you can form the product  $AB$  does not mean that you can form the product  $BA$ . Clearly,  $AB \neq BA$  in general, so that **matrix multiplication is not commutative**.

### EXAMPLE 2 Multiplying matrices

Find the product  $AB$  and the product  $BA$ , where

$$A = \begin{bmatrix} 2 & 1 \\ -3 & 0 \\ 1 & 2 \end{bmatrix} \quad B = \begin{bmatrix} -1 & 6 & 5 & -2 \\ 3 & 0 & 1 & -4 \end{bmatrix}$$

Matrix  $A$  is  $3 \times 2$ , and matrix  $B$  is  $2 \times 4$ , so the product  $AB$  can be formed, resulting in a  $3 \times 4$  matrix. The calculations are shown below, with the elements used to form the element  $AB_{11}$  and the element  $AB_{32}$  outlined in colour.

$$\begin{bmatrix} \boxed{2} & \boxed{1} \\ -3 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \boxed{-1} & \boxed{6} & 5 & -2 \\ 3 & 0 & 1 & -4 \end{bmatrix} = \begin{bmatrix} \boxed{2(-1) + 1(3)} & 2(6) + 1(0) & 2(5) + 1(1) & 2(-2) + 1(-4) \\ -3(-1) + 0(3) & -3(6) + 0(0) & -3(5) + 0(1) & -3(-2) + 0(-4) \\ 1(-1) + 2(3) & \boxed{1(6) + 2(0)} & 1(5) + 2(1) & 1(-2) + 2(-4) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 12 & 11 & -8 \\ 3 & -18 & -15 & 6 \\ 5 & 6 & 7 & -10 \end{bmatrix}$$

#### Practice Exercise

1. Find the product  $AB$ .

$$A = \begin{bmatrix} -1 & 4 \\ 8 & -2 \\ 0 & 12 \end{bmatrix} \quad B = \begin{bmatrix} 5 & -3 \\ -7 & 10 \end{bmatrix}$$

In trying to form the product  $BA$ , we see that  $B$  has four columns and  $A$  has three rows. Since these numbers are not the same, the product cannot be formed. ■

■ Note that, if the second matrix had been to the left of the first, then the product would have had four rows and four columns.

$$\begin{aligned} \begin{bmatrix} -1 & 9 & 3 & -2 \\ 2 & 0 & -7 & 1 \end{bmatrix} \begin{bmatrix} 6 & -2 \\ 1 & 0 \\ 3 & -5 \\ 3 & 9 \end{bmatrix} &= \begin{bmatrix} -1(6) + 9(1) + 3(3) + (-2)(3) & -1(-2) + 9(0) + 3(-5) + (-2)(9) \\ 2(6) + 0(1) + (-7)(3) + 1(3) & 2(-2) + 0(0) + (-7)(-5) + 1(9) \end{bmatrix} \\ &= \begin{bmatrix} -6 + 9 + 9 - 6 & 2 + 0 - 15 - 18 \\ 12 + 0 - 21 + 3 & -4 + 0 + 35 + 9 \end{bmatrix} \\ &= \begin{bmatrix} 6 & -31 \\ -6 & 40 \end{bmatrix} \end{aligned}$$

### EXAMPLE 3 Multiplying matrices

The product of two matrices below may be formed because the first matrix has four columns and the second matrix has four rows. The matrix is formed as shown.

### IDENTITY MATRIX

There are two special matrices of particular importance in the multiplication of matrices. The first of these is the **identity matrix  $I$** , which is a square matrix with 1's for elements of the principal diagonal with all other elements zero. (The principal diagonal starts with the element  $a_{11}$ .) It has the property that if it is multiplied by another square matrix with the same number of rows and columns, then the second matrix equals the product matrix.

### EXAMPLE 4 Identity matrix

Show that  $AI = IA = A$  for the matrix

$$A = \begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix}$$

Since  $A$  is  $2 \times 2$ , we choose  $I$  to be  $2 \times 2$ . Therefore, for this case,

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{elements of principal diagonal are 1's}$$

Forming the indicated products, we have results as follows:

$$\begin{aligned} AI &= \begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2(1) + (-3)(0) & 2(0) + (-3)(1) \\ 4(1) + 1(0) & 4(0) + 1(1) \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix} \\ IA &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1(2) + 0(4) & 1(-3) + 0(1) \\ 0(2) + 1(4) & 0(-3) + 1(1) \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix} \end{aligned}$$

Therefore, we see that  $AI = IA = A$ .

### INVERSE OF A MATRIX

For a given square matrix  $A$ , its **inverse  $A^{-1}$**  is the other important special matrix. *The matrix  $A$  and its inverse  $A^{-1}$  have the property that*

$$AA^{-1} = A^{-1}A = I$$

(16.5)

If the product of two square matrices equals the identity matrix, the matrices are called inverses of each other. Under certain conditions, the inverse of a given square matrix may not exist. In the next section, we develop the procedure for finding the inverse of a square matrix, and the section that follows shows how the inverse is used in the solution of systems of equations. At this point, we simply show that the product of certain matrices equals the identity matrix and that therefore these matrices are inverses of each other.

**EXAMPLE 5 Inverse matrix**

For the given matrices  $A$  and  $B$ , show that  $AB = BA = I$ , and therefore that  $B = A^{-1}$ :

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 7 \end{bmatrix} \quad B = \begin{bmatrix} 7 & 3 \\ 2 & 1 \end{bmatrix}$$

**Practice Exercise**

2. Show that  $AB = BA = I$ .

$$A = \begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$$

Forming the products  $AB$  and  $BA$ , we have the following:

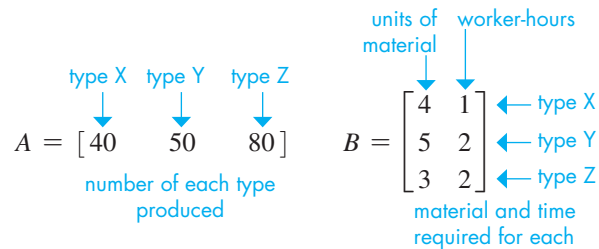
$$AB = \begin{bmatrix} 1 & -3 \\ -2 & 7 \end{bmatrix} \begin{bmatrix} 7 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 7 - 6 & 3 - 3 \\ -14 + 14 & -6 + 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 7 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ -2 & 7 \end{bmatrix} = \begin{bmatrix} 7 - 6 & -21 + 21 \\ 2 - 2 & -6 + 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Since  $AB = I$  and  $BA = I$ ,  $B = A^{-1}$  and  $A = B^{-1}$ . ■

**EXAMPLE 6 Matrix multiplication—application**

A company makes three types of automobile parts. In one day, it produces 40 of type X, 50 of type Y, and 80 of type Z. Required for production are 4 units of material and 1 worker-hour for type X, 5 units of material and 2 worker-hours for type Y, and 3 units of material and 2 worker-hours for type Z. By representing the number of each type produced as matrix  $A$  and the material and time requirements as matrix  $B$ , we have



The product  $AB$  gives the total number of units of material and the total number of worker-hours needed for the day's production in a one-row, two-column matrix:

$$AB = [40 \quad 50 \quad 80] \begin{bmatrix} 4 & 1 \\ 5 & 2 \\ 3 & 2 \end{bmatrix} = [160 + 250 + 240 \quad 40 + 100 + 160] = [650 \quad 300]$$

The result is annotated with 'total units of material' above the first element and 'total worker-hours' above the second element.

Therefore, 650 units of material and 300 worker-hours are required. ■

**EXAMPLE 7 Multiplication of matrices—application to robotic arms**

Consider the two-dimensional robotic arm in Fig. 16.4, consisting of two links, each one 2.0 m long. The first link has been rotated  $45^\circ$  with respect to the  $x$ -axis, and the second link has been rotated  $45^\circ$  with respect to the first link. We can find the coordinates of the end of the arm by applying transformation matrices to the origin of our coordinate system.

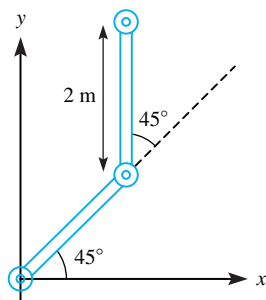


Fig. 16.4

We start by representing a point  $(x, y)$  as the vector (a  $3 \times 1$  matrix)  $\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$ . In particular, the origin is represented by the vector  $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ . If the coordinate system is rotated

■ See the chapter introduction.

by an angle  $\theta$  and then translated by  $m$  units in the  $x$ -direction and  $n$  units in the  $y$ -direction, the new coordinates of the origin are found by the matrix multiplication

$$\begin{bmatrix} \cos \theta & -\sin \theta & m \\ \sin \theta & \cos \theta & n \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

For our example,  $\theta = 45^\circ$ ,  $m = 2$  (the length of a link), and  $n = 0$ , so the transformation matrix becomes

$$\begin{bmatrix} \cos 45^\circ & -\sin 45^\circ & 2 \\ \sin 45^\circ & \cos 45^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 2 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since the process is repeated twice (once for each link), the location of the end of the arm is given by

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 2 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 2 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 2 + \sqrt{2} \\ 1 & 0 & \sqrt{2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 + \sqrt{2} \\ \sqrt{2} \\ 1 \end{bmatrix}$$

which represents the point  $(2 + \sqrt{2}, \sqrt{2})$ .

This method can be generalized to robotic arms in three dimensions by including more variables. Moreover, transformation matrices are also useful for computer graphics and CAD. ■

### Some Properties of Matrix Algebra

We have seen that *matrix multiplication is not commutative* (see Example 2); that is,  $AB \neq BA$  in general. This differs from the multiplication of real numbers. Another difference is that it is possible that  $AB = O$ , even though neither  $A$  nor  $B$  is  $O$ . Moreover, the only number that does not have a multiplicative inverse is 0, yet there are many nonzero matrices that do not have an inverse. There are also some similarities, for example,  $AI = A$ , where  $I$  and the number 1 are equivalent. Also, *the distributive property*  $A(B + C) = AB + AC$  holds for matrix multiplication.

## EXERCISES 16.2

In Exercises 1 and 2, make the given changes in the indicated examples of this section and then perform the indicated multiplications.

- In Example 2, interchange columns 1 and 2 in matrix  $A$  and then do the multiplication.
- In Example 5, in  $A$  change  $-2$  to  $2$  and  $-3$  to  $3$ , in  $B$  change  $2$  to  $-2$  and  $3$  to  $-3$ , and then do the multiplications.

In Exercises 3–14, perform the indicated multiplications.

- $\begin{bmatrix} 4 & -2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 2 & 6 \end{bmatrix}$
- $\begin{bmatrix} -\frac{1}{2} & 6 & -\frac{2}{3} \end{bmatrix} \begin{bmatrix} 4 & 8 \\ \frac{1}{3} & -\frac{1}{2} \\ 0 & 9 \end{bmatrix}$
- $\begin{bmatrix} 2 & -3 & 1 \\ 0 & 7 & -3 \end{bmatrix} \begin{bmatrix} 90 \\ -25 \\ 50 \end{bmatrix}$
- $\begin{bmatrix} 0 & -1 & 2 \\ 4 & 11 & 2 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 1 & 2 \\ 6 & 1 \end{bmatrix}$
- $\begin{bmatrix} -8 & \frac{3}{4} \\ \frac{7}{2} & -8 \\ -6 & \frac{4}{5} \end{bmatrix} \begin{bmatrix} \frac{1}{4} & -3 \\ 2 & 5 \end{bmatrix}$
- $\begin{bmatrix} 12 & -47 \\ 43 & -18 \\ 36 & -22 \end{bmatrix} \begin{bmatrix} 25 \\ 66 \end{bmatrix}$

- $\begin{bmatrix} -1 & 7 \\ 3 & 5 \\ 10 & -1 \\ -5 & 12 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 5 & -3 \end{bmatrix}$
- $\begin{bmatrix} 5 & 4 \end{bmatrix} \begin{bmatrix} 4 & -4 \\ -5 & 5 \end{bmatrix}$
- $\begin{bmatrix} 2 & -3 \\ 5 & -1 \end{bmatrix} \begin{bmatrix} 3 & 0 & -1 \\ 7 & -5 & 8 \end{bmatrix}$
- $\begin{bmatrix} -7 & 8 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} -90 & 100 \\ 10 & 40 \end{bmatrix}$
- $\begin{bmatrix} -9.2 & 2.3 & 0.5 \\ -3.8 & -2.4 & 9.2 \end{bmatrix} \begin{bmatrix} 6.5 & -5.2 \\ 4.9 & 1.7 \\ -1.8 & 6.9 \end{bmatrix}$
- $\begin{bmatrix} 1 & 2 & -6 & 6 & 1 \\ -2 & 4 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 5 \\ 2 \end{bmatrix}$

In Exercises 15–18, find, if possible,  $AB$  and  $BA$ . If it is not possible, explain why.

15.  $A = \begin{bmatrix} 1 & -3 & 8 \end{bmatrix}$   $B = \begin{bmatrix} -1 \\ 5 \\ 7 \end{bmatrix}$

16.  $A = \begin{bmatrix} -3 & 2 & 0 \\ 1 & -4 & 5 \end{bmatrix}$   $B = \begin{bmatrix} -2 & 0 \\ 4 & -6 \\ 5 & 1 \end{bmatrix}$

17.  $A = \begin{bmatrix} -10 & 25 & 40 \\ 42 & -5 & 0 \end{bmatrix}$   $B = \begin{bmatrix} 6 \\ -15 \\ 12 \end{bmatrix}$

18.  $A = \begin{bmatrix} -2 & 1 & 7 \\ 3 & -1 & 0 \\ 0 & 2 & -1 \end{bmatrix}$   $B = \begin{bmatrix} 4 & -1 & 5 \end{bmatrix}$

In Exercises 19–22, show that  $AI = IA = A$ .

19.  $A = \begin{bmatrix} 1 & 8 \\ -2 & 2 \end{bmatrix}$       20.  $A = \begin{bmatrix} -15 & 28 \\ -5 & 64 \end{bmatrix}$

21.  $A = \begin{bmatrix} 1 & 3 & -5 \\ 2 & 0 & 1 \\ 1 & -2 & 4 \end{bmatrix}$       22.  $A = \begin{bmatrix} -1 & 2 & 0 \\ 4 & -3 & 1 \\ 2 & 1 & 3 \end{bmatrix}$

In Exercises 23–26, determine whether or not  $B = A^{-1}$ .

23.  $A = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$   $B = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$

24.  $A = \begin{bmatrix} 3 & -4 \\ 5 & -7 \end{bmatrix}$   $B = \begin{bmatrix} 7 & -4 \\ 5 & -2 \end{bmatrix}$

25.  $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -5 & 7 \\ -1 & 3 & -5 \end{bmatrix}$   $B = \begin{bmatrix} 4 & -1 & 1 \\ 3 & -2 & -1 \\ 1 & -1 & -1 \end{bmatrix}$

26.  $A = \begin{bmatrix} 1 & -1 & 3 \\ 3 & -4 & 8 \\ -2 & 3 & -4 \end{bmatrix}$   $B = \begin{bmatrix} 8 & -5 & -4 \\ 4 & -2 & -1 \\ -1 & 1 & 1 \end{bmatrix}$

In Exercises 27–30, determine by matrix multiplication whether or not  $A$  is the proper matrix of solution values.

27.  $3x - 2y = -1$   $4x + y = 6$   $A = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

28.  $4x + y = -5$   $3x + 4y = 6$   $A = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

29.  $3x + y + 2z = 1$   $x - 3y + 4z = -3$   $2x + 2y + z = 1$   $A = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$

30.  $2x - y + z = 7$   $x - 3y + 2z = 6$   $3x + y - z = 8$   $A = \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}$

In Exercises 31–34, perform the indicated matrix multiplications, using the following matrices. For matrix  $A$ ,  $A^2 = A \times A$ .

$A = \begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix}$   $B = \begin{bmatrix} 1 & -2 & -6 \\ -3 & 2 & 9 \\ 2 & 0 & -3 \end{bmatrix}$   $C = \begin{bmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{bmatrix}$

31. Show that  $A^2 = A$ .

32. Show that  $C^2 = O$ .

33. Show that  $B^3 = B$

34. Show that  $A^3B^3 = AB$

In Exercises 35–46, perform the indicated matrix multiplications.

35. For matrices  $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$  and  $B = \begin{bmatrix} c & d \\ d & c \end{bmatrix}$ , show that  $AB = BA$ .

36. For matrix  $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ , show that  $A^2 - 4A - 5I = O$ .

37. Using two rows and columns, show that  $(-I)^2 = I$ .

38. For  $J = \begin{bmatrix} j & 0 \\ 0 & j \end{bmatrix}$ , where  $j = \sqrt{-1}$ , show that  $J^2 = -I, J^3 = -J$ , and  $J^4 = I$ . Explain the similarity with  $j^2, j^3$ , and  $j^4$ .

39. Show that  $A^2 - I = (A + I)(A - I)$  for  $A = \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix}$ .

40. In the study of polarized light, the matrix product  $\begin{bmatrix} 1 & 0 \\ 0 & -j \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$  occurs ( $j = \sqrt{-1}$ ). Find this product.

41. In studying the motion of electrons, one of the Pauli spin matrices used is  $\sigma_y = \begin{bmatrix} 0 & -j \\ j & 0 \end{bmatrix}$ , where  $j = \sqrt{-1}$ . Show that  $\sigma_y^2 = I$ .

42. In analysing the motion of a robotic mechanism, the following matrix multiplication is used. Perform the multiplication and evaluate each element of the result. (See Example 7.)

$$\begin{bmatrix} \cos 60^\circ & -\sin 60^\circ & 0 \\ \sin 60^\circ & \cos 60^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$$

43. In an *ammeter*, nearly all the electric current flows through a *shunt*, and the remaining known fraction of current is measured by the metre. See Fig. 16.5. From the given matrix equation, find voltage  $V_2$  and current  $i_2$  in terms of  $V_1, i_1$ , and resistance  $R$ , whichever may be applicable.

$$\begin{bmatrix} V_2 \\ i_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{R} & 1 \end{bmatrix} \begin{bmatrix} V_1 \\ i_1 \end{bmatrix}$$

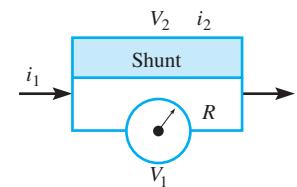


Fig. 16.5

44. In the theory related to the reproduction of colour photography, the equation

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 1.0 & 0.1 & 0 \\ 0.5 & 1.0 & 0.1 \\ 0.3 & 0.4 & 1.0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

is found. The  $X, Y$ , and  $Z$  represent the red, green, and blue densities of the reproductions, respectively, and the  $x, y$ , and  $z$  represent the red, green, and blue densities, respectively, of the subject. Give the equations relating  $X, Y$ , and  $Z$  and  $x, y$ , and  $z$ .

45. The path of an earth satellite can be written as

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 7.10 & -1 \\ 1 & 7.23 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = [5.13 \times 10^8]$$

where distances are in kilometres. What type of curve is represented? (See Section 14.1.)



46. Using Kirchoff's laws on the circuit shown in Fig. 16.6, the following matrix equation is found. By matrix multiplication, find the resulting system of equations.

$$\begin{bmatrix} R_1 + R_2 & -R_2 & 0 \\ -R_2 & R_2 + R_3 + R_4 & -R_4 \\ 0 & -R_4 & R_4 + R_5 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} V_1 \\ 0 \\ -V_2 \end{bmatrix}$$

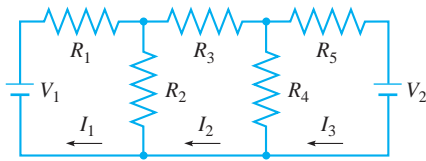


Fig. 16.6

Answers to Practice Exercises

1.  $\begin{bmatrix} -33 & 43 \\ 54 & -44 \\ -84 & 120 \end{bmatrix}$       2.  $AB = BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

## 16.3 Finding the Inverse of a Matrix

Inverse of a  $2 \times 2$  Matrix • Gauss–Jordan Method • Inverse on a Calculator

In this section we discuss two methods for finding the inverse of a square matrix. The first method is straightforward, but it can only be used for  $2 \times 2$  matrices.

### The Inverse of a $2 \times 2$ Matrix

1. Evaluate the determinant of the matrix. If the determinant is zero, the inverse does not exist.
2. Interchange the elements on the principal diagonal.
3. Change the signs of the off-diagonal elements.
4. Divide each resulting element by the determinant.

This method is illustrated in the following example.

#### EXAMPLE 1 Inverse of a $2 \times 2$ matrix—method 1

Find the inverse of the matrix  $A = \begin{bmatrix} 2 & -3 \\ 4 & -7 \end{bmatrix}$ .

First, we find the determinant of the original matrix, which means we evaluate

$$\begin{vmatrix} 2 & -3 \\ 4 & -7 \end{vmatrix} = -14 - (-12) = -2$$

Since the determinant is not zero, the inverse exists. Next, we interchange the elements on the principal diagonal and change the signs of the off-diagonal elements. This gives us the matrix

$$\begin{bmatrix} -7 & 3 \\ -4 & 2 \end{bmatrix} \begin{array}{l} \leftarrow \text{signs changed} \\ \leftarrow \text{elements interchanged} \end{array}$$

We now divide each element of the second matrix by  $-2$  (the determinant). This gives

$$A^{-1} = \frac{1}{-2} \begin{bmatrix} -7 & 3 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} \frac{-7}{-2} & \frac{3}{-2} \\ \frac{-4}{-2} & \frac{2}{-2} \end{bmatrix} = \begin{bmatrix} \frac{7}{2} & -\frac{3}{2} \\ 2 & -1 \end{bmatrix} \leftarrow \text{inverse}$$

Checking by multiplication gives

$$AA^{-1} = \begin{bmatrix} 2 & -3 \\ 4 & -7 \end{bmatrix} \begin{bmatrix} \frac{7}{2} & -\frac{3}{2} \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 7 - 6 & -3 + 3 \\ 14 - 14 & -6 + 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Since  $AA^{-1} = I$ , the matrix  $A^{-1}$  is the proper inverse matrix. ■

#### Practice Exercise

1. Find the inverse:

$$A = \begin{bmatrix} 3 & -8 \\ 1 & -2 \end{bmatrix}$$

■ See Exercise 37.

■ The Gauss–Jordan method is named for the German mathematician Karl Gauss (1777–1855) and the German geodesist Wilhelm Jordan (1842–1899).

The second method, called the *Gauss–Jordan method*, is applicable for square matrices of any size.

### The Inverse of a Matrix by the Gauss–Jordan Method

1. Set up the given matrix and the identity matrix of the same size side by side.
2. Transform the given matrix into the identity matrix by performing any of the following allowable **row operations**:
  - a. Any two rows may be interchanged.
  - b. Every element in any row may be multiplied by any number other than zero.
  - c. Any row may be replaced by a row whose elements are the sum of a nonzero multiple of itself and a nonzero multiple of another row.

Work one column at a time, transforming the columns in order from left to right.
3. At every step, perform the same row operations on the identity matrix. The resulting matrix will be the required inverse.

Note that these are **row operations**, not column operations, and that they are the operations used in solving a system of equations by addition and subtraction.

### EXAMPLE 2 $2 \times 2$ Inverse—Gauss–Jordan method

Find the inverse of the matrix

$$A = \begin{bmatrix} 2 & -3 \\ 4 & -7 \end{bmatrix} \quad \text{this is the same matrix as in Example 1}$$

First, we set up the given matrix with the identity matrix side by side.

$$\left[ \begin{array}{cc|cc} 2 & -3 & 1 & 0 \\ 4 & -7 & 0 & 1 \end{array} \right]$$

The vertical line simply shows the separation of the two matrices.

We wish to transform the left matrix into the identity matrix. Therefore, the first requirement is a 1 for element  $a_{11}$ . Therefore, we divide all elements of the first row by 2. This gives the following setup:

$$\left[ \begin{array}{cc|cc} 1 & -\frac{3}{2} & \frac{1}{2} & 0 \\ 4 & -7 & 0 & 1 \end{array} \right]$$

Next, we want to have a zero for element  $a_{21}$ . Therefore, we subtract 4 times each element of row 1 from the corresponding element in row 2, replacing the elements of row 2. This gives us the following setup:

$$\left[ \begin{array}{cc|cc} 1 & -\frac{3}{2} & \frac{1}{2} & 0 \\ 4 - 4(1) & -7 - 4(-\frac{3}{2}) & 0 - 4(\frac{1}{2}) & 1 - 4(0) \end{array} \right] \quad \text{or} \quad \left[ \begin{array}{cc|cc} 1 & -\frac{3}{2} & \frac{1}{2} & 0 \\ 0 & -1 & -2 & 1 \end{array} \right]$$

Next, we want to have 1, not  $-1$ , for element  $a_{22}$ . Therefore, we multiply each element of row 2 by  $-1$ . This gives

$$\left[ \begin{array}{cc|cc} 1 & -\frac{3}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 2 & -1 \end{array} \right]$$

Finally, we want zero for element  $a_{12}$ . Therefore, we add  $\frac{3}{2}$  times each element of row 2 to the corresponding elements of row 1, replacing row 1. This gives

$$\left[ \begin{array}{cc|cc} 1 + \frac{3}{2}(0) & -\frac{3}{2} + \frac{3}{2}(1) & \frac{1}{2} + \frac{3}{2}(2) & 0 + \frac{3}{2}(-1) \\ 0 & 1 & 2 & -1 \end{array} \right] \quad \text{or} \quad \left[ \begin{array}{cc|cc} 1 & 0 & \frac{7}{2} & -\frac{3}{2} \\ 0 & 1 & 2 & -1 \end{array} \right]$$

#### Practice Exercise

2. Find the inverse using the Gauss–Jordan

method:  $A = \begin{bmatrix} 3 & -8 \\ 1 & -2 \end{bmatrix}$

#### LEARNING TIP

As in Example 2, (1) always work one column at a time, from left to right, and (2) never undo the work in a previously completed column.

At this point, we have transformed the given matrix into the identity matrix, and the identity matrix into the inverse. Therefore, the matrix to the right of the vertical bar in the last setup is the required inverse. Thus,

$$A^{-1} = \begin{bmatrix} \frac{7}{2} & -\frac{3}{2} \\ 2 & -1 \end{bmatrix}$$

As was done in Example 2, it is generally best to make the element on the principal diagonal for the column 1 first and then make all other elements in the column 0. Let us consider two more examples.

### EXAMPLE 3 $2 \times 2$ Inverse—Gauss–Jordan method

Find the inverse of the matrix  $\begin{bmatrix} -3 & 6 \\ 4 & 5 \end{bmatrix}$ .

original setup

$$\left[ \begin{array}{cc|cc} -3 & 6 & 1 & 0 \\ 4 & 5 & 0 & 1 \end{array} \right]$$

row 1 divided by  $-3$

$$\left[ \begin{array}{cc|cc} 1 & -2 & -\frac{1}{3} & 0 \\ 4 & 5 & 0 & 1 \end{array} \right]$$

row 2 divided by 13

$$\left[ \begin{array}{cc|cc} 1 & -2 & -\frac{1}{3} & 0 \\ 0 & 13 & \frac{4}{3} & 1 \end{array} \right]$$

$-4$  times row 1 added to row 2

$$\left[ \begin{array}{cc|cc} 1 & -2 & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{4}{39} & \frac{1}{13} \end{array} \right]$$

$2$  times row 2 added to row 1

$$\left[ \begin{array}{cc|cc} 1 & 0 & -\frac{5}{39} & \frac{2}{13} \\ 0 & 1 & \frac{4}{39} & \frac{1}{13} \end{array} \right]$$

$I \quad A^{-1}$

Therefore,  $A^{-1} = \begin{bmatrix} -\frac{5}{39} & \frac{2}{13} \\ \frac{4}{39} & \frac{1}{13} \end{bmatrix}$ , which can be checked by multiplication.

### EXAMPLE 4 $3 \times 3$ Inverse—Gauss–Jordan method

Find the inverse of the matrix  $\begin{bmatrix} 1 & 2 & -1 \\ 3 & 5 & -1 \\ -2 & -1 & -2 \end{bmatrix}$ .

original setup

$$\left[ \begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 3 & 5 & -1 & 0 & 1 & 0 \\ -2 & -1 & -2 & 0 & 0 & 1 \end{array} \right]$$

$-3$  times row 1 added to row 2

$$\left[ \begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -1 & 2 & -3 & 1 & 0 \\ -2 & -1 & -2 & 0 & 0 & 1 \end{array} \right]$$

$2$  times row 1 added to row 3

$$\left[ \begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -1 & 2 & -3 & 1 & 0 \\ 0 & 3 & -4 & 2 & 0 & 1 \end{array} \right]$$

row 2 multiplied by  $-1$

$$\left[ \begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & -1 & 0 \\ 0 & 3 & -4 & 2 & 0 & 1 \end{array} \right]$$

$-2$  times row 2 added to row 1

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 3 & -5 & 2 & 0 \\ 0 & 1 & -2 & 3 & -1 & 0 \\ 0 & 3 & -4 & 2 & 0 & 1 \end{array} \right]$$

$-3$  times row 2 added to row 3

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 3 & -5 & 2 & 0 \\ 0 & 1 & -2 & 3 & -1 & 0 \\ 0 & 0 & 2 & -7 & 3 & 1 \end{array} \right]$$

row 3 divided by 2

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 3 & -5 & 2 & 0 \\ 0 & 1 & -2 & 3 & -1 & 0 \\ 0 & 0 & 1 & -\frac{7}{2} & \frac{3}{2} & \frac{1}{2} \end{array} \right]$$

$2$  times row 3 added to row 2

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 3 & -5 & 2 & 0 \\ 0 & 1 & 0 & -4 & 2 & 0 \\ 0 & 0 & 1 & -\frac{7}{2} & \frac{3}{2} & \frac{1}{2} \end{array} \right]$$

$-3$  times row 3 added to row 1

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{11}{2} & -\frac{5}{2} & -\frac{3}{2} \\ 0 & 1 & 0 & -4 & 2 & 1 \\ 0 & 0 & 1 & -\frac{7}{2} & \frac{3}{2} & \frac{1}{2} \end{array} \right]$$

$I \quad A^{-1}$

Therefore, the required inverse matrix is

$$\begin{bmatrix} \frac{11}{2} & -\frac{5}{2} & -\frac{3}{2} \\ -4 & 2 & 1 \\ -\frac{7}{2} & \frac{3}{2} & \frac{1}{2} \end{bmatrix}$$

which may be checked by multiplication.

In the next example, we discuss entering a matrix, displaying it, and finding its inverse on a typical calculator. The manual for any particular model should be used to see how the various operations are done on it.

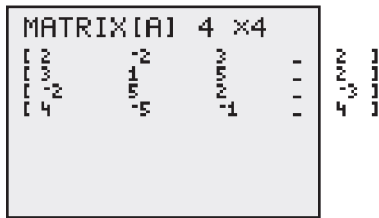


Fig. 16.7

**EXAMPLE 5**  $4 \times 4$  inverse on a calculator

By using a graphing calculator, find the inverse of the matrix

$$A = \begin{bmatrix} 2 & -2 & 3 & 2 \\ 3 & 1 & 5 & 2 \\ -2 & 5 & 2 & -3 \\ 4 & -5 & -1 & 4 \end{bmatrix}$$

Using the *matrix* feature, in Fig. 16.7 we show the window when the elements of the matrix are entered. Scrolling by use of the arrow key will probably be required to enter all values.

To obtain the inverse, we enter [A] and then use the  $x^{-1}$  key. The inverse matrix is found to be

$$A^{-1} = \begin{bmatrix} 0.435 & -0.652 & 1.652 & 1.348 \\ -0.739 & 0.609 & -0.609 & -0.391 \\ 0.391 & -0.087 & 0.087 & -0.087 \\ -1.261 & 1.391 & -2.391 & -1.609 \end{bmatrix}$$

The elements of  $A^{-1}$  can be checked directly on the calculator by showing that the product  $AA^{-1} = I$  (if the rounded values are used, the values of  $I$  will be approximate but should be sufficiently close to the necessary 1's and 0's).

■ Note that a calculator entry such as  $1E-10$  means that it is zero.

**EXERCISES 16.3**

In Exercises 1 and 2, make the given changes in the indicated examples of this section and then find the matrix inverses.

- In Example 1, change the element  $-7$  to  $-5$  and then find the inverse using the same method.
- In Example 3, change the element  $-3$  to  $-2$  and then find the inverse using the same method.

In Exercises 3–10, find the inverse of each of the given matrices by the method of Example 1 of this section.

- $\begin{bmatrix} 2 & -5 \\ -2 & 4 \end{bmatrix}$
- $\begin{bmatrix} -6 & 3 \\ 3 & -2 \end{bmatrix}$
- $\begin{bmatrix} -1 & 5 \\ 4 & 10 \end{bmatrix}$
- $\begin{bmatrix} 8 & -1 \\ -4 & -5 \end{bmatrix}$
- $\begin{bmatrix} 0 & -4 \\ 2 & 6 \end{bmatrix}$
- $\begin{bmatrix} 7 & -3 \\ -6 & 4 \end{bmatrix}$
- $\begin{bmatrix} -50 & -45 \\ 26 & 80 \end{bmatrix}$
- $\begin{bmatrix} 7.2 & -3.6 \\ -1.3 & -5.7 \end{bmatrix}$

In Exercises 11–24, find the inverse of each of the given matrices by transforming the identity matrix, as in Examples 2–4.

- $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$
- $\begin{bmatrix} 1 & 5 \\ -1 & -4 \end{bmatrix}$
- $\begin{bmatrix} 2 & 4 \\ -1 & -1 \end{bmatrix}$
- $\begin{bmatrix} -2 & 6 \\ 3 & -4 \end{bmatrix}$
- $\begin{bmatrix} 2 & 5 \\ -1 & 2 \end{bmatrix}$
- $\begin{bmatrix} -2 & 3 \\ -3 & 5 \end{bmatrix}$

- $\begin{bmatrix} 25 & 30 \\ -10 & -14 \end{bmatrix}$
- $\begin{bmatrix} 1 & -3 & -2 \\ -2 & 7 & 3 \\ 1 & -1 & -3 \end{bmatrix}$
- $\begin{bmatrix} 1 & 3 & 2 \\ -2 & -5 & -1 \\ 2 & 4 & 0 \end{bmatrix}$
- $\begin{bmatrix} 2 & 4 & 0 \\ 3 & 4 & -2 \\ -1 & 1 & 2 \end{bmatrix}$
- $\begin{bmatrix} 1 & -3 \\ 7 & -5 \end{bmatrix}$
- $\begin{bmatrix} 1 & 2 & -1 \\ 3 & 7 & -5 \\ -1 & -2 & 0 \end{bmatrix}$
- $\begin{bmatrix} 1 & 3 & 4 \\ -1 & -4 & -2 \\ 4 & 9 & 20 \end{bmatrix}$
- $\begin{bmatrix} -2 & 6 & 1 \\ 0 & 3 & -3 \\ 4 & -7 & 3 \end{bmatrix}$

In Exercises 25–34, find the inverse of each of the given matrices by using a graphing calculator, as in Example 5. The matrices in Exercises 27–29 are the same as those in Exercises 21–23.

- $\begin{bmatrix} 2 & 8 \\ -1 & 6 \end{bmatrix}$
- $\begin{bmatrix} 20 & -45 \\ -12 & 24 \end{bmatrix}$
- $\begin{bmatrix} 1 & 3 & 2 \\ -2 & -5 & -1 \\ 2 & 4 & 0 \end{bmatrix}$
- $\begin{bmatrix} 1 & 3 & 4 \\ -1 & -4 & -2 \\ 4 & 9 & 20 \end{bmatrix}$
- $\begin{bmatrix} 2 & 4 & 0 \\ 3 & 4 & -2 \\ -1 & 1 & 2 \end{bmatrix}$
- $\begin{bmatrix} 10 & -5 & 30 \\ -2 & 4 & -5 \\ 20 & 5 & 5 \end{bmatrix}$

31.  $\begin{bmatrix} 1 & -2 & 1 & 0 \\ 1 & -2 & 2 & -3 \\ 0 & 1 & -1 & 1 \\ -2 & 3 & -2 & 3 \end{bmatrix}$       32.  $\begin{bmatrix} 3 & -2 & -1 & 4 \\ 2 & 0 & 5 & 1 \\ -1 & 2 & 1 & -2 \\ 4 & 1 & 3 & 5 \end{bmatrix}$
33.  $\begin{bmatrix} 0.2 & 1.2 & -0.8 & -0.5 \\ -0.4 & 3.0 & -1.6 & 0.4 \\ 1.0 & -2.4 & 3.2 & 1.5 \\ -0.1 & 0.4 & 0.0 & 3.0 \end{bmatrix}$
34.  $\begin{bmatrix} 12.5 & -2.6 & 1.2 & 7.6 \\ -4.6 & 10.0 & -4.7 & -6.8 \\ 5.7 & -3.7 & 7.3 & 11.0 \\ 8.8 & 6.8 & 14.0 & 4.7 \end{bmatrix}$

In Exercises 35–42, solve the given problems.

35. Show that the matrix  $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$  has no inverse.
36. Find the determinant of the matrix  $\begin{bmatrix} 1 & -2 & 0 \\ -2 & 4 & 8 \\ 3 & -6 & 6 \end{bmatrix}$ . Explain what this tells us about its inverse.
37. For the matrix  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ , show that  $\frac{1}{ad - bc} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ . This verifies the method of Example 1.
38. Describe the relationship between the elements of the matrix  $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$  and the elements of its inverse.
39. The matrix  $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 4 & -3 \\ 1 & -3 & 2 \end{bmatrix}$  is *symmetric* (note the elements on opposite sides of the main diagonal are equal). Show that  $A^{-1}$  is also symmetric.

40. For the *four-terminal network* shown in Fig. 16.8, it can be shown that the voltage matrix  $V$  is related to the coefficient matrix  $A$  and the current matrix  $I$  by  $V = A^{-1}I$ , where

$$V = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad I = \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

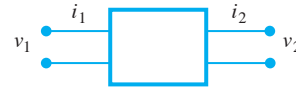


Fig. 16.8

Find the individual equations for  $v_1$  and  $v_2$  that give each in terms of  $i_1$  and  $i_2$ .

41. The rotations of a robot arm such as that shown in Fig. 16.9 are often represented by matrices. The values represent trigonometric functions of the angles of rotation. For the following rotation matrix  $R$ , find  $R^{-1}$ .

$$R = \begin{bmatrix} 0.8 & 0.0 & -0.6 \\ 0.0 & 1.0 & 0.0 \\ 0.6 & 0.0 & 0.8 \end{bmatrix}$$

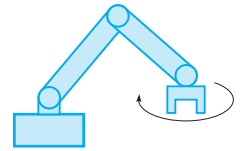


Fig. 16.9

42. In cryptography, one type of code makes use of a matrix (the encoding matrix) to encode a message. The receiver of the message decodes it using the inverse of the matrix (the decoding matrix). Given the following encoding matrix, find the decoding matrix.

$$A = \begin{bmatrix} 6 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$

Answers to Practice Exercises

1.  $\begin{bmatrix} -1 & 4 \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix}$       2.  $\begin{bmatrix} -1 & 4 \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix}$

## 16.4 Matrices and Linear Equations

Using Inverse Matrices • Solving Systems of Equations • Matrix Solution on a Calculator

As we stated at the beginning of Section 16.1, matrices can be used to solve systems of equations, and in this section, we show one method by which this is done. As we develop this method, it will be apparent that there is a great deal of numerical work involved. However, methods such as this one are easily programmed for use on a computer, which can do the arithmetic work very rapidly. Also, most graphing calculators can perform these operations, and we will show an example at the end of the section in which a calculator is used to solve a system of four equations more readily than with earlier methods. It is the *method* of solving the system of equations that is of primary importance here.

Let us consider the system of equations

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}$$

Recalling the definition of equality of matrices, we can write this system as

$$\begin{bmatrix} a_1x + b_1y \\ a_2x + b_2y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \tag{16.6}$$

If we let

$$A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \end{bmatrix} \quad C = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \quad (16.7)$$

the left side of Eq. (16.6) can be written as the product of matrices  $A$  and  $X$ .

$$AX = C \quad (16.8)$$

If we now multiply (on the left) each side of this matrix equation by  $A^{-1}$  (assuming that it exists), we have

$$A^{-1}AX = A^{-1}C$$

Since  $A^{-1}A = I$ , we have

$$IX = A^{-1}C$$

However,  $IX = X$ . Therefore,

$$X = A^{-1}C \quad (16.9)$$

### LEARNING TIP

We can solve a system of linear equations by multiplying the one-column matrix of the constants on the right of the equation by the inverse of the matrix of the coefficients. The result is a one-column matrix whose elements are the required values for the solution. Also, note that

$$X = A^{-1}C \quad \text{and not} \quad CA^{-1}$$

as the order of matrix multiplication must be carefully followed.

### EXAMPLE 1 Matrix solution for a system with two equations

Use matrices to solve the system of equations

$$2x - y = 7$$

$$5x - 3y = 18$$

We set up the matrix of coefficients and the matrix of constants as

$$A = \begin{bmatrix} 2 & -1 \\ 5 & -3 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 7 \\ 18 \end{bmatrix}$$

By either of the methods of the previous section, we can determine the inverse of matrix  $A$  to be

$$A^{-1} = \begin{bmatrix} 3 & -1 \\ 5 & -2 \end{bmatrix}$$

We now form the matrix product  $A^{-1}C$ .

$$A^{-1}C = \begin{bmatrix} 3 & -1 \\ 5 & -2 \end{bmatrix} \begin{bmatrix} 7 \\ 18 \end{bmatrix} = \begin{bmatrix} 21 - 18 \\ 35 - 36 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

Since  $X = A^{-1}C$ , this means that

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

Therefore, the required solution is  $x = 3$  and  $y = -1$ , which checks when these values are substituted into the original equations. ■

### EXAMPLE 2 Matrix solution for two equations—application to circuits

For the electric circuit shown in Fig. 16.10, the equations used to find the currents (in amperes)  $i_1$  and  $i_2$  are

$$\begin{aligned} 2.30i_1 + 6.45(i_1 + i_2) &= 15.0 & \text{or} & & 8.75i_1 + 6.45i_2 &= 15.0 \\ 1.25i_2 + 6.45(i_1 + i_2) &= 12.5 & & & 6.45i_1 + 7.70i_2 &= 12.5 \end{aligned}$$

### Practice Exercise

- Use matrices to solve the system of equations
 
$$x - 3y = 6$$

$$2x + y = 5$$

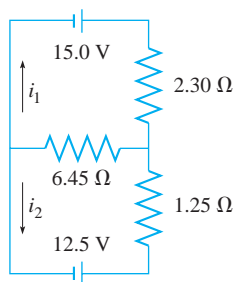


Fig. 16.10

Using matrices to solve this system of equations, we set up the matrix  $A$  of coefficients, the matrix  $C$  of constants, and the matrix  $X$  of currents as

$$A = \begin{bmatrix} 8.75 & 6.45 \\ 6.45 & 7.70 \end{bmatrix} \quad C = \begin{bmatrix} 15.0 \\ 12.5 \end{bmatrix} \quad X = \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

We now find the inverse of  $A$  as

$$A^{-1} = \frac{1}{8.75(7.70) - 6.45(6.45)} \begin{bmatrix} 7.70 & -6.45 \\ -6.45 & 8.75 \end{bmatrix} = \begin{bmatrix} 0.2988 & -0.2503 \\ -0.2503 & 0.3395 \end{bmatrix}$$

Therefore,

$$\begin{aligned} X = A^{-1}C &= \begin{bmatrix} 0.2988 & -0.2503 \\ -0.2503 & 0.3395 \end{bmatrix} \begin{bmatrix} 15.0 \\ 12.5 \end{bmatrix} \\ &= \begin{bmatrix} 0.2988(15.0) - 0.2503(12.5) \\ -0.2503(15.0) + 0.3395(12.5) \end{bmatrix} = \begin{bmatrix} 1.35 \\ 0.49 \end{bmatrix} \end{aligned}$$

Therefore, the required currents are  $i_1 = 1.35$  A and  $i_2 = 0.49$  A. These values check when substituted into the original equations. ■

### EXAMPLE 3 Matrix solution—three equations

Use matrices to solve the system of equations

$$x + 4y - z = 4$$

$$x + 3y + z = 8$$

$$2x + 6y + z = 13$$

Setting up matrices  $A$ ,  $C$ , and  $X$ , we have

$$A = \begin{bmatrix} 1 & 4 & -1 \\ 1 & 3 & 1 \\ 2 & 6 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 4 \\ 8 \\ 13 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

To give another example of finding the inverse of a  $3 \times 3$  matrix, we briefly show the steps for finding  $A^{-1}$ :

$$\begin{array}{c} \left[ \begin{array}{ccc|ccc} 1 & 4 & -1 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 & 1 & 0 \\ 2 & 6 & 1 & 0 & 0 & 1 \end{array} \right] \rightarrow \left[ \begin{array}{ccc|ccc} 1 & 4 & -1 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & -1 & 0 \\ 0 & -2 & 3 & -2 & 0 & 1 \end{array} \right] \rightarrow \left[ \begin{array}{ccc|ccc} 1 & 0 & 7 & -3 & 4 & 0 \\ 0 & 1 & -2 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 2 & -1 \end{array} \right] \\ \left[ \begin{array}{ccc|ccc} 1 & 4 & -1 & 1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 1 & 0 \\ 2 & 6 & 1 & 0 & 0 & 1 \end{array} \right] \rightarrow \left[ \begin{array}{ccc|ccc} 1 & 0 & 7 & -3 & 4 & 0 \\ 0 & 1 & -2 & 1 & -1 & 0 \\ 0 & -2 & 3 & -2 & 0 & 1 \end{array} \right] \rightarrow \left[ \begin{array}{ccc|ccc} 1 & 0 & 7 & -3 & 4 & 0 \\ 0 & 1 & 0 & 1 & 3 & -2 \\ 0 & 0 & 1 & 0 & 2 & -1 \end{array} \right] \\ \left[ \begin{array}{ccc|ccc} 1 & 4 & -1 & 1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 1 & 0 \\ 0 & -2 & 3 & -2 & 0 & 1 \end{array} \right] \rightarrow \left[ \begin{array}{ccc|ccc} 1 & 0 & 7 & -3 & 4 & 0 \\ 0 & 1 & -2 & 1 & -1 & 0 \\ 0 & 0 & -1 & 0 & -2 & 1 \end{array} \right] \rightarrow \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & -10 & 7 \\ 0 & 1 & 0 & 1 & 3 & -2 \\ 0 & 0 & 1 & 0 & 2 & -1 \end{array} \right] \end{array}$$

$$\text{Thus, } A^{-1} = \begin{bmatrix} -3 & -10 & 7 \\ 1 & 3 & -2 \\ 0 & 2 & -1 \end{bmatrix} \quad \text{and}$$

$$X = A^{-1}C = \begin{bmatrix} -3 & -10 & 7 \\ 1 & 3 & -2 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 8 \\ 13 \end{bmatrix} = \begin{bmatrix} -12 - 80 + 91 \\ 4 + 24 - 26 \\ 0 + 16 - 13 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$$

This means that  $x = -1$ ,  $y = 2$ , and  $z = 3$ . ■



**EXAMPLE 4** Matrix solution—three equations

Use matrices to solve the system of equations

$$\begin{aligned}x + 2y - z &= -4 \\3x + 5y - z &= -5 \\-2x - y - 2z &= -5\end{aligned}$$

Setting up matrices  $A$ ,  $C$ , and  $X$ , we have

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 5 & -1 \\ -2 & -1 & -2 \end{bmatrix} \quad C = \begin{bmatrix} -4 \\ -5 \\ -5 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Finding  $A^{-1}$  (see Example 3) and solving for  $X$ , we have

$$\begin{aligned}A^{-1} &= \begin{bmatrix} \frac{11}{2} & -\frac{5}{2} & -\frac{3}{2} \\ -4 & 2 & 1 \\ -\frac{7}{2} & \frac{3}{2} & \frac{1}{2} \end{bmatrix} \\X = A^{-1}C &= \begin{bmatrix} \frac{11}{2} & -\frac{5}{2} & -\frac{3}{2} \\ -4 & 2 & 1 \\ -\frac{7}{2} & \frac{3}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -4 \\ -5 \\ -3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}\end{aligned}$$

This means that  $x = -2$ ,  $y = 1$ , and  $z = 4$ . ■

**EXAMPLE 5** Matrix solution on a calculator—four equations

Use a calculator to perform the necessary matrix operations in solving the following system of equations:

$$\begin{aligned}2r + 4s - t + u &= 5 \\r - 2s + 3t - u &= -4 \\3r + s + 2t - 4u &= 8 \\4r + 5s - t + 3u &= -1\end{aligned}$$

First, we set up matrices  $A$ ,  $X$ , and  $C$ :

$$A = \begin{bmatrix} 2 & 4 & -1 & 1 \\ 1 & -2 & 3 & -1 \\ 3 & 1 & 2 & -4 \\ 4 & 5 & -1 & 3 \end{bmatrix} \quad X = \begin{bmatrix} r \\ s \\ t \\ u \end{bmatrix} \quad C = \begin{bmatrix} 5 \\ -4 \\ 8 \\ -1 \end{bmatrix}$$

It is now necessary only to enter matrices  $A$  and  $C$  in the calculator and find the matrix product  $A^{-1}C$ . (There is no need to record or display  $A^{-1}$ .) The solution is

$$r = -2 \quad s = 3 \quad t = 0.5 \quad u = -2.5$$

This solution can be checked on the calculator by storing the resulting matrix  $X$  as matrix  $B$  and finding the matrix product  $AB$ , which should equal matrix  $C$ . The product  $AB$  is equivalent to substituting each value into the original equations, as shown in Eq. (16.8). ■

**EXERCISES 16.4**

In Exercises 1 and 2, make the given changes in the indicated examples of this section and then solve the systems of equations.

- In Example 1, change the 18 to 19 and then solve the system of equations.
- In Example 3, change the 8 to 7 and the 13 to 12 and then solve the system of equations.

In Exercises 3–10, solve the given systems of equations by using the inverse of the coefficient matrix. The numbers in parentheses refer to exercises from Section 16.3, where the inverses may be checked.

- |  |   |
|--|---|
| 3. $2x - 5y = -14$ (3)<br>$-2x + 4y = 11$                          | 4. $-x + 5y = 4$ (5)<br>$4x + 10y = -4$                                 |
| 5. $x + 2y = 7$ (11)<br>$2x + 3y = 11$                             | 6. $2x + 4y = -9$ (13)<br>$-x - y = 2$                                  |
| 7. $2x + 5y = -6$ (15)<br>$-x + 2y = -6$                           | 8. $x - 3y - 2z = -8$ (19)<br>$-2x + 7y + 3z = 19$<br>$x - y - 3z = -3$ |
| 9. $x + 3y + 2z = 5$ (21)<br>$-2x - 5y - z = -1$<br>$2x + 4y = -2$ | 10. $2x + 4y = -2$ (23)<br>$3x + 4y - 2z = -6$<br>$-x + y + 2z = 5$     |

In Exercises 11–18, solve the given systems of equations by using the inverse of the coefficient matrix.

- |  |   |
|--|---|
| 11. $2x - 3y = 3$<br>$4x - 5y = 4$                                     | 12. $x + 2y = 3$<br>$3x + 4y = 11$                                    |
| 13. $2.5x + 2.8y = -3.0$<br>$3.5x - 1.6y = 9.6$                        | 14. $12x - 5y = -400$<br>$31x + 25y = 180$                            |
| 15. $x + 2y + 2z = -4$<br>$4x + 9y + 10z = -18$<br>$-x + 3y + 7z = -7$ | 16. $x - 4y - 2z = -7$<br>$-x + 5y + 5z = 18$<br>$3x - 7y + 10z = 38$ |
| 17. $2x + 4y + z = 5$<br>$-2x - 2y - z = -6$<br>$-x + 2y + z = 0$      | 18. $4x + y = 2$<br>$-2x - y + 3z = -18$<br>$2x + y - z = 8$          |

In Exercises 19–26, solve the given systems of equations by using the inverse of the coefficient matrix. You may use a calculator to perform the necessary matrix operations. See Example 5.

- |  |  |
|--|--|
| 19. $2x - y - z = 7$<br>$4x - 3y + 2z = 4$<br>$3x + 5y + z = -10$  | 20. $6x + 2y + 9z = 13$<br>$7x + 6y - 6z = 6$<br>$5x - 4y + 3z = 15$                                     |
| 21. $u - 3v - 2w = 9$<br>$3u + 2v + 6w = 20$<br>$4u - v + 3w = 25$   | 22. $2x + y - z = 1$<br>$3x - 2y - 8z = -3$<br>$x + 3y + z = 10$   |
| 23. $x - 5y + 2z - t = -18$<br>$3x + y - 3z + 2t = 17$<br>$4x - 2y + z - t = -1$<br>$-2x + 3y - z + 4t = 11$   | 24. $2p + q + 5r + s = 5$<br>$p + q - 3r - 4s = -1$<br>$3p + 6q - 2r + s = 8$<br>$2p + 2q + 2r - 3s = 2$ |
| 25. $2v + 3w + x - y - 2z = 6$<br>$6v - 2w - x + 3y - z = 21$<br>$v + 3w - 4x + 2y + 3z = -9$<br>$3v - w - x + 7y + 4z = 5$<br>$v + 6w + 6x - 4y - z = -4$ |  |

26.  $4x - y + 2z - 2t + u = -15$   
 $8x + y - z + 4t - 2u = 26$   
 $2x - 6y - 2z + t - u = 10$   
 $2x + 5y + z - 3t + 8u = -22$   
 $4x - 3y + 2z + 4t + 2u = -4$

In Exercises 27–39, solve the indicated systems of equations using the inverse of the coefficient matrix. In Exercises 33–38, it is necessary to set up the appropriate equations.

- For the following system of equations, solve for  $x^2$  and  $y$  using the matrix methods of this section, and then solve for  $x$  and  $y$ .  
 $x^2 + y = 2$   
 $2x^2 - y = 10$
- For the following system of equations, solve for  $x^2$  and  $y^2$  using the matrix methods of this section, and then solve for  $x$  and  $y$ .  
 $x^2 - y^2 = 8$   
 $x^2 + y^2 = 10$
- Solve any pair of the system of equations  $2x - y = 4$ ,  $3x + y = 1$ , and  $x - 2y = 5$ . Show that the solution is valid for any pair chosen. What conclusions can be drawn about the graphs of the three equations?
- In solving the system of equations  $3x - 4y = 5$ ,  $8y - 6x = 7$ , what conclusion can be drawn?

31. Forces  $F_1$  and  $F_2$  hold up a beam that weighs 2540 N, as shown in Fig. 16.11. The equations used to find the forces are  
 $F_1 \sin 47.2^\circ + F_2 \sin 64.4^\circ = 2540$   
 $F_1 \cos 47.2^\circ - F_2 \cos 64.4^\circ = 0$   
Find the magnitude of each force.

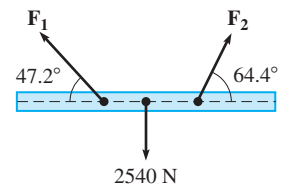


Fig. 16.11

32. In applying Kirchhoff's laws (see Exercise 36 on page 175) to the circuit shown in Fig. 16.12, the following equations are found. Determine the indicated currents (in A).

$$I_A + I_B + I_C = 0$$

$$2I_A - 5I_B = 6$$

$$5I_B - I_C = -3$$

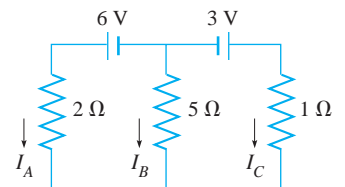


Fig. 16.12

- Two batteries in an electric circuit have a combined voltage of  $18\text{ V}$ , and one battery produces  $6\text{ V}$  less than twice the other. What is the voltage of each?
- An investment of  $\$5000$  is invested partly at  $3.00\%$  and the remainder at  $3.50\%$ . If the total annual income is  $\$167$ , how much is invested at each rate?
- What volume of each of a  $20\%$  acid solution and a  $50\%$  acid solution should be combined to form  $48\text{ mL}$  of a  $25\%$  solution?
- Two computer programs together require  $51.8$  megabytes of memory. If one program requires  $2.0$  megabytes more than the other, what are the memory requirements of each?

37. A research chemist wants to make 10.0 L of gasoline containing 2.0% of a new experimental additive. Gasoline without additive and two mixtures of gasoline with additive, one with 5.0% and the other with 6.0%, are to be used. If four times as much gasoline without additive as the 5.0% mixture is to be used, how much of each is needed?
38. A river tour boat takes 5.0 h to cruise downstream and 7.0 h for the return upstream. If the river flows at 4.0 km/h, how fast does the boat travel in still water, and how far downstream does the boat go before starting the return trip?
39. A logging company in British Columbia harvests in four different regions in order to supply its clients with Douglas fir, lodgepole pine, western red cedar, and ponderosa pine. Considering species

mix and yield per region, the number of hectares that must be logged in each region for a certain week can be found by solving the system of equations

$$\begin{aligned} 324x + 240y + 224z + 274t &= 3010 \\ 80y + 28z + 20t &= 378 \\ 28z + 98t &= 434 \\ 36x &= 90 \end{aligned}$$

Find  $x$ ,  $y$ ,  $z$ , and  $t$ .

#### Answer to Practice Exercise

1.  $x = 3$ ,  $y = -1$

## 16.5 Gaussian Elimination

**Gaussian Elimination Form • Solving Systems of Equations • Number of Possible Solutions • Consistent and Inconsistent Systems**

$$\begin{aligned} x - 3y - z &= 1 \\ y + 2z &= 5 \\ z &= 2 \end{aligned}$$

We now show a general method that can be used to solve a system of linear equations. The procedure used in this method is similar to that used in finding the inverse of a matrix in Section 16.3. It is known as **Gaussian elimination**, and as noted in the chapter introduction, it was developed in the early 1800s by Karl Gauss. Today, it is commonly used in computer programs for the solutions of systems of linear equations.

When using this method, we first rewrite the system of equations in a different form. In doing so, there are very specific operations that may be used on the equations. The following example illustrates the form that is to be obtained.

### EXAMPLE 1 Gaussian elimination form

In solving the system of linear equations to the left, we see that the third equation directly gives us the value  $z = 2$ . Since the second equation contains only  $y$  and  $z$ , we can now substitute  $z = 2$  into the second equation to get  $y = 1$ . Then we can find  $x$  by substituting  $y = 1$  and  $z = 2$  into the first equation, and get  $x = 6$ . Therefore, after writing the system of equations in the triangular form to the left, the solution is completed by back substitution, starting with the last equation. ■

### Solving Linear Systems by Gaussian Elimination

1. Transform the given system of equations into the triangular form shown in Example 1 by performing any of the following allowable operations:
  - a. Any two equations may be interchanged.
  - b. Both sides of an equation may be multiplied by a nonzero constant.
  - c. A nonzero multiple of one equation may be added to another equation.
2. If the last equation shows the value of one of the unknowns, the others are found by back substitution. If that is not the case, either the system has an unlimited number of solutions (see Example 4) or the system is inconsistent and there is no solution (see Example 5).

Using three linear equations in three unknowns as an example, by using the above operations, we can change the system

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned} \tag{16.10}$$

into the equivalent system

$$\begin{aligned} x + b_4y + c_4z &= d_4 \\ y + c_5z &= d_5 \\ z &= d_6 \end{aligned} \tag{16.11}$$

The solution is completed by substituting the value of  $z$  into the second equation, and then substituting the values of  $y$  and  $z$  into the first equation, as in Example 1.

**EXAMPLE 2 Solving a system of two equations**

Solve the given system of equations by Gaussian elimination.

Given system of equations (equations and solution at left)

We want the coefficient of  $x$  in the first equation to be 1. Therefore we divide the first equation by 2, the coefficient of  $x$ .

We next eliminate  $x$  in the second equation by subtracting 3 times the first equation from the second equation.

Now, we solve the second equation for  $y$  by dividing by  $-\frac{7}{2}$ .

To find the value of  $x$ , we substitute the value of  $y = \frac{6}{7}$  into the first equation.

The solution is  $x = \frac{11}{7}$ ,  $y = \frac{6}{7}$ , which checks when substituted into the original equations. ■

■ This is in general form of Eq. (16.10).

$$\begin{aligned} 2x + y &= 4 \\ 3x - 2y &= 3 \end{aligned}$$

$$\begin{aligned} x + \frac{1}{2}y &= 2 \\ 3x - 2y &= 2 \end{aligned}$$

$$\begin{aligned} x + \frac{1}{2}y &= 2 \\ - \frac{7}{2}y &= -3 \end{aligned}$$

$$\begin{aligned} x + \frac{1}{2}y &= 2 \\ y &= \frac{6}{7} \end{aligned}$$

$$\begin{aligned} x + \frac{1}{2}\left(\frac{6}{7}\right) &= 2 \\ x &= \frac{11}{7} \end{aligned}$$

This is in general form of Eq. (16.11).

**EXAMPLE 3 Solving a system of three equations**

Solve the given system of equations by Gaussian elimination.

Given system of equations (equations and solution at left)

The coefficient of  $x$  in the first equation is already 1, so we can proceed to subtract 2 times the first equation from the second equation and add 3 times the first equation to the third equation to eliminate  $x$  from the second and third equations.

To get the coefficient of  $y$  equal to 1 in the second equation, divide it by  $-7$ .

To eliminate  $y$  from the third equation, subtract 11 times the second equation from the third equation.

Solve for  $z$  by multiplying the third equation by  $\frac{7}{25}$  (or dividing by  $\frac{25}{7}$ ).

The value of  $y$  is found by substituting  $z = 3$  into the second equation.

The value of  $x$  is found by substituting  $y = 1$  and  $z = 3$  back into the first equation.

Thus,  $x = -2$ ,  $y = 1$ ,  $z = 3$ , and this solution checks in the original equations. ■

■ This is in the general form of Eq. (16.10).

$$\begin{aligned} x + 3y - 2z &= -5 \\ 2x - y + 4z &= 7 \\ -3x + 2y - 3z &= -1 \end{aligned}$$

$$\begin{aligned} x + 3y - 2z &= -5 \\ - 7y + 8z &= 17 \\ 11y - 9z &= -16 \end{aligned}$$

$$\begin{aligned} x + 3y - 2z &= -5 \\ y - \frac{8}{7}z &= -\frac{17}{7} \\ 11y - 9z &= -16 \end{aligned}$$

$$\begin{aligned} x + 3y - 2z &= -5 \\ y - \frac{8}{7}z &= -\frac{17}{7} \\ \frac{25}{7}z &= \frac{75}{7} \end{aligned}$$

$$\begin{aligned} x + 3y - 2z &= -5 \\ y - \frac{8}{7}z &= -\frac{17}{7} \\ z &= 3 \end{aligned}$$

$$\begin{aligned} y - \frac{8}{7}(3) &= -\frac{17}{7} \\ y &= 1 \end{aligned}$$

$$\begin{aligned} x + 3(1) - 2(3) &= -5 \\ x &= -2 \end{aligned}$$

■ This is in the form of Eq. (16.11).

**EXAMPLE 4 Solving a system with an unlimited number of solutions**

Solve the given system of equations by Gaussian elimination.

Given system of equations (equations and solution at left)

Since the first equation does not contain  $x$ , which means that  $a_1 = 0$ , we cannot divide by  $a_1$ . Therefore, interchange the first and second equations and then divide the new first equation by 2.

Eliminate  $x$  in the third equation by subtracting 4 times the first equation from the third equation.

■ This is in general form of Eq. (16.10).

$$\begin{aligned} 4y + z &= 2 \\ 2x + 6y - 2z &= 3 \\ 4x + 8y - 5z &= 4 \end{aligned}$$

$$\begin{aligned} x + 3y - z &= \frac{3}{2} \\ 4y + z &= 2 \\ 4x + 8y - 5z &= 4 \end{aligned}$$

$$\begin{aligned} x + 3y - z &= \frac{3}{2} \\ 4y + z &= 2 \\ 4x + 8y - 5z &= 4 \end{aligned}$$

$$\begin{aligned} x + 3y - z &= \frac{3}{2} \\ 4y + z &= 2 \end{aligned}$$

■ This is the form that Eq. (16.11) takes in this case.

$$\begin{array}{r} -4y - z = -2 \\ \hline x + 3y - z = \frac{3}{2} \\ y + \frac{1}{4}z = \frac{1}{2} \\ -4y - z = -2 \\ \hline x + 3y - z = \frac{3}{2} \\ y + \frac{1}{4}z = \frac{1}{2} \\ \hline 0 = 0 \end{array}$$

$$\begin{array}{r} x + 3y - z = \frac{3}{2} \\ y = \frac{1}{2} - \frac{1}{4}z \\ \hline x + 3\left(\frac{1}{2} - \frac{1}{4}z\right) - z = \frac{3}{2} \\ x = \frac{7}{4}z \end{array}$$

Make the coefficient of  $y$  in the second equation equal to 1 by dividing the second equation by 4.

Eliminate  $y$  from the third equation by adding 4 times the second equation to the third equation.

Since the third equation,  $0 = 0$ , is correct, we can continue. Although there is no specific value for  $z$ , it is possible to express both  $x$  and  $y$  in terms of  $z$ .

Solve the second equation for  $y$ . In this case, it is expressed in terms of  $z$ . We no longer need to include the third equation.

Substitute the solution for  $y$  in the first equation and solve for  $x$  in terms of  $z$ .

We have expressed the solution as  $x = \frac{7}{4}z$  and  $y = \frac{1}{2} - \frac{1}{4}z$ . Since both  $x$  and  $y$  are expressed in terms of  $z$  and there is no specific value of  $z$ , the value of  $z$  can be chosen arbitrarily. This means **there is an unlimited number of solutions**. For example, if  $z = 4$ , then  $x = 7$  and  $y = -\frac{1}{2}$ . If  $z = -2$ , then  $x = -\frac{7}{2}$  and  $y = 1$ . ■

**Possible Number of Solutions**

When we solved systems of linear equations in Chapter 5, we found that not all systems have unique solutions, as they do in Examples 2 and 3. In Example 4, we illustrated the use of Gaussian elimination on a system of equations for which the solution is not unique.

In Example 4, one of the equations became  $0 = 0$ , and there was an unlimited number of solutions. If any of the equations of a system becomes  $0 = a$ ,  $a \neq 0$ , then the system is inconsistent and there is no solution. Example 5 illustrates such a system.

If a system has more unknowns than equations, or it can be written this way, as in Example 4, it usually has an unlimited number of solutions. It is possible, however, that such a system is inconsistent.

If a system has more equations than unknowns, it is inconsistent unless enough equations become  $0 = 0$  such that at least one solution is found. The following example illustrates two systems in which there are more equations than unknowns.

**Practice Exercise**

- Use Gaussian elimination to solve the system of equations  
 $2x - y = 7$   
 $4x + 3y = -1$

**EXAMPLE 5 Consistent and inconsistent systems**

Solve the following systems of equations by Gaussian elimination.

The solutions are shown at the left. We note that each system has three equations and two unknowns.

In the solution of the first system, the third equation becomes  $0 = 0$ , and only two equations are needed to find the solution  $x = 1, y = 2$ .

In the solution of the second system, the third equation becomes  $0 = -4$ , which means the system is inconsistent and there is no solution.

Graphing the systems, note that in Fig. 16.13, each of the lines passes through the point  $(1, 2)$ , whereas in Fig. 16.14, there is no point common to the three lines.

<p>First system</p> $\begin{array}{r} x + 2y = 5 \\ 3x - y = 1 \\ 4x + y = 6 \\ \hline x + 2y = 5 \\ -7y = -14 \\ -7y = -14 \\ \hline x + 2y = 5 \\ y = 2 \\ -7y = -14 \\ \hline x + 2y = 5 \\ y = 2 \\ 0 = 0 \\ \hline x = 1 \end{array}$	<p>Second system</p> $\begin{array}{r} x + 2y = 5 \\ 3x - y = 1 \\ 4x + y = 2 \\ \hline x + 2y = 5 \\ -7y = -14 \\ -7y = -18 \\ \hline x + 2y = 5 \\ y = 2 \\ -7y = -18 \\ \hline x + 2y = 5 \\ y = 2 \\ 0 = -4 \end{array}$
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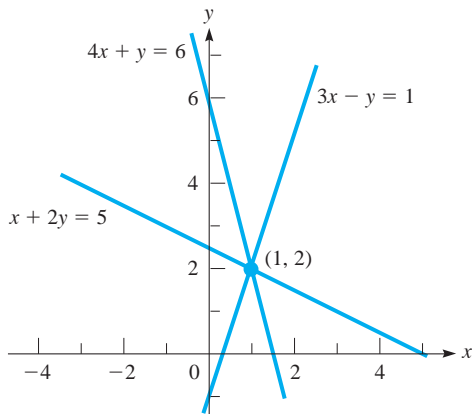


Fig. 16.13

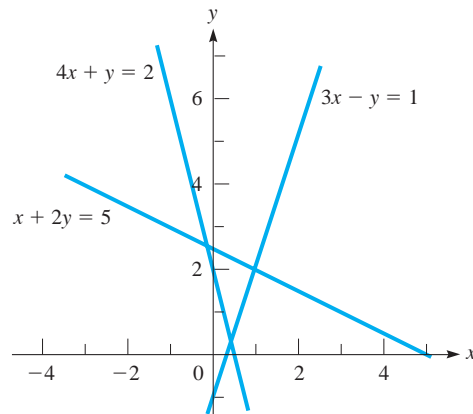


Fig. 16.14

## EXERCISES 16.5

In Exercises 1 and 2, make the given changes in the indicated examples of this section, and then solve the resulting systems using Gaussian elimination.

- In Example 2, change the signs of both coefficients of  $x$  to  $-$ .
- In Example 5, change the third equation of the second system to  $5x + 3y = 11$ .

In Exercises 3–28, solve the given systems of equations by Gaussian elimination. If there is an unlimited number of solutions, find two of them.

- $x + 2y = 4$   
 $3x - y = 5$
- $5x - 3y = 2$   
 $-2x + 4y = 3$
- $2x - 3y + z = 4$   
 $6y - 4x - 2z = 9$
- $x + 3y + 3z = -3$   
 $2x + 2y + z = -5$   
 $-2x - y + 4z = 6$
- $w + 2x - y + 3z = 12$   
 $2w - 2y - z = 3$   
 $3x - y - z = -1$   
 $-w + 2x + y + 2z = 3$
- $x - 4y + z = 2$   
 $3x - y + 4z = -4$
- $2x - y + z = 5$   
 $3x + 2y - 2z = 4$   
 $5x + 8y - 8z = 5$
- $x + 3y + z = 4$   
 $2x - 6y - 3z = 10$   
 $4x - 9y + 3z = 4$
- $2x - 4y = 7$   
 $3x + 5y = -6$   
 $9x - 7y = 15$
- $3x + 5y = -2$   
 $24x - 18y = 13$   
 $15x - 33y = 19$   
 $6x + 68y = -33$
- $4x - 8y - 8z = 12$   
 $10x + 5y + 15z = 20$   
 $-6x - 3y - 3z = 15$   
 $3x + 3y - 2z = 2$
- $2x + y = 1$   
 $5x + 2y = 1$
- $-3x + 2y = 4$   
 $4x + y = -5$
- $3s + 4t - u = -5$   
 $2u - 6s - 8t = 10$
- $3x - y + 2z = 3$   
 $4x - 2y + z = 3$   
 $6x + 6y + 3z = 4$
- $3x - 2y = -11$   
 $5x + y = -1$   
 $2x + 3y = 10$   
 $x - 5y = -21$
- $4x + z = 6$   
 $2x - y - 2z = -2$
- $3u + 6v + 2w = -2$   
 $u + 3v - 4w = 2$   
 $2u - 3v - 2w = -2$
- $30x + 20y - 10z = 30$   
 $4x - 2y - 6z = 4$   
 $-5x + 20y - 25z = -5$
- $4x - y = 5$   
 $2x + 2y = 3$   
 $6x - 4y = 7$   
 $2x + y = 4$
- $x + 3y - z = 1$   
 $3x - y + 4z = 4$   
 $-2x + 2y + 3z = 17$   
 $3x + 7y + 5z = 23$
- $2x - y - 2z - t = 4$   
 $4x + 2y + 3z + 2t = 3$   
 $-2x - y + 4z = -2$

- $s + 2t - 3u = 2$   
 $6t + 3s - 9u = 6$   
 $7s + 14t - 21u = 13$
- $x + 2y - 3z + 2t = 3$   
 $4y - 6z + 4t = 1$
- $r - s - 3t - u = 1$   
 $2r + 4s - 2u = 2$   
 $r + 5s + 3t - u = 1$   
 $3r + 4s - 2t = 0$   
 $r + 2t - 3u = 3$
- $x + 2y - 3z = 4$   
 $2x - y - 6z + 2t = 2$   
 $x + 3y + 3z - t = 1$

In Exercises 29 and 30, solve the given problems using Gaussian elimination.

- Solve the system  $a_1x + b_1y = c_1$ ,  $a_2x + b_2y = c_2$  and show that the result is the same as that obtained using determinants as in Section 5.5. See page 160.
- Solve the system  $x + 2y = 6$ ,  $2x + ay = 4$  and show that the solution depends on the value of  $a$ . What value of  $a$  does the solution show may not be used?

In Exercises 31–34, set up systems of equations and solve by Gaussian elimination.

- One computer can perform  $x$  calculations per second, and a second computer can perform  $y$  calculations per second. If each operates for 0.200 s, 25.0 million calculations are performed. If the first operates for 0.400 s and the second for 0.300 s, 43.2 million calculations are performed. Find  $x$  and  $y$ .
- The voltage across an electric resistor equals the current (in A) times the resistance (in  $\Omega$ ). If a current of 3.00 A passes through each of two resistors, the sum of the voltages is 10.5 V. If 2.00 A passes through the first resistor and 4.00 A passes through the second resistor, the sum of the voltages is 13.0 V. Find the resistances.
- Three machines together produce 650 parts each hour. Twice the production of the second machine is 10 parts/h more than the sum of the production of the other two machines. If the first operates for 3.00 h and the others operate for 2.00 h, 1550 parts are produced. Find the production rate of each machine.
- A total of \$12 000 is invested, part at 6.5%, part at 6.0%, and part at 5.5%, yielding a total annual interest of \$726. The income from the 6.5% part yields \$128 less than that for the other two parts combined. How much is invested at each rate?

## Answer to Practice Exercise

- $x = 2$ ,  $y = -3$

## 16.6 Higher-Order Determinants

Minors • Expansion by Minors •  
Properties of Determinants •  
Solving Systems of Equations

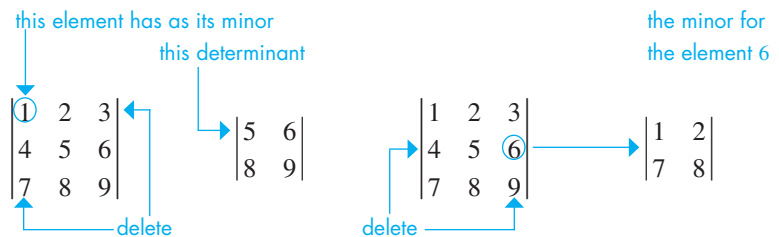
In Chapter 5, we limited our discussion of determinants to those of the second and third orders. We now show some methods of evaluating higher-order determinants. We will then be able to use Cramer's rule to solve systems of equations of higher order.

From Section 5.7, we recall the definition of a third-order determinant. Then we rewrite it in a form that can be generalized to determinants of any order. We have

$$\begin{aligned}
 \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} &= a_1 b_2 c_3 + a_3 b_1 c_2 + a_2 b_3 c_1 - a_3 b_2 c_1 - a_1 b_3 c_2 - a_2 b_1 c_3 \\
 &= a_1 (b_2 c_3 - b_3 c_2) - a_2 (b_1 c_3 - b_3 c_1) + a_3 (b_1 c_2 - b_2 c_1) \\
 &= a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}
 \end{aligned} \tag{16.12}$$

In Eq. (16.12), the third-order determinant is expanded as products of the elements of the first column and second-order determinants, known as *minors*. In general, *the minor of an element of a determinant is the determinant that results by deleting the row and column in which the element lies.*

**EXAMPLE 1** Minors



The following theorem generalizes the expansion by minors expressed in Eq. (16.12) for a third-order determinant to determinants of any order.

**Expansion of a Determinant by Minors**

*The value of a determinant of order n may be found by forming the n products of the elements of any column (or row) and their minors. A product is given a plus sign if the sum of the number of the column and the number of the row in which the element lies is even, and a minus sign if this sum is odd. The algebraic sum of these terms is the value of the determinant.*

**EXAMPLE 2** Expansion by minors

In evaluating the following determinant, note that the third column has two zeros. This means that expanding by the third column will require less numerical work. Therefore,

$$\begin{aligned}
 \begin{vmatrix} 3 & -2 & 0 & 2 \\ 1 & 0 & -1 & 4 \\ -3 & 1 & 2 & -2 \\ 2 & -1 & 0 & -1 \end{vmatrix} &= + (0) \begin{vmatrix} 1 & 0 & 4 \\ -3 & 1 & -2 \\ 2 & -1 & -1 \end{vmatrix} - (-1) \begin{vmatrix} 3 & -2 & 2 \\ -3 & 1 & -2 \\ 2 & -1 & -1 \end{vmatrix} + (2) \begin{vmatrix} 3 & -2 & 2 \\ 1 & 0 & 4 \\ 2 & -1 & -1 \end{vmatrix} - (0) \begin{vmatrix} 3 & -2 & 2 \\ 1 & 0 & 4 \\ -3 & 1 & -2 \end{vmatrix} \\
 &= \begin{vmatrix} 3 & -2 & 2 \\ -3 & 1 & -2 \\ 2 & -1 & -1 \end{vmatrix} + 2 \begin{vmatrix} 3 & -2 & 2 \\ 1 & 0 & 4 \\ 2 & -1 & -1 \end{vmatrix} \\
 &= \left[ 3 \begin{vmatrix} 1 & -2 \\ -1 & -1 \end{vmatrix} - (-3) \begin{vmatrix} -2 & 2 \\ -1 & -1 \end{vmatrix} + 2 \begin{vmatrix} -2 & 2 \\ 1 & -2 \end{vmatrix} \right] + 2 \left[ -(-2) \begin{vmatrix} 1 & 4 \\ 2 & -1 \end{vmatrix} + 0 \begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix} - (-1) \begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix} \right] \\
 &\quad \text{expanding first determinant by first column} \qquad \qquad \qquad \text{expanding second determinant by second column} \\
 &= [3(-1 - 2) + 3(2 + 2) + 2(4 - 2)] + 2[2(-1 - 8) + (12 - 2)] \\
 &= [-9 + 12 + 4] + 2[-18 + 10] = 7 + 2(-8) = -9
 \end{aligned}$$



### PROPERTIES OF DETERMINANTS

We can expand any determinant by minors, but even a fourth-order determinant may require many calculations. We now state some properties of determinants with which they can be evaluated, often with less work than with minors, and not requiring a calculator or computer:

1. If each element above or each element below the principal diagonal of a determinant is zero, then the product of the elements of the principal diagonal is the value of the determinant.
2. If all corresponding rows and columns of a determinant are interchanged, the value of the determinant is unchanged.

#### EXAMPLE 3 Properties of determinants 1 and 2

$$(a) \begin{vmatrix} 2 & 1 & 8 \\ 0 & -5 & 9 \\ 0 & 0 & -6 \end{vmatrix} = 2(-5)(-6) = 60 \quad \text{property 1}$$

$$(b) \begin{vmatrix} 1 & 3 & -1 \\ 2 & 0 & 4 \\ -2 & 5 & -6 \end{vmatrix} \quad \begin{array}{l} \text{we obtain the} \\ \text{determinant} \end{array} \quad \begin{vmatrix} 1 & 2 & -2 \\ 3 & 0 & 5 \\ -1 & 4 & -6 \end{vmatrix} \quad \begin{array}{l} \text{property 2} \\ \text{the value of each} \\ \text{determinant is } -18 \end{array}$$

3. If two columns (or rows) of a determinant are identical, the value of the determinant is zero.
4. If two columns (or rows) of a determinant are interchanged, the value of the determinant is changed in sign.

#### EXAMPLE 4 Properties of determinants 3 and 4

$$(a) \begin{array}{l} \text{identical} \\ \text{rows} \end{array} \begin{vmatrix} 3 & 5 & 2 \\ -4 & 6 & 9 \\ -4 & 6 & 9 \end{vmatrix} = 0 \quad \text{property 3}$$

$$(b) \begin{vmatrix} 3 & 0 & 2 \\ 1 & 1 & 5 \\ 2 & 1 & 3 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} 2 & 0 & 3 \\ 5 & 1 & 1 \\ 3 & 1 & 2 \end{vmatrix} \quad \begin{array}{l} \text{property 4} \\ \text{the value of the first is } -8, \text{ and} \\ \text{the value of the second is } 8 \end{array}$$

5. If all elements of a column (or row) are multiplied by the same number  $k$ , the value of the determinant is multiplied by  $k$ .
6. If all the elements of any column (or row) are multiplied by the same number  $k$ , and the resulting numbers are added to the corresponding elements of another column (or row), the value of the determinant is unchanged.

#### EXAMPLE 5 Properties of determinants 5 and 6

$$(a) \begin{vmatrix} -1 & 0 & 6 \\ 6 & 3 & -6 \\ 0 & 5 & 3 \end{vmatrix} = 3 \begin{vmatrix} -1 & 0 & 6 \\ 2 & 1 & -2 \\ 0 & 5 & 3 \end{vmatrix} \quad \begin{array}{l} \text{property 5} \\ \text{the value of the first is } 141, \\ \text{and the value of the second is } 47 \\ 141 = 3(47) \end{array}$$

- (b) The value of the following determinant is unchanged if we multiply each element of the first row by 2 and add these numbers to the corresponding elements of the second row. The value of each determinant is  $-37$ . The great value in using property 6 is that we can purposely place zeros in the resulting determinant.

$$\begin{vmatrix} 4 & -1 & 3 \\ 2 & 2 & 1 \\ 1 & 0 & -3 \end{vmatrix} \quad \begin{array}{l} 4 \times 2 = 8 \\ -1 \times 2 = -2 \\ 3 \times 2 = 6 \end{array} \quad \begin{vmatrix} 4 & -1 & 3 \\ 2+8 & 2+(-2) & 1+6 \\ 1 & 0 & -3 \end{vmatrix} = \begin{vmatrix} 4 & -1 & 3 \\ 10 & 0 & 7 \\ 1 & 0 & -3 \end{vmatrix} \quad \text{or} \quad \begin{vmatrix} 4 & -1 & 3 \\ 10 & 0 & 7 \\ 1 & 0 & -3 \end{vmatrix} = \begin{vmatrix} 4 & -1 & 3 \\ 2 & 2 & 1 \\ 1 & 0 & -3 \end{vmatrix}$$

**EXAMPLE 6** Evaluating a determinant using properties 1–6

Using these properties, we evaluate the following determinant.

$$\begin{vmatrix} 3 & 2 & -1 & 1 \\ -1 & 1 & 2 & 3 \\ 2 & 2 & 1 & 4 \\ 0 & -1 & -2 & 2 \end{vmatrix} = \begin{vmatrix} 0 & 5 & 5 & 10 \\ -1 & 1 & 2 & 3 \\ 2 & 2 & 1 & 4 \\ 0 & -1 & -2 & 2 \end{vmatrix}$$

Each element of the second row is multiplied by 3, and the resulting numbers are added to the corresponding elements of the first row. Here, we have used property 6. In this way, a zero has been placed in column 1, row 1.

$$= \begin{vmatrix} 0 & 5 & 5 & 10 \\ -1 & 1 & 2 & 3 \\ 0 & 4 & 5 & 10 \\ 0 & -1 & -2 & 2 \end{vmatrix}$$

Each element of the second row is multiplied by 2, and the resulting numbers are added to the corresponding elements of the third row. Again, we have used property 6. Also, a zero has been placed in the first column, third row. We now have three zeros in the first column.

$$= -(-1) \begin{vmatrix} 5 & 5 & 10 \\ 4 & 5 & 10 \\ -1 & -2 & 2 \end{vmatrix}$$

Expand the determinant by the first column. We have now reduced the determinant to a third-order determinant.

$$= 5 \begin{vmatrix} 1 & 1 & 2 \\ 4 & 5 & 10 \\ -1 & -2 & 2 \end{vmatrix}$$

Factor 5 from each element of the first row. Here, we are using property 5.

$$= 5(2) \begin{vmatrix} 1 & 1 & 1 \\ 4 & 5 & 5 \\ -1 & -2 & 1 \end{vmatrix}$$

Factor 2 from each element of the third column. Again, we are using property 5. Also, by doing this, we have reduced the size of the numbers, and the resulting numbers are somewhat easier to work with.

$$= 10 \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ -1 & -2 & 1 \end{vmatrix}$$

Each element of the first row is multiplied by  $-4$ , and the resulting numbers are added to the corresponding elements of the second row. Here, we are using property 6. We have placed a zero in the first column, second row.

$$= 10 \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & -1 & 2 \end{vmatrix}$$

Each element of the first row is added to the corresponding element of the third row. Again, we have used property 6. A zero has been placed in the first column, third row. We now have two zeros in the first column.

$$= 10(1) \begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix}$$

Expand the determinant by the first column.

$$= 10(2 + 1) = 30$$

Expand the second-order determinant.

A somewhat more systematic method is to place zeros below the principal diagonal and then use property 1. ■

The methods of this section and that of expansion by minors illustrate the ways in which determinants were evaluated before the extensive use of computers and calculators. As we noted in Chapter 5, most graphing calculators can be used to quickly evaluate determinants.

**SOLVING SYSTEMS OF LINEAR EQUATIONS BY DETERMINANTS**

We can use the expansion of determinants by minors when solving systems of linear equations. **Cramer's rule** for solving systems of linear equations, as stated in Section 5.7, is valid for any system of  $n$  equations in  $n$  unknowns.

**LEARNING TIP**

With the use of these six properties, determinants of higher order can be evaluated much more easily. The technique is to

*obtain zeros in a given column (or row) in all positions except one.*

We can then expand by this column (or row), thereby reducing the order of the determinant. Property 6 is probably the most valuable for obtaining the zeros.

■ To display and evaluate a determinant on a calculator, see page 171.

**EXAMPLE 7** Solving a system of four equations

Solve the following system of equations:

$$x + 2y + z = 5$$

$$2x + z + 2t = 1$$

$$x - y + 3z + 4t = -6$$

$$4x - y - 2t = 0$$

expanding by fourth row

$$\begin{aligned}
 x &= \frac{\begin{vmatrix} 5 & 2 & 1 & 0 \\ 1 & 0 & 1 & 2 \\ -6 & -1 & 3 & 4 \\ 0 & -1 & 0 & -2 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 1 & 0 \\ 2 & 0 & 1 & 2 \\ 1 & -1 & 3 & 4 \\ 4 & -1 & 0 & -2 \end{vmatrix}} = \frac{- (0) \begin{vmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \\ -1 & 3 & 4 \end{vmatrix} + (-1) \begin{vmatrix} 5 & 1 & 0 \\ 1 & 1 & 2 \\ -6 & 3 & 4 \end{vmatrix} - (0) \begin{vmatrix} 5 & 2 & 0 \\ 1 & 0 & 2 \\ -6 & -1 & 4 \end{vmatrix} + (-2) \begin{vmatrix} 5 & 2 & 1 \\ 1 & 0 & 1 \\ -6 & -1 & 3 \end{vmatrix}}{(1) \begin{vmatrix} 0 & 1 & 2 \\ -1 & 3 & 4 \\ -1 & 0 & -2 \end{vmatrix} - 2 \begin{vmatrix} 2 & 1 & 2 \\ 1 & 3 & 4 \\ 4 & 0 & -2 \end{vmatrix} + (1) \begin{vmatrix} 2 & 0 & 2 \\ 1 & -1 & 4 \\ 4 & -1 & -2 \end{vmatrix} - (0) \begin{vmatrix} 2 & 0 & 1 \\ 1 & -1 & 3 \\ 4 & -1 & 0 \end{vmatrix}} \\
 &= \frac{-(-26) - 2(-14)}{1(0) - 2(-18) + 1(18)} = \frac{26 + 28}{36 + 18} = \frac{54}{54} = 1
 \end{aligned}$$

We could use minors or the properties of determinants for the evaluations. Noting the two zeros in the fourth row of the numerator, we expanded by minors. For the denominator, which we need to evaluate only once, we used minors of the first row, although we could have used the properties to create zeros in the first column.

In solving for  $y$ , we again note the two zeros in the fourth row:

$$\begin{aligned}
 y &= \frac{\begin{vmatrix} 1 & 5 & 1 & 0 \\ 2 & 1 & 1 & 2 \\ 1 & -6 & 3 & 4 \\ 4 & 0 & 0 & -2 \end{vmatrix}}{54} = \frac{-4 \begin{vmatrix} 5 & 1 & 0 \\ 1 & 1 & 2 \\ -6 & 3 & 4 \end{vmatrix} + (-2) \begin{vmatrix} 1 & 5 & 1 \\ 2 & 1 & 1 \\ 1 & -6 & 3 \end{vmatrix}}{54} \\
 &= \frac{-4(-26) - 2(-29)}{54} = \frac{104 + 58}{54} = \frac{162}{54} = 3
 \end{aligned}$$

Substituting  $x = 1$  and  $y = 3$  in the first equation gives us  $z = -2$ . Then substituting  $x = 1$  and  $y = 3$  in the fourth equation gives us  $t = 1/2$ . Therefore, the required solution is  $x = 1, y = 3, z = -2, t = 1/2$ . We can check the solution by substituting in the second or third equation (we used the first and fourth to *find* values of  $z$  and  $t$ .) ■

**EXERCISES 16.6**

In Exercises 1 and 2, make the given changes in the indicated examples of this section. Then evaluate the resulting determinants.

- In Example 4(b), change the third column of the first determinant to  $0 \ 0 \ 3$ .
- In Example 5(a), change the third column of the first determinant to  $2 \ -2 \ 6$ .

In Exercises 3–6, evaluate each determinant by inspection. Observation will allow evaluation by using the properties of this section.

- $\begin{vmatrix} 4 & -5 & 8 \\ 0 & 3 & -8 \\ 0 & 0 & -5 \end{vmatrix}$
- $\begin{vmatrix} 3 & 0 & 0 \\ 0 & 10 & 0 \\ -9 & -1 & -5 \end{vmatrix}$

5. 
$$\begin{vmatrix} 3 & -2 & 4 & 2 \\ 5 & -1 & 2 & -1 \\ 3 & -2 & 4 & 2 \\ 0 & 3 & -6 & 0 \end{vmatrix}$$

6. 
$$\begin{vmatrix} -12 & -24 & -24 & 15 \\ 12 & 32 & 32 & -35 \\ -22 & 18 & 18 & 18 \\ 44 & 0 & 0 & -26 \end{vmatrix}$$

In Exercises 7–10, use the given value of the determinant at the right and the properties of this section to evaluate the following determinants.

7. 
$$\begin{vmatrix} 2 & 1 & -3 \\ -4 & 3 & 1 \\ 1 & -2 & -3 \end{vmatrix}$$

8. 
$$\begin{vmatrix} 2 & -3 & 1 \\ -4 & 1 & 3 \\ 2 & -6 & -4 \end{vmatrix} = 40$$

9. 
$$\begin{vmatrix} 2 & -3 & -1 \\ -4 & 1 & -3 \\ 1 & -3 & 2 \end{vmatrix}$$

10. 
$$\begin{vmatrix} 2 & -4 & 1 \\ -3 & 1 & -3 \\ 1 & 3 & -2 \end{vmatrix}$$

In Exercises 11–20, evaluate the given determinants by expansion by minors.

11. 
$$\begin{vmatrix} 3 & 0 & 0 \\ -2 & 1 & 4 \\ 4 & -2 & 5 \end{vmatrix}$$

12. 
$$\begin{vmatrix} 10 & 0 & -3 \\ -2 & -4 & 1 \\ 3 & 0 & 2 \end{vmatrix}$$

13. 
$$\begin{vmatrix} 3 & 1 & 0 \\ -2 & 3 & -1 \\ 4 & 2 & 5 \end{vmatrix}$$

14. 
$$\begin{vmatrix} -40 & 30 & -20 \\ -8 & 8 & 16 \\ -15 & 75 & -45 \end{vmatrix}$$

15. 
$$\begin{vmatrix} 4 & 3 & 6 & 0 \\ 3 & 0 & 0 & 4 \\ 5 & 0 & 1 & 2 \\ 2 & 1 & 1 & 7 \end{vmatrix}$$

16. 
$$\begin{vmatrix} 6 & -3 & -6 & 3 \\ -2 & 1 & 2 & -1 \\ 18 & 7 & -1 & 5 \\ 0 & -1 & 10 & 10 \end{vmatrix}$$

17. 
$$\begin{vmatrix} 1 & 3 & -3 & 5 \\ 4 & 2 & 1 & 2 \\ 3 & 2 & -2 & 2 \\ 0 & 1 & 2 & -1 \end{vmatrix}$$

18. 
$$\begin{vmatrix} -2 & 2 & 1 & 3 \\ 1 & 4 & 3 & 1 \\ 4 & 3 & -2 & -2 \\ 3 & -2 & 1 & 5 \end{vmatrix}$$

19. 
$$\begin{vmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 1 \\ 1 & 0 & -1 & 1 & -1 \\ -2 & 0 & -1 & 2 & 1 \\ 1 & 0 & 2 & -1 & -2 \end{vmatrix}$$

20. 
$$\begin{vmatrix} -1 & 3 & 5 & 0 & -5 \\ 0 & 1 & 7 & 3 & -2 \\ 5 & -2 & -1 & 0 & 3 \\ -3 & 0 & 2 & -1 & 3 \\ 6 & 2 & 1 & -4 & 2 \end{vmatrix}$$

In Exercises 21–30, use the determinants for Exercises 11–20 and evaluate each using the properties of determinants. Do not evaluate directly more than one second-order determinant.

In Exercises 31–34, solve the given systems of equations by determinants. Evaluate by expansion by minors.

31. 
$$\begin{cases} x + t = 0 \\ 3x + y + z = -1 \\ 2y - z + 3t = 1 \\ 2z - 3t = 1 \end{cases}$$

32. 
$$\begin{cases} 2x + y + z = 4 \\ 2y - 2z - t = 3 \\ 3y - 3z + 2t = 1 \\ 6x - y + t = 0 \end{cases}$$

33. 
$$\begin{cases} x + 2y - z = 6 \\ y - 2z - 3t = -5 \\ 3x - 2y + t = 2 \\ 2x + y + z - t = 0 \end{cases}$$

34. 
$$\begin{cases} 2p + 3r + s = 4 \\ p - 2r - 3s + 4t = -1 \\ 3p + r + s - 5t = 3 \\ -p + 2r + s + 3t = 2 \end{cases}$$

In Exercises 35–38, solve the given systems of equations by determinants. Evaluate by using the properties of determinants.

35. 
$$\begin{cases} 2x + y + z = 2 \\ 3y - z + 2t = 4 \\ y + 2z + t = 0 \\ 3x + 2z = 4 \end{cases}$$

36. 
$$\begin{cases} 2x + y + z = 0 \\ x - y + 2t = 2 \\ 2y + z + 4t = 2 \\ 5x + 2z + 2t = 4 \end{cases}$$

37. 
$$\begin{cases} D + E + 2F = 1 \\ 2D - E + G = -2 \\ D - E - F - 2G = 4 \\ 2D - E + 2F - G = 0 \end{cases}$$

38. 
$$\begin{cases} 3x + y + t = 0 \\ 3z + 2t = 8 \\ 6x + 2y + 2z + t = 3 \\ 3x - y - z - t = 0 \end{cases}$$

In Exercises 39–42, make the indicated changes in the determinant at the right, and then solve the indicated problem. Assume the elements are nonzero, unless otherwise specified.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

39. Evaluate the determinant if  $a = c$ ,  $d = f$ , and  $g = i$ .

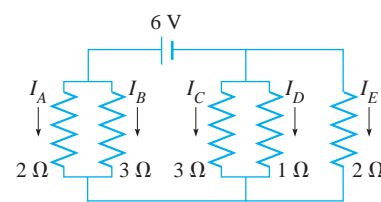
40. Evaluate the determinant if  $b = c = f = 0$ .

41. By what factor is the value of the determinant changed if all elements are doubled?

42. How is the value changed if  $a$  is added to  $g$ ,  $b$  added to  $h$ , and  $c$  added to  $i$ ?

In Exercises 43–48, solve the given problems by using determinants.

43. In applying Kirchhoff's laws (see Exercise 42 on page 165) to the circuit shown in Fig. 16.15, the following equations are found. Determine the indicated currents (in A).



$$\begin{cases} I_A + I_B + I_C + I_D + I_E = 0 \\ -2I_A + 3I_B = 0 \\ 3I_B - 3I_C = 6 \\ -3I_C + I_D = 0 \\ -I_D + 2I_E = 0 \end{cases}$$

Fig. 16.15

44. In analysing the forces  $A$ ,  $B$ ,  $C$ , and  $D$  shown on the beam in Fig. 16.16, the following equations are used. Find these forces.

$$\begin{cases} A + B = 850 \\ A + B + 400 = 0.8C + 0.6D \\ 0.6C = 0.8D \\ 5A - 5B + 4C - 3D = 0 \end{cases}$$

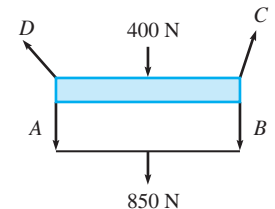


Fig. 16.16

45. In analysing the motion of four equally spaced particles along a string, the equation at the right is found. Here,  $C$  depends on the string and the mass of each object. Solve for  $C$  ( $C > 0$ ).
- $$\begin{vmatrix} C & -1 & 0 & 0 \\ -1 & C & -1 & 0 \\ 0 & -1 & C & -1 \\ 0 & 0 & -1 & C \end{vmatrix} = 0$$
46. An alloy is to be made from four other alloys containing copper (Cu), nickel (Ni), zinc (Zn), and Iron (Fe). The first is 80% Cu and 20% Ni. The second is 60% Cu, 20% Ni, and 20% Zn. The third is 30% Cu, 60% Ni, and 10% Fe. The fourth is 20% Ni, 40% Zn, and 40% Fe. How much of each is needed so that the final alloy has 56 g Cu, 28 g Ni, 10 g Zn, and 6 g Fe?
47. In testing for air pollution, a given air sample contained 6.0 parts per million (ppm) of four pollutants, sulfur dioxide ( $\text{SO}_2$ ), nitric oxide (NO), nitrogen dioxide ( $\text{NO}_2$ ), and carbon monoxide (CO). The ppm of CO was 10 times that of  $\text{SO}_2$ , which in turn equaled those of NO and  $\text{NO}_2$ . There was a total of 0.8 ppm of  $\text{SO}_2$  and NO. How many ppm of each were present in the air sample?
48. Three computer programs,  $A$ ,  $B$ , and  $C$ , use 15% of a computer's 6.0 GB (gigabyte) hard-drive memory. If two additional programs are added, one requiring the same memory as  $A$  and the other half the memory of  $C$ , 22% of the memory will be used. However, if two other programs are added to  $A$ ,  $B$ , and  $C$ , one requiring half the memory of  $B$  and the other the same memory as  $C$ , 25% of the memory will be used. How many megabytes of memory are required for each of  $A$ ,  $B$ , and  $C$ ?

## CHAPTER 16 EQUATIONS

### Basic laws for matrices

$$A + B = B + A \quad (\text{commutative law}) \quad (16.1)$$

$$A + (B + C) = (A + B) + C \quad (\text{associative law}) \quad (16.2)$$

$$k(A + B) = kA + kB \quad (16.3)$$

$$A + O = A \quad (16.4)$$

### Inverse matrix

$$AA^{-1} = A^{-1}A = I \quad (16.5)$$

### Solving systems of equations by matrices

$$\begin{bmatrix} a_1x + b_1y \\ a_2x + b_2y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \quad (16.6)$$

$$A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \end{bmatrix} \quad C = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \quad (16.7)$$

$$AX = C \quad (16.8)$$

$$X = A^{-1}C \quad (16.9)$$

### Gaussian elimination

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 & x + b_4y + c_4z &= d_4 \\ a_2x + b_2y + c_2z &= d_2 & y + c_5z &= d_5 \end{aligned} \quad (16.10) \quad (16.11)$$

$$a_3x + b_3y + c_3z = d_3 \quad z = d_6$$

### Higher-order determinants

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \quad (16.12)$$

## CHAPTER 16 REVIEW EXERCISES

In Exercises 1–6, determine the values of the literal numbers.

$$1. \begin{bmatrix} 2a \\ a - b \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \end{bmatrix} \quad 2. \begin{bmatrix} x - y \\ 2x + 2z \\ 4y + z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$$

$$3. \begin{bmatrix} 2x & 3y & 2z \\ x + y & 2y + z & z - x \end{bmatrix} = \begin{bmatrix} 4 & -9 & 5 \\ a & b & c \end{bmatrix}$$

$$4. \begin{bmatrix} a + bj & b \\ aj & b - aj \end{bmatrix} = \begin{bmatrix} 6j & 2d \\ 2cj & ej^2 \end{bmatrix} \quad (j = \sqrt{-1})$$

$$5. \begin{bmatrix} \cos \pi & \sin \frac{\pi}{6} \\ x + y & x - y \end{bmatrix} = \begin{bmatrix} x & y \\ a & b \end{bmatrix}$$

$$6. \begin{bmatrix} \ln e & \log 100 \\ a^2 & b^2 \end{bmatrix} = \begin{bmatrix} a + b & a - b \\ x & y \end{bmatrix}$$



In Exercises 61–64, use the determinants for Exercises 57–60 and evaluate each using properties of determinants. Do not evaluate directly more than one second-order determinant.

In Exercises 65 and 66, use the matrix  $N$ .

$$N = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

65. Show that  $N^{-1} = -N$ .      66. Show that  $N^2 = -I$ .

In Exercises 67 and 68, solve the given problems.

67. For any real number  $n$ , show that  $\begin{bmatrix} n & 1+n \\ 1-n & -n \end{bmatrix}^2 = I$ .

68. For the matrix  $N = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ , find (a)  $N^2$ , (b)  $N^3$ , (c)  $N^4$ . What is  $N^{20}$ ? Explain.

In Exercises 69–72, use matrices  $A$  and  $B$ .

$$A = \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix} \quad B = \begin{bmatrix} -3 & 1 \\ 2 & -1 \end{bmatrix}$$

69. Show that  $(A + B)(A - B) \neq A^2 - B^2$ .  
 70. Show that  $(A + B)^2 \neq A^2 + 2AB + B^2$ .  
 71. Show that the inverse of  $2A$  is  $A^{-1}/2$ .  
 72. Show that the inverse of  $B/2$  is  $2B^{-1}$ .

In Exercises 73–76, solve the given systems of equations by use of matrices as in Section 16.4.

73. Two electric resistors,  $R_1$  and  $R_2$ , are tested with currents and voltages such that the following equations are found:

$$2R_1 + 3R_2 = 26$$

$$3R_1 + 2R_2 = 24$$

Find the resistances  $R_1$  and  $R_2$  (in  $\Omega$ ).

74. A company produces two products, each of which is processed in two departments. Considering the worker time available, the numbers  $x$  and  $y$  of each product produced each week can be found by solving the system of equations

$$4.0x + 2.5y = 1200$$

$$3.2x + 4.0y = 1200$$

Find  $x$  and  $y$ .

75. A beam is supported as shown in Fig. 16.17. Find the magnitudes of the force  $F$  and the tension  $T$  by solving the following system of equations:

$$0.500F = 0.866T$$

$$0.866F + 0.500T = 350$$

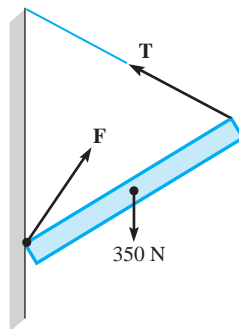


Fig. 16.17

76. To find the electric currents (in A) indicated in Fig. 16.18, it is necessary to solve the following equations.

$$\begin{aligned} I_A + I_B + I_C &= 0 \\ 5I_A - 2I_B &= -4 \\ 2I_B - I_C &= 0 \end{aligned}$$

Find  $I_A$ ,  $I_B$ , and  $I_C$ .

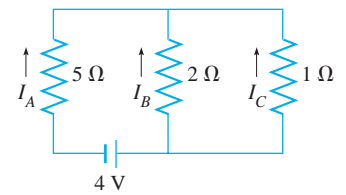


Fig. 16.18

In Exercises 77–80, solve the system of equations in Exercises 73–76 by Gaussian elimination.

In Exercises 81–85, solve the given problems by setting up the necessary equations and solving them by any appropriate method of this chapter.

81. A crime suspect passes an intersection in a car travelling at 180 km/h. The police pass the intersection 3.0 min later in a car travelling at 225 km/h. How long is it before the police overtake the suspect?  
 82. A contractor needs a backhoe and a generator for two different jobs. Renting the backhoe for 5.0 h and the generator for 6.0 h costs \$425 for one job. On the other job, renting the backhoe for 2.0 h and the generator for 8.0 h costs \$310. What are the hourly charges for the backhoe and the generator?  
 83. By mass, three alloys have the following percentages of lead, zinc, and copper:

	Lead	Zinc	Copper
Alloy A	60%	30%	10%
Alloy B	40%	30%	30%
Alloy C	30%	70%	

How many grams of each of alloys A, B, and C must be mixed to get 100 g of an alloy that is 44% lead, 38% zinc, and 18% copper?

84. On a 1014-km trip from Halifax to Ottawa that took a total of 5.0 h, a person took a shuttle to the airport, then a plane, and finally a taxi to reach the final destination. The shuttle took twice as long as the taxi, and the time for connections was as long as all other legs of the trip combined. The shuttle averaged 61 km/h, the plane averaged 640 km/h, and the taxi averaged 40 km/h. How long did each leg of the trip and the connections take?  
 85. In a Markov chain brand-switching model for three brands of detergent, the long-run market share of each brand (given as a proportion) is found by solving the system of equations

$$-0.3x + 0.5y + 0.32z = 0$$

$$0.2x - 0.8y + 0.3z = 0$$

$$0.1x + 0.3y - 0.62z = 0$$

$$x + y + z = 1$$

Find  $x$ ,  $y$ , and  $z$ .



In Exercises 86–89, perform the indicated matrix operations.

86. An automobile maker has two assembly plants at which cars with either 4, 6, or 8 cylinders and with either standard or automatic transmission are assembled. The annual production at the first plant of cars with the number of cylinders–transmission type (standard, automatic) is as follows:

4: 12 000, 15 000; 6: 24 000, 8000; 8: 4000, 30 000

At the second plant the annual production is

4: 15 000, 20 000; 6: 12 000, 3000; 8: 2000, 22 000

Set up matrices for this production and by matrix addition, find the matrix for the total production by the number of cylinders and type of transmission.

87. Set up a matrix representing the information given in Exercise 83. A given shipment contains 500 g of alloy A, 800 g of alloy B, and 700 g of alloy C. Set up a matrix for this information. By multiplying these matrices, obtain a matrix that gives the total weight of lead, zinc, and copper in the shipment.

88. The matrix equation

$$\begin{bmatrix} R_1 & -R_2 \\ -R_2 & R_1 \end{bmatrix} + R_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$$

may be used to represent the system of equations relating the currents and resistances of the circuit in Fig. 16.19. Find this system of equations by performing the indicated matrix operations.

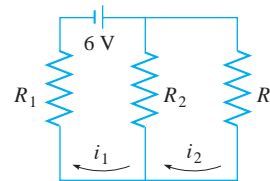


Fig. 16.19

89. A person prepared a meal of the following items, each having the given number of grams of protein, carbohydrates, and fat, respectively. Beef stew: 25, 21, 22; coleslaw: 3, 10, 10; (light) ice cream: 7, 25, 6. If the kilojoule count of each gram of protein, carbohydrate, and fat is 17 kJ/g, 16 kJ/g, and 37 kJ/g, respectively, find the total kilojoule count of each item by matrix multiplication.

### Writing Exercise

90. A hardware company has 60 different retail stores in which 1500 different products are sold. Write a paragraph or two explaining why matrices provide an efficient method of inventory control, and what matrix operations in this chapter would be of use.

## CHAPTER 16 PRACTICE TEST

1. For matrices  $A$  and  $B$ , find  $A - 2B$ .

$$A = \begin{bmatrix} 3 & -1 & 4 \\ 2 & 0 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 4 & 5 \\ -1 & -2 & 3 \end{bmatrix}$$

2. Evaluate the literal symbols.

$$\begin{bmatrix} 2x & x - y & z \\ x + z & 2y & y + z \end{bmatrix} = \begin{bmatrix} 6 & -2 & 4 \\ a & b & c \end{bmatrix}$$

3. For matrices  $C$  and  $D$ , find  $CD$  and  $DC$ .

$$C = \begin{bmatrix} 1 & 0 & 4 \\ 2 & -2 & 1 \\ -1 & 3 & 2 \end{bmatrix} \quad D = \begin{bmatrix} 2 & -2 \\ 4 & -5S \\ 6 & 1 \end{bmatrix}$$

4. Determine whether or not  $B = A^{-1}$ .

$$A = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \quad B = \begin{bmatrix} -2 & 5 \\ -1 & 2 \end{bmatrix}$$

5. For matrix  $C$  of Problem 3, find  $C^{-1}$ .

6. Solve by using the inverse of the coefficient matrix.

$$\begin{aligned} 2x - 3y &= 11 \\ x + 2y &= 2 \end{aligned}$$

7. Solve the system of equations in Problem 6 by Gaussian elimination.

8. Evaluate the following determinant by expansion by minors.

$$\begin{vmatrix} 2 & -4 & -3 \\ -3 & 6 & 2 \\ 5 & -1 & 5 \end{vmatrix}$$

9. Evaluate the determinant in Problem 8 by using the properties of determinants.

10. Solve the following system of equations by using the inverse of the coefficient matrix. You may use a calculator to perform the necessary matrix operations.

$$\begin{aligned} 7x - 2y + z &= 6 \\ 2x + 3y - 4z &= 6 \\ 4x - 5y + 2z &= 10 \end{aligned}$$

11. Fifty shares of stock A and 30 shares of stock B cost \$2600. Thirty shares of stock A and 40 shares of stock B cost \$2000. What is the price per share of each stock? Solve by setting up the appropriate equations and then using the inverse of the coefficient matrix.

# Inequalities

Having devoted a great deal of time to the solution of equations and systems of equations, we now turn our attention to solving inequalities and systems of inequalities. In doing so, we will find it necessary to find *all values* of the variable or variables that satisfy the inequality or system of inequalities.

There are numerous technical applications of inequalities. For example, in electricity it might be necessary to find the values of a current that are *greater than* a specified value. In designing a link in a robotic mechanism, it might be necessary to find the forces that are *less than* a specified value. Computers can be programmed to switch from one part of a program to another, based upon a result that is greater than (or less than) some given value.

Systems of linear *equations* have been studied for better than 2000 years, but almost no attention was given to systems of linear *inequalities* until World War II in the 1940s. Problems of deploying personnel and aircraft effectively and allocating supplies efficiently led the U.S. Air Force to have a number of scientists, economists, and mathematicians look for solutions. From this, a procedure for analysing such problems was devised in 1947 by George Danzig and his colleagues. Their system involved using systems of linear inequalities and is today called *linear programming*. We introduce the basic method of linear programming in the final section of this chapter.

Here, we see that a mathematical method was developed as a result of a military need. Today, linear programming is widely used in business and industry in order to set production levels for maximizing profits and minimizing costs.

# 17

## LEARNING OUTCOMES

After completion of this chapter, the student should be able to:

- Distinguish between a conditional and an absolute inequality
- Graph solutions of inequalities on the number line
- Represent solutions of inequalities in interval notation
- Apply the properties of inequalities to solve linear inequalities
- Solve linear inequalities with three members
- Solve nonlinear inequalities algebraically and graphically
- Solve inequalities involving absolute values
- Solve inequalities and systems of inequalities with two variables graphically
- Understand the concepts of constraint, objective function, and feasible point in the context of linear programming
- Solve linear programs with two variables graphically
- Solve application problems involving inequalities

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◀ In Section 17.6, we use inequalities to show how an airline can minimize operating costs when determining how many planes it needs.



## 17.1 Properties of Inequalities

**Solution of an Inequality • Conditional Inequality • Absolute Inequality • Properties of Inequalities**

In Chapter 1, we first introduced the signs of inequality. To this point, only a basic understanding of their meanings has been necessary to show certain intervals associated with a variable. In this section, we review the meanings and develop certain basic properties of inequalities. We also show the meaning of the solution of an inequality and how it is shown on the number line.

The expression  $a < b$  is read as “ $a$  is less than  $b$ ,” and the expression  $a > b$  is read as “ $a$  is greater than  $b$ .” *These signs define what is known as the sense (indicated by the direction of the sign) of the inequality.* Two inequalities are said to have the same sense if the signs of inequality point in the same direction. They are said to have the opposite sense if the signs of inequality point in opposite directions. *The two sides of the inequality are called members of the inequality.*

### EXAMPLE 1 Sense of an inequality

The inequalities  $x + 3 > 2$  and  $x + 1 > 0$  have the same sense, as do the inequalities  $3x - 1 < 4$  and  $x^2 - 1 < 3$ .

The inequalities  $x - 4 < 0$  and  $x > -4$  have the opposite sense, as do the inequalities  $2x + 4 > 1$  and  $3x^2 - 7 < 1$ . ■

The **solution** of an inequality consists of all values of the variable that make the inequality a true statement. Most inequalities with which we deal are **conditional inequalities**, which are true for some, but not all, values of the variable. Also, some inequalities are true for all values of the variable, and they are called **absolute inequalities**. A solution of an inequality consists of only real numbers, as the terms *greater than* or *less than* have not been defined for complex numbers.

### LEARNING TIP

When we have an equation like  $x - 1 = 0$ , we know that  $x$  is equal to the single numerical value of 1, so that 1 is the unique solution to the equation. In contrast, when we have an inequality like  $x + 1 > 0$ , then  $x$  is not a single number. As we have seen in Example 2,  $x$  can be any number in the interval  $(-1, \infty)$ . Every value of  $x$  within this interval is a solution of the inequality.

### EXAMPLE 2 Conditional and absolute inequalities

The inequality  $x + 1 > 0$  is true for all values of  $x$  greater than  $-1$ . Therefore, the values of  $x$  that satisfy this inequality are written as  $x > -1$  or, using interval notation (as introduced in Section 3.2), as  $(-1, \infty)$ . Considering that the inequality  $x + 1 > 0$  holds only for  $x$  in the interval  $(-1, \infty)$ , it is a *conditional inequality*.

The inequality  $x^2 + 1 > 0$  is true for all real values of  $x$ , since  $x^2$  is never negative. It is an *absolute inequality*. ■

There are occasions when it is convenient to combine an inequality with an equality. For such purposes, the symbols  $\leq$ , meaning *less than or equal to*, and  $\geq$ , meaning *greater than or equal to*, are used.

### EXAMPLE 3 Greater (or less) than or equal to

If we wish to state that  $x$  is positive, we can write  $x > 0$ . However, the value zero is not included in the solution. If we wish to state that  $x$  is not negative, we write  $x \geq 0$ . Here, zero is part of the solution.

In order to state that  $x$  is less than or equal to  $-5$ , we write  $x \leq -5$ . ■

In the sections that follow, we will solve inequalities. It is often useful to show the solution on the number line. The next example shows how this is done.

### EXAMPLE 4 Graphing solutions of inequalities

(a) To graph  $x > 2$ , or  $(2, \infty)$ , we draw a small open circle at 2 on the number line (which is equivalent to the  $x$ -axis). Then we draw a solid line to the right of the point and with an arrowhead pointing to the right, indicating all values greater than 2. See Fig. 17.1(a). The *open circle* shows that the point is not part of the indicated solution.



(a)



(b)

Fig. 17.1

- (b) To graph  $x \leq 1$ , or  $(-\infty, 1]$ , we follow the same basic procedure as in part (a), except that we use a solid circle and the arrowhead points to the left. See Fig. 17.1(b). The *solid circle* shows that the point is part of the indicated solution. ■

### PROPERTIES OF INEQUALITIES

We now show the basic operations performed on inequalities. These are the same operations as those performed on equations, but in certain cases the results take on a different form. *The following are the properties of inequalities:*

1. *The sense of an inequality is not changed when the same number is added to—or subtracted from—both members of the inequality.* Symbolically, this may be stated as “if  $a > b$ , then  $a + c > b + c$  and  $a - c > b - c$ .”

#### EXAMPLE 5 Illustrations of property 1

Using property 1 on the inequality  $9 > 6$ , we have the following results:

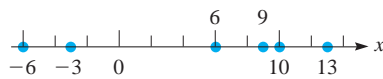


Fig. 17.2

$$\begin{array}{ll}
 9 > 6 & 9 > 6 \\
 \text{add 4 to each member} & \text{subtract 12 from each member} \\
 9 + 4 > 6 + 4 & 9 - 12 > 6 - 12 \\
 13 > 10 & -3 > -6
 \end{array}$$

In Fig. 17.2, we see that 9 is to the right of 6, 13 is to the right of 10, and  $-3$  is to the right of  $-6$ . ■

2. *The sense of an inequality is not changed if both members are multiplied or divided by the same positive number.* Symbolically, this is stated as “if  $a > b$ , then  $ac > bc$ , and  $a/c > b/c$ , provided that  $c > 0$ .”

#### EXAMPLE 6 Illustrations of property 2

Using property 2 on the inequality  $8 < 15$ , we have the following results:

$$\begin{array}{ll}
 8 < 15 & 8 < 15 \\
 \text{multiply both members by 2} & \text{divide both members by 2} \\
 2(8) < 2(15) & \frac{8}{2} < \frac{15}{2} \\
 16 < 30 & 4 < \frac{15}{2}
 \end{array}$$

3. *The sense of an inequality is reversed if both members are multiplied or divided by the same negative number.* Symbolically, this is stated as “if  $a > b$ , then  $ac < bc$ , and  $a/c < b/c$ , provided that  $c < 0$ .”

#### COMMON ERROR

In using this property of inequalities, be very careful to note that *the inequality sign remains the same if both members are multiplied or divided by a positive number*, but that *the inequality sign changes if both members are multiplied or divided by a negative number*. Most of the errors made in dealing with inequalities occur when using this property.

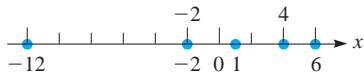


Fig. 17.3

**EXAMPLE 7** Be careful in using property 3

Using property 3 on the inequality  $4 > -2$ , we have the following results:

$$\begin{array}{ccc}
 4 > -2 & & 4 > -2 \\
 \downarrow \text{multiply both} & \text{reversed} & \downarrow \text{reversed} \\
 \text{members by } -3 & & \\
 -3(4) < -3(-2) & & \frac{4}{-2} < \frac{-2}{-2} \\
 \uparrow & \uparrow & \text{divide both} \\
 -12 < 6 & & \text{members by } -2 \\
 & & -2 < 1
 \end{array}$$

In Fig. 17.3, we see that 4 is to the right of  $-2$  but that  $-12$  is to the left of 6 and that  $-2$  is to the left of 1. This is consistent with reversing the sense of the inequality when it is multiplied by  $-3$  and when it is divided by  $-2$ . ■

**Practice Exercises**

For the inequality  $-6 < 3$ , state the inequality that results.

1. Multiply both members by 4.
2. Divide both members by  $-3$ .

4. If both members of an inequality are positive numbers and  $n$  is a positive number, then the inequality formed by taking the  $n$ th power of each member, or the  $n$ th root of each member, is in the same sense as the given inequality. Symbolically, this is stated as “if  $a > b$ , then  $a^n > b^n$ , and  $\sqrt[n]{a} > \sqrt[n]{b}$ , provided that  $n > 0$ ,  $a > 0$ ,  $b > 0$ .”

**EXAMPLE 8** Illustrations of property 4

Using property 4 on the inequality  $16 > 9$ , we have

$$\begin{array}{ccc}
 16 > 9 & & 16 > 9 \\
 \text{square both members} & & \text{take square root of both members} \\
 16^2 > 9^2 & & \sqrt{16} > \sqrt{9} \\
 256 > 81 & & 4 > 3
 \end{array}$$

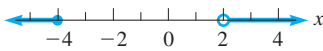
Many inequalities have more than two members. In fact, inequalities with three members are common, and care must be used in stating these inequalities.

**EXAMPLE 9** Inequalities with three members

- (a) To state that 5 is less than 6, and also greater than 2, we may write  $2 < 5 < 6$ , or  $6 > 5 > 2$ . (Generally, the *less than* form is preferred.)
- (b) To state that a number  $x$  may be greater than  $-1$  and also less than or equal to 3, we write  $-1 < x \leq 3$ , or as the interval  $(-1, 3]$ . (It can also be written as  $x > -1$  and  $x \leq 3$ .) This is shown in Fig. 17.4(a). Note the use of the open circle and the solid circle.
- (c) By writing  $x \leq -4$  or  $x > 2$ , we state that  $x$  is less than or equal to  $-4$ , or greater than 2. In interval notation we write  $(-\infty, -4] \cup (2, \infty)$ . See Fig. 17.4(b). ■



(a)



(b)

Fig. 17.4

**COMMON ERROR**

If  $x$  is less than or equal to  $-4$  or greater than 2, it may not be written as the single expression  $2 < x \leq -4$ . This statement would say that  $x$  is less than or equal to  $-4$ , while at the same time being greater than 2, and no such numbers exist. Moreover, it would follow that  $-4$  is greater than 2, which is absurd.

**LEARNING TIP**

Note carefully that **and** is used when the solution consists of values that make **both** statements true. The word **or** is used when the solution consists of values that make **either** statement true.

**Practice Exercise**

3. Graph the inequality  $-1 < x \leq 3$  on the number line.

**EXAMPLE 10** Meaning of and/or

The inequality  $x^2 - 3x + 2 > 0$  is satisfied if  $x$  is either greater than 2 or less than 1. This is written as  $x > 2$  or  $x < 1$ , or as  $(-\infty, 1) \cup (2, \infty)$ . Once again, it is incorrect to write the single expression  $1 > x > 2$ . However, it is correct to say that the inequality is *not* satisfied for  $1 \leq x \leq 2$ , or in  $[1, 2]$ . In other words, the inequality does not hold for values of  $x$  greater than or equal to 1 and less than or equal to 2. ■

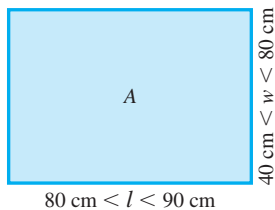


Fig. 17.5

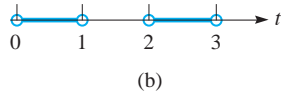
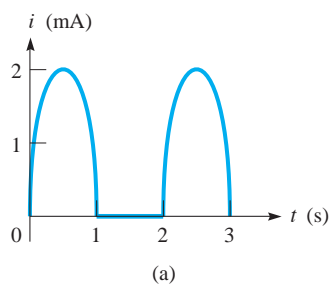


Fig. 17.6

**EXAMPLE 11** Setting up an inequality—application

The design of a rectangular solar panel shows that the length  $l$  is between 80 cm and 90 cm and the width  $w$  is between 40 cm and 80 cm. See Fig. 17.5. Find the values of area the panel may have.

Since  $l$  is to be less than 90 cm and  $w$  less than 80 cm, the area must be less than  $(90 \text{ cm})(80 \text{ cm}) = 7200 \text{ cm}^2$ . Also, since  $l$  is to be greater than 80 cm and  $w$  greater than 40 cm, the area must be greater than  $(80 \text{ cm})(40 \text{ cm}) = 3200 \text{ cm}^2$ . Therefore, the area  $A$  is in the interval  $(3200 \text{ cm}^2, 7200 \text{ cm}^2)$ , or

$$3200 \text{ cm}^2 < A < 7200 \text{ cm}^2$$

This means the area is greater than  $3200 \text{ cm}^2$  and less than  $7200 \text{ cm}^2$ . ■

**EXAMPLE 12** Setting up an inequality—application

A semiconductor *diode* has the property that an electric current flows through it in only one direction. If it is an alternating-current circuit, the current in the circuit flows only during the half-cycle when the diode allows it to flow. If a source of current given by  $i = 2 \sin \pi t$  ( $i$  in mA,  $t$  in seconds) is connected in series with a diode, write the inequalities for the current and the time. Assume that the source is on for 3.0 s and a positive current passes through the diode.

We are to find the values of  $t$  that correspond to  $i > 0$ . From the properties of the sine function, we know that  $2 \sin \pi t$  has a period of  $2\pi/\pi = 2.0$  s. Therefore, the current is zero for  $t = 0, 1.0$  s,  $2.0$  s, and  $3.0$  s.

The source current is positive for  $0 < t < 1.0$  s and for  $2.0 < t < 3.0$  s.

The source current is negative for  $1.0 < t < 2.0$  s.

Therefore, in the circuit

$$i > 0 \text{ for } 0 < t < 1.0 \text{ s and } 2.0 \text{ s} < t < 3.0 \text{ s}$$

$$i = 0 \text{ for } t = 0, 1.0 \text{ s} \leq t \leq 2.0 \text{ s}$$

A graph of the current in the circuit as a function of time is shown in Fig. 17.6(a). In Fig. 17.6(b), the values of  $t$  for which  $i > 0$  are shown. ■

**EXERCISES 17.1**

In Exercises 1–4, make the given changes in the indicated examples of this section and then perform the indicated operations.

- In Example 2, in the first paragraph, change the  $>$  to  $<$  and then complete the meaning of the resulting inequality as in the first sentence. Rewrite the meaning as in the second line.
- In Example 4(b), change  $\leq$  to  $>$  and then graph the resulting inequality.
- In Example 7, change the inequality to  $-2 > -4$  and then perform the two operations shown in colour.
- In Example 9(b), change the  $-1$  to  $-3$  and the  $3$  to  $1$  and then write the two forms in which an inequality represents the statement.

In Exercises 5–12, for the inequality  $4 < 9$ , state the inequality that results when the given operations are performed on both members.

- Add 3.
- Multiply by 5.
- Divide by  $-1$ .
- Square both.
- Subtract 16.
- Multiply by  $-2$ .
- Divide by  $0.5$ .
- Take square roots.

In Exercises 13–24, give the inequalities equivalent to the following statements about the number  $x$ .

- Greater than  $-2$
- Less than  $0.7$
- Less than or equal to  $45$
- Greater than or equal to  $-6$
- Greater than  $1$  and less than  $7$
- Greater than or equal to  $-200$  and less than  $650$
- Less than  $-9$ , or greater than or equal to  $-4$
- Less than or equal to  $8$ , or greater than or equal to  $12$
- Less than  $1$ , or greater than  $3$  and less than or equal to  $5$
- Greater than or equal to  $0$  and less than or equal to  $2$ , or greater than  $5$
- Greater than  $-2$  and less than  $2$ , or greater than or equal to  $3$  and less than  $4$
- Less than  $-4$ , or greater than or equal to  $0$  and less than or equal to  $1$ , or greater than or equal to  $5$



In Exercises 25–28, give verbal statements equivalent to the given inequalities involving the number  $x$ .

25.  $0 < x \leq 2$                       26.  $x < 5$  or  $x > 7$   
 27.  $x < -10$  or  $10 \leq x < 20$   
 28.  $-1 \leq x < 3$  or  $5 < x < 7$

In Exercises 29–44, express the given inequalities in interval notation and graph them on the number line.

29.  $x < 3$                                 30.  $x \geq -1$   
 31.  $x \leq 1$  or  $x > 3$                 32.  $x < -300$  or  $x \geq 0$   
 33.  $0 \leq x < 5$                         34.  $-4 < y < -2$   
 35.  $x \geq -3$  and  $x < 5$             36.  $x > 4$  and  $x < 3$   
 37.  $x < -1$  or  $1 \leq x < 4$   
 38.  $-3 < x < 0$  or  $x > 3$   
 39.  $-3 < x < -1$  or  $1 < x \leq 3$   
 40.  $1 < x \leq 2$  or  $3 \leq x < 4$   
 41.  $t < -0.3$  or  $t > -0.3$   
 42.  $x < 1$  or  $1 < x \leq 4$   
 43.  $(x \leq 5$  or  $x \geq 8)$  and  $(3 < x < 10)$   
 44.  $(x < 7$  and  $x > 2)$  or  $(x > 10$  or  $x < 1)$

In Exercises 45–48, answer the given questions about the inequality  $0 < a < b$ .

45. Is  $a^2 < b^2$  a conditional inequality or an absolute inequality?  
 46. Is  $|a - b| < b - a$ ?  
 47. If each member of the inequality  $2 > 1$  is multiplied by  $a - b$ , is the result  $2(a - b) > (a - b)$ ?  
 48. What is wrong with the following sequence of steps?  
 $a < b$ ,  $ab < b^2$ ,  $ab - b^2 < 0$ ,  $b(a - b) < 0$ ,  $b < 0$

In Exercises 49–52, solve the given problems.


49. Write the relationship between  $(|x| + |y|)$  and  $|x + y|$  if  $x > 0$  and  $y < 0$ .  
 50. Write the relationship between  $|xy|$  and  $|x||y|$  if  $x > 0$  and  $y < 0$ .  
 51. Explain the error in the following “proof” that  $3 < 2$ :  
 (1)  $1/8 < 1/4$     (2)  $0.5^3 < 0.5^2$     (3)  $\log 0.5^3 < \log 0.5^2$   
 (4)  $3 \log 0.5 < 2 \log 0.5$     (5)  $3 < 2$

52. If  $x \neq y$ , show that  $x^2 + y^2 > 2xy$ .

In Exercises 53–60, some applications of inequalities are shown.

53. An electron microscope can magnify an object from 2000 times to 1 000 000 times. Assuming that these values are exact, express these magnifications  $M$  as an inequality and graph them.  
 54. A busy person glances at a digital clock that shows 9:36. Another glance a short time later shows the clock at 9:44. Express the amount of time  $t$  (in min) that could have elapsed between glances by use of inequalities. Graph these values of  $t$ .  
 55. An earth satellite put into orbit near the earth’s surface will have an elliptic orbit if its velocity  $v$  is between 29 000 km/h and 40 000 km/h. Write this as an inequality and graph these values of  $v$ .  
 56. Fossils found in Jurassic rocks indicate that dinosaurs flourished during the Jurassic geological period, 140 MY (million years ago) to 200 MY. Write this as an inequality, with  $t$  representing past time. Graph the values of  $t$ .  
 57. In executing a program, a computer must perform a set of calculations. Any one of the calculations takes no more than 2565 steps. Express the number  $n$  of steps required for a given calculation by an inequality. (Note that  $n$  is a positive integer.)  
 58. The velocity  $v$  of an ultrasound wave in soft human tissue may be represented as  $1550 \pm 60$  m/s, where the  $\pm 60$  m/s gives the possible variation in the velocity. Express the possible velocities by an inequality.  
 59. The electric intensity  $E$  within a charged spherical conductor is zero. The intensity on the surface and outside the sphere equals a constant  $k$  divided by the square of the distance  $r$  from the centre of the sphere. State these relations for a sphere of radius  $a$  by using inequalities, and graph  $E$  as a function of  $r$ .  
 60. If the current from the source in Example 12 is  $i = 5 \cos 4\pi t$  and the diode allows only negative current to flow, write the inequalities and draw the graph for the current in the circuit as a function of time for  $0 \leq t \leq 1$  s.

### Answers to Practice Exercises

1.  $-24 < 12$     2.  $2 > -1$     3. 

## 17.2 Solving Linear Inequalities

### Linear Inequalities • Inequalities with Three Members

Using the properties and definitions discussed in Section 17.1, we can now proceed to solve inequalities. In this section, we solve linear inequalities in one variable. Similar to linear functions as defined in Chapter 5, a **linear inequality** is one in which each term contains only one variable and the exponent of each variable is 1. We will consider linear inequalities in two variables in Section 17.5.

#### LEARNING TIP

The procedure for solving a linear inequality in one variable is the same as the one used for solving linear equations, with the added restriction that every time we multiply or divide by a negative number, we have to change the sense of the inequality. In other words, the objective is to isolate the variable by performing the same operations on each member of the inequality, keeping in mind the basic properties given in Section 17.1.



**EXAMPLE 1** Solutions using the basic operations

In each of the following inequalities, by performing the indicated operation, we isolate  $x$  and thereby solve the inequality.

$x + 2 < 4$	$\frac{x}{2} > 4$	$2x \leq 4$
Subtract 2 from each member.	Multiply each member by 2.	Divide each member by 2.
$x < 2$	$x > 8$	$x \leq 2$
$(-\infty, 2)$	$(8, \infty)$	$(-\infty, 2]$

Each solution can be checked by substituting any number in the indicated interval into the original inequality. For example, any value less than 2 will satisfy the first inequality, whereas 2 or any number less than 2 will satisfy the third inequality. ■

**EXAMPLE 2** Solving a linear inequality

Solve the following inequality:  $3 - 2x \geq 15$ .

We have the following solution:

$$\begin{array}{ll}
 3 - 2x \geq 15 & \text{original inequality} \\
 -2x \geq 12 & \text{subtract 3 from each member} \\
 \text{inequality reversed} \longrightarrow \downarrow & \\
 x \leq -6 & \text{divide each member by } -2 \\
 (-\infty, -6] &
 \end{array}$$

Again, carefully note that *the sign of inequality was reversed when each number was divided by  $-2$* . We check the solution by substituting  $-7$  in the original inequality, obtaining  $17 \geq 15$ . ■

**EXAMPLE 3** Solving a linear inequality

Solve the inequality  $2x \leq 3 - x$ .

The solution proceeds as follows:

$$\begin{array}{ll}
 2x \leq 3 - x & \text{original inequality} \\
 3x \leq 3 & \text{add } x \text{ to each member} \\
 x \leq 1 & \text{divide each member by 3} \\
 (-\infty, 1] &
 \end{array}$$

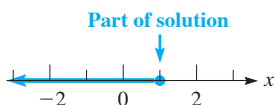


Fig. 17.7

This solution checks and is represented in Fig. 17.7, as we showed in Section 17.1.

This inequality could have been solved by combining  $x$ -terms on the right. In doing so, we would obtain  $1 \geq x$ . Since this might be misread, it is best to combine the variable terms on the left, as we did above. ■

**EXAMPLE 4** Solving a linear inequality

Solve the inequality  $\frac{3}{2}(1 - x) > \frac{1}{4} - x$ .

$$\begin{array}{ll}
 \frac{3}{2}(1 - x) > \frac{1}{4} - x & \text{original inequality} \\
 6(1 - x) > 1 - 4x & \text{multiply each member by 4} \\
 6 - 6x > 1 - 4x & \text{remove parentheses} \\
 -6x > -5 - 4x & \text{subtract 6 from each member} \\
 -2x > -5 & \text{add } 4x \text{ to each member} \\
 \downarrow & \\
 x < \frac{5}{2} & \text{divide each member by } -2 \\
 \left(-\infty, \frac{5}{2}\right) &
 \end{array}$$

**Practice Exercise**

1. Solve the inequality  $2(3 - x) > 5 + 4x$ .

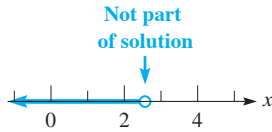


Fig. 17.8

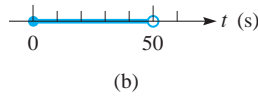
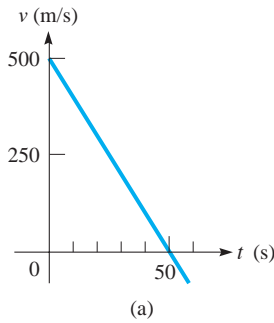


Fig. 17.9

Note that the sense of the inequality was reversed when we divided by  $-2$ . This solution is shown in Fig. 17.8. Any value of  $x < 5/2$  checks when substituted into the original inequality. ■

The following example illustrates an application that involves the solution of an inequality.

**EXAMPLE 5 Linear inequality—application**

The velocity  $v$  (in m/s) of a missile in terms of the time  $t$  (in s) is given by  $v = 490 - 9.8t$ . For how long is the velocity positive? (Since velocity is a vector, this can also be interpreted as asking “how long is the missile moving upward?”)

In terms of inequalities, we are asked to find the values of  $t$  for which  $v > 0$ . This means that we must solve the inequality  $490 - 9.8t > 0$ . The solution is as follows:

$$\begin{aligned} 490 - 9.8t &> 0 && \text{original inequality} \\ -9.8t &> -490 && \text{subtract 490 from each member} \\ t &< 50 \text{ s} && \text{divide each member by } -9.8 \end{aligned}$$

Negative values of  $t$  have no meaning in this problem. Checking  $t = 0$ , we find that  $v = 490$  m/s. Therefore, the complete solution is  $0 \leq t < 50$  s, or  $[0, 50)$  s).

In Fig. 17.9(a), we show the graph of  $v = 490 - 9.8t$ , and in Fig. 17.9(b), we show the solution  $0 \leq t < 50$  s on the number line (which is really the  $t$ -axis in this case). Note that the values of  $v$  are above the  $t$ -axis for those values of  $t$  that are part of the solution. This shows the relationship of the graph of  $v$  as a function of  $t$ , and the solution as graphed on the number line (the  $t$ -axis). ■

**INEQUALITIES WITH THREE MEMBERS**

**EXAMPLE 6 Solving an inequality with three members**

Solve  $-1 < 2x + 3 < 6$ .

We have the following solution:

$$\begin{aligned} -1 < 2x + 3 < 6 &&& \text{original inequality} \\ -4 < 2x < 3 &&& \text{subtract 3 from each member} \\ -2 < x < \frac{3}{2} &&& \text{divide each member by 2} \\ (-2, \frac{3}{2}) &&& \end{aligned}$$

The solution is shown in Fig. 17.10. ■

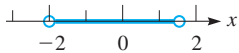


Fig. 17.10

**EXAMPLE 7 Solving an inequality with three members**

Solve the inequality  $2x < x - 4 \leq 3x + 8$ .

Since we cannot isolate  $x$  in the middle member (or in any member), we rewrite the inequality as

$$2x < x - 4 \quad \text{and} \quad x - 4 \leq 3x + 8$$

We then solve each of the inequalities, keeping in mind that the solution must satisfy both of them. Therefore, we have

$$\begin{aligned} 2x < x - 4 \quad \text{and} \quad x - 4 &\leq 3x + 8 \\ -2x &\leq 12 \\ x < -4 & \quad \quad \quad x \geq -6 \end{aligned}$$

We see that the solution is  $x < -4$  and  $x \geq -6$ , which can also be written as  $-6 \leq x < -4$ , or  $[-6, -4)$ . Either of these latter two forms is generally preferred since they are more concise and more easily interpreted. The solution checks and is shown in Fig. 17.11. ■



Fig. 17.11

**Practice Exercise**

- Solve the inequality  $-2 \leq 4x - 3 < 5$ .

**EXAMPLE 8** Inequality with three members—application

In emptying a wastewater tank, one pump can remove no more than 40 L/min. If it operates for 8.0 min and a second pump operates for 5.0 min, what must be the pumping rate of the second pump if 480 L are to be removed?

Let  $x$  = the pumping rate of the first pump and  $y$  = the pumping rate of the second pump. Since the first operates for 8.0 min and the second for 5.0 min to remove 480 L, we have

$$\begin{array}{rcl} \text{first} & \text{second} & \\ \text{pump} & \text{pump} & \text{total} \leftarrow \text{amounts pumped} \\ 8.0x & + & 5.0y = 480 \end{array}$$

Since we know that the first pump can remove no more than 40 L/min, which means that  $0 \leq x \leq 40$  L/min, we solve for  $x$ , then substitute in this inequality:

$$\begin{array}{rcl} x & = & 60 - 0.625y & \text{solve for } x \\ 0 & \leq & 60 - 0.625y \leq 40 & \text{substitute in inequality} \\ -60 & \leq & -0.625y \leq -20 & \text{subtract 60 from each member} \\ 96 & \geq & y \geq 32 & \text{divide each member by } -0.625 \\ 32 & \leq & y \leq 96 \text{ L/min} & \text{use } \leq \text{ symbol (optional step)} \\ & & [32 \text{ L/min, } 96 \text{ L/min}] & \end{array}$$



Fig. 17.12

This means that the second pump must be able to pump at least 32 L/min and no more than 96 L/min. See Fig. 17.12.

Although this was a three-member inequality combined with equalities, the solution was done in the same way as with a two-member inequality. ■

**EXERCISES 17.2**

In Exercises 1–4, make the given changes in the indicated examples of this section and then perform the indicated operations.

- In Example 2, change the 3 to 21 and then solve the resulting inequality.
- In Example 4, change the  $1/4$  to  $7/4$  and then solve and display the resulting inequality.
- In Example 6, change the  $+$  in the middle member to  $-$  and then solve the resulting inequality. Graph the solution.
- In Example 7, change the middle member to  $-x + 6$  and then solve the resulting inequality. Graph the solution.

In Exercises 5–28, solve the given inequalities. Graph each solution.

- $x - 3 > -4$
- $x + 2 \leq 6$
- $\frac{1}{2}x < 32$
- $-4t > 12$
- $3x - 5 \leq -11$
- $\frac{1}{3}x + 2 \geq 1$
- $6 - y > 8$
- $32 - 5x < -8$
- $\frac{4x - 5}{2} \leq x$
- $1.50 - 5.24x > 3.75 + 2.25x$
- $180 - 6(T + 12) > 14T + 285$
- $-2[x - (3 - 2x)] > \frac{1 - 5x}{3} + 2$

- $2.50(1.50 - 3.40x) < 3.84 - 8.45x$
- $(2x - 7)(x + 1) \leq 4 - x(1 - 2x)$
- $\frac{1}{3} - \frac{L}{2} < L + \frac{3}{2}$
- $\frac{x}{5} - 2 > \frac{2}{3}(x + 3)$
- $-1 < 2x + 1 < 3$
- $2 < 3R + 1 \leq 8$
- $-4 \leq 1 - x < -1$
- $0 \leq 3 - 2x \leq 6$
- $2x < x - 1 \leq 3x + 5$
- $x + 19 \leq 25 - x < 2x$
- $2s - 3 < s - 5 < 3s - 3$
- $0 < 1 - x \leq 3$  or  $-1 < 2x - 3 < 5$

In Exercises 29–46, solve the given problems by setting up and solving appropriate inequalities. Graph each solution. When necessary, round to 3 significant digits.

- Determine the values of  $x$  that are in the domain of the function  $f(x) = \sqrt{2x - 10}$ .
- Determine the values of  $x$  that are in the domain of the function  $f(x) = 1/\sqrt{3 - 0.5x}$ .
- Determine the values of  $x$  that are in the domain of the function  $f(x) = \log(2x + 10)$ .
- Determine the values of  $x$  that are in the domain of the function  $f(x) = \ln(7x + 4)$ .

33. For what values of  $k$  are the roots of the equation  $x^2 - kx + 9 = 0$  imaginary?
34. For what values of  $k$  are the roots of the equation  $2x^2 + 3x + k = 0$  real and unequal?
35. A contractor is considering two similar jobs, each of which is estimated to take  $n$  hours to complete. One pays \$350 plus \$15 per hour, and the other pays \$25 per hour. For what values of  $n$  will the contractor make more at the second position?
36. In designing plastic pipe, if the inner radius  $r$  is increased by 5.00 cm, and the inner cross-sectional area is increased by between  $125 \text{ cm}^2$  and  $175 \text{ cm}^2$ , what are the possible inner radii of the pipe?
37. The relation between the temperature in degrees Fahrenheit  $F$  and degrees Celsius  $C$  is  $9C = 5(F - 32)$ . What temperatures  $F$  correspond to temperatures between  $10^\circ\text{C}$  and  $20^\circ\text{C}$ ?
38. The voltage drop  $V$  (in volts) across a resistor is the product of the current  $i$  (in A) and the resistance  $R$  (in  $\Omega$ ). Find the possible voltage drops across a variable resistor  $R$  if the minimum and maximum resistances are  $1.6 \text{ k}\Omega$  and  $3.6 \text{ k}\Omega$ , respectively, and the current is constant at  $2.5 \text{ mA}$ .
39. A rectangular PV (photovoltaic) solar panel is designed to be  $1.42 \text{ m}$  long and supply  $130 \text{ W/m}^2$  of power. What must the width of the panel be in order to supply between  $100 \text{ W}$  and  $150 \text{ W}$ ?
40. A beam is supported at each end, as shown in Fig. 17.13. Analysing the forces leads to the equation  $F_1 = 13 - 3d$ . For what values of  $d$  is  $F_1$  more than  $6 \text{ N}$ ?

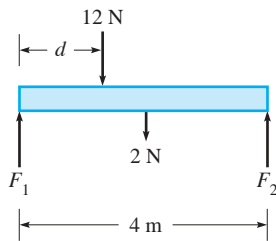


Fig. 17.13

41. The mass  $m$  (in g) of silver plate on a dish is increased by electroplating. The mass of silver on the plate is given by  $m = 125 + 15.0t$ , where  $t$  is the time (in h) of electroplating. For what values of  $t$  is  $m$  between  $131 \text{ g}$  and  $164 \text{ g}$ ?
42. For a ground temperature of  $T_0$  (in  $^\circ\text{C}$ ), the temperature  $T$  (in  $^\circ\text{C}$ ) at a height  $h$  (in m) above the ground is given approximately by  $T = T_0 - 0.010h$ . If the ground temperature is  $25^\circ\text{C}$ , for what heights is the temperature above  $10^\circ\text{C}$ ?
43. During a given rush hour, the numbers of vehicles shown in Fig. 17.14 go in the indicated directions in a one-way-street section of a city. By finding the possible values of  $x$  and the equation relating  $x$  and  $y$ , find the possible values of  $y$ .
44. The minimum legal speed on a certain highway is  $70 \text{ km/h}$ , and the maximum legal speed is  $110 \text{ km/h}$ . What legal distances can a motorist travel in  $4 \text{ h}$  on this highway without stopping?
45. The route of a rapid transit train is  $40 \text{ km}$  long, and the train makes five stops of equal length. If the train is actually moving for  $1 \text{ h}$  and each stop must be at least  $2 \text{ min}$ , what are the lengths of the stops if the train maintains an average speed of at least  $30 \text{ km/h}$ , including stop times?
46. An oil company plans to install eight storage tanks, each with a capacity of  $x$  litres, and five additional tanks, each with a capacity of  $y$  litres, such that the total capacity of all tanks is  $440\,000 \text{ L}$ . If capacity  $y$  will be at least  $40\,000 \text{ L}$ , what are the possible values of capacity  $x$ ?

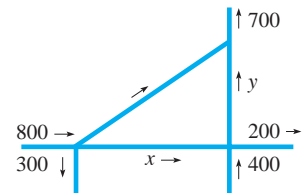


Fig. 17.14

Answers to Practice Exercises

1.  $x < 1/6$
2.  $1/4 \leq x < 2$

## 17.3 Solving Nonlinear Inequalities

Critical Values • Using Linear Factors • Polynomial and Rational Inequalities • Solving Inequalities Graphically

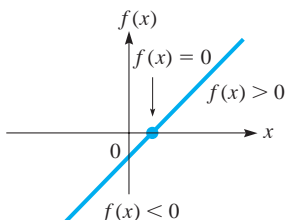


Fig. 17.15

In this section, we develop methods of solving inequalities with polynomials, rational expressions (expressions involving fractions), and nonalgebraic expressions. To develop the basic method for solving these types of inequalities, we now take another look at a linear inequality.

In Fig. 17.15, we see that all values of the linear function  $f(x) = ax + b$  ( $a \neq 0$ ) are positive on one side of the point at which  $f(x) = 0$ , and all values of  $f(x)$  are negative on the opposite side of the same point. This means that we can solve a linear inequality by expressing it with zero on the right and then finding the sign of the resulting function on either side of zero.

### EXAMPLE 1 Sign of a function with zero on the right

Solve the inequality  $2x - 5 > 1$ .

Finding the equivalent inequality with zero on the right, we have  $2x - 6 > 0$ . Setting the left member equal to zero, we have

$$2x - 6 = 0 \quad \text{for } x = 3$$

which means  $f(x) = 2x - 6$  has one sign for  $x < 3$ , and the other sign for  $x > 3$ . Testing values in these intervals, we find, for example, that

$$f(x) = -2 \quad \text{for } x = 2 \quad \text{and} \quad f(x) = +2 \quad \text{for } x = 4$$

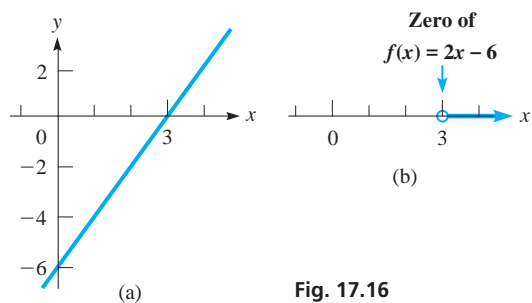


Fig. 17.16

Therefore, the solution to the original inequality is  $x > 3$ , or  $(3, \infty)$ . The solution in Fig. 17.16(b) corresponds to the positive values of  $f(x)$  in Fig. 17.16(a).

We could have solved this inequality by methods of the previous section, but the important idea here is to use the **sign of the function**, with zero on the right.

We can extend this method to solving inequalities with polynomials of higher degree. We first find the equivalent inequality with zero on the right and then *factor the function on the left into linear factors and any quadratic factors that lead to complex roots*. As all values of  $x$  are considered, each linear factor can change sign at the value for which it is zero. The quadratic factors do not change sign.

This method is especially useful in solving inequalities involving fractions. If a linear factor occurs in the numerator, the function is zero at the value of  $x$  for which the linear factor is zero. If such a factor appears in the denominator, the function is undefined where the factor is zero. *The values of  $x$  for which a function is zero or undefined are called the critical values of the function*. As all negative and positive values of  $x$  are considered (starting with negative numbers of large absolute value, proceeding through zero, and ending with large positive numbers), a function can change sign only at a critical value.

We now outline the method of solving an inequality by using the critical values of the function. Several examples of the method follow.

### Critical Values

#### Using Critical Values to Solve an Inequality

1. Determine the equivalent inequality with zero on the right.
2. Find all linear factors of the function.
3. To find the critical values, set each linear factor equal to zero and solve for  $x$ .
4. Determine the sign of the function to the left of the leftmost critical value, between critical values, and to the right of the rightmost critical value.
5. Those intervals in which the function has the proper sign satisfy the inequality.

#### EXAMPLE 2 Solving a quadratic inequality

Solve the inequality  $x^2 - 3 > 2x$ .

We first find the equivalent inequality with zero on the right. Therefore, we have  $x^2 - 2x - 3 > 0$ . We then factor the left member and have

$$(x + 1)(x - 3) > 0$$

Setting each factor equal to zero, we find the left critical value is  $-1$  and the right critical value is  $3$ . All values of  $x$  to the left of  $-1$  give the same sign for the function. All values between  $-1$  and  $3$  give the function the same sign. All values of  $x$  to the right of  $3$  give the same sign to the function. Therefore, **we must determine the sign of  $f(x)$  for each of the intervals  $x < -1$ ,  $-1 < x < 3$ , and  $x > 3$** .

For  $x < -1$ , both factors are negative, which means their product is positive, or  $(x + 1)(x - 3) > 0$ . For  $-1 < x < 3$ , the left factor is positive, and the right factor is negative, which means their product is negative, or  $(x + 1)(x - 3) < 0$ . For  $x > 3$ , both factors are positive, which means their product is positive, or  $(x + 1)(x - 3) > 0$ .

Summarizing these results for  $f(x) = (x + 1)(x - 3)$ , we have

$$\text{If } x < -1, f(x) > 0; \text{ if } -1 < x < 3, f(x) < 0; \text{ if } x > 3, f(x) > 0$$

Therefore, the solution to the inequality is  $x < -1$  or  $x > 3$  or, in interval notation,  $(-\infty, -1) \cup (3, \infty)$ .

The solution that is shown in Fig. 17.17(b) corresponds to the positive values of the function  $f(x) = x^2 - 2x - 3$ , shown in Fig. 17.17(a).

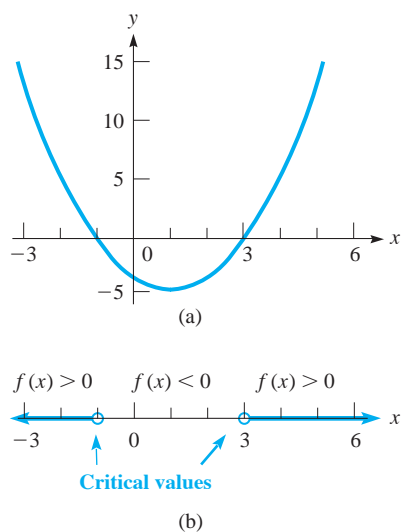


Fig. 17.17

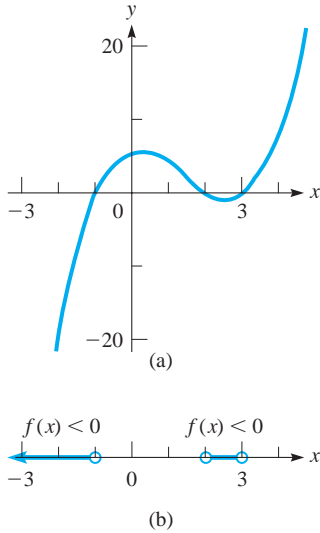


Fig. 17.18

**EXAMPLE 3 Solving a cubic inequality**

Solve the inequality  $x^3 - 4x^2 + x + 6 < 0$ .

By methods developed in Chapter 15, we factor the function on the left and obtain  $(x + 1)(x - 2)(x - 3) < 0$ . The critical values are  $-1, 2, 3$ . We wish to determine the sign of the left member for the intervals  $x < -1$ ,  $-1 < x < 2$ ,  $2 < x < 3$ , and  $x > 3$ . The following table shows each interval, the sign of each factor in each interval, and the resulting sign of the function.

$$f(x) = (x + 1)(x - 2)(x - 3)$$

Interval	$(x + 1)(x - 2)(x - 3)$			Sign of $f(x)$
$x < -1$ $(-\infty, -1)$	-	-	-	-
$-1 < x < 2$ $(-1, 2)$	+	-	-	+
$2 < x < 3$ $(2, 3)$	+	+	-	-
$x > 3$ $(3, \infty)$	+	+	+	+

Since we want values for  $f(x) < 0$ , the solution is  $(-\infty, -1)$  or  $(2, 3)$ , that is,  $x < -1$  or  $2 < x < 3$ . The solution shown in Fig. 17.18(b) corresponds to the values of  $f(x) < 0$  in Fig. 17.18(a).

The following example illustrates an applied situation that involves the solution of an inequality.

**EXAMPLE 4 Solving a quadratic inequality—application**

The force  $F$  (in N) acting on a cam varies according to the time  $t$  (in s), and it is given by the function  $F = 2t^2 - 12t + 20$ . For what values of  $t$ ,  $0 \leq t \leq 6$  s, is the force at least 4 N?

For a force of at least 4 N, we know that  $F \geq 4$  N, or  $2t^2 - 12t + 20 \geq 4$ . This means we are to solve the inequality  $2t^2 - 12t + 16 \geq 0$ , and the solution is as follows:

$$\begin{aligned} 2t^2 - 12t + 16 &\geq 0 \\ t^2 - 6t + 8 &\geq 0 \\ (t - 2)(t - 4) &\geq 0 \end{aligned}$$

The critical values are  $t = 2$  and  $t = 4$ , which lead to the following table:

Interval	$(t - 2)(t - 4)$		Sign of $(t - 2)(t - 4)$
$0 \leq t < 2$ $[0, 2)$	-	-	+
$2 < t < 4$ $(2, 4)$	+	-	-
$4 < t \leq 6$ $(4, 6]$	+	+	+

We see that the values of  $t$  that satisfy the *greater than* part of the problem are  $[0, 2)$  and  $(4, 6]$ . Since we know that  $(t - 2)(t - 4) = 0$  for  $t = 2$  and  $t = 4$ , the solution is  $[0, 2]$  or  $[4, 6]$ , that is,

$$0 \leq t \leq 2 \text{ s or } 4 \text{ s} \leq t \leq 6 \text{ s}$$

The graph of  $f(t) = 2t^2 - 12t + 16$  is shown in Fig. 17.19(a). The solution, shown in Fig. 17.19(b), corresponds to the values of  $f(t)$  that are zero or positive or for which  $F \geq 4$  N.

**Practice Exercise**

- Solve the inequality  $x^2 - x - 42 < 0$ .

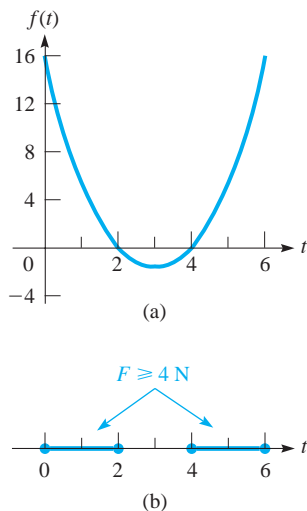


Fig. 17.19

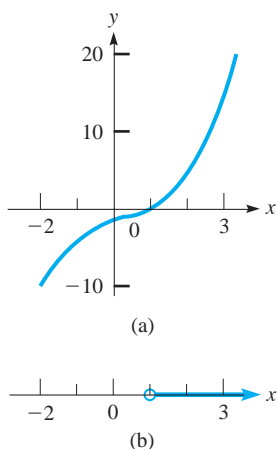


Fig. 17.20

We note here that if the cam rotates in 6-s intervals, the force on the cam is periodic, varying from 2 N to 20 N. ■

### EXAMPLE 5 Solving a cubic inequality

Solve the inequality  $x^3 - x^2 + x - 1 > 0$ .

In order to factor this function, we can use the methods of Chapter 15, or we might note that it is factorable by grouping (see Section 6.2). Either method leads to the inequality

$$(x^2 + 1)(x - 1) > 0$$

In this case, we have a quadratic factor  $x^2 + 1$  that leads to imaginary roots ( $j$  and  $-j$ ) and is never negative. This means we have only one linear factor and therefore only one critical value.

Setting  $x - 1 = 0$ , we get the critical value  $x = 1$ . Since  $x - 1$  is positive for  $x > 1$  and negative for  $x < 1$ , the solution to the inequality is  $x > 1$ , or  $(1, \infty)$ .

The solution is shown in Fig. 17.20(b), and we see that this solution corresponds to the positive values of  $f(x) = x^3 - x^2 + x - 1$  shown in Fig. 17.20(a). ■

### EXAMPLE 6 Solving a rational inequality

Find the values of  $x$  for which  $\sqrt{\frac{x-3}{x+4}}$  represents a real number.

For the expression to represent a real number, the fraction under the radical must be greater than or equal to zero. This means we must solve the inequality

$$\frac{x-3}{x+4} \geq 0$$

The critical values are found from the factors that are in the numerator or in the denominator. Thus, the critical values are  $-4$  and  $3$ . Considering now the *greater than* part of the  $\geq$  sign, we set up the following table:

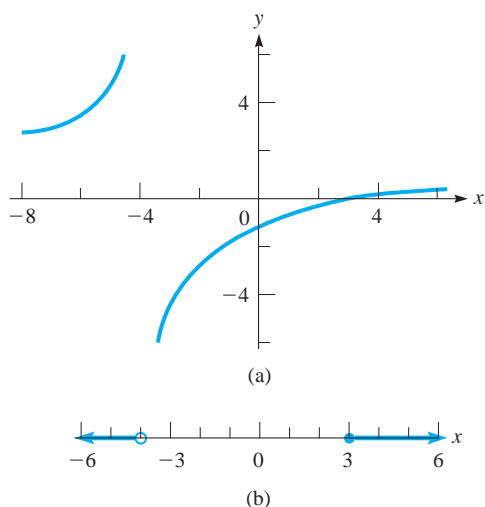


Fig. 17.21

Interval	$\frac{x-3}{x+4}$	Sign of $\frac{x-3}{x+4}$
$x < -4$ $(-\infty, -4)$	$\frac{-}{-}$	+
$-4 < x < 3$ $(-4, 3)$	$\frac{-}{+}$	-
$x > 3$ $(3, \infty)$	$\frac{+}{+}$	+

Thus, the values that satisfy the *greater than* part of the problem are  $(-\infty, -4)$  or  $(3, \infty)$ . Now, considering the equality part of the  $\geq$  sign, we note that  $x = 3$  is valid, for the fraction is zero. However,

**if  $x = -4$ , we have division by zero, and therefore  $x$  may not equal  $-4$ .**

Therefore, the solution of the inequality is  $(-\infty, -4)$  or  $[3, \infty)$ , that is,  $x < -4$  or  $x \geq 3$ . This means these are the values for which the original expression represents a real number. The graph of  $f(x) = \frac{x-3}{x+4}$  is shown in Fig. 17.21(a), and the graph of the solution is shown in Fig. 17.21(b). ■

#### Practice Exercise

2. Solve the inequality  $\frac{x+3}{2x-5} \geq 0$ .



**COMMON ERROR**

Note that to solve a rational inequality, we **do not** start by multiplying by the least common denominator, as we did with equations involving fractions. Since we do not know the value of  $x$ , we can't multiply both sides by anything whose sign depends on  $x$ . When we do not know if the denominator is positive or negative, we do not know if we need to change the sense of the inequality or not. It is for this reason that we solve rational inequalities using critical values.

Note also that to avoid division by zero, the solution **never** includes the critical value that makes the denominator zero.

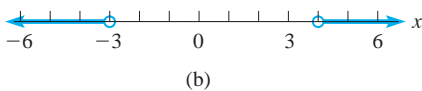
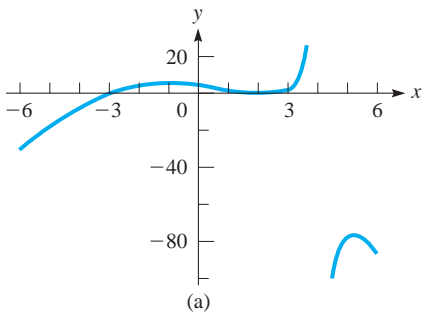


Fig. 17.22

**EXAMPLE 7 Solving a rational inequality**

Solve the inequality  $\frac{(x - 2)^2(x + 3)}{4 - x} < 0$ .

The critical values are  $-3, 2$ , and  $4$ . Thus, we have the following table:

Interval	$\frac{(x - 2)^2(x + 3)}{4 - x}$	Sign of $\frac{(x - 2)^2(x + 3)}{4 - x}$
$x < -3$ $(-\infty, -3)$	$\frac{+ -}{+}$	-
$-3 < x < 2$ $(-3, 2)$	$\frac{+ +}{+}$	+
$2 < x < 4$ $(2, 4)$	$\frac{+ +}{+}$	+
$x > 4$ $(4, \infty)$	$\frac{+ +}{-}$	-

Thus, the solution is  $(-\infty, -3)$  or  $(4, \infty)$ , or  $x < -3$  or  $x > 4$ . The graph of the function is shown in Fig. 17.22(a), and the graph of the solution is shown in Fig. 17.22(b). ■

**LEARNING TIP**

When a rational inequality involves more than one nonzero term, the first step is to get a zero on the right and combine terms using the least common denominator so as to obtain a single rational expression. We can then proceed to use critical values as before, without multiplying both sides by the least common denominator. Example 8 illustrates the procedure.

**EXAMPLE 8 Solving a rational inequality**

Solve the inequality  $\frac{x + 3}{x - 1} > 2$ .

We first subtract 2 from each member and combine terms on the left. We have

$$\begin{aligned} \frac{x + 3}{x - 1} - 2 &> 0 && \text{subtract 2 from both members} \\ \frac{x + 3 - 2(x - 1)}{x - 1} &> 0 && \text{combine over the common denominator} \\ \frac{5 - x}{x - 1} &> 0 && \text{this is the form to use} \end{aligned}$$

The critical values are 1 and 5, and we have the following table of signs:

Interval	$(5 - x)/(x - 1)$	Sign of $(5 - x)/(x - 1)$
$x < 1$ $(-\infty, 1)$	$+ / -$	-
$1 < x < 5$ $(1, 5)$	$+ / +$	+
$x > 5$ $(5, \infty)$	$- / +$	-

Thus, the solution is  $(1, 5)$ , or  $1 < x < 5$ . The graph of  $f(x) = (5 - x)/(x - 1)$  is shown in Fig. 17.23(a), and the graph of the solution is shown in Fig. 17.23(b). ■

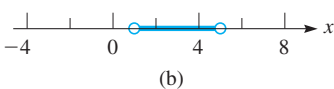
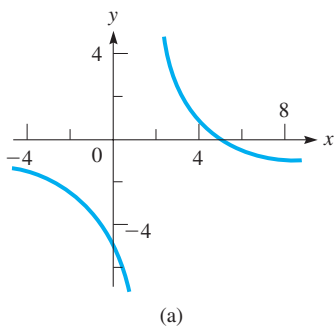


Fig. 17.23

## SOLVING INEQUALITIES GRAPHICALLY

We can get an approximate solution of an inequality from the graph of a function, including functions that are not factorable or not algebraic. The method follows several of the earlier examples. First, write the equivalent inequality with zero on the right and then graph this function. Those values of  $x$  corresponding to the proper values of  $y$  (either above or below the  $x$ -axis) are those that satisfy the inequality.

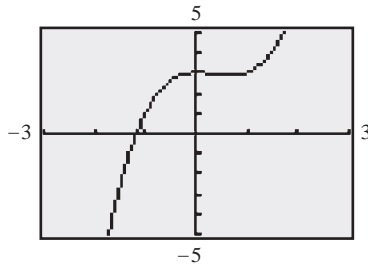


Fig. 17.24

## EXAMPLE 9 Solving an inequality graphically

Use a graphing calculator to solve the inequality  $x^3 > x^2 - 3$ .

Finding the equivalent inequality with zero on the right, we have  $x^3 - x^2 + 3 > 0$ . On the calculator, we then let  $y_1 = x^3 - x^2 + 3$  and graph this function. This gives us the calculator display shown in Fig. 17.24.

We can see that the curve crosses the  $x$ -axis only once, between  $x = -2$  and  $x = -1$ . Using the *zero* feature, we find that this value is approximately  $x = -1.17$ . Since we want positive values of  $x$  that correspond to positive values of  $y$ , we see that the solution is  $x > -1.17$ , or  $(-1.17, \infty)$ . ■

## EXERCISES 17.3

In Exercises 1–4, make the given changes in the indicated examples of this section and then solve the resulting inequalities.

- In Example 2, change the  $-$  sign before the 3 to  $+$  and change  $2x$  to  $4x$ , then solve the resulting inequality, and graph the solution.
- In Example 5, change both  $-$  signs to  $+$ , then solve the resulting inequality, and graph the solution.
- In Example 7, change the exponent on  $(x - 2)$  from 2 to 3, then solve the resulting inequality, and graph the solution.
- In Example 8, on the right, change the 2 to 3, then solve the resulting inequality, and graph the solution.

In Exercises 5–32, solve the given inequalities. Graph each solution.

- $x^2 - 16 < 0$
- $2x^2 \leq 4x$
- $2x^2 - 12 \leq -5x$
- $x^2 + 4x \leq -4$
- $R^2 + 4 > 0$
- $x^3 + x^2 - 2x > 0$
- $s^3 + 2s^2 - s \geq 2$
- $3x^2 + 5x \geq 2$
- $\frac{2x - 3}{x + 6} \leq 0$
- $\frac{x^2 - 6x - 7}{x + 5} > 0$
- $\frac{x}{x + 1} > 1$
- $\frac{T - 8}{3 - T} \leq 0$
- $\frac{6 - x}{3 - x - 4x^2} \geq 0$
- $x^2 + 3x \geq 0$
- $x^2 - 4x > 21$
- $9t^2 + 6t > -1$
- $6x^2 + 1 < 5x$
- $x^4 + 2 < 1$
- $x^3 - 2x^2 + x \geq 0$
- $n^4 - 2n^3 + 8n + 12 \leq 7n^2$
- $12x^2 + x > 1$
- $\frac{x + 15}{x - 9} > 0$
- $\frac{(x - 2)^2(5 - x)}{(4 - x)^3} \leq 0$
- $\frac{2p}{p - 1} > 3$
- $\frac{3x + 1}{x + 3} \geq 0$
- $\frac{4 - x}{3 + 2x - x^2} > 0$

$$31. \frac{x^4(9 - x)(x - 5)(2 - x)}{(4 - x)^5} > 0 \quad 32. \frac{2}{x - 3} < 4$$

In Exercises 33–36, determine the values of  $x$  for which the radicals represent real numbers.

$$33. \sqrt{(x - 1)(x + 2)} \quad 34. \sqrt{x^2 - 3x}$$

$$35. \sqrt{-x - x^2} \quad 36. \sqrt{\frac{x^3 + 6x^2 + 8x}{3 - x}}$$

In Exercises 37–44, solve the given inequalities graphically by using a graphing calculator. Approximate the critical values to the nearest 0.01. See Example 9.

- $x^3 - x > 2$
- $0.5x^3 < 3 - 2x^2$
- $x^4 < x^2 - 2x - 1$
- $3x^4 + x + 1 > 5x^2$
- $2^x > x + 2$
- $\log x < 1 - 2x^2$
- $\sin x < 0.1x^2 - 1$
- $4 \cos 2x > 2x - 3$

In Exercises 45–52, use inequalities to solve the given problems. Where necessary, approximate the critical values to the nearest 0.01.

- Is  $x^2 > x$  for all  $x$ ? Explain.
- Is  $x > 1/x$  for all  $x$ ? Explain.
- Find an inequality of the form  $ax^2 + bx + c < 0$  with  $a > 0$  for which the solution is  $-1 < x < 4$ .
- Find an inequality of the form  $ax^3 + bx < 0$  with  $a > 0$  for which the solution is  $x < -1$  or  $0 < x < 1$ .
- Algebraically find the values of  $x$  for which  $2^{x+2} > 3^{2x-3}$ .
- Graphically find the values of  $x$  for which  $2 \log_2 x < \log_3(x + 1)$ .
- For what values of real numbers  $a$  and  $b$  does the inequality  $(x - a)(x - b) < 0$  have real solutions?
- Algebraically find the intervals for which  $f(x) = 2x^4 - 5x^3 + 3x^2$  is positive and those for which it is negative. Using only this information, draw a rough sketch of the graph of the function.

In Exercises 53–64, answer the given questions by solving the appropriate inequalities.

53. The electric power  $P$  (in W) delivered to part of a circuit is given by  $P = 6i - 4i^2$ , where  $i$  is the current (in A). For what positive values of  $i$  is the power greater than 2 W?
54. The mass  $m$  (in Mg) of fuel in a rocket after launch is  $m = 2000 - t^2 - 140t$ , where  $t$  is the time (in min). During what period of time is the mass of fuel greater than 500 Mg?
55. The weekly sales  $S$  (in thousands of units) of a certain product  $t$  weeks after it is introduced to the market are given by  $S = 200t / (t^2 + 64)$ . When will sales be at least 10 000 units per week?
56. The object distance  $p$  (in cm) and image distance  $q$  (in cm) for a camera of focal length 3.00 cm is given by  $p = 3.00q / (q - 3.00)$ . For what values of  $q$  is  $p > 12.0$  cm?
57. The total capacitance  $C$  of capacitors  $C_1$  and  $C_2$  in series is  $C^{-1} = C_1^{-1} + C_2^{-1}$ . If  $C_2 = 4.00 \mu\text{F}$ , find  $C_1$  if  $C > 1.00 \mu\text{F}$ .
58. A rectangular field is to be enclosed by a fence and divided down the middle by another fence. The middle fence costs \$4/m and the other fence costs \$8/m. If the area of the field is to be 8000 m<sup>2</sup>, and the cost of the fence cannot exceed \$4000, what are the possible dimensions of the field?
59. The weight  $w$  (in N) of an object  $h$  metres above the surface of earth is  $w = r^2 w_0 / (r + h)^2$ , where  $r$  is the radius of earth and  $w_0$  is the weight of the object at sea level. Given that  $r = 6380$  km, if an object weighs 200 N at sea level, for what altitudes is its weight less than 100 N?
60. One type of machine part costs \$10 each, and a second type costs \$20 each. How many of the second type can be purchased if at least 30 of the first type are purchased and a total of \$1000 is spent?
61. The length of a rectangular microprocessor chip is 2.0 mm more than its width. If its area is less than 35 mm<sup>2</sup>, what values are possible for the width if it must be at least 3.0 mm?
62. A laser source is 2.0 cm from the nearest point  $P$  on a flat mirror, and the laser beam is directed at a point  $Q$  that is on the mirror and is  $x$  cm from  $P$ . The beam is then reflected to the receiver, which is  $x$  cm from  $Q$ . What is  $x$  if the total length of the beam is greater than 6.5 cm? See Fig. 17.25.

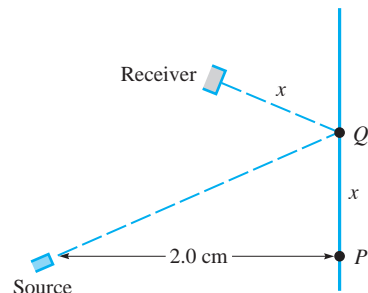


Fig. 17.25 Source

63. A plane takes off from Winnipeg and flies due east at 620 km/h. At the same time, a second plane takes off from the surface of Lake Winnipeg 310 km due north of Winnipeg and flies due north at 560 km/h. For how many hours are the planes less than 1000 km apart?
64. An open box (no top) is formed from a piece of cardboard 8.00 cm square by cutting equal squares from the corners, turning up the resulting sides, and taping the edges together. Find the edges of the squares that are cut out in order that the volume of the box is greater than 32.0 cm<sup>3</sup>. See Fig. 17.26.

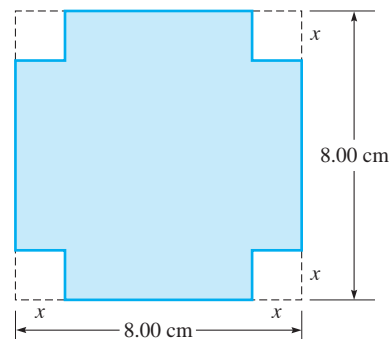


Fig. 17.26

### Answers to Practice Exercises

1.  $-6 < x < 7$ , or  $(-6, 7)$       2.  $x \leq -3$  or  $x > 5/2$ ,  
or  $(-\infty, -3]$  or  $(\frac{5}{2}, \infty)$

## 17.4 Inequalities Involving Absolute Values

### Absolute Value Greater Than Given Value • Absolute Value Less Than Given Value

Inequalities involving absolute values are often useful in later topics in mathematics such as calculus and in applications such as the accuracy of measurements. In this section, we show the meaning of such inequalities and how they are solved.

If we wish to write the inequality  $|x| > 1$  without absolute-value signs, we must note that we are considering values of  $x$  that are *numerically* larger than 1. Thus, we may write this inequality in the equivalent form  $x < -1$  or  $x > 1$ . We now note that *the original inequality, with an absolute-value sign, can be written in terms of two equivalent inequalities, neither involving absolute values*. If we are asked to write the inequality  $|x| < 1$  without absolute-value signs, we write  $-1 < x < 1$ , since we are considering values of  $x$  numerically less than 1.

Following reasoning similar to this, whenever absolute values are involved in inequalities, the following two relations allow us to write equivalent inequalities without absolute values. For  $n > 0$ ,

$$\text{If } |f(x)| > n, \text{ then } f(x) < -n \text{ or } f(x) > n. \quad (17.1)$$

$$\text{If } |f(x)| < n, \text{ then } -n < f(x) < n. \quad (17.2)$$

**EXAMPLE 1** Absolute value less than a number

Solve the inequality  $|x - 3| < 2$ .

Here, we want values of  $x$  such that  $x - 3$  is numerically smaller than 2, or the values of  $x$  within 2 units of  $x = 3$ . These are given by the inequality  $1 < x < 5$ . Now, using Eq. (17.2), we have

$$-2 < x - 3 < 2$$

By adding 3 to all three members of this inequality, we have

$$1 < x < 5$$

or  $(1, 5)$ , which is the proper interval. See Fig. 17.27. ■

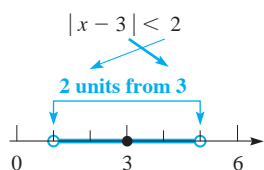


Fig. 17.27

**EXAMPLE 2** Absolute value greater than a number

Solve the inequality  $|2x - 1| > 5$ .

By using Eq. (17.1), we have

$$2x - 1 < -5 \quad \text{or} \quad 2x - 1 > 5$$

Completing the solution, we have

$$\begin{aligned} 2x &< -4 \quad \text{or} \quad 2x > 6 && \text{add 1 to each member} \\ x &< -2 \quad \quad \quad x > 3 && \text{divide each member by 2} \\ (-\infty, -2) & \quad \quad (3, \infty) \end{aligned}$$

This means that the given inequality is satisfied for  $x < -2$  or for  $x > 3$ , or in  $(-\infty, -2)$  or  $(3, \infty)$ . We must be very careful to remember that *we cannot write this as*  $3 < x < -2$ . The solution is shown in Fig. 17.28.

The meaning of this inequality is that the numerical value of  $2x - 1$  is greater than 5. By considering values in these intervals, we can see that this is true for values of  $x$  less than  $-2$  or greater than 3. ■

**EXAMPLE 3** Absolute value greater than or equal to a number

Solve the inequality  $2\left|\frac{2x}{3} + 1\right| \geq 4$ .

The solution is as follows:

$$\begin{aligned} 2\left|\frac{2x}{3} + 1\right| &\geq 4 && \text{original inequality} \\ \left|\frac{2x}{3} + 1\right| &\geq 2 && \text{divide each member by 2} \\ \frac{2x}{3} + 1 &\leq -2 \quad \text{or} \quad \frac{2x}{3} + 1 \geq 2 && \text{using Eq. (17.1)} \\ 2x + 3 &\leq -6 \quad \quad \quad 2x + 3 \geq 6 \\ 2x &\leq -9 \quad \quad \quad 2x \geq 6 \\ x &\leq -\frac{9}{2} \quad \quad \quad x \geq \frac{3}{2} && \text{solution} \\ \left(-\infty, -\frac{9}{2}\right] & \quad \quad \quad \left[\frac{3}{2}, \infty\right) \end{aligned}$$

This solution is shown in Fig. 17.29. Note that the sign of equality does not change the method of solution. It simply indicates that  $-\frac{9}{2}$  and  $\frac{3}{2}$  are included in the solution. ■

**Practice Exercise**

1. Solve the inequality  $|2x - 9| > 3$ .

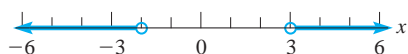


Fig. 17.28

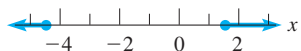


Fig. 17.29

**EXAMPLE 4** Absolute value less than a numberSolve the inequality  $|3 - 2x| < 3$ .

We have the following solution:

$$\begin{array}{ll}
 |3 - 2x| < 3 & \text{original inequality} \\
 -3 < 3 - 2x < 3 & \text{using Eq. (17.2)} \\
 -6 < -2x < 0 & \\
 3 > x > 0 & \text{divide by } -2 \text{ and reverse signs of inequality} \\
 0 < x < 3 & \text{solution} \\
 (0, 3) & 
 \end{array}$$



Fig. 17.30

The meaning of the inequality is that the numerical value of  $3 - 2x$  is less than 3. This is true for values of  $x$  between 0 and 3. The solution is shown in Fig. 17.30. ■

**EXAMPLE 5** Absolute value—application

A technician measures an electric current and reports that it is 0.036 A with a possible error of  $\pm 0.002$  A. Write this result for the current  $i$ , using an inequality with absolute values.

The statement of the problem tells us that the current is no less than 0.034 A and no more than 0.038 A. Another way of stating this is that the numerical difference between the true value of  $i$  (unknown exactly) and the measured value, 0.036 A, is less than or equal to 0.002 A. Using an absolute-value inequality, this is written as

$$|i - 0.036| \leq 0.002$$

where values are in amperes.

We can see that this inequality is correct by using (Eq. 17.2):

$$\begin{array}{ll}
 -0.002 \leq i - 0.036 \leq 0.002 & \\
 0.034 \leq i \leq 0.038 & \text{add 0.036 to each member}
 \end{array}$$

This verifies that  $i$  should not be less than 0.034 A or more than 0.038 A. The solution is shown in Fig. 17.31. ■

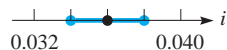


Fig. 17.31

**Practice Exercise**

2. Solve the inequality  $|4 - x| \leq 2$ .

**EXERCISES 17.4**

In Exercises 1 and 2, make the given changes in the indicated examples of this section and then solve the resulting inequalities.

- In Example 2, change the  $>$  to  $<$ , solve the resulting inequality, and graph the solution.
- In Example 4, change the  $<$  to  $>$ , solve the resulting inequality, and graph the solution.

In Exercises 3–24, solve the given inequalities. Graph each solution.

- $|x - 4| < 1$
- $|x + 1| < 3$
- $|5x + 4| > 6$
- $\left|\frac{1}{2}N - 1\right| > 1$
- $1 + |6x - 5| \leq 5$
- $|30 - 42x| \leq 0$
- $|3 - 4x| > 3$
- $3 + |3x + 1| \geq 5$
- $\left|\frac{t + 1}{5}\right| < 5$
- $\left|\frac{2x - 9}{4}\right| < 1$
- $|20x + 85| \leq 43$
- $|2.6x - 9.1| > 10.4$
- $2|x - 24| > 84$
- $3|4 - 3x| \leq 10$

- $8 + 3|3 - 2x| < 11$
- $5 - 4|1 - 7x| > 13$
- $4|2 - 5x| \geq 6$
- $2.5|7.1 - 2.0x| \leq 6.5$
- $\left|\frac{3R}{5} + 1\right| < 8$
- $\left|\frac{4x}{3} - 5\right| \geq 7$
- $\left|6.5 - \frac{x}{2}\right| \geq 2.3$
- $\left|\frac{2w + 5}{w + 1}\right| \geq 1$

In Exercises 25–28, solve the given quadratic inequalities.

- $|x^2 + x - 4| > 2$   
(After using Eq. 17.1, you will have two inequalities. The solution includes the values of  $x$  that satisfy *either* of the inequalities.)
- $|x^2 + 3x - 1| > 3$  (See Exercise 25.)
- $|x^2 + x - 4| < 2$   
(Use Eq. 17.2, then treat the resulting inequality as two inequalities of the form  $f(x) > -n$  and  $f(x) < n$ . The solution includes the values of  $x$  that satisfy *both* of the inequalities.)
- $|x^2 + 3x - 1| < 3$  (See Exercise 27.)

In Exercises 29–34, solve the given problems.

29. Solve for  $x$  if  $|x| < a$  and  $a \leq 0$ . Explain.
30. Solve for  $x$  if  $|x - 1| < 4$  and  $x \geq 0$ .
31. Solve for  $x$ :  $a - |bx| < c$  given that  $a - c > 0$ .
32. Solve for  $x$ :  $1 < |x - 2| < 3$ .
33. The thickness  $t$  (in km) of earth's crust varies and can be described as  $|t - 27| \leq 23$ . What are the minimum and maximum values of the thickness of earth's crust?
34. The temperature  $T$  (in  $^{\circ}\text{C}$ ) at which a certain integrated-circuit computer chip is designed to operate is given by the inequality  $|T - 110| \leq 10$ . What are the minimum and maximum temperatures of this range?

In Exercises 35–41, use inequalities involving absolute values to solve the given problems.

35. The production  $p$  (in barrels) of oil at a refinery is estimated at  $2\,000\,000 \pm 200\,000$ . Express  $p$  using an inequality with absolute values and describe the production in a verbal statement.
36. The Mach number  $M$  of a moving object is the ratio of its velocity  $v$  to the velocity of sound  $v_s$ , and  $v_s$  varies with temperature. A jet travelling at 1650 km/h changes its altitude from 500 m to 5500 m. At 500 m (with the temperature at  $27^{\circ}\text{C}$ ),  $v_s = 1250$  km/h, and at 5500 m ( $-3^{\circ}\text{C}$ ),  $v_s = 1180$  km/h. Express the range of  $M$ , using an inequality with absolute values.
37. The diameter  $d$  of a certain type of tubing is 3.675 cm with a tolerance of 0.002 cm. Express this as an inequality with absolute values.
38. The moon is at a maximum distance of  $4.07 \times 10^5$  km from earth and at a minimum distance of  $3.57 \times 10^5$  km. Express this as an inequality with absolute values.

39. The Peterborough Lift Lock in Peterborough, Ontario (the highest hydraulic lift lock in the world), raises and lowers boats as the mass  $m$  of its two identical caissons changes from 1542 t to 1672.6 t. Express the range of  $m$  using an inequality with absolute values.
40. The voltage  $V$  in a certain circuit is given by  $V = 6.0 - 200i$ , where  $i$  is the current (in A). For what values of the current is the absolute value of the voltage less than 2.0 V?
41. A rocket is fired from a plane flying horizontally at 3000 m. The height  $h$  (in m) of the rocket above the plane is given by  $h = 190t - 4.9t^2$ , where  $t$  is the time (in s) of flight of the rocket. When is the rocket more than 1300 m above or below the plane? Round to 3 significant digits. See Fig. 17.32.

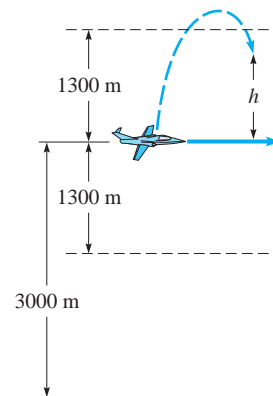


Fig. 17.32

#### Answers to Practice Exercises

1.  $x < 3$  or  $x > 6$ , or  $(-\infty, 3)$  or  $(6, \infty)$
2.  $2 \leq x \leq 6$ , or  $[2, 6]$

## 17.5 Graphical Solution of Inequalities with Two Variables

### Points Above and Below a Curve • Use of a Dashed Curve or a Solid Curve

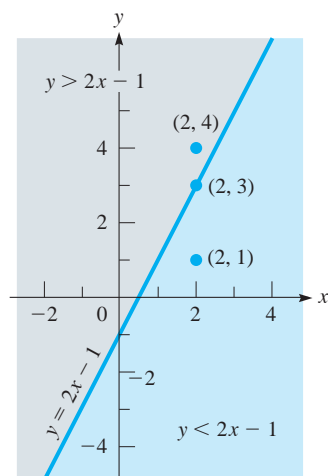


Fig. 17.33

To this point, we have considered inequalities with one variable and certain methods of solving them. We may also graphically solve inequalities involving two variables, such as  $x$  and  $y$ . In this section, we consider the solution of such inequalities.

Let us consider the function  $y = f(x)$ . We know that the coordinates of points on the graph satisfy the equation  $y = f(x)$ . In addition, the graph divides the plane into two regions. Points in the region above the graph satisfy the inequality  $y > f(x)$ , whereas points in the region below the graph satisfy the inequality  $y < f(x)$ . Consider the following example.

#### EXAMPLE 1 Checking points above and below a line

Consider the linear function  $y = 2x - 1$ , the graph of which is shown in Fig. 17.33. This equation is satisfied for points on the line. For example, the point  $(2, 3)$  is on the line, and we have  $3 = 2(2) - 1 = 3$ . Therefore, for points on the line, we have  $y = 2x - 1$ , or  $y - 2x + 1 = 0$ .

The point  $(2, 4)$  is above the line, since we have  $4 > 2(2) - 1$ , or  $4 > 3$ . Therefore, for points above the line, we have  $y > 2x - 1$ , or  $y - 2x + 1 > 0$ . In the same way, for points below the line,  $y < 2x - 1$  or  $y - 2x + 1 < 0$ . We note this is true for the point  $(2, 1)$ , since  $1 < 2(2) - 1$ , or  $1 < 3$ .

The line for which  $y = 2x - 1$  and the regions for which  $y > 2x - 1$  and for which  $y < 2x - 1$  are shown in Fig. 17.33.

Summarizing,

$$\begin{aligned} y > 2x - 1 & \quad \text{for points above the line} \\ y = 2x - 1 & \quad \text{for points on the line} \\ y < 2x - 1 & \quad \text{for points below the line} \end{aligned}$$

The illustration in Example 1 leads us to the graphical method of indicating the points that satisfy an inequality with two variables.

### Procedure for Solving an Inequality with Two Variables

1. Solve the inequality for  $y$  by isolating  $y$ .
2. Consider the equation  $y = f(x)$  obtained by changing the inequality sign to an equality sign. Determine the graph of  $y = f(x)$  and draw it as a solid or dashed curve, as follows:
  - a. If the original inequality is strict ( $>$  or  $<$ ), draw  $f(x)$  as a dashed curve. This indicates that the graph of  $f(x)$  is **not** part of the solution.
  - b. If the original inequality is not strict ( $\geq$  or  $\leq$ ), draw  $f(x)$  as a solid curve. This indicates that the graph of  $f(x)$  is part of the solution.
3. The solution to the inequality consists of all points in an entire region of the plane, which is shaded in as follows:
  - a. Shade in the region **above the curve** if the inequality is of the form  $y > f(x)$  or  $y \geq f(x)$ .
  - b. Shade in the region **below the curve** if the inequality is of the form  $y < f(x)$  or  $y \leq f(x)$ .

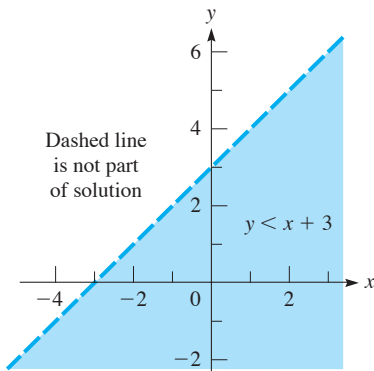


Fig. 17.34

### EXAMPLE 2 Sketching an inequality of the form $y < f(x)$

Draw a sketch of the graph of the inequality  $y < x + 3$ .

The variable  $y$  is already isolated, so the first step is to draw the function  $y = x + 3$ . Since the original inequality is strict, the function is represented by a dashed line, as shown in Fig. 17.34. The solution to the inequality is then shaded in as the region *below* the line.

### EXAMPLE 3 Sketching an inequality—application

After a snowstorm, it is estimated that it will take 30 min to plow each kilometre of Route 15 and 45 min to plow each kilometre of Route 80. If no more than 60 plowing-hours are available, what combinations of Route 15 and Route 80 can be plowed?

Let  $x$  = kilometres of Route 15 that can be plowed and  $y$  = kilometres of Route 80 that can be plowed. The time to plow along each route is the product of the time for each kilometre and the number of kilometres to be plowed. This gives us

$$\begin{aligned} & \begin{array}{ccc} \text{time to plow Rt. 15} & \text{time to plow Rt. 80} & \text{max. available time} \\ (0.50 \text{ h/km})(x \text{ km}) & + & (0.75 \text{ h/km})(y \text{ km}) \leq 60 \text{ h} \end{array} \\ & \begin{array}{ccc} \uparrow & & \uparrow \\ 30 \text{ min} & & 45 \text{ min} \end{array} \\ & 0.50x + 0.75y \leq 60 \\ & y \leq 80 - 0.67x \end{aligned}$$

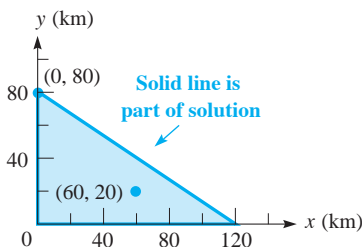


Fig. 17.35

Noting that negative values of  $x$  and  $y$  do not have meaning, we have the graph in Fig. 17.35, shading in the region below the line since we have  $y < 80 - 0.67x$  for that region. Any point in the shaded region, or on the axes or the line around the shaded region, gives a solution. The *solid line* indicates that points on it are part of the solution.



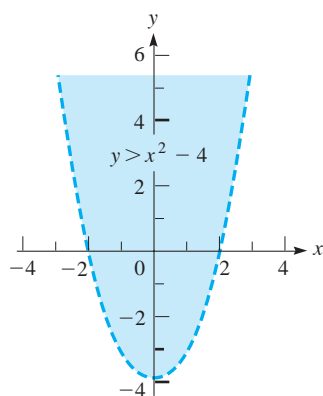


Fig. 17.36

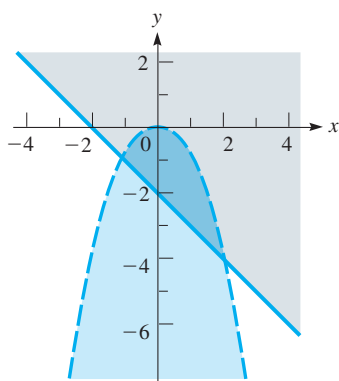


Fig. 17.37

The point  $(0, 80)$ , for example, is a solution and tells us that 80 km of Route 80 can be plowed if none of Route 15 is plowed. In this case, all 60 h of plowing time are used for Route 80. Another possibility is shown by the point  $(60, 20)$ , which indicates that 60 km of Route 15 and 20 km of Route 80 can be plowed. In this case, not all of the 60 plowing hours are used. ■

#### EXAMPLE 4 Sketching a nonlinear inequality with two variables

Draw a sketch of the graph of the inequality  $y > x^2 - 4$ . Although the graph of  $y = x^2 - 4$  is not a straight line, the method of solution is the same. We graph the function  $y = x^2 - 4$  as a dashed curve, since it is not part of the solution, as shown in Fig. 17.36. We then shade in the region above the curve to indicate the points that satisfy the inequality. ■

#### EXAMPLE 5 Solution of a system of inequalities

Draw a sketch of the region that is defined by the system of inequalities  $y \geq -x - 2$  and  $y + x^2 < 0$ .

Similar to the solution of a system of equations, *the solution of a system of inequalities is any pair of values  $(x, y)$  that satisfies both inequalities.* This means we want the region common to both inequalities. In Fig. 17.37, we first shade in the region above the line  $y = -x - 2$  and then shade in the region below the parabola  $y = -x^2$ . The region defined by this system is the darkly shaded region below the parabola that is also above and on the line.

Note that the region defined by  $y \geq -x - 2$  or  $y + x^2 < 0$  consists of both shaded regions and all points on the line since that is where either of the inequalities is true. ■

Most graphing calculators can be used to display the solution of an inequality or of a system of inequalities involving two variables. The manual should be consulted to determine how to do this on any particular model of calculator. Some examples can be found in the exercises.

## EXERCISES 17.5

In Exercises 1 and 2, make the given changes in the indicated examples of this section and then draw the graph of the resulting inequality.

- In Example 2, change  $x + 3$  to  $3 - x$  and then draw the graph of the resulting inequality.
- In Example 4, change  $x^2 - 4$  to  $4 - x^2$  and then draw the graph of the resulting inequality.

In Exercises 3–22, draw a sketch of the graph of the given inequality.

- $y > x - 1$
- $y < 3x - 2$
- $y \geq 2x + 5$
- $y \leq 15 - 3x$
- $3x + 2y + 6 > 0$
- $x + 4y - 8 < 0$
- $y < x^2$
- $y \leq 2x^2 - 3$
- $2x^2 - 4x - y > 0$
- $y \leq x^3 - 1$
- $y < 32x - x^4$
- $y \leq \sqrt{2x + 5}$
- $y > \frac{10}{x^2 + 1}$
- $y < \ln x$
- $y > 1 + \sin 2x$
- $y > |x| - 3$
- $-4 < y \leq -2$
- $y > |x + 3|$
- $|x| < |y|$
- $|x - 3y| > 2$

In Exercises 23–32, draw a sketch of the graph of the region in which the points satisfy the given system of inequalities.

- $y > x$   
 $y > 1 - x$
- $y \leq 2x$   
 $y \geq x - 1$
- $y \leq 2x^2$   
 $y > x - 2$
- $y > \frac{1}{2}x^2$   
 $y \leq 4x - x^2$
- $y < 4 - \frac{x}{\sqrt{16 - x^2}}$
- $y \geq 0$   
 $y \leq \sin x$   
 $0 \leq x \leq 3\pi$
- $y > 0$   
 $y > 1 - x$   
 $y < e^x$
- $|y + 2| < 5$   
 $|x - 3| \leq 2$
- $16x + 3y - 12 > 0$   
 $y > x^2 - 2x - 3$   
 $|2x - 3| < 3$
- $2x + y < 5$
- $4x - y > 1$
- $y \geq 1 - x^2$
- $y < |4 - 2x|$
- $y > 2x - 1$   
 $y < x^4 - 8$
- $y < 3 - x$   
 $y > 3x - x^3$

In Exercises 33–42, use a graphing calculator to display the solution of the given inequality or system of inequalities.

39.  $y > x^2 + 2x - 8$   
 $y < \frac{1}{x} - 2$
40.  $y > -2x^2$   
 $y < 1 - e^{-x}$
41.  $y \leq |2x - 3|$   
 $y > 1 - 2x^2$
42.  $y \geq |4 - x^2|$   
 $y < 2 \ln |x|$

In Exercises 43–48, solve the given problems.

43. By an inequality, define the region below the line  $4x - 2y + 5 = 0$ .
44. By an inequality, define the region that is bounded by or includes the parabola  $x^2 - 2y = 0$ , and that contains the point  $(1, 0.4)$ .
45. For  $Ax + By > C$ , if  $B < 0$ , would you shade above or below the line?
46. Find a system of inequalities that would describe the region within the triangle with vertices  $(0, 0)$ ,  $(0, 4)$ , and  $(2, 0)$ .
47. Draw a graph of the solution of the system  $y \geq 2x^2 - 6$  and  $y = x - 3$ .
48. Draw a graph of the solution of the system  $y < |x + 2|$  and  $y = x^2$ .

In Exercises 49–54, set up the necessary inequalities and sketch the graph of the region in which the points satisfy the indicated system of inequalities.

49. A telephone company is installing two types of fibre-optic cable in an area. It is estimated that no more than 300 m of type

A cable, and at least 200 m but no more than 400 m of type B cable, are needed. Graph the possible lengths of cable that are needed.

50. A refinery can produce gasoline and diesel fuel, in amounts of any combination, except that equipment restricts total production to 7500 L/day. Graph the different possible production combinations of the two fuels.
51. The elements of an electric circuit dissipate  $P$  watts of power. The power  $P_R$  dissipated by a resistor in the circuit is given by  $P_R = Ri^2$ , where  $R$  is the resistance (in  $\Omega$ ) and  $i$  is the current (in A). Graph the possible values of  $P$  and  $i$  for  $P > P_R$  and  $R = 0.5 \Omega$ .
52. The cross-sectional area  $A$  (in  $m^2$ ) of a certain trapezoid culvert in terms of its depth  $d$  (in m) is  $A = 2d + d^2$ . Graph the possible values of  $d$  and  $A$  if  $A$  is between  $1 m^2$  and  $2 m^2$ .
53. One pump can remove wastewater at the rate of 250 L/min, and a second pump works at the rate of 150 L/min. Graph the possible values of the time (in min) that each of these pumps operates such that together they pump more than 15 000 L.
54. A rectangular computer chip is being designed such that its perimeter is no more than 30 mm, its width at least 3 mm, and its length at least 8 mm. Graph the possible values of the width  $w$  and the length  $l$ .

## 17.6 Linear Programming

**Constraints • Objective Function • Feasible Point • Vertices of the Region of Feasible Points**

■ This section is an introduction to linear programming. Methods for applications involving many decision variables can be found in a linear programming or operations research text.

An important area in which inequalities with two or more variables are used is in the branch of mathematics known as **linear programming** (in this context, “programming” does not mean computer programming). This subject, which we mentioned in the chapter introduction, is widely applied in industry, business, economics, and technology. The analysis of many social problems can also be made by use of linear programming.

Linear programming is used to analyse problems such as those related to maximizing profits, minimizing costs, or the use of materials with certain constraints of production. We begin by introducing the basic terminology of linear programming through an example.

### EXAMPLE 1 Maximum value of $F$

Find the maximum value of  $F$ , where  $F = 2x + 3y$  and  $x$  and  $y$  are subject to the conditions that

$$\begin{aligned}x &\geq 0, y \geq 0 \\x + y &\leq 6 \\x + 2y &\leq 8\end{aligned}$$

The variables  $x$  and  $y$  are the **decision variables** of the problem. Their values are restricted by the four given inequalities, known as the **constraints**. The function  $F$  to be maximized is known as the **objective function**.

We now graph this set of inequalities, as shown in Fig. 17.38. Each point in the shaded region (including the line segments on the edges) satisfies all the constraints and is known as a **feasible point**.

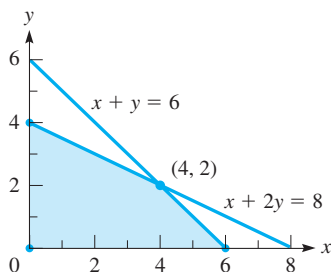


Fig. 17.38

The maximum value of  $F$  must be found at one of the feasible points. Testing for values at the vertices of the region, we have the values in the following table:

Point	(0, 0)	(6, 0)	(4, 2)	(0, 4)
Value of $F$	0	12	14	12

**Practice Exercise**

- In Example 1, what is the maximum value of  $F$  if the third constraint is changed to  $x + y \leq 5$ ?

If we evaluate  $F$  at any other feasible point, we will find that  $F < 14$ . Therefore, the maximum value of  $F$  under the given constraints is 14. Since (4, 2) is the feasible point with the largest value of the objective function, we say that (4, 2) is the **optimal solution**. ■

In Example 1, we found the maximum value of a *linear* objective function, subject to *linear* constraints (thus the name *linear programming*). We found the maximum value of  $F$ , subject to the given constraints, to be at one of the vertices of the region of feasible points. In fact, in the theory of linear programming the following result is established.

**Optimal Solution of a Linear Programming Problem**

If a linear programming problem has an optimal solution, then it will occur at a vertex of the region of feasible points.

If the problem has more than one solution, all points along a line segment connecting two vertices will be optimal.

In either case, the value of the objective function is unique.

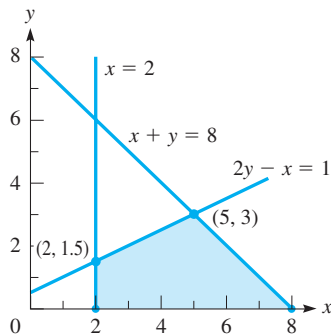


Fig. 17.39

**EXAMPLE 2 Maximum and minimum values of  $F$**

Find the maximum and minimum values of the objective function  $F = 3x + y$ , subject to the constraints

$$\begin{aligned} x &\geq 2, y \geq 0 \\ x + y &\leq 8 \\ 2y - x &\leq 1 \end{aligned}$$

The constraints are graphed as shown in Fig. 17.39. We then locate the vertices and evaluate  $F$  at these vertices as follows:

Vertex	(2, 0)	(8, 0)	(5, 3)	(2, 1.5)
Value of $F$	6	24	18	7.5

Therefore, we see that the maximum value of  $F$  is 24, and the minimum value of  $F$  is 6. If we check the value of  $F$  at any other feasible point, we will find a value between 6 and 24. ■

The following two examples show the use of linear programming in finding a maximum value and a minimum value in applied situations.

**EXAMPLE 3 Linear programming—application**

A company makes two types of stereo speaker systems, their good-quality system and their highest-quality system. The production of these systems requires assembly of the speaker system itself and the production of the cabinets in which they are installed. The good-quality system requires 3 worker-hours for speaker assembly and 2 worker-hours for cabinet production for each complete system. The highest-quality system requires 4 worker-hours for speaker assembly and 6 worker-hours for cabinet production for each complete system. Available skilled labour allows for a maximum of 480 worker-hours per week for speaker assembly and a maximum of 540 worker-hours per week for cabinet production. It is anticipated that all systems will be sold and that the profit will be

■ The loudspeaker was developed in the early 1920s by the U.S. inventor Kellogg Rice.

\$30 for each good-quality system and \$75 for each highest-quality system. How many of each system should be produced to provide the greatest profit?

First, let  $x =$  the number of good-quality systems and  $y =$  the number of highest-quality systems made in one week. Thus, the profit  $P$  is given by

$$P = 30x + 75y$$

We know that negative numbers are not valid for either  $x$  or  $y$ , and therefore we have  $x \geq 0$  and  $y \geq 0$ .

The number of available worker-hours per week for each part of the production also restricts the number of systems that can be made. In the speaker-assembly shop,  $3x$  worker-hours are needed for each good-quality system and  $4y$  worker-hours are needed for each highest-quality system. The constraint that only 480 hours are available in the speaker-assembly shop means that  $3x + 4y \leq 480$ .

In the cabinet shop, it takes  $2x$  worker-hours for a good-quality system and  $6y$  worker-hours for a highest-quality system. The constraint that only 540 hours are available in the cabinet shop means that  $2x + 6y \leq 540$ .

Therefore, we want to maximize the profit  $P = 30x + 75y$ , which is the objective function, under the constraints

$$\begin{aligned} x \geq 0, y \geq 0 & \quad \text{number of systems produced cannot be negative} \\ 3x + 4y \leq 480 & \quad \text{worker-hours for speaker assembly} \\ 2x + 6y \leq 540 & \quad \text{worker-hours for cabinet production} \end{aligned}$$

The constraints are graphed as shown in Fig. 17.40. We locate the vertices and evaluate the profit  $P$  at these points as follows:

Vertex	(0, 0)	(160, 0)	(72, 66)	(0, 90)
Profit (\$)	0	4800	7110	6750

Therefore, we see that the greatest profit of \$7110 is made by producing 72 good-quality systems and 66 highest-quality systems.

A way of showing the number of each system to be made for the greatest profit is to assume values of the profit  $P$  and graph these lines. For example, for  $P = \$3000$  or  $P = \$6000$ , we have the lines shown in Fig. 17.40. Both amounts of profit are possible with various combinations of speaker systems being produced. However, we note that the line for  $P = \$6000$  passes through feasible points farther from the origin. It is also clear that the lines for the profits are parallel and that *the greatest profit attainable is given by the line passing through A*, where  $3x + 4y = 480$  and  $2x + 6y = 540$  intersect. This also illustrates why the greatest profit is found at one of the vertices of the region of feasible points.

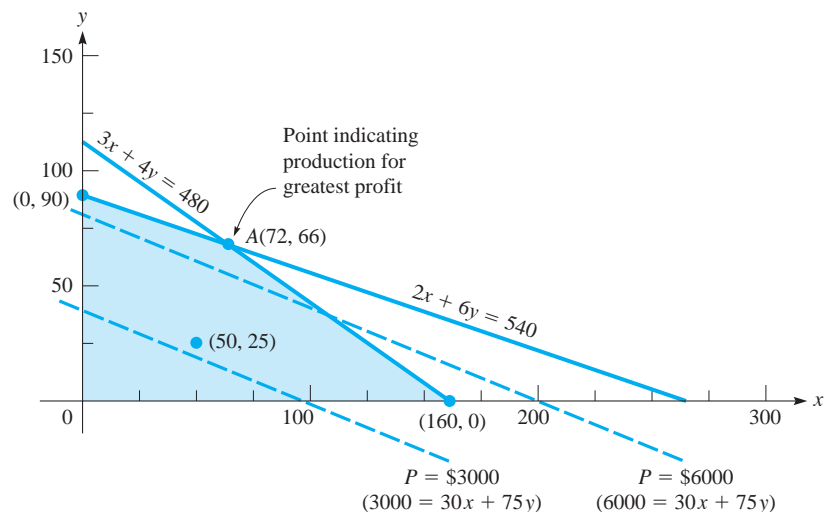


Fig. 17.40

■ See the chapter introduction.

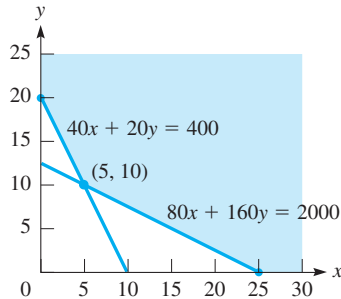


Fig. 17.41

**EXAMPLE 4** Linear programming—application

An airline plans to open new routes and use two types of planes, A and B, on these routes. It is expected there would be at least 400 first-class passengers and 2000 economy-class passengers on these routes each day. Plane A costs \$18 000/day to operate and has seats for 40 first-class and 80 economy-class passengers. Plane B costs \$16 000/day to operate and has seats for 20 first-class and 160 economy-class passengers. How many of each type of plane should be used for these routes to minimize operating costs? (Assume that the planes will be used only on these routes.)

We first let  $x$  = the number of A planes and  $y$  = the number of B planes to be used. Then the operating cost  $C = 18\,000x + 16\,000y$  is the objective function. The constraints are

$$\begin{aligned} x \geq 0, y \geq 0 & \quad \text{number of planes cannot be negative} \\ 40x + 20y \geq 400 & \quad \text{at least 400 first-class passengers} \\ 80x + 160y \geq 2000 & \quad \text{at least 2000 economy-class passengers} \end{aligned}$$

The constraints are graphed as shown in Fig. 17.41. We see that the region of feasible points is unlimited (there could be more than 2400 passengers), but we still want to evaluate the operating cost  $C$  at the vertices. Evaluating  $C$  at these points, we have

Vertex	(25, 0)	(5, 10)	(0, 20)
Cost (\$)	450 000	250 000	320 000

Therefore, 5 type-A planes and 10 type-B planes should be used on these routes to keep the operating costs at a minimum of \$250 000. ■

The problems in linear programming that arise in business and industry involve many more variables than in the simplified examples in this text. They are solved by computers using matrices, but the basic idea is the same as that shown in this section.

**EXERCISES 17.6**

In Exercises 1 and 2, make the given changes in the indicated examples of this section and then find the indicated values.

- In Example 1, change the last constraint to  $2x + y \leq 8$ . Then graph the feasible points and find the maximum value of  $F$ .
- In Example 2, change the last constraint to  $3y - x \leq 4$ . Then graph the feasible points and find the maximum and minimum values of  $F$ .

In Exercises 3–16, find the indicated maximum and minimum values by the linear programming method of this section. For Exercises 5–16, the constraints are shown below the objective function.

- Graphing the constraints of a linear programming problem shows the consecutive vertices of the region of feasible points to be  $(0, 0)$ ,  $(12, 0)$ ,  $(10, 7)$ ,  $(0, 5)$ , and  $(0, 0)$ . What are the maximum and minimum values of the objective function  $F = 3x + 4y$  in this region?
- Graphing the constraints of a linear programming problem shows the consecutive vertices of the region of feasible points to be  $(1, 3)$ ,  $(8, 0)$ ,  $(9, 7)$ ,  $(5, 8)$ ,  $(0, 6)$ , and  $(1, 3)$ . What are the maximum and minimum values of the objective function  $F = 2x + 5y$  in this region?

- |  |  |  |
|--|--|--|
| <p><b>5.</b> Maximum <math>P</math>:<br/> <math>P = 3x + 5y</math><br/> <math>x \geq 0, y \geq 0</math><br/> <math>2x + y \leq 6</math></p>  | <p><b>6.</b> Maximum <math>P</math>:<br/> <math>P = 2x + 7y</math><br/> <math>x \geq 1, y \geq 0</math><br/> <math>x + 4y \leq 8</math></p>  | <p><b>7.</b> Maximum <math>P</math>:<br/> <math>P = 5x + 9y</math><br/> <math>x \geq 0, y \geq 0</math><br/> <math>x + 2y \leq 6</math></p>                                      |
| <p><b>8.</b> Minimum <math>C</math>:<br/> <math>C = 10x + 20y</math><br/> <math>x \geq 0, y \geq 0</math><br/> <math>3x + 5y \geq 30</math></p>  | <p><b>9.</b> Minimum <math>C</math>:<br/> <math>C = 4x + 6y</math><br/> <math>x \geq 0, y \geq 0</math><br/> <math>x + y \geq 5</math><br/> <math>x + 2y \geq 7</math></p>                           | <p><b>10.</b> Minimum <math>C</math>:<br/> <math>C = 6x + 4y</math><br/> <math>x \geq 0, y \geq 0</math><br/> <math>2x + y \geq 6</math><br/> <math>x + y \geq 5</math></p>      |
| <p><b>11.</b> Maximum and minimum <math>F</math>:<br/> <math>F = x + 3y</math><br/> <math>x \geq 1, y \geq 2</math><br/> <math>y - x \leq 3</math><br/> <math>y + 2x \leq 8</math></p> | <p><b>12.</b> Maximum and minimum <math>F</math>:<br/> <math>F = 3x - y</math><br/> <math>x \geq 1, y \geq 0</math><br/> <math>x + 4y \leq 8</math><br/> <math>4x + y \leq 8</math></p>              | <p><b>13.</b> Maximum <math>P</math>:<br/> <math>P = 9x + 2y</math><br/> <math>x \geq 0, y \geq 0</math><br/> <math>2x + 5y \leq 10</math><br/> <math>4x + 3y \leq 12</math></p> |
| <p><b>14.</b> Minimum <math>C</math>:<br/> <math>C = 3x + 8y</math><br/> <math>x \geq 0, y \geq 0</math><br/> <math>6x + y \geq 6</math><br/> <math>x + 4y \geq 4</math></p>           | <p><b>15.</b> Minimum <math>C</math>:<br/> <math>C = 6x + 4y</math><br/> <math>y \geq 2</math><br/> <math>x + y \leq 12</math><br/> <math>x + 2y \geq 12</math><br/> <math>2x + y \geq 12</math></p> | <p><b>16.</b> Maximum <math>P</math>:<br/> <math>P = 3x + 4y</math><br/> <math>2x + y \geq 2</math><br/> <math>x + 2y \geq 2</math><br/> <math>x + y \leq 2</math></p>           |

In Exercises 17–22, solve the given linear programming problems.

17. A person wants to invest no more than \$9000, part at 6% and part at 5%, with no more than twice as much being invested at 6% as at 5%. How much should be invested at each rate to maximize the income from the investments?
18. An oil refinery refines types A and B of crude oil and can refine as much as 4000 barrels each week. Type A crude has 2 kg of impurities per barrel, type B has 3 kg of impurities per barrel, and the refinery can handle no more than 9000 kg of these impurities each week. How much of each type should be refined in order to maximize profits, if the profit is \$4/barrel for type A and \$5/barrel for type B?
19. A manufacturer produces a business calculator and a graphing calculator. Each calculator is assembled in two sets of operations, where each operation is in production 8 h during each day. The average time required for a business calculator in the first operation is 3 min, and 6 min is required in the second operation. The graphing calculator averages 6 min in the first operation and 4 min in the second operation. All calculators can be sold; the profit for a business calculator is \$8, and the profit for a graphing calculator is \$10. How many of each type of calculator should be made each day in order to maximize profit?
20. Using the information given in Example 3, with the one change that the profit on each good-quality speaker system is \$60 (instead of \$30), how many of each system should be made? Explain why this one change in data makes such a change in the solution.

21. Brands A and B of breakfast cereal are both enriched with vitamins P and Q. The necessary information about these cereals is as follows:

	Cereal A	Cereal B	RDA
Vitamin P	1 unit/serving	2 units/serving	10 units
Vitamin Q	5 units/serving	3 units/serving	30 units
Cost	12¢/serving	18¢/serving	

(RDA is the *Recommended Dietary Allowance*.) Find the amounts of each cereal that together satisfies the RDA of vitamins P and Q at the lowest cost. (There are 30 g of cereal in one serving.)

22. A computer company makes parts A and B in each of two different plants. It costs \$4000 per day to operate the first plant and \$5000 per day to operate the second plant. Each day the first plant produces 100 of part A and 200 of part B, while at the second plant 250 of part A and 100 of part B are produced. How many days should each plant operate to produce 2000 of each part and keep operating costs at a minimum?

#### Answer to Practice Exercise

1. 13 (at (2, 3))

## CHAPTER 17 EQUATIONS

$$\text{If } |f(x)| > n, \text{ then } f(x) < -n \text{ or } f(x) > n. \quad (17.1)$$

$$\text{If } |f(x)| < n, \text{ then } -n < f(x) < n. \quad (17.2)$$

## CHAPTER 17 REVIEW EXERCISES

In Exercises 1–24, solve each of the given inequalities algebraically. Graph each solution.

- $2x - 12 > 0$
- $2.4(T - 4.0) \geq 5.5 - 2.4T$
- $5 - 3x < 0$
- $6 \leq 4x - 2 < 9$
- $4 < 2x - 1 < 11$
- $2x < x + 1 < 4x + 7$
- $2 \leq \frac{4n - 2}{3} < 3$
- $3x < 2x + 1 < x - 5$
- $5x^2 + 9x < 2$
- $x^2 + 2x > 63$
- $6n^2 - n > 35$
- $2x^3 + 4 \leq x^2 + 8x$
- $\frac{(2x - 1)(3 - x)}{x + 4} > 0$
- $x^4 + x^2 \leq 0$
- $\frac{8}{x} < 2$
- $\frac{1}{x - 2} < \frac{1}{4}$
- $\frac{8 - R}{2R + 1} \leq 0$
- $\frac{(3 - x)^2}{2x + 7} \leq 0$

$$19. |3x + 2| \leq 4 \quad 20. |4 - 3x| \geq 7$$

$$21. |3 - 5x| > 7 \quad 22. \left| 2 - \frac{y}{2} \right| \leq 0$$

$$23. |x - 30| > 48 \quad 24. 2|2x - 9| < 8$$

In Exercises 25–28, use a graphing calculator to solve the given inequalities. Graph the appropriate function and from the graph determine the solution. Approximate roots to the nearest 0.01.

- $x^3 + x + 1 < 0$
- $\frac{2}{R + 2} > 3$
- $e^{-t} > 0.5$
- $\sin 2x < 0.8 \quad (0 < x < 4)$

In Exercises 29–40, draw a sketch of the region in which the points satisfy the given inequality or system of inequalities.

- $y > 12 - 3x$
- $y < \frac{1}{2}x + 2$
- $2y - 3x - 4 \leq 0$
- $3y - x + 6 \geq 0$



33.  $y > 6 - 2x^2$
34.  $y \leq \frac{6}{x^2 - 4}$
35.  $y - |x + 1| < 0$
36.  $2y + 2x^3 + 6x > 3$
37.  $y > x + 1$   
 $y < 4 - x^2$
38.  $y > 2x - x^2$   
 $y \geq -2$
39.  $y \leq \frac{4}{x^2 + 1}$   
 $y < x - 3$
40.  $y < \cos \frac{1}{2}x$   
 $y > \frac{1}{2}e^x$   
 $-\pi < x < \pi$

In Exercises 41–48, use a graphing calculator to display the region in which the points satisfy the given inequality or system of inequalities.

41.  $y < 3x + 5$
42.  $y > 2 - \frac{1}{4}x$
43.  $y > 8 + 7x - x^2$
44.  $y < x^3 + 4x^2 - x - 4$
45.  $y < 32x - x^4$
46.  $y > 2x - 1$   
 $y < 6 - 3x^2$
47.  $y > 1 - x \sin 2x$   
 $y < 5 - x^2$
48.  $y > |x - 1|$   
 $y < 4 + \ln x$

In Exercises 49–52, determine the values of  $x$  for which the given radicals represent real numbers.

49.  $\sqrt{3 - x}$
50.  $\sqrt{x + 5}$
51.  $\sqrt{x^2 + 4x}$
52.  $\sqrt{\frac{x - 1}{x + 2}}$

In Exercises 53–56, find the indicated maximum and minimum values by the method of linear programming. The constraints are shown below the objective function.

53. Maximum  $P$ :  
 $P = 2x + 9y$   
 $x \geq 0, y \geq 0$   
 $x + 4y \leq 13$   
 $3y - x \leq 8$
54. Maximum  $P$ :  
 $P = x + 2y$   
 $x \geq 1, y \geq 0$   
 $3x + y \leq 6$   
 $2x + 3y \leq 8$
55. Minimum  $C$ :  
 $C = 3x + 4y$   
 $x \geq 0, y \geq 1$   
 $2x + 3y \geq 6$   
 $4x + 2y \geq 5$
56. Minimum  $C$ :  
 $C = 2x + 4y$   
 $x \geq 0, y \geq 0$   
 $x + 3y \geq 6$   
 $4x + 7y \geq 18$

In Exercises 57–82, solve the given problems using inequalities. (All data are accurate to at least two significant digits.)

57. Under what conditions is  $|a + b| < |a| + |b|$ ?
58. Is  $|a - b| < |a| + |b|$  always true? Explain.
59. If  $a > 0, a \neq 1$ , show that  $a + a^{-1} > 2$ .
60. Solve for  $x$ :  $a + |bx| < c$ , given that  $a - c < 0$ .
61. Form an inequality of the form  $ax^2 + bx + c < 0$  with  $a > 0$  for which the solution is  $-2 < x < 5$ .
62. By means of an inequality, define the region above the line  $x - 3y - 6 = 0$ .
63. Draw a graph of the system  $y < 1 - x^2$  and  $y = x^2$ .
64. Find a system of inequalities that would describe the region within the quadrilateral with vertices  $(0, 0)$ ,  $(4, 4)$ ,  $(0, -3)$ , and  $(4, -3)$ .

65. Find the values for which  $f(x) = (x - 2)(x - 3)$  is positive, zero, and negative. Use this information along with  $f(0)$  and  $f(5)$  to make a rough sketch of the graph of  $f(x)$ .
66. Follow the same instructions as in Exercise 65 for the function  $f(x) = (x - 2)/(x - 3)$ .
67. If two adjacent sides of a square design on a sheet of metal are extended 6.0 cm and 10.0 cm, respectively, how long is each side of the square if the perimeter of the rectangle is at least twice as long as the perimeter of the square? See Fig. 17.42.

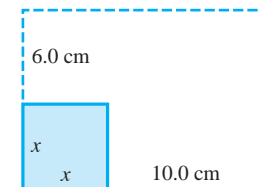


Fig. 17.42

68. The value  $V$  (in \$) of each building lot in a development is estimated as  $V = 65\,000 + 5000t$ , where  $t$  is the time in years from now. For how long is the value of each lot no more than \$90\,000?
69. The cost  $C$  of producing two of one type of calculator and five of a second type is \$50. If the cost of producing each of the second type is between \$5 and \$8, what are the possible costs of producing each of the first type?
70. Ottawa is 450 km from Québec City. One car starts from Ottawa for Québec City one hour before a second car. The first car averages 60 km/h, and the second car averages 80 km/h for the trip. For what times after the first car starts is the second car ahead of the first car?
71. The pressure  $p$  (in kPa) at a depth  $d$  (in m) in the ocean is given by  $p = 101 + 10.1d$ . For what values of  $d$  is  $p > 500$  kPa?
72. After conducting tests, it was determined that the stopping distance  $x$  (in m) of a car travelling at 90 km/h was  $|x - 95| \leq 10$ . Express this inequality without absolute values and find the interval of stopping distances that were found in the tests.
73. A heating unit with 80% efficiency and a second unit with 90% efficiency deliver 360 MJ of heat to an office complex. If the first unit consumes an amount of fuel that contains no more than 261 MJ, what is the MJ content of the fuel consumed by the second unit?
74. A rectangular parking lot is to have a perimeter of 100 m and an area no greater than  $600 \text{ m}^2$ . What are the possible dimensions of the lot?
75. The electric power  $P$  (in W) dissipated in a resistor is given by  $P = Ri^2$ , where  $R$  is the resistance (in  $\Omega$ ) and  $i$  is the current (in A). For a given resistor,  $R = 12.0 \Omega$ , and the power varies between 2.50 W and 8.00 W. Find the values of the current.
76. The reciprocal of the total resistance of two electric resistances in parallel equals the sum of the reciprocals of the resistances. If a  $2.0\text{-}\Omega$  resistance is in parallel with a resistance  $R$ , with a total resistance greater than  $0.5 \Omega$ , find  $R$ .
77. The efficiency  $E$  (in %) of a certain gasoline engine is given by  $E = 100(1 - r^{-0.4})$ , where  $r$  is the compression ratio for the engine. For what values of  $r$  is  $E > 50\%$ ?
78. A rocket is fired such that its height  $h$  (in km) is given by  $h = 41t - t^2$ . For what values of  $t$  (in min) is the height greater than 400 km?



79. In developing a new product, a company estimates that it will take no more than 1200 min of computer time for research and no more than 1000 min of computer time for development. Graph the possible combinations of the computer times that are needed.
80. A natural-gas supplier has a maximum of 120 worker-hours per week for delivery and for customer service. Graph the possible combinations of times available for these two services.
81. A company produces two types of cell phones, the regular model and the deluxe model. For each regular model produced, there is a profit of \$8, and for each deluxe model the profit is \$15. The same amount of materials is used to make each model, but the supply is sufficient only for 450 cell phones per day. The deluxe model requires twice the time to produce as the regular model. If only regular models were made, there would be time enough to produce 600 per day. Assuming all cell phones will be sold, how many of each model should be produced if the profit is to be a maximum?
82. A company that manufactures DVD/ CD players gets two different parts, A and B, from two different suppliers. Each package of parts from the first supplier costs \$2.00 and contains 6 of each type of part. Each package of parts from the second supplier costs \$1.50 and contains 4 of A and 8 of B. How many packages should be bought from each supplier to keep the total cost to a minimum, if production requirements are 600 of A and 900 of B?

### Writing Exercise

83. In planning a new city development, an engineer uses a rectangular coordinate system to locate points within the development. A park in the shape of a quadrilateral has corners at  $(0, 0)$ ,  $(0, 20)$ ,  $(40, 20)$ , and  $(20, 40)$  (measurements in metres). Write two or three paragraphs explaining how to describe the park region with inequalities and find these inequalities.

## CHAPTER 17 PRACTICE TEST

1. State conditions on  $x$  and  $y$  in terms of inequalities if the point  $(x, y)$  is in the second quadrant.

In Problems 2–7, solve the given inequalities algebraically and graph each solution.

2.  $\frac{-x}{2} \geq 3$

3.  $3x + 1 < -5$

4.  $-1 < 1 - 2x < 5$

5.  $\frac{x^2 + x}{x - 2} \leq 0$

6.  $|2x + 1| \geq 3$

7.  $|2 - 3x| < 8$

8. Sketch the region in which the points satisfy the following system of inequalities:

$$y < x^2$$

$$y \geq x + 1$$

9. Determine the values of  $x$  for which  $\sqrt{x^2 - x - 6}$  represents a real number.

10. The length of a rectangular lot is 20 m more than its width. If the area is to be at least  $4800 \text{ m}^2$ , what values may the width be?

11. Type A wire costs \$0.10 per metre, and type B wire costs \$0.20 per metre. Show the possible combinations of lengths of wire that can be purchased for less than \$5.00.

12. The range of the visible spectrum in terms of the wavelength  $\lambda$  of light ranges from about  $\lambda = 400 \text{ nm}$  (violet) to about  $\lambda = 700 \text{ nm}$  (red). Express these values using an inequality with absolute values.

13. Solve the inequality  $x^2 > 12 - x$  graphically.

14. By using linear programming, find the maximum value of the objective function  $P = 5x + 3y$ , subject to the following constraints:  $x \geq 0$ ,  $y \geq 0$ ,  $2x + 3y \leq 12$ ,  $4x + y \leq 8$ .

# Variation

In 1969, as millions watched on television, men walked on the moon for the first time. Today, spacecraft on science missions still travel from the earth to the moon. The 2012 Gravity Recovery and Interior Laboratory (GRAIL) mission placed two spacecraft twins into the same orbit around the moon. The objective of the mission is to obtain an accurate map of the moon's gravitational field. This map will help increase understanding of the moon's interior and, ultimately, its origin and development.

Numerous discoveries in science, engineering, and mathematics have made it possible for spacecraft to travel to the moon. Yet it can be argued that the first step was Newton's formulation of the *universal law of gravitation* in the 1680s, based on earlier discoveries regarding the motion of the planets and the moon. We introduce the universal law of gravitation in Section 18.2 as an example of a scientific law that can be stated in terms of *variation*, the principal topic of this chapter. Using the language of variation, we can describe how one variable changes as other related variables change.

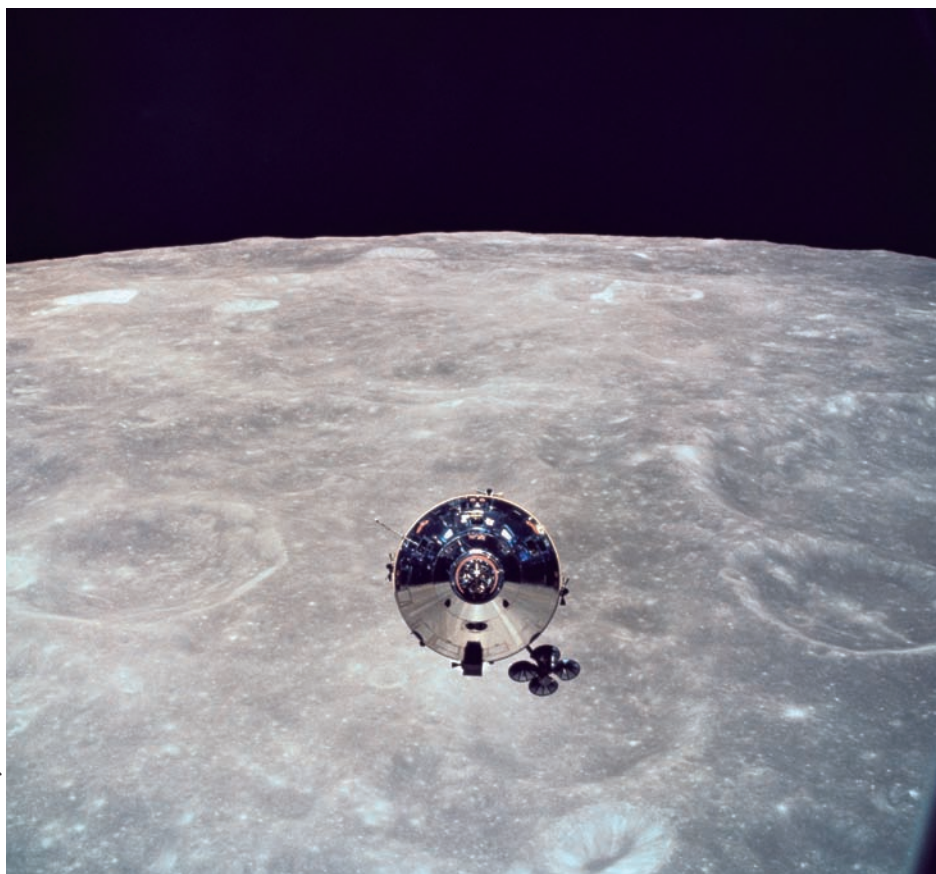
We begin this chapter by reviewing the meanings of *ratio* and *proportion*, which were first introduced in Chapter 1. These concepts then lead us to the concept of variation, which allows us to establish different types of relationships between variables. Many of these relationships are based on experimentation and observation, just as in the case of gravitation. Applications of variation are found in all areas of technology. It is used in acoustics, biology, chemistry, computer technology, economics, electronics, environmental technology, hydrodynamics, mechanics, navigation, optics, physics, space technology, thermodynamics, and other fields.

# 18

## LEARNING OUTCOMES

After completion of this chapter, the student should be able to:

- Solve application problems involving ratios and proportions
- Set up relationships between variables in terms of direct variation, inverse variation, or joint variation
- Solve application problems involving the different types of variation



◀ In Section 18.2 we use Newton's universal law of gravitation to analyse the forces acting on a spacecraft travelling to the moon.

## 18.1 Ratio and Proportion

### Ratio • Proportion

In order to develop the meaning of *variation*, we now review and expand our discussion of *ratio* and *proportion*. First, from Chapter 1, recall that *the quotient  $a/b$  is the ratio of  $a$  to  $b$* . Therefore, a fraction is a ratio.

A measurement is the ratio of the measured magnitude to an accepted unit of measurement. For example, measuring the length of an object as 5 cm means it is five times as long as the accepted unit of length, the centimetre. Other examples of ratios are density (weight/volume), relative density (density of object/density of water), and pressure (force/area). Thus, ratios compare quantities of the same kind (for example, the trigonometric ratios) or express the division of magnitudes of different quantities (such a ratio is also called a **rate**).

#### EXAMPLE 1 Units of measurement when calculating ratios

The approximate airline distance from Toronto to Los Angeles is 3500 km, and the approximate airline distance from Toronto to Miami is 2000 km. The ratio of these distances is

$$\frac{3500 \text{ km}}{2000 \text{ km}} = \frac{7}{4}$$

Since both units are in kilometres, the resulting ratio is a dimensionless number.

If a jet travels from Toronto to Los Angeles in 5 h, its average speed is

$$\frac{3500 \text{ km}}{5 \text{ h}} = 700 \text{ km/h}$$

In this case, we must attach the proper units to the resulting ratio. ■

As we noted in Example 1, we must be careful to attach the proper units to the resulting ratio. Generally, the ratio of measurements of the same kind should be expressed as a dimensionless number. Consider the following example.

#### EXAMPLE 2 Ratio of measurements of the same kind

The length of a certain room is 8 m, and the width of the room is 6 m. Therefore, the ratio of the length to the width is  $\frac{8}{6}$ , or  $\frac{4}{3}$ .

If the width of the room is expressed as 6000 mm, we have the ratio  $8 \text{ m}/6000 \text{ mm} = 1 \text{ m}/750 \text{ mm}$ . However, this does not clearly show the ratio. It is better and more meaningful first to change the units of one of the measurements to the units of the other measurement. Changing the length from 8 m to 8000 mm, we express the ratio as  $\frac{4}{3}$ , as we saw above. From this ratio, we can easily see that the length is  $\frac{4}{3}$  as long as the width. ■

Dimensionless ratios are often used in definitions in mathematics and in technology. For example, the irrational number  $\pi$  is the dimensionless ratio of the circumference of a circle to its diameter. The specific gravity of a substance is the ratio of its density to the density of water. Other illustrations are found in the exercises for this section.

From Chapter 1, also recall that *an equation stating that two ratios are equal is called a proportion*. By this definition, a proportion is

$$\frac{a}{b} = \frac{c}{d} \quad (18.1)$$

Consider the following example.

■ The first jet-propelled airplane was flown in Germany in 1928.

**EXAMPLE 3** Proportion—scale of a map application

On a certain map, 1 cm represents 10 km. Therefore, on this map we have a ratio of distances of 1 cm/10 km. To find the distance represented by 3.5 cm on the map, we can set up the proportion

$$\begin{array}{c} \text{map distances} \\ \frac{3.5 \text{ cm}}{x} = \frac{1 \text{ cm}}{10 \text{ km}} \\ \text{land distances} \\ (10x) \left( \frac{3.5}{x} \right) = 10x \left( \frac{1}{10} \right) \quad \text{multiply each side by LCD} = 10x \\ 35 = x \quad \text{or} \quad x = 35 \text{ km} \end{array}$$

The ratio 1 cm/10 km is the *scale* of the map and has a special meaning, relating map distances in centimetres to land distances in kilometres. In a case like this, we should not change either unit to the other, even though they are both units of length. ■

In Section 1.3, we discussed a method for converting measurements from one set of units to another. As we see in the next example, unit conversion can also be done by setting up a proportion.

**EXAMPLE 4** Changing units of measurement by using a proportion

Given that 1 in. = 2.54 cm (in. is the symbol for the unit of length the *inch*), what is the length in centimetres of the diagonal of a flat computer screen that is 17.0 in. long? See Fig. 18.1.

If we equate the ratio of known lengths to the ratio of the given length to the required length, we can find the required length by solving the resulting proportion (which is an equation). This gives us

$$\begin{aligned} \frac{1 \text{ in.}}{2.54 \text{ cm}} &= \frac{17.0 \text{ in.}}{x \text{ cm}} \\ x &= (17.0)(2.54) \\ &= 43.2 \text{ cm} \end{aligned}$$

Therefore, the diagonal of the flat computer screen is 17.0 in., or 43.2 cm. ■

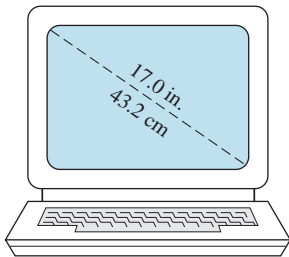


Fig. 18.1

**EXAMPLE 5** Proportion—application

The magnitude of an electric field  $E$  is the ratio of the force  $F$  on a charge  $q$  to the magnitude of  $q$ . We can write this as  $E = F/q$ . If we know the force exerted on a particular charge at some point in the field, we can determine the force that would be exerted on another charge placed at the same point. For example, if we know that a force of 10 nN is exerted on a charge of 4.0 nC, we can then determine the force that would be exerted on a charge of 6.0 nC by the proportion

$$\begin{array}{c} \text{forces at point} \\ \frac{10 \times 10^{-9}}{4.0 \times 10^{-9}} = \frac{F}{6.0 \times 10^{-9}} \\ \text{charges at point} \\ F = \frac{(6.0 \times 10^{-9})(10 \times 10^{-9})}{4.0 \times 10^{-9}} \\ = 15 \times 10^{-9} \\ = 15 \text{ nN} \end{array}$$

**Practice Exercise**

- In a certain electric field a force of 21 nN is exerted on a charge of 6.0 nC. At the same point, what is the force on a charge of 16 nC?

## Practice Exercise

2. An alloy is 7 parts zinc and 5 parts lead. How much lead is there in 48 g of the alloy?

## EXAMPLE 6 Proportion—application

An alloy is 5 parts tin and 3 parts lead. How many grams of each are in 40 g of the alloy?

First, let  $x$  = the number of grams of tin in 40 g of the alloy. Next, we note that there are 8 total parts of alloy, of which 5 are tin. Thus, 5 is to 8 as  $x$  is to 40. Therefore,

$$\begin{array}{l} \text{parts tin} \longrightarrow \frac{5}{8} = \frac{x}{40} \longleftarrow \text{grams of tin} \\ \text{total parts} \longrightarrow \end{array}$$

$$x = 40 \left( \frac{5}{8} \right) = 25 \text{ g}$$

There are 25 g of tin and 15 g of lead. The ratio 25/15 is the same as 5/3. ■

## EXERCISES 18.1

In Exercises 1 and 2, make the given changes in the indicated examples of this section and then solve the indicated problem.

- In Example 3, change 10 km to 16 km.
- In Example 6, change 5 parts tin to 7 parts tin.

In Exercises 3–10, express the ratios in the simplest form.

- 18 V to 3 V
- 63 m<sup>2</sup> to 18 m<sup>2</sup>
- 96 h to 3 days
- 120 s to 4 min
- 6500 cL to 2.6 L
- 25 cm<sup>2</sup> to 75 mm<sup>2</sup>
- 0.14 kg to 3500 mg
- 2000 μm to 6 mm

In Exercises 11–24, find the required ratios.

- The *efficiency* of a power amplifier is defined as the ratio of the power output to the power input. Find the efficiency of an amplifier for which the power output is 2.6 W and the power input is 9.6 W.
- A virus  $3.0 \times 10^{-5}$  cm long appears to be 1.2 cm long through a microscope. What is the *magnification* (ratio of image length to object length) of the microscope?
- The *coefficient of friction* for two contacting surfaces is the ratio of the frictional force between them to the perpendicular force that presses them together. If it takes 450 N to overcome friction to move a 1.10-kN crate along the floor, what is the coefficient of friction between the crate and the floor? See Fig. 18.2.

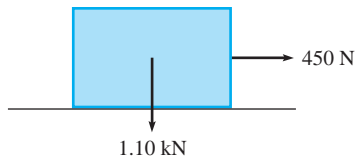


Fig. 18.2

- The *Mach number* of a moving object is the ratio of its velocity to the velocity of sound (1200 km/h). Find the Mach number of a jet travelling at 2000 km/h.
- The *capacitance*  $C$  of a capacitor is defined as the ratio of its charge  $q$  (in C) to the voltage  $V$  in volts. Find  $C$  (in F) for which  $q = 5.00 \mu\text{C}$  and  $V = 200 \text{ V}$ . (1 F = 1 C/1 V.)
- The *atomic mass* of an atom of carbon is defined to be 12 u. The ratio of the atomic mass of an atom of oxygen to that of an atom of carbon is  $\frac{4}{3}$ . What is the atomic mass of an atom of oxygen? (The symbol u represents the *unified atomic mass unit*, where  $1 \text{ u} = 1.66 \times 10^{-27} \text{ kg}$ .)

- An important design feature of an aircraft wing is its *aspect ratio*. It is defined as the ratio of the square of the span of the wing (wingtip to wingtip) to the total area of the wing. If the span of the wing for a certain aircraft is 10.0 m and the area is 18.0 m<sup>2</sup>, find the aspect ratio.
- For an automobile engine, the ratio of the cylinder volume to compressed volume is the *compression ratio*. If the cylinder volume of 820 cm<sup>3</sup> is compressed to 110 cm<sup>3</sup>, find the compression ratio.
- The *specific gravity* (or relative density) of a substance is the ratio of its density to the density of water. If the density of water is 1.00 g/cm<sup>3</sup>, find the specific gravity of a western Canada crude oil that has density of 931 kg/m<sup>3</sup>.
- The *percent grade* of a road is the ratio of vertical rise to the horizontal change in distance (expressed in percent). If a highway rises 75 m for each 1.2 km along the horizontal, what is the percent grade?
- The *percent error* in a measurement is the ratio of the error in the measurement to the measurement itself, expressed as a percent. When writing a computer program, the memory remaining is determined as 2450 bytes and then it is correctly found to be 2540 bytes. What is the percent error in the first reading?
- The electric *current* in a given circuit is the ratio of the voltage to the resistance. What is the current (1 V/1 Ω = 1 A) for a circuit where the voltage is 24.0 mV and the resistance is 10.0 Ω?
- The *mass* of an object is the ratio of its weight to the acceleration  $g$  due to gravity. If a space probe weighs 8.46 kN on earth, where  $g = 9.80 \text{ m/s}^2$ , find its mass.
- Power* is defined as the ratio of work done to the time required to do the work. If an engine performs 3.65 kJ of work in 15.0 s, find the power developed by the engine.

In Exercises 25–28, find the required quantities from the given proportions.

- According to Boyle's law, the relation  $p_1/p_2 = V_1/V_2$  holds for pressures  $p_1$  and  $p_2$  and volumes  $V_1$  and  $V_2$  of a gas at constant temperature. Find  $V_1$  if  $p_1 = 36.6 \text{ kPa}$ ,  $p_2 = 84.4 \text{ kPa}$ , and  $V_2 = 0.0447 \text{ m}^3$ .



26. For two connected gears, the relation

$$\frac{d_1}{d_2} = \frac{N_1}{N_2}$$

holds, where  $d$  is the diameter of the gear and  $N$  is the number of teeth. Find  $N_1$  if  $d_1 = 2.60$  cm,  $d_2 = 11.7$  cm, and  $N_2 = 45$ . The ratio  $N_2/N_1$  is called the *gear ratio*. See Fig. 18.3.

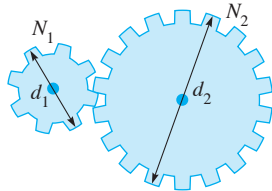


Fig. 18.3

27. In an electric instrument called a “Wheatstone bridge,” electric resistances are related by

$$\frac{R_1}{R_2} = \frac{R_3}{R_4}$$

Find  $R_2$  if  $R_1 = 6.00 \Omega$ ,  $R_3 = 62.5 \Omega$ , and  $R_4 = 15.0 \Omega$ . See Fig. 18.4.

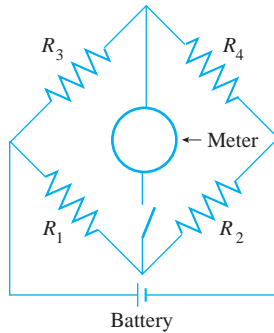


Fig. 18.4

28. In a transformer, an electric current in one coil of wire induces a current in a second coil. For a transformer,

$$\frac{i_1}{i_2} = \frac{t_2}{t_1}$$

where  $i$  is the current and  $t$  is the number of windings in each coil. In a neon sign amplifier,  $i_1 = 1.2$  A and the *turns ratio*  $t_2/t_1 = 160$ . Find  $i_2$ . See Fig. 18.5.

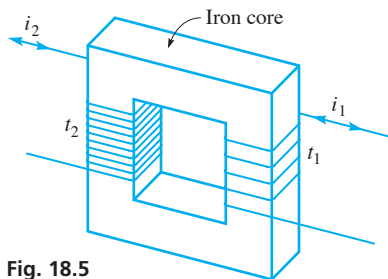


Fig. 18.5

In Exercises 29–48, answer the given questions by setting up and solving the appropriate proportions.

29. Given that  $10\,000 \text{ m}^2 = 1 \text{ ha}$ , what area in hectares (ha) is  $4500 \text{ m}^2$ ?
30. Given that  $1 \text{ W} \cdot \text{h} = 3.6 \text{ kJ}$ , what power in kilojoules is  $250 \text{ W} \cdot \text{h}$ ?
31. Given that  $0.01 \text{ d} = 864 \text{ s}$ , what time in seconds is  $2.75 \text{ d}$ ?
32. Given that  $360^\circ = 2\pi \text{ rad}$ , what angle in degrees is  $5.00 \text{ rad}$ ?

33. Given that  $10^4 \text{ cm}^2 = 10^6 \text{ mm}^2$ , what area in square centimetres is  $2.50 \times 10^5 \text{ mm}^2$ ?
34. How many metres per second are equivalent to  $45.0 \text{ km/h}$ ?
35. How many kilolitres per hour are equivalent to  $540 \text{ mL/min}$ ?
36. The water/cement ratio of a concrete mix is the ratio of the mass of water in the mix to the mass of cement in the mix. If the ratio of a mix is specified as  $0.40$ , how much water should be added to a mixture with  $25 \text{ kg}$  of cement?
37. A particular type of automobile engine produces  $62\,500 \text{ cm}^3$  of carbon monoxide in  $2.00 \text{ min}$ . How much carbon monoxide is produced in  $45.0 \text{ s}$ ?
38. An airplane consumes  $140 \text{ L}$  of gasoline in flying  $680 \text{ km}$ . Under similar conditions, how far can it fly on  $240 \text{ L}$ ?
39. Two separate sections of a roof have the same slope. If the rise and run on one section are, respectively,  $3.0 \text{ m}$  and  $6.3 \text{ m}$ , what is the run on the other section if its rise is  $4.2 \text{ m}$ ?
40. When a bullet is fired from a loosely held rifle, the ratio of the mass of the bullet to that of the rifle equals the negative of the reciprocal of the ratio of the velocity of the bullet to that of the rifle. If a  $3.0 \text{ kg}$  rifle fires a  $5.0 \text{ g}$  bullet and the velocity of the bullet is  $300 \text{ m/s}$ , what is the recoil velocity of the rifle?
41. A board  $7.5 \text{ m}$  long is cut into two pieces, the lengths of which are in the ratio  $2/3$ . Find the lengths of the pieces.
42. If  $c/d$  is in *inverse ratio* to  $a/b$ , then  $a/b = d/c$  (see Exercises 27 and 28). The current  $i$  (in A) in an electric circuit is in inverse ratio to the resistance  $R$  (in  $\Omega$ ). If  $i = 0.25 \text{ mA}$  when  $R = 2.8 \Omega$ , what is  $i$  when  $R = 7.2 \Omega$ ?
43. By mass, the ratio of chlorine to sodium in table salt is  $35.46$  to  $23.00$ . How much sodium is contained in  $50.00 \text{ kg}$  of salt?
44. An industrial cleaner is diluted 2 parts of cleaner to 5 parts of water. What volume (in mL) of cleaner should be used to get  $350 \text{ mL}$  of diluted solution?
45. In testing for quality control, it was found that 17 of every 500 computer chips produced by a company in a day were defective. If a total of 595 defective parts were found, what was the total number of chips produced during that day?
46. An electric current of  $0.772 \text{ mA}$  passes into two wires in which it is divided into currents in the ratio of  $2.83$  to  $1.09$ . What are the currents in the two wires?
47. The ratio of the width to the height of an HDTV screen is  $16/9$ . What is the length of the diagonal of an HDTV screen if it is  $98.0 \text{ cm}$  wide?
48. Of the earth’s water area, the Pacific Ocean covers  $46.0\%$ , and the Atlantic Ocean covers  $23.9\%$ . Together they cover a total of  $2.53 \times 10^8 \text{ km}^2$ . What is the area of each?

#### Answers to Practice Exercises

1.  $56 \text{ nN}$       2.  $20 \text{ g}$

## 18.2 Variation

Direct Variation • Inverse Variation • Joint Variation • Calculating Constant of Proportionality

■ Charles' law is named for the French physicist Jacques Charles (1746–1823).

Scientific laws are often stated in terms of ratios and proportions. For example, Charles' law can be stated as “for a perfect gas under constant pressure, the ratio of any two volumes this gas may occupy equals the ratio of the absolute temperatures.” Symbolically, this could be stated as  $V_1/V_2 = T_1/T_2$ . Thus, if the ratio of the volumes and one of the values of the temperature are known, we can easily find the other temperature.

By multiplying both sides of the proportion of Charles' law by  $V_2/T_1$ , we can change the form of the proportion to  $V_1/T_1 = V_2/T_2$ . This statement says that the ratio of the volume to the temperature (for constant pressure) is constant. Thus, if any pair of values of volume and temperature is known, the ratio of  $V_1/T_1$  can be calculated. This ratio of  $V_1/T_1$  can be called a constant  $k$ , which means that Charles' law can be written as  $V/T = k$ . We now have the statement that the ratio of the volume to temperature is always constant; or, as it is normally stated, “The volume is proportional to the temperature.” Therefore, we write  $V = kT$ , the clearest and most informative statement of Charles' law.

Thus, *for any two quantities always in the same proportion, we say that one is **proportional to** (or **varies directly as**) the second. To show that  $y$  is proportional to  $x$  (or varies directly as  $x$ ), we write*

Direct Variation

$$y = kx \quad (18.2)$$

where  $k$  is the **constant of proportionality**. We can also write  $y \propto x$ . This type of relationship is known as **direct variation**.

### EXAMPLE 1 Direct variation—applications

- (a) The circumference  $c$  of a circle is proportional to (varies directly as) the radius  $r$ . We write this as  $c \propto r$ , or  $c = kr$ . Since we know that  $c = 2\pi r$  for a circle, we know that in this case the constant of proportionality is  $k = 2\pi$ .
- (b) The fact that the resistance  $R$  of a wire varies directly as (is proportional to) its length  $s$  is written as  $R \propto s$ , or  $R = ks$ . As the length of the wire increases (or decreases), this equation tells us that the resistance increases (or decreases) proportionally. In this case, the constant of proportionality is different for different wires, although it remains constant for any given wire. ■

It is very common that, when two quantities are related, the product of the two quantities remains constant. In such a case,  $yx = k$ , or

Inverse Variation

$$y = \frac{k}{x} \quad (18.3)$$

This is read as “ $y$  varies inversely as  $x$ ” or “ $y$  is inversely proportional to  $x$ .” This type of relationship is known as **inverse variation**.

### EXAMPLE 2 Inverse variation—application

Boyle's law states that “at a given temperature, the pressure  $p$  of an ideal gas varies inversely as the volume  $V$ .” We write this as  $p = k/V$ . In this case, as the volume of the gas increases, the pressure decreases, or as the volume decreases, the pressure increases. ■

■ Boyle's law is named for the English physicist Robert Boyle (1627–1691).

In Fig. 18.6(a), the graph of the equation for direct variation  $y = kx$ , where  $x \geq 0$ , is shown. It is a straight line of slope  $k$  ( $k > 0$ ) and  $y$ -intercept of 0. We see that  $y$



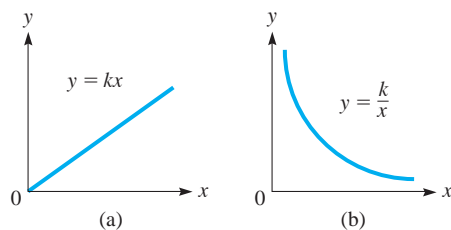


Fig. 18.6

increases as  $x$  increases, or  $y$  decreases as  $x$  decreases. In Fig. 18.6(b), the graph of the equation for inverse variation  $y = k/x$  ( $k > 0, x > 0$ ) is shown. It is a *hyperbola* (a different form of the equation from that of Example 4 of Section 14.1). As  $x$  increases,  $y$  decreases, or as  $x$  decreases,  $y$  increases.

For many relationships, one quantity varies as a specific power of another quantity. The terms *varies directly* and *varies inversely* as are used in the following example with a specific power of the independent variable.

### EXAMPLE 3 Direct variation with powers—applications

- (a) The statement that the volume  $V$  of a sphere varies directly as the cube of its radius is written as  $V = kr^3$ . In this case, we know that  $k = 4\pi/3$ . We see that as the radius increases, the volume increases much more rapidly. For example, if  $r = 2.00$  cm,  $V = 33.5$  cm<sup>3</sup>, and if  $r = 3.00$  cm,  $V = 113$  cm<sup>3</sup>.
- (b) A company finds that the number  $n$  of units of a product that are sold is inversely proportional to the square of the price  $p$  of the product. This is written as  $n = k/p^2$ . As the price is raised, the number of units that are sold decreases much more rapidly. ■

One quantity may vary as the product of two or more other quantities. Such variation is called **joint variation**. We write

Joint Variation

$$y = kxz$$

(18.4)

to show that  $y$  varies jointly as  $x$  and  $z$ .

### EXAMPLE 4 Joint variation—application

The cost  $C$  of a piece of sheet metal varies jointly as the area  $A$  of the piece and the cost  $c$  per unit area. This is written as  $C = kAc$ . Here,  $C$  increases as the product  $Ac$  increases, and decreases as the product  $Ac$  decreases. ■

Direct, inverse, and joint variations may be combined. A given relationship may be a combination of two or all three of these types of variation.

### EXAMPLE 5 Combined variation—application

Newton's *universal law of gravitation* can be stated as follows: "The force  $F$  of gravitation between two objects varies jointly as the masses  $m_1$  and  $m_2$  of the objects, and inversely as the square of the distance  $r$  between their centres." We write this as

$$F = \frac{Gm_1m_2}{r^2}$$

← force varies jointly as masses and  
← inversely as the square of the distance

where  $G$  is the constant of proportionality. ■

■ Newton's universal law of gravitation was formulated by the great English mathematician and physicist Isaac Newton (1642–1727).

#### Practice Exercise

- Express the relationship that  $y$  varies directly as the square of  $x$  and inversely as  $z$ .

### Procedure for Solving Problems Involving Variation

- Use the given statement to set up a general equation in terms of the variables and the constant of proportionality.
- Substitute one complete set of known values into the general equation and solve the resulting equation to find the constant of proportionality.
- Substitute the constant of proportionality into the general equation to find the specific equation relating the variables.
- The specific equation can then be used to find the value of any one of the variables given any set of the others.

**COMMON ERROR**

Note the use of the word *and* in Example 5. It is used to indicate that  $F$  varies in more than one way, but *it is not interpreted as addition*.

**EXAMPLE 6 Find one value—given others**

If  $y$  varies inversely as  $x$ , and  $x = 15$  when  $y = 4$ , find the value of  $y$  when  $x = 12$ .  
The solution is as follows:

$$y = \frac{k}{x} \quad \text{general equation showing inverse variation}$$

$$4 = \frac{k}{15}, \quad k = 60 \quad \text{evaluate constant of proportionality}$$

$$y = \frac{60}{x} \quad \text{specific equation relating } y \text{ and } x$$

$$y = \frac{60}{12} = 5 \quad \text{evaluating } y \text{ for } x = 12 \quad \blacksquare$$

**Practice Exercise**

2. If  $y$  varies directly as  $x$ , and  $y = 5$  when  $x = 20$ , find the value of  $y$  when  $x = 12$ .

**EXAMPLE 7 Variation as a square root—application**

The frequency  $f$  of vibration of a wire varies directly as the square root of the tension  $T$  of the wire. If  $f = 420$  Hz when  $T = 1.14$  N, find  $f$  when  $T = 3.40$  N.

The steps in making this evaluation are outlined below:

$$f = k\sqrt{T} \quad \text{set up general equation: } f \text{ varies directly as } \sqrt{T}$$

$$420 \text{ Hz} = k\sqrt{1.14 \text{ N}} \quad \text{substitute given set of values and evaluate } k$$

$$k = 393 \text{ Hz/N}^{1/2}$$

$$f = 393 \text{ Hz/N}^{1/2} \sqrt{T} \quad \text{substitute value of } k \text{ to get specific equation}$$

$$f = 393 \text{ Hz/N}^{1/2} \sqrt{3.40 \text{ N}} \quad \text{evaluate } f \text{ for } T = 3.40 \text{ N}$$

$$f = 725 \text{ Hz} \quad \blacksquare$$

**LEARNING TIP**

In applied situations such as Example 7, the constant of proportionality usually has a set of units associated with it. As long as all values used for the same variable are expressed in the same units, the units for the final variable evaluated will be the appropriate ones.

**EXAMPLE 8 Variation as a square—application**

The heat  $H$  developed in an electric resistor varies jointly as the time  $t$  and the square of the current  $i$  in the resistor. If  $H_0$  joules of heat are developed in  $t_0$  seconds with  $i_0$  amperes passing through the resistor, how much heat is developed if both the time and the current are doubled?

$$H = kt^2 \quad \text{set up general equation}$$

$$H_0 \text{ J} = k(t_0 \text{ s})(i_0 \text{ A})^2 \quad \text{substitute given values and evaluate } k$$

$$k = \frac{H_0}{t_0 i_0^2} \text{ J}/(\text{s} \cdot \text{A}^2)$$

$$H = \frac{H_0 t^2}{t_0 i_0^2} \quad \text{substitute for } k \text{ to get specific equation}$$

We are asked to determine  $H$  when both the time and the current are doubled. This means we are to substitute  $t = 2t_0$  and  $i = 2i_0$ . Making this substitution,

$$H = \frac{H_0(2t_0)(2i_0)^2}{t_0 i_0^2} = \frac{8H_0 t_0 i_0^2}{t_0 i_0^2} = 8H_0$$

Since  $H = H_0$  when  $t = t_0$  and  $i = i_0$ , when time and current are doubled, the heat developed is eight times that for the original values of  $i$  and  $t$ .  $\blacksquare$

■ See the chapter introduction.

■ The first landing on the moon was by the crew of the U. S. spacecraft *Apollo 11* in July 1969.

### EXAMPLE 9 Application of Newton's universal law

In Example 5, we stated Newton's universal law of gravitation. This law was formulated in the late seventeenth century, but it has numerous modern space-age applications. Use this law to solve the following problem.

A spacecraft is travelling from the earth to the moon, which are 390 000 km apart. The mass of the moon is 0.0123 that of the earth. How far from the earth is the gravitational force of the earth on the spacecraft equal to the gravitational force of the moon on the spacecraft?

From Example 5, we have the gravitational force between two objects as

$$F = \frac{Gm_1m_2}{r^2}$$

where the constant of proportionality  $G$  is the same for any two objects. Since we want the force between the earth and the spacecraft to equal the force between the moon and the spacecraft, we have

$$\frac{Gm_s m_e}{r^2} = \frac{Gm_s m_m}{(390\,000 - r)^2}$$

where  $m_s$ ,  $m_e$ , and  $m_m$  are the masses of the spacecraft, the earth, and the moon, respectively;  $r$  is the distance from the earth to the spacecraft; and  $390\,000 - r$  is the distance from the moon to the spacecraft. Since  $m_m = 0.0123m_e$ , we have

$$\frac{Gm_s m_e}{r^2} = \frac{Gm_s (0.0123m_e)}{(390\,000 - r)^2}$$

$$\frac{1}{r^2} = \frac{0.0123}{(390\,000 - r)^2} \quad \text{divide each side by } Gm_s m_e$$

$$(390\,000 - r)^2 = 0.0123r^2 \quad \text{multiply each side by LCD}$$

$$390000^2 - 780000r + r^2 = 0.0123r^2$$

$$390000^2 - 780000r + 0.9877r^2 = 0$$

$$r = \frac{780000 \pm \sqrt{780000^2 - 4(0.9877)(390000^2)}}{2(0.9877)}$$

$$r_1 = 439000 \text{ km}$$

$$r_2 = 351000 \text{ km}$$

The only valid solution is  $r^2 = 351000$ , so that the spacecraft is 351 000 km from the earth and 39 000 km from the moon when the gravitational forces are equal. See Fig. 18.7.

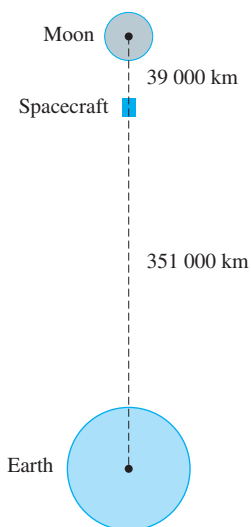


Fig. 18.7

## EXERCISES 18.2

In Exercises 1–4, make the given changes in the indicated examples of this section and solve the indicated problems.

- In Example 1(a), change radius  $r$  to diameter  $d$ , write the appropriate equation, and find the value of the constant of proportionality.
- In Example 6, change “inversely as  $x$ ” to “inversely as the square of  $x$ ” and then solve the resulting problem.
- In Example 7, change 1.14 N to 1.35 N and then solve the resulting problem.

- In Example 8, change “both the time and current are doubled” to “the time is halved and the current is doubled” and then solve the resulting problem.

In Exercises 5–12, set up the general equations from the given statements.

- The speed  $v$  at which a galaxy is moving away from earth varies directly as its distance  $r$  from earth.
- The demand  $D$  for a product varies inversely as its price  $P$ .

7. The electric resistance  $R$  of a wire varies inversely as the square of its diameter  $d$ .
8. The volume  $V$  of silt carried by a river is proportional to the sixth power of the velocity  $v$  of the river.
9. In a tornado, the pressure  $p$  that a roof will withstand is inversely proportional to the square root of the area  $A$  of the roof.
10. During an adiabatic (no heat loss or gain) expansion of a gas, the pressure  $p$  is inversely proportional to the  $3/2$  power of the volume  $V$ .
11. The stiffness  $S$  of a beam varies jointly as its width  $w$  and the cube of its depth  $d$ .
12. The average electric power  $P$  entering a load varies jointly as the resistance  $R$  of the load and the square of the effective voltage  $V$ , and inversely as the square of the impedance  $Z$ .

In Exercises 13–16, express the meaning of the given equation in a verbal statement, using the language of variation. ( $k$  and  $\pi$  are constants.)

13.  $A = \pi r^2$
14.  $s = \frac{k}{t^{1.2}}$
15.  $f = \frac{kL}{\sqrt{m}}$
16.  $V = \pi r^2 h$

In Exercises 17–20, give the specific equation relating the variables after evaluating the constant of proportionality for the given set of values.

17.  $V$  varies directly as the square of  $H$ , and  $V = 2$  when  $H = 64$ .
18.  $n$  is inversely proportional to the square of  $p$ , and  $n = \frac{1}{27}$  when  $p = 3$ .
19.  $p$  is proportional to  $q$  and inversely proportional to the cube of  $r$ , and  $p = 6$  when  $q = 3$  and  $r = 2$ .
20.  $v$  is proportional to  $t$  and the square root of  $s$ , and  $v = 80$  when  $s = 4$  and  $t = 5$ .

In Exercises 21–28, find the required value by setting up the general equation and then evaluating.

21. Find  $y$  when  $x = 10$  if  $y$  varies directly as  $x$ , and  $y = 20$  when  $x = 8$ .
22. Find  $y$  when  $x = 5$  if  $y$  varies directly as the square of  $x$ , and  $y = 6$  when  $x = 8$ .
23. Find  $s$  when  $t = 720$  if  $s$  is inversely proportional to  $t$ , and  $s = 800$  when  $t = 0.030$ .
24. Find  $p$  for  $q = 0.8$  if  $p$  is inversely proportional to the square of  $q$ , and  $p = 18$  when  $q = 0.2$ .
25. Find  $y$  for  $x = 6$  and  $z = 5$  if  $y$  varies directly as  $x$  and inversely as  $z$ , and  $y = 60$  when  $x = 4$  and  $z = 10$ .
26. Find  $r$  when  $n = 16$  if  $r$  varies directly as the square root of  $n$  and  $r = 4$  when  $n = 25$ .
27. Find  $f$  when  $p = 2$  and  $c = 4$  if  $f$  varies jointly as  $p$  and the cube of  $c$ , and  $f = 8$  when  $p = 4$  and  $c = 0.1$ .
28. Find  $v$  when  $r = 2$ ,  $s = 3$ , and  $t = 4$  if  $v$  varies jointly as  $r$  and  $s$  and inversely as the square of  $t$ , and  $v = 8$  when  $r = 2$ ,  $s = 6$ , and  $t = 6$ .

In Exercises 29 and 30,  $A$  varies directly as  $x$ , and  $B$  varies directly as  $x$ , although not in the same proportion as  $A$ . All numbers are positive.

29. Show that  $A + B$  varies directly as  $x$ .
30. Show that  $\sqrt{AB}$  varies directly as  $x$ .

In Exercises 31–60, solve the given applied problems involving variation.

31. The time  $t$  it takes to download a program onto a computer hard drive is proportional to the memory  $M$  required for the program. If it takes 45.0 min to download a program with 27.5 MB (mega-bytes) of memory required, what is the rate of downloading in kB (kilobytes) per second?
32. The volume  $V$  of carbon dioxide ( $\text{CO}_2$ ) that is exhausted from a room in a given time varies directly as the initial volume  $V_0$  that is present. If  $75 \text{ m}^3$  of  $\text{CO}_2$  are removed in 1 h from a room with an initial volume of  $160 \text{ m}^3$ , how much is removed in 1 h if the initial volume is  $130 \text{ m}^3$ ?
33. The amount of heat  $H$  required to melt ice is proportional to the mass  $m$  of ice that is melted. If it takes  $2.93 \times 10^5 \text{ J}$  to melt 875 g of ice, how much heat is required to melt 625 g?
34. In electroplating, the mass  $m$  of the material deposited varies directly as the time  $t$  during which the electric current is on. Set up the equation for this relationship if 2.50 g are deposited in 5.25 h.
35. Hooke's law states that the force needed to stretch a spring is proportional to the amount the spring is stretched. If 10.0 N stretches a certain spring 4.00 cm, how much will the spring be stretched by a force of 6.00 N?
36. The rate  $H$  of heat removal by an air conditioner is proportional to the electric power input  $P$ . The constant of proportionality is the *performance coefficient*. Find the performance coefficient of an air conditioner for which  $H = 1.8 \text{ kW}$  and  $P = 720 \text{ W}$ .
37. The energy  $E$  available daily from a solar collector varies directly as the percent  $p$  that the sun shines during the day. If a collector provides 1200 kJ for 75% sunshine, how much does it provide for a day during which there is 35% sunshine?
38. The distance  $d$  that can be seen from horizon to horizon from an airplane varies directly as the square root of the altitude  $h$  of the airplane. If  $d = 213 \text{ km}$  for  $h = 3950 \text{ m}$ , find  $d$  for  $h = 5250 \text{ m}$ .
39. The time  $t$  required to empty a wastewater-holding tank is inversely proportional to the cross-sectional area  $A$  of the drainage pipe. If it takes 2.0 h to empty a tank with a drainage pipe for which  $A = 48 \text{ cm}^2$ , how long will it take to empty the tank if  $A = 68 \text{ cm}^2$ ?
40. The time  $t$  required to make a particular trip is inversely proportional to the average speed  $v$ . If a jet takes 2.75 h at an average speed of 520 km/h, how long will it take at an average speed of 620 km/h? Explain the meaning of the constant of proportionality.
41. In a physics experiment, a given force was applied to three objects. The mass  $m$  and the resulting acceleration  $a$  were recorded as follows:

$m$ (g)	2.0	3.0	4.0
$a$ ( $\text{cm/s}^2$ )	30	20	15

- (a) Is the relationship  $a = f(m)$  one of direct or inverse variation? Explain. (b) Find  $a = f(m)$ .

42. The lift  $L$  of each of three model airplane wings of width  $w$  was measured and recorded as follows:

$w$ (cm)	20	40	60
$L$ (N)	10	40	90

- (a) Is the relationship  $L = f(w)$  one of direct or inverse variation? Explain. (b) Find  $L = f(w)$ .
43. The power  $P$  required to propel a ship varies directly as the cube of the speed  $s$  of the ship. If 3.88 MW will propel a ship at 19.3 km/h, what power is required to propel it at 24.2 km/h?
44. The  $f$ -number lens setting of a camera varies directly as the square root of the time  $t$  that the film is exposed. If the  $f$ -number is 8 (written as  $f/8$ ) for  $t = 0.0200$  s, find the  $f$ -number for  $t = 0.0098$  s.
45. The force  $F$  on the blade of a wind generator varies jointly as the blade area  $A$  and the square of the wind velocity  $v$ . Find the equation relating  $F$ ,  $A$ , and  $v$  if  $F = 76.5$  N when  $A = 0.372$  m<sup>2</sup> and  $v = 9.42$  m/s.
46. The escape velocity  $v$  a spacecraft needs to leave the gravitational field of a planet varies directly as the square root of the product of the planet's radius  $R$  and its acceleration due to gravity  $g$ . For Mars and earth,  $R_M = 0.533R_e$  and  $g_M = 0.400g_e$ . Find  $v_M$  for Mars if  $v_e = 11.2$  km/s.
47. The force  $F$  between two parallel wires carrying electric currents is inversely proportional to the distance  $d$  between the wires. If a force of 0.750 N exists between wires that are 1.25 cm apart, what is the force between them if they are separated by 1.75 cm?
48. The velocity  $v$  of a pulse travelling in a string varies directly as the square root of the tension  $T$  in the string. If the velocity of a pulse in a string is 150 m/s when the tension is 90.0 N, find the velocity when the tension is 135 N.
49. The average speed  $s$  of oxygen molecules in the air is directly proportional to the square root of the absolute temperature  $T$ . If the speed of the molecules is 460 m/s at 273 K, what is the speed at 300 K?
50. The time  $t$  required to test a computer memory unit varies directly as the square of the number  $n$  of memory cells in the unit. If a unit with 4800 memory cells can be tested in 15.0 s, how long does it take to test a unit with 8400 memory cells?
51. The electric resistance  $R$  of a wire varies directly as its length  $l$  and inversely as its cross-sectional area  $A$ . Find the relation between resistance, length, and area for a wire that has a resistance of 0.200  $\Omega$  for a length of 60.0 m and cross-sectional area of 0.007 80 cm<sup>2</sup>.
52. The general gas law states that the pressure  $p$  of an ideal gas varies directly as the thermodynamic temperature  $T$  and inversely as the volume  $V$ . If  $p = 610$  kPa for  $V = 10.0$  cm<sup>3</sup> and  $T = 290$  K, find  $V$  for  $p = 400$  kPa and  $T = 400$  K.
53. The power  $P$  in an electric circuit varies jointly as the resistance  $R$  and the square of the current  $I$ . If the power is 10.0 W when the current is 0.500 A and the resistance is 40.0  $\Omega$ , find the power if the current is 2.00 A and the resistance is 20.0  $\Omega$ .

54. The difference  $m_1 - m_2$  in magnitudes (visual brightnesses) of two stars varies directly as the base 10 logarithm of the ratio  $b_2/b_1$  of their actual brightnesses. For two particular stars, if  $b_2 = 100b_1$  for  $m_1 = 7$  and  $m_2 = 2$ , find the equation relating  $m_1$ ,  $m_2$ ,  $b_1$ , and  $b_2$ .
55. The power gain  $G$  by a parabolic microwave dish varies directly as the square of the diameter  $d$  of the opening and inversely as the square of the wavelength  $\lambda$  of the wave carrier. Find the equation relating  $G$ ,  $d$ , and  $\lambda$  if  $G = 5.5 \times 10^4$  for  $d = 2.9$  m and  $\lambda = 3.0$  cm.
56. The intensity  $I$  of sound varies directly as the power  $P$  of the source and inversely as the square of the distance  $r$  from the source. Two sound sources are separated by a distance  $d$ , and one has twice the power output of the other. Where should an observer be located on a line between them such that the intensity of each sound is the same?
57. The  $x$ -component of the acceleration of an object moving around a circle with constant angular velocity  $\omega$  varies jointly as  $\cos \omega t$  and the square of  $\omega$ . If the  $x$ -component of the acceleration is  $-11.4$  cm/s<sup>2</sup> when  $t = 1.00$  s for  $\omega = 0.524$  rad/s, find the  $x$ -component of the acceleration when  $t = 2.00$  s.
58. The tangent of the proper banking angle  $\theta$  of the road for a car making a turn is directly proportional to the square of the car's velocity  $v$  and inversely proportional to the radius  $r$  of the turn. If  $7.75^\circ$  is the proper banking angle for a car travelling at 20.0 m/s around a turn of radius 300 m, what is the proper banking angle for a car travelling at 30.0 m/s around a turn of radius 250 m? See Fig. 18.8.

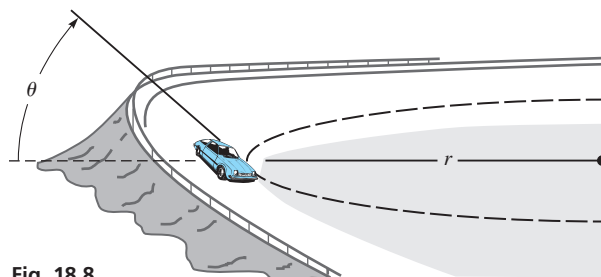


Fig. 18.8

59. The acoustical intensity  $I$  of a sound wave is proportional to the square of the pressure amplitude  $p$  and inversely proportional to the velocity  $v$  of the wave. If  $I = 0.474$  W/m<sup>2</sup> for  $p = 20.0$  Pa and  $v = 346$  m/s, find  $I$  if  $p = 15.0$  Pa and  $v = 320$  m/s.
60. To cook a certain vegetable mix in a microwave oven, the instructions are to cook 120 g for 2.5 min or 240 g for 3.5 min. Assuming the cooking time  $t$  is proportional to some power (not necessarily integral) of the weight  $w$ , use logarithms to find  $t$  as a function of  $w$ .

#### Answers to Practice Exercises

1.  $y = kx^2/z$     2.  $y = 3$

## CHAPTER 18 EQUATIONS

$$\text{Proportion} \quad \frac{a}{b} = \frac{c}{d} \quad (18.1)$$

$$\text{Direct variation} \quad y = kx \quad (18.2)$$

$$\text{Inverse variation} \quad y = \frac{k}{x} \quad (18.3)$$

$$\text{Joint variation} \quad y = kxz \quad (18.4)$$

## CHAPTER 18 REVIEW EXERCISES

In Exercises 1–14, find the indicated ratios.

- 4 Mg to 20 kg
- 300 nm to 6  $\mu\text{m}$
- 375 mL to 25 cL
- 12 ks to 2 h
- The number  $\pi$  equals the ratio of the circumference  $c$  of a circle to its diameter  $d$ . To check the value of  $\pi$ , a technician used computer simulation to measure the circumference and diameter of a metal cylinder and found the values to be  $c = 4.2736$  cm and  $d = 1.3603$  cm. What value of  $\pi$  did the technician get?
- The ratio of the diagonal  $d$  of a square to the side  $s$  of the square is  $\sqrt{2}$ . The diagonal and side of the face of a glass cube are found to be  $d = 35.375$  mm and  $s = 25.014$  mm. What is the value of  $\sqrt{2}$  found from these measurements?
- The mechanical advantage of a lever is the ratio of the output force  $F_o$  to the input force  $F_i$ . Find the mechanical advantage if  $F_o = 28$  kN and  $F_i = 5000$  N.
- For an automobile, the ratio of the number  $n_1$  of teeth on the ring gear to the number  $n_2$  of teeth on the pinion gear is the *rear axle ratio* of the car. Find this ratio if  $n_1 = 64$  and  $n_2 = 20$ .
- The pressure  $p$  exerted on a surface is the ratio of the force  $F$  on the surface to its area  $A$ . Find the pressure on a square patch, 2.25 cm on a side, on a tank if the force on the patch is 37.4 N.
- The electric resistance  $R$  of a resistor is the ratio of the voltage  $V$  across the resistor to the current  $i$  in the resistor. Find  $R$  if  $V = 0.632$  V and  $i = 2.03$  mA.
- The *heat of vaporization* of a substance is the amount of heat needed to change a unit amount from liquid to vapour. Experimentation shows 7910 J are needed to change 3.50 g of water to steam. What is the heat of vaporization of water?
- The ratio of the effective length of a column to its radius of gyration is the *slenderness ratio*. What is the slenderness ratio of a column of effective length 3.15 m and radius of gyration 60.0 mm?
- The ratio of the commission for selling a home to the selling price of the home is the commission rate. What is this rate if the commission of \$20 900 is charged for selling a home priced at \$380 000?
- A total cholesterol level of 200 mg/dL is considered too high, but many doctors consider the ratio of the total cholesterol to HDL (high density lipids—the “good” cholesterol) a more important measure of risk to coronary heart disease. What is this

ratio if the total cholesterol is 179 mg/dL and the HDL is 39 mg/dL? (Average for women is about 4.5, and for men is about 5.0.)

In Exercises 15 and 16, given that  $a/b = c/d$  ( $b$  and  $d$  not zero), show that the indicated proportions are correct.

$$15. \frac{a+b}{b} = \frac{c+d}{d} \quad 16. \frac{a-b}{b} = \frac{c-d}{d}$$

In Exercises 17–32, answer the given questions by setting up and solving the appropriate proportions.

- On a map of Australia, 37 mm represents 300 km. If the distance on the map between Melbourne and Hobart, Tasmania, is 78 mm, how far is Hobart from Melbourne?
- Given that 1.000 kg = 1000 g, what is the mass in kilograms of a 14.0-g computer disk?
- Given that 1.00 kJ =  $10^6$  mJ, how much heat in kilojoules is produced by a heating element that produces 2660 mJ?
- Given that 1.00 L = 1000 cm<sup>3</sup>, what capacity in litres has a cubical box that is 3.23 cm along an edge?
- A computer printer can print 50 pages in 2 min 5 s. How many pages can it print in 10.0 min?
- A solar heater with a collector area of 58.0 m<sup>2</sup> is required to heat 2560 kg of water. Under the same conditions, how much water can be heated by a rectangular solar collector 9.50 m by 8.75 m?
- The dosage of a certain medicine is 25 mL for each 10 kg of the patient’s weight. What is the dosage for a person weighing 56 kg?
- A woman invests \$50 000 and a man invests \$20 000 in a partnership. If profits are to be shared in the ratio that each invested in the partnership, how much does each receive from \$10 500 in profits?
- On a certain blueprint, a measurement of 25.0 m is represented by 2.00 mm. What is the actual distance between two points if they are 5.75 mm apart on the blueprint?
- The chlorine concentration in a water supply is 0.12 part per million. How much chlorine is there in a cylindrical holding tank 4.22 m in radius and 5.82 m high filled from the water supply?
- One fibre-optic cable carries 60.0% as many messages as another fibre-optic cable. Together they carry 12 000 messages. How many does each carry?



28. Two types of roadbed material, one 50% rock and the other 100% rock, are used in the ratio of 4 to 1 to form a roadbed. If a total of 150 Mg are used, how much rock is in the roadbed?
29. To neutralize 80.0 kg of sodium hydroxide, 98.0 kg of sulfuric acid are needed. How much sodium hydroxide can be neutralized with 37.0 kg of sulfuric acid?
30. For analysis, 105 mg of DNA sample is divided into two vials in the ratio of 2 to 5. How many milligrams are in each vial?
31. A total of 322 bolts are in two containers. The ratio of the number of bolts in the first container to the number in the second container is  $5/9$ . How many are in each container?
32. A gasoline company sells octane-87 gas and octane-91 gas in the ratio of 9 to 2. How many of each octane are sold of a total of 16.5 million litres?

In Exercises 33–36, give the specific equation relating the variables after evaluating the constant of proportionality for the given set of values.

33.  $y$  varies directly as the square of  $x$ , and  $y = 27$  when  $x = 3$ .
34.  $f$  varies inversely as  $l$ , and  $f = 5$  when  $l = 8$ .
35.  $v$  is directly proportional to  $x$  and inversely proportional to the cube of  $y$ , and  $v = 10$  when  $x = 5$  and  $y = 4$ .
36.  $r$  varies jointly as  $u$ ,  $v$ , and the square of  $w$ , and  $r = 8$  when  $u = 2$ ,  $v = 4$ , and  $w = 3$ .

In Exercises 37–77, solve the given applied problems.

37. For the lever balanced at the fulcrum, the relation

$$\frac{F_1}{F_2} = \frac{L_2}{L_1}$$

holds, where  $F_1$  and  $F_2$  are forces on opposite sides of the fulcrum at distances  $L_1$  and  $L_2$  (see Fig. 18.9). Find  $L_2$  if  $F_1 = 4.50$  N,  $F_2 = 6.75$  N, and  $L_1 = 17.5$  cm.

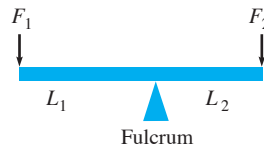


Fig. 18.9

38. A company finds that the volume  $V$  of sales of a certain item and the price  $P$  of the item are related by

$$\frac{P_1}{P_2} = \frac{V_2}{V_1}$$

Find  $V_2$  if  $P_1 = \$8.00$ ,  $P_2 = \$6.00$ , and  $V_1 = 3000$  items per week.

39. The image height  $h$  and object height  $H$  for the lens shown in Fig. 18.10 are related to the image distance  $q$  and object distance  $p$  by

$$\frac{h}{H} = \frac{q}{p}$$

Find  $q$  if  $h = 24$  cm,  $H = 84$  cm, and  $p = 36$  cm.

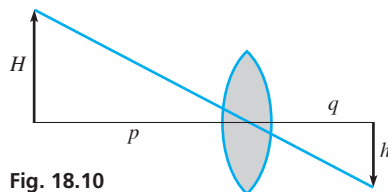


Fig. 18.10

40. For two pulleys connected by a belt, the relation

$$\frac{d_1}{d_2} = \frac{n_2}{n_1}$$

holds, where  $d$  is the diameter of the pulley and  $n$  is the number of revolutions per unit time it makes. Find  $n_2$  if  $d_1 = 4.60$  cm,  $d_2 = 8.30$  cm, and  $n_1 = 18.0$  r/min.

41. An apartment owner charges rent  $R$  proportional to the floor area  $A$  of the apartment. Find the equation relating  $R$  and  $A$  if an apartment of  $100$  m<sup>2</sup> rents for \$850/month.
42. The number  $r$  of aluminum cans that can be made by recycling  $n$  used cans is proportional to  $n$ . How many cans can be made from 50 000 used cans if  $r = 115$  cans for  $n = 125$  cans?
43. Under certain conditions, the rate of increase  $v$  of bacteria is proportional to the number  $N$  of bacteria present. Find  $v$  for  $N = 7500$  bacteria, if  $v = 800$  bacteria/h for  $N = 4000$  bacteria.
44. The index of refraction  $n$  of a medium varies inversely as the velocity of light  $v$  within it. For quartz,  $n = 1.46$  and  $v = 2.05 \times 10^8$  m/s. What is  $n$  for a diamond, in which  $v = 1.24 \times 10^8$  m/s?
45. The force  $F$  needed to tighten a bolt varies as the length  $L$  of the wrench handle. See Fig. 18.11. If  $F = 250$  N for  $L = 22$  cm, and  $F = 550$  N for  $L = 10$  cm, determine the equation relating  $F$  and  $L$ .

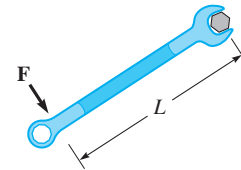


Fig. 18.11

46. In the two-base method of barometric levelling, bases are established at two points of known elevation, one at the lowest point and one at the highest point. An altimeter at an unknown elevation is read at the same time that both the upper and lower base altimeters are read. The relation

$$\frac{H_R - H_L}{H_U - H_L} = \frac{R_R - R_L}{R_U - R_L}$$

holds, where  $H_L$ ,  $H_U$ , and  $H_R$  are the elevations at the lower base, upper base, and roving altimeter stations (in mm), and  $R_L$ ,  $R_U$ , and  $R_R$  are the altimeter readings at the lower base, upper base, and roving altimeter stations (in m). Find  $H_R$  if  $H_U = 412.3$  m,  $H_L = 107.5$  m,  $R_U = 360$  mm,  $R_L = 108$  m, and  $R_R = 135$  mm. See Fig. 18.12.

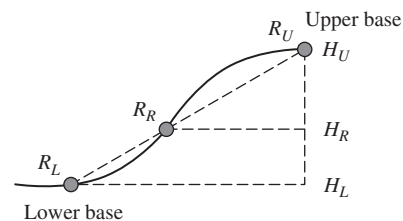


Fig. 18.12



47. The period  $T$  of a pendulum varies directly as the square root of its length  $L$ . If  $T = \pi/2$  s for  $L = 61.0$  cm, find  $T$  for  $L = 122$  cm.
48. The component of velocity  $v_x$  of an object moving in a circle with constant angular velocity  $\omega$  varies jointly with  $\omega$  and  $\sin \omega t$ . If  $\omega = \pi/6$  rad/s, and  $v_x = -4\pi$  cm/s when  $t = 1.00$  s, find  $v_x$  when  $t = 9.00$  s.
49. The charge  $C$  on a capacitor varies directly as the voltage  $V$  across it. If the charge is  $6.3 \mu\text{C}$  with a voltage of  $220$  V across a capacitor, what is the charge on it with a voltage of  $150$  V across it?
50. The amount of natural gas burned is proportional to the amount of oxygen consumed. If  $24.0$  kg of oxygen is consumed in burning  $15.0$  kg of natural gas, how much air, which is  $23.2\%$  oxygen by weight, is consumed to burn  $50.0$  kg of natural gas?
51. The power  $P$  of a gas engine is proportional to the area  $A$  of the piston. If an engine with a piston area of  $50.0$  cm<sup>2</sup> can develop  $22.5$  kW, what power is developed by an engine with a piston area of  $40.0$  cm<sup>2</sup>?
52. The decrease in temperature above a region is directly proportional to the altitude above the region. If the temperature  $T$  at the base of the Rock of Gibraltar is  $22.0^\circ\text{C}$  and a plane  $3.50$  km above notes that the temperature is  $1.0^\circ\text{C}$ , what is the temperature at the top of Gibraltar, the altitude of which is  $430$  m? (Assume there are no other temperature effects.)
53. The distance  $d$  an object falls under the influence of gravity varies directly as the square of the time  $t$  of fall. A chunk of ice falls from the top of the CN Tower and travels  $19.62$  m in  $2.00$  s. It hits the ground after  $10.43$  s. How tall is the tower?
54. The kinetic energy  $E$  of a moving object varies jointly as the mass  $m$  of the object and the square of its velocity  $v$ . If a  $5.00$ -kg object, travelling at  $10.0$  m/s, has a kinetic energy of  $250$  J, find the kinetic energy of an  $8.00$ -kg object moving at  $50.0$  m/s.
55. In a particular computer design,  $N$  numbers can be sorted in a time proportional to the square of  $\log N$ . How many times longer does it take to sort  $8000$  numbers than to sort  $2000$  numbers?
56. The velocity  $v$  of a jet of fluid flowing from an opening in the side of a container is proportional to the square root of the depth  $d$  of the opening. If the velocity of the jet from an opening at a depth of  $1.22$  m is  $4.88$  m/s, what is the velocity of a jet from an opening at a depth of  $7.62$  m? See Fig. 18.13.

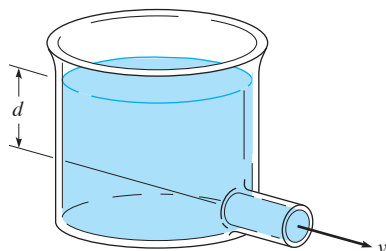


Fig. 18.13

57. In an electric circuit containing an inductance  $L$  and a capacitance  $C$ , the resonant frequency  $f$  is inversely proportional to the square root of the capacitance. If the resonant frequency in a circuit is  $25.0$  Hz and the capacitance is  $95.0 \mu\text{F}$ , what is the resonant frequency of this circuit if the capacitance is  $25.0 \mu\text{F}$ ?

58. The rate of emission  $R$  of radiant energy from the surface of a body is proportional to the fourth power of the thermodynamic temperature  $T$ . Given that a  $25.0$ -W (the rate of emission) lamp has an operating temperature of  $2500$  K, what is the operating temperature of a similar  $40.0$ -W lamp?
59. The frequency  $f$  of a radio wave is inversely proportional to its wavelength  $\lambda$ . The constant of proportionality is the velocity of the wave, which equals the speed of light. Find this velocity if an FM radio wave has a frequency of  $90.9$  MHz and a wavelength of  $3.29$  m.
60. The acceleration of gravity  $g$  on a satellite in orbit around the earth varies inversely as the square of its distance  $r$  from the centre of the earth. If  $g = 8.7$  m/s<sup>2</sup> for a satellite at an altitude of  $400$  km above the surface of the earth, find  $g$  if it is  $1000$  km above the surface. The radius of the earth is  $6.4 \times 10^6$  m.
61. If the weight  $w$  of an airplane varies directly as the cube of its length  $L$ , and  $w = 15\,400$  N for  $L = 15$  m, what would be the weight of a plane that is  $550$  cm long?
62. The thrust  $T$  of a propeller varies jointly as the square of the number  $n$  of revolutions per second and the fourth power of its diameter  $d$ . What is the effect on  $T$  if  $n$  is doubled and  $d$  is halved?
63. Using *holography* (a method of producing an image without using a lens), an image of concentric circles is formed. The radius  $r$  of each circle varies directly as the square root of the wavelength  $\lambda$  of the light used. If  $r = 3.56$  cm for  $\lambda = 575$  nm, find  $r$  if  $\lambda = 483$  nm.
64. A metal circular ring has a circular cross section of radius  $r$ . If  $R$  is the radius of the ring (measured to the middle of the cross section), the volume  $V$  of metal in the ring varies directly as  $R$  and the square of  $r$ . If  $V = 2550$  mm<sup>3</sup> for  $r = 2.32$  mm and  $R = 24.0$  mm, find  $V$  for  $r = 3.50$  mm and  $R = 32.0$  mm. See Fig. 18.14.

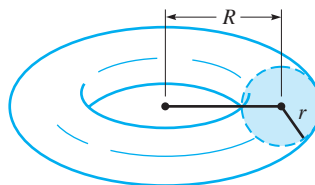


Fig. 18.14

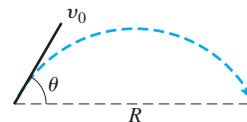


Fig. 18.15

65. The range  $R$  of a projectile varies jointly as the square of its initial velocity  $v_0$  and the sine of twice the angle  $\theta$  from the horizontal at which it is fired. See Fig. 18.15. A bullet for which  $v_0 = 850$  m/s and  $\theta = 22.0^\circ$  has a range of  $5.12 \times 10^4$  m. Find the range if  $v_0 = 750$  m/s and  $\theta = 43.2^\circ$ .
66. Kepler's third law of planetary motion states that the square of the period of any planet is proportional to the cube of the mean radius (about the sun) of that planet, with the constant of proportionality being the same for all planets. Using the fact that the period of the earth is  $1$  year and its mean radius is  $150 \times 10^6$  km, calculate the mean radius for Venus, given that its period is  $7.38$  months.
67. The stopping distance  $d$  of a car varies directly as the square of the velocity  $v$  of the car when the brakes are applied. A car moving at  $48$  km/h can stop in  $15$  m. What is the stopping distance for the car if it is moving at  $85$  km/h?

68. The load  $L$  that a helical spring can support varies directly as the cube of its wire diameter  $d$  and inversely as its coil diameter  $D$ . A spring for which  $d = 0.120$  cm and  $D = 0.953$  cm can support 45.0 N. What is the coil diameter of a similar spring that supports 78.5 N and for which  $d = 0.156$  cm?
69. The volume rate of flow  $R$  of blood through an artery varies directly as the fourth power of the radius  $r$  of the artery and inversely as the distance  $d$  along the artery. If an operation is successful in effectively increasing the radius of an artery by 25% and decreasing its length by 2%, by how much is the volume rate of flow increased?
70. The safe, uniformly distributed load  $L$  on a horizontal beam, supported at both ends, varies jointly as the width  $w$  and the square of the depth  $d$  and inversely as the distance  $D$  between supports. Given that one beam has double the dimensions of another, how many times heavier is the safe load it can support than the first can support?
71. A bank statement exactly 30 years old is discovered. It states, "This 10-year-old account is now worth \$185.03 and pays 4% interest compounded annually." An investment with annual compound interest varies directly as  $1 + r$  to the power  $n$ , where  $r$  is the interest rate expressed as a decimal and  $n$  is the number of years of compounding. What was the value of the original investment, and what is it worth now?
72. The distance  $s$  that an object falls due to gravity varies jointly as the acceleration  $g$  due to gravity and the square of the time  $t$  of fall. The acceleration due to gravity on the moon is 0.172 of that on earth. If a rock falls for  $t_0$  seconds on earth, how many times farther would the rock fall on the moon in  $3t_0$  seconds?
73. The heat loss  $L$  through fibreglass insulation varies directly as the time  $t$  and inversely as the thickness  $d$  of the fibreglass. If the loss through 20.0 cm of fibreglass is 1.20 MJ in 30 min, what is the loss through 15.0 cm in 1 h 30 min?
74. A quantity important in analysing the rotation of an object is its *moment of inertia*  $I$ . For a ball bearing, the moment of inertia varies directly as its mass  $m$  and the square of its radius  $r$ . Find the general expression for  $I$  if  $I = 39.9 \text{ g} \cdot \text{cm}^2$  for  $m = 63.8$  g and  $r = 1.25$  cm.
75. In the study of polarized light, the intensity  $I$  is proportional to the square of the cosine of the angle  $\theta$  of transmission. If  $I = 0.025 \text{ W/m}^2$  for  $\theta = 12.0^\circ$ , find  $I$  for  $\theta = 20.0^\circ$ .
76. The force  $F$  that acts on a pendulum bob is proportional to the mass  $m$  of the bob and the sine of the angle  $\theta$  the pendulum makes with the vertical. If  $F = 0.120$  N for  $m = 0.350$  kg and  $\theta = 2.00^\circ$ , find  $F$  for  $m = 0.750$  kg and  $\theta = 3.50^\circ$ .
77. On a map of Canada, 12.5 mm represents 200 km. If the distance on the map between Edmonton and Calgary is 18 mm, how far is Edmonton from Calgary?

### Writing Exercise

78. A fruit-packing company plans to reduce the size of its fruit juice can (a right circular cylinder) by 10% and keep the price of each can the same (effectively raising the price). The radius and the height of the new can are to be equally proportional to those of the old can. Write one or two paragraphs explaining how to determine the percent decrease in the radius and the height of the old can that is required to make the new can.

## CHAPTER 18 PRACTICE TEST

- Express the ratio of 180 s to 4 min in simplest form.
- A person 1.8 m tall is photographed, and the film image is 20.0 mm high. Under the same conditions, how tall is a person whose film image is 14.5 mm high?
- The change  $L$  in length of a copper rod varies directly as the change  $T$  in temperature of the rod. Set up an equation for this relationship if  $L = 2.7$  cm for  $T = 150^\circ\text{C}$ .
- Given that 1.00 in. = 2.54 cm, what length in inches is 7.24 cm?
- The perimeter of a rectangular solar panel is 210.0 cm. The ratio of the length to the width is 7 to 3. What are the dimensions of the panel?
- The difference  $p$  in pressure in a fluid between that at the surface and that at a point below varies jointly as the density  $d$  of the fluid and the depth  $h$  of the point. The density of water is  $1000 \text{ kg/m}^3$ , and the density of alcohol is  $800 \text{ kg/m}^3$ . This difference in pressure at a point 0.200 m below the surface of water is 1.96 kPa. What is the difference in pressure at a point 0.300 m below the surface of alcohol? (All data are accurate to 3 significant digits.)
- The crushing load  $L$  of a pillar varies directly as the fourth power of its radius  $r$  and inversely as the square of its length  $l$ . If one pillar has twice the radius and three times the length of a second pillar, what is the ratio of the crushing load of the first pillar to that of the second pillar?

# 19

## LEARNING OUTCOMES

After completion of this chapter, the student should be able to:

- Determine if a sequence forms an arithmetic sequence
- Find the terms, common difference, number of terms, or sum of an arithmetic sequence
- Determine if a sequence forms a geometric sequence
- Find the terms, common ratio, number of terms, or sum of a finite geometric sequence
- Find the sum of an infinite geometric series
- Find the fraction equal to a given repeating decimal
- Evaluate expressions involving factorials
- Expand binomials to any power, using Pascal's triangle and the binomial theorem
- Obtain terms of a binomial series
- Solve application problems involving sequences and series

► Sequences are basic to many calculations in business, including compound interest. In Section 19.2, we show such a calculation.

# Sequences and the Binomial Theorem

If a person is saving for the future and invests \$1000 at 5%, compounded annually, the value of the investment 40 years later would be about \$7040. Even better, if the interest is compounded daily, which is a common method today, it would be worth about \$7390 in 40 years. A person saving for retirement would do much better by putting aside \$1000 each year and letting the interest accumulate. If \$1000 is invested each year at 5%, compounded annually, the total investment would be worth about \$126 840 after 40 years. If the interest is compounded daily, it would be worth about \$131 000 in 40 years. Obviously, if more is invested, or the interest rate is higher, the value of these investments would be higher.

Each of these values can be found quickly since the values of the annual investments form what is called a *geometric sequence*, and formulas can be formed for such sums. Such formulas involving compound interest are widely used in calculating values such as monthly car payments, home mortgages, and annuities.

A *sequence* is a set of numbers arranged in some specific way and usually follows a pattern. Sequences have been of interest to people for centuries. Records dating back to at least 1700 B.C.E. show calculations involving sequences. Euclid (see page 55) in his *Elements* dealt with sequences in about 300 B.C.E. More advanced forms of sequences were used extensively in the study of advanced mathematics in the 1700s and 1800s. These advances in mathematics have been very important in many areas of science and technology.

Of the many types of sequences, we study certain basic ones in this chapter. Included are those used in the expansion of a binomial to a power. We show applications in areas such as physics and chemistry in studying radioactivity, biology in studying population growth, and, of course, in business when calculating interest.



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## 19.1 Arithmetic Sequences

Arithmetic Sequence •  
Common Difference •  
Finite and Infinite Sequences •  
Sum of  $n$  Terms

A **sequence** is an ordered list of numbers, each of which is said to be a **term** of the sequence. While there are sequences with no relationship between their terms, sequences that are useful follow a pattern. In this chapter we consider only sequences of real numbers or of literal numbers representing real numbers.

An **arithmetic sequence** (or **arithmetic progression**) is a set of numbers in which each number after the first can be obtained from the preceding one by adding to it a fixed number called the **common difference**. This definition can be expressed in terms of the *recursion formula*

$$a_n = a_{n-1} + d \quad (19.1)$$

where  $a_n$  is any term,  $a_{n-1}$  is the preceding term, and  $d$  is the common difference.

### EXAMPLE 1 Illustrations of arithmetic sequences

- (a) The sequence 2, 5, 8, 11, 14, . . . , is an arithmetic sequence with a common difference  $d = 3$ . We can obtain any term by adding 3 to the previous term. We see that the fifth term is  $a_5 = a_4 + d$ , or  $14 = 11 + 3$ .
- (b) The sequence 7, 2, -3, -8, . . . , is an arithmetic sequence with  $d = -5$ . We can get any term after the first by adding  $-5$  to the previous term.

The three dots after the 14 in part (a) and after  $-8$  in part (b) mean that the sequences continue. ■

If we know the first term of an arithmetic sequence, we can find any other term by adding the common difference enough times to get the desired term. This, however, is very inefficient, and there is a general way of finding a particular term.

If  $a_1$  is the first term and  $d$  is the common difference, the second term is  $a_1 + d$ , the third term is  $a_1 + 2d$ , and so on. For the  $n$ th term, we need to add  $d$  to the first term  $n - 1$  times. Therefore, *the  $n$ th term,  $a_n$ , of the arithmetic sequence is given by*

$n$ th term

$$a_n = a_1 + (n - 1)d \quad (19.2)$$

Equation (19.2) can be used to find any given term in any arithmetic sequence. We can refer to  $a_n$  as the *last term* of an arithmetic sequence if no terms beyond it are included in the sequence. Such a sequence is called a **finite** sequence. If the terms in a sequence continue without end, the sequence is called an **infinite** sequence.

### EXAMPLE 2 Finding a specified term

Find the 10th term of the arithmetic sequence 2, 5, 8, . . . .

By subtracting any term from the following term, we find the common difference  $d = 3$ , and see that the first term  $a_1 = 2$ . Therefore, the 10th term,  $a_{10}$ , is

$$\begin{aligned} a_{10} &= \overset{a_1}{2} + \overset{n}{(10 - 1)} \overset{d}{3} = 2 + (9)(3) \\ &= 29 \end{aligned}$$

The three dots after the 8 show that the sequence continues. With no additional information given, this indicates that it is an infinite arithmetic sequence. ■

**EXAMPLE 3** Finding the common difference

Find the common difference between successive terms of the arithmetic sequence for which the third term is 5 and the 34th term is  $-119$ .

In order to use Eq. (19.2), we could calculate  $a_1$ , but we can also treat the third term as  $a_1$  and the 34th term as  $a_{32}$ . This gives us the same sequence from 5 to  $-119$ . Therefore, using the values  $a_1 = 5$ ,  $a_{32} = -119$ , and  $n = 32$ , we can find the value of  $d$ . Substituting these values in Eq. (19.2) gives

$$\begin{aligned} -119 &= 5 + (32 - 1)d \\ 31d &= -124 \\ d &= -4 \end{aligned}$$

There is no information as to whether this is a finite or an infinite sequence. The solution is the same in either case. ■

**EXAMPLE 4** Finding the number of terms

How many integers between 10 and 1000 are divisible by 6?

We must first find the smallest and the largest numbers in this range that are divisible by 6. These numbers are 12 and 996. Obviously, the common difference between one multiple of 6 and the next is 6. Thus, we can solve this as an arithmetic sequence with  $a_1 = 12$ ,  $a_n = 996$ , and  $d = 6$ . Substituting these values in Eq. (19.2), we have

$$\begin{aligned} 996 &= 12 + (n - 1)6 \\ 6n &= 990 \\ n &= 165 \end{aligned}$$

Thus, 165 numbers between 10 and 1000 are divisible by 6.

All the positive multiples of 6 are included in the infinite arithmetic sequence 6, 12, 18, . . . , whereas those between 10 and 1000 are included in the finite arithmetic sequence 12, 18, 24, . . . , 996. ■

**EXAMPLE 5** Arithmetic sequence—application

A package delivery company uses a metal (very low friction) ramp to slide packages from the sorting area down to the loading area below. If a given package is pushed such that it starts down the ramp at 25 cm/s, and the package accelerates as it slides down the ramp such that it gains 35 cm/s during each second, after how many seconds is its velocity 305 cm/s? See Fig. 19.1.

Here, we see that the velocity (in cm/s) of the package after each second is

$$60, 95, 130, \dots, 305, \dots$$

Therefore,  $a_1 = 60$  (the 25 cm/s was at the beginning, that is, after 0 s),  $d = 35$ ,  $a_n = 305$ , and we are to find  $n$ , which in this case represents the number of seconds during which the velocity increases. Therefore,

$$\begin{aligned} 305 &= 60 + (n - 1)(35) \\ 245 &= 35n - 35 \\ 35n &= 280 \\ n &= 8.0 \end{aligned}$$

This means that the velocity of a package sliding down the ramp is 305 cm/s after 8.0 s. ■

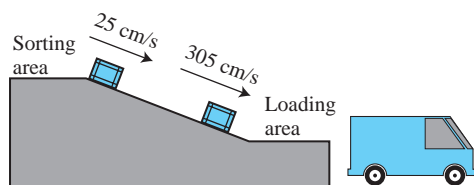


Fig. 19.1

**Practice Exercise**

1. Find  $d$  if  $a_1 = 4$  and  $a_{13} = 34$ .

**SUM OF  $n$  TERMS**

Another important quantity related to an arithmetic sequence is the sum of the first  $n$  terms. We can indicate this sum by starting the sum either with the first term or with the last term, as shown by these two equations:

$$S_n = a_1 + (a_1 + d) + (a_1 + 2d) + \cdots + (a_n - d) + a_n$$

or

$$S_n = a_n + (a_n - d) + (a_n - 2d) + \cdots + (a_1 + d) + a_1$$

If we now add the corresponding members of these two equations, we obtain the result

$$2S_n = (a_1 + a_n) + (a_1 + a_n) + (a_1 + a_n) + \cdots + (a_1 + a_n) + (a_1 + a_n)$$

Each term on the right in parentheses has the same expression  $(a_1 + a_n)$ , and there are  $n$  such terms. This tells us that *the sum of the first  $n$  terms is given by*

$$S_n = \frac{n}{2}(a_1 + a_n) \quad (19.3)$$

The use of Eq. (19.3) is illustrated in the following examples.

**EXAMPLE 6 Finding the sum of terms**

Find the sum of the first 1000 positive integers.

The first 1000 positive integers form a finite arithmetic sequence for which  $a_1 = 1$ ,  $a_{1000} = 1000$ ,  $n = 1000$ , and  $d = 1$ . Substituting these values in Eq. (19.3) (in which we do not use the value of  $d$ ), we have

$$\begin{aligned} S_{1000} &= \frac{1000}{2}(1 + 1000) = 500(1001) \\ &= 500\,500 \end{aligned}$$

■ Karl Friedrich Gauss (1777–1856) is considered one of the greatest mathematicians of all time. A popular story about him is that at the age of 10 his schoolmaster asked his class to add all the numbers from 1 to 100, assuming it would take some time to do. He had barely finished stating the problem when Gauss handed in the correct answer of 5050. He had done it essentially as we did the solution in Example 6.

**EXAMPLE 7 Finding the sum of terms**

Find the sum of the first ten terms of the arithmetic sequence in which the first term is 4 and the common difference is  $-5$ .

We are to find  $S_n$ , given that  $n = 10$ ,  $a_1 = 4$ , and  $d = -5$ . Since Eq. (19.3) uses the value of  $a_n$  but not the value of  $d$ , we first find  $a_{10}$  by using Eq. (19.2). This gives us

$$\begin{aligned} a_{10} &= 4 + (10 - 1)(-5) = 4 - 45 \\ &= -41 \end{aligned}$$

Now, we can solve for  $S_{10}$  by using Eq. (19.3):

$$\begin{aligned} S_{10} &= \frac{10}{2}(4 - 41) = 5(-37) \\ &= -185 \end{aligned}$$

**Practice Exercise**

2. Find  $S_9$  if  $d = -2$  and  $a_9 = 1$ .

If any three of the five values  $a_1$ ,  $a_n$ ,  $n$ ,  $d$ , and  $S_n$  are given for a particular arithmetic sequence, the other two may be found from Eqs. (19.2) and (19.3). Consider the following example.



**EXAMPLE 8** Finding  $n$  and  $a_n$ 

For an arithmetic sequence, given that  $a_1 = 2$ ,  $d = 1.5$ , and  $S_n = 72$ , find  $n$  and  $a_n$ .

First, we substitute the given values in Eqs. (19.2) and (19.3) in order to identify what is known, and how to proceed. Making these substitutions, we have

$$a_n = 2 + (n - 1)(1.5) \quad \text{substituting in Eq. (19.2)}$$

$$72 = \frac{n}{2}(2 + a_n) \quad \text{substituting in Eq. (19.3)}$$

$$72 = \frac{n}{2}[2 + 2 + (n - 1)(1.5)] \quad \text{substituting first equation into second equation}$$

$$= 2n + 0.75n(n - 1)$$

$$288 = 8n + 3n^2 - 3n \quad \text{multiplying each term by 4 to clear decimals}$$

$$3n^2 + 5n - 288 = 0$$

$$n = \frac{-5 \pm \sqrt{25 - 4(3)(-288)}}{6} = \frac{-5 \pm \sqrt{3481}}{6} = \frac{-5 \pm 59}{6}$$

Since  $n$  must be a positive integer, we find that  $n = (-5 + 59)/6 = 9$ . Therefore, since  $n = 9$ ,  $a_9 = 2 + (9 - 1)(1.5) = 14$ . The values  $n = 9$  and  $a_9 = 14$  check with the given values. ■

**EXAMPLE 9** Sum of terms—application

The voltage across a resistor increases such that during each second the increase is 0.002 mV less than during the previous second. Given that the increase during the first second is 0.350 mV, what is the total voltage increase during the first 10.0 s?

We are asked to find the sum of the voltage increases 0.350 mV, 0.348 mV, 0.346 mV, . . . , so as to include ten increases. This means we want the sum of an arithmetic sequence for which  $a_1 = 0.350$ ,  $d = -0.002$ , and  $n = 10$ . Since we need  $a_n$  to use Eq. (19.3), we first calculate it, using Eq. (19.2):

$$a_{10} = 0.350 + (10 - 1)(-0.002) = 0.332 \text{ mV}$$

Now, we use Eq. (19.3) to find the sum with  $a_1 = 0.350$ ,  $a_{10} = 0.332$ , and  $n = 10$ :

$$S_{10} = \frac{10}{2}(0.350 + 0.332) = 3.410 \text{ mV}$$

Thus, the total voltage increase is 3.410 mV. ■

**EXERCISES 19.1**

In Exercises 1–4, make the given changes in the indicated examples of this section and then solve the resulting problems.

- In Example 3, change  $-119$  to  $-88$  and then find the common difference.
- In Example 4, change 6 to 5 and then find out how many numbers there are.
- In Example 6, change 1000 to 500 and then find the sum.
- In Example 7, change  $-5$  to 5 and then find the sum.

In Exercises 5–8, write the first five terms of the arithmetic sequence with the given values.

- $a_1 = 4$ ,  $d = 2$
- $a_1 = 6$ ,  $d = -\frac{1}{2}$

$$7. a_1 = 2.5, a_5 = -1.5$$

$$8. a_2 = -2, a_5 = 43$$

In Exercises 9–16, find the  $n$ th term of the arithmetic sequence with the given values.

$$9. 1, 4, 7, \dots; n = 8$$

$$10. -6, -4, -2, \dots; n = 10$$

$$11. \frac{7\pi}{4}, \frac{3\pi}{2}, \frac{5\pi}{4}, \dots; n = 17$$

$$12. 2, 0.5, -1, \dots; n = 25$$

$$13. a_1 = -0.7, d = 0.4, n = 80$$

$$14. a_1 = \frac{3}{2}, d = \frac{1}{6}, n = 601$$

$$15. a_1 = b, d = 2b, n = 25$$

$$16. a_1 = -c, d = 3c, n = 30$$



In Exercises 17–20, find the sum of the  $n$  terms of the indicated arithmetic sequence.

17.  $n = 20, a_1 = 4, a_{20} = 40$

18.  $n = 8, a_1 = -12, d = -2$

19.  $-2, -\frac{5}{2}, -3, \dots; n = 10$

20.  $3k, \frac{10}{3}k, \frac{11}{3}k, \dots; n = 40$

In Exercises 21–32, find any of the values of  $a_1, d, a_n, n$ , or  $S_n$  that are missing for an arithmetic sequence.

21.  $a_1 = 5, d = 8, a_n = 45$

22.  $a_1 = -2, n = 60, a_n = 28$

23.  $a_1 = \frac{5}{3}, n = 20, S_{20} = \frac{40}{3}$

24.  $a_1 = 0.1, a_n = -5.9, S_n = -8.7$

25.  $d = 3, n = 30, S_{30} = 1875$

26.  $d = 9, a_n = 86, S_n = 455$

27.  $a_1 = 7.4, d = -0.5, a_n = -23.1$

28.  $a_1 = -\frac{9}{7}, n = 19, a_{19} = -\frac{36}{7}$

29.  $a_1 = 5k, d = 0.5k, S_n = 104k$

30.  $d = -2c, n = 50, S_{50} = 0$

31.  $a_1 = -c, a_n = \frac{b}{2}, S_n = 2b - 4c$

32.  $a_1 = 3b, n = 7, d = \frac{b}{3}$

In Exercises 33–60, find the indicated quantities for the appropriate arithmetic sequence.

33.  $a_6 = 560, a_{10} = 720$  (find  $a_1, d, S_n$  for  $n = 10$ )

34.  $a_{17} = -91, a_2 = -73$  (find  $a_1, d, S_n$  for  $n = 40$ )

35. Is  $\ln 3, \ln 6, \ln 12, \dots$  an arithmetic sequence? Explain. If it is, what is the fifth term?

36. Is  $\sin 2^\circ, \sin 4^\circ, \sin 6^\circ, \dots$  an arithmetic sequence? Explain. If it is, what is the fifth term?

37. Find a formula with variable  $n$  for the  $n$ th term of the arithmetic sequence with  $a_1 = 3$  and  $a_{n+1} = a_n + 2$  for  $n = 1, 2, 3, \dots$

38. If  $a, b$ , and  $c$  are the first three terms of an arithmetic sequence, find their sum in terms of  $b$  only.

39. The terms  $a, m, b$  form an arithmetic sequence. Express a formula for the common difference  $d$  in terms of  $a$  and  $b$ . (The term  $m$  is the *arithmetic mean* of the terms  $a$  and  $b$ .)

40. The terms  $a, m, n, b$  form an arithmetic sequence. Express  $m$  and  $n$  in terms of  $a$  and  $b$ . (The terms  $m$  and  $n$  are *arithmetic means* of the terms  $a$  and  $b$ .)

41. Write the first five terms of the arithmetic sequence for which the second term is  $b$  and the third term is  $c$ .

42. If the sum of the first two terms of an arithmetic sequence equals the sum of the first three terms, find the sum of the first five terms.

43. Find the sum of the first 100 positive integers. (See the margin note on page 513.)

44. Find the number of multiples of 8 between 99 and 999.

45. Find  $x$  if  $3 - x, -x$ , and  $\sqrt{9 - 2x}$  are the first three terms of an arithmetic sequence.

46. Is  $x, x + 2y, 2x + 3y, \dots$  an arithmetic sequence? If it is, find the sum of the first 100 terms.

47. A beach now has an area of  $9500 \text{ m}^2$  but is eroding such that it loses  $100 \text{ m}^2$  more of its area each year than during the previous year. If it lost  $400 \text{ m}^2$  during the last year, what would you expect its area to be 8 years from now?

48. During a period of heavy rains, on a given day  $4500 \text{ m}^3/\text{s}$  of water was being released from a dam. In order to minimize downstream flooding, engineers then reduced the releases by  $500 \text{ m}^3/\text{s}$  each day thereafter. How much water was released during the first week of these releases?

49. At a logging camp, 15 layers of logs are so piled that there are 20 logs in the bottom layer, and each layer has 1 less log than the layer below it. How many logs are in the pile?

50. In order to prevent an electric current surge in a circuit, the resistance  $R$  in the circuit is stepped down by  $4.0 \Omega$  after each  $0.1 \text{ s}$ . If the voltage  $V$  is constant at  $120 \text{ V}$ , do the resulting currents  $I$  (in A) form an arithmetic sequence if  $V = IR$ ?

51. There are 12 seats in the first row around a semicircular stage. Each row behind the first has 4 more seats than the row in front of it. How many rows of seats are there if there is a total of 300 seats?

52. A bank loan of  $\$8000$  is repaid in annual payments of  $\$1000$  plus 10% interest on the unpaid balance. What is the total amount of interest paid?

53. A car depreciates  $\$1800$  during the first year after it is bought. Each year thereafter it depreciates  $\$150$  less than the year before. How many years after it was bought will it be considered to have no value, and what was the original cost?

54. The sequence of ships' bells is as follows: 12:30 A.M. one bell is rung, and each half hour later one more bell is rung than the previous time until eight bells are rung. The sequence is then repeated starting at 4:30 A.M., again until eight bells are rung. This pattern is followed throughout the day. How many bells are rung in 1 day?

55. If a stone released from a spacecraft near the surface of Mars falls  $1.85 \text{ m}$  during the first second,  $5.55 \text{ m}$  during the second second,  $9.25 \text{ m}$  during the third second,  $12.95 \text{ m}$  during the fourth second, and so on, how far from the surface is the spacecraft if it takes the stone  $10.0 \text{ s}$  to reach the surface?

56. In preparing a bid for constructing a new building, a contractor determines that the foundation and basement will cost  $\$605,000$  and the first floor will cost  $\$360,000$ . Each floor above the first will cost  $\$15,000$  more than the one below it. How much will the building cost if it is to be 18 floors high?

57. Derive a formula for  $S_n$  in terms of  $n, a_1$ , and  $d$ .

58. A *harmonic sequence* is a sequence of numbers whose reciprocals form an arithmetic sequence. Is a harmonic sequence also an arithmetic sequence? Explain.

59. Show that the sum of the first  $n$  positive integers is  $\frac{1}{2}n(n + 1)$ .

60. Show that the sum of the first  $n$  positive odd integers is  $n^2$ .

### Answers to Practice Exercises

1.  $d = 2.5$     2.  $S_9 = 81$

## 19.2 Geometric Sequences

Geometric Sequence • Common Ratio • Sum of  $n$  Terms

A second type of important sequence of numbers is the **geometric sequence** (or **geometric progression**). In a geometric sequence, each number after the first can be obtained from the preceding one by multiplying it by a fixed number, called the **common ratio**. We can express this definition in terms of the *recursion formula*

$$a_n = ra_{n-1} \quad (19.4)$$

where  $a_n$  is any term,  $a_{n-1}$  is the preceding term, and  $r$  is the common ratio. One important application of geometric sequences is in computing compound interest on savings accounts. Other applications are found in areas such as biology and physics.

### EXAMPLE 1 Illustrations of geometric sequences

- (a) The sequence 2, 4, 8, 16, . . . , is a geometric sequence with a common ratio of 2. Any term after the first can be obtained by multiplying the previous term by 2. We see that the fourth term  $a_4 = ra_3$ , or  $16 = 2(8)$ .
- (b) The sequence 9,  $-3$ , 1,  $-1/3$ , . . . , is a geometric sequence with a common ratio of  $-1/3$ . We can obtain any term after the first by multiplying the previous term by  $-1/3$ . ■

If we know the first term, we can find any other term by multiplying by the common ratio a sufficient number of times. When we do this for a general geometric sequence, we can find the  $n$ th term in terms of the first term  $a_1$ , the common ratio  $r$ , and  $n$ . Thus, the second term is  $a_1r$ , the third term is  $a_1r^2$ , and so forth. In general, the expression for the  $n$ th term is

$n$ th term

$$a_n = a_1r^{n-1} \quad (19.5)$$

### EXAMPLE 2 Finding a specified term

Find the eighth term of the geometric sequence 8, 4, 2, . . . .

By dividing any term by the previous term, we find the common ratio to be  $\frac{1}{2}$ . From the terms given, we see that  $a_1 = 8$ . From the statement of the problem, we know that  $n = 8$ . Thus, we substitute into Eq. (19.5) to find  $a_8$ :

$$a_8 = 8 \left( \frac{1}{2} \right)^{8-1} = \frac{8}{2^7} = \frac{1}{16} \quad \blacksquare$$

### EXAMPLE 3 Finding the common ratio

Find the common ratio  $r$  if  $a_1 = \frac{8}{625}$  and  $a_{10} = -\frac{3125}{64}$ .

Using Eq. (19.5), we have

$$-\frac{3125}{64} = \frac{8}{625}r^{10-1} = \frac{8}{625}r^9 \quad \text{substituting in Eq. (19.5)}$$

$$r^9 = -\frac{(3125)(625)}{(64)(8)} \quad \text{solving for } r$$

$$r = \left[ -\frac{(3125)(625)}{(64)(8)} \right]^{1/9} = -2.5 \quad \blacksquare$$

**EXAMPLE 4** Finding a specified term

Find the seventh term of the geometric sequence for which the second term is 3, the fourth term is 9, and  $r > 0$ .

We can find  $r$  if we let  $a_1 = 3$ ,  $a_3 = 9$ , and  $n = 3$ . (At this point, we are considering a sequence made up of 3, the next number, and 9. These are the second, third, and fourth terms of the original sequence.) Thus,

$$9 = 3r^2, \quad r = \sqrt{3} \quad (\text{since } r > 0)$$

We can now find  $a_1$  of the original sequence by considering just the first two terms of the sequence,  $a_1$  and  $a_2 = 3$ :

$$3 = a_1(\sqrt{3})^{2-1}, \quad a_1 = \sqrt{3}$$

We can now find the seventh term, using  $a_1 = \sqrt{3}$ ,  $r = \sqrt{3}$ , and  $n = 7$ :

$$a_7 = \sqrt{3}(\sqrt{3})^{7-1} = \sqrt{3}(3^3) = 27\sqrt{3}$$

We could have shortened this procedure one step by letting the second term be the first term of a new sequence of six terms. If the first term is of no importance in itself, this is acceptable. ■

**EXAMPLE 5** Geometric sequence—application

In an experiment, 22.0% of a substance changes chemically each 10.0 min. If there is originally 120 g of the substance, how much will remain after 45.0 min?

Let  $P_n$  = the portion of the substance remaining after  $n$  minutes. From the statement of the problem, we know that  $r = 0.780$ , since 78.0% remains after each 10.0-min period. We also know that  $a_1 = 120$  g. This means that  $P_n = 120(0.780)^{n/10.0}$ . It is necessary to divide by 10.0 because the ratio is given for a 10.0-min period. In order to find  $P_n$  when  $n = 45.0$  min, we write

$$\begin{aligned} P_{45} &= 120(0.780)^{45.0/10.0} = 120(0.780)^{4.50} \\ &= 39.2 \text{ g} \end{aligned}$$

This means that 39.2 g remain after 45.0 min. Note that the power 4.50 represents 4.50 ten-minute periods. ■

**SUM OF  $n$  TERMS**

A general expression for the sum  $S_n$  of the first  $n$  terms of a geometric sequence may be found by directly forming the sum and multiplying this equation by  $r$ :

$$\begin{aligned} S_n &= a_1 + a_1r + a_1r^2 + \cdots + a_1r^{n-1} \\ rS_n &= a_1r + a_1r^2 + a_1r^3 + \cdots + a_1r^n \end{aligned}$$

If we now subtract the second of these equations from the first, we get  $S_n - rS_n = a_1 - a_1r^n$ . All other terms cancel by subtraction. Now, factoring  $S_n$  from the terms on the left and  $a_1$  from the terms on the right, we solve for  $S_n$ . Thus, *the sum  $S_n$  of the first  $n$  terms of a geometric sequence is*

$$S_n = \frac{a_1(1 - r^n)}{1 - r} \quad (r \neq 1) \quad (19.6)$$

**Practice Exercise**

- Find the sixth term of the geometric sequence for which  $a_1 = 81$  and  $r = -1/3$ .

**EXAMPLE 6** Finding the sum of terms

Find the sum of the first seven terms of the geometric sequence for which the first term is 2 and the common ratio is  $1/2$ .

We are to find  $S_n$ , given that  $a_1 = 2$ ,  $r = \frac{1}{2}$ , and  $n = 7$ . Using Eq. (19.6), we have

$$S_7 = \frac{2\left(1 - \left(\frac{1}{2}\right)^7\right)}{1 - \frac{1}{2}} = \frac{2\left(1 - \frac{1}{128}\right)}{\frac{1}{2}} = 4\left(\frac{127}{128}\right) = \frac{127}{32}$$

**Practice Exercise**

2. Find the sum of the six terms of the geometric sequence in Practice Exercise 1.

**EXAMPLE 7** Finding the sum of terms—application

If \$100 is invested each year at 5% interest compounded annually, what would be the total amount of the investment after 10 years (before the 11th deposit is made)?

After 1 year, the amount invested will have added to it the interest for the year. Therefore, for the last (10th) \$100 invested, its value will become

$$\$100(1 + 0.05) = \$100(1.05) = \$105$$

The next to last \$100 will have interest added twice. After 1 year, its value becomes  $\$100(1.05)$ , and after 2 years it is  $\$100(1.05)(1.05) = \$100(1.05)^2$ . In the same way, the value of the first \$100 becomes  $\$100(1.05)^{10}$ , since it will have interest added 10 times. This means that we are to find the sum of the sequence

$$100(1.05) + 100(1.05)^2 + 100(1.05)^3 + \cdots + 100(1.05)^{10}$$

1 year  
in account
2 years  
in account
3 years  
in account
10 years  
in account

or

$$100[1.05 + (1.05)^2 + (1.05)^3 + \cdots + (1.05)^{10}]$$

For the sequence in the brackets, we have  $a_1 = 1.05$ ,  $r = 1.05$ , and  $n = 10$ . Thus,

$$S_{10} = \frac{1.05[1 - (1.05)^{10}]}{1 - 1.05} = 13.2068$$

The total value of the \$100 investments is  $100(13.2068) = \$1320.68$ . We see that \$320.68 in interest has been earned. ■

**EXERCISES 19.2**

In Exercises 1 and 2, make the given changes in the indicated examples of this section and then solve the given problems.

- In Example 4, change “seventh” to “10th.”
- In Example 6, change  $\frac{1}{2}$  to  $\frac{1}{3}$  and then find the sum.

In Exercises 3–6, write down the first five terms of the geometric sequence with the given values.

- $a_1 = 6400$ ,  $r = 0.25$
- $a_1 = 0.09$ ,  $r = -\frac{2}{3}$
- $a_1 = \frac{1}{6}$ ,  $r = 3$
- $a_1 = -3$ ,  $r = 2$

In Exercises 7–14, find the  $n$ th term of the geometric sequence with the given values.

- $\frac{1}{2}$ , 1, 2, ...;  $n = 9$
- 10, 1, 0.1, ...;  $n = 8$
- 125, -25, 5, ...;  $n = 7$
- 0.1, 0.3, 0.9, ...;  $n = 5$

- $a = -2700$ ,  $r = -\frac{1}{3}$ ,  $n = 10$
- $a_1 = 48$ ,  $r = \frac{1}{2}$ ,  $n = 12$
- $10^{100}$ ,  $-10^{98}$ ,  $10^{96}$ , ...;  $n = 51$
- $-2, 4k, -8k^2, \dots$ ;  $n = 6$

In Exercises 15–20, find the sum of the first  $n$  terms of the indicated geometric sequence with the given values.

- $a_1 = \frac{1}{8}$ ,  $r = 4$ ,  $n = 5$
- 162, -54, 18, ...;  $n = 6$
- 384, 192, 96, ...;  $n = 7$
- $a_1 = 9$ ,  $a_n = -243$ ,  $n = 4$
- $a_1 = 96$ ,  $r = -\frac{k}{2}$ ,  $n = 10$
- $\log 2, \log 4, \log 16, \dots$ ;  $n = 6$

In Exercises 21–28, find any of the values of  $a_1$ ,  $r$ ,  $a_n$ ,  $n$ , or  $S_n$  that are missing.

- $a_1 = \frac{1}{16}$ ,  $r = 4$ ,  $n = 6$
- $r = 0.2$ ,  $a_n = 0.00032$ ,  $n = 7$

23.  $r = \frac{3}{2}$ ,  $n = 5$ ,  $S_5 = 211$   
 24.  $r = -\frac{1}{2}$ ,  $a_n = \frac{1}{8}$ ,  $n = 7$   
 25.  $a_n = 27$ ,  $n = 4$ ,  $S_4 = 40$   
 26.  $a_1 = 3$ ,  $n = 7$ ,  $a_7 = 192$   
 27.  $a_1 = 75$ ,  $r = \frac{1}{5}$ ,  $a_n = \frac{3}{625}$   
 28.  $r = -2$ ,  $n = 6$ ,  $S_6 = 42$

In Exercises 29–58, find the indicated quantities.

29. Is  $3, 3^{x+1}, 3^{2x+1}, \dots$  a geometric sequence? Explain. If it is, find  $a_{20}$ .
30. The first two terms of a geometric sequence are added together to form the first term of a second sequence. The third and fourth terms of the first sequence are added to form the second term of the second sequence, and so on. If  $n$  is even for the first sequence, is the second sequence geometric?
31. Show that  $a, \sqrt{ab}, b$  are three successive terms of a geometric sequence ( $a > 0, b > 0$ ).
32. Find  $x$  if  $\sqrt{x}, 4, \sqrt{15x+4}$  are the first three terms of a geometric sequence.
33. Find  $x$  if  $2, 6, 2x+8$  are successive terms of a geometric sequence.
34. Write the first five terms of a geometric sequence of which the first two terms are  $a$  and  $b$ .
35. Using Eq. (19.6), find the sum of the terms of a geometric sequence if  $r = -1$ , and  $n$  is (a) odd, (b) even.
36. If a positive number  $c$  is added to each term of a geometric sequence, is the resulting sequence geometric?
37. The sum of the first three terms of a geometric sequence equals seven times the first term. Find the common ratio.
38. The sum of the first three terms of an arithmetic sequence is 3. What are the numbers if their squares form a geometric sequence?
39. Each stroke of a pump removes 8.2% of the remaining air from a container. What percent of the air remains after 50 strokes?
40. In 2009, the population of Canada was 33.7 million. According to a low growth scenario, it is estimated that the population will increase at an average annual rate of 0.65% until 2036. What is the estimated 2036 population, according to this scenario?
41. An electric current decreases by 12.5% each  $1.00 \mu\text{s}$ . If the initial current is  $3.27 \text{ mA}$ , what is the current after  $8.20 \mu\text{s}$ ?
42. A copying machine is set to reduce the dimensions of material copied by 10%. A drawing  $12.0 \text{ cm}$  wide is reduced, and then the copies are in turn reduced. What is the width of the drawing on the sixth reduction?
43. How much is an investment of  $\$250$  worth after 8 years if it earns annual interest of 7.2% compounded monthly? (7.2% annual interest compounded monthly means that 0.6% ( $7.2\%/12$ ) interest is added each month.)
44. A chemical spill pollutes a stream. A monitoring device finds 620 ppm (parts per million) of the chemical  $1.0 \text{ km}$  below the spill, and the readings decrease by 12.5% for each kilometre farther downstream. How far downstream is the reading 100 ppm?
45. Measurements show that the temperature of a distant star is presently  $9800^\circ\text{C}$  and is decreasing by 10% every 800 years. What will its temperature be in 4000 years?
46. The chlorine in a swimming pool was measured (in ppm, parts per million) to be 1.80, 1.53, 1.30, and 1.10 on 4 successive days. Noting that these values approximate a geometric sequence, what would be the reading 3 days after the last reading?
47. The strength of a signal in a fibre-optic cable decreases 12% for every 15 km along the cable. What percent of the signal remains after 100 km?
48. A series of deposits, each of value  $A$  and made at equal time intervals, earns an interest rate of  $i$  for the time interval. The deposits have a total value of  $A(1+i) + A(1+i)^2 + A(1+i)^3 + \dots + A(1+i)^n$  after  $n$  time intervals (just before the next deposit). Find a formula for this sum.
49. A thermometer is removed from hot water at  $100.0^\circ\text{C}$  into a room at  $20.0^\circ\text{C}$ . The temperature difference  $D$  between the thermometer and the air decreases by 35.0% each minute. What is the temperature reading on the thermometer 10.0 min later?
50. The power on a space satellite is supplied by a radioactive isotope. On a given satellite, the power decreases by 0.2% each day. What percent of the initial power remains after 1 year?
51. If you decided to save money by putting away  $1\text{¢}$  on a given day,  $2\text{¢}$  one week later,  $4\text{¢}$  a week later, and so on, how much would you have to put away 6 months (26 weeks) after putting away the  $1\text{¢}$ ?
52. How many direct ancestors (parents, grandparents, and so on) does a person have in the 10 generations that preceded him or her (assuming that no ancestor appears in more than one line of descent)?
53. Derive a formula for  $S_n$  in terms of  $a_1, r$ , and  $a_n$ .
54. Write down several terms of a general geometric sequence. Then take the logarithm of each term. Explain why the resulting sequence is an arithmetic sequence.
55. Do the squares of the terms of a geometric sequence also form a geometric sequence? Explain.
56. If  $a_1, a_2, a_3, \dots$  is an arithmetic sequence, explain why  $2^{a_1}, 2^{a_2}, 2^{a_3}, \dots$  is a geometric sequence.
57. The numbers  $8, x, y$  form an arithmetic sequence, and the numbers  $x, y, 36$  form a geometric sequence. Find all of the possible sequences.
58. For  $f(x) = ab^x$ , with  $b > 0, b \neq 1$ , and  $a \neq 0$ , is  $f(1), f(3), f(5)$  an arithmetic sequence or a geometric sequence? What is the common difference (or common ratio)?

#### Answers to Practice Exercises

1.  $a_6 = -1/3$     2.  $S_6 = 182/3$

## 19.3 Infinite Geometric Series

Series • Infinity • Limit • Repeating Decimal

In the previous sections, we developed formulas for the sum of the first  $n$  terms of an arithmetic sequence and of a geometric sequence. *The sum of the terms of a sequence is called a series.*

### EXAMPLE 1 Illustrations of series

- (a) The sum of the terms of the arithmetic sequence 2, 5, 8, 11, 14, . . . is the series  $2 + 5 + 8 + 11 + 14 + \dots$
- (b) The sum of terms of the geometric sequence  $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots$  is the series  $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$  ■

The series associated with a finite sequence will sum up to a real number. The series associated with an infinite arithmetic sequence will not sum up to a real number, as the terms being added become larger and larger numerically. The sum is unbounded, as we can see in Example 1(a). The series associated with an infinite geometric sequence may or may not sum up to a real number, as we now show.

Let us consider the sum of the first  $n$  terms of the infinite geometric sequence  $1, \frac{1}{2}, \frac{1}{4}, \dots$ . This is the sum of the  $n$  terms of the associated geometric series

$$1 + \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^{n-1}}$$

Here,  $a_1 = 1$  and  $r = \frac{1}{2}$ , and we find that we get the values of  $S_n$  for the given values of  $n$  in the following table:

$n$	2	3	4	5	6	7	8	9	10
$S_n$	$\frac{3}{2}$	$\frac{7}{4}$	$\frac{15}{8}$	$\frac{31}{16}$	$\frac{63}{32}$	$\frac{127}{64}$	$\frac{255}{128}$	$\frac{511}{256}$	$\frac{1023}{512}$

$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \xrightarrow{\quad}$ 
 $\xleftarrow{\quad} 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64}$

The series for  $n = 4$  and  $n = 7$  are shown. We see that as  $n$  gets larger, the numerator of each fraction comes closer to being twice the denominator. In fact, we find that if we continue to compute  $S_n$  as  $n$  becomes larger,  $S_n$  can be found as close to the value 2 as desired, although it will never actually reach the value 2. For example, if  $n = 100$ ,  $S_{100} = 2 - 1.6 \times 10^{-30}$ , which could be written as

$$1.999\ 999\ 999\ 999\ 999\ 999\ 999\ 999\ 998\ 4$$

to 32 significant digits. For the sum of the first  $n$  terms of a geometric sequence

$$S_n = a_1 \frac{1 - r^n}{1 - r}$$

the term  $r^n$  becomes exceedingly small if  $|r| < 1$ , and if  $n$  is sufficiently large, this term is effectively zero. *If this term were exactly zero*, the sum would be

$$S_n = 1 \frac{1 - 0}{1 - \frac{1}{2}} = 2$$

■ The symbol  $\infty$  for infinity was first used by the English mathematician John Wallis (1616–1703).

The only problem is that we cannot find any number large enough for  $n$  to make  $(\frac{1}{2})^n$  zero. There is, however, an accepted notation for this. This notation is

$$\lim_{n \rightarrow \infty} r^n = 0 \quad (\text{if } |r| < 1)$$

and it is read as “the limit, as  $n$  approaches infinity, of  $r$  to the  $n$ th power is zero.”

**COMMON ERROR**

The symbol  $\infty$  is read as **infinity**, but **it must not be thought of as a number**. It is simply a symbol that stands for a *process* of considering numbers that become large without bound. The number called the **limit** of the sums is simply the number the sums get closer and closer to, as  $n$  is considered to approach infinity. This notation and terminology are of particular importance in calculus.

If we consider values of  $r$  such that  $|r| < 1$  and let the values of  $n$  become unbounded, we find that  $\lim_{n \rightarrow \infty} r^n = 0$ . The formula for *the sum of the terms of an infinite geometric series then becomes*

$$S = \frac{a_1}{1 - r} \quad (|r| < 1) \quad (19.7)$$

where  $a_1$  is the first term and  $r$  is the common ratio. If  $|r| \geq 1$ ,  $S$  is unbounded in value.

**EXAMPLE 2** Sum of an infinite geometric series

Find the sum of the infinite geometric series

$$4 - \frac{1}{2} + \frac{1}{16} - \frac{1}{128} + \dots$$

Here, we see that  $a_1 = 4$ . We find  $r$  by dividing any term by the previous term, and we find that  $r = -\frac{1}{8}$ . We then find the sum by substituting in Eq. (19.7). This gives us

$$\begin{aligned} S &= \frac{4}{1 - \left(-\frac{1}{8}\right)} = \frac{4}{1 + \frac{1}{8}} \\ &= \frac{4}{1} \times \frac{8}{9} = \frac{32}{9} \end{aligned}$$

**Practice Exercise**

- Find the sum of the infinite geometric series  $9 + 3 + 1 + 1/3 + \dots$

**EXAMPLE 3** Writing a repeating decimal as a fraction

Find the fraction that has its decimal form  $0.121\ 212\ \dots$

This decimal form can be considered as being

$$0.12 + 0.0012 + 0.000\ 012 + \dots$$

which means that we have an infinite geometric series in which  $a_1 = 0.12$  and  $r = 0.01$ . Thus,

$$\begin{aligned} S &= \frac{0.12}{1 - 0.01} = \frac{0.12}{0.99} \\ &= \frac{4}{33} \end{aligned}$$

**Practice Exercise**

- Find the fraction that has as its decimal form  $0.272727\ \dots$

Therefore, the decimal  $0.121\ 212\ \dots$  and the fraction  $\frac{4}{33}$  represent the same number. ■

The decimal in Example 3 is called a **repeating decimal** because *a particular sequence of digits in the decimal form repeats endlessly*. This example verifies the theorem that any repeating decimal represents a rational number. However, not all repeating decimals start repeating immediately. If the numbers never do repeat, the decimal represents an irrational number. For example, there are no repeating decimals that represent  $\pi$ ,  $\sqrt{2}$ , or  $e$ . The decimal form of the number  $e$  does repeat at one point, but the repetition stops. As we noted in Section 12.5, the decimal form of  $e$  to 16 decimal places is 2.7 1828 1828 4590 452. We see that the sequence of digits 1828 repeats only once.



■ A calculator can be used to change a repeating decimal to a fraction.

**EXAMPLE 4** Repeating decimal

Find the fraction that has its decimal form the repeating decimal  $0.503\ 453\ 453\ 45\ \dots$ .

We first separate the decimal into the beginning, nonrepeating part, and the infinite repeating decimal, which follows. Thus, we have

$$0.503\ 453\ 453\ 45\ \dots = 0.50 + 0.003\ 453\ 453\ 45\ \dots$$

This means that we are to add  $\frac{50}{100}$  to the fraction that represents the sum of the terms of the infinite geometric series  $0.003\ 45 + 0.000\ 003\ 45 + \dots$ . For this series,  $a_1 = 0.003\ 45$  and  $r = 0.001$ . We find this sum to be

$$S = \frac{0.003\ 45}{1 - 0.001} = \frac{0.003\ 45}{0.999} = \frac{115}{33\ 300} = \frac{23}{6660}$$

Therefore,

$$0.503\ 453\ 45\ \dots = \frac{5}{10} + \frac{23}{6660} = \frac{5(666) + 23}{6660} = \frac{3353}{6660}$$

**EXAMPLE 5** Infinite geometric series—application

Each swing of a certain pendulum bob is 95% as long as the previous swing. How far does the bob travel in coming to rest if the first swing is 40.0 cm long?

We are to find the sum of the terms of an infinite geometric series for which  $a_1 = 40.0$  and  $r = 95\% = \frac{19}{20}$ . Substituting these values into Eq. (19.7), we obtain

$$S = \frac{40.0}{1 - \frac{19}{20}} = \frac{40.0}{\frac{1}{20}} = (40.0)(20) = 800\text{ cm}$$

The pendulum bob travels 800 cm in coming to rest.

**EXERCISES 19.3**

In Exercises 1 and 2, make the given changes in the indicated examples of this section, and then solve the resulting problems.

- In Example 2, change all  $-$  signs to  $+$  in the series.
- In Example 3, change the decimal form to  $0.012012012\ \dots$ .

In Exercises 3–6, find the indicated quantity for an infinite geometric series.

- $a_1 = 4$ ,  $r = \frac{1}{2}$ ,  $S = ?$
- $a_1 = 68$ ,  $r = -\frac{1}{3}$ ,  $S = ?$
- $a_1 = 0.5$ ,  $S = 0.625$ ,  $r = ?$
- $S = 4 + 2\sqrt{2}$ ,  $r = \frac{1}{\sqrt{2}}$ ,  $a_1 = ?$

In Exercises 7–14, find the sums of the given infinite geometric series.

- $20 - 1 + 0.05 - \dots$
- $9 + 8.1 + 7.29 + \dots$
- $1 + \frac{7}{8} + \frac{49}{64} + \dots$
- $6 - 4 + \frac{8}{3} - \dots$
- $1 + 10^{-4} + 10^{-8} + \dots$
- $1000 - 300 + 90 - \dots$
- $(2 + \sqrt{3}) + 1 + (2 - \sqrt{3}) + \dots$
- $(1 + \sqrt{2}) - 1 + (\sqrt{2} - 1) - \dots$

In Exercises 15–28, find the fractions equal to the given decimals.

- |                                   |  |
|-----------------------------------|--|
| 15. $0.333\ 33\ \dots$            | 16. $0.555\ 55\ \dots$                 |
| 17. $0.499\ 999\ \dots$           | 18. $0.999\ 999\ \dots$                |
| 19. $0.404\ 040\ \dots$           | 20. $0.070\ 707\ \dots$                |
| 21. $0.181\ 818\ \dots$           | 22. $0.336\ 336\ 336\ \dots$           |
| 23. $0.027\ 327\ 327\ 3\ \dots$   | 24. $0.822\ 22\ \dots$                 |
| 25. $0.366\ 666\ \dots$           | 26. $0.664\ 242\ 42\ \dots$            |
| 27. $0.100\ 841\ 841\ 841\ \dots$ | 28. $0.184\ 561\ 845\ 618\ 456\ \dots$ |

In Exercises 29–39, solve the given problems by use of the sum of an infinite geometric series.

- There are two infinite geometric series with  $a_1 = 50$  and  $a_3 = 2$ . Find the sum of each.
- Explain why there is no infinite geometric series with  $a_1 = 5$  and  $S = 2$ .
- Liquid is continuously collected in a wastewater-holding tank such that during a given hour only 92.0% as much liquid is collected as in the previous hour. If 28.0 L are collected in the first hour, what must be the minimum capacity of the tank?

- 32. If 75% of all aluminum cans are recycled, what is the total number of recycled cans that can be made from 400 000 cans that are recycled over and over until all the aluminum from these cans is used up? (Assume no aluminum is lost in the recycling process.)
- 33. The amounts of plutonium-237 that decay each day because of radioactivity form a geometric sequence. Given that the amounts that decay during each of the first 4 days are 5.882 g, 5.782 g, 5.684 g, and 5.587 g, respectively, what total amount will decay?
- 34. A helium-filled balloon rose 36.0 m in 1.0 min. Each minute after that, it rose 75% as much as in the previous minute. What was its maximum height?
- 35. A bicyclist travelling at 10 m/s then coasts to a stop as the bicycle travels 0.90 as far each second as in the previous second. How far does the bicycle travel in coasting to a stop?

- 36. A square has sides of 20 cm. Another square is inscribed in the first square by joining the midpoints of the sides. Assuming that such inscribed squares can be formed endlessly, find the sum of the areas of all the squares and explain how the sum is found. See Fig. 19.2.

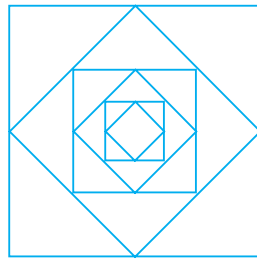


Fig. 19.2

- 37. Find  $x$  if the sum of the terms of the infinite geometric series  $1 + 2x + 4x^2 + \dots$  is  $2/3$ .
- 38. Find the sum of the terms of the infinite series  $1 + 2x + 3x^2 + 4x^3 + \dots$  for  $|x| < 1$ . (Hint: Use  $S - xS$ .)
- 39. In a “torture test,” a light switch is turned on and off until it fails. For a certain switch, the probability that the switch will fail after it has been turned on or off 1000 times is given by
 
$$(0.999)^{1000}(0.001) + (0.999)^{1001}(0.001)$$

$$+ (0.999)^{1002}(0.001) + \dots$$

Find this probability.

Answers to Practice Exercises

- 1.  $S = 27/2$
- 2.  $3/11$

## 19.4 The Binomial Theorem

Properties of the Binomial  $(a + b)^n$  • Factorial Notation • Binomial Theorem • Pascal's Triangle • Binomial Series

We end this chapter by studying another useful and important series, the **binomial series**. The binomial series is the infinite series generalization of the **binomial formula**, which we develop first. The binomial formula allows us to expand binomials to any positive integer power without direct multiplication. It is used in technical applications and in many areas of mathematics.

By direct multiplication, we may obtain the following expansions of the binomial  $a + b$ :

$$(a + b)^0 = 1$$

$$(a + b)^1 = a + b$$

$$(a + b)^2 = a^2 + 2ab + b^2$$

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

$$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

Inspection shows these expansions have certain properties, and it can be shown that these properties are valid for the expansion of  $(a + b)^n$ , where  $n$  is any positive integer.

### Properties of the Binomial $(a + b)^n$

1. There are  $n + 1$  terms.
2. The first term is  $a^n$ , and the final term is  $b^n$ .
3. Progressing from the first term to the last, the exponent of  $a$  decreases by 1 from term to term, the exponent of  $b$  increases by 1 from term to term, and the sum of the exponents of  $a$  and  $b$  in each term is  $n$ .
4. If the coefficient of any term is multiplied by the exponent of  $a$  in that term and this product is divided by the number of that term, we obtain the coefficient of the next term.
5. The coefficients of terms equidistant from the ends are equal.

**EXAMPLE 1** Using the basic binomial properties

Using the basic properties, develop the expansion for  $(a + b)^5$ .

Since the exponent of the binomial is 5, we have  $n = 5$ .

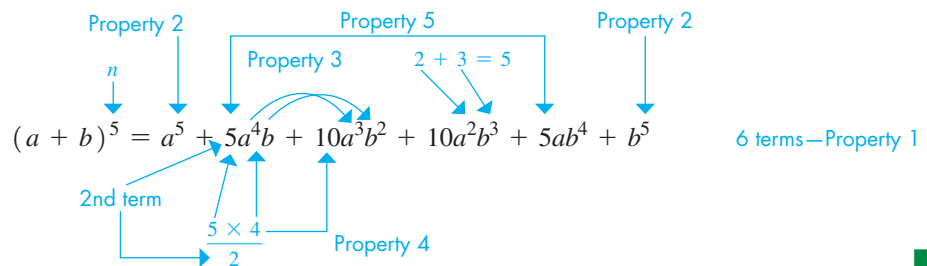
From Property 1, we know that there are six terms.

From Property 2, we know that the first term is  $a^5$  and the final term is  $b^5$ .

From Property 3, we know that the factors of  $a$  and  $b$  in terms 2, 3, 4, and 5 are  $a^4b$ ,  $a^3b^2$ ,  $a^2b^3$ , and  $ab^4$ , respectively.

From Property 4, we obtain the coefficients of terms 2, 3, 4, and 5. In the first term,  $a^5$ , the coefficient is 1. Multiplying by 5, the power of  $a$ , and dividing by 1, the number of the term, we obtain 5, which is the coefficient of the second term. Thus, the second term is  $5a^4b$ . Again using Property 4, we obtain the coefficient of the third term. The coefficient of the second term is 5. Multiplying by 4 and dividing by 2, we obtain 10. This means that the third term is  $10a^3b^2$ .

From Property 5, we know that the coefficient of the fifth term is the same as the second and that the coefficient of the fourth term is the same as the third. These properties are illustrated in the following diagram:



It is not necessary to use the above properties directly to expand a given binomial. If they are applied to  $(a + b)^n$ , a general formula for the expansion of a binomial may be obtained. In developing and stating the general formula, it is convenient to use the **factorial notation**  $n!$ , where

$$n! = n(n - 1)(n - 2) \cdots (2)(1) \quad (19.8)$$

We see that  $n!$ , read “ $n$  factorial,” represents the product of the first  $n$  positive integers. We also define  $0! = 1$ . (See Exercise 39.)

**EXAMPLE 2** Evaluating factorials

(a)  $3! = (3)(2)(1) = 6$

(b)  $5! = (5)(4)(3)(2)(1) = 120$

(c)  $8! = (8)(7)(6)(5)(4)(3)(2)(1) = 40\,320$

(d)  $3! + 5! = 6 + 120 = 126$

(e)  $\frac{4!}{2!} = \frac{(4)(3)(2)(1)}{(2)(1)} = 12$

**COMMON ERROR**

In evaluating factorials, we must remember that they represent products of numbers. In parts (d) and (e) of Example 2, we see that

$$3! + 5! \text{ is not } (3 + 5)! \quad \text{and} \quad 4!/2! \text{ is not } (4/2)!.$$

**Practice Exercise**

1. Evaluate  $9!/7!$ .

**THE BINOMIAL FORMULA**

Based on the binomial properties, the **binomial theorem** states that the following **binomial formula** is valid for all positive integer values of  $n$  (the binomial theorem is proven through advanced methods).

$$(a + b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \dots + b^n \quad (19.9)$$

**EXAMPLE 3 Using the binomial formula**

Using the binomial formula, expand  $(2x + 3)^6$ .

In using the binomial formula for  $(2x + 3)^6$ , we use  $2x$  for  $a$ ,  $3$  for  $b$ , and  $6$  for  $n$ . Thus,

$$\begin{aligned} (2x + 3)^6 &= (2x)^6 + 6(2x)^5(3) + \frac{\overset{n=6}{\downarrow} \overset{n-1}{\downarrow} (6)(5)}{\underset{2!}{\uparrow}} (2x)^4(3)^2 + \frac{\overset{n}{\downarrow} \overset{n-1}{\downarrow} \overset{n-2}{\downarrow} (6)(5)(4)}{\underset{3!}{\uparrow}} (2x)^3(3)^3 + \frac{\overset{(6)(5)(4)(3)}{\downarrow}}{\underset{4!}{\uparrow}} (2x)^2(3)^4 + \frac{\overset{(6)(5)(4)(3)(2)}{\downarrow}}{\underset{5!}{\uparrow}} (2x)(3)^5 + 3^6 \\ &= 64x^6 + 576x^5 + 2160x^4 + 4320x^3 + 4860x^2 + 2916x + 729 \end{aligned}$$

**Pascal's Triangle**

■ Pascal's triangle is named after the French scientist, mathematician, and philosopher Blaise Pascal (1623–1662). However, it was already known in China and in medieval Islam during the eleventh century.

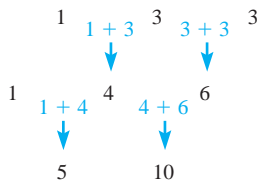


Fig. 19.3

For the first few integral powers of a binomial  $a + b$ , the coefficients can be obtained by setting them up in the following pattern, known as **Pascal's triangle**.

$n = 0$	1							
$n = 1$	1	1						
$n = 2$		1	2	1				
$n = 3$		1	3	3	1			
$n = 4$		1	4	6	4	1		
$n = 5$		1	5	10	10	5	1	
$n = 6$		1	6	15	20	15	6	1

see expansions on page 523

We note that the first and last coefficients shown in each row are 1, and the second and next-to-last coefficients are equal to  $n$ . Other coefficients are obtained by adding the two nearest coefficients in the row above, as illustrated in Fig. 19.3 for the indicated section of Pascal's triangle. This pattern may be continued indefinitely, although use of Pascal's triangle is cumbersome for high values of  $n$ .

**EXAMPLE 4 Using Pascal's triangle**

Using Pascal's triangle, expand  $(5s - 2t)^4$ .

Here, we note that  $n = 4$ . Thus, the coefficients of the five terms are 1, 4, 6, 4, and 1, respectively. Also, here we use  $5s$  for  $a$  and  $-2t$  for  $b$ . We are expanding this expression as  $[(5s) + (-2t)]^4$ . Therefore,

$$\begin{aligned} (5s - 2t)^4 &= (5s)^4 + 4(5s)^3(-2t) + 6(5s)^2(-2t)^2 + 4(5s)(-2t)^3 + (-2t)^4 \\ &= 625s^4 - 1000s^3t + 600s^2t^2 - 160st^3 + 16t^4 \end{aligned}$$

In certain uses of a binomial expansion, it is not necessary to obtain all terms. Only the first few terms are required. The following example illustrates finding the first four terms of an expansion.

**EXAMPLE 5** Using the binomial formula

Find the first four terms of the expansion of  $(x + 7)^{12}$ .

Here, we use  $x$  for  $a$ ,  $7$  for  $b$ , and  $12$  for  $n$ . Thus, from the binomial formula, we have

$$\begin{aligned}(x + 7)^{12} &= x^{12} + 12x^{11}(7) + \frac{(12)(11)}{2}x^{10}(7)^2 + \frac{(12)(11)(10)}{(2)(3)}x^9(7)^3 + \cdots \\ &= x^{12} + 84x^{11} + 3234x^{10} + 75\,460x^9 + \cdots\end{aligned}$$

If we let  $a = 1$  and  $b = x$  in the binomial formula, we obtain the **binomial series**

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots \quad (19.10)$$

**Practice Exercise**

2. Find the first three terms of the expansion of  $(x - 4)^9$ .

## ■ Binomial Series

which, through advanced methods, can be shown to be valid for any real number  $n$  if  $|x| < 1$ . When  $n$  is either negative or a fraction, we obtain an infinite series. In such a case, we calculate as many terms as may be needed, although such a series is not obtainable through direct multiplication. The binomial series may be used to develop important expressions that are used in applications and more advanced mathematics topics.

**EXAMPLE 6** Binomial series—application

In the analysis of forces on beams, the expression  $1/(1 + m^2)^{3/2}$  is used. Use the binomial series to find the first four terms of the expansion.

Using negative exponents, we have

$$1/(1 + m^2)^{3/2} = (1 + m^2)^{-3/2}$$

Now, in using Eq. (19.10), we have  $n = -3/2$  and  $x = m^2$ :

$$(1 + m^2)^{-3/2} = 1 + \left(-\frac{3}{2}\right)(m^2) + \frac{\left(-\frac{3}{2}\right)\left(-\frac{3}{2} - 1\right)}{2!}(m^2)^2 + \frac{\left(-\frac{3}{2}\right)\left(-\frac{3}{2} - 1\right)\left(-\frac{3}{2} - 2\right)}{3!}(m^2)^3 + \cdots$$

Therefore,

$$\frac{1}{(1 + m^2)^{3/2}} = 1 - \frac{3}{2}m^2 + \frac{15}{8}m^4 - \frac{35}{16}m^6 + \cdots$$

**EXAMPLE 7** Evaluation using the binomial series

Approximate the value of  $0.97^7$  by use of the binomial series.

We note that  $0.97 = 1 - 0.03$ , which means  $0.97^7 = [1 + (-0.03)]^7$ . Using four terms of the binomial series, we have

$$\begin{aligned}0.97^7 &= [1 + (-0.03)]^7 \\ &= 1 + 7(-0.03) + \frac{7(6)}{2!}(-0.03)^2 + \frac{7(6)(5)}{3!}(-0.03)^3 \\ &= 1 - 0.21 + 0.0189 - 0.000\,945 = 0.807\,955\end{aligned}$$

From a calculator, we find that  $0.97^7 = 0.807\,983$  (to six decimal places), which means these values agree to 4 significant digits with a value  $0.8080$ . Greater accuracy can be found using the binomial series if more terms are used.

## EXERCISES 19.4

In Exercises 1 and 2, make the given changes in the indicated examples of this section and then solve the given problems.

- In Example 3, change the exponent from 6 to 5.
- In Example 7, change 0.97 to 0.98.

In Exercises 3–12, expand and simplify the given expressions by use of the binomial formula.

- $(t + 4)^3$
- $(2x - 1)^4$
- $(6 + 0.1)^5$
- $(n + 2\pi)^5$
- $(2a - b^2)^6$
- $(x - 2)^3$
- $(x^2 + 5)^4$
- $(xy - z)^5$
- $(1 - j)^6$  ( $j = \sqrt{-1}$ )
- $\left(\frac{a}{x} + x\right)^6$

In Exercises 13–16, expand and simplify the given expressions by use of Pascal's triangle.

- $(5x - 3)^4$
- $(2a + 1)^6$
- $(x - 4)^5$
- $(x - 3)^7$

In Exercises 17–24, find the first four terms of the indicated expansions.

- $(x + 2)^{10}$
- $(2a - 1)^7$
- $(x^{1/2} - 4y)^{12}$
- $\left(b^2 + \frac{1}{2b}\right)^{20}$
- $(x - 4)^8$
- $(3b + 2)^9$
- $(2a - x^{-1})^{11}$
- $\left(2x^2 + \frac{y}{3}\right)^{15}$

In Exercises 25–28, approximate the value of the given expression to three decimal places by using three terms of the appropriate binomial series. Check using a calculator.

- $1.05^6$
- $\sqrt[3]{1.045}$
- $\sqrt{0.927}$
- $0.98^{-7}$

In Exercises 29–36, find the first four terms of the indicated expansions by use of the binomial series.

- $(1 + x)^8$
- $(1 - 3x)^{-2}$
- $\sqrt{1 + x}$
- $\frac{1}{\sqrt{9 - 9x}}$
- $(1 + x)^{-1/3}$
- $(1 - 2\sqrt{x})^9$
- $\frac{1}{\sqrt{1 - x}}$
- $\sqrt{4 + x^2}$

In Exercises 37–40, solve the given problems involving factorials.

- Using a calculator, evaluate (a)  $17! + 4!$ , (b)  $21!$ , (c)  $17! \times 4!$ , and (d)  $68!$ .
- Using a calculator, evaluate (a)  $8! - 7!$ , (b)  $8!/7!$ , (c)  $8! \times 7!$ , and (d)  $56!$ .
- Show that  $n! = n \times (n - 1)!$  for  $n \geq 2$ . To use this equation for  $n = 1$ , explain why it is necessary to define  $0! = 1$ .
- Show that  $\frac{(n + 1)!}{(n - 2)!} = n^3 - n$  for  $n \geq 2$ . See Exercise 39.

In Exercises 41–44, find the indicated terms by use of the following information. The  $r + 1$  term of the expansion of  $(a + b)^n$  is given by

$$\frac{n(n-1)(n-2)\cdots(n-r+1)}{r!} a^{n-r} b^r$$

- The term involving  $b^5$  in  $(a + b)^8$
- The term involving  $y^6$  in  $(x + y)^{10}$
- The fifth term of  $(2x - 3b)^{12}$
- The sixth term of  $(\sqrt{a} - \sqrt{b})^{14}$

In Exercises 45–56, solve the given problems.

- Explain why  $n!$  ends in a zero if  $n > 4$ .
- Expand  $[(a + b) + c]^3$  using the binomial theorem. Group terms as indicated.
- In the expansion of  $(a - x)^n$ , where  $n$  is a positive integer, show that the sum of the coefficients is zero.
- If  $x$  is very small, show that  $(1 + x)^{-n}$  is approximately  $(1 - nx)$ .
- Approximate  $\sqrt{6}$  to hundredths by noting that  $\sqrt{6} = \sqrt{4(1.5)} = 2\sqrt{1 + 0.5}$  and using four terms of the appropriate binomial series.
- Approximate  $\sqrt[3]{10}$  by using the method of Exercise 49.
- A company purchases a piece of equipment for  $A$  dollars, and the equipment depreciates at a rate of  $r$  each year. Its value  $V$  after  $n$  years is  $V = A(1 - r)^n$ . Expand this expression for  $n = 5$ .
- In finding the rate of change of emission of energy from the surface of a body at temperature  $T$ , the expression  $(T + h)^4$  is used. Expand this expression.
- In the theory associated with the magnetic field due to an electric current, the expression  $1 - \frac{x}{\sqrt{a^2 + x^2}}$  is found. By expanding  $(a^2 + x^2)^{-1/2}$ , find the first three nonzero terms that could be used to approximate the given expression.
- In the theory related to the dispersion of light, the expression  $1 + \frac{A}{1 - \lambda_0^2/\lambda^2}$  arises. (a) Find the first four terms of the expansion of  $(1 - x)^{-1}$ . (b) Find the same expansion by using long division. (c) Let  $x = \lambda_0^2/\lambda^2$  and write the original expression in expanded form, using the results of (a) and (b).
- In finding the rate at which the resistance of a wire changes with its radius, the expression  $\frac{k}{(r + h)^2} - \frac{k}{r^2}$  is used. Use three terms of the binomial expansion of  $k(r + h)^{-2}$  to approximate this expression.
- Find the first four terms of the expansion of  $(1 + x)^{-1}$  and then divide  $1 + x$  into 1. Compare the results.

## Answers to Practice Exercises

- 72
- $x^9 - 36x^8 + 576x^7 - \cdots$

## CHAPTER 19 EQUATIONS

<b>Arithmetic sequences</b>	Recursion formula	$a_n = a_{n-1} + d$	(19.1)
	$n$ th term	$a_n = a_1 + (n - 1)d$	(19.2)
	Sum of $n$ terms	$S_n = \frac{n}{2}(a_1 + a_n)$	(19.3)
<b>Geometric sequences</b>	Recursion formula	$a_n = ra_{n-1}$	(19.4)
	$n$ th term	$a_n = a_1 r^{n-1}$	(19.5)
	Sum of $n$ terms	$S_n = \frac{a_1(1 - r^n)}{1 - r} \quad (r \neq 1)$	(19.6)
<b>Sum of a geometric series</b>		$S = \frac{a_1}{1 - r} \quad ( r  < 1)$	(19.7)
<b>Factorial notation</b>		$n! = n(n - 1)(n - 2) \cdots (2)(1)$	(19.8)
<b>Binomial formula</b>		$(a + b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \cdots + b^n$	(19.9)
<b>Binomial series</b>		$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$	(19.10)

## CHAPTER 19 REVIEW EXERCISES

In Exercises 1–8, find the indicated term of each sequence.

- 1, 6, 11, ... (17th)
- 1, -3, -7, ... (21st)
- 500, 100, 20, ... (9th)
- 0.025, 0.01, 0.004, ... (7th)
- 1, 3.5, 8, ... (16th)
- $-1, -\frac{5}{3}, -\frac{7}{3}, \dots$  (25th)
- $\frac{3}{4}, \frac{1}{2}, \frac{1}{3}, \dots$  (7th)
- $5^{-2}, 5^0, 5^2, \dots$  (7th)

In Exercises 9–12, find the sum of each sequence with the indicated values.

- $a_1 = -4, n = 15, a_{15} = 17$  (arith.)
- $a_1 = 300, d = -\frac{20}{3}, n = 10$
- $a_1 = 16, r = -\frac{1}{2}, n = 14$
- $a_1 = 64, a_n = 729, n = 7$  (geom.,  $r > 0$ )

In Exercises 13–24, find the indicated quantities for the appropriate sequences.

- $a_1 = 17, d = -2, n = 9, S_9 = ?$
- $d = \frac{4}{3}, a_1 = -3, a_n = 17, n = ?$
- $a_1 = 4, r = \sqrt{2}, n = 7, a_7 = ?$
- $a_n = \frac{49}{8}, r = -\frac{2}{7}, S_n = \frac{1911}{32}, a_1 = ?$
- $a_1 = 80, a_n = -25, S_n = 220, d = ?$
- $a_1 = 2, d = 0.2, n = 11, S_{11} = ?$
- $n = 6, r = -0.25, S_6 = 204.75, a_6 = ?$

- $a_1 = 10, r = 0.1, S_n = 11.111, n = ?$
- $a_1 = -1, a_n = 32, n = 12, S_{12} = ?$  (arith.)
- $a_1 = 100, a_n = 6400, S_n = 32\,500, n = ?$  (arith.)
- $a_1 = 1, n = 7, a_7 = 64, S_7 = ?$
- $a_1 = \frac{1}{4}, n = 6, a_6 = 8, S_6 = ?$

In Exercises 25–28, find the sums of the given infinite geometric series.

- $0.9 + 0.6 + 0.4 + \cdots$
- $1280 - 320 + 80 - \cdots$
- $1 + 1.02^{-1} + 1.02^{-2} + \cdots$
- $3 - \sqrt{3} + 1 - \cdots$

In Exercises 29–32, find the fractions equal to the given decimals.

- 0.030 303 ...
- 0.363 363 ...
- 0.072 727 2 ...
- 0.253 993 993 99 ...

In Exercises 33–36, expand and simplify the given expression. In Exercises 37–40, find the first four terms of the appropriate expansion.

- $(x - 2)^4$
- $(x^2 + 4)^5$
- $(a + 2e)^{10}$
- $(p^2 - \frac{q}{6})^9$
- $(3 + 0.1)^4$
- $(3n^{1/2} - a)^6$
- $(\frac{x}{4} - y)^{12}$
- $(2s^2 - \frac{3}{2}t^{-1})^{14}$



In Exercises 41–48, find the first four terms of the indicated expansions by use of the binomial series.

41.  $(1 + x)^{12}$                       42.  $(1 - 2x)^{10}$   
 43.  $\sqrt{1 + x^2}$                       44.  $(4 - 4\sqrt{x})^{-1}$   
 45.  $\sqrt{1 - a^2}$                       46.  $\sqrt{1 + 2b^4}$   
 47.  $(2 - 4x)^{-3}$                     48.  $(1 + 4x)^{-1/4}$

In Exercises 49–92, solve the given problems by use of an appropriate sequence or expansion. All numbers are accurate to at least 2 significant digits.

49. Find the sum of the first 1000 positive even integers.  
 50. How many integers divisible by 4 lie between 23 and 121?  
 51. What is the fifth term of the arithmetic sequence in which the first term is  $a$  and the second term is  $b$ ?  
 52. Find three consecutive numbers in an arithmetic sequence such that their sum is 15 and the sum of their squares is 77.  
 53. For a geometric sequence, is it possible that  $a_3 = 6$ ,  $a_5 = 9$ , and  $a_7 = 12$ ?  
 54. Find three consecutive numbers in a geometric sequence such that their product is 64 and the sum of their squares is 84.  
 55. Find a formula with variable  $n$  of the arithmetic sequence with  $a_1 = -5$ ,  $a_{n+1} = a_n - 3$ , for  $n = 1, 2, 3, \dots$   
 56. Find the ninth term of the sequence  $(a + 2b)$ ,  $b$ ,  $-a$ ,  $\dots$   
 57. Approximate the value of  $(1.06)^{-6}$  by using three terms of the appropriate binomial series. Check using a calculator.  
 58. Approximate the value of  $\sqrt[4]{0.94}$  by using three terms of the appropriate binomial series. Check using a calculator.  
 59. Approximate the value of  $\sqrt{30}$  by noting that  $\sqrt{30} = \sqrt{25(1.2)} = 5\sqrt{1 + 0.2}$  and using three terms of the appropriate binomial series.  
 60. Approximate  $\sqrt[3]{29.7}$  by using the *method* of Exercise 59.  
 61. Each stroke of a pile driver moves a post 2 cm less than the previous stroke. If the first stroke moves the post 24 cm, which stroke moves the post 4 cm?  
 62. During each hour, an exhaust fan removes 15.0% of the carbon dioxide present in the air in a room at the beginning of the hour. What percent of the carbon dioxide remains after 10.0 h?  
 63. Each 1.0 mm of a filter through which light passes reduces the intensity of the light by 12%. How thick should the filter be to reduce the intensity of the light to 20%?  
 64. A pile of dirt and ten holes are in a straight line. It is 20 m from the dirt pile to the nearest hole, and the holes are 8 m apart. If a backhoe takes two trips to fill each hole, how far must it travel in filling all the holes if it starts and ends at the dirt pile?  
 65. A roof support with equally spaced vertical pieces is shown in Fig. 19.4. Find the total length of the vertical pieces if the shortest one is 254 mm long.  
 66. During each microsecond, the current in an electric circuit decreases by 9.3%. If the initial current is 2.45 mA, how long does it take to reach 0.50 mA?  
 67. A machine that costs \$8600 depreciates 1.0% in value each month. What is its value 5 years after it was purchased?  
 68. The level of chemical pollution in a lake is 4.50 ppb (parts per billion). If the level increases by 0.20 ppb in the following month and by 5.0% less each month thereafter, what will be the maximum level?  
 69. A piece of paper 0.015 cm thick is cut in half. These two pieces are then placed one on the other and cut in half. If this is repeated such that the paper is cut in half 40 times, how high will the pile be?  
 70. After the power is turned off, an object on a nearly frictionless surface slows down such that it travels 99.9% as far during 1 s as during the previous second. If it travels 100 cm during the first second after the power is turned off, how far does it travel while stopping?  
 71. Under gravity, an object falls 4.9 m during the first second, 14.7 m during the second second, 24.5 m during the third second, and so on. How far will it fall during the 20th second?  
 72. For the object in Exercise 71, what is the total distance fallen during the first 20 s?  
 73. An object suspended on a spring is oscillating up and down. If the first oscillation is 10.0 cm and each oscillation thereafter is 9/10 of the preceding one, find the total distance the object travels in coming to rest.  
 74. During each oscillation, a pendulum swings through 85% of the distance of the previous oscillation. If the pendulum swings through 80.8 cm in the first oscillation, through what total distance does it move in 12 oscillations?  
 75. A person invests \$1000 each year at the beginning of the year. What is the total value of these investments after 20 years if they earn 7.5% annual interest, compounded semiannually?  
 76. A well driller charges \$10.00 for drilling the first metre of a well and for each metre thereafter charges 0.20% more than for the preceding metre. How much is charged for drilling a 150-m well?  
 77. When new, an article cost \$250. It was then sold at successive yard sales at 40% of the previous price. What was the price at the fourth yard sale?  
 78. Each side of an equilateral triangle is 2 cm in length. The midpoints are joined to form a second equilateral triangle. The midpoints of the second triangle are joined to form a third equilateral triangle. Find the sum of all of the perimeters of the triangles if this process is continued indefinitely. See Fig. 19.5.

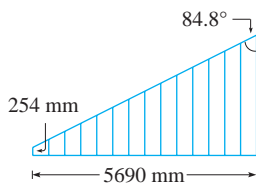


Fig. 19.4

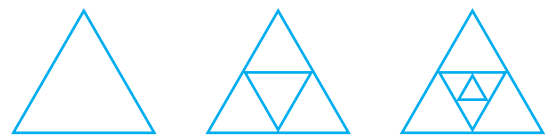


Fig. 19.5

79. In testing a type of insulation, the temperature in a room was made to fall to  $2/3$  of the initial temperature after 1.0 h, to  $2/5$  of the initial temperature after 2.0 h, to  $2/7$  of the initial temperature after 3.0 h, and so on. If the initial temperature was  $50.0^\circ\text{C}$ , what was the temperature after 12.0 h? (This is an illustration of a *harmonic sequence*.)

80. Two competing businesses make the same item and sell it initially for \$100. One increases the price by \$8 each year for 5 years, and the other increases the price by 8% each year for 5 years. What is the difference in price after 5 years?
81. In hydrodynamics, while studying compressible fluid flow, the expression  $\left(1 + \frac{a-1}{2}m^2\right)^{a/(a-1)}$  arises. Find the first three terms of the expansion of this expression.
82. The maximum tension  $T$  of a suspension bridge cable is given by
- $$T = \frac{wL}{2} \sqrt{1 + \left(\frac{L}{4s}\right)^2}$$
- Find the first three terms of the expansion of this expression.
83. During 1 year a beach eroded 1.2 m to a line 48.3 m from the wall of a building. If the erosion is 0.1 m more each year than the previous year, when will the waterline reach the wall?
84. On a highway with a steep incline, the runaway truck ramp is constructed so that a vehicle that has lost its brakes can stop. The ramp is designed to slow a truck in succeeding 20-m distances by 10 km/h, 12 km/h, 14 km/h, . . . . If the ramp is 160 m long, will it stop a truck moving at 120 km/h when it reaches the ramp?
85. Each application of an insecticide destroys 75% of a certain insect. How many applications are needed to destroy at least 99.9% of the insects?
86. A house valued at \$375 000 increases in value 5.00%/year for the next 6 years and then decreases 3.00%/year for the following 4 years. What is its value at the end of the 10 years?

87. Oil pumped from a certain oil field decreases 10% each year. How long will it take for production to be 10% of the first year's production?
88. A wire hung between two poles is parabolic in shape. To find the length of wire between two points on the wire, the expression  $\sqrt{1 + 0.08x^2}$  is used. Find the first three terms of the binomial expansion of this expression.
89. Show that the middle term of a finite arithmetic sequence equals  $S/n$ , if  $n$  is odd.
90. The sum of three terms of a geometric sequence is  $-3$ , and the second is 6 more than the first. Find these terms.
91. Do the reciprocals of the terms of a geometric sequence form a geometric sequence? Explain.
92. The terms  $a, a + 12, a + 24$  form an arithmetic sequence, and the terms  $a, a + 24, a + 12$  form a geometric sequence. Find these sequences.

### Writing Exercise

93. Derive a formula for the value  $V$  after 1 year of an amount  $A$  invested at  $r\%$  (as a decimal) annual interest, compounded  $n$  times during the year. If  $A = \$1000$  and  $r = 0.10$  (10%), write two or three paragraphs explaining why the amount of interest increases as  $n$  increases and stating your approach to finding the maximum possible amount of interest.

## CHAPTER 19 PRACTICE TEST

- Write the first five terms of the sequence for which (a)  $a_1 = 8$  and  $d = -1/2$ ; (b)  $a_1 = 8$  and  $r = -1/2$ .
- Find the sum of the first seven terms of the sequence  $6, -2, \frac{2}{3}, \dots$ .
- For a given sequence,  $a_1 = 6$ ,  $d = 4$ , and  $S_n = 126$ . Find  $n$ .
- Find the fraction equal to the decimal  $0.454\ 545\ \dots$ .
- Find the first three terms of the expansion of  $\sqrt{1 - 4x}$ .
- Expand and simplify the expression  $(2x - y)^5$ .
- What is the value after 20 years of an investment of \$2500 if it draws 5% annual interest compounded annually?
- Find the sum of the first 100 even integers.
- A ball is dropped from a height of 8.00 m, and on each rebound, it rises to  $1/2$  of the height it last fell. If it bounces indefinitely, through what total distance will it move?

# Additional Topics in Trigonometry

As the use of electricity became widespread in the 1880s, there was a serious debate over the best way to distribute electric power. The American inventor Thomas Edison favoured the use of direct current because it was safer and did not vary with time. Another American inventor and engineer, George Westinghouse, favoured alternating current because the voltage could be stepped up and down with transformers during transmission.

Also favouring the use of alternating current was Nikola Tesla, an American (born in Croatia) electrical engineer, and he had a strong influence on the fact that alternating current came to be used for transmission. Tesla developed many electrical devices, among them the *polyphase generator* that allowed alternating current to be transmitted with constant instantaneous power. Using this type of generator, power losses are greatly reduced in transmission lines, which allows for smaller conductors, and the power can be generated far from where it is used.

Three-phase systems are used in most commercial electric generators, and in Section 20.2 we show how a relationship involving trigonometric functions can be used to show a basic property of the current produced by a three-phase generator. Many such relationships among the trigonometric functions can be found from the definitions and other known relationships.

The trigonometric relationships that we develop in this chapter are important for a number of reasons. In fact, we already made use of some of them in Section 10.4 when we graphed certain trigonometric functions, and in Chapter 9 in deriving the law of cosines. In calculus, certain problems use trigonometric relationships, even including some in which these functions do not appear in the initial problem or final answer. Also, they are useful in a number of technical applications in areas such as electronics, optics, solar energy, and robotics.

Later in the chapter, we see that various trigonometric relationships are used in solving equations with trigonometric functions. Also, we develop the concept of the inverse trigonometric functions that were introduced in Chapter 4.

# 20

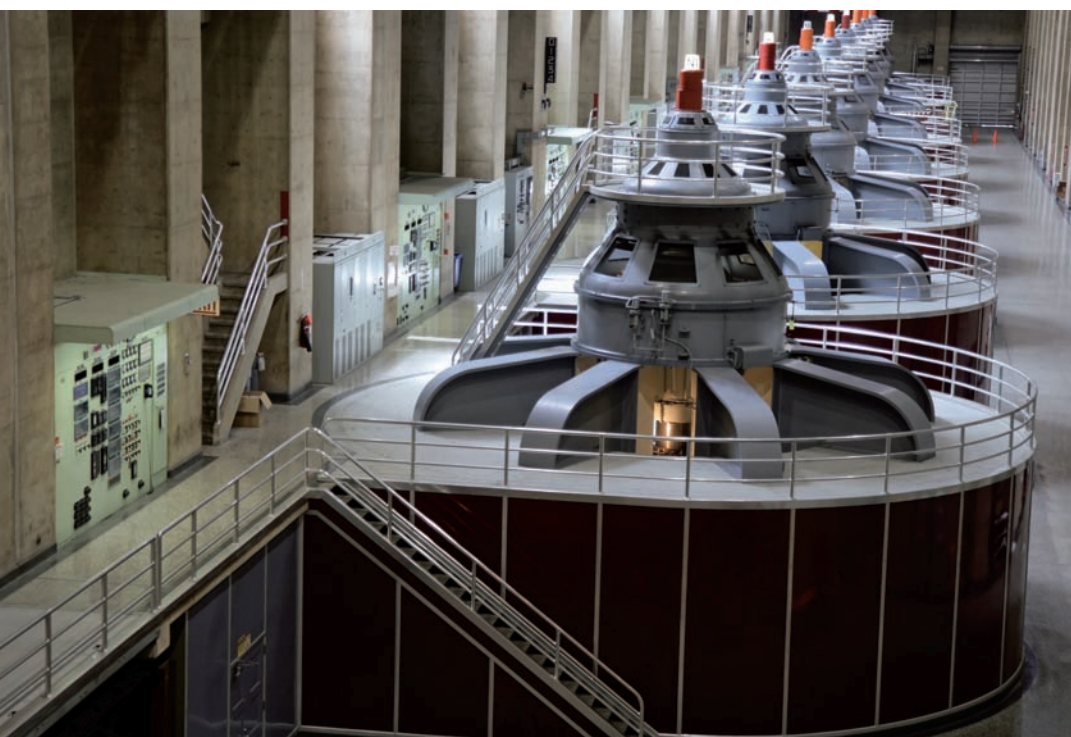
## LEARNING OUTCOMES

After completion of this chapter, the student should be able to:

- Recognize the basic trigonometric identities and use them to prove other trigonometric identities and simplify trigonometric expressions
- Recognize and apply the formulas for trigonometric functions of sums and differences of angles
- Recognize and apply the formulas for trigonometric functions of half and double angles
- Solve trigonometric equations
- Evaluate inverse trigonometric functions in their defined range
- Find algebraic expressions for expressions involving inverse trigonometric functions
- Solve application problems involving basic trigonometric identities, other trigonometric formulas, and inverse trigonometric functions

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◀ In Section 20.2, we show how trigonometric relationships are used in analysing the voltage produced by a three-phase generator, the most widely used type of polyphase generator of alternating current.



## 20.1 Fundamental Trigonometric Identities

Trigonometric Identities • Basic Identities • Proving Trigonometric Identities

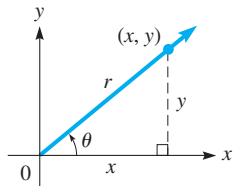


Fig. 20.1

From Chapters 4 and 8, from the definitions, recall that  $\sin \theta = y/r$  and  $\csc \theta = r/y$  (see Fig. 20.1). Since  $y/r = 1/(r/y)$ , we see that  $\sin \theta = 1/\csc \theta$ . These definitions hold for *any* angle, which means that  $\sin \theta = 1/\csc \theta$  is true for *any* angle. *This type of relation, which is true for any value of the variable, is called an **identity**.* Of course, values where division by zero would be indicated are excluded.

In this section, we develop several important identities among the trigonometric functions. We also show how the basic identities are used to verify other identities.

From the definitions, we have

$$\sin \theta \csc \theta = \frac{y}{r} \times \frac{r}{y} = 1 \quad \text{or} \quad \sin \theta = \frac{1}{\csc \theta} \quad \text{or} \quad \csc \theta = \frac{1}{\sin \theta}$$

$$\cos \theta \sec \theta = \frac{x}{r} \times \frac{r}{x} = 1 \quad \text{or} \quad \cos \theta = \frac{1}{\sec \theta} \quad \text{or} \quad \sec \theta = \frac{1}{\cos \theta}$$

$$\tan \theta \cot \theta = \frac{y}{x} \times \frac{x}{y} = 1 \quad \text{or} \quad \tan \theta = \frac{1}{\cot \theta} \quad \text{or} \quad \cot \theta = \frac{1}{\tan \theta}$$

$$\frac{\sin \theta}{\cos \theta} = \frac{y/r}{x/r} = \frac{y}{x} = \tan \theta \quad \frac{\cos \theta}{\sin \theta} = \frac{x/r}{y/r} = \frac{x}{y} = \cot \theta$$

Also, from the definitions and the Pythagorean theorem in the form of  $x^2 + y^2 = r^2$ , we arrive at the following identities.

By dividing the Pythagorean relation through by  $r^2$ , we have

$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1, \quad \text{which leads us to} \quad \cos^2 \theta + \sin^2 \theta = 1$$

By dividing the Pythagorean relation by  $x^2$ , we have

$$1 + \left(\frac{y}{x}\right)^2 = \left(\frac{r}{x}\right)^2, \quad \text{which leads us to} \quad 1 + \tan^2 \theta = \sec^2 \theta$$

By dividing the Pythagorean relation by  $y^2$ , we have

$$\left(\frac{x}{y}\right)^2 + 1 = \left(\frac{r}{y}\right)^2, \quad \text{which leads us to} \quad \cot^2 \theta + 1 = \csc^2 \theta$$

### Basic Identities

The term  $\cos^2 \theta$  is the common way of writing  $(\cos \theta)^2$ , and it means to square the value of the cosine of the angle. Obviously, the same holds true for the other functions.

Summarizing these results, we have the following important identities:

$$\sin \theta = \frac{1}{\csc \theta} \quad (20.1)$$

$$\cos \theta = \frac{1}{\sec \theta} \quad (20.2)$$

$$\tan \theta = \frac{1}{\cot \theta} \quad (20.3)$$

$$\tan \theta = \frac{\sin \theta}{\cos \theta} \quad (20.4)$$

$$\cot \theta = \frac{\cos \theta}{\sin \theta} \quad (20.5)$$

$$\sin^2 \theta + \cos^2 \theta = 1 \quad (20.6)$$

$$1 + \tan^2 \theta = \sec^2 \theta \quad (20.7)$$

$$1 + \cot^2 \theta = \csc^2 \theta \quad (20.8)$$

In using the basic identities,  $\theta$  may stand for any angle or number or expression representing an angle or a number.

**EXAMPLE 1** Using the basic identities

$$(a) \sin ax = \frac{1}{\csc ax} \text{ using Eq. (20.1)} \quad (b) \tan 157^\circ = \frac{\sin 157^\circ}{\cos 157^\circ} \text{ using Eq. (20.4)}$$

$$(c) \sin^2 \frac{\pi}{4} + \cos^2 \frac{\pi}{4} = 1 \text{ using Eq. (20.6)}$$

**EXAMPLE 2** Checking identities with numerical values

Let us check the last two illustrations of Example 1 for the given values of  $\theta$ .

(a) Using a calculator, we find that

$$\frac{\sin 157^\circ}{\cos 157^\circ} = \frac{0.390\ 731\ 128\ 5}{-0.920\ 504\ 853\ 5} = -0.424\ 474\ 816\ 2 \text{ and}$$

$$\tan 157^\circ = -0.424\ 474\ 816\ 2$$

We see that  $\sin 157^\circ / \cos 157^\circ = \tan 157^\circ$ .

(b) Checking Example 1(c), refer to Example 2 on page 123, or to Fig. 20.2.

$$\sin \frac{\pi}{4} = \sin 45^\circ = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \quad \text{and} \quad \cos \frac{\pi}{4} = \cos 45^\circ = \frac{\sqrt{2}}{2}$$

$$\sin^2 \frac{\pi}{4} + \cos^2 \frac{\pi}{4} = \left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{1}{2} + \frac{1}{2} = 1$$

We see that this checks with Eq. (20.6) for these values.

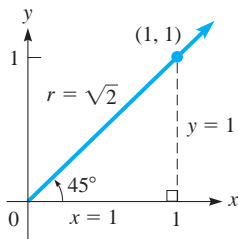


Fig. 20.2

**Practice Exercise**

1. Multiply and simplify  $\cos x (\sec x + \tan x)$ .

**EXAMPLE 3** Simplifying basic expressions

(a) Multiply and simplify the expression  $\sin \theta \tan \theta (\csc \theta + \cot \theta)$ .

$$\begin{aligned} \sin \theta \tan \theta (\csc \theta + \cot \theta) &= \sin \theta \tan \theta \csc \theta + \sin \theta \tan \theta \cot \theta \\ &= \left(\frac{1}{\csc \theta}\right) \tan \theta \csc \theta + \sin \theta \left(\frac{1}{\cot \theta}\right) \cot \theta && \text{using Eqs. (20.1) and (20.4)} \\ &= \tan \theta + \sin \theta \end{aligned}$$

(b) Factor and simplify the expression  $\tan^3 \alpha + \tan \alpha$ .

$$\begin{aligned} \tan^3 \alpha + \tan \alpha &= \tan \alpha (\tan^2 \alpha + 1) \\ &= \tan \alpha \sec^2 \alpha && \text{using Eq. (20.7)} \end{aligned}$$

We see that when algebraic operations are performed on trigonometric terms, they are handled in just the same way as with algebraic terms. This is true for the basic operations of addition, subtraction, multiplication, and division, as well as other operations used in simplifying expressions, such as factoring and taking roots.

**PROVING TRIGONOMETRIC IDENTITIES**

A great many identities exist among the trigonometric functions. We are going to use the basic identities that have been developed in Eqs. (20.1) through (20.8), along with a few additional ones developed in later sections to prove the validity of still other identities.

In proving identities, look for combinations that appear in, or are very similar to, those in the basic identities. This is illustrated in the following examples.

**EXAMPLE 4** Proving a trigonometric identity

In proving the identity

$$\sin x = \frac{\cos x}{\cot x}$$

**LEARNING TIP**

*The ability to prove trigonometric identities depends to a large extent on being very familiar with the basic identities so that you can recognize them in somewhat different forms.*

If you do not learn these basic identities well, you will have difficulty in following the examples and doing the exercises. The more readily you recognize these forms, the more easily you will be able to prove such identities.



we know that  $\cot x = \frac{\cos x}{\sin x}$ . Since  $\sin x$  appears on the left, substituting for  $\cot x$  on the right will eliminate  $\cot x$  and introduce  $\sin x$ . This should help us proceed in proving the identity. Thus,

**Practice Exercise**

2. Prove the identity  $\frac{\cos x \csc x}{\cot^2 x} = \tan x$ .

$$\begin{aligned} \sin x &= \frac{\cos x}{\cot x} = \frac{\cos x}{\frac{\cos x}{\sin x}} = \frac{\cos x}{1} \times \frac{\sin x}{\cos x} \\ &= \sin x \quad \text{cancel } \cos x \text{ factors} \end{aligned}$$

Eq. (20.5)
invert

By showing that the right side may be changed exactly to  $\sin x$ , the expression on the left side, we have proved the identity. ■

### Proving Trigonometric Identities

There is no specific procedure for proving trigonometric identities. However, the following points are important:

- The basic identities that may be useful must be readily recognized, keeping in mind equivalent forms of the same identity. For example,  $\sin^2 \theta = 1 - \cos^2 \theta$  is an equivalent form of the identity  $\sin^2 \theta + \cos^2 \theta = 1$ .
- Algebraic operations such as substituting, factoring, and simplifying fractions are frequently used, and they must be done carefully and correctly.
- Either side of the identity can be changed to the form on the other side. It is usually easier to change the form of the more complicated side to the same form as the less complicated side. (Although it is possible to prove an identity by working on both sides at the same time, we will not explore that method here.)
- Substitutions should be selected looking ahead and keeping in mind the side that is not being changed, which is the goal.
- Multiplying both numerator and denominator of an expression by the same quantity can be useful to obtain a difference of squares. For example, if an expression contains  $1 - \sin \theta$ , multiplication by  $1 + \sin \theta$  gives  $1 - \sin^2 \theta$ , which can be replaced by  $\cos^2 \theta$ .
- A given identity can be proved through a variety of procedures.

### EXAMPLE 5 A trigonometric identity—two different solutions

Prove that  $\tan \theta \csc \theta = \sec \theta$ .

In proving this identity, we know that  $\tan \theta = \frac{\sin \theta}{\cos \theta}$  and also that  $\frac{1}{\cos \theta} = \sec \theta$ . Thus, by substituting for  $\tan \theta$ , we introduce  $\cos \theta$  in the denominator, which is equivalent to introducing  $\sec \theta$  in the numerator. Therefore, changing only the left side, we have

$$\begin{aligned} \tan \theta \csc \theta &= \frac{\sin \theta}{\cos \theta} \csc \theta = \frac{\sin \theta}{\cos \theta} \frac{1}{\sin \theta} \\ &= \frac{1}{\cos \theta} \quad \text{cancel } \sin \theta \text{ factors} \\ &= \sec \theta \quad \text{using Eq. (20.2)} \end{aligned}$$

Eq. (20.4)
Eq. (20.1)

Having changed the left side into the form on the right side, we have proven the identity.

Many variations of the preceding steps are possible. Also, we could have changed the right side to obtain the form on the left. For example,

$$\begin{aligned}\tan \theta \csc \theta &= \sec \theta = \frac{1}{\cos \theta} && \text{using Eq. (20.2)} \\ &= \frac{\sin \theta}{\cos \theta \sin \theta} = \frac{\sin \theta}{\cos \theta} \frac{1}{\sin \theta} && \text{multiply numerator and denominator by } \sin \theta \text{ and rewrite} \\ &= \tan \theta \csc \theta && \text{using Eqs. (20.4) and (20.1)}\end{aligned}$$

### EXAMPLE 6 A trigonometric identity—simplifying the left side

Prove the identity  $\frac{\sec^2 y}{\cot y} - \tan^3 y = \tan y$ .

Here, we simplify the left side. We can remove  $\cot y$  from the denominator, since  $\cot y = 1/\tan y$ . Also, the presence of  $\sec^2 y$  suggests the use of Eq. (20.7). Therefore, we have

$$\begin{aligned}\frac{\sec^2 y}{\cot y} - \tan^3 y &= \frac{\sec^2 y}{\frac{1}{\tan y}} - \tan^3 y = \sec^2 y \tan y - \tan^3 y \\ &\stackrel{\text{Eq. (20.3)}}{=} \tan y (\sec^2 y - \tan^2 y) = \tan y (1) \\ &= \tan y \quad \text{Eq. (20.7)}\end{aligned}$$

Here, we have used Eq. (20.7) in the form  $\sec^2 y - \tan^2 y = 1$ .

### EXAMPLE 7 A trigonometric identity—difference of squares

Prove the identity  $\frac{1 - \sin x}{\sin x \cot x} = \frac{\cos x}{1 + \sin x}$ .

The combination  $1 - \sin x$  suggests  $1 - \sin^2 x$ , since multiplying  $(1 - \sin x)$  by  $(1 + \sin x)$  gives  $1 - \sin^2 x$ , which can then be replaced by  $\cos^2 x$ . Thus, changing only the left side, we have

$$\begin{aligned}\frac{1 - \sin x}{\sin x \cot x} &= \frac{(1 - \sin x)(1 + \sin x)}{\sin x \cot x (1 + \sin x)} && \text{multiply numerator and denominator by } 1 + \sin x \\ &= \frac{1 - \sin^2 x}{\sin x \left(\frac{\cos x}{\sin x}\right) (1 + \sin x)} = \frac{\cos^2 x}{\cos x (1 + \sin x)} && \text{Eq. (20.6)} \\ &\stackrel{\text{cancel } \sin x}{=} \frac{\cos x}{1 + \sin x} && \text{cancel } \cos x\end{aligned}$$

### EXAMPLE 8 A trigonometric identity—application

In finding the radiation rate of an accelerated electric charge, it is necessary to show that  $\sin^3 \theta = \sin \theta - \sin \theta \cos^2 \theta$ . Show this by changing the left side.

Since each term on the right has a factor of  $\sin \theta$ , we see that we can proceed by writing  $\sin^3 \theta$  as  $\sin \theta (\sin^2 \theta)$ . Then the factor  $\sin^2 \theta$  and the  $\cos^2 \theta$  on the right suggest the use of Eq. (20.6). Thus, we have

$$\begin{aligned}\sin^3 \theta &= \sin \theta (\sin^2 \theta) = \sin \theta (1 - \cos^2 \theta) \\ &= \sin \theta - \sin \theta \cos^2 \theta && \text{multiplying}\end{aligned}$$

Since we substituted for  $\sin^2 \theta$ , we used Eq. (20.6) in the form  $\sin^2 \theta = 1 - \cos^2 \theta$ .



**EXAMPLE 9** Simplifying a trigonometric expression

Simplify the expression  $\frac{\csc x}{\tan x + \cot x}$ .

We proceed with a simplification in a manner similar to proving an identity, although we do not know what the result should be. Following is one procedure for this simplification:

$$\begin{aligned}
 \frac{\csc x}{\tan x + \cot x} &= \frac{\csc x}{\tan x + \frac{1}{\tan x}} = \frac{\csc x}{\frac{\tan^2 x + 1}{\tan x}} \\
 &\stackrel{\text{Eq. (20.3)}}{=} \frac{\csc x \tan x}{\tan^2 x + 1} \stackrel{\text{Eq. (20.1)}}{=} \frac{\csc x \tan x}{\sec^2 x} \stackrel{\text{Eq. (20.4)}}{=} \frac{1 \cdot \frac{\sin x}{\cos x}}{\frac{1}{\cos^2 x}} \\
 &\stackrel{\text{Eq. (20.7)}}{=} \frac{\csc x \tan x}{\sec^2 x} \stackrel{\text{Eq. (20.2)}}{=} \frac{\frac{\sin x}{\cos x} \cdot \frac{\cos^2 x}{1}}{1} = \cos x
 \end{aligned}$$

A graphing calculator can be used to check an identity or a simplification. This is done by graphing the function on each side of an identity, or the initial expression and the final expression for simplification. If the two graphs are the same, the identity or simplification can be seen to be correct, although it would not be a proof. Examples are left as exercises throughout the chapter.

**EXERCISES 20.1**

In Exercises 1 and 2, make the given changes in the indicated examples of this section and then prove the resulting identities.

- In Example 4, change the right side to  $\tan x / \sec x$ .
- In Example 6, change the first term on the left to  $\sin y / \cos^3 y$ .

In Exercises 3–6, use a calculator to check the indicated basic identities for the given angles.

- Eq. (20.4) for  $\theta = 56^\circ$
- Eq. (20.5) for  $\theta = 280^\circ$
- Eq. (20.6) for  $\theta = 4\pi/3$
- Eq. (20.7) for  $\theta = 5\pi/6$

In Exercises 7–12, multiply and simplify. In Exercises 13–18, factor and simplify.

- $\cos x(\tan x - \sec x)$
- $\tan y(\cot y + 3 \cos y)$
- $\cos \theta \cot \theta(\sec \theta - 2 \tan \theta)$
- $(\csc x - 1)(\csc x + 1)$
- $\tan u(\cot u + \tan u)$
- $\cos^2 t(1 + \tan^2 t)$
- $\sin x + \sin x \tan^2 x$
- $\sec \theta + \sec^3 \theta$
- $\sin^3 t \cos t + \sin t \cos^3 t$
- $\tan^2 u \sec^2 u - \tan^4 u$
- $\csc^4 y - 1$
- $\sin x + \sin x \cot^2 x$

In Exercises 19–42, prove the given identities.

- $\frac{\sin x}{\tan x} = \cos x$
- $\frac{\csc \theta}{\sec \theta} = \cot \theta$
- $\sin x \sec x = \tan x$
- $\cot \theta \sec \theta = \csc \theta$
- $\csc^2 x(1 - \cos^2 x) = 1$
- $\sec \theta(1 - \sin^2 \theta) = \cos \theta$

- $\sin x(1 + \cot^2 x) = \csc x$
- $\csc x(\csc x - \sin x) = \cot^2 x$
- $\cos \theta \cot \theta + \sin \theta = \csc \theta$
- $\csc x \sec x - \tan x = \cot x$
- $\cot \theta \sec^2 \theta - \cot \theta = \tan \theta$
- $\sin y + \sin y \cot^2 y = \csc y$
- $\tan x + \cot x = \sec x \csc x$
- $\tan x + \cot x = \tan x \csc^2 x$
- $\cos^2 x - \sin^2 x = 1 - 2 \sin^2 x$
- $\frac{1 + \cos x}{\sin x} = \frac{\sin x}{1 - \cos x}$
- $\frac{\sin \theta}{\csc \theta} + \frac{\cos \theta}{\sec \theta} = 1$
- $\frac{\sec \theta}{\cos \theta} - \frac{\tan \theta}{\cot \theta} = 1$
- $2 \sin^4 x - 3 \sin^2 x + 1 = \cos^2 x(1 - 2 \sin^2 x)$
- $\frac{\sin^2 \theta + 2 \cos \theta - 1}{\sin^2 \theta + 3 \cos \theta - 3} = \frac{1}{1 - \sec \theta}$
- $\frac{1}{2} \sin \pi t \left( \frac{\sin \pi t}{1 - \cos \pi t} + \frac{1 - \cos \pi t}{\sin \pi t} \right) = 1$
- $\frac{\cot \omega t}{\sec \omega t - \tan \omega t} - \frac{\cos \omega t}{\sec \omega t + \tan \omega t} = \sin \omega t + \csc \omega t$

41.  $1 + \sin^2 x + \sin^4 x + \dots = \sec^2 x$  ( $-\frac{\pi}{2} < x < \frac{\pi}{2}$ )  
 42.  $1 - \tan^2 x + \tan^4 x - \dots = \cos^2 x$  ( $-\frac{\pi}{4} < x < \frac{\pi}{4}$ )

In Exercises 43–50, simplify the given expressions. The result will be one of  $\sin x$ ,  $\cos x$ ,  $\tan x$ ,  $\cot x$ ,  $\sec x$ , or  $\csc x$ .

43.  $\frac{\tan x \csc^2 x}{1 + \tan^2 x}$       44.  $\frac{\cos x - \cos^3 x}{\sin x - \sin^3 x}$   
 45.  $\cot x (\sec x - \cos x)$       46.  $\sin x (\tan x + \cot x)$   
 47.  $\frac{\tan x + \cot x}{\csc x}$       48.  $\frac{1 + \tan x}{\sin x} - \sec x$   
 49.  $\frac{\cos x + \sin x}{1 + \tan x}$       50.  $\frac{\sec x - \cos x}{\tan x}$

In Exercises 51–54, use a graphing calculator to verify the given identities by comparing the graphs of each side.

51.  $\sin x (\csc x - \sin x) = \cos^2 x$   
 52.  $\cos y (\sec y - \cos y) = \sin^2 y$   
 53.  $\frac{\sec x + \csc x}{1 + \tan x} = \csc x$       54.  $\frac{\cot x + 1}{\cot x} = 1 + \tan x$

In Exercises 55–58, use a graphing calculator to determine whether the given equations are identities.

55.  $\sec \theta \tan \theta \csc \theta = \tan^2 \theta + 1$   
 56.  $\sin x \cos x \tan x = \cos^2 x - 1$   
 57.  $\frac{2 \cos^2 x - 1}{\sin x \cos x} = \tan x - \cot x$   
 58.  $\cos^3 x \csc^3 x \tan^3 x = \csc^2 x - \cot^2 x$

In Exercises 59–62, solve the given problems involving trigonometric identities.

59. When designing a solar-energy collector, it is necessary to account for the latitude and longitude of the location, the angle of the sun, and the angle of the collector. In doing this, the equation  $\cos \theta = \cos A \cos B \cos C + \sin A \sin B$  is used. If  $\theta = 90^\circ$ , show that  $\cos C = -\tan A \tan B$ .  
 60. The path of a point on the circumference of a circle, such as a point on the rim of a bicycle wheel as it rolls along, traces out a curve called a *cycloid*. To find the distance through which the point moves, it is necessary to simplify the expression  $(1 - \cos \theta)^2 + \sin^2 \theta$ . Perform this simplification.

61. Show that the length  $l$  of the straight brace shown in Fig. 20.3 can be found from the equation

$$l = \frac{a(1 + \tan \theta)}{\sin \theta}$$

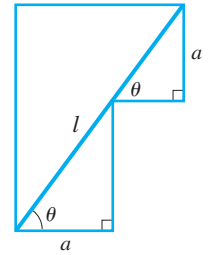


Fig. 20.3

62. In determining the path of least time between two points under certain conditions, it is necessary to show that

$$\sqrt{\frac{1 + \cos \theta}{1 - \cos \theta}} \sin \theta = 1 + \cos \theta$$

Show this by transforming the left-hand side.

In Exercises 63–68, solve the given problems.

63. Explain how to transform  $\sin \theta \tan \theta + \cos \theta$  into  $\sec \theta$ .  
 64. Explain how to transform  $\tan^2 \theta \cos^2 \theta + \cot^2 \theta \sin^2 \theta$  into 1.  
 65. Show that  $\sin^2 x (1 - \sec^2 x) + \cos^2 x (1 + \sec^4 x)$  has a constant value.  
 66. Show that  $\cot y \csc y \sec y - \csc y \cos y \cot y$  has a constant value.  
 67. Prove that  $\sec^2 \theta + \csc^2 \theta = \sec^2 \theta \csc^2 \theta$  by expressing each function in terms of its  $x$ ,  $y$ , and  $r$  definition.  
 68. Prove that  $\frac{\csc \theta}{\tan \theta + \cot \theta} = \cos \theta$  by expressing each function in terms of its  $x$ ,  $y$ , and  $r$  definition.

In Exercises 69–72, use the given substitutions to show that the given equations are valid. In each,  $0 < \theta < \pi/2$ .

69. If  $x = \cos \theta$ , show that  $\sqrt{1 - x^2} = \sin \theta$ .  
 70. If  $x = 3 \sin \theta$ , show that  $\sqrt{9 - x^2} = 3 \cos \theta$ .  
 71. If  $x = 2 \tan \theta$ , show that  $\sqrt{4 + x^2} = 2 \sec \theta$ .  
 72. If  $x = 4 \sec \theta$ , show that  $\sqrt{x^2 - 16} = 4 \tan \theta$ .

Answers to Practice Exercises

1.  $1 + \sin x$       2.  $\frac{\cos x \cdot \frac{1}{\sin x}}{\cot^2 x} = \frac{\cot x}{\cot^2 x} = \tan x$

## 20.2 The Sum and Difference Formulas

**Formulas for  $\sin(\alpha + \beta)$  and  $\cos(\alpha + \beta)$  • Formulas for  $\sin(\alpha - \beta)$  and  $\cos(\alpha - \beta)$  • Formulas for  $\tan(\alpha + \beta)$  and  $\tan(\alpha - \beta)$**

There are other important relations among the trigonometric functions. The most important and useful relations are those that involve twice an angle and half an angle. To obtain these relations, in this section we derive the expressions for the sine and cosine of the sum and difference of two angles. These expressions will lead directly to the desired relations of double and half angles that we will derive in the following sections.

■ For reference, Eq. (12.13) is  $r_1(\cos \theta_1 + j \sin \theta_1) r_2(\cos \theta_2 + j \sin \theta_2) = r_1 r_2 [\cos(\theta_1 + \theta_2) + j \sin(\theta_1 + \theta_2)]$

Equation (12.13) gives the polar (or trigonometric) form of the product of two complex numbers. We can use this formula to derive the expressions for the sine and cosine of the sum and difference of two angles.

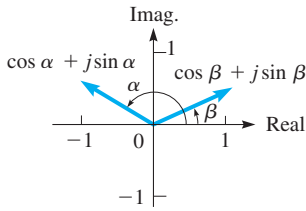


Fig. 20.4

Using Eq. (12.13) to find the product of the complex numbers  $\cos \alpha + j \sin \alpha$  and  $\cos \beta + j \sin \beta$ , which are represented in Fig. 20.4, we have

$$(\cos \alpha + j \sin \alpha)(\cos \beta + j \sin \beta) = \cos(\alpha + \beta) + j \sin(\alpha + \beta)$$

Expanding the left side, and then switching sides, we have

$$\cos(\alpha + \beta) + j \sin(\alpha + \beta) = (\cos \alpha \cos \beta - \sin \alpha \sin \beta) + j(\sin \alpha \cos \beta + \cos \alpha \sin \beta)$$

Since two complex numbers are equal if their real parts are equal and their imaginary parts are equal, we have the following formulas:

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta \quad (20.9)$$

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta \quad (20.10)$$

**EXAMPLE 1** Verifying the  $\sin(\alpha + \beta)$  formula

Verify that  $\sin 90^\circ = 1$ , by finding  $\sin(60^\circ + 30^\circ)$ .

$$\begin{aligned} \sin 90^\circ &= \sin(60^\circ + 30^\circ) = \sin 60^\circ \cos 30^\circ + \cos 60^\circ \sin 30^\circ && \text{using Eq. (20.9)} \\ &= \frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} + \frac{1}{2} \times \frac{1}{2} && \text{for values, see} \\ &= \frac{3}{4} + \frac{1}{4} = 1 && \text{Section 4.3} \end{aligned}$$

**COMMON ERROR**

It should be obvious from this example that  $\sin(\alpha + \beta)$  is not equal to  $\sin \alpha + \sin \beta$ , which is something that many students simply assume before they become familiar with the formulas and ideas of this section. If we used such a formula, we would get  $\sin 90^\circ = \frac{1}{2}\sqrt{3} + \frac{1}{2} = 1.366$  for the combination  $(60^\circ + 30^\circ)$ . This is not possible since the values of the sine never exceed 1 in value. Also, if we used the combination  $(45^\circ + 45^\circ)$ , we would get 1.414, a different value for the same number,  $\sin 90^\circ$ .

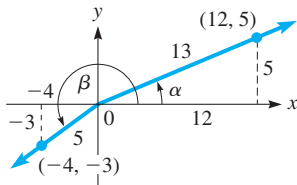


Fig. 20.5

**EXAMPLE 2** Using  $\cos(\alpha + \beta)$  with numerical values

Given that  $\sin \alpha = \frac{5}{13}$  ( $\alpha$  in the first quadrant) and  $\sin \beta = -\frac{3}{5}$  (for  $\beta$  in the third quadrant), find  $\cos(\alpha + \beta)$ .

Since  $\sin \alpha = \frac{5}{13}$  for  $\alpha$  in the first quadrant, from Fig. 20.5, we have  $\cos \alpha = \frac{12}{13}$ . Also, since  $\sin \beta = -\frac{3}{5}$  for  $\beta$  in the third quadrant, from Fig. 20.5, we also have  $\cos \beta = -\frac{4}{5}$ .

Then, by using Eq. (20.10), we have

$$\begin{aligned} \cos(\alpha + \beta) &= \cos \alpha \cos \beta - \sin \alpha \sin \beta \\ &= \frac{12}{13} \left( -\frac{4}{5} \right) - \frac{5}{13} \left( -\frac{3}{5} \right) \\ &= -\frac{48}{65} + \frac{15}{65} = -\frac{33}{65} \end{aligned}$$

From Eqs. (20.9) and (20.10), we can easily find expressions for  $\sin(\alpha - \beta)$  and  $\cos(\alpha - \beta)$ . This is done by finding  $\sin[\alpha + (-\beta)]$  and  $\cos[\alpha + (-\beta)]$ . Thus, we have

$$\sin(\alpha - \beta) = \sin[\alpha + (-\beta)] = \sin \alpha \cos(-\beta) + \cos \alpha \sin(-\beta)$$

Since  $\cos(-\beta) = \cos \beta$  and  $\sin(-\beta) = -\sin \beta$  (see Eq. 8.7 on page 247), we have

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta \quad (20.11)$$

In the same manner, we find that

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta \quad (20.12)$$

### EXAMPLE 3 Using the $\cos(\alpha - \beta)$ formula

Find  $\cos 15^\circ$  from  $\cos(45^\circ - 30^\circ)$ .

$$\begin{aligned} \cos 15^\circ &= \cos(45^\circ - 30^\circ) = \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ && \text{using Eq. (20.12)} \\ &= \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \times \frac{1}{2} = \frac{\sqrt{6} + \sqrt{2}}{4} && \text{(exact)} \\ &= 0.966 \end{aligned}$$

### EXAMPLE 4 $\sin(\alpha + \beta)$ formula—application

In analysing the motion of an object oscillating up and down at the end of a spring, the expression  $\sin(\omega t + \alpha) \cos \alpha - \cos(\omega t + \alpha) \sin \alpha$  occurs. Simplify this expression.

If we let  $x = \omega t + \alpha$ , the expression becomes  $\sin x \cos \alpha - \cos x \sin \alpha$ , which is the form for  $\sin(x - \alpha)$ . Therefore,

$$\begin{aligned} \sin(\omega t + \alpha) \cos \alpha - \cos(\omega t + \alpha) \sin \alpha &= \sin x \cos \alpha - \cos x \sin \alpha = \sin(x - \alpha) \\ &= \sin(\omega t + \alpha - \alpha) = \sin \omega t \end{aligned}$$

### EXAMPLE 5 Using the $\cos(\alpha + \beta)$ formula

Evaluate  $\cos 23^\circ \cos 67^\circ - \sin 23^\circ \sin 67^\circ$ .

We note that this expression fits the form of the right side of Eq. (20.10), so

$$\begin{aligned} \cos 23^\circ \cos 67^\circ - \sin 23^\circ \sin 67^\circ &= \cos(23^\circ + 67^\circ) \\ &= \cos 90^\circ \\ &= 0 \end{aligned}$$

Again, we are able to evaluate this expression by *recognizing the form* of the given expression. Evaluation by a calculator will verify the result. ■

By dividing the right side of Eq. (20.9) by that of Eq. (20.10), we can determine expressions for  $\tan(\alpha + \beta)$ , and by dividing the right side of Eq. (20.11) by that of Eq. (20.12), we can determine an expression for  $\tan(\alpha - \beta)$ . The derivation of these formulas is Exercise 33 of this section. These formulas can be written together, as

$$\tan(\alpha \pm \beta) = \frac{\tan \alpha \pm \tan \beta}{1 \mp \tan \alpha \tan \beta} \quad (20.13)$$

The formula for  $\tan(\alpha + \beta)$  uses the upper signs, and the formula for  $\tan(\alpha - \beta)$  uses the lower signs.

Certain trigonometric identities can be proven by the formulas derived in this section. The following examples illustrate this use of these formulas.

#### Practice Exercise

1. Evaluate

$$\sin 115^\circ \cos 25^\circ - \cos 115^\circ \sin 25^\circ.$$

**EXAMPLE 6** Proving a trigonometric identity with  $\tan(\alpha \pm \beta)$

Show that  $\tan(\alpha + \beta) \tan(\alpha - \beta) = \frac{\tan^2 \alpha - \tan^2 \beta}{1 - \tan^2 \alpha \tan^2 \beta}$ .

Using Eq. (20.13), we have

$$\begin{aligned} \tan(\alpha + \beta) \tan(\alpha - \beta) &= \left( \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} \right) \left( \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta} \right) \\ &= \frac{\tan^2 \alpha - \tan^2 \beta}{1 - \tan^2 \alpha \tan^2 \beta} \end{aligned}$$

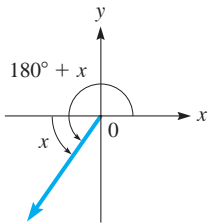


Fig. 20.6

**EXAMPLE 7** Using the  $\sin(\alpha + \beta)$  formula

Prove that  $\sin(180^\circ + x) = -\sin x$ .

$$\begin{aligned} \sin(180^\circ + x) &= \sin 180^\circ \cos x + \cos 180^\circ \sin x && \text{using Eq. (20.9)} \\ &= (0) \cos x + (-1) \sin x && \sin 180^\circ = 0, \cos 180^\circ = -1 \\ &= -\sin x \end{aligned}$$

Although  $x$  may or may not be an acute angle, this agrees with the results for the sine of a third-quadrant angle, as discussed in Section 8.2. See Fig. 20.6.

**EXAMPLE 8** A trigonometric identity using  $\sin(\alpha - \beta)$

Simplify the expression  $\frac{\sin(\alpha - \beta)}{\sin \alpha \sin \beta}$ .

$$\begin{aligned} \frac{\sin(\alpha - \beta)}{\sin \alpha \sin \beta} &= \frac{\sin \alpha \cos \beta - \cos \alpha \sin \beta}{\sin \alpha \sin \beta} && \text{using Eq. (20.11)} \\ &= \frac{\sin \alpha \cos \beta}{\sin \alpha \sin \beta} - \frac{\cos \alpha \sin \beta}{\sin \alpha \sin \beta} \\ &= \frac{\cos \beta}{\sin \beta} - \frac{\cos \alpha}{\sin \alpha} \\ &= \cot \beta - \cot \alpha && \text{using Eq. (20.5)} \end{aligned}$$

**Practice Exercise**

- Simplify  $\tan(180^\circ + x)$ .

■ See the chapter introduction.

**EXAMPLE 9** Application using  $\sin(\alpha - \beta)$

Alternating electric current is produced essentially by a coil of wire rotating in a magnetic field, and this is the basis for designing generators of alternating current. A three-phase generator uses three coils of wire and thereby produces three electric currents at the same time. This is the most widely used type of *polyphase generator* as mentioned in the chapter introduction.

The voltages induced in a three-phase generator can be represented as

$$E_1 = E_0 \sin \omega t \quad E_2 = E_0 \sin \left( \omega t - \frac{2\pi}{3} \right) \quad E_3 = E_0 \sin \left( \omega t - \frac{4\pi}{3} \right)$$

where  $E_0$  is the maximum voltage and  $\omega$  is the angular velocity of rotation. Show that the sum of these voltages at any time  $t$  is zero.

Setting up the sum  $E_1 + E_2 + E_3$  and using Eq. (20.11), we have

$$\begin{aligned} E_1 + E_2 + E_3 &= E_0 \left[ \sin \omega t + \sin \left( \omega t - \frac{2\pi}{3} \right) + \sin \left( \omega t - \frac{4\pi}{3} \right) \right] \\ &= E_0 \left( \sin \omega t + \sin \omega t \cos \frac{2\pi}{3} - \cos \omega t \sin \frac{2\pi}{3} + \sin \omega t \cos \frac{4\pi}{3} - \cos \omega t \sin \frac{4\pi}{3} \right) \\ &= E_0 \left[ \sin \omega t + (\sin \omega t) \left( -\frac{1}{2} \right) - (\cos \omega t) \left( \frac{1}{2} \sqrt{3} \right) + (\sin \omega t) \left( -\frac{1}{2} \right) - (\cos \omega t) \left( -\frac{1}{2} \sqrt{3} \right) \right] \\ &= E_0 \left[ (\sin \omega t) \left( 1 - \frac{1}{2} - \frac{1}{2} \right) + (\cos \omega t) \left( \frac{1}{2} \sqrt{3} - \frac{1}{2} \sqrt{3} \right) \right] = 0 \end{aligned}$$

## EXERCISES 20.2

In Exercises 1 and 2, make the given changes in the indicated examples of this section and then solve the given problems.

- In Example 2, change  $\frac{5}{13}$  to  $\frac{12}{13}$  and then find the value of  $\cos(\alpha + \beta)$ .
- In Example 7, change  $180^\circ + x$  to  $180^\circ - x$  and then determine what other changes result.

In Exercises 3–6, determine the values of the given functions as indicated. If using a calculator, round to 3 significant digits.

- Find  $\sin 105^\circ$  by using  $105^\circ = 60^\circ + 45^\circ$ .
- Find  $\tan 75^\circ$  by using  $75^\circ = 30^\circ + 45^\circ$ .
- Find  $\cos 15^\circ$  by using  $15^\circ = 60^\circ - 45^\circ$ .
- Find  $\sin 15^\circ$  by using  $15^\circ = 45^\circ - 30^\circ$ .

In Exercises 7–10, evaluate the given functions with the following information:  $\sin \alpha = 4/5$  ( $\alpha$  in first quadrant) and  $\cos \beta = -12/13$  ( $\beta$  in second quadrant).

- $\sin(\alpha + \beta)$
- $\tan(\beta - \alpha)$
- $\cos(\alpha + \beta)$
- $\sin(\alpha - \beta)$

In Exercises 11–20, simplify the given expressions.

- $\sin x \cos 2x + \sin 2x \cos x$
- $\sin 3x \cos x - \sin x \cos 3x$
- $\cos \pi \cos x + \sin \pi \sin x$
- $\frac{\tan(x - y) + \tan x}{1 - \tan(x - y) \tan x}$
- $\sin(360^\circ - x)$
- $\cos(2\pi - x)$
- $\tan(x - \pi)$
- $\sin(x + \pi/2)$
- $\sin 3x \cos(3x - \pi) - \cos 3x \sin(3x - \pi)$
- $\cos(x + \pi) \cos(x - \pi) + \sin(x + \pi) \sin(x - \pi)$

In Exercises 21–24, evaluate each expression by first changing the form. Verify each by use of a calculator.

- $\sin 122^\circ \cos 32^\circ - \cos 122^\circ \sin 32^\circ$
- $\cos 250^\circ \cos 70^\circ + \sin 250^\circ \sin 70^\circ$
- $\cos \frac{\pi}{5} \cos \frac{3\pi}{10} - \sin \frac{\pi}{5} \sin \frac{3\pi}{10}$
- $\frac{\tan 18^\circ + \tan 27^\circ}{1 - \tan 18^\circ \tan 27^\circ}$

In Exercises 25–28, prove the given identities.

- $\sin(x + y) \sin(x - y) = \sin^2 x - \sin^2 y$
- $\cos(x + y) \cos(x - y) = \cos^2 x - \sin^2 y$
- $\cos(\alpha + \beta) + \cos(\alpha - \beta) = 2 \cos \alpha \cos \beta$
- $\tan(90^\circ + x) = -\cot x$  (Explain why Eq. 20.13 cannot be used for this, but Eqs. 20.9 and 20.10 can be used.)

In Exercises 29–32, verify each identity by comparing the graph of the left side with the graph of the right side on a graphing calculator.

- $\cos(30^\circ + x) = \frac{\sqrt{3} \cos x - \sin x}{2}$
- $\sin(120^\circ - x) = \frac{\sqrt{3} \cos x + \sin x}{2}$

$$31. \tan\left(\frac{\pi}{4} + x\right) = \frac{1 + \tan x}{1 - \tan x} \quad 32. \cos\left(\frac{\pi}{2} - x\right) = \sin x$$

In Exercises 33–36, derive the given equations as indicated. Equations (20.14)–(20.16) are known as the product formulas.

- By dividing the right side of Eq. (20.9) by that of Eq. (20.10), and dividing the right side of Eq. (20.11) by that of Eq. (20.12), derive Eq. (20.13).

$$\tan(\alpha \pm \beta) = \frac{\tan \alpha \pm \tan \beta}{1 \mp \tan \alpha \tan \beta} \quad (20.13)$$

(Hint: Divide numerator and denominator by  $\cos \alpha \cos \beta$ .)

- By adding Eqs. (20.9) and (20.11), derive the equation  $\sin \alpha \cos \beta = \frac{1}{2}[\sin(\alpha + \beta) + \sin(\alpha - \beta)]$  (20.14)
- By adding Eqs. (20.10) and (20.12), derive the equation  $\cos \alpha \cos \beta = \frac{1}{2}[\cos(\alpha + \beta) + \cos(\alpha - \beta)]$  (20.15)
- By subtracting Eq. (20.10) from Eq. (20.12), derive  $\sin \alpha \sin \beta = \frac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)]$  (20.16)

In Exercises 37–40, derive the given equations by letting  $\alpha + \beta = x$  and  $\alpha - \beta = y$ , which leads to  $\alpha = \frac{1}{2}(x + y)$  and  $\beta = \frac{1}{2}(x - y)$ . The resulting equations are known as the factor formulas.

- Use Eq. (20.14) and the substitutions above to derive the equation  $\sin x + \sin y = 2 \sin \frac{1}{2}(x + y) \cos \frac{1}{2}(x - y)$  (20.17)
- Use Eqs. (20.9) and (20.11) and the substitutions above to derive the equation  $\sin x - \sin y = 2 \sin \frac{1}{2}(x - y) \cos \frac{1}{2}(x + y)$  (20.18)
- Use Eq. (20.15) and the substitutions above to derive the equation  $\cos x + \cos y = 2 \cos \frac{1}{2}(x + y) \cos \frac{1}{2}(x - y)$  (20.19)
- Use Eq. (20.16) and the substitutions above to derive the equation  $\cos x - \cos y = -2 \sin \frac{1}{2}(x + y) \sin \frac{1}{2}(x - y)$  (20.20)

In Exercises 41–50, solve the given problems.

- Show that  $\frac{\sin 2x}{\sin x} = 2 \cos x$ . (Hint:  $\sin 2x = \sin(x + x)$ .)
- Explain how the exact value of  $\sin 75^\circ$  can be found using either Eq. (20.9) or Eq. (20.11).
- Express  $\cos(A + B + C)$  in terms of  $\sin A$ ,  $\sin B$ ,  $\sin C$ ,  $\cos A$ ,  $\cos B$ , and  $\cos C$ .
- The design of a certain three-phase alternating-current generator uses the fact that the sum of the currents  $I \cos(\theta + 30^\circ)$ ,  $I \cos(\theta + 150^\circ)$ , and  $I \cos(\theta + 270^\circ)$  is zero. Verify this.
- For voltages  $V_1 = 20 \sin 120\pi t$  and  $V_2 = 20 \cos 120\pi t$ , show that  $V = V_1 + V_2 = 20\sqrt{2} \sin(120\pi t + \pi/4)$ .
- The displacements  $y_1$  and  $y_2$  of two waves travelling through the same medium are given by  $y_1 = A \sin 2\pi(t/T - x/\lambda)$  and  $y_2 = A \sin 2\pi(t/T + x/\lambda)$ . Find an expression for the displacement  $y_1 + y_2$  of the combination of the waves.
- An alternating electric current  $i$  is given by the equation  $i = i_0 \sin(\omega t + \alpha)$ . Show that this can be written as  $i = i_1 \sin \omega t + i_2 \cos \omega t$ , where  $i_1 = i_0 \cos \alpha$  and  $i_2 = i_0 \sin \alpha$ .

48. A weight  $w$  is held in equilibrium by forces  $F$  and  $T$ , as shown in Fig. 20.7. Equations relating  $w$ ,  $F$ , and  $T$  are

$$F \cos \theta = T \sin \alpha$$

$$w + F \sin \theta = T \cos \alpha$$

$$\text{Show that } w = \frac{T \cos(\theta + \alpha)}{\cos \theta}.$$

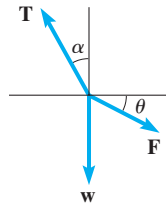


Fig. 20.7

49. For the two bevel gears shown in Fig. 20.8, the equation

$$\tan \alpha = \frac{\sin \beta}{R + \cos \beta}$$

is used. Here,  $R$  is the ratio of gear 1 to gear 2. Show that

$$R = \frac{\sin(\beta - \alpha)}{\sin \alpha}.$$

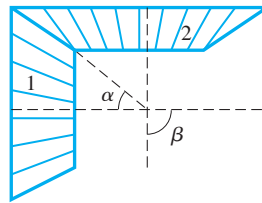


Fig. 20.8

50. In the analysis of the angles of incidence  $i$  and reflection  $r$  of a light ray subject to certain conditions, the following expression is found:

$$E_2 \left( \frac{\tan r}{\tan i} + 1 \right) = E_1 \left( \frac{\tan r}{\tan i} - 1 \right)$$

$$\text{Show that } E_2 = E_1 \frac{\sin(r - i)}{\sin(r + i)}.$$

### Answers to Practice Exercises

1. 1    2.  $\tan x$

## 20.3 Double-Angle Formulas

Formula for  $\sin 2\alpha$  • Formulas for  $\cos 2\alpha$  • Formula for  $\tan 2\alpha$

If we let  $\beta = \alpha$  in Eqs. (20.9) and (20.10), we can derive the important double-angle formulas. By making this substitution in Eq. (20.9), we have

$$\sin(\alpha + \alpha) = \sin(2\alpha) = \sin \alpha \cos \alpha + \cos \alpha \sin \alpha = 2 \sin \alpha \cos \alpha$$

Using the same substitution in Eq. (20.10), we have

$$\cos(\alpha + \alpha) = \cos(2\alpha) = \cos \alpha \cos \alpha - \sin \alpha \sin \alpha = \cos^2 \alpha - \sin^2 \alpha$$

Again, using this substitution in the  $\tan(\alpha + \beta)$  form of Eq. (20.13), we have

$$\tan(\alpha + \alpha) = \tan(2\alpha) = \frac{\tan \alpha + \tan \alpha}{1 - \tan \alpha \tan \alpha} = \frac{2 \tan \alpha}{1 - \tan^2 \alpha}$$

Then using the basic identity Eq. (20.6), other forms of the equation for  $\cos 2\alpha$  may be derived. Summarizing these forms, we have

$$\sin 2\alpha = 2 \sin \alpha \cos \alpha \quad (20.21)$$

$$\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha \quad (20.22)$$

$$= 2 \cos^2 \alpha - 1 \quad (20.23)$$

$$= 1 - 2 \sin^2 \alpha \quad (20.24)$$

$$\tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha} \quad (20.25)$$

These double-angle formulas are widely used in applications of trigonometry, especially in calculus. They should be recognized quickly in any of the above forms.

### EXAMPLE 1 Using double-angle formulas

- (a) If  $\alpha = 30^\circ$ , we have

$$\cos 60^\circ = \cos 2(30^\circ) = \cos^2 30^\circ - \sin^2 30^\circ = \left(\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{1}{2}\right)^2 = \frac{1}{2} \quad \text{using Eq. (20.22)}$$

- (b) If  $\alpha = 3x$ , we have

$$\sin 6x = \sin 2(3x) = 2 \sin 3x \cos 3x \quad \text{using Eq. (20.21)}$$



(c) If  $2\alpha = x$ , we may write  $\alpha = x/2$ , which means that

$$\sin x = \sin 2\left(\frac{x}{2}\right) = 2 \sin \frac{x}{2} \cos \frac{x}{2} \quad \text{using Eq. (20.21)}$$

(d) If  $\alpha = \frac{\pi}{6}$ , we have

$$\tan \frac{\pi}{3} = \tan 2\left(\frac{\pi}{6}\right) = \frac{2 \tan \frac{\pi}{6}}{1 - \tan^2\left(\frac{\pi}{6}\right)} = \frac{2(\sqrt{3}/3)}{1 - (\sqrt{3}/3)^2} = \sqrt{3} \quad \text{using Eq. (20.25)} \blacksquare$$

### EXAMPLE 2 Simplification using the $\cos 2\alpha$ formula

Simplify the expression  $\cos^2 2x - \sin^2 2x$ .

Since this is the difference of the square of the cosine of an angle and the square of the sine of the same angle, it fits the right side of Eq. (20.22). Therefore, letting  $\alpha = 2x$ , we have

$$\cos^2 2x - \sin^2 2x = \cos 2(2x) = \cos 4x \quad \blacksquare$$

### EXAMPLE 3 Using the $\sin 2\alpha$ formula—application

To find the area  $A$  of a right triangular tract of land, a surveyor may use the formula  $A = \frac{1}{4}c^2 \sin 2\theta$ , where  $c$  is the hypotenuse and  $\theta$  is *either* of the acute angles. Derive this formula.

In Fig. 20.9, we see that  $\sin \theta = a/c$  and  $\cos \theta = b/c$ , which gives us

$$a = c \sin \theta \quad \text{and} \quad b = c \cos \theta$$

The area is given by  $A = \frac{1}{2}ab$ , which leads to the solution

$$\begin{aligned} A &= \frac{1}{2}ab = \frac{1}{2}(c \sin \theta)(c \cos \theta) \\ &= \frac{1}{2}c^2 \sin \theta \cos \theta = \frac{1}{2}c^2 \left(\frac{1}{2} \sin 2\theta\right) \quad \text{using Eq. (20.21)} \\ &= \frac{1}{4}c^2 \sin 2\theta \end{aligned}$$

In using Eq. (20.21), we divided both sides by 2 to get  $\sin \theta \cos \theta = \frac{1}{2} \sin 2\theta$ .

If we had labelled the upper acute angle in Fig. 20.9 as  $\theta$ , we would have  $a = c \cos \theta$  and  $b = c \sin \theta$ . Using these values in the formula for the area gives the same solution.  $\blacksquare$

### EXAMPLE 4 Verifying values

(a) Verifying the values of  $\sin 90^\circ$ , using the functions of  $45^\circ$ , we have

$$\sin 90^\circ = \sin 2(45^\circ) = 2 \sin 45^\circ \cos 45^\circ = 2\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = 1 \quad \text{using Eq. (20.21)}$$

(b) Using Eq. (20.25),  $\tan 142^\circ = \frac{2 \tan 71^\circ}{1 - \tan^2 71^\circ}$ . Using a calculator to verify this, we have

$$\tan 142^\circ = -0.781\,285\,626\,5 \quad \text{and} \quad \frac{2 \tan 71^\circ}{1 - \tan^2 71^\circ} = -0.781\,285\,626\,5 \quad \blacksquare$$

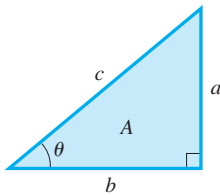
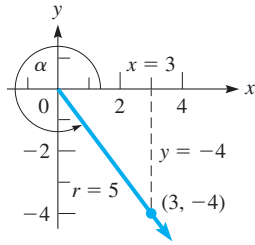
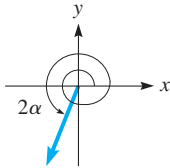


Fig. 20.9



(a)



(b)

Fig. 20.10

**EXAMPLE 5** Evaluation using the  $\sin 2\alpha$  formula

Knowing that  $\cos \alpha = 3/5$  for an angle in the fourth quadrant, we see from Fig. 20.10(a) that  $\sin \alpha = -4/5$ . Therefore, we have

$$\begin{aligned} \sin 2\alpha &= 2 \sin \alpha \cos \alpha && \text{Eq. (20.21)} \\ &= 2\left(-\frac{4}{5}\right)\left(\frac{3}{5}\right) = -\frac{24}{25} \end{aligned}$$

In Fig. 20.10(b), angle  $2\alpha$  is shown. It is a third-quadrant angle, which verifies the sign of the result. (Since  $\cos \alpha = 3/5$ ,  $\alpha = 307^\circ$  and  $2\alpha = 614^\circ$ , which is a third-quadrant angle.) ■

**EXAMPLE 6** Simplification using  $\cos 2\alpha$

Simplify the expression  $\frac{2}{1 + \cos 2x}$ .

$$\begin{aligned} \frac{2}{1 + \cos 2x} &= \frac{2}{1 + (2 \cos^2 x - 1)} && \text{using Eq. (20.23)} \\ &= \frac{2}{2 \cos^2 x} = \sec^2 x && \text{using Eq. (20.2)} \end{aligned}$$

**Practice Exercise**

1. Simplify  $\frac{\sin 2x}{\cos^2 x - \sin^2 x}$ .

**EXERCISES 20.3**

In Exercises 1–4, make the given changes in the indicated examples of this section and then solve the resulting problems.

1. In Example 1(d), change  $\frac{\pi}{6}$  to  $\frac{\pi}{3}$  and then evaluate  $\tan \frac{2\pi}{3}$ .
2. In Example 2, change  $2x$  to  $3x$  and then simplify.
3. In Example 5, change  $3/5$  to  $4/5$  and then evaluate  $\sin 2\alpha$ .
4. In Example 6, change the  $+$  in the denominator to  $-$  and then simplify the expression.

In Exercises 5–8, determine the values of the indicated functions in the given manner.

5. Find  $\sin 60^\circ$  by using the functions of  $30^\circ$ .
6. Find  $\sin 120^\circ$  by using the functions of  $60^\circ$ .
7. Find  $\tan 120^\circ$  by using the functions of  $60^\circ$ .
8. Find  $\cos 60^\circ$  by using the functions of  $30^\circ$ .

In Exercises 9–14, use a calculator to verify the values found by using the double-angle formulas. Round to 3 significant digits.

9. Find  $\sin 258^\circ$  directly and by using functions of  $129^\circ$ .
10. Find  $\tan 84^\circ$  directly and by using functions of  $42^\circ$ .
11. Find  $\cos 96^\circ$  directly and by using functions of  $48^\circ$ .
12. Find  $\cos 276^\circ$  directly and by using functions of  $138^\circ$ .
13. Find  $\tan \frac{2\pi}{5}$  directly and by using functions of  $\frac{\pi}{5}$ .
14. Find  $\sin (0.2\pi)$  directly and by using functions of  $0.1\pi$ .

In Exercises 15–18, evaluate the indicated functions with the given information.

15. Find  $\sin 2x$  if  $\cos x = \frac{4}{5}$  (in first quadrant).
16. Find  $\cos 2x$  if  $\sin x = -\frac{12}{13}$  (in third quadrant).

17. Find  $\tan 2x$  if  $\sin x = 0.5$  (in second quadrant).
18. Find  $\sin 4x$  if  $\sin x = 0.6$  (in first quadrant).

In Exercises 19–30, simplify the given expressions.

- |   |   |
|---|---|
| 19. $4 \sin 4x \cos 4x$                               | 20. $4 \sin^2 x \cos^2 x$                             |
| 21. $1 - 2 \sin^2 4x$                                 | 22. $\frac{4 \tan 4\theta}{1 - \tan^2 4\theta}$       |
| 23. $2 \cos^2 \frac{1}{2}x - 1$                       | 24. $2 \sin \frac{1}{2}x \cos \frac{1}{2}x$           |
| 25. $4 \sin^2 2x - 2$                                 | 26. $\cos 3x \sin 3x$                                 |
| 27. $\frac{\sin 4\theta}{\sin 2\theta}$               | 28. $\cos^4 u - \sin^4 u$                             |
| 29. $\frac{\sin 3x}{\sin x} - \frac{\cos 3x}{\cos x}$ | 30. $\frac{\cos 3x}{\sin x} + \frac{\sin 3x}{\cos x}$ |

In Exercises 31–40, prove the given identities.

31.  $\cos^2 \alpha - \sin^2 \alpha = 2 \cos^2 \alpha - 1$
32.  $\cos^2 \alpha - \sin^2 \alpha = 1 - 2 \sin^2 \alpha$
33.  $\frac{\cos x - \tan x \sin x}{\sec x} = \cos 2x$
34.  $2 + \frac{\cos 2\theta}{\sin^2 \theta} = \csc^2 \theta$
35.  $\frac{\sin 2\theta}{1 + \cos 2\theta} = \tan \theta$
36.  $\frac{2 \tan \alpha}{1 + \tan^2 \alpha} = \sin 2\alpha$
37.  $1 - \cos 2\theta = \frac{2}{1 + \cot^2 \theta}$
38.  $\frac{\cos^3 \theta + \sin^3 \theta}{\cos \theta + \sin \theta} = 1 - \frac{1}{2} \sin 2\theta$

39.  $\ln(1 - \cos 2x) - \ln(1 + \cos 2x) = 2 \ln \tan x$   
 40.  $\log(20 \sin^2 \theta + 10 \cos 2\theta) = 1$

In Exercises 41–44, verify each identity by comparing the graph of the left side with the graph of the right side on a graphing calculator.

41.  $\tan 2\theta = \frac{2}{\cot \theta - \tan \theta}$       42.  $\frac{1 - \tan^2 x}{\sec^2 x} = \cos 2x$   
 43.  $(\sin x + \cos x)^2 = 1 + \sin 2x$   
 44.  $2 \csc 2x \tan x = \sec^2 x$

In Exercises 45–60, solve the given problems.

45. Express  $\sin 3x$  in terms of  $\sin x$  only.  
 46. Express  $\cos 3x$  in terms of  $\cos x$  only.  
 47. Express  $\cos 4x$  in terms of  $\cos x$  only.  
 48. Express  $\sin 4x$  in terms of  $\sin x$  and  $\cos x$ .  
 49. Show that  $\cos 2x + \sin 2x \tan x$  reduces to 1.  
 50. For an acute angle  $\theta$ , show that  $2 \sin \theta > \sin 2\theta$ .  
 51. Without graphing, determine the amplitude and period of the function  $y = 4 \sin x \cos x$ . Explain.  
 52. Without graphing, determine the amplitude and period of the function  $y = \cos^2 x - \sin^2 x$ .  
 53. The path of a bouncing ball is given by  $y = \sqrt{(\sin x + \cos x)^2}$ . Show that this path can also be shown as  $y = \sqrt{1 + \sin 2x}$ . Use a calculator to show that this can also be shown as  $y = |\sin x + \cos x|$ .  
 54. The equation for the trajectory of a missile fired into the air at an angle  $\alpha$  with velocity  $v_0$  is  $y = x \tan \alpha - \frac{g}{2v_0^2 \cos^2 \alpha} x^2$ . Here,  $g$  is the acceleration due to gravity. On the right of the equal sign, combine terms and simplify.  
 55. The CN Tower in Toronto is 553 m high, and it has an observation deck at the 335-m level. How far from the top of the CN

Tower must a 553-m-high helicopter be in order that the angle subtended at the helicopter by the part of the tower above the deck equals the angle subtended by the part of the tower below the deck?

56. The cross section of a radio-wave reflector is defined by  $x = \cos 2\theta$ ,  $y = \sin \theta$ . Find the relation between  $x$  and  $y$  by eliminating  $\theta$ .  
 57. To find the horizontal range  $R$  of a projectile, the equation  $R = vt \cos \alpha$  is used, where  $\alpha$  is the angle between the line of fire and the horizontal,  $v$  is the initial velocity of the projectile, and  $t$  is the time of flight. It can be shown that  $t = (2v \sin \alpha)/g$ , where  $g$  is the acceleration due to gravity. Show that  $R = (v^2 \sin 2\alpha)/g$ . See Fig. 20.11.

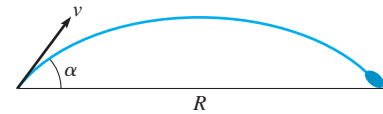


Fig. 20.11

58. In analysing light reflection from a cylinder onto a flat surface, the expression  $3 \cos \theta - \cos 3\theta$  arises. Show that this equals  $2 \cos \theta \cos 2\theta + 4 \sin \theta \sin 2\theta$ .  
 59. The instantaneous electric power  $P$  in an inductor is given by the equation  $P = vi \sin \omega t \sin(\omega t - \pi/2)$ . Show that this equation can be written as  $P = -\frac{1}{2}vi \sin 2\omega t$ .  
 60. In the study of the stress at a point in a bar, the equation  $s = a \cos^2 \theta + b \sin^2 \theta - 2t \sin \theta \cos \theta$  arises. Show that this equation can be written as  $s = \frac{1}{2}(a + b) + \frac{1}{2}(a - b) \cos 2\theta - t \sin 2\theta$ .

#### Answer to Practice Exercise

1.  $\tan 2x$

## 20.4 Half-Angle Formulas

Formula for  $\sin(\alpha/2)$  • Formula for  $\cos(\alpha/2)$

If we let  $\theta = \alpha/2$  in the identity  $\cos 2\theta = 1 - 2 \sin^2 \theta$  and then solve for  $\sin(\alpha/2)$ ,

$$\sin \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{2}} \quad (20.26)$$

Also, with the same substitution in the identity  $\cos 2\theta = 2 \cos^2 \theta - 1$ , which is then solved for  $\cos(\alpha/2)$ , we have

$$\cos \frac{\alpha}{2} = \pm \sqrt{\frac{1 + \cos \alpha}{2}} \quad (20.27)$$

In each of Eqs. (20.26) and (20.27), *the sign chosen depends on the quadrant in which  $\frac{\alpha}{2}$  lies.*

**EXAMPLE 1** Evaluation using the  $\cos(\alpha/2)$  formula

We can find  $\cos 165^\circ$  by using the relation

$$\begin{aligned}\cos 165^\circ &= -\sqrt{\frac{1 + \cos 330^\circ}{2}} && \text{using Eq. (20.27)} \\ &= -\sqrt{\frac{1 + 0.866}{2}} = -0.966\end{aligned}$$

Here, the minus sign is used, since  $165^\circ$  is in the second quadrant, and the cosine of a second-quadrant angle is negative. ■

**EXAMPLE 2** Evaluation using the  $\sin(\alpha/2)$  formula

Simplify  $\sqrt{\frac{1 - \cos 114^\circ}{2}}$  by expressing the result in terms of one-half the given angle. Then, using a calculator, show that the values are equal.

We note that the given expression fits the form of the right side of Eq. (20.26), which means that

$$\sqrt{\frac{1 - \cos 114^\circ}{2}} = \sin \frac{1}{2}(114^\circ) = \sin 57^\circ$$

Using a calculator shows that

$$\sqrt{\frac{1 - \cos 114^\circ}{2}} = 0.838\,670\,567\,9 \quad \text{and} \quad \sin 57^\circ = 0.838\,670\,567\,9$$

which verifies the equation for these values. ■

**EXAMPLE 3** Simplification using the  $\cos(\alpha/2)$  formula

Simplify the expression  $\sqrt{\frac{9 + 9 \cos 6x}{2}}$ .

$$\begin{aligned}\sqrt{\frac{9 + 9 \cos 6x}{2}} &= \sqrt{\frac{9(1 + \cos 6x)}{2}} = 3\sqrt{\frac{1 + \cos 6x}{2}} \\ &= 3 \cos \frac{1}{2}(6x) && \text{using Eq. (20.27) with } \alpha = 6x \\ &= 3 \cos 3x\end{aligned}$$

Noting the original expression, we see that  $\cos 3x$  cannot be negative. ■

**EXAMPLE 4** A trigonometric identity with  $\sin(\alpha/2)$ —application

In the kinetic theory of gases, the expression

$$\begin{aligned}\sqrt{(1 - \cos \alpha)^2 + \sin^2 \alpha} &\text{ is found. Show that this expression equals } 2 \sin \frac{1}{2} \alpha: \\ \sqrt{(1 - \cos \alpha)^2 + \sin^2 \alpha} &= \sqrt{1 - 2 \cos \alpha + \cos^2 \alpha + \sin^2 \alpha} && \text{expanding} \\ &= \sqrt{1 - 2 \cos \alpha + 1} && \text{using Eq. (20.26)} \\ &= \sqrt{2 - 2 \cos \alpha} \\ &= \sqrt{2(1 - \cos \alpha)} && \text{factoring}\end{aligned}$$

This last expression is very similar to that for  $\sin \frac{1}{2} \alpha$ , except that no 2 appears in the denominator. Therefore, multiplying the numerator and the denominator under the radical by 2 leads to the solution:

$$\begin{aligned}\sqrt{2(1 - \cos \alpha)} &= \sqrt{\frac{4(1 - \cos \alpha)}{2}} = 2\sqrt{\frac{1 - \cos \alpha}{2}} \\ &= 2 \sin \frac{1}{2} \alpha && \text{using Eq. (20.26)}\end{aligned}$$

Noting the original expression, we see that  $\sin \frac{1}{2} \alpha$  cannot be negative. ■

**Practice Exercise**

1. Simplify:  $\sqrt{\frac{25 - 25 \cos 4x}{2}}$

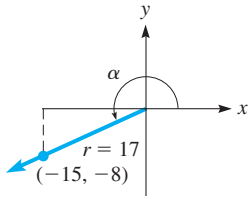


Fig. 20.12

**EXAMPLE 5** Evaluation using the  $\cos(\alpha/2)$  formula

Given that  $\tan \alpha = \frac{8}{15}$  ( $180^\circ < \alpha < 270^\circ$ ), find  $\cos(\frac{\alpha}{2})$ .

Knowing that  $\tan \alpha = \frac{8}{15}$  for a third-quadrant angle, we determine from Fig. 20.12 that  $\cos \alpha = -\frac{15}{17}$ . This means

$$\begin{aligned}\cos \frac{\alpha}{2} &= -\sqrt{\frac{1 + (-15/17)}{2}} = -\sqrt{\frac{2}{34}} && \text{using Eq. (20.27)} \\ &= -\frac{1}{17}\sqrt{17} = -0.2425\end{aligned}$$

Since  $180^\circ < \alpha < 270^\circ$ , we know that  $90^\circ < \frac{\alpha}{2} < 135^\circ$ , and therefore  $\frac{\alpha}{2}$  is in the second quadrant. Since the cosine is negative for second-quadrant angles, we use the negative value of the radical. ■

**EXAMPLE 6** Simplification—using the  $\cos(\alpha/2)$  formula

Show that  $2 \cos^2 \frac{x}{2} - \cos x = 1$ .

The first step is to substitute for  $\cos \frac{x}{2}$ , which will result in each term on the left being in terms of  $x$  and no  $\frac{x}{2}$  terms will exist. This will allow us to combine terms. We have for the left side

$$\begin{aligned}2 \cos^2 \frac{x}{2} - \cos x &= 2 \left( \frac{1 + \cos x}{2} \right) - \cos x && \text{using Eq. (20.27) with} \\ &= 1 + \cos x - \cos x = 1 && \text{both sides squared}\end{aligned}$$

**Practice Exercise**

2. Find the formula for  $\csc(x/2)$ .

**EXAMPLE 7** Formulas for other functions of  $\alpha/2$ 

We can find relations for other functions of  $\frac{\alpha}{2}$  by expressing these functions in terms of  $\sin(\frac{\alpha}{2})$  and  $\cos(\frac{\alpha}{2})$ . For example,

$$\begin{aligned}\sec \frac{\alpha}{2} &= \frac{1}{\cos \frac{\alpha}{2}} = \pm \frac{1}{\sqrt{\frac{1 + \cos \alpha}{2}}} && \text{using Eq. (20.27)} \\ &= \pm \sqrt{\frac{2}{1 + \cos \alpha}}\end{aligned}$$

**EXERCISES 20.4**

In Exercises 1 and 2, make the given changes in the indicated examples of this section and then solve the given problem.

- In Example 2, change the  $-$  sign in the numerator to  $+$ .
- In Example 5, change  $\frac{8}{15}$  ( $180^\circ < \alpha < 270^\circ$ ) to  $-\frac{8}{15}$  ( $270^\circ < \alpha < 360^\circ$ ).

In Exercises 3–8, use the half-angle formulas to evaluate the given functions. When using a calculator, round to 3 significant digits.

- $\cos 15^\circ$
- $\sin 22.5^\circ$
- $\sin 105^\circ$
- $\cos 112.5^\circ$
- $\cos \frac{7\pi}{8}$
- $\sin \frac{11\pi}{12}$

In Exercises 9–12, simplify the given expressions by giving the results in terms of one-half the given angle. Then use a calculator to verify the result.

- $\sqrt{\frac{1 - \cos 236^\circ}{2}}$
- $\sqrt{\frac{1 + \cos 98^\circ}{2}}$

- $\sqrt{1 + \cos 164^\circ}$
- $\sqrt{2 - 2 \cos 328^\circ}$

In Exercises 13–20, simplify the given expressions.

- $\sqrt{\frac{1 - \cos 6x}{2}}$
- $\sqrt{\frac{4 + 4 \cos 8\beta}{2}}$
- $\sqrt{8 + 8 \cos 4x}$
- $\sqrt{2 - 2 \cos 16x}$
- $\sqrt{4 - 4 \cos 10\theta}$
- $\sqrt{18 + 18 \cos 1.4x}$
- $2 \sin^2 \frac{x}{2} + \cos x$
- $2 \cos^2 \frac{\theta}{2} \sec \theta$

In Exercises 21–24, evaluate the indicated functions.

- Find the value of  $\sin(\frac{\alpha}{2})$  if  $\cos \alpha = \frac{12}{13}$  ( $0^\circ < \alpha < 90^\circ$ ).
- Find the value of  $\cos(\frac{\alpha}{2})$  if  $\sin \alpha = -\frac{4}{5}$  ( $180^\circ < \alpha < 270^\circ$ ).

23. Find the value of  $\cos\left(\frac{\alpha}{2}\right)$  if  $\tan \alpha = -0.292$  ( $90^\circ < \alpha < 180^\circ$ ).  
 24. Find the value of  $\sin\left(\frac{\alpha}{2}\right)$  if  $\cos \alpha = 0.471$  ( $270^\circ < \alpha < 360^\circ$ ).

In Exercises 25–28, derive the required expressions.

25. Derive an expression for  $\csc\left(\frac{\alpha}{2}\right)$  in terms of  $\sec \alpha$ .  
 26. Derive an expression for  $\sec\left(\frac{\alpha}{2}\right)$  in terms of  $\sec \alpha$ .  
 27. Derive an expression for  $\tan\left(\frac{\alpha}{2}\right)$  in terms of  $\sin \alpha$  and  $\cos \alpha$ .  
 28. Derive an expression for  $\cot\left(\frac{\alpha}{2}\right)$  in terms of  $\sin \alpha$  and  $\cos \alpha$ .

In Exercises 29–32, prove the given identities.

29.  $\sin \frac{\alpha}{2} = \frac{1 - \cos \alpha}{2 \sin \frac{\alpha}{2}}$       30.  $\cos \frac{\theta}{2} = \frac{\sin \theta}{2 \sin \frac{\theta}{2}}$

31.  $2 \cos \frac{x}{2} = (1 + \cos x) \sec \frac{x}{2}$

32.  $\cos^2 \frac{x}{2} \left[ 1 + \left( \frac{\sin x}{1 + \cos x} \right)^2 \right] = 1$

In Exercises 33–36, verify each identity by comparing the graph of the left side with the graph of the right side on a graphing calculator.

33.  $2 \sin^2 \frac{\alpha}{2} - \cos^2 \frac{\alpha}{2} = \frac{1 - 3 \cos \alpha}{2}$

34.  $\cos^2 \frac{A}{2} - \sin^2 \frac{A}{2} = \frac{\sin 2A}{2 \sin A}$

35.  $2 \sin^2 \frac{\theta}{2} = \frac{\sin^2 \theta}{1 + \cos \theta}$       36.  $\tan \frac{\alpha}{2} = \frac{\sin \alpha}{1 + \cos \alpha}$

In Exercises 37–44, use the half-angle formulas to solve the given problems.

37. Find  $\tan \theta$  if  $\sin(\theta/2) = 3/5$ .  
 38. In a right triangle with sides and angles as shown in Fig. 20.13, show that  $\sin^2 \frac{A}{2} = \frac{c-b}{2c}$ .

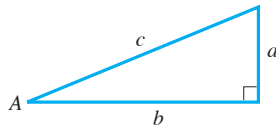


Fig. 20.13

39. In finding the path of a sliding particle, the expression  $\sqrt{8 - 8 \cos \theta}$  is used. Simplify this expression.  
 40. In designing a track for a railway system, the equation  $d = 4r \sin^2 \frac{A}{2}$  is used. Solve for  $d$  in terms of  $\cos A$ .  
 41. In electronics, in order to find the *root-mean-square current* in a circuit, it is necessary to express  $\sin^2 \omega t$  in terms of  $\cos 2\omega t$ . Show how this is done.  
 42. In studying interference patterns of radio signals, the expression  $2E^2 - 2E^2 \cos(\pi - \theta)$  arises. Show that this can be written as  $4E^2 \cos^2(\theta/2)$ .  
 43. The index of refraction  $n$ , the angle  $A$  of a prism, and the minimum angle of deflection  $\phi$  are related by  $n = \frac{\sin \frac{1}{2}(A + \phi)}{\sin \frac{1}{2}A}$ .

See Fig. 20.14. Show that an equivalent expression is

$$n = \sqrt{\frac{1 - \cos A \cos \phi + \sin A \sin \phi}{1 - \cos A}}$$

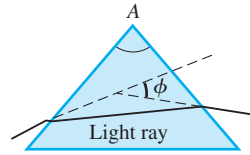


Fig. 20.14

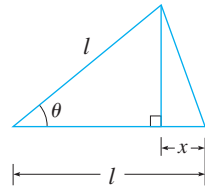


Fig. 20.15

44. For the structure shown in Fig. 20.15, show that  $x = 2l \sin^2 \frac{1}{2} \theta$ .

Answers to Practice Exercises

1.  $5 \sin 2x$       2.  $\pm \sqrt{\frac{2}{1 - \cos x}}$

## 20.5 Solving Trigonometric Equations

Solve for the Function and Then the Angle • Use of Algebraic Methods and Trigonometric Identities • Use of a Calculator

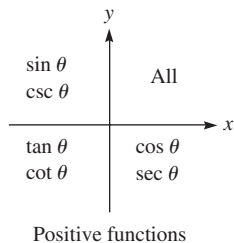


Fig. 20.16

One of the most important uses of the trigonometric identities is in the solution of equations involving trigonometric functions. The solution of this type of equation consists of the angles that satisfy the equation. When solving for the angle, we generally first solve for a value of a function of the angle and then find the angle from this value of the function.

When equations are written in terms of more than one function, the identities provide a way of changing many of them to equations or factors involving only one function of the same angle. Thus, *the solution is found by using algebraic methods and trigonometric identities and values.* From Chapter 8, recall that we must be careful regarding the sign of the value of a trigonometric function in finding the angle. Fig. 20.16 shows again the quadrants in which the functions are positive. Functions not listed are negative.

**EXAMPLE 1** Solving a linear equation

Solve the equation  $2 \cos \theta - 1 = 0$  for all values of  $\theta$  such that  $0 \leq \theta < 2\pi$ .

Solving the equation for  $\cos \theta$ , we obtain  $\cos \theta = \frac{1}{2}$ . The problem asks for all values of  $\theta$  from 0 to  $2\pi$  that satisfy the equation. We know that the cosines of angles in the first and fourth quadrants are positive. Also, we know that  $\cos \frac{\pi}{3} = \frac{1}{2}$ , which means that  $\frac{\pi}{3}$  is the reference angle. Therefore, the solution proceeds as follows:

$$\begin{aligned} 2 \cos \theta - 1 &= 0 \\ 2 \cos \theta &= 1 && \text{solve for } \cos \theta \\ \cos \theta &= \frac{1}{2} \\ \theta &= \frac{\pi}{3}, \frac{5\pi}{3} && \theta \text{ in quadrants I and IV} \end{aligned}$$

**Practice Exercise**

1. Solve for  $x$  ( $0 \leq x < 2\pi$ ):  
 $2 \sin x + 1 = 0$

**EXAMPLE 2** Solution using a trigonometric identity and factoring

Solve the equation  $2 \cos^2 x - \sin x - 1 = 0$  ( $0 \leq x < 2\pi$ ).

By use of the identity  $\sin^2 x + \cos^2 x = 1$ , this equation may be put in terms of  $\sin x$  only. Thus, we have

$$\begin{aligned} 2(1 - \sin^2 x) - \sin x - 1 &= 0 && \text{use identity} \\ -2 \sin^2 x - \sin x + 1 &= 0 && \text{multiply and simplify} \\ 2 \sin^2 x + \sin x - 1 &= 0 \\ (2 \sin x - 1)(\sin x + 1) &= 0 && \text{factor} \end{aligned}$$

Setting each factor equal to zero, we find  $\sin x = 1/2$  or  $\sin x = -1$ . For the domain 0 to  $2\pi$ ,  $\sin x = 1/2$  gives  $x = \pi/6, 5\pi/6$ , and  $\sin x = -1$  gives  $x = 3\pi/2$ . Therefore,

$$x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$$

These values check when substituted in the original equation. ■

**Graphical Solutions**

As with algebraic equations, graphical solutions of trigonometric equations are approximate, whereas algebraic solutions often give exact solutions. As before, we collect all terms on the left of the equal sign, with zero on the right. We then graph the function on the left to find its zeros by finding the values of  $x$  where the graph crosses (or is tangent to) the  $x$ -axis.

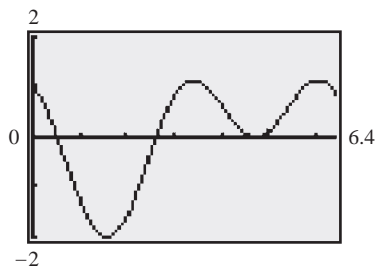


Fig. 20.17

**EXAMPLE 3** Solution using a calculator

Graphically solve the equation  $2 \cos^2 x - \sin x - 1 = 0$  ( $0 \leq x < 2\pi$ ) by using a graphing calculator. (This is the same equation as in Example 2.)

Since all the terms of the equation are on the left, with zero on the right, we now set  $y = 2 \cos^2 x - \sin x - 1$ . We then enter this function in the graphing calculator as  $Y_1$ , and the graph is displayed in Fig. 20.17. Since angles are expressed in radians and  $2\pi = 6.3$ , Xmax was chosen as 6.4. Using the *trace* and *zoom* features (or the *zero* feature, or by letting  $y_1 = 2 \cos^2 x - \sin x - 1$ ,  $y_2 = 0$  and using the *intersect* feature) of a graphing calculator, we find that  $y = 0$  for

$$x = 0.52, 2.62, 4.71$$

These values are the same as in Example 2. We note that for  $x = 4.71$ , the curve touches the  $x$ -axis but does not cross it. This means it is *tangent* to the  $x$ -axis. ■



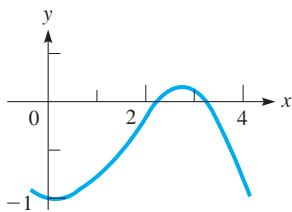


Fig. 20.18

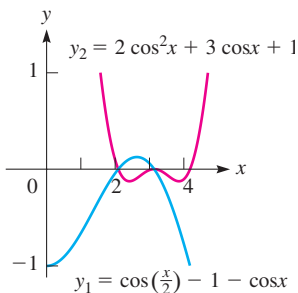


Fig. 20.19

**LEARNING TIP**

When checking solutions, we must use the original equation and not one that has been derived after an algebraic step (such as squaring both sides) that could have changed the original function.

**EXAMPLE 4** Solution by squaring—extraneous solutions

Solve the equation  $\cos(x/2) = 1 + \cos x$  ( $0 \leq x < 2\pi$ ).

By using the half-angle formula for  $\cos(x/2)$  and then squaring both sides of the resulting equation, this equation can be solved:

$$\pm \sqrt{\frac{1 + \cos x}{2}} = 1 + \cos x \quad \text{using identity}$$

$$\frac{1 + \cos x}{2} = 1 + 2 \cos x + \cos^2 x \quad \text{squaring both sides}$$

$$2 \cos^2 x + 3 \cos x + 1 = 0 \quad \text{simplifying}$$

$$(2 \cos x + 1)(\cos x + 1) = 0 \quad \text{factoring}$$

$$\cos x = -\frac{1}{2}, -1$$

$$x = \frac{2\pi}{3}, \frac{4\pi}{3}, \pi$$

In finding this solution, we squared both sides of the original equation. In doing this, we may have introduced extraneous solutions (see Section 14.3). Thus, we must check each solution in the original equation to see if it is valid. Hence,

$$\cos \frac{\pi}{3} \stackrel{?}{=} 1 + \cos \frac{2\pi}{3} \quad \text{or} \quad -\frac{1}{2} \stackrel{?}{=} 1 + \left(-\frac{1}{2}\right) \quad \text{or} \quad \frac{1}{2} = \frac{1}{2}$$

$$\cos \frac{2\pi}{3} \stackrel{?}{=} 1 + \cos \frac{4\pi}{3} \quad \text{or} \quad -\frac{1}{2} \stackrel{?}{=} 1 + \left(-\frac{1}{2}\right) \quad \text{or} \quad -\frac{1}{2} \neq \frac{1}{2}$$

$$\cos \frac{\pi}{2} \stackrel{?}{=} 1 + \cos \pi \quad \text{or} \quad 0 \stackrel{?}{=} 1 - 1 \quad \text{or} \quad 0 = 0$$

Thus, the apparent solution  $x = \frac{4\pi}{3}$  is not a solution of the original equation. The correct solutions are  $x = \frac{2\pi}{3}$  and  $x = \pi$ .

We can see that these values agree with the values for  $x$  for which the graph of  $y_1 = \cos(x/2) - 1 - \cos x$  crosses the  $x$ -axis in Fig. 20.18. In Fig. 20.19 the graph of this function and the graph of  $y_2 = 2 \cos^2 x + 3 \cos x + 1$  are compared. We see  $x = 0$  for both curves at  $x = 2\pi/3$  and  $x = \pi$ , but that the curve for  $y_2$  is quite different from that of  $y_1$ , and  $y_2 = 0$  for  $x = 4\pi/3$ . When we squared both sides of the equation, we introduced a zero that is not a solution to the original equation. ■

**EXAMPLE 5** A trigonometric equation—application

The vertical displacement  $y$  of an object at the end of a spring, which itself is being moved up and down, is given by  $y = 3.50 \sin t + 1.20 \sin 2t$ . Find the first two values of  $t$  (in seconds) for which  $y = 0$ .

Using the double-angle formula for  $\sin 2t$  leads to the solution.

$$3.50 \sin t + 1.20 \sin 2t = 0 \quad \text{setting } y = 0$$

$$3.50 \sin t + 2.40 \sin t \cos t = 0 \quad \text{using identities}$$

$$\sin t(3.50 + 2.40 \cos t) = 0 \quad \text{factoring}$$

$$\sin t = 0 \quad \text{or} \quad \cos t = -1.46$$

$$t = 0.00, 3.14, \dots$$

Since  $\cos t$  cannot be numerically larger than 1, there are no values of  $t$  for which  $\cos t = -1.46$ . Thus, the required times are  $t = 0.00$  s, 3.14 s.

We can see that these values agree with the values of  $t$  for which the graph of  $y = 3.50 \sin t + 1.20 \sin 2t$  crosses the  $t$ -axis in Fig. 20.20. ■

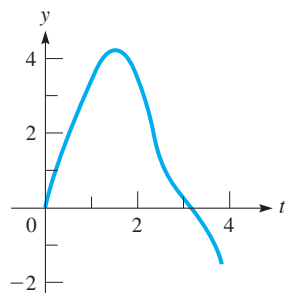


Fig. 20.20

**EXAMPLE 6** An equation with a double angle

Solve the equation  $\tan 2\theta - \cot 2\theta = 0$  ( $0 \leq \theta < 2\pi$ ).

We first solve for  $2\theta$  and then for  $\theta$ .

$$\begin{aligned} \tan 2\theta - \frac{1}{\tan 2\theta} &= 0 && \text{using } \cot 2\theta = \frac{1}{\tan 2\theta} \\ \tan^2 2\theta &= 1 && \text{multiplying by } \tan 2\theta \text{ and adding 1 to each side} \\ \tan 2\theta &= \pm 1 && \text{taking square roots} \end{aligned}$$

For  $0 \leq \theta < 2\pi$ , we must have values of  $2\theta$  such that  $0 \leq 2\theta < 4\pi$ . Therefore,

$$2\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \frac{11\pi}{4}, \frac{13\pi}{4}, \frac{15\pi}{4}$$

This means that the solutions are

$$\theta = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8}$$

These values satisfy the original equation. Since we multiplied through by  $\tan 2\theta$  in the solution, any value of  $\theta$  that leads to  $\tan 2\theta = 0$  would not be valid since this would indicate division by zero in the original equation.

We see that these solutions agree with the values of  $\theta$  for which the graph of  $y = \cos 2\theta - \cot 2\theta$  crosses the  $\theta$ -axis in Fig. 20.21. ■

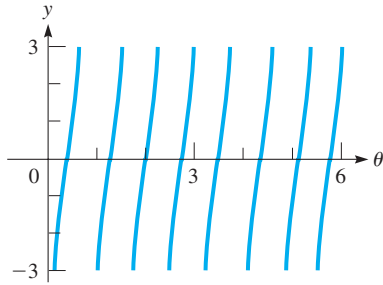


Fig. 20.21

**Practice Exercise**

2. Solve for  $x$  ( $0 \leq x < 2\pi$ ):  
 $\sec^2 x + 2 \tan x = 0$

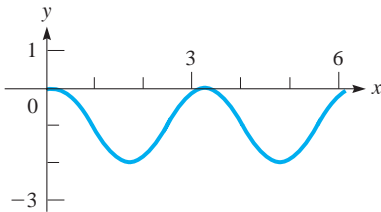


Fig. 20.22

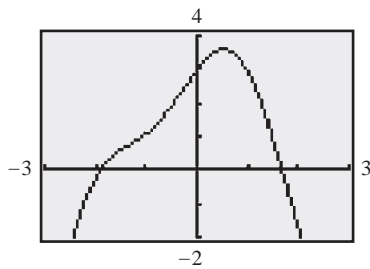


Fig. 20.23

**EXAMPLE 7** Solution using a trigonometric identity

Solve the equation  $\cos 3x \cos x + \sin 3x \sin x = 1$  ( $0 \leq x < 2\pi$ ).

The left side of this equation is of the general form  $\cos(A - x)$ , where  $A = 3x$ . Therefore,

$$\cos 3x \cos x + \sin 3x \sin x = \cos(3x - x) = \cos 2x$$

The original equation becomes

$$\cos 2x = 1$$

This equation is satisfied if  $2x = 0$  or  $2x = 2\pi$ . The solutions are  $x = 0$  and  $x = \pi$ . Only through recognition of the proper trigonometric form can we readily solve this equation.

We see that these solutions agree with the two values of  $x$  for which the graph of  $y = \cos 3x \cos x + \sin 3x \sin x - 1$  touches the  $x$ -axis in Fig. 20.22. ■

**EXAMPLE 8** Solution using a calculator

Solve the equation  $\sin 2x + 3 = x^2$ .

Although we can substitute for  $\sin 2x$ , we cannot express  $x^2$  in terms of a trigonometric function. However, we can graphically find an approximate solution.

Collecting terms on the left, we have  $\sin 2x + 3 - x^2 = 0$ . Then we let  $y = \sin 2x + 3 - x^2$ . Entering this function into the calculator as  $Y_1$ , we find from its graph where  $y$  is zero. See Fig. 20.23. Using the *zero* feature, we find that  $x = -1.90$  and  $x = 1.67$  are the only values for which  $y = 0$ .

The solution can also be found by letting  $y_1 = \sin 2x + 3$ ,  $y_2 = x^2$ , and using the *intersect* feature of a graphing calculator. ■

## EXERCISES 20.5

In Exercises 1–4, make the given changes in the indicated examples of this section and then solve the resulting problems.

- In Example 1, change  $2 \cos \theta$  to  $\tan \theta$ .
- In Example 2, change  $2 \cos^2 x$  to  $2 \sin^2 x$ .
- In Example 4, change  $\cos \frac{1}{2}x$  to  $\sin \frac{1}{2}x$  and on the right of the equal sign change the  $+$  to  $-$ .
- In Example 7, change the  $+$  to  $-$ .

In Exercises 5–20, solve the given trigonometric equations analytically (using identities when necessary for exact values when possible) for values of  $x$  for  $0 \leq x < 2\pi$ .

- $\sin x - 1 = 0$
- $2 \cos x + 1 = 0$
- $\sin(x - \frac{\pi}{4}) = \cos(x - \frac{\pi}{4})$
- $4 \tan x + 2 = 3(1 + \tan x)$
- $4 \cos^2 x - 1 = 0$
- $3 \tan^2 x - 1 = 0$
- $2 \sin^2 x - \sin x = 0$
- $\sin 4x - \sin 2x = 0$
- $\sin 2x \sin x + \cos x = 0$
- $\sin x - \sin \frac{x}{2} = 0$
- $2 \cos^2 x - 2 \cos 2x - 1 = 0$
- $\tan^2 x + 6 = 5 \tan x$
- $4 \tan x - \sec^2 x = 0$
- $\sin x \sin \frac{1}{2}x = 1 - \cos x$
- $\sin 2x \cos x - \cos 2x \sin x = 0$
- $\cos 3x \cos x - \sin 3x \sin x = 0$

In Exercises 21–38, solve the given trigonometric equations analytically. Use values of  $x$  for  $0 \leq x < 2\pi$ .

- $\tan x + 1 = 0$
- $2 \sin x + 1 = 0$
- $3 - 4 \cos x = 7 - (2 - \cos x)$
- $7 \sin x - 2 = 3(2 - \sin x)$
- $4 \sin^2 x - 3 = 0$
- $|\sin x| = \frac{1}{2}$
- $\sin 4x - \cos 2x = 0$
- $3 \cos x - 6 \cos^2 x = 0$
- $2 \sin x = \tan x$
- $\cos 2x + \sin^2 x = 0$
- $\sin^2 x - 2 \sin x = 1$
- $2 \cos^2 2x + 1 = 3 \cos 2x$
- $\tan x + 3 \cot x = 4$
- $\tan^2 x + 4 = 2 \sec^2 x$
- $\sin 2x + \cos 2x = 0$
- $2 \sin 4x + \csc 4x = 3$
- $2 \sin 2x - \cos x \sin^3 x = 0$
- $\tan^4 x = 1$

In Exercises 39–52, solve the indicated equations analytically.

- $\sin 3x + \sin x = 0$  (Hint: See Eq. 20.17.)
- $\cos 3x - \cos x = 0$  (Hint: See Eq. 20.20.)
- Is there any positive acute angle  $\theta$  for which  $\sin \theta + \cos \theta + \tan \theta + \cot \theta + \sec \theta + \csc \theta = 1$ ? Explain.
- Use a graphing calculator to determine the minimum value of the function to the left of the equal sign in Exercise 41 (for a positive acute angle).
- Solve the system of equations  $r = \sin \theta$ ,  $r = \sin 2\theta$ , for  $0 \leq \theta < 2\pi$ .
- Solve the system of equations  $r = \sin \theta$ ,  $r = \cos 2\theta$ , for  $0 \leq \theta < 2\pi$ .

- The acceleration due to gravity  $g$  (in  $\text{m/s}^2$ ) varies with latitude, approximately given by  $g = 9.7805(1 + 0.0053 \sin^2 \theta)$ , where  $\theta$  is the latitude in degrees. Find  $\theta$  for  $g = 9.8000$ .

- Under certain conditions, the electric current  $i$  (in A) in the circuit shown in Fig. 20.24 is given below. For what value of  $t$  (in s) is the current first equal to zero?

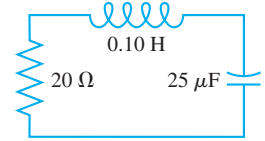


Fig. 20.24

$$i = -e^{-100t}(32.0 \sin 624t + 0.200 \cos 624t)$$

- The vertical displacement  $y$  (in m) of the end of a robot arm is given by  $y = 2.30 \cos 0.1t - 1.35 \sin 0.2t$ . Find the first four values of  $t$  (in s) for which  $y = 0$ .
- In finding the maximum illuminance from a point source of light, it is necessary to solve the equation  $\cos \theta \sin 2\theta - \sin^3 \theta = 0$ . Find  $\theta$  if  $0 < \theta < 90^\circ$ .
- To find the acute angle  $\theta$  subtended by a certain object on a camera film, it is necessary to solve the equation  $\frac{p^2 \tan \theta}{0.0063 + p \tan \theta} = 1.6$ , where  $p$  is the distance from the camera to the object. Find  $\theta$  if  $p = 4.8$  m.
- The velocity of a certain piston is maximum when the acute crank angle  $\theta$  satisfies the equation  $8 \cos \theta + \cos 2\theta = 0$ . Find this angle.
- Resolve a force of 500.0 N into two components, perpendicular to each other, for which the sum of their magnitudes is 700.0 N, by using the angle between a component and the resultant.
- A search and rescue helicopter flew from Gander, Newfoundland, and travelled 160 km east. It then turned and flew due south, finally making a final turn to fly directly back to Gander. If the total distance flown was 480 km, how long were the final two legs of the flight? Solve by setting up and solving an appropriate trigonometric equation. (The Pythagorean theorem may be used only as a check.)

In Exercises 53–60, solve the given equations graphically.

- $3 \sin x - x = 0$
- $4 \cos x + 3x = 0$
- $2 \sin 2x = x^2 + 1$
- $\sqrt{x} - \sin 3x = 1$
- $2 \ln x = 1 - \cos 2x$
- $e^x = 1 + \sin x$
- In finding the frequencies of vibration of a vibrating wire, the equation  $x \tan x = 2.00$  occurs. Find  $x$  if  $0 < x < \pi/2$ .
- An equation used in astronomy is  $\theta - e \sin \theta = M$ . Solve for  $\theta$  for  $e = 0.25$  and  $M = 0.75$ .

### Answers to Practice Exercises

- $x = 7\pi/6, 11\pi/6$
- $x = 3\pi/4, 7\pi/4$

## 20.6 The Inverse Trigonometric Functions

Inverse Functions • Inverse Trigonometric Functions • Ranges of the Inverse Trigonometric Functions

When we studied the exponential and logarithmic functions, we often changed an expression from one form to the other. The exponential function  $y = b^x$ , written in logarithmic form with  $x$  as a function of  $y$ , is  $x = \log_b y$ . Then we wrote the logarithmic function as  $y = \log_b x$ , since it is standard practice to use  $y$  as the dependent variable and  $x$  as the independent variable.

As we discussed in Chapter 13, these two functions, the exponential function  $y = b^x$  and the logarithmic function  $y = \log_b x$ , are **inverse functions**. This means that if we solve for the independent variable in terms of the dependent variable in one function, we will arrive at the functional relationship expressed by the other. It also means that, for every value of  $x$ , there is only one corresponding value of  $y$ .

Just as we are able to solve  $y = b^x$  for the exponent by writing it in logarithmic form, at times it is necessary to solve for the independent variable (the angle) in trigonometric functions. Therefore, we define the **inverse sine function**

$$y = \sin^{-1} x \quad \left( -\frac{\pi}{2} \leq y \leq \frac{\pi}{2} \right) \quad (20.28)$$

where  $y$  is the angle whose sine is  $x$ . This means that  $x$  is the value of the sine of the angle  $y$ , or  $x = \sin y$ . (It is necessary to show the range as  $-\pi/2 \leq y \leq \pi/2$ , as we will see shortly.)

### COMMON ERROR

In Eq. (20.28), the  $-1$  is **not** an exponent.

The  $-1$  in  $\sin^{-1} x$  is the notation showing the inverse function. We introduced this notation in Chapter 4 when finding the angle with a known value of one of the functions.

The notations  $\text{Arcsin } x$ ,  $\arcsin x$ , and  $\text{Sin}^{-1} x$  are also used to designate the inverse sine. Some calculators use the  $\boxed{\text{inv}}$  key and then the  $\boxed{\text{sin}}$  key to find values of the inverse sine. However, since most calculators use the notation  $\sin^{-1}$  as a second function on the  $\boxed{\text{sin}}$  key, we will continue to use  $\sin^{-1} x$  for the inverse sine.

Similar definitions are used for the other inverse trigonometric functions. They also have meanings similar to that of Eq. (20.28).

### EXAMPLE 1 Meaning of inverse trigonometric functions

- (a)  $y = \cos^{-1} x$  is read as “ $y$  is the angle whose cosine is  $x$ .” In this case,  $x = \cos y$ .  
 (b)  $y = \tan^{-1} 2x$  is read as “ $y$  is the angle whose tangent is  $2x$ .” In this case,  $2x = \tan y$ .  
 (c)  $y = \csc^{-1} (1 - x)$  is read as “ $y$  is the angle whose cosecant is  $1 - x$ .” In this case,  $1 - x = \csc y$ , or  $x = 1 - \csc y$ . ■

We have seen that  $y = \sin^{-1} x$  means that  $x = \sin y$ . From our previous work with the trigonometric functions, we know that there is an unlimited number of possible values of  $y$  for a given value of  $x$  in  $x = \sin y$ . Consider the following example.

### EXAMPLE 2 Values of trigonometric functions

- (a) For  $x = \sin y$ , we know that

$$\sin \frac{\pi}{6} = \frac{1}{2} \quad \text{and} \quad \sin \frac{5\pi}{6} = \frac{1}{2}$$

In fact,  $x = \frac{1}{2}$  also for values of  $y$  of  $-\frac{7\pi}{6}$ ,  $\frac{13\pi}{6}$ ,  $\frac{17\pi}{6}$ , and so on.

- (b) For  $x = \cos y$ , we know that

$$\cos 0 = 1 \quad \text{and} \quad \cos 2\pi = 1$$

In fact,  $\cos y = 1$  for  $y$  equal to any even multiple of  $\pi$ . ■

From Chapter 3, we know that *to have a properly defined function, there must be only one value of the dependent variable for a given value of the independent variable.* (A *relation*, on the other hand, may have more than one such value.) Therefore, as in Eq. (20.28), in order to have only one value of  $y$  for each value of  $x$  in the domain of the inverse trigonometric functions, it is not possible to include all values of  $y$  in the range. For this reason, *the range of each of the inverse trigonometric functions is defined as follows:*

$$\begin{aligned} -\frac{\pi}{2} \leq \sin^{-1} x \leq \frac{\pi}{2} & \quad 0 \leq \cos^{-1} x \leq \pi & \quad -\frac{\pi}{2} < \tan^{-1} x < \frac{\pi}{2} & \quad 0 < \cot^{-1} x < \pi \\ 0 \leq \sec^{-1} x \leq \pi & \quad \left( \sec^{-1} x \neq \frac{\pi}{2} \right) & \quad -\frac{\pi}{2} \leq \csc^{-1} x \leq \frac{\pi}{2} & \quad (\csc^{-1} x \neq 0) \end{aligned} \quad (20.29)$$

We must choose a value of  $y$  in the range as defined in Eq. (20.29) that corresponds to a given value of  $x$  in the domain. We will discuss the domains and the reasons for these definitions following the next two examples.

### EXAMPLE 3 Range of inverse sine and cosine

(a)  $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$  *first-quadrant angle*

This is the only value of the function that lies within the defined range. The value  $\frac{5\pi}{6}$  is not correct, even though  $\sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}$ , since  $\frac{5\pi}{6}$  lies outside the defined range.

(b)  $\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$  *second-quadrant angle*

Other values such as  $\frac{4\pi}{3}$  and  $-\frac{2\pi}{3}$  are not correct, since they are not within the defined range for the function  $\cos^{-1} x$ . ■

### EXAMPLE 4 Range of inverse tangent

$$\tan^{-1}(-1) = -\frac{\pi}{4} \quad \text{fourth-quadrant angle}$$

This is the only value within the defined range for the function  $\tan^{-1} x$ . We must remember that *when  $x$  is negative for  $\sin^{-1} x$  and  $\tan^{-1} x$ , the value of  $y$  is a fourth-quadrant angle, expressed as a negative angle.* This is a direct result of the definition. (The single exception is  $\sin^{-1}(-1) = -\pi/2$ , which is a quadrantal angle and is not in the fourth quadrant.) ■

In choosing these values to be the ranges of the inverse trigonometric functions, we first note that the *domain* of  $y = \sin^{-1} x$  and  $y = \cos^{-1} x$  are each  $-1 \leq x \leq 1$ , since the sine and cosine functions take on only these values. Therefore, for each value in this domain, we use only one value of  $y$  in the range of the function. Although the domain of  $y = \tan^{-1} x$  is all real numbers, we still use only one value of  $y$  in the range.

The ranges of the inverse trigonometric functions are chosen so that if  $x$  is positive, the resulting value is an angle in the first quadrant. However, care must be taken in choosing the range for negative values of  $x$ .

Since the sine of a second-quadrant angle is positive, we cannot choose these angles for  $\sin^{-1} x$  for negative values of  $x$ . Therefore, we chose fourth-quadrant angles in the form of negative angles in order to have a continuous range of values for  $\sin^{-1} x$ . The range for  $\tan^{-1} x$  is chosen in the same way for similar reasons. However, since the cosine of a fourth-quadrant angle is positive, the range for  $\cos^{-1} x$  cannot be the same. To keep a continuous range of values for  $\cos^{-1} x$ , second-quadrant angles are used.

Values for the other functions are chosen such that the result is also an angle in the first quadrant if  $x$  is positive. As for negative values of  $x$ , it rarely makes any

difference, since either positive values of  $x$  arise, or we can use one of the other functions. Our definitions, however, are those that are generally used.

The graphs of the inverse trigonometric functions can be used to show the domains and ranges. We can obtain the graph of the inverse sine function by first sketching the sine curve  $x = \sin y$  along the  $y$ -axis. We then mark the specific part of this curve for which  $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$  as the graph of the inverse sine function. The graphs of the other inverse trigonometric functions are found in the same manner. In Figs. 20.25, 20.26, and 20.27, the graphs  $x = \sin y$ ,  $x = \cos y$ , and  $x = \tan y$ , respectively, are shown. The heavier, coloured portions indicate the graphs of the respective inverse trigonometric functions.

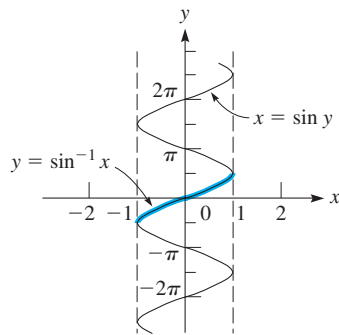


Fig. 20.25

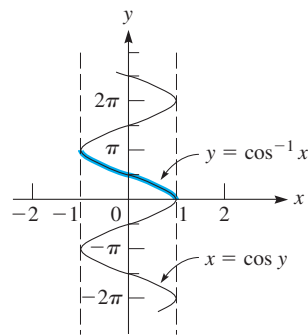


Fig. 20.26

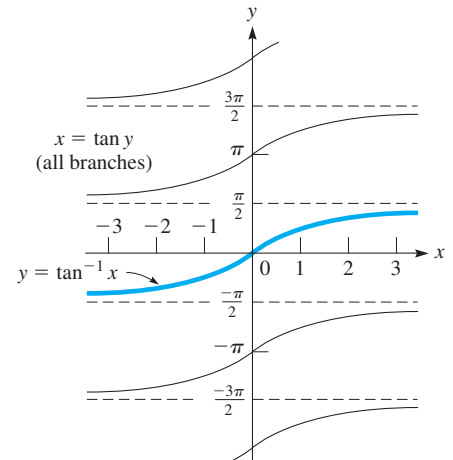


Fig. 20.27

The following examples further illustrate the values and meanings of the inverse trigonometric functions.

**EXAMPLE 5** Values of inverse trigonometric functions

- (a)  $\sin^{-1}(-\sqrt{3}/2) = -\pi/3$
- (b)  $\cos^{-1}(-1) = \pi$
- (c)  $\tan^{-1} 0 = 0$
- (d)  $\tan^{-1}(\sqrt{3}) = \pi/3$

Using a calculator in radian mode, we find the following values:

- (e)  $\sin^{-1} 0.6294 = 0.6808$
- (f)  $\sin^{-1}(-0.1568) = -0.1574$
- (g)  $\cos^{-1}(-0.8026) = 2.5024$
- (h)  $\tan^{-1}(-1.9268) = -1.0921$

We note that in each case the calculator gives the value of the inverse function in the defined range for the given function. ■

**EXAMPLE 6** Given an inverse function, solve for  $x$

Given that  $y = \pi - \sec^{-1} 2x$ , solve for  $x$ .

We first find the expression for  $\sec^{-1} 2x$  and then use the meaning of the inverse secant. The solution follows:

$$\begin{aligned}
 y &= \pi - \sec^{-1} 2x \\
 \sec^{-1} 2x &= \pi - y && \text{isolate } \sec^{-1} 2x \\
 2x &= \sec(\pi - y) && \text{use meaning of inverse secant} \\
 x &= \frac{1}{2} \sec y && \sec(\pi - y) = -\sec y
 \end{aligned}$$

As  $\sec 2x$  and  $2 \sec x$  are different functions,  $\sec^{-1} 2x$  and  $2 \sec^{-1} x$  are also different functions. Since the values of  $\sec^{-1} 2x$  are restricted, so are the resulting values of  $y$ . ■

**Practice Exercise**

1. Evaluate  $\tan^{-1}(-\sqrt{3}/3)$ .

**EXAMPLE 7** Solve for an angle—application

The instantaneous power  $P$  in an electric inductor is given by the equation  $P = vi \sin \omega t \cos \omega t$ . Solve for  $t$ .

Noting the product  $\sin \omega t \cos \omega t$  suggests using  $\sin 2\alpha = 2 \sin \alpha \cos \alpha$ . Then, using the meaning of the inverse sine, we can complete the solution:

$$\begin{aligned} P &= vi \sin \omega t \cos \omega t \\ &= \frac{1}{2} vi \sin 2\omega t && \text{using double-angle formula} \\ \sin 2\omega t &= \frac{2P}{vi} \\ 2\omega t &= \sin^{-1}\left(\frac{2P}{vi}\right) && \text{using meaning of inverse sine} \\ t &= \frac{1}{2\omega} \sin^{-1}\left(\frac{2P}{vi}\right) \end{aligned}$$

If we know the value of one of the inverse functions, we can find the trigonometric functions of the angle. If general relations are desired, a representative triangle is very useful. The following examples illustrate these methods.

**EXAMPLE 8** Angle in terms of inverse functions

(a) Find  $\cos(\sin^{-1} 0.5)$ .

Knowing that the values of inverse trigonometric functions are *angles*, we see that  $\sin^{-1} 0.5$  is a first-quadrant angle. Thus, we find  $\sin^{-1} 0.5 = \pi/6$ . The problem is now to find  $\cos(\pi/6)$ . This is, of course,  $\sqrt{3}/2$ , or 0.8660. Thus,

$$\cos(\sin^{-1} 0.5) = \cos(\pi/6) = 0.8660.$$

(b)  $\sin(\cot^{-1} 1) = \sin(\pi/4)$  first-quadrant angle

$$= \frac{\sqrt{2}}{2} = 0.7071$$

(c)  $\tan[\cos^{-1}(-1)] = \tan \pi$  quadrantal angle

$$= 0$$

(d)  $\cos[\sin^{-1}(-0.2395)] = 0.9709$  using a calculator

**Practice Exercise**

2. Evaluate  $\sin[\cos^{-1}(-0.5)]$ .

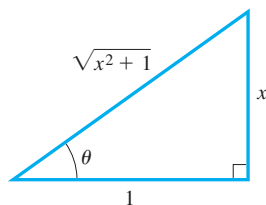


Fig. 20.28

**EXAMPLE 9** Trigonometric function of an inverse trigonometric function

Find  $\sin(\tan^{-1} x)$ .

We know that  $\tan^{-1} x$  is another way of stating “the angle whose tangent is  $x$ .” Thus, let us draw a right triangle (as in Fig. 20.28) and label one of the acute angles as  $\theta$ , the side opposite  $\theta$  as  $x$ , and the side adjacent to  $\theta$  as 1. In this way, we see that, by definition,  $\tan \theta = \frac{x}{1}$ , or  $\theta = \tan^{-1} x$ , which means  $\theta$  is the desired angle. By the Pythagorean theorem, the hypotenuse of this triangle is  $\sqrt{x^2 + 1}$ . Now, we find that  $\sin \theta$ , which is the same as  $\sin(\tan^{-1} x)$ , is  $x/\sqrt{x^2 + 1}$ . Thus,

$$\sin(\tan^{-1} x) = \frac{x}{\sqrt{x^2 + 1}}$$



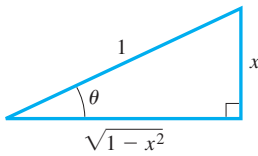


Fig. 20.29

**EXAMPLE 10** Trigonometric function of an inverse trigonometric function

Find  $\cos(2 \sin^{-1} x)$ .

From Fig. 20.29, we see that  $\theta = \sin^{-1} x$ . From the double-angle formulas, we have

$$\cos 2\theta = 1 - 2 \sin^2 \theta$$

Thus, since  $\sin \theta = x$ , we have

$$\cos(2 \sin^{-1} x) = 1 - 2x^2$$

**EXAMPLE 11** Inverse trigonometric function—application

A triangular brace of sides  $a$ ,  $b$ , and  $c$  supports a shelf, as shown in Fig. 20.30. Find the expression for the angle between sides  $b$  and  $c$ .

The law of cosines leads to the solution:

$$a^2 = b^2 + c^2 - 2bc \cos A \quad \text{law of cosines}$$

$$2bc \cos A = b^2 + c^2 - a^2 \quad \text{solving for } \cos A$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$A = \cos^{-1}\left(\frac{b^2 + c^2 - a^2}{2bc}\right) \quad \text{using meaning of inverse cosine}$$

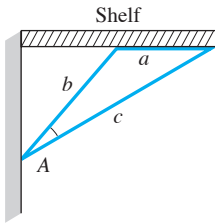


Fig. 20.30

**EXERCISES 20.6**

In Exercises 1–4, make the given changes in the indicated examples of this section and then solve the resulting problems.

1. In Example 1(b), change  $2x$  to  $3A$ .
2. In Example 3(a), change  $1/2$  to  $-1$ .
3. In Example 8(a), change  $0.5$  to  $1$ .
4. In Example 9, change  $\sin$  to  $\cos$ .

In Exercises 5–10, write down the meaning of each of the given equations. See Example 1.

- |                              |                               |
|------------------------------|-------------------------------|
| 5. $y = \cot^{-1} 3x$        | 6. $y = \csc^{-1} 4x$         |
| 7. $y = 2 \sin^{-1} x$       | 8. $y = 3 \tan^{-1} x$        |
| 9. $y = 5 \cos^{-1}(2x - 1)$ | 10. $y = 4 \sin^{-1}(3x + 2)$ |

In Exercises 11–28, evaluate the given expressions exactly.

- |                               |                                |                              |
|-------------------------------|--------------------------------|------------------------------|
| 11. $\cos^{-1} 0.5$           | 12. $\sin^{-1} 1$              | 13. $\tan^{-1} 1$            |
| 14. $\cos^{-1} 2$             | 15. $\tan^{-1}(-\sqrt{3})$     | 16. $\sin^{-1}(-0.5)$        |
| 17. $\sec^{-1} 0.5$           | 18. $\cot^{-1} \sqrt{3}$       | 19. $\sin^{-1}(-\sqrt{2}/2)$ |
| 20. $\cos^{-1}(-\sqrt{3}/2)$  | 21. $\sin(\tan^{-1} \sqrt{3})$ |                              |
| 22. $\tan[\sin^{-1}(2/3)]$    |                                |                              |
| 23. $\cos^{-1}[\cos(-\pi/4)]$ | 24. $\tan^{-1}[\tan(2\pi/3)]$  |                              |
| 25. $\cos[\tan^{-1}(-5)]$     | 26. $\sec[\cos^{-1}(-0.5)]$    |                              |
| 27. $\cos(2 \sin^{-1} 1)$     | 28. $\sin(2 \tan^{-1} 2)$      |                              |

In Exercises 29–32, find the exact value of  $x$ .

- |  |   |
|--|---|
| 29. $\tan^{-1} x = \sin^{-1} \frac{2}{5}$    | 30. $\cot^{-1} x = \cos^{-1} \frac{1}{3}$ |
| 31. $\sec^{-1} x = -\sin^{-1}(-\frac{1}{2})$ | 32. $\sin^{-1} x = -\tan^{-1}(-1)$        |

In Exercises 33–40, use a calculator to evaluate the given expressions.

- |                                |                                |
|--------------------------------|--------------------------------|
| 33. $\tan^{-1}(-3.7321)$       | 34. $\cos^{-1}(-0.6561)$       |
| 35. $\sin^{-1} 0.0219$         | 36. $\tan^{-1} 0.2846$         |
| 37. $\tan[\cos^{-1}(-0.6281)]$ | 38. $\cos[\tan^{-1}(-7.2256)]$ |
| 39. $\sin[\tan^{-1}(-0.2297)]$ | 40. $\tan[\sin^{-1}(-0.3019)]$ |

In Exercises 41–46, solve the given equations for  $x$ .

- |                                |                             |
|--------------------------------|-----------------------------|
| 41. $y = \sin 3x$              | 42. $y = \cos(2x - \pi)$    |
| 43. $y = \tan^{-1}(x/4)$       | 44. $y = 2 \sin^{-1}(x/6)$  |
| 45. $1 - y = \cos^{-1}(1 - x)$ | 46. $2y = \cot^{-1} 3x - 5$ |

In Exercises 47–54, find an algebraic expression for each of the given expressions.

- |                                       |                                       |
|---------------------------------------|---------------------------------------|
| 47. $\tan(\sin^{-1} x)$               | 48. $\sin(\cos^{-1} x)$               |
| 49. $\sin(\sin^{-1} x + \cos^{-1} y)$ | 50. $\cos(\sin^{-1} x - \cos^{-1} y)$ |
| 51. $\sec(\csc^{-1} 3x)$              | 52. $\tan(\sin^{-1} 2x)$              |
| 53. $\sin(2 \sin^{-1} x)$             | 54. $\cos(2 \tan^{-1} x)$             |

In Exercises 55–60, solve the given problems with the use of the inverse trigonometric functions.

55. Is  $\sin^{-1}(\sin x) = x$  for all  $x$ ? Explain.
56. Show that the area  $A$  of a segment of a circle of radius  $r$ , bounded by a chord at a distance  $d$  from the centre, is given by  $A = r^2 \cos^{-1}(d/r) - d\sqrt{r^2 - d^2}$ .
57. In the analysis of ocean tides, the equation  $y = A \cos 2(\omega t + \phi)$  is used. Solve for  $t$ .

58. For an object of weight  $w$  on an inclined plane that is at an angle  $\theta$  to the horizontal, the equation relating  $w$  and  $\theta$  is  $\mu w \cos \theta = w \sin \theta$ , where  $\mu$  is the coefficient of friction between the surfaces in contact. Solve for  $\theta$ .
59. The electric current in a certain circuit is given by  $i = I_m[\sin(\omega t + \alpha) \cos \phi + \cos(\omega t + \alpha) \sin \phi]$ . Solve for  $t$ .
60. The time  $t$  as a function of the displacement  $d$  of a piston is given by  $t = \frac{1}{2\pi f} \cos^{-1} \frac{d}{A}$ . Solve for  $d$ .

In Exercises 61 and 62, prove that the given expressions are equal. Use the relation for  $\sin(\alpha + \beta)$  and show that the sine of the sum of the angles on the left equals the sine of the angle on the right.

61.  $\sin^{-1} \frac{3}{5} + \sin^{-1} \frac{5}{13} = \sin^{-1} \frac{56}{65}$
62.  $\tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{2} = \frac{\pi}{4}$

In Exercises 63–66, evaluate the given expressions.

63.  $\sin^{-1} 0.5 + \cos^{-1} 0.5$       64.  $\tan^{-1} \sqrt{3} + \cot^{-1} \sqrt{3}$
65.  $\sin^{-1} x + \sin^{-1}(-x)$       66.  $\sin^{-1} x + \cos^{-1} x$

In Exercises 67–70, solve for the angle  $A$  for the given triangles in the given figures in terms of the given sides and angles.

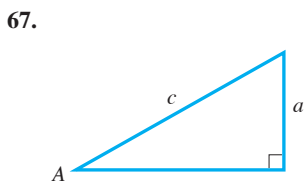


Fig. 20.31

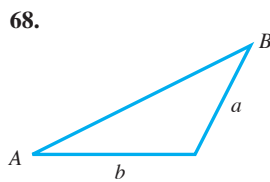


Fig. 20.32

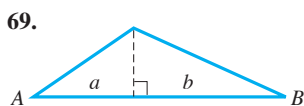


Fig. 20.33

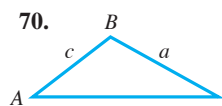


Fig. 20.34

In Exercises 71–74, solve the given problems.

71. The Terry Fox monument in Thunder Bay, Ontario, is a 2.7-m bronze statue that stands on a granite base. From a point at a horizontal distance  $d$  from the monument, the angles of elevation of the top of the statue and the top of the base are  $\alpha$  and  $\beta$ , respectively. Show that
- $$\alpha = \tan^{-1} \left( \frac{2.7}{d} + \tan \beta \right).$$
72. Explain why  $\sin^{-1} 2x$  is not equal to  $2 \sin^{-1} x$ .
73. If a TV camera is  $x$  m from a launch pad of a 50-m rocket that is  $y$  m above the ground, find an expression for  $\theta$ , the angle subtended by the rocket at the camera lens.
74. Show that the length  $L$  of the pulley belt shown in Fig. 20.35 is
- $$L = 24 + 11\pi + 10 \sin^{-1} \frac{5}{13}.$$

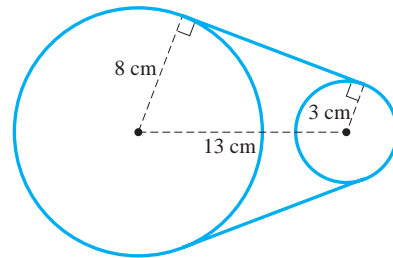


Fig. 20.35

### Answers to Practice Exercises

1.  $-\pi/6$     2.  $\sqrt{3}/2$

## CHAPTER 20 EQUATIONS

### Basic trigonometric identities

$$\sin \theta = \frac{1}{\csc \theta} \quad (20.1)$$

$$\cos \theta = \frac{1}{\sec \theta} \quad (20.2)$$

$$\tan \theta = \frac{1}{\cot \theta} \quad (20.3)$$

$$\tan \theta = \frac{\sin \theta}{\cos \theta} \quad (20.4)$$

$$\cot \theta = \frac{\cos \theta}{\sin \theta} \quad (20.5)$$

$$\sin^2 \theta + \cos^2 \theta = 1 \quad (20.6)$$

$$1 + \tan^2 \theta = \sec^2 \theta \quad (20.7)$$

$$1 + \cot^2 \theta = \csc^2 \theta \quad (20.8)$$

### Sum and difference identities

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta \quad (20.9)$$

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta \quad (20.10)$$

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta \quad (20.11)$$

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta \quad (20.12)$$

$$\tan(\alpha \pm \beta) = \frac{\tan \alpha \pm \tan \beta}{1 \mp \tan \alpha \tan \beta} \quad (20.13)$$

## Double-angle formulas

$$\sin 2\alpha = 2 \sin \alpha \cos \alpha \quad (20.21)$$

$$\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha \quad (20.22)$$

$$= 2 \cos^2 \alpha - 1 \quad (20.23)$$

$$= 1 - 2 \sin^2 \alpha \quad (20.24)$$

$$\tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha} \quad (20.25)$$

## Half-angle formulas

$$\sin \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{2}} \quad (20.26)$$

$$\cos \frac{\alpha}{2} = \pm \sqrt{\frac{1 + \cos \alpha}{2}} \quad (20.27)$$

## Inverse trigonometric functions

$$y = \sin^{-1} x \left( -\frac{\pi}{2} \leq y \leq \frac{\pi}{2} \right) \quad (20.28)$$

$$-\frac{\pi}{2} \leq \sin^{-1} x \leq \frac{\pi}{2} \quad 0 \leq \cos^{-1} x \leq \pi \quad -\frac{\pi}{2} < \tan^{-1} x < \frac{\pi}{2} \quad 0 < \cot^{-1} x < \pi$$

$$0 \leq \sec^{-1} x \leq \pi \quad \left( \sec^{-1} x \neq \frac{\pi}{2} \right) \quad -\frac{\pi}{2} \leq \csc^{-1} x \leq \frac{\pi}{2} \quad (\csc^{-1} x \neq 0) \quad (20.29)$$

## CHAPTER 20 REVIEW EXERCISES

In Exercises 1–8, determine the values of the indicated functions in the given manner.

- Find  $\sin 120^\circ$  by using  $120^\circ = 90^\circ + 30^\circ$ .
- Find  $\cos 30^\circ$  by using  $30^\circ = 90^\circ - 60^\circ$ .
- Find  $\sin 315^\circ$  by using  $315^\circ = 360^\circ - 45^\circ$ .
- Find  $\tan \frac{5\pi}{4}$  by using  $\frac{5\pi}{4} = \pi + \frac{\pi}{4}$ .
- Find  $\cos \pi$  by using  $\pi = 2\left(\frac{\pi}{2}\right)$ .
- Find  $\sin 180^\circ$  by using  $180^\circ = 2(90^\circ)$ .
- Find  $\tan 60^\circ$  by using  $60^\circ = 2(30^\circ)$ .
- Find  $\cos 45^\circ$  by using  $45^\circ = \frac{1}{2}(90^\circ)$ .

In Exercises 9–16, simplify the given expressions by using one of the basic formulas of the chapter. Then use a calculator to verify the result by finding the value of the original expression and the value of the simplified expression.

- $\sin 14^\circ \cos 38^\circ + \cos 14^\circ \sin 38^\circ$
- $\cos^2 148^\circ - \sin^2 148^\circ$
- $2 \sin \frac{\pi}{12} \cos \frac{\pi}{12}$
- $1 - 2 \sin^2 \frac{\pi}{8}$
- $\cos 73^\circ \cos (-142^\circ) + \sin 73^\circ \sin (-142^\circ)$
- $\cos 3^\circ \cos 215^\circ - \sin 3^\circ \sin 215^\circ$
- $\frac{4 \tan 12^\circ}{1 - \tan^2 12^\circ}$
- $\sqrt{\frac{1 - \cos 166^\circ}{2}}$

In Exercises 17–24, simplify each of the given expressions. Expansion of any term is not necessary; recognition of the proper form leads to the proper result.

- $\sin 2x \cos 3x + \cos 2x \sin 3x$
- $\cos 7x \cos 3x + \sin 7x \sin 3x$

- $8 \sin 6x \cos 6x$
- $2 - 4 \sin^2 6x$
- $\sqrt{2 + 2 \cos 2x}$
- $\frac{\tan x + \tan 2x}{1 - \tan x \tan 2x}$
- $\cos^2 2x - \sin^2 2x$
- $\sqrt{32 - 32 \cos 4x}$

In Exercises 25–32, evaluate the given expressions.

- $\sin^{-1}(-1)$
- $\sec^{-1} \sqrt{2}$
- $\cos^{-1} 0.9659$
- $\tan^{-1}(-6.249)$
- $\tan[\sin^{-1}(-0.5)]$
- $\cos[\tan^{-1}(-\sqrt{3})]$
- $\sin^{-1}[\sin(7\pi/6)]$
- $\cos^{-1}[\tan(-\pi/4)]$

In Exercises 33–36, simplify the given expressions.

- $\frac{\sec y}{\cos y} - \frac{\tan y}{\cot y}$
- $\frac{\sin 2\theta}{2 \csc \theta} - \cos^3 \theta$
- $\sin x (\csc x - \sin x)$
- $\cos y (\sec y - \cos y)$

In Exercises 37–44, prove the given identities.

- $\frac{\sec^4 x - 1}{\tan^2 x} = 2 + \tan^2 x$
- $\cos^2 y - \sin^2 y = \frac{1 - \tan^2 y}{1 + \tan^2 y}$
- $2 \csc 2x \cot x = 1 + \cot^2 x$
- $\sin x \cot^2 x = \csc x - \sin x$
- $\frac{1 - \sin^2 \theta}{1 - \cos^2 \theta} = \cot^2 \theta$
- $\frac{\cos 2\theta}{\cos^2 \theta} = 1 - \tan^2 \theta$

43.  $\sin \frac{\theta}{2} \cos \frac{\theta}{2} = \frac{\sin \theta}{2}$

44.  $\cos(x - y) \cos y - \sin(x - y) \sin y = \cos x$

In Exercises 45–52, simplify the given expressions. The result will be one of  $\sin x$ ,  $\cos x$ ,  $\tan x$ ,  $\cot x$ ,  $\sec x$ , or  $\csc x$ .

45.  $\frac{\sec x}{\sin x} - \sec x \sin x$

46.  $\cos x \cot x + \sin x$

47.  $\sin x \tan x + \cos x$

48.  $\frac{\tan x \csc x}{\sin x} - \cot x$

49.  $\frac{\sin x \cot x + \cos x}{2 \cot x}$

50.  $\frac{1 + \cos 2x}{2 \cos x}$

51.  $\frac{\sin 2x \sec x}{2}$

52.  $(\sec x + \tan x)(1 - \sin x)$

In Exercises 53–60, verify each identity by comparing the graph of the left side with the graph of the right side on a graphing calculator.

53.  $\frac{\cos \theta - \sin \theta}{\cos \theta + \sin \theta} = \frac{\cot \theta - 1}{\cot \theta + 1}$

54.  $\sin 3y \cos 2y - \cos 3y \sin 2y = \sin y$

55.  $\sin 4x(\cos^2 2x - \sin^2 2x) = \frac{\sin 8x}{2}$

56.  $\csc 2x + \cot 2x = \cot x$

57.  $\frac{\sin x}{\csc x - \cot x} = 1 + \cos x$

58.  $\cos x - \sin \frac{x}{2} = \left(1 - 2 \sin \frac{x}{2}\right) \left(1 + \sin \frac{x}{2}\right)$

59.  $\tan \frac{\alpha}{2} = \csc \alpha - \cot \alpha$

60.  $\sec \frac{x}{2} + \csc \frac{x}{2} = \frac{2(\sin \frac{x}{2} + \cos \frac{x}{2})}{\sin x}$

In Exercises 61–64, solve for  $x$ .

61.  $y = 2 \cos 2x$

62.  $y - 2 = 2 \tan\left(x - \frac{\pi}{2}\right)$

63.  $y = \frac{\pi}{4} - 3 \sin^{-1} 5x$

64.  $2y = \sec^{-1} 4x - 2$

In Exercises 65–76, solve the given equations for  $x$  ( $0 \leq x < 2\pi$ ).

65.  $3(\tan x - 2) = 1 + \tan x$

66.  $5 \sin x = 3 - (\sin x + 2)$

67.  $2(1 - 2 \sin^2 x) = 1$

68.  $\sec x = 2 \tan^2 x$

69.  $2 \sin^2 \theta + 3 \cos \theta - 3 = 0$

70.  $2 \sin 2x + 1 = 0$

71.  $\sin x = \sin \frac{x}{2}$

72.  $\cos 2x = \sin(-x)$

73.  $\sin 2x = \cos 3x$

74.  $\cos 3x \cos x + \sin 3x \sin x = 0$

75.  $\sin^2\left(\frac{x}{2}\right) - \cos x + 1 = 0$

76.  $\sin x + \cos x = 1$

In Exercises 77–80, determine whether the equality is an identity or an equation. If it is an identity, prove it. If it is an equation, solve it for  $0 \leq x < 2\pi$ .

77.  $\tan x + \cot x = \csc x \sec x$

78.  $\tan x - \sin^2 x = \cos^2 x - \sec^2 x$

79.  $\sin x \cos x - 1 = \cos x - \sin x$

80.  $2 \tan x = \sin 2x \sec^2 x$

In Exercises 81–84, solve the given equations graphically.

81.  $x + \ln x - 3 \cos^2 x = 2$

82.  $e^{\sin x} - 2 = x \cos^2 x$

83.  $2 \tan^{-1} x + x^2 = 3$

84.  $3 \sin^{-1} x = 6 \sin x + 1$

In Exercises 85–90, find an algebraic expression for each of the given expressions.

85.  $\tan(\cot^{-1} x)$

86.  $\cos(\csc^{-1} x)$

87.  $\sin(2 \cos^{-1} x)$

88.  $\cos(\pi - \tan^{-1} x)$

89.  $\cos(\sin^{-1} x + \tan^{-1} y)$

90.  $\sin(\cos^{-1} x - \tan^{-1} y)$

In Exercises 91–94, use the given substitutions to show that the equations are valid for  $0 \leq \theta < \pi/2$ .

91. If  $x = 2 \cos \theta$ , show that  $\sqrt{4 - x^2} = 2 \sin \theta$ .

92. If  $x = 2 \sec \theta$ , show that  $\sqrt{x^2 - 4} = 2 \tan \theta$ .

93. If  $x = \tan \theta$ , show that  $\frac{x}{\sqrt{1 + x^2}} = \sin \theta$ .

94. If  $x = \cos \theta$ , show that  $\frac{\sqrt{1 - x^2}}{x} = \tan \theta$ .

In Exercises 95–114, use the methods and formulas of this chapter to solve the given problems.

95. Prove  $(\cos \theta + j \sin \theta)^2 = \cos 2\theta + j \sin 2\theta$  ( $j = \sqrt{-1}$ ).

96. Prove  $(\cos \theta + j \sin \theta)^3 = \cos 3\theta + j \sin 3\theta$  ( $j = \sqrt{-1}$ ).

97. Solve the inequality  $\sin 2x > 2 \cos x$  for  $0 \leq x < 2\pi$ .

98. Show that  $(\cos 2\alpha + \sin^2 \alpha) \sec^2 \alpha$  has a constant value.

99. Show that  $y = A \sin 2t + B \cos 2t$  may be written as  $y = C \sin(2t + \alpha)$ , where  $C = \sqrt{A^2 + B^2}$  and  $\tan \alpha = B/A$ . (Hint: Let  $A/C = \cos \alpha$  and  $B/C = \sin \alpha$ .)

100. In a right triangle with sides  $a$  and  $b$  and hypotenuse  $c$ , show that  $\sin(A/2) = \sqrt{(c - b)/2c}$ , where angle  $A$  is opposite  $a$ .

101. Forces  $\mathbf{A}$  and  $\mathbf{B}$  act on a bolt such that  $\mathbf{A}$  makes an angle  $\theta$  with the  $x$ -axis and  $\mathbf{B}$  makes an angle  $\theta$  with the  $y$ -axis as shown in Fig. 20.36. The resultant  $\mathbf{R}$  has components  $R_x = A \cos \theta - B \sin \theta$  and  $R_y = A \sin \theta + B \cos \theta$ . Using these components, show that  $R = \sqrt{A^2 + B^2}$ .

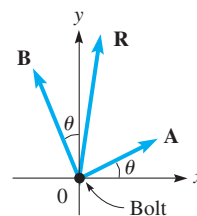


Fig. 20.36

102. For a certain alternating-current generator, the expression  $I \cos \theta + I \cos(\theta + 2\pi/3) + I \cos(\theta + 4\pi/3)$  arises. Simplify this expression.

103. Some comets follow a parabolic path that can be described by the equation  $r = (k/2) \csc^2(\theta/2)$ , where  $r$  is the distance to the sun and  $k$  is a constant. Show that this equation can be written as  $r = k/(1 - \cos \theta)$ .
104. In studying the interference of light waves, the identity  $\frac{\sin \frac{3}{2}x}{\sin \frac{1}{2}x} \sin x = \sin x + \sin 2x$  is used. Prove this identity. (Hint:  $\sin \frac{3}{2}x = \sin(x + \frac{1}{2}x)$ .)
105. In the study of chemical spectroscopy, the equation  $\omega t = \sin^{-1} \frac{\theta - \alpha}{R}$  arises. Solve for  $\theta$ .
106. The power  $P$  in a certain electric circuit is given by  $P = 2.5[\cos \alpha \sin(\omega t + \phi) - \sin \alpha \cos(\omega t + \phi)]$ . Solve for  $t$ .
107. In surveying, when determining an azimuth (a measure used for reference purposes), it might be necessary to simplify the expression  $(2 \cos \alpha \cos \beta)^{-1} - \tan \alpha \tan \beta$ . Perform this operation by expressing it in the simplest possible form when  $\alpha = \beta$ .
108. In analysing the motion of an automobile universal joint, the equation  $\sec^2 A - \sin^2 B \tan^2 A = \sec^2 C$  is used. Show that this equation is true if  $\tan A \cos B = \tan C$ .
109. The instantaneous power  $P$  in a certain electric circuit is given by  $P = VI \cos \phi \cos^2 \omega t - VI \sin \phi \cos \omega t \sin \omega t$ . Simplify this expression.
110. In studying waveforms, a sawtooth wave may often be approximated by  $y = \sin \pi x + \frac{1}{2} \sin 2\pi x + \frac{1}{3} \sin 3\pi x$ . For what values of  $x$ ,  $0 \leq x < 2$ , is the sum of the first two terms equal to zero?
111. A roof truss is in the shape of an isosceles triangle of height 3.2 m. If the total length of the three members is 25.6 m, what is the length of a rafter? Solve by setting up and solving an appropriate trigonometric equation. (The Pythagorean theorem may be used only as a check.)
112. The angle of elevation of the top of the Peace Tower on Parliament Hill in Ottawa from a point on level ground 77 m from the tower is twice the angle of elevation of the top from a point 120 m farther away. Find the height of the Peace Tower.
113. If a plane surface inclined at angle  $\theta$  moves horizontally, the angle for which the lifting force of the air is a maximum is found by solving the equation  $2 \sin \theta \cos^2 \theta - \sin^3 \theta = 0$ , where  $0 < \theta < 90^\circ$ . Solve for  $\theta$ .
114. To determine the angle between two sections of a certain robot arm, the equation  $1.20 \cos \theta + 0.135 \cos 2\theta = 0$  is to be solved. Find the required angle  $\theta$  if  $0^\circ < \theta < 180^\circ$ .

### Writing Exercise

115. In checking the angles of a section of a bridge support, an engineer finds the expression  $\cos(2 \sin^{-1} 0.40)$ . Write a paragraph explaining how the value of this expression can be found without the use of a calculator.

## CHAPTER 20 PRACTICE TEST

- Prove that  $\sec \theta - \frac{\tan \theta}{\csc \theta} = \cos \theta$ .
- Solve for  $x$  ( $0 \leq x < 2\pi$ ) analytically, using trigonometric relations where necessary:  $\sin 2x + \sin x = 0$ .
- Find an algebraic expression for  $\cos(\sin^{-1} x)$ .
- The electric current as a function of the time for a particular circuit is given by  $i = 8.00e^{-20t}(1.73 \cos 10.0t - \sin 10.0t)$ . Find the time (in s) when the current is first zero.
- Prove that  $\frac{\tan \alpha + \tan \beta}{\tan \alpha - \tan \beta} = \frac{\sin(\alpha + \beta)}{\sin(\alpha - \beta)}$ .
- Prove that  $\cot^2 x - \cos^2 x = \cot^2 x \cos^2 x$ .
- Find  $\cos \frac{1}{2}x$  if  $\sin x = -\frac{3}{5}$  and  $270^\circ < x < 360^\circ$ .
- The intensity of a certain type of polarized light is given by  $I = I_0 \sin 2\theta \cos 2\theta$ . Solve for  $\theta$ .
- Solve graphically:  $x - 2 \cos x = 5$ .
- Find the exact value of  $x$ :  $\cos^{-1} x = -\tan^{-1}(-1)$ .

# 21

## LEARNING OUTCOMES

After completion of this chapter, the student should be able to:

- Find the distance between two points in the coordinate plane
- Find the slope and the inclination of a line
- Understand the relationship between the slopes of parallel and perpendicular lines
- Determine the equation and properties of a circle, a parabola, an ellipse, and a hyperbola
- Sketch the graph of a line, a circle, a parabola, an ellipse, or a hyperbola
- Find the equation of a conic from the definition of its locus
- Identify the conic section from the general second-degree equation
- Recognize conic sections as intersections of planes and cones
- Obtain a new coordinate system by translation and/or rotation of axes
- Convert from rectangular coordinates to polar coordinates and vice versa
- Graph functions in polar coordinates
- Use algebra to solve geometric problems
- Solve application problems involving lines, conics, and polar coordinates

# Plane Analytic Geometry

The development of geometry made little progress from the time of the ancient Greeks until the 1600s. Then in the 1630s, two French mathematicians, Rene Descartes and Pierre de Fermat, independently introduced algebra into the study of geometry. Each used algebraic notation and a coordinate system, and this allowed them to analyse the properties of curves. Fermat briefly indicated his methods in a letter he wrote in 1636. However, in 1637 Descartes included a much more complete work as an appendix to his *Discourse on Method*, and he is therefore now considered the founder of *analytic geometry*.

At the time, it was known, for example, that a projectile follows a parabolic path and that the orbits of the planets are ellipses. There was a renewed interest in curves of various kinds because of their applications. However, Descartes was primarily interested in studying the relationship of algebra to geometry. In doing so, he started the development of analytic geometry, which has many applications and also was very important in the invention of calculus that came soon thereafter. In turn, through these advances in mathematics, many more areas of science and technology were able to be developed.

The underlying principle of analytic geometry is the relationship of an algebraic equation and the geometric properties of the curve that represents the equation. In this chapter, we develop equations for a number of important curves and find their properties through an analysis of their equations. Most important among these curves are the *conic sections*, which we briefly introduced in Chapter 14.

As we have noted, analytic geometry has applications in the study of projectile motion and planetary orbits. Other important applications range from the design of gears, airplane wings, and automobile headlights to the construction of bridges and nuclear towers.

▼ In Section 21.4, we show an important application of analytic geometry in the design of a TV satellite dish.





## 21.1 Basic Definitions

The Distance Formula • The Slope of a Line • Inclination • Parallel Lines • Perpendicular Lines

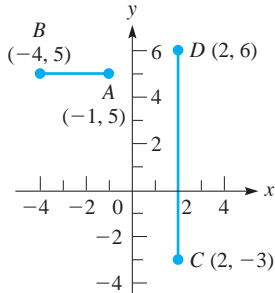


Fig. 21.1

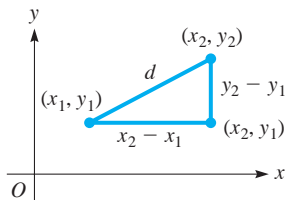


Fig. 21.2

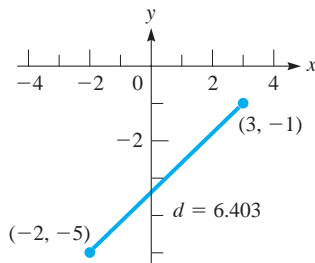


Fig. 21.3

### LEARNING TIP

In Eq. (21.1), it makes no difference which point is chosen as  $(x_1, y_1)$  and which is chosen as  $(x_2, y_2)$ , since the differences in the  $x$ -coordinates and the  $y$ -coordinates are squared. For instance, in Example 2, we obtain the same value for the distance when we calculate it as

$$\begin{aligned} d &= \sqrt{[3 - (-2)]^2 + [(-1) - (-5)]^2} \\ &= \sqrt{5^2 + 4^2} = \sqrt{41} = 6.40 \end{aligned}$$

### Practice Exercise

- Find the distance between  $(-2, 6)$  and  $(5, -3)$ .

As we have noted, analytic geometry deals with the relationship between an algebraic equation and the geometric curve it represents. In this section, we develop certain basic concepts that will be needed for future use in establishing the proper relationships between an equation and a curve.

### THE DISTANCE FORMULA

The first of these concepts involves the distance between any two points in the coordinate plane. If these points lie on a line parallel to the  $x$ -axis, the distance from the first point  $(x_1, y)$  to the second point  $(x_2, y)$  is  $|x_2 - x_1|$ . The absolute value is used since we are interested only in the magnitude of the distance. Therefore, we could also denote the distance as  $|x_1 - x_2|$ . Similarly, the distance between two points  $(x, y_1)$  and  $(x, y_2)$  that lie on a line parallel to the  $y$ -axis is  $|y_2 - y_1|$  or  $|y_1 - y_2|$ .

#### EXAMPLE 1 Distance between points that share one coordinate

The line segment joining  $A(-1, 5)$  and  $B(-4, 5)$  in Fig. 21.1 is parallel to the  $x$ -axis. Therefore, the distance between these points is

$$d = |-4 - (-1)| = 3 \quad \text{or} \quad d = |-1 - (-4)| = 3$$

Also, in Fig. 21.1, the line segment joining  $C(2, -3)$  and  $D(2, 6)$  is parallel to the  $y$ -axis. The distance  $d$  between these points is

$$d = |6 - (-3)| = 9 \quad \text{or} \quad d = |-3 - 6| = 9$$

We now wish to find the length of a line segment joining any two points in the plane. If these points are on a line that is not parallel to either axis (see Fig. 21.2), we use the Pythagorean theorem to find the distance between them. By making a right triangle with the line segment joining the points as the hypotenuse and line segments parallel to the axes as legs, we have the **distance formula**, which gives the distance between any two points in the plane. This formula is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \quad (21.1)$$

Here, we choose the positive square root since we are concerned only with the magnitude of the length of the line segment.

#### EXAMPLE 2 Distance formula

The distance between  $(3, -1)$  and  $(-2, -5)$  is given by

$$\begin{aligned} d &= \sqrt{[(-2) - 3]^2 + [(-5) - (-1)]^2} \\ &= \sqrt{(-5)^2 + (-4)^2} = \sqrt{25 + 16} \\ &= \sqrt{41} = 6.40 \end{aligned}$$

See Fig. 21.3.

### THE SLOPE OF A LINE

Another important quantity for a line is its *slope*, which we defined in Chapter 5. Here, we review its definition and develop its meaning in more detail.

The **slope** of a line through two points is defined as the difference in the  $y$ -coordinates (rise) divided by the difference in the  $x$ -coordinates (run). Therefore, the slope,



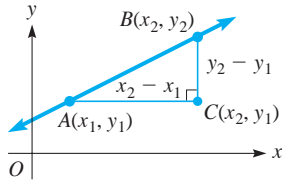


Fig. 21.4

$m$ , which gives a measure of the direction of a line, is defined as

$$m = \frac{y_2 - y_1}{x_2 - x_1} \tag{21.2}$$

See Fig. 21.4. ■

**LEARNING TIP**

Again, in Eq. (21.2) we may interpret either of the points as  $(x_1, y_1)$  and the other as  $(x_2, y_2)$ , although we *must be careful to place the  $y_2 - y_1$  in the numerator and  $x_2 - x_1$  in the denominator.*

When the line is horizontal,  $y_2 - y_1 = 0$  and  $m = 0$ . When the line is vertical,  $x_2 - x_1 = 0$ , and the slope is undefined.

**LEARNING TIP**

The larger the numerical value of the slope of a line, the more nearly vertical is the line. Also, a line rising to the right has a positive slope, and a line falling to the right has a negative slope.

**EXAMPLE 3 Slope of a line**

The slope of a line through  $(3, -5)$  and  $(-2, -6)$  is

$$m = \frac{-6 - (-5)}{-2 - 3} = \frac{-6 + 5}{-5} = \frac{1}{5}$$

See Fig. 21.5. Again, we may interpret either of the points as  $(x_1, y_1)$  and the other as  $(x_2, y_2)$ . We can also obtain the slope of this same line from

$$m = \frac{-5 - (-6)}{3 - (-2)} = \frac{1}{5}$$

**EXAMPLE 4 Magnitude and sign of the slope**

(a) The line in Example 3 has a positive slope, which is numerically small. From Fig. 21.5, it can be seen that the line rises slightly to the right.

(b) The line joining  $(3, 4)$  and  $(4, -6)$  has a slope of

$$m = \frac{4 - (-6)}{3 - 4} = -10 \quad \text{or} \quad m = \frac{-6 - 4}{4 - 3} = -10$$

This line falls sharply to the right, as shown in Fig. 21.6. ■

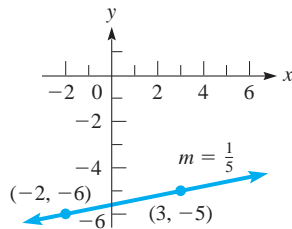


Fig. 21.5

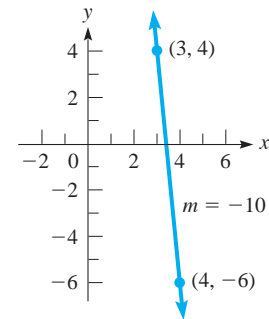


Fig. 21.6

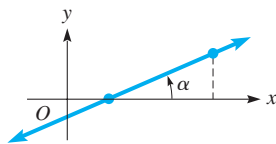


Fig. 21.7

If a given line is extended indefinitely in either direction, it must cross the  $x$ -axis at some point unless it is parallel to the  $x$ -axis. The angle measured from the  $x$ -axis in a positive direction to the line is called the **inclination** of the line (see Fig. 21.7). The inclination of a line parallel to the  $x$ -axis is defined to be zero. An alternate definition of slope, in terms of the inclination  $\alpha$ , is

$$m = \tan \alpha \quad (0^\circ \leq \alpha < 180^\circ) \tag{21.3}$$

Since the slope can be defined in terms of any two points on the line, we can choose the  $x$ -intercept and any other point. Therefore, from the definition of the tangent of an angle, we see that Eq. (21.3) is in agreement with Eq. (21.2).

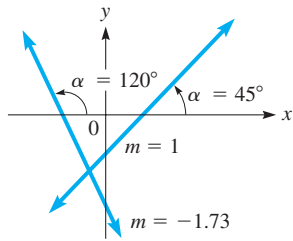


Fig. 21.8

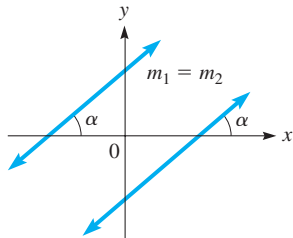


Fig. 21.9

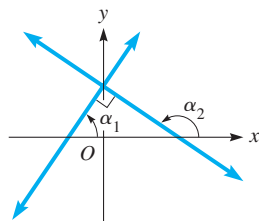


Fig 21.10

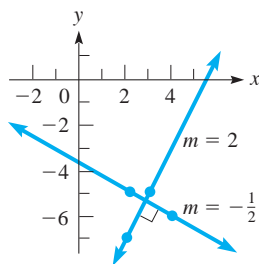


Fig. 21.11

**Practice Exercise**

2. What is the slope of a line perpendicular to the line through  $(-1, 6)$  and  $(-3, -3)$ ?

**EXAMPLE 5** Inclination of a line

- (a) The slope of a line with an inclination of  $45^\circ$  is

$$m = \tan 45^\circ = 1.00$$

- (b) If a line has a slope of  $-1.73$ , we know that  $\tan \alpha = -1.73$ . Since  $\tan \alpha$  is negative,  $\alpha$  must be a second-quadrant angle. The reference angle is  $\tan^{-1} 1.73 = 60^\circ$ , and therefore  $\alpha = 180^\circ - 60^\circ = 120^\circ$ . See Fig. 21.8.

We see that if the inclination is an acute angle, the slope is positive and the line rises to the right. If the inclination is negative, the slope is negative and the line falls to the right. ■

Any two parallel lines crossing the  $x$ -axis have the same inclination. Therefore, as shown in Fig. 21.9, the *slopes of parallel lines are equal*. This can be stated as

$$m_1 = m_2 \quad (\text{for } \parallel \text{ lines}) \quad (21.4)$$

If two lines are perpendicular, this means that there must be  $90^\circ$  between their inclinations (Fig. 21.10). The relation between their inclinations is

$$\begin{aligned} \alpha_2 &= \alpha_1 + 90^\circ \\ 90^\circ - \alpha_2 &= -\alpha_1 \end{aligned}$$

If neither line is vertical (the slope of a vertical line is undefined) and we take the tangent in this last relation, we have

$$\tan(90^\circ - \alpha_2) = \tan(-\alpha_1)$$

or

$$\cot \alpha_2 = -\tan \alpha_1$$

since a function of the complement of an angle equals the cofunction of that angle (see page 128) and since  $\tan(-\alpha) = -\tan \alpha$  (see page 247). But  $\cot \alpha = 1/\tan \alpha$ , which means  $1/\tan \alpha_2 = -\tan \alpha_1$ . Using the inclination definition of slope, we have *as the relation between slopes of perpendicular lines*,

$$m_2 = -\frac{1}{m_1} \text{ or } m_1 m_2 = -1 \quad \text{for } \perp \text{ lines} \quad (21.5)$$

**EXAMPLE 6** Perpendicular lines

The line through  $(3, -5)$  and  $(2, -7)$  has a slope of

$$m_1 = \frac{-5 + 7}{3 - 2} = 2$$

The line through  $(4, -6)$  and  $(2, -5)$  has a slope of

$$m_2 = \frac{-6 - (-5)}{4 - 2} = -\frac{1}{2}$$

Since the slopes of the two lines are negative reciprocals, we know that the lines are perpendicular. See Fig. 21.11. ■

Using the formulas for distance and slope, we can show certain basic geometric relationships. The following examples illustrate the use of the formulas and thereby show the use of algebra in solving problems that are basically geometric. They illustrate the methods of analytic geometry.

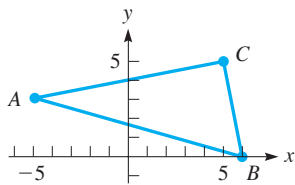


Fig. 21.12

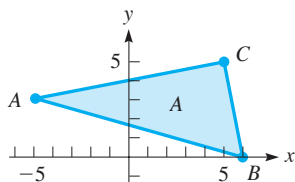


Fig. 21.13

### EXAMPLE 7 Use of algebra in geometric problems

- (a) Show that the line segments joining  $A(-5, 3)$ ,  $B(6, 0)$ , and  $C(5, 5)$  form a right triangle. See Fig. 21.12.

If these points are vertices of a right triangle, the slopes of two of the sides must be negative reciprocals. This would show perpendicularity. These slopes are

$$m_{AB} = \frac{3 - 0}{-5 - 6} = -\frac{3}{11} \quad m_{AC} = \frac{3 - 5}{-5 - 5} = \frac{1}{5} \quad m_{BC} = \frac{0 - 5}{6 - 5} = -5$$

We see that the slopes of  $AC$  and  $BC$  are negative reciprocals, which means  $AC \perp BC$ . From this we conclude that the triangle is a right triangle.

- (b) Find the area of the triangle in part (a). See Fig. 21.13.

Since the right angle is at  $C$ , the legs of the triangle are  $AC$  and  $BC$ . The area is one-half the product of the lengths of the legs of a right triangle. The lengths of the legs are

$$d_{AC} = \sqrt{(-5 - 5)^2 + (3 - 5)^2} = \sqrt{104} = 2\sqrt{26}$$

$$d_{BC} = \sqrt{(6 - 5)^2 + (0 - 5)^2} = \sqrt{26}$$

Therefore, the area is  $A = \frac{1}{2}(2\sqrt{26})(\sqrt{26}) = 26$ . ■

## EXERCISES 21.1

In Exercises 1–4, make the given changes in the indicated examples of this section and then solve the resulting problems.

- In Example 2, change  $(-2, -5)$  to  $(-2, 5)$ .
- In Example 3, change  $(3, -5)$  to  $(-3, -5)$ .
- In Example 5(b), change  $-1.73$  to  $-0.577$ .
- In Example 7(a), change  $B(6, 0)$  to  $B(-4, -2)$ .

In Exercises 5–14, find the distance between the given pairs of points.

- $(3, 8)$  and  $(-1, -2)$
- $(-1, 3)$  and  $(-8, -4)$
- $(4, -5)$  and  $(4, -8)$
- $(-3, 7)$  and  $(2, 10)$
- $(-12, 20)$  and  $(32, -13)$
- $(23, -9)$  and  $(-25, 11)$
- $(\sqrt{32}, -\sqrt{18})$  and  $(-\sqrt{50}, \sqrt{8})$
- $(e, -\pi)$  and  $(-2e, -\pi)$
- $(1.22, -3.45)$  and  $(-1.07, -5.16)$
- $(a, h^2)$  and  $(a + h, (a + h)^2)$

In Exercises 15–24, find the slopes of the lines through the points in Exercises 5–14.

In Exercises 25–28, find the slopes of the lines with the given inclinations.

- $30^\circ$
- $62.5^\circ$
- $163^\circ$
- $93.5^\circ$

In Exercises 29–32, find the inclinations of the lines with the given slopes.

- 0.364
- 0.824
- 6.69
- 0.721

In Exercises 33–36, determine whether the lines through the two pairs of points are parallel or perpendicular.

- $(6, -1)$  and  $(4, 3)$ ;  $(-5, 2)$  and  $(-7, 6)$
- $(-3, 9)$  and  $(4, 4)$ ;  $(9, -1)$  and  $(4, -8)$
- $(-1, -4)$  and  $(2, 3)$ ;  $(-5, 2)$  and  $(-19, 8)$
- $(-a, -2b)$  and  $(3a, 6b)$ ;  $(2a, -6b)$  and  $(5a, 0)$

In Exercises 37–40, determine the value of  $k$ .

- The distance between  $(-1, 3)$  and  $(11, k)$  is 13.
- The distance between  $(k, 0)$  and  $(0, 2k)$  is 10.
- Points  $(6, -1)$ ,  $(3, k)$ , and  $(-3, -7)$  are on the same line.
- The points in Exercise 39 are the vertices of a right triangle, with the right angle at  $(3, k)$ .

In Exercises 41–44, show that the given points are vertices of the given geometric figures.

- $(2, 3)$ ,  $(4, 9)$ , and  $(-2, 7)$  are the vertices of an isosceles triangle.
- $(-1, 3)$ ,  $(3, 5)$ , and  $(5, 1)$  are the vertices of a right triangle.

43.  $(-5, -4)$ ,  $(7, 1)$ ,  $(10, 5)$ , and  $(-2, 0)$  are the vertices of a parallelogram.

44.  $(-5, 6)$ ,  $(0, 8)$ ,  $(-3, 1)$ , and  $(2, 3)$  are the vertices of a square.

In Exercises 45–48, find the indicated areas and perimeters.

45. Find the area of the triangle in Exercise 42.

46. Find the area of the square in Exercise 44.

47. Find the perimeter of the triangle in Exercise 41.

48. Find the perimeter of the parallelogram in Exercise 43.

In Exercises 49–52, use the following definition to find the midpoints between the given points on a straight line.

The *midpoint* between points  $(x_1, y_1)$  and  $(x_2, y_2)$  on a straight line is the point

$$\left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

49.  $(-4, 9)$  and  $(6, 1)$

50.  $(-1, 6)$  and  $(-13, -8)$

51.  $(-12.4, 25.7)$  and  $(6.8, -17.3)$

52.  $(2.6, 5.3)$  and  $(-4.2, -2.7)$

In Exercises 53–62, solve the given problems.

53. Find the relation between  $x$  and  $y$  such that  $(x, y)$  is always 3 units from the origin.

54. Find the relation between  $x$  and  $y$  such that  $(x, y)$  is always equidistant from the  $y$ -axis and  $(2, 0)$ .

55. Show that the diagonals of a square are perpendicular to each other. (*Hint:* Use  $(0, 0)$ ,  $(a, 0)$ , and  $(0, a)$  as three of the vertices.)

56. The centre of a circle is  $(2, -3)$ , and one end of a diameter is  $(-1, 2)$ . What are the coordinates of the other end of the diameter?

57. A line segment has a slope of 3 and one endpoint at  $(-2, 5)$ . If the other endpoint is on the  $x$ -axis, what are its coordinates?

58. The points  $(-1, 3)$ ,  $(5, x)$ , and  $(2, 4)$  are collinear (on the same line). Find  $x$ .

59. Find the coordinates of the point on the  $y$ -axis that is equidistant from  $(-3, -5)$  and  $(2, 4)$ .

60. The points  $(-6, 2)$ ,  $(5, -1)$ , and  $(a, a)$  ( $a > 0$ ) are vertices of a right triangle, with the right angle at  $(a, a)$ . Find  $a$ .

61. An equilateral triangle has one vertex at the origin, another vertex on the  $x$ -axis, and the third vertex in the first quadrant. Find the slopes of the sides of the triangle.

62. A person is working out on a treadmill inclined at 12% (the slope of the treadmill expressed in percent). What is the angle between the treadmill and the horizontal?

#### Answers to Practice Exercises

1.  $d = \sqrt{130} = 11.4$     2.  $m = -2/9$

## 21.2 The Straight Line

Point-Slope Form of Equation •  
Vertical Line • Horizontal Line •  
Slope-Intercept Form of Equation •  
General Form of Equation

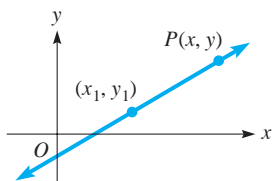


Fig. 21.14

In Chapter 5, we derived the *slope-intercept form* of the equation of the straight line. Here, we extend the development to include other forms of the equation of a straight line. Also, other methods of finding and applying these equations are shown. For completeness, we review some of the material in Chapter 5.

Using the definition of slope, we can derive the general type of equation that represents a straight line. This is another basic method of analytic geometry. That is, equations of a particular form can be shown to represent a particular type of curve. When we recognize the form of the equation, we know the kind of curve it represents. As we have seen, this is of great assistance in sketching the graph.

A straight line can be defined as a *curve* with a constant slope. This means that the value for the slope is the same for any two different points on the line that might be chosen. Thus, considering point  $(x_1, y_1)$  on a line to be fixed (Fig. 21.14) and another point  $P(x, y)$  that *represents* any other point on the line, we have

$$m = \frac{y - y_1}{x - x_1}$$

which can be written as

$$y - y_1 = m(x - x_1) \quad (21.6)$$

Eq. (21.6) is the **point-slope form** of the equation of a straight line. It is useful when we know the slope of a line and some point through which the line passes.

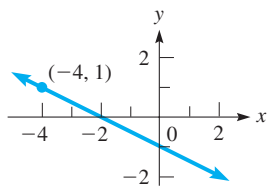


Fig. 21.15

**EXAMPLE 1 Point-slope form**

Find the equation of the line that passes through  $(-4, 1)$  with a slope of  $-1/2$ . See Fig. 21.15.

Substituting in Eq. (21.6), we find that

$$y - 1 = \left(-\frac{1}{2}\right) [x - (-4)]$$

↑ slope  
↑ coordinates

Simplifying, we have

$$2y - 2 = -x - 4$$

$$2y + x + 2 = 0$$

**EXAMPLE 2 Equation of a line through two points**

Find the equation of the line through  $(2, -1)$  and  $(6, 2)$ .

We first find the slope of the line through these points:

$$m = \frac{2 + 1}{6 - 2} = \frac{3}{4}$$

Then by using either of the two known points and Eq. (21.6), we can find the equation of the line (see Fig. 21.16):

$$y - (-1) = \frac{3}{4}(x - 2)$$

$$4y + 4 = 3x - 6$$

$$4y - 3x + 10 = 0$$

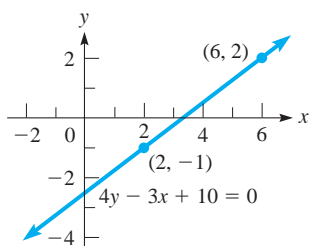


Fig. 21.16

Eq. (21.6) can be used for any line except for one parallel to the  $y$ -axis. Such a line has an undefined slope. However, it does have the property that all points on it have the same  $x$ -coordinate, regardless of the  $y$ -coordinate. We represent a line parallel to the  $y$ -axis as

$x = a$     see Fig. 21.17    (21.7)

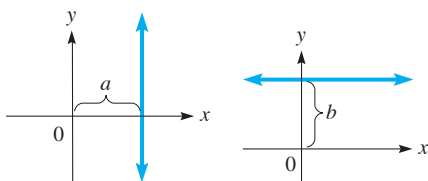


Fig. 21.17

Fig. 21.18

A line parallel to the  $x$ -axis has a slope of zero. From Eq. (21.6), we find its equation is  $y = y_1$ . To keep the same form as Eq. (21.7), we write this as

$y = b$     see Fig. 21.18    (21.8)

**EXAMPLE 3 Vertical line—horizontal line**

(a) The line  $x = 2$  is a line parallel to the  $y$ -axis and 2 units to the right of it. This line is shown in Fig. 21.19.

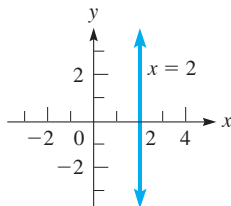


Fig. 21.19

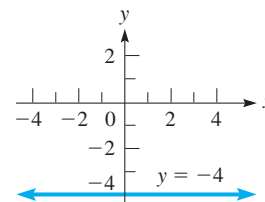


Fig. 21.20

(b) The line  $y = -4$  is a line parallel to the  $x$ -axis and 4 units below it. This line is shown in Fig. 21.20.

**Practice Exercise**

- Find the equation of the straight line that passes through  $(-5, 2)$  and  $(-1, -6)$ .

If we choose the special point  $(0, b)$ , which is the  $y$ -intercept of the line, as the point to use in Eq. (21.6), we have  $y - b = m(x - 0)$ , or

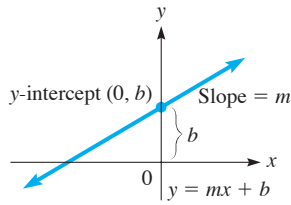


Fig. 21.21

$$y = mx + b \quad (21.9)$$

Eq. (21.9) is the **slope-intercept form** of the equation of a straight line, and we first derived it in Chapter 5. Its primary usefulness lies in the fact that once we find the equation of a line and then write it in slope-intercept form, we know that the slope of the line is the coefficient of the  $x$ -term and that it crosses the  $y$ -axis at the coordinate indicated by the constant term. See Fig. 21.21.

#### EXAMPLE 4 Slope-intercept form

Find the slope and the  $y$ -intercept of the straight line for which the equation is  $2y + 4x - 5 = 0$ .

We write this equation in slope-intercept form:

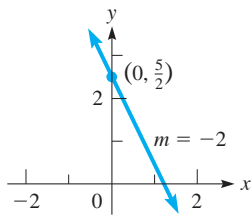


Fig. 21.22

$$\begin{aligned} 2y &= -4x + 5 \\ \text{slope} &\quad \quad \quad \text{y-coordinate of intercept} \\ y &= -2x + \frac{5}{2} \end{aligned}$$

Since the coefficient of  $x$  in this form is  $-2$ , the slope is  $-2$ . The constant on the right is  $5/2$ , which means that the  $y$ -intercept is  $(0, 5/2)$ . See Fig. 21.22. ■

#### EXAMPLE 5 Straight line—application

The pressure  $p_0$  at the surface of a body of water (due to the atmosphere) is 101 kPa. The pressure  $p$  at a depth of 10.0 m is 199 kPa. In general, the pressure difference  $p - p_0$  below the surface varies directly as the depth  $h$ . Find the equation of  $p$  as a function of  $h$ , and sketch its graph.

The solution is as follows:

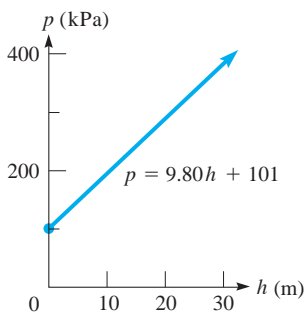


Fig. 21.23

$$\begin{aligned} p - p_0 &= kh && \text{direct variation} \\ 199 - 101 &= k(10.0) && \text{substitute given values} \\ k &= 9.80 \text{ kPa/m} \\ p - 101 &= 9.80h && \text{substitute in first equation} \\ p &= 9.80h + 101 \end{aligned}$$

We see that this is the equation of a straight line. The slope is 9.80, and the  $p$ -intercept is  $(0, 101)$ . Negative values do not have any physical meaning. The graph is shown in Fig. 21.23. ■

From Eqs. (21.6) and (21.9), and from the examples of this section, we see that the equation of the straight line has certain characteristics: We have a term in  $y$ , a term in  $x$ , and a constant term if we simplify as much as possible. *This form is represented by the equation*

$$Ax + By + C = 0 \quad (21.10)$$

■ In Eq. (21.10),  $A$  and  $B$  cannot both be zero.

which is known as the **general form** of the equation of the straight line. We saw this form before in Chapter 5. Now we have shown why it represents a straight line.

**EXAMPLE 6** Equation of a line in general form

Find the general form of the equation of the line parallel to the line  $3x + 2y - 6 = 0$  and that passes through the point  $(-1, 2)$ .

Since the line whose equation we want is parallel to the line  $3x + 2y - 6 = 0$ , it has the same slope. Thus, writing  $3x + 2y - 6 = 0$  in slope-intercept form,

$$2y = -3x + 6 \quad \text{solving for } y$$

$$y = -\frac{3}{2}x + 3$$

Since the slope of  $3x + 2y - 6 = 0$  is  $-3/2$ , the slope of the required line is also  $-3/2$ . Using  $m = -3/2$ , the point  $(-1, 2)$ , and the point-slope form, we have

$$y - 2 = -\frac{3}{2}(x + 1)$$

$$2y - 4 = -3(x + 1)$$

$$3x + 2y - 1 = 0$$

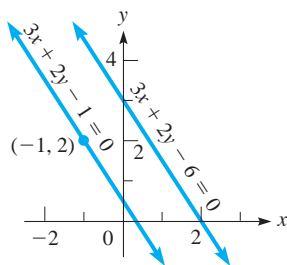


Fig. 21.24

This is the general form of the equation. Both lines are shown in Fig. 21.24. ■

In many physical situations, a linear relationship exists between variables. A few examples of this are (1) the distance travelled by an object and the elapsed time, when the velocity is constant, (2) the amount a spring stretches and the force applied, (3) the change in electric resistance and the change in temperature, (4) the force applied to an object and the resulting acceleration, and (5) the pressure at a certain point within a liquid and the depth of the point.

**EXAMPLE 7** Straight line—application

For a period of 6.0 s, the velocity  $v$  of a rocket varies linearly with the elapsed time  $t$ . If  $v = 40$  m/s when  $t = 1.0$  s and  $v = 55$  m/s when  $t = 4.0$  s, find the equation relating  $v$  and  $t$  and graph the function. From the graph, find the initial velocity and the velocity after 6.0 s. What is the meaning of the slope of the line?

With  $v$  as the dependent variable and  $t$  as the independent variable, the slope is

$$m = \frac{v_2 - v_1}{t_2 - t_1}$$

Using the information given in the statement of the problem, we have

$$m = \frac{55 - 40}{4.0 - 1.0} = 5.0$$

Then using the point-slope form of the equation of a straight line, we have

$$v - 40 = 5.0(t - 1.0)$$

$$v = 5.0t + 35$$

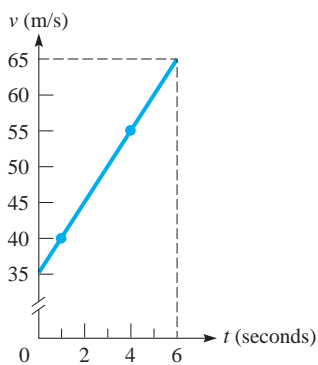


Fig. 21.25

The given values are sufficient to graph the line in Fig. 21.25. There is no need to include negative values of  $t$ , since they have no physical meaning. We see that the line crosses the  $v$ -axis at 35. This means that the initial velocity (for  $t = 0$ ) is 35 m/s. Also, when  $t = 6.0$  s, we see that  $v = 65$  m/s.

The slope is the ratio of the change in velocity to the change in time. This is the rocket's *acceleration*. Here, the speed of the rocket increases 5.0 m/s each second. We can express this acceleration as  $5.0 \text{ (m/s)}/\text{s} = 5.0 \text{ m/s}^2$ . ■

**Practice Exercise**

- Write the general form of the equation of the straight line that passes through  $(0, -1)$  with a slope of 3.



## EXERCISES 21.2

In Exercises 1–4, make the given changes in the indicated examples of this section and then solve the resulting problems.

- In Example 1, change  $(-4, 1)$  to  $(4, -1)$ .
- In Example 2, change  $(2, -1)$  to  $(-2, 1)$ .
- In Example 4, change the  $+$  before  $4x$  to  $-$ .
- In Example 6, change the  $+$  before  $2y$  to  $-$ .

In Exercises 5–20, find the equation of each of the lines with the given properties. Sketch the graph of each line.

- Passes through  $(-3, 8)$  with a slope of 4.
- Passes through  $(-2, -1)$  with a slope of  $-2$ .
- Passes through  $(2, -5)$  and  $(4, 2)$ .
- Has an  $x$ -intercept  $(4, 0)$  and a  $y$ -intercept  $(0, -6)$ .
- Passes through  $(-7, 12)$  with an inclination of  $45^\circ$ .
- Has a  $y$ -intercept  $(0, -2)$  and an inclination of  $120^\circ$ .
- Passes through  $(5.3, -2.7)$  and is parallel to the  $x$ -axis.
- Passes through  $(-15, 9)$  and is perpendicular to the  $x$ -axis.
- Is parallel to the  $y$ -axis and is 3 units to the left of it.
- Is parallel to the  $x$ -axis and is 4.1 units below it.
- Perpendicular to line with slope of  $-3$ ; passes through  $(1, -2)$ .
- Parallel to line through  $(-1, 7)$  and  $(3, 1)$ ; passes through  $(1, 2)$ .
- Has equal intercepts and passes through  $(5, 2)$ .
- Is perpendicular to the line  $6.0x - 2.4y - 3.9 = 0$  and passes through  $(7.5, -4.7)$ .
- Has a slope of  $-3$  and passes through the intersection of the lines  $5x - y = 6$  and  $x + y = 12$ .
- Passes through the point of intersection of  $2x + y - 3 = 0$  and  $x - y - 3 = 0$  and through the point  $(4, -3)$ .

In Exercises 21–28, reduce the equations to slope-intercept form and find the slope and the  $y$ -intercept. Sketch each line.

- |                          |                            |
|--------------------------|----------------------------|
| 21. $4x - y = 8$         | 22. $2x - 3y - 6 = 0$      |
| 23. $3x + 5y - 10 = 0$   | 24. $4y = 6x - 9$          |
| 25. $3x - 2y - 1 = 0$    | 26. $4x + 2y - 5 = 0$      |
| 27. $11.2x + 1.6 = 3.2y$ | 28. $11.5x + 4.60y = 5.98$ |

In Exercises 29–36, determine whether the given lines are parallel, perpendicular, or neither.

- $3x - 2y + 5 = 0$  and  $4y = 6x - 1$
- $8x - 4y + 1 = 0$  and  $4x + 2y - 3 = 0$
- $6x - 3y - 2 = 0$  and  $x + 2y - 4 = 0$
- $3y - 2x = 4$  and  $6x - 9y = 5$
- $5x + 2y - 3 = 0$  and  $10y = 7 - 4x$
- $48y - 36x = 71$  and  $52x = 17 - 39y$
- $4.5x - 1.8y = 1.7$  and  $2.4x + 6.0y = 0.3$
- $3.5y = 4.3 - 1.5x$  and  $3.6x + 8.4y = 1.7$

In Exercises 37–60, solve the given problems. Exercises 49–60 show some applications of straight lines.

- Find  $k$  if the lines  $4x - ky = 6$  and  $6x + 3y + 2 = 0$  are parallel.
- Find  $k$  if the lines given in Exercise 37 are perpendicular.
- Find  $k$  if the lines  $3x - y = 9$  and  $kx + 3y = 5$  are perpendicular. Explain how this value is found.
- Find  $k$  such that the line through  $(k, 2)$  and  $(3, 1 - k)$  is perpendicular to the line  $x - 2y = 5$ . Explain your method.
- Find the slope of the line joining points on the graph of  $y = x^2$  that have  $x$ -coordinates of  $-a$  and  $b$  ( $a > 0, b > 0$ ).
- Show that the intercept form  $\frac{x}{a} + \frac{y}{b} = 1$  is the equation of a line with  $x$ -intercept  $(a, 0)$  and  $y$ -intercept  $(0, b)$ .
- Find the distance from  $(4, 1)$  to the line  $4x - 3y + 12 = 0$ .
- Find the acute angle between the lines  $x + y = 3$  and  $2x - 5y = 4$ .
- Show that the following lines intersect to form a parallelogram.  $8x + 10y = 3$ ;  $2x - 3y = 5$ ;  $4x - 6y = -3$ ;  $5y + 4x = 1$ .
- For nonzero values of  $a, b$ , and  $c$ , find the intercepts of the line  $ax + by + c = 0$ .
- For nonzero values of  $a, b, c$ , and  $d$ , show that (a) lines  $ax + by + c = 0$  and  $ax + by + d = 0$  are parallel, and (b) lines  $ax + by + c = 0$  and  $bx - ay + d = 0$  are perpendicular.
- Find the equation of the line with positive intercepts that passes through  $(3, 2)$  and forms with the axes a triangle of area 12.
- In the 1700s, the French physicist Réaumur established a temperature scale on which the freezing point of water was  $0^\circ$  and the boiling point was  $80^\circ$ . Set up an equation for the Celsius temperature  $T$  (freezing point  $0^\circ$ , boiling point  $100^\circ$ ) as a function of the Réaumur temperature  $R$ .
- The voltage  $V$  across part of an electric circuit is given by  $V = E - iR$ , where  $E$  is a battery voltage,  $i$  is the current, and  $R$  is the resistance. If  $E = 6.00$  V and  $V = 4.35$  V for  $i = 9.17$  mA, find  $V$  as a function of  $i$ . Sketch the graph. ( $i$  and  $V$  may be negative.)
- The velocity of sound  $v$  increases  $0.607$  m/s for each increase in temperature  $T$  of  $1.00^\circ\text{C}$ . If  $v = 343$  m/s for  $T = 20.0^\circ\text{C}$ , express  $v$  as a function of  $T$ .
- An acid solution is made from  $x$  litres of a 20% solution and  $y$  litres of a 30% solution. If the final solution contains 20 L of acid, find the equation relating  $x$  and  $y$ .
- A wall is 15 cm thick. At the outside, the temperature is  $3^\circ\text{C}$ , and at the inside, it is  $23^\circ\text{C}$ . If the temperature changes at a constant rate through the wall, write an equation of the temperature  $T$  in the wall as a function of the distance  $x$  from the outside to the inside of the wall. What is the meaning of the slope of the line?
- An oil-storage tank is emptied at a constant rate. At 10 A.M., 1800 barrels remain, and at 2 P.M., 600 barrels remain. If pumping started at 8 A.M., find the equation relating the number of barrels  $n$  at time  $t$  (in  $h$ ) from 8 A.M. When will the tank be empty?

55. The power output  $P$  (in W) of a computer chip operating at  $120^\circ\text{C}$  is proportional to  $120 - T_S$ , where  $T_S$  is the temperature of the surroundings. If  $P = 1.0$  W for  $T_S = 80^\circ\text{C}$ , find the equation relating  $P$  and  $T_S$ .
56. The length of a rectangular solar cell is 10 cm more than the width  $w$ . Express the perimeter  $p$  of the cell as a function of  $w$ . What is the meaning of the slope of the line?
57. A light beam is reflected off the edge of an optic fibre at an angle of  $0.0032^\circ$ . The diameter of the fibre is  $48\ \mu\text{m}$ . Find the equation of the reflected beam with the  $x$ -axis (at the centre of the fibre) and the  $y$ -axis as shown in Fig. 21.26.

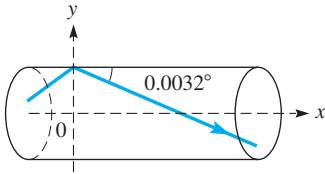


Fig. 21.26

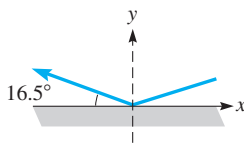


Fig. 21.27

58. A police report stated that a bullet caromed upward off a floor at an angle of  $16.5^\circ$  with the floor, as shown in Fig. 21.27. What is the equation of the bullet's path after impact?
59. A survey of the traffic on a particular highway showed that the number of cars passing a particular point each minute varied linearly from 6:30 A.M. to 8:30 A.M. on workday mornings. The study showed that an average of 45 cars passed the point in 1 min at 7 A.M. and that 115 cars passed in 1 min at 8 A.M. If  $n$  is the number of cars passing the point in 1 min, and  $t$  is the number of minutes after 6:30 A.M., find the equation relating  $n$  and  $t$  and graph the equation. From the graph, determine  $n$  at 6:30 A.M. and at 8:30 A.M. What is the meaning of the slope of the line?
60. In a research project on cancer, a tumour was determined to weigh 30 mg when first discovered. While being treated, it grew smaller by 2 mg each month. Find the equation relating the weight  $w$  of the tumour as a function of the time  $t$  in months. Graph the equation.

In Exercises 61–64, treat the given nonlinear functions as linear functions in order to sketch their graphs. At times, this can be useful in showing certain values of a function. For example,  $y = 2 + 3x^2$  can be shown as a straight line by graphing  $y$  as a function of  $x^2$ . A table of values for this graph is shown along with the corresponding graph in Fig. 21.28.

$x$	0	1	2	3	4	5
$x^2$	0	1	4	9	16	25
$y$	2	5	14	29	50	77

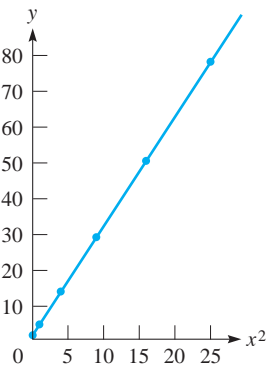


Fig. 21.28

61. The number  $n$  of memory cells of a certain computer that can be tested in  $t$  seconds is given by  $n = 1200\sqrt{t}$ . Sketch  $n$  as a function of  $\sqrt{t}$ .
62. The force  $F$  (in N) applied to a lever to balance a certain weight on the opposite side of the fulcrum is given by  $F = 40/d$ , where  $d$  is the distance (in m) of the force from the fulcrum. Sketch  $F$  as a function of  $1/d$ .
63. A spacecraft is launched such that its altitude  $h$  (in km) is given by  $h = 300 + 2t^{3/2}$  for  $0 \leq t < 100$  s. Sketch this as a linear function.
64. The current  $i$  (in A) in a certain electric circuit is given by  $i = 6(1 - e^{-t})$ . Sketch this as a linear function.

In Exercises 65–68, show that the given nonlinear functions are linear when plotted on semilogarithmic or logarithmic paper. In Section 13.7, we noted that graphs on this paper often become straight lines.

65. A function of the form  $y = ax^n$  is straight when plotted on logarithmic paper, since  $\log y = \log a + n \log x$  is in the form of a straight line. The variables are  $\log y$  and  $\log x$ ; the slope can be found from  $(\log y - \log a) / \log x = n$ , and the intercept is  $\log a$ . (To get the slope from the graph, we calculate  $(\log y - \log a) / \log x$  for some set values of  $x$  and  $y$ . The  $\log y$ -intercept is found where  $\log x = 0$ , and this occurs when  $x = 1$ .) Plot  $y = 3x^4$  on logarithmic paper to verify this analysis.
66. A function of the form  $y = a(b^x)$  is a straight line on semilogarithmic paper, since  $\log y = \log a + x \log b$  is in the form of a straight line. The variables are  $\log y$  and  $x$ , the slope is  $\log b$ , and the intercept is  $\log a$ . (To get the slope from the graph, we calculate  $(\log y - \log a) / x$  for some set values of  $x$  and  $y$ . The intercept is read directly off the graph where  $x = 0$ .) Plot  $y = 3(2^x)$  on semilogarithmic paper to verify this analysis.
67. If experimental data are plotted on logarithmic paper and the points lie on a straight line, it is possible to determine the function (see Exercise 65). The following data come from an experiment to determine the functional relationship between the pressure  $p$  and the volume  $V$  of a gas undergoing an adiabatic (no heat loss) change. From the graph on logarithmic paper, determine  $p$  as a function of  $V$ .

$V$ ( $\text{m}^3$ )	0.100	0.500	2.00	5.00	10.0
$p$ (kPa)	20.1	2.11	0.303	0.0840	0.0318

68. If experimental data are plotted on semilogarithmic paper, and the points lie on a straight line, it is possible to determine the function (see Exercise 66). The following data come from an experiment designed to determine the relationship between the voltage across an inductor and the time after the switch is opened. Determine  $V$  as a function of  $t$ .

$V$ (V)	40	15	5.6	2.2	0.8
$t$ (ms)	0.0	20	40	60	80

Answers to Practice Exercises

1.  $2x + y + 8 = 0$     2.  $3x - y - 1 = 0$

## 21.3 The Circle

Standard Equation • Symmetry • General Equation

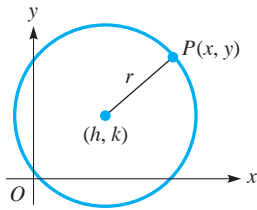


Fig. 21.29

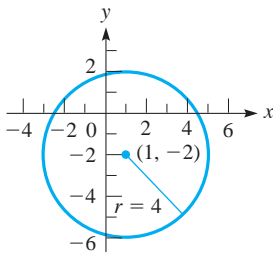


Fig. 21.30

### LEARNING TIP

It is important to pay close attention to signs when finding the coordinates of the centre. (This will also be relevant in later sections of the chapter.) We must have a minus sign before each of the coordinates. If the expression contains a plus sign, we must rewrite it to have a minus sign.

For instance, in Example 1, we were able to identify the  $x$ -coordinate of the centre directly as  $+1$  because of the minus sign before it. However, we had to write  $+2$  as  $-(-2)$  in order to identify the  $y$ -coordinate of the centre as  $-2$ .

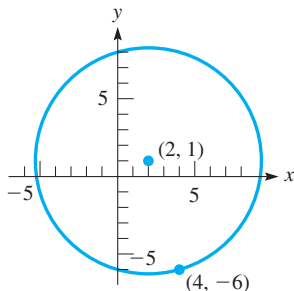


Fig. 21.31

We have found that we can obtain a general equation that represents a straight line by considering a fixed point on the line and then a general point  $P(x, y)$  that can represent any other point on the same line. Mathematically, we can state this as “the line is the **locus** of a point  $P(x, y)$  that moves from a fixed point with constant direction.” That is, the point  $P(x, y)$  can be considered a variable point that moves along the line.

In this way, we can define a number of important curves. A **circle** is defined as the locus of a point  $P(x, y)$  that moves so that it is always equidistant from a fixed point. We call this fixed distance the **radius**, and we call the fixed point the **centre** of the circle. Thus, using this definition, calling the fixed point  $(h, k)$  and the radius  $r$ , we have

$$\sqrt{(x - h)^2 + (y - k)^2} = r$$

or, by squaring both sides, we have

$$(x - h)^2 + (y - k)^2 = r^2 \quad (21.11)$$

Eq. (21.11) is called the **standard equation** of a circle with centre at  $(h, k)$  and radius  $r$ . See Fig. 21.29.

### EXAMPLE 1 Standard equation

The equation  $(x - 1)^2 + (y + 2)^2 = 16$  represents a circle with centre at  $(1, -2)$  and a radius of 4. We determine these values by considering the equation of the circle to be in the form of Eq. (21.11) as

$$(x - 1)^2 + [y - (-2)]^2 = 4^2$$

form requires - signs  
coordinates of centre      radius

This circle is shown in Fig. 21.30. ■

### EXAMPLE 2 Find the equation of a circle

Find the equation of the circle with centre at  $(2, 1)$  and that passes through  $(4, -6)$ .

In Eq. (21.11), we can determine the equation of this circle if we can find  $h$ ,  $k$ , and  $r$ . From the given information, the centre is  $(2, 1)$ , which means  $h = 2$  and  $k = 1$ . To find  $r$ , we use the fact that *all points on the circle must satisfy the equation of the circle*. The point  $(4, -6)$  must satisfy Eq. (21.11), with  $h = 2$  and  $k = 1$ . This means

$$(4 - 2)^2 + (-6 - 1)^2 = r^2 \quad \text{or} \quad r^2 = 53$$

Therefore, the equation of the circle is

$$(x - 2)^2 + (y - 1)^2 = 53$$

This circle is shown in Fig. 21.31. ■

### Practice Exercise

- Find the centre and radius of the circle  $(x + 7)^2 + (y - 2)^2 = 1$ .

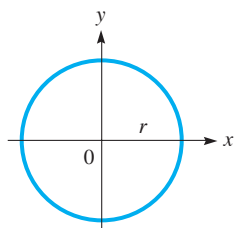


Fig. 21.32

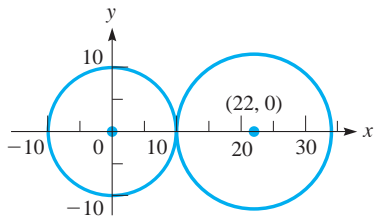


Fig. 21.33

If the centre of the circle is at the origin, which means that the coordinates of the centre are  $(0, 0)$ , the equation of the circle (see Fig. 21.32) becomes

$$x^2 + y^2 = r^2 \quad (21.12)$$

The following example illustrates an application using this type of circle and one with its centre not at the origin.

### EXAMPLE 3 Equations of circles—application

A student is drawing a friction drive in which two circular discs are in contact with each other. They are represented by circles in the drawing. The first has a radius of 10.0 cm, and the second has a radius of 12.0 cm. What is the equation of each circle if the origin is at the centre of the first circle and the positive  $x$ -axis passes through the centre of the second circle? See Fig. 21.33.

Since the centre of the smaller circle is at the origin, we can use Eq. (21.12). Given that the radius is 10.0 cm, we have as its equation

$$x^2 + y^2 = 100$$

The fact that the two discs are in contact tells us that they meet at the point  $(10.0, 0)$ . Knowing that the radius of the larger circle is 12.0 cm tells us that its centre is at  $(22.0, 0)$ . Thus, using Eq. (21.11) with  $h = 22.0$ ,  $k = 0$ , and  $r = 12.0$ , we get

$$(x - 22.0)^2 + (y - 0)^2 = 12.0^2$$

or

$$(x - 22.0)^2 + y^2 = 144$$

as the equation of the larger circle. ■

### Symmetry

A circle with its centre at the origin exhibits an important property of the graphs of many equations. It is **symmetric to the  $x$ -axis and also to the  $y$ -axis**. Symmetry to the  $x$ -axis can be thought of as meaning that the lower half of the curve is a reflection of the upper half, and conversely. It can be shown that *if  $-y$  can replace  $y$  in an equation without changing the equation, the graph of the equation is symmetric to the  $x$ -axis*. Symmetry to the  $y$ -axis is similar. *If  $-x$  can replace  $x$  in the equation without changing the equation, the graph is symmetric to the  $y$ -axis*.

This type of circle is also symmetric to the origin as well as being symmetric to both axes. The meaning of symmetry to the origin is that the origin is the midpoint of any two points  $(x, y)$  and  $(-x, -y)$  that are on the curve. Thus, *if  $-x$  can replace  $x$  and  $-y$  can replace  $y$  at the same time, without changing the equation, the graph of the equation is symmetric to the origin*.

### EXAMPLE 4 Symmetry

The equation of the circle with its centre at the origin and with a radius of 6 is  $x^2 + y^2 = 36$ .

The symmetry of this circle can be shown analytically by the substitutions mentioned above. Replacing  $x$  by  $-x$ , we obtain  $(-x)^2 + y^2 = 36$ . Since  $(-x)^2 = x^2$ , this equation can be rewritten as  $x^2 + y^2 = 36$ . Since this substitution did not change the equation, the graph is symmetric to the  $y$ -axis.

Replacing  $y$  by  $-y$ , we obtain  $x^2 + (-y)^2 = 36$ , which is the same as  $x^2 + y^2 = 36$ . This means that the curve is symmetric to the  $x$ -axis.

Replacing  $x$  by  $-x$  and simultaneously replacing  $y$  by  $-y$ , we obtain  $(-x)^2 + (-y)^2 = 36$ , which is the same as  $x^2 + y^2 = 36$ . This means that the curve is symmetric to the origin. This circle is shown in Fig. 21.34. ■

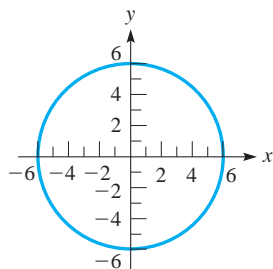


Fig. 21.34

If we multiply out each of the terms in Eq. (21.11), we may combine the resulting terms to obtain

$$x^2 - 2hx + h^2 + y^2 - 2ky + k^2 = r^2$$

$$x^2 + y^2 - 2hx - 2ky + (h^2 + k^2 - r^2) = 0 \tag{21.13}$$

Since each of  $h$ ,  $k$ , and  $r$  is constant for any given circle, the coefficients of  $x$  and  $y$  and the term within parentheses in Eq. (21.13) are constants. Eq. (21.13) can then be written as

$$x^2 + y^2 + Dx + Ey + F = 0 \tag{21.14}$$

Eq. (21.14) is called the **general equation of the circle**. It tells us that any equation that can be written in that form will represent a circle.

**LEARNING TIP**

If we know the general equation of a circle, we can find the centre and radius by writing the equation in standard form. To do so, we *must complete the square both in the  $x$ -terms and in the  $y$ -terms* (see Section 7.2).

**EXAMPLE 5 General equation**

Find the centre and radius of the circle with general equation

$$x^2 + y^2 - 6x + 8y - 24 = 0$$

We write the given equation in standard form by completing the squares. This is done by first writing the equation in the form

$$(x^2 - 6x \quad) + (y^2 + 8y \quad) = 24$$

To complete the square of the  $x$ -terms, we take half of  $-6$ , which is  $-3$ , square it, and add the result, 9, to each side of the equation. In the same way, we complete the square of the  $y$ -terms by adding 16 to each side of the equation, which gives

$$(x^2 - 6x + 9) + (y^2 + 8y + 16) = 24 + 9 + 16$$

$$(x - 3)^2 + (y + 4)^2 = 49$$

$$(x - 3)^2 + (y - (-4))^2 = 7^2$$

↑ coordinates of centre
↑ radius

Thus, the centre is  $(3, -4)$ , and the radius is 7 (see Fig. 21.35).

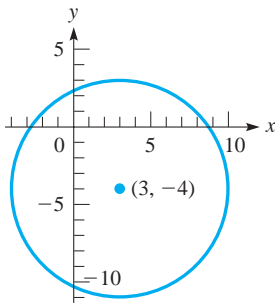


Fig. 21.35

**Practice Exercise**

- Find the centre and radius of the circle  $x^2 + y^2 - 8x + 6y + 21 = 0$ .

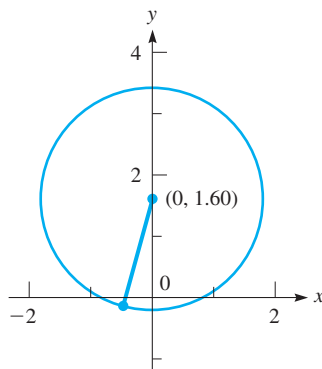


Fig. 21.36

**EXAMPLE 6 Circle—application**

A certain pendulum is found to swing through an arc of the circle  $3x^2 + 3y^2 - 9.60y - 2.80 = 0$ . What is the length (in m) of the pendulum, and from what point is it swinging?

We see that this equation represents a circle by dividing through by 3. This gives us  $x^2 + y^2 - 3.20y - 2.80/3 = 0$ . The length of the pendulum is the radius of the circle, and the point from which it swings is the centre. These are found as follows:

$$x^2 + (y^2 - 3.20y + 1.60^2) = 1.60^2 + 2.80/3$$

complete squares in both  $x$ - and  $y$ -terms

$$x^2 + (y - 1.60)^2 = 3.493$$

standard form

Since  $\sqrt{3.493} = 1.87$ , the length of the pendulum is 1.87 m. The point from which it is swinging is  $(0, 1.60)$ . See Fig. 21.36.

Replacing  $x$  by  $-x$ , the equation does not change. Replacing  $y$  by  $-y$ , the equation does change (the  $3.20y$  term changes sign). Thus, the circle is symmetric only to the  $y$ -axis.

In Section 14.1 we noted that the equation of a circle does not represent a function since there are two values of  $y$  for most values of  $x$  in the domain. However, the circle is a combination of two functions, each of which has a semicircle as its graph. The top and bottom semicircle functions can be found using the quadratic formula, as illustrated in the following example.

### EXAMPLE 7 A circle as a graph of two functions

Determine the two semicircle functions of the circle  $3x^2 + 3y^2 + 6y - 20 = 0$ .

To fit the form of a quadratic equation in  $y$ , we write

$$3y^2 + 6y + (3x^2 - 20) = 0$$

Now, using the quadratic formula to solve for  $y$ , we let

$$a = 3 \quad b = 6 \quad c = 3x^2 - 20$$

$$y = \frac{-6 \pm \sqrt{6^2 - 4(3)(3x^2 - 20)}}{2(3)}$$

which means we get the *two functions*

$$y_1 = \frac{-6 + \sqrt{276 - 36x^2}}{6} \quad \text{and} \quad y_2 = \frac{-6 - \sqrt{276 - 36x^2}}{6}$$

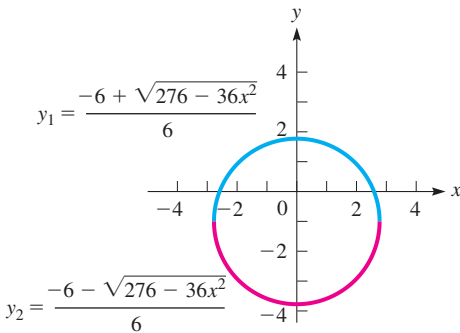


Fig. 21.37

The two functions are graphed together in Fig. 21.37. These two functions would be entered into a graphing calculator in order to display a circle. ■

## EXERCISES 21.3

In Exercises 1–4, make the given changes in the indicated examples of this section and then solve the resulting problems.

- In Example 1, change  $(y + 2)^2$  to  $(y + 1)^2$ .
- In Example 2, change  $(2, 1)$  to  $(-2, 1)$ .
- In Example 5, change the  $+$  before  $8y$  to  $-$ .
- In Example 7, change the  $+$  before  $6y$  to  $-$ .

In Exercises 5–8, determine the centre and the radius of each circle.

- $(x - 2)^2 + (y - 1)^2 = 25$
- $(x - 3)^2 + (y + 4)^2 = 49$
- $4(x + 1)^2 + 4y^2 = 121$
- $9x^2 + 9(y - 6)^2 = 64$

In Exercises 9–24, find the standard and general equations of each of the circles from the given information.

- Centre at  $(0, 0)$ , radius 3
- Centre at  $(0, 0)$ , radius 12
- Centre at  $(2, 2)$ , radius 4
- Centre at  $(\frac{3}{2}, -2)$ , radius  $\frac{5}{2}$
- Centre at  $(12, -15)$ , radius 18
- Centre at  $(-3, -5)$ , radius  $2\sqrt{3}$
- The origin and  $(-6, 8)$  are ends of a diameter.
- The points  $(3, 8)$  and  $(-3, 0)$  are the ends of a diameter.
- Concentric with the circle  $(x - 2)^2 + (y - 1)^2 = 4$  and passes through  $(4, -1)$

18. Concentric with the circle  $x^2 + y^2 + 2x - 8y + 8 = 0$  and passes through  $(-2, 3)$

- Centre at  $(-3, 5)$  and tangent to the line  $y = 10$
- Centre at  $(-7, 1)$  and tangent to the  $y$ -axis
- Tangent to both axes and the lines  $y = 4$  and  $x = -4$
- Tangent to the lines  $y = 2$  and  $y = 8$ , centre on the line  $y = x$
- Centre at the origin, tangent to the line  $x + y = 2$
- Centre at  $(5, 12)$ , tangent to the line  $y = 2x - 3$

In Exercises 25–36, determine the centre and radius of each circle. Sketch each circle.

- $x^2 + (y - 3)^2 = 4$
- $(x - 2)^2 + (y + 3)^2 = 49$
- $4(x + 1)^2 + 4(y - 5)^2 = 81$
- $4(x + 7)^2 + 4(y + 11)^2 = 169$
- $x^2 + y^2 - 2x - 8 = 0$
- $x^2 + y^2 - 4x - 6y - 12 = 0$
- $x^2 + y^2 + 4.20x - 2.60y = 3.51$
- $x^2 + y^2 + 22x + 14y = 26$
- $4x^2 + 4y^2 - 16y = 9$
- $9x^2 + 9y^2 + 18y = 7$
- $2x^2 + 2y^2 - 4x - 8y - 1 = 0$
- $3x^2 + 3y^2 - 12x + 4 = 0$



In Exercises 37–40, determine whether the circles with the given equations are symmetric to either axis or to the origin.

37.  $x^2 + y^2 = 100$   
 38.  $x^2 + y^2 - 4x - 5 = 0$   
 39.  $3x^2 + 3y^2 + 24y = 8$   
 40.  $5x^2 + 5y^2 - 10x + 20y = 3$

In Exercises 41–62, solve the given problems.

41. Determine whether the circle  $x^2 - 6x + y^2 - 7 = 0$  crosses the  $x$ -axis.  
 42. Find the points of intersection of the circle  $x^2 + y^2 - x - 3y = 0$  and the line  $y = x - 1$ .  
 43. Find the locus of a point  $P(x, y)$  that moves so that its distance from  $(2, 4)$  is twice its distance from  $(0, 0)$ . Describe the locus.  
 44. Find the equation of the locus of a point  $P(x, y)$  that moves so that the line joining it and  $(2, 0)$  is always perpendicular to the line joining it and  $(-2, 0)$ . Describe the locus.  
 45. Use a graphing calculator to view the circle  $x^2 + y^2 + 5y - 4 = 0$ .  
 46. Use a graphing calculator to view the circle  $2x^2 + 2y^2 + 2y - x - 1 = 0$ .  
 47. What type of graph is represented by each of the following equation?  
 (a)  $y = \sqrt{9 - (x - 2)^2}$  (b)  $y = -\sqrt{9 - (x - 2)^2}$   
 (c) Are the equations in parts (a) and (b) functions? Explain.  
 48. What type of graph is represented by each of the following equations?  
 (a)  $x^2 + (y - 1)^2 = 0$  (b)  $x^2 + (y - 1)^2 = -1$   
 49. Is the point  $(0.1, 3.1)$  inside, outside, or on the circle  $x^2 + y^2 - 2x - 4y + 3 = 0$ ?  
 50. Find the equation of the line along which the diameter of the circle  $x^2 + y^2 - 2x - 4y - 4 = 0$  lies, if the diameter is parallel to the line  $3x + 5y = 4$ .  
 51. For the equation  $(x - h)^2 + (y - k)^2 = p$ , how does the value of  $p$  indicate whether the graph is a circle, a point, or does not exist?  
 52. Determine whether the graph of  $x^2 + y^2 - 2x + 10y + 29 = 0$  is a circle, a point, or does not exist.  
 53. The inner and outer circles of the cross section of a pipe are represented by the equations  $2.00x^2 + 2.00y^2 = 5.73$  and  $2.80x^2 + 2.80y^2 = 8.91$ . How thick (in cm) is the pipe wall?  
 54. A 4-m pole leaning against a wall slips to the ground. Find the equation that represents the path of the midpoint of the pole. See Fig. 21.38.  
 55. In a hoisting device, two of the pulley wheels may be represented by  $x^2 + y^2 = 14.5$  and  $x^2 + y^2 - 19.6y + 86.0 = 0$ . How far apart (in cm) are the wheels?  
 56. The design of a machine part shows it as a circle represented by the equation  $x^2 + y^2 = 42.5$ , with a circular hole represented by  $x^2 + y^2 + 3.06y - 1.24 = 0$  cut out. What is the least distance (in cm) from the edge of the hole to the edge of the machine part?  
 57. A wire is rotating in a circular path through a magnetic field to induce an electric current in the wire. The wire is rotating at 60.0 Hz, with a constant velocity of 37.7 m/s. Taking the origin at the centre of the circle of rotation, find the equation of the path of the wire.  
 58. A communications satellite remains stationary at an altitude of 36 200 km over a point on the earth's equator. It therefore rotates once each day about the earth's centre. Its velocity is constant, but the horizontal and vertical components,  $v_H$  and  $v_V$ , of the velocity constantly change. Show that the equation relating  $v_H$  and  $v_V$  (in km/h) is that of a circle. The radius of the earth is 6370 km.  
 59. Find the equation describing the rim of a circular porthole 0.80 m in diameter if the top is 2.0 m below the surface of the water. Take the origin at the water surface directly above the centre of the porthole.  
 60. An earthquake occurred  $37^\circ$  north of east of a seismic recording station. If the tremors travel at 4.8 km/s and were recorded 25 s later at the station, find the equation of the circle that represents the tremor recorded at the station. Take the station to be at the centre of the coordinate system.  
 61. In analysing the strain on a beam, *Mohr's circle* is often used. To form it, normal strain is plotted as the  $x$ -coordinate and shear strain is plotted as the  $y$ -coordinate. The centre of the circle is midway between the minimum and maximum values of normal strain on the  $x$ -axis. Find the equation of Mohr's circle if the minimum normal strain is  $100 \times 10^{-6}$  and the maximum normal strain is  $900 \times 10^{-6}$  (strain is unitless). Sketch the graph.  
 62. An architect designs a Norman window, which has the form of a semicircle surmounted on a rectangle, as in Fig. 21.39. Find the area (in  $\text{m}^2$ ) of the window if the circular part is on the circle  $x^2 + y^2 - 3.00y + 1.25 = 0$ .

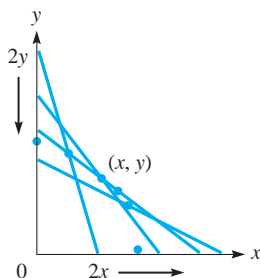


Fig. 21.38

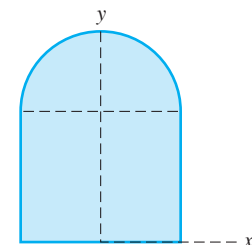


Fig. 21.39

### Answers to Practice Exercises

1.  $C(-7, 2)$ ,  $r = 1$     2.  $C(4, -3)$ ,  $r = 2$



## 21.4 The Parabola

**Directrix • Focus • Axis • Vertex • Standard Equation • Axis along x-axis • Axis along y-axis**

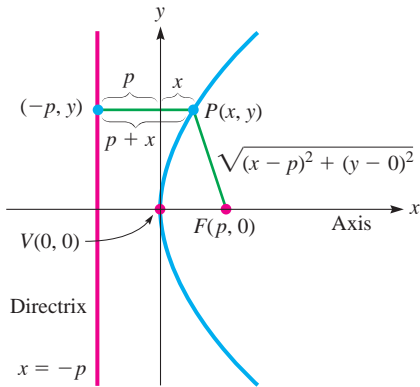


Fig. 21.40

In Chapter 7, we showed that the graph of a quadratic function is a *parabola*. We now define the parabola more generally and find the general form of its equation.

A **parabola** is defined as the locus of a point  $P(x, y)$  that moves so that it is always equidistant from a given line (the **directrix**) and a given point (the **focus**). The line through the focus that is perpendicular to the directrix is the **axis** of the parabola. The point midway between the focus and directrix is the **vertex**.

Using the definition, we now find the equation of the parabola with the focus at  $(p, 0)$  and the directrix  $x = -p$ . With these choices, we find a general equation of a parabola with its vertex at the origin.

From the definition, the distance from  $P(x, y)$  on the parabola to the focus  $(p, 0)$  must equal the distance from  $P(x, y)$  to the directrix  $x = -p$ . The distance from  $P$  to the focus is found from the distance formula. The distance from  $P$  to the directrix is the perpendicular distance and is along a line parallel to the  $x$ -axis. These distances are shown in Fig. 21.40. Therefore, we have

$$\begin{aligned} \sqrt{(x-p)^2 + (y-0)^2} &= x + p \\ (x-p)^2 + y^2 &= (x+p)^2 && \text{squaring both sides} \\ x^2 - 2px + p^2 + y^2 &= x^2 + 2px + p^2 \end{aligned}$$

Simplifying, we obtain

$$y^2 = 4px \tag{21.15}$$

Eq. (21.15) is called the **standard form** of the equation of a parabola with its axis along the  $x$ -axis and the vertex at the origin. Its symmetry to the  $x$ -axis can be proven since  $(-y)^2 = 4px$  is the same as  $y^2 = 4px$ .

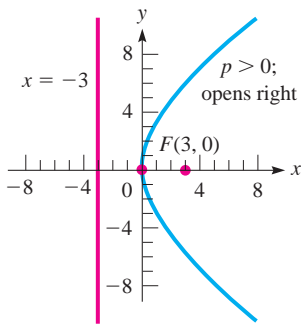


Fig. 21.41

### EXAMPLE 1 Focus and directrix of a parabola

Find the coordinates of the focus and the equation of the directrix and sketch the graph of the parabola  $y^2 = 12x$ .

Since the equation of this parabola fits the form of Eq. (21.15), we know that the vertex is at the origin. The coefficient of 12 tells us that

$$4p = 12, \quad p = 3$$

The focus is the point  $(3, 0)$ , and the directrix is the line  $x = -3$ , as shown in Fig. 21.41. ■

### EXAMPLE 2 Find the equation, given the directrix and the vertex

If the focus is to the left of the origin, with the directrix an equal distance to the right, the coefficient of the  $x$ -term is negative. This tells us that the parabola opens to the left, rather than to the right, as in the case when the focus is to the right of the origin. For example, the parabola  $y^2 = -8x$  has its vertex at the origin, its focus at  $(-2, 0)$ , and the line  $x = 2$  as its directrix. We determine this from the equation as follows:

$$y^2 = -8x \quad \begin{matrix} \downarrow \\ 4p = -8, \\ p = -2 \end{matrix}$$

Since  $p = -2$ , we find

the focus is  $(-2, 0)$

the directrix is the line  $x = -(-2)$ , or  $x = 2$

The parabola opens to the left, as shown in Fig. 21.42. ■

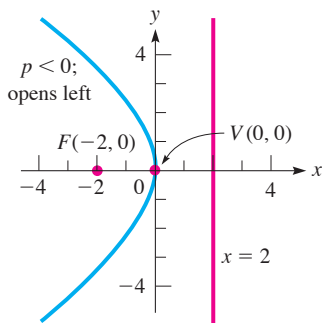


Fig. 21.42

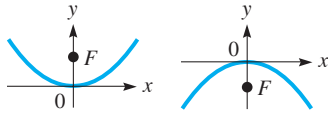


Fig. 21.43

If we chose the focus as the point  $(0, p)$  and the directrix as the line  $y = -p$  (see Fig. 21.43), we would find that the resulting equation is

$$x^2 = 4py \tag{21.16}$$

This is the standard form of the equation of a parabola with the  $y$ -axis as its axis and the vertex at the origin. Its symmetry to the  $y$ -axis can be proved, since  $(-x)^2 = 4py$  is the same as  $x^2 = 4py$ .

**Standard Equations of a Parabola with Vertex at  $(0, 0)$**

Equation:	$y^2 = 4px$ (21.15)	$x^2 = 4py$ (21.16)
Axis:	$x$ -axis	$y$ -axis
Focus:	$(p, 0)$	$(0, p)$
Directrix:	$x = -p$	$y = -p$
Symmetry:	$x$ -axis	$y$ -axis

**EXAMPLE 3 Standard form—axis along the  $y$ -axis**

- (a) The parabola  $x^2 = 4y$  fits the form of Eq. (21.16). Therefore, its axis is along the  $y$ -axis, and its vertex is at the origin. From the equation, we find the value of  $p$ , which in turn tells us the location of the vertex and the directrix.

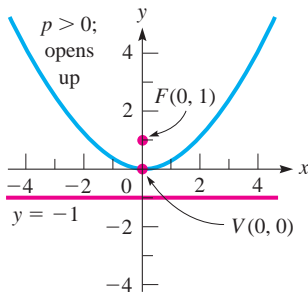


Fig. 21.44

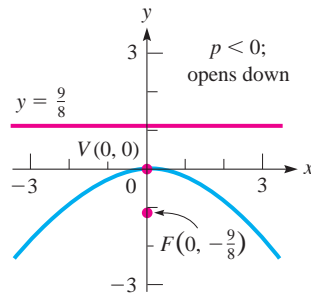


Fig. 21.45

$$x^2 = 4y \quad 4p = 4, \quad p = 1$$

Focus  $(0, p)$  is  $(0, 1)$ ; directrix  $y = -p$  is  $y = -1$ . The parabola is shown in Fig. 21.44, and we see in this case that it opens upward.

- (b) The parabola  $2x^2 = -9y$  fits the form of Eq. (21.16) if we write it in the form

$$x^2 = -\frac{9}{2}y$$

Here, we see that  $4p = -9/2$ . Therefore, its axis is along the  $y$ -axis, and its vertex is at the origin. Since  $4p = -9/2$ , we have

$$p = -\frac{9}{8} \quad \text{focus} \left(0, -\frac{9}{8}\right) \quad \text{directrix } y = \frac{9}{8}$$

The parabola opens downward, as shown in Fig. 21.45. ■

**EXAMPLE 4 Parabola—application**

In calculus, it can be shown that an electromagnetic wave (light wave, television signal, etc.) parallel to the axis of a parabolic reflector will pass through the focus of the parabola. Applications of this property of a parabola are many, including a satellite television dish and a spotlight (the light from the focus is reflected as a beam).

A parabolic satellite television dish is 46.0 cm across and 4.50 cm deep, as shown in Fig. 21.46. Find where the receiver should be located to receive all of the waves reflected off the parabolic surface.

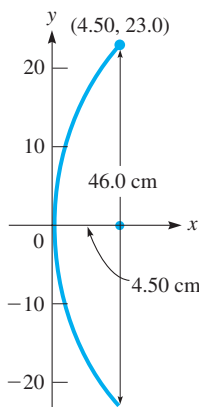


Fig. 21.46

■ See the chapter introduction.

With the vertex at the origin and the axis along the  $x$ -axis, we will use the general form  $y^2 = 4px$  of the parabola. Since the parabolic opening is 46.0 cm across and 4.50 cm

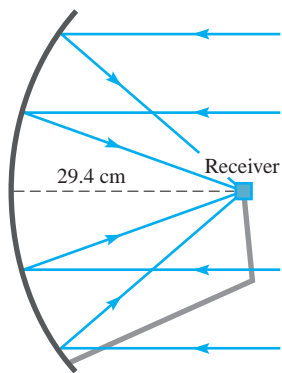


Fig. 21.47

**Practice Exercises**

Find the coordinates of the focus and the equation of the directrix for each parabola.

1.  $y^2 = -24x$     2.  $x^2 = 40y$

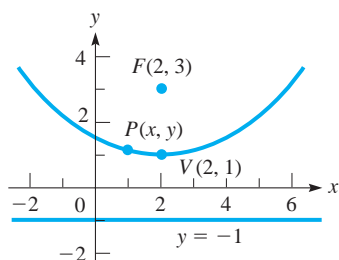


Fig. 21.48

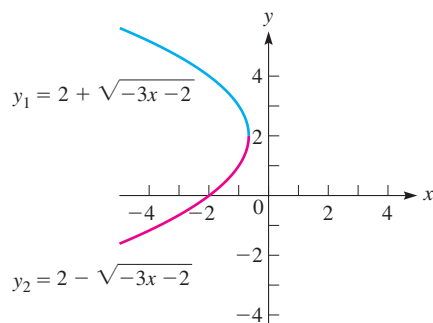


Fig. 21.49

deep, the point  $(4.50, 23.0)$  will be on any of the parabolic cross sections. Therefore, we find  $p$  by substituting  $(4.50, 23.0)$  in the equation. This means

$$23.0^2 = 4p(4.50), \quad p = 29.4 \text{ cm}$$

and the receiver should be placed 29.4 cm from the vertex as shown in Fig. 21.47. The equation of the parabolic cross section is  $y^2 = 118x$ . ■

Eqs. (21.15) and (21.16) give the general forms of the equation of a parabola with vertex at the origin and focus on one of the coordinate axes. We now use the definition to find the equation of a parabola that has its vertex at a point other than the origin.

**EXAMPLE 5 Find an equation from the definition**

Using the definition of the parabola, find the equation of the parabola with its focus at  $(2, 3)$  and its directrix the line  $y = -1$ . See Fig. 21.48.

Choosing a general point  $P(x, y)$  on the parabola and equating the distances from this point to  $(2, 3)$  and to the line  $y = -1$ , we have

$$\begin{aligned} \sqrt{(x-2)^2 + (y-3)^2} &= y+1 \\ \text{distance } P \text{ to } F &= \text{distance } P \text{ to } y = -1 \\ (x-2)^2 + (y-3)^2 &= (y+1)^2 && \text{squaring both sides} \\ x^2 - 4x + 4 + y^2 - 6y + 9 &= y^2 + 2y + 1 \\ 8y &= 12 - 4x + x^2 \end{aligned}$$

We note that this type of equation has appeared frequently in earlier chapters. The  $x$ -term and the constant (12 in this case) are characteristic of a parabola that does not have its vertex at the origin if the directrix is parallel to the  $x$ -axis. ■

A parabola whose axis is along the  $y$ -axis or parallel to it, as in Examples 3 and 5, is the graph of the quadratic function obtained when solving for  $y$ . In contrast, a parabola whose axis is along the  $x$ -axis or parallel to it, as in Examples 1, 2, and 4, is not a function but a combination of two functions. This is similar to Example 7 in Section 2.3. These cases are shown in the next example.

**EXAMPLE 6 A parabola as a graph of one or two functions**

We can solve for  $y$  from the last equation in Example 5 to obtain the function

$$y = (12 - 4x + x^2)/8$$

The graph of this single function is the same parabola shown in Fig. 21.48.

The equation  $y^2 + 3x - 4y + 6 = 0$  represents a parabola that is not the graph of a single function. If we solve for  $y$  using the quadratic formula, we get

$$\begin{aligned} a &= 1 & b &= -4 & c &= 3x + 6 \\ y &= \frac{4 \pm \sqrt{(-4)^2 - 4(1)(3x + 6)}}{2} \end{aligned}$$

Therefore, we obtain the two functions

$$y_1 = 2 + \sqrt{-3x - 2} \quad \text{and} \quad y_2 = 2 - \sqrt{-3x - 2}$$

The two functions are graphed together in Fig. 21.49. Both functions would be entered into a graphing calculator in order to display this parabola. ■

We see that *the equation of a parabola is characterized by the square of either  $x$  or  $y$  (but not both) and a first power term of the other variable*. We will consider the parabola further in Sections 21.7, 21.8, and 21.9.

The parabola has numerous technical applications. The reflection property illustrated in Example 4 has other important applications, such as the design of a radar antenna. The path of a projectile is parabolic. The cables of a suspension bridge are parabolic.

## EXERCISES 21.4

In Exercises 1–4, make the given changes in the indicated examples of this section and then solve the resulting problems. In each, find the focus and directrix, and sketch the parabola.

- In Example 1, change  $12x$  to  $20x$ .
- In Example 2, change  $-8x$  to  $-20x$ .
- In Example 3(a), change  $4y$  to  $-6y$ .
- In Example 3(b), change  $-9y$  to  $7y$ .

In Exercises 5–16, determine the coordinates of the focus and the equation of the directrix of the given parabolas. Sketch each curve.

- $y^2 = 4x$
- $y^2 = -4x$
- $x^2 = 72y$
- $x^2 = -4y$
- $2y^2 - 5x = 0$
- $y = 0.48x^2$
- $y^2 = 16x$
- $y^2 = -36x$
- $x^2 = y$
- $x^2 + 12y = 0$
- $3x^2 = 8y$
- $x = 7.6y^2$

In Exercises 17–30, find the equations of the parabolas satisfying the given conditions. The vertex of each is at the origin.

- Focus  $(3, 0)$
- Focus  $(0, 0.4)$
- Focus  $(0, -0.5)$
- Focus  $(2.5, 0)$
- Directrix  $y = -0.16$
- Directrix  $x = 20$
- Directrix  $x = -84$
- Directrix  $y = 2.3$
- Axis  $x = 0$ , passes through  $(-1, 8)$
- Symmetric to  $x$ -axis, passes through  $(2, -1)$
- Passes through  $(3, 5)$  and  $(3, -5)$
- Passes through  $(6, -1)$  and  $(-6, -1)$
- Passes through  $(3, 3)$  and  $(12, 6)$
- Passes through  $(-5, -5)$  and  $(-10, -20)$

In Exercises 31–58, solve the given problems.

- Sketch the graph of the inequality  $x^2 < 8y$ .
- Sketch the graph of the inequality  $9y \leq 4x^2$ .
- At what point(s) do the parabolas  $y^2 = 2x$  and  $x^2 = -16y$  intersect?
- Using trigonometric identities, show that the parametric equations  $x = \sin t$ ,  $y = 2(1 - \cos^2 t)$  are the equations of a parabola.
- Find the equation of the parabola with focus  $(6, 1)$  and directrix  $x = 0$ , by use of the definition. Sketch the curve.
- Find the equation of the parabola with focus  $(1, 1)$  and directrix  $y = 5$ , by use of the definition. Sketch the curve.
- Use a graphing calculator to view the parabola  $y^2 + 2x + 8y + 13 = 0$ .
- Use a graphing calculator to view the parabola  $y^2 - 2x - 6y + 19 = 0$ .
- The equation of a parabola with vertex  $(h, k)$  and axis parallel to the  $x$ -axis is  $(y - k)^2 = 4p(x - h)$ . (This is shown in Section 21.7.) Sketch the parabola for which  $(h, k)$  is  $(2, -3)$  and  $p = 2$ .

- The equation of a parabola with vertex  $(h, k)$  and axis parallel to the  $y$ -axis is  $(x - h)^2 = 4p(y - k)$ . (This is shown in Section 21.7.) Sketch the parabola for which  $(h, k)$  is  $(-1, 2)$  and  $p = -3$ .
- The chord of a parabola that passes through the focus and is parallel to the directrix is called the *latus rectum* of the parabola. Find the length of the latus rectum of the parabola  $y^2 = 4px$ .
- Find the equation of the circle that has the focus and the vertex of the parabola  $x^2 = 8y$  as the ends of a diameter.
- For either standard form of the equation of a parabola, describe what happens to the shape of the parabola as  $|p|$  increases.
- To ensure proper drainage, the surface of a lacrosse field with synthetic turf is parabolic. The field is 55.0 m wide and 66.0 cm higher in the centre than at the sides. Find the equation that represents the surface if the origin is at the centre of the field.
- The Lions' Gate Bridge in Vancouver, British Columbia, is a suspension bridge, and its supporting cables are parabolic. See Fig. 21.50. With the origin at the low point of the cable, what equation represents the cable if the supporting towers are 473 m apart and the maximum sag is 55.0 m?

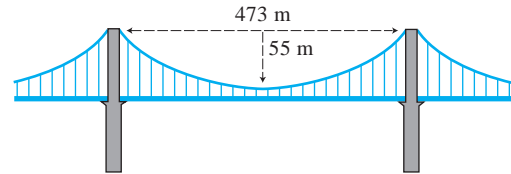


Fig. 21.50

- Each arch of the Allen Lambert Galleria in Brookfield Place in Toronto is a parabolic arch 26 m high at the centre and 13.7 m wide at the base. What equation represents the arch if the origin is at the top of the arch?
- The rate of development of heat  $H$  (in W) in a resistor of resistance  $R$  (in  $\Omega$ ) of an electric circuit is given by  $H = Ri^2$ , where  $i$  is the current (in A) in the resistor. Sketch the graph of  $H$  vs.  $i$ , if  $R = 6.0 \Omega$ .
- What is the length of the horizontal bar across the parabolically shaped window shown in Fig. 21.51?

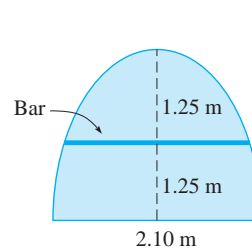


Fig. 21.51

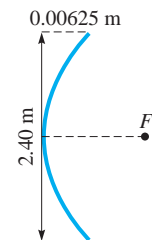


Fig. 21.52

- The primary mirror in the Hubble space telescope has a parabolic cross section, which is shown in Fig. 21.52. What is the focal length (vertex to focus) of the mirror?

50. A rocket is fired horizontally from a plane. Its horizontal distance  $x$  and vertical distance  $y$  from the point at which it was fired are given by  $x = v_0 t$  and  $y = \frac{1}{2} g t^2$ , where  $v_0$  is the initial velocity of the rocket,  $t$  is the time, and  $g$  is the acceleration due to gravity. Express  $y$  as a function of  $x$  and show that it is the equation of a parabola.
51. To launch a spacecraft to the moon, it is first put into orbit around earth and then into a parabolic path towards the moon. Assume the parabolic path is represented by  $x^2 = 4py$  and the spacecraft is later observed at  $(10, 3)$  (units in thousands of km) after launch. Will this path lead directly to the moon at  $(110, 340)$ ?
52. Under certain load conditions, a beam fixed at both ends is approximately parabolic in shape. If a beam is 4.0 m long and the deflection in the middle is 2.0 cm, find an equation to represent the shape of the beam.
53. A spotlight with a parabolic reflector is 15.0 cm wide and 6.50 cm deep. See Fig. 21.53 and Example 4. Where should the filament of the bulb be located so as to produce a beam of light?

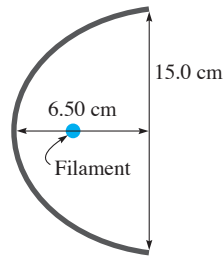


Fig. 21.53

54. A wire is fastened 12.0 m up on each of two telephone poles that are 60.0 m apart. Halfway between the poles, the wire is 10.0 m above the ground. Assuming the wire is parabolic, find the height of the wire 15.0 m from either pole.
55. The total annual fraction  $f$  of energy supplied by solar energy to a home is given by  $f = 0.065\sqrt{A}$ , where  $A$  is the area of the solar collector. Sketch the graph of  $f$  as a function of  $A$  ( $0 < A \leq 200 \text{ m}^2$ ).
56. The velocity  $v$  (in m/s) of a jet of water flowing from an opening in the side of a certain container is given by  $v = 4.4\sqrt{h}$ , where  $h$  is the depth (in m) of the opening. Sketch a graph of  $v$  vs.  $h$ .
57. A small island is 4 km north of a straight shoreline. A ship channel is equidistant between the island and the shoreline. Write an equation for the channel.
58. Under certain circumstances, the maximum power  $P$  (in W) in an electric circuit varies as the square of the voltage of the source  $E_0$  and inversely as the internal resistance  $R_i$  (in  $\Omega$ ) of the source. If 10 W is the maximum power for a source of 2.0 V and internal resistance of 0.10  $\Omega$ , sketch the graph of  $P$  vs.  $E_0$  if  $R_i$  remains constant.

Answers to Practice Exercises

1.  $F(-6, 0)$ ; dir.  $x = 6$     2.  $F(0, 10)$ ; dir.  $y = -10$

## 21.5 The Ellipse

Foci • Vertices • Major Axis • Minor Axis • Standard Equation—Axis along  $x$ -Axis • Standard Equation—Axis along  $y$ -Axis

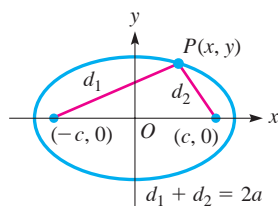


Fig. 21.54

The next important curve is the ellipse. An **ellipse** is defined as the locus of a point  $P(x, y)$  that moves so that the sum of its distances from two fixed points is constant. These fixed points are the **foci** of the ellipse. Letting this sum of distances be  $2a$  and the foci be the points  $(-c, 0)$  and  $(c, 0)$ , we have

$$\sqrt{(x - c)^2 + y^2} + \sqrt{(x + c)^2 + y^2} = 2a$$

See Fig. 21.54. The ellipse has its centre at the origin so that  $c$  is the length of the line segment from the centre to a focus. We will also see that  $a$  has a special meaning. Now, from Section 14.4, we see that in order to remove the radicals we should move one radical to the right and then square each side. This leads to the following steps:

$$\begin{aligned} \sqrt{(x + c)^2 + y^2} &= 2a - \sqrt{(x - c)^2 + y^2} \\ (x + c)^2 + y^2 &= 4a^2 - 4a\sqrt{(x - c)^2 + y^2} + (\sqrt{(x - c)^2 + y^2})^2 \\ x^2 + 2cx + c^2 + y^2 &= 4a^2 - 4a\sqrt{(x - c)^2 + y^2} + x^2 - 2cx + c^2 + y^2 \\ 4a\sqrt{(x - c)^2 + y^2} &= 4a^2 - 4cx \\ a\sqrt{(x - c)^2 + y^2} &= a^2 - cx \\ a^2(x^2 - 2cx + c^2 + y^2) &= a^4 - 2a^2cx + c^2x^2 \\ (a^2 - c^2)x^2 + a^2y^2 &= a^2(a^2 - c^2) \end{aligned}$$

We now define  $a^2 - c^2 = b^2$ . (The reason for this will be clear shortly.) Therefore,

$$b^2x^2 + a^2y^2 = a^2b^2$$

Dividing through by  $a^2b^2$ , we have

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{21.17}$$

A graphical analysis of this equation is found below.

The  $x$ -intercepts are  $(-a, 0)$  and  $(a, 0)$ . This means that  $2a$  (the sum of distances used in the derivation) is also the distance between the  $x$ -intercepts. The points  $(a, 0)$  and  $(-a, 0)$  are the **vertices** of the ellipse, and the line between them is the **major axis** [see Fig. 21.55(a)]. Thus,  $a$  is the length of the **semimajor axis**.

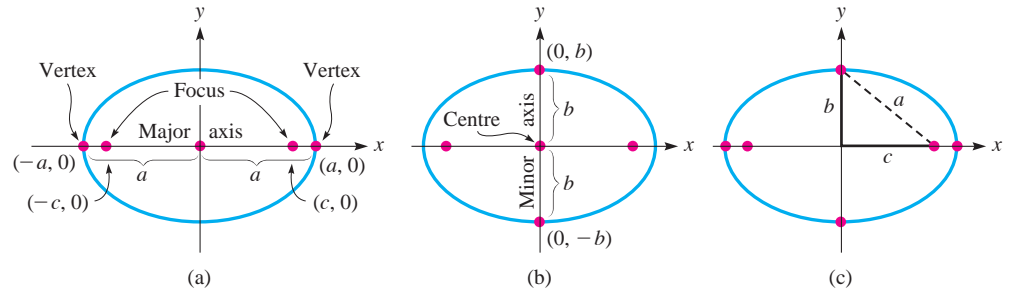


Fig. 21.55

We can now state that Eq. (21.17) is called the **standard equation** of the ellipse with its major axis along the  $x$ -axis and its centre at the origin.

The  $y$ -intercepts of this ellipse are  $(0, -b)$  and  $(0, b)$ . The line joining these intercepts is called the **minor axis** of the ellipse (Fig. 21.55(b)), which means  $b$  is the length of the **semiminor axis**. The intercept  $(0, b)$  is equidistant from  $(-c, 0)$  and  $(c, 0)$ . Since the sum of the distances from these points to  $(0, b)$  is  $2a$ , the distance  $(c, 0)$  to  $(0, b)$  must be  $a$ . Thus, we have a right triangle with line segments of lengths  $a, b$ , and  $c$ , with  $a$  as hypotenuse (Fig. 21.55(c)). Therefore,

$$a^2 = b^2 + c^2 \tag{21.18}$$

is the relation between distances  $a, b$ , and  $c$ . This also shows why  $b$  was defined as it was in the derivation of Eq. (21.17).

If we choose points on the  $y$ -axis as the foci, the standard equation of the ellipse, with its centre at the origin and its major axis along the  $y$ -axis, is

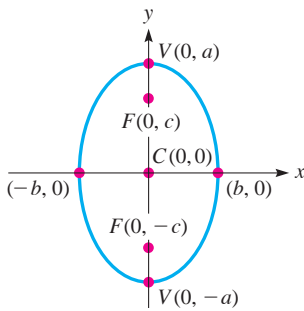


Fig. 21.56

$$\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1 \tag{21.19}$$

See Fig. 21.56.

We now summarize the properties of ellipses with centre at the origin.

**Standard Equation of an Ellipse with Centre at (0, 0)**

Equation:	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (21.17)	$\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1$ (21.19)
	$a > b > 0$	$a > b > 0$

Geometric relationship:	$a^2 = b^2 + c^2$	$a^2 = b^2 + c^2$
Major axis:	$x$ -axis (length $2a$ )	$y$ -axis (length $2a$ )
Minor axis:	$y$ -axis (length $2b$ ) $(0, -b)$ to $(0, b)$	$x$ -axis (length $2b$ ) $(-b, 0)$ to $(b, 0)$
Vertices:	$(-a, 0), (a, 0)$	$(0, -a), (0, a)$
Foci:	$(-c, 0), (c, 0)$	$(0, -c), (0, c)$
Symmetry:	$x$ -axis $y$ -axis origin	$x$ -axis $y$ -axis origin



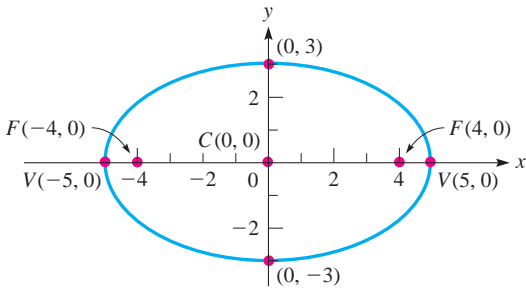


Fig. 21.57

**EXAMPLE 1** Standard equation—major axis along the  $x$ -axis

The ellipse  $\frac{x^2}{25} + \frac{y^2}{9} = 1$  fits the form of Eq. (21.17) because the larger square, 25, appears under  $x^2$ . Therefore,  $a^2 = 25$  and  $b^2 = 9$ , or  $a = 5$  and  $b = 3$ . This means that the vertices are  $(5, 0)$  and  $(-5, 0)$ , and the minor axis extends from  $(0, -3)$  to  $(0, 3)$ . See Fig. 21.57.

We find  $c$  from the relation  $c^2 = a^2 - b^2$ . This means that  $c^2 = 16$  and the foci are  $(4, 0)$  and  $(-4, 0)$ . ■

**EXAMPLE 2** Standard equation—major axis along the  $y$ -axis

The equation of the ellipse

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

$\begin{matrix} \uparrow & & \uparrow \\ b^2 & & a^2 \end{matrix}$

has the larger denominator, 9, under the  $y^2$ . Therefore, it fits the form of Eq. (21.19) with  $a^2 = 9$  and  $b^2 = 4$ . This means that the vertices are  $(0, 3)$  and  $(0, -3)$ , and the minor axis extends from  $(-2, 0)$  to  $(2, 0)$ . In turn, we know that  $c^2 = a^2 - b^2 = 9 - 4 = 5$  and that the foci are  $(0, \sqrt{5})$  and  $(0, -\sqrt{5})$ . This ellipse is shown in Fig. 21.58. ■

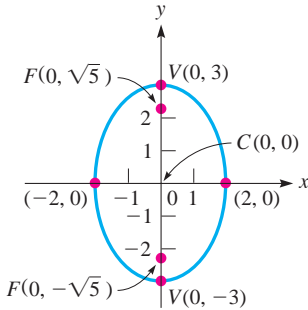


Fig. 21.58

**EXAMPLE 3** Find vertices, foci, minor axis

Find the coordinates of the vertices, the ends of the minor axis, and the foci of the ellipse  $4x^2 + 16y^2 = 64$ .

This equation must be put in standard form first, which we do by dividing through by 64. When this is done, we obtain

$$\frac{x^2}{16} + \frac{y^2}{4} = 1$$

$\leftarrow$  form requires + and 1

We see that  $a^2 = 16$  and  $b^2 = 4$ , which tells us that  $a = 4$  and  $b = 2$ . Then  $c = \sqrt{16 - 4} = \sqrt{12} = 2\sqrt{3}$ . Since  $a^2$  appears under  $x^2$ , the vertices are  $(4, 0)$  and  $(-4, 0)$ . The ends of the minor axis are  $(0, 2)$  and  $(0, -2)$ , and the foci are  $(2\sqrt{3}, 0)$  and  $(-2\sqrt{3}, 0)$ . See Fig. 21.59. ■

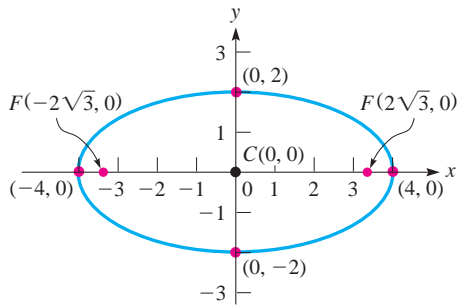


Fig. 21.59

**EXAMPLE 4** Ellipse—application

A satellite to study the earth's atmosphere has a minimum altitude of 1000 km and a maximum altitude of 3200 km. If the path of the satellite about the earth is an ellipse with the centre of the earth at one focus, what is the equation of its path? Assume that the radius of the earth is 6400 km.

We set up the coordinate system such that the centre of the ellipse is at the origin and the centre of the earth is at the right focus, as shown in Fig. 21.60. We know that the distance between vertices is

$$2a = 3200 + 6400 + 6400 + 1000 = 17\,000 \text{ km}$$

$$a = 8500 \text{ km}$$

The distance from the right focus to the right vertex is 7400 km. This tells us

$$c = a - 7400 = 8500 - 7400 = 1100 \text{ km}$$

We can now calculate  $b^2$  as

$$b^2 = a^2 - c^2 = 8500^2 - 1100^2 = 7.10 \times 10^7 \text{ km}^2$$

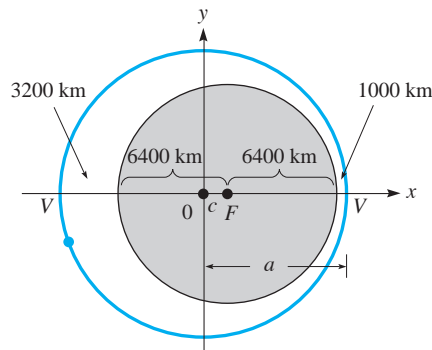


Fig. 21.60



Since  $a^2 = 8500^2 = 7.23 \times 10^7 \text{ km}^2$ , the equation is

$$\frac{x^2}{7.23 \times 10^7} + \frac{y^2}{7.10 \times 10^7} = 1$$

or

$$7.10x^2 + 7.23y^2 = 5.13 \times 10^8$$

**EXAMPLE 5 Find the equation given points**

Find the equation of the ellipse with its centre at the origin and an end of its minor axis at  $(2, 0)$  and which passes through  $(-1, \sqrt{6})$ .

Since the centre is at the origin and an end of the minor axis is at  $(2, 0)$ , we know that the ellipse is of the form of Eq. (21.19) and that  $b = 2$ . Thus, we have

$$\frac{y^2}{a^2} + \frac{x^2}{2^2} = 1$$

In order to find  $a^2$ , we use the fact that the ellipse passes through  $(-1, \sqrt{6})$ . This means that these coordinates satisfy the equation of the ellipse. This gives

$$\frac{(\sqrt{6})^2}{a^2} + \frac{(-1)^2}{4} = 1, \quad \frac{6}{a^2} = \frac{3}{4}, \quad a^2 = 8$$

Therefore, the equation of the ellipse, shown in Fig. 21.61, is

$$\frac{y^2}{8} + \frac{x^2}{4} = 1$$

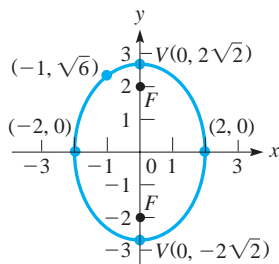


Fig. 21.61

**Practice Exercises**

Find the vertices and foci of each ellipse.

1.  $\frac{x^2}{9} + y^2 = 1$
2.  $25x^2 + 4y^2 = 25$

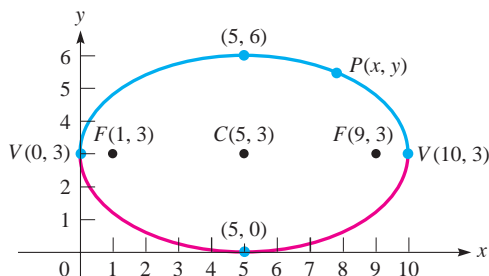
The following example illustrates the use of the definition of the ellipse to find the equation of an ellipse with its centre at a point other than the origin.

**EXAMPLE 6 Find the equation from the definition—two functions**

Using the definition, find the equation of the ellipse with foci at  $(1, 3)$  and  $(9, 3)$ , with major axis of 10.

Recalling that the sum of distances in the definition equals the length of the major axis, we now use the same method as in the derivation of Eq. (21.17).

$$\begin{aligned} \sqrt{(x-1)^2 + (y-3)^2} + \sqrt{(x-9)^2 + (y-3)^2} &= 10 && \text{use definition of ellipse} \\ \sqrt{(x-1)^2 + (y-3)^2} &= 10 - \sqrt{(x-9)^2 + (y-3)^2} && \text{isolate a radical} \\ x^2 - 2x + 1 + y^2 - 6y + 9 &= 100 - 20\sqrt{(x-9)^2 + (y-3)^2} + 100 - 20\sqrt{(x-9)^2 + (y-3)^2} && \text{square both sides and simplify} \\ x^2 - 2x + 1 + y^2 - 6y + 9 &= 200 - 40\sqrt{(x-9)^2 + (y-3)^2} + 200 - 40\sqrt{(x-9)^2 + (y-3)^2} \\ 20\sqrt{(x-9)^2 + (y-3)^2} &= 399 - 40\sqrt{(x-9)^2 + (y-3)^2} && \text{isolate radical} \\ 5\sqrt{(x-9)^2 + (y-3)^2} &= 99 - 10\sqrt{(x-9)^2 + (y-3)^2} && \text{divide by 4} \\ 25(x^2 - 18x + 81 + y^2 - 6y + 9) &= 9801 - 3960\sqrt{(x-9)^2 + (y-3)^2} + 3960\sqrt{(x-9)^2 + (y-3)^2} && \text{square both sides} \\ 9x^2 - 90x + 25y^2 - 150y + 225 &= 0 && \text{simplify} \end{aligned}$$



The additional  $x$ - and  $y$ -terms are characteristic of the equation of an ellipse whose centre is not at the origin (see Fig. 21.62).

The ellipse is shown in Fig. 21.62, with each half drawn in a different colour to represent a different function. Solving the original equation for  $y$ , the two functions are  $y = \frac{15 \pm 3\sqrt{10x - x^2}}{5}$ .

Fig. 21.62

■ The Polish astronomer Nicolaus Copernicus (1473–1543) is credited as being the first to suggest that the earth revolved about the sun, rather than the previously held belief that the earth was the centre of the universe.

We see that *the equation of an ellipse is characterized by both an  $x^2$ -term and a  $y^2$ -term, having different coefficients (in value but not in sign)*. We note that the coefficients of the squared terms differ, whereas for the circle, they are the same. We will consider the ellipse further in Sections 21.7, 21.8, and 21.9.

The ellipse has many applications. The orbits of the planets about the sun are elliptical. Gears, cams, and springs are often elliptical in shape. Arches are often constructed in the form of a semiellipse. Ellipses are also important for blood pattern analysis in forensic science, for parameter estimation in statistics, and in computer vision.

## EXERCISES 21.5

In Exercises 1 and 2, make the given changes in the indicated examples of this section and then solve the given problems.

- In Example 1, change the 9 to 36; find the vertices, ends of the minor axis, and foci; and sketch the ellipse.
- In Example 2, interchange the 4 and 9; find the vertices, ends of the minor axis, and foci; and sketch the ellipse.

In Exercises 3–16, find the coordinates of the vertices and foci of the given ellipses. Sketch each curve.

- $\frac{x^2}{4} + \frac{y^2}{1} = 1$
- $\frac{x^2}{100} + \frac{y^2}{64} = 1$
- $\frac{x^2}{25} + \frac{y^2}{144} = 1$
- $\frac{x^2}{49} + \frac{y^2}{81} = 1$
- $\frac{4x^2}{25} + \frac{y^2}{4} = 1$
- $x^2 + \frac{9y^2}{25} = 1$
- $4x^2 + 9y^2 = 324$
- $x^2 + 36y^2 = 144$
- $49x^2 + 4y^2 = 196$
- $y^2 = 25(1 - x^2)$
- $y^2 = 8(2 - x^2)$
- $2x^2 + 3y^2 = 600$
- $4x^2 + 25y^2 = 0.25$
- $9x^2 + 4y^2 = 0.09$

In Exercises 17–28, find the equations of the ellipses satisfying the given conditions. The centre of each is at the origin.

- Vertex (15, 0), focus (9, 0)
- Minor axis 8, vertex (0, -5)
- Focus (0, 8), major axis 34
- Vertex (0, 13), focus (0, -5)
- End of minor axis (0, 12), focus (8, 0)
- Sum of lengths of major and minor axes 18, focus (3, 0)
- Vertex (8, 0), passes through (2, 3)
- Focus (0, 2), passes through (-1,  $\sqrt{3}$ )
- Passes through (2, 2) and (1, 4)
- Passes through (-2, 2) and (1,  $\sqrt{6}$ )
- The sum of distances from (x, y) to (6, 0) and (-6, 0) is 20.
- The sum of distances from (x, y) to (0, 2) and (0, -2) is 5.

In Exercises 29–56, solve the given problems.

- Find any point(s) of intersection of the graphs of the ellipse  $4x^2 + 9y^2 = 40$  and the parabola  $y^2 = 4x$ .
- Find the equation of the circle that has the same centre as the ellipse  $4x^2 + 9y^2 = 36$  and is internally tangent to the ellipse.
- Find the equation of the ellipse with foci (-2, 1) and (4, 1) and a major axis of 10, by use of the definition. Sketch the curve.

- Find the equation of the ellipse with foci (1, 4) and (1, 0) that passes through (4, 4), by use of the definition. Sketch the curve.
- Use a graphing calculator to view the ellipse  $4x^2 + 3y^2 + 16x - 18y + 31 = 0$ .
- Use a graphing calculator to view the ellipse  $4x^2 + 8y^2 + 4x - 24y + 1 = 0$ .
- The equation of an ellipse with centre (h, k) and major axis parallel to the x-axis is  $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ . (This is shown in Section 21.7.) Sketch the ellipse that has a major axis of 6, a minor axis of 4, and for which (h, k) is (2, -1).
- The equation of an ellipse with centre (h, k) and major axis parallel to the y-axis is  $\frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1$ . (This is shown in Section 21.7.) Sketch the ellipse that has a major axis of 8, a minor axis of 6, and for which (h, k) is (1, 3).
- For what values of k does the ellipse  $x^2 + k^2 = 1$  have its vertices on the y-axis? Explain how these values are found.
- For what value of k does the ellipse  $x^2 + k^2y^2 = 25$  have a focus at (3, 0)? Explain how this value is found.
- Show that the ellipse  $2x^2 + 3y^2 - 8x - 4 = 0$  is symmetric to the x-axis.
- Show that the ellipse  $5x^2 + y^2 - 3y - 7 = 0$  is symmetric to the y-axis.
- Graph the inequality  $100x^2 + 49y^2 \leq 4900$ .
- Graph the inequality  $5x^2 + 4y^2 > 20$ .
- Show that the parametric equations  $x = 2 \sin t$  and  $y = 3 \cos t$  define an ellipse.
- For what values of k does  $y^2 = 1 + kx^2$  represent an ellipse with foci on the y-axis?
- The electric power P (in W) dissipated in a resistance R (in  $\Omega$ ) is given by  $P = Ri^2$ , where i is the current (in A) in the resistor. Find the equation for the total power of 64 W dissipated in two resistors, with resistances 2.0  $\Omega$  and 8.0  $\Omega$ , respectively, and with currents  $i_1$  and  $i_2$ , respectively. Sketch the graph, assuming that negative values of current are meaningful.
- The eccentricity e of an ellipse is defined as  $e = c/a$ . A cam in the shape of an ellipse can be described by the equation  $x^2 + 9y^2 = 81$ . Find the eccentricity of this elliptical cam.

47. A space object (dubbed 2003 UB313), larger and more distant than Pluto, was discovered in 2003. In its elliptical orbit with the sun at one focus, it is  $5.6 \times 10^9$  km from the sun at the closest, and  $14.4 \times 10^9$  km at the farthest. What is the eccentricity of its orbit? See Exercise 46.
48. Halley's Comet has an elliptical orbit with  $a = 17.94$  AU (AU is astronomical unit,  $1 \text{ AU} = 1.5 \times 10^8$  km) and  $b = 4.552$  AU, with the sun at one focus. What is the closest that the comet comes to the sun? Explain your method.
49. A draftsman draws a series of triangles with a base from  $(-3, 0)$  to  $(3, 0)$  and a perimeter of 14 cm (all measurements in centimetres). Find the equation of the curve on which all of the third vertices of the triangles are located.
50. A lithotripter is used to break up a kidney stone by placing the kidney stone at one focus of an ellipsoid end-section and a source of shock waves at the focus of the other end-section. If the vertices of the end-sections are 30.0 cm apart and a minor axis of the ellipsoid is 6.0 cm, how far apart are the foci? In a lithotripter, the shock waves are reflected as the sound waves noted in Exercise 51.
51. An ellipse has a focal property such that a light ray or sound wave emanating from one focus will be reflected through the other focus. Many buildings and structures are built with elliptical ceilings or walls, so that a sound from one focus is easily heard at the other focus. (Some examples include: Statuary Hall in the U.S. Capitol, the Echo Wall in the Temple of Heaven in Beijing, the Whispering Wall at the Barossa Reservoir in South Australia, and the Whispering Wall on the grounds of Parliament Hill in Ottawa.) If a building has a ceiling whose cross sections are part of an ellipse that can be described by the equation  $36x^2 + 225y^2 = 8100$  (measurements in metres), how far apart must two persons stand in order to whisper to each other using this focal property?
52. An airplane wing is designed such that a certain cross section is an ellipse 2.80 m wide and 0.40 m thick. Find the equation that can be used to describe the perimeter of this cross section.
53. A road passes through a tunnel with a semielliptical cross section 19.6 m wide and 5.5 m high at the centre. What is the height of the tallest vehicle that can pass through the tunnel at a point 6.7 m from the centre? See Fig. 21.63.

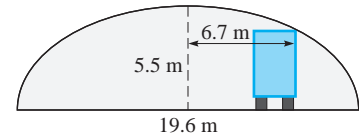


Fig. 21.63

54. An architect designs a window in the shape of an ellipse 1.50 m wide and 1.10 m high. Find the perimeter of the window from the formula  $p = \pi(a + b)$ . This formula gives a good *approximation* for the perimeter when  $a$  and  $b$  are nearly equal.
55. The ends of a horizontal tank 20.0 m long are ellipses, which can be described by the equation  $9x^2 + 20y^2 = 180$ , where  $x$  and  $y$  are measured in metres. The area of an ellipse is  $A = \pi ab$ . Find the volume of the tank.
56. A laser beam 6.80 mm in diameter is incident on a plane surface at an angle of  $62.0^\circ$ , as shown in Fig. 21.64. What is the elliptical area that the laser covers on the surface? (See Exercise 55.)

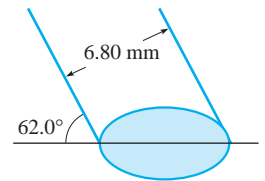


Fig. 21.64

### Answers to Practice Exercises

- $V(3, 0)$ ,  $V(-3, 0)$ ;  $F(2\sqrt{2}, 0)$ ,  $F(-2\sqrt{2}, 0)$
- $V(0, 5/2)$ ,  $V(0, -5/2)$ ;  $F(0, \sqrt{21}/2)$ ,  $F(0, -\sqrt{21}/2)$

## 21.6 The Hyperbola

**Foci • Vertices • Asymptotes •  
Transverse Axis • Conjugate Axis •  
Standard Equation—Axis along  $x$ -Axis •  
Standard Equation—Axis along  $y$ -Axis •  
 $xy = c$  Hyperbola**

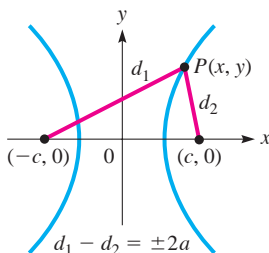


Fig. 21.65

A **hyperbola** is defined as the locus of a point  $P(x, y)$  that moves so that the difference of the distances from two fixed points (the **foci**) is constant. We choose the foci to be  $(-c, 0)$  and  $(c, 0)$  (see Fig. 21.65), and the constant difference to be  $2a$ . As with the ellipse,  $c$  is the length of the line segment from the centre to a focus, and  $a$  (as we will see) is the length of the line segment from the centre to a vertex. Therefore,

$$\sqrt{(x + c)^2 + y^2} - \sqrt{(x - c)^2 + y^2} = 2a$$

Following the same procedure as with the ellipse, the equation of the hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad (21.20)$$

When we derive this equation, *we have a definition of the relation between  $a$ ,  $b$ , and  $c$  that is different from that for the ellipse.* This relation is

$$c^2 = a^2 + b^2 \quad (21.21)$$

In Eq. (21.20), if we let  $y = 0$ , we find that the  $x$ -intercepts are  $(-a, 0)$  and  $(a, 0)$ , just as they are for the ellipse. *These are the vertices of the hyperbola.* For  $x = 0$ , we find that we have imaginary solutions for  $y$ , which means there are no points on the curve that correspond to a value of  $x = 0$ .

To find the meaning of  $b$ , we solve Eq. (21.20) for  $y$  in the special form:

$$\begin{aligned} \frac{y^2}{b^2} &= \frac{x^2}{a^2} - 1 \\ &= \frac{x^2}{a^2} - \frac{a^2x^2}{a^2x^2} = \frac{x^2}{a^2} \left( 1 - \frac{a^2}{x^2} \right) \\ y^2 &= \frac{b^2x^2}{a^2} \left( 1 - \frac{a^2}{x^2} \right) && \text{multiply through by } b^2 \text{ and} \\ &&& \text{take the square root of each side} \\ y &= \pm \frac{bx}{a} \sqrt{1 - \frac{a^2}{x^2}} \end{aligned} \tag{21.22}$$

We note that, if large values of  $x$  are assumed in Eq. (21.22), the quantity under the radical becomes approximately 1. In fact, the larger  $x$  becomes, the nearer to 1 this expression becomes since the  $x^2$  in the denominator of  $a^2/x^2$  makes this term nearly zero. Thus, for large values of  $x$ , Eq. (21.22) is approximately

$$y = \pm \frac{bx}{a} \tag{21.23}$$

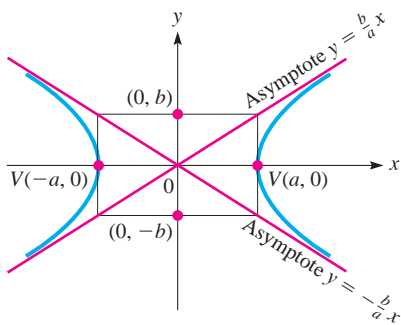


Fig. 21.66

Eq. (21.23) is seen to represent two straight lines, each of which passes through the origin. One has a slope of  $b/a$ , and the other has a slope of  $-b/a$ . *These lines are called the asymptotes of the hyperbola.* An asymptote is a line that the curve approaches as one of the variables approaches some particular value. The graph of the tangent function also has asymptotes, as we saw in Fig. 10.23. We can designate this limiting procedure with notation introduced in Chapter 19 by

$$\text{As } x \rightarrow \pm \infty, y \rightarrow \pm \frac{bx}{a}$$

The easiest way to sketch a hyperbola is to draw its asymptotes and then draw the hyperbola out from each vertex so that it comes closer and closer to each asymptote as  $x$  becomes numerically larger. To draw the asymptotes, first draw a small rectangle  $2a$  by  $2b$  with the origin at the centre, as shown in Fig. 21.66. Then straight lines, the asymptotes, are drawn through opposite vertices of the rectangle. This shows us that the significance of the value of  $b$  is in the slopes of the asymptotes.

Eq. (21.20) is called the **standard equation of the hyperbola with its centre at the origin**. It has a **transverse axis** of length  $2a$  along the  $x$ -axis and a **conjugate axis** of length  $2b$  along the  $y$ -axis. This means that  $a$  represents the length of the semitransverse axis and  $b$  represents the length of the semiconjugate axis. See Fig. 21.67. From the definition of  $c$ , it is the length of the line segment from the centre to a focus. Also,  $c$  is the length of the semidiagonal of the rectangle, as shown in Fig. 21.67. This shows us the geometric meaning of the relationship among  $a$ ,  $b$ , and  $c$  given in Eq. (21.21).

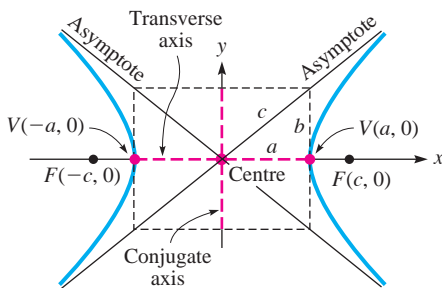


Fig. 21.67

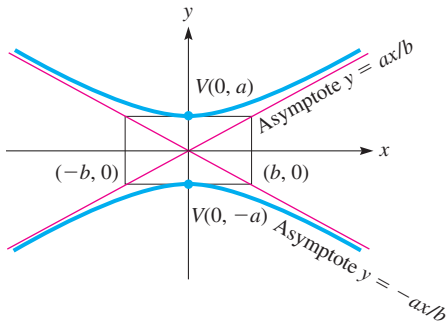


Fig. 21.68

If the transverse axis is along the y-axis and the conjugate axis is along the x-axis, the equation of the hyperbola with its centre at the origin (see Fig. 21.68) is

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \tag{21.24}$$

In this case, a similar argument as the one leading to Eq. (21.23) shows that the equations of the asymptotes are  $y = \pm (a/b)x$ .

We now summarize the properties of hyperbolas with centre at the origin.

**Standard Equation of a Hyperbola with Centre at (0, 0)**

Equation:  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  (21.20)  $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$  (21.24)

Geometric relationship:  $c^2 = a^2 + b^2$   $c^2 = a^2 + b^2$

Transverse axis: x-axis (length 2a) y-axis (length 2a)

Conjugate axis: y-axis (length 2b) x-axis (length 2b)  
 (0, -b) to (0, b) (-b, 0) to (b, 0)

Vertices: (-a, 0), (a, 0) (0, -a), (0, a)

Foci: (-c, 0), (c, 0) (0, -c), (0, c)

Asymptotes:  $y = \pm \frac{bx}{a}$   $y = \pm \frac{ax}{b}$

Symmetry: x-axis y-axis origin  
 y-axis y-axis origin  
 origin origin

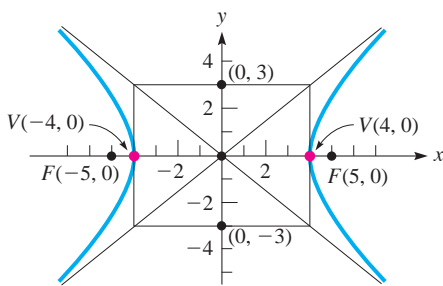


Fig. 21.69

**EXAMPLE 1 Standard equation—transverse axis on the x-axis**

The hyperbola  $\frac{x^2}{16} - \frac{y^2}{9} = 1$   
 $a^2 \uparrow \quad \quad \uparrow b^2$

fits the form of Eq. (21.20). We know that it fits Eq. (21.20) and not Eq. (21.24) since the  $x^2$ -term is the positive term with 1 on the right. From the equation, we see that  $a^2 = 16$  and  $b^2 = 9$ , or  $a = 4$  and  $b = 3$ . In turn, this means the vertices are (4, 0) and (-4, 0) and the conjugate axis extends from (0, -3) to (0, 3).

Since  $c^2 = a^2 + b^2$ , we find that  $c^2 = 25$ , or  $c = 5$ . The foci are (-5, 0) and (5, 0).

Drawing the rectangle and the asymptotes in Fig. 21.69, we then sketch in the hyperbola from each vertex toward each asymptote. ■

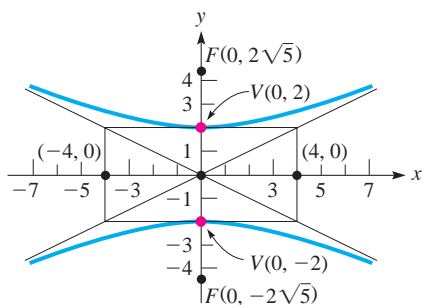


Fig. 21.70

**EXAMPLE 2 Standard equation—transverse axis on the y-axis**

The hyperbola  $\frac{y^2}{4} - \frac{x^2}{16} = 1$   
 $a^2 \uparrow \quad \quad \uparrow b^2$

has vertices at (0, -2) and (0, 2). Its conjugate axis extends from (-4, 0) to (4, 0). The foci are (0, -2√5) and (0, 2√5). We find this directly from the equation since the  $y^2$ -term is the positive term with 1 on the right. This means the equation fits the form of Eq. (21.24) with  $a^2 = 4$  and  $b^2 = 16$ . Also,  $c^2 = 20$ , which means that  $c = \sqrt{20} = 2\sqrt{5}$ . The hyperbola is shown in Fig. 21.70. ■

**EXAMPLE 3 Find vertices, foci**

Determine the coordinates of the vertices and foci of the hyperbola

$$4x^2 - 9y^2 = 36$$

First, by dividing through by 36, we have

$$\frac{x^2}{9} - \frac{y^2}{4} = 1$$

← form requires  
- and 1

From this form, we see that  $a^2 = 9$  and  $b^2 = 4$ . In turn, this tells us that  $a = 3$ ,  $b = 2$ , and  $c = \sqrt{9 + 4} = \sqrt{13}$ . Since  $a^2$  appears under  $x^2$ , the equation fits the form of Eq. (21.20). Therefore, the vertices are  $(-3, 0)$  and  $(3, 0)$  and the foci are  $(-\sqrt{13}, 0)$  and  $(\sqrt{13}, 0)$ . The hyperbola is shown in Fig. 21.71. ■

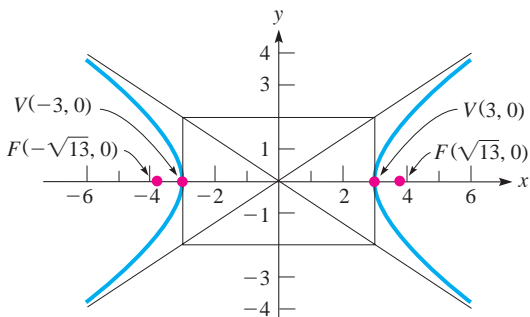


Fig. 21.71

**EXAMPLE 4 Hyperbola—application**

In physics, it is shown that where the velocity of a fluid is greatest, the pressure is the least. In designing an experiment to study this effect in the flow of water, a pipe is constructed such that its lengthwise cross section is hyperbolic. The pipe is 1.0 m long, 0.2 m in diameter at the narrowest point in the middle, and 0.4 m in diameter at each end. What is the equation that represents the cross section of the pipe as shown in Fig. 21.72?

As shown, the hyperbola has its transverse axis along the  $y$ -axis and its centre at the origin. This means the general equation is given by Eq. (21.24). Since the radius at the middle of the pipe is 0.1 m, we know that  $a = 0.1$  m. Also, since it is 1.0 m long and the radius at the end is 0.2 m, we know the point  $(0.5, 0.2)$  is on the hyperbola. This point must satisfy the equation

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \quad \text{Eq. (21.24)}$$

point  $(0.5, 0.2)$  satisfies equation

$$a = 0.1 \rightarrow \frac{0.2^2}{0.1^2} - \frac{0.5^2}{b^2} = 1$$

$$4 - \frac{0.25}{b^2} = 1, \quad 3b^2 = 0.25, \quad b^2 = 0.083$$

$$\frac{y^2}{0.1^2} - \frac{x^2}{0.083} = 1 \quad \text{substituting } a = 0.1, b^2 = 0.083 \text{ in Eq. (21.24)}$$

$$100y^2 - 12x^2 = 1 \quad \text{equation of cross section} \quad \blacksquare$$

**Practice Exercises**

Find the vertices and foci of each hyperbola.

1.  $\frac{x^2}{36} + \frac{y^2}{13} = 1$
2.  $4y^2 - x^2 = 4$

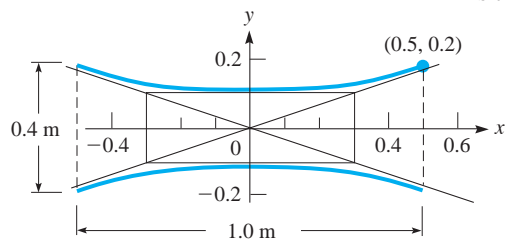


Fig. 21.72

**EXAMPLE 5 The hyperbola as a graph of two functions**

A hyperbola represented by either Eq. (21.20) or Eq. (21.24) is the graph of two functions, which are found when we solve the equation for  $y$ . One represents the upper half of the hyperbola, and the other represents the lower half. For the hyperbola in Example 4, the functions are

$$y = \pm \sqrt{(12x^2 + 1)/100}$$

Both functions would have to be entered into a graphing calculator to view the hyperbola shown in Fig. 21.72. ■

Equations (21.20) and (21.24) give us the standard forms of the equation of the hyperbola with its centre at the origin and its foci on one of the coordinate axes. There is another important equation form that represents a hyperbola:



$$xy = c$$

(21.25)

The asymptotes of this hyperbola are the coordinate axes, and the foci are on the line  $y = x$  if  $c$  is positive or on the line  $y = -x$  if  $c$  is negative.

The hyperbola represented by Eq. (21.25) is symmetric to the origin, for if  $-x$  replaces  $x$  and  $-y$  replaces  $y$  at the same time, we obtain  $(-x)(-y) = c$ , or  $xy = c$ . The equation is unchanged. However, if  $-x$  replaces  $x$  or  $-y$  replaces  $y$ , but not both, the sign on the left is changed. This means it is not symmetric to either axis. Here,  $c$  represents a constant, and it is not related to the focus. The following two examples deal with hyperbolas of this type.

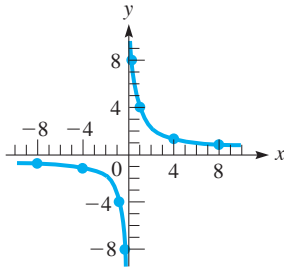


Fig. 21.73

### EXAMPLE 6 $xy = c$ Hyperbola

Plot the graph of the hyperbola  $xy = 4$ .

We find the values in the table below and then plot the appropriate points. Here, it is permissible to use a limited number of points, since we know the equation represents a hyperbola. Therefore, using  $y = 4/x$ , we obtain the values

$x$	-8	-4	-1	$-\frac{1}{2}$	$\frac{1}{2}$	1	4	8
$y$	$-\frac{1}{2}$	-1	-4	-8	8	4	1	$\frac{1}{2}$

Note that neither  $x$  nor  $y$  may equal zero. The hyperbola is shown in Fig. 21.73.

If the constant on the right is negative (for example, if  $xy = -4$ ), then the two branches of the hyperbola are in the second and fourth quadrants. ■

$f$ (THz)	750	600	500	430
$\lambda$ (nm)	400	500	600	700

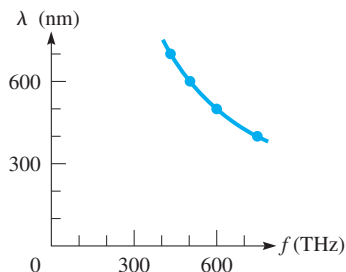


Fig. 21.74

### EXAMPLE 7 $xy = c$ Hyperbola—application

For a light wave, the product of its frequency  $f$  of vibration and its wavelength  $\lambda$  is a constant, and this constant is the speed of light  $c$ . For green light, for which  $f = 600$  THz,  $\lambda = 500$  nm. Graph  $\lambda$  as a function of  $f$  for any light wave.

From the statement above, we know that  $f\lambda = c$ , and from the given values, we have

$$(600 \text{ THz})(500 \text{ nm}) = (6.0 \times 10^{14} \text{ Hz})(5.0 \times 10^{-7} \text{ m}) = 3.0 \times 10^8 \text{ m/s}$$

which means  $c = 3.0 \times 10^8$  m/s. We are to sketch  $f\lambda = 3.0 \times 10^8$ . Solving for  $\lambda$  as  $\lambda = 3.0 \times 10^8/f$ , **we have the table at the left** (only positive values have meaning). See Fig. 21.74. (Violet light has wavelengths of about 400 nm, orange light has wavelengths of about 600 nm, and red light has wavelengths of about 700 nm.) ■

We can conclude that *the equation of a hyperbola is characterized by the presence of both an  $x^2$ -term and a  $y^2$ -term, having different signs, or by the presence of an  $xy$ -term with no squared terms.* We will consider the equation of the hyperbola further in Sections 21.7, 21.8, and 21.9.

The hyperbola has some very useful applications. The LORAN radio navigation system is based on the use of hyperbolic paths. Some reflecting telescopes use hyperbolic mirrors. The paths of comets that never return to pass by the sun are hyperbolic. Some applications are illustrated in the exercises.

■ The first reasonable measurement of the speed of light was made by the Danish astronomer Olaf Roemer (1644–1710). He measured the time required for light to come from the moons of Jupiter across the earth's orbit.



## EXERCISES 21.6

In Exercises 1 and 2, make the given changes in the indicated examples of this section and then solve the resulting problems.

- In Example 2, interchange the denominators of 4 and 16; find the vertices, ends of the conjugate axis, and foci. Sketch the curve.
- In Example 3, change  $-9y^2$  to  $-y^2$  and then follow the same instructions as in Exercise 1.

In Exercises 3–16, find the coordinates of the vertices and the foci of the given hyperbolas. Sketch each curve.

- $\frac{x^2}{25} - \frac{y^2}{144} = 1$
- $\frac{x^2}{16} - \frac{y^2}{4} = 1$
- $\frac{y^2}{9} - \frac{x^2}{1} = 1$
- $\frac{y^2}{4} - \frac{x^2}{21} = 1$
- $\frac{4x^2}{25} - \frac{y^2}{4} = 1$
- $\frac{9y^2}{25} - x^2 = 1$
- $4x^2 - y^2 = 4$
- $x^2 - 9y^2 = 1$
- $2y^2 - 5x^2 = 10$
- $9y^2 - 16x^2 = 144$
- $y^2 = 4(x^2 + 1)$
- $y^2 = 9(x^2 - 1)$
- $4x^2 - y^2 = 0.64$
- $9y^2 - x^2 = 0.36$

In Exercises 17–28, find the equations of the hyperbolas satisfying the given conditions. The centre of each is at the origin.

- Vertex  $(3, 0)$ , focus  $(5, 0)$
- Vertex  $(0, 1)$ , focus  $(0, \sqrt{3})$
- Conjugate axis = 48, vertex  $(0, 10)$
- Sum of lengths of transverse and conjugate axes 28, focus  $(10, 0)$
- Passes through  $(2, 3)$ , focus  $(2, 0)$
- Passes through  $(8, \sqrt{3})$ , vertex  $(4, 0)$
- Passes through  $(5, 4)$  and  $(3, \frac{4}{5}\sqrt{5})$
- Passes through  $(1, 2)$  and  $(2, 2\sqrt{2})$
- Asymptote  $y = 2x$ , vertex  $(1, 0)$
- Asymptote  $y = -4x$ , vertex  $(0, 4)$
- The difference of distances to  $(x, y)$  from  $(10, 0)$  and  $(-10, 0)$  is 12.
- The difference of distances to  $(x, y)$  from  $(0, 4)$  and  $(0, -4)$  is 6.

In Exercises 29–54, solve the given problems.

- Sketch the graph of the hyperbola  $xy = 2$ .
- Sketch the graph of the hyperbola  $xy = -4$ .
- Show that the parametric equations  $x = \sec t$ ,  $y = \tan t$  define a hyperbola.
- Show that all hyperbolas  $\frac{x^2}{\cos^2 \theta} - \frac{y^2}{\sin^2 \theta} = 1$  have foci at  $(\pm 1, 0)$  for all values of  $\theta$ .
- Find any points of intersection of the ellipse  $2x^2 + y^2 = 17$  and the hyperbola  $y^2 - x^2 = 5$ .
- Find any points of intersection of the hyperbolas  $x^2 - 3y^2 = 22$  and  $xy = 5$ .
- Find the equation of the hyperbola with foci  $(1, 2)$  and  $(11, 2)$ , and a transverse axis of 8, by use of the definition. Sketch the curve.

- Find the equation of the hyperbola with vertices  $(-2, 4)$  and  $(-2, -2)$ , and a conjugate axis of 4, by use of the definition. Sketch the curve.
- Use a graphing calculator to view the hyperbola  $x^2 - 4y^2 + 4x + 32y - 64 = 0$ .
- Use a graphing calculator to view the hyperbola  $5y^2 - 4x^2 + 8x + 40y + 56 = 0$ .
- The equation of a hyperbola with centre  $(h, k)$  and transverse axis parallel to the  $x$ -axis is  $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ . (This is shown in Section 21.7.) Sketch the hyperbola that has a transverse axis of 4, a conjugate axis of 6, and for which  $(h, k)$  is  $(-3, 2)$ .
- The equation of a hyperbola with centre  $(h, k)$  and transverse axis parallel to the  $y$ -axis is  $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$ . (This is shown in Section 21.7.) Sketch the hyperbola that has a transverse axis of 2, a conjugate axis of 8, and for which  $(h, k)$  is  $(5, 0)$ .
- Two concentric (same centre) hyperbolas are called conjugate hyperbolas if the transverse and conjugate axes of one are, respectively, the conjugate and transverse axes of the other. What is the equation of the hyperbola conjugate to the hyperbola in Exercise 18?
- As with an ellipse, the *eccentricity*  $e$  of a hyperbola is defined as  $e = c/a$ . Find the eccentricity of the hyperbola  $2x^2 - 3y^2 = 24$ .
- Graph the inequality  $100x^2 - 49y^2 \leq 4900$ .
- Draw a sketch of the graph of the region in which the points satisfy the system of inequalities  $xy < 4$ ,  $y > x$ .
- Find the equation of the hyperbola that has the same foci as the ellipse  $\frac{x^2}{169} + \frac{y^2}{144} = 1$  and passes through  $(4\sqrt{2}, 3)$ .
- Explain how a branch of a hyperbola differs from a parabola.
- The cross section of the roof of a storage building shown in Fig. 21.75 is hyperbolic, with the horizontal beam passing through the focus. Find the equation of the hyperbola such that its centre is at the origin.

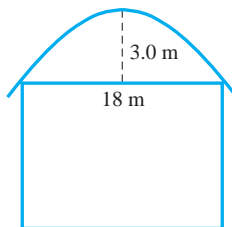


Fig. 21.75

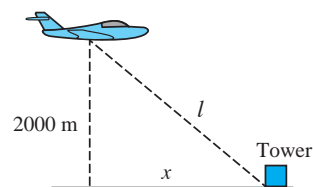


Fig. 21.76

- A plane is flying at a constant altitude of 2000 m. Show that the equation relating the horizontal distance  $x$  and the direct-line distance  $l$  from a control tower to the plane is that of a hyperbola. Sketch the graph of  $l$  as a function of  $x$ . See Fig. 21.76.
- A jet travels 600 km at a speed of  $v$  km/h for  $t$  hours. Graph the equation relating  $v$  as a function of  $t$ .

50. A drain pipe 100 m long has an inside diameter  $d$  (in m) and an outside diameter  $D$  (in m). If the volume of material of the pipe itself is  $0.50 \text{ m}^3$ , what is the equation relating  $d$  and  $D$ ? Graph  $D$  as a function of  $d$ .
51. Ohm's law in electricity states that the product of the current  $i$  and the resistance  $R$  equals the voltage  $V$  across the resistance. If a battery of  $6.00 \text{ V}$  is placed across a variable resistor  $R$ , find the equation relating  $i$  and  $R$  and sketch the graph of  $i$  as a function of  $R$ .
52. A ray of light directed at one focus of a hyperbolic mirror is reflected toward the other focus. Find the equation that represents the hyperbolic mirror shown in Fig. 21.77.
53. A radio signal is sent simultaneously from stations A and B 600 km apart on the coast of New South Wales, Australia. A ship receives the signal from A 1.20 ms before it receives the signal from B. Given that radio signals travel at  $300 \text{ km/ms}$ , draw a graph showing the possible locations of the ship. This problem illustrates the basis of LORAN (Long Range Navigation).
54. Maximum intensity for monochromatic (single-color) light from two sources occurs where the difference in distances from the sources is an integral number of wavelengths. Find the equation of the curves of maximum intensity in a thin film between the sources where the difference in paths is two wavelengths and the sources are four wavelengths apart. Let the sources be on the  $x$ -axis and the origin midway between them. Use units of one wavelength for both  $x$  and  $y$ .

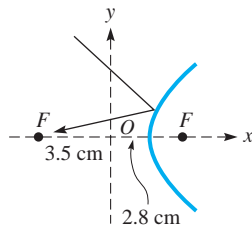


Fig. 21.77

Answers to Practice Exercises

1.  $V(6, 0), V(-6, 0); F(7, 0), F(-7, 0)$
2.  $V(0, 1), V(0, -1); F(0, \sqrt{5}), F(0, -\sqrt{5})$

## 21.7 Translation of Axes

Centres of Curves Not at the Origin • Axes of Curves Parallel to the Coordinate Axes • Translation of Axes

The equations we have considered for the parabola, the ellipse, and the hyperbola are those for which the centre of the ellipse or hyperbola, or the vertex of the parabola, is at the origin. In this section, we consider, without specific use of the definition, the equations of these curves for the cases in which the axis of the curve is parallel to one of the coordinate axes. This is done by **translation of axes**.

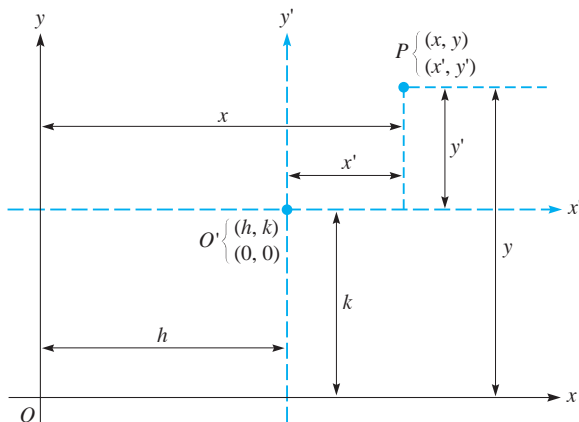


Fig. 21.78

In Fig. 21.78, we choose a point  $(h, k)$  in the  $xy$ -coordinate plane as the origin of another coordinate system, the  $x'y'$ -coordinate system. The  $x'$ -axis is parallel to the  $x$ -axis and the  $y'$ -axis is parallel to the  $y$ -axis. Every point now has two sets of coordinates  $(x, y)$  and  $(x', y')$ . We see that

$$x = x' + h \quad \text{and} \quad y = y' + k \tag{21.26}$$

Eq. (21.26) can also be written in the form

$$x' = x - h \quad \text{and} \quad y' = y - k \tag{21.27}$$

### EXAMPLE 1 Find the equation—given vertex and focus

Find the equation of the parabola with vertex  $(2, 4)$  and focus  $(4, 4)$ .

If we let the origin of the  $x'y'$ -coordinate system be the point  $(2, 4)$ , then the point  $(4, 4)$  is the point  $(2, 0)$  in the  $x'y'$ -system. This means  $p = 2$  and  $4p = 8$ . See Fig. 21.79. In the  $x'y'$ -system, the equation is

$$(y')^2 = 8(x')$$

Using Eq. (21.27), we have

$$(y - 4)^2 = 8(x - 2)$$

as the equation of the parabola in the  $xy$ -coordinate system. ■

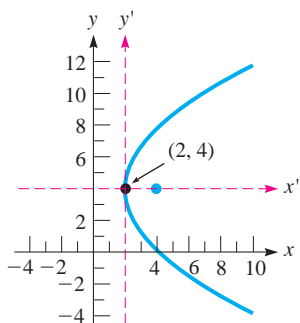


Fig. 21.79

Following the method of Example 1, by writing the equation of the curve in the  $x'y'$ -system and then using Eq. (21.27), we have the following more general forms of the equations of the parabola, ellipse, and hyperbola:

Parabola, vertex $(h, k)$ :	$(y - k)^2 = 4p(x - h)$	(axis parallel to $x$ -axis)	(21.28)
	$(x - h)^2 = 4p(y - k)$	(axis parallel to $y$ -axis)	(21.29)
Ellipse, centre $(h, k)$ :	$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$	(major axis parallel to $x$ -axis)	(21.30)
	$\frac{(y - k)^2}{a^2} + \frac{(x - h)^2}{b^2} = 1$	(major axis parallel to $y$ -axis)	(21.31)
Hyperbola, centre $(h, k)$ :	$\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1$	(transverse axis parallel to $x$ -axis)	(21.32)
	$\frac{(y - k)^2}{a^2} - \frac{(x - h)^2}{b^2} = 1$	(transverse axis parallel to $y$ -axis)	(21.33)

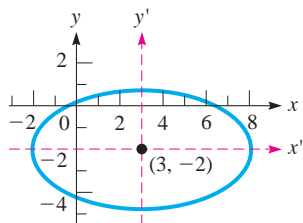


Fig. 21.80

**EXAMPLE 2** Describing a curve given its equation

Describe the curve defined by the equation

$$\frac{(x - 3)^2}{25} + \frac{(y + 2)^2}{9} = 1$$

We see that this equation fits the form of Eq. (21.30) with  $h = 3$  and  $k = -2$ . It is the equation of an ellipse with its centre at  $(3, -2)$  and its major axis parallel to the  $x$ -axis. The semimajor axis is  $a = 5$ , and the semiminor axis is  $b = 3$ . The ellipse is shown in Fig. 21.80. ■

**EXAMPLE 3** Finding the centre of a hyperbola

Find the centre of the hyperbola  $2x^2 - y^2 - 4x - 4y - 4 = 0$ .

To analyse this curve, we first complete the square in the  $x$ -terms and in the  $y$ -terms. This will allow us to recognize properly the choice of  $h$  and  $k$ .

$$\begin{aligned} 2x^2 - 4x - y^2 - 4y &= 4 \\ 2(x^2 - 2x \quad) - (y^2 + 4y \quad) &= 4 \\ 2(x^2 - 2x + 1) - (y^2 + 4y + 4) &= 4 + 2 - 4 \end{aligned}$$

We note here that when we added 1 to complete the square of the  $x$ -terms within the parentheses, *we were actually adding 2 to the left side*. Thus, we added 2 to the right side. Similarly, when we added 4 to the  $y$ -terms within the parentheses, *we were actually subtracting 4 from the left side*. Continuing, we have

$$\begin{aligned} 2(x - 1)^2 - (y + 2)^2 &= 2 \\ \frac{(x - 1)^2}{1} - \frac{(y + 2)^2}{2} &= 1 \end{aligned}$$

coordinates of centre  $(1, -2)$

Therefore, the centre of the hyperbola is  $(1, -2)$ . See Fig. 21.81. ■

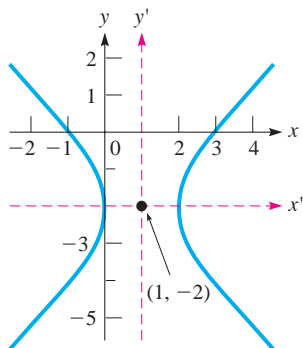


Fig. 21.81

**Practice Exercise**

- Find the centre and foci of the ellipse  $2x^2 + 3y^2 - 8x + 18y + 29 = 0$ .

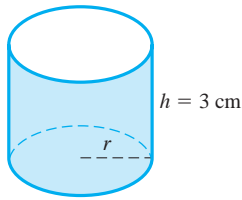


Fig. 21.82

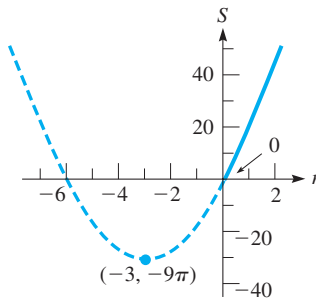


Fig. 21.83

**EXAMPLE 4** Translation of axes—application

Cylindrical glass beakers are to be made with a height of 3 cm. Express the surface area in terms of the radius of the base and sketch the curve.

The total surface area  $S$  of a beaker is the sum of the area of the base and the lateral surface area of the side. In general,  $S$  in terms of the radius  $r$  of the base and height  $h$  of the side is  $S = \pi r^2 + 2\pi rh$ . Since  $h = 3$  cm, we have

$$S = \pi r^2 + 6\pi r$$

which is the desired relationship. See Fig. 21.82.

To analyse the equation relating  $S$  and  $r$ , we complete the square of the  $r$  terms:

$$\begin{aligned} S &= \pi(r^2 + 6r) \\ S + 9\pi &= \pi(r^2 + 6r + 9) && \text{complete the square} \\ S + 9\pi &= \pi(r + 3)^2 && \text{vertex } (-3, -9\pi) \\ (r + 3)^2 &= \frac{1}{\pi}(S + 9\pi) \end{aligned}$$

This represents a parabola with vertex  $(-3, -9\pi)$ . Since  $4p = 1/\pi$ , then  $p = 1/(4\pi)$ , and the focus is  $(-3, \frac{1}{4\pi} - 9\pi)$ , as shown in Fig. 21.83. The part of the graph for negative  $r$  is dashed since only positive values have meaning. ■

**EXERCISES 21.7**

In Exercises 1 and 2, make the given changes in the indicated examples of this section and then solve the resulting problem.

- In Example 2, change  $y + 2$  to  $y - 2$  and change the sign before the second term from  $+$  to  $-$ . Then describe and sketch the curve.
- In Example 3, change the fourth and fifth terms from  $-4y - 4$  to  $+6y - 9$  and then find the centre.

In Exercises 3–10, describe the curve represented by each equation. Identify the type of curve and its centre (or vertex if it is a parabola). Sketch each curve.

- $(y - 2)^2 = 4(x + 1)$
- $\frac{(x + 4)^2}{4} + \frac{(y - 1)^2}{1} = 1$
- $\frac{(x - 1)^2}{4} - \frac{(y - 2)^2}{9} = 1$
- $(y + 5)^2 = -8(x - 2)$
- $\frac{(x + 1)^2}{1} + \frac{y^2}{9} = 1$
- $\frac{(y - 4)^2}{16} - \frac{(x + 2)^2}{4} = 1$
- $(x + 3)^2 = -12(y - 1)$
- $\frac{x^2}{0.16} + \frac{(y + 1)^2}{0.25} = 1$

In Exercises 11–22, find the equation of each of the curves described by the given information.

- Parabola: vertex  $(-1, 3)$ , focus  $(3, 3)$
- Parabola: focus  $(2, -5)$ , directrix  $y = 3$
- Parabola: axis, directrix are coordinate axes, focus  $(12, 0)$
- Parabola: vertex  $(4, 4)$ , vertical directrix, passes through  $(0, 1)$
- Ellipse: centre  $(-2, 2)$ , focus  $(-5, 2)$ , vertex  $(-7, 2)$
- Ellipse: centre  $(0, 3)$ , focus  $(12, 3)$ , major axis 26 units
- Ellipse: centre  $(-2, 1)$ , vertex  $(-2, 5)$ , passes through  $(0, 1)$

- Ellipse: foci  $(1, -2)$  and  $(1, 10)$ , minor axis 5 units
- Hyperbola: vertex  $(-1, 1)$ , focus  $(-1, 4)$ , centre  $(-1, 2)$
- Hyperbola: foci  $(2, 1)$  and  $(8, 1)$ , conjugate axis 6 units
- Hyperbola: vertices  $(2, 1)$  and  $(-4, 1)$ , focus  $(-6, 1)$
- Hyperbola: centre  $(1, -4)$ , focus  $(1, 1)$ , transverse axis 8 units

In Exercises 23–40, determine the centre (or vertex, if the curve is a parabola) of the given curve. Sketch each curve.

- $x^2 + 2x - 4y - 3 = 0$
- $y^2 - 2x - 2y - 9 = 0$
- $x^2 + 4y = 24$
- $x^2 + 4y^2 = 32y$
- $4x^2 + 9y^2 + 24x = 0$
- $2x^2 + y^2 + 8x = 8y$
- $9x^2 - y^2 + 8y = 7$
- $2x^2 - 4x = 9y - 2$
- $5x^2 - 4y^2 + 20x + 8y = 4$
- $0.04x^2 + 0.16y^2 = 0.01y$
- $4x^2 - y^2 + 32x + 10y + 35 = 0$
- $2x^2 + 2y^2 - 24x + 16y + 95 = 0$
- $16x^2 + 25y^2 - 32x + 100y - 284 = 0$
- $9x^2 - 16y^2 - 18x + 96y - 279 = 0$
- $5x^2 - 3y^2 + 95 = 40x$
- $5x^2 + 12y + 18 = 2y^2$
- $9x^2 + 9y^2 + 14 = 6x + 24y$
- $4y^2 + 29 = 15x + 12y$

In Exercises 41–52, solve the given problems.

- Find the equation of the hyperbola with asymptotes  $x - y = -1$  and  $x + y = -3$  and vertex  $(3, -1)$ .
- The circle  $x^2 + y^2 + 4x - 5 = 0$  passes through the foci and the ends of the minor axis of an ellipse that has its major axis along the  $x$ -axis. Find the equation of the ellipse.

43. The vertex and focus of one parabola are, respectively, the focus and vertex of a second parabola. Find the equation of the first parabola, if  $y^2 = 4x$  is the equation of the second.
44. Identify the curve represented by  $4y^2 - x^2 - 6x - 2y = 14$  and view it on a graphing calculator.
45. What is the general form of the equation of a *family* of parabolas if each vertex and focus is on the  $x$ -axis?
46. What is the general form of the equation of a *family* of ellipses with foci on the  $y$ -axis if each passes through the origin?
47. An electric current (in A) is  $i = 2 + \sin\left(2\pi t - \frac{\pi}{3}\right)$ . What is the equation for the current if the origin of the  $(t', i')$  system is taken as  $\left(\frac{1}{6}, 2\right)$  of the  $(t, i)$  system?
48. The stopping distance  $d$  (in m) of a car travelling at  $v$  km/h is represented by  $d = 0.005v^2 + 0.2v$ . Where is the vertex of the parabola that represents  $d$ ?
49. The stream from a fire hose follows a parabolic curve and reaches a maximum height of 18 m at a horizontal distance of 28 m from the nozzle. Find the equation that represents the stream, with the origin at the nozzle. Sketch the graph.
50. For a constant capacitive reactance and a constant resistance, sketch the graph of the impedance and inductive reactance (as abscissas) for an alternating-current circuit. (See Section 12.7.)

51. Two wheels in a friction drive assembly are equal ellipses, as shown in Fig. 21.84. They are always in contact, with the left wheel fixed in position and the right wheel able to move horizontally. Find the equation that can be used to represent the circumference of each wheel in the position shown.

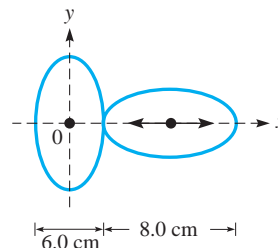


Fig. 21.84

52. An agricultural test station is to be divided into rectangular sections, each with a perimeter of 480 m. Express the area  $A$  of each section in terms of its width  $w$  and identify the type of curve represented. Sketch the graph of  $A$  as a function of  $w$ . For what value of  $w$  is  $A$  the greatest?

#### Answer to Practice Exercise

1.  $C(2, -3)$ ,  $F(1, -3)$ ,  $F(3, -3)$

## 21.8 The Second-Degree Equation

Second-Degree Equation •  
Coefficients  $A$ ,  $B$ ,  $C$  Determine the  
Type of Curve • Conic Sections

The equations of the circle, parabola, ellipse, and hyperbola are all special cases of the same general equation. In this section, we discuss this equation and how to identify the particular form it takes when it represents a specific type of curve.

Each of these curves can be represented by a **second-degree equation** of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \quad (21.34)$$

The coefficients of the second-degree equation terms determine the type of curve that results. Recalling the discussions of the general forms of the equations of the circle, parabola, ellipse, and hyperbola from the previous sections of this chapter, Eq. (21.34) represents the indicated curve for given conditions of  $A$ ,  $B$ , and  $C$ , as follows:

#### Identifying a Curve from the Coefficients of the Second-Degree Equation

1. If  $A = C$ ,  $B = 0$ , a circle.
2. If  $A \neq C$  (but they have the same sign),  $B = 0$ , an ellipse.
3. If  $A$  and  $C$  have different signs,  $B = 0$ , a hyperbola.
4. If  $A = 0$ ,  $C = 0$ ,  $B \neq 0$ , a hyperbola.
5. If either  $A = 0$  or  $C = 0$  (but not both),  $B = 0$ , a parabola. (Special cases such as a single point or no real locus can also result.)

Another conclusion about Eq. (21.34) is that, if either  $D \neq 0$  or  $E \neq 0$  (or both), the centre of the curve (or vertex of a parabola) is not at the origin. If  $B \neq 0$ , the axis of the curve has been rotated. We have considered only one such case (the hyperbola  $xy = c$ ) so far in this chapter. In the following section, *rotation of axes* is taken up, and if  $B \neq 0$ , we will see that the type of curve depends on the value  $B^2 - 4AC$ . The following examples illustrate how the type of curve is identified from the equation according to the five criteria listed above.

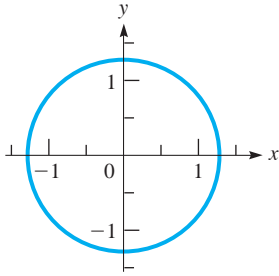


Fig. 21.85

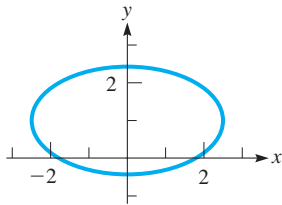


Fig. 21.86

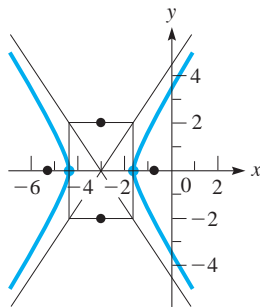


Fig. 21.87

### Practice Exercises

Identify the type of curve represented by each equation.

- $2y^2 - 4y + 5 = 4x - x^2$
- $x^2 + y^2 - 8y + 12x = x^2 - y^2$

### EXAMPLE 1 Identify a circle from an equation

The equation  $2x^2 = 3 - 2y^2$  represents a circle. This can be seen by putting the equation in the form of Eq. (21.34). This form is

$$2x^2 + 2y^2 - 3 = 0$$

$A = 2$      $C = 2$

We see that  $A = C$ . Also, since there is no  $xy$ -term, we know that  $B = 0$ . This means that the equation represents a circle. If we write it as  $x^2 + y^2 = \frac{3}{2}$ , we see that it fits the form of Eq. (21.12). The circle is shown in Fig. 21.85. ■

### EXAMPLE 2 Identify an ellipse from an equation

The equation  $3y^2 = 6y - x^2 + 3$  represents an ellipse. Before analysing the equation, we should put it in the form of Eq. (21.34). For this equation, this form is

$$x^2 + 3y^2 - 6y - 3 = 0$$

$A = 1$      $C = 3$

Here, we see that  $B = 0$ ,  $A$  and  $C$  have the same sign, and  $A \neq C$ . Therefore, it is an ellipse. The  $-6y$  term indicates that the centre of the ellipse is not at the origin. This ellipse is shown in Fig. 21.86. ■

### EXAMPLE 3 Identify a hyperbola from an equation

Identify the curve represented by  $2x^2 + 12x = y^2 - 14$ . Determine the appropriate quantities for the curve, and sketch the graph.

Writing this equation in the form of Eq. (21.34), we have

$$2x^2 - y^2 + 12x + 14 = 0$$

$A = 2$      $C = -1$

We identify this equation as representing a hyperbola, since  $A$  and  $C$  have different signs and  $B = 0$ . We now write it in the standard form of a hyperbola:

$$\begin{aligned}
 2x^2 + 12x - y^2 &= -14 \\
 2(x^2 + 6x \quad) - y^2 &= -14 && \text{complete the square} \\
 2(x^2 + 6x + 9) - y^2 &= -14 + 18 \\
 2(x + 3)^2 - y^2 &= 4 \\
 \frac{(x + 3)^2}{2} - \frac{y^2}{4} &= 1 && \text{centre is } (-3, 0) \\
 \frac{x'^2}{2} - \frac{y'^2}{4} &= 1
 \end{aligned}$$

Thus, we see that the centre  $(h, k)$  of the hyperbola is the point  $(-3, 0)$ . Also,  $a = \sqrt{2}$  and  $b = 2$ . This means that the vertices are  $(-3 + \sqrt{2}, 0)$  and  $(-3 - \sqrt{2}, 0)$ , and the conjugate axis extends from  $(-3, 2)$  to  $(-3, -2)$ . Also,  $c^2 = 2 + 4 = 6$ , which means that  $c = \sqrt{6}$ . The foci are  $(-3 + \sqrt{6}, 0)$  and  $(-3 - \sqrt{6}, 0)$ . The graph is shown in Fig. 21.87. ■

### EXAMPLE 4 Identify a parabola from an equation

Identify the curve represented by  $4y^2 - 23 = 4(4x + 3y)$  and find the appropriate important quantities.

Writing the equation in the form of Eq. (21.34), we have

$$4y^2 - 16x - 12y - 23 = 0$$

Therefore, we recognize the equation as representing a parabola, since  $A = 0$  and  $B = 0$ . Now, writing the equation in the standard form of a parabola, we have

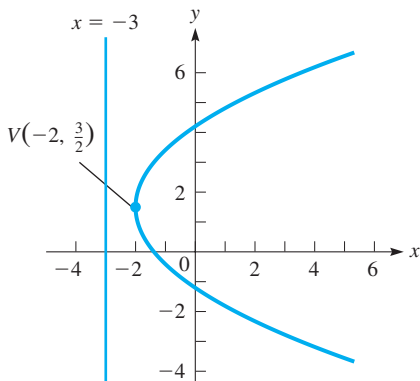


Fig. 21.88

$$\begin{aligned}
 4y^2 - 12y &= 16x + 23 \\
 4(y^2 - 3y) &= 16x + 23 && \text{complete the square} \\
 4\left(y^2 - 3y + \frac{9}{4}\right) &= 16x + 23 + 9 \\
 4\left(y - \frac{3}{2}\right)^2 &= 16(x + 2) \\
 \left(y - \frac{3}{2}\right)^2 &= 4(x + 2) \quad \text{or} \quad y'^2 = 4x' && \text{vertex } \left(-2, \frac{3}{2}\right)
 \end{aligned}$$

We now note that the vertex is  $(-2, 3/2)$  and that  $p = 1$ . This means that the focus is  $(-1, 3/2)$  and the directrix is  $x = -3$ . The graph is shown in Fig. 21.88. ■

In Chapter 14, when these curves were first introduced, they were referred to as **conic sections**. If a plane is passed through a cone, the intersection of the plane and the cone results in one of these curves; the curve formed depends on the angle of the plane with respect to the axis of the cone. This is shown in Fig. 21.89.

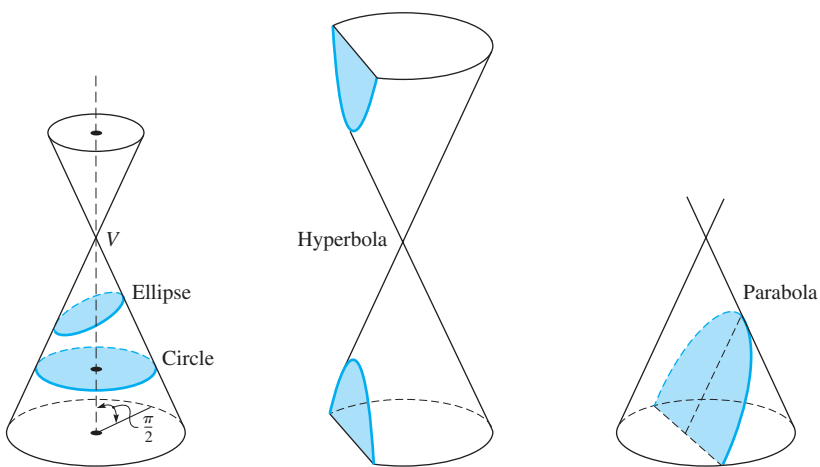


Fig. 21.89

## EXERCISES 21.8

In Exercises 1 and 2, make the given changes in the indicated examples of this section and then solve the indicated problem.

- In Example 1, change the  $-$  before the  $2y^2$  to  $+$  and then determine what type of curve is represented.
- In Example 3, change  $y^2 - 14$  to  $14 - y^2$  and then determine what type of curve is represented.

In Exercises 3–24, identify each of the equations as representing either a circle, a parabola, an ellipse, a hyperbola, or none of these.

- $x^2 + 2y^2 - 2 = 0$
- $2x^2 - y^2 - 1 = 0$
- $8x^2 + 2y^2 = 6y(1 - y)$
- $x^2 - y = 0$
- $y(y + x^2) = 4$
- $x(x - 3) = y(1 - 2y^2)$

- $2.2x^2 - x - y = 1.6$
- $x^2 = (y - 1)(y + 1)$
- $36x^2 = 12y(1 - 3y) + 1$
- $y(3 - 2x) = x(5 - 2y)$
- $2xy + x - 3y = 6$
- $2x(x - y) = y(3 - y - 2x)$
- $15x^2 = x(x - 12) + 4y(y - 6)$
- $(x + 1)^2 + (y + 1)^2 = 2(x + y + 1)$
- $(2x + y)^2 = 4x(y - 2) - 16$
- $x(y + 3x) = x^2 + xy - y^2 + 1$
- $4x(x - 1) = 2x^2 - 2y^2 + 3$
- $2x^2 + 4y^2 = y + 2x$
- $3.2x^2 = 2.1y(1 - 2y)$
- $y = 3(1 - 2x)(1 + 2x)$
- $x(13 - 5x) = 5y^2$
- $(y + 1)^2 = x^2 + y^2 - 1$



In Exercises 25–30, identify the curve represented by each of the given equations. Determine the appropriate important quantities for the curve and sketch the graph.

25.  $x^2 = 8(y - x - 2)$       26.  $x^2 = 6x - 4y^2 - 1$   
 27.  $y^2 = 2(x^2 - 2x - 2y)$       28.  $4x^2 + 4 = 9 - 8x - 4y^2$   
 29.  $y^2 + 42 = 2x(10 - x)$       30.  $x^2 - 4y = y^2 + 4(1 - x)$

In Exercises 31–36, identify the type of curve for each equation, and then view it on a graphing calculator.

31.  $x^2 + 2y^2 - 4x + 12y + 14 = 0$   
 32.  $4y^2 - x^2 + 40y - 4x + 60 = 0$   
 33.  $4(y^2 - 4x - 2) = 5(4y - 5)$   
 34.  $2(2x^2 - y) = 8 - y^2$   
 35.  $4(y^2 + 6y + 1) = x(x - 4) - 24$   
 36.  $8x + 31 - xy = y(y - 2 - x)$

In Exercises 37–42, use the given values to determine the type of curve represented.

37. For the equation  $x^2 + ky^2 = a^2$ , what type of curve is represented if (a)  $k = 1$ , (b)  $k < 0$ , and (c)  $k > 0$  ( $k \neq 1$ )?  
 38. For the equation  $\frac{x^2}{4 - k} - \frac{y^2}{k} = 1$ , what type of curve is represented if (a)  $k < 0$  and (b)  $0 < k < 4$ ? (For  $k > 4$ , see Exercise 40.)  
 39. In Eq. (21.34), if  $A > C > 0$  and  $B = D = E = F = 0$ , describe the locus of the equation.  
 40. For the equation in Exercise 38, describe the locus of the equation if  $k > 4$ .

41. In Eq. (21.34), if  $A = B = C = 0$ ,  $D \neq 0$ ,  $E \neq 0$ , and  $F \neq 0$ , describe the locus of the equation.  
 42. In Eq. (21.34), if  $A = -C \neq 0$ ,  $B = D = E = 0$ , and  $F = C$ , describe the locus of the equation if  $C > 0$ .

In Exercises 43–48, determine the type of curve from the given information.

43. The diagonal brace in a rectangular metal frame is 3.0 cm longer than the length of one of the sides. Determine the type of curve represented by the equation relating the lengths of the sides of the frame.  
 44. One circular solar cell has a radius that is 2.0 cm less than the radius  $r$  of a second circular solar cell. Determine the type of curve represented by the equation relating the total area  $A$  of both cells and  $r$ .  
 45. A flashlight emits a cone of light onto the floor. What type of curve is the perimeter of the lighted area on the floor, if the floor cuts completely through the cone of light?  
 46. What type of curve is the perimeter of the lighted area on the floor of the cone of light in Exercise 45, if the upper edge of the cone is parallel to the floor?  
 47. A supersonic jet creates a conical shock wave behind it. What type of curve is outlined on the surface of a lake by the shock wave if the jet is flying horizontally?  
 48. In Fig. 21.89, if the plane cutting the cones passes through the intersection of the upper and lower cones, what type of curve is the intersection of the plane and cones?

### Answers to Practice Exercises

1. Ellipse      2. Parabola

## 21.9 Rotation of Axes

Angle of Rotation of Axes •  
 Value of  $B^2 - 4AC$  Determines the Curve

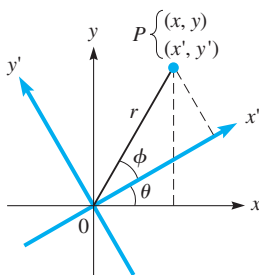


Fig. 21.90

In this chapter, we have discussed the circle, parabola, ellipse, and hyperbola, and in the last section, we showed how these curves are represented by the second-degree equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \quad (21.34)$$

The discussion of these curves included their equations with the centre (or vertex for a parabola) at the origin. However, as noted in the last section, except for the special case of the hyperbola  $xy = c$ , we did not show what happens when the axes are rotated about the origin.

If a set of axes is rotated about the origin through an angle  $\theta$ , as shown in Fig. 21.90, we say that there has been a **rotation of axes**. In this case, each point  $P$  in the plane has two sets of coordinates,  $(x, y)$  in the original system and  $(x', y')$  in the rotated system.

If we now let  $r$  equal the distance from the origin  $O$  to point  $P$  and let  $\phi$  be the angle between the  $x'$ -axis and the line  $OP$ , we have

$$x' = r \cos \phi \quad y' = r \sin \phi \quad (21.35)$$

$$x = r \cos(\theta + \phi) \quad y = r \sin(\theta + \phi) \quad (21.36)$$

Using the cosine and sine of the sum of two angles, we can write Eq. (21.36) as

$$\begin{aligned} x &= r \cos \phi \cos \theta - r \sin \phi \sin \theta \\ y &= r \cos \phi \sin \theta + r \sin \phi \cos \theta \end{aligned} \quad (21.37)$$

Now, using Eq. (21.35), we have

$$\begin{aligned}x &= x' \cos \theta - y' \sin \theta \\y &= x' \sin \theta + y' \cos \theta\end{aligned}\quad (21.38)$$

In our derivation, we have used the special case when  $\theta$  is acute and  $P$  is in the first quadrant of both sets of axes. When simplifying equations of curves using Eq. (21.38), we find that a rotation through a positive acute angle  $\theta$  is sufficient. It can be shown, however, that Eq. (21.38) holds for any  $\theta$  and position of  $P$ .

### EXAMPLE 1 Rotation through $45^\circ$

Transform  $x^2 - y^2 + 8 = 0$  by rotating the axes through  $45^\circ$ .

When  $\theta = 45^\circ$ , the rotation Eq. (21.38) becomes

$$x = x' \cos 45^\circ - y' \sin 45^\circ = \frac{x'}{\sqrt{2}} - \frac{y'}{\sqrt{2}}$$

$$y = x' \sin 45^\circ + y' \cos 45^\circ = \frac{x'}{\sqrt{2}} + \frac{y'}{\sqrt{2}}$$

Substituting into the equation  $x^2 - y^2 + 8 = 0$  gives

$$\begin{aligned}\left(\frac{x'}{\sqrt{2}} - \frac{y'}{\sqrt{2}}\right)^2 - \left(\frac{x'}{\sqrt{2}} + \frac{y'}{\sqrt{2}}\right)^2 + 8 &= 0 \\ \frac{1}{2}x'^2 - x'y' + \frac{1}{2}y'^2 - \frac{1}{2}x'^2 - x'y' - \frac{1}{2}y'^2 + 8 &= 0 \\ x'y' &= 4\end{aligned}$$

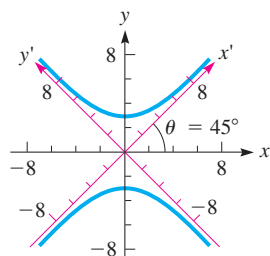


Fig. 21.91

The graph and both sets of axes are shown in Fig. 21.91. The original equation represents a hyperbola. In this example, we have shown that the  $xy = c$  type of hyperbola is obtained by a  $45^\circ$  rotation of the standard form given by Eq. (21.34). ■

In Section 21.8, when we showed the type of conic section represented by the second-degree equation Eq. (21.34), the standard forms of the parabola, ellipse, and hyperbola required that there be no  $xy$ -term, which means that  $B = 0$ . Remember, the  $xy = c$  type of hyperbola does not fit the standard form of a hyperbola.

Therefore, we see that if we can remove the  $xy$ -term from the second-degree equation of the form in Eq. (21.34), the analysis of the graph is simplified. By a proper rotation of axes, we find that Eq. (21.34) can be transformed into an equation containing no  $xy$ -term. We now show how this is done.

By substituting Eq. (21.38) into Eq. (21.34) and then simplifying, we have

$$\begin{aligned}(A \cos^2 \theta + B \sin \theta \cos \theta + C \sin^2 \theta)x'^2 + [B \cos 2\theta - (A - C) \sin 2\theta]x'y' \\ + (A \sin^2 \theta - B \sin \theta \cos \theta + C \cos^2 \theta)y'^2 + (D \cos \theta + E \sin \theta)x' \\ + (E \cos \theta - D \sin \theta)y' + F = 0\end{aligned}\quad (21.39)$$

If there is to be no  $x'y'$ -term, its coefficient must be zero. This means that  $B \cos 2\theta - (A - C) \sin 2\theta = 0$ , or

Angle of Rotation

$$\tan 2\theta = \frac{B}{A - C} \quad (A \neq C) \quad (21.40)$$

Eq. (21.40) gives the angle of rotation except when  $A = C$ . In this case, the coefficient of the  $x'y'$ -term is  $B \cos 2\theta$ , which is zero if  $2\theta = 90^\circ$ . Thus,

$$\theta = 45^\circ \quad (A = C) \quad (21.41)$$

Consider the following example.

■ For reference:

$$\text{Eq. (21.2) is } \cos \theta = \frac{1}{\sec \theta}$$

$$\text{Eq. (20.7) is } 1 + \tan^2 \theta = \sec^2 \theta$$

$$\text{Eq. (20.26) is } \sin \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{2}}$$

$$\text{Eq. (20.27) is } \cos \frac{\alpha}{2} = \pm \sqrt{\frac{1 + \cos \alpha}{2}}$$

### EXAMPLE 2 Rotating axes—eliminating the $xy$ -term

By rotation of axes, transform  $8x^2 + 4xy + 5y^2 = 9$  into a form without an  $xy$ -term. Identify and sketch the curve.

Here,  $A = 8$ ,  $B = 4$ , and  $C = 5$ . Therefore, using Eq. (21.40), we have

$$\tan 2\theta = \frac{4}{8 - 5} = \frac{4}{3}$$

Since  $\tan 2\theta$  is positive, we may take  $2\theta$  as an acute angle, which means  $\theta$  is also acute. For the transformation, we need  $\sin \theta$  and  $\cos \theta$ . We find these values by first finding the value of  $\cos 2\theta$  and then using the half-angle formulas:

$$\cos 2\theta = \frac{1}{\sec 2\theta} = \frac{1}{\sqrt{1 + \tan^2 2\theta}} = \frac{1}{\sqrt{1 + \left(\frac{4}{3}\right)^2}} = \frac{3}{5} \quad \text{using Eqs. (20.2) and (20.7)}$$

Now, using the half-angle formulas, Eqs. (20.26) and (20.27), we have

$$\sin \theta = \sqrt{\frac{1 - \cos 2\theta}{2}} = \sqrt{\frac{1 - \frac{3}{5}}{2}} = \frac{1}{\sqrt{5}} \quad \cos \theta = \sqrt{\frac{1 + \cos 2\theta}{2}} = \sqrt{\frac{1 + \frac{3}{5}}{2}} = \frac{2}{\sqrt{5}}$$

Here,  $\theta$  is about  $26.6^\circ$ . Now, substituting these values into Eq. (21.38), we have

$$x = x' \left( \frac{2}{\sqrt{5}} \right) - y' \left( \frac{1}{\sqrt{5}} \right) = \frac{2x' - y'}{\sqrt{5}} \quad y = x' \left( \frac{1}{\sqrt{5}} \right) + y' \left( \frac{2}{\sqrt{5}} \right) = \frac{x' + 2y'}{\sqrt{5}}$$

Now, substituting into the equation  $8x^2 + 4xy + 5y^2 = 9$  gives

$$8 \left( \frac{2x' - y'}{\sqrt{5}} \right)^2 + 4 \left( \frac{2x' - y'}{\sqrt{5}} \right) \left( \frac{x' + 2y'}{\sqrt{5}} \right) + 5 \left( \frac{x' + 2y'}{\sqrt{5}} \right)^2 = 9$$

$$8(4x'^2 - 4x'y' + y'^2) + 4(2x'^2 + 3x'y' - 2y'^2) + 5(x'^2 + 4x'y' + 4y'^2) = 45$$

$$45x'^2 + 20y'^2 = 45$$

$$\frac{x'^2}{1} + \frac{y'^2}{\frac{9}{4}} = 1$$

This is an ellipse with semimajor axis of  $3/2$  and semiminor axis of  $1$ . See Fig. 21.92. ■

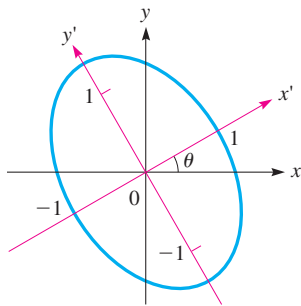


Fig. 21.92

In Example 2,  $\tan 2\theta$  was positive, and we made  $2\theta$  and  $\theta$  positive. If, when using Eq. (21.40),  $\tan 2\theta$  is negative, we then make  $2\theta$  obtuse ( $90^\circ < 2\theta < 180^\circ$ ). In this case,  $\cos 2\theta$  will be negative, but  $\theta$  will be acute ( $45^\circ < \theta < 90^\circ$ ).

In Section 21.8, we identified a conic section by inspecting the values of  $A$  and  $C$  when  $B = 0$ . If  $B \neq 0$ , these curves are identified as follows:

#### Identifying a Curve from the Value of $B^2 - 4AC$

1. If  $B^2 - 4AC = 0$ , a parabola
2. If  $B^2 - 4AC < 0$ , an ellipse
3. If  $B^2 - 4AC > 0$ , a hyperbola

Special cases such as a point, parallel or intersecting lines, or no curve may result.

#### Practice Exercises

Identify the type of curve represented by each equation.

1.  $3x^2 - 2xy + 3y^2 = 8$
2.  $x^2 - 4xy + y^2 = -5$
3.  $4x^2 + 4xy + y^2 - 24x + 38y - 19 = 0$

To see why the result above is true, we note that if  $A'$ ,  $B'$ , and  $C'$  represent the coefficients of  $x'^2$ ,  $x'y'$ , and  $y'^2$ , respectively, in Eq. (21.39), then it can be shown that  $B^2 - 4AC = B'^2 - 4A'C'$ . This means that if the axes have been rotated so that  $B' = 0$ , then  $B^2 - 4AC = -4A'C'$ . In other words, the sign of  $B^2 - 4AC$  is determined by the values of  $A'$  and  $C'$ , and these are used to identify a conic as in Section 21.8. For example, if the conic is a hyperbola,  $A'$  and  $C'$  differ in sign, so their product is negative and  $-4A'C' > 0$ , which gives  $B^2 - 4AC > 0$ . The other cases are analysed similarly.

In Section 21.7, we showed the use of translation of axes in writing an equation in standard form if  $B = 0$ . In this section, we have seen how rotation of axes is used to eliminate the  $xy$ -term. It is possible that both a translation of axes and a rotation of axes are needed to write an equation in standard form, as seen in the following example.

### EXAMPLE 3 Translation after rotation

For the equation  $16x^2 - 24xy + 9y^2 + 20x - 140y - 300 = 0$ , identify the curve and simplify it to standard form. Sketch the graph.

With  $A = 16$ ,  $B = -24$ , and  $C = 9$ , using Eq. (21.40), we have

$$\tan 2\theta = \frac{-24}{16 - 9} = -\frac{24}{7}$$

In this case,  $\tan 2\theta$  is negative, and we take  $2\theta$  to be an obtuse angle. We then find that  $\cos 2\theta = -7/25$ . In turn, we find that  $\sin \theta = 4/5$  and  $\cos \theta = 3/5$ . Here,  $\theta$  is about  $53.1^\circ$ . Using these values in Eq. (21.38), we find that

$$x = \frac{3x' - 4y'}{5} \quad y = \frac{4x' + 3y'}{5}$$

Substituting these into the original equation and simplifying, we get

$$y'^2 - 4x' - 4y' - 12 = 0$$

This equation represents a parabola with its axis parallel to the  $x'$ -axis. The vertex is found by completing the square:

$$(y' - 2)^2 = 4(x' + 4)$$

The vertex is the point  $(-4, 2)$  in the  $x'y'$ -rotated system. Therefore,

$$y''^2 = 4x''$$

is the equation in the  $x''y''$ -rotated and then translated system. The graph and the coordinate systems are shown in Fig. 21.93. ■

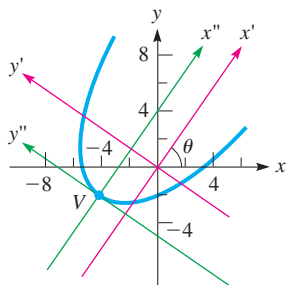


Fig. 21.93

## EXERCISES 21.9

In Exercises 1–4, transform the given equations by rotating the axes through the given angle. Identify and sketch each curve.

- $x^2 - y^2 = 25$ ,  $\theta = 45^\circ$
- $x^2 + y^2 = 16$ ,  $\theta = 60^\circ$
- $8x^2 - 4xy + 5y^2 = 36$ ,  $\theta = \tan^{-1} 2$
- $2x^2 + 24xy - 5y^2 = 8$ ,  $\theta = \tan^{-1} \frac{3}{4}$

In Exercises 5–10, identify the type of curve that each equation represents by evaluating  $B^2 - 4AC$ .

- $x^2 + 2xy + x - y - 3 = 0$
- $8x^2 - 4xy + 2y^2 + 7 = 0$
- $x^2 - 2xy + y^2 + 3y = 0$
- $4xy + 3y^2 - 8x + 16y + 19 = 0$
- $13x^2 + 10xy + 13y^2 + 6x - 42y - 27 = 0$
- $x^2 - 4xy + 4y^2 + 36x + 28y + 24 = 0$

In Exercises 11–16, transform each equation to a form without an  $xy$ -term by a rotation of axes. Identify and sketch each curve.

- $x^2 + 2xy + y^2 - 2x + 2y = 0$
- $5x^2 - 6xy + 5y^2 = 32$
- $3x^2 + 4xy = 4$

- $9x^2 - 24xy + 16y^2 - 320x - 240y = 0$
- $11x^2 - 6xy + 19y^2 = 20$
- $x^2 + 4xy - 2y^2 = 6$

In Exercises 17 and 18, transform each equation to a form without an  $xy$ -term by a rotation of axes. Then transform the equation to a standard form by a translation of axes. Identify and sketch each curve.

- $16x^2 - 24xy + 9y^2 - 60x - 80y + 400 = 0$
- $73x^2 - 72xy + 52y^2 + 100x - 200y + 100 = 0$

In Exercises 19 and 20, solve the given problems.

- (a) In Eq. (21.34), if  $A$  and  $C$  have opposite signs, what type of curve is represented? (b) If  $B \neq 0$ , and either  $A$  or  $C$  is zero, what type of curve is represented?
- An elliptical cam can be represented by the equation  $x^2 - 3xy + 5y^2 - 13 = 0$ . Through what angle is the cam rotated from its standard position?

### Answers to Practice Exercises

- Ellipse
- Hyperbola
- Parabola

## 21.10 Polar Coordinates

**Pole • Polar Axis • Polar Coordinates • Polar and Rectangular Coordinates**

■ The Swiss mathematician Jakob Bernoulli (1654–1705) was among the first to make significant use of polar coordinates.

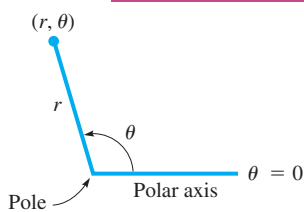


Fig. 21.94

### COMMON ERROR

When working with polar coordinates, angles are measured in radians. Make sure that your calculator is in radian mode.

When using polar coordinates, we generally label the lines for some of the values of  $\theta$ ; namely, those for  $\theta = 0$  (the polar axis),  $\theta = \pi/2$  (equivalent to the positive  $y$ -axis),  $\theta = \pi$  (equivalent to the negative  $x$ -axis),  $\theta = 3\pi/2$  (equivalent to the negative  $y$ -axis), and possibly others. In Fig. 21.95, these lines and those for multiples of  $\pi/6$  are shown. Also, the circles for  $r = 1$ ,  $r = 2$ , and  $r = 3$  are shown in this figure.

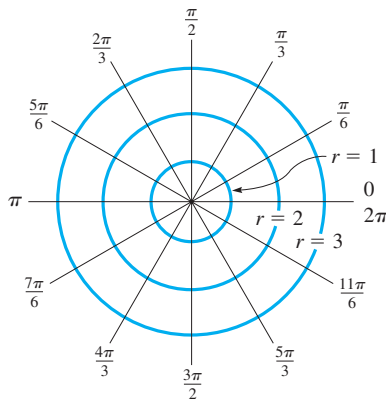


Fig. 21.95

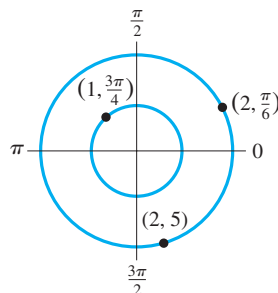


Fig. 21.96

### EXAMPLE 1 Polar coordinates of a point

- If  $r = 2$  and  $\theta = \pi/6$ , we have the point shown in Fig. 21.96. The point corresponds to  $(\sqrt{3}, 1)$  in rectangular coordinates.
- The polar coordinate point  $(1, 3\pi/4)$  is also shown. It is equivalent to  $(-\sqrt{2}/2, \sqrt{2}/2)$  in rectangular coordinates.
- The polar coordinate point  $(2, 5)$  is also shown. It is equivalent approximately to  $(0.6, -1.9)$  in rectangular coordinates. Remember, the 5 is an angle in radians. ■

One difference between rectangular coordinates and polar coordinates is that, for each point in the plane, there are limitless possibilities for the polar coordinates of that point. For example, the point  $(2, \frac{\pi}{6})$  can also be represented by  $(2, \frac{13\pi}{6})$  since the angles  $\frac{\pi}{6}$  and  $\frac{13\pi}{6}$  are coterminal. We also remove one restriction on  $r$  that we imposed in the definition of the trigonometric functions. That is,  $r$  is allowed to take on positive and negative values.

### EXAMPLE 2 Polar coordinates—negative $r$

The polar coordinates  $(3, 2\pi/3)$  and  $(3, -4\pi/3)$  represent the same point. However, the point  $(-3, 2\pi/3)$  is on the opposite side of the pole from  $(3, 2\pi/3)$ , 3 units from the pole. Another possible set of polar coordinates for the point  $(-3, 2\pi/3)$  is  $(3, 5\pi/3)$ . These points are shown in Fig. 21.97. ■

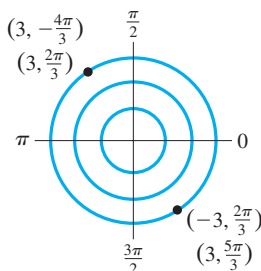


Fig. 21.97

### LEARNING TIP

If  $r$  is negative,  $\theta$  is located as before, but **the point is found  $r$  units from the pole but on the opposite side** from that on which it is positive.

When locating and plotting a point in polar coordinates, it is generally best to *first locate the terminal side of  $\theta$ , and then measure  $r$  along this terminal side*. This is illustrated in the following example.

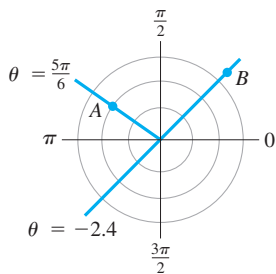


Fig. 21.98

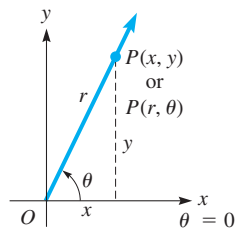


Fig. 21.99

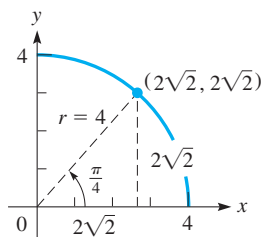


Fig. 21.100

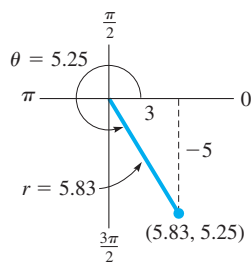


Fig. 21.101

■ Calculators are programmed to make conversions between rectangular and polar coordinates.

### Practice Exercise

1. Transform the polar coordinates  $(-4, 5\pi/6)$  into rectangular coordinates.

■ The cyclotron was invented in 1931 at the University of California. It was the first accelerator to deflect particles into circular paths.

### EXAMPLE 3 Locate the terminal side first

Plot the points  $A(2, 5\pi/6)$  and  $B(-3.2, -2.4)$  in the polar coordinate system.

To locate  $A$ , we determine the terminal side of  $\theta = 5\pi/6$  and then determine  $r = 2$ . See Fig. 21.98.

To locate  $B$ , we find the terminal side of  $\theta = -2.4$ , measuring clockwise from the polar axis (and recalling that  $\pi = 3.14 = 180^\circ$ ). Then we locate  $r = -3.2$  on the opposite side of the pole. See Fig. 21.98.

We will find that points with negative values of  $r$  occur frequently when plotting curves in polar coordinates. ■

### POLAR AND RECTANGULAR COORDINATES

The relationships between the polar coordinates of a point and the rectangular coordinates of the same point come from the definitions of the trigonometric functions. Those most commonly used are (see Fig. 21.99)

$$x = r \cos \theta \quad y = r \sin \theta \quad (21.42)$$

$$\tan \theta = \frac{y}{x} \quad r = \sqrt{x^2 + y^2} \quad (21.43)$$

As we have done in previous chapters, when  $\theta$  is not in the first quadrant, we first find the reference angle and then use it to find  $\theta$ , depending on the quadrant.

The following examples show the use of Eqs. (21.42) and (21.43) in changing coordinates in one system to coordinates in the other system. Also, these equations are used to transform equations from one system to the other.

### EXAMPLE 4 Polar to rectangular coordinates

Using Eqs. (21.42) and (21.43), we can transform the polar coordinates  $(4, \pi/4)$  into the rectangular coordinates  $(2\sqrt{2}, 2\sqrt{2})$ , since

$$x = 4 \cos \frac{\pi}{4} = 4 \left( \frac{\sqrt{2}}{2} \right) = 2\sqrt{2} \quad \text{and} \quad y = 4 \sin \frac{\pi}{4} = 4 \left( \frac{\sqrt{2}}{2} \right) = 2\sqrt{2}$$

See Fig. 21.100. ■

### EXAMPLE 5 Rectangular to polar coordinates

Using Eq. (21.43), we can transform the rectangular coordinates  $(3, -5)$  into polar coordinates, as follows:

$$\begin{aligned} \tan \theta_{\text{ref}} &= \frac{5}{3}, \theta_{\text{ref}} = 1.03, \theta = 2\pi - 1.03 = 5.25 \quad (\text{or } -1.03) \\ r &= \sqrt{3^2 + (-5)^2} = 5.83 \end{aligned}$$

We know that  $\theta$  is a fourth-quadrant angle since  $x$  is positive and  $y$  is negative. Therefore, the point  $(3, -5)$  in rectangular coordinates can be expressed as the point  $(5.83, 5.25)$  in polar coordinates (see Fig. 21.101). Other polar coordinates for the point are also possible. ■

### EXAMPLE 6 Rectangular to polar equation of a curve

If an electrically charged particle enters a magnetic field at right angles to the field, the particle follows a circular path. This fact is used in the design of nuclear particle accelerators.



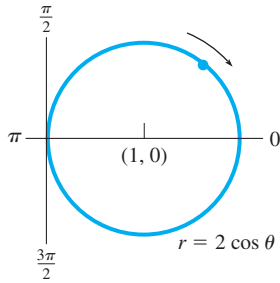


Fig. 21.102

A proton (positively charged) enters a magnetic field such that its path may be described by the rectangular equation  $x^2 + y^2 = 2x$ , where measurements are in metres. Find the polar equation of this circle.

We change this equation expressed in the rectangular coordinates  $x$  and  $y$  into an equation expressed in the polar coordinates  $r$  and  $\theta$  by using the relations  $r^2 = x^2 + y^2$  and  $x = r \cos \theta$ , as follows:

$$\begin{aligned} x^2 + y^2 &= 2x && \text{rectangular equation} \\ r^2 &= 2r \cos \theta && \text{substitute} \\ r &= 2 \cos \theta && \text{divided by } r \end{aligned}$$

This is the polar equation of the circle, which is shown in Fig. 21.102. ■

### EXAMPLE 7 Polar to rectangular equation of a curve

Find the rectangular equation of the rose  $r = 4 \sin 2\theta$ .

Using the trigonometric identity  $\sin 2\theta = 2 \sin \theta \cos \theta$  and Eqs. (21.42) and (21.43) leads to the solution:

$$\begin{aligned} r &= 4 \sin 2\theta && \text{polar equation} \\ &= 4(2 \sin \theta \cos \theta) = 8 \sin \theta \cos \theta && \text{using identity} \\ \sqrt{x^2 + y^2} &= 8 \left(\frac{y}{r}\right) \left(\frac{x}{r}\right) = \frac{8xy}{r^2} = \frac{8xy}{x^2 + y^2} && \text{using Eqs. (21.42) and (21.43)} \\ x^2 + y^2 &= \frac{64x^2y^2}{(x^2 + y^2)^2} && \text{squaring both sides} \\ (x^2 + y^2)^3 &= 64x^2y^2 && \text{simplifying} \end{aligned}$$

Plotting the graph of this equation from the rectangular equation would be complicated. However, as we will see in the next section, plotting this graph in polar coordinates is quite simple. The curve is shown in Fig. 21.103. ■

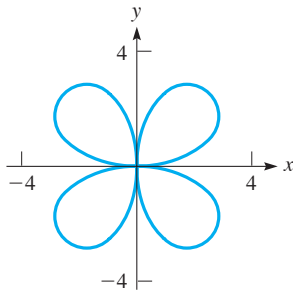


Fig. 21.103

#### Practice Exercise

- Find the polar equation of the circle  $x^2 + y^2 + 2x = 0$ .

## EXERCISES 21.10

In Exercises 1–4, make the given changes in the indicated examples of this section and then solve the indicated problems.

- In Example 2, change  $2\pi/3$  to  $\pi/3$  and then find another set of coordinates for each point, similar to those shown for the points in the example.
- In Example 4, change  $\pi/4$  to  $\pi/6$ .
- In Example 5, change 3 to  $-3$  and  $-5$  to 5.
- In Example 7, change  $\sin$  to  $\cos$ .

In Exercises 5–16, plot the given polar coordinate points on polar coordinate paper.

- $\left(3, \frac{\pi}{6}\right)$
- $(2, \pi)$
- $\left(\frac{5}{2}, -\frac{2\pi}{5}\right)$
- $\left(5, -\frac{\pi}{3}\right)$
- $\left(-8, \frac{7\pi}{6}\right)$
- $\left(-5, \frac{\pi}{4}\right)$
- $\left(-3, -\frac{5\pi}{4}\right)$
- $\left(-4, -\frac{5\pi}{3}\right)$
- $(2, 2)$
- $(-6, -6)$
- $(0.5, -8.4)$
- $(-2.2, 18.8)$

In Exercises 17–20, find a set of polar coordinates for each of the points for which the rectangular coordinates are given.

- $(\sqrt{3}, 1)$
- $(-1, -1)$
- $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$
- $(-10, 8)$

In Exercises 21–24, find the rectangular coordinates for each of the points for which the polar coordinates are given.

- $\left(8, \frac{4\pi}{3}\right)$
- $(-4, -\pi)$
- $(3.0, -0.40)$
- $(-1.0, 1.0)$

In Exercises 25–36, find the polar equation of each of the given rectangular equations.

- $x = 3$
- $y = -x$
- $x + 2y + 3 = 0$
- $x^2 + y^2 = 0.81$
- $x^2 + (y - 2)^2 = 4$
- $x^2 - y^2 = 0.01$
- $x^2 + 4y^2 = 4$
- $y^2 = 4x$
- $x^2 + y^2 = 6y$
- $xy = 9$
- $x^3 + y^3 - 4xy = 0$
- $y = \frac{2x}{x^2 + 1}$



In Exercises 37–48, find the rectangular equation of each of the given polar equations. In Exercises 37–44, identify the curve that is represented by the equation.

37.  $r = \sin \theta$                       38.  $r = 4 \cos \theta$   
 39.  $r \cos \theta = 4$                     40.  $r \sin \theta = -2$   
 41.  $r = \frac{2}{\cos \theta - 3 \sin \theta}$             42.  $r = e^{r \cos \theta} \csc \theta$   
 43.  $r = 4 \cos \theta + 2 \sin \theta$         44.  $r \sin(\theta + \pi/6) = 3$   
 45.  $r = 2(1 + \cos \theta)$               46.  $r = 1 - \sin \theta$   
 47.  $r^2 = \sin 2\theta$                     48.  $r^2 = 16 \cos 2\theta$

In Exercises 49–60, solve the given problems. All coordinates given are polar coordinates.

49. Is the point  $(2, 3\pi/4)$  on the curve  $r = 2 \sin 2\theta$ ?  
 50. Is the point  $(1/2, 3\pi/2)$  on the curve  $r = \sin(\theta/3)$ ?  
 51. Show that the polar coordinate equation  $r = a \sin \theta + b \cos \theta$  represents a circle by changing it to a rectangular equation.  
 52. Find the distance between the points  $(3, \pi/6)$  and  $(4, \pi/2)$  by using the law of cosines.  
 53. The centre of a regular hexagon is at the pole with one vertex at  $(2, \pi)$ . What are the coordinates of the other vertices?  
 54. Find the distance between the points  $(4, \pi/6)$  and  $(5, 5\pi/3)$ .  
 55. Under certain conditions, the  $x$ - and  $y$ -components of a magnetic field  $B$  are given by the equations

$$B_x = \frac{-ky}{x^2 + y^2} \quad \text{and} \quad B_y = \frac{kx}{x^2 + y^2}$$

Write these equations in terms of polar coordinates.

56. In designing a domed roof for a building, an architect uses the equation  $x^2 + \frac{y^2}{k^2} = 1$ , where  $k$  is a constant. Write this equation in polar form.  
 57. The shape of a cam can be described by the polar equation  $r = 3 - \sin \theta$ . Find the rectangular equation for the shape of the cam.  
 58. The polar equation of the path of a weather satellite of the earth is  $r = \frac{7600}{1 + 0.14 \cos \theta}$ , where  $r$  is measured in kilometres. Find the rectangular equation of the path of this satellite. The path is an ellipse, with the earth at one of the foci.  
 59. The control tower of an airport is taken to be at the pole, and the polar axis is taken as due east in a polar coordinate graph. How far apart (in km) are planes, at the same altitude, if their positions on the graph are  $(6.10, 1.25)$  and  $(8.45, 3.74)$ ?  
 60. The perimeter of a certain type of machine part can be described by the equation  $r = a \sin \theta + b \cos \theta$  ( $a > 0, b > 0$ ). Explain why all such machine parts are circular.

#### Answers to Practice Exercises

1.  $(2\sqrt{3}, -2)$       2.  $r = -2 \cos \theta$

## 21.11 Curves in Polar Coordinates

Curve Sketched from Equations •  
 Plotting Polar Coordinate Curves •  
 Calculator Display of Polar Curves

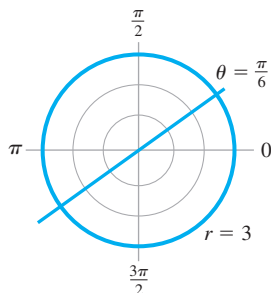


Fig. 21.104

The basic method for finding a curve in polar coordinates is the same as in rectangular coordinates. We assume values of the independent variable—in this case,  $\theta$ —and then find the corresponding values of the dependent variable  $r$ . These points are then plotted and joined, thereby forming the curve that represents the relation in polar coordinates.

Before using the basic method, it is useful to point out that certain basic curves can be sketched directly from the equation. This is done by noting the meaning of each of the polar coordinate variables,  $r$  and  $\theta$ . This is illustrated in the following example.

#### EXAMPLE 1 Curves sketched from equations

- (a) The graph of the polar equation  $r = 3$  is a circle of radius 3, with centre at the pole. This can be seen to be the case, since  $r = 3$  for all possible values of  $\theta$ . It is not necessary to find specific points for this circle, which is shown in Fig. 21.104.  
 (b) The graph of  $\theta = \pi/6$  is a straight line through the pole. It represents all points for which  $\theta = \pi/6$  for all possible values of  $r$ . This line is shown in Fig. 21.104. ■

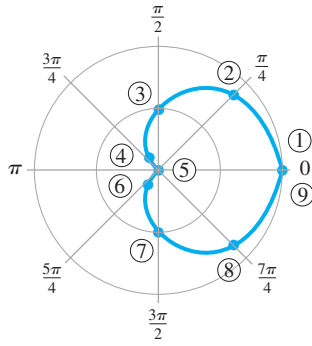


Fig. 21.105

**EXAMPLE 2 Plotting a cardioid**

Plot the graph of  $r = 1 + \cos \theta$ .

We find the following values of  $r$  corresponding to the chosen values of  $\theta$ .

$\theta$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	$\pi$	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	$2\pi$
$r$	2	1.7	1	0.3	0	0.3	1	1.7	2
Point Number	1	2	3	4	5	6	7	8	9

We now see that the points are repeating, and it is unnecessary to find additional points. This curve is called a **cardioid** and is shown in Fig. 21.105. ■

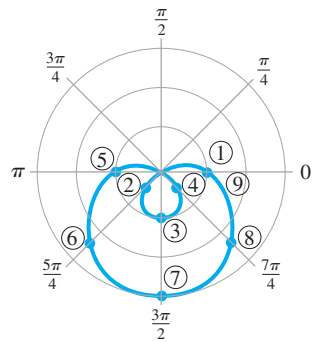


Fig. 21.106

**EXAMPLE 3 Plotting a limaçon**

Plot the graph of  $r = 1 - 2 \sin \theta$ .

Choosing values of  $\theta$  and then finding the corresponding values of  $r$ , we find the following table of values.

$\theta$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	$\pi$	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	$2\pi$
$r$	1	-0.4	-1	-0.4	1	2.4	3	2.4	1
Point Number	1	2	3	4	5	6	7	8	9

Particular care should be taken in plotting the points for which  $r$  is negative. This curve is known as a **limaçon** and is shown in Fig. 21.106. ■

**EXAMPLE 4 Polar curve—application**

A cam is shaped such that the edge of the upper “half” is represented by the equation  $r = 2.0 + \cos \theta$  and the lower “half” by the equation  $r = \frac{3.0}{2.0 - \cos \theta}$ , where measurements are in centimetres. Plot the curve that represents the shape of the cam.

We get the points for the edge of the cam by using values of  $\theta$  from 0 to  $\pi$  for the upper “half” and from  $\pi$  to  $2\pi$  for the lower “half”. The table of values follows:

$\theta$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	$\pi$
$r = 2.0 + \cos \theta$	3.0	2.7	2.0	1.3	1.0
Point Number	1	2	3	4	5

$\theta$	$\pi$	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	$2\pi$
$r = \frac{3.0}{2.0 - \cos \theta}$	1.0	1.1	1.5	2.3	3.0
Point Number	6	7	8	9	10

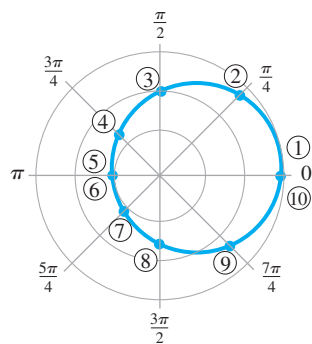


Fig. 21.107

The upper “half” is part of a limaçon and the lower “half” is a semiellipse. The cam is shown in Fig. 21.107. ■

**Practice Exercises**

Determine the type of curve represented by each polar coordinate equation.

- $\theta = 2\pi/5$
- $r = -4$

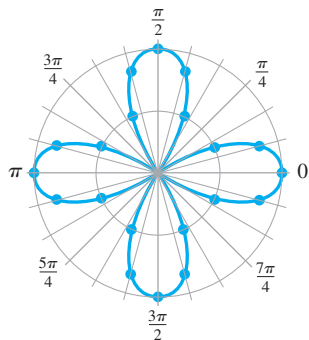


Fig. 21.108

**EXAMPLE 5** Plotting a rose

Plot the graph of  $r = 2 \cos 2\theta$ .

In finding values of  $r$ , we must be careful first to multiply the values of  $\theta$  by 2 before finding the cosine of the angle. Also, for this reason, we take values of  $\theta$  as multiples of  $\pi/12$ , so as to get enough useful points. The tables of value follows:

$\theta$	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$
$r$	2	1.7	1	0	-1	-1.7	-2

$\theta$	$\frac{7\pi}{12}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	$\frac{11\pi}{12}$	$\pi$
$r$	-1.7	-1	0	1	1.7	2

For values of  $\theta$  starting with  $\pi$ , the values of  $r$  repeat. We have a four-leaf **rose**, as shown in Fig. 21.108. ■

**EXAMPLE 6** Plotting a lemniscate

Plot the graph of  $r^2 = 9 \cos 2\theta$ .

Choosing the indicated values of  $\theta$ , we get the values of  $r$  as shown in the following table:

$\theta$	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$	...	$\frac{3\pi}{4}$	$\frac{7\pi}{8}$	$\pi$
$r$	$\pm 3$	$\pm 2.5$	0		0	$\pm 2.5$	$\pm 3$

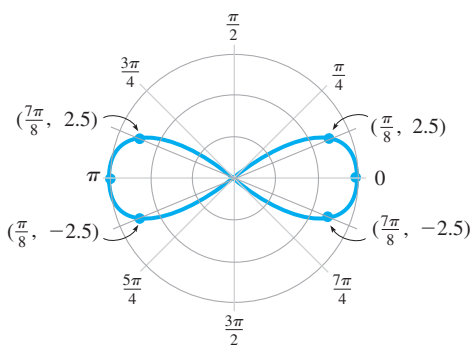


Fig. 21.109

There are no values of  $r$  corresponding to values of  $\theta$  in the range  $\pi/4 < \theta < 3\pi/4$ , since twice these angles are in the second and third quadrants and the cosine is negative for such angles. The value of  $r^2$  cannot be negative. Also, the values of  $r$  repeat for  $\theta > \pi$ . The figure is called a **lemniscate** and is shown in Fig. 21.109. ■

**EXAMPLE 7** Calculator display of a polar curve

View the graph of  $r = 1 - 2 \cos \theta$  on a graphing calculator.

Using the *mode* feature, a polar curve is displayed using the polar graph option or the parametric graph option, depending on the calculator. (Review the manual for the calculator.) The graph displayed will be the same with either method.

With the polar graph option, the function is entered directly. The values for the viewing window are determined by settings for  $x$ ,  $y$  and the angles  $\theta$  that will be used. These values are set in a manner similar to those used for parametric equations. (See page 315 for an example of graphing parametric equations.)

With the parametric graph option, to graph  $r = f(\theta)$ , we note that  $x = r \cos \theta$  and  $y = r \sin \theta$ . This tells us that

$$\begin{aligned} x &= f(\theta) \cos \theta \\ y &= f(\theta) \sin \theta \end{aligned}$$

Thus, for  $r = 1 - 2 \cos \theta$ , by using

$$\begin{aligned} x &= (1 - 2 \cos \theta) \cos \theta \\ y &= (1 - 2 \cos \theta) \sin \theta \end{aligned}$$

the graph can be displayed, as shown in Fig. 21.110. ■

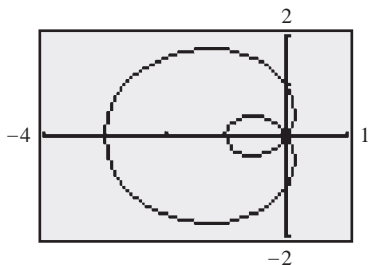


Fig 21.110

## EXERCISES 21.11

In Exercises 1–4, make the given changes in the indicated examples of this section and then make the indicated graphs.

- In Example 1(b), change  $\pi/6$  to  $5\pi/6$ .
- In Example 3, change the  $-$  before the  $2 \sin \theta$  to  $+$ .
- In Example 5, change  $\cos$  to  $\sin$ .
- In Example 7, change the  $-$  before the  $2 \cos \theta$  to  $+$ .

In Exercises 5–32, plot the curves of the given polar equations in polar coordinates.

- |   |  |
|---|--|
| 5. $r = 5$                                    | 6. $r = -2$                                    |
| 7. $\theta = 3\pi/4$                          | 8. $\theta = -1.5$                             |
| 9. $r = 4 \sec \theta$                        | 10. $r = 4 \csc \theta$                        |
| 11. $r = 2 \sin \theta$                       | 12. $r = 3 \cos \theta$                        |
| 13. $1 - r = \cos \theta$ (cardioid)          | 14. $r + 1 = \sin \theta$ (cardioid)           |
| 15. $r = 2 - \cos \theta$ (limaçon)           | 16. $r = 2 + 3 \sin \theta$ (limaçon)          |
| 17. $r = 4 \sin 2\theta$ (rose)               | 18. $r = 2 \sin 3\theta$ (rose)                |
| 19. $r^2 = 4 \sin 2\theta$ (lemniscate)       | 20. $r^2 = 2 \sin \theta$                      |
| 21. $r = 2^\theta$ (spiral)                   | 22. $r = 1.5^{-\theta}$ (spiral)               |
| 23. $r = 4 \sin 3\theta $                     | 24. $r = 2 \sin \theta \tan \theta$ (cissoid)  |
| 25. $r = \frac{3}{2 - \cos \theta}$ (ellipse) | 26. $r = \frac{2}{1 - \cos \theta}$ (parabola) |
| 27. $r - 2r \cos \theta = 6$ (hyperbola)      |  |
| 28. $3r - 2r \sin \theta = 6$ (ellipse)       |  |
| 29. $r = 4 \cos \frac{1}{2} \theta$           | 30. $r = 2 + \cos 3\theta$                     |
| 31. $r = 2[1 - \sin(\theta - \pi/4)]$         | 32. $r = 4 \tan \theta$                        |

In Exercises 33–42, view the curves of the given polar equations on a graphing calculator.

- $r = \theta$  ( $-20 \leq \theta \leq 20$ )
- $r = 0.5^{\sin \theta}$
- $r = 2 \sec \theta + 1$
- $r = 2 \cos(\cos 2\theta)$
- $r = 3 \cos 4\theta$
- $r = 3 \sin 5\theta$
- $r = 2 \cos \theta + 3 \sin \theta$
- $r = 1 + 3 \cos \theta - 2 \sin \theta$
- $r + 2 = \cos 2\theta$
- $2r \cos \theta + r \sin \theta = 2$

In Exercises 43–54, solve the given problems and sketch or display the indicated curves.

- What is the graph of  $\tan \theta = 1$ ?
- Find the polar equation of the line through the polar points  $(1, 0)$  and  $(2, \pi/2)$ .
- Using a graphing calculator, show that the curves  $r = 2 \sin \theta$  and  $r = 2 \cos \theta$  intersect at right angles. Proper window settings are necessary.

- Using a graphing calculator, determine what type of graph is displayed by  $r = 3 \sec^2(\theta/2)$ .
- An architect designs a patio shaped such that it can be described as the area within the polar curve  $r = 4.0 - \sin \theta$ , where measurements are in metres. Sketch the curve that represents the perimeter of the patio.
- The radiation pattern of a certain television transmitting antenna can be represented by  $r = 120(1 + \cos \theta)$ , where distances (in km) are measured from the antenna. Sketch the radiation pattern.
- The joint between two links of a robot arm moves in an elliptical path (in cm), given by  $r = \frac{25}{10 + 4 \cos \theta}$ . Sketch the path.
- A missile is fired at an airplane and is always directed toward the airplane. The missile is travelling at twice the speed of the airplane. An equation that describes the distance  $r$  between the missile and the airplane is  $r = \frac{70 \sin \theta}{(1 - \cos \theta)^2}$ , where  $\theta$  is the angle between their directions at all times. See Fig. 21.111. This is a *relative pursuit curve*. Sketch the graph of this equation for  $\pi/4 \leq \theta \leq \pi$ .

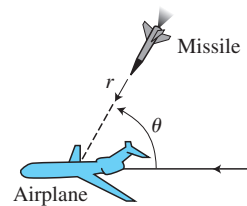


Fig. 21.111

- In studying the photoelectric effect, an equation used for the rate  $R$  at which photoelectrons are ejected at various angles  $\theta$  is  $R = \frac{\sin^2 \theta}{(1 - 0.5 \cos \theta)^2}$ . Sketch the graph.
- In order to display the graph of the rectangular equation  $4(x^6 + 3x^4y^2 + 3x^2y^4 + y^6 - x^4 - 2x^2y^2 - y^4) + y^2 = 0$  on a computer screen, it is first transformed into polar coordinates. Transform the equation and then sketch the graph. (*Hint:* The equation can be written as  $4(x^2 + y^2)^3 - 4(x^2 + y^2)^2 + y^2 = 0$ .)
- Noting the graphs in Exercises 17, 18, 37, and 38, what conclusion do you draw about the value of  $n$  and the graph of  $y = a \sin n\theta$  or  $y = a \cos n\theta$ ?
- (a) Solve the equations  $r = 2 \sin \theta$  and  $r = 1 + \sin \theta$  simultaneously to find a point of intersection. (b) Display the graphs of the equations on a graphing calculator and note another common point, the coordinates of which do not satisfy both equations. Explain why this occurs.

### Answers to Practice Exercises

- Straight line
- Circle

## CHAPTER 21 EQUATIONS

<b>Distance formula</b>	Fig. 21.2	$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$	(21.1)
<b>Slope</b>	Fig. 21.4	$m = \frac{y_2 - y_1}{x_2 - x_1}$	(21.2)
	Fig. 21.7	$m = \tan \alpha \quad (0^\circ \leq \alpha < 180^\circ)$	(21.3)
	Fig. 21.9	$m_1 = m_2$ (for $\parallel$ lines)	(21.4)
	Fig. 21.10	$m_2 = -\frac{1}{m_1}$ or $m_1 m_2 = -1$ (for $\perp$ lines)	(21.5)
<b>Straight line</b>	Fig. 21.14	$y - y_1 = m(x - x_1)$	(21.6)
	Fig. 21.17	$x = a$	(21.7)
	Fig. 21.18	$y = b$	(21.8)
	Fig. 21.21	$y = mx + b$	(21.9)
		$Ax + By + C = 0$	(21.10)
<b>Circle</b>	Fig. 21.29	$(x - h)^2 + (y - k)^2 = r^2$	(21.11)
	Fig. 21.32	$x^2 + y^2 = r^2$	(21.12)
		$x^2 + y^2 + Dx + Ey + F = 0$	(21.14)
<b>Parabola</b>	Fig. 21.40	$y^2 = 4px$	(21.15)
	Fig. 21.43	$x^2 = 4py$	(21.16)
<b>Ellipse</b>	Fig. 21.55	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	(21.17)
	Fig. 21.55	$a^2 = b^2 + c^2$	(21.18)
	Fig. 21.56	$\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1$	(21.19)
<b>Hyperbola</b>	Fig. 21.65	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	(21.20)
	Fig. 21.65	$c^2 = a^2 + b^2$	(21.21)
	Fig. 21.66	$y = \pm \frac{bx}{a}$ (asymptotes)	(21.23)
	Fig. 21.68	$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$	(21.24)
	Fig. 21.73	$xy = c$	(21.25)
<b>Translation of axes</b>	Fig. 21.78	$x = x' + h$ and $y = y' + k$	(21.26)
		$x' = x - h$ and $y' = y - k$	(21.27)
<b>Parabola, vertex <math>(h, k)</math></b>		$(y - k)^2 = 4p(x - h)$ (axis parallel to $x$ -axis)	(21.28)
		$(x - h)^2 = 4p(y - k)$ (axis parallel to $y$ -axis)	(21.29)
<b>Ellipse, centre <math>(h, k)</math></b>		$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$ (major axis parallel to $x$ -axis)	(21.30)
		$\frac{(y - k)^2}{a^2} + \frac{(x - h)^2}{b^2} = 1$ (major axis parallel to $y$ -axis)	(21.31)
<b>Hyperbola, centre <math>(h, k)</math></b>		$\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1$ (transverse axis parallel to $x$ -axis)	(21.32)
		$\frac{(y - k)^2}{a^2} - \frac{(x - h)^2}{b^2} = 1$ (transverse axis parallel to $y$ -axis)	(21.33)

<b>Second-degree equation</b>		$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$	(21.34)
<b>Rotation of axes</b>	Fig. 21.90	$x = x'\cos\theta - y'\sin\theta$	(21.38)
		$y = x'\sin\theta - y'\cos\theta$	(21.38)
<b>Angle of rotation</b>		$\tan 2\theta = \frac{B}{A - C} \quad (A \neq C)$	(21.40)
		$\theta = 45^\circ \quad (A = C)$	(21.41)
<b>Polar coordinates</b>	Fig. 21.99	$x = r \cos\theta \quad y = r \sin\theta$	(21.42)
		$\tan\theta = \frac{y}{x} \quad r = \sqrt{x^2 + y^2}$	(21.43)

## CHAPTER 21 REVIEW EXERCISES

In Exercises 1–12, find the equation of the indicated curve, subject to the given conditions. Sketch each curve.

- Straight line: passes through  $(1, -7)$  with a slope of 4
- Straight line: passes through  $(-1, 5)$  and  $(-2, -3)$
- Straight line: perpendicular to  $3x - 2y + 8 = 0$  and has a y-intercept of  $(0, -1)$
- Straight line: parallel to  $2x - 5y + 1 = 0$  and has an x-intercept of  $(2, 0)$
- Circle: concentric with  $x^2 + y^2 = 6x$ , passes through  $(4, -3)$
- Circle: tangent to lines  $x = 3$  and  $x = 9$ , centre on line  $y = 2x$
- Parabola: focus  $(3, 0)$ , vertex  $(0, 0)$
- Parabola: vertex  $(0, 0)$ , passes through  $(1, 1)$  and  $(-2, 4)$
- Ellipse: vertex  $(10, 0)$ , focus  $(8, 0)$ , tangent to  $x = -10$
- Ellipse: centre  $(0, 0)$ , passes through  $(0, 3)$  and  $(2, 1)$
- Hyperbola:  $V(0, 13)$ ,  $C(0, 0)$ , conj. axis of 24
- Hyperbola: vertex  $(0, 8)$ , asymptotes  $y = 2x$ ,  $y = -2x$

In Exercises 13–26, find the indicated quantities for each of the given equations. Sketch each curve.

- $x^2 + y^2 + 6x - 7 = 0$ , centre and radius
- $2x^2 + 2y^2 + 4x - 8y - 15 = 0$ , centre and radius
- $x^2 = -20y$ , focus and directrix
- $y^2 = 24x$ , focus and directrix
- $4x^2 + y^2 = 1$ , vertices and foci
- $2y^2 - 9x^2 = 18$ , vertices and foci
- $2x^2 - 5y^2 = 0.25$ , vertices and foci
- $2x^2 + 25y^2 = 800$ , vertices and foci
- $x^2 - 8x - 4y - 16 = 0$ , vertex and focus
- $y^2 - 4x + 4y + 24 = 0$ , vertex and directrix
- $4x^2 + y^2 - 16x + 2y + 13 = 0$ , centre
- $x^2 - 2y^2 + 4x + 4y + 6 = 0$ , centre
- $x^2 - 2xy + y^2 + 4x + 4y = 0$ , vertex
- $3x^2 - 3xy + 7y^2 - 5 = 0$ , centre

In Exercises 27–34, plot the given curves in polar coordinates.

- $r = 4(1 + \sin\theta)$
- $r = 1 - 3\cos\theta$
- $r = 4\cos 3\theta$
- $r = 3\sin\theta - 4\cos\theta$
- $r = \frac{3}{\sin\theta + 2\cos\theta}$
- $r = \frac{1}{2(\sin\theta - 1)}$
- $r = 2\sin\left(\frac{\theta}{2}\right)$
- $r = 1 - \cos 2\theta$

In Exercises 35–38, find the polar equation of each of the given rectangular equations.

- $y = 2x$
- $2xy = 1$
- $x^2 + xy + y^2 = 2$
- $x^2 + (y + 3)^2 = 16$

In Exercises 39–42, find the rectangular equation of each of the given polar equations.

- $r = 2\sin 2\theta$
- $r^2 = 9\sin\theta$
- $r = \frac{4}{2 - \cos\theta}$
- $r = 4\tan\theta \sec\theta$

In Exercises 43–48, determine the number of real solutions of the given systems of equations by sketching the indicated curves. (See Section 14.1.)

- $x^2 + y^2 = 9$   
 $4x^2 + y^2 = 16$
- $x^2 + y^2 - 4y - 5 = 0$   
 $y^2 - 4x^2 - 4 = 0$
- $y = 2\sin x$   
 $y = 2 - x^2$
- $y = e^x$   
 $x^2 - y^2 = 1$
- $x^2 - 4y^2 + 2x - 3 = 0$   
 $y^2 - 4x - 4 = 0$
- $y = 4\ln x$   
 $xy = 6$

In Exercises 49–58, view the curves of the given equations on a graphing calculator.

- $x^2 + 3y + 2 - (1 + x)^2 = 0$
- $y^2 = 4x + 6$
- $2x^2 + 2y^2 + 4y - 3 = 0$
- $2x^2 + (y - 3)^2 - 5 = 0$
- $x^2 - 4y^2 + 4x + 24y - 48 = 0$
- $x^2 + 2xy + y^2 - 3x + 8y = 0$
- $r = 3\cos(3\theta/2)$
- $r = 5 - 2\sin 4\theta$
- $r = 2 - 3\csc\theta$
- $r = 2\sin(\cos 3\theta)$

In Exercises 59–62, find the equation of the locus of a point  $P(x, y)$  that moves as stated.

59. Always 4 units from  $(3, -4)$
60. Passes through  $(7, -5)$  with a constant slope of  $-2$
61. The sum of its distances from  $(1, -3)$  and  $(7, -3)$  is 8.
62. The difference of its distances from  $(3, -1)$  and  $(3, -7)$  is 4.

In Exercises 63–106, solve the given problems.

63. Considering Eq. (21.30) of an ellipse, describe the graph if  $a = b$ .
64. Show that the ellipse  $x^2 + 9y^2 = 9$  has the same foci as the hyperbola  $x^2 - y^2 = 4$ .
65. The points  $(-2, -5)$ ,  $(3, -3)$ , and  $(13, x)$  are collinear. Find  $x$ .
66. For the polar coordinate point  $(-5, \pi/4)$ , find another set of polar coordinates such that  $r < 0$  and  $-2\pi < \theta < 0$ .
67. Find the distance between the polar coordinate points  $(3, \pi/6)$  and  $(6, -\pi/3)$ .
68. Show that the parametric equations  $y = \cot \theta$  and  $x = \csc \theta$  define a hyperbola.
69. In two ways, show that the line segments joining  $(-3, 11)$ ,  $(2, -1)$ , and  $(14, 4)$  form a right triangle.
70. Find the equation of the circle that passes through  $(3, -2)$ ,  $(-1, -4)$ , and  $(2, -5)$ .
71. Graph the inequality  $y > 4(x + 2)^2$ .
72. Graph the inequality  $4x^2 + 9(y - 2)^2 < 36$ .
73. What type of curve is represented by  $(x + jy)^2 + (x - jy)^2 = 2?$  ( $j = \sqrt{-1}$ )
74. For the ellipse in Fig. 21.112, show that the product of the slopes  $PA$  and  $PB$  is  $-b^2/a^2$ .

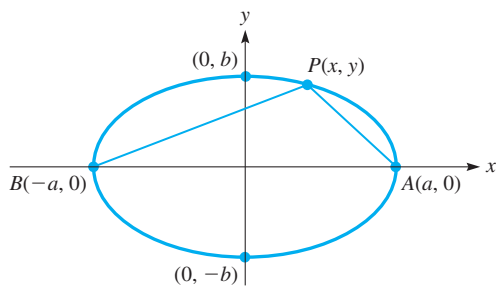


Fig. 21.112

75. Find the area of the square that can be inscribed in the ellipse  $7x^2 + 2y^2 = 18$ .
76. Using a graphing calculator, determine the number of points of intersection of the polar curves  $r = 4|\cos 2\theta|$  and  $r = 6 \sin[\cos(\cos 3\theta)]$ .
77. By means of the definition of a parabola, find the equation of the parabola with focus at  $(3, 1)$  and directrix the line  $y = -3$ . Find the same equation by the method of translation of axes.
78. For what value(s) of  $k$  does  $x^2 - ky^2 = 1$  represent an ellipse with vertices on the  $y$ -axis?

79. The total resistance  $R_T$  of two resistances in series in an electric circuit is the sum of the resistances. If a variable resistor  $R$  is in series with a  $2.5\text{-}\Omega$  resistor, express  $R_T$  as a function of  $R$  and sketch the graph.
80. The acceleration of an object is defined as the change in velocity  $v$  divided by the corresponding change in time  $t$ . Find the equation relating the velocity  $v$  and time  $t$  for an object for which the acceleration is  $6.0 \text{ m/s}^2$  and  $v = 5.0 \text{ m/s}$  when  $t = 0 \text{ s}$ .
81. The velocity  $v$  of a crate sliding down a ramp is given by  $v = v_0 + at$ , where  $v_0$  is the initial velocity,  $a$  is the acceleration, and  $t$  is the time. If  $v_0 = 1.92 \text{ m/s}$  and  $v = 6.20 \text{ m/s}$  when  $t = 5.50 \text{ s}$ , find  $v$  as a function of  $t$ . Sketch the graph.
82. An airplane touches down when landing at  $150 \text{ km/h}$ . Its velocity  $v$  while coming to a stop is given by  $v = 150 - 20000t$ , where  $t$  is the time in hours. Sketch the graph of  $v$  vs.  $t$ .
83. It takes  $2.010 \text{ kJ}$  of heat to raise the temperature of  $1.000 \text{ kg}$  of steam by  $1.000^\circ\text{C}$ . In a steam generator, a total of  $y \text{ kJ}$  is used to raise the temperature of  $50.00 \text{ kg}$  of steam from  $100^\circ\text{C}$  to  $T^\circ\text{C}$ . Express  $y$  as a function of  $T$  and sketch the graph.
84. The temperature in a certain region is  $27^\circ\text{C}$ , and at an altitude of  $2500 \text{ m}$  above the region it is  $12^\circ\text{C}$ . If the equation relating the temperature  $T$  and the altitude  $h$  is linear, find the equation.
85. The radar gun on a police helicopter  $170 \text{ m}$  above a multilane highway is directed vertically down onto the highway. If the radar gun signal is cone-shaped with a vertex angle of  $14^\circ$ , what area of the highway is covered by the signal?
86. The Niagara Sky Wheel in Niagara Falls, Ontario, has a diameter of  $50.5 \text{ m}$ . Find the equation representing its circumference. Place the origin of the coordinate system  $2.50 \text{ m}$  below the bottom of the wheel and the centre of the wheel on the  $y$ -axis.
87. The arch of a small bridge across a stream is parabolic. If, at water level, the span of the arch is  $80 \text{ m}$  and the maximum height above water level is  $20 \text{ m}$ , what is the equation that represents the arch? Choose the most convenient point for the origin of the coordinate system.
88. A laser source is  $2.00 \text{ cm}$  from a spherical surface of radius  $3.00 \text{ cm}$ , and the laser beam is tangent to the surface. By placing the centre of the sphere at the origin, and the source on the positive  $x$ -axis, find the equation of the line along which the beam shown in Fig. 21.113 is directed.

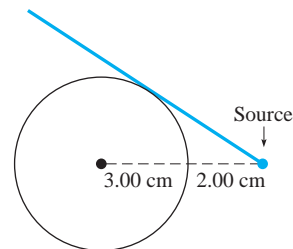


Fig. 21.113

89. The top horizontal cross section of a dam is parabolic. The open area within this cross section is  $80 \text{ m}$  across and  $50 \text{ m}$  from front to back. Find the equation of the edge of the open area with the vertex at the origin of the coordinate system and the axis along the  $x$ -axis.



90. The *quality factor*  $Q$  of a series resonant electric circuit with resistance  $R$ , inductance  $L$ , and capacitance  $C$  is given by  $Q = \frac{1}{R} \sqrt{\frac{L}{C}}$ . Sketch the graph of  $Q$  and  $L$  for a circuit in which  $R = 1000 \Omega$  and  $C = 4.00 \mu\text{F}$ .
91. At very low temperatures, certain metals have an electric resistance of zero. This phenomenon is called *superconductivity*. A magnetic field also affects the superconductivity. A certain level of magnetic field  $H_T$ , the threshold field, is related to the thermodynamic temperature  $T$  by  $H_T/H_0 = 1 - (T/T_0)^2$ , where  $H_0$  and  $T_0$  are specifically defined values of magnetic field and temperature. Sketch the graph of  $H_T/H_0$  vs.  $T/T_0$ .
92. A rectangular parking lot is to have a perimeter of 600 m. Express the area  $A$  in terms of the width  $w$  and sketch the graph.
93. The electric power  $P$  (in W) supplied by a battery is given by  $P = 12.0i - 0.500i^2$ , where  $i$  is the current (in A). Sketch the graph of  $P$  vs.  $i$ .
94. The technical ring of the Olympic Stadium in Montreal, Quebec, is in the shape of an ellipse 175 m long and 104 m wide. Find the area of the technical ring. ( $A = \pi ab$  for an ellipse.)
95. A specialty electronics company makes an ultrasonic device to repel animals. It emits a 20–25 kHz sound (above those heard by people), which is unpleasant to animals. The sound covers an elliptical area starting at the device, with the longest dimension extending 36 m from the device and the focus of the area 5 m from the device. Find the area covered by the signal. ( $A = \pi ab$ )
96. A study indicated that the fraction  $f$  of cells destroyed by various dosages  $d$  of X rays is given by the graph in Fig. 21.114. Assuming that the curve is a quarter-ellipse, find the equation relating  $f$  and  $d$  for  $0 \leq f \leq 1$  and  $0 < d \leq 10$  units.

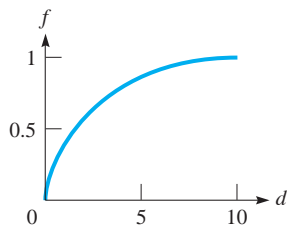


Fig. 21.114

97. A machine-part designer wishes to make a model for an elliptical cam by placing two pins in a design board, putting a loop of string over the pins, and marking off the outline by keeping the string taut. (Note that the definition of the ellipse is being used.) If the cam is to measure 10 cm by 6 cm, how long should the loop of string be and how far apart should the pins be?
98. Soon after reaching the vicinity of the moon, *Apollo 11* (the first spacecraft to land a man on the moon) went into an elliptical lunar orbit. The closest the craft was to the moon in this orbit was 110 km, and the farthest it was from the moon was 310 km. What was the equation of the path if the centre of the moon was at one of the foci of the ellipse? Assume that the major axis is along the  $x$ -axis and that the centre of the ellipse is at the origin. The radius of the moon is 1740 km.

99. The vertical cross section of the cooling tower of a nuclear power plant is hyperbolic, as shown in Fig. 21.115. Find the radius  $r$  of the smallest circular horizontal cross section.

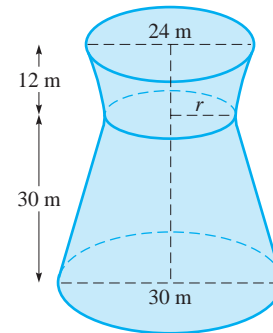


Fig. 21.115

100. Tremors from an earthquake are recorded at the Giralia seismograph station 36 s before they are recorded at the Morawa seismograph station, both in Western Australia. If the stations are 740 km apart and the shock waves from the tremors travel at 5.0 km/s, what is the curve on which lies the point where the earthquake occurred?
101. An electronic instrument located at point  $P$  records the sound of a rifle shot and the impact of the bullet striking the target at the same instant. Show that  $P$  lies on a branch of a hyperbola. (The bullet travels faster than the speed of sound.)
102. A 20-m rope passes over a pulley 4 m above the ground, and a crate on the ground is attached at one end. The other end of the rope is held at a level of 1 m above the ground and is drawn away from the pulley. Express the height of the crate over the ground in terms of the distance the person is from directly below the crate. Sketch the graph of distance and height. See Fig. 21.116. (Neglect the thickness of the crate.)

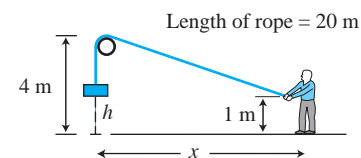


Fig. 21.116

103. A satellite makes one revolution per day around the centre of the earth. The projection on the earth of its path can be approximated by the curve  $r^2 = R^2 \cos 2(\theta + \frac{\pi}{2})$ , where  $R$  is the radius of the earth. Sketch the path of the projection.
104. The vertical cross sections of two pipes as drawn on a drawing board are shown in Fig. 21.117. Find the polar equation of each.

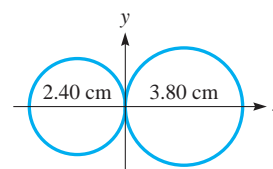


Fig. 21.117

105. The path of a certain plane is  $r = 200(\sec \theta + \tan \theta)^{-5} / \cos \theta$ ,  $0 < \theta < \pi/2$ . Sketch the path and check it on a graphing calculator. (This path is the same as for the plane in Exercise 96 on page 340.)
106. The sound produced by a jet engine was measured at a distance of 100 m in all directions. The loudness of the sound  $d$  (in decibels) was found to be  $d = 115 + 10 \cos \theta$ , where the  $0^\circ$  line for the angle  $\theta$  is directed in front of the engine. Sketch the graph of  $d$  vs.  $\theta$  in polar coordinates (use  $d$  as  $r$ ).

### Writing Exercise

107. Under a force that varies inversely as the square of the distance from an attracting object (such as the sun exerts on the earth), it can be shown that the equation of the path an object follows is given in general by

$$\frac{1}{r} = a + b \cos \theta$$

where  $a$  and  $b$  are constants for a particular path. First, transform this equation into rectangular coordinates. Then write one or two paragraphs explaining why this equation represents one of the conic sections, depending on the values of  $a$  and  $b$ . It is through this kind of analysis that we know the paths of the planets and comets are conic sections.

## CHAPTER 21 PRACTICE TEST

- (a) Find the distance between  $(4, -1)$  and  $(6, 3)$ . (b) Find the slope of the line perpendicular to the line segment joining the points in part (a).
- Identify the type of curve represented by the equation  $2(x^2 + x) = 1 - y^2$ .
- Sketch the graph of the straight line  $4x - 2y + 5 = 0$  by finding its slope and  $y$ -intercept.
- Find the polar equation of the curve whose rectangular equation is  $x^2 = 2x - y^2$ .
- Find the vertex and the focus of the parabola  $x^2 = -12y$ . Sketch the graph.
- Find the equation of the circle with centre at  $(-1, 2)$  and that passes through  $(2, 3)$ .
- Find the equation of the straight line that passes through  $(-4, 1)$  and  $(2, -2)$ .
- Where is the focus of a parabolic reflector that is 12.0 cm across and 4.00 cm deep?
- A hallway 6.0 m wide has a ceiling whose cross section is a semiellipse. The ceiling is 3.0 m high at the walls and 4.0 m high at the centre. Find the height of the ceiling 1.0 m from each wall.
- Plot the polar curve  $r = 3 + \cos \theta$ .
- Find the centre and vertices of the conic section  $4y^2 - x^2 - 4x - 8y - 4 = 0$ . Show completely the sketch of the curve.
- (a) What type of curve is represented by  $8x^2 - 4xy + 5y^2 = 36$ ? (b) Through what angle must the curve in part (a) be rotated in order that there is no  $x'y'$ -term?

# Introduction to Statistics

After the invention of the steam engine in the late 1700s by the Scottish engineer James Watt, the production of machine-made goods became widespread during the 1800s. However, it was not until the 1920s that much attention was paid to the quality control of the goods being produced. In 1924, Walter Shewhart of Bell Telephone Laboratories used a statistical chart for controlling product variables; in the 1940s, quality control was used in much of wartime production.

Quality control is one of the modern uses of *statistics*, the branch of mathematics in which data are collected, displayed, analysed, and interpreted. Today it is nearly impossible to read a newspaper or watch television news without seeing some type of study, in areas such as medicine or politics, that involves statistics. Other areas in which statistical methods are used include biology, physics, psychology, sociology, reliability engineering, actuarial science, economics, business, and education, to name but a few.

The first significant use of statistics was made in the 1660s by John Graunt, and in the 1690s by Edmund Halley (of Halley's Comet), when each published some conclusions about the population in England based on mortality tables. There was little development of statistics until the 1800s, when statistical measures became more widely used. For example, important contributions were made by the scientist Francis Galton, who used statistics in the study of human heredity, and by the nurse Florence Nightingale, who used statistical graphs to show that more soldiers died in the Crimean War (in the 1850s) from unsanitary conditions than from combat wounds.

In using statistics, we generally collect and summarize data (using methods from *descriptive statistics*) to make inferences based on those data (using methods from *inferential statistics*). The first two sections of this chapter are dedicated to descriptive statistics. After a discussion regarding the normal distribution, we dedicate the rest of the chapter to introducing some basic concepts of inferential statistics, including confidence intervals, statistical process control, and regression.

## 22

### LEARNING OUTCOMES

After completion of this chapter, the student should be able to:

- Understand the basic concepts of population, sample, parameter, statistic, and variable
- Construct frequency, relative frequency, and cumulative frequency tables for data
- Draw a histogram, a frequency polygon, and an ogive
- Calculate measures of central tendency (mean, median, and mode) and measures of spread (range and standard deviation)
- Use Chebychev's theorem to draw conclusions about a data set
- Calculate relative frequencies using the normal distribution
- Construct large sample confidence intervals for a population mean or for a population proportion
- Plot an  $\bar{x}$  control chart, an  $R$  control chart, and a  $p$  control chart
- Find the equation of the least-squares line that best fits a given set of data
- Find the equation of a curve that best fits a given set of data by transforming the independent variable and using linear least squares

David P. Lewis/Shutterstock

◀ In Section 22.4 we see how statistical analysis was used in the design of the 12.9-km-long Confederation Bridge that joins New Brunswick and Prince Edward Island.



## 22.1 Tabular and Graphical Representation of Data

Population • Sample • Variables  
 Array • Class • Frequency Distribution  
 Table • Relative Frequency • Frequency  
 Polygon • Histogram • Cumulative  
 Frequency • Ogive

In statistics, a **population** is the complete collection of elements (people, DVDs, households, temperatures) that are of interest and about which information is desired. Typically, a researcher is interested in a numerical property of the population called a **parameter**. For example, if the population is all DVDs produced by a certain manufacturer over the course of a week, a parameter would be the proportion of defective DVDs in that lot. If the population is all Internet users, a parameter would be the average number of hours spent each week on the Internet.

Because of constraints on time, money, and other scarce resources, conclusions about the population are usually drawn after observing only a subset of the population, called a **sample**. Quantities computed from samples are called **statistics**. Statistics are used to estimate parameters in the population. For example, the average in a sample of Internet users can be used to estimate the average among all users. Similarly, the proportion of defectives in a sample can be used to estimate the proportion of defectives in the complete lot.

We are usually only interested in some of the characteristics that elements of the population have in common. A **variable** is any characteristic whose value changes from individual to individual in the population. A **quantitative variable** has a value that represents a numerical measurement. Examples of quantitative variables are weight, length, voltage, pressure, and number of children in a family. When the value of a variable is non-numerical, it is called a qualitative variable, or an **attribute**. Examples of attributes are colour, gender, and quality (measured as defective or nondefective).

Values of variables that have been recorded constitute data. *Data that have been collected but not yet organized are called **raw data**.* In order to obtain useful information from the data, it is necessary to organize it in some way. Normally, a first step in organizing the data is to arrange the numerical values in ascending (or descending) order, forming what is called an **array**.

### EXAMPLE 1 Illustrating an array

Each user in a sample of 50 home computer users was asked to estimate carefully the number of hours they spent each week on-line on the Internet. Following are the estimates.

12, 20, 15, 14, 7, 10, 12, 25, 18, 5, 10, 24, 16, 3, 12, 14, 28, 8, 13, 18,  
 15, 8, 11, 15, 14, 22, 14, 19, 6, 10, 18, 4, 16, 24, 18, 5, 13, 20, 12, 12,  
 25, 11, 8, 12, 20, 5, 10, 15, 13, 8

As we can see, no clear pattern can be seen from the raw data. Arranging these in numerical order to form an array, we can summarize the array by showing the number of persons reporting each estimate as follows:

(hours-persons) 3-1, 4-1, 5-3, 6-1, 7-1, 8-4, 10-4, 11-2, 12-6,  
 13-3, 14-4, 15-4, 16-2, 18-4, 19-1, 20-3, 22-1, 24-2, 25-2, 28-1 ■

In Example 1, although a pattern is somewhat clearer from the summarized array than from the raw data, a still clearer pattern is found by *grouping the data*. In the process of grouping, the detail of the raw data is lost, but the advantage is that a much clearer overall pattern of the data can be obtained.

The grouping of data is done by first defining what values are to be included in each group and then tabulating the number of values that are in each group. *Each group is called a **class**, and the number of values in each class is called the **frequency**. The table is called a **frequency distribution table**.* This is illustrated in the following example.

**EXAMPLE 2** Frequency distribution table

In Example 1, we note that the estimates vary from 3 h to 28 h. We see that if we form *classes* of 0–4 h, 5–9 h, etc., we will have five possible estimates in each class and that there will be six classes. This gives us the following frequency distribution table of values showing the number of persons (frequency) reporting the indicated estimate of the hours spent on the Internet.

<i>Estimate</i> (hours)	0–4	5–9	10–14	15–19	20–24	25–29
<i>Frequency</i> (persons)	2	9	19	11	6	3

This table shows us the frequency distribution. The 0, 5, 10, and so on are the *lower class limits*, and the 4, 9, 14, and so on are the *upper class limits*. Each class includes five values, which is the *class width*. As with the class limits we have chosen, it is generally preferable to have the same width for each class.

We can see from this frequency distribution table that the pattern of hours on the Internet by the persons responding to the survey is clearer. ■

**Guidelines for Constructing Frequency Tables**

- Make sure that classes are mutually exclusive, so that each observation belongs to one, and only one, class.
- Use between 5 and 20 classes. The principal consideration is that the relevant characteristics of the data should be clear.
- Ensure that all classes (except for open-ended classes) have the same width.
- Use class limits with convenient numbers. The first lower class limit is selected as the lowest value, or as a convenient number less than the lowest value.
- Be sure to include all classes, even if their frequency is zero.

At times, it is also helpful to know the **relative frequency** of the class, which is the *frequency of the class divided by the total frequency of all classes*. The relative frequency can be expressed as a fraction, decimal, or percent.

**EXAMPLE 3** Relative frequency

The relative frequency of each class for the data in Example 2 can be shown as in the following table:

<i>Estimated Hours on Internet</i>	<i>Frequency</i>	<i>Relative Frequency (%)</i>
0–4	2	4
5–9	9	18
10–14	19	38
15–19	11	22
20–24	6	12
25–29	3	6
Total	50	100

$2/50 = 0.04 = 4\%$

**Practice Exercise**

- Assuming the data in Example 1 are divided into classes of 0–3 h, 4–7 h, etc., for the 8–11 h class find:
  - the frequency
  - the relative frequency

Using graphs is a very convenient method of representing frequency distributions. There are several useful types of graphs for such distributions. *Among the most important of these are the histogram and the frequency polygon.*

In order to represent *grouped* data, where the raw data values are generally not all the same within a given class, we find it necessary to use a representative value for each class. For this, we use the **class mark**, which is found by dividing the sum of the lower and upper class limits by 2. The following examples illustrate the use of a class mark with a histogram and a frequency polygon.

■ A graphing calculator may be used to display histograms and frequency polygons.



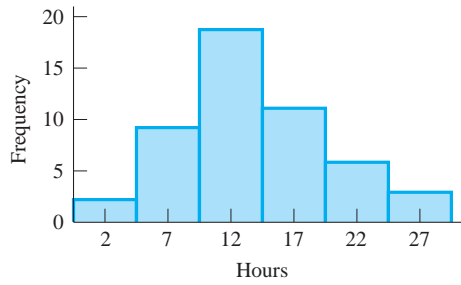


Fig. 22.1

Computer spreadsheets are very useful for this type of analysis.

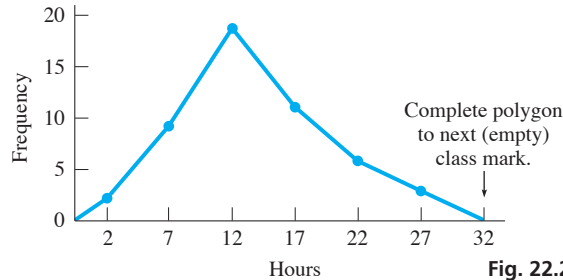


Fig. 22.2

**EXAMPLE 4 Histogram**

A histogram represents a particular set of data by displaying each class of the data as a rectangle. Each rectangle is labelled at the centre of its base by the class mark. The width of each rectangle represents the class width, and the height of the rectangle represents the frequency of the class.

For the data in Example 2 on estimated hours on the Internet, the class marks are  $(0 + 4)/2 = 2$ ,  $(5 + 9)/2 = 7$ , and so on. Using these values, a histogram representing these data is shown in Fig. 22.1.

**EXAMPLE 5 Frequency polygon**

A frequency polygon is used to represent a set of data by plotting the class marks as abscissas ( $x$ -values) and the frequencies as ordinates ( $y$ -values). The resulting points are joined by straight-line segments.

A frequency polygon representing the data in Example 2 on estimated hours on the Internet is shown in Fig. 22.2.

If the polygon is not completed as shown in Fig. 22.2, and the figure starts at the first class mark, and ends at the last class mark, it is then referred to as a broken-line graph.

Another way of analysing data is to use cumulative totals. The way this is generally done is to change the frequency into a “less than” cumulative frequency. To do this, we add the class frequencies, starting at the lowest class boundary. The graphical display that is generally used for cumulative frequency is called an ogive (pronounced oh-jive).

**EXAMPLE 6 Cumulative frequency—ogive**

For the data on estimated hours on the Internet in Examples 2 and 3, the cumulative frequency is shown in the following table:

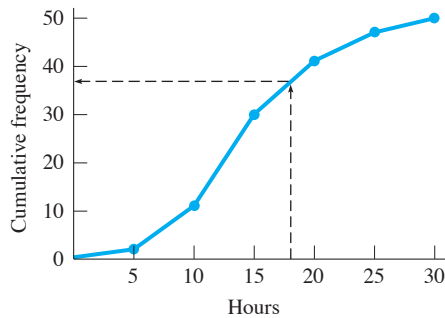


Fig. 22.3

Estimated Hours on Internet	Cumulative Frequency
Less than 5	2
Less than 10	11
Less than 15	30
Less than 20	41
Less than 25	47
Less than 30	50

The ogive showing the cumulative frequency for the values in this table is shown in Fig. 22.3. The vertical scale shows the frequency, and the horizontal scale shows the class boundaries.

One important use of an ogive is to determine the number of values above or below a certain value. For example, to approximate the number of respondents that use the Internet less than 18 hours per week, we draw a line from the horizontal axis to the ogive and then to the vertical axis as shown in Fig. 22.3. From this, we see that about 37 respondents use the Internet less than 18 hours per week.

If the data with which we are dealing have only a limited number of values and we do not divide the data into classes, we can still use the methods we have developed, using the specific values rather than class values. Consider the following example.

**LEARNING TIP**

Note that in a cumulative frequency table, the name of the class can vary. For example, the first class in Example 6 could be defined as less than 5, or up to but not including 5, or less than or equal to 4. What is important is that the classes be cumulative and that there be no overlap nor ambiguity between the classes.

**EXAMPLE 7** Graphs using specific values

A test station measured the loudness of the sound of jet aircraft taking off from a certain airport. The decibel (dB) readings measured to the nearest 5 dB for the first 20 jets were as follows:

110, 95, 100, 115, 105, 110, 120, 110, 115, 105,  
90, 95, 105, 110, 100, 115, 105, 120, 95, 110

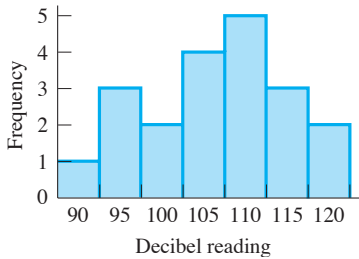


Fig. 22.4

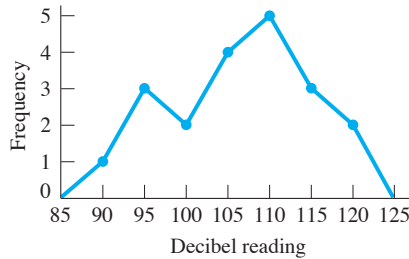


Fig. 22.5

Since there are only seven different values for the 20 readings, the best idea of the pattern is found by using these values, as shown in the following frequency table:

<i>dB Reading</i>	90	95	100	105	110	115	120
<i>Frequency</i>	1	3	2	4	5	3	2

The histogram for this table is shown in Fig. 22.4, and the frequency polygon is shown in Fig. 22.5. ■

**EXERCISES 22.1**

In Exercises 1–4, divide the data in Example 1 into five classes of hours (0–5, 6–11, etc.) on the Internet and then do the following:

1. Form a frequency distribution table.
2. Find the relative frequencies.
3. Draw a histogram.
4. Draw an ogive.

In Exercises 5–12, use the following data. An automobile company tested a new engine, and found the following results in twenty tests of the number of litres of gasoline used by a certain model for each 100 km travelled.

5.3, 5.8, 5.6, 5.4, 5.9, 5.4, 6.0, 5.8, 5.8, 5.4,  
6.3, 5.6, 5.7, 5.6, 5.7, 5.9, 5.5, 6.1, 5.9, 5.8

5. Form an ordered array and summarize it by finding the frequency of each number of litres used.
6. Find the relative frequency of each number of litres used.
7. Form a frequency distribution table with five classes.
8. Find the relative frequencies for the data in the frequency distribution table in Exercise 7.
9. Draw a histogram for the data of Exercise 7.
10. Draw a frequency polygon for the data of Exercise 7.
11. Form a cumulative frequency table for the data of Exercise 7.
12. Draw an ogive for the data of Exercise 7.

In Exercises 13–32, find the indicated quantities.

13. In testing a computer system, the number of instructions it could perform in 1 ns was measured at different points in a program. The numbers of instructions were recorded as follows:  
19, 21, 22, 25, 22, 20, 18, 21, 20, 19, 22, 21, 19, 23, 21  
Form a frequency distribution table for these values.
14. For the data of Exercise 13, draw a histogram.
15. For the data of Exercise 13, draw a frequency polygon.

16. For the data of Exercise 13, form a relative frequency distribution table.
17. A strobe light is designed to flash every 2.25 s at a certain setting. Sample bulbs were tested with the following results:

<i>Time (s) between Flashes (class mark)</i>	2.21	2.22	2.23	2.24
<i>Number of Bulbs</i>	2	7	18	41

<i>Time (s)</i>	2.25	2.26	2.27	2.28	2.29
<i>No. Bulbs</i>	56	32	8	3	3

Draw a histogram for these data.

18. For the data of Exercise 17, draw a frequency polygon.
19. For the data of Exercise 17, form a cumulative frequency distribution table.
20. For the data of Exercise 17, draw an ogive.
21. In testing a braking system, the distance required to stop a car from 110 km/h was measured in 120 trials. The results are shown in the following distribution table:

<i>Stopping Distance (m)</i>	47–49	50–52	53–55	56–58
<i>Times Car Stopped</i>	2	15	32	36

<i>Stopping Distance (m)</i>	59–61	62–64	65–67
<i>Times Car Stopped</i>	24	10	1

Form a relative frequency distribution table for these data.

22. For the data in Exercise 21, form a cumulative frequency distribution table.
23. For the data of Exercise 21, draw an ogive.
24. From the ogive in Exercise 23, estimate the number of cars that stopped in less than 57 m.





the values and then dividing by the number of values. (The mean is the number most people call the “average.” However, in statistics the word *average* has the more general meaning of a measure of central tendency.)

### EXAMPLE 2 Arithmetic mean

The arithmetic mean of the numbers given in Example 1 is determined by finding the sum of all the numbers and dividing by 15. Therefore, by letting  $\bar{x}$  (read as “x bar”) represent the mean, we have

$$\begin{aligned}\bar{x} &= \frac{5 + 2 + 6 + 4 + 7 + 4 + 7 + 2 + 8 + 9 + 4 + 11 + 9 + 1 + 3}{15} \\ &= \frac{82}{15} = 5.5\end{aligned}$$

Thus, the mean is 5.5. ■

If we wish to find the arithmetic mean of a large number of values, and if some of them appear more than once, the calculation can be simplified. The mean can be calculated by multiplying each value by its frequency, adding these results, and then dividing by the total number of values (the sum of the frequencies). Letting  $\bar{x}$  represent the mean of the values  $x_1, x_2, \dots, x_n$ , which occur with frequencies  $f_1, f_2, \dots, f_n$ , respectively, we have

■ This is called a *weighted mean* since each value is given a weight based on the number of times it occurs.

$$\bar{x} = \frac{x_1 f_1 + x_2 f_2 + \cdots + x_n f_n}{f_1 + f_2 + \cdots + f_n} \quad (22.1)$$

### EXAMPLE 3 Arithmetic mean using frequencies

Using Eq. (22.1) to find the arithmetic mean of the numbers of Example 1, we first set up a table of values and their respective frequencies, as follows:

Values	1	2	3	4	5	6	7	8	9	11
Frequency	1	2	1	3	1	1	2	1	2	1

We now calculate the arithmetic mean  $\bar{x}$  by using Eq. (22.1):

$$\begin{aligned}\bar{x} &= \frac{1(1) + 2(2) + 3(1) + 4(3) + 5(1) + 6(1) + 7(2) + 8(1) + 9(2) + 11(1)}{1 + 2 + 1 + 3 + 1 + 1 + 2 + 1 + 2 + 1} \\ &= \frac{82}{15} = 5.5\end{aligned}$$

We see that this agrees with the result of Example 2. ■

Summations such as those in Eq. (22.1) occur frequently in statistics and other branches of mathematics. In order to simplify writing these sums, the symbol  $\Sigma$  is used to indicate the process of summation. ( $\Sigma$  is the Greek capital letter sigma.)  $\Sigma x$  means the sum of the  $x$ 's.

**EXAMPLE 4** Summation symbol  $\Sigma$ 

We can show the sum of the numbers  $x_1, x_2, x_3, \dots, x_n$  as

$$\sum x = x_1 + x_2 + x_3 + \cdots + x_n$$

If these numbers are 3, 7, 2, 6, 8, 4, and 9, we have

$$\sum x = 3 + 7 + 2 + 6 + 8 + 4 + 9 = 39$$

Using the summation symbol  $\Sigma$ , we can write Eq. (22.1) for the arithmetic mean as

$$\bar{x} = \frac{x_1f_1 + x_2f_2 + x_3f_3 + \cdots + x_nf_n}{f_1 + f_2 + f_3 + \cdots + f_n} = \frac{\sum xf}{\sum f} \quad (22.1)$$

The summation notation  $\sum x$  is an abbreviated form of the more general notation  $\sum_{i=1}^n x_i$ . This more general form can be used to indicate the sum of the first  $n$  numbers of a sequence or to indicate the sum of a certain set within the sequence. For example, for a set of at least 5 numbers,  $\sum_{i=3}^5 x_i$  indicates the sum of the third through the fifth of these numbers (in Example 4,  $\sum_{i=3}^5 x_i = 16$ ). We will use the abbreviated form  $\sum x$  to indicate the sum of all the numbers being considered.

**EXAMPLE 5** Arithmetic mean using frequencies

We find the arithmetic mean of the Internet hours in Example 1 of Section 22.1 (page 616) by

$$\begin{aligned} \bar{x} &= \frac{\sum xf}{\sum f} = \frac{3(1) + 4(1) + 5(3) + \cdots + 12(6) + \cdots + 28(1)}{50} \\ &= \frac{687}{50} = 13.7 \text{ h} \quad (\text{rounded off to tenths}) \end{aligned}$$

**Mode**

Another measure of central tendency is *the mode*, which is the value that appears most frequently. If two or more values appear with the same greatest frequency, each is a mode. If no value is repeated, there is no mode.

**EXAMPLE 6** Mode

(a) The mode of the numbers in Example 1 is 4, since it appears three times, and no other value appears more than twice.

(b) The modes of the numbers

$$1, 2, 2, 4, 5, 5, 6, 7$$

are 2 and 5, since each appears twice and no other number is repeated.

(c) There is no mode for the values

$$1, 2, 5, 6, 7, 9$$

since none of the values is repeated.

**EXAMPLE 7** Measures of central tendency

To find the frictional force between two specially designed surfaces, the force to move a block with one surface along an inclined plane with the other surface is measured ten times. The results, with forces in newtons, are

$$2.2, 2.4, 2.1, 2.2, 2.5, 2.2, 2.4, 2.7, 2.1, 2.5$$

Find the mean, median, and mode of these forces.

**Practice Exercises**

For the following numbers, find the indicated value:

12, 17, 16, 12, 14, 18, 14, 12, 15, 18

1. The median
2. The arithmetic mean

■ The arithmetic mean is one of a number of statistical measures that can be found on a calculator.

**LEARNING TIP**

- The mean is useful for many statistical methods and is used extensively. Nevertheless, be aware that the mean is very sensitive to extreme observations so that a single extreme observation can change the value of the mean dramatically and give the wrong impression about the data.
- The median is not affected by extreme observations. Therefore, it is a good choice as a measure of centre in the presence of extreme values.
- The mean, the median, and the mode coincide when the distribution of data is symmetric. In those cases, the median or the mode (which are easy to calculate) can be used as estimates of the mean.

To find the mean, we sum the values of the forces and divide this total by 10. This gives

$$\begin{aligned}\bar{F} &= \frac{\sum F}{10} = \frac{2.2 + 2.4 + 2.1 + 2.2 + 2.5 + 2.2 + 2.4 + 2.7 + 2.1 + 2.5}{10} \\ &= \frac{23.3}{10} = 2.33 \text{ N}\end{aligned}$$

The median is found by arranging the values in order and finding the middle value. The values in order are

$$2.1, 2.1, 2.2, 2.2, 2.2, 2.4, 2.4, 2.5, 2.5, 2.7$$

Since there are 10 values, we see that the fifth value is 2.2 and the sixth is 2.4. The value midway between these is 2.3, which is the median. Therefore, the median force is 2.3 N.

The mode is 2.2 N, since this value appears three times, which is more than any other value. ■

**MEASURES OF SPREAD**

Measures of central tendency on their own are not very informative. They do not tell us whether values are grouped closely together or how spread out they are. Therefore, we also need some measure of the deviation, or spread, of the values from the centre. If the spread is small and the numbers are grouped closely together, the measure of central tendency is more reliable and descriptive of the data than in the case in which the spread is greater.

In statistics, there are several measures of spread that may be defined. The simplest one is the **range**, which is the difference between the highest value and the lowest value in the data set. For example, the range of the data in Example 7 is  $2.7 - 2.1 = 0.6$ . We will see how the range is applied to statistical process control in Section 22.5.

The most widely used measure of spread is the *standard deviation*. The **standard deviation** of a set of **sample** values is defined by the equation

$$s = \sqrt{\frac{\sum (x - \bar{x})^2}{n - 1}} \quad (22.2)$$

The definition of  $s$  shows that the following steps are used in computing its value.

**Steps for Calculating Standard Deviation**

1. Find the arithmetic mean  $\bar{x}$  of the numbers of the set.
2. Subtract the mean from each number of the set.
3. Square these differences.
4. Find the sum of these squares.
5. Divide this sum by  $n - 1$ .
6. Find the square root of this result.

Following the steps shown above, we use Eq. (22.2) for the calculation of standard deviation in the following examples.

**Range****Standard Deviation****LEARNING TIP**

The standard deviation  $s$  is a positive number. It is a *deviation from the mean*, regardless of whether the individual numbers are greater than or less than the mean. Numbers close together will have a small standard deviation, whereas numbers further apart have a larger standard deviation. Therefore, *the standard deviation becomes larger as the spread of data increases.*

**EXAMPLE 8** Standard deviation—using Eq. (22.2)

Find the standard deviation of the following numbers: 1, 5, 4, 2, 6, 2, 1, 1, 5, 3.  
A table of the necessary values is shown below, and steps 1–6 are indicated:

$x$	step 2 $x - \bar{x}$	step 3 $(x - \bar{x})^2$
1	-2	4
5	2	4
4	1	1
2	-1	1
6	3	9
2	-1	1
1	-2	4
1	-2	4
5	2	4
3	0	0
<u>30</u>		<u>32</u>

$$\bar{x} = \frac{30}{10} = 3 \quad \text{step 1}$$

$$\frac{\sum (x - \bar{x})^2}{n - 1} = \frac{32}{10 - 1} = \frac{32}{9} \quad \text{step 5}$$

$$s = \sqrt{\frac{32}{9}} = 1.9 \quad \text{step 6 (rounded off to tenths)}$$

step 4

**LEARNING TIP**

The *population standard deviation*, represented by the Greek letter  $\sigma$  (read “sigma”), is computed using  $n$  in the denominator of Eq. (22.2) instead of  $n - 1$ . The  $n - 1$  in the denominator of Eq. (22.2) adjusts  $s$  so that it gives good estimates of the parameter  $\sigma$  when the standard deviation can only be measured from a sample.

Since we generally use data coming from samples, in this text we will always use Eq. (22.2), and we will refer to the sample standard deviation simply as the standard deviation.

If some of the values in the data are repeated, we can use the frequency of those values that occur more than once in calculating the standard deviation. This is illustrated in the following example.

**EXAMPLE 9** Standard deviation using frequencies

Find the standard deviation of the numbers in Example 1.

Since several of the numbers appear more than once, it is helpful to use the frequency of each number in the table, as follows:

$x$	$f$	$xf$	step 2 $x - \bar{x}$	step 3 $(x - \bar{x})^2$	$(x - \bar{x})^2 f$
1	1	1	-4.5	20.25	20.25
2	2	4	-3.5	12.25	24.50
3	1	3	-2.5	6.25	6.25
4	3	12	-1.5	2.25	6.75
5	1	5	-0.5	0.25	0.25
6	1	6	0.5	0.25	0.25
7	2	14	1.5	2.25	4.50
8	1	8	2.5	6.25	6.25
9	2	18	3.5	12.25	24.50
11	<u>1</u>	<u>11</u>	5.5	30.25	30.25
	<u>15</u>	<u>82</u>			<u>123.75</u>

$$\bar{x} = \frac{82}{15} = 5.5 \quad \text{step 1}$$

$$\frac{\sum (x - \bar{x})^2 f}{n - 1} = \frac{123.75}{15 - 1} = \frac{123.75}{14} \quad \text{step 5}$$

$$s = \sqrt{\frac{123.75}{14}} = 3.0 \quad \text{step 6}$$

step 4

It is possible to reduce the computational work required to find the standard deviation. Algebraically, it can be shown (although we will not do so here) that the following equation is another form of Eq. (22.2) and therefore gives the same results.

$$s = \sqrt{\frac{n(\sum x^2) - (\sum x)^2}{n(n - 1)}} \quad (22.3)$$

Although the form of this equation appears more involved, it does reduce the amount of calculation that is necessary. Consider the following examples.

**EXAMPLE 10** Standard deviation—using Eq. (22.3)

Using Eq. (22.3), find  $s$  for the numbers in Example 8.

$x$	$x^2$
1	1
5	25
4	16
2	4
6	36
2	4
1	1
1	1
5	25
3	9
30	122

$$\begin{aligned}
 n &= 10 \\
 \sum x^2 &= 122 \\
 (\sum x)^2 &= 30^2 = 900 \\
 s &= \sqrt{\frac{10(122) - 900}{10(9)}} = 1.9
 \end{aligned}$$

**Practice Exercise**

3. Find the standard deviation of the first eight numbers in Example 8.

**COMMON ERROR**

It is a common error to confuse  $\sum x^2$  and  $(\sum x)^2$  in Eq. (22.3). Note that for  $\sum x^2$ , we square the  $x$  values and then add the squares, whereas for  $(\sum x)^2$ , we first add the  $x$  values and then square the sum.

**EXAMPLE 11** Standard deviation—application

An ammeter measures the electric current in a circuit. In an ammeter, two resistances are connected in parallel, with most of the current passing through a very low resistance called the *shunt*. The resistance of each shunt in a sample of 100 shunts was measured. The results were grouped, and the class mark and frequency for each class are shown in the following table. Calculate the arithmetic mean and the standard deviation of the resistances of the shunts.

■ Statistical measures such as  $\bar{x}$ ,  $\sum x$ ,  $\sum x^2$ ,  $s_x$ ,  $\sigma_x$ , and  $n$  can be obtained directly on a scientific or a graphing calculator.

$R$ (ohms)	$f$	$Rf$	$R^2f$
0.200	1	0.200	0.0400
0.210	3	0.630	0.1323
0.220	5	1.100	0.2420
0.230	10	2.300	0.5290
0.240	17	4.080	0.9792
0.250	40	10.000	2.5000
0.260	13	3.380	0.8788
0.270	6	1.620	0.4374
0.280	3	0.840	0.2352
0.290	2	0.580	0.1682
	100	24.730	6.1421

$$\begin{aligned}
 \bar{R} &= \frac{24.730}{100} = 0.2473 \Omega \\
 n &= 100 \\
 \sum R^2 &= 6.1421 \\
 (\sum R)^2 &= 24.730^2 \\
 s &= \sqrt{\frac{100(6.1421) - 24.730^2}{100(99)}} = 0.0163
 \end{aligned}$$

The arithmetic mean of the resistances is  $0.2473 \Omega$ , with a standard deviation of  $0.0163 \Omega$ .

**EXAMPLE 12** Standard deviation—application

Find the standard deviation of the estimated hours on the Internet as grouped in Example 2 of Section 22.1 (page 617). In doing this, we assume that each value in the class is the same as the class mark. The method is not exact, but with a large set of numbers, it provides a good approximation with less arithmetic work.

Interval	$x$	$f$	$xf$	$x^2f$
0–4	2	2	4	8
5–9	7	9	63	441
10–14	12	19	228	2736
15–19	17	11	187	3179
20–24	22	6	132	2904
25–29	27	3	81	2187
		50	695	11 455

$$\begin{aligned}
 n &= 50 \\
 \sum x^2 &= 11\,455 \\
 (\sum x)^2 &= 695^2 \\
 s &= \sqrt{\frac{50(11\,455) - 695^2}{50(49)}} = 6.1
 \end{aligned}$$

Thus,  $s = 6.1$  h. ■

The mean and the standard deviation together can help us draw conclusions about the values in a data set. Thanks to a result known as **Chebychev's theorem**, we can state the percentage of data values that must be within a specific number of standard deviations from the mean.

### Chebychev's Theorem

For **any** data set (population or sample), the proportion of observations that must be within  $k$  standard deviations of the mean is always at least  $1 - \frac{1}{k^2}$  ( $k > 1$ ).

For the particular values  $k = 2, 3$ , and  $4$ , here is what the statement of the theorem implies:

- At least 75% of observations are within two standard deviations of the mean.
- At least 89% of observations are within three standard deviations of the mean.
- At least 94% of observations are within four standard deviations of the mean.

Note that since Chebychev's theorem is so general, it will underestimate the percentages for some distributions. In Section 22.3, we will obtain more precise percentages for the important case of the normal distribution.

### EXAMPLE 13 An application of Chebychev's theorem

A sample of computers of a certain brand had a mean time of 38 months without a hardware malfunction, with a standard deviation of 2.5 months. What percentage of the computers in the sample lasted between 33 and 43 months without a hardware malfunction?

We can write  $33 = 38 - 2(2.5)$ , and  $43 = 38 + 2(2.5)$ , so 33 and 43 are 2 standard deviations away from the mean, and we use Chebychev's theorem with  $k = 2$ . Therefore, at least 75% of the computers in the sample lasted between 33 and 43 months without a hardware malfunction. ■

In using the statistical measures we have discussed, we must be careful in using and interpreting such measures. Consider the following example.

### EXAMPLE 14 Interpreting statistical measures

- (a) The numbers 1, 2, 3, 4, 5 have a mean of 3, a median of 3, and a standard deviation of 1.6. These values fairly well describe the centre and distribution of the numbers in the set.
- (b) The numbers 1, 2, 3, 4, 100 have a mean of 22, a median of 3, and a standard deviation of 44. The large difference between the median and the mean and the very large range of values within one standard deviation of the mean (–22 to 66) indicate that this set of measures does not describe this set of numbers well. In a case like this, the 100 should be checked to see if it is in error. ■

Example 14 illustrates how statistical measures can be misleading in the presence of extreme values. Misleading statistics can also come from the process of data collection.



Consider the probable results of a survey to find the percent of persons in favour of raising income taxes for the wealthy if the survey is taken at the entrance to a welfare office or if it is taken at the entrance to a stock brokerage firm. There are many other considerations in the proper use and interpretation of statistical measures.

## EXERCISES 22.2

In Exercises 1–4, delete the 5 from the data numbers given for Example 1 and then do the following with the resulting data.

- Find the median.
- Find the arithmetic mean using the definition, as in Example 2.
- Find the arithmetic mean using Eq. (22.1), as in Example 3.
- Find the mode, as in Example 6.

In Exercises 5 and 6, use the data in Example 8. Change the first 1 to 6 and the first 2 to 7 and then find the standard deviation of the resulting data as directed.

- Find  $s$  from the definition, as in Example 8.
- Find  $s$  using Eq. 22.3, as in Example 10.
- In Example 13, change 33 to 28 and 43 to 48 and then find the percentage.

In Exercises 8–15, use the following sets of numbers.

A: 3, 6, 4, 2, 5, 4, 7, 6, 3, 4, 6, 4, 5, 7, 3

B: 25, 26, 23, 24, 25, 28, 26, 27, 23, 28, 25

C: 0.48, 0.53, 0.49, 0.45, 0.55, 0.49, 0.47, 0.55, 0.48, 0.57, 0.51, 0.46, 0.53, 0.50, 0.49, 0.53

D: 105, 108, 103, 108, 106, 104, 109, 104, 110, 108, 108, 104, 113, 106, 107, 106, 107, 109, 105, 111, 109, 108

In Exercises 8–11, determine (a) the mean, (b) the median, and (c) the mode of the numbers of the given set.

8. Set A      9. Set B      10. Set C      11. Set D

In Exercises 12–15, find the standard deviation  $s$  for the indicated set of numbers (a) using Eq. (22.2), and (b) using Eq. (22.3).

12. Set A      13. Set B      14. Set C      15. Set D

In Exercises 16–30, the required data are those in Exercises 22.1.

- Find the mean, the median, and the mode of L/100 km of fuel usage in Exercise 5.
- Find the standard deviation of L/100 km of fuel usage in Exercise 5 using Eq. (22.2).
- Find the standard deviation of L/100 km of fuel usage in Exercise 5 using Eq. (22.3).
- Find the mean, the median, and the mode of computer instructions in Exercise 13.
- Find the range and standard deviation of computer instructions in Exercise 13.
- Find the mean and the median of strobe light times in Exercise 17.
- Find the standard deviation of strobe light times in Exercise 17.
- Find the mean and median of stopping distances in Exercise 21. (Use the class mark for each class.)

- Find the standard deviation of stopping distances in Exercise 21. (Use the class mark for each class.)
- Find the mean, the median, and the mode of X-ray dosages in Exercise 25.
- Find the range and the standard deviation of X-ray dosages in Exercise 25.
- Find the mean, the median, and the mode of battery lives in Exercise 27.
- Find the standard deviation of battery lives in Exercise 27.
- Find the mean and the median of cable diameters in Exercise 29.
- Find the standard deviation of cable diameters in Exercise 29.

In Exercises 31–47, find the indicated measure of central tendency or of spread.

- The weekly salaries (in dollars) for the workers in a small factory are as follows:  
600, 750, 625, 575, 525, 700, 550,  
750, 625, 800, 700, 575, 600, 700  
Find the median and the mode of the salaries.
- Find the mean salary for the salaries in Exercise 31.
- Find the range and the standard deviation of the salaries in Exercise 31.
- In a particular month, the electrical usage, rounded to the nearest 400 MJ, of 1000 homes in a certain city was summarized as follows:

Usage	2000	2400	2800	3200	3600	4000	4400	4800
No. Homes	22	80	106	185	380	122	90	15

Find the mean of the electrical usage.

- Find the median and mode of electrical usage in Exercise 34.
- Find the standard deviation of electrical usage in Exercise 34.
- A test of air pollution in a city gave the following readings of the concentration of sulfur dioxide (in parts per million) for 18 consecutive days:  
0.14, 0.18, 0.27, 0.19, 0.15, 0.22, 0.20, 0.18, 0.15,  
0.17, 0.24, 0.23, 0.22, 0.18, 0.32, 0.26, 0.17, 0.23  
Find the median and the mode of these readings.
- Find the mean of the readings in Exercise 37.
- Find the range and the standard deviation of air pollution data in Exercise 37.
- The following data give the mean number of days of rain for Vancouver, British Columbia, for the 12 months of the year.  
20, 17, 17, 14, 12, 11, 7, 8, 9, 16, 19, 22  
Find the standard deviation.

41. The *midrange*, another measure of central tendency, is found by finding the sum of the lowest and the highest values and dividing this sum by 2. Find the midrange of the salaries in Exercise 31.
42. Find the midrange of the sulfur dioxide readings in Exercise 37. (see Exercise 41.)
43. Add \$100 to each of the salaries in Exercise 31. Then find the median, mean, and mode of the resulting salaries. State any conclusion that might be drawn from the results.
44. Multiply each of the salaries in Exercise 31 by 2. Then find the median, mean, and mode of the resulting salaries. State any conclusion that might be drawn from the results.
45. Change the final salary in Exercise 31 to \$4000, with all other salaries being the same. Then find the mean of these salaries. State any conclusion that might be drawn from the result. (The \$4000 here is called an *outlier*, which is an extreme value.)
46. Find the median and mode of the salaries indicated in Exercise 45. State any conclusion that might be drawn from the results.
47. The *k% trimmed mean* is a measure of central tendency that avoids the influence of extreme observations while still using

most of the observations in the data set. It is computed by finding the mean of the data after the smallest  $k\%$  and the largest  $k\%$  of the data have been discarded. Find the 10% trimmed mean of X-ray dosages in Exercise 25 of Section 22.1.

In Exercises 48–50, solve the given problems.

48. Use Chebychev's theorem to find the percentage of values that are between 175 and 195 in a data set with mean 185 and standard deviation 5.
49. Use Chebychev's theorem to find the percentage of values that are between 55.7 and 68.3 in a data set with mean 62 and standard deviation 2.1.
50. The mean compressive strength of a sample of steel beams was 40 000 N/cm<sup>2</sup>, with a standard deviation of 450 N/cm<sup>2</sup>. What percent of the beams had compressive strength between 38 650 and 41 350 N/cm<sup>2</sup>?

#### Answers to Practice Exercises

1. 14.5    2. 14.8    3. 2.0

## 22.3 Normal Distributions

### Normal Distribution of a Population • Standard Normal Distribution • Standard Score (z-Score) • Sampling Distributions

■ The first derivation of the normal distribution is due to Abraham de Moivre (1667–1754), who was interested in approximating quantities arising in gambling problems. It was also derived independently by Pierre-Simon Laplace (1749–1847), and by Carl Friedrich Gauss (1777–1855), both in the context of measurement errors.

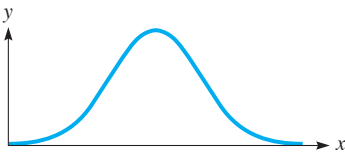


Fig. 22.6

In this section we discuss the **normal distribution**, the most important and most widely used distribution in statistics. The normal distribution is a continuous distribution, so we begin by discussing some generalities of continuous distributions.

In Section 22.1 we learned that for variables that take a limited number of values, the relative frequency of a value is obtained by dividing the frequency of that value by the total frequency of all values. Let us now consider variables that can be regarded as having an infinite number of possible values, such as weights, lengths, or durations for a very large population. (Such variables are said to be *continuous*.) When using the same procedure as before, the denominator becomes infinite, giving a relative frequency of zero for all values. How are we then to compute the relative frequency of intervals, if the relative frequency of all values is zero?

The answer lies in establishing a correspondence between relative frequency and *area*. To each continuous variable we associate a function, which we can graph as a curve on the plane. The relative frequency of a particular interval corresponds to the area under the curve in that interval. Because the relative frequency for the complete population must be 1, the total area under the curve must be 1 (100% of the data).

The **normal distribution** is associated with the symmetric, bell-shaped curve shown in Fig. 22.6. Using advanced methods, its equation is found to be

$$y = \frac{e^{-(x-\mu)^2/2\sigma^2}}{\sigma\sqrt{2\pi}} \quad (22.4)$$

Here,  $\mu$  is the *population mean* and  $\sigma$  is the *population standard deviation*, and  $\pi$  and  $e$  are the familiar numbers first used in Chapters 2 and 12, respectively.

From Eq. (22.4), we can see that any particular normal distribution depends on the values of  $\mu$  and  $\sigma$ . The horizontal location of the curve depends on  $\mu$ , and the shape (how spread out the curve is) depends on  $\sigma$ , but the bell shape remains. This is illustrated in general in the following example.

**EXAMPLE 1** Location and spread of a normal distribution

In Fig. 22.7, for the left curve,  $\mu = 10$  and  $\sigma = 5$ , whereas for the right curve,  $\mu = 20$  and  $\sigma = 10$ .

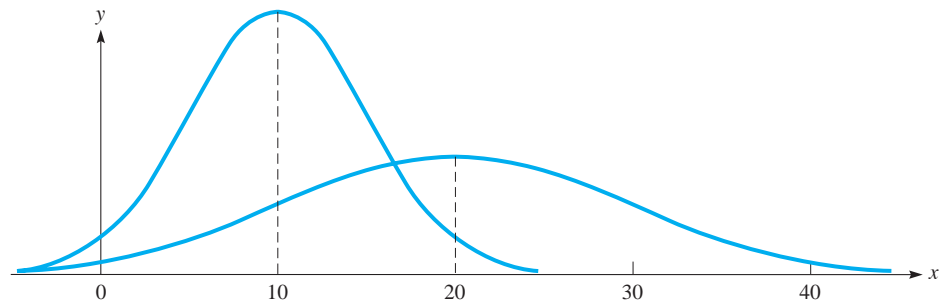


Fig. 22.7

**EXAMPLE 2** The normal distribution applied to reliability

The normal distribution has important applications in probabilistic reliability techniques for bridge design. For instance, load factors developed for the design of the Confederation Bridge are discussed in the article “Design criteria and load and resistance factors for the Confederation Bridge,” by J. G. MacGregor et al. (*Can. J. Civ. Eng.*, 24, 882–897 (1997)). Quantities that were found to be normally distributed (or whose logarithms were found to be normally distributed) arose in the analysis of dead loads, live loads due to vehicles, wind loads, temperature loads, and ice loads. To give a specific example, after analysing records of daily average temperatures in the region for 46 years, it was concluded that the 3-day temperature drop that was equalled or exceeded 100 times in 100 years is distributed normally, with mean 26.9°C and standard deviation 3.2°C. This distribution is shown in Fig. 22.8.

■ See the chapter introduction.

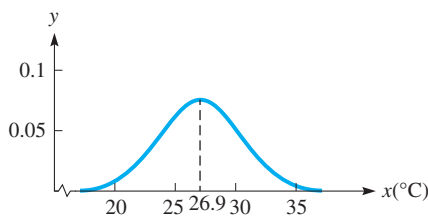


Fig. 22.8

**Properties of the Normal Curve**

- The curve is symmetric about the mean.
- The curve is always above the  $x$ -axis. ( $y$  is always positive.)
- The  $x$ -axis is a horizontal asymptote. (As  $x$  increases numerically,  $y$  becomes very small.)
- The total area under the curve is 1.
- The curve is bell-shaped, as seen in Figs. 22.6–22.8.

■ A more complete and rigorous treatment of the material covered in this and the remaining sections of this chapter would require the study of probability theory, which is beyond the scope of this introductory chapter.

Fig. 22.9 shows areas under a normal curve with mean  $\mu$  and standard deviation  $\sigma$  for particular regions. Using the correspondence between relative frequency and area, we can use the given information to find the percentage of data values that fall within one, two, and three standard deviations from the mean, as summarized below.

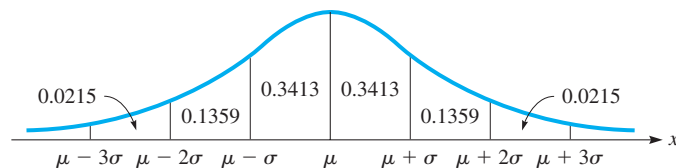


Fig. 22.9

### Important Percentages for the Normal Curve

- About 68% of values are within one standard deviation from the mean—that is, between  $\mu - \sigma$  and  $\mu + \sigma$ .
- About 95% of values are within two standard deviations from the mean—that is, between  $\mu - 2\sigma$  and  $\mu + 2\sigma$ .
- Almost all values (99.74%) are within three standard deviations from the mean—that is, between  $\mu - 3\sigma$  and  $\mu + 3\sigma$ .

We can compare these percentages with those given by Chebychev's theorem in Section 22.2—namely, that at least 75% of observations are within two standard deviations from the mean and that at least 89% of observations are within three standard deviations from the mean. We see that Chebychev's theorem heavily underestimates the true percentages in the case of the normal distribution.

We can use the areas from Fig. 22.9 to calculate relative frequencies for other intervals. This is illustrated in the following example.

### EXAMPLE 3 Relative frequencies and areas under the curve—application

Let us consider the normal distribution of 3-day temperature drops from Example 2, so that the mean and standard deviation are  $\mu = 26.9^\circ\text{C}$  and  $\sigma = 3.2^\circ\text{C}$ , respectively. The percentage of values that lie between one standard deviation below the mean and two standard deviations above the mean is 81.85%, since the area between  $\mu - \sigma$  and  $\mu + 2\sigma$  is  $0.3412 + 0.3413 + 0.1359 = 0.8185$ . We have

$$\mu - \sigma = 26.9 - 3.2 = 23.7 \text{ and } \mu + 2\sigma = 26.9 + 2(3.2) = 33.3$$

Therefore, about 81.85% of values for this distribution are within  $23.7^\circ\text{C}$  and  $33.3^\circ\text{C}$ . ■

### STANDARD NORMAL DISTRIBUTION

As we have just seen, there are innumerable possible normal distributions. However, there is one of particular interest. *The standard normal distribution is the normal distribution for which the mean is 0 and the standard deviation is 1.* Making these substitutions in Eq. (22.4), we have

$$y = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \quad (22.5)$$

as the equation of the standard normal distribution curve. All the properties of a normal curve are satisfied by the standard normal distribution. In particular, since the mean is 0, the curve is symmetric with respect to the y-axis. The curve is also bell-shaped, as seen in Fig. 22.10. As we discuss below, areas under the standard normal curve are used to find relative frequencies for all other normal curves.

We can find the relative frequency of values for any normal distribution by use of the **standard score**  $z$  (or  $z$ -score), which is defined as

$$z = \frac{x - \mu}{\sigma} \quad (22.6)$$

For the standard normal distribution, where  $\mu = 0$ , if we let  $x = \sigma$ , then  $z = 1$ . If we let  $x = 2\sigma$ ,  $z = 2$ . Therefore, we can see that *a value of  $z$  tells us the number of standard deviations the given value of  $x$  is above or below the mean.* From the discussion above, we can see that the value of  $z$  can tell us the area under the curve between the mean and the value of  $x$  corresponding to that value of  $z$ . In turn, *this tells us the relative frequency of all values between the mean and the value of  $x$ .*

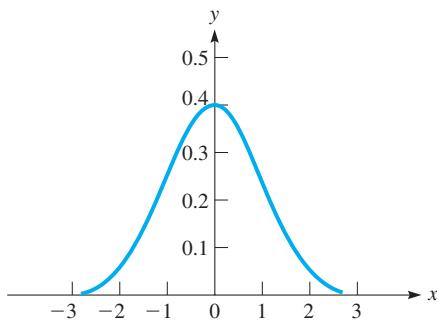


Fig. 22.10

Table 22.1 gives the area under the standard normal distribution curve between zero and the given values of  $z$ . The table includes values only to  $z = 3$  since nearly all of the area is between  $z = -3$  and  $z = 3$ . Since the curve is symmetric to the  $y$ -axis, the values shown are also valid for negative values of  $z$ .

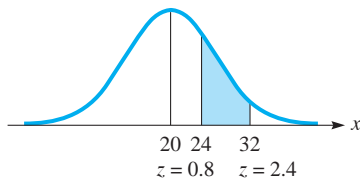
**Table 22.1 Standard Normal ( $z$ ) Distribution**

$z$	Area	$z$	Area	$z$	Area
0.0	0.0000	1.0	0.3413	2.0	0.4772
0.1	0.0398	1.1	0.3643	2.1	0.4821
0.2	0.0793	1.2	0.3849	2.2	0.4861
0.3	0.1179	1.3	0.4032	2.3	0.4893
0.4	0.1554	1.4	0.4192	2.4	0.4918
0.5	0.1915	1.5	0.4332	2.5	0.4938
0.6	0.2257	1.6	0.4452	2.6	0.4953
0.7	0.2580	1.7	0.4554	2.7	0.4965
0.8	0.2881	1.8	0.4641	2.8	0.4974
0.9	0.3159	1.9	0.4713	2.9	0.4981
1.0	0.3413	2.0	0.4772	3.0	0.4987

The following examples illustrate the use of Eq. (22.6) and  $z$ -scores.

**EXAMPLE 4 Normal score ( $z$ -score)**

For a normal distribution curve based on values of  $\mu = 20$  and  $\sigma = 5$ , find the area between  $x = 24$  and  $x = 32$ . To find this area, we use Eq. (22.6) to find the corresponding values of  $z$  and then find the difference between the areas associated with these  $z$ -scores. These  $z$ -scores are



**Fig. 22.11**

$$z = \frac{24 - 20}{5} = 0.8 \quad \text{and} \quad z = \frac{32 - 20}{5} = 2.4$$

For  $z = 0.8$ , the area is 0.2881, and for  $z = 2.4$ , the area is 0.4918. Therefore, the area between  $x = 24$  and  $x = 32$  (see Fig. 22.11) is

$$0.4918 - 0.2881 = 0.2037$$

This means that the relative frequency of the values between  $x = 24$  and  $x = 32$  is 20.37%. If we have a large set of measured values with  $\mu = 20$  and  $\sigma = 5$ , we should expect that about 20% of them are between  $x = 24$  and  $x = 32$ . ■

**Practice Exercise**

- For values of  $\mu = 40$  and  $\sigma = 8$ , find the area between  $x = 36$  and  $x = 48$ .

**EXAMPLE 5  $z$ -score—application**

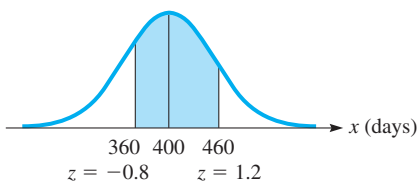
The lifetimes of a certain type of watch battery are normally distributed. The mean lifetime is 400 days, and the standard deviation is 50 days. For a sample of 5000 new batteries, determine how many batteries are expected to last (a) between 360 days and 460 days, (b) more than 320 days, and (c) less than 280 days.

- For this distribution,  $\mu = 400$  days and  $\sigma = 50$  days. Using Eq. (22.6), we find the  $z$ -scores for  $x = 360$  days and  $x = 460$  days. They are

$$z = \frac{360 - 400}{50} = -0.8 \quad \text{and} \quad z = \frac{460 - 400}{50} = 1.2$$

For  $z = -0.8$ , the area is to the left of the mean, and since the curve is symmetric about the mean, we use the  $z = 0.8$  value of the area and add it to the area for  $z = 1.2$ . Therefore, the area is

$$0.2881 + 0.3849 = 0.6730$$



**Fig. 22.12**

See Fig. 22.12. This means that 67.30% of the 5000 batteries, or 3365 of the batteries, are expected to last between 360 and 460 days. Because of variability within samples, not every sample will have exactly 3365 batteries that will last between

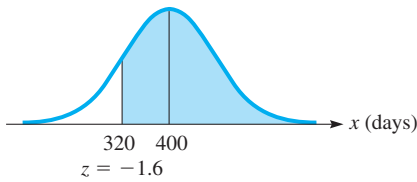


Fig. 22.13

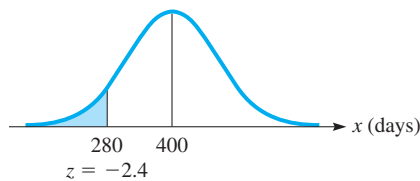


Fig. 22.14

360 and 460 days. On average, however, a sample of 5000 batteries will have 3365 that will last that amount of time.

- (b) To determine the number of batteries that will last more than 320 days, we first find the  $z$ -score for  $x = 320$ . It is  $z = (320 - 400)/50 = -1.6$ . This means we want the total area to the right of  $z = -1.6$ . In this case, we add the area for  $z = 1.6$  to the total area to the right of the mean. Since the total area under the curve is 1.0000, the total area on either side of the mean is 0.5000. Therefore, the area to the right of  $z = -1.6$  is  $0.4452 + 0.5000 = 0.9452$ . See Fig. 22.13. This means that  $0.9452 \times 5000 = 4726$  batteries are expected to last more than 320 days.
- (c) To find the number of batteries that will last less than 280 days, we first find that  $z = (280 - 400)/50 = -2.4$  for  $x = 280$ . Since we want the total area to the left of  $z = -2.4$ , we subtract the area for  $z = 2.4$  from 0.5000, the total area to the left of the mean. Since the area for  $z = 2.4$  is 0.4918, the total area to the left of  $z = -2.4$  is  $0.5000 - 0.4918 = 0.0082$ . See Fig. 22.14. Therefore,  $0.0082 \times 5000 = 41$  batteries are expected to last less than 280 days. ■

We now summarize the procedure for finding relative frequencies using  $z$ -scores.

### Finding Relative Frequencies Using $z$ -Scores

1. Sketch the normal curve, labeling the mean and the given  $x$  values. Identify the desired relative frequency as an area under the curve.
2. Use Eq. (22.6) to find the  $z$ -score for each  $x$ . Identify the desired relative frequency as an area under the standard normal curve.
3. Look up the absolute value of each  $z$ -score in Table 22.1 to find its associated area.
4. Depending on the situation, proceed as follows:

Area	Procedure
Between two $z$ -scores of the same sign	Subtract the smaller area from the larger one
Between two $z$ -scores of different sign	Add both areas together
To the right of a positive $z$ -score or to the left of a negative $z$ -score	Subtract the area from 0.5
To the right of a negative $z$ -score or to the left of a positive $z$ -score	Add the area to 0.5

## SAMPLING DISTRIBUTIONS

In Example 4, we assumed that the lifetimes of the batteries were normally distributed. Of course, for any set of 5000 batteries, or any number of batteries for that matter, the lifetimes that actually occur will not follow a normal distribution *exactly*. There will be some variation from the normal distribution, but for a large sample, this variation should be small. The mean and the standard deviation for any sample will vary somewhat from that of the population. When we consider the relative frequency distribution of the sample means obtained from all possible samples of the same size, we obtain what is called the **sampling distribution** of the sample means.

In the study of probability, it is shown that if we select all possible samples of size  $n$  from a population with a mean  $\mu$  and standard deviation  $\sigma$ , the mean of the sample means is also  $\mu$ . Also, *the standard deviation of the sample means*, denoted by  $\sigma_{\bar{x}}$ , and called **the standard error of the mean**, is

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} \quad (22.7)$$



Moreover, when  $n$  is large (*large* in this situation is usually considered to be over 30) the sampling distribution of the sample means is approximately normal. In other words, the normal curve approximates the relative frequency distribution of the sample means, so that areas under the normal curve can be used to approximate relative frequencies of the sample means.

The normal distribution also approximates the sampling distribution of quantities calculated from samples when the variable of interest is an attribute. For instance, suppose that we take samples of  $n$  items from a very large population containing a proportion  $p$  of defective items. If  $n$  is large (in this case,  $np$  and  $n(1 - p)$  **must both be at least 5**), then the sampling distribution of the sample proportion  $\hat{p}$  of defective items in each sample is also approximately normal. In this situation, the mean of the sample proportions is  $p$ , and the standard error of  $\hat{p}$  is

$$\sigma_{\hat{p}} = \sqrt{\frac{p(1-p)}{n}} \quad (22.8)$$

We can see from Eqs. (22.7) and (22.8) that as the sample size gets larger, the less variation there will be in the mean or the proportion obtained from a sample.

#### COMMON ERROR

It is a common error to forget the  $\sqrt{n}$  term in the denominator of the standard error formulas. It is very important to include it since it is what guarantees that variation between samples decreases as the sample size increases.

#### EXAMPLE 6 Sampling distribution of the sample mean

For the sample of 5000 watch batteries in Example 5, we know that  $\sigma = 50$  days. Therefore, the standard error of the mean  $\bar{x}$  is  $50/\sqrt{5000} = 0.7$  day. This means that of all samples of 5000 batteries, about 68% should have a mean lifetime of  $400 \pm 0.7$  day (between 399.3 days and 400.7 days). Considering the significant digits of these values, 68% (within one standard deviation) or even 95% (within two standard deviations) of the sample values of  $\bar{x}$  would not vary by more than 1 day. ■

#### EXAMPLE 7 Sampling distribution of the sample proportion

Of the items in a very large population, 10% are defective. Samples of size 200 are taken from this population, and the proportion of defectives in each sample is recorded. The normal approximation applies since  $np = 200(0.1) = 20$  and  $n(1 - p) = 200(0.9) = 180$ . Therefore, the sample proportion  $\hat{p}$  is approximately normally distributed, with mean 0.1 and standard error  $\sqrt{(0.10)(0.90)/200} = 0.021$ , or 2.1%. This means that of all samples of size 200 taken from this population, about 68% would have a sample proportion of defectives of  $0.1 \pm 0.021$  (i.e., between 7.9% and 12.1% of the sample would be defective in 68% of samples). ■

## EXERCISES 22.3

In Exercises 1–4, make the given changes in the indicated examples of this section and then solve the indicated problem.

- In Example 1, change the second  $\sigma$  from 10 to 5 and then describe the curve that would result in terms of either or both curves shown in Fig. 22.7.
- In Example 3, change two to three and then find the resulting percent of values and the corresponding interval.
- In Example 4, change  $x = 32$  to  $x = 33$  and then find the resulting area.
- In Example 5(b), change 320 to 360 and then find the resulting number of batteries.



In Exercises 5–8, use a graphing calculator to display the indicated graph of Eq. (22.4).

5. Display the graph of the normal distribution of values for which  $\mu = 10$  and  $\sigma = 5$ . Compare with the graph shown in Fig. 22.7.
6. Display the graph of the normal distribution of values for which  $\mu = 20$  and  $\sigma = 10$ . Compare with the graph shown in Fig. 22.7.
7. Sketch a graph of a normal distribution of values for which  $\mu = 100$  and  $\sigma = 10$ . Then compare with the graph displayed by a graphing calculator.
8. Sketch a graph of a normal distribution of values for which  $\mu = 100$  and  $\sigma = 30$ . Then compare with the graph displayed by a graphing calculator. How does this graph differ from that of Exercise 7?

In Exercises 9–12, use the following data and refer to Fig. 22.9. A sample of 200 bags of cement are weighed as a quality check. Over a long period, it has been found that the mean value and standard deviation for this size bag are known and that the weights are normally distributed. Determine how many bags within this sample are expected to have weights that satisfy the following conditions.

9. Within one standard deviation of the mean
10. Within two standard deviations of the mean
11. Between the mean and two standard deviations above the mean
12. Between one standard deviation below the mean and three standard deviations above the mean

In Exercises 13–16, use the following data. Each AA battery in a sample of 500 batteries is checked for its voltage. It has been previously established for this type of battery (when newly produced) that the voltages are distributed normally with  $\mu = 1.50$  V and  $\sigma = 0.05$  V.

13. How many batteries are expected to have voltages between 1.45 V and 1.55 V?
14. How many batteries are expected to have voltages between 1.52 V and 1.58 V?
15. What percent of the batteries are expected to have voltages below 1.54 V?
16. What percent of the batteries are expected to have voltages above 1.64 V?

In Exercises 17–22, use the following data. The lifetimes of a certain type of automobile tire have been found to be distributed normally with a mean lifetime of 100 000 km and a standard deviation of 10 000 km. Answer the following questions for a sample of 5000 of these tires.

17. How many tires are expected to last between 85 000 km and 100 000 km?

18. How many tires are expected to last between 95 000 km and 115 000 km?
19. How many tires are expected to last more than 118 000 km?
20. If the manufacturer guarantees to replace all tires that do not last 75 000 km, what percent of the tires may have to be replaced under this guarantee?
21. What is the standard error in the mean for all samples of 5000 of these tires? Explain the meaning of this result.
22. What percent of the samples of 5000 of these tires should have a mean lifetime of more than 100 282 km?

In Exercises 23–32, solve the given problems.

23. Find the standard error of the proportion of defective items in samples of size 500 taken from a very large population of which 12% of the items are defective. Explain the meaning of this result.
24. Of the 300 mL bottles filled by a certain filling machine, 1% contain less than 290 mL of juice. If samples of 600 bottles produced by this machine are selected, find the standard error of the sample proportion of bottles that contain less than 290 mL. Explain the meaning of this result.
25. With 75.8% of the area under the normal curve to the right of  $z$ , find the  $z$ -value.
26. With 21% of the area under the normal curve between  $z_1$  and  $z_2$ , to the right of  $z_1 = 0.8$ , find  $z_2$ .
27. With 59% of the area under the normal curve between  $z_1$  and  $z_2$ , to the left of  $z_2 = 1.1$ , find  $z_1$ .
28. With 5.8% of the area under the normal curve between  $z_1$  and  $z_2$ , to the left of  $z_2 = 2.0$ , find  $z_1$ .
29. For the strobe light times in Exercise 17 of Section 22.1, find the percent of times within one standard deviation of the mean. From Exercises 21 and 22 of Section 22.2, we find that  $\bar{x} = 2.248$  s and  $s = 0.014$  s. Compare the results with that of a normal distribution.
30. Follow the same instructions as in Exercise 29 for the fibre-optic diameters in Exercise 29 of Section 22.1. From Exercises 29 and 30 of Section 22.2,  $\bar{x} = 0.00595$  mm and  $s = 0.00022$  mm.
31. Follow the same instructions as in Exercise 29 for the hours estimated on the Internet in Example 1 of Section 22.1. From Examples 5 and 12 of Section 22.2, we find that  $\bar{x} = 13.7$  h and  $s = 6.1$  h.
32. Discuss the results found in Exercises 29 and 31, considering the methods used to find the mean and the standard deviation.

#### Answer to Practice Exercise

1.  $z = 0.5328$

## 22.4 Confidence Intervals

Estimators • Confidence Level • Confidence Intervals for Means • Margin of Error • Confidence Intervals for Proportions • Determining Sample Size

In this section we begin our study of inferential statistics, where information from a sample is used to make statements about a whole population. We focus on the problem of estimation of parameters, under the assumption that data are available for a random sample taken from a very large population.

When a parameter is being estimated, the estimate can be a single number (called a **point estimate**), or it can be a range of numbers (called a **confidence interval**). Consider the following example.

**EXAMPLE 1** Two kinds of estimates

Consider the data on Internet hours in Example 1 of Section 22.1. If the sample of 50 users constitutes a random sample of a large population of users of home computers, then the information obtained from the sample can be used to estimate the population mean.

On the one hand, the sample mean  $\bar{x} = 13.7$  is a point estimate of the population mean. Because it is a single number, this estimate does not convey information about the reliability of the estimation.

On the other hand, the interval  $13.7 \pm 1.7 = (12.0 \text{ h}, 15.4 \text{ h})$  is a 95% confidence interval estimate of the population mean. This means that we are 95% confident that the true value of the population mean is between 12.0 h and 15.4 h. It is preferable to use a confidence interval because the length of the interval and the level of confidence attached to it give us an idea about the reliability of our estimation, in a sense that will be made precise below. ■

All confidence intervals are calculated by first selecting a **confidence level**, which measures the degree of certainty that the confidence interval will contain the population parameter. The most common values for the confidence level are 90%, 95%, and 99%, with the most common one being 95%. A confidence level of 95% means that, of all possible samples of size  $n$  taken from the same population, 95% of them will give an interval that will contain the population parameter, and 5% of them will not. For a particular sample, it is not possible to know whether it is one of the successful ones or not.

**COMMON ERROR**

It is a common error to interpret the level of confidence as measuring the likelihood that the parameter of interest will fall within a particular interval. There is nothing random about the parameter; its value is a constant (unfortunately unknown to us), and either our interval covers it or it does not. The randomness lies in the sample, so the level of confidence is the likelihood that a **sample of size  $n$**  will cover the parameter. At the 95% confidence level, 95% of the samples will, and 5% of the samples will not. We acknowledge that there is a 5% risk that the particular interval obtained will not cover the parameter.

The method for constructing a confidence interval depends on the parameter(s) being estimated and on the characteristics of the sample. We will concentrate our attention on *large sample confidence intervals for a single mean and a single proportion*. We will obtain general formulas for these two cases using the normal distribution theory developed in the previous section.

**LARGE SAMPLE CONFIDENCE INTERVALS FOR THE MEAN**

Suppose that we have a random sample of size  $n$  ( $n$  large) from a population with unknown mean  $\mu$ . We construct a 95% confidence interval for the mean  $\mu$ , under the assumption that the population standard deviation  $\sigma$  is known.

In Section 22.3, we learned that the sampling distribution of the sample mean is approximately normal, with mean  $\mu$  and standard error  $\sigma/\sqrt{n}$ . It is found from tables that the area under the standard normal curve between  $-1.96$  and  $1.96$  is  $0.95$ , so that 95% of the observations from a normal distribution fall within 1.96 standard deviations of the mean. Applying this to the sampling distribution of the sample mean, we have that 95% of all samples of size  $n$  will have a sample mean  $\bar{x}$  that falls within 1.96 standard errors of the true mean. In other words, in 95% of samples, the sample mean  $\bar{x}$  satisfies the inequality

$$\mu - 1.96 \cdot \frac{\sigma}{\sqrt{n}} < \bar{x} < \mu + 1.96 \cdot \frac{\sigma}{\sqrt{n}}$$

We can manipulate this inequality in order to transform it into a statement about the unknown population mean  $\mu$ . We have

$$\begin{aligned} \mu - 1.96 \cdot \frac{\sigma}{\sqrt{n}} < \bar{x} < \mu + 1.96 \cdot \frac{\sigma}{\sqrt{n}} && \text{original inequality} \\ -1.96 \cdot \frac{\sigma}{\sqrt{n}} < \bar{x} - \mu < 1.96 \cdot \frac{\sigma}{\sqrt{n}} && \text{subtract } \mu \text{ from each member} \\ -\bar{x} - 1.96 \cdot \frac{\sigma}{\sqrt{n}} < -\mu < -\bar{x} + 1.96 \cdot \frac{\sigma}{\sqrt{n}} && \text{subtract } \bar{x} \text{ from each member} \\ \bar{x} + 1.96 \cdot \frac{\sigma}{\sqrt{n}} > \mu > \bar{x} - 1.96 \cdot \frac{\sigma}{\sqrt{n}} && \text{multiply by } -1 \text{ (reverse the inequality)} \\ \bar{x} - 1.96 \cdot \frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + 1.96 \cdot \frac{\sigma}{\sqrt{n}} && \end{aligned}$$

This last inequality is equivalent to the original one and is therefore satisfied for 95% of samples of size  $n$ . In other words,

$$\left( \bar{x} - 1.96 \cdot \frac{\sigma}{\sqrt{n}}, \bar{x} + 1.96 \cdot \frac{\sigma}{\sqrt{n}} \right) \quad (22.9)$$

is a 95% confidence interval for  $\mu$ .

The endpoints of the interval in Eq. (22.9) are often written in the form

$$\bar{x} \pm E, \text{ with } E = 1.96 \cdot \frac{\sigma}{\sqrt{n}} \quad (22.10)$$

The quantity  $E$  is called the **margin of error**, and it represents the largest estimated difference between the estimate and the true value of the parameter.

### EXAMPLE 2 A 95% confidence interval— $\sigma$ known

A random sample of size  $n = 100$  is taken from a population with  $\sigma = 2.3$ . Construct a 95% confidence interval for the population mean  $\mu$  if the sample mean is  $\bar{x} = 32.8$ .

We substitute the given values of  $\bar{x}$ ,  $\sigma$ , and  $n$  into Eq. (22.10). The resulting 95% confidence interval is

$$\bar{x} \pm 1.96 \cdot \frac{\sigma}{\sqrt{n}} = 32.8 \pm 1.96 \cdot \frac{2.3}{\sqrt{100}} = 32.8 \pm 0.451 = (32.3, 33.3)$$

Because the interval obtained is narrow, the estimation is quite precise. ■

When the standard deviation  $\sigma$  is unknown and the sample size is large, the sample standard deviation  $s$  can be used to estimate the population standard deviation  $\sigma$ . Therefore, a 95% confidence interval for  $\mu$  when the standard deviation  $\sigma$  is unknown is given by

$$\bar{x} \pm E, \text{ with } E = 1.96 \cdot \frac{s}{\sqrt{n}} \quad (22.11)$$

**Practice Exercise**

- Find a 95% confidence interval for a mean  $\mu$  if a sample with  $n = 45$  gives  $\bar{x} = 97.6$  and  $s = 3.2$ .

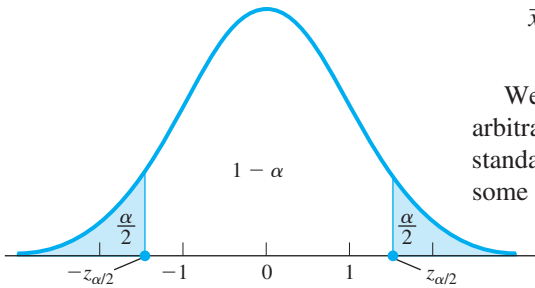


Fig. 22.15

**EXAMPLE 3 A 95% confidence interval— $\sigma$  estimated**

Let us derive the confidence interval for the mean stated in Example 1, obtained from the Internet hours data from Example 1 of Section 22.1. The sample mean and sample standard deviation for the 50 observations are  $\bar{x} = 13.7$  h and  $s = 6.1$  h, respectively. Substituting these values into Eq. (22.11), the resulting 95% confidence interval is

$$\bar{x} \pm 1.96 \frac{s}{\sqrt{n}} = 13.7 \pm 1.96 \cdot \frac{6.1}{\sqrt{50}} = 13.7 \pm 1.69 = (12.0 \text{ h}, 15.4 \text{ h}) \quad \blacksquare$$

We now derive the formula for a large sample confidence interval for a mean for an arbitrary confidence level  $1 - \alpha$ . Here we have written the confidence level as is standard for general formulas, expressing it as a decimal whose value is  $1 - \alpha$  for some small  $\alpha$  (for example,  $\alpha = 0.05$  for a 95% confidence interval).

Let  $z_{\alpha/2}$  denote the score such that the area under the standard normal curve between  $-z_{\alpha/2}$  and  $z_{\alpha/2}$  is  $1 - \alpha$ . See Fig. 22.15 and Table 22.2. By replacing 1.96 with  $z_{\alpha/2}$  in the inequalities leading to Eq. (22.10), we obtain the following general formula.

**Large Sample Confidence Interval for a Mean**

A  $(1 - \alpha)100\%$  confidence interval for the mean  $\mu$  when the sample size is large ( $n \geq 30$ ) is given by

$$\bar{x} \pm E, \text{ where } E = z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}} \quad \text{if } \sigma \text{ is known}$$

$$E = z_{\alpha/2} \cdot \frac{s}{\sqrt{n}} \quad \text{if } \sigma \text{ is unknown} \quad (22.12)$$

Table 22.2 Common Values of  $z_{\alpha/2}$

Confidence Level ( $1 - \alpha$ )100%	$\alpha/2$	$z_{\alpha/2}$
90%	0.05	1.645
95%	0.025	1.96
99%	0.005	2.575

**Practice Exercise**

- Find a 90% confidence interval for a mean  $\mu$  if a sample with  $n = 45$  gives  $\bar{x} = 97.6$  and  $s = 3.2$ .

**EXAMPLE 4 A 99% confidence interval**

Construct a 99% confidence interval for the Internet hours data from Example 1 of Section 22.1. Recall that  $\bar{x} = 13.7$  h,  $s = 6.1$  h, and  $n = 50$ .

From Table 22.2 we get that  $z_{\alpha/2} = 2.575$ . We substitute the given values into Eq. (22.12). The desired 99% confidence interval is

$$\bar{x} \pm z_{\alpha/2} \frac{s}{\sqrt{n}} = 13.7 \pm 2.575 \cdot \frac{6.1}{\sqrt{50}} = 13.7 \pm 2.22 = (11.5 \text{ h}, 15.9 \text{ h}) \quad \blacksquare$$

**COMMON ERROR**

Always state a confidence interval together with its confidence level. **A confidence interval by itself is meaningless.**

Comparing the intervals in Example 3 and Example 4, we see that by increasing the confidence level, the margin of error increased as well, so that what we have gained in confidence, we have lost in precision. By examining the formula for the margin of error in Eq. (22.12), we can find the general relationship between confidence level, margin of error, and sample size.

**Confidence Level, Margin of Error, and Sample Size Relationships**

- For a fixed sample size, a higher confidence level implies a larger margin of error. What we gain in confidence we lose in precision.
- For a fixed sample size, a smaller margin of error implies a lower confidence level. What we gain in precision we lose in confidence.
- The only way we can increase the confidence level while at the same time decrease the margin of error is to increase the sample size.

The formula for  $E$  in Eq. (22.12) for known  $\sigma$  can be used to determine the sample size needed for a desired margin of error at a fixed confidence level. Suppose that a maximum margin of error  $E$  is desired with a confidence level  $1 - \alpha$ . We have

$$E = z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}} \quad \text{from Eq. (22.12)}$$

$$\sqrt{n} = \frac{z_{\alpha/2}\sigma}{E} \quad \text{multiplying both sides by } \frac{\sqrt{n}}{E}$$

After squaring both sides, we get

$$n = \left[ \frac{z_{\alpha/2}\sigma}{E} \right]^2 \quad (22.13)$$

When the result of Eq. (22.13) is not an integer, it must always be **rounded up** to the next integer in order to guarantee the prescribed margin of error.

### EXAMPLE 5 Determining the sample size

An estimate of the mean direct-current output voltage of a certain kind of AC adaptor is desired. If it can be assumed that  $\sigma = 0.04$  V, find the sample size necessary to estimate that mean with a margin of error of 0.01 V with 95% confidence.

Substituting  $\sigma = 0.04$ ,  $E = 0.01$ , and  $z_{\alpha/2} = 1.96$ , we get

$$n = \left[ \frac{1.96(0.04)}{0.01} \right]^2 = 61.5$$

Therefore, a sample of 62 adaptors is necessary. ■

## LARGE SAMPLE CONFIDENCE INTERVALS FOR A PROPORTION

We now analyse the problem of estimating an unknown population proportion with confidence. We consider a large population such that a proportion  $p$  of its elements share a certain attribute. For example,  $p$  could be the proportion of defective items in a lot, or the proportion of incorrect entries in an account, or the proportion of components that will last a certain number of hours, or the proportion of voters who will vote for a certain candidate in the next election.

Suppose that a random sample of size  $n$  is taken. We calculate the sample proportion  $\hat{p}$  by dividing the number of elements in the sample that share the attribute by the sample size  $n$ . We further assume that  $n$  is large, so that  $np \geq 5$  and  $n(1 - p) \geq 5$ . (Since  $p$  is unknown, we require that  $n\hat{p} \geq 5$  and  $n(1 - \hat{p}) \geq 5$ .)

As we saw in Section 22.3, under these conditions, the sampling distribution of  $\hat{p}$  is approximately normal with mean  $p$  and standard error  $\sqrt{p(1 - p)/n}$ . Therefore, if  $z_{\alpha/2}$  is the value that leaves an area of  $1 - \alpha$  between  $-z_{\alpha/2}$  and  $z_{\alpha/2}$ , then in  $(1 - \alpha)100\%$  of samples,  $\hat{p}$  satisfies

$$p - z_{\alpha/2}\sqrt{\frac{p(1 - p)}{n}} < \hat{p} < p + z_{\alpha/2}\sqrt{\frac{p(1 - p)}{n}}$$

Manipulating this inequality and eliminating some terms because of the large sample size gives the following general formula for a  $(1 - \alpha)100\%$  confidence interval for  $p$ .

### Large Sample Confidence Interval for a Proportion

Let  $\hat{p}$  be the sample proportion obtained from a sample of size  $n$  such that  $n\hat{p} \geq 5$  and  $n(1 - \hat{p}) \geq 5$ . A  $(1 - \alpha)100\%$  confidence interval for the population proportion  $p$  is given by

$$\hat{p} \pm E, \text{ where } E = z_{\alpha/2} \sqrt{\frac{\hat{p}(1 - \hat{p})}{n}} \quad (22.14)$$

#### EXAMPLE 6 A 90% confidence interval for $p$

A manufacturer wants to estimate the proportion of defective parts in a large lot produced by a particular machine. In a sample of 350 parts, 41 of them were found to be defective. Construct a 90% confidence interval for the proportion of defective parts in the lot.

We have

$$\hat{p} = \frac{41}{350}, n\hat{p} = 41 \geq 5, \text{ and } n(1 - \hat{p}) = 309 \geq 5$$

Therefore, we can apply Eq. (22.14) with  $z_{\alpha/2} = 1.645$  (see Table 22.2). The desired 90% confidence interval is

$$\hat{p} \pm z_{\alpha/2} \sqrt{\frac{\hat{p}(1 - \hat{p})}{n}} = \frac{41}{350} \pm 1.645 \sqrt{\frac{\frac{41}{350} \cdot \frac{309}{350}}{350}} = 0.117 \pm 0.028 = (0.089, 0.145)$$

As we did with the mean, we can obtain the required sample size for a desired margin of error at a fixed confidence level by using the formula for margin of error from Eq. (22.14) and solving for  $n$ . For a confidence level  $1 - \alpha$ , we get

$$n = \hat{p}(1 - \hat{p}) \left[ \frac{z_{\alpha/2}}{E} \right]^2 \quad (22.15)$$

Note that Eq. (22.15) requires an estimate for  $\hat{p}$ . (It can be obtained from past data or from a pilot study.) When no such estimate is available, we can use the fact that  $\hat{p}(1 - \hat{p})$  is maximized when  $\hat{p} = \frac{1}{2}$ , so we use this worst-case scenario estimate in Eq. (22.15). The required sample size for a confidence level  $1 - \alpha$  becomes

$$n = \frac{1}{4} \left[ \frac{z_{\alpha/2}}{E} \right]^2 \quad (22.16)$$

#### EXAMPLE 7 Determining sample size

Suppose that the manufacturer from Example 6 wishes to estimate the proportion of defectives with a maximum error of 0.025 with 90% confidence.

(a) How large a sample will he need if no information from the past is used?

For this case we use Eq. (22.16) with  $E = 0.025$  and  $z_{\alpha/2} = 1.645$ . The required sample size is  $n = \frac{1}{4} \left[ \frac{z_{\alpha/2}}{E} \right]^2 = \frac{1}{4} \left[ \frac{1.645}{0.025} \right]^2 = 1082.41$ , so that 1083 parts must be sampled.

(b) How large a sample will he need if the information from the sample of 350 parts is used? (Recall that  $\hat{p} = \frac{41}{350}$ .)

If  $\hat{p} = \frac{41}{350}$  is known from the past, Eq. (22.15) gives

$$n = \hat{p}(1 - \hat{p}) \left[ \frac{z_{\alpha/2}}{E} \right]^2 = \frac{41}{350} \cdot \frac{309}{350} \left[ \frac{1.645}{0.025} \right]^2 = 447.8$$

Therefore, 448 parts must be sampled. Note how information about the possible size of  $\hat{p}$  substantially reduced the size of the required sample.



## EXERCISES 22.4

In Exercises 1–5, make the given changes in the indicated examples of this section and then solve the resulting problems.

- In Example 2, change  $n = 100$  to  $n = 140$  and then find the indicated confidence interval.
- In Example 4, change the confidence level from 99% to 90% and then find the indicated confidence interval.
- In Example 5, change  $\sigma = 0.04$  to  $\sigma = 0.03$  and find the required sample size.
- In Example 6, change the 41 to a 62 and find the indicated confidence interval.
- In Example 7(a), change 90% to 95% and find the required sample size.

In Exercises 6–8, use the following data. A random sample of size  $n = 300$  is taken from a large population with  $\sigma = 25.9$ . The sample mean is  $\bar{x} = 247.1$ .

- Construct a 95% confidence interval for the population mean  $\mu$ .
- Construct a 99% confidence interval for the population mean  $\mu$ .
- How large a sample must be taken so that a 95% confidence interval for  $\mu$  will have a maximum margin of error  $E = 2.4$ ?

In Exercises 9–12, use the following data. A random sample of size  $n = 215$  is taken from a large population, and 38 are found to be defective.

- Construct a 95% confidence interval for the population proportion of defectives  $p$ .
- Construct a 99% confidence interval for the population proportion of defectives  $p$ .
- How large a sample must be taken so that a 95% confidence interval for  $p$  will have a maximum margin of error of 4.5%? Assume that the information from the sample is used.
- How large a sample must be taken so that a 95% confidence interval for  $p$  will have a maximum margin of error of 4.5%? Assume that no prior information is used.

In Exercises 13–15, use the following information. A random sample of size  $n$  is taken from a large population with  $\sigma = 3.14$ . The sample mean is  $\bar{x} = 83.7$ .

- Find a 90% confidence interval if the sample size  $n$  is 50.
- Find a 90% confidence interval if the sample size  $n$  is 80. Compare the interval with the interval obtained in Exercise 13. How does increasing the sample size affect the margin of error?
- Find a 95% confidence interval if the sample size is 80. Compare the interval with the interval obtained in Exercise 14. How does increasing the confidence level affect the margin of error?

In Exercises 16–22, solve the given problems.

- A sample of 70 washing machines of a certain brand had a mean replacement time of 9.1 years, with a standard deviation of 2.7 years. Find a 95% confidence interval for the mean replacement time of all washing machines of this brand.
- A test station measured the loudness of a random sample of 45 jets taking off from a certain airport. The mean was found to be 107.2 dB, with a standard deviation of 9.2 dB. Find a 90% confidence interval for the mean loudness of all jets taking off from this airport.
- An airline wishes to estimate the mean time passengers have to wait for their luggage when arriving at a large airport. How many passengers must be sampled so that a 95% confidence interval for the true mean waiting time  $\mu$  will have a maximum margin of error of 30 seconds? A similar study done in the past had a standard deviation of 2.16 minutes.
- A toy manufacturer wishes to estimate the mean time it takes an adult to assemble a certain “easy to assemble” toy. How many adults must be sampled so that a 99% confidence interval for the true mean assembly time  $\mu$  will have a maximum margin of error of 2.0 minutes? The standard deviation of assembly time for a similar model is known to be 5.9 minutes.
- From a random sample of 60 bicycle helmets subjected to an impact test, 13 helmets showed some damage from the test. Find a 95% confidence interval for the true proportion of helmets that would show damage from this test.
- Suppose that we want to estimate the proportion of drivers that exceed the 100 km/h speed limit by more than 10 km in a certain stretch of highway. How large a sample must be taken so that a 95% confidence interval for the true proportion  $p$  will have a maximum margin of error of 4%? Assume that no prior information is used.
- Following are two confidence interval estimates of the true mean contents of certain 306 mL jars of sauce:

$$(306.2, 307.4) \quad (306.3, 307.3)$$

The confidence level for one interval is 90%, and the confidence level for the other is 95%, with both intervals constructed from the same sample data. Which of the intervals is the 90% confidence interval? Explain.

### Answers to Practice Exercises

- (96.7, 98.5) with 95% confidence
- (96.8, 98.4) with 90% confidence

## 22.5 Statistical Process Control

### Control Charts • Central Line • Range • Upper and Lower Control Limits

One of the most important uses of statistics in industry is statistical process control (SPC), which is used to maintain and improve product quality. Samples are tested during the production at specified intervals to determine whether the production process needs adjustment to meet quality requirements.



■ This is intended only as a brief introduction to this topic. A complete development requires at least a chapter in a statistics book.

■ Minor variations may be expected, for example, from very small fluctuations in voltage, temperature, or material composition. Special causes resulting in an out-of-control process could include line stoppage, material defect, or an incorrect applied pressure.

A particular industrial process is considered to be *in control* if it is stable and predictable, and sample measurements fall within upper and lower control limits. The process is *out of control* if it has an unpredictable amount of variation and there are sample measurements outside the control limits due to special causes.

### EXAMPLE 1 Process—in control—out of control

A manufacturer of 1.5-V batteries states that the voltage of its batteries is no less than 1.45 V or greater than 1.55 V and has designed the manufacturing process to meet these specifications.

If all samples of batteries that are tested have voltages in the proper range with only expected minor variations, the production process is *in control*.

However, if some samples have batteries with voltages out of the proper range, the process is *out of control*. This would indicate some special cause for the problem, such as an improperly operating machine or an impurity getting into the process. The process would probably be halted until the cause is determined. ■

## CONTROL CHARTS

An important device used in SPC is the *control chart*. It is used to show a trend of a production characteristic over time. In this section we study one type of **control chart for measurements** (when the variable involved is quantitative), and one type of **control chart for attributes** (when the variable involved is qualitative). In both cases, samples are observed at specified intervals of time to see if the sample values are within acceptable limits. The sample values are plotted on a chart to check for trends and abnormalities in the production process.

In making a control chart for measurements, we must determine what the mean should be. For a stable process for which previous data are known, it can be based on a production specification or on previous data. For a new or recently modified process, it may be necessary to use present data, although the value may have to be revised for future charts. On a control chart, *this value is used as the population mean,  $\mu$* .

It is also necessary to establish the upper and lower control limits. The standard generally used is that 99.7% of the sample measurements should fall within these control limits. This assumes a normal distribution, and we note that this is within three sample standard deviations of the population mean. We will establish these limits by use of a table or a formula that has been made using statistical measures developed in a more complete coverage of quality control. This does follow the normal practice of using a formula or a more complete table in setting up the control limits.

In Fig. 22.16, we show a sample control chart, and on the following pages, we illustrate how control charts are made.

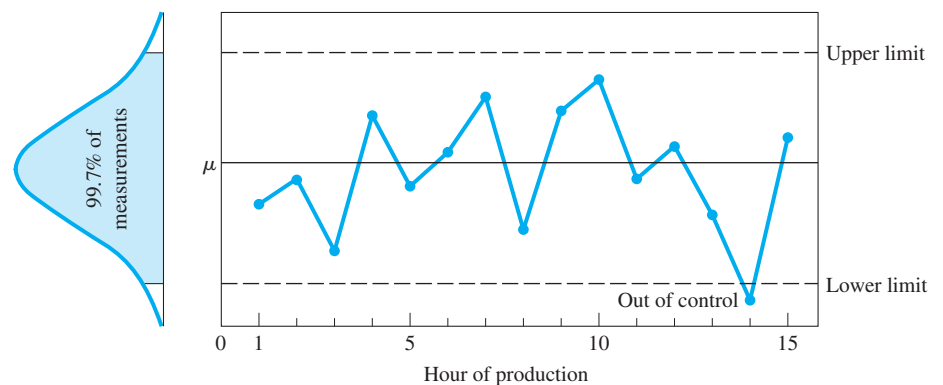


Fig. 22.16

**EXAMPLE 2** Making  $\bar{x}$  and  $R$  control charts

A pharmaceutical company makes a capsule of a prescription drug that contains 500 mg of the drug, according to the label. In a newly modified process of making the capsule, five capsules are tested every 15 min to check the amount of the drug in each capsule. Testing over a 5-h period gave the following results for the 20 subgroups of samples.

Subgroup	Amount of Drug (in mg) of Five Capsules					Mean $\bar{x}$	Range $R$
1	503	501	498	507	502	502.2	9
2	497	499	500	495	502	498.6	7
3	496	500	507	503	502	501.6	11
4	512	503	488	500	497	500.0	24
5	504	505	500	508	502	503.8	8
6	495	495	501	497	497	497.0	6
7	503	500	507	499	498	501.4	9
8	494	498	497	501	496	497.2	7
9	502	504	505	500	502	502.6	5
10	500	502	500	496	497	499.0	6
11	502	498	510	503	497	502.0	13
12	497	498	496	502	500	498.6	6
13	504	500	495	498	501	499.6	9
14	500	499	498	501	494	498.4	7
15	498	496	502	501	505	500.4	9
16	500	503	504	499	505	502.2	6
17	487	496	499	498	494	494.8	12
18	498	497	497	502	497	498.2	5
19	503	501	500	498	504	501.2	6
20	496	494	503	502	501	499.2	9
					Sum	9998.0	174
					Mean	499.9	8.7

From this table of values, we can make an  $\bar{x}$  control chart and an  $R$  control chart. The  $\bar{x}$  chart maintains a check on the average quality level, whereas the  $R$  chart maintains a check on the dispersion of the production process. These two control charts are often plotted together and referred to as the  $\bar{x}$ - $R$  chart.

In order to define the **central line** of the  $\bar{x}$  chart, which ideally is equivalent to the value of the population mean  $\mu$ , we use the mean of the sample means  $\bar{\bar{x}}$ . For the central line of the  $R$  chart, we use  $\bar{R}$ . From the table, we see that

$$\bar{\bar{x}} = 499.9 \text{ mg} \quad \text{and} \quad \bar{R} = 8.7 \text{ mg}$$

The **upper control limit** (UCL) and the **lower control limit** (LCL) for each chart are defined in terms of the mean range  $\bar{R}$  and an appropriate constant taken from a table of control chart factors. These factors, which are related to the sample size  $n$ , are determined by statistical considerations found in a more complete coverage of quality control. At the left is a brief table of control chart factors (Table 22.3).

**Table 22.3** Control Chart Factors

$n$	$d_2$	$A$	$A_2$	$D_1$	$D_2$	$D_3$	$D_4$
5	2.326	1.342	0.577	0.000	4.918	0.000	2.115
6	2.534	1.225	0.483	0.000	5.078	0.000	2.004
7	2.704	1.134	0.419	0.205	5.203	0.076	1.924

The UCL and LCL for the  $\bar{x}$  chart are found as follows:

$$UCL(\bar{x}) = \bar{\bar{x}} + A_2\bar{R} = 499.9 + 0.577(8.7) = 504.9 \text{ mg} \quad (\text{using Table 22.3 with } n = 5)$$

$$LCL(\bar{x}) = \bar{\bar{x}} - A_2\bar{R} = 499.9 - 0.577(8.7) = 494.9 \text{ mg}$$

The UCL and LCL for the  $R$  chart are found as follows:

$$LCL(R) = D_3\bar{R} = 0.000(8.7) = 0.0 \text{ mg}$$

$$UCL(R) = D_4\bar{R} = 2.115(8.7) = 18.4 \text{ mg}$$

Using these central lines and control limit lines, we now plot the  $\bar{x}$  control chart in Fig. 22.17 and the  $R$  control chart in Fig. 22.18.

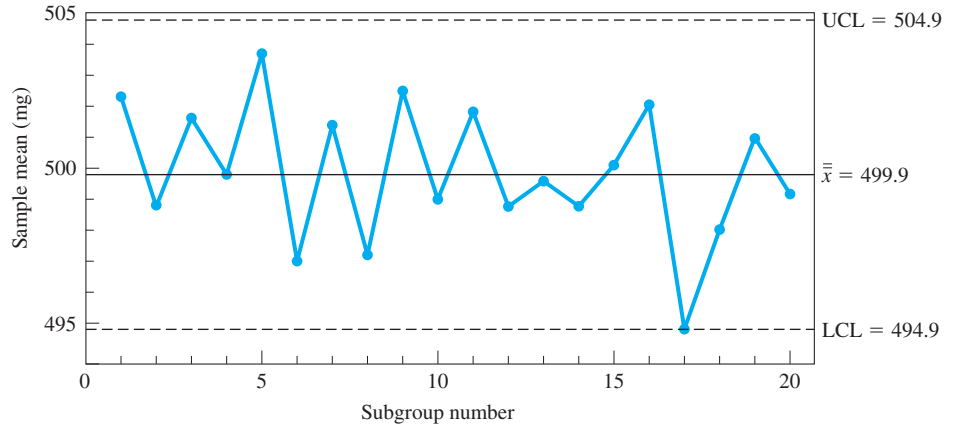


Fig. 22.17

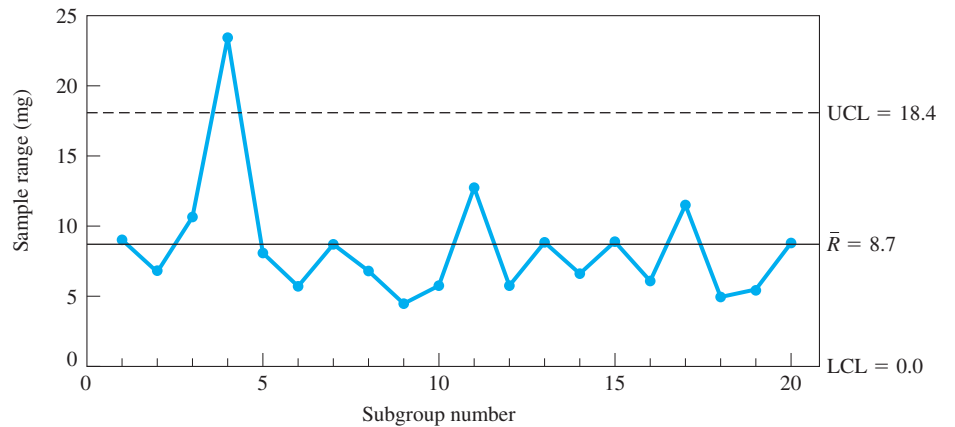


Fig. 22.18

This would be considered a *well-centred process* since  $\bar{x} = 499.9$  mg, which is very near the target value of 500.0 mg. We do note, however, that subgroup 17 was at a control limit and this might have been due to some special cause, such as the use of a substandard mixture of ingredients. We also note that the process was *out of control* due to some special cause since the range of subgroup 4 was above the upper control limit. We should keep in mind that there are numerous considerations, including human factors, that should be taken into account when making and interpreting control charts and that this is only a very brief introduction to this important industrial use of statistics. ■

We now discuss a control chart for attributes, for the case when each item tested is classified as being either acceptable or not acceptable. To monitor such an attribute in a production process, we obtain the *proportion* of defective parts by *dividing the number of defective parts in a sample by the total number of parts in the sample*, and then make a *p control chart*. This is illustrated in the following example.

**EXAMPLE 3** Making a  $p$  control chart

A manufacturer of video discs has 1000 DVDs checked each day for defects (surface scratches, for example). The data for this procedure for 25 days are shown in the table at the left.

The central line for the  $p$  control chart is ideally equal to the true proportion of defectives in the population. This value is usually estimated from the data, using the average sample proportion  $\bar{p}$ , which in this case is

$$\bar{p} = \frac{490}{25\,000} = 0.0196$$

The control limits are each three standard deviations from  $\bar{p}$ . If  $n$  is the size of each sample, we obtain the standard error of  $\bar{p}$  using Eq. (22.8) (with  $\bar{p}$  in place of  $p$ ). We get

$$\sigma_{\bar{p}} = \sqrt{\frac{\bar{p}(1 - \bar{p})}{n}} = \sqrt{\frac{0.0196(1 - 0.0196)}{1000}} = 0.004\,38$$

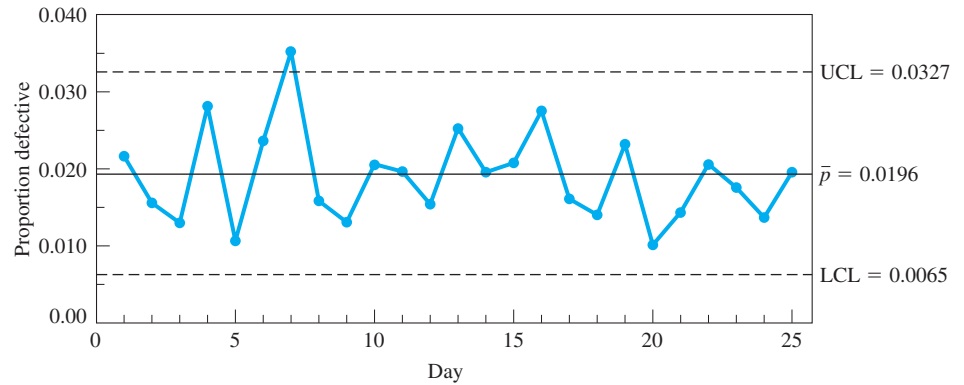
Therefore, the control limits are

$$UCL(p) = 0.0196 + 3(0.004\,38) = 0.0327$$

$$LCL(p) = 0.0196 - 3(0.004\,38) = 0.0065$$

Using this central line and these control limit lines, we now plot the  $p$  control chart in Fig. 22.19.

Day	Defective DVDs	Proportion Defective
1	22	0.022
2	16	0.016
3	14	0.014
4	18	0.018
5	12	0.012
6	25	0.025
7	36	0.036
8	16	0.016
9	14	0.014
10	22	0.022
11	20	0.020
12	17	0.017
13	26	0.026
14	20	0.020
15	22	0.022
16	28	0.028
17	17	0.017
18	15	0.015
19	25	0.025
20	12	0.012
21	16	0.016
22	22	0.022
23	19	0.019
24	16	0.016
25	20	0.020
Sum	490	



**Fig. 22.19**

**Practice Exercise**

- In Example 3, change Day 9 datum from 14 to 24 defective parts. Then find  $UCL(p)$  and  $LCL(p)$ .

According to the proportion mean of 0.0196, the process produces about 2% defective DVDs. We note that the process was out of control on Day 7. An adjustment to the production process was probably made to remove the special cause of the additional defective DVDs. ■

### EXERCISES 22.5

In Exercises 1–4, in Example 2, change the first subgroup to 497, 499, 502, 493, and 498 and then proceed as directed.

1. Find UCL( $\bar{x}$ ) and LCL( $\bar{x}$ ).
2. Find LCL( $R$ ) and UCL( $R$ ).
3. How would the  $\bar{x}$  control chart differ from Fig. 22.17?
4. How would the  $R$  control chart differ from Fig. 22.18?

In Exercises 5–8, use the following data.

Five automobile engines are taken from the production line each hour and tested for their torque (in N · m) when rotating at a constant frequency. The measurements of the sample torques for 20 h of testing are as follows:

Hour	Torques (in N · m) of Five Engines				
1	366	352	354	360	362
2	370	374	362	366	356
3	358	357	365	372	361
4	360	368	367	359	363
5	352	356	354	348	350
6	366	361	372	370	363
7	365	366	361	370	362
8	354	363	360	361	364
9	361	358	356	364	364
10	368	366	368	358	360
11	355	360	359	362	353
12	365	364	357	367	370
13	360	364	372	358	365
14	348	360	352	360	354
15	358	364	362	372	361
16	360	361	371	366	346
17	354	359	358	366	366
18	362	366	367	361	357
19	363	373	364	360	358
20	372	362	360	365	367

5. Find the central line, UCL, and LCL for the mean.
6. Find the central line, UCL, and LCL for the range.
7. Plot an  $\bar{x}$  chart.
8. Plot an  $R$  chart.

In Exercise 9–12, use the following data.

Five AC adaptors that are used to charge batteries of a cellular phone are taken from the production line each 15 minutes and tested for their direct-current output voltage. The output voltages for 24 sample subgroups are as follows:

Subgroup	Output Voltages of Five Adaptors				
1	9.03	9.08	8.85	8.92	8.90
2	9.05	8.98	9.20	9.04	9.12
3	8.93	8.96	9.14	9.06	9.00
4	9.16	9.08	9.04	9.07	8.97
5	9.03	9.08	8.93	8.88	8.95
6	8.92	9.07	8.86	8.96	9.04
7	9.00	9.05	8.90	8.94	8.93
8	8.87	8.99	8.96	9.02	9.03
9	8.89	8.92	9.05	9.10	8.93
10	9.01	9.00	9.09	8.96	8.98
11	8.90	8.97	8.92	8.98	9.03
12	9.04	9.06	8.94	8.93	8.92
13	8.94	8.99	8.93	9.05	9.10
14	9.07	9.01	9.05	8.96	9.02
15	9.01	8.82	8.95	8.99	9.04
16	8.93	8.91	9.04	9.05	8.90
17	9.08	9.03	8.91	8.92	8.96
18	8.94	8.90	9.05	8.93	9.01
19	8.88	8.82	8.89	8.94	8.88
20	9.04	9.00	8.98	8.93	9.05
21	9.00	9.03	8.94	8.92	9.05
22	8.95	8.95	8.91	8.90	9.03
23	9.12	9.04	9.01	8.94	9.02
24	8.94	8.99	8.93	9.05	9.07

9. Find the central line, UCL, and LCL for the mean.
10. Find the central line, UCL, and LCL for the range.
11. Plot an  $\bar{x}$  chart.
12. Plot an  $R$  chart.

In Exercises 13–16, use the following information.

For a production process for which there is a great deal of data since its last modification, the population mean  $\mu$  and population standard deviation  $\sigma$  are assumed known. For such a process, we have the following values (using additional statistical analysis):

$\bar{x}$  chart: central line =  $\mu$ , UCL =  $\mu + A\sigma$ , LCL =  $\mu - A\sigma$

$R$  chart: central line =  $d_2\sigma$ , UCL =  $D_2\sigma$ , LCL =  $D_1\sigma$

The values of  $A$ ,  $d_2$ ,  $D_2$ , and  $D_1$  are found in the table of control chart factors in Example 2 (Table 22.3).

13. In the production of robot links and tests for their lengths, it has been found that  $\mu = 2.725$  cm and  $\sigma = 0.032$  cm. Find the central line, UCL, and LCL for the mean if the sample subgroup size is 5.
14. For the robot link samples of Exercise 13, find the central line, UCL, and LCL for the range.
15. After bottling, the volume of soft drink in six sample bottles is checked each 10 minutes. For this process  $\mu = 750.0$  mL and  $\sigma = 2.2$  mL. Find the central line, UCL, and LCL for the range.
16. For the bottling process of Exercise 15, find the central line, UCL, and LCL for the mean.

In Exercises 17 and 18, use the following data.

A telephone company rechecks the entries for 1000 of its new customers each week for name, address, and phone number. The data collected regarding the number of new accounts with errors, along with the proportion of these accounts with errors, is given in the following table for a 20-wk period:

Week	Accounts with Errors	Proportion with Errors
1	52	0.052
2	36	0.036
3	27	0.027
4	58	0.058
5	44	0.044
6	21	0.021
7	48	0.048
8	63	0.063
9	32	0.032
10	38	0.038
11	27	0.027
12	43	0.043
13	22	0.022
14	35	0.035
15	41	0.041
16	20	0.020
17	28	0.028
18	37	0.037
19	24	0.024
20	42	0.042
Total	738	

17. For a  $p$  chart, find the values for the central line, UCL, and LCL.

18. Plot a  $p$  chart.

#### Answers to Practice Exercise

1.  $UCL(p) = 0.0333$ ,  $LCL(p) = 0.0067$

In Exercises 19 and 20, use the following data.

A maker of electric fuses checks 500 fuses each day for defects. The number of defective fuses, along with the proportion of defective fuses for 24 days, is shown in the following table.

Day	Number Defective	Proportion Defective
1	26	0.052
2	32	0.064
3	37	0.074
4	16	0.032
5	28	0.056
6	31	0.062
7	42	0.084
8	22	0.044
9	31	0.062
10	28	0.056
11	24	0.048
12	35	0.070
13	30	0.060
14	34	0.068
15	39	0.078
16	26	0.052
17	23	0.046
18	33	0.066
19	25	0.050
20	25	0.050
21	32	0.064
22	23	0.046
23	34	0.068
24	20	0.040
Total	696	

19. For a  $p$  chart, find the values for the central line, UCL, and LCL.

20. Plot a  $p$  chart.

## 22.6 Linear Regression

Regression • Linear Regression • Method of Least Squares • Deviation • Least-Squares Line

We have considered statistical methods for dealing with one variable. We now discuss how to find an equation relating two variables for which a set of points is known.

In this section, we show a method of finding the equation of a straight line that passes through a set of data points, and in this way we *fit* the line to the points. In general, *the fitting of a curve to a set of points is called regression*. Fitting a straight line to a set of points is *linear regression*, and fitting some other type of curve is called *nonlinear regression*. We consider nonlinear regression in the next section.

Some of the reasons for using regression to find the equation of a curve that passes through a set of points, and thereby “fit” the curve to the points, are (1) to express a concise relationship between the variables, (2) to use the equation to predict certain fundamental results, (3) to determine the reliability of certain sets of data, and (4) to use the data for testing certain theoretical concepts.

For a given set of several (at least 5 or 6) points for representing pairs of data values, we cannot reasonably expect that the curve of any given equation will pass through all of the points *exactly*. Therefore, when we fit the curve of an equation to the points, we

are finding the curve that best approximates passing through the points. It is possible that the curve that best fits the data will not actually pass directly through any of the points, although it should come reasonably close to most of them. Consider the following example.

### EXAMPLE 1 Fitting a line to a set of points

All the students enrolled in a mathematics course took an entrance test. To study the reliability of this test as an indicator of future success, an instructor tabulated the test scores of ten students (selected at random), along with their course averages at the end of the course, and made a graph of the data. See the table below and Fig. 22.20.

Student	Entrance Test Score, Based on 40	Course Average, Based on 100
A	29	63
B	33	88
C	22	77
D	17	67
E	26	70
F	37	93
G	30	72
H	32	81
I	23	47
J	30	74

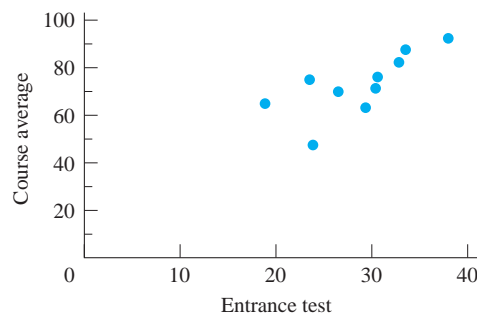


Fig. 22.20

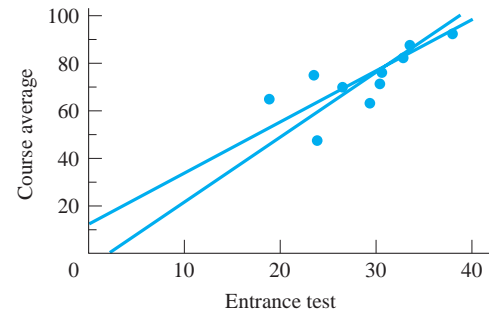


Fig. 22.21

We now ask whether there is a functional relationship between the test scores and the course grades. Certainly, no clear-cut relationship exists, but in general we see that the higher the test score, the higher the course grade. This leads to the possibility that there might be some straight line, from which none of the points would vary too significantly. If such a line could be found, then it could be the basis of predictions as to the possible success a student might have in the course, on the basis of his or her grade on the entrance test. Assuming that such a straight line exists, the problem is to find the equation of this line. Fig. 22.21 shows two such possible lines. ■

There are a number of different methods of determining the straight line that best fits the given data points. We employ the method that is most widely used: the **method of least squares**. *The basic principle of this method is that the sum of the squares of the deviations of all data points from the best line (in accordance with this method) has the least value possible. By **deviation**, we mean the difference between the y-value of the line and the y-value for the point (of original data) for a particular value of x.*



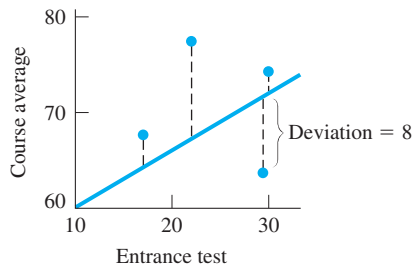


Fig. 22.22

**EXAMPLE 2** Deviation

In Fig. 22.22, the deviations of some of the points of Example 1 are shown. The point  $(29, 63)$  (student A of Example 1) has a deviation of 8 from the indicated line in the figure. Thus, we square the value of this deviation to obtain 64. In order to find the equation of the straight line that best fits the given points, the method of least squares requires that the sum of all such squares be a minimum. ■

In applying this method of least squares, it is necessary to use the equation of a straight line and the coordinates of the points of the data. The deviations of all of these data points are determined, and these values are then squared. It is then necessary to determine the constants for the slope  $m$  and the  $y$ -intercept  $b$  in the equation of a straight line  $y = mx + b$  for which the sum of the squared values is a minimum. To do this requires certain methods of advanced mathematics. Using those methods, it is shown that *the equation of the least-squares line*

$$y = mx + b \quad (22.17)$$

can be found by calculating the values of the slope  $m$  and the  $y$ -intercept  $b$  by using the formulas

$$m = \frac{n \sum xy - (\sum x)(\sum y)}{n \sum x^2 - (\sum x)^2} \quad (22.18)$$

and

$$b = \frac{(\sum x^2)(\sum y) - (\sum xy)(\sum x)}{n \sum x^2 - (\sum x)^2} \quad (22.19)$$

In Eqs. (22.18) and (22.19), *the  $x$ 's and  $y$ 's are the values of the coordinates of the points in the given data*, and  $n$  is the number of points of data. We can reduce the calculational work in finding the values of  $m$  and  $b$  by noting that the denominators in Eqs. (22.18) and (22.19) are the same. Therefore, in using a calculator, the value of this denominator can be stored in memory.

**COMMON ERROR**

It is a common error to confuse  $\sum x^2$  and  $(\sum x)^2$  in the denominators of Eqs. (22.18) and (22.19). Note that for  $\sum x^2$ , we square the  $x$  values and then add the squares, whereas for  $(\sum x)^2$ , we first add the  $x$  values and then square the sum.

**EXAMPLE 3** Finding the equation of the least-squares line

Find the equation of the least-squares line for the points indicated in the following table. Graph the line and data points on the same graph.

$x$	1	2	3	4	5
$y$	3	6	6	8	12

We see from Eqs. (22.18) and (22.19) that we need the sums of  $x$ ,  $y$ ,  $xy$ , and  $x^2$  in order to find  $m$  and  $b$ . Thus, we set up a table for these values, along with the necessary calculations, as follows:

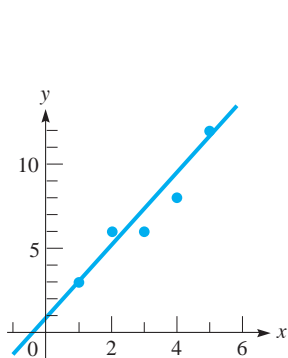


Fig. 22.23

$x$	$y$	$xy$	$x^2$
1	3	3	1
2	6	12	4
3	6	18	9
4	8	32	16
5	12	60	25
sums → 15	35	125	55
↑ $\sum x$	↑ $\sum y$	↑ $\sum xy$	↑ $\sum x^2$

$$n = 5 \quad (5 \text{ points})$$

$$m = \frac{5(125) - (15)(35)}{5(55) - (15)^2} = \frac{100}{50} = 2$$

$$b = \frac{(55)(35) - (125)(15)}{50} = \frac{50}{50} = 1$$

This means that the equation of the least-squares line is  $y = 2x + 1$ . This line and the data points are shown in Fig. 22.23. ■

#### EXAMPLE 4 Finding the equation of the least-squares line

Find the least-squares line for the data of Example 1.

Here, the  $x$ -values will be the entrance-test scores and the  $y$ -values are the course averages.

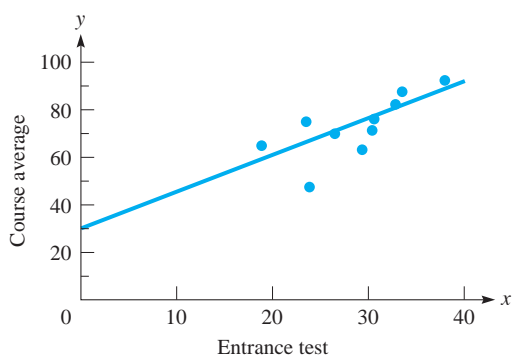


Fig. 22.24

$x$	$y$	$xy$	$x^2$
29	63	1827	841
33	88	2904	1089
22	77	1694	484
17	67	1139	289
26	70	1820	676
37	93	3441	1369
30	72	2160	900
32	81	2592	1024
23	47	1081	529
30	74	2220	900
279	732	20 878	8101

$$n = 10$$

$$m = \frac{10(20\,878) - 279(732)}{10(8101) - 279^2} = 1.44$$

$$b = \frac{8101(732) - 20\,878(279)}{10(8101) - 279^2} = 33.1$$

Thus, the equation of the least-squares line is  $y = 1.44x + 33.1$ . The line and data points are shown in Fig. 22.24. This line best fits the data, although the fit is obviously approximate. It can be used to predict the approximate course average that a student might be expected to attain, based on the entrance test. ■

#### EXAMPLE 5 Least-squares line—application

In a research project to determine the amount of a drug that remains in the bloodstream after a given dosage, the amounts  $y$  (in mg of drug/dL of blood) were recorded after  $t$  hours, as shown in the following table. Find the least-squares line for these data, expressing  $y$  as a function of  $t$ . Sketch the graph of the line and data points.

■ A graphing calculator, a spreadsheet, or computer software can be used to find the slope and the intercept and to display the least-squares line.

The calculations are as follows:

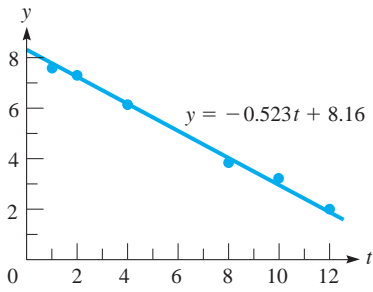


Fig. 22.25

$t$ (h)	1.0	2.0	4.0	8.0	10.0	12.0
$y$ (mg/dL)	7.6	7.2	6.1	3.8	2.9	2.0

$$n = 6$$

$$m = \frac{6(129.8) - 37.0(29.6)}{6(329) - 37.0^2} = -0.523$$

$$b = \frac{(329)(29.6) - (129.8)(37.0)}{6(329) - 37.0^2} = 8.16$$

$t$	$y$	$ty$	$t^2$
1.0	7.6	7.6	1.0
2.0	7.2	14.4	4.0
4.0	6.1	24.4	16.0
8.0	3.8	30.4	64.0
10.0	2.9	29.0	100
12.0	2.0	24.0	144
37.0	29.6	129.8	329

The equation of the least-squares line is  $y = -0.523t + 8.16$ . The line and the data points are shown in Fig. 22.25. This line is useful in determining the effectiveness of the drug. It can also be used to determine when additional medication may be administered. ■

## EXERCISES 22.6

In Exercises 1–14, find the equation of the least-squares line for the given data. Graph the line and data points on the same graph.

1. In Example 3, replace the  $y$ -values with 3, 7, 9, 9, and 12. Then follow the instructions above.

2. 

$x$	1	2	3	4	5	6	7
$y$	10	17	28	37	49	56	72

3. 

$x$	20	26	30	38	48	60
$y$	160	145	135	120	100	90

4. 

$x$	1	3	6	5	8	10	4	7	3	8
$y$	15	12	10	8	9	2	11	9	11	7

5. In Example 5, change the  $y$  (mg of drug/dL of blood) values to 8.7, 8.4, 7.7, 7.3, 5.7, and 5.2. Then proceed to find  $y$  as a function of  $t$ , as in Example 5.

6. The velocity  $v$  (in m/s) of a falling object was found each second by use of an electronic device, as shown in the following table. Find  $v$  as a function of  $t$ .

$t$ (s)	1.00	2.00	3.00	4.00	5.00	6.00	7.00
$v$ (m/s)	9.70	19.5	29.5	39.4	49.2	58.9	68.6

7. In an electrical experiment, the following data were found for the values of current and voltage for a particular element of the circuit. Find the voltage  $V$  as a function of the current  $i$ .

Current (mA)	15.0	10.8	9.30	3.55	4.60
Voltage (V)	3.00	4.10	5.60	8.00	10.50

8. A particular muscle was tested for its speed of shortening as a function of the force applied to it. The results appear below. Find the speed as a function of the force.

Force (N)	60.0	44.2	37.3	24.2	19.5
Speed (m/s)	1.25	1.67	1.96	2.56	3.05

9. The altitude  $h$  (in m) of a rocket was measured at several positions at a horizontal distance  $x$  (in m) from the launch site, shown in the table. Find the least-squares line for  $h$  as a function of  $x$ .

$x$ (m)	0	500	1000	1500	2000	2500
$h$ (m)	0	1130	2250	3360	4500	5600

10. In testing an air-conditioning system, the temperature  $T$  in a building was measured during the afternoon hours with the results shown in the table. Find the least-squares line for  $T$  as a function of the time  $t$  from noon.

$t$ (h)	0.0	1.0	2.0	3.0	4.0	5.0
$T$ (°C)	20.5	20.6	20.9	21.3	21.7	22.0

11. The pressure  $p$  was measured along an oil pipeline at different distances from a reference point, with results as shown. Find the least-squares line for  $p$  as a function of  $x$ .

$x$ (m)	0	50	100	150	200
$p$ (kPa)	4370	4240	4070	3970	3840

12. The heat loss  $L$  per hour through various thicknesses of a particular type of insulation was measured as shown in the table. Find the least-squares line for  $L$  as a function of  $t$ .

$t$ (m)	3.0	4.0	5.0	6.0	7.0
$L$ (MJ)	5.90	4.80	3.90	3.10	2.45

13. In an experiment on the photoelectric effect, the frequency of light being used was measured as a function of the stopping potential (the voltage just sufficient to stop the photoelectric effect) with the results given below. Find the least-squares line for  $V$  as a function of  $f$ . The frequency for  $V = 0$  is known as the *threshold frequency*. From the graph determine the threshold frequency.

$f$ (PHz)	0.550	0.605	0.660	0.735	0.805	0.880
$V$ (V)	0.350	0.600	0.850	1.10	1.45	1.80

14. If gas is cooled under conditions of constant volume, it is noted that the pressure falls nearly proportionally as the temperature. If this were to happen until there was no pressure, the theoretical temperature for this case is referred to as *absolute zero*. In an elementary experiment, the following data were found for pressure and temperature under constant volume.

$T$ (°C)	0.0	20	40	60	80	100
$p$ (kPa)	133	143	153	162	172	183

Find the least-squares line for  $p$  as a function of  $T$ , and from the graph determine the value of absolute zero found in this experiment.

The linear coefficient of correlation, a measure of the strength of the linear relationship of two variables, is defined by  $r = m(s_x/s_y)$ , where  $s_x$  and  $s_y$  are the standard deviations of the  $x$ -values and  $y$ -values, respectively. Due to its definition, the values of  $r$  lie in the range  $-1 \leq r \leq 1$ . If  $r$  is near 1, the correlation is considered good. For the values of  $r$  between  $-0.5$  and  $+0.5$ , the correlation is poor. If  $r$  is near  $-1$ , the variables are said to be negatively correlated; that is, one increases as the other decreases. In Exercises 15–18, compute  $r$  for the given data.

15. Exercise 1  
16. Exercise 2  
17. Exercise 4  
18. Example 1

## 22.7 Nonlinear Regression

Nonlinear Regression • Least-Squares Curve • Types of Curves on Calculator

If the experimental points do not appear to be on a straight line, but we recognize them as being approximately on some other type of curve, then nonlinear regression must be used to fit a curve to the data points. For example, if the points are apparently on a parabola, we would want to fit a quadratic equation instead of a line.

Often, the method of linear least squares can be adapted to fit curves. The method is to create new variables from the available data. By using the appropriate transformation on the data, the curved function can be written as a linear function of the new variables. In particular, we consider nonlinear transformations  $f(x)$  of the  $x$  variable, and extend the least-squares line to

$$y = m[f(x)] + b \tag{22.20}$$

Here,  $f(x)$  must be calculated first, and then the problem can be treated as a least-squares line to find the values of  $m$  and  $b$ . Some of the functions  $f(x)$  that may be considered for use are  $x^2$ ,  $1/x$ ,  $10^x$ , and  $\ln x$ .

### EXAMPLE 1 Fitting $y = mx^2 + b$ to a set of points

Find the least-squares curve  $y = mx^2 + b$  for the following points:

$x$	0	1	2	3	4	5
$y$	1	5	12	24	53	76

In using Eq. (22.12),  $f(x) = x^2$ . Our first step is to calculate values of  $x^2$ , and then we use  $x^2$  as we used  $x$  in finding the equation of the least-squares line.

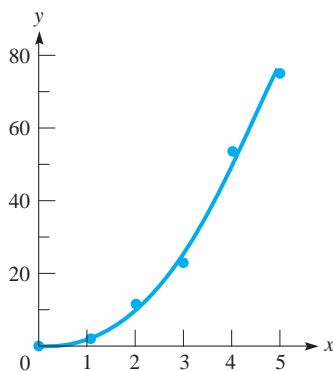


Fig. 22.26

$x$	$f(x) = x^2$	$y$	$x^2y$	$(x^2)^2$
0	0	1	0	0
1	1	5	5	1
2	4	12	48	16
3	9	24	216	81
4	16	53	848	256
5	25	76	1900	625
	55	171	3017	979

$$n = 6$$

$$m = \frac{6(3017) - 55(171)}{6(979) - 55^2} = 3.05$$

$$b = \frac{(979)(171) - (3017)(55)}{6(979) - 55^2} = 0.52$$

Therefore, the required equation is  $y = 3.05x^2 + 0.52$ . The graph of this equation and the data points are shown in Fig. 22.26.

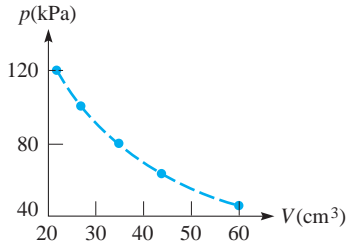


Fig. 22.27

**EXAMPLE 2** Fitting  $y = m(1/x) + b$  to a set of points

In a physics experiment, the pressure  $p$  and volume  $V$  of a gas were measured at constant temperature. When the points were plotted, they were seen to approximate the hyperbola  $y = c/x$ . Find the least-squares approximation to the hyperbola  $y = m(1/x) + b$  for the given data. See Fig. 22.27.

$p$ (kPa)	$V$ (cm <sup>3</sup> )	$x$ (= $V$ )	$f(x) = \frac{1}{x}$	$y$ (= $p$ )	$(\frac{1}{x})y$	$(\frac{1}{x})^2$
120.0	21.0	21.0	0.047 619 0	120.0	5.714 285 7	0.002 267 6
99.2	25.0	25.0	0.040 000 0	99.2	3.968 000 0	0.001 600 0
81.3	31.8	31.8	0.031 446 5	81.3	2.556 603 8	0.000 988 9
60.6	41.1	41.1	0.024 330 9	60.6	1.474 452 6	0.000 592 0
42.7	60.1	60.1	0.016 638 9	42.7	0.710 482 5	0.000 276 9
			0.160 035 3	403.8	14.423 824 6	0.005 725 4

(Calculator note: The final digits for the values shown may vary depending on the calculator and how the values are used. Here, all individual values are shown with 8 digits (rounded off), although more digits were used. The value of  $1/x$  was found from the value of  $x$ , with the 8 digits shown. However, the values of  $(1/x)y$  and  $(1/x)^2$  were found from the value of  $1/x$ , using the extra digits. The sums were found using the rounded-off values shown. However, since the data contain only 3 digits, any variation in the final digits for  $1/x$ ,  $(1/x)y$ , or  $(1/x)^2$ , will not matter.)

$$m = \frac{5(14.423\ 824\ 6) - 0.160\ 035\ 3(403.8)}{5(0.005\ 725\ 4) - 0.160\ 035\ 3^2} = 2490$$

$$b = \frac{(0.005\ 725\ 4)(403.8) - (14.423\ 824\ 6)(0.160\ 035\ 3)}{5(0.005\ 725\ 4) - 0.160\ 035\ 3^2} = 1.2$$

The equation of the hyperbola  $y = m(1/x) + b$  is

$$y = \frac{2490}{x} + 1.2$$

This hyperbola and data points are shown in Fig. 22.28. ■

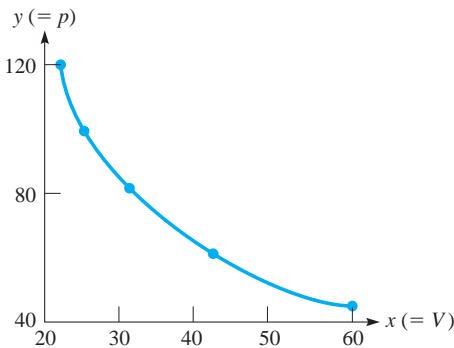


Fig. 22.28

**EXAMPLE 3** Fitting  $y = m(10^x) + b$  to a set of points

It has been found experimentally that the tensile strength of brass (a copper-zinc alloy) increases (within certain limits) with the percent of zinc. The following table shows the values that have been found. See Fig. 22.29.

Tensile Strength (GPa)	0.32	0.36	0.40	0.44	0.48
Percent of Zinc	0	5	13	22	34

Fit a curve of the form  $y = m(10^x) + b$  to the data. Let  $x$  = tensile strength and  $y$  = percent of zinc.

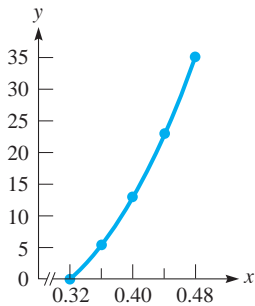


Fig. 22.29

$x$	$f(x) = 10^x$	$y$	$(10^x)y$	$(10^x)^2$
0.32	2.089 296 1	0	0.000 000	4.365 158 3
0.36	2.290 867 7	5	11.454 338	5.248 074 6
0.40	2.511 886 4	13	32.654 524	6.309 573 4
0.44	2.754 228 7	22	60.593 031	7.585 775 8
0.48	3.019 951 7	34	102.678 36	9.120 108 4
	12.666 230 6	74	207.380 25	32.628 690 5

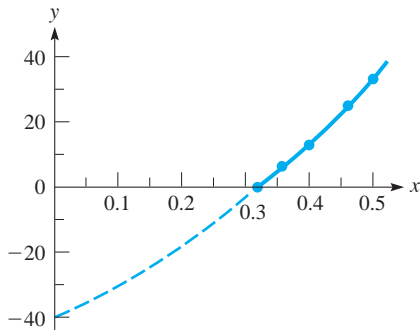


Fig. 22.30

(See the note on calculator use in Example 2.)

$$m = \frac{5(207.380\ 25) - 12.666\ 230\ 6(74)}{5(32.628\ 690\ 5) - 12.666\ 230\ 6^2} = 36.8$$

$$b = \frac{32.628\ 690\ 5(74) - 207.380\ 25(12.666\ 230\ 6)}{5(32.628\ 690\ 5) - 12.666\ 230\ 6^2} = -78.3$$

The equation of the curve is  $y = 36.8(10^x) - 78.3$ . It must be remembered that for practical purposes,  $y$  must be positive. The graph of the equation is shown in Fig. 22.30, with the solid portion denoting the meaningful part of the curve. The points of the data are also shown. ■

As we noted in the previous section, a graphing calculator, a spreadsheet, or computer software can be used to determine the equation of the regression curve, and to display its graph. The different models that can be considered include

- Linear:  $y = ax + b$
- Quadratic:  $y = ax^2 + bx + c$
- Cubic:  $y = ax^3 + bx^2 + cx + d$
- Quartic:  $y = ax^4 + bx^3 + cx^2 + dx + e$
- Logarithmic:  $y = a + b \ln x$
- Exponential:  $y = ab^x$
- Power:  $y = ax^b$
- Logistic:  $y = \frac{c}{1 - ae^{-bx}}$
- Sinusoidal:  $y = a \sin(bx + c) + d$

## EXERCISES 22.7

In the following exercises, find the equation of the indicated least-squares curve. Sketch the curve and plot the data points on the same graph.

- In Example 1, replace the  $y$ -values with 2, 3, 10, 25, 44, and 65. Then follow the instructions above.
- For the points in the following table, find the least-squares curve  $y = m\sqrt{x} + b$ .

$x$	0	4	8	12	16
$y$	1	9	11	14	15

- In Example 2, change the  $V$  (volume of the gas) values to 19.9, 24.5, 29.4, 39.4, and 56.0. Then find  $y$  ( $= p$ ) as a function of  $x$  ( $= V$ ), as in Example 2.
- In Example 3, change the  $y$  (percent of zinc) values to 2, 8, 15, 23, and 32. Then find  $y$  as a function of  $x$  (tensile strength), as in Example 3.
- The following data were found for the distance  $y$  that an object rolled down an inclined plane in time  $t$ . Determine the least-squares curve  $y = mt^2 + b$ . Compare the equation with that using the *quadratic regression* feature on a graphing calculator.

$t$ (s)	1.0	2.0	3.0	4.0	5.0
$y$ (cm)	6.0	23	55	98	148

- The increase in length  $y$  of a certain metallic rod was measured in relation to particular increases  $x$  in temperature. Find the least-squares curve  $y = mx^2 + b$ .

$x$ (°C)	50.0	100	150	200	250
$y$ (cm)	1.00	4.40	9.40	16.4	24.0

- The pressure  $p$  at which Freon, a refrigerant, vapourizes for temperature  $T$  is given in the following table. Find the least-squares curve  $p = mT^2 + b$ .

$T$ (°C)	0	10	20	30	40
$p$ (kPa)	480	600	830	1040	1400

- A fraction  $f$  of annual hot-water loads at a certain facility are heated by solar energy. The fractions  $f$  for certain values of the collector area  $A$  are given in the following table. Find the least-squares curve  $f = m\sqrt{A} + b$ .

$A$ (m <sup>2</sup> )	0	12	27	56	90
$f$	0.0	0.2	0.4	0.6	0.8

- The makers of a special blend of coffee found that the demand for the coffee depended on the price charged. The price  $P$  per pound and the monthly sales  $S$  are shown in the following table. Find the least-squares curve  $P = m(1/S) + b$ .

$S$ (thousands)	240	305	420	480	560
$P$ (dollars)	5.60	4.40	3.20	2.80	2.40

10. The resonant frequency  $f$  of an electric circuit containing a  $4\text{-}\mu\text{F}$  capacitor was measured as a function of an inductance  $L$  in the circuit. The following data were found. Find the least-squares curve  $f = m(1/\sqrt{L}) + b$ .

$L$ (H)	1.0	2.0	4.0	6.0	9.0
$f$ (Hz)	490	360	250	200	170

11. The displacement  $y$  of an object at the end of a spring at given times  $t$  is shown in the following table. Find the least-squares curve  $y = me^{-t} + b$ .

$t$ (s)	0.0	0.5	1.0	1.5	2.0	3.0
$y$ (cm)	6.1	3.8	2.3	1.3	0.7	0.3

12. The average daily temperatures  $T$  (in  $^{\circ}\text{C}$ ) for each month in Montreal (Environment Canada Archives) are given in the following table:

$t$	J	F	M	A	M	J	J	A	S	O	N	D
$T(^{\circ}\text{C})$	-9	-7	-1	7	14	19	22	21	16	9	2	-6

Find the least-squares curve  $T = m \cos\left[\frac{\pi}{6}(t - 0.5)\right] + b$ . Assume the average temperature is for the 15th of each month. Then the values of  $t$  (in months) are 0.5, 1.5, . . . , 11.5.

## CHAPTER 22 EQUATIONS

### Arithmetic mean

$$\bar{x} = \frac{x_1f_1 + x_2f_2 + \cdots + x_nf_n}{f_1 + f_2 + \cdots + f_n} \quad (22.1)$$

### Standard deviation

$$s = \sqrt{\frac{\sum (x - \bar{x})^2}{n - 1}} \quad (22.2)$$

$$s = \sqrt{\frac{n(\sum x^2) - (\sum x)^2}{n(n - 1)}} \quad (22.3)$$

### Normal distribution

$$y = \frac{e^{-(x-\mu)^2/2\sigma^2}}{\sigma\sqrt{2\pi}} \quad (22.4)$$

### Standard normal distribution

$$y = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \quad (22.5)$$

### Standard ( $z$ ) score

$$z = \frac{x - \mu}{\sigma} \quad (22.6)$$

### Standard error of $\bar{x}$

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} \quad (22.7)$$

### Standard error of $\hat{p}$

$$\sigma_{\hat{p}} = \sqrt{\frac{p(1-p)}{n}} \quad (22.8)$$

### $(1 - \alpha)$ 100% confidence interval for $\mu$

$$\bar{x} \pm E, \text{ where } E = z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}} \quad \text{if } \sigma \text{ is known} \quad (22.12)$$

$$E = z_{\alpha/2} \cdot \frac{s}{\sqrt{n}} \quad \text{if } \sigma \text{ is unknown}$$

### Sample size required ( $\mu$ )

$$n = \left[ \frac{z_{\alpha/2}\sigma}{E} \right]^2 \quad (22.13)$$

### $(1 - \alpha)$ 100% confidence interval for $p$

$$\hat{p} \pm E, \text{ where } E = z_{\alpha/2} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \quad (22.14)$$



**Sample sized required ( $p$ )****An estimate  $\hat{p}$  is available**

$$n = \hat{p}(1 - \hat{p}) \left[ \frac{z_{\alpha/2}}{E} \right]^2 \quad (22.15)$$

**No estimate  $\hat{p}$  is available**

$$n = \frac{1}{4} \left[ \frac{z_{\alpha/2}}{E} \right]^2 \quad (22.16)$$

**Least-squares lines**

$$y = mx + b \quad (22.17)$$

$$m = \frac{n \sum xy - (\sum x)(\sum y)}{n \sum x^2 - (\sum x)^2} \quad (22.18)$$

$$b = \frac{(\sum x^2)(\sum y) - (\sum xy)(\sum x)}{n \sum x^2 - (\sum x)^2} \quad (22.19)$$

**Nonlinear curves**

$$y = m[f(x)] + b \quad (22.20)$$

**CHAPTER 22 REVIEW EXERCISES**

In Exercises 1–10, use the following data. An airline's records showed that the percent of on-time flights each day for a 20-day period was as follows:

72, 75, 76, 70, 77, 73, 80, 75, 82, 85,  
77, 78, 74, 86, 72, 77, 67, 78, 69, 80

- Determine the median.
- Determine the mode.
- Determine the mean.
- Determine the standard deviation.
- Construct a frequency distribution table with five classes and a lowest class limit of 67.
- Draw a frequency polygon for the data in Exercise 5.
- Draw a histogram for the data in Exercise 5.
- Construct a relative frequency table for the data in Exercise 5.
- Construct a cumulative frequency table for the data of Exercise 5.
- Draw an ogive for the data of Exercise 5.

In Exercises 11–16, use the following data: An important property of oil is its coefficient of viscosity, which gives a measure of how well it flows. In order to determine the viscosity of a certain motor oil, a refinery took samples from 12 different storage tanks and tested them at 50°C. The results (in pascal-seconds) were 0.24, 0.28, 0.29, 0.26, 0.27, 0.26, 0.25, 0.27, 0.28, 0.26, 0.26, 0.25.

- Find the mean.
- Find the median.
- Find the standard deviation.
- Draw a histogram.
- Draw a frequency polygon.
- Determine the range.

In Exercises 17–26, use the following data: A sample of wind generators was tested for power output when the wind speed was 30 km/h. The following table gives the class marks of the powers produced and the number of generators in each class.

Power (W)	650	660	670	680	690
No. Generators	3	2	7	12	27

Power (W)	700	710	720	730
No. Generators	34	15	16	5

- Find the median.
- Find the mean.
- Find the mode.
- Draw a histogram.
- Find the standard deviation.
- Draw a frequency polygon.
- Make a cumulative frequency table.
- Draw an ogive.
- The mean and the standard deviation obtained from the raw data are  $\bar{x} = 696$  and  $s = 17.7$ . Find a 95% confidence interval for the true mean power output produced by all generators of this kind when wind is 30 km/h.
- How many more observations should be taken so that a 95% confidence interval for the true mean power output will have a maximum margin of error  $E = 2.5$  W?

In Exercises 27–30, use the following data: A Geiger counter records the presence of high-energy nuclear particles. Even though no apparent radioactive source is present, some particles will be recorded. These are primarily cosmic rays, which are caused by very high-energy

particles from outer space. In an experiment to measure the amount of cosmic radiation, the number of counts were recorded during 200 5-s intervals. The following table gives the number of counts and the number of 5-s intervals having this number of counts. Draw a frequency curve for these data.

Counts	0	1	2	3	4	5	6	7	8	9	10
Intervals	3	10	25	45	29	39	26	11	7	2	3

- 27. Find the median.
- 28. Find the mean.
- 29. Draw a histogram.
- 30. Make a relative frequency table.

In Exercises 31–36, use the following data: Police radar on a city street recorded the speeds of 110 cars in a 65 km/h zone. The following table shows the class marks of the speeds recorded and the number of cars in each class.

Speed (km/h)	40	45	50	55	60	65	70	75	80	85
No. cars	3	4	4	5	8	22	48	10	4	2

- 31. Find the mean.
- 32. Find the median
- 33. Find the standard deviation.
- 34. Draw an ogive.
- 35. The mean and the standard deviation obtained from the raw data are  $\bar{x} = 66$  and  $s = 9.1$ . Find a 90% confidence interval for the true mean speed in this zone.
- 36. How many more observations should be taken so that a 90% confidence interval for the true mean speed will have a maximum margin of error  $E = 1.0$  km/h?

In Exercises 37–38, solve the given problems.

- 37. Use Chebychev’s theorem to find the percentage of values that are between 27.8 and 36.2 in a data set with mean 32 and standard deviation 2.1.
- 38. Use Chebychev’s theorem to find the percentage of values that are between 174.2 and 189.8 in a data set with mean 182 and standard deviation 2.6.

In Exercises 39–42, use the following data. A random sample of size  $n = 185$  is taken from a large population, and 36 are found to be defective.

- 39. Construct a 95% confidence interval for the population proportion of defectives  $p$ .
- 40. Construct a 90% confidence interval for the population proportion of defectives  $p$ .
- 41. How large a sample must be taken so that a 90% confidence interval for  $p$  will have a maximum margin of error of 3.2%? Assume that the information from the sample is used.
- 42. How large a sample must be taken so that a 90% confidence interval for  $p$  will have a maximum margin of error of 3.2%? Assume that no prior information is used.

In Exercises 43 and 44, use the following information: A company that makes electric light bulbs tests 500 bulbs each day for defects.

The number of defective bulbs, along with the proportion of defective bulbs for 20 days, is shown in the following table.

Day	Number Defective	Proportion Defective
1	23	0.046
2	31	0.062
3	19	0.038
4	27	0.054
5	29	0.058
6	39	0.078
7	26	0.052
8	17	0.034
9	28	0.056
10	33	0.066
11	22	0.044
12	29	0.058
13	20	0.040
14	35	0.070
15	21	0.042
16	32	0.064
17	25	0.050
18	23	0.046
19	29	0.058
20	32	0.064
Total	540	

- 43. For a  $p$  chart, find the values of the central line, UCL, and LCL.
- 44. Plot a  $p$  chart.

In Exercises 45 and 46, use the following information: Five ball bearings are taken from the production line every 15 min and their diameters are measured. The diameters of the sample ball bearings for 16 successive subgroups are given in the following table.

Subgroup	Diameters (mm) of Five Ball Bearings				
1	4.98	4.92	5.02	4.91	4.93
2	5.03	5.01	4.94	5.06	5.07
3	5.05	5.03	5.00	5.02	4.96
4	5.01	4.92	4.91	4.99	5.03
5	4.92	4.97	5.02	4.95	4.94
6	5.02	4.95	5.01	5.07	5.15
7	4.93	5.03	5.02	4.96	4.99
8	4.85	4.91	4.88	4.92	4.90
9	5.02	4.95	5.06	5.04	5.06
10	4.98	4.98	4.93	5.01	5.00
11	4.90	4.97	4.93	5.05	5.02
12	5.03	5.05	4.92	5.03	4.98
13	4.90	4.96	5.00	5.02	4.97
14	5.09	5.04	5.05	5.02	4.97
15	4.88	5.00	5.02	4.97	4.94
16	5.02	5.09	5.03	4.99	5.03

- 45. Plot an  $\bar{x}$  chart.
- 46. Plot an  $R$  chart.

In Exercises 47–50, use the following data: After analysing data for a long period of time, it was determined that samples of 500 readings of an organic pollutant for an area are distributed normally. For this pollutant,  $\mu = 2.20 \mu\text{g}/\text{m}^3$  and  $\sigma = 0.50 \mu\text{g}/\text{m}^3$ .

47. In a sample, how many readings are expected to be between  $1.50 \mu\text{g}/\text{m}^3$  and  $2.50 \mu\text{g}/\text{m}^3$ ?
48. In a sample, how many readings are expected to be between  $2.50 \mu\text{g}/\text{m}^3$  and  $3.50 \mu\text{g}/\text{m}^3$ ?
49. In a sample, how many readings are expected to be above  $1.00 \mu\text{g}/\text{m}^3$ ?
50. In a sample, how many readings are expected to be below  $2.00 \mu\text{g}/\text{m}^3$ ?

In Exercises 51–60, find the indicated least-squares curve. Sketch the curve and data points on the same graph.

51. In a certain experiment, the resistance  $R$  of a certain resistor was measured as a function of the temperature  $T$ . The data found are shown in the following table. Find the least-squares line, expressing  $R$  as a function of  $T$ .

$T$ ( $^{\circ}\text{C}$ )	0.0	20.0	40.0	60.0	80.0	100
$R$ ( $\Omega$ )	25.0	26.8	28.9	31.2	32.8	34.7

52. An air-pollution monitoring station took samples of air each hour during the later morning hours and tested each sample for the number  $n$  of parts per million (ppm) of carbon monoxide. The results are shown in the table, where  $t$  is the number of hours after 6 A.M. Find the least-squares line for  $n$  as a function of  $t$ .

$t$ (h)	0.0	1.0	2.0	3.0	4.0	5.0	6.0
$n$ (ppm)	8.0	8.2	8.8	9.5	9.7	10.0	10.7

53. The *Mach number* of a moving object is the ratio of its speed to the speed of sound (1200 km/h). The following table shows the speed  $s$  of a jet aircraft, in terms of Mach numbers, and the time  $t$  after it starts to accelerate. Find the least-squares line of  $s$  as a function of  $t$ .

$t$ (min)	0.00	0.60	1.20	1.80	2.40	3.00
$s$ (Mach number)	0.88	0.97	1.03	1.11	1.19	1.25

54. In an experiment to determine the relation between the load  $x$  on a spring and the length  $y$  of the spring, the following data were found. Find the least-squares line that expresses  $y$  as a function of  $x$ .

Load (kg)	0.0	1.0	2.0	3.0	4.0	5.0
Length (cm)	10.0	11.2	12.3	13.4	14.6	15.9

55. The distance  $s$  of a missile above the ground at time  $t$  after being released from a plane is given by the following table. Find the least-squares curve of the form  $s = mt^2 + b$  for these data.

$t$ (s)	0.0	3.0	6.0	9.0	12.0	15.0	18.0
$s$ (m)	3000	2960	2820	2600	2290	1900	1410

56. In an elementary experiment that measured the wavelength  $L$  of sound as a function of the frequency  $f$ , the following results were obtained.

Frequency (Hz)	240	320	400	480	560
Wavelength (cm)	140	107	81.0	70.0	60.0

Find the least-squares curve of the form  $L = m(1/f) + b$  for these data.

57. Measurements were made on the current  $i$  (in A) in an electric circuit as a function of the time  $t$  (in s). The circuit contained a resistance of  $5.00 \Omega$  and an inductance of  $10.0 \text{ H}$ . The following data were found.

$t$ (s)	0.00	2.00	4.00	6.00	8.00
$i$ (A)	0.00	2.52	3.45	3.80	3.92

Find the least-squares curve  $i = m(e^{-0.500t}) + b$  for these data. From the equation, determine the value of the current as  $t$  approaches infinity.

58. The power  $P$  (in W) generated by a wind turbine was measured for various wind velocities  $v$  (in km/h), as shown in the following table.

$v$ (km/h)	10	15	20	25	30	40
$P$ (W)	75	250	600	1200	2100	4800

Find the least-squares curve of the form  $P = mv^3 + b$  for these data.

59. The vertical distance  $y$  of the cable of a suspension bridge above the surface of the bridge is measured at a horizontal distance  $x$  along the bridge from its centre. See Fig. 22.31. The results are as follows:

$x$ (m)	0	100	200	300	400	500
$y$ (m)	15	17	23	33	47	65

Plot these points and choose an appropriate function  $f(x)$  for  $y = m[f(x)] + b$ . Then find the equation of the least-squares curve.

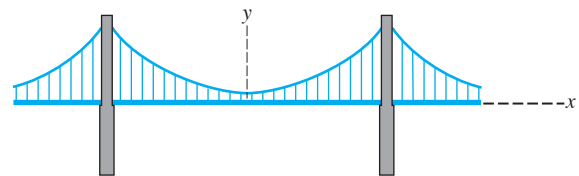


Fig. 22.31

60. After being heated, the temperature  $T$  of an insulated liquid is measured at times  $t$  as follows:

$t$ (h)	0	2	4	6	8	10
$T$ ( $^{\circ}\text{C}$ )	100	85	72	63	54	48

Plot these points and choose an appropriate function  $f(x)$  for  $y = m[f(x)] + b$ . Then find the equation of the least-squares curve.

In Exercises 61–64, use a graphing calculator, a spreadsheet, or software to solve the given problems related to the following data. Using aerial photography, the area  $A$  (in  $\text{km}^2$ ) of an oil spill as a function of the time  $t$  (in h) after the spill was found to be as follows:

$t$ (h)	1.0	2.0	4.0	6.0	8.0	10.0
$A$ ( $\text{km}^2$ )	1.4	2.5	4.7	6.8	8.8	10.2

- Find the linear equation  $y = ax + b$  to fit these data.
- Find the quadratic equation  $y = ax^2 + bx + c$  to fit these data.
- Find the power equation  $y = ax^b$  to fit these data.
- Find the value of the linear coefficient of correlation  $r$  for these data.

In Exercises 65–70, solve the given problems.

- With 30.5% of the area under the normal curve between  $z_1 = 0.5$  and  $z_2$ , to the right of  $z_1$ , find  $z_2$ .
- With 79.8% of the area under the normal curve between  $z_1$  and  $z_2 = 2.1$ , find  $z_1$ .

- The  $n$ th root of the product of  $n$  positive numbers is the *geometric mean* of the numbers. Find the geometric mean of the carbon monoxide readings in Exercise 52.
- One use of the geometric mean (see Exercise 67) is to find an average ratio. By finding the geometric mean, find the average Mach number for the jet in Exercise 53.
- Show that Eqs. (22.18) and (22.19) satisfy the equation  $\bar{y} = m\bar{x} + b$ .
- Given that  $\sum (x - \bar{x})^2 = \sum x^2 - n\bar{x}^2$ , derive Eq. (22.3) from Eq. (22.2).

### Writing Exercise

- A research institute is planning a study of the effect of education on the income of workers. Write two or three paragraphs explaining what data should be collected and which of the measures discussed in this chapter would be useful in analysing the data.

## CHAPTER 22 PRACTICE TEST

In Problems 1–3, use the following set of numbers.

5, 6, 1, 4, 9, 5, 7, 3, 8, 10, 5, 8, 4, 9, 6

- Find the median.
- Find the mode.
- Draw a histogram with five classes and the lowest class limit at 1.

In Problems 4–8, use the following data: Two machine parts are considered satisfactorily assembled if their total thickness (to the nearest 0.01 cm) is between or equal to 0.92 cm and 0.94 cm. One hundred assemblies are tested, and the class mark of the thicknesses and the number of assemblies in each class are given in the following table.

Total Thickness (cm)	0.90	0.91	0.92	0.93	0.94	0.95	0.96
Number	3	9	31	38	12	5	2

- Find the mean.
- Find the standard deviation.
- Draw a frequency polygon.
- Make a relative frequency table.
- Draw an ogive (less than).
- A random sample of size  $n = 175$  is taken from a large population so that  $\bar{x} = 143$  and  $s = 5.2$ . Construct a 95% confidence interval for the population mean  $\mu$ .
- A random sample of size  $n = 120$  is taken from a large population and 33 are found defective. Construct a 99% confidence interval for the proportion  $p$  of defectives in the population.

- For a set of values that are normally distributed, what percent of them is below the value (greater than the mean) for which the  $z$ -score is 0.2257?
- The machine-part assemblies in Problems 4–8 were tested in groups of five each hour for 20 hours. Explain, in general, how to use the data from the test subgroups to plot an  $R$  chart.
- Find the equation of the least-squares line for the points indicated in the following table. Graph the line and data points on the same graph.

$x$	1	3	5	7	9
$y$	5	11	17	20	27

- The velocity (in m/s) of an object moving down an inclined plane was measured as a function of the distance (in m) it moved, with the following results:

Distance (m)	1.00	3.00	5.00	7.00	9.00
Velocity (m/s)	1.10	1.90	2.50	2.90	3.30

Find the equation of the least-squares curve of the form  $y = m\sqrt{x} + b$ , which expresses the velocity as a function of the distance.

# The Derivative

In the 1600s, the ability to understand and predict the motion of planets, projectiles, and other moving objects became the focus of many scientists and mathematicians. For example, the velocity of a falling object changes from one instant to the next, and particular interest was devoted to determining this *instantaneous velocity*. It was also noted that the geometric problem of finding the slope of a curve *at a specific point* was really equivalent to finding the instantaneous velocity of an object, since each involved an *instantaneous rate of change*.

It was well known at the time how to find an *average velocity* (distance travelled divided by time taken) and a slope (difference in  $y$ -values divided by difference in  $x$ -values). However, these methods do not work in finding an instantaneous velocity or slope since they would result in a division by zero. Therefore, a method for finding the slope of a tangent line to a curve at a given point was a major point of interest in mathematics in the 1600s.

This interest in motion and slope of a tangent line led to the development of *calculus*. In this chapter, we start developing *differential calculus*, which is concerned with determining the instantaneous rates of change of one quantity with respect to another rather than just estimating average changes over a large interval. This could include applications like instantaneous electric current (rate of change of electric charge per unit time), instantaneous power (rate of change of work per unit time) of a machine producing a force varying in time, or the rate of change of light intensity with respect to the distance from the source. In Chapter 25, we will study *integral calculus*, which involves finding the function for which the rate of change is known.

Isaac Newton, the English mathematician and physicist, and Gottfried Leibnitz, a German mathematician and philosopher, are credited with the creation of the basic methods of calculus in the 1660s and 1670s. Others, including the French mathematician Pierre de Fermat, are known to have developed some of the topics related to calculus in the mid-1600s. In the 1700s and 1800s, many mathematicians further developed and refined the concepts of calculus.

The topic of this chapter, the *derivative*, is the basic concept of differential calculus that is used to measure an instantaneous rate of change. We will show some of the applications of the derivative in fields such as physical science and engineering in this chapter, and develop several important applications in the next chapter.



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## 23

### LEARNING OUTCOMES

After completion of this chapter, the student should be able to:

- Determine the continuity of a function at a single point or over an interval
- Define the concept of a limit
- Evaluate a limit of a function
- Calculate average slope equations
- Find a derivative using the delta method/definition of derivative
- Interpret instantaneous slope
- Use derivative formulas to calculate derivatives of polynomials, and powers, products, and quotients of functions
- Apply implicit differentiation
- Calculate higher derivatives of functions

◀ Determining how fast an object is moving is important in physics and other areas of technology. In Section 23.4, we develop the method of finding the instantaneous velocity of a moving object.



## 23.1 Limits

Continuity • Limit of a Function • Limit as  $x$  Approaches Infinity

### Continuity

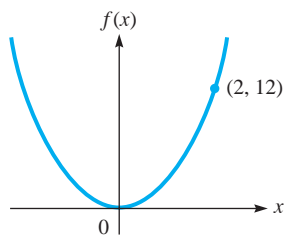


Fig. 23.1

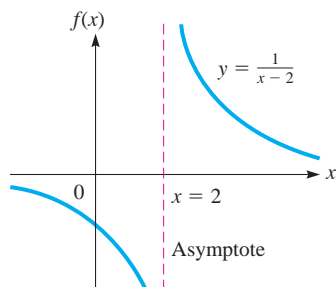


Fig. 23.2

### LEARNING TIP

From a graphical perspective, a function is continuous over an interval if you can draw the graph of the function entirely without ever lifting your pencil from the paper.

If a “break” occurs in the graph because of an undefined point or instantaneous “jump” in the value of the function, or any domain gaps in the interval, then the function is discontinuous for some value or values in that interval.

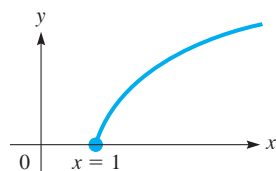


Fig. 23.3

### CONTINUITY OF A FUNCTION

The use of *limits* in dealing with functions requires an understanding of the projected behaviour of the function *near* a certain point, not the behaviour exactly at that point. We have already encountered limits when studying infinite geometric series and the asymptotes of a hyperbola. They will be very useful in the application of curve sketching and the definition of instantaneous slope of a curve.

To develop the concept of a limit, we first consider the **continuity** of a function. *For a function to be **continuous at a point**, the function must exist at the point, and any small change in  $x$  must produce only a small change in  $f(x)$ .* In fact, if the function is continuous, the change in  $f(x)$  can be made as small as we wish by restricting the change in  $x$  sufficiently. Also, *a function is said to be **continuous over an interval** if it is continuous at each point in the interval.*

If the domain of a function includes a point and an interval on only one side of this point, it is *continuous at the point* if the definition of continuity holds for that part of the domain.

#### EXAMPLE 1 Function discontinuous for all $x$

The function  $f(x) = 3x^2$  is continuous for all values of  $x$ . That is,  $f(x)$  is defined for all values of  $x$ , and a small change in  $x$  for any given value of  $x$  produces only a small change in  $f(x)$ . If we choose  $x = 2$ , and then let  $x$  change by 0.1, 0.01, and so on, we obtain the values in the following table:

$x$	2	2.1	2.01	2.001
$f(x)$	12	13.23	12.1203	12.012 003
Change in $x$		0.1	0.01	0.001
Change in $f(x)$		1.23	0.1203	0.012 003

We see that the change in  $f(x)$  is smaller for smaller changes in  $x$ . This shows that  $f(x)$  is continuous at  $x = 2$ . Since this type of result would be obtained for any other  $x$  we may choose, we see that  $f(x)$  is continuous over the interval of all values of  $x$ . The graph of the function  $f(x) = 3x^2$  is shown in Fig. 23.1. ■

#### EXAMPLE 2 Function discontinuous at a point

The function  $f(x) = \frac{1}{x-2}$  is not continuous at  $x = 2$ . When we substitute 2 for  $x$ , we have division by zero. This means the function is not defined for the value  $x = 2$ . The condition of continuity—that the function must exist—is not satisfied. The graph of the function is shown in Fig. 23.2. ■

#### EXAMPLE 3 Continuity and graph of a function

Looking at the graph of the function  $f(x) = 3x^2$  in Fig. 23.1, we see that there are no breaks in the curve representing this function. In Example 1, we determined that this function is continuous for all values of  $x$ .

Now, looking at the graph of the function  $f(x) = \frac{1}{x-2}$  in Fig. 23.2, we see that there is a break in the curve at  $x = 2$ . In Example 2, we determined that this function is not defined at  $x = 2$ , and is therefore not continuous at  $x = 2$ . We note that it is a hyperbola with an asymptote  $x = 2$ . ■

#### EXAMPLE 4 Continuity on an interval and discontinuity at a point

- (a) For the function represented in Fig. 23.3, the solid circle at  $x = 1$  shows that the point is on the graph. Since it is continuous to the right of the point, it is also continuous at the point. Thus, the function is continuous for  $x \geq 1$ .

**Practice Exercise**

1. Determine the continuity of the function  $f(x) = \frac{3}{x(x+3)}$ .

- (b) The function represented by the graph in Fig. 23.4 is not continuous at  $x = 1$ . The function is defined (by the solid circle point) for  $x = 1$ . However, a small change from  $x = 1$  may result in a change of at least 1.5 in  $f(x)$ , regardless of how small a change in  $x$  is made. The small change condition is not satisfied.
- (c) The function represented by the graph in Fig. 23.5 is not continuous for  $x = -2$ . The open circle shows that the point is not part of the graph, and therefore  $f(x)$  is not defined for  $x = -2$ .

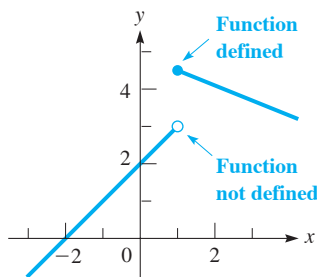


Fig. 23.4

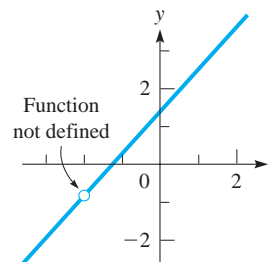


Fig. 23.5

**EXAMPLE 5** Function defined differently over the domain

- (a) We can define the function in Fig. 23.4 as

$$f(x) = \begin{cases} x + 2 & \text{for } x < 1 \\ -\frac{1}{2}x + 5 & \text{for } x \geq 1 \end{cases}$$

where we note that the equation differs for different parts of the domain, and at  $x = 1$ , an instantaneous “jump” occurs in the graph, making it discontinuous at that point. This function is continuous on the intervals  $-\infty < x < 1$  and  $1 < x < \infty$  because no gaps or jumps exist in those intervals, but the function is discontinuous at the point  $x = 1$ .

- (b) The graph of the function

$$g(x) = \begin{cases} 2x - 1 & \text{for } x \leq 2 \\ -x + 5 & \text{for } x > 2 \end{cases}$$

is shown in Fig. 23.6. We see that it is a continuous function even though the equation for  $x \leq 2$  is different from that for  $x > 2$ .

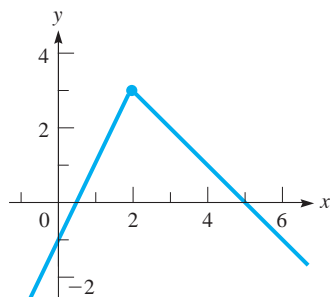


Fig. 23.6

**LEARNING TIP**

The notation used in limits can be interpreted as follows:

Symbol	Meaning
$x \rightarrow 2$	“as $x$ approaches 2”; $x$ can be any number as arbitrarily close to 2 as you want, but <b>cannot be exactly 2</b>
$x \rightarrow 2^+$	“ $x$ approaches 2 from above”; $x$ can be any number as arbitrarily close to 2 as you want, but it <b>must be larger than 2</b>
$x \rightarrow 2^-$	“ $x$ approaches 2 from below”; $x$ can be any number as arbitrarily close to 2 as you want, but it <b>must be less than 2</b>

**EXAMPLE 6** Behaviour of a function as  $x$  approaches 2

Consider the behaviour of  $f(x) = 2x + 1$  as  $x \rightarrow 2$ .

Since we are not to use  $x = 2$ , we use a calculator to set up tables in order to determine values of  $f(x)$ , as  $x$  gets close to 2:

$x$	1.000	1.500	1.900	1.990	1.999
$f(x)$	3.000	4.000	4.800	4.980	4.998
$x$	3.000	2.500	2.100	2.010	2.001
$f(x)$	7.000	6.000	5.200	5.020	5.002

values approach 5

We see that  $f(x)$  approaches 5, as  $x$  approaches 2, from above 2 and from below 2.



## LIMIT OF A FUNCTION

In Example 6, since  $f(x) \rightarrow 5$  as  $x \rightarrow 2$ , the number 5 is called the limit of  $f(x)$  as  $x \rightarrow 2$ . This leads to the meaning of the limit of a function. In general, *the limit of a function  $f(x)$  is that value which the function approaches as  $x$  approaches the given value  $a$* . This is written as

$$\lim_{x \rightarrow a} f(x) = L \quad (23.1)$$

where  $L$  is the value of the limit of the function.

An important conclusion can be drawn from the limit in Example 6. The function  $f(x)$  is a continuous function, and  $f(2)$  equals the value of the limit as  $x \rightarrow 2$ . In general, it is true that

*if  $f(x)$  is continuous at  $x = a$ , then the limit as  $x \rightarrow a$  equals  $f(a)$ .*

In fact, looking back at our definition of continuity, we see that this is what the definition means.

A function is continuous at the point  $x = a$   
if

$$\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a} f(x) = f(a)$$

Although we can evaluate the limit for a continuous function as  $x \rightarrow a$  by evaluating  $f(a)$ , it is possible that a function is not continuous at  $x = a$  and yet the limit exists and can be determined. Thus, we must be able to determine the value of a limit without finding  $f(a)$ . The following example illustrates the evaluation of such a limit.

## EXAMPLE 7 Limit of a function using a table of values

Find  $\lim_{x \rightarrow 2} \frac{2x^2 - 3x - 2}{x - 2}$ .

We note immediately that the function is not continuous at  $x = 2$ , for division by zero is indicated. Thus, we cannot evaluate the limit by substituting  $x = 2$  into the function. Using a calculator to set up tables we determine the value that  $f(x)$  approaches, as  $x$  approaches 2:

$x$	1.000	1.500	1.900	1.990	1.999	
$f(x)$	3.000	4.000	4.800	4.980	4.998	values approach 5
$x$	3.000	2.500	2.100	2.010	2.001	
$f(x)$	7.000	6.000	5.200	5.020	5.002	

We see that the values obtained are identical to those in Example 6. Since  $f(x) \rightarrow 5$  as  $x \rightarrow 2$ , we have

$$\lim_{x \rightarrow 2} \frac{2x^2 - 3x - 2}{x - 2} = 5$$

Therefore, we see that the limit exists at  $x \rightarrow 2$ , although the function does not exist at  $x = 2$ . ■

The reason the functions in Examples 6 and 7 have the same limit as  $x$  approaches 2 is shown in the following example.

### EXAMPLE 8 Factoring in limits

The function  $y = \frac{2x^2 - 3x - 2}{x - 2}$  in Example 7 is the same as the function  $y = 2x + 1$  in Example 6, except when  $x = 2$ . By factoring the numerator of the function of Example 7, and then cancelling, we have

$$\frac{2x^2 - 3x - 2}{x - 2} = \frac{(2x + 1)(x - 2)}{x - 2} = 2x + 1$$

The cancellation for this expression is valid, as long as  $x$  does not equal 2, for we have division by zero at  $x = 2$ . Also, in finding the limit as  $x \rightarrow 2$ , we do not use the value  $x = 2$ . Therefore,

$$\lim_{x \rightarrow 2} \frac{2x^2 - 3x - 2}{x - 2} = \lim_{x \rightarrow 2} (2x + 1) = 5$$

The limits of the two functions are equal, since, again, in finding the limit as  $x$  approaches 2, we do not let  $x = 2$ . The graphs of the two functions are shown in Fig. 23.7(a) and (b). We can see from the graphs that the limits are the same, although one of the functions is not continuous.

If  $f(x) = 5$  for  $x = 2$  is added to the definition of the function in Example 7, it is then the same as  $f(x) = 2x + 1$ , and its graph is that in Fig. 23.7(b).

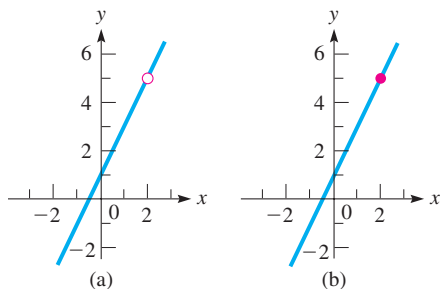


Fig. 23.7

The limit of the function in Example 7 was determined by calculating values near  $x = 2$  and by means of an algebraic change in the function. This illustrates that limits may be found through the meaning and definition and through other procedures when the function is not continuous. The following is a step-by-step guide to evaluating limits. It is intended as an introduction to the most common limits that could be encountered, but it is by no means complete. The steps that produce an analytical result are listed in order of increasing difficulty, so it is important to exhaust the simpler methods before moving on to the more difficult ones. Always remember that algebraic reduction in the initial stages of any problem can greatly simplify it.

#### Procedure for Evaluating Limits

1. Substitute the limiting value of  $x = a$  into the function. If you obtain a real result, and the function is continuous at that point, you have the limit  $L$ .
2. If you obtain a  $\frac{0}{0}$  indeterminate limit, try factoring the expression.
3. If you obtain a  $\frac{\infty}{\infty}$  indeterminate limit, try dividing both the numerator and the denominator by the highest power of  $x$  that appears in either the numerator or denominator.
4. If you see square roots over individual terms in either the numerator or denominator, try conjugate multiplication of both the numerator and denominator.
5. If you see square roots over an entire numerator or denominator, try rearranging the function so that it is entirely under a root sign.
6. Construct a table of values for values of  $x$  approaching the limiting value from above and below. If the limits from above and below agree, you have the limit  $L$ .

Remember, checking the valid domains for  $x$  in the cases where square roots are present is always important.

**EXAMPLE 9** Limit for a continuous function

Find  $\lim_{x \rightarrow 4} (x^2 - 7)$ .

Since the function  $f(x) = x^2 - 7$  is continuous at  $x = 4$ , we may evaluate the limit by substituting  $x = 4$  in the function. For  $f(x) = x^2 - 7$ , we have in this case  $f(4) = 4^2 - 7 = 9$ . This means that

$$\lim_{x \rightarrow 4} (x^2 - 7) = 9 \quad \blacksquare$$

**EXAMPLE 10** Limit for a discontinuous function, using factoring

Find  $\lim_{t \rightarrow 2} \left( \frac{t^2 - 4}{t - 2} \right)$ .

Since  $\frac{2^2 - 4}{2 - 2} = \frac{0}{0}$ , which is indeterminate, we try factoring:

$$\frac{t^2 - 4}{t - 2} = \frac{(t - 2)(t + 2)}{t - 2} = t + 2$$

Since this expression is valid as long as  $t \neq 2$ , we find that

$$\lim_{t \rightarrow 2} \left( \frac{t^2 - 4}{t - 2} \right) = \lim_{t \rightarrow 2} (t + 2) = 4$$

Again, we do not have to be concerned with the fact that the cancellation is not valid for  $t = 2$ . In finding the limit, we do not consider the value of  $f(t)$  at  $t = 2$ .  $\blacksquare$

**Practice Exercise**

2. Find  $\lim_{x \rightarrow 3} \frac{x^2 - 25}{x - 5}$ .

**LIMITS AS  $x$  APPROACHES INFINITY**

Limits as  $x$  approaches infinity are also of importance. However, when dealing with these limits, we must remember the following.

**COMMON ERROR**

When we say the limit is infinite ( $\infty$ ), we mean the value of the function gets larger and larger, without bound. Infinity is not a specific value, a place, nor a stop sign; it is a *concept*.  $\infty$  does not represent a real number, and algebraic operations may not be performed on it.

Therefore, when we write  $x \rightarrow \infty$ , we know that we are to consider values of  $x$  that are becoming larger and larger without bound. We encountered this concept in Chapter 19 when discussing infinite geometric series, and in Chapter 21 when discussing the asymptotes of a hyperbola. The following examples illustrate the evaluation of this type of limit for algebraic expressions.

**EXAMPLE 11** Limit as  $x$  approaches infinity

Find  $\lim_{x \rightarrow \infty} \frac{x^2 + 1}{2x^2 + 3}$ .

We note that as  $x \rightarrow \infty$ , both the numerator and the denominator become large without bound. Therefore, we make a table to see how  $f(x)$  behaves as  $x$  becomes very large:

$x$	1	10	100	1000
$f(x)$	0.4	0.497 536 945 8	0.499 975 003 7	0.499 999 75 <span style="color: blue;">values approach 0.5</span>

From this table, we see that  $f(x) \rightarrow 0.5$  as  $x \rightarrow \infty$ .

This limit can also be found through algebraic operations and an examination of the resulting algebraic form. If we divide both the numerator and the denominator of the

function by  $x^2$ , which is the largest power of  $x$  that appears in either the numerator or the denominator, we have

$$\frac{x^2 + 1}{2x^2 + 3} = \frac{1 + \frac{1}{x^2}}{2 + \frac{3}{x^2}}$$

$\leftarrow$  terms  $\rightarrow 0$   
 $\leftarrow$  as  $x \rightarrow \infty$

Here, we see that  $1/x^2$  and  $3/x^2$  both approach zero as  $x \rightarrow \infty$ . This means that the numerator approaches 1 and the denominator approaches 2. Therefore,

$$\lim_{x \rightarrow \infty} \frac{x^2 + 1}{2x^2 + 3} = \lim_{x \rightarrow \infty} \frac{1 + \frac{1}{x^2}}{2 + \frac{3}{x^2}} = \frac{1}{2}$$

### Practice Exercise

3. Find  $\lim_{x \rightarrow \infty} \frac{x+2}{2x-7}$ .

### EXAMPLE 12 Limits with conjugate multiplication

Find  $\lim_{x \rightarrow 9} \frac{\sqrt{x} - 3}{x - 9}$ .

$$\lim_{x \rightarrow 9} \frac{\sqrt{x} - 3}{x - 9} = \frac{\sqrt{9} - 3}{9 - 9} = \frac{0}{0} \quad \text{indeterminate}$$

We could attempt factoring. However, although factoring the denominator as a difference of squares is possible, it is not easy to do. Let us instead try rationalizing the numerator by multiplying both top and bottom of the function by the conjugate. Remember that the conjugate of the expression  $(\sqrt{x} - 3)$  is  $(\sqrt{x} + 3)$ . The only difference between an expression and its conjugate is that one of the signs for the terms is switched, so that when the expression is multiplied by its conjugate, a **difference of squares** will result. Some terms should cancel in this method.

$$\begin{aligned} \lim_{x \rightarrow 9} \frac{\sqrt{x} - 3}{x - 9} &= \lim_{x \rightarrow 9} \frac{\sqrt{x} - 3}{x - 9} \cdot \frac{\sqrt{x} + 3}{\sqrt{x} + 3} \\ &= \lim_{x \rightarrow 9} \frac{(\sqrt{x})^2 + 3\sqrt{x} - 3\sqrt{x} - 9}{(x - 9)(\sqrt{x} + 3)} \\ &= \lim_{x \rightarrow 9} \frac{x - 9}{(x - 9)(\sqrt{x} + 3)} \\ &= \lim_{x \rightarrow 9} \frac{1}{\sqrt{x} + 3} \\ \lim_{x \rightarrow 9} \frac{\sqrt{x} - 3}{x - 9} &= \frac{1}{\sqrt{9} + 3} = \frac{1}{6} \end{aligned}$$

### EXAMPLE 13 Limit with a square root over the entire function

Find  $\lim_{x \rightarrow \infty} \frac{\sqrt{x^2 - 8x}}{2x + 1}$ .

If you can move the square root to encompass the whole function, the limit can move inside the square root sign, and the limit can be evaluated, so that

$$\lim_{x \rightarrow a} \sqrt{f(x)} = \sqrt{\lim_{x \rightarrow a} f(x)}$$

This function can be rearranged so that the square root applies to the whole function.

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{\sqrt{x^2 - 8x}}{2x + 1} &= \lim_{x \rightarrow \infty} \frac{\sqrt{x^2 - 8x}}{\sqrt{(2x + 1)^2}} = \lim_{x \rightarrow \infty} \frac{\sqrt{x^2 - 8x}}{\sqrt{4x^2 + 4x + 1}} \\ &= \lim_{x \rightarrow \infty} \sqrt{\frac{x^2 - 8x}{4x^2 + 4x + 1}} = \sqrt{\lim_{x \rightarrow \infty} \frac{x^2 - 8x}{4x^2 + 4x + 1}} \end{aligned}$$

We can now evaluate the limit inside, and take the square root of the answer at the end.

$$= \sqrt{\lim_{x \rightarrow \infty} \frac{\frac{x^2}{x^2} - \frac{8x}{x^2}}{4\frac{x^2}{x^2} + \frac{4x}{x^2} + \frac{1}{x^2}}} = \sqrt{\lim_{x \rightarrow \infty} \frac{1 - \frac{8}{x}}{4 + \frac{4}{x} + \frac{1}{x^2}}} = \sqrt{\frac{1 - 0}{4 + 0 + 0}} = \frac{1}{2}$$

**EXAMPLE 14** Limit does not exist

In trying to find

$$\lim_{x \rightarrow 2} \frac{1}{x - 2}$$

we note that  $f(x)$  is not defined for  $x = 2$ , since we would have division by zero. Therefore, we set up the following table to see how  $f(x)$  behaves as  $x \rightarrow 2$ :

$x$	3	2.5	2.1	2.01	2.001	$f(x) \rightarrow +\infty$
$f(x)$	1	2	10	100	1000	

$x$	1	1.5	1.9	1.99	1.999	$f(x) \rightarrow -\infty$
$f(x)$	-1	-2	-10	-100	-1000	

We see that  $f(x)$  gets larger as  $x \rightarrow 2$  from above 2 and  $f(x)$  gets smaller (large negative values) as  $x \rightarrow 2$  from below 2. This may be written as  $f(x) \rightarrow +\infty$  as  $x \rightarrow 2^+$  and  $f(x) \rightarrow -\infty$  as  $x \rightarrow 2^-$ , but we must remember that  $\infty$  is not a real number. Therefore, the limit as  $x \rightarrow 2$  does not exist. The graph of this function is shown in Fig. 23.2, which is shown again for reference.

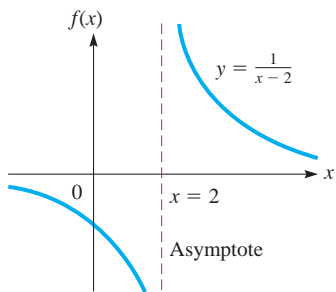


Fig. 23.2

**EXAMPLE 15** Limit as a value approaches infinity

The efficiency of an engine is given by  $\eta = 1 - Q_2/Q_1$ , where  $Q_1$  is the heat taken in and  $Q_2$  is the heat ejected by the engine. ( $Q_1 - Q_2$  is the work done by the engine.) If, in an engine cycle,  $Q_2 = 500$  kJ, find  $\eta$  as  $Q_1$  becomes large without bound.

We are to find

$$\lim_{Q_1 \rightarrow \infty} \left(1 - \frac{500}{Q_1}\right)$$

As  $Q_1$  becomes larger and larger,  $500/Q_1$  becomes smaller and smaller and approaches zero. This means  $f(Q_1) \rightarrow 1$  as  $Q_1 \rightarrow \infty$ . Thus,

$$\lim_{Q_1 \rightarrow \infty} \left(1 - \frac{500}{Q_1}\right) = 1$$

We can verify our reasoning and the value of the limit by making a table of values for  $Q_1$  and  $\eta$  as  $Q_1$  becomes large:

$Q_1$	500	5000	50 000	500 000	
$\eta$	0	0.9	0.99	0.999	values approach 1

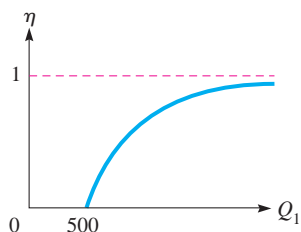


Fig. 23.8

Again, we see that  $\eta \rightarrow 1$  as  $Q_1 \rightarrow \infty$ . See Fig. 23.8.

This is primarily a theoretical consideration, as there are obvious practical limitations as to how much heat can be supplied to an engine. An engine for which  $\eta = 1$  would operate at 100% efficiency. ■

**EXAMPLE 16 Value of the function exists—limit does not**

Find  $\lim_{x \rightarrow 0} (x\sqrt{x-3})$ .

We see that  $x\sqrt{x-3} = 0$  if  $x = 0$ , but this function does not have real values for any values of  $x$  less than 3, other than  $x = 0$ . Therefore, *since  $x$  cannot approach zero,  $f(x)$  does not approach zero, and the limit does not exist.* The point of this example is that even if  $f(a)$  exists, we cannot evaluate the limit simply by finding  $f(a)$ , unless  $f(x)$  is continuous at  $x = a$ . In this case,  $f(a)$  exists, but zero is the only value less than  $x = 3$  for which  $f(x)$  is defined, and therefore the limit does not exist. ■

■ Calculus was not on a sound mathematical basis until limits were properly developed by the French mathematician Augustin-Louis Cauchy (1789–1857) and others in the mid-1800s.

The definitions and development of continuity and of a limit presented in this section are not mathematically rigorous. However, the development is consistent with a more rigorous development, and the concept of a limit is the principal concern.

**EXERCISES 23.1**

In Exercises 1–4, make the given changes in the indicated examples of this section. Then solve the resulting problems.

1. In Example 2, change the denominator to  $x + 2$  and then determine the continuity.
2. In Example 8, change the numerator to  $3x^2 - 5x - 2$  and find the resulting limit. Disregard references to Examples 6 and 7.
3. In Example 10, change the denominator to  $t + 2$  and then find the limit as  $t \rightarrow -2$ .
4. In Example 11, change the numerator to  $4x^2 + 1$  and find the resulting limit.

In Exercises 5–10, determine the values of  $x$  for which the function is continuous. If the function is not continuous, determine the reason.

- |                                  |                                   |
|----------------------------------|-----------------------------------|
| 5. $f(x) = 3x - 2$               | 6. $f(x) = 9 - x^2$               |
| 7. $f(x) = \frac{2}{x^2 - x}$    | 8. $f(x) = \frac{1}{\sqrt{x}}$    |
| 9. $f(x) = \sqrt{\frac{x}{x-2}}$ | 10. $f(x) = \frac{\sqrt{x+2}}{x}$ |

In Exercises 11–16, determine the values of  $x$  for which the function, as represented by the graphs in Fig. 23.9, is continuous. If the function is not continuous, determine the reason.

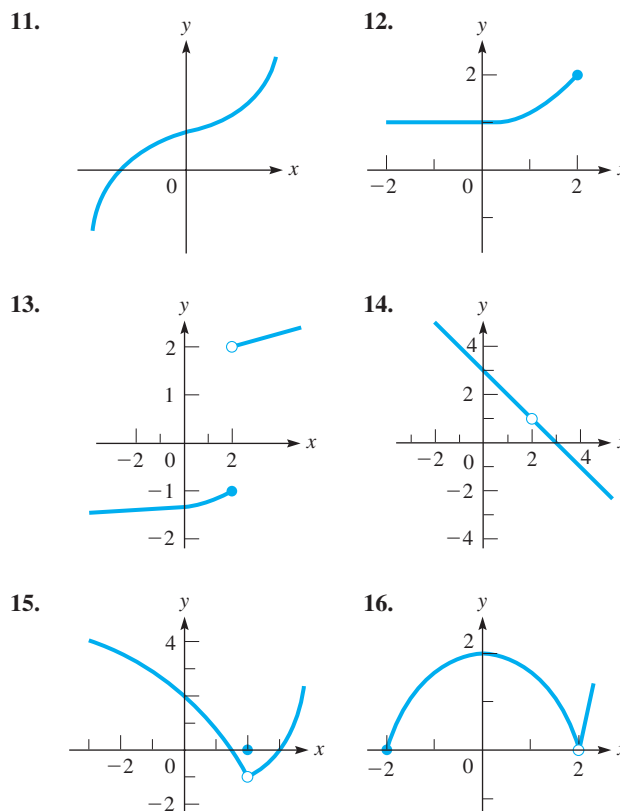



Fig. 23.9

In Exercises 17–20, for the function shown in the graph for the indicated exercise, find (a)  $f(2)$ , and (b)  $\lim_{x \rightarrow 2} f(x)$ .

17. Exercise 13                      18. Exercise 14  
 19. Exercise 15                      20. Exercise 16

 In Exercises 21–24, graph the function and determine the values of  $x$  for which the functions are continuous. Explain.

21.  $f(x) = \begin{cases} x^2 & \text{for } x < 2 \\ 2 & \text{for } x \geq 2 \end{cases}$

22.  $f(x) = \begin{cases} \frac{x^3 - x^2}{x - 1} & \text{for } x \neq 1 \\ 1 & \text{for } x = 1 \end{cases}$

23.  $f(x) = \begin{cases} \frac{2x^2 - 18}{x - 3} & \text{for } x < 3 \text{ or } x > 3 \\ 12 & \text{for } x = 3 \end{cases}$

24.  $f(x) = \begin{cases} \frac{x + 2}{x^2 - 4} & \text{for } x < -2 \\ \frac{x}{8} & \text{for } x > -2 \end{cases}$

In Exercises 25–30, evaluate the indicated limits by evaluating the function for values shown in the table and observing the values that are obtained. Do not change the form of the function.

25. Find  $\lim_{x \rightarrow 1} \frac{x^3 - x}{x - 1}$ .

$x$	0.900	0.990	0.999	1.001	1.010	1.100
$f(x)$						

26. Find  $\lim_{x \rightarrow -3} \frac{x^3 + 2x^2 - 2x + 3}{x + 3}$ .

$x$	-3.100	-3.010	-3.001	-2.999	-2.990	-2.900
$f(x)$						

27. Find  $\lim_{x \rightarrow 2} \frac{2 - \sqrt{x + 2}}{x - 2}$ .

$x$	1.900	1.990	1.999	2.001	2.010	2.100
$f(x)$						

28. Find  $\lim_{x \rightarrow 0} \frac{e^x - 1}{x}$ .

$x$	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$						

29. Find  $\lim_{x \rightarrow \infty} \frac{2x + 1}{5x - 3}$ .

$x$	10	100	1000	10 000
$f(x)$				

30. Find  $\lim_{x \rightarrow \infty} \frac{1 - x^2}{8x^2 + 5}$ .

$x$	10	100	1000	10 000
$f(x)$				

In Exercises 31–48, evaluate the indicated limits by direct evaluation as in Examples 9–16. Change the form of the function where necessary.

31.  $\lim_{x \rightarrow 3} (3x - 2)$       32.  $\lim_{x \rightarrow 4} \sqrt{x^2 - 7}$       33.  $\lim_{x \rightarrow 0} \frac{x^2 + x}{x}$

34.  $\lim_{v \rightarrow 2} \frac{4v^2 - 8v}{v - 2}$       35.  $\lim_{x \rightarrow -1} \frac{x^2 - 1}{3x + 3}$       36.  $\lim_{x \rightarrow 3} \frac{x^2 - 2x - 3}{3 - x}$

37.  $\lim_{h \rightarrow 3} \frac{h^3 - 27}{h - 3}$       38.  $\lim_{x \rightarrow 1/3} \frac{3x - 1}{3x^2 + 5x - 2}$

39.  $\lim_{x \rightarrow 1} \frac{(2x - 1)^2 - 1}{2x - 2}$       40.  $\lim_{x \rightarrow 4} \frac{|x - 4|}{x - 4}$

41.  $\lim_{p \rightarrow -1} \sqrt{p}(p + 1.3)$       42.  $\lim_{x \rightarrow 1} (x - 1)\sqrt{x^2 - 4}$

43.  $\lim_{x \rightarrow 1} \frac{\sqrt{x} - 1}{x - 1}$       44.  $\lim_{x \rightarrow 8} \frac{x - 8}{\sqrt[3]{x} - 2}$

45.  $\lim_{x \rightarrow \infty} \frac{3x^2 + 4.5}{x^2 - 1.5}$       46.  $\lim_{x \rightarrow \infty} \frac{x - 1}{7x + 4}$

47.  $\lim_{t \rightarrow \infty} \frac{\sqrt{t^2 + 16}}{t + 1}$       48.  $\lim_{x \rightarrow \infty} \frac{1 - 2x^2}{(4x + 3)^2}$

In Exercises 49 and 50, evaluate the function at 0.1, 0.01, and 0.001 from both sides of the value it approaches. In Exercises 51 and 52, evaluate the function for values of  $x$  of 10, 100, and 1000. From these values, determine the limit. Then, by using an appropriate change of algebraic form, evaluate the limit directly and compare values.

49.  $\lim_{x \rightarrow 0} \frac{x^2 - 3x}{x}$       50.  $\lim_{x \rightarrow 3} \frac{2x^2 - 6x}{x - 3}$

51.  $\lim_{x \rightarrow \infty} \frac{2x^2 + x}{x^2 - 3}$       52.  $\lim_{x \rightarrow \infty} \frac{x^2 + 5}{\sqrt{64x^4 + 1}}$

In Exercises 53–56, solve the given problems involving limits.

53. A certain object, after being heated, cools at such a rate that its temperature  $T$  (in °C) decreases 10% each minute. If the object is originally heated to 100°C, find  $\lim_{t \rightarrow 10} T$  and  $\lim_{t \rightarrow \infty} T$ , where  $t$  is the time (in min).

54. The area  $A$  (in mm<sup>2</sup>) of the pupil of a certain person's eye is given by  $A = \frac{36 + 24b^3}{1 + 4b^3}$ , where  $b$  is the brightness (in lumens) of the light source. Between what values does  $A$  vary?

55. Velocity can be found by dividing the displacement  $s$  of an object by the elapsed time  $t$  in moving through the displacement. In a certain experiment, the following values were measured for the displacements and elapsed times for the motion of an object. Determine the limiting value of the velocity.

$s$ (cm)	0.480 000	0.280 000	0.029 800	0.002 998 0	0.000 299 98
$t$ (s)	0.200 000	0.100 000	0.010 000	0.001 000 0	0.000 100 00



56. A 5-Ω resistor and a variable resistor of resistance  $R$  are placed in parallel. The expression for the resulting resistance  $R_T$  is given by  $R_T = \frac{5R}{5 + R}$ . Determine the limiting value of  $R_T$  as  $R \rightarrow \infty$ .

In Exercises 57–60, construct a table of values to evaluate the indicated limits.

57. Approximate  $\lim_{x \rightarrow 2} \frac{2^x - 4}{x - 2}$ .      58. Approximate  $\lim_{x \rightarrow 1} \frac{4^x - 4}{x - 1}$ .

59.  $\lim_{x \rightarrow 0} (1 + x)^{1/x}$  (Do you recognize the limiting value?)

60.  $\lim_{x \rightarrow 0} \frac{\sin x}{x}$  (Use radian mode.)

In Exercises 61–66, solve the given problems involving one-sided limits.

61. For the function displayed in Exercise 13, find

(a)  $\lim_{x \rightarrow 2^-} f(x)$       (b)  $\lim_{x \rightarrow 2^+} f(x)$       (c)  $\lim_{x \rightarrow 2} f(x)$

62. For the function displayed in Exercise 16, find

(a)  $\lim_{x \rightarrow -2^-} f(x)$       (b)  $\lim_{x \rightarrow -2^+} f(x)$       (c)  $\lim_{x \rightarrow -2} f(x)$

63. Find  $\lim_{x \rightarrow 4^-} x\sqrt{16 - x^2}$ .

64. Explain why  $\lim_{x \rightarrow 0^+} 2^{1/x} \neq \lim_{x \rightarrow 0^-} 2^{1/x}$ .

65. For  $f(x) = \frac{x}{|x|}$ , find  $\lim_{x \rightarrow 0^-} f(x)$  and  $\lim_{x \rightarrow 0^+} f(x)$ . Is  $f(x)$  continuous at  $x = 0$ ? Explain.

66. In Einstein's theory of relativity, the length  $L$  of an object moving at a velocity  $v$  is  $L = L_0\sqrt{1 - \frac{v^2}{c^2}}$ , where  $c$  is the speed of light and  $L_0$  is the length of the object at rest. Find  $\lim_{v \rightarrow c^-} L$  and explain why a limit from the left is used.

Answers to Practice Exercises

1. Discontinuous at  $x = -3$  and  $x = 0$     2. 10    3. 0.5

## 23.2 The Slope of a Tangent to a Curve

### Slope of a Secant Line • Slope of a Tangent Line

Having developed the basic operations with functions and the concept of a limit, we now turn our attention to a graphical interpretation of the rate of change of a function. This interpretation, basic to an understanding of calculus, deals with the slope of a line tangent to the curve of a function.

Consider the points  $P(x_1, y_1)$  and  $Q(x_2, y_2)$  in Fig. 23.10. From Chapter 21, we know that the slope of the line through these points is given by

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$

This, however, represents the slope of the line through  $P$  and  $Q$  (a *secant line*) and no other line. We can also refer to this as the average rate of change over the interval, or the *average slope over the interval* from  $x_1$  to  $x_2$ . If we now allow  $Q$  to be a point closer to  $P$ , the slope of  $PQ$  will more closely approximate the slope of a line drawn tangent to the curve at  $P$  (see Fig. 23.11). In fact, the closer  $Q$  is to  $P$ , the better this approximation becomes. It is not possible to allow  $Q$  to coincide with  $P$ , for then it would not be possible to define the slope of  $PQ$  in terms of two points. *The slope of the tangent line, often referred to as the slope of the curve, is the limiting value of the slope of the secant line  $PQ$  as  $Q$  approaches  $P$ .*

#### EXAMPLE 1 Limit of the slopes of secant lines

Find the slope of a line tangent to the curve  $y = x^2 + 3x$  at the point  $P(2, 10)$  by finding the limit of the slopes of the secant lines  $PQ$  as  $Q$  approaches  $P$ .

Let point  $Q$  have the  $x$ -values of 3.0, 2.5, 2.1, 2.01, and 2.001. Then, using a calculator, we tabulate the necessary values. Since  $P$  is the point  $(2, 10)$ ,  $x_1 = 2$  and  $y_1 = 10$ . Thus, using the values of  $x_2$ , we tabulate the values of  $y_2$ ,  $y_2 - 10$ ,  $x_2 - 2$ , and the slope  $m$ :

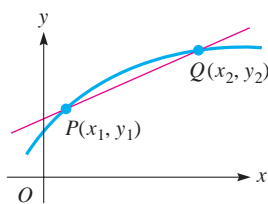


Fig. 23.10

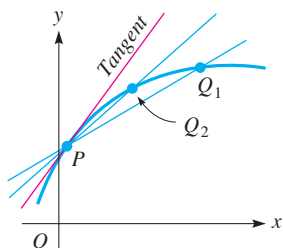


Fig. 23.11

Point	$Q_1$	$Q_2$	$Q_3$	$Q_4$	$Q_5$	$P$
$x_2$	3.0	2.5	2.1	2.01	2.001	2
$y_2$	18.0	13.75	10.71	10.0701	10.007 001	10
$y_2 - 10$	8.0	3.75	0.71	0.0701	0.007 001	
$x_2 - 2$	1.0	0.5	0.1	0.01	0.001	
$m = \frac{y_2 - 10}{x_2 - 2}$	8.0	7.5	7.1	7.01	7.001	

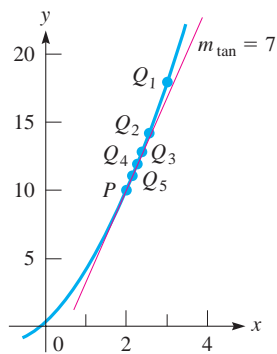


Fig. 23.12

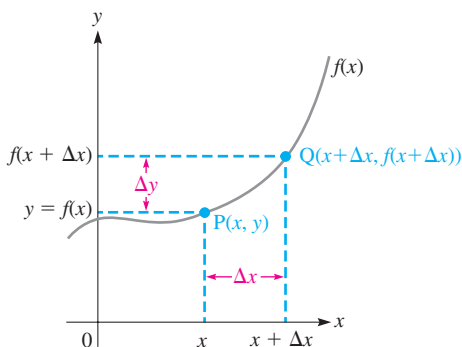


Fig. 23.13

We see that the slope of  $PQ$  approaches the value of 7 as  $Q$  approaches  $P$ . Therefore, the slope of the tangent line at  $(2, 10)$  is 7. See Fig. 23.12. ■

Let a general point  $P$  be represented by  $P(x, y)$ . Using the appropriate notation, the coordinates of any other point  $Q(x_2, y_2)$  on the curve of  $f(x)$  can be expressed in terms of the coordinates of  $P$ . Define  $\Delta x$  and  $\Delta y$  as the  $x$ - and  $y$ -distances between the points  $P$  and  $Q$ . Then

$$\Delta x = x_2 - x \tag{23.2}$$

$$x_2 = x + \Delta x \tag{23.3}$$

For the function  $y = f(x)$ , the point  $P(x, y)$  can be written as  $P(x, f(x))$  and the point  $Q(x_2, y_2)$  can be written as  $Q(x + \Delta x, f(x + \Delta x))$ . See Fig. 23.13.

You can see from the  $y$ -values in Fig. 23.13 that  $y + \Delta y = f(x + \Delta x)$ . Since this equation contains the quantities  $\Delta x$  and  $\Delta y$ , we can use it as a starting point for calculating the average slope, which we define as  $\frac{\Delta y}{\Delta x}$ .

To calculate the average slope of any function over an interval between any two points, we can use the delta method.

### Delta Method for Finding an Average Slope

1. Write  $y + \Delta y = f(x + \Delta x)$ . Substitute  $x + \Delta x$  into the function and simplify if possible.
2. Rearrange the equation to isolate  $\Delta y$ . Do this by subtracting  $y$  (the original function) from both sides of the equation. Make sure that  $y$  is expressed as a function of  $x$  during this subtraction. Simplify if possible.
3. Divide both sides of the equation by  $\Delta x$  to get  $\frac{\Delta y}{\Delta x}$ . This equation represents the average slope between any two points for the function. The first point is  $(x, y)$  and the interval width in question is  $\Delta x$ .

### EXAMPLE 2 Average slope over an interval

For the function  $y = x^2 + 1$ , find the average slope from  $x = 1$  to  $x = 1.1$  using the delta method.

$$\begin{aligned}
 y + \Delta y &= f(x + \Delta x) && \text{step 1} \\
 &= (x + \Delta x)^2 + 1 && \text{substitute } x + \Delta x \text{ into } f \\
 &= x^2 + 2x\Delta x + \Delta x^2 + 1 \\
 \Delta y &= x^2 + 2x\Delta x + \Delta x^2 + 1 - y && \text{step 2} \\
 &= x^2 + 2x\Delta x + \Delta x^2 + 1 - (x^2 + 1) \\
 &= 2x\Delta x + \Delta x^2 \\
 \frac{\Delta y}{\Delta x} &= \frac{2x\Delta x + \Delta x^2}{\Delta x} && \text{step 3} \\
 \frac{\Delta y}{\Delta x} &= 2x + \Delta x
 \end{aligned}$$

This is the general equation for the *average slope* between any two points for this function. If  $x = 1$  and  $\Delta x = 1.1 - 1 = 0.1$  then

$$\frac{\Delta y}{\Delta x} = 2(1) + 0.1$$

$$\frac{\Delta y}{\Delta x} = 2.1$$

Using Eq. (23.3), along with the definition of slope, we can express the average slope over the interval from  $P$  to  $Q$  as

$$m_{PQ} = \frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x} \quad (23.4)$$

Notice that this summarizes the delta method in determining average slope. By our previous discussion, as  $Q$  approaches  $P$ , the slope of the tangent line will be approximated by Eq. (23.4).

### EXAMPLE 3 Slope of the tangent line at a specific point

Find the slope of a line tangent to the curve of  $y = x^2 + 3x$  at the point  $(2, 10)$ . (This is the same slope as in Example 1.)

As in Example 1, point  $P$  has the coordinates  $(2, 10)$ . Thus, the coordinates of any other point  $Q$  on the curve can be expressed as  $(2 + \Delta x, f(2 + \Delta x))$ . See Fig. 23.14. The slope of  $PQ$  then becomes

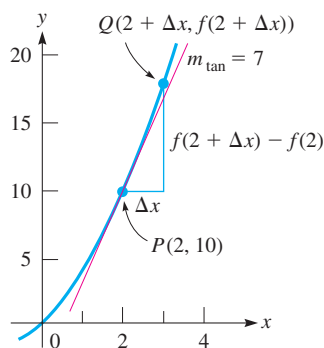


Fig. 23.14

$$\begin{aligned} m_{PQ} &= \frac{f(2 + \Delta x) - f(2)}{\Delta x} = \frac{[(2 + \Delta x)^2 + 3(2 + \Delta x)] - [2^2 + 3(2)]}{\Delta x} \\ &= \frac{(4 + 4\Delta x + \Delta x^2 + 6 + 3\Delta x) - (4 + 6)}{\Delta x} \\ &= \frac{7\Delta x + \Delta x^2}{\Delta x} = 7 + \Delta x \end{aligned}$$

From this expression, we can see that  $m_{PQ} \rightarrow 7$  as  $\Delta x \rightarrow 0$ . Therefore, we can see that the slope of the tangent line is

$$m_{\text{tan}} = \lim_{\Delta x \rightarrow 0} m_{PQ} = 7$$

We see that this result agrees with that found in Example 1.

We can define the slope of a tangent line at a single point  $P$  by letting the point  $Q$  approach  $P$ . As  $Q$  approaches  $P$ , we have  $\Delta x \rightarrow 0$ . Consequently, we add a final step to the delta method outlined above.

### Delta Method for Finding a Tangent Line Slope

$$4. m_{\text{tan}} = \lim_{\Delta x \rightarrow 0} m_{PQ} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$$

### EXAMPLE 4 Slope of the tangent line at a general point

Find the slope of a line tangent to the curve of  $y = 4x - x^2$  at a point  $(x, y)$  using the delta method.

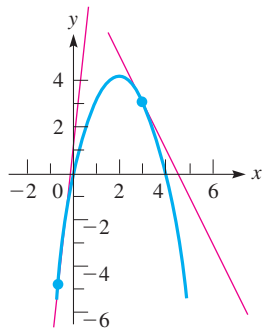


Fig. 23.15

■ Graphing calculators have a feature for drawing a line tangent to a curve.

### Practice Exercise

1. Find the expression for the slope of a line tangent to the curve of  $y = 4x^2$  at  $(x, y)$ .

### LEARNING TIP

The ratio of the change in  $f(x)$  to the change in  $x$  is the average rate of change of  $f(x)$  with respect to  $x$ .

As  $\Delta x \rightarrow 0$ , the limit of the ratio of the change in  $f(x)$  to the change in  $x$  is the instantaneous rate of change of  $f(x)$  with respect to  $x$ .

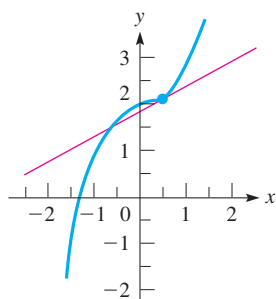


Fig. 23.16

$$\begin{aligned} m_{PQ} &= \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ m_{PQ} &= \frac{[4(x + \Delta x) - (x + \Delta x)^2] - (4x - x^2)}{\Delta x} \\ m_{PQ} &= \frac{4x + 4\Delta x - x^2 - 2x \cdot \Delta x - \Delta x^2 - 4x + x^2}{\Delta x} \\ m_{PQ} &= \frac{4\Delta x - 2x \cdot \Delta x - \Delta x^2}{\Delta x} \\ m_{PQ} &= 4 - 2x - \Delta x \end{aligned}$$

Here, we see that  $m_{PQ} \rightarrow 4 - 2x$  as  $\Delta x \rightarrow 0$ . Therefore,

$$m_{\text{tan}} = 4 - 2x$$

This method has an advantage over that used in Example 2. We now have a general expression for the slope of a line tangent to the curve for any value  $x$ . If  $x = -1$ ,  $m_{\text{tan}} = 6$  and if  $x = 3$ ,  $m_{\text{tan}} = -2$ . The tangent lines are shown in Fig. 23.15. ■

### EXAMPLE 5 Evaluating the slope of a tangent line

Find the expression for the slope of a line tangent to the curve of  $y = x^3 + 2$  at the general point  $(x, y)$  and use this expression to find the slope when  $x = 1/2$ .

$$\begin{aligned} m_{PQ} &= \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ m_{PQ} &= \frac{[(x + \Delta x)^3 + 2] - (x^3 + 2)}{\Delta x} \\ m_{PQ} &= \frac{x^3 + 3x^2 \cdot \Delta x + 3x \cdot \Delta x^2 + \Delta x^3 + 2 - x^3 - 2}{\Delta x} \\ m_{PQ} &= \frac{3x^2 \cdot \Delta x + 3x \cdot \Delta x^2 + \Delta x^3}{\Delta x} \\ m_{PQ} &= 3x^2 + 3x \cdot \Delta x + \Delta x^2 \end{aligned}$$

Taking the limit as  $\Delta x \rightarrow 0$  yields the slope of the tangent line.

$$\begin{aligned} m_{\text{tan}} &= \lim_{\Delta x \rightarrow 0} (3x^2 + 3x \cdot \Delta x + \Delta x^2) \\ m_{\text{tan}} &= 3x^2 \end{aligned}$$

When  $x = 1/2$ , we find that the slope of the tangent is  $3(1/4) = 3/4$ . The curve and this tangent line are shown in Fig. 23.16. ■

### EXAMPLE 6 Slope as instantaneous rate of change

In Example 1, consider points  $P(2, 10)$  and  $Q(2.5, 13.75)$ . From  $P$  to  $Q$ ,  $x$  changes by 0.5 unit and  $f(x)$  changes by 3.75 units. This means the average change in  $f(x)$  for a 1-unit change in  $x$  is  $3.75/0.5 = 7.5$  units. However, this is not the rate at which  $f(x)$  is changing with respect to  $x$  at most points within this interval.

At point  $P$ , the slope of 7 of the tangent line tells us that  $f(x)$  is changing 7 units for a 1-unit change in  $x$ . However, **this is an instantaneous rate of change at point  $P$**  and tells the rate at which  $f(x)$  is changing with respect to  $x$  at  $P$ . ■

## EXERCISES 23.2

In Exercises 1 and 2, make the given changes in the indicated examples of this section and then find the indicated slopes.

- In Example 3, change the point  $(2, 10)$  to  $(3, 18)$ .
- In Example 4, change the  $4x$  to  $3x$ .

In Exercises 3–6, use the method of Example 1 to calculate the slope of the line tangent to the curve of each of the given functions. Let  $Q_1, Q_2, Q_3,$  and  $Q_4$  have the indicated  $x$ -values. Sketch the curve and tangent lines.

- $y = x^2$ ;  $P$  is  $(2, 4)$ ; let  $Q$  have  $x$ -values of 1.5, 1.9, 1.99, 1.999.
- $y = 1 - \frac{1}{2}x^2$ ;  $P$  is  $(2, -1)$ ; let  $Q$  have  $x$ -values of 1.5, 1.9, 1.99, 1.999.
- $y = 2x^2 + 5x$ ;  $P$  is  $(-2, -2)$ ; let  $Q$  have  $x$ -values of  $-1.5, -1.9, -1.99, -1.999$ .
- $y = x^3 + 1$ ;  $P$  is  $(-1, 0)$ ; let  $Q$  have  $x$ -values of  $-0.5, -0.9, -0.99, -0.999$ .

In Exercises 7–18, use the delta method to find a general expression for the slope of a tangent line to each of the indicated curves. Then find the slopes for the given values of  $x$ . Sketch the curves and tangent lines.

- $y = x^2$ ;  $x = 2, x = -1$
- $y = 1 - \frac{1}{2}x^2$ ;  $x = 2, x = -2$
- $y = 2x^2 + 5x$ ;  $x = -2, x = 0.5$
- $y = 4.5 - 3x^2$ ;  $x = 0, x = 2$
- $y = x^2 + 4x + 2\pi$ ;  $x = -3, x = 2$
- $y = 2x^2 - 4x$ ;  $x = 1, x = 1.5$
- $y = 6x - x^2$ ;  $x = -2, x = 3$
- $y = x^3 - 2x$ ;  $x = -1, x = 0, x = 1$
- $y = 1.5x^4$ ;  $x = 0, x = 0.5, x = 1$
- $y = 4 - x^4$ ;  $x = 0, x = 1, x = 2$
- $y = x^5$ ;  $x = 0, x = 0.5, x = 1$
- $y = \frac{1}{x}$ ;  $x = 0.5, x = 1, x = 2$

In Exercises 19–22, use the tangent feature of a graphing calculator to display the curve and the tangent line for the given values of  $x$ .

- $y = 2x - 3x^2$ ;  $x = 0, x = 0.5$
- $y = 3x - x^3$ ;  $x = -2, x = 0, x = 2$

$$21. y = \frac{1}{3}x^6; x = 0, x = 0.5, x = 1$$

$$22. y = \frac{2}{x+1}; x = -0.5, x = 0, x = 1$$

In Exercises 23–26, find the average rate of change of  $y$  with respect to  $x$  from  $P$  to  $Q$ . Then compare this with the instantaneous rate of change of  $y$  with respect to  $x$  at  $P$  by finding  $m_{\tan}$  at  $P$ .

$$23. y = x^2 + 2; P(2, 6), Q(2.1, 6.41)$$

$$24. y = 1 - 2x^2; P(1, -1), Q(1.1, -1.42)$$

$$25. y = 9 - x^3; P(2, 1), Q(2.1, -0.261)$$

$$26. y = x^3 - 6x; P(3, 9), Q(3.1, 11.191)$$

In Exercises 27 and 28, find the point(s) where the slope of a tangent line to the given curve has the given value. In Exercises 29–33, solve the given problems.

$$27. y = 2x^2, m_{\tan} = -4$$

$$28. y = x^3 + 3x^2, m_{\tan} = 9$$

29. Find the slope of a line perpendicular to the tangent of the curve of  $y = 8 - 3x^2$  where  $x = -1$ .

30. Find the slopes of the tangent lines to the curve  $y = \frac{1}{3}x^3 + x$  at points where  $x = 1$  and  $x = 2$ . Then find the acute angle between these lines at the point where they cross.

31. In a computer game, at one point an airplane is diving along the curve of  $y = -2x^2 + 10$ . What is the angle of the dive (with the vertical) when  $x = 1.5$ ?

32. At an amusement park, a waterslide follows the curve of  $y = 8/(x+1)$ , for  $x = 0$  m to  $x = 7$  m. What is the angle (with the horizontal) of the slide when  $x = 4.00$  m?

33. A structural support panel follows the curve  $y = x^3 + 1$  (all dimensions are in metres) from  $x = -2.00$  m to  $x = 2.00$  m. Determine the slope of the structural panel at the point  $x = -1.00$  m.

### Answer to Practice Exercise

$$1. m_{\tan} = 8x$$

## 23.3 The Derivative

**Definition of the Derivative • Differentiation • Procedure for Finding a Derivative**

In the preceding section, we found the slope of a line tangent to a curve at a point  $(x, f(x))$  by calculating the limit (if it exists) of the difference  $f(x + \Delta x) - f(x)$  divided by  $\Delta x$  as  $\Delta x \rightarrow 0$ . For brevity, we can define  $h = \Delta x = x_2 - x$ . We can then write the slope of the tangent line as

$$m_{\tan} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \quad (23.5)$$

The limit on the right is defined as *the derivative of  $f(x)$  at  $x$* . This is one of the fundamental definitions of calculus.

Keep in mind that  $\frac{dy}{dx}$  is simply a shorter way of writing  $\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$ . There are other ways of representing the derivative:

Derivative					
Slope of Tangent Line	$\frac{dy}{dx}$	$\frac{d}{dx}(y)$	$f'(x)$	$D_x y$	$y'$
Slope at a Single Point					
Instantaneous Slope					

Here  $\frac{d}{dx}$  and  $D_x$  represent the *process* of finding the derivative with respect to a change in variable  $x$ .

*The process of finding a derivative is called **differentiation**.*

A four-step procedure for finding the derivative of a function by use of the definition is outlined here.

### Summary of the Delta Process for Finding the Derivative of a Function

1. Find  $f(x + h)$ .
2. Subtract  $f(x)$  from  $f(x + h)$ .
3. Divide the result of step 2 by  $h$ .
4. For the result of step 3, find the limit (if it exists) as  $h \rightarrow 0$ .

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \quad (23.6)$$

### EXAMPLE 1 Using the definition to find a derivative

Find the derivative of  $y = 2x^2 + 3x$  by using the definition.

With  $y = f(x)$ , using the above procedure to find the derivative  $f'(x)$ , we have the following:

$$f(x+h) = 2(x+h)^2 + 3(x+h) \quad \text{step 1}$$

$$\begin{aligned} f(x+h) - f(x) &= 2(x+h)^2 + 3(x+h) - (2x^2 + 3x) && \text{step 2} \\ &= 2x^2 + 4xh + 2h^2 + 3x + 3h - 2x^2 - 3x \\ &= 4xh + 3h + 2h^2 \end{aligned}$$

$$\frac{f(x+h) - f(x)}{h} = \frac{4hx + 3h + 2h^2}{h} = 4x + 3 + 2h \quad \text{step 3}$$

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \rightarrow 0} (4x + 3 + 2h) = 4x + 3 && \text{step 4} \\ f'(x) &= 4x + 3 \end{aligned}$$

We see that the derivative of the function  $2x^2 + 3x$  is the function  $4x + 3$ . From the meanings of a slope of a tangent line and the derivative, this means we can find the slope of a tangent line for any point on the curve of  $y = 2x^2 + 3x$  by substituting the  $x$ -coordinate into the expression  $4x + 3$ . For example, the slope of a tangent line is 5 if  $x = 1/2$  (at the point  $(1/2, 2)$ ). See Fig. 23.17. ■

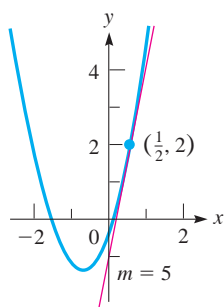


Fig. 23.17

**Practice Exercise**

1. Using the definition, find the derivative of  $y = 5x - x^2$ .

Since the derivative of a function is itself a function, it is possible that it may not be defined for all values of  $x$ . If the value  $x_0$  is in the domain of the derivative, then the function is said to be **differentiable** at  $x_0$ . The examples that follow illustrate functions that are not differentiable for all values of  $x$ .

**EXAMPLE 2** Using the definition—derivative of a fraction

Find the derivative of  $y = \frac{3}{x+2}$  by using the definition.

$$f(x+h) = \frac{3}{x+h+2} \quad \text{step 1}$$

$$f(x+h) - f(x) = \frac{3}{x+h+2} - \frac{3}{x+2} \quad \text{step 2}$$

$$f(x+h) - f(x) = \frac{3(x+2) - 3(x+h+2)}{(x+h+2)(x+2)} = \frac{-3h}{(x+h+2)(x+2)}$$

$$\frac{f(x+h) - f(x)}{h} = \frac{-3h}{h(x+h+2)(x+2)} = \frac{-3}{(x+h+2)(x+2)} \quad \text{step 3}$$

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{-3}{(x+h+2)(x+2)} = \frac{-3}{(x+2)^2} \quad \text{step 4}$$

$$f'(x) = \frac{-3}{(x+2)^2}$$

Note that neither the function nor the derivative is defined for  $x = -2$ . This means the function is not differentiable at  $x = -2$ . ■

**COMMON ERROR**

In Example 2, it was necessary to combine fractions in the process of finding the derivative. Such algebraic operations must be done with care. *One of the most common sources of errors is the improper handling of fractions.*

**EXAMPLE 3** Derivative—proper handling of fractions

Find the derivative of  $y = 4x^3 + \frac{5}{x}$  by using the definition.

$$f(x+h) = 4(x+h)^3 + \frac{5}{x+h}$$

$$f(x+h) - f(x) = 4(x+h)^3 + \frac{5}{x+h} - \left(4x^3 + \frac{5}{x}\right)$$

$$= 4(x^3 + 3x^2h + 3xh^2 + h^3) - 4x^3 + \frac{5}{x+h} - \frac{5}{x}$$

$$= 4x^3 + 12x^2h + 12xh^2 + 4h^3 - 4x^3 + \frac{5x - 5(x+h)}{x(x+h)}$$

$$= 12x^2h + 12xh^2 + 4h^3 - \frac{5h}{x(x+h)}$$

$$\frac{f(x+h) - f(x)}{h} = \frac{12x^2h + 12xh^2 + 4h^3}{h} - \frac{5h}{hx(x+h)}$$

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \left(12x^2 + 12xh + 4h^2 - \frac{5}{x(x+h)}\right) = 12x^2 - \frac{5}{x^2}$$

$$f'(x) = 12x^2 - \frac{5}{x^2}$$

Note that this function is not differentiable for  $x = 0$ . ■

**LEARNING TIP**

The algebra is easier if the fractions are combined separately from the other terms.

**Practice Exercise**

2. Using the definition, find the derivative of  $y = 3/x$ .



**EXAMPLE 4** Notation for evaluating the derivative

In Example 2,  $y = \frac{3}{x+2}$ , and we found that the derivative is  $\frac{-3}{(x+2)^2}$ . Therefore, we may write

$$y' = \frac{-3}{(x+2)^2} \quad \text{or} \quad \frac{dy}{dx} = \frac{-3}{(x+2)^2}$$

instead of  $f'(x) = \frac{-3}{(x+2)^2}$  as we did in Example 2.

If we wish to find the value of the derivative at some point, such as  $(-1, 3)$ , we write

$$\begin{aligned} \frac{dy}{dx} &= \frac{-3}{(x+2)^2} \\ \left. \frac{dy}{dx} \right|_{x=-1} &= \frac{-3}{(-1+2)^2} = -3 \end{aligned}$$

Note that only the  $x$ -coordinate was needed to evaluate the derivative. ■

■ Most graphing calculators have a feature for evaluating derivatives. In particular, the *numerical derivative* feature evaluates derivatives through numerical approximations.

**EXAMPLE 5** Using the definition—derivative of the square root

Find  $dy/dx$  for the function  $y = \sqrt{x}$  by using the definition.

When finding this derivative, we will employ the method of conjugate multiplication for evaluating limits. See Section 23.1 regarding this approach.

$$\begin{aligned} f(x+h) &= \sqrt{x+h} \\ f(x+h) - f(x) &= \sqrt{x+h} - \sqrt{x} \\ \frac{f(x+h) - f(x)}{h} &= \frac{\sqrt{x+h} - \sqrt{x}}{h} \\ \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \frac{0}{0} \text{ (indeterminate)} \\ \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \\ &= \lim_{h \rightarrow 0} \frac{(\sqrt{x+h})^2 - (\sqrt{x})^2}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{h \rightarrow 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{h \rightarrow 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{h \rightarrow 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} \\ &= \frac{1}{\sqrt{x} + \sqrt{x}} \\ \frac{dy}{dx} &= \frac{1}{2\sqrt{x}} \end{aligned}$$

**LEARNING TIP**

One might ask why, when finding a derivative, we take the limit as  $h$  approaches zero and not just let  $h$  equal zero. If we did this, we would find that the ratio  $[f(x+h) - f(x)]/h$  is exactly  $0/0$ , which requires division by zero. As we know, this is indeterminate, and therefore  $h$  **cannot equal zero**. However, it can equal any value as near zero as necessary. This idea is basic in the meaning of the word *limit*.

The domain of  $f(x)$  is  $x \geq 0$ . However, since  $x$  appears in the denominator of the derivative, the domain of the derivative is  $x > 0$ . Thus, the function is differentiable for  $x > 0$ . ■

## EXERCISES 23.3

In Exercises 1 and 2, make the given changes in the indicated examples of this section and then find the derivative by using the definition.

- In Example 1, change  $2x^2$  to  $4x^2$ .
- In Example 2, change  $x + 2$  in the denominator to  $x - 2$ .

In Exercises 3–26, find the derivative of each of the functions by using the definition.

- |                                  |   |
|----------------------------------|---|
| 3. $y = 3x - 1$                  | 4. $y = 6x + 3$                               |
| 5. $y = 1 - 2x$                  | 6. $y = 2.3 - 5x$                             |
| 7. $y = x^2 - 1$                 | 8. $y = 4^2 - x^2$                            |
| 9. $y = \pi x^2$                 | 10. $y = -6x^2$                               |
| 11. $y = x^2 - 7x$               | 12. $y = x^2 + 4ex$                           |
| 13. $y = 8x - 2x^2$              | 14. $y = 3x - \frac{1}{2}x^2$                 |
| 15. $y = x^3 + 4x - 3\pi$        | 16. $y = 2x - 4x^3$                           |
| 17. $y = \frac{\sqrt{3}}{x + 2}$ | 18. $y = \frac{3}{5x + 3}$                    |
| 19. $y = x + \frac{4}{3x}$       | 20. $y = \frac{x}{x - 1}$                     |
| 21. $y = \frac{2}{x^2}$          | 22. $y = \frac{2e^2}{x^2 + 4}$                |
| 23. $y = x^4 + x^3 + x^2 + x$    | 24. $y = \frac{1}{3}x^3 + \frac{1}{2}x^2 + x$ |
| 25. $y = x^4 - \frac{2}{x}$      | 26. $y = \frac{1}{6x} + \frac{5}{x^2}$        |

In Exercises 27–30, find the derivative of each function by using the definition. Then evaluate the derivative at the given point.

- |  |   |
|--|---|
| 27. $y = 3x^2 - 2x$ ; $(-1, 5)$        | 28. $y = 9x - x^3$ ; $(2, 10)$          |
| 29. $y = \frac{11}{3x + 2}$ ; $(3, 1)$ | 30. $y = x^2 - \frac{2}{x}$ ; $(-2, 5)$ |

In Exercises 31–34, find the derivative of each function by using the definition. Then determine the values for which the function is differentiable.

- |                             |                             |
|-----------------------------|-----------------------------|
| 31. $y = 1 + \frac{2}{x}$   | 32. $y = \frac{5x}{x - 4}$  |
| 33. $y = \frac{3}{x^2 - 1}$ | 34. $y = \frac{2}{x^2 + 1}$ |

In Exercises 35–44, solve the given problems.

- Find the point(s) on the curve of  $y = x^2 - 4x$  for which the slope of a tangent line is 6.
- Find the point(s) on the curve of  $y = 1/(x + 1)$  for which the slope of the tangent line is  $-1$ .
- At what point on the curve of  $y = 2x^2 - 16x$  is there a tangent line that is horizontal?
- At what point on the curve of  $y = 9 - 2x^2$  is there a tangent line that is parallel to the line  $12x - 2y + 7 = 0$ ?
- Find  $dy/dx$  for  $y = \sqrt{x + 1}$  by the method of Example 5. For what values of  $x$  is the function differentiable? Explain.
- Find  $dy/dx$  for  $y = \sqrt{x^2 + 3}$  by the method of Example 5.
- By noting the derivative for the function in Exercise 23, guess as to a formula for the derivative of  $y = x^n$ , where  $n$  is a positive integer.
- By noting the derivative for the function in Exercise 26, guess as to a formula for the derivative of  $y = x^n$ , where  $n$  is a negative integer. Compare the result with that of Exercise 41.
- A ski run follows the curve of  $y = 0.01x^2 - 0.4x + 4$  from  $x = 0$  m to  $x = 20.0$  m. What is the angle between the ski run and the horizontal when  $x = 10.0$  m? (Round to 3 significant digits.)
- The cross section of a hill can be approximated by the curve of  $y = 0.3x - 0.00003x^3$  from  $x = 0$  m to  $x = 100$  m. The top of the hill is level. How high is the hill? (Round to 3 significant digits.)

### Answers to Practice Exercises

1.  $y' = 5 - 2x$     2.  $y' = -3/x^2$

## 23.4 The Derivative as an Instantaneous Rate of Change

**Instantaneous Rate of Change • Instantaneous Velocity • Instantaneous Rate of Change of the Dependent Variable with Respect to the Independent Variable**

In Section 23.2, we saw that the slope of a line tangent to a curve at point  $P$  was the limiting value of the slope of the line through points  $P$  and  $Q$  as  $Q$  approaches  $P$ . In Section 23.3, we defined the limit of the ratio  $(f(x + h) - f(x))/h$  as  $h \rightarrow 0$  as the derivative. Therefore, the first meaning we have given to *the derivative is the slope of a line tangent to a curve*, as we noted in Example 1 of Section 23.3. The following example further illustrates this meaning of the derivative.

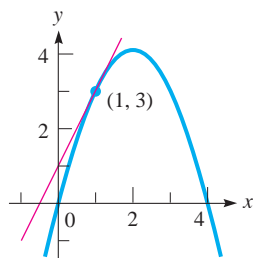


Fig. 23.18

**LEARNING TIP**

The derivative gives the *instantaneous rate of change* of  $f(x)$  with respect to  $x$ .

**EXAMPLE 1** Slope of a tangent line

Find the slope of the line tangent to the curve of  $y = 4x - x^2$  at the point  $(1, 3)$ .

As we have noted, from Sections 23.2 and 23.3, we know that we must first find the derivative and then evaluate it at the given point.

$$\begin{aligned} f(x+h) &= 4(x+h) - (x+h)^2 \\ f(x+h) - f(x) &= 4(x+h) - (x+h)^2 - (4x - x^2) \\ &= 4x + 4h - x^2 - 2xh - h^2 - 4x + x^2 = 4h - 2xh - h^2 \end{aligned}$$

$$\frac{f(x+h) - f(x)}{h} = \frac{4h - 2xh - h^2}{h} = 4 - 2x - h$$

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} (4 - 2x - h) = 4 - 2x$$

$$\frac{dy}{dx} = 4 - 2x$$

$$\left. \frac{dy}{dx} \right|_{(1,3)} = 4 - 2(1) = 2$$

The slope of the tangent line at  $(1, 3)$  is 2. Note that only  $x = 1$  was needed for the evaluation. The curve and tangent line are shown in Fig. 23.18. ■

At the end of Section 23.2, we discussed the idea that the ratio  $(f(x+h) - f(x))/h$  gives the rate of change of  $f(x)$  with respect to  $x$ . In defining the derivative as the limit of this ratio as  $h \rightarrow 0$ , it is a measure of the rate of change of  $f(x)$  with respect to  $x$  at point  $P$ . However,  $P$  may represent any point, which means the value of the derivative changes from one point on a curve to another point.

**EXAMPLE 2** Rate of change of  $f(x)$  for an exact value of  $x$ 

In Examples 1 and 3 of Section 23.2,  $f(x)$  is changing at the rate of 7 units for a 1-unit change in  $x$ , **when  $x$  equals exactly 2**. In Example 4 of Section 23.2,  $f(x)$  is increasing 6 units for a 1-unit change in  $x$ , **when  $x$  equals exactly  $-1$** , and  $f(x)$  is decreasing 2 units for a 1-unit increase in  $x$ , **when  $x$  equals exactly 3**. ■

This gives us a more general meaning of the derivative. If a functional relationship exists between any two variables, then one can be taken to be varying with respect to the other, and the derivative gives us the instantaneous rate of change. There are many applications of this principle, one of which is the velocity of an object. We consider now the case of motion along a straight line, called *rectilinear motion*.

The **average velocity** of an object is found by dividing the change in displacement by the time interval required for this change. As the time interval approaches zero, the limiting value of the average velocity gives the value of the **instantaneous velocity**. Using symbols for the derivative, *the instantaneous velocity of an object moving in rectilinear motion at a specified time  $t$  is given by*

Instantaneous Velocity

$$v = \lim_{h \rightarrow 0} \frac{s(t+h) - s(t)}{h} \quad (23.7)$$

where  $s(t)$  is the displacement as a function of the time  $t$ , and  $h$  is the time interval that approaches zero. In this case, the derivative has units of displacement divided by units of time, and we can denote it as  $ds/dt$ .

**EXAMPLE 3** Instantaneous velocity

Find the instantaneous velocity, when  $t = 4$  s (exactly), of a falling object for which the distance  $s$  (in m) fallen is the displacement given by  $s = 4.90t^2$ , by calculating average velocities between  $t = 3.5$  s, 3.9 s, 3.99 s, 3.999 s, and  $t = 4$  s, and then noting the apparent limiting value as  $h \rightarrow 0$ .

■ See the chapter introduction.

The values of  $h$  are found by subtracting the given times from 4 s. Also, since  $s = 78.40$  m for  $t = 4$  s, the differences in distance are found by subtracting the values of  $s$  for the given times from 78.40 m. Therefore, we have

$t$ (s)	3.5	3.9	3.99	3.999
$s$ (m)	60.025	74.529	78.00849	78.360805
$78.40 - s$ (m)	18.375	3.871	0.3915	0.039195
$h = 4 - t$ (s)	0.5	0.1	0.01	0.001
$v = \frac{78.40 - s}{h}$ (m/s)	36.750	38.710	39.150	39.155

We can see that the value of  $v$  is approaching 39.20 m/s, which is therefore the instantaneous velocity when  $t = 4$  s. ■

**EXAMPLE 4** Instantaneous velocity from the derivative

Find the expression for the instantaneous velocity of the object of Example 3, for which  $s = 4.90t^2$ , where  $s$  is the displacement (in m) and  $t$  is the time (in s). Determine the instantaneous velocity for  $t = 2$  s and  $t = 4$  s.

■ See the chapter introduction.

The required expression is the derivative of  $s$  with respect to  $t$ .

$$\begin{aligned} f(t+h) &= 4.90(t+h)^2 \\ f(t+h) - f(t) &= 4.90(t+h)^2 - 4.90t^2 = 9.80th + 4.90h^2 \\ \frac{f(t+h) - f(t)}{h} &= \frac{9.80th + 4.90h^2}{h} = 9.80t + 4.90h \end{aligned}$$

expression for instantaneous velocity

$$v = \lim_{h \rightarrow 0} \frac{f(t+h) - f(t)}{h} = \lim_{h \rightarrow 0} (9.80t + 4.90h) = 9.80t$$

**Practice Exercise**

1. Find the instantaneous velocity of an object moving such that  $s = 6.00t^2$  for  $t = 3.00$  s. Here,  $s$  is the displacement (in m) and  $t$  is the time (in s).

$$\left. \frac{ds}{dt} \right|_{t=2} = 9.80(2) = 19.6 \text{ m/s} \quad \text{and} \quad \left. \frac{ds}{dt} \right|_{t=4} = 9.80(4) = 39.2 \text{ m/s}$$

We see that the second result agrees with that found in Example 3. ■

**LEARNING TIP**

The derivative can be interpreted as the instantaneous rate of change of the dependent variable with respect to the independent variable. This is true for a differentiable function, no matter what the variables represent.

By finding  $\lim_{h \rightarrow 0} (f(x+h) - f(x))/h$ , we can find the instantaneous rate of change of  $f(x)$  with respect to  $x$ . The expression  $\lim_{h \rightarrow 0} (f(t+h) - f(t))/h$  gives the instantaneous velocity, or instantaneous rate of change of displacement *with respect to time*.

**EXAMPLE 5** Instantaneous rate of change of volume

A spherical balloon is being inflated. Find the expression for the instantaneous rate of change of the volume with respect to the radius. Evaluate this instantaneous rate of change for a radius of 2.00 m.

$$\begin{aligned} V &= \frac{4}{3}\pi r^3 && \text{volume of sphere} \\ f(r+h) &= \frac{4}{3}\pi (r+h)^3 && \text{find derivative} \\ f(r+h) - f(r) &= \frac{4}{3}\pi (r+h)^3 - \frac{4}{3}\pi r^3 \\ f(r+h) - f(r) &= \frac{4}{3}\pi (r^3 + 3r^2h + 3rh^2 + h^3 - r^3) = \frac{4}{3}\pi (3r^2h + 3rh^2 + h^3) \end{aligned}$$

$$\begin{aligned}\frac{f(r+h) - f(r)}{h} &= \frac{4\pi}{3} \left( \frac{3r^2h + 3rh^2 + h^3}{h} \right) = \frac{4\pi}{3} (3r^2 + 3rh + h^2) \\ \frac{dV}{dr} &= \lim_{h \rightarrow 0} \frac{f(r+h) - f(r)}{h} = \lim_{h \rightarrow 0} \left( \frac{4\pi}{3} (3r^2 + 3rh + h^2) \right) = 4\pi r^2 \\ \left. \frac{dV}{dr} \right|_{r=2.00 \text{ m}} &= 4\pi (2.00)^2 = 16.0\pi = 50.3 \text{ m}^2 \quad \text{instantaneous rate of change} \\ & \quad \text{when } r = 2.00 \text{ m}\end{aligned}$$

The instantaneous rate of change of the volume with respect to the radius ( $dV/dr$ ) for  $r = 2.00$  m is  $50.3 \text{ m}^3/\text{m}$  (this way of showing the units is more meaningful).

As  $r$  increases,  $dV/dr$  also increases. This should be expected as the volume of a sphere varies directly as the cube of the radius. ■

### EXAMPLE 6 Instantaneous rate of change of power

The power  $P$  produced by an electric current  $i$  in a resistor varies directly as the square of the current. Given that 1.20 W of power are produced by a current of 0.500 A in a certain resistor, find an expression for the instantaneous rate of change of power with respect to current. Evaluate this rate of change for  $i = 2.50$  A.

We must first find the functional relationship between power and current, by solving the indicated problem in variation:

$$P = ki^2 \quad 1.20 = k(0.500)^2 \quad k = 4.80 \text{ W/A}^2 \quad P = 4.8i^2$$

Now, knowing the function, we may determine the expression for the instantaneous rate of change of  $P$  with respect to  $i$  by finding the derivative.

$$\begin{aligned}f(i+h) &= 4.8(i+h)^2 \\ f(i+h) - f(i) &= 4.8(i+h)^2 - 4.8i^2 = 4.8(2ih + h^2) \\ \frac{f(i+h) - f(i)}{h} &= \frac{4.8(2ih + h^2)}{h} = 4.8(2i + h) \quad \text{expression for instantaneous} \\ & \quad \text{rate of change} \\ \frac{dP}{di} &= \lim_{h \rightarrow 0} \frac{f(i+h) - f(i)}{h} = \lim_{h \rightarrow 0} [4.8(2i + h)] = 9.6i \\ \left. \frac{dP}{di} \right|_{i=2.50 \text{ A}} &= 9.60(2.50) = 24.0 \text{ W/A} \quad \text{instantaneous rate of change} \\ & \quad \text{when } i = 2.50 \text{ A}\end{aligned}$$

This tells us that when  $i = 2.50$  A, the rate of change of power with respect to current is 24.0 W/A. Also, we see that the larger the current is, the greater is the increase in power. This should be expected, since the power varies directly as the square of the current. ■

#### Practice Exercise

2. In Example 6, change 1.20 W of power to 1.60 W of power, and 0.500 A of current to 0.800 A of current, and then find the instantaneous rate of change of power with respect to current for  $i = 3.00$  A.

## EXERCISES 23.4

In Exercises 1 and 2, make the given changes in the indicated examples of this section and then solve the resulting problems.

- In Example 4, change  $4.90t^2$  to  $14.00t - 4.90t^2$  and then evaluate the instantaneous velocity at the indicated times.
- In Example 5, change “volume” in the second line to “surface area” and then evaluate the rate of change for  $r = 2.00$  m.

In Exercises 3–6, find the slope of a line tangent to the curve of the given equation at the given point. Sketch the curve and the tangent line.

- $y = x^2 - 1$ ;  $(2, 3)$
- $y = 2x - x^2$ ;  $(-1, -3)$
- $y = \frac{16}{3x+1}$ ;  $(-3, -2)$
- $y = 3 - \frac{16}{x^2}$ ;  $(2, -1)$

In Exercises 7–10, calculate the instantaneous velocity for the indicated value of the time (in s) of an object for which the displacement (in m) is given by the indicated function. Use the method of Example 3 and calculate values of the average velocity for the given values of  $t$  and note the apparent limit as the time interval approaches zero.

- $s = 4t + 10$  when  $t = 3$ ; use values of  $t$  of 2.0, 2.5, 2.9, 2.99, 2.999
- $s = 6 - 3t$  when  $t = 4$ ; use values of  $t$  of 3.0, 3.5, 3.9, 3.99, 3.999
- $s = 3t^2 - 4t$  when  $t = 2$ ; use values of  $t$  of 1.0, 1.5, 1.9, 1.99, 1.999
- $s = 40t - 4.9t^2$  when  $t = 0.5$ ; use values of  $t$  of 0.4, 0.45, 0.49, 0.499, 0.4999

In Exercises 11–14, use the definition to find an expression for the instantaneous velocity of an object moving with rectilinear motion according to the given functions (the same as those for Exercises 7–10) relating  $s$  (in m) and  $t$  (in s). Then calculate the instantaneous velocity for the given value of  $t$ , rounding answers to 3 significant digits.

11.  $s = 4t + 10$ ;  $t = 3$       12.  $s = 6 - 3t$ ;  $t = 4$   
 13.  $s = 3t^2 - 4t$ ;  $t = 2$       14.  $s = 40t - 4.9t^2$ ;  $t = 0.5$

In Exercises 15–20, use the definition to find an expression for the instantaneous velocity of an object moving with rectilinear motion according to the given functions relating  $s$  and  $t$ .

15.  $s = 48t + 12$     16.  $s = 3t^2 - 2t^3$     17.  $s = 12t^2 - t^3$   
 18.  $s = s_0 + v_0t + \frac{1}{2}at^2$  ( $s_0$ ,  $v_0$ , and  $a$  are constants.)  
 19.  $s = 3t - \frac{6}{5t}$       20.  $s = \frac{2t}{t + 2}$

In Exercises 21–24, use the definition to find an expression for the instantaneous acceleration of an object moving with rectilinear motion according to the given functions. The instantaneous acceleration of an object is defined as the instantaneous rate of change of the velocity with respect to time. Here,  $v$  is the velocity,  $s$  is the displacement, and  $t$  is the time.

21.  $v = 6t^2 - 4t + 2$       22.  $v = \sqrt{6t + 1}$   
 23.  $s = t^3 + 15t$  (Find  $v$ , then find  $a$ .)  
 24.  $s = s_0 + v_0t - \frac{1}{2}at^2$  ( $s_0$ ,  $v_0$ , and  $a$  are constants.)  
 (Find  $v$ , then find  $a$ .)

In Exercises 25–42, find the indicated instantaneous rates of change. Unless specified otherwise, round answers to 3 significant digits.

25. A metal circular ring is being cooled. Find the rate at which the circumference changes if the radius is decreasing at the rate of 0.00150 cm/min.
26. Liquid is poured into a tank with vertical sides and a square horizontal cross section of edge 6.25 cm. Find the instantaneous rate of change of volume with respect to the depth  $h$ .
27. The distance  $s$  (in m) above the ground for a projectile fired vertically upward with a velocity of 44.0 m/s as a function of time  $t$  (in s) is given by  $s = 44.0t - 4.90t^2$ . Find  $t$  for  $v = 0$ .
28. For the projectile in Exercise 27, find  $v$  for  $t = 4.00$  s and for  $t = 5.00$  s. What conclusion can be drawn?
29. The electric current  $i$  at a point in an electric circuit is the instantaneous rate of change of the electric charge  $q$  that passes the point, with respect to the time  $t$ . Find  $i$  in a circuit for which  $q = 30 - 2t$ .
30. A load  $L$  (in N) is distributed along a beam 10.0 m long such that  $L = 5x - 0.5x^2$ , where  $x$  is the distance from one end of the beam. Find the expression for the instantaneous rate of change of  $L$  with respect to  $x$ .
31. A rectangular metal plate contracts while cooling. Find the expression for the instantaneous rate of change of the area  $A$  of the plate with respect to its width  $w$ , if the length of the plate is constantly three times as long as the width.

32. A circular oil spill is increasing in size. Find the instantaneous rate of change of the area  $A$  of the spill with respect to its radius  $r$  for  $r = 245$  m.
33. The total power  $P$  (in W) transmitted by an AM radio station is given by  $P = 500 + 250m^2$ , where  $m$  is the modulation index. Find the instantaneous rate of change of  $P$  with respect to  $m$  for  $m = 0.920$ .
34. The bottom of a soft-drink can is being designed as an inverted spherical segment, the volume of which is  $V = \frac{1}{6}\pi h^3 + 2.00\pi h$ , where  $h$  is the depth (in cm) of the segment. Find the instantaneous rate of change of  $V$  with respect to  $h$  for  $h = 0.600$  cm.
35. The total solar radiation  $H$  (in W/m<sup>2</sup>) on a particular surface during an average clear day is given by  $H = \frac{5000}{t^2 + 10}$ , where  $t$  ( $-6 \leq t \leq 6$ ) is the number of hours from noon (6 A.M. is equivalent to  $t = -6$  h). Find the instantaneous rate of change of  $H$  with respect to  $t$  at 3 P.M.
36. For the solar radiator in Exercise 35, find the average rate of change of  $H$  between 2 P.M. and 4 P.M. Compare with the instantaneous rate of change at 3 P.M.
37. The value (in thousands of dollars) of a certain car is given by the function  $V = \frac{48}{t + 3}$ , where  $t$  is measured in years. Find a general expression for the instantaneous rate of change of  $V$  with respect to  $t$  and evaluate this expression when  $t = 3$  years.
38. For the car in Exercise 37, find the average rate of change of  $V$  between  $t = 2$  years and  $t = 4$  years. Compare with the instantaneous rate of change for  $t = 3$  years.

39. Oil in a certain machine is stored in a conical reservoir, for which the radius and height are both 4.00 cm (see Fig. 23.19). Find the instantaneous rate of change of the volume  $V$  of oil in the reservoir with respect to the depth  $d$  of the oil.

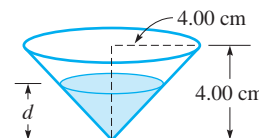


Fig. 23.19

40. The time  $t$  required to test a computer memory unit is directly proportional to the square of the number  $n$  of memory cells in the unit. For a particular type of unit,  $n = 6400$  for  $t = 25.0$  s. Find the instantaneous rate of change of  $t$  with respect to  $n$  for this type of unit for  $n = 8000$ .
41. A *holograph* (an image formed without using a lens) of concentric circles is formed. The radius  $r$  of each circle varies directly as the square root of the wavelength  $\lambda$  of the light used. If  $r = 3.72$  cm for  $\lambda = 592$  nm, find the expression for the instantaneous rate of change of  $r$  with respect to  $\lambda$ .
42. The force  $F$  between two electric charges varies inversely as the square of the distance  $r$  between them. For two charged particles,  $F = 0.120$  N for  $r = 0.0600$  m. Find the instantaneous rate of change of  $F$  with respect to  $r$  for  $r = 0.120$  m.

#### Answers to Practice Exercises

1. 36.0 m/s    2. 15.0 W/A

## 23.5 Derivatives of Polynomials

Derivative of a Constant • Derivative of a Power of  $x$  • Derivative of a Constant Times a Function of  $x$  • Derivative of a Sum of Functions of  $x$

Previously, we have been able to compute derivatives of functions using either a table of values or the delta method (i.e., definition of the derivative) approach. With the table of values technique, the construction of the table can be tedious. Moreover, it is limited, in the sense that it applies to a region around only one point of the function, and it requires an estimation of an emerging pattern of behaviour from the values. In contrast, the derivative resulting from the delta method applies to *any* point on the curve, but the technique is very time-consuming. Nevertheless, if we apply the method to general forms of functions, we can obtain *derivative formulas* that are valid at any point where the function is differentiable and take much less time to compute. In this section, we derive the formulas that are used for finding the derivatives of polynomial functions of the form  $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$ .

First, we find the derivative of a constant. Letting  $f(x) = c$  and then using the definition of a derivative, we find that  $f(x+h) - f(x) = c - c = 0$ . This means that

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \left( \frac{0}{h} \right) = 0$$

From this, we conclude that *the derivative of a constant is zero*. This result holds for all constants. Therefore, if  $y = c$ , then  $dy/dx = 0$ , or

$$\frac{dc}{dx} = 0 \quad (23.8)$$

Derivative of a Constant

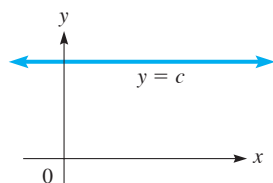


Fig. 23.20

Graphically, this means that for any function of the type  $y = c$ , the slope is always zero. We know that  $y = c$  represents a straight line parallel to the  $x$ -axis. From the definition of slope, we know that any line parallel to the  $x$ -axis has a slope of zero. See Fig. 23.20. We see that the two results are consistent.

Next, we find the derivative of a positive integral power of  $x$ . If  $f(x) = x^n$ , where  $n$  is a positive integer, by using the binomial theorem we have

$$\begin{aligned} f(x+h) - f(x) &= (x+h)^n - x^n = x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + h^n - x^n \\ f(x+h) - f(x) &= nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + h^n \\ \frac{f(x+h) - f(x)}{h} &= nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \cdots + h^{n-1} \\ \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} &= nx^{n-1} \end{aligned}$$

Thus, *the derivative of the  $n$ th power of  $x$  is*

$$\frac{d(x^n)}{dx} = nx^{n-1} \quad (23.9)$$

Derivative of a Power of  $x$

### LEARNING TIP

To find the derivative of a power of  $x$ , we multiply the exponent of  $x$  by the numerical coefficient and then decrease the exponent by 1.

The derivative of any function that is a power of  $x$  can be found with this formula, known as the power rule. The power rule is also valid for non-integer powers.

### EXAMPLE 1 Derivative of a constant

Find the derivative of the function  $y = -5$ .

Since  $-5$  is a constant, applying Eq. (23.8), we have

$$\frac{dy}{dx} = \frac{d(-5)}{dx} = 0$$

■



**EXAMPLE 2** Derivative of  $y = x$ 

Find the derivative of the function  $y = x$ .

In using Eq. (23.9), we have  $n = 1$  since  $x = x^1$ . This means

$$\frac{dy}{dx} = \frac{d(x)}{dx} = (1)x^{1-1} = (1)(x^0)$$

Since  $x^0 = 1$ , we have

$$\frac{dy}{dx} = 1$$

Thus, the derivative of  $y = x$  is 1, which means that the slope of the line  $y = x$  is always 1. This is consistent with our previous discussion of the slope of a straight line. ■

**EXAMPLE 3** Derivative of a power of  $r$ 

Find the derivative of the function  $v = r^{10}$ .

Here, the dependent variable is  $v$ , and the independent variable is  $r$ . Therefore,

$$\begin{aligned}\frac{dv}{dr} &= \frac{d(r^{10})}{dr} = 10r^{10-1} \\ &= 10r^9\end{aligned}$$

**LEARNING TIP**

From here on, we will be using functions that are combinations of simpler functions. Therefore, we must denote some functions by symbols other than  $f(x)$ . We will often be using  $u$  and  $v$ .

Next, we find the derivative of a constant times a function of  $x$ . We denote this function as  $u$ , or to show directly that it is a function of  $x$ , as  $u(x)$ . In finding the derivative of  $c \cdot u$  with respect to  $x$ , we have

$$\begin{aligned}\frac{d}{dx}(c \cdot u) &= \lim_{h \rightarrow 0} \frac{c \cdot u(x+h) - c \cdot u(x)}{h} \\ \frac{d(c \cdot u)}{dx} &= c \cdot \lim_{h \rightarrow 0} \frac{u(x+h) - u(x)}{h} = c \cdot \frac{du}{dx}\end{aligned}$$

Therefore, *the derivative of the product of a constant and a differentiable function of  $x$  is the product of the constant and the derivative of the function of  $x$ .* This is written as

**Derivative of a Constant  
Times a Function**

$$\frac{d(c \cdot u)}{dx} = c \cdot \frac{du}{dx} \quad (23.10)$$

**EXAMPLE 4** Derivative of a constant times a power of  $x$ 

Find the derivative of  $y = 3x^2$ .

In this case,  $c = 3$  and  $u = x^2$ . Thus,  $du/dx = 2x$ . Therefore,

$$\begin{aligned}\frac{dy}{dx} &= \frac{d(3x^2)}{dx} = 3 \cdot \frac{d(x^2)}{dx} = 3(2x) \\ \frac{dy}{dx} &= 6x\end{aligned}$$

**Practice Exercise**

- Find the derivative of  $y = 6x^3$ .

**COMMON ERROR**

Occasionally, the derivative of a constant times a function of  $x$  is confused with the derivative of a constant that stands alone. It is necessary to distinguish clearly between a constant that multiplies a function and an isolated constant.

If the types of functions for which we have found derivatives are added, the result is a polynomial function with more than one term. The derivative is found by letting  $y = u + v$ , where  $u$  and  $v$  are functions of  $x$ . Using the definition, we have

$$\begin{aligned} y &= u(x) + v(x) \\ \frac{dy}{dx} &= \frac{d}{dx}[u(x) + v(x)] = \lim_{h \rightarrow 0} \frac{[u(x+h) + v(x+h)] - [u(x) + v(x)]}{h} \\ \frac{dy}{dx} &= \lim_{h \rightarrow 0} \left[ \frac{u(x+h) - u(x)}{h} + \frac{v(x+h) - v(x)}{h} \right] \\ \frac{dy}{dx} &= \lim_{h \rightarrow 0} \left[ \frac{u(x+h) - u(x)}{h} \right] + \lim_{h \rightarrow 0} \left[ \frac{v(x+h) - v(x)}{h} \right] = \frac{du}{dx} + \frac{dv}{dx} \end{aligned}$$

This tells us that *the derivative of the sum of differentiable functions of  $x$  is the sum of the derivatives of the functions*. This is written as

### Derivative of a Sum

$$\frac{d(u + v)}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad (23.11)$$

The combination of the four rules obtained above allows us to take derivatives of any polynomial of the form  $f(x) = a_0x^n + a_1x^{n-1} + \dots + a_n$ .

### EXAMPLE 5 Evaluation of a derivative

Evaluate the derivative of  $f(x) = 2x^4 - 6x^2 - 8x - 9$  at  $(-2, 15)$ .

First, finding the derivative, we have

$$\begin{aligned} f'(x) &= \frac{d(2x^4)}{dx} - \frac{d(6x^2)}{dx} - \frac{d(8x)}{dx} - \frac{d(9)}{dx} \\ f'(x) &= 8x^3 - 12x - 8 \end{aligned}$$

We now evaluate this derivative for  $x = -2$ .

$$f'(x) \Big|_{x=-2} = 8(-2)^3 - 12(-2) - 8 = -48$$

### EXAMPLE 6 Slope of a tangent line

Find the slope of the line tangent to the curve of  $y = 4x^7 - x^4$  at the point  $(1, 3)$ .

We must find and then evaluate the derivative for the value  $x = 1$ :

$$\begin{aligned} \frac{dy}{dx} &= 28x^6 - 4x^3 && \text{find derivative} \\ \frac{dy}{dx} \Big|_{x=1} &= 28(1) - 4(1) = 24 && \text{evaluate derivative} \end{aligned}$$

Thus, the slope of the tangent line is 24. Again, we note that *the substitution  $x = 1$  must be made after the differentiation has been performed*. The curve and the tangent line at  $(1, 3)$  are shown in Fig. 23.21. Note that a slope of 24 results in a steep tangent line.

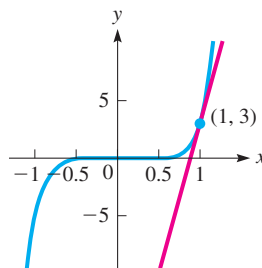


Fig. 23.21

## Practice Exercise

2. Find the expression for the instantaneous velocity if  $s = 7t^4 - 4t + 5$ , where  $s$  is the displacement and  $t$  is the time.

## LEARNING TIP

Method for all general derivatives:

1. Identify the form.
2. Use the appropriate derivative formula to find the derivative.
3. Simplify algebraically and factor the solution.

## EXAMPLE 7 Evaluation of a derivative—application

For each 4.00-s cycle, the displacement  $s$  (in cm) of a piston is given by  $s = t^3 - 6t^2 + 8t$ , where  $t$  is the time. Find the instantaneous velocity of the piston for  $t = 2.60$  s.

$$s = t^3 - 6t^2 + 8t$$

$$\frac{ds}{dt} = 3t^2 - 12t + 8 \quad \text{find derivative}$$

$$\left. \frac{ds}{dt} \right|_{t=2.6} = 3(2.60)^2 - 12(2.60) + 8 \quad \text{evaluate derivative}$$

$$= -2.92 \text{ cm/s}$$

The piston is moving at  $-2.92$  cm/s (it is moving in a negative direction) when  $t = 2.60$  s. ■

## EXERCISES 23.5

In Exercises 1–4, make the given changes in the indicated examples of this section, and then solve the resulting problem.

1. In Example 3, change the exponent 10 to 9.
2. In Example 4, change the coefficient 3 to 4.
3. In Example 5, change  $2x^4$  to  $2x^3$  and  $-8x$  to  $+8x$ .
4. In Example 6, change  $4x^7 - x^4$  to  $4x^4 - x^7$  and evaluate the slope.

In Exercises 5–20, find the derivative of each of the given functions.

5.  $y = x^5$
6.  $y = x^{12}$
7.  $f(x) = -4x^9$
8.  $y = -7x^6$
9.  $y = 5x^4 - 3\pi$
10.  $s = 3t^5 + 4t - \frac{8}{t^2}$
11.  $y = x^2 + 2x$
12.  $y = x^3 - \frac{3}{2}x^2$
13.  $p = 5r^3 - 2r + 1$
14.  $y = 6x^2 - 6x + 5$
15.  $y = 25x^8 - 34x^5 - \sqrt{x}$
16.  $u = 4v^4 - 12v + 9$
17.  $f(x) = -6x^7 + 5x^3 + \pi^2$
18.  $y = 13x^4 - 6x^3 - x - x^2$
19.  $y = \frac{1}{3}x^3 + \frac{1}{2}x^2 - \frac{1}{x}$
20.  $f(z) = -\frac{1}{4}z^8 + \frac{1}{2}z^4 - 2^3$

In Exercises 21–24, evaluate the derivative of each of the given functions at the given point.

21.  $y = 6x^2 - 8x + 1$ ;  $(2, 9)$
22.  $s = 2t^3 - 5t^2$ ;  $(-1, -7)$
23.  $y = 2x^3 + 9x - 7$ ;  $(-2, -41)$
24.  $y = x^4 - 9x^2 - 5x$ ;  $(3, -15)$

In Exercises 25–28, find the slope of a line tangent to the curve of each of the given functions for the given values of  $x$ .

25.  $y = 2x^6 - 4x^2$  ( $x = -1$ )
26.  $y = 3x^3 - \frac{9}{x}$  ( $x = 1$ )
27.  $y = 35x - 2x^4$  ( $x = 2$ )
28.  $y = x^4 - \frac{1}{2}x^2 + 2$  ( $x = -2$ )

In Exercises 29–32, determine an expression for the instantaneous velocity of objects moving with rectilinear motion according to the functions given, if  $s$  represents displacement in terms of time  $t$ .

29.  $s = 6t^5 - 5t + 2$
30.  $s = 20 + 60t - 4.9t^2$

31.  $s = 2 - 6t - 2t^3$
32.  $s = s_0 + v_0t + \frac{1}{2}at^2$

In Exercises 33–36,  $s$  represents the displacement and  $t$  represents the time for objects moving with rectilinear motion, according to the given functions. Find the instantaneous velocity for the given times, rounded to 3 significant digits.

33.  $s = 2t^3 - 4t^2$ ;  $t = 4$
34.  $s = 8t^2 - 10t + 6$ ;  $t = 5$
35.  $s = 120 + 80t - 16t^2$ ;  $t = 2.5$
36.  $s = 0.5t^4 - 1.5t^2 + 2.5$ ;  $t = 3$

In Exercises 37–57, solve the given problems by finding the appropriate derivative.

37. For what value(s) of  $x$  is the tangent to the curve of  $y = 3x^2 - 6x$  parallel to the  $x$ -axis? (That is, where is the slope zero?)
38. Find the value of  $a$  if the tangent to the curve of  $y = ax^2 + 2x$  has a slope of  $-4$  for  $x = 2$ .
39. For what point(s) on the curve of  $y = 3x^2 - 4x$  is the slope of a tangent line equal to 8?
40. Explain why the curve  $y = 5x^3 + 4x - 3$  does not have a tangent line with a slope less than 4.
41. Find the point at which a tangent line to the parabola  $y = 2x^2 - 7x$  is perpendicular to the line  $x - 3y = 16$ .
42. Display the graphs of  $y = x^2$  and its derivative on a graphing calculator. State any conclusions you can draw from the relationship of the two graphs.
43. For what value(s) of  $x$  is the slope of a line tangent to the curve of  $y = 4x^2 + 3x$  equal to the slope of a line tangent to the curve of  $y = 5 - 2x^2$ ?
44. For what value(s) of  $t$  is the instantaneous velocity of an object moving according to  $s = 5t - 2t^2$  equal to the instantaneous velocity of an object moving according to  $s = 3t^2 + 4$ ?
45. A metal cylinder is heated and then cools. If the radius always equals the height, find the expression for the instantaneous rate of change of the volume  $V$  with respect to the radius  $r$ .
46. A rectangular solid block of ice is melting such that the height is always twice the edge of the square base. Find the expression for the instantaneous rate of change of surface area  $A$  with respect to the edge of the base  $s$ .

47. The electric power  $P$  (in W) as a function of the current  $i$  (in A) in a certain circuit is given by  $P = 16i^2 + 60i$ . Find the instantaneous rate of change of  $P$  with respect to  $i$  for  $i = 0.750$  A.
48. The torque  $T$  on the arm of a robotic control mechanism varies directly as the cube of the diameter  $d$  of the arm. If  $T = 845 \text{ N} \cdot \text{m}$  for  $d = 0.925$  cm, find the expression for the instantaneous rate of change of  $T$  with respect to  $d$ .
49. The resistance  $R$  (in  $\Omega$ ) of a certain wire as a function of the temperature  $T$  (in  $^\circ\text{C}$ ) is given by  $R = 16.0 + 0.450T + 0.0125T^2$ . Find the instantaneous rate of change of  $R$  with respect to  $T$  when  $T = 115^\circ\text{C}$ .
50. The deflection  $d$  of a diving board  $x$  m from the fixed end at the pool side is given by  $d = kx^2(3L - x)$ , where  $L$  is the length of the diving board and  $k$  is a positive constant. Find the expression for the instantaneous rate of change of  $d$  with respect to  $x$ .
51. The tensile strength  $S$  (in N) of a certain material as a function of the temperature  $T$  (in  $^\circ\text{C}$ ) is  $S = 1600 - 0.000\,022T^2$ . Find the instantaneous rate of change of  $S$  with respect to  $T$  for  $T = 65.0^\circ\text{C}$ .
52. A tank containing 6000 L of water drains out in 30.0 min. The volume  $V$  of water in the tank after  $t$  min of draining is  $V = 6000(1 - t/30)^2$ . Find the instantaneous time rate of change of  $V$  after 15.0 min of draining.
53. The altitude  $h$  (in m) of a jet as a function of the horizontal distance  $x$  (in km) it has travelled is given by  $h = 0.000\,104x^4 - 0.0417x^3 + 4.21x^2 - 8.33x$ . Find the instantaneous rate of change of  $h$  with respect to  $x$  for  $x = 125$  km.

54. The force  $F$  (in N) exerted by a cam on a lever is given by  $F = x^4 - 12x^3 + 46x^2 - 60x + 25$ , where  $x$  ( $1 \leq x \leq 5$ ) is the distance (in cm) from the centre of rotation of the cam to the edge of the cam in contact with the lever (see Fig. 23.22). Find the instantaneous rate of change of  $F$  with respect to  $x$  when  $x = 4.00$  cm.

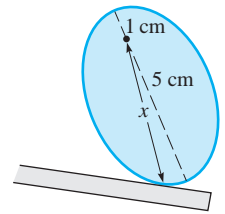


Fig. 23.22

55. Two ball bearings wear down such that the radius  $r$  of one is constantly 1.20 mm less than the radius of the other. Find the instantaneous rate of change of the total volume  $V_T$  of the two ball bearings with respect to  $r$  for  $r = 3.30$  mm.
56. An open-top container is to be made from a rectangular piece of cardboard 6.00 cm by 8.00 cm. Equal squares of side  $x$  are to be cut from each corner, then the sides are to be bent up and taped together. Find the instantaneous rate of change of the volume  $V$  of the container with respect to  $x$  for  $x = 1.75$  cm.
57. The head loss  $h_L$  (in m) for a fluid travelling in a pipe at average velocity  $v$  (in m/s) is given by  $h_L = \frac{Kv^2}{2g}$ , where  $K$  is a friction loss coefficient (constant), and  $g = 9.81 \text{ m/s}^2$ . Determine the rate of change of head loss with respect to velocity when the velocity is 3.25 m/s and  $K = 0.500$ .

#### Answers to Practice Exercises

1.  $dy/dx = 18x^2$     2.  $v = 28t^3 - 4$

## 23.6 Derivatives of Products and Quotients of Functions

### Products and Quotients of Functions • Derivative of a Product • Derivative of a Quotient

In the previous section, we considered polynomial functions. For functions that are not polynomials, some are products of simpler functions, some are quotients of simpler functions, and others are powers of simpler functions. In this section, we develop the formula for the derivative of a product of functions and the formula for the quotient of functions.

#### EXAMPLE 1 Product, quotient, and powers of functions

The functions  $f(x) = x^2 + 2$  and  $g(x) = 3 - 2x$  can be combined to form functions of the type mentioned above, as we now illustrate:

$$p(x) = f(x) \cdot g(x) = (x^2 + 2)(3 - 2x) \quad \text{product of functions}$$

$$q(x) = \frac{g(x)}{f(x)} = \frac{3 - 2x}{x^2 + 2} \quad \text{quotient of functions}$$

$$F(x) = [g(x)]^2 = (3 - 2x)^2 \quad \text{power of a function} \quad \blacksquare$$

The first two forms will be discussed in this section and the final form will be discussed in Section 23.7.

If  $u$  and  $v$  are differentiable functions of  $x$ , the derivative of the function  $u \cdot v$  is found by letting  $y = u \cdot v$ , and applying the definition as follows:

$$y = u(x) \cdot v(x)$$

$$\frac{dy}{dx} = \frac{d}{dx}[u(x) \cdot v(x)] = \lim_{h \rightarrow 0} \frac{u(x+h) \cdot v(x+h) - u(x) \cdot v(x)}{h}$$

By adding and subtracting  $u(x+h) \cdot v(x)$  in the numerator of the last fraction, we can put it in a form that includes the derivatives of  $u(x)$  and  $v(x)$ . Therefore,

$$\begin{aligned} \frac{d}{dx}[u(x) \cdot v(x)] &= \lim_{h \rightarrow 0} \frac{u(x+h) \cdot v(x+h) - u(x+h) \cdot v(x) + u(x+h) \cdot v(x) - u(x) \cdot v(x)}{h} \\ \frac{d(u \cdot v)}{dx} &= \lim_{h \rightarrow 0} \left[ u(x+h) \frac{v(x+h) - v(x)}{h} + v(x) \frac{u(x+h) - u(x)}{h} \right] = u(x) \cdot \frac{dv(x)}{dx} + v(x) \cdot \frac{du(x)}{dx} \end{aligned}$$

We conclude that *the derivative of the product of two differentiable functions equals the first function times the derivative of the second function plus the second function times the derivative of the first function*. This is written as

### Derivative of a Product

$$\frac{d(u \cdot v)}{dx} = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx} \quad (23.12)$$

### EXAMPLE 2 Derivative of a product of functions

Find the derivative of the product function in Example 1.

$$\begin{aligned} p(x) &= (x^2 + 2)(3 - 2x) & u &= x^2 + 2 & v &= 3 - 2x \\ \frac{d(u \cdot v)}{dx} &= u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx} \\ & \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\ p'(x) &= (x^2 + 2) \cdot (-2) + (3 - 2x) \cdot (2x) = -2x^2 - 4 + 6x - 4x^2 \\ p'(x) &= -6x^2 + 6x - 4 \end{aligned}$$

We could have multiplied the functions first, and then taken the derivative as a polynomial. ■

### EXAMPLE 3 Derivative of a product of functions with variable $r$

Find the derivative of  $f(r) = 8r^3(2r^2 - 4r)$  using the product rule.

Recognize that the function  $f(r) = 8r^3(2r^2 - 4r)$  is of the form  $y = u \cdot v$  with  $u = 8r^3$  and  $v = 2r^2 - 4r$ . Therefore,

$$\begin{aligned} \frac{d(u \cdot v)}{dr} &= u \cdot \frac{dv}{dr} + v \cdot \frac{du}{dr} \quad \text{product rule} \\ \frac{df}{dr} &= 8r^3 \cdot (4r - 4) + (2r^2 - 4r) \cdot (24r^2) \\ \frac{df}{dr} &= 32r^4 - 32r^3 + 48r^4 - 96r^3 \\ \frac{df}{dr} &= 80r^4 - 128r^3 \\ \frac{df}{dr} &= 16r^3(5r - 8) \end{aligned}$$

We will now find the derivative of the quotient of two differentiable functions by applying the definition to the function  $y = u/v$ , as shown below.

$$\frac{dy}{dx} = \frac{d\left(\frac{u(x)}{v(x)}\right)}{dx} = \lim_{h \rightarrow 0} \frac{\frac{u(x+h)}{v(x+h)} - \frac{u(x)}{v(x)}}{h} = \lim_{h \rightarrow 0} \frac{v(x) \cdot u(x+h) - u(x) \cdot v(x+h)}{h \cdot v(x+h) \cdot v(x)}$$

### Practice Exercise

- Find the derivative of  $y = (3 - 2x^2)(x^4 - 1)$ . Do not multiply factors together before finding the derivative.

### LEARNING TIP

When the powers over the functions are larger than 1, it may be very tedious to multiply out and expand the functions first. The product rule will be very useful in combination with another rule, the chain rule, which discusses finding derivatives of powers of functions. This concept is covered in Section 23.7.

We can put the last fraction in a form that includes the derivatives of  $u(x)$  and  $v(x)$  by subtracting and adding  $u(x) \cdot v(x)$  in the numerator. Therefore,

$$\begin{aligned}\frac{d}{dx}\left[\frac{u(x)}{v(x)}\right] &= \lim_{h \rightarrow 0} \frac{v(x) \cdot u(x+h) - v(x+h) \cdot u(x) + v(x) \cdot u(x) - u(x) \cdot v(x+h)}{h \cdot v(x+h) \cdot v(x)} \\ \frac{d}{dx}(u/v) &= \lim_{h \rightarrow 0} \frac{v(x) \cdot \frac{u(x+h) - u(x)}{h} - u(x) \cdot \frac{v(x+h) - v(x)}{h}}{v(x+h) \cdot v(x)} \\ \frac{d}{dx}(u/v) &= \frac{v(x) \cdot \frac{du(x)}{dx} - u(x) \cdot \frac{dv(x)}{dx}}{[v(x)]^2}\end{aligned}$$

Therefore, *the derivative of the quotient of two differentiable functions equals the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.*

### Derivative of a Quotient

$$\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v \cdot \frac{du}{dx} - u \cdot \frac{dv}{dx}}{v^2} \quad (23.13)$$

#### EXAMPLE 4 Derivative of a quotient of functions

Find the derivative of the quotient function in Example 1.

$$\begin{aligned}q(x) &= \frac{3 - 2x}{x^2 + 2} & u &= 3 - 2x & v &= x^2 + 2 \\ & & v \cdot \frac{du}{dx} & - & u \cdot \frac{dv}{dx} & \\ & & \downarrow & & \downarrow & \\ q'(x) &= \frac{(x^2 + 2) \cdot (-2) - (3 - 2x) \cdot (2x)}{(x^2 + 2)^2} = \frac{-2x^2 - 4 - 6x + 4x^2}{(x^2 + 2)^2} \\ q'(x) &= \frac{2(x^2 - 3x - 2)}{(x^2 + 2)^2}\end{aligned}$$

#### COMMON ERROR

In the first expression for  $q'(x)$ , be careful not to cancel the factor of  $(x^2 + 2)$ , as it is not a factor of both terms of the numerator.

#### Practice Exercise

2. Find the derivative of  $y = \frac{2x^2}{x^4 - 1}$ .

#### LEARNING TIP

Note that when taking derivatives of quotients, it is sometimes easier to rewrite the problem as a product, with the denominator changing to have a negative power when moved to the numerator. In other words,  $y = \frac{u}{v}$  can be written in product form as  $y = u \cdot v^{-1}$ .

#### EXAMPLE 5 Derivative of a quotient—application

The stress  $S$  on a hollow tube is given by

$$S = \frac{16DT}{\pi(D^4 - d^4)}$$

where  $T$  is the tension,  $D$  is the outer diameter, and  $d$  is the inner diameter of the tube. Find the expression for the instantaneous rate of change of  $S$  with respect to  $D$ , with the other values being constant.

We are to find the derivative of  $S$  with respect to  $D$ , and it is found as follows:

$$\begin{aligned}\frac{dS}{dD} &= \frac{\pi(D^4 - d^4) \cdot (16T) - 16DT \cdot (\pi)(4D^3)}{\pi^2(D^4 - d^4)^2} = \frac{16\pi T(D^4 - d^4 - 4D^4)}{\pi^2(D^4 - d^4)^2} \\ \frac{dS}{dD} &= \frac{-16T(3D^4 + d^4)}{\pi(D^4 - d^4)^2}\end{aligned}$$

**EXAMPLE 6** Evaluation of a derivative

Evaluate the derivative of  $y = \frac{3x^2 + x}{1 - 4x}$  at  $(2, -2)$ .

$$\begin{aligned} \frac{dy}{dx} &= \frac{(1 - 4x)(6x + 1) - (3x^2 + x)(-4)}{(1 - 4x)^2} \\ &= \frac{6x + 1 - 24x^2 - 4x + 12x^2 + 4x}{(1 - 4x)^2} \\ &= \frac{-12x^2 + 6x + 1}{(1 - 4x)^2} \\ \left. \frac{dy}{dx} \right|_{x=2} &= \frac{-12(2^2) + 6(2) + 1}{[1 - 4(2)]^2} = \frac{-35}{49} \\ &= -\frac{5}{7} \end{aligned}$$

**EXERCISES 23.6**

In Exercises 1 and 2, make the given changes in the indicated examples of this section and then solve the resulting problems.

- In Example 2, change  $u$  to  $5 - 3x^2$ .
- In Example 6, change the numerator to  $3x^2 - x$  and then find and evaluate the derivative at  $x = 2$ .

In Exercises 3–8, find the derivative of each function by using Eq. (23.12). Do not find the product before finding the derivative.

- $y = 6x(3x^2 - 5x)$
- $y = 2x^3(3x^4 + x)$
- $s = (3t + 2)(2t - 5)$
- $f(x) = (3x - 2)(4x^2 + 3)$
- $y = (x^4 - 3x^2 + 3)(1 - 2x^3)$
- $y = (x^3 - 6x)(2 - 4x^3)$

In Exercises 9–12, find the derivative of each function by using Eq. (23.12). Then multiply out each function and find the derivative by treating it as a polynomial. Compare the results.

- $y = (2x - 7)(5 - 2x)$
- $f(s) = (5s^2 + 2)(2s^2 - 1)$
- $V = (h^3 - 1)(2h^2 - h - 1)$
- $y = (3x^2 - 4x + 1)(5 - 6x^2)$

In Exercises 13–24, find the derivative of each function by using Eq. (23.13).

- $y = \frac{x}{2x + 3}$
- $u = \frac{4}{v^2}$
- $y = \frac{\pi}{2x^2 + 1}$
- $R = \frac{5i + 2}{2i + 3}$
- $y = \frac{6x^2}{3 - 2x}$
- $y = \frac{e^2}{3x^2 - 5x}$
- $y = \frac{2x - 1}{3x^2 + 2}$
- $P = \frac{2i^2}{4 - 3i}$
- $f(x) = \frac{3x + 8}{x^2 + 4x + 2}$
- $y = \frac{33x}{4x^5 - 3x - 4}$
- $y = \frac{2x^2 - x - 1}{x^3 + 2x^2}$
- $y = \frac{3x^3 - x}{2x^2 - 5x + 4}$

In Exercises 25–32, evaluate the derivatives of the given functions for the given values of  $x$ .

- $y = (3x - 1)(4 - 7x), x = 3$
- $y = (3x^2 - 5)(2x^2 - 1), x = -1$
- $y = (2x^2 - x + 1)(4 - 2x - x^2), x = -3$
- $y = (4x^4 + 0.5x^2 + 1)(3x - 2x^2), x = 0.5$
- $y = \frac{3x - 5}{2x + 3}, x = -2$
- $y = \frac{2x^2 - 5x}{3x + 2}, x = 2$
- $S = \frac{2n^3 - 3n + 8}{2n - 3n^4}, n = -1$
- $y = \frac{2x^3 - x^2 - 2}{4x + 3}, x = 0.5$

In Exercises 33–57, solve the given problems by finding the appropriate derivatives, and round answers to 3 significant digits unless otherwise specified.

- What text equation from Section 23.5 is equivalent to the product rule if one of the functions  $u$  and  $v$  is a constant?
- By use of the quotient rule, derive a formula for the derivative of the function  $1/v(x)$ .
- Using the product rule, find the point(s) on the curve of  $y = (2x^2 - 1)(1 - 4x)$  for which the tangent line is  $y = 4x - 1$ .
- Do the curves of  $y = x^2$  and  $y = 1/x^2$  cross at right angles? Explain.
- If  $f(x)$  is a differentiable function, find an expression for the derivative of  $y = x^2 \cdot f(x)$ .
- If  $f(x)$  is a differentiable function, find an expression for the derivative of  $y = f(x)/x^2$ .
- Find the derivative of  $y = \frac{x^2(1 - 2x)}{3x - 7}$  in each of the following two ways. (1) Do not multiply out the numerator before finding the derivative. (2) Multiply out the numerator before finding the derivative. Compare the results.



40. Find the derivative of  $y = 4x^2 - \frac{1}{x-1}$  in each of the following two ways. (1) Do not combine the terms over a common denominator before finding the derivative. (2) Combine the terms over a common denominator before finding the derivative. Compare the results.
41. Find the slope of a line tangent to the curve of the function  $y = (4x + 1)(x^4 - 1)$  at the point  $(-1, 0)$ . Do not multiply the factors together before taking the derivative.
42. Find the slope of a line tangent to the curve of the function  $y = (3x + 4)(1 - 4x)$  at the point  $(2, -70)$ . Do not multiply the factors together before taking the derivative.
43. For what value(s) of  $x$  is the slope of a tangent to the curve of  $y = \frac{x}{x^2 + 1}$  equal to zero? View the graph on a graphing calculator to verify the values found.
44. Determine the sign of the derivative of the function  $y = \frac{2x - 1}{1 - x^2}$  for the following values of  $x$ :  $-2, -1, 0, 1, 2$ . Is the slope of a tangent line to this curve ever negative? View the graph on a graphing calculator to verify your conclusion.
45. The power  $P$  (in W) in an electric circuit is the product of the voltage  $V$  and the current  $I$  (in A). If  $V$  and  $I$  vary with time  $t$  (in s) ( $0 \leq t \leq 0.2$  s), according to  $V = 0.0480 - 1.20t^2$  and  $I = 2.00 - 0.800t$ , find  $dP/dt$  when  $t = 0.150$  s.
46. The thermodynamic temperature  $T$  (in K) varies jointly as the pressure  $p$  (in Pa) and volume  $V$  (in  $m^3$ ). Find the expression for  $dT/dt$  if  $p = 1200(1 - 0.0025t)$  and  $V = 2.50(1 + 0.0048t^2)$ , where  $t$  is the time (in s).

47. If a constant current of 2 A passes through the *current divider* parallel resistors shown in Fig. 23.23, the current  $i$  is given by  $i = 8R/(7R + 12)$ , where  $R$  is a variable resistor. Find  $di/dR$ .

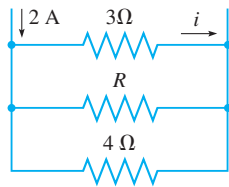


Fig. 23.23

48. The sales  $S$  of a product as a function of the time  $t$  (in weeks) is given by  $S = \frac{2300(5t + 1)}{2t + 5}$ . Find the instantaneous rate of change of  $S$  with respect to  $t$  for  $t = 9$  weeks.
49. During each cycle, the vertical displacement  $s$  of the end of a robot arm is given by  $s = (t^2 - 8t)(2t^2 + t + 1)$ , where  $t$  is the time. Find the expression for the instantaneous velocity of the end of the robot arm. See Fig. 23.24.

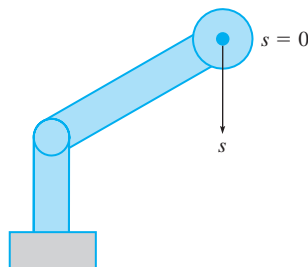


Fig. 23.24

50. The concentration  $c$  (in mg/L) of a certain drug in the bloodstream is found to be  $c = 25t/(t^2 + 5)$ , where  $t$  is the time (in h) after the drug is taken. Find  $dc/dt$ .
51. A computer, using data from a refrigeration plant, estimated that in the event of a power failure the temperature  $T$  (in  $^{\circ}\text{C}$ ) in the freezers would be given by  $T = \frac{2t}{0.05t + 1} - 20$ , where  $t$  is the number of hours after the power failure. Find the time rate of change of temperature after 6.00 h.
52. The voltage  $V$  across a resistor in an electric circuit is the product of the resistance and the current. If the current  $I$  (in A) varies with time  $t$  (in s) according to the relation  $I = 5.00 + 0.01t^2$  and the resistance varies with time according to the relation  $R = 15.00 - 0.10t$ , find the time rate of change of the voltage when  $t = 5.00$  s.
53. The frictional radius  $r_f$  of a disc clutch is given by the equation  $r_f = \frac{2(R^2 + Rr + r^2)}{3(R + r)}$ , where  $R$  and  $r$  are the outer radius and the inner radius of the clutch, respectively. Find the derivative of  $r_f$  with respect to  $R$  with  $r$  constant.
54. In thermodynamics, an equation relating the thermodynamic temperature  $T$ , the pressure  $p$ , and the volume  $V$  of a gas is  $T = \left(p + \frac{a}{V^2}\right)\left(\frac{V - b}{R}\right)$ , where  $a$ ,  $b$ , and  $R$  are constants. Find the derivative of  $T$  with respect to  $V$ , assuming  $p$  is constant.
55. The electric power  $P$  produced by a certain source is given by  $P = \frac{E^2r}{R^2 + 2Rr + r^2}$ , where  $E$  is the voltage of the source,  $R$  is the resistance of the source, and  $r$  is the resistance in the circuit. Find the derivative of  $P$  with respect to  $r$ , assuming that the other quantities remain constant.
56. In the theory of lasers, the power  $P$  radiated is given by the equation  $P = \frac{kf^2}{\omega^2 - 2\omega f + f^2 + a^2}$ , where  $f$  is the field frequency and  $a$ ,  $k$ , and  $\omega$  are constants. Find the derivative of  $P$  with respect to  $f$ .
57. The pressure  $p$  (in Pa) exerted at a depth  $h$  (in m) on the wall of a particular dam is given by  $p = \frac{9800h^2}{h + 1}$ . Determine the rate at which pressure is changing at the instant the depth is 48.0 m.

Answers to Practice Exercises

1.  $dy/dx = -12x^5 + 12x^3 + 4x$     2.  $\frac{dy}{dx} = \frac{-4x(x^4 + 1)}{(x^4 - 1)^2}$

## 23.7 The Derivative of a Power of a Function

Composite Function • Chain Rule • Derivative of a Power of a Function • Power Rule Extended to All Rational Exponents

In Example 1 of Section 23.6, we illustrated  $y = (3 - 2x)^2$  as the power of a function of  $x$ , where  $3 - 2x$  is the function. If we let  $u = 3 - 2x$ , we can write  $y = u^2$ , and in this way,  $y$  is a function of  $u$ , and  $u$  is a function of  $x$ . This means that  $y$  is a function of a function of  $x$ , which is called a **composite function**.

Since we will often need to find the derivative of a power of a function, which is a composite function, let us look at the derivative of  $y = (3 - 2x)^2$ .

### EXAMPLE 1 Developing the chain rule

Find the derivative of  $y = (3 - 2x)^2$ .

We will use the product rule by treating  $(3 - 2x)^2$  as  $(3 - 2x)(3 - 2x)$ . The reason for doing it this way will be shown.

$$\begin{aligned}y &= (3 - 2x)^2 = (3 - 2x)(3 - 2x) \\ \frac{dy}{dx} &= (3 - 2x) \cdot (-2) + (3 - 2x) \cdot (-2) \\ \frac{dy}{dx} &= 2(3 - 2x)(-2)\end{aligned}$$

We want to leave the answer in this form in order to compare with a result we can get by letting  $y = u^2$  and  $u = 3 - 2x$ :

$$\begin{aligned}y &= u^2, & \frac{dy}{du} &= 2u \\ u &= 3 - 2x, & \frac{du}{dx} &= -2 \\ \left(\frac{dy}{du}\right) \cdot \left(\frac{du}{dx}\right) &= 2u(-2) = 2(3 - 2x)(-2)\end{aligned}$$

We see that this result is the same as the first result, and therefore for this function we see that  $\frac{dy}{dx} = \left(\frac{dy}{du}\right) \cdot \left(\frac{du}{dx}\right)$ . ■

As for the function in Example 1, it can be shown that for a differentiable composite function for which  $y$  is a function of  $u$  and  $u$  is a function of  $x$ ,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \quad (23.14)$$

#### Chain Rule

Eq. (23.14) is known as the **chain rule** for derivatives.

Using Eq. (23.14) for  $y = u^n$ , where  $u$  is a differentiable function of  $x$ , we have

$$\frac{dy}{dx} = \frac{d(u^n)}{du} \cdot \frac{du}{dx} \quad \text{or}$$

$$\frac{du^n}{dx} = nu^{n-1} \left(\frac{du}{dx}\right) \quad (23.15)$$

#### Derivative of a Power of a Function of $x$

#### LEARNING TIP

The chain rule is one of the most important derivative formulas in calculus, as it will be useful in many applications. It is critical for implicit differentiation (Section 23.8) and any differentiation involving functions expressed in terms of other functions.

We use Eq. (23.15) to find the derivative of a power of a differentiable function of  $x$ .

### EXAMPLE 2 Derivative of the power of a function

Find the derivative of  $y = (3 - 2x)^3$ .

For this function,  $n = 3$  and  $u = 3 - 2x$ . Therefore,  $du/dx = -2$ . This means

$$\begin{aligned}\frac{du^n}{dx} &= n \quad u \quad n-1 \quad \left(\frac{du}{dx}\right) \\ &\quad \downarrow \quad \downarrow \quad \downarrow \quad \swarrow \\ \frac{dy}{dx} &= 3(3 - 2x)^2(-2) \\ \frac{dy}{dx} &= -6(3 - 2x)^2\end{aligned}$$

#### Practice Exercise

- Find the derivative of  $y = (5x + 2)^4$ .

**COMMON ERROR**

When finding derivatives requiring the chain rule, the  $du/dx$  factor is often forgotten. The derivative is incomplete and therefore incorrect without this factor.

**EXAMPLE 3 Do not forget  $du/dx$** 

Find the derivative of  $p(x) = (1 - 3x^2)^4$ .

In this example,  $n = 4$  and  $u = 1 - 3x^2$ , and  $du/dx = -6x$ .

$$\begin{aligned} \frac{du^n}{dx} &= n \cdot u^{n-1} \left( \frac{du}{dx} \right) \\ p'(x) &= 4(1 - 3x^2)^3(-6x) \\ p'(x) &= -24x(1 - 3x^2)^3 \end{aligned}$$

**EXAMPLE 4 Product rule combined with power rule**

Find the derivative of  $y = 2x^3(3 - x^3)^4$ .

Here, we must *use the product rule in combination with the power rule*.

$$\frac{dy}{dx} = 2x^3 \cdot [4(3 - x^3)^3(-3x^2)] + (3 - x^3)^4 \cdot [2(3x^2)]$$

$$\frac{dy}{dx} = -24x^5(3 - x^3)^3 + 6x^2(3 - x^3)^4 = 6x^2(3 - x^3)^3[-4x^3 + (3 - x^3)]$$

$$\frac{dy}{dx} = 6x^2(3 - 5x^3)(3 - x^3)^3$$

**Practice Exercise**

2. Find the derivative of  $y = 5x(2x + 7)^3$ .

**LEARNING TIP**

We see that in finding the derivative, we multiply the function by the rational exponent and subtract 1 from it to find the exponent of the function in the derivative. *This is the same rule as derived for positive integral exponents in Eq. (23.15).*

■ For reference, Eq. (23.9) is  $\frac{dx^n}{dx} = nx^{n-1}$ .

Until now, we have derived formulas for derivatives of differentiable functions of  $x$  raised to positive integral powers. We now show that these formulas are also valid for any rational number used as an exponent. If we raise each side of  $y = u^{p/q}$  to the  $q$ th power, we have  $y^q = u^p$ . Applying the power rule, we have

$$\begin{aligned} qy^{q-1} \left( \frac{dy}{dx} \right) &= pu^{p-1} \left( \frac{du}{dx} \right) \\ \frac{dy}{dx} &= \frac{pu^{p-1} (du/dx)}{qy^{q-1}} = \frac{p}{q} \frac{u^{p-1}}{(u^{p/q})^{q-1}} \frac{du}{dx} = \frac{p}{q} \frac{u^{p-1}}{u^{p-p/q}} \frac{du}{dx} \\ \frac{dy}{dx} &= \frac{p}{q} u^{p-1-p+(p/q)} \frac{du}{dx} \end{aligned}$$

Thus,

$$\frac{du^{p/q}}{dx} = \frac{p}{q} u^{(p/q)-1} \frac{du}{dx} \quad (23.16)$$

In deriving Eqs. (23.15) and (23.16), we used Eq. (23.9), and we noted it was valid for positive integral exponents. We can show that Eq. (23.9) is also valid for negative exponents by using the quotient rule on  $1/x^n$ , which is the same as  $x^{-n}$ . Therefore,

*the power rule for derivatives, Eq. (23.15), can be extended to include all rational exponents, positive or negative.*

This, of course, includes all integral exponents, positive and negative. Also, we note that Eq. (23.9) is equivalent to Eq. (23.15) with  $u = x$  (since  $du/dx = 1$ ).

**EXAMPLE 5** Derivative of the square root of a polynomial

We can now find the derivative of  $y = \sqrt{x^2 + 1}$ .

By using Eq. (23.16) or Eq. (23.15) and writing the square root as the fractional exponent  $1/2$ , we can derive the result:

$$\begin{aligned} y &= (x^2 + 1)^{1/2} \\ \frac{dy}{dx} &= \frac{1}{2}(x^2 + 1)^{-1/2}(2x) \\ \frac{dy}{dx} &= \frac{x}{(x^2 + 1)^{1/2}} \end{aligned}$$

To avoid introducing apparently significant factors into the numerator, we do not usually rationalize such fractions. ■

Having shown that we may use fractional exponents to find derivatives of roots of functions of  $x$ , we may also use them to find derivatives of roots of  $x$  itself.

**EXAMPLE 6** Derivative using a fractional exponent

Find the derivative of  $y = 6\sqrt[3]{x^2}$ .

We can write this function as  $y = 6x^{2/3}$ . In finding the derivative, we may use Eq. (23.9) with  $n = \frac{2}{3}$ . This gives us

$$\begin{aligned} y &= 6x^{2/3} \\ \frac{dy}{dx} &= 6\left(\frac{2}{3}\right)x^{-1/3} = \frac{4}{x^{1/3}} \end{aligned}$$

We could also use Eq. (23.15) with  $u = x$  and  $n = \frac{2}{3}$ . This gives us

$$\frac{dy}{dx} = 6\left(\frac{2}{3}\right)x^{-1/3}\left(\frac{du}{dx} = \frac{dx}{dx} = 1\right) = \frac{4}{x^{1/3}}$$

This shows us why Eq. (23.9) is equivalent to Eq. (23.15) with  $u = x$ .

Note that the domain of the function is all real numbers, but the function is not differentiable for  $x = 0$ . ■

In the following examples, we illustrate the use of Eqs. (23.9) and (23.15) for the case in which  $n$  is a negative exponent. *Special care must be taken in the case of a negative exponent*, so carefully note the caution in each example.

**EXAMPLE 7** Derivative using a negative exponent

The electric resistance  $R$  of a wire varies inversely as the square of its radius  $r$ . For a given wire,  $R = 4.66 \Omega$  for  $r = 0.150$  mm. Find the derivative of  $R$  with respect to  $r$  for this wire.

Since  $R$  varies inversely as the square of  $r$ , we have  $R = k/r^2$ . Then, using the fact that  $R = 4.66 \Omega$  for  $r = 0.150$  mm, we have

$$4.66 = \frac{k}{(0.150)^2}, \quad k = 0.105 \Omega \cdot \text{mm}^2$$

which means that

$$R = \frac{0.105}{r^2}$$

**LEARNING TIP**

Note that we first rewrite the function in a different, more useful form. This is often an important step before taking the derivative.

**Practice Exercise**

3. Find the derivative of  $y = \sqrt[3]{4 - 9x}$ .

We could find the derivative by the quotient rule. However, when the numerator is constant, the derivative is easily found by using negative exponents and the power rule.

$$R = \frac{0.105}{r^2} = 0.105r^{-2}$$

$$\frac{dR}{dr} = 0.105(-2)r^{-3}$$

$$\frac{dR}{dr} = -\frac{0.210}{r^3}$$

Here, we used Eq. (23.9) directly. ■

### COMMON ERROR

When subtracting 1 from a negative exponent, be sure to *decrease* the value (e.g.,  $-2 - 1 = -3$ , or  $-5 - 1 = -6$ ).

### EXAMPLE 8 The chain rule using a negative exponent

Find the derivative of  $y = \frac{1}{(1 - 4x)^5}$ .

$$y = \frac{1}{(1 - 4x)^5} = (1 - 4x)^{-5} \quad \text{use negative exponent}$$

$$\frac{dy}{dx} = (-5)(1 - 4x)^{-6}(-4) \quad \text{use Eq. (23.15)}$$

$$\frac{dy}{dx} = \frac{20}{(1 - 4x)^6} \quad \text{express result with positive exponent} \quad \blacksquare$$

### Practice Exercise

4. Find the derivative of  $y = \frac{3}{(6x + 5)^4}$ .

We now see the value of fractional exponents in calculus. They are useful in many algebraic operations, but they are almost essential in calculus. Without fractional exponents, it would be necessary to develop additional formulas to find the derivatives of radical expressions. In order to find the derivative of an algebraic function, we need only those formulas we have already developed. Often, it is necessary to combine these formulas, as we saw in Example 4. Actually, most derivatives are combinations. The problem in finding the derivative is **recognizing the form of the function** with which you are dealing. When you have recognized the form, completing the problem is only a matter of mechanics and algebra. You should now see the importance of being able to handle algebraic operations with ease.

### EXAMPLE 9 Using the chain rule and the quotient rule

Using the quotient rule, find the derivative of  $y = \frac{x}{(x^2 - 9)^3}$  and evaluate it at  $x = 4$ .

Notice that this function is a quotient because of the division of the two functions, but the denominator by itself is a power of a function and will require the chain rule to differentiate.

Recognize that the function  $y = \frac{x}{(x^2 - 9)^3}$  is of the form  $y = \frac{u}{v}$ , with  $u = x$  and  $v = (x^2 - 9)^3$ . In order to use the quotient rule, we need the derivative of each function by itself.

$$\begin{array}{l}
 u = x \\
 \frac{du}{dx} = 1 \\
 \\
 v = (x^2 - 9)^3 \\
 \frac{dv}{dx} = 3(x^2 - 9)^2(2x) \quad \text{using chain rule} \\
 \frac{dv}{dx} = 6x(x^2 - 9)^2 \\
 \\
 \frac{dy}{dx} = \frac{v \cdot \frac{du}{dx} - u \cdot \frac{dv}{dx}}{v^2} \quad \text{quotient rule} \\
 \frac{dy}{dx} = \frac{(x^2 - 9)^3 \cdot (1) - x \cdot 6x(x^2 - 9)^2}{(x^2 - 9)^6} \\
 \frac{dy}{dx} = \frac{(x^2 - 9)^3 - 6x^2(x^2 - 9)^2}{(x^2 - 9)^6}
 \end{array}$$

We can factor out a common factor of  $(x^2 - 9)^2$  in the numerator, and it will divide out with the similar factor in the denominator:

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{(x^2 - 9)^2}{(x^2 - 9)^6} [(x^2 - 9) - 6x^2] \\
 \frac{dy}{dx} &= \frac{-1(5x^2 + 9)}{(x^2 - 9)^4}
 \end{aligned}$$

Now evaluating at  $x = 4$ ,

$$\begin{aligned}
 \left. \frac{dy}{dx} \right|_{x=4} &= \frac{-1(5 \cdot 4^2 + 9)}{(4^2 - 9)^4} \\
 \left. \frac{dy}{dx} \right|_{x=4} &= \frac{-89}{2401} = -0.0371
 \end{aligned}$$

### EXAMPLE 10 Using the chain rule and the product rule

Using the product rule, find the derivative of  $y = \frac{x}{(x^2 - 9)^3}$  and evaluate it at  $x = 4$ . This is the same function as in Example 9.

Notice that this function is a quotient because of the division of the two functions, but we can change it into a product by bringing the denominator to the top of the fraction with a negative power before differentiating.

$$y = x(x^2 - 9)^{-3}$$

Recognize that the function  $y = x(x^2 - 9)^{-3}$  is of the form  $y = u \cdot v$ , with  $u = x$  and  $v = (x^2 - 9)^{-3}$ . In order to use the product rule, we need the derivatives of each function by itself.

$$\begin{array}{l}
 u = x \\
 \frac{du}{dx} = 1 \\
 \\
 v = (x^2 - 9)^{-3} \\
 \frac{dv}{dx} = -3(x^2 - 9)^{-4}(2x) \quad \text{using chain rule} \\
 \frac{dv}{dx} = -6x(x^2 - 9)^{-4}
 \end{array}$$

$$\frac{d(u \cdot v)}{dx} = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx} \quad \text{product rule}$$

$$\frac{dy}{dx} = x \cdot (-6x)(x^2 - 9)^{-4} + (x^2 - 9)^{-3} \cdot (1)$$

$$\frac{dy}{dx} = -6x^2(x^2 - 9)^{-4} + (x^2 - 9)^{-3}$$

#### LEARNING TIP

In cases where the denominator of a quotient is a variable or a function with a power other than 1, it is sometimes easier to bring the denominator to the top of the fraction with a negative power. This creates a product rule, instead of a quotient rule, which is then combined with the chain rule. The simpler formula sometimes produces a solution for the derivative that requires less factoring to express it in its simplest form.

We can factor out a common factor between the two terms:

$$\frac{dy}{dx} = (x^2 - 9)^{-4}[-6x^2 + (x^2 - 9)]$$

$$\frac{dy}{dx} = (x^2 - 9)^{-4}[-5x^2 - 9]$$

$$\frac{dy}{dx} = \frac{-1(5x^2 + 9)}{(x^2 - 9)^4}$$

Now evaluating at  $x = 4$ ,

$$\left. \frac{dy}{dx} \right|_{x=4} = \frac{-1(5 \cdot 4^2 + 9)}{(4^2 - 9)^4}$$

$$\left. \frac{dy}{dx} \right|_{x=4} = \frac{-89}{2401} = -0.0371$$

This is the same answer as in Example 9, though with somewhat simpler factoring involved. ■

### EXAMPLE 11 The chain rule inside a product rule

Find the derivative of  $y = (4x + 1)^2(2x^2 + 3)^3$ .

Notice that  $y$  is a product because of the multiplication of the two functions, but each function by itself is a power of a function and will require the chain rule to differentiate.

Recognize that the function  $y = (4x + 1)^2(2x^2 + 3)^3$  is of the form  $y = u \cdot v$ , with  $u = (4x + 1)^2$  and  $v = (2x^2 + 3)^3$ . In order to use the product rule, we need the derivatives of each function by itself.

$$u = (4x + 1)^2 \qquad v = (2x^2 + 3)^3$$

$$\frac{du}{dx} = 2(4x + 1)^1(4) \qquad \frac{dv}{dx} = 3(2x^2 + 3)^2(4x)$$

$$\frac{du}{dx} = 8(4x + 1) \qquad \frac{dv}{dx} = 12x(2x^2 + 3)^2$$

$$\frac{d(u \cdot v)}{dx} = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx} \qquad \text{product rule}$$

$$\frac{dy}{dx} = (4x + 1)^2 \cdot (12x)(2x^2 + 3)^2 + (2x^2 + 3)^3 \cdot 8(4x + 1)$$

There are some common algebraic factors between the two terms:

$$\frac{dy}{dx} = 4(4x + 1)(2x^2 + 3)^2[3x(4x + 1) + (2x^2 + 3) \cdot 2]$$

$$\frac{dy}{dx} = 4(4x + 1)(2x^2 + 3)^2[12x^2 + 3x + 4x^2 + 6]$$

$$\frac{dy}{dx} = 4(4x + 1)(2x^2 + 3)^2[16x^2 + 3x + 6] \qquad \blacksquare$$

### EXAMPLE 12 Evaluation of a derivative

Evaluate the derivative of  $y = \frac{x}{\sqrt{1 - 4x}}$  for  $x = -2$ .

Here we have a quotient, but the denominator function has a power not equal to 1. Therefore, we will take the denominator to the top as a negative power and use a product rule with a chain rule embedded in it.



$$\begin{aligned}
 y &= x \cdot (1 - 4x)^{-1/2} \\
 \frac{dy}{dx} &= x \cdot \left(-\frac{1}{2}\right) (1 - 4x)^{-3/2} \cdot (-4) + (1 - 4x)^{-1/2} \cdot (1) \\
 \frac{dy}{dx} &= \left(\frac{1}{2}\right) (1 - 4x)^{-3/2} [4x + 2(1 - 4x)^1] \\
 \frac{dy}{dx} &= \left(\frac{1}{2}\right) (1 - 4x)^{-3/2} [4x + 2 - 8x] \\
 \frac{dy}{dx} &= \left(\frac{1}{2}\right) (1 - 4x)^{-3/2} [2 - 4x] \\
 \frac{dy}{dx} &= (1 - 4x)^{-3/2} [1 - 2x] \\
 \frac{dy}{dx} &= \frac{1 - 2x}{(1 - 4x)^{3/2}}
 \end{aligned}$$

Now, evaluating the derivative at  $x = -2$ , we have

$$\left. \frac{dy}{dx} \right|_{x=-2} = \frac{1 - 2(-2)}{[1 - 4(-2)]^{3/2}} = \frac{1 + 4}{(1 + 8)^{3/2}} = \frac{5}{9^{3/2}} = \frac{5}{27}$$

## EXERCISES 23.7

In Exercises 1–4, make the given changes in the indicated examples of this section and then find the derivatives.

- In Example 3, change  $1 - 3x^2$  to  $2 + 3x^3$ .
- In Example 4, change  $3 - x^3$  to  $2 + x^5$ .
- In Example 5, change  $x^2 + 1$  to  $2 - 3x^2$ .
- In Example 8, change the exponent 5 to 3.

In Exercises 5–32, find the derivative of each of the given functions.

- $y = \sqrt{x}$
- $y = \sqrt[4]{x^3}$
- $v = \frac{3}{5t^2}$
- $y = \frac{2}{x^4}$
- $y = \frac{3}{\sqrt[3]{x}}$
- $y = \frac{55}{\sqrt[5]{x^2}}$
- $y = x\sqrt{x} - \frac{6}{x}$
- $f(x) = 2x^{-3} - 3x^{-2}$
- $y = (x^2 + 1)^5$
- $y = (1 - 2x)^4$
- $y = 2.25(7 - 4x^3)^8$
- $s = 3(8t^2 - 7)^6$
- $y = (2x^3 - 3)^{1/3}$
- $y = 8(1 - 6x)^{1.5}$
- $f(y) = \frac{3}{(4 - y^2)^4}$
- $y = \frac{\pi^3}{\sqrt{1 - 3x}}$
- $y = 4(2x^4 - 5)^{0.75}$
- $r = 5(3\theta^6 - 4)^{2/3}$
- $y = \sqrt[4]{1 - 8x^2}$
- $y = 9\sqrt[3]{4x^6 + 2}$
- $u = v\sqrt{8v + 5}$
- $y = x^2(1 - 3x)^5$

- $y = \frac{2\sqrt{1 - 6x}}{x^3}$
- $R = \frac{2T^2}{\sqrt[3]{1 + 4T}}$
- $y = \frac{2x\sqrt{x + 2}}{x + 4}$
- $y = 8\sqrt{1 + \sqrt{x}}$
- $f(R) = \sqrt{\frac{2R + 1}{4R + 1}}$
- $y = \left(\frac{2x + 1}{3x - 2}\right)^2$

In Exercises 33–36, evaluate the derivatives of the given functions for the given values of  $x$ .

- $y = \sqrt{3x + 4}$ ,  $x = 7$
- $y = (4 - x^2)^{-1}$ ,  $x = -1$
- $y = \frac{\sqrt{x}}{1 - x}$ ,  $x = 4$
- $y = x^2\sqrt[3]{3x + 2}$ ,  $x = 2$

In Exercises 37–60, solve the given problems by finding the appropriate derivatives, and round answers to 3 significant digits unless otherwise specified.

- Find the derivative of  $x^{3/2}$  by writing  $x^{3/2} = x(x^{1/2})$  and using the product rule.
- Find the derivative of  $x^{3/2}$  by writing  $x^{3/2} = x^2/x^{1/2}$  and using the quotient rule.
- Find the derivative of  $y = 1/x^3$  as (a) a quotient and (b) a negative power of  $x$  and show that the results are the same.
- Let  $y = [u(x)]^2$  and find  $dy/dx$ , treating  $[u(x)]^2$  as the product  $u(x)u(x)$ . (See Example 1.)
- Find any values of  $x$  for which the derivative of  $y = \frac{x^2}{\sqrt{x^2 + 1}}$  is zero. View the curve of the function on a graphing calculator to verify the values found.

42. Find any values of  $x$  for which the derivative of  $y = \frac{x}{\sqrt{4x-1}}$  is zero. View the curve of the function on a graphing calculator to verify the values found.
43. Is the line  $x + 3y - 12 = 0$  ever perpendicular to a tangent to the graph of  $y = \sqrt{2x+3}$ ?
44. Explain why the graph of  $y_1 = \sqrt{x} + a$  cannot be tangent to the graph of  $y_2 = \sqrt{4-2x}$ , regardless of the value of  $a$ .
45. Find the slope of a line tangent to the parabola  $y^2 = 4x$  at the point  $(1, 2)$ .
46. Find the slope of a line tangent to the circle  $x^2 + y^2 = 25$  at the point  $(4, 3)$ .
47. The lowest flying speed  $v$  (in m/s) at which a certain airplane can fly varies directly as the square root of the wing load  $w$  (in Pa). If  $v = 27.0$  m/s for  $w = 775$  Pa, find the derivative of  $v$  with respect to  $w$ .
48. During and after a period of rain, the depth  $h$  (in m) of water behind a certain dam was given by  $h = 75.0(x+2)/(x+3)$ , where  $x$  is the number of days after the start of the rain period. Find  $dh/dx$  for  $x = 2.50$  days.
49. The displacement  $s$  (in cm) of a linkage joint of a robot is given by  $s = (8t - t^2)^{2/3}$ , where  $t$  is the time (in s). Find the velocity of the joint for  $t = 6.25$  s.
50. Water is slowly rising in a horizontal drainage pipe. The width  $w$  of the water as a function of the depth  $h$  is  $w = \sqrt{2rh - h^2}$ , where  $r$  is the radius of the pipe. Find  $dw/dh$  for  $h = 225$  mm and  $r = 600$  mm.
51. When the volume of a gas changes very rapidly, an approximate relation is that the pressure  $p$  varies inversely as the  $3/2$  power of the volume. If  $p$  is 300 kPa when  $V = 100$  cm<sup>3</sup>, find the derivative of  $p$  with respect to  $V$ . Evaluate this derivative for  $V = 100$  cm<sup>3</sup>.
52. The power gain  $G$  of a certain antenna is inversely proportional to the square of the wavelength  $\lambda$  (in m) of the carrier wave. If  $G = 5.00 \times 10^4$  for  $\lambda = 0.110$  m, find the derivative of  $G$  with respect to  $\lambda$  for  $\lambda = 0.110$  m.
53. In deep water, the velocity of a wave is  $v = k\sqrt{\frac{l}{a} + \frac{a}{l}}$ , where  $a$  and  $k$  are constants and  $l$  is the length of the wave. For what value of  $l$  is  $dv/dl = 0$ ?
54. Due to air friction, the drag  $F$  on a plane is  $F = c_1v^2 + c_2v^{-2}$ , where  $v$  is the plane's velocity and  $c_1$  and  $c_2$  are positive constants. For what values of  $v$  is  $dF/dv = 0$ ?
55. The total solar radiation  $H$  (in W/m<sup>2</sup>) on a certain surface during an average clear day is given by
- $$H = \frac{4000}{\sqrt{t^6 + 100}} \quad (-6 < t < 6)$$
- where  $t$  is the number of hours from noon. Find the rate at which  $H$  is changing with time at 4 P.M.
56. In determining the time for a laser beam to go from  $S$  to  $P$  (see Fig. 23.25), which are in different mediums, it is necessary to find the derivative of the time

$$t = \frac{\sqrt{a^2 + x^2}}{v_1} + \frac{\sqrt{b^2 + (c-x)^2}}{v_2}$$

with respect to  $x$ , where  $a, b, c, v_1$ , and  $v_2$  are constants. Here,  $v_1$  and  $v_2$  are the velocities of the laser in each medium. Find this derivative.

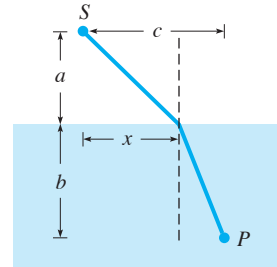


Fig. 23.25

57. The radio waveguide wavelength  $\lambda_r$  is related to its free-space wavelength  $\lambda$  by

$$\lambda_r = \frac{2a\lambda}{\sqrt{4a^2 - \lambda^2}}$$

where  $a$  is a constant. Find  $d\lambda_r/d\lambda$ .

58. The current  $I$  in a circuit containing a resistance  $R$  and an inductance  $L$  is found from the expression

$$I = \frac{V}{\sqrt{R^2 + (\omega L)^2}}$$

Find the expression for the instantaneous rate of change of current with respect to  $L$ , assuming that the other quantities remain constant.

59. The length  $l$  of a rectangular microprocessor chip is 2 mm longer than its width  $w$ . Find the derivative of the length of the diagonal  $D$  with respect to  $w$ .
60. The trapezoidal engineering support structure shown in Fig. 23.26 has an internal support of length  $l$ . Find the derivative of  $l$  with respect to  $x$ .

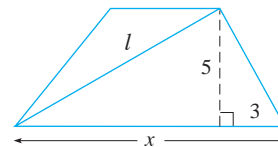


Fig. 23.26

Answers to Practice Exercises

1.  $dy/dx = 20(5x+2)^3$     2.  $dy/dx = 5(2x+7)^2(8x+7)$   
 3.  $dy/dx = -3(4-9x)^{-2/3}$     4.  $dy/dx = -72(6x+5)^{-5}$

## 23.8 Differentiation of Implicit Functions

### Implicit Function • Differentiating Term by Term

To this point, all of the functions that have been differentiated have been of the form  $y = f(x)$ , where  $y$  is explicitly defined as a function of  $x$ . Sometimes, however, a relationship does not define the dependent variable explicitly in terms of the independent variable.

**Explicit functions** require direct substitution for the value of the variable  $x$ , and the value of  $y$  can be determined exactly. **Implicit functions** are defined by relations of the form  $F(x, y) = 0$ . This means that in the equation defining the relationship between  $x$  and  $y$ , sometimes  $y$  cannot be isolated algebraically as a single term. Often, implicit relations are not functions, in that there may be more than one value of  $y$  for each value of  $x$ . Even with this difficulty, the derivatives of implicitly defined relations can still be calculated using a process called **implicit differentiation**.

#### EXAMPLE 1 Illustrations of implicit functions

- (a) The equation  $3x + 4y = 5$  is an equation that defines a function, although it is not in explicit form. In solving for  $y$  as  $y = -\frac{3}{4}x + \frac{5}{4}$ , we have the explicit form of the function.
- (b) The equation  $y^2 + x = 3$  is an equation that defines two functions, although we do not have the explicit forms. When we solve for  $y$ , we obtain the explicit functions  $y = \sqrt{3 - x}$  and  $y = -\sqrt{3 - x}$ .
- (c) The equation  $y^5 + xy^2 + 3x^2 = 5$  defines  $y$  as a function of  $x$ , although we cannot actually solve for the explicit algebraic form of the function.
- (d) The equation  $x^2 + y^2 + 4 = 0$  is not satisfied by any pair of real values of  $x$  and  $y$ . ■

Even when it is possible to determine the explicit form of a function given in implicit form, it is not always desirable to do so. In some cases, the implicit form is more convenient than the explicit form.

The derivative of an implicit function may be found directly without having to solve for the explicit function. Often, the chain rule must be used (since powers of  $y$  are powers of a function of  $x$ ).

The following two steps can be used to determine the derivative at any point  $\frac{dy}{dx}$  for implicit relations that are expressed in terms of  $x$  (the variable) and  $y$  (the relation).

■ For reference, Eq. (23.15) is

$$\frac{du^n}{dx} = nu^{n-1} \frac{du}{dx}$$

#### Implicit Differentiation

1. **Differentiate with respect to  $x$  both sides of the implicit equation.** This means applying the process  $d/dx$  to every term of the equation. Remember to use the chain rule every time you encounter a relation  $y$ . Every time that you differentiate a function  $y$ , you will introduce the  $dy/dx$  factor through using the chain rule Eq. (23.15). Moreover, remember that  $y$  is a function of  $x$ , so you will need to use the product rule or the quotient rule if  $y$  (or  $y^n$ ) is multiplied or divided by a function of  $x$ .
2. **Rearrange algebraically to isolate  $dy/dx$ .** It may appear in multiple terms, so algebraic factoring will be a common requirement. For implicitly defined relations, the derivative is itself an implicitly defined relation of  $x$  and  $y$ .

**EXAMPLE 2** Implicit derivative

Find  $dy/dx$  if  $y^2 + 2x^2 = 5$ .

Here, we find the derivative of each term and then solve for  $dy/dx$ . Thus,

$$\begin{aligned}\frac{d(y^2)}{dx} + \frac{d(2x^2)}{dx} &= \frac{d(5)}{dx} \\ 2y^{2-1} \cdot \frac{dy}{dx} + 2\left(2x^{2-1} \cdot \frac{dx}{dx}\right) &= 0 \\ 2y \cdot \frac{dy}{dx} + 4x &= 0 \\ \frac{dy}{dx} &= -\frac{2x}{y}\end{aligned}$$

**EXAMPLE 3** Implicit derivative involving a product

Find  $dy/dx$  if  $3y^4 + xy^2 + 2x^3 - 6 = 0$ .

In finding the derivative, we note that the second term is a product of two functions of  $x$ , and we must use the product rule for derivatives on it. Thus, we have

$$\begin{aligned}\frac{d(3y^4)}{dx} + \frac{d(x \cdot y^2)}{dx} + \frac{d(2x^3)}{dx} - \frac{d(6)}{dx} &= \frac{d(0)}{dx} \\ 12y^3 \cdot \frac{dy}{dx} + \left[ x \left( 2y \cdot \frac{dy}{dx} \right) + y^2(1) \right] + 6x^2 - 0 &= 0 \\ 12y^3 \cdot \frac{dy}{dx} + 2xy \cdot \frac{dy}{dx} + y^2 + 6x^2 &= 0 \quad \text{solve for } \frac{dy}{dx} \\ (12y^3 + 2xy) \frac{dy}{dx} &= -y^2 - 6x^2 \\ \frac{dy}{dx} &= \frac{-y^2 - 6x^2}{12y^3 + 2xy}\end{aligned}$$

**EXAMPLE 4** Implicit derivative—product and power

Find  $dy/dx$  if  $2x^3y + (y^2 + x)^3 = x^4$ .

In this case, we use the product rule on the first term and the power rule on the second term:

$$\begin{aligned}\frac{d(2x^3 \cdot y)}{dx} + \frac{d(y^2 + x)^3}{dx} &= \frac{d(x^4)}{dx} \\ 2x^3 \left( \frac{dy}{dx} \right) + y(6x^2) + 3(y^2 + x)^2 \left( 2y \cdot \frac{dy}{dx} + 1 \right) &= 4x^3 \\ 2x^3 \cdot \frac{dy}{dx} + 6x^2y + 3(y^2 + x)^2 \left( 2y \cdot \frac{dy}{dx} \right) + 3(y^2 + x)^2 &= 4x^3 \\ [2x^3 + 6y(y^2 + x)^2] \frac{dy}{dx} &= 4x^3 - 6x^2y - 3(y^2 + x)^2 \\ \frac{dy}{dx} &= \frac{4x^3 - 6x^2y - 3(y^2 + x)^2}{2x^3 + 6y(y^2 + x)^2}\end{aligned}$$

**Practice Exercise**

1. Find  $dy/dx$  if  $2y^3 + xy + 1 = 0$ .

**EXAMPLE 5** Slope of a tangent line

Find the slope of a tangent line to the curve  $x^3 + y^3 - 9xy = 0$  at the point  $(2, 4)$ . This curve is known as a *folium*, which dates back to Descartes in the 1630s. See Fig. 23.27.

Here, we are to find  $dy/dx$  and evaluate it for  $x = 2$  and  $y = 4$ .

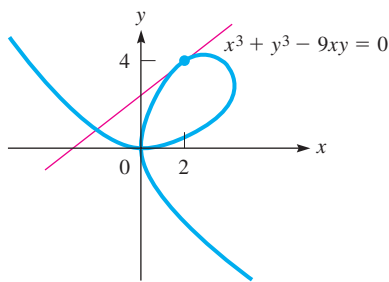


Fig. 23.27

$$\frac{d(x^3)}{dx} + \frac{d(y^3)}{dx} - \frac{d(9x \cdot y)}{dx} = \frac{d(0)}{dx}$$

$$3x^2 + 3y^2 \cdot \frac{dy}{dx} - 9\left(x \cdot \frac{dy}{dx} + y\right) = 0$$

$$(3y^2 - 9x) \frac{dy}{dx} = 9y - 3x^2$$

$$\frac{dy}{dx} = \frac{3y - x^2}{y^2 - 3x} \quad \left. \frac{dy}{dx} \right|_{(2,4)} = \frac{3(4) - 2^2}{4^2 - 3(2)} = \frac{8}{10} = \frac{4}{5}$$

Therefore, the slope of the tangent line at  $(2, 4)$  is  $4/5$ . ■

**EXERCISES 23.8**

In Exercises 1 and 2, make the given changes in the indicated examples of this section and then find  $dy/dx$ .

- In Example 2, change  $y^2$  to  $y^3$ .
- In Example 3, change  $xy^2$  to  $x^2y$ .

In Exercises 3–6, terms that might appear in an implicit function are shown. Differentiate each term with respect to  $x$ .

- $x^2y$
- $2xy^3$
- $\frac{1}{xy}$
- $\frac{y^2}{x+1}$

In Exercises 7–26, find  $dy/dx$  by differentiating implicitly. When applicable, express the result in terms of  $x$  and  $y$ .

- $3x + 2y = 5$
- $4y - 3x^2 = x$
- $x^2 - 4y^2 - 9 = 0$
- $y^5 = x^2 - 1$
- $y^2 + y = x^2 - 4$
- $y + 3xy - 4 = 0$
- $x^2 = \frac{x-y}{x+y}$
- $\frac{3x^2}{y^2+1} + y = 3x + 1$
- $(2y-x)^4 + x^2 = y + 3$
- $2(x^2+1)^3 + (y^2+1)^2 = 17$
- $(2x+1)(1-3y) + y^2 = 13$
- $6x - 3y = 4$
- $x^5 - 5y = 6 - x$
- $x^2 + 2y^2 - 11 = 0$
- $x^{2/3} + y^{2/3} = 5$
- $2y^3 - y = 7 - x^4$
- $xy^3 + 3y + x^2 = 2\pi^2$
- $y^2x - \frac{5y}{x+1} + 3x = 4$
- $\sqrt{xy} = \frac{x}{4} + \frac{1}{y^2}$
- $(y^2+2)^3 = x^4y + e^2$

In Exercises 27–32, evaluate the derivatives of the given functions at the given points.

- $3x^3y^2 - 2y^3 = -4$ ;  $(1, 2)$
- $2y + 5 - x^2 - y^3 = 0$ ;  $(2, -1)$

- $5y^4 + 7 = x^4 - 3y$ ;  $(3, -2)$
- $(xy - y^2)^3 = 5y^2 + 22$ ;  $(4, 1)$
- $xy^2 + 3x^2 - y^2 + 15 = 0$ ;  $(-1, 3)$
- $2(x+y)^3 - y^2/x = 15$ ;  $(4, -2)$

In Exercises 33–48, solve the given problems by using implicit differentiation, and round answers to 3 significant digits if applicable.

- At what point(s) does the graph of  $x^2 + y^2 = 4x$  have a horizontal tangent?
- Show that if  $P(x, y)$  is any point on the circle  $x^2 + y^2 = a^2$ , then a tangent line at  $P$  is perpendicular to a line through  $P$  and the origin.
- Show that two tangents to the curve  $x^2 + xy + y^2 = 7$  at the points where it crosses the  $x$ -axis are parallel.
- At what point(s) is the tangent to the curve  $y^2 = 2x^3$  perpendicular to the line  $4x - 3y + 1 = 0$ ?
- Find the slope of a line tangent to the curve of the implicit function  $xy + y^2 + 2 = 0$  at the point  $(-3, 1)$ . Use the derivative evaluation feature of a graphing calculator to check your result.
- Show that the graphs of  $2x^2 + y^2 = 24$  and  $y^2 = 8x$  are perpendicular at the point  $(2, 4)$ . Display the graphs on a graphing calculator.
- In an  $RLC$  circuit, the angular frequency  $\omega$  at which the circuit resonates is given by  $\omega^2 = 1/LC - R^2/L^2$ . Find  $d\omega/dL$ .
- A lens is described by the ellipse  $x^2 - xy + y^2 = 7$ . Find the slope of a light ray perpendicular to the lens at  $(-1, 2)$ .
- The pressure  $p$ , volume  $V$ , and temperature  $T$  of a gas are related by  $pV = n(RT + ap - bp/T)$ , where  $a$ ,  $b$ ,  $n$ , and  $R$  are constants. For constant  $V$ , find  $dp/dT$ .
- Oil moves through a pipeline such that the distance  $s$  it moves and the time  $t$  are related by  $s^3 - t^2 = 7t$ . Find the velocity of the oil for  $s = 4.01$  m and  $t = 5.25$  s.

43. The shelf support shown in Fig. 23.28 is 0.750 m long. Find the expression for  $dy/dx$  in terms of  $x$  and  $y$ .

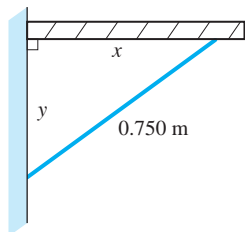


Fig. 23.28

44. An open (no top) right circular cylindrical container of radius  $r$  and height  $h$  has a total surface area of  $936 \text{ cm}^2$ . Find  $dr/dh$  in terms of  $r$  and  $h$ .
45. Two resistors, with resistances  $r$  and  $r + 2$ , are connected in parallel. Their combined resistance  $R$  is related to  $r$  by the equation  $r^2 = 2rR + 2R - 2r$ . Find  $dR/dr$ .
46. The polar moment of inertia  $I$  of a rectangular slab of concrete is given by  $I = \frac{1}{12}(b^3h + bh^3)$ , where  $b$  and  $h$  are the base and the height, respectively, of the slab. If  $I$  is constant, find the expression for  $db/dh$ .

47. A formula relating the length  $L$  and radius of gyration  $r$  of a steel column is  $24C^3Sr^3 = 40C^3r^3 + 9LC^2r^2 - 3L^3$ , where  $C$  and  $S$  are constants. Find  $dL/dr$ .

48. A computer is programmed to draw the graph of the implicit function  $(x^2 + y^2)^3 = 64x^2y^2$  (see Fig. 23.29 and Example 7 on page 605). Find the slope of a line tangent to this curve at  $(2.00, 0.56)$  and at  $(2.00, 3.07)$ .

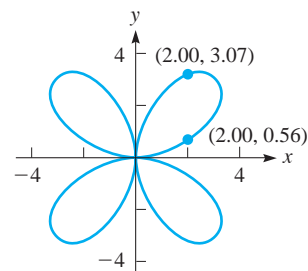


Fig. 23.29

#### Answer to Practice Exercise

1.  $dy/dx = -y/(6y^2 + x)$

## 23.9 Higher Derivatives

First Derivative • Second Derivative •  
Higher Derivatives • Instantaneous  
Acceleration

### The Second Derivative

Since the derivative of a function is itself a function, we may take its derivative. A *higher-order derivative* refers to repeating the differentiation process. This will become relevant to curve sketching, maximum and minimum problems, motion problems, and other applications.

The derivative of a function is called the **first derivative**. To find the **second derivative**, just differentiate the first derivative equation. To find the third derivative, just differentiate the second derivative, and so on (provided that each derivative is defined). The second derivative, third derivative, and so on are collectively known as **higher derivatives**, or **successive derivatives**.

Repeated use of the derivative process leads to some cumbersome notation. Consequently, many other more concise notations have been adopted to decrease the amount of writing required. Those used in this text are summarized in the chart below:

First Derivative	$\frac{dy}{dx}$	$\frac{dy}{dx}$	$f'(x)$	$D_x y$	$y'$
Second Derivative	$\frac{d}{dx}\left(\frac{dy}{dx}\right)$	$\frac{d^2y}{dx^2}$	$f''(x)$	$D_x^2 y$	$y''$
Third Derivative	$\frac{d}{dx}\left(\frac{d}{dx}\left(\frac{dy}{dx}\right)\right)$	$\frac{d^3y}{dx^3}$	$f'''(x)$	$D_x^3 y$	$y'''$
Fourth Derivative	$\frac{d}{dx}\left(\frac{d}{dx}\left(\frac{d}{dx}\left(\frac{dy}{dx}\right)\right)\right)$	$\frac{d^4y}{dx^4}$	$f^{(4)}(x)$	$D_x^4 y$	$y^{(4)}$
Fifth Derivative	$\frac{d}{dx}\left(\frac{d}{dx}\left(\frac{d}{dx}\left(\frac{d}{dx}\left(\frac{dy}{dx}\right)\right)\right)\right)$	$\frac{d^5y}{dx^5}$	$f^{(5)}(x)$	$D_x^5 y$	$y^{(5)}$

Note that the “primed” notation in the last column is a common method of denoting higher-order derivatives because it is the most concise. The disadvantage of “primed” notation is that it does not explicitly state the variable of differentiation.

**EXAMPLE 1** Higher derivatives of a function

Find the higher derivatives of  $y = 5x^3 - 2x$ .

We find the first derivative as

$$\frac{dy}{dx} = 15x^2 - 2 \quad \text{or} \quad y' = 15x^2 - 2$$

Next, we obtain the second derivative by finding the derivative of the first derivative:

$$\frac{d^2y}{dx^2} = 30x \quad \text{or} \quad y'' = 30x$$

Continuing to find the successive derivatives, we have

$$\frac{d^3y}{dx^3} = 30 \quad \text{or} \quad y''' = 30$$

$$\frac{d^4y}{dx^4} = 0 \quad \text{or} \quad y^{(4)} = 0$$

Since the third derivative is a constant, the fourth derivative and all successive derivatives will be zero. This can be shown as  $d^n y/dx^n = 0$  for  $n \geq 4$ . ■

**EXAMPLE 2** Higher derivatives of a function

Find the higher derivatives of  $f(x) = x(x^2 - 1)^2$ .

Using the product rule, Eq. (23.12), to find the first derivative, we have

$$\begin{aligned} f'(x) &= x(2)(x^2 - 1)(2x) + (x^2 - 1)^2(1) \\ f'(x) &= (x^2 - 1)(4x^2 + x^2 - 1) = (x^2 - 1)(5x^2 - 1) \\ f'(x) &= 5x^4 - 6x^2 + 1 \end{aligned}$$

Continuing to find the higher derivatives, we have

$$f''(x) = 20x^3 - 12x$$

$$f'''(x) = 60x^2 - 12$$

$$f^{(4)}(x) = 120x$$

$$f^{(5)}(x) = 120$$

$$f^{(n)}(x) = 0 \quad \text{for } n \geq 6$$

All derivatives after the fifth derivative are equal to zero. ■

**EXAMPLE 3** Evaluation of a second derivative

Evaluate the second derivative of  $y = \frac{2}{1-x}$  for  $x = -2$ .

We write the function as  $y = 2(1-x)^{-1}$  and then find the derivatives:

$$y = 2(1-x)^{-1}$$

$$\frac{dy}{dx} = 2(-1)(1-x)^{-2}(-1) = 2(1-x)^{-2}$$

$$\frac{d^2y}{dx^2} = 2(-2)(1-x)^{-3}(-1) = 4(1-x)^{-3} = \frac{4}{(1-x)^3}$$

Evaluating the second derivative for  $x = -2$ , we have

$$\left. \frac{d^2y}{dx^2} \right|_{x=-2} = \frac{4}{(1+2)^3} = \frac{4}{27}$$

■ For reference, Eq. (23.12) is

$$\frac{d(uv)}{dx} = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}$$

■ Note that when using the prime ( $f'(x)$ ) notation the  $n$ th derivative may be shown as  $f^{(n)}(x)$ .



The function is not differentiable for  $x = 1$ . Also, if we continue to find higher derivatives, the expressions will not become zero, as in Examples 1 and 2. ■

#### EXAMPLE 4 Second derivative of an implicit function

Find  $y''$  for the implicit function defined by  $2x^2 + 3y^2 = 6$ .

Differentiating with respect to  $x$ , we have

$$\begin{aligned} 2(2x) + 3(2yy') &= 0 \\ 4x + 6yy' &= 0 \quad \text{or} \quad 2x + 3yy' = 0 \end{aligned} \quad (1)$$

Before differentiating again, we see that  $3yy'$  is a **product**, and we note that the derivative of  $y'$  is  $y''$ . Thus, differentiating again, we have

#### Practice Exercise

1. Find the second derivative of

$$y = \frac{3}{x^2 + 4}.$$

$$\begin{aligned} 2 + 3yy'' + 3y'(y') &= 0 \\ 2 + 3yy'' + 3(y')^2 &= 0 \end{aligned} \quad (2)$$

differentiation of  $3yy'$

Now, solving Eq. (1) for  $y'$  and substituting this into Eq. (2), we have

$$\begin{aligned} y' &= -\frac{2x}{3y} \\ 2 + 3yy'' + 3\left(-\frac{2x}{3y}\right)^2 &= 0 \\ 2 + 3yy'' + \frac{4x^2}{3y^2} &= 0 \\ 6y^2 + 9y^3y'' + 4x^2 &= 0 \\ y'' &= \frac{-4x^2 - 6y^2}{9y^3} = \frac{-2(2x^2 + 3y^2)}{9y^3} \end{aligned}$$

Since  $2x^2 + 3y^2 = 6$ , we have

$$y'' = \frac{-2(6)}{9y^3} = -\frac{4}{3y^3} \quad \blacksquare$$

As mentioned earlier, higher derivatives are useful in certain applications. This is particularly true of the second derivative. The first and second derivatives are used in the next chapter for several types of applications, and higher derivatives are used when we discuss infinite series in Chapter 30. An important technical application of the second derivative is shown in the example that follows.

In Section 23.4, we briefly discussed the instantaneous velocity of an object, and in the exercises we mentioned acceleration. From that discussion, recall that the instantaneous velocity is the time rate of change of the displacement, and that **the instantaneous acceleration is the time rate of change of the instantaneous velocity**. Therefore, the acceleration is found from the second derivative of the displacement with respect to time.

#### EXAMPLE 5 Instantaneous acceleration

For the first 12 s after launch, the height  $s$  (in m) of a certain rocket is given by  $s = 10\sqrt{t^2 + 25} - 50$ . Find the vertical acceleration of the rocket when  $t = 10.0$  s.

Since the velocity is found from the first derivative and the acceleration is found from the second derivative, we must find the second derivative and evaluate it for  $t = 10.0$  s.

$$\begin{aligned}
 s &= 10\sqrt{t^4 + 25} - 50 \\
 v &= \frac{ds}{dt} = 10\left(\frac{1}{2}\right)(t^4 + 25)^{-1/2}(4t^3) = \frac{20t^3}{(t^4 + 25)^{1/2}} \\
 a &= \frac{dv}{dt} = \frac{d^2s}{dt^2} = \frac{(t^4 + 25)^{1/2}(60t^2) - 20t^3\left(\frac{1}{2}\right)(t^4 + 25)^{-1/2}(4t^3)}{t^4 + 25} \\
 a &= \frac{(t^4 + 25)(60t^2) - 40t^6}{(t^4 + 25)^{3/2}} = \frac{20t^6 + 1500t^2}{(t^4 + 25)^{3/2}} \\
 a &= \frac{20t^2(t^4 + 75)}{(t^4 + 25)^{3/2}}
 \end{aligned}$$

multiply numerator and denominator by  $(t^4 + 25)^{1/2}$

Finding the value of the acceleration when  $t = 10.0$  s, we have

$$a|_{t=10.0} = \frac{20(10.0)^2(10.0^4 + 75)}{(10.0^4 + 25)^{3/2}} = 20.1 \text{ m/s}^2$$

In many applications, it is important to find at which values of  $x$  a specific derivative equals zero. If we factor derivatives whenever possible, the roots (zeros) of such derivatives can be easily obtained.

#### EXAMPLE 6 The zeros of a higher derivative

Determine all values of  $x$  for which  $y'' = 0$  for the function

$$y = \frac{1}{12}x^4 - \frac{5}{6}x^3 + 3x^2 + 2x - 1.$$

$$y' = \frac{1}{12}(4x^3) - \frac{5}{6}(3x^2) + 3(2x) + 2$$

$$y' = \frac{1}{3}x^3 - \frac{5}{2}x^2 + 6x + 2$$

$$y'' = \frac{1}{3}(3x^2) - \frac{5}{2}(2x) + 6$$

$$y'' = x^2 - 5x + 6$$

Since  $y'' = 0$ ,

$$0 = x^2 - 5x + 6$$

We can factor the quadratic in order to solve:

$$0 = (x - 2)(x - 3)$$

so  $x = 2$  or  $x = 3$

## EXERCISES 23.9

In Exercises 1 and 2, make the given changes in the indicated examples of this section and then solve the resulting problems.

1. In Example 1, change  $2x$  to  $2x^2$ .
2. In Example 3, in the denominator change  $1 - x$  to  $1 + 2x$ .

In Exercises 3–10, find all the higher derivatives of the given functions.

3.  $y = x^3 + x^2$
4.  $f(x) = 3x - x^4$
5.  $f(x) = x^3 - 6x^4$
6.  $s = 8t^5 + 5t^4$
7.  $y = (1 - 2x)^4$
8.  $f(x) = (3x + 2)^3$
9.  $f(r) = r(4r + 9)^3$
10.  $y = x(x - 1)^3$

In Exercises 11–30, find the second derivative of each of the given functions.

11.  $y = 2x^7 - x^6 - 3x$

12.  $y = 6x - 2x^5$

13.  $y = 2x + \sqrt{x}$

14.  $r = 3\theta^2 - \frac{20}{\sqrt{\theta}}$

15.  $f(x) = \sqrt[4]{8x - 3}$

16.  $f(x) = \sqrt[3]{6x + 5}$

17.  $f(p) = \frac{4.8\pi}{\sqrt{1 + 2p}}$

18.  $f(x) = \frac{7.5}{\sqrt{3 - 4x}}$

19.  $y = 2(2 - 5x)^4$

20.  $y = (4x + 1)^6$

21.  $y = (3x^2 - 1)^5$

22.  $y = 3(2x^3 + 3)^4$

23.  $f(x) = \frac{2\pi^2}{6 - x}$

24.  $f(R) = \frac{1 - 3R}{1 + 3R}$

25.  $u = \frac{v^2}{v + 15}$

26.  $y = \frac{x}{\sqrt{1 - x^2}}$

27.  $x^2 - y^2 = 9$

28.  $xy + y^2 = 4$

29.  $x^2 - xy = 1 - y^2$

30.  $4xy = y^2 + 2e^3$

In Exercises 31–36, evaluate the second derivative of the given function for the given value of  $x$ .

31.  $f(x) = \sqrt{x^2 + 9}, x = 4$

32.  $f(x) = x - \frac{2}{x^3}, x = -1$

33.  $y = 3x^{2/3} - \frac{2}{x}, x = -8$

34.  $y = 3(1 + 2x)^4, x = \frac{1}{2}$

35.  $v = t(8 - t)^5, t = 2$

36.  $y = \frac{x}{2 - 3x}, x = -\frac{1}{3}$

In Exercises 37–40, find the acceleration of an object for which the displacement  $s$  (in m) is given as a function of the time  $t$  (in s) for the given value of  $t$ .

37.  $s = 26.0t - 4.90t^2, t = 3.00$  s

38.  $s = 3(1 + 2t)^4, t = 0.500$  s

39.  $s = \frac{16}{0.5t + 1}, t = 2.00$  s

40.  $s = 250\sqrt{6t + 1}, t = 4.00$  s

In Exercises 41–53, solve the given problems by finding the appropriate derivatives; round answers to 3 significant digits.

41. Show that  $\frac{d^2}{dx^2}(uv) = u \frac{d^2v}{dx^2} + 2 \frac{du}{dx} \frac{dv}{dx} + \frac{d^2u}{dx^2} v$ .

42. Show that  $\frac{d^6(x^6)}{dx^6} = 6!$ .

43. What is the instantaneous rate of change of the first derivative of  $y$  with respect to  $x$  for  $y = (1 - 2x)^4$  for  $x = 1$ ?

44. What is the instantaneous rate of change of the first derivative of  $y$  with respect to  $x$  for  $2xy + y = 1$  for  $x = 0.5$ ?

45. If the population of a city is  $P(t) = 8000(1 + 0.02t + 0.005t^2)$  ( $t$  is in years from 2000), what is the acceleration in the size of the population?

46. The potential  $V$  (in V) of a certain electric charge is given by  $V = 6/(t + 1)$ , where  $t$  is the time (in s). Find  $d^2V/dt^2$ .

47. A bullet is fired vertically upward. Its distance  $s$  (in m) above the ground is given by  $s = 655t - 4.90t^2$ , where  $t$  is the time (in s). Find the acceleration of the bullet.

48. In testing the brakes on a new model car, it was found that the distance  $s$  (in m) it traveled after the brakes were applied was given by  $s = 19.2 - 0.400t^3$ , where  $t$  is the time (in s). What were the velocity and acceleration for  $t = 4.00$  s?

49. The voltage  $V$  induced in an inductor in an electric circuit is given by  $V = L(d^2q/dt^2)$ , where  $L$  is the inductance (in H). Find the expression for the voltage induced in a 1.60-H inductor if  $q = \sqrt{2t + 1} - 1$ .

50. How fast is the rate of change of solar radiation changing on the surface in Exercise 35 of Section 23.4 at 3 P.M.?

51. The deflection  $y$  (in m) of a 5.00-m beam as a function of the distance  $x$  (in m) from one end is  $y = 0.0001(x^5 - 25x^2)$ . Find the value of  $d^2y/dx^2$  (the rate of change at which the slope of the beam changes) where  $x = 3.00$  m.

52. The force  $F$  (in N) on an object is  $F = 12 dv/dt + 2.0v + 5.0$ , where  $v$  is the velocity (in m/s) and  $t$  is the time (in s). If the displacement is  $s = 25t^{0.60}$ , find  $F$  for  $t = 3.50$  s.

53. A robotic arm moves according to the displacement  $s$  (in m) equation  $s = \frac{1}{6}t^4 - \frac{7}{6}t^3 - 2t^2 + 3$ , where  $t$  is time (in s), for  $t > 0$ . Determine the times at which the acceleration of the robotic arm will be zero.

### Answer to Practice Exercise

1.  $y'' = \frac{18x^2 - 24}{(x^2 + 4)^3}$

## CHAPTER 23 EQUATIONS

### Limit of function

$$\lim_{x \rightarrow a} f(x) = L \quad (23.1)$$

### Difference in $x$ -coordinates

$$\Delta x = x_2 - x_1 \quad (23.2)$$

$$x_2 = x_1 + \Delta x \quad (23.3)$$

**Average slope** 
$$m_{PQ} = \frac{f(x + \Delta x) - f(x)}{(x + \Delta x) - x} = \frac{f(x + \Delta x) - f(x)}{\Delta x} \quad (23.4)$$

**Slope of the tangent line** 
$$m_{\tan} = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h} \quad (23.5)$$

**Definition of derivative** 
$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h} \quad (23.6)$$

**Instantaneous velocity** 
$$v = \lim_{h \rightarrow 0} \frac{s(t + h) - s(t)}{h} \quad (23.7)$$

**Derivatives of polynomials** 
$$\frac{dc}{dx} = 0 \quad (23.8)$$

$$\frac{dx^n}{dx} = nx^{n-1} \quad (23.9)$$

$$\frac{d(c \cdot u)}{dx} = c \cdot \frac{du}{dx} \quad (23.10)$$

$$\frac{d(u + v)}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad (23.11)$$

**Derivative of a product** 
$$\frac{d(u \cdot v)}{dx} = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx} \quad (23.12)$$

**Derivative of a quotient** 
$$\frac{\frac{du}{v}}{\frac{dv}{dx}} = \frac{v \cdot \frac{du}{dx} - u \cdot \frac{dv}{dx}}{v^2} \quad (23.13)$$

**Chain rule** 
$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \quad (23.14)$$

**Derivative of power** 
$$\frac{du^n}{dx} = nu^{n-1} \left( \frac{du}{dx} \right) \quad (23.15)$$

$$\frac{du^{p/q}}{dx} = \frac{p}{q} u^{(p/q)-1} \frac{du}{dx} \quad (23.16)$$

## CHAPTER 23 REVIEW EXERCISES

In Exercises 1–12, evaluate the given limits.

1.  $\lim_{x \rightarrow 4} (8 - 3x)$       2.  $\lim_{x \rightarrow 3} (2x^2 - 10)$

4.  $\lim_{x \rightarrow 3} \sqrt{2x^2 - 18}$       5.  $\lim_{x \rightarrow 2} \frac{4x - 8}{x^2 - 4}$

7.  $\lim_{x \rightarrow 3} \frac{x^2 - 5x + 6}{x^2 - 2x - 3}$       8.  $\lim_{x \rightarrow 4} \frac{2 - \sqrt{x}}{x - 4}$

10.  $\lim_{x \rightarrow \infty} \frac{3x^3 - 5x}{6x^2 + 3}$       11.  $\lim_{x \rightarrow \infty} \frac{x - 2x^3}{(1 + x)^3}$

3.  $\lim_{x \rightarrow -2} \frac{|x + 2|}{x + 2}$

6.  $\lim_{x \rightarrow 5} \frac{x^2 - 25}{3x - 15}$

9.  $\lim_{x \rightarrow \infty} \frac{2 + \frac{1}{x+4}}{3 - \frac{1}{x^2}}$

12.  $\lim_{x \rightarrow \infty} \frac{\sqrt{4x^2 + 3}}{x + 5}$

In Exercises 13–20, use the definition of the derivative/delta method to find the derivative of each of the given functions.

13.  $y = 7 + 5x$       14.  $y = 6x - 2$       15.  $y = 6 - 2x^2$

16.  $y = 12x^2 - x^3$       17.  $y = \frac{2}{x^2}$       18.  $y = \frac{x}{1 - 4x}$

19.  $y = \sqrt{x + 5}$       20.  $y = \frac{1}{\sqrt{x}}$

In Exercises 21–36, find the derivative of each of the given functions.

21.  $y = 2x^7 - 3x^2 + 5$

22.  $y = 8x^7 - 2^5 - x$

23.  $y = 4\sqrt{x} - \frac{3}{x} + \sqrt{3}$

24.  $R = \frac{3}{T^2} - 8\sqrt[4]{T}$

25.  $f(y) = \frac{3y}{1 - 5y}$

26.  $y = \frac{2x - 1}{x^2 + 1}$

27.  $y = (2 - 7x)^4$

28.  $y = (2x^2 - 3)^6$

29.  $y = \frac{3\pi}{(5 - 2x^2)^{3/4}}$

30.  $f(Q) = \frac{70}{(3Q + 1)^3}$

31.  $v = \sqrt{1 + \sqrt{1 + \sqrt{1 + 8s}}}$

32.  $y = (x - 1)^3(x^2 - 2)^2$

33.  $y = \frac{\sqrt{4x + 3}}{2x}$

34.  $R = \frac{\sqrt{t} + 4}{\sqrt{t} - 4}$

35.  $(2x - 3y)^3 = x^2 - y$

36.  $x^2y^2 = x^2 + y^2$

In Exercises 37–40, evaluate the derivatives of the given functions for the given values of  $x$ .

37.  $y = \frac{4}{x} + 2\sqrt[3]{x}$ ,  $x = 8$

38.  $y = (3x - 5)^4$ ,  $x = -2$

39.  $y = 2x\sqrt{12x + 7}$ ,  $x = 1.5$

40.  $y = \frac{\sqrt{2x^2 + 1}}{3x}$ ,  $x = 2$

In Exercises 41–44, find the second derivative of each of the given functions.

41.  $y = 3x^4 - \frac{1}{x}$

42.  $y = \sqrt{1 - 8x}$

43.  $s = \frac{1 - 3t}{1 + 4t}$

44.  $y = 2x(6x + 5)^4$

In Exercises 45–86, solve the given problems, and round answers to 3 significant digits unless otherwise specified.

45. As  $x$  approaches  $0^+$ , which of the functions  $1/x$ ,  $1/x^2$ , and  $1/\sqrt{x}$  increases most rapidly (all become infinite)?

46. The parabola  $y = ax^2 + bx + c$  passes through  $(1, 2)$  and is tangent to the line  $y = x$  at the origin. Find  $a$ ,  $b$ , and  $c$ .

47. Find the acute angle between tangent lines to the parabolas  $y = x^2$  and  $y = (x - 2)^2$  at the point where they intersect.

48. Find the point(s) on the curve of  $y = \frac{x}{x^2 + 1}$  where the tangent line is horizontal.

49. View the graph of  $y = \frac{2(x^2 - 4)}{x - 2}$  on a graphing calculator with window values such that  $y$  can be evaluated exactly for  $x = 2$ . (Xmin =  $-1$  (or  $0$ ), Xmax =  $4$ , Ymin =  $0$ , Ymax =  $10$  will probably work.) Using the trace feature, determine the value of  $y$  for  $x = 2$ . Comment on the accuracy of the view and the value found.

50. A continuous function  $f(x)$  is positive at  $x = 0$  and negative for  $x = 1$ . How many solutions does  $f(x) = 0$  have between  $x = 0$  and  $x = 1$ ? Explain.

51. The velocity  $v$  (in m/s) of a weight falling in water is given by  $v = \frac{6(t + 5)}{t + 1}$ , where  $t$  is the time (in s). What are (a) the initial velocity and (b) the terminal velocity (as  $t \rightarrow \infty$ )?

52. Two lenses of focal lengths  $f_1$  and  $f_2$ , separated by a distance  $d$ , are used in the study of lasers. The combined focal length  $f$  of this lens combination is  $f = \frac{f_1 f_2}{f_1 + f_2 - d}$ . If  $f_2$  and  $d$  remain constant, find the limiting value of  $f$  as  $f_1$  continues to increase in value.

53. Find the slope of a line tangent to the curve of  $y = 7x^4 - x^3$  at  $(-1, 8)$ .

54. Find the slope of a line tangent to the curve of  $y = \sqrt[3]{3 - 8x}$  at  $(-3, 3)$ .

55. Find the point(s) at which a tangent line to the graph of  $y = 1/\sqrt{3x^2 + 3}$  is parallel to the  $x$ -axis.

56. Find the point(s) on the graph of  $y = 2(1 - 3x)^2$  at which a tangent line is parallel to the line  $y = -2x + 5$ .

57. If \$5000 is invested at interest rate  $i$ , compounded quarterly, in two years it will grow to an amount  $A$  given by  $A = 5000(1 + 0.250i)^8$ . Find  $dA/di$ .

58. The temperature  $T$  (in  $^{\circ}\text{C}$ ) of a rotating machine part that has been in operation for  $t$  hours is given by  $T = \frac{100(t + 1)}{t + 5}$ . Find  $dT/dt$  when  $t = 4.00$  h.

59. Find the equations for (a) the velocity and (b) the acceleration if the displacement  $s$  (in m) of an object as a function of the time  $t$  (in s) is given by  $s = \sqrt{1 + 8t}$ .

60. Find the values of the velocity and acceleration for the object in Exercise 59 for  $t = 3.00$  s.

61. The cable of a 200-m suspension bridge can be represented by  $y = 0.0015x^2 + C$ . At one point, the tension is directed along the line  $y = 0.3x - 10$ . Find the value of  $C$ .

62. The displacement  $s$  (in cm) of a piston during each 8.00-s cycle is given by  $s = 8t - t^2$ , where  $t$  is the time (in s). For what value(s) of  $t$  is the velocity of the piston 4.00 cm/s?

63. The reliability  $R$  of a computer system measures the probability that the system will be operating properly after  $t$  hours. For one system,  $R = 1 - kt + \frac{k^2 t^2}{2} - \frac{k^3 t^3}{6}$ , where  $k$  is a constant. Find the expression for the instantaneous rate of change of  $R$  with respect to  $t$ .

64. The distance  $s$  (in m) travelled by a subway train after the brakes are applied is given by  $s = 20t - 2t^2$ , where  $t$  is the time (in s). How far does it travel, after the brakes are applied, in coming to a stop?

65. The electric field  $E$  at a distance  $r$  from a point charge is  $E = k/r^2$ , where  $k$  is a constant. Find an expression for the instantaneous rate of change of the electric field with respect to  $r$ .

66. The velocity of an object moving with constant acceleration can be found from the equation  $v = \sqrt{v_0^2 + 2as}$ , where  $v_0$  is the initial velocity,  $a$  is the acceleration, and  $s$  is the distance travelled. Find  $dv/ds$ .

67. The voltage induced in an inductor  $L$  is given by  $E = L(dI/dt)$ , where  $I$  is the current in the circuit and  $t$  is the time. Find the voltage induced in a 0.4-H inductor if the current  $I$  (in A) is related to the time (in s) by  $I = t(0.01t + 1)^3$ .
68. In studying the energy used by a mechanical robotic device, the equation  $v = \frac{z}{\alpha(1 - z^2) - \beta}$  is used. If  $\alpha$  and  $\beta$  are constants, find  $dv/dz$ .
69. The frictional radius  $r_f$  of a collar used in a braking system is given by  $r_f = \frac{2(R^3 - r^3)}{3(R^2 - r^2)}$ , where  $R$  is the outer radius and  $r$  is the inner radius. Find  $dr_f/dR$  if  $r$  is constant.
70. Water is being drained from a pond such that the volume  $V$  (in  $\text{m}^3$ ) of water in the pond after  $t$  hours is given by  $V = 5000(60 - t)^2$ . Find the rate at which the pond is being drained after 4.00 h.
71. The energy output  $E$  of an electric heater is a function of the time  $t$  (in s) given by  $E = t(1 + 2t)^2$  for  $t < 10.0$  s. Find the power  $dE/dt$  (in W) generated by the heater for  $t = 8.00$  s.
72. The amount  $n$  (in g) of a compound formed during a chemical change is  $n = \frac{8t}{2t^2 + 3}$ , where  $t$  is the time (in s). Find  $dn/dt$  for  $t = 4.00$  s. What is the meaning of the result?
73. The deflection  $y$  of a 10-m beam is  $y = kx(x^4 + 450x^2 - 950)$ , where  $k$  is a constant and  $x$  is the horizontal distance from one end. Find the expression for the instantaneous rate of change of  $y$  with respect to  $x$ .
74. The kinetic energy  $K$  (in J) of a rotating flywheel varies directly as the square of its angular velocity  $\omega$  (in rad/s). If  $K = 125$  J for  $\omega = 75.0$  rad/s, find  $dK/d\omega$  for  $\omega = 155$  rad/s.
75. The frequency  $f$  of a certain electronic oscillator is given by  $f = \frac{1}{2\pi\sqrt{C(L + 2)}}$ , where  $C$  is a capacitance and  $L$  is an inductance. If  $C$  is constant, find  $df/dL$ .
76. The volume  $V$  of fluid produced in the retina of the eye in reaction to exposure to light of intensity  $I$  is given by  $V = \frac{aI^2}{b - I}$ , where  $a$  and  $b$  are constants. Find  $dV/dI$ .
77. The temperature  $T$  (in  $^\circ\text{C}$ ) in a freezer as a function of the time  $t$  (in h) is given by  $T = \frac{10(1 - t)}{0.5t + 1}$ . Find  $dT/dt$ .

78. Under certain conditions, the efficiency  $\eta$  (in %) of an internal combustion engine is given by

$$\eta = 100 \left( 1 - \frac{1}{(V_1/V_2)^{0.4}} \right)$$

where  $V_1$  and  $V_2$  are the maximum and minimum volumes of air in a cylinder, respectively. Assuming that  $V_2$  is kept constant, find the expression for the instantaneous rate of change of efficiency with respect to  $V_1$ .

79. The deflection  $y$  of a cantilever beam (clamped at one end and free at the other end) is  $y = \frac{w}{24EI}(6L^2x^2 - 4Lx^3 + x^4)$ . Here,  $L$  is the length of the beam, and  $w$ ,  $E$ , and  $I$  are constants. Find the first four derivatives of  $y$  with respect to  $x$ . (Each of these derivatives is useful in analysing the properties of the beam.)

80. The number  $n$  of grams of a compound formed during a certain chemical reaction is given by  $n = \frac{2t}{t + 1}$ , where  $t$  is the time (in min). Evaluate  $d^2n/dt^2$  (the rate of increase of the amount of the compound being formed) when  $t = 4.00$  min.
81. The area of a rectangular patio is to be  $75.0 \text{ m}^2$ . Express the perimeter  $p$  of the patio as a function of its width  $w$  and find  $dp/dw$ .
82. A water tank is being designed in the shape of a right circular cylinder with a volume of  $100 \text{ m}^3$ . Find the expression for the instantaneous rate of change of the total surface area  $A$  of the tank with respect to the radius  $r$  of the base.
83. An arch over a walkway can be described by the first-quadrant part of the parabola  $y = 4 - x^2$ . In order to determine the size and shape of rectangular objects that can pass under the arch, express the area  $A$  of a rectangle inscribed under the parabola in terms of  $x$ . Find  $dA/dx$ .
84. A computer analysis showed that a specialized piece of machinery has a value (in dollars) given by  $V = 1\,500\,000/(2t + 10)$ , where  $t$  is the number of years after the purchase. Calculate the value of  $dV/dt$  and  $d^2V/dt^2$  for  $t = 5.00$  years. What is the meaning of these values?
85. An airplane flies over an observer with a velocity of 400 km/h and at an altitude of 500 m. If the plane flies horizontally in a straight line, find the rate at which the distance  $r$  from the observer to the plane is changing 0.600 min after the plane passes over the observer. See Fig. 23.30.

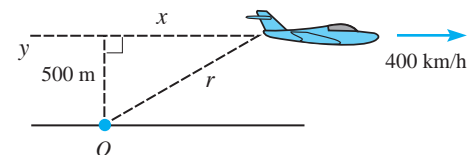


Fig. 23.30

86. The radius of curvature of  $y = f(x)$  at the point  $(x, y)$  on the curve of  $y = f(x)$  is given by

$$R = \frac{[1 + (y')^2]^{3/2}}{|y''|}$$

A certain roadway follows the parabola  $y = 1.2x - x^2$  for  $0 < x < 1.2$ , where  $x$  is measured in kilometres. Find  $R$  for  $x = 0.200$  km and  $x = 0.600$  km. See Fig. 23.31.

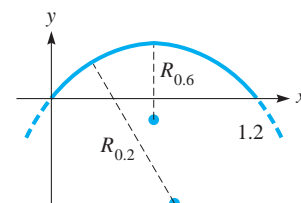


Fig. 23.31

### Writing Exercise

87. An engineer designing military rockets uses a computer simulation to find the path of a rocket as  $y = f(x)$  and the path of an aircraft to be  $y = g(x)$ . Write two or three paragraphs explaining how the engineer can determine the angle at which the path of the rocket crosses the path of the aircraft.

## CHAPTER 23 PRACTICE TEST

1. Find  $\lim_{x \rightarrow 1} \frac{x^2 - x}{x^2 - 1}$ .
2. Find  $\lim_{x \rightarrow \infty} \frac{1 - 4x^2}{x + 2x^2}$ .
3. Find the slope of a line tangent to the curve of  $y = 3x^2 - \frac{4}{x^2}$  at  $(2, 11)$ .
4. The displacement  $s$  (in cm) of a pumping machine piston in each cycle is given by  $s = t\sqrt{10 - 2t}$ , where  $t$  is the time (in s). Find the velocity of the piston for  $t = 4.00$  s.
5. Find  $dy/dx$ :  $y = 4x^6 - 2x^4 + \pi^3$
6. Find  $dy/dx$ :  $y = 2x(5 - 3x)^4$
7. Find  $dy/dx$ :  $(1 + y^2)^3 - x^2y = 7x$ .
8. Under certain conditions, due to the presence of a charge  $q$ , the electric potential  $V$  along a line is given by
 
$$V = \frac{kq}{\sqrt{x^2 + b^2}}$$
 where  $k$  is a constant and  $b$  is the minimum distance from the charge to the line. Find the expression for the instantaneous rate of change of  $V$  with respect to  $x$ .
9. Find the second derivative of  $y = \frac{2x}{3x + 2}$ .
10. By using the definition, find the derivative of  $y = 5x - 2x^2$  with respect to  $x$ .



# Applications of the Derivative

Following the work of Newton and Leibniz, the development of the calculus proceeded rapidly but in a rather disorganized way. Much of the progress in the late 1600s and early 1700s was due to a desire to solve applied problems, particularly in some areas of physics. These included problems such as finding velocities in more complex types of motion, accurately measuring time by use of a pendulum, and finding the equation of a uniform cable hanging under its own weight.

A number of mathematicians, most of whom also studied in various areas of physics, contributed to these advances in calculus. Among them was the Swiss mathematician Leonhard Euler, the most prolific mathematician of all time. Throughout the mid- to late 1700s, he used the idea of a function to better organize the study of algebra, trigonometry, and calculus. In doing so, he fully developed the use of calculus on problems from physics in areas such as planetary motion, mechanics, and optics.

Euler had a nearly unbelievable memory and ability to calculate. At an early age, he memorized the entire *Aeneid* by the Roman poet Virgil and was able to recite it from memory at age 70. In his head he solved major problems related to the motion of the moon that Newton had not been able to solve. At one time, he was given two solutions to a problem that differed in the 50th decimal place, and he determined, in his head, which was correct. Although blind for the last 17 years of his life, it was one of his most productive periods. From memory, he dictated many of his articles (he wrote a total of over 70 volumes) until his sudden death in 1783.

We have noted some of the problems in technology in which the derivative plays a key role in the solution. Another important type is finding the maximum values or minimum values of functions. Such values are useful, for example, in finding the maximum possible income from production or the least amount of material needed in making a product. In this chapter, we consider several of these kinds of applications of the derivative.

## 24

### LEARNING OUTCOMES

After completion of this chapter, the student should be able to:

- Find the equation of a line tangent or normal to a given curve
- Solve equations using Newton's method
- Find the velocity and acceleration of an object undergoing curvilinear motion
- Solve related rates problems
- Use derivatives to describe important features of the graph of a function such as maxima, minima, points of inflection, and concavity
- Sketch a curve using information about the function and its derivatives
- Solve applied maximum and minimum problems
- Use differentials to estimate errors in measurement
- Obtain the linear approximation of a function



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◀ In Section 24.7, we see how to use the derivative in the design of cylindrical containers such as storage tanks.

## 24.1 Tangents and Normals

### Tangent Line • Normal Line

The first application of the derivative we consider involves finding the equation of a line *tangent* to a given curve and the equation of a line *normal* (perpendicular) to a given curve.

#### TANGENT LINE

To find the equation of a line tangent to a curve at a given point, we first find the derivative, which is then evaluated at the point. This gives us the slope of the tangent line to the curve at the point. Then, by using the point-slope form of the equation of a straight line, we find the equation of the tangent line. The following examples illustrate the method.

#### EXAMPLE 1 Equation of a tangent line

Find the equation of the line tangent to the parabola  $y = x^2 - 1$  at the point  $(-2, 3)$ . Finding the derivative and evaluating it at  $x = -2$ , we have

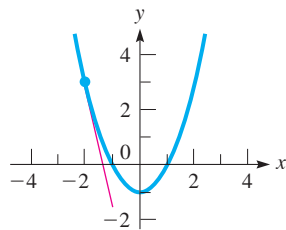


Fig. 24.1

$$\frac{dy}{dx} = 2x \quad \text{derivative}$$

$$\left. \frac{dy}{dx} \right|_{x=-2} = -4 \quad \text{evaluate derivative at } (-2, 3)$$

$$y - 3 = -4(x + 2) \quad \text{point-slope form of straight line}$$

$$y = -4x - 5 \quad \text{equation of tangent line}$$

The parabola and the tangent line are shown in Fig. 24.1. ■

#### EXAMPLE 2 Tangent line to an implicit function

Find the equation of the line tangent to the ellipse  $4x^2 + 9y^2 = 40$  at the point  $(1, 2)$ . Treating the equation as an implicit function, we have the following solution.

$$8x + 18yy' = 0 \quad \text{find derivative}$$

$$y' = -\frac{4x}{9y}$$

$$y' \big|_{(1,2)} = -\frac{4}{18} = -\frac{2}{9} \quad \text{evaluate derivative to find slope of tangent line}$$

$$y - 2 = -\frac{2}{9}(x - 1) \quad \text{point-slope form of tangent line}$$

$$9y - 18 = -2x + 2$$

$$2x + 9y - 20 = 0 \quad \text{standard form of tangent line}$$

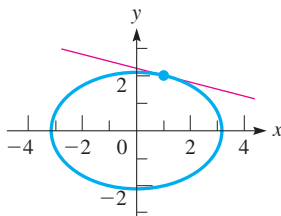


Fig. 24.2

The ellipse and the tangent line  $2x + 9y - 20 = 0$  are shown in Fig. 24.2. ■

#### NORMAL LINE

If we wish to obtain the equation of a line normal (perpendicular to a tangent) to a curve, we must recall that the slopes of perpendicular lines are negative reciprocals. Thus, the derivative is found and evaluated at the specified point. Since this gives the slope of a tangent line, ***we take the negative reciprocal of this number to find the slope of the normal line.*** Then by using the point-slope form of the equation of a straight line, we find the equation of the normal. The following examples illustrate the method.

■ About 1700, the word *normal* was adapted from the Latin word *normalis*, which was being used for *perpendicular*.

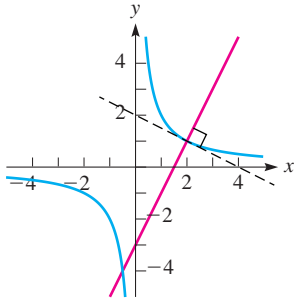


Fig. 24.3

**EXAMPLE 3** Equation of a normal line

Find the equation of a line normal to the hyperbola  $y = 2/x$  at the point  $(2, 1)$ .

Taking the derivative and evaluating it for  $x = 2$ , we have

$$\frac{dy}{dx} = -\frac{2}{x^2}, \quad \left. \frac{dy}{dx} \right|_{x=2} = -\frac{1}{2}$$

Therefore, the slope of the normal line at  $(2, 1)$  is 2. The equation of the normal line is then

$$\begin{aligned} y - 1 &= 2(x - 2) \\ y &= 2x - 3 \end{aligned}$$

The hyperbola and normal line are shown in Fig. 24.3. ■

Many of the applications of tangents and normals are geometric. However, there are certain applications in technology, and one of these is shown in the following example. Others are shown in the exercises.

**EXAMPLE 4** Normal line—application

In Fig. 24.4, the cross section of a parabolic solar reflector is shown, along with an incident ray of light and the reflected ray. The angle of incidence  $i$  is equal to the angle of reflection  $r$  where both angles are measured with respect to the normal to the surface. If the incident ray strikes at the point where the slope of the normal is  $-1$  and the equation of the parabola is  $4y = x^2$ , what is the equation of the normal line?

If the slope of the normal line is  $-1$ , then the slope of a tangent line is  $-\left(\frac{1}{-1}\right) = 1$ . Therefore, we know that the value of the derivative at the point of reflection is 1. This allows us to find the coordinates of the point:

$$\begin{aligned} 4y &= x^2 \\ 4 \frac{dy}{dx} &= 2x, & \frac{dy}{dx} &= \frac{1}{2}x & \text{find derivative} \\ 1 &= \frac{1}{2}x & & & \text{substitute } \frac{dy}{dx} = 1 \\ x &= 2 \end{aligned}$$

This means that the  $x$ -coordinate of the point of reflection is 2. We can find the  $y$ -coordinate by substituting  $x = 2$  into the equation of the parabola. Thus, the point is  $(2, 1)$ . Since the slope is  $-1$ , the equation is

$$\begin{aligned} y - 1 &= (-1)(x - 2) \\ y &= -x + 3 \end{aligned}$$

If the incident ray is vertical, for which  $i = 45^\circ$  at the point  $(2, 1)$ , the reflected ray passes through  $(0, 1)$ , the focus of the parabola. See Fig. 24.5. This illustrates the important reflection property of a parabola that *any incident ray parallel to its axis passes through the focus*. We first noted this property in our discussion of the parabola in Example 4 of Section 21.4. ■

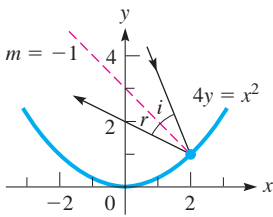


Fig. 24.4

**Practice Exercise**

- Find the equation of the line normal to  $y = 4 - x^2$  at  $(3, -5)$ .

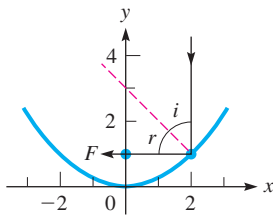


Fig. 24.5

**EXERCISES 24.1**

In Exercises 1 and 2, make the given changes in the indicated examples of this section and then solve the resulting problems.

- In Example 2, change  $4x^2 + 9y^2$  to  $x^2 + 4y^2$ , change 40 to 17, and then find the equation of the tangent line.
- In Example 3, change  $2/x$  to  $3/(x + 1)$  and then find the equation of the normal line.

In Exercises 3–6, find the equations of the lines tangent to the indicated curves at the given points. In Exercises 3 and 6, sketch the curve and the tangent line. In Exercises 4 and 5, use a graphing calculator to view the curve and the tangent line.

- $y = x^2 + 2$  at  $(2, 6)$
- $y = \frac{1}{3}x^3 - 5x$  at  $(3, -6)$
- $y = \frac{1}{x^2 + 1}$  at  $(1, \frac{1}{2})$
- $x^2 + y^2 = 25$  at  $(3, 4)$

In Exercises 7–10, find the equations of the lines normal to the indicated curves at the given points. In Exercises 7 and 10, sketch the curve and the normal line. In Exercises 8 and 9, use a graphing calculator to view the curve and the normal line.

7.  $y = 6x - 2x^2$  at  $(2, 4)$       8.  $y = 8 - x^3$  at  $(-1, 9)$   
 9.  $y = \frac{6}{(x^2 + 1)^2}$  at  $(1, \frac{3}{2})$       10.  $4x^2 - y^2 = 20$  at  $(-3, 4)$ .

In Exercises 11–14, find the equations of the lines tangent or normal to the given curves and with the given slopes. View the curves and lines on a graphing calculator.

11.  $y = x^2 - 2x$ , tangent line with slope 2  
 12.  $y = \sqrt{2x - 9}$ , tangent line with slope 1  
 13.  $y = (2x - 1)^3$ , normal line with slope  $-\frac{1}{24}$ ,  $x > 0$   
 14.  $y = \frac{1}{2}x^4 + 1$ , normal line with slope 4

In Exercises 15–30, solve the given problems involving tangent and normal lines.

15. Find the equations of the tangent and normal lines to the parabola with vertex at  $(0, 3)$  and focus at  $(0, 0)$ , where  $x = -1$ . Graph the curve and lines.  
 16. Find the equations of the tangent and normal lines to the ellipse with focus at  $(4, 0)$ , vertex at  $(5, 0)$ , and centre at the origin, where  $x = 2$ . Use a graphing calculator to view the curve and lines.  
 17. Show that the line tangent to the graph of  $y = x + 2x^2 - x^4$  at  $(1, 2)$  is also tangent at  $(-1, 0)$ .  
 18. Show that the graphs of  $y^2 = 4x + 4$  and  $y^2 = 4 - 4x$  cross at right angles.  
 19. Without actually finding the points of intersection, explain why the parabola  $y^2 = 4x$  and the ellipse  $2x^2 + y^2 = 6$  intersect at right angles. (*Hint:* Call a point of intersection  $(a, b)$ .)  
 20. Find the  $y$ -intercept of the line normal to the curve  $y = x^{3/4}$ , where  $x = 16$ .  
 21. Show that the equation of the tangent line to the circle  $x^2 + y^2 = a^2$  at the point  $(x_1, y_1)$  is  $x_1x + y_1y = a^2$ .  
 22. At what point on the curve  $y = x^4$  does the normal line have a slope of 16?  
 23. Heat flows normal to isotherms, curves along which the temperature is constant. Find the line along which heat flows through the point  $(2, 1)$  if the isotherm is the graph of  $2x^2 + y^2 = 9$ .  
 24. The sparks from an emery wheel to sharpen blades fly off tangent to the wheel. Find the equation along which sparks fly from a wheel described by  $x^2 + y^2 = 25$ , at  $(3, 4)$ .

25. A certain suspension cable with supports on the same level is closely approximated as being parabolic in shape. If the supports are 80 m apart and the sag at the centre is 10 m, what is the equation of the line along which the tension acts (tangentially) at the right support? (Choose the origin of the coordinate system at the lowest point of the cable.)  
 26. In a video game, airplanes move from left to right along the path described by  $y = 2 + 1/x$ . They can shoot rockets tangent to the direction of flight at targets on the  $x$ -axis located at  $x = 1, 2, 3$ , and 4. Will a rocket fired from  $(1, 3)$  hit a target?  
 27. In an electric field, the lines of force are perpendicular to the curves of equal electric potential. In a certain electric field, a curve of equal potential is  $y = \sqrt{2x^2 + 8}$ . If the line along which the force acts on an electron has an inclination of  $135^\circ$ , find its equation.  
 28. A radio wave reflects from a reflecting surface in the same way as a light wave (see Example 4). A certain horizontal radio wave reflects off a parabolic reflector such that the reflected wave is  $43.60^\circ$  below the horizontal, as shown in Fig. 24.6. If the equation of the parabola is  $y^2 = 8x$ , what is the equation of the normal line through the point of reflection?

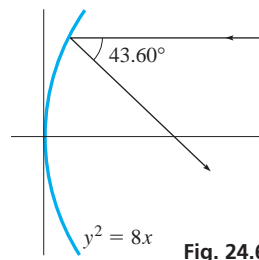


Fig. 24.6

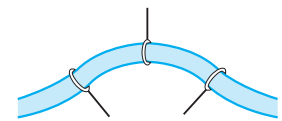


Fig. 24.7

29. In designing a flexible tubing system, the supports for the tubing must be perpendicular to the tubing. If a section of the tubing follows the curve  $y = \frac{4}{x^2 + 1}$  ( $-2 \text{ dm} < x < 2 \text{ dm}$ ), along which lines must the supports be directed if they are located at  $x = -1, x = 0$ , and  $x = 1$ ? See Fig. 24.7.  
 30. On a particular drawing, a pulley wheel can be described by the equation  $x^2 + y^2 = 100$  (units in cm). The pulley belt is directed along the lines  $y = -10$  and  $4y - 3x - 50 = 0$  when first and last making contact with the wheel. What are the first and last points on the wheel where the belt makes contact?

#### Answer to Practice Exercise

1.  $x - 6y = 33$

## 24.2 Newton's Method for Solving Equations

### Iterative Method • Newton's Method for Solving Equations

As we know, finding the roots of an equation  $f(x) = 0$  is very important in mathematics and in many types of applications, and we have developed methods of solving many types of equations in the previous chapters. However, for a great many algebraic and nonalgebraic equations, there is no method for finding the roots exactly.

■ Newton's method is another mathematical development by the English mathematician and physicist Isaac Newton (1642–1727).

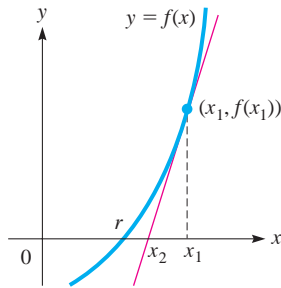


Fig. 24.8

We have shown that the roots of an equation can be found with great accuracy on a graphing calculator. In this section, we develop **Newton's method**, which uses the derivative to find approximately, but also with great accuracy, the real roots of many kinds of algebraic and nonalgebraic equations.

Newton's method is an **iterative method**, which starts with a reasonable guess for the root, and then yields a new and better approximation. This, in turn, is used to obtain an even better approximation, and so on until an approximate answer with the required degree of accuracy is obtained. Iterative methods are easily programmable for use on a computer.

Let us consider a section of the curve of  $y = f(x)$  that (a) crosses the  $x$ -axis, (b) always has either a positive slope or a negative slope, and (c) has a slope that either becomes greater or becomes less as  $x$  increases. See Fig. 24.8. (When either (b) or (c) is not satisfied, Newton's method may fail to find the root. See Exercises 20 and 23.) The curve in the figure crosses the  $x$ -axis at  $x = r$ , which means that  $x = r$  is a root of the equation  $f(x) = 0$ . If  $x_1$  is sufficiently close to  $r$ , a line tangent to the curve at  $[x_1, f(x_1)]$  will cross the  $x$ -axis at a point  $(x_2, 0)$ , with  $x_2$  closer to  $r$  than  $x_1$ .

We know that the slope of the tangent line is the value of the derivative at  $x_1$ , or  $m_{\text{tan}} = f'(x_1)$ . Therefore, the equation of the tangent line is

$$y - f(x_1) = f'(x_1)(x - x_1)$$

For the point  $(x_2, 0)$  on this line, we have

$$-f(x_1) = f'(x_1)(x_2 - x_1)$$

Solving for  $x_2$ , we have

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

Here,  $x_2$  is a second approximation to the root. We can repeat the process starting with  $x_2$  in place of  $x_1$ , obtaining an even better approximation  $x_3$ . The repetition of this process is what we call Newton's method, which we now summarize.

### Newton's Method

Let  $x = r$  be a root of the equation  $f(x) = 0$ , and let  $x_1$  be a first approximation of  $r$ . We obtain approximations  $x_2, x_3, \dots$  by using

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad (24.1)$$

for  $n = 1, 2, \dots$ . The number of iterations depends on the required accuracy.

The initial approximation  $x_1$  (which must be close to  $r$ ) may be found by either of the following procedures:

1. Choose  $x_1$  inside an interval  $(a, b)$  where  $f(a)$  and  $f(b)$  have opposite signs, with  $x_1$  closer to the endpoint where the function is closer to zero.
2. Sketch the graph of the function and choose  $x_1$  as an estimate of the  $x$ -intercept of the function. For some equations, it may be easier to choose  $x_1$  as an estimate of the intersection of two functions.

### EXAMPLE 1 Using Newton's method

Find the root of  $x^2 - 3x + 1 = 0$  between  $x = 0$  and  $x = 1$ .

Here,  $f(x) = x^2 - 3x + 1$ ,  $f(0) = 1$ , and  $f(1) = -1$ . This indicates that the root may be near the middle of the interval. Therefore, we choose  $x_1 = 0.5$ .

The derivative is

$$f'(x) = 2x - 3$$

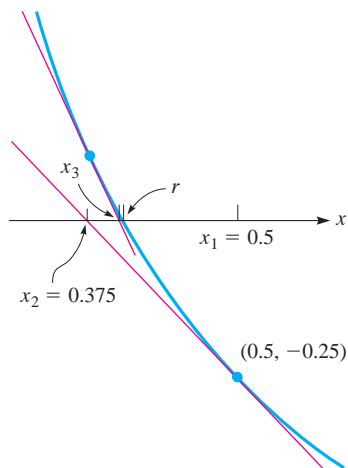


Fig. 24.9

**Practice Exercise**

1. In Example 1, let  $x_1 = 0.3$ , and find  $x_2$ .

■ An explanation and an example of Newton's method due to Thomas Simpson (of Simpson's rule) and dating back to 1740 can be found in the text's companion website.

Therefore,  $f(0.5) = -0.25$  and  $f'(0.5) = -2$ , which gives us

$$x_2 = 0.5 - \frac{-0.25}{-2} = 0.375$$

This is a second approximation, which is closer to the actual value of the root. See Fig. 24.9. A second iteration using  $x_2 = 0.375$ ,  $f(0.375) = 0.015625$ , and  $f'(0.375) = -2.25$  gives us

$$x_3 = 0.375 - \frac{0.015625}{-2.25} = 0.381944$$

We can check this particular result by using the quadratic formula. This tells us the root is  $x = 0.3819660$ . Our result using Newton's method is good to three decimal places, and additional accuracy may be obtained by using the method again as many times as needed. ■

**EXAMPLE 2 Newton's method—application**

A spherical water-storage tank holds  $500.0 \text{ m}^3$ . If the outside diameter is  $10.0000 \text{ m}$ , what is the thickness of the metal of which the tank is made?

Let  $x =$  the thickness of the metal. We know that the outside radius of the tank is  $5.0000 \text{ m}$ . Therefore, using the formula for the volume of a sphere, we have

$$\frac{4\pi}{3}(5.0000 - x)^3 = 500.0$$

$$125.0 - 75.00x + 15.00x^2 - x^3 = 119.366$$

$$x^3 - 15.00x^2 + 75.00x - 5.634 = 0$$

$$f(x) = x^3 - 15.00x^2 + 75.00x - 5.634$$

$$f'(x) = 3x^2 - 30.00x + 75.00$$

Since  $f(0) = -5.634$  and  $f(0.1) = 1.717$ , the root may be closer to  $0.1$  than to  $0.0$ . Therefore, we let  $x_1 = 0.07$ . Setting up a table, we have these values:

$n$	$x_n$	$f(x_n)$	$f'(x_n)$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	0.07	-0.457157	72.9147	0.0762697508
2	0.0762697508	-0.000581145	72.7293587	0.0762777413

Since  $x_2 = x_3 = 0.0763$  to four decimal places, the thickness is  $0.0763 \text{ m}$ . This means the inside radius of the tank is  $4.9237 \text{ m}$ , and this value gives an inside volume of  $500.0 \text{ m}^3$ . In using a calculator, the values in the table are more easily found if the values of  $x_n$ ,  $f(x_n)$ , and  $f'(x_n)$  are stored in memory for each step. ■

**EXAMPLE 3 Graphically locating  $x_1$** 

Solve the equation  $x^2 - 1 = \sqrt{4x - 1}$ .

We can see approximately where the root is by sketching the graphs of  $y_1 = x^2 - 1$  and  $y_2 = \sqrt{4x - 1}$ , as shown in Fig. 24.10. We see that they intersect between  $x = 1$  and  $x = 2$ . Therefore, we choose  $x_1 = 1.5$ . With

$$f(x) = x^2 - 1 - \sqrt{4x - 1}$$

$$f'(x) = 2x - \frac{2}{\sqrt{4x - 1}}$$

we now find the values in the following table:

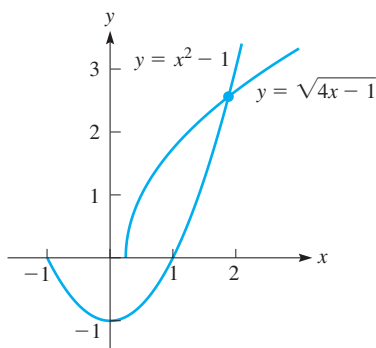


Fig. 24.10



$n$	$x_n$	$f(x_n)$	$f'(x_n)$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.5	-0.986 067 98	2.105 572 8	1.968 313 4
2	1.968 313 4	0.252 568 59	3.173 759 8	1.888 733 2
3	1.888 733 2	0.007 052 69	2.996 295 7	1.886 379 4
4	1.886 379 4	0.000 006 20	2.991 026 5	1.886 377 3

Since  $x_5 = x_4 = 1.88638$  to five decimal places, this is the required solution. (Here, rounded-off values of  $x_n$  are shown, although additional digits were carried and used.)

## EXERCISES 24.2

In Exercises 1–4, find the indicated roots of the given quadratic equations by finding  $x_3$  from Newton's method. Compare this root with that obtained by using the quadratic formula.

- In Example 1, change the middle term from  $-3x$  to  $-5x$  and use the same  $x_1$ .
- $2x^2 - x - 2 = 0$  (between 1 and 2)
- $3x^2 - 5x - 1 = 0$  (between -1 and 0)
- $x^2 + 4x + 2 = 0$  (between -4 and -3)

In Exercises 5–16, find the indicated roots of the given equations to at least four decimal places by using Newton's method.

- $x^3 - 6x^2 + 10x - 4 = 0$  (between 0 and 1)
- $x^3 - 3x^2 - 2x + 3 = 0$  (between 0 and 1)
- $x^3 + 6x^2 + 9x + 2 = 0$  (the smallest root)
- $2x^3 + 2x^2 - 11x + 3 = 0$  (the largest root)
- $x^4 - x^3 - 3x^2 - x - 4 = 0$  (between 2 and 3)
- $2x^4 - 2x^3 - 5x^2 - x - 3 = 0$  (between -2 and -1)
- $x^4 - 2x^3 - 8x - 16 = 0$  (the negative root)
- $3x^4 - 3x^3 - 11x^2 - x - 4 = 0$  (the negative root)
- $2x^2 = \sqrt{2x + 1}$  (the positive real solution)
- $x^3 = \sqrt{x + 1}$  (the real solution)
- $x = \frac{1}{\sqrt{x + 2}}$  (the real solution)
- $x^{3/2} = \frac{1}{2x + 1}$  (the real solution)

In Exercises 17–31, determine the required values to at least 4 decimal places by using Newton's method.

- Find all the real roots of  $x^3 - 2x^2 - 5x + 4 = 0$ .
- Find all the real roots of  $x^4 - 2x^3 + 3x^2 + x - 7 = 0$ .
- Explain how to find  $\sqrt[3]{4}$  by using Newton's method.
- Explain why Newton's method does not work for finding the root of  $x^3 - 3x = 5$  if  $x_1$  is chosen as 1.
- Use Newton's method to find an expression for  $x_{n+1}$ , in terms of  $x_n$  and  $a$ , for the equation  $x^2 - a = 0$ . Such an equation can be used to find  $\sqrt{a}$ .

- Use Newton's method on  $f(x) = x^{1/3}$  with  $x_1 = 1$ . Calculate  $x_2, x_3$ , and  $x_4$ . What is happening as successive approximations are calculated?
- To calculate reciprocals without dividing, a computer programmer applied Newton's method to the equation  $1/x - a = 0$ . Show that  $x_2 = 2x_1 - ax_1^2$ . From this, determine the expression for  $x_n$ .
- The altitude  $h$  (in m) of a rocket is given by  $h = -2t^3 + 84t^2 + 480t + 10$ , where  $t$  is the time (in s) of flight. When does the rocket hit the ground?
- A solid sphere of specific gravity  $s$  sinks in water to a depth  $h$  (in cm) given by  $0.00926h^3 - 0.0833h^2 + s = 0$ . Find  $h$  for  $s = 0.786$  (when the diameter of the sphere is 6.00 cm).
- A dome in the shape of a spherical segment is to be placed over the top of a sports stadium. If the radius  $r$  of the dome is to be 60.0 m and the volume  $V$  within the dome is  $180\,000\text{ m}^3$ , find the height  $h$  of the dome. See Fig. 24.11. ( $V = \frac{1}{6}\pi h(h^2 + 3r^2)$ .)

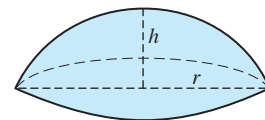


Fig. 24.11

- The capacitances (in  $\mu\text{F}$ ) of three capacitors in series are  $C$ ,  $C + 1.00$ , and  $C + 2.00$ . If their combined capacitance is  $1.00\ \mu\text{F}$ , their individual values can be found by solving the equation

$$\frac{1}{C} + \frac{1}{C + 1.00} + \frac{1}{C + 2.00} = 1.00$$

Find these capacitances.

- An oil-storage tank has the shape of a right circular cylinder with a hemisphere at each end. See Fig. 24.12. If the volume of the tank is  $50.0\text{ m}^3$  and the length  $l$  is 4.00 m, find the radius  $r$ .

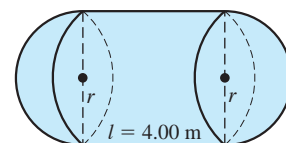


Fig. 24.12



29. A rectangular block of plastic with edges 2.00 cm, 2.00 cm, and 4.00 cm is heated until its volume doubles. By how much does each edge increase if each increases by the same amount?

30. Water flow in a wide channel approaches a bump. See Fig. 24.13. The water depth  $x$  (in m) over the bump is the largest root of the equation

$$x^3 - 1.015x^2 + 0.115 = 0.$$

Find  $x$  if it is between 0.750 m and 1.20 m.

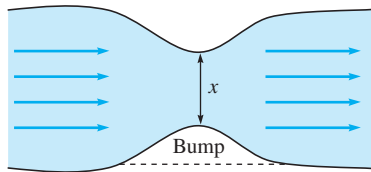


Fig. 24.13

31. The vibration analysis of a three-storey building requires that the following equation be solved:

$$\lambda^3 - 0.0012\lambda^2 + 9 \times 10^{-7}\lambda - 7.2 \times 10^{-11} = 0.$$

Find the root that lies between 0 and  $4 \times 10^{-4}$  to 3 significant digits.

#### Answer to Practice Exercise

1.  $x_2 = 0.379167$

## 24.3 Curvilinear Motion

Vectors and Curvilinear Motion •  
Parameter • Parametric Form •  
Acceleration

When velocity was introduced in Section 23.4, the discussion was limited to rectilinear motion, or motion along a straight line. A more general discussion of velocity is necessary when we discuss the motion of an object in a plane. There are many important applications of motion in a plane, a principal one being the motion of a projectile.

An important concept in developing this topic is that of a vector. The necessary fundamentals related to vectors are taken up in Chapter 9. Although vectors can be used to represent many physical quantities, we will restrict our attention to their use in describing the velocity and acceleration of an object moving in a plane along a specified path. Such motion is called **curvilinear motion**.

In describing an object undergoing curvilinear motion, it is common to express the  $x$ - and  $y$ -coordinates of its position separately as functions of time. Equations given in this form—that is,  $x$  and  $y$  both given in terms of a third variable (in this case,  $t$ )—are said to be in **parametric form**, which we encountered in Section 10.6. The third variable,  $t$ , is called the **parameter**. Each value of the parameter  $t$  determines a point  $(x(t), y(t))$ , and the set of all such points determines the graph of the curve. The graph can be sketched by plotting enough points to indicate its shape. In some cases, we can eliminate the parameter  $t$  to obtain an equation in  $x$  and  $y$ , which we can then graph.

To find the velocity of an object whose coordinates are given in parametric form, we find its  $x$ -component of velocity  $v_x$  by determining  $dx/dt$  and its  $y$ -component of velocity  $v_y$  by determining  $dy/dt$ . These are then evaluated, and the resultant velocity is found from  $v = \sqrt{v_x^2 + v_y^2}$ . The direction in which the object is moving is found from  $\tan \theta = v_y/v_x$ .

### EXAMPLE 1 Parametric form—resultant velocity

If the horizontal distance  $x$  that an object has moved is given by  $x = 3t^2$  and the vertical distance  $y$  is given by  $y = 1 - t^2$ , find the resultant velocity when  $t = 2$ .

To find the resultant velocity, we must find  $v$  and  $\theta$ , by first finding  $v_x$  and  $v_y$ . After the derivatives are found, they are evaluated for  $t = 2$ . Therefore,

$$v_x = \frac{dx}{dt} = 6t \qquad v_x|_{t=2} = 12 \qquad \text{find velocity components}$$

$$v_y = \frac{dy}{dt} = -2t \qquad v_y|_{t=2} = -4$$

$$v = \sqrt{12^2 + (-4)^2} = 12.6 \qquad \text{magnitude of velocity}$$

$$\tan \theta = \frac{-4}{12} \qquad \theta = -18.4^\circ \qquad \text{direction of motion}$$

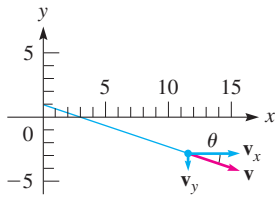


Fig. 24.14

To sketch the graph of the path, we can eliminate the parameter  $t$  by substituting  $t^2 = x/3$  (from the equation for  $x$ ) into the equation for  $y$ . Therefore,  $y = 1 - \frac{x}{3}$ , and the path of the object is a line. The path and velocity vectors are shown in Fig. 24.14.

**EXAMPLE 2** Parametric form—resultant velocity

Find the velocity and direction of motion when  $t = 2$  of an object moving such that its  $x$ - and  $y$ -coordinates of position are given by  $x = 1 + 2t$  and  $y = t^2 - 3t$ .

$$v_x = \frac{dx}{dt} = 2 \qquad v_x|_{t=2} = 2 \qquad \text{find velocity components}$$

$$v_y = \frac{dy}{dt} = 2t - 3 \qquad v_y|_{t=2} = 1$$

$$v|_{t=2} = \sqrt{2^2 + 1^2} = 2.24 \qquad \text{magnitude of velocity}$$

$$\tan \theta = \frac{1}{2} \qquad \theta = 26.6^\circ \qquad \text{direction of motion}$$

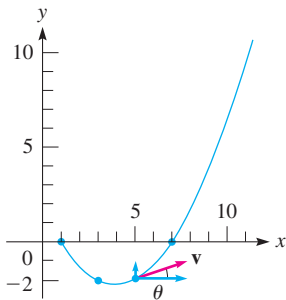


Fig. 24.15

We graph the path by plotting the following points:

$t$	0	1	2	3
$x$	1	3	5	7
$y$	0	-2	-2	0

The path and the velocity are shown in Fig. 24.15.

**COMMON ERROR**

Note that to find the resultant velocity, **we first find the necessary derivatives and then evaluate them**. This procedure should always be followed. When a derivative is to be found, it is incorrect to take the derivative of the evaluated expression (which is a constant).

**Acceleration** is the time rate of change of velocity. Therefore, if the velocity, or its components, is known as a function of time, the acceleration of an object can be found by taking the derivative of the velocity with respect to time. If the displacement is known, the acceleration is found by finding the second derivative with respect to time. Finding the acceleration of an object is illustrated in the following example.

**EXAMPLE 3** Parametric form—resultant acceleration

Find the magnitude and direction of the acceleration when  $t = 2$  for an object that is moving such that its  $x$ - and  $y$ -coordinates of position are given by the parametric equations  $x = t^3$  and  $y = 1 - t^2$ .

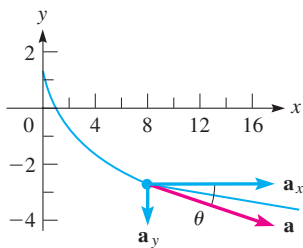


Fig. 24.16

$$v_x = \frac{dx}{dt} = 3t^2 \qquad a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2} = 6t \qquad a_x|_{t=2} = 12$$

↑ take second derivatives to find acceleration components ↓

$$v_y = \frac{dy}{dt} = -2t \qquad a_y = \frac{dv_y}{dt} = \frac{d^2y}{dt^2} = -2 \qquad a_y|_{t=2} = -2$$

$$a|_{t=2} = \sqrt{12^2 + (-2)^2} = 12.2 \qquad \text{magnitude of acceleration}$$

$$\tan \theta = \frac{a_y}{a_x} = -\frac{2}{12} \qquad \theta = -9.5^\circ \qquad \text{direction of acceleration}$$

The quadrant in which  $\theta$  lies is determined from the fact that  $a_y$  is **negative** and  $a_x$  is **positive**. Thus,  $\theta$  must be a fourth-quadrant angle (see Fig. 24.16).

## Practice Exercise

1. In Example 3, solve for the acceleration when  $t = 2$ , if  $x = 0.8t^{5/2}$ , instead of  $x = t^3$ .

We now summarize the equations used to find the velocity and acceleration of an object for which the displacement is a function of time. They indicate how to find the components, as well as the magnitude and direction, of each.

$$v_x = \frac{dx}{dt} \qquad v_y = \frac{dy}{dt} \qquad \text{velocity components} \qquad (24.2)$$

$$a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2} \qquad a_y = \frac{dv_y}{dt} = \frac{d^2y}{dt^2} \qquad \text{acceleration components} \qquad (24.3)$$

$$v = \sqrt{v_x^2 + v_y^2} \qquad a = \sqrt{a_x^2 + a_y^2} \qquad \text{magnitude} \qquad (24.4)$$

$$\tan \theta_v = \frac{v_y}{v_x} \qquad \tan \theta_a = \frac{a_y}{a_x} \qquad \text{direction} \qquad (24.5)$$

## COMMON ERROR

If the curvilinear path an object follows is given with  $y$  as a function of  $x$ , **the velocity (or acceleration) is found by taking derivatives of each term of the equation with respect to time**. It is assumed that both  $x$  and  $y$  are functions of time, although these functions are not stated. When finding derivatives, we must be careful in using the power rule, Eq. (23.15), so that the factor  $du/dt$  is not neglected.

■ For reference, Eq. (23.15) rewritten for  $u$  as a function of  $t$  is

$$\frac{du^n}{dt} = nu^{n-1} \frac{du}{dt}$$

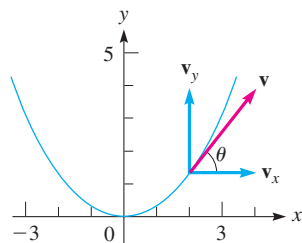


Fig. 24.17

In the following examples, we illustrate the use of Eqs. (24.2) to (24.5) in applied situations for which we know the equation of the path of the motion. Again, we must be careful to find the direction of the vector as well as its magnitude in order to have a complete solution.

## EXAMPLE 4 Velocity at a point along a path

In a physics experiment, a small sphere is constrained to move along a parabolic path described by  $y = \frac{1}{3}x^2$ . If the horizontal velocity  $v_x$  is constant at 6.00 cm/s, find the velocity at the point  $(2.00, 1.33)$ . See Fig. 24.17.

Since both  $y$  and  $x$  change with time, both can be considered to be functions of time. Therefore, we can take derivatives of  $y = \frac{1}{3}x^2$  with respect to time.

$$\frac{dy}{dt} = \frac{1}{3} \left( 2x \frac{dx}{dt} \right) \longleftarrow \frac{dx^2}{dt} = 2x \frac{dx}{dt}$$

$$v_y = \frac{2}{3} x v_x \qquad \text{using Eq. (24.2)}$$

$$v_y = \frac{2}{3} (2.00) (6.00) = 8.00 \text{ cm/s} \qquad \text{substituting}$$

$$v = \sqrt{6.00^2 + 8.00^2} = 10.0 \text{ cm/s} \qquad \text{magnitude (Eq. 24.4)}$$

$$\tan \theta = \frac{8.00}{6.00}, \quad \theta = 53.1^\circ \qquad \text{direction (Eq. 24.5)} \quad \blacksquare$$

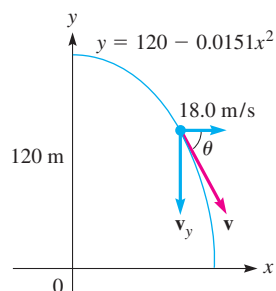


Fig. 24.18

## EXAMPLE 5 Velocity and acceleration—application to projectile motion

A helicopter is flying at 18.0 m/s and at an altitude of 120 m when a rescue marker is released from it. The marker maintains a horizontal velocity and follows a path given by  $y = 120 - 0.0151x^2$ , as shown in Fig. 24.18. Find the magnitude and direction of the velocity and of the acceleration of the marker 3.00 s after release.

From the given information, we know that  $v_x = dx/dt = 18.0$  m/s. Taking derivatives with respect to time leads to this solution:

$$\begin{aligned}
 y &= 120 - 0.0151x^2 \\
 \frac{dy}{dt} &= -0.0302x \frac{dx}{dt} && \text{taking derivatives} \\
 v_y &= -0.0302xv_x && \text{using Eq. (24.2)} \\
 x &= (3.00)(18.0) = 54.0 \text{ m} && \text{evaluating at } t = 3.00 \text{ s} \\
 v_y &= -0.0302(54.0)(18.0) = -29.35 \text{ m/s} \\
 v &= \sqrt{18.0^2 + (-29.35)^2} = 34.4 \text{ m/s} && \text{magnitude} \\
 \tan \theta &= \frac{-29.35}{18.0}, \quad \theta = -58.5^\circ && \text{direction}
 \end{aligned}$$

The velocity is 34.4 m/s and is directed at an angle of  $58.5^\circ$  below the horizontal.

To find the acceleration, we return to the equation  $v_y = -0.0302xv_x$ . Since  $v_x$  is constant, we can substitute 18.0 for  $v_x$  to get

$$v_y = -0.5436x$$

Again taking derivatives with respect to time, we have

$$\begin{aligned}
 \frac{dv_y}{dt} &= -0.5436 \frac{dx}{dt} \\
 a_y &= -0.5436v_x && \text{using Eqs. (24.3) and (24.2)} \\
 a_y &= -0.5436(18.0) = -9.78 \text{ m/s}^2 && \text{evaluating}
 \end{aligned}$$

We know that  $v_x$  is constant, which means that  $a_x = 0$ . Therefore, the acceleration is  $9.78 \text{ m/s}^2$  and is directed vertically downward. ■

## EXERCISES 24.3

In Exercises 1 and 2, make the given changes in the indicated examples of this section and then solve the resulting problems.

- In Example 1, change  $x = 3t^2$  to  $x = 4t^2$ .
- In Example 4, change  $y = x^2/3$  to  $y = x^2/4$  and  $(2.00, 1.33)$  to  $(2.00, 1.00)$ .

In Exercises 3–6, given that the  $x$ - and  $y$ -coordinates of a moving particle are given by the indicated parametric equations, find the magnitude and direction of the velocity for the specific value of  $t$  to 3 significant digits. Sketch the curves and show the velocity and its components.

- $x = 3t, y = 1 - t, t = 4$
- $x = \frac{5t}{2t + 1}, y = 0.1(t^2 + t), t = 2$
- $x = t(2t + 1)^2, y = \frac{6}{\sqrt{4t + 3}}, t = 0.5$
- $x = \sqrt{1 + 2t}, y = t - t^2, t = 4$

In Exercises 7–10, use the parametric equations and values of  $t$  of Exercises 3–6 to find the magnitude and direction of the acceleration in each case.

In Exercises 11–28, find the indicated velocities and accelerations.

- The water from a fire hose follows a path described by  $y = 2.0 + 0.80x - 0.20x^2$  (units are in metres). If  $v_x$  is constant at 5.0 m/s, find the resultant velocity at the point  $(4.0, 2.0)$ .
- A roller mechanism follows a path described by  $y = \sqrt{4x + 1}$ , where units are in metres. If  $v_x = 2x$ , find the resultant velocity (in m/s) at the point  $(2.0, 3.0)$ .
- A float is used to test the flow pattern of a stream. It follows a path described by  $x = 0.20t^2, y = -0.10t^3$  ( $x$  and  $y$  in m,  $t$  in min). Find the acceleration of the float after 2.0 min.
- A radio-controlled model car is operated in a parking lot. The coordinates (in m) of the car are given by  $x = 3.5 + 2.0t^2$  and  $y = 8.5 + 0.25t^3$ , where  $t$  is the time (in s). Find the acceleration of the car after 2.5 s.
- An astronaut on Mars drives a golf ball that moves according to the equations  $x = 25t$  and  $y = 15t - 3.7t^2$  ( $x$  and  $y$  in metres,  $t$  in seconds). Find the resultant velocity and acceleration of the golf ball for  $t = 6.0$  s.
- A package of relief supplies is dropped and moves according to the parametric equations  $x = 45t$  and  $y = -4.9t^2$  ( $x$  and  $y$  in m,  $t$  in s). Find the velocity and acceleration when  $t = 3.0$  s.

17. A spacecraft moves along a path described by the parametric equations  $x = 10(\sqrt{1+t^4} - 1)$ ,  $y = 40t^{3/2}$  for the first 100 s after launch. Here,  $x$  and  $y$  are measured in metres, and  $t$  is measured in seconds. Find the magnitude and direction of the velocity of the spacecraft 10.0 s and 100 s after launch.
18. An electron moves in an electric field according to the equations  $x = 8.0/\sqrt{1+t^2}$  and  $y = 8.0t/\sqrt{1+t^2}$  ( $x$  and  $y$  in Mm and  $t$  in s). Find the velocity of the electron when  $t = 0.50$  s.
19. In a computer game, an airplane starts at  $(1.00, 4.00)$  (in cm) on the curve  $y = 3.00 + x^{-1.50}$  and moves with a constant horizontal velocity of 1.20 cm/s. What is the plane's velocity after 0.500 s?
20. In an aerobic exercise machine, weights are lifted and a person's hands are constrained to move along arcs of the ellipse  $16x^2 + 9y^2 = 9$  (in m). If the person's hands move upward at 0.100 m/s, and start at  $y = 0$ , at what velocity is each moving after 1.50 s?
21. Find the resultant acceleration of the spacecraft in Exercise 17 for the specified times.
22. A ski jump is designed to follow the path given by the equations  $x = 3.50t^2$  and  $y = 20.0 + 0.120t^4 - 3.00\sqrt{t^4 + 1}$  ( $0 \leq t \leq 4.00$  s) ( $x$  and  $y$  in m,  $t$  in s). Find the velocity and acceleration of a skier when  $t = 4.00$  s. See Fig. 24.19.
23. A rocket follows a path given by  $y = x - \frac{1}{90}x^3$  (distances in km). If the horizontal velocity is given by  $v_x = x$ , find the magnitude and direction of the velocity when the rocket hits the ground (assume level terrain) if time is in minutes.
24. A ship is moving around an island on a route described by  $y = 3.0x^2 - 0.20x^3$ . If  $v_x = 1.2$  km/h, find the velocity of the ship where  $x = 3.5$  km.
25. A computer's hard disk is 88.9 mm in diameter and rotates at 3600 r/min. With the centre of the disk at the origin, find the velocity components of a point on the rim for  $x = 30.5$  mm, if  $y > 0$  and  $v_x > 0$ .
26. A robot arm joint moves in an elliptical path (horizontal major axis 8.0 cm, minor axis 4.0 cm, centre at origin). For  $y > 0$  and  $-2 \text{ cm} < x < 2 \text{ cm}$ , the joint moves such that  $v_x = 2.5$  cm/s. Find its velocity for  $x = -1.5$  cm.
27. An airplane ascends such that its gain  $h$  in altitude is proportional to the square root of the change  $x$  in horizontal distance travelled. If  $h = 280$  m for  $x = 400$  m and  $v_x$  is constant at 350 m/s, find the velocity at this point.
28. A meteor travelling toward the earth has a velocity inversely proportional to the square root of the distance from the earth's centre. State how its acceleration is related to its distance from the centre of the earth.

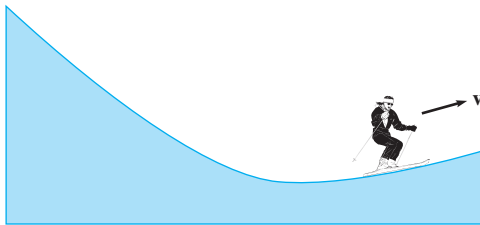


Fig. 24.19

## Answer to Practice Exercise

$$1. a = 4.70, \theta = -25.2^\circ$$

## 24.4 Related Rates

### Derivatives with Respect to Time • Rates of Change are Related

Often, variables vary with respect to time, and are therefore implicitly functions of time. If a relation is known to exist relating them, the time rate of change of one can be expressed in terms of the time rate of change of the other(s). This is done by taking the derivative with respect to time of the expression relating the variables, even if  $t$  does not appear in the expression, as in Examples 4 and 5 of Section 24.3. Since the time rates of change are related, this is referred to as a **related-rates** problem. Consider the following examples.

#### COMMON ERROR

Remember that all variables in related rates problems are functions of time, even if  $t$  does not appear explicitly in the expression. Once again, a common error is not to include the derivative with respect to  $t$  as a factor when applying the power rule or the chain rule.

#### EXAMPLE 1 Related rates—voltage and temperature

The voltage  $E$  of a certain thermocouple as a function of the temperature  $T$  (in  $^\circ\text{C}$ ) is given by  $E = 2.800T + 0.006T^2$ . If the temperature is increasing at the rate of  $1.00^\circ\text{C}/\text{min}$ , how fast is the voltage increasing when  $T = 100^\circ\text{C}$ ?

Since we are asked to find the time rate of change of voltage, we first take derivatives with respect to time. This gives us

$$\frac{dE}{dt} = 2.800 \frac{dT}{dt} + 0.012T \frac{dT}{dt} \leftarrow \frac{d}{dt}(0.006T^2) = 0.006 \left( 2T \frac{dT}{dt} \right)$$

again being careful to include the factor  $dT/dt$ . From the given information, we know that  $dT/dt = 1.00^\circ\text{C}/\text{min}$  and that we wish to know  $dE/dt$  when  $T = 100^\circ\text{C}$ . Thus,

$$\left. \frac{dE}{dt} \right|_{T=100} = 2.800(1.00) + 0.012(100)(1.00) = 4.00 \text{ V/min}$$

### COMMON ERROR

When working with related rates, the derivative must be taken **before** given values are substituted. A common error is to substitute before taking derivatives, so that the functions appear constant when they are not. For instance, in Example 1 we are finding the time rate of change of voltage for a specified value of  $T$ . Different values of  $T$  will lead to different values of  $dE/dt$ .

### EXAMPLE 2 Related rates—distances

The distance  $q$  that an image is from a certain lens in terms of  $p$ , the distance of the object from the lens, is given by

$$q = \frac{10p}{p - 10}$$

If the object distance is increasing at the rate of  $0.200 \text{ cm/s}$ , how fast is the image distance changing when  $p = 15.0 \text{ cm}$ ? See Fig. 24.20.

Taking derivatives with respect to time, we have

$$\frac{dq}{dt} = \frac{(p - 10) \left( 10 \frac{dp}{dt} \right) - 10p \left( \frac{dp}{dt} \right)}{(p - 10)^2} = \frac{-100 \frac{dp}{dt}}{(p - 10)^2}$$

don't forget the  $\frac{dp}{dt}$

Now, substituting  $p = 15.0$  and  $dp/dt = 0.200$ , we have

$$\begin{aligned} \left. \frac{dq}{dt} \right|_{p=15} &= \frac{-100(0.200)}{(15.0 - 10)^2} \\ &= -0.800 \text{ cm/s} \end{aligned}$$

Thus, the image distance is decreasing (the significance of the minus sign) at the rate of  $0.800 \text{ cm/s}$  when  $p = 15.0 \text{ cm}$ .

In many related-rate problems, the function is not given but must be set up according to the statement of the problem. The following examples illustrate this type of problem.

### EXAMPLE 3 Related rates—volume and radius

A spherical balloon is being blown up such that its volume increases at the constant rate of  $2.00 \text{ m}^3/\text{min}$ . Find the rate at which the radius is increasing when it is  $3.00 \text{ m}$ . See Fig. 24.21.

We are asked to find the relation between the rate of change of the volume of a sphere with respect to time and the corresponding rate of change of the radius with respect to time. Therefore, we are to **take derivatives of the expression for the volume of a sphere with respect to time**:

$$V = \frac{4}{3}\pi r^3 \quad \text{volume of sphere}$$

$$\frac{dV}{dt} = 4\pi r^2 \left( \frac{dr}{dt} \right) \quad \text{take derivatives with respect to time}$$

$$2.00 = 4\pi (3.00)^2 \left( \frac{dr}{dt} \right) \quad \text{substitute } \frac{dV}{dt} = 2.00 \text{ m}^3/\text{min} \text{ and } r = 3.00 \text{ m}$$

$$\begin{aligned} \left. \frac{dr}{dt} \right|_{r=3} &= \frac{1}{18.0\pi} \\ &= 0.0177 \text{ m/min} \end{aligned} \quad \text{solve for } \frac{dr}{dt}$$

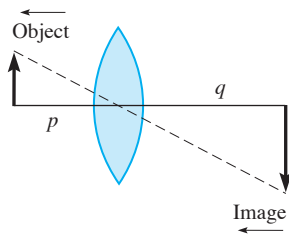


Fig. 24.20

#### Practice Exercise

- In Example 2, change each 10 to 12 and then solve.

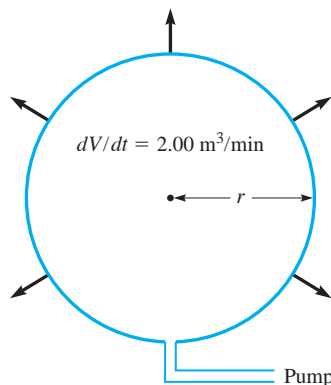


Fig. 24.21

**EXAMPLE 4** Related rates—force and distance

The force  $F$  of gravity of Earth on a spacecraft varies inversely as the square of the distance  $r$  of the spacecraft from the centre of Earth. A particular spacecraft weighs 4500 N on the launchpad ( $F = 4500$  N for  $r = 6370$  km). Find the rate at which  $F$  changes later as the spacecraft moves away from the Earth at the rate of 12 km/s, where  $r = 8500$  km.

We begin by setting up the equation relating  $F$  and  $r$ . Note that we substitute known values of  $F$  and  $r$  in order to obtain the constant of proportionality. As we said earlier, the values from the related rates problem are only substituted after taking derivatives.

$$\begin{aligned}
 F &= \frac{k}{r^2} && \text{inverse variation} \\
 4500 &= \frac{k}{6370^2} && \text{substitute } F = 4500 \text{ N, } r = 6370 \text{ km} \\
 k &= 1.83 \times 10^{11} \text{ N} \cdot \text{km}^2 && \text{solve for } k \\
 F &= \frac{1.83 \times 10^{11}}{r^2} && \text{substitute for } k \text{ in equation} \\
 \frac{dF}{dt} &= (1.83 \times 10^{11})(-2)(r^{-3}) \frac{dr}{dt} && \text{take derivatives with respect to time} \\
 &= \frac{-3.66 \times 10^{11}}{r^3} \frac{dr}{dt} \\
 \left. \frac{dF}{dt} \right|_{r=8500 \text{ km}} &= \frac{-3.66 \times 10^{11}}{8500^3} (12) && \text{evaluate derivative for } r = 8500 \text{ km, } dr/dt = 12 \text{ km/s} \\
 &= -7.2 \text{ N/s} && \text{gravitational force is decreasing} \quad \blacksquare
 \end{aligned}$$

These examples show the following method of solving a related-rates problem.

**Steps for Solving Related-Rates Problems**

1. Identify the variables and rates in the problem.
2. If possible, make a sketch showing the variables.
3. Determine the equation relating the variables.
4. Differentiate with respect to time.
5. Solve for the required rate.
6. Evaluate the required rate.

**EXAMPLE 5** Related rates—distances

Two cruise ships leave Vancouver, British Columbia, at noon. Ship A travels west at 12.0 km/h (before leaving toward Alaska), and ship B travels south at 16.0 km/h (toward Seattle). How fast are they separating at 2 P.M.?

In Fig. 24.22, we let  $x$  = the distance travelled by A and  $y$  = the distance travelled by B. We can find the distance between them,  $z$ , from the Pythagorean theorem. Therefore, we are to find  $dz/dt$  for  $t = 2.00$  h. Even though there are three variables, each is a function of time. This means we can find  $dz/dt$  by taking derivatives of each term with respect to time. This gives us

$$\begin{aligned}
 z^2 &= x^2 + y^2 && \text{using the Pythagorean theorem (step 3)} \\
 2z \frac{dz}{dt} &= 2x \frac{dx}{dt} + 2y \frac{dy}{dt} && \text{taking derivatives with respect to time (step 4)} \\
 \frac{dz}{dt} &= \frac{x(dx/dt) + y(dy/dt)}{z} && \text{solve for } \frac{dz}{dt} \quad \text{(step 5)}
 \end{aligned}$$

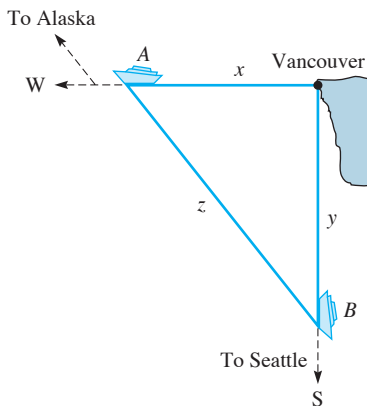


Fig. 24.22



At 2 P.M., we have the values

(step 6)

$$\begin{aligned} x &= 24.0 \text{ km}, & y &= 32.0 \text{ km}, & z &= 40.0 \text{ km} & d = rt \text{ and Pythagorean theorem} \\ dx/dt &= 12.0 \text{ km/h}, & dy/dt &= 16.0 \text{ km/h} & & & \text{from statement of problem} \\ \left. \frac{dz}{dt} \right|_{z=40} &= \frac{(24.0)(12.0) + (32.0)(16.0)}{40.0} = 20.0 \text{ km/h} & & & & & \text{substitute values} \quad \blacksquare \end{aligned}$$

## EXERCISES 24.4

In Exercises 1 and 2, make the given changes in the indicated examples of this section and then solve the resulting problems.

- In Example 1, change 0.006 to 0.012.
- In Example 3, change “volume” to “surface area” and change  $\text{m}^3/\text{min}$  to  $\text{m}^2/\text{min}$ .

In Exercises 3–6, assume that all variables are implicit functions of time  $t$ . Find the indicated rates.

- $y = 5x^2 - 4x$ ;  $dx/dt = 0.5$  when  $x = 5$ ; find  $dy/dt$ .
- $x^2 + 2y^2 = 9$ ;  $dx/dt = 3$  when  $x = 1$  and  $y = 2$ ; find  $dy/dt$ .
- $x^2 + 3y^2 + 2y = 10$ ;  $dx/dt = 2$  when  $x = 3$  and  $y = -1$ ; find  $dy/dt$ .
- $z = 2x^2 - 3xy$ ;  $dx/dt = -2$  and  $dy/dt = 3$  when  $x = 1$  and  $y = 4$ ; find  $dz/dt$ .

In Exercises 7–44, solve the problems in related rates.

- How fast is the slope of a tangent to the curve  $y = 2/(x + 1)$  changing where  $x = 3.0$  if  $dx/dt = 0.50$  unit/s?
- How fast is the slope of a tangent to the curve  $y = 2(1 - 2x)^2$  changing where  $x = 1.5$  if  $dx/dt = 0.75$  unit/s?
- The velocity  $v$  (in m/s) of a pulse travelling in a certain string is a function of the tension  $T$  (in N) in the string given by  $v = 18\sqrt{T}$ . Find  $dv/dt$  if  $dT/dt = 0.20$  N/s when  $T = 25$  N.
- The force  $F$  (in N) on the blade of a certain wind generator as a function of the wind velocity  $v$  (in m/s) is given by  $F = 0.024v^2$ . Find  $dF/dt$  if  $dv/dt = 0.75$   $\text{m/s}^2$  when  $v = 28$  m/s.
- The electric resistance  $R$  (in  $\Omega$ ) of a certain resistor as a function of the temperature  $T$  (in  $^\circ\text{C}$ ) is  $R = 4.000 + 0.003T^2$ . If the temperature is increasing at the rate of  $0.100^\circ\text{C/s}$ , find how fast the resistance changes when  $T = 150^\circ\text{C}$ .
- The kinetic energy  $K$  (in J) of an object is given by  $K = \frac{1}{2}mv^2$ , where  $m$  is the mass (in kg) of the object and  $v$  is its velocity. If a 250-kg wrecking ball accelerates at  $5.00$   $\text{m/s}^2$ , how fast is the kinetic energy changing when  $v = 30.0$  m/s?
- The length  $L$  (in cm) of a pendulum is slowly decreasing at the rate of  $0.100$  cm/s. What is the time rate of change of the period  $T$  (in s) of the pendulum when  $L = 16.0$  cm, if the equation relating the period and length is  $T = \pi\sqrt{L/245}$ ?
- The voltage  $V$  that produces a current  $I$  (in A) in a wire of radius  $r$  (in mm) is  $V = 0.030I/r^2$ . If the current increases at  $0.020$  A/s in a wire of  $0.040$  mm radius, find the rate at which the voltage is increasing.
- A plane flying at an altitude of  $2.0$  km is at a direct distance  $D = \sqrt{4.0 + x^2}$  from an airport control tower, where  $x$  is the horizontal distance to the tower. If the plane’s speed is  $350$  km/h, how fast is  $D$  changing when  $x = 6.2$  km?
- A variable resistor  $R$  and an  $8\text{-}\Omega$  resistor in parallel have a combined resistance  $R_T$  given by  $R_T = \frac{8R}{8 + R}$ . If  $R$  is changing at  $0.30$   $\Omega/\text{min}$ , find the rate at which  $R_T$  is changing when  $R = 6.0$   $\Omega$ .
- The radius  $r$  of a ring of a certain holograph (an image produced without using a lens) is given by  $r = \sqrt{0.40\lambda}$ , where  $\lambda$  is the wavelength of the light being used. If  $\lambda$  is changing at the rate of  $0.10 \times 10^{-7}$  m/s when  $\lambda = 6.0 \times 10^{-7}$  m, find the rate at which  $r$  is changing.
- An earth satellite moves in a path that can be described by  $\frac{x^2}{72.5} + \frac{y^2}{71.5} = 1$ , where  $x$  and  $y$  are in thousands of kilometres. If  $dx/dt = 12\,900$  km/h for  $x = 3200$  km and  $y > 0$ , find  $dy/dt$ .
- The magnetic field  $B$  due to a magnet of length  $l$  at a distance  $r$  is given by  $B = \frac{k}{[r^2 + (l/2)^2]^{3/2}}$ , where  $k$  is a constant for a given magnet. Find the expression for the time rate of change of  $B$  in terms of the time rate of change of  $r$ .
- An approximate relationship between the pressure  $p$  and volume  $V$  of the vapour in a diesel engine cylinder is  $pV^{1.4} = k$ , where  $k$  is a constant. At a certain instant,  $p = 4200$  kPa,  $V = 75$   $\text{cm}^3$ , and the volume is increasing at the rate of  $850$   $\text{cm}^3/\text{s}$ . What is the time rate of change of the pressure at this instant?
- A swimming pool with a rectangular surface  $18.0$  m long and  $12.0$  m wide is being filled at the rate of  $0.80$   $\text{m}^3/\text{min}$ . At one end it is  $1.0$  m deep, and at the other end it is  $2.5$  m deep, with a constant slope between ends. How fast is the height of water rising when the depth of water at the deep end is  $1.0$  m?
- An engine cylinder  $15.0$  cm deep is being bored such that the radius increases by  $0.100$  mm/min. How fast is the volume  $V$  of the cylinder changing when the diameter is  $9.50$  cm?
- Fatty deposits have decreased the circular cross-sectional opening of a person’s artery. A test drug reduces these deposits such that the radius of the opening increases at the rate of  $0.020$  mm/month. Find the rate at which the area of the opening increases when  $r = 1.2$  mm.
- A computer program increases the side of a square image on the screen at the rate of  $0.25$  cm/s. Find the rate at which the area of the image increases when the edge is  $6.50$  cm.
- A metal cube dissolves in acid such that an edge of the cube decreases by  $0.500$  mm/min. How fast is the volume of the cube changing when the edge is  $8.20$  mm?
- A metal sphere is placed in seawater to study the corrosive effect of seawater. If the surface area decreases at  $35$   $\text{cm}^2/\text{year}$  due to corrosion, how fast is the radius changing when it is  $12$  cm?

27. A uniform layer of ice covers a spherical water-storage tank. As the ice melts, the volume  $V$  of ice decreases at a rate that varies directly as the surface area  $A$ . Show that the outside radius decreases at a constant rate.
28. A light in a garage is 2.90 m above the floor and 3.65 m behind the door. If the garage door descends vertically at 0.45 m/s, how fast is the door's shadow moving toward the garage when the door is 0.60 m above the floor?
29. One statement of Boyle's law is that the pressure of a gas varies inversely as the volume for constant temperature. If a certain gas occupies  $650 \text{ cm}^3$  when the pressure is 230 kPa and the volume is increasing at the rate of  $20.0 \text{ cm}^3/\text{min}$ , how fast is the pressure changing when the volume is  $810 \text{ cm}^3$ ?
30. The tuning frequency  $f$  of an electronic tuner is inversely proportional to the square root of the capacitance  $C$  in the circuit. If  $f = 920 \text{ kHz}$  for  $C = 3.5 \text{ pF}$ , find how fast  $f$  is changing at this frequency if  $dC/dt = 0.30 \text{ pF/s}$ .
31. A spherical metal object is ejected from an earth satellite and reenters the atmosphere. It heats up (until it burns) so that the radius increases at the rate of  $5.00 \text{ mm/s}$ . What is the time rate of change of volume when the radius is  $225 \text{ mm}$ ?
32. The acceleration due to the gravity  $g$  on a spacecraft is inversely proportional to its distance from the centre of the earth. At the surface of the earth,  $g = 9.80 \text{ m/s}^2$ . Given that the radius of the earth is  $6370 \text{ km}$ , how fast is  $g$  changing on a spacecraft approaching the earth at  $1400 \text{ m/s}$  at a distance of  $41\,000 \text{ km}$  from the surface?
33. The intensity  $I$  of heat varies directly as the strength of the source and inversely as the square of the distance from the source. If an object approaches a heated object of strength 8.00 units at the rate of  $50.0 \text{ cm/s}$ , how fast is the intensity changing when it is  $100 \text{ cm}$  from the source?
34. The speed of sound  $v$  (in m/s) is  $v = 331\sqrt{T/273}$ , where  $T$  is the temperature (in K). If the temperature is  $303 \text{ K}$  ( $30^\circ\text{C}$ ) and is rising at  $2.0^\circ\text{C/h}$ , how fast is the speed of sound rising?
35. As a space shuttle moves into space, an astronaut's weight decreases. An astronaut weighing  $650 \text{ N}$  at sea level has a weight of  $w = 650\left(\frac{6400}{6400+h}\right)$  at  $h$  kilometres above sea level. If the shuttle is moving away from earth at  $6.0 \text{ km/s}$ , at what rate is  $w$  changing when  $h = 1200 \text{ km}$ ?
36. The oil reservoir for the lubricating mechanism of a machine is in the shape of an inverted pyramid. It is being filled at the rate of  $8.00 \text{ cm}^3/\text{s}$  and the top surface is increasing at the rate of  $6.00 \text{ cm}^2/\text{s}$ . When the depth of oil is  $6.50 \text{ cm}$  and the top surface area is  $22.5 \text{ cm}^2$ , how fast is the level increasing?
37. A tank in the shape of an inverted cone has a height of  $3.60 \text{ m}$  and a radius at the top of  $1.15 \text{ m}$ . Water is flowing into the tank at the rate of  $0.500 \text{ m}^3/\text{min}$ . How fast is the level rising when it is  $1.80 \text{ m}$  deep?
38. The top of a ladder  $4.00 \text{ m}$  long is slipping down a vertical wall at the constant rate of  $0.750 \text{ m/s}$ . How fast is the bottom of the ladder moving along the ground away from the wall when it is  $2.50 \text{ m}$  from the wall?

39. A supersonic jet leaves an airfield travelling due east at  $1600 \text{ km/h}$ . A second jet leaves the same airfield at the same time and travels  $1800 \text{ km/h}$  along a line north of east such that it remains due north of the first jet. After a half-hour, how fast are the jets separating?
40. A car passes over a bridge at  $15.0 \text{ m/s}$  at the same time a boat passes under the bridge at a point  $10.5 \text{ m}$  directly below the car. If the boat is moving perpendicularly to the bridge at  $4.00 \text{ m/s}$ , how fast are the car and the boat separating  $5.00 \text{ s}$  later?
41. A rope attached to a boat is being pulled in at a rate of  $2.50 \text{ m/s}$ . If the water is  $5.00 \text{ m}$  below the level at which the rope is being drawn in, how fast is the boat approaching the wharf when  $13.0 \text{ m}$  of rope are yet to be pulled in? See Fig. 24.23.

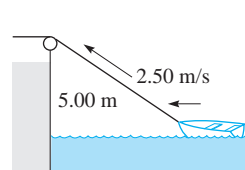


Fig. 24.23

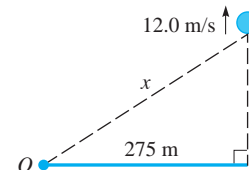


Fig. 24.24

42. A weather balloon leaves the ground  $275 \text{ m}$  from an observer and rises vertically at  $12.0 \text{ m/s}$ . How fast is the line of sight from the observer to the balloon increasing when the balloon is  $450 \text{ m}$  high? See Fig. 24.24.
43. A man  $1.80 \text{ m}$  tall approaches a street light  $4.50 \text{ m}$  above the ground at the rate of  $1.50 \text{ m/s}$ . How fast is the end of the man's shadow moving when he is  $3.00 \text{ m}$  from the base of the light? See Fig. 24.25.

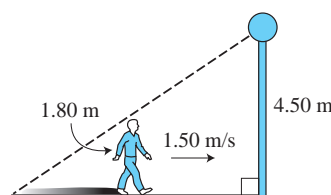


Fig. 24.25

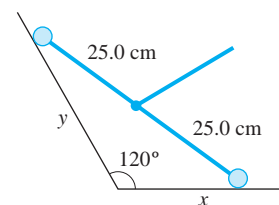


Fig. 24.26

44. A roller mechanism, as shown in Fig. 24.26, moves such that the right roller is always in contact with the bottom surface and the left roller is always in contact with the left surface. If the right roller is moving to the right at  $1.50 \text{ cm/s}$  when  $x = 10.0 \text{ cm}$ , how fast is the left roller moving?

### Answer to Practice Exercise

1.  $-3.20 \text{ cm/s}$

## 24.5 Using Derivatives in Curve Sketching

Function Increasing and Decreasing • Relative Maximum and Minimum Points • First and Second Derivative Tests • Concave Up and Concave Down • Points of Inflection

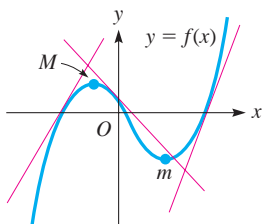


Fig. 24.27

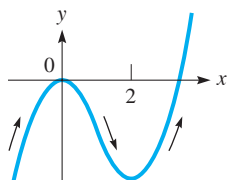


Fig. 24.28

Maximum Points  
Minimum Points

First-Derivative Test  
for Maxima and Minima

For the function in Fig. 24.27, we see that as  $x$  increases (from left to right),  $y$  also increases until point  $M$  is reached. From  $M$  to  $m$ ,  $y$  decreases. To the right of  $m$ ,  $y$  again increases. Also, any tangent line left of  $M$  or right of  $m$  has a positive slope, and any tangent line between  $M$  and  $m$  has a negative slope. Since the derivative gives us the slope of a tangent line, we see that *as  $x$  increases,  $y$  increases if the derivative is positive and decreases if the derivative is negative.* This can be stated as

$$f(x) \text{ increases if } f'(x) > 0 \text{ and } f(x) \text{ decreases if } f'(x) < 0$$

*It is always assumed that  $x$  is increasing.* Also, we assume in our present analysis that  $f(x)$  and its derivatives are continuous over the interval of interest.

### EXAMPLE 1 Function increasing and decreasing

Find those values of  $x$  for which the function  $f(x) = x^3 - 3x^2$  is increasing and those values for which it is decreasing.

We find these values by finding the derivative and then finding the values of  $x$  for which it is positive or negative by solving inequalities. Therefore,

$$f'(x) = 3x^2 - 6x = 3x(x - 2)$$

$$f(x) \text{ is increasing if } 3x(x - 2) > 0, \text{ and decreasing if } 3x(x - 2) < 0$$

We now solve these inequalities by methods in Chapter 17 (see page 477). Finding the **critical values** of  $x = 0$  and  $x = 2$ , we have the following:

$$\text{If } x < 0, 3x(x - 2) > 0 \text{ or } f'(x) > 0. \quad f(x) \text{ increasing}$$

$$\text{If } 0 < x < 2, 3x(x - 2) < 0 \text{ or } f'(x) < 0. \quad f(x) \text{ decreasing}$$

$$\text{If } x > 2, 3x(x - 2) > 0 \text{ or } f'(x) > 0. \quad f(x) \text{ increasing}$$

The solution of the inequality  $3x(x - 2) > 0$  is  $x < 0$  or  $x > 2$ , which means that  $f(x)$  is increasing for these values. Also,  $3x(x - 2) < 0$  for  $0 < x < 2$ , which means that  $f(x)$  is decreasing for these values. See Fig. 24.28 for the graph of  $f(x) = x^3 - 3x^2$ . ■

The points  $M$  and  $m$  in Fig. 24.27 are called a **relative maximum point** and a **relative minimum point**, respectively. *This means that  $M$  has a greater  $y$ -value than any other point near it and that  $m$  has a smaller  $y$ -value than any point near it.* This does not necessarily mean that  $M$  has the greatest  $y$ -value of any point on the curve or that  $m$  has the least  $y$ -value of any point on the curve. However, the points  $M$  and  $m$  have the greatest or least values of  $y$  for that part of the curve (that is why we use the word *relative*). Examination of Fig. 24.27 verifies this point. *The characteristic of both  $M$  and  $m$  is that the derivative is zero at each point.* (We see that this is so since a tangent line would have a slope of zero at each.) *This is how relative maximum and relative minimum points are located. The derivative is found and then set equal to zero. The solutions of the resulting equation give the  $x$ -coordinates of the maximum and minimum points.*

It remains now to determine whether a given value of  $x$ , for which the derivative is zero, is the coordinate of a maximum or a minimum point (or neither, which is also possible). From the discussion of increasing and decreasing values for  $y$ , we see that *the derivative changes sign from plus to minus when passing through a relative maximum point and from minus to plus when passing through a relative minimum point.* Thus, we find maximum and minimum points by determining those values of  $x$  for which the derivative is zero and by properly analysing the sign change of the derivative. If the sign of the derivative does not change, it is neither a maximum nor a minimum point. This is known as the **first-derivative test for maxima and minima**.

In Fig. 24.29, a diagram for the first-derivative test is shown. The test for a relative maximum is shown in Fig. 24.29(a), and that for a relative minimum is shown in

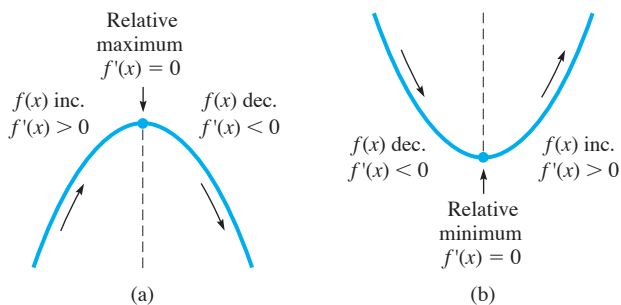


Fig. 24.29

Fig. 24.29(b). For the curves shown in Fig. 24.29,  $f(x)$  and  $f'(x)$  are continuous throughout the interval shown. (Although  $f(x)$  must be continuous,  $f'(x)$  may be discontinuous at the maximum point or the minimum point, and the sign changes of the first-derivative test remain valid.)

**EXAMPLE 2** Relative maximum and minimum points

Find any relative maximum points and any relative minimum points of the graph of the function

$$y = 3x^5 - 5x^3$$

Finding the derivative and setting it equal to zero, we have

$$y' = 15x^4 - 15x^2 = 15x^2(x^2 - 1) = 15x^2(x - 1)(x + 1)$$

Therefore,

$$15x^2(x - 1)(x + 1) = 0 \quad \text{for } x = 0, \quad x = 1, \quad x = -1$$

Thus, the sign of the derivative is the same for all points to the left of  $x = -1$ . For these values,  $y' > 0$  (thus,  $y$  is increasing). For values of  $x$  between  $-1$  and  $0$ ,  $y' < 0$ . For values of  $x$  between  $0$  and  $1$ ,  $y' < 0$ . For values of  $x$  greater than  $1$ ,  $y' > 0$ . Thus, the curve has a maximum at  $(-1, 2)$  and a minimum at  $(1, -2)$ . The point  $(0, 0)$  is neither a maximum nor a minimum, since the sign of the derivative did not change at this value of  $x$ . The graph of  $y = 3x^5 - 5x^3$  is shown in Fig. 24.30.

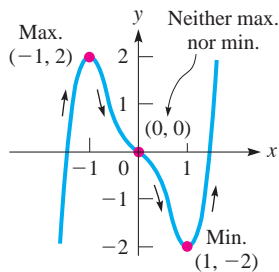


Fig. 24.30

**Practice Exercise**

- Find the relative maximum and minimum points on the graph of  $y = x^3 - 3x^2 - 9x$ .

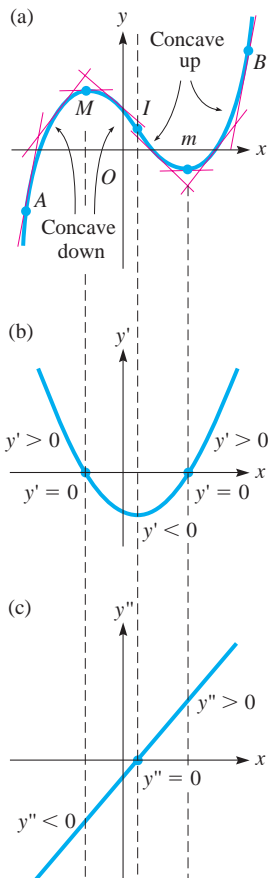


Fig. 24.31

We now look again at the slope of a tangent drawn to a curve. In Fig. 24.31(a), consider the *change* in the values of the slope of a tangent at a point as the point moves from  $A$  to  $B$ . At  $A$  the slope is positive, and as the point moves toward  $M$ , the slope remains positive but becomes smaller until it becomes zero at  $M$ . To the right of  $M$ , the slope is negative and becomes more negative until it reaches  $I$ . Therefore, *from  $A$  to  $I$ , the slope continually decreases*. To the right of  $I$ , the slope remains negative but increases until it becomes zero again at  $m$ . To the right of  $m$ , the slope becomes positive and increases to point  $B$ . Therefore, *from  $I$  to  $B$ , the slope continually increases*. We say that *the curve is concave down from  $A$  to  $I$  and concave up from  $I$  to  $B$* .

The curve in Fig. 24.31(b) is that of the derivative, and it therefore indicates the values of the slope of  $f(x)$ . If the slope changes, we are dealing with the rate of change of slope or the rate of change of the derivative. This function is the second derivative. The curve in Fig. 24.31(c) is that of the second derivative. We see that *where the second derivative of a function is negative, the slope is decreasing, or the curve is concave down (opens down)*. *Where the second derivative is positive, the slope is increasing, or the curve is concave up (opens up)*. This may be summarized as follows:

- If  $f''(x) > 0$ , the curve is concave up.
- If  $f''(x) < 0$ , the curve is concave down.

We can also now use this information in the determination of maximum and minimum points. By the nature of the definition of maximum and minimum points and of concavity, it is apparent that *a curve is concave down at a maximum point and concave up at a minimum point*. We can see these properties when we make a close analysis of the curve in Fig. 24.31. Therefore, at  $x = a$ ,

$$\text{if } f'(a) = 0 \text{ and } f''(a) < 0,$$

then  $f(x)$  has a relative maximum at  $x = a$ , or

$$\text{if } f'(a) = 0 \text{ and } f''(a) > 0,$$

then  $f(x)$  has a relative minimum at  $x = a$ .

**Second-Derivative Test for Maxima and Minima**

These statements constitute what is known as the **second-derivative test for maxima and minima**. This test is often easier to use than the first-derivative test. However, it can happen that  $y'' = 0$  at a maximum or minimum point, and in such cases it is necessary that we use the first-derivative test.

**COMMON ERROR**

In using the second-derivative test, we should note that  $f''(x)$  is **negative at a maximum** point and **positive at a minimum** point. This is contrary to a natural inclination to think of "maximum" and "positive" together or "minimum" and "negative" together.

**Points of Inflection**

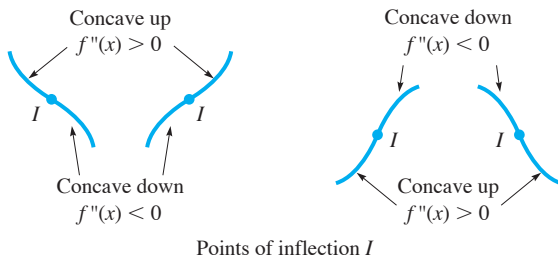


Fig. 24.32

The points at which the curve changes from concave up to concave down, or from concave down to concave up, are known as **points of inflection**. Thus, point  $I$  in Fig. 24.31 is a point of inflection. Inflection points are found by determining those values of  $x$  for which the second derivative changes sign. This is analogous to finding maximum and minimum points by the first-derivative test. In Fig. 24.32, various types of points of inflection are illustrated.

Throughout this analysis it has been assumed that  $f(x)$  and its derivatives are continuous functions. To show that this is necessary, see Exercise 54 of this section.

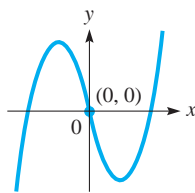


Fig. 24.33

**EXAMPLE 3 Concavity—points of inflection**

Determine the concavity and find any points of inflection on the graph of the function  $y = x^3 - 3x$ .

This requires an inspection and analysis of the second derivative. Therefore, we find the first two derivatives:

$$y' = 3x^2 - 3$$

$$y'' = 6x$$

The second derivative is positive where the function is concave up, and this occurs if  $x > 0$ . The curve is concave down for  $x < 0$ , since  $y''$  is negative. Thus,  $(0, 0)$  is a point of inflection, since the concavity changes there. The graph of  $y = x^3 - 3x$  is shown in Fig. 24.33.

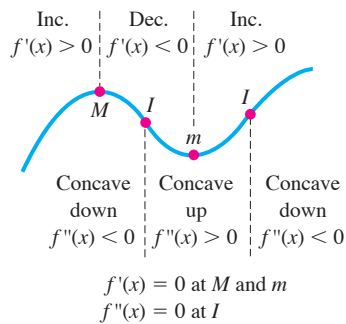


Fig. 24.34

At this point, we summarize the information found from the derivatives of a function  $f(x)$ . See Fig. 24.34.

**Finding Features of a Graph Using Derivatives**

Let  $f$  and its derivatives be continuous. Then,

$f'(x) > 0$  where  $f(x)$  increases;  $f'(x) < 0$  where  $f(x)$  decreases.

$f''(x) > 0$  where the graph of  $f(x)$  is concave up;  $f''(x) < 0$  where the graph of  $f(x)$  is concave down.

If  $f'(x) = 0$  at  $x = a$ , there is a relative maximum point if  $f'(x)$  changes from + to - or if  $f''(a) < 0$ .

If  $f'(x) = 0$  at  $x = a$ , there is a relative minimum point if  $f'(x)$  changes from - to + or if  $f''(a) > 0$ .

If  $f''(x) = 0$  at  $x = a$ , there is a point of inflection if  $f''(x)$  changes from + to - or from - to +.

The following examples illustrate how the above information is put together to obtain the graph of a function.



**EXAMPLE 4** Sketching a curve using derivatives

Sketch the graph of  $y = 6x - x^2$ .

Finding the first two derivatives, we have

$$y' = 6 - 2x = 2(3 - x)$$

$$y'' = -2$$

We now note that  $y' = 0$  for  $x = 3$ . For  $x < 3$ , we see that  $y' > 0$ , which means that  $y$  is increasing over this interval. Also, for  $x > 3$ , we note that  $y' < 0$ , which means that  $y$  is decreasing over this interval.

Since  $y'$  changes from positive on the left of  $x = 3$  to negative on the right of  $x = 3$ , the curve has a maximum point where  $x = 3$ . Since  $y = 9$  for  $x = 3$ , this maximum point is  $(3, 9)$ .

Since  $y'' = -2$ , this means that its value remains constant for all values of  $x$ . Therefore, there are no points of inflection, and the curve is concave down for all values of  $x$ . This also shows that the point  $(3, 9)$  is a maximum point.

Summarizing, we know that  $y$  is increasing for  $x < 3$ ,  $y$  is decreasing for  $x > 3$ , there is a maximum point at  $(3, 9)$ , and the curve is always concave down. Using this information, we sketch the curve shown in Fig. 24.35.

From the equation, we know this curve is a parabola. We could also find the maximum point from the material of Section 7.4 or Section 21.7. However, using derivatives we can find this kind of important information about the graphs of a great many types of functions. ■

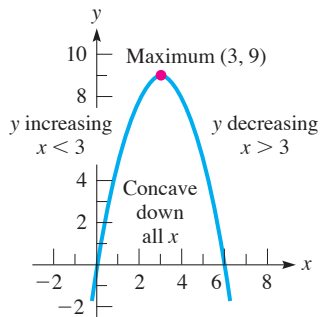


Fig. 24.35

**Practice Exercise**

2. Find the point(s) of inflection on the graph of  $y = x^3 - 3x^2 - 9x$ .

**EXAMPLE 5** Sketching a curve using derivatives

Sketch the graph of  $y = 2x^3 + 3x^2 - 12x$ .

Finding the first two derivatives, we have

$$y' = 6x^2 + 6x - 12 = 6(x + 2)(x - 1)$$

$$y'' = 12x + 6 = 6(2x + 1)$$

We note that  $y' = 0$  when  $x = -2$  and  $x = 1$ . Using these values in the second derivative, we find that  $y''$  is negative ( $-18$ ) for  $x = -2$  and  $y''$  is positive ( $+18$ ) when  $x = 1$ . When  $x = -2$ ,  $y = 20$ ; and when  $x = 1$ ,  $y = -7$ . Therefore,  $(-2, 20)$  is a relative maximum point, and  $(1, -7)$  is a relative minimum point.

Next, we see that  $y' > 0$  if  $x < -2$  or  $x > 1$ . Also,  $y' < 0$  for the interval  $-2 < x < 1$ . Therefore,  $y$  is increasing if  $x < -2$  or  $x > 1$ , and  $y$  is decreasing if  $-2 < x < 1$ .

Now, we note that  $y'' = 0$  when  $x = -\frac{1}{2}$ ,  $y'' < 0$  when  $x < -\frac{1}{2}$ , and  $y'' > 0$  when  $x > -\frac{1}{2}$ . When  $x = -\frac{1}{2}$ ,  $y = \frac{13}{2}$ . Therefore, there is a point of inflection at  $(-\frac{1}{2}, \frac{13}{2})$ , the curve is concave down if  $x < -\frac{1}{2}$ , and the curve is concave up if  $x > -\frac{1}{2}$ .

Finally, by locating the points  $(-2, 20)$ ,  $(-\frac{1}{2}, \frac{13}{2})$ , and  $(1, -7)$ , we draw the curve up to  $(-2, 20)$  and then down to  $(-\frac{1}{2}, \frac{13}{2})$ , with the curve concave down. **Continuing down, but concave up**, we draw the curve to  $(1, -7)$ , at which point we start up and continue up. We now know the key points and the shape of the curve. See Fig. 24.36. For more precision, additional points may be used. ■

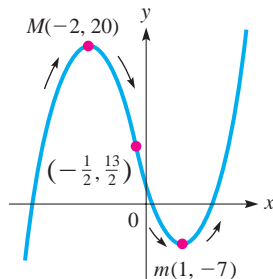


Fig. 24.36

**EXAMPLE 6** Sketching a curve using derivatives

Sketch the graph of  $y = x^5 - 5x^4$ .

The first two derivatives are

$$y' = 5x^4 - 20x^3 = 5x^3(x - 4)$$

$$y'' = 20x^3 - 60x^2 = 20x^2(x - 3)$$

We now see that  $y' = 0$  when  $x = 0$  and  $x = 4$ . For  $x = 0$ ,  $y'' = 0$  also, which means **we cannot use the second-derivative test** for maximum and minimum points for  $x = 0$  in this case. For  $x = 4$ ,  $y'' > 0$  ( $+320$ ), which means that  $(4, -256)$  is a relative minimum point.

Next, we note that

$$y' > 0 \text{ for } x < 0 \text{ or } x > 4 \quad y' < 0 \text{ for } 0 < x < 4$$

Thus, by the first-derivative test, there is a relative maximum point at  $(0, 0)$ . Also,  $y$  is increasing for  $x < 0$  or  $x > 4$  and decreasing for  $0 < x < 4$ .

The second derivative indicates that there is a point of inflection at  $(3, -162)$ . It also indicates that the curve is concave down for  $x < 3$  ( $x \neq 0$ ) and concave up for  $x > 3$ . There is no point of inflection at  $(0, 0)$  since the second derivative does not change sign at  $x = 0$ .

From this information, we sketch the curve in Fig. 24.37. ■

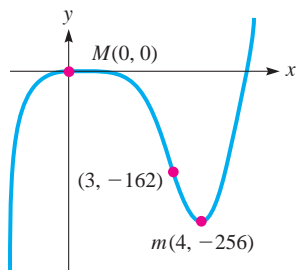


Fig. 24.37

Graphing calculators have a feature by which we can find the  $x$ -value for which a function is maximum (or minimum) over a specified interval. To find a *relative* maximum (or minimum) value, care must be used in choosing the  $x$ -interval so as not to include a value for which the function may be greater (or less) than the value at the relative maximum (or minimum).

## EXERCISES 24.5

In Exercises 1–4, make the given changes in the indicated examples of this section and then solve the resulting problems.

- In Example 1, after the  $-$  sign change  $3x^2$  to  $6x^2$  and then find the values of  $x$  for which  $f(x)$  is increasing and those for which it is decreasing.
- In Example 2, change the  $5x^3$  to  $15x$  and then find the relative maximum and minimum points.
- In Example 3, change the  $-$  sign to  $+$  and then determine the concavity and find any points of inflection.
- In Example 4, change the  $6x$  to  $8x$  and then sketch the graph as in the example.

In Exercises 5–8, find those values of  $x$  for which the given functions are increasing and those values of  $x$  for which they are decreasing.

- $y = x^2 + 2x$
- $y = 2 + 6x - 3x^2$
- $y = 2x^3 + 3x^2 - 36x - 10$
- $y = x^4 - 6x^2$

In Exercises 9–12, find any relative maximum or minimum points of the given functions. (These are the same functions as in Exercises 5–8.)

- $y = x^2 + 2x$
- $y = 2 + 6x - 3x^2$
- $y = 2x^3 + 3x^2 - 36x - 10$
- $y = x^4 - 6x^2$

In Exercises 13–16, find the values of  $x$  for which the given function is concave up, the values of  $x$  for which it is concave down, and any points of inflection. (These are the same functions as in Exercises 5–8.)

- $y = x^2 + 2x$
- $y = 2 + 6x - 3x^2$
- $y = 2x^3 + 3x^2 - 36x - 10$
- $y = x^4 - 6x^2$

In Exercises 17–20, sketch the graphs of the given functions by determining the appropriate information and points from the first and second derivatives (see Exercises 5–16).

- $y = x^2 + 2x$
- $y = 2 + 6x - 3x^2$
- $y = 2x^3 + 3x^2 - 36x - 10$
- $y = x^4 - 6x^2$

In Exercises 21–32, sketch the graphs of the given functions by determining the appropriate information and points from the first and second derivatives.

- $y = 12x - 2x^2$
- $y = 4x^2 - 16x - 20$
- $y = 2x^3 + 6x^2 - 5$
- $y = x^3 - 9x^2 + 15x + 1$
- $y = x^3 + 3x^2 + 3x + 2$
- $y = x^3 - 12x + 12$
- $y = 4x^3 - 24x^2 + 36x$
- $y = x(x - 4)^3$
- $y = 4x^3 - 3x^4 + 6$
- $y = x^5 - 20x^2$
- $y = x^5 - 5x$
- $y = x^4 + 32x + 2$

In Exercises 33 and 34, view the graphs of  $y$ ,  $y'$ , and  $y''$  together on a graphing calculator. State how the graphs of  $y'$  and  $y''$  are related to the graph of  $y$ .

- $y = x^3 - 12x$
- $y = 24x - 9x^2 - 2x^3$

In Exercises 35–38, describe the indicated features of the given graphs.

- Display the graph of  $y = x^3 + cx$  for  $c = -3, -1, 1$ , and  $3$  on a graphing calculator. Describe how the graph changes as  $c$  varies.
- Follow the instructions of Exercise 35 for the function  $y = x^4 + cx^2$ .
- Display the graph of  $y = 2x^5 - 7x^3 + 8x$  on a graphing calculator. Describe the relative locations of the left relative maximum point and the relative minimum points.



38. Describe the following features of the graph in Fig. 24.38 between or at the points  $A, B, C, D, E, F,$  and  $G$ . (a) Increasing and decreasing, (b) relative maximum and minimum points, (c) concavity, and (d) points of inflection.

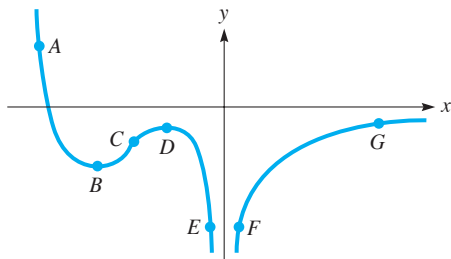


Fig. 24.38

In Exercises 39–50, sketch the indicated curves by the methods of this section. You may check the graphs by using a graphing calculator.

39. A batter hits a baseball that follows a path given by  $y = x - 0.025x^2$ , where distances are in metres. Sketch the graph of the path of the baseball.
40. The angle  $\theta$  (in degrees) of a robot arm with the horizontal as a function of the time  $t$  (in s) is given by  $\theta = 10 + 12t^2 - 2t^3$ . Sketch the graph for  $0 \leq t \leq 6$  s.
41. The power  $P$  (in W) in a certain electric circuit is given by  $P = 4i - 0.5i^2$ . Sketch the graph of  $P$  vs.  $i$ .
42. A computer analysis shows that the thrust  $T$  (in kN) of an experimental rocket motor is  $T = 20 + 9t - 4t^3$ , where  $t$  is the time (in min) after the motor is activated. Sketch the graph for the first 2 min.
43. For the force  $F$  exerted on the cam on the lever in Exercise 54 on page 686, sketch the graph of  $F$  vs.  $x$ . (Hint: Use methods of Chapter 15 to analyse the first derivative.)
44. The solar-energy power  $P$  (in W) produced by a certain solar system does not rise and fall uniformly during a cloudless day because of the system's location. An analysis of records shows that  $P = -0.45(2t^5 - 45t^4 + 350t^3 - 1000t^2)$ , where  $t$  is the time (in h) during which power is produced. Show that, during the solar-power production, the production flattens (inflection) in the middle and then peaks before shutting down. (Hint: The derivative is zero at integer values of  $t$ .)
45. An electric circuit is designed such that the resistance  $R$  (in  $\Omega$ ) is a function of the current  $i$  (in mA) according to  $R = 75 - 18i^2 + 8i^3 - i^4$ . Sketch the graph if  $R \geq 0$  and  $i$  can be positive or negative.
46. A horizontal 12-m beam is deflected by a load such that it can be represented by the equation  $y = 0.0004(x^3 - 12x^2)$ . Sketch the curve followed by the beam.

47. The altitude  $h$  (in m) of a certain rocket is given by  $h = -t^3 + 54t^2 + 480t + 20$ , where  $t$  is the time (in s) of flight. Sketch the graph of  $h = f(t)$ .
48. An analysis of data showed that the mean density  $d$  (in  $\text{mg}/\text{cm}^3$ ) of a calcium compound in the bones of women was given by  $d = 0.00181x^3 - 0.289x^2 + 12.2x + 30.4$ , where  $x$  represents the ages of women ( $20 < x < 80$  years). (A woman probably has osteoporosis if  $d < 115 \text{ mg}/\text{cm}^3$ .) Sketch the graph.
49. A rectangular box is made from a piece of cardboard 8 cm by 12 cm by cutting equal squares from each corner and bending up the sides. See Fig. 24.39. Express the volume of the box as a function of the side of the square that is cut out and then sketch the curve of the resulting equation.

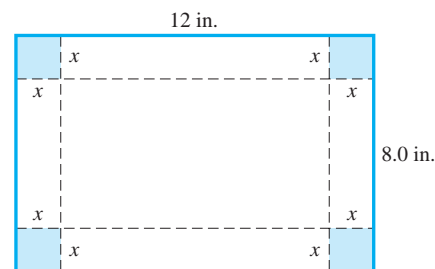


Fig. 24.39

50. A rectangular planter with a square end is to be made from  $8.0 \text{ m}^2$  of redwood. Express the volume of soil the planter can hold as a function of the side of the square of the end. Sketch the curve of the resulting function.

In Exercises 51–53, sketch a continuous curve that has the given characteristics.

51.  $f(1) = 0; f'(x) > 0$  for all  $x; f''(x) < 0$  for all  $x$
52.  $f(0) = 1; f'(x) < 0$  for all  $x; f''(x) < 0$  for  $x < 0; f''(x) > 0$  for  $x > 0$
53.  $f(-1) = 0; f(2) = 2; f'(x) < 0$  for  $x < -1; f'(x) > 0$  for  $x > -1; f''(x) < 0$  for  $0 < x < 2; f''(x) > 0$  for  $x < 0$  or  $x > 2$
54. Display the graph of  $f(x) = x^{2/3}$  for  $-2 < x < 2$  on a graphing calculator. Determine the continuity of  $f(x), f'(x),$  and  $f''(x)$ . Discuss the concavity of the curve in relation to the minimum point. (See the paragraph before Example 3 on page 729.)

#### Answers to Practice Exercises

1. Max.  $(-1, 5)$ , Min.  $(3, -27)$     2. Infl.  $(1, -11)$

## 24.6 More on Curve Sketching

Intercepts • Symmetry •  
Behaviour as  $x$  Becomes Large •  
Vertical Asymptotes •  
Domain and Range • Derivatives

When graphing functions in earlier chapters, we often used information that was obtainable from the function itself. We will now use this type of information, along with that found from the derivatives, to graph functions. We will find, in graphing any particular function, some types of information are of more value than others. The features we will consider in this section are as follows:

### Features to Be Used in Graphing Functions

1. *Intercepts* Points for which the graph crosses (or is tangent to) each axis.
2. *Symmetry* For a review of symmetry, see Example 4 and the text before it on page 574.
3. *Behaviour as  $x$  Becomes Large* We will find what happens to the function as  $x \rightarrow \infty$  and as  $x \rightarrow -\infty$  in a way similar to that when we discussed the asymptotes of a hyperbola on page 588.
4. *Vertical Asymptotes* We will find that we can find vertical asymptotes similar to those noted on pages 100 and 308 by finding values that make factors in the denominator zero.
5. *Domain and Range* For a review of domain and range, see pages 91–93.
6. *Derivatives* We will find information from the first two derivatives as we did in the previous section (pages 727–731).

### EXAMPLE 1 Sketch a graph

Sketch the graph of  $y = \frac{8}{x^2 + 4}$ .

*Intercepts* If  $x = 0$ ,  $y = 2$ , which means  $(0, 2)$  is an intercept. If  $y = 0$ , there is no corresponding value of  $x$ , since  $8/(x^2 + 4)$  is a fraction greater than zero for all  $x$ . This also indicates that all points on the curve are above the  $x$ -axis.

*Symmetry* The curve is symmetric to the  $y$ -axis since

$$y = \frac{8}{(-x)^2 + 4} \text{ is the same as } y = \frac{8}{x^2 + 4}.$$

The curve is not symmetric to the  $x$ -axis since

$$-y = \frac{8}{x^2 + 4} \text{ is not the same as } y = \frac{8}{x^2 + 4}.$$

The curve is not symmetric to the origin since

$$-y = \frac{8}{(-x)^2 + 4} \text{ is not the same as } y = \frac{8}{x^2 + 4}.$$

The value in knowing the symmetry is that we should find those portions of the curve on either side of the  $y$ -axis as reflections of the other. It is possible to use this fact directly or to use it as a check.

*Behaviour as  $x$  Becomes Large* We note that as  $x \rightarrow \infty$ ,  $y \rightarrow 0$  since  $8/(x^2 + 4)$  is always a fraction that is greater than zero but which becomes smaller as  $x$  becomes larger. Therefore, we see that  $y = 0$  is an asymptote. From either the symmetry or the function, we also see that  $y \rightarrow 0$  as  $x \rightarrow -\infty$ .

*Vertical Asymptotes* From the discussion of the hyperbola, recall that an asymptote is a line that a curve approaches. We have already noted that  $y = 0$  is an asymptote for this curve. This asymptote, the  $x$ -axis, is a horizontal line. *Vertical asymptotes, if any exist, are found by determining those values of  $x$  for which the denominator of any term is zero.* Such a value of  $x$  makes  $y$  undefined. Since  $x^2 + 4$  cannot be zero, this curve has no vertical asymptotes.

*Domain and Range* Since the denominator  $x^2 + 4$  cannot be zero,  $x$  can take on any value. This means the domain of the function is all values of  $x$ . Also, we have noted that  $8/(x^2 + 4)$  is a fraction greater than zero. Since  $x^2 + 4$  is 4 or greater,  $y$  is 2 or less. This tells us that the range of the function is  $0 < y \leq 2$ .

*Derivatives* We now determine what the derivatives can also tell us about the curve. We start with the first derivative.

■ The curve for Example 1 is a special case (with  $a = 1$ ) of the curve known as the *witch of Agnesi*. Its general form is

$$y = \frac{8a^3}{x^2 + 4a^2}$$

It is named for the Italian mathematician Maria Gaetana Agnesi (1718–1799). She wrote the first text that contained analytic geometry, differential and integral calculus, series (see Chapter 30), and differential equations (see Chapter 31). The word *witch* was used due to a mistranslation from Italian to English.

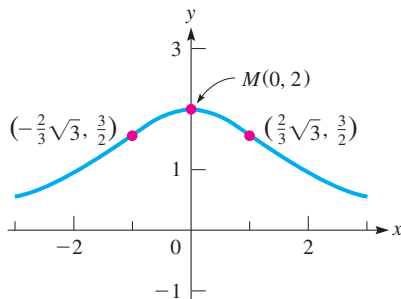


Fig. 24.40

$$y = \frac{8}{(x^2 + 4)} = 8(x^2 + 4)^{-1}$$

$$y' = -8(x^2 + 4)^{-2}(2x)$$

$$= \frac{-16x}{(x^2 + 4)^2}$$

Since  $(x^2 + 4)^2$  is positive for all values of  $x$ , the sign of  $y'$  is determined by the numerator. Thus, we note that  $y' = 0$  for  $x = 0$  and that  $y' > 0$  for  $x < 0$  and  $y' < 0$  for  $x > 0$ . The curve, therefore, is increasing for  $x < 0$ , is decreasing for  $x > 0$ , and has a maximum point at  $(0, 2)$ .

Now, finding the second derivative, we have

$$y'' = \frac{(x^2 + 4)^2(-16) + 16x(2)(x^2 + 4)(2x)}{(x^2 + 4)^4} = \frac{-16(x^2 + 4) + 64x^2}{(x^2 + 4)^3}$$

$$= \frac{48x^2 - 64}{(x^2 + 4)^3} = \frac{16(3x^2 - 4)}{(x^2 + 4)^3}$$

We note that  $y''$  is negative for  $x = 0$ , which confirms that  $(0, 2)$  is a maximum point. Also, points of inflection are found for the values of  $x$  satisfying  $3x^2 - 4 = 0$ . Thus,  $(-\frac{2}{3}\sqrt{3}, \frac{3}{2})$  and  $(\frac{2}{3}\sqrt{3}, \frac{3}{2})$  are points of inflection. The curve is concave up if  $x < -\frac{2}{3}\sqrt{3}$  or  $x > \frac{2}{3}\sqrt{3}$ , and the curve is concave down if  $-\frac{2}{3}\sqrt{3} < x < \frac{2}{3}\sqrt{3}$ .

Putting this information together, we sketch the curve shown in Fig. 24.40. Note that this curve could have been sketched primarily by use of the fact that  $y \rightarrow 0$  as  $x \rightarrow +\infty$  and as  $x \rightarrow -\infty$  and the fact that a maximum point exists at  $(0, 2)$ . However, the other parts of the analysis, such as symmetry and concavity, serve as checks and make the curve more accurate. ■

### EXAMPLE 2 Sketch a graph

Sketch the graph of  $y = x + \frac{4}{x}$ .

**Intercepts** If we set  $x = 0$ ,  $y$  is undefined. This means that the curve is not *continuous* at  $x = 0$  and there are no  $y$ -intercepts. If we set  $y = 0$ ,  $x + 4/x = (x^2 + 4)/x$  cannot be zero since  $x^2 + 4$  cannot be zero. Therefore, there are no intercepts. This may seem to be of little value, but we must realize *this curve does not cross either axis*. This will be of value when we sketch the curve in Fig. 24.41.

**Symmetry** In testing for symmetry, we find that the curve is not symmetric to either axis. However, this curve does possess symmetry to the origin. This is determined by the fact that when  $-x$  replaces  $x$  and at the same time  $-y$  replaces  $y$ , the equation does not change.

**Behaviour as  $x$  Becomes Large** As  $x \rightarrow +\infty$  and as  $x \rightarrow -\infty$ ,  $y \rightarrow x$  since  $4/x \rightarrow 0$ . Thus,  $y = x$  is an asymptote of the curve.

**Vertical Asymptotes** As we noted in Example 1, vertical asymptotes exist for values of  $x$  for which  $y$  is undefined. In this equation,  $x = 0$  makes the second term on the right undefined, and therefore  $y$  is undefined. In fact, as  $x \rightarrow 0$  from the positive side,  $y \rightarrow +\infty$ , and as  $x \rightarrow 0$  from the negative side,  $y \rightarrow -\infty$ . This is derived from the sign of  $4/x$  in each case.

**Domain and Range** Since  $x$  cannot be zero, the domain of the function is all  $x$  except zero. As for the range, the analysis from the derivatives will show it to be  $y \leq -4$ ,  $y \geq 4$ .

**Derivatives** Finding the first derivative, we have

$$y' = 1 - \frac{4}{x^2} = \frac{x^2 - 4}{x^2}$$

The  $x^2$  in the denominator indicates that the sign of the first derivative is the same as its numerator. The numerator is zero if  $x = -2$  or  $x = 2$ . If  $x < -2$  or  $x > 2$ , then

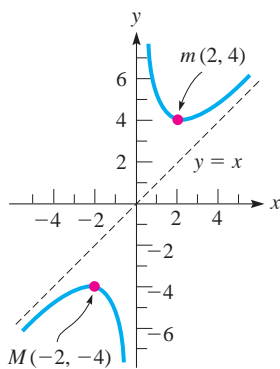


Fig. 24.41

$y' > 0$ ; and if  $-2 < x < 2, x \neq 0, y' < 0$ . Thus,  $y$  is increasing if  $x < -2$  or  $x > 2$ , and also  $y$  is decreasing if  $-2 < x < 2$ , except at  $x = 0$  ( $y$  is undefined). Also,  $(-2, -4)$  is a relative maximum point, and  $(2, 4)$  is a relative minimum point. The second derivative is  $y'' = 8/x^3$ . This cannot be zero, but it is negative if  $x < 0$  and positive if  $x > 0$ . Thus, the curve is concave down if  $x < 0$  and concave up if  $x > 0$ . Using this information, we have the curve shown in Fig. 24.41. ■

### EXAMPLE 3 Sketch a graph

Sketch the graph of  $y = \frac{1}{\sqrt{1-x^2}}$ .

*Intercepts* If  $x = 0, y = 1$ . If  $y = 0, 1/\sqrt{1-x^2}$  would have to be zero, but it cannot since it is a fraction with 1 as the numerator for all values of  $x$ . Thus,  $(0, 1)$  is the only intercept.

*Symmetry* The curve is symmetric to the  $y$ -axis.

*Behaviour as  $x$  Becomes Large* The values of  $x$  cannot be considered beyond 1 or  $-1$ , for any value of  $x < -1$  or  $x > 1$  gives imaginary values for  $y$ . Thus, the curve does not exist for values of  $x < -1$  or  $x > 1$ .

*Vertical Asymptotes* If  $x = 1$  or  $x = -1, y$  is undefined. In each case, as  $x \rightarrow 1$  and as  $x \rightarrow -1, y \rightarrow +\infty$ .

*Domain and Range* From the analysis of  $x$  becoming large and of the vertical asymptotes, we see that the domain is  $-1 < x < 1$ . Also, since  $\sqrt{1-x^2}$  is 1 or less,  $1/\sqrt{1-x^2}$  is 1 or more, which means the range is  $y \geq 1$ .

*Derivatives*

$$y' = -\frac{1}{2}(1-x^2)^{-3/2}(-2x) = \frac{x}{(1-x^2)^{3/2}}$$

We see that  $y' = 0$  if  $x = 0$ . If  $-1 < x < 0, y' < 0$ , and also if  $0 < x < 1, y' > 0$ . Thus, the curve is decreasing if  $-1 < x < 0$  and increasing if  $0 < x < 1$ . There is a minimum point at  $(0, 1)$ .

$$\begin{aligned} y'' &= \frac{(1-x^2)^{3/2} - x(\frac{3}{2})(1-x^2)^{1/2}(-2x)}{(1-x^2)^3} = \frac{(1-x^2) + 3x^2}{(1-x^2)^{5/2}} \\ &= \frac{2x^2 + 1}{(1-x^2)^{5/2}} \end{aligned}$$

The second derivative cannot be zero since  $2x^2 + 1$  is positive for all values of  $x$ . The second derivative is also positive for all permissible values of  $x$ , which means the curve is concave up for these values.

Using this information, we sketch the graph in Fig. 24.42. ■

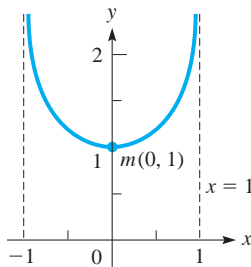


Fig. 24.42

### EXAMPLE 4 Sketch a graph

Sketch the graph of  $y = \frac{x}{x^2 - 4}$ .

*Intercepts* If  $x = 0, y = 0$ , and if  $y = 0, x = 0$ . The only intercept is  $(0, 0)$ .

*Symmetry* The curve is not symmetric to either axis. However, since  $-y = -x/[(x)^2 - 4]$  is the same as  $y = x/(x^2 - 4)$ , it is symmetric to the origin.

*Behaviour as  $x$  Becomes Large* As  $x \rightarrow +\infty$  and as  $x \rightarrow -\infty, y \rightarrow 0$ . This means that  $y = 0$  is an asymptote.

*Vertical Asymptotes* If  $x = -2$  or  $x = 2, y$  is undefined. As  $x \rightarrow -2, y \rightarrow -\infty$  if  $x < -2$  since  $x^2 - 4$  is positive, and  $y \rightarrow +\infty$  if  $x > -2$  since  $x^2 - 4$  is negative. As  $x \rightarrow 2, y \rightarrow -\infty$  if  $x < 2$ , and  $y \rightarrow +\infty$  if  $x > 2$ .

*Domain and Range* The domain is all real values of  $x$  except  $-2$  and  $2$ . As for the range, if  $x < -2, y < 0$  (the numerator is negative and the denominator is positive). If  $x > 2, y > 0$  (both numerator and denominator are positive). Since  $(0, 0)$  is an intercept, we see that the range is all values of  $y$ .

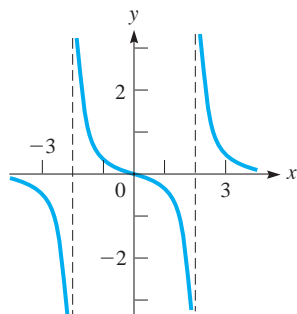


Fig. 24.43

Derivatives

$$y' = \frac{(x^2 - 4)(1) - x(2x)}{(x^2 - 4)^2} = -\frac{x^2 + 4}{(x^2 - 4)^2}$$

Since  $y' < 0$  for all values of  $x$  except  $-2$  and  $2$ , the curve is decreasing for all values in the domain.

Now, finding the second derivative, we have

$$\begin{aligned} y'' &= -\frac{(x^2 - 4)^2(2x) - (x^2 + 4)(2)(x^2 - 4)(2x)}{(x^2 - 4)^4} \\ &= -\frac{2x(x^2 - 4) - 4x(x^2 + 4)}{(x^2 - 4)^3} = \frac{2x^3 + 24x}{(x^2 - 4)^3} = \frac{2x(x^2 + 12)}{(x^2 - 4)^3} \end{aligned}$$

The sign of  $y''$  depends on  $x$  and  $(x^2 - 4)^3$ . If  $x < -2$ ,  $y'' < 0$ . If  $-2 < x < 0$ ,  $y'' > 0$ . If  $0 < x < 2$ ,  $y'' < 0$ . If  $x > 2$ ,  $y'' > 0$ . This means the curve is concave down for  $x < -2$  or  $0 < x < 2$  and is concave up for  $-2 < x < 0$  or  $x > 2$ .

The curve is sketched in Fig. 24.43. ■

## EXERCISES 24.6

In Exercises 1–18, use the method of the examples of this section to sketch the indicated curves.

1. In Example 2, change the  $+$  to  $-$  before  $4/x$  and then proceed.

- |                                   |                               |                                       |
|-----------------------------------|-------------------------------|---------------------------------------|
| 2. $y = \frac{4}{x^2}$            | 3. $y = \frac{2}{x+1}$        | 4. $y = \frac{x}{x-2}$                |
| 5. $y = x^2 + \frac{2}{x}$        | 6. $y = x + \frac{4}{x^2}$    | 7. $y = x - \frac{1}{x}$              |
| 8. $y = 3x + \frac{1}{x^3}$       | 9. $y = \frac{x^2}{x+1}$      | 10. $y = \frac{9x^2}{x^2+9}$          |
| 11. $y = \frac{1}{x^2-1}$         | 12. $y = \frac{x^2-1}{x^3}$   | 13. $y = \frac{4}{x} - \frac{4}{x^2}$ |
| 14. $y = 4x + \frac{1}{\sqrt{x}}$ | 15. $y = x\sqrt{1-x^2}$       | 16. $y = \frac{x-1}{x^2-2x}$          |
| 17. $y = \frac{9x}{9-x^2}$        | 18. $y = \frac{x^2-4}{x^2+4}$ |                                       |

In Exercises 19–30, solve the given problems.

19. Display the graph of  $y = \frac{cx}{1+c^2x^2}$  on a graphing calculator for  $c = -3, -1, 1, \text{ and } 3$ . Describe how the graph changes as  $c$  varies.
20. Display the graph of  $y = x + \frac{4}{x^n}$  on a graphing calculator for  $n = 1, 2, 3, \text{ and } 4$ . Describe how the graph changes as  $n$  varies.
21. In Exercise 10, first divide the numerator by the denominator. Does this simplify any of the graphing features?
22. The concentration  $C$  (in  $\text{mg}/\text{cm}^3$ ) of a certain drug in a patient's bloodstream is  $C = \frac{0.15t}{t^2+1}$ , where  $t$  is the time (in h) after the drug is administered. Sketch the graph.
23. The combined capacitance  $C_T$  (in  $\mu\text{F}$ ) of a  $6\text{-}\mu\text{F}$  capacitance and a variable capacitance  $C$  in series is given by  $C_T = \frac{6C}{6+C}$ . Sketch the graph.
24. Sketch a graph of  $p$  vs.  $V$  from van der Waal's equation (see Exercise 54 of Section 23.6), assuming the following values:  $R = T = a = 1$  and  $b = 0$ . For many gases the value of  $a$  is much greater than that of  $b$ . Even though the values of  $R = T = 1$  are not realistic, the *shape* of the curve will be correct for the assumed value of  $b = 0$ .
25. The reliability  $R$  of a computer model is found to be  $R = \frac{200}{\sqrt{t^2 + 40000}}$ , where  $t$  is the time of operation in hours. ( $R = 1$  is perfect reliability, and  $R = 0.5$  means there is a 50% chance of a malfunction.) Sketch the graph.
26. The electric power  $P$  (in W) produced by a source is given by  $P = \frac{36R}{R^2 + 2R + 1}$ , where  $R$  is the resistance in the circuit. Sketch the graph.
27. Assuming that a raindrop is always spherical and as it falls its radius increases from 1 mm to  $r$  mm, its velocity  $v$  (in mm/s) is  $v = k\left(r - \frac{1}{r^3}\right)$ . With  $k = 1$ , sketch the graph.
28. If a positive electric charge of  $+q$  is placed between two negative charges of  $-q$  that are two units apart, Coulomb's law states that the force  $F$  on the positive charge is  $F = -\frac{kq^2}{x^2} + \frac{kq^2}{(x-2)^2}$ , where  $x$  is the distance from one of the negative charges. Let  $kq^2 = 1$  and sketch the graph for  $0 < x < 2$ .
29. A cylindrical oil drum is to be made such that it will contain 20 kL. Sketch the area of sheet metal required for construction as a function of the radius of the drum.
30. A fence is to be constructed to enclose a rectangular area of 20 000  $\text{m}^2$ . A previously constructed wall is to be used for one side. Sketch the length of fence to be built as a function of the side of the fence parallel to the wall. See Fig. 24.44.

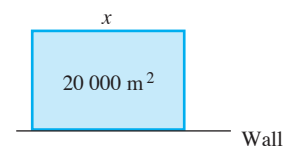


Fig. 24.44

## 24.7 Applied Maximum and Minimum Problems

Steps in Solving Maximum and Minimum Problems • Setting Up and Solving Problems

■ The first gasoline-engine automobile was built by the German engineer Karl Benz (1844 – 1929) in the 1880s.

Problems from various applied situations frequently occur that require finding a maximum or minimum value of some function. If the function is known, the methods we have already discussed can be used directly. This is discussed in the following example.

### EXAMPLE 1 Finding the maximum efficiency

An automobile manufacturer, in testing a new engine on one of its new models, found that the efficiency  $\eta$  (in %) of the engine as a function of the speed  $s$  (in km/h) of the car was given by  $\eta = 0.768s - 0.00004s^3$ . What is the maximum efficiency of the engine?

In order to find a maximum value, we find the derivative of  $\eta$  with respect to  $s$ :

$$\frac{d\eta}{ds} = 0.768 - 0.00012s^2$$

We then set the derivative equal to zero in order to find the value of  $s$  for which a maximum may occur:

$$\begin{aligned} 0.768 - 0.00012s^2 &= 0 \\ 0.00012s^2 &= 0.768 \\ s^2 &= 6400 \\ s &= 80.0 \text{ km/h} \end{aligned}$$

We know that  $s$  must be positive to have meaning in this problem. Therefore, the apparent solution of  $s = -80$  is discarded. The second derivative is

$$\frac{d^2\eta}{ds^2} = -0.00024s$$

which is negative for any positive value of  $s$ . Therefore, we have a maximum for  $s = 80.0$ . Substituting  $s = 80.0$  in the function for  $\eta$ , we obtain

$$\eta = 0.768(80.0) - 0.00004(80.0^3) = 61.44 - 20.48 = 40.96$$

The maximum efficiency is about 41.0%, which occurs for  $s = 80.0$  km/h. ■

In many problems for which a maximum or minimum value is to be found, the function is not given. To solve such a problem, we use these steps:

### Steps in Solving Applied Maximum and Minimum Problems

1. Determine the quantity  $Q$  to be maximized or minimized.
2. If possible, draw a figure illustrating the problem.
3. Write an equation for  $Q$  in terms of another variable of the problem.
4. Take the derivative of the function in step 3.
5. Set the derivative equal to zero, and solve the resulting equation.
6. Check as to whether the value found in step 5 makes  $Q$  a maximum or a minimum. This might be clear from the statement of the problem, or it might require one of the derivative tests.
7. Be sure the stated answer is the one the problem required. Some problems require the maximum or minimum value, and others require values of other variables that give the maximum or minimum value.



**LEARNING TIP**

*The principal difficulty that arises in these problems is finding the proper function.* We must carefully read the problem to find the information needed to set up the function.

The following examples illustrate several types of stated problems involving maximum and minimum values.

**EXAMPLE 2** Setting up an equation and finding the maximum

Find the number that exceeds its square by the greatest amount.

The quantity to be maximized is the difference  $D$  between a number  $x$  and its square  $x^2$ . Therefore, the required function is

$$D = x - x^2$$

Since we want  $D$  to be a maximum, we find  $dD/dx$ , which is

$$\frac{dD}{dx} = 1 - 2x$$

Setting the derivative equal to zero and solving for  $x$ , we have

$$0 = 1 - 2x, \quad x = \frac{1}{2}$$

The second derivative gives  $d^2D/dx^2 = -2$ , which tells us that the second derivative is always negative. This means that whenever the first derivative is zero, it represents a maximum. In many problems, it is not necessary to test for maximum or minimum, since the nature of the problem will indicate which must be the case. For example, in this problem, we know that numbers greater than 1 do not exceed their squares at all. The same is true for all negative numbers. Thus, the difference must reach a maximum for some number between 0 and 1. The answer is  $x = 1/2$ . ■

**EXAMPLE 3** Setting up an equation using geometry

A rectangular corral is to be enclosed with 1600 m of fencing. Find the maximum possible area of the corral.

There are limitless possibilities for rectangles of a perimeter of 1600 m and differing areas. See Fig. 24.45. For example, if the sides are 700 m and 100 m, the area is 70 000 m<sup>2</sup>, or if the sides are 600 m and 200 m, the area is 120 000 m<sup>2</sup>. Therefore, we set up a function for the area of a rectangle in terms of its sides  $x$  and  $y$ :

$$A = xy$$

Another important fact is that the perimeter of the corral is 1600 m. Therefore,  $2x + 2y = 1600$ . Solving for  $y$ , we have  $y = 800 - x$ . By using this expression for  $y$ , we can express the area in terms of  $x$  only. This gives us

$$A = x(800 - x) = 800x - x^2$$

We complete the solution as follows:

$$\begin{aligned} \frac{dA}{dx} &= 800 - 2x && \text{take derivative} \\ 800 - 2x &= 0 && \text{set derivative equal to zero} \\ x &= 400 \text{ m} \end{aligned}$$

The second derivative is  $d^2A/dx^2 = -2$ , which is always negative. Therefore, we have a maximum for  $x = 400$ . This means that  $x = 400$  m and  $y = 400$  m give the maximum area of 160 000 m<sup>2</sup> for the corral. ■

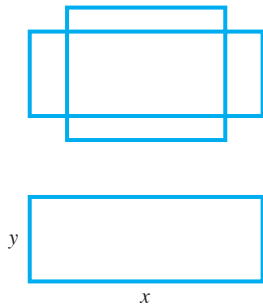


Fig. 24.45



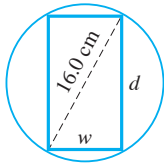


Fig. 24.46

**Practice Exercise**

1. In Example 4, change 16.0 cm to 12.0 cm and solve the resulting problem.

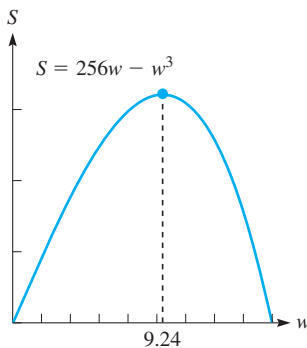


Fig. 24.47

**EXAMPLE 4** Setting up an equation using variation

The strength  $S$  of a beam with a rectangular cross section is directly proportional to the product of the width  $w$  and the square of the depth  $d$ . Find the dimensions of the strongest beam that can be cut from a log with a circular cross section that is 16.0 cm in diameter. See Fig. 24.46.

The solution proceeds as follows:

$$\begin{aligned}
 S &= kwd^2 && \text{direct variation} \\
 d^2 &= 256 - w^2 && \text{Pythagorean theorem} \\
 S &= kw(256 - w^2) && \text{substituting} \\
 &= k(256w - w^3) && S = f(w) \\
 \frac{dS}{dw} &= k(256 - 3w^2) && \text{take derivative} \\
 0 &= k(256 - 3w^2) && \text{set derivative equal to zero} \\
 3w^2 &= 256 && \text{solve for } w \\
 w &= \frac{16}{\sqrt{3}} = 9.24 \text{ cm} \\
 d &= \sqrt{256 - \frac{256}{3}} && \text{solve for } d \\
 &= 13.1 \text{ cm}
 \end{aligned}$$

This means that the strongest beam is about 9.24 cm wide and 13.1 cm deep. Since  $d^2S/dw^2 = -6kw$ , which is negative for  $w > 0$  (the only values with meaning in this problem), these dimensions give the maximum strength for the beam.

The solution can also be checked from the graph of  $S$  as a function of  $w$  (which we have graphed in Fig. 24.47 for  $k = 1$  since the value of  $k$  does not affect the solution). The function  $S$  reaches a maximum at  $w = 9.24$  cm (the maximum strength value will depend on  $k$ ). ■

**EXAMPLE 5** Setting up an equation using the distance formula

Find the point on the parabola  $y = x^2$  that is nearest to the point  $(6, 3)$ .

In this example, we must set up a function for the distance between a general point  $(x, y)$  on the parabola and the point  $(6, 3)$ . The relation is

$$D = \sqrt{(x - 6)^2 + (y - 3)^2}$$

However, to make it easier to take derivatives, we will square both sides of this expression. If a function is a minimum, then so is its square. We will also use the fact that the point  $(x, y)$  is on  $y = x^2$  by replacing  $y$  by  $x^2$ . Thus, we have

$$\begin{aligned}
 D^2 &= (x - 6)^2 + (x^2 - 3)^2 \\
 &= x^2 - 12x + 36 + x^4 - 6x^2 + 9 \\
 &= x^4 - 5x^2 - 12x + 45
 \end{aligned}$$

$$\begin{aligned}
 \frac{dD^2}{dx} &= 4x^3 - 10x - 12 && \text{take derivative} \\
 0 &= 2x^3 - 5x - 6 && \text{set derivative equal to zero}
 \end{aligned}$$

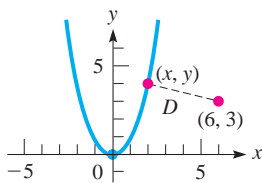


Fig. 24.48

There is exactly one positive solution to this equation (only one sign change). Testing the possible integer solutions (1, 2, 3, and 6), we find that  $x = 2$  is the solution. Thus, the required point on the parabola is  $(2, 4)$ . (See Fig. 24.48.) The second derivative is  $d(D^2)/dx^2 = 6x^2 - 5$ , which is positive at  $x = 2$ . Therefore, the distance is minimized. ■

**EXAMPLE 6** Setting up an equation using total value

A company determines that it can sell 1000 units of a product per month if the price is \$5 for each unit. It also estimates that for each 1 cent reduction in the unit price, 10 more units can be sold. Under these conditions, what is the maximum possible income and what price per unit gives this income?

If we let  $x =$  the number of units over 1000 sold, the total number of units sold is  $1000 + x$ . The price for each unit is \$5 less 1¢ (\$0.01) for each block of 10 units over 1000 that are sold. Thus, the price for each unit is

$$5 - 0.01\left(\frac{x}{10}\right) \quad \text{or} \quad 5 - 0.001x \text{ dollars}$$

The income  $I$  is the number of units sold times the price of each unit. Therefore,

$$I = (1000 + x)(5 - 0.001x)$$

Multiplying and finding the first derivative, we have

$$I = 5000 + 4x - 0.001x^2$$

$$\frac{dI}{dx} = 4 - 0.002x \quad \text{take derivative}$$

$$0 = 4 - 0.002x \quad \text{set derivative equal to zero}$$

$$x = 2000$$

We note that if  $x < 2000$ , the derivative is positive, and if  $x > 2000$ , the derivative is negative. Therefore, if  $x = 2000$ ,  $I$  is at a maximum. This means that the maximum income is derived if 2000 units over 1000 are sold, or 3000 units in all. This in turn means that the maximum income is \$9000 and the price per unit is \$3. These values are found by substituting  $x = 2000$  into the expression for  $I$  and for the price. ■

**EXAMPLE 7** Setting up an equation using geometry

Find the dimensions of a 700-kL cylindrical oil-storage tank that can be made with the least cost of sheet metal, assuming there is no wasted sheet metal.

Analysing the wording of the problem carefully, we see that we are to minimize the surface area of a right circular cylinder with a volume of 700 kL. Therefore, we set up expressions for the surface area and the volume ( $700 \text{ kL} = 700 \text{ m}^3$ ):

$$A = 2\pi r^2 + 2\pi r h \quad V = 700 = \pi r^2 h \quad (\text{see Fig. 24.49})$$

We can express the equation for the area in terms of  $r$  only by solving the second equation for  $h$  and substituting in the first equation. This gives us

$$h = \frac{700}{\pi r^2}, \quad A = 2\pi r^2 + 2\pi r \left(\frac{700}{\pi r^2}\right) = 2\pi r^2 + \frac{1400}{r}$$

In order to find the minimum value of  $A$ , we find  $dA/dr$  and set it equal to zero:

$$\frac{dA}{dr} = 4\pi r - \frac{1400}{r^2}, \quad 4\pi r - \frac{1400}{r^2} = 0, \quad \frac{4\pi r^3 - 1400}{r^2} = 0$$

$$4\pi r^3 - 1400 = 0, \quad r^3 = \frac{1400}{4\pi} \quad \text{numerator must} = 0$$

$$r = \sqrt[3]{\frac{1400}{4\pi}} = 4.81 \text{ m} \quad h = \frac{700}{\pi r^2} = 9.62 \text{ m}$$

Since  $dA/dr$  changes sign from negative to positive at  $r = 4.81 \text{ m}$ ,  $A$  is a minimum. ■

**Practice Exercise**

2. In Example 6, change 1 cent to 2 cents and solve the resulting problem.

■ See the chapter introduction.

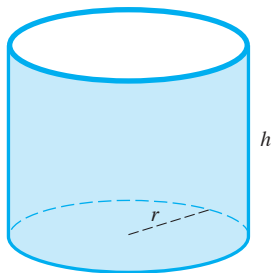


Fig. 24.49

**EXAMPLE 8** Setting up an equation using variation

The illuminance of a light source (in lux) at any point equals the strength of the source (in lumens) divided by the square of the distance (in metres) from the source. Two sources, of strengths 8 lumens and 1 lumen, are 100 m apart. Determine at what point between them the illuminance is the least, assuming that the illuminance at any point is the sum of the illuminances of the two sources.

Let  $I$  = the sum of the illuminances and  
 $x$  = the distance from the source of strength 8

Then we find that

$$I = \frac{8}{x^2} + \frac{1}{(100 - x)^2}$$

is the function relating the illuminance and the distance from the source of strength 8. We must now take a derivative of  $I$  with respect to  $x$ , set it equal to zero, and solve for  $x$  to find the point at which the illuminance is a minimum:

$$\frac{dI}{dx} = -\frac{16}{x^3} + \frac{2}{(100 - x)^3} = \frac{-16(100 - x)^3 + 2x^3}{x^3(100 - x)^3}$$

This function will be zero if the numerator is zero. Therefore, we have

$$2x^3 - 16(100 - x)^3 = 0 \quad \text{or} \quad x^3 = 8(100 - x)^3$$

Taking cube roots of each side, we have

$$x = 2(100 - x) \quad \text{or} \quad x = 66.7 \text{ m}$$

The illuminance is a minimum 66.7 m from the 8-lumen source. ■

**EXERCISES 24.7**

In the following exercises, solve the given maximum and minimum problems.

- In Example 3, change 1600 m to 2400 m and then proceed.
- In Example 7, change 700 kL to 800 kL and then proceed.
- The height (in m) of a flare shot upward from the ground is given by  $s = 34.3t - 4.9t^2$ , where  $t$  is the time (in s). What is the greatest height to which the flare goes?
- A small oil refinery estimates that its daily profit  $P$  (in dollars) from refining  $x$  barrels of oil is  $P = 8x - 0.02x^2$ . How many barrels should be refined for maximum daily profit, and what is the maximum profit?
- The power output  $P$  of a battery of voltage  $E$  and internal resistance  $R$  is  $P = EI - RI^2$ , where  $I$  is the current. Find the current for which the power is a maximum.
- In a random sample of 20 machine parts manufactured by a certain company, 3 are defective. Find the maximum likelihood estimate of the proportion of machine parts produced by this company that are defective by maximizing the likelihood function  $L(p) = 1140p^3(1 - p)^{17}$ .
- A dc-generator with an internal resistance  $r$  develops  $V$  volts. If the variable resistance in the circuit is  $R$ , the power generated is  $P = \frac{V^2}{r + R}$ . What resistance  $R$  gives the maximum power?
- If an airplane is moving at velocity  $v$ , the drag  $D$  on the plane is  $D = av^2 + b/v^2$ , where  $a$  and  $b$  are positive constants. Find the value(s) of  $v$  for which the drag is the least.
- A company projects that its total savings  $S$  (in dollars) by converting to a solar-heating system with a solar-collector of area  $A$  (in  $\text{m}^2$ ) will be  $S = 360A - 0.10A^3$ . Find the area that should give the maximum savings and find the amount of the maximum savings.
- The altitude  $h$  (in m) of a jet that goes into a dive and then again turns upward is given by  $h = 16t^3 - 240t^2 + 10\,000$ , where  $t$  is the time (in s) after the start of the dive. What is the altitude of the jet when it turns up out of the dive?
- If a resistance  $R$  and inductance  $L$  are in parallel with a capacitance  $C$ , the impedance is  $Z = \sqrt{\frac{R^2 + \omega^2 L^2}{\omega^2 C^2 R^2 + (\omega^2 LC - 1)^2}}$ , where  $\omega$  is the angular frequency of the circuit impedance. For what value(s) of  $C$  is  $Z$  a maximum, if  $R$  and  $L$  are constant?
- Test results show that, when coughing, the velocity  $v$  of the wind in a person's windpipe is  $v = kr^2(a - r)$ , where  $a$  is the radius of the windpipe when not coughing and  $k$  is a constant. Find  $r$  for the maximum value of  $v$ .
- An alpha particle moves through a magnetic field along the parabolic path  $y = x^2 - 4$ . Determine the closest that the particle comes to the origin.

14. The electric potential  $V$  on the line  $3x + 2y = 6$  is given by  $V = 3x^2 + 2y^2$ . At what point on this line is the potential a minimum?
15. In deep water, the velocity of a wave is  $v = k\sqrt{\frac{l}{a} + \frac{a}{l}}$ , where  $a$  and  $k$  are constants, and  $l$  is the length of the wave. What is the length of the wave that results in the minimum velocity?
16. The sum of the length  $l$  and width  $w$  of a rectangular table top is to be 240 cm. Determine  $l$  and  $w$  if the area of the table top is to be a maximum.
17. A rectangular hole is to be cut in a wall for a vent. If the perimeter of the hole is 48 cm and the length of the diagonal is a minimum, what are the dimensions of the hole?
18. When two electric resistors  $R_1$  and  $R_2$  are in series, their total resistance (the sum) is  $32 \Omega$ . If the same resistors are in parallel, their total resistance (the reciprocal of which equals the sum of the reciprocals of the individual resistances) is the maximum possible for two such resistors. What is the resistance of each?
19. A rectangular microprocessor chip is designed to have an area of  $25 \text{ mm}^2$ . What must its dimensions be if its perimeter is to be a minimum?
20. The rectangular animal display area in a zoo is enclosed by chain-link fencing and divided into two areas by internal fencing parallel to one of the sides. What dimensions will give the maximum area for the display if a total of 240 m of fencing are used?
21. For the microprocessor chip of Exercise 19, show that the chip will always be a square for a given value of the area  $A$ .
22. A rectangular storage area is to be constructed along the side of a tall building. A security fence is required along the remaining three sides of the area. What is the maximum area that can be enclosed with 800 m of fencing?
23. Ship  $A$  is travelling due east at 18.0 km/h as it passes a point 40.0 km due south of ship  $B$ , which is travelling due south at 16.0 km/h. How much later are the ships nearest each other?
24. An architect is designing a rectangular building in which the front wall costs twice as much per linear metre as the other three walls. The building is to cover  $1350 \text{ m}^2$ . What dimensions must it have such that the cost of the walls is a minimum?
25. A computer is programmed to display a slowly changing right triangle with its hypotenuse always equal to 12.0 cm. What are the legs of the triangle when it has its maximum area?
26. Canada Post regulations require that the sum of the three dimensions of a rectangular package not exceed 3 m. What are the dimensions of the largest rectangular box with square ends that can be mailed?
27. Referring to Exercise 26, a cylinder-shaped package is considered to have width and height equal to its diameter. What are the dimensions of the largest cylindrical package that may be sent through the mail?
28. An architect designs a window in the shape of a rectangle surmounted by an equilateral triangle. If the perimeter of the window is to be 6.00 m, what dimensions of the rectangle give the window the largest area?
29. The printed area of a rectangular poster is  $384 \text{ cm}^2$ , with margins of 4.00 cm on each side and margins of 6.00 cm at the top and bottom. Find the dimensions of the poster with the smallest area.

30. A conical funnel, with a very small opening, is being designed such that the slant height of the cone is 4.00 cm. What is the maximum volume of liquid that the funnel will be able to hold?
31. A culvert designed with a semicircular cross section of diameter 2.40 m is redesigned to have an isosceles trapezoidal cross section by inscribing the trapezoid in the semicircle. See Fig. 24.50. What is the length of the bottom base  $b$  of the trapezoid if its area is to be maximum?

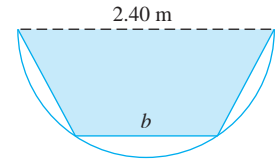


Fig. 24.50

32. A 36-cm-wide sheet of metal is bent into a rectangular trough as shown in Fig. 24.51. What dimensions give the maximum water flow?

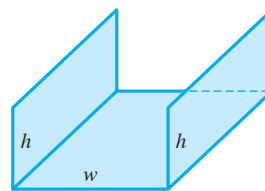


Fig. 24.51

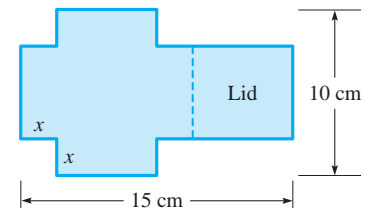


Fig. 24.52

33. A box with a lid is to be made from a rectangular piece of cardboard 10 cm by 15 cm, as shown in Fig. 24.52. Two equal squares of side  $x$  are to be removed from one end, and two equal rectangles are to be removed from the other end so that the tabs can be folded to form the box with a lid. Find  $x$  such that the volume of the box is a maximum.
34. A lap pool (a pool for swimming laps) is designed to be seven times as long as it is wide. If the area of the sides and bottom is  $90.0 \text{ m}^2$ , what are the dimensions of the pool if the volume of water it can hold is a maximum?
35. What is the maximum slope of the curve  $y = 6x^2 - x^3$ ?
36. What is the minimum slope of the curve  $y = x^5 - 10x^2$ ?
37. The deflection  $y$  of a beam of length  $L$  at a horizontal distance  $x$  from one end is given by  $y = k(2x^4 - 5Lx^3 + 3L^2x^2)$ , where  $k$  is a constant. For what value of  $x$  does the maximum deflection occur?
38. The electric power  $P$  (in W) produced by a certain battery is given by  $P = \frac{144r}{(r + 0.6)^2}$ , where  $r$  is the resistance in the circuit. For what value of  $r$  is the power a maximum?
39. For raising a load, the efficiency  $E$  (in %) of a screw with square threads is  $E = \frac{100T(1 - fT)}{T + f}$ , where  $f$  is the coefficient of friction and  $T$  is the tangent of the pitch angle of the screw. If  $f = 0.25$ , what acute angle makes  $E$  the greatest?
40. Computer simulation shows that the drag  $F$  (in N) on a certain airplane is  $F = 0.00500v^2 + 3.00 \times 10^8/v^2$ , where  $v$  is the velocity (in km/h) of the plane. For what velocity is the drag the least?
41. Factories  $A$  and  $B$  are 8.0 km apart, with factory  $B$  emitting eight times the pollutants into the air as factory  $A$ . If the number  $n$  of particles of pollutants is inversely proportional to the square of the distance from a factory, at what point between  $A$  and  $B$  is the pollution the least?

42. The potential energy  $E$  of an electric charge  $q$  due to another charge  $q_1$  at a distance of  $r_1$  is proportional to  $q_1$  and inversely proportional to  $r_1$ . If charge  $q$  is placed directly between two charges of 2.00 nC and 1.00 nC that are separated by 10.0 mm, find the point at which the total potential energy (the sum due to the other two charges) of  $q$  is a minimum.

43. An open box is to be made from a square piece of cardboard whose sides are 20.0 cm long, by cutting equal squares from the corners and bending up the sides. Determine the side of the square that is to be cut out so that the volume of the box may be a maximum. See Fig. 24.53.

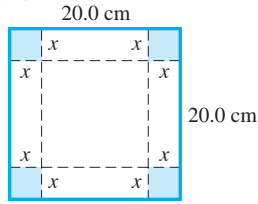


Fig. 24.53

44. A cone-shaped paper cup is to hold  $100 \text{ cm}^3$  of water. Find the height and radius of the cup that can be made from the least amount of paper.

45. A race track 400 m long is to be built around an area that is a rectangle with a semicircle at each end. Find the open side of the rectangle if the area of the rectangle is to be a maximum. See Fig. 24.54.

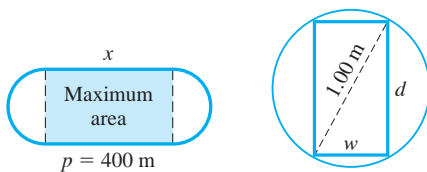


Fig. 24.54

Fig. 24.55

46. A beam of rectangular cross section is to be cut from a log 1.00 m in diameter. The stiffness of the beam varies directly as the width and the cube of the depth. What dimensions will give the beam maximum stiffness? See Fig. 24.55.

47. A company finds that there is a net profit of \$10 for each of the first 1000 units produced each week. For each unit over 1000 produced, there is 2 cents less profit per unit. How many units should be produced each week to net the greatest profit?

48. On a computer simulation, a target plane is projected to be at  $(1.20, 6.05)$  (distances in km), and a rocket is fired along the path  $y = 8.00 - 2.00x^2$ . How far from the target does the rocket pass from the target plane's projected position?

49. An oil pipeline is to be built from a refinery to a tanker loading area. The loading area is 10.0 km downstream from the refinery and on the opposite side of a river 2.5 km wide. The pipeline is to run along the river and then cross to the loading area. If the pipeline costs \$50 000 per kilometre alongside the river and

\$80 000 per kilometre across the river, find the point  $P$  (see Fig. 24.56) at which the pipeline should be turned to cross the river if construction costs are to be a minimum.

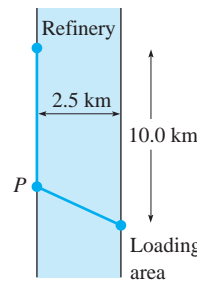


Fig. 24.56

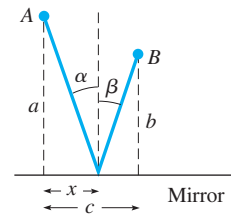


Fig. 24.57

50. A light ray follows a path of least time. If a ray starts at point  $A$  (see Fig. 24.57) and is reflected off a plane mirror to point  $B$ , show that the angle of incidence  $\alpha$  equals the angle of reflection  $\beta$ . (*Hint:* Set up the expression in terms of  $x$ , which will lead to  $\sin \alpha = \sin \beta$ .)

51. A rectangular building covering  $7000 \text{ m}^2$  is to be built on a rectangular lot as shown in Fig. 24.58. If the building is to be 10.0 m from the lot boundary on each side and 20.0 m from the boundary in front and back, find the dimensions of the building if the area of the lot is a minimum.

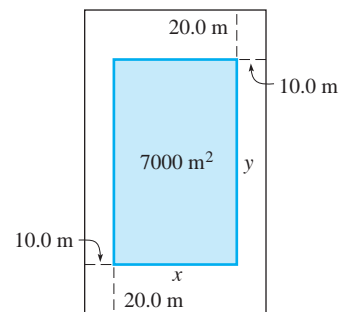


Fig. 24.58

52. A cylindrical cup (no top) is designed to hold  $375 \text{ cm}^3$  (375 mL). There is no waste in the material used for the sides. However, there is waste in that the bottom is made from a square  $2r$  on a side. What are the most economical dimensions for a cup made under these conditions?

Answers to Practice Exercises

1. 6.93 cm by 9.80 cm    2. \$6125 at \$3.50 per unit

## 24.8 Differentials and Linear Approximations

Differential • Increment • Estimating Errors in Measurement • Linear Approximations

■ The symbol  $dy/dx$  for the derivative was first used by Leibniz (see page 660).

To this point, we have used the  $dy/dx$  notation for the derivative of  $y$  with respect to  $x$ , but we have not considered it to be a ratio. In this section, we define the quantities  $dy$  and  $dx$ , called *differentials*, such that their ratio is equal to the derivative. Then we show that differentials have applications in approximating errors in measurement and in approximating values of functions. Also, in the next chapter, we use the differential notation in the development of integration, which is the inverse process of differentiation.

## DIFFERENTIALS

We define the **differential** of a function  $y = f(x)$  as

$$dy = f'(x) dx \quad (24.6)$$

In Eq. (24.6), the quantity  $dy$  is the differential of  $y$ , and  $dx$  is the differential of  $x$ . In this way, we can interpret the derivative as the ratio of the differential of  $y$  to the differential of  $x$ .

## EXAMPLE 1 Differential of a polynomial

Find the differential of  $y = 3x^5 - x$ .

Since  $f(x) = 3x^5 - x$ , we find  $f'(x) = 15x^4 - 1$ . This means that

$$dy = (15x^4 - 1) dx$$

$\xrightarrow{f'(x)}$   
 $\xrightarrow{\text{the differential of } x}$

## EXAMPLE 2 Differential of a rational expression

Find the differential of  $s = \frac{4t}{t^2 + 4}$ .

$$\begin{aligned} ds &= \frac{(t^2 + 4)(4) - (4t)(2t)}{(t^2 + 4)^2} dt && \text{using derivative quotient rule} \\ &= \frac{4t^2 + 16 - 8t^2}{(t^2 + 4)^2} dt = \frac{-4t^2 + 16}{(t^2 + 4)^2} dt \\ &= \frac{-4(t^2 - 4)}{(t^2 + 4)^2} dt && \text{don't forget the } dt \end{aligned}$$

## Practice Exercise

1. Find the differential of  $y = (2x - 1)^4$ .

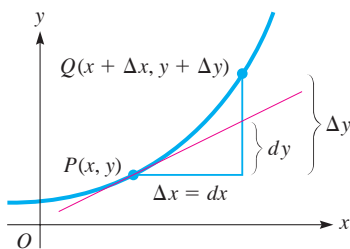


Fig. 24.59

To understand more about differentials and their applications, we use the delta notation introduced in Chapter 23. Recall that we defined  $\Delta x$  as the difference between two values of  $x$ . (Here we call  $\Delta x$  the **increment** in  $x$ .) By choosing  $\Delta x = dx$ , if  $dx$  is small, then the differential in  $y$ ,  $dy$ , closely approximates the increment in  $y$ ,  $\Delta y$ .

This is the basis of the applications of the differential, and to understand this better, let us look at Fig. 24.59. We see that points  $P(x, y)$  and  $Q(x + \Delta x, y + \Delta y)$  lie on the graph of  $f(x)$  and that the increments of  $x$  and  $y$  between the points are  $\Delta x$  and  $\Delta y$ . At point  $P$ ,  $f'(x) = dy/dx$ , which means that the slope of a tangent line at  $P$  is indicated by  $dy/dx$ . With  $dx = \Delta x$ , we see that as  $dx$  becomes smaller,  $dy$  more nearly approximates  $\Delta y$ , which is the actual difference in the  $y$ -values. Therefore, for small values of  $\Delta x$ ,  $dy$  can be used to approximate  $\Delta y$ .

## EXAMPLE 3 Calculating the increment and the differential

Calculate  $\Delta y$  and  $dy$  for  $y = x^3 - 2x$  for  $x = 3$  and  $\Delta x = 0.1$ .

First, we find  $\Delta y$  by calculating  $f(3.1) - f(3)$ . Therefore,

$$\Delta y = f(3.1) - f(3) = [3.1^3 - 2(3.1)] - [3^3 - 2(3)] = 2.591$$

The differential of  $y$  is

$$dy = (3x^2 - 2)dx$$

Since  $dx = \Delta x$ , we have

$$dy = [3(9) - 2](0.1) = 2.5$$

Thus,  $\Delta y = 2.591$  and  $dy = 2.5$ . In this case,  $dy$  is very nearly equal to  $\Delta y$ .



## ESTIMATING ERRORS IN MEASUREMENT

The fact that  $dy$  can be used to approximate  $\Delta y$  is useful in finding the error in a result from a measurement, if the data are in error, or the equivalent problem of finding the change in the result if a change is made in the data. Even though such changes can be found by using a calculator, the differential can be used to set up a general expression for the change of a particular function.

## EXAMPLE 4 Calculating the error in measurement

The edge of a cube of gold was measured to be 3.850 cm. From this value, the volume was found. Later, it was discovered that the value of the edge was 0.020 cm too small. By approximately how much was the volume in error?

The volume  $V$  of a cube, in terms of an edge  $s$ , is  $V = s^3$ . Since we wish to find the change in  $V$  for a given change in  $s$ , we want the value of  $dV$  for  $s = 3.850$  cm and  $ds = 0.020$  cm.

First, finding the general expression for  $dV$ , we have

$$dV = 3s^2 ds$$

Now, evaluating this expression for the given values, we have

$$dV = 3(3.850)^2(0.020) = 0.89 \text{ cm}^3$$

In this case, the volume was in error by about  $0.89 \text{ cm}^3$ . As long as  $ds$  is small compared with  $s$ , we can calculate an error or change in the volume of a cube by calculating the value of  $3s^2 ds$ . ■

Often, when considering the error of a given value or result, *the actual numerical value of the error*, the **absolute error**, is not as important as its size in relation to the size of the quantity itself. *The ratio of the absolute error to the size of the quantity itself is known as the relative error*, which is commonly expressed as a percent.

## EXAMPLE 5 Absolute error and relative error

Referring to Example 4, we see that the absolute error in the edge was 0.020 cm. The relative error in the edge was

$$\frac{ds}{s} = \frac{0.020}{3.85} = 0.0052 = 0.52\%$$

The absolute error in the volume was  $0.89 \text{ cm}^3$ , and the original value of the volume was  $3.850^3 = 57.07 \text{ cm}^3$ . This means the relative error in the volume was

$$\frac{dV}{V} = \frac{0.89}{57.07} = 0.016 = 1.6\% \quad \blacksquare$$

## LINEAR APPROXIMATIONS

Continuing our discussion related to Fig. 24.59, on the curve of the function  $f(x)$  at the point  $(a, f(a))$ , the slope of a tangent line is  $f'(a)$ . See Fig. 24.60. For a point  $(x, y)$  on the tangent line, by using the point-slope form for the equation of a straight line, Eq. (21.6), we have

$$f(x) \approx f(a) + f'(a)(x - a)$$

For the points on  $y = f(x)$  near  $x = a$ , we can use the tangent line to approximate the function, as shown in Fig. 24.60. Therefore, the approximation  $f(x) \approx L(x)$  is the **linear approximation** of  $f(x)$  near  $x = a$ , where the function

$$L(x) = f(a) + f'(a)(x - a) \quad (24.7)$$

is called the **linearization** of  $f(x)$  at  $x = a$ .

## Practice Exercise

2. In Example 4, approximate the error in the total surface area  $A$  of the cube.

■ For reference, Eq. (21.6) is  $y - y_1 = m(x - x_1)$ .

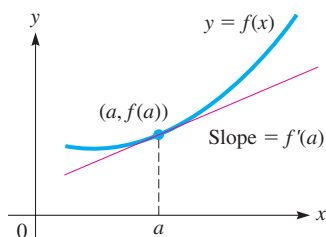


Fig. 24.60



**EXAMPLE 6** Linearizing a function

Find the linearization of the function  $f(x) = \sqrt{2x+1}$  at  $x = 4$ . Use it to approximate  $\sqrt{9.06}$ .

The solution is as follows:

$$f(x) = \sqrt{2x+1}$$

$$f'(x) = \frac{1}{2}(2x+1)^{-1/2}(2) = \frac{1}{\sqrt{2x+1}}$$

find the derivative

$$f(4) = \sqrt{2(4)+1} = 3 \quad f'(4) = \frac{1}{\sqrt{2(4)+1}} = \frac{1}{3}$$

evaluate  $f(4)$  and  $f'(4)$

$$L(x) = 3 + \frac{1}{3}(x-4) = \frac{x+5}{3}$$

Use Eq. (24.7)

$$\sqrt{2x+1} \approx \frac{x+5}{3}$$

$2x+1 = 9.06$   
 $x = 4.03$

$$\sqrt{9.06} \approx \frac{4.03+5}{3} = \frac{9.03}{3} = 3.01$$

by calculator,  
 $\sqrt{9.06} = 3.009983389$

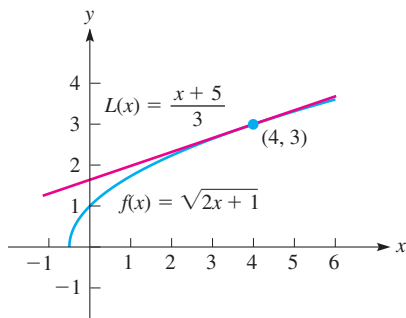


Fig. 24.61

In Fig. 24.61, the function  $f(x) = \sqrt{2x+1}$  and the tangent line are shown. We see that the tangent line gives a good approximation of the function when  $x$  is near 4. ■

**EXERCISES 24.8**

In Exercises 1–4, make the given changes in the indicated examples of this section and then solve the resulting problems.

- In Example 2, change the  $t^2$  to  $t^3$ .
- In Example 3, change the  $-$  before  $2x$  to  $+$ .
- In Example 5, change 0.020 to 0.025.
- In Example 6, change  $x = 4$  to  $x = 12$ ; approximate  $\sqrt{25.06}$ .

In Exercises 5–16, find the differential of each of the given functions.

- $y = x^5 + x$
- $y = 3x^2 + 6$
- $V = \frac{2}{r^5} + 3\pi^2$
- $y = 2\sqrt{x} - \frac{1}{8x}$
- $s = 2(3t^2 - 5)^4$
- $y = 5(4 + 3x)^{1/3}$
- $y = \frac{12}{3x^2 + 1}$
- $R = \sqrt{\frac{u}{1 + 2u}}$
- $y = x^2(1 - x)^3$
- $y = 6x\sqrt{1 - 4x}$
- $y = \frac{x}{5x + 2}$
- $y = \frac{3x + 1}{\sqrt{2x - 1}}$

In Exercises 17–20, find the values of  $\Delta y$  and  $dy$  for the given values of  $x$  and  $dx$ .

- $y = 7x^2 + 4x$ ,  $x = 4$ ,  $\Delta x = 0.2$
- $y = (x^2 + 2x)^3$ ,  $x = 7$ ,  $\Delta x = 0.02$
- $y = x\sqrt{1 + 4x}$ ,  $x = 12$ ,  $\Delta x = 0.06$
- $y = \frac{x}{\sqrt{6x - 1}}$ ,  $x = 3.5$ ,  $\Delta x = 0.025$

In Exercises 21–24, find the linearization  $L(x)$  of the given functions for the given values of  $a$ .

- $f(x) = x^2 + 2x$ ,  $a = 0$
- $f(x) = 2\sqrt[3]{x}$ ,  $a = 8$
- $f(x) = \frac{1}{2x+1}$ ,  $a = -1$
- $g(x) = x\sqrt{2x+8}$ ,  $a = -2$

In Exercises 25–36, solve the given problems by finding the appropriate differential. Round answers to 3 significant digits.

- If a spacecraft circles the earth at an altitude of 250 km, how much farther does it travel in one orbit than an airplane that circles the earth at a low altitude? The radius of the earth is 6370 km.
- Approximate the amount of paint needed to apply one coat of paint 0.50 mm thick on a hemispherical dome 55 m in diameter.
- The radius of a circular manhole cover is measured to be  $40.6 \pm 0.05$  cm (this means the possible error in the radius is 0.05 cm). Estimate the possible relative error in the area of the top of the cover.
- The side of a square microprocessor chip is measured as 0.950 cm, and later it is measured as 0.952 cm. What is the difference in the calculations of the area due to the difference in the measurements of the side? See Fig. 24.62.

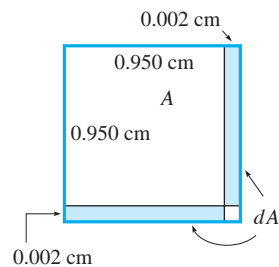


Fig. 24.62

29. The wavelength  $\lambda$  of light is inversely proportional to its frequency  $f$ . If  $\lambda = 685$  nm for  $f = 4.38 \times 10^{14}$  Hz, find the change in  $\lambda$  if  $f$  increases by  $0.20 \times 10^{14}$  Hz. (These values are for red light.)
30. The velocity of an object rolling down a certain inclined plane is given by  $v = \sqrt{40 + 4.9h}$ , where  $h$  is the distance (in m) travelled along the plane by the object. What is the increase in velocity (in m/s) of an object in moving from 20.0 m to 20.5 m along the plane? What is the relative change in the velocity?
31. The radius  $r$  of a holograph is directly proportional to the square root of the wavelength  $\lambda$  of the light used. Show that  $dr/r = \frac{1}{2}d\lambda/\lambda$ .
32. The gravitational force  $F$  of the earth on an object is inversely proportional to the square of the distance  $r$  of the object from the centre of the earth. Show that  $dF/F = -2dr/r$ .
33. Show that an error of 2% in the measurement of the radius of a DVD results in an error of approximately 4% in the calculation of the area.
34. Show that an error of 2% in the measurement of the radius of a ball bearing results in an error of approximately 6% in the calculation of the volume.
35. Calculate  $\sqrt{4.05}$ , using differentials.
36. Explain how to evaluate  $2.03^4$ .
- In Exercises 37–42, solve the given linearization problems.
37. Show that the linearization of  $f(x) = (1+x)^k$  at  $x = 0$  is  $L(x) = 1 + kx$ .
38. Use the result shown in Exercise 37 to approximate the value of  $f(x) = \frac{1}{\sqrt{1+x}}$  near zero.
39. Linearize  $f(x) = \sqrt{2-x}$  for  $a = 1$  and use it to approximate the value of  $\sqrt{1.9}$ .
40. Explain how to evaluate  $\sqrt[3]{8.03}$ , using linearization.
41. The capacitance  $C$  (in  $\mu\text{F}$ ) in an element of an electronic tuner is given by  $C = \frac{3.6}{\sqrt{1+2V}}$ , where  $V$  is the voltage. Linearize  $C$  for  $V = 4.0$  V.
42. A 16- $\Omega$  resistor is put in parallel with a variable resistor of resistance  $R$ . The combined resistance of the two resistors is  $R_T = \frac{16R}{16+R}$ . Linearize  $R_T$  for  $R = 4.0$   $\Omega$ .

### Answers to Practice Exercises

1.  $dy = 8(2x - 1)^3 dx$     2.  $dA = 0.92 \text{ cm}^2$

## CHAPTER 24 EQUATIONS

### Newton's method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad (24.1)$$

### Curvilinear motion

$$v_x = \frac{dx}{dt} \quad v_y = \frac{dy}{dt} \quad (24.2)$$

$$a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2} \quad a_y = \frac{dv_y}{dt} = \frac{d^2y}{dt^2} \quad (24.3)$$

$$v = \sqrt{v_x^2 + v_y^2} \quad a = \sqrt{a_x^2 + a_y^2} \quad (24.4)$$

$$\tan \theta_v = \frac{v_y}{v_x} \quad \tan \theta_a = \frac{a_y}{a_x} \quad (24.5)$$

### Curve sketching and maximum and minimum values

$f'(x) > 0$  where  $f(x)$  increases;  $f'(x) < 0$  where  $f(x)$  decreases.

$f''(x) > 0$  where the graph of  $f(x)$  is concave up;  $f''(x) < 0$  where the graph of  $f(x)$  is concave down.

If  $f'(x) = 0$  at  $x = a$ , there is a relative maximum point if  $f'(x)$  changes from + to - or if  $f''(a) < 0$ .

If  $f'(x) = 0$  at  $x = a$ , there is a relative minimum point if  $f'(x)$  changes from - to + or if  $f''(a) > 0$ .

If  $f''(x) = 0$  at  $x = a$ , there is a point of inflection if  $f''(x)$  changes from + to - or from - to +.

### Differential

$$dy = f'(x)dx \quad (24.6)$$

### Linearization

$$L(x) = f(a) + f'(a)(x - a) \quad (24.7)$$

## CHAPTER 24 REVIEW EXERCISES

In Exercises 1–6, find the equations of the tangent and normal lines.

- Find the equation of the line tangent to the parabola  $y = 3x - x^2$  at the point  $(-1, -4)$ . Graph the curve and the line.
- Find the equation of the line tangent to the curve  $y = x^2 - \frac{6}{x}$  at the point  $(3, 7)$ .
- Find the equation of the line normal to  $x^2 - 4y^2 = 9$  at the point  $(5, 2)$ . Graph the curve and the line.
- Find the equation of the line normal to  $y = \frac{4}{\sqrt{x-2}}$  at the point  $(6, 2)$ .
- Find the equation of the line tangent to the curve  $y = \sqrt{x^2 + 3}$  and that has a slope of  $\frac{1}{2}$ .
- Find the equation of the line normal to the curve  $y = \frac{1}{2x+1}$  and that has a slope of  $\frac{1}{2}$  if  $x \geq 0$ .

In Exercises 7–12, find the indicated velocities and accelerations. Round answers to 3 significant digits.

- Given that the  $x$ - and  $y$ -coordinates of a moving particle are given as a function of time  $t$  by the parametric equations  $x = \sqrt{t} + t$ ,  $y = \frac{1}{12}t^3$ , find the magnitude and direction of the velocity when  $t = 4$ .
- If the  $x$ - and  $y$ -coordinates of a moving object as functions of time  $t$  are given by  $x = 0.1t^2 + 1$ ,  $y = \sqrt{4t + 1}$ , find the magnitude and direction of the velocity when  $t = 6$ .
- An object moves along the curve  $y = 0.5x^2 + x$  such that  $v_x = 0.5\sqrt{x}$ . Find  $v_y$  at  $(2, 4)$ .
- A particle moves along the curve of  $y = \frac{1}{x+2}$  with a constant velocity in the  $x$ -direction of 4 cm/s. Find  $v_y$  at  $(2, \frac{1}{4})$ .
- Find the magnitude and direction of the acceleration for the particle in Exercise 7.
- Find the magnitude and direction of the acceleration for the particle in Exercise 10.

In Exercises 13–16, find the indicated roots of the given equations to at least four decimal places by use of Newton's method.

- $x^3 - 3x^2 - x + 2 = 0$  (between 0 and 1)
- $2x^3 - 4x^2 - 9 = 0$  (between 2 and 3)
- $x^2 = \frac{3}{x}$  (the real solution)
- $\sqrt{x-2} = \frac{1}{(x-6)^2}$  (the two real solutions)

In Exercises 17–24, sketch the graphs of the given functions by information obtained from the function as well as information obtained from the derivatives.

- |                                    |                              |
|------------------------------------|------------------------------|
| 17. $y = 4x^2 + 16x$               | 18. $y = x^3 + 2x^2 + x + 1$ |
| 19. $y = 27x - x^3$                | 20. $y = x(6-x)^3$           |
| 21. $y = x^4 - 32x$                | 22. $y = x^5 - 20x^2 + 10$   |
| 23. $y = \frac{x^2}{\sqrt{x^2-1}}$ | 24. $y = x^3 + \frac{3}{x}$  |

In Exercises 25–28, find the differential of each of the given functions.

- |                              |                                  |
|------------------------------|----------------------------------|
| 25. $y = 4x^3 + \frac{1}{x}$ | 26. $y = \frac{1}{(2x-1)^2}$     |
| 27. $y = x\sqrt[3]{1-3x}$    | 28. $s = \sqrt{\frac{2+t}{2-t}}$ |

In Exercises 29 and 30, evaluate  $\Delta y - dy$  for the given functions and values.

- $y = 4x^3 - 12$ ,  $x = 2$ ,  $\Delta x = 0.1$
- $y = 6x^2 - x$ ,  $x = 3$ ,  $\Delta x = 0.2$

In Exercises 31 and 32, find the linearization of the given functions for the given values of  $a$ .

- $f(x) = \sqrt{x^4 + 3x^2 + 8}$ ,  $a = 2$
- $f(x) = x^2(x+1)^4$ ,  $a = -2$

In Exercises 33–40, solve the given problems by finding the appropriate differentials.

- A weather balloon 3.50 m in radius becomes covered with a uniform layer of ice 1.20 cm thick. What is the volume of the ice?
- The total power  $P$  (in W) transmitted by an AM radio transmitter is  $P = 460 + 230m^2$ , where  $m$  is the modulation index. What is the change in power if  $m$  changes from 0.86 to 0.89?
- A cylindrical silo with a flat top is 12.0 m in diameter and is 12.0 m high. By how much is the volume changed if the radius is increased by 0.100 m and the height is unchanged?
- A ski slope follows a path that can be represented by  $y = 0.010x^2 - 0.86x + 24$ . What is the change in the slope of the path when  $x$  changes from 26 m to 28 m?
- The impedance  $Z$  of an electric circuit as a function of the resistance  $R$  and the reactance  $X$  is given by  $Z = \sqrt{R^2 + X^2}$ . Derive an expression of the relative error in impedance for an error in  $R$  and a given value of  $X$ .
- Evaluate  $\sqrt{8.94}$ .
- Evaluate  $3.02^5$ .
- Show that the relative error in the calculation of the volume of a sphere is approximately three times the relative error in the measurement of the radius.

In Exercises 41–80, solve the given problems.

- The parabolas  $y = x^2 + 2$  and  $y = 4x - x^2$  are tangent to each other. Find the equation of the line tangent to them at the point of tangency.
- Find the equation of the line tangent to the curve of  $y = x^4 - 8x$  and perpendicular to the line  $4y - x + 5 = 0$ .
- The deflection  $y$  (in m) of a beam at a horizontal distance  $x$  (in m) from one end is given by  $y = k(x^4 - 30x^3 + 1000x)$ , where  $k$  is a constant. Observing the equation and using Newton's method, find the values of  $x$  where the deflection is zero, if the beam is 10.000 m long.

44. The edges of a rectangular water tank are 3.00 m, 5.00 m, and 8.00 m. By Newton's method, determine by how much each edge should be increased equally to double the volume of the tank.
45. A parachutist descends (after the parachute opens) in a path that can be described by  $x = 8.0t$  and  $y = -0.15t^2$ , where distances are in metres and time is in seconds. Find the parachutist's velocity upon landing if the landing occurs when  $t = 12$  s.
46. One of the curves on an automobile test track can be described by  $y = 225/x$ , where dimensions are in metres. For a car approaching the curve at a constant velocity of 120 km/h, find the  $x$ - and  $y$ -components of the velocity at  $(10.0, 22.5)$ .
47. A person walks 250 m north and turns west. Walking at the rate of 1.0 m/s, at what rate is the distance between the person and the starting point increasing 1.0 min after turning west?
48. A glass prism for refracting light has equilateral triangular ends and vertical cross sections, and a volume of  $45 \text{ cm}^3$ . Find the edge of one of the ends such that the total surface area is a minimum.
49. In Fig. 24.63, the tension  $T$  supports the 40.0-N weight. The relation between the tension  $T$  and the deflection  $d$  is  $d = \frac{1000}{\sqrt{T^2 - 400}}$ . If the tension is increasing at 2.00 N/s when  $T = 28.0$  N, how fast is the deflection (in cm) changing?

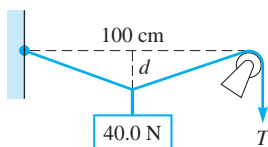


Fig. 24.63

50. The impedance  $Z$  (in  $\Omega$ ) in a particular electric circuit is given by  $Z = \sqrt{48 + R^2}$ , where  $R$  is the resistance. If  $R$  is increasing at a rate of  $0.45 \text{ } \Omega/\text{min}$  for  $R = 6.5 \text{ } \Omega$ , find the rate at which  $Z$  is changing.
51. By using the methods of this chapter to graph  $y = x^2 - \frac{2}{x}$ , graph the solution of the inequality  $x^2 > \frac{2}{x}$ .
52. Display the graphs of  $y_1 = x^2 - \frac{2}{x}$  and  $y_2 = x^2$  on the same screen of a graphing calculator, and explain why  $y_2$  can be considered to be a nonlinear asymptote of  $y_1$ .
53. An analysis of the power output  $P$  (in  $\text{kW}/\text{m}^3$ ) of a certain turbine showed that it depended on the flow rate  $r$  (in  $\text{m}^3/\text{s}$ ) of water to the turbine according to the equation  $P = 0.030r^3 - 2.6r^2 + 71r - 200$  ( $6.0 \leq r \leq 30 \text{ m}^3/\text{s}$ ). Determine the rate for which  $P$  is a maximum.
54. The altitude  $h$  (in m) of a certain rocket as a function of the time  $t$  (in s) after launching is given by  $h = 550t - 4.9t^2$ . What is the maximum altitude the rocket attains?
55. Sketch the continuous curve having these characteristics:  
 $f(0) = 2$     $f'(x) < 0$  for  $x < 0$     $f''(x) > 0$  for all  $x$   
 $f'(x) > 0$  for  $x > 0$
56. Sketch a continuous curve having these characteristics:  
 $f(0) = 1$     $f'(0) = 0$     $f''(x) < 0$  for  $x < 0$   
 $f'(x) > 0$  for  $|x| > 0$     $f''(x) > 0$  for  $x > 0$
57. A horizontal cylindrical oil tank (the length is parallel to the ground) of radius 2.00 m is being emptied. Find how fast the width  $w$  of the oil surface is changing when the depth  $h$  is 0.500 m and changing at the rate of  $0.0500 \text{ m}/\text{min}$ .

58. The current  $I$  (in A) in a circuit with a resistance  $R$  (in  $\Omega$ ) and a battery whose voltage is  $E$  and whose internal resistance is  $r$  (in  $\Omega$ ) is given by  $I = E/(R + r)$ . If  $R$  changes at the rate of  $0.250 \text{ } \Omega/\text{min}$ , how fast is the current changing when  $R = 6.25 \text{ } \Omega$ , if  $E = 3.10 \text{ V}$  and  $r = 0.230 \text{ } \Omega$ ?
59. The radius of a circular oil spill is increasing at the rate of  $15 \text{ m}/\text{min}$ . How fast is the area of the spill changing when the radius is 400 m?
60. A special insulation strip is to be sealed completely around three edges of a rectangular solar panel. If 200 cm of the strip are used, what is the maximum area of the panel?
61. A baseball diamond is a square 90.0 ft on a side. See Fig. 24.64. As a player runs from first base toward second base at  $18.0 \text{ ft}/\text{s}$ , at what rate is the player's distance from home plate increasing when the player is 40.0 ft from first base? (1 ft = 0.3048 m)

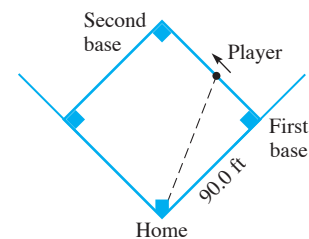


Fig. 24.64

62. A swimming pool with a rectangular surface of  $130 \text{ m}^2$  is to have a cement border area that is 4.00 m wide at each end and 2.75 m wide at the sides. Find the surface dimensions of the pool if the total area covered is to be a minimum.
63. A study showed that the percent  $y$  of persons surviving burns to  $x$  percent of the body is given by  $y = \frac{300}{0.0005x^2 + 2} - 50$ . Linearize this function with  $a = 50$  and sketch the graphs of  $y$  and  $L(x)$ .
64. A company estimates that the sales  $S$  (in dollars) of a new product will be  $S = 5000t/(t + 4)^2$ , where  $t$  is the time (in months) after it is put into production. Sketch the graph of  $S$  vs.  $t$ .
65. An airplane flying horizontally at 2400 m is moving toward a radar installation at 1110 km/h. If the plane is directly over a point on the ground 8.00 km from the radar installation, what is its actual speed? See Fig. 24.65.

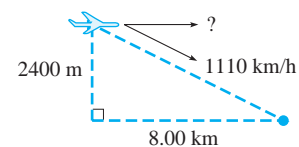


Fig. 24.65

66. The base of a conical machine part is being milled such that the height is decreasing at the rate of  $0.050 \text{ cm}/\text{min}$ . If the part originally had a radius of 1.0 cm and a height of 3.0 cm, how fast is the volume changing when the height is 2.8 cm?
67. The reciprocal of the total capacitance  $C_T$  of electrical capacitances in series equals the sum of the reciprocals of the individual capacitances. If the sum of two capacitances is  $12 \text{ } \mu\text{F}$ , find their values if their total capacitance in series is a maximum.

68. A cable is to be from point  $A$  to point  $B$  on a wall and then to point  $C$ . See Fig. 24.66. Where is  $B$  located if the total length of cable is a minimum?

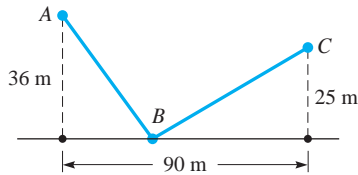


Fig. 24.66

69. A box with a square base and an open top is to be made of  $27 \text{ dm}^2$  of cardboard. What is the maximum volume that can be contained within the box?
70. A car is travelling east at  $90.0 \text{ km/h}$ , and an airplane is travelling south at  $450.0 \text{ km/h}$  at an elevation of  $2.00 \text{ km}$ . At one instant the plane is directly above the car. At what rate are they separating  $15.0 \text{ min}$  later?
71. A person in a boat  $4 \text{ km}$  from the nearest point  $P$  on a straight shoreline wants to go to point  $A$  on the shoreline,  $5 \text{ km}$  from  $P$ . If the person can row at  $3 \text{ km/h}$  and walk at  $5 \text{ km/h}$ , at what point on the shoreline should the boat land in order that point  $A$  can be reached in the least time? See Fig. 24.67.

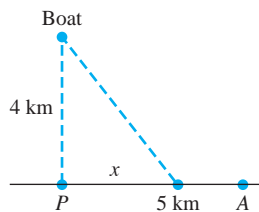


Fig. 24.67

72. A machine part is to be in the shape of a circular sector of radius  $r$  and central angle  $\theta$ . Find  $r$  and  $\theta$  if the area is one unit and the perimeter is a minimum. See Fig. 24.68.

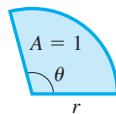


Fig. 24.68

73. An open drawer for small tools is to be made from a rectangular piece of heavy sheet metal  $36.0 \text{ cm}$  by  $30.0 \text{ cm}$ , by cutting out equal squares from two corners and bending up the three sides, as shown in Fig. 24.69. Find the side of the square that should be cut out so that the volume of the drawer is a maximum.

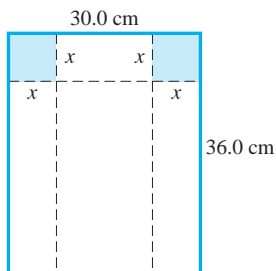


Fig. 24.69

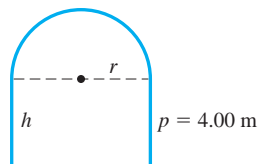


Fig. 24.70

74. A Norman window has the form of a rectangle surmounted by a semicircle. Find the dimensions (radius of circular part and height of rectangular part) of the window that will admit the most light if the perimeter of the window is  $4.00 \text{ m}$ . See Fig. 24.70.
75. A pile of sand in the shape of a cone has a radius that always equals the altitude. If  $3.00 \text{ m}^3$  of sand are poured onto the pile each minute, how fast is the radius increasing when the pile is  $2.50 \text{ m}$  high?
76. A book is designed such that its (rectangular) pages have  $2.5\text{-cm}$  margins at the top and bottom,  $1.5\text{-cm}$  margins on the sides, and a total area of  $320 \text{ cm}^2$ . What are the page dimensions that give the maximum printed area?
77. A specially made cylindrical container is made of stainless steel sides and bottom and a silver top. If silver is 10 times as expensive as stainless steel, what are the most economical dimensions of the container if it is to hold  $314 \text{ cm}^3$ ?
78. An object is moving in a horizontal circle of a radius  $2.00 \text{ m}$  at the rate of  $2.00 \text{ rad/s}$ . If the object is on the end of a string and the string breaks after  $1.05 \text{ s}$ , causing the object to travel along a line tangent to the circle, what is the equation of the path of the object after the string breaks? (Choose the origin of the coordinate system at the centre of the circle and assume the object started on the positive  $x$ -axis moving counterclockwise.)
79. A builder is designing a storage building with a total volume of  $38.3 \text{ m}^3$ , a rectangular base, and a flat roof. The width is to be  $0.75$  of the length. The cost per square foot is  $\$6.00$  for the floor,  $\$9.00$  for the sides, and  $\$4.50$  for the roof. What dimensions will minimize the cost of the building?
80. City  $B$  is  $16.0 \text{ km}$  east and  $12.0 \text{ km}$  north of city  $A$ . City  $A$  is  $8.00 \text{ km}$  due south of a river that is  $1.00 \text{ km}$  wide. A road is to be built between  $A$  and  $B$  that crosses straight across the river. See Fig. 24.71. Where should the bridge be located so that the road between  $A$  and  $B$  is as short as possible?

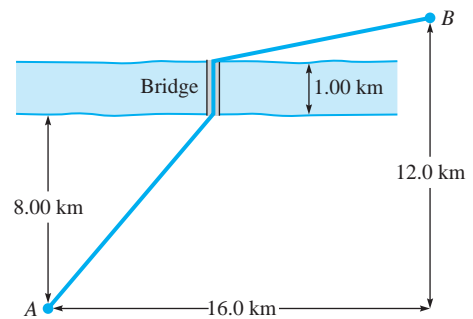


Fig. 24.71

### Writing Exercise

81. A container manufacturer makes various sizes of closed cylindrical plastic containers for shipping liquid products. Write two or three paragraphs explaining how to determine the ratio of the height to radius of the container such that the least amount of plastic is used for each size. Include the reason why it is not necessary to specify the volume of the container in finding this ratio.

## CHAPTER 24 PRACTICE TEST

- Find the equation of the line tangent to the curve  $y = x^4 - 3x^2$  at the point  $(1, -2)$ .
- For  $y = 3x^2 - x$ , evaluate (a)  $\Delta y$ , (b)  $dy$ , and (c)  $\Delta y - dy$  for  $x = 3$  and  $\Delta x = 0.1$ .
- If the  $x$ - and  $y$ -coordinates of a moving object as functions of time are given by the parametric equations  $x = 3t^2$ ,  $y = 2t^3 - t^2$ , find the magnitude and direction of the acceleration when  $t = 2$ .
- The electric power (in W) produced by a certain source is given by  $P = \frac{144r}{(r + 0.6)^2}$ , where  $r$  is the resistance (in  $\Omega$ ) in the circuit. For what value of  $r$  is the power a maximum?
- Find the root of the equation  $x^2 - \sqrt{4x + 1} = 0$  between 1 and 2 to four decimal places by use of Newton's method. Use  $x_1 = 1.5$  and find  $x_3$ .
- Linearize the function  $y = \sqrt{2x + 4}$  for  $a = 6$ .
- Sketch the graph of  $y = x^3 + 6x^2$  by finding the values of  $x$  for which the function is increasing, decreasing, concave up, and concave down and by finding any maximum points, minimum points, and points of inflection.
- Sketch the graph of  $y = \frac{4}{x^2} - x$  by finding the same information as required in Problem 7, as well as intercepts, symmetry, behaviour as  $x$  becomes large, vertical asymptotes, and the domain and range.
- Trash is being compacted into a cubical volume. The edge of the cube is decreasing at the rate of 0.10 m/s. When an edge of the cube is 1.25 m, how fast is the volume changing?
- A rectangular field is to be fenced and then divided in half by a fence parallel to two opposite sides. If a total of 6000 m of fencing is used, what is the maximum area that can be fenced?



# 25

## LEARNING OUTCOMES

After completion of this chapter, the student should be able to:

- Understand the concept of the antiderivative of a function
- Integrate basic functions such as constants, polynomials, and powers
- Evaluate the constant of integration
- Approximate the area under a curve using inscribed rectangles
- Find the area under a curve using a definite integral
- Use the trapezoidal rule and Simpson's rule to approximate a definite integral

# Integration

Finding areas of geometric figures had been studied by the ancient Greeks, and they had found how to find the area of any polygon. Also, they were able to find the area of a curved figure by inscribing polygons in the figure and then letting the number of sides of the polygon increase. There was little more progress in finding such areas until the 1600s when analytic geometry was developed.

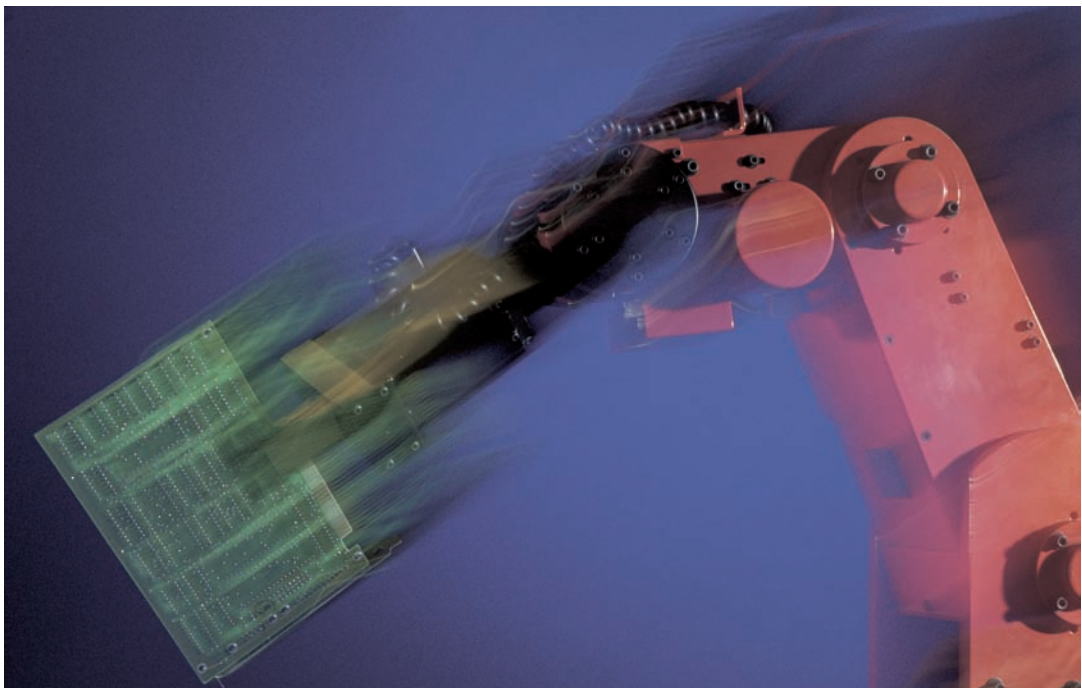
Several mathematicians of the 1600s studied both the area problem and the tangent problem that was discussed in Chapter 23. These included the French mathematician Pierre de Fermat and the English mathematician Isaac Barrow, both of whom developed a few formulas by which tangents and areas could be found. However, Newton and Leibniz found that these two problems were related and determined general methods of finding them. For these reasons, Newton and Leibniz are credited with the creation of calculus.

Finding tangents and finding areas appear to be very different, but they are closely related. As we will show, finding an area uses the inverse process of finding the slope of a tangent line, which we have shown can be interpreted as an instantaneous rate of change.

In physical and technical applications, we often find information that gives us the instantaneous rate of change of a variable. With such information, we have to reverse the process of differentiation in order to find the function when we know its derivative. This procedure is known as *integration*, which is the inverse process of differentiation.

This means that areas are found by integration. There are also many applications of integration in science and technology. A few of these applications will be illustrated in this chapter, and several specific applications will be developed in the next chapter.

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► In Section 25.2, integration is used to find the displacement of a robot arm as a function of time.



## 25.1 Antiderivatives

### Antidifferentiation • Antiderivative

We now show how to reverse the process of finding a derivative or a differential. *This reverse process is known as **antidifferentiation**.* In the next section, we formalize the process, and it is only the basic idea that is the topic of this section.

#### EXAMPLE 1 Find a function, knowing its derivative

Find a function for which the derivative is  $8x^3$ . That is, find an antiderivative of  $8x^3$ .

When finding the derivative of a constant times a power of  $x$ , we multiply the constant coefficient by the power of  $x$  and reduce the power by 1. Therefore, in this case, the power of  $x$  must have been 4 before the differentiation was performed.

If we let the derivative function be  $f(x) = 8x^3$  and then let its antiderivative function be  $F(x) = ax^4$  (by increasing the power of  $x$  in  $f(x)$  by 1), we can find the value of  $a$  by equating the derivative of  $F(x)$  to  $f(x)$ . This gives us

$$F'(x) = 4ax^3 = 8x^3, \quad 4a = 8, \quad a = 2$$

This means that  $F(x) = 2x^4$ . ■

#### EXAMPLE 2 Antiderivative of a polynomial

Find the antiderivative of  $v^2 + 2v$ .

For  $v^2$ , we know that the power of  $v$  required in an antiderivative is 3. Also, to make the coefficient correct, we must multiply by  $\frac{1}{3}$ .  $2v$  should be recognized as the derivative of  $v^2$ . Therefore, we have as an antiderivative  $\frac{1}{3}v^3 + v^2$ . ■

In Examples 1 and 2, *we could add any constant to the antiderivative shown and still have a correct antiderivative.* This is true because the derivative of a constant is zero. This is discussed in the next section, and we will not show any such constants in this section.

When we find an antiderivative of a function, we obtain another function. Thus, *we can define an **antiderivative** of the function  $f(x)$  to be a function  $F(x)$  such that  $F'(x) = f(x)$ .*

#### EXAMPLE 3 Antiderivative of a square root and a power with a negative exponent

Find an antiderivative of the function  $f(x) = \sqrt{x} - \frac{2}{x^3}$ .

Since we wish to find an antiderivative of  $f(x)$ , we know that  $f(x)$  is the derivative of the required function.

Considering the term  $\sqrt{x}$ , we first write it as  $x^{1/2}$ . To have  $x$  to the  $\frac{1}{2}$  power in the derivative, we must have  $x$  to the  $\frac{3}{2}$  power in the antiderivative. Knowing that the derivative of  $x^{3/2}$  is  $\frac{3}{2}x^{1/2}$ , we write  $x^{1/2}$  as  $\frac{2}{3}(\frac{3}{2}x^{1/2})$ . Thus, the first term of the antiderivative is  $\frac{2}{3}x^{3/2}$ .

As for the term  $-2/x^3$ , we write it as  $-2x^{-3}$ . This we recognize as the derivative of  $x^{-2}$ , or  $1/x^2$ .

This means that an antiderivative of the function

$$\begin{array}{ccc}
 \text{function} & f(x) = \sqrt{x} - \frac{2}{x^3} & \leftarrow \text{derivative} \\
 \downarrow & \text{is the function} & \uparrow \\
 \text{antiderivative} \rightarrow & F(x) = \frac{2}{3}x^{3/2} + \frac{1}{x^2} & \text{function}
 \end{array}$$

A great many functions of which we must find an antiderivative are not polynomials or simple powers of  $x$ . It is these functions that may cause more difficulty in the general process of antidifferentiation. Pay special attention to the following examples, for they illustrate a type of problem that you will find to be very important.

#### Practice Exercise

1. Find an antiderivative of  $x^3 + 4x$ .

## Practice Exercise

2. Find an antiderivative of  $5(4x - 3)^4(4)$ .

## LEARNING TIP

Note that when finding the antiderivative of a power of a function, the factor representing the derivative of the function does not appear in the antiderivative. Nevertheless, it must be present in the expression containing the power if we are to find a proper antiderivative.

For instance, in Example 4, the factor  $3x^2$  does not appear in the antiderivative. However, since the factor is part of the derivative of  $(x^3 - 1)^3$ , it must be present in the original function for  $(x^3 - 1)^3$  to be a correct antiderivative.

## EXAMPLE 4 Antiderivative of a power of a function

Find an antiderivative of the function  $f(x) = 3(x^3 - 1)^2(3x^2)$ .

Noting that we have a power of  $x^3 - 1$  in the derivative, it is reasonable that the antiderivative may include a power of  $x^3 - 1$ . Since, in the derivative,  $x^3 - 1$  is raised to the power 2, the antiderivative would then have  $x^3 - 1$  raised to the power 3. Noting that the derivative of  $(x^3 - 1)^3$  is  $3(x^3 - 1)^2(3x^2)$ , the desired antiderivative is

$$F(x) = (x^3 - 1)^3$$

## EXAMPLE 5 Antiderivative of a power of a function

Find an antiderivative of the function  $f(x) = (2x + 1)^{1/2}$ .

Here, we note a power of  $2x + 1$  in the derivative, which implies that the antiderivative has a power of  $2x + 1$ . Since in finding a derivative, 1 is subtracted from the power of  $2x + 1$ , we should add 1 in finding the antiderivative. Thus, we should have  $(2x + 1)^{3/2}$  as part of the antiderivative. Finding a derivative of  $(2x + 1)^{3/2}$ , we obtain  $\frac{3}{2}(2x + 1)^{1/2}(2) = 3(2x + 1)^{1/2}$ . This differs from the given derivative by the factor of 3. Thus, if we write  $(2x + 1)^{1/2} = \frac{1}{3}[3(2x + 1)^{1/2}]$ , we have the required antiderivative as

$$F(x) = \frac{1}{3}(2x + 1)^{3/2}$$

Checking, the derivative of  $\frac{1}{3}(2x + 1)^{3/2}$  is  $\frac{1}{3}(\frac{3}{2})(2x + 1)^{1/2}(2) = (2x + 1)^{1/2}$ .

## EXERCISES 25.1

In Exercises 1–4, make the given changes in the indicated examples of this section and then solve the resulting problems.

- In Example 1, change the coefficient 8 to 12.
- In Example 2, change  $v^2$  to  $v^3$ .
- In Example 3, change  $2/x^3$  to  $3/x^4$ .
- In Example 5, change  $2x$  to  $4x$ .

In Exercises 5–12, determine the value of  $a$  that makes  $F(x)$  an antiderivative of  $f(x)$ .

- $f(x) = 3x^2, F(x) = ax^3$
- $f(x) = 5x^4, F(x) = ax^5$
- $f(x) = 18x^5, F(x) = ax^6$
- $f(x) = 40x^7, F(x) = ax^8$
- $f(x) = 9\sqrt{x}, F(x) = ax^{3/2}$
- $f(x) = 10x^{1/4}, F(x) = ax^{5/4}$
- $f(x) = \frac{1}{x^2}, F(x) = \frac{a}{x}$
- $f(x) = \frac{6}{x^4}, F(x) = \frac{a}{x^3}$

In Exercises 13–38, find antiderivatives of the given functions.

- $f(x) = \frac{5}{2}x^{3/2}$
- $f(x) = \frac{4}{3}x^{1/3}$
- $f(t) = 6t^3 + 12$
- $f(x) = 12x^5 + 2x$
- $f(x) = 2x^2 - x$
- $f(x) = 6x^2 - 5$
- $f(x) = 2\sqrt{x} + 3$
- $f(s) = 9\sqrt[3]{s} - 3$
- $f(x) = -\frac{7}{x^6}$
- $f(x) = \frac{8}{x^5} - \pi$
- $f(v) = 4v + 3\pi^2$
- $f(x) = \frac{1}{2\sqrt{x}} + \sqrt{3}$
- $f(x) = x^2 - 4 + x^{-2}$
- $f(x) = x\sqrt{x} - x^{-3}$
- $f(x) = 6(2x + 1)^5(2)$
- $f(R) = 3(R^2 + 1)^2(2R)$

- $f(p) = 4(p^2 - 1)^3(2p)$
- $f(x) = 5(2x^4 + 1)^4(8x^3)$
- $f(x) = x^3(2x^4 + 1)^4$
- $f(x) = x(1 - x^2)^7$
- $f(x) = \frac{3}{2}(6x + 1)^{1/2}(6)$
- $f(y) = \frac{5}{4}(1 - y)^{1/4}(-1)$
- $f(x) = (3x + 1)^{1/3}$
- $f(x) = (4x + 3)^{1/2}$
- $f(x) = \frac{-2}{(2x + 1)^2}$
- $f(s) = \frac{4s}{(1 - s^2)^3}$

In Exercises 39 and 40, answer the given questions.

- Why is  $(x + 5)^3$  a correct antiderivative of  $3(x + 5)^2$ , whereas  $(2x + 5)^3$  is not a correct antiderivative of  $3(2x + 5)^2$ ?
- Is  $\frac{1}{(x + 5)^3}$  a correct antiderivative of  $\frac{1}{3(x + 5)^2}$ ?

## Answers to Practice Exercises

- $\frac{1}{4}x^4 + 2x^2$
- $F(x) = (4x - 3)^5$

## 25.2 The Indefinite Integral

Indefinite Integral • Integration • Evaluating the Constant of Integration

In the previous section, in developing the basic technique of finding an antiderivative, we noted that the results given are not unique. That is, we could have added any constant to the answers and the result would still have been correct. Again, this is the case since the derivative of a constant is zero.

### EXAMPLE 1 Antiderivatives are not unique

The derivatives of  $x^3$ ,  $x^3 + 4$ ,  $x^3 - 7$ , and  $x^3 + 4\pi$  are all  $3x^2$ . This means that any of the functions listed, as well as innumerable others, would be a proper answer to the problem of finding an antiderivative of  $3x^2$ . ■

From Section 24.8, we know that the differential of a function  $F(x)$  can be written as  $d[F(x)] = F'(x)dx$ . Therefore, since finding a differential of a function is closely related to finding the derivative, so is the antiderivative closely related to the process of finding the function for which the differential is known.

The notation used for finding the general form of the antiderivative, the **indefinite integral**, is written in terms of the differential. Thus, the indefinite integral of a function  $f(x)$ , for which  $dF(x)/dx = f(x)$ , or  $dF(x) = f(x)dx$ , is defined as

$$\int f(x) dx = F(x) + C \quad (25.1)$$

Here,  $f(x)$  is called the **integrand**,  $F(x) + C$  is the **indefinite integral**, and  $C$  is an arbitrary constant, called the **constant of integration**. It represents any of the constants that may be attached to an antiderivative to have a proper result. We must have additional information beyond a knowledge of the differential to assign a specific value to  $C$ . The symbol  $\int$  is the **integral sign**, and it indicates that the inverse of the differential is to be found. Determining the indefinite integral is called **integration**, which we can see is essentially the same as finding an antiderivative.

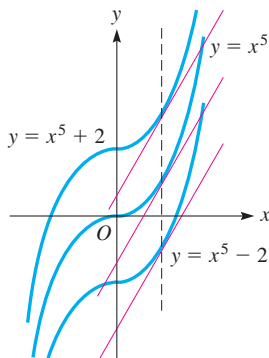


Fig. 25.1

### EXAMPLE 2 Indefinite integral of a polynomial

In performing the integration

$$\int 5x^4 dx = x^5 + C$$

↑ constant of integration  
↑ integrand  
↑ indefinite integral

we might think that the inclusion of this constant  $C$  would affect the derivative of the function  $x^5$ . However, the only effect of the  $C$  is to raise or lower the curve. The slope of  $y = x^5 + 2$ ,  $y = x^5 - 2$ , or any function of the form  $y = x^5 + C$  is the same for any given value of  $x$ . As Fig. 25.1 shows, tangents drawn to the curves are all parallel for the same value of  $x$ . ■

#### COMMON ERROR

When evaluating an indefinite integral, it is a common error to omit the constant of integration.

At this point, we shall derive some basic formulas for integration. Since we know  $d(cu)/dx = c(du/dx)$ , where  $u$  is a function of  $x$  and  $c$  is a constant, we can write

$$\int c du = c \int du = cu + C \quad (25.2)$$

Also, since the derivative of a sum of functions equals the sum of the derivatives, we write

$$\int (du + dv) = u + v + C \quad (25.3)$$

To find the differential of a power of a function, we multiply by the power, subtract 1 from it, and multiply by the differential of the function. *To find the integral, we reverse this procedure to get the power formula for integration:*

#### LEARNING TIP

We must be able to recognize the proper form and the component parts to use these formulas. Unless you do this and have a good knowledge of differentiation, you will have trouble using Eq. (25.4). Most of the difficulty, if it exists, arises from an improper identification of  $du$ .

$$\int u^n du = \frac{u^{n+1}}{n+1} + C \quad (n \neq -1) \quad (25.4)$$

#### EXAMPLE 3 Integration using the power rule

Integrate  $\int 6x dx$ .

We must identify  $u$ ,  $n$ ,  $du$ , and any multiplying constants. Noting that 6 is a multiplying constant, we identify  $x$  as  $u$ , which means  $dx$  must be  $du$  and  $n = 1$ .

$$\int 6x dx = 6 \int x^1 dx = 6 \left( \frac{x^2}{2} \right) + C = 3x^2 + C$$

do not forget the constant of integration

We see that our result checks since the differential of  $3x^2 + C$  is  $6x dx$ . ■

#### EXAMPLE 4 Integration of a polynomial using the power rule

Integrate  $\int (5x^3 - 6x^2 + 1) dx$ .

Here, we must use a combination of Eqs. (25.2), (25.3), and (25.4). Therefore,

$$\begin{aligned} \int (5x^3 - 6x^2 + 1) dx &= \int 5x^3 dx + \int (-6x^2) dx + \int dx \\ &= 5 \int x^3 dx - 6 \int x^2 dx + \int dx \end{aligned}$$

In the first integral,  $u = x$ ,  $n = 3$ , and  $du = dx$ . In the second,  $u = x$ ,  $n = 2$ , and  $du = dx$ . The third uses Eq. (25.2) directly, with  $c = 1$  and  $du = dx$ . This means

$$\begin{aligned} 5 \int x^3 dx - 6 \int x^2 dx + \int dx &= 5 \left( \frac{x^4}{4} \right) - 6 \left( \frac{x^3}{3} \right) + x + C \\ &= \frac{5}{4} x^4 - 2x^3 + x + C \end{aligned} \quad \blacksquare$$

#### Practice Exercise

1. Integrate  $\int (6x^2 - 5) dx$ .

**EXAMPLE 5** Integrating a square root and a power with a negative exponent

Integrate  $\int \left( \sqrt{r} - \frac{1}{r^3} \right) dr$ .

In order to use Eq. (25.4), we must first write  $\sqrt{r} = r^{1/2}$  and  $1/r^3 = r^{-3}$ :

$$\begin{aligned} \int \left( \sqrt{r} - \frac{1}{r^3} \right) dr &= \int r^{1/2} dr - \int r^{-3} dr = \frac{1}{\frac{3}{2}} r^{3/2} - \frac{1}{-2} r^{-2} + C \\ &= \frac{2}{3} r^{3/2} + \frac{1}{2} r^{-2} + C = \frac{2}{3} r^{3/2} + \frac{1}{2r^2} + C \end{aligned}$$

**EXAMPLE 6** Integrating a power of a function

Integrate  $\int (x^2 + 1)^3 (2x dx)$ .

We first note that  $n = 3$ , for this is the power involved in the function being integrated. If  $n = 3$ , then  $x^2 + 1$  must be  $u$ . If  $u = x^2 + 1$ , then  $du = 2x dx$ . Thus, the integral is in proper form for integration *as it stands*. Using the power formula,

$$\int (x^2 + 1)^3 (2x dx) = \frac{(x^2 + 1)^4}{4} + C$$

Showing the use of  $u$  directly, we can write the integration as

$$\int (x^2 + 1)^3 (2x dx) = \int u^3 du = \frac{1}{4} u^4 + C = \frac{(x^2 + 1)^4}{4} + C$$

**LEARNING TIP**

*It must be emphasized that the entire quantity  $(2x dx)$  must be equated to  $du$ . Normally,  $u$  and  $n$  are recognized first, and then  $du$  is derived from  $u$ .*

**EXAMPLE 7** Integrating a power of a function

Integrate  $\int x^2 \sqrt{x^3 + 2} dx$ .

We first note that  $n = \frac{1}{2}$  and  $u$  is then  $x^3 + 2$ . Since  $u = x^3 + 2$ ,  $du = 3x^2 dx$ . Now, we group  $3x^2 dx$  as  $du$ . Since there is no 3 under the integral sign, we introduce one. In order not to change the numerical value, we also introduce  $\frac{1}{3}$ , normally before the integral sign.

$$\int x^2 \sqrt{x^3 + 2} dx = \frac{1}{3} \int 3x^2 \sqrt{x^3 + 2} dx = \frac{1}{3} \int \sqrt{x^3 + 2} (3x^2 dx)$$

Here, we indicate the proper grouping to have the proper form of Eq. (25.4):

$$\begin{aligned} \int x^2 \sqrt{x^3 + 2} dx &= \frac{1}{3} \int \sqrt{x^3 + 2} (3x^2 dx) = \frac{1}{3} \left( \frac{2}{3} \right) (x^3 + 2)^{3/2} + C \\ &= \frac{2}{9} (x^3 + 2)^{3/2} + C \end{aligned}$$

The  $1/\frac{2}{3}$  was written as  $\frac{2}{3}$ , since this form is more convenient with fractions.

With  $u = x^3 + 2$  and using  $u$  directly in the integration, we can write

$$\begin{aligned} \int x^2 \sqrt{x^3 + 2} dx &= \int (x^3 + 2)^{1/2} (x^2 dx) \\ &= \int u^{1/2} \left( \frac{1}{3} du \right) = \frac{1}{3} \int u^{1/2} du && \text{integrating in terms of } u \\ &= \frac{1}{3} \left( \frac{2}{3} \right) u^{3/2} + C = \frac{2}{9} u^{3/2} + C \\ &= \frac{2}{9} (x^3 + 2)^{3/2} + C && \text{substituting } x^3 + 2 = u \end{aligned}$$

**Practice Exercise**

2. Integrate  $\int 8x \sqrt{1 - 2x^2} dx$ .

■ We have integrated certain basic functions. Other methods are used to integrate other types of functions, and some of these are discussed in Chapter 28. Also, many functions cannot be integrated.

**COMMON ERROR**

Because a constant factor may be moved across the integral sign, we can always introduce a constant required to complete  $du$ , and also its reciprocal (so as not to change the numerical value of the integral), as in Example 7. We note, however, that **only constant factors may be moved across the integral sign**. If  $du$  is missing a variable factor, we **cannot introduce the variable factor and move its variable reciprocal across the integral sign**. For example, when integrating  $\int (x^2 + 1)^2 dx$ , we cannot integrate by setting  $u = x^2 + 1$ ,  $du = 2x dx$  because we cannot introduce the factor  $x$  and take  $1/x$  out of the integral. Instead, we need to square the binomial and integrate  $\int (x^4 + 2x^2 + 1) dx$ .

**EVALUATING THE CONSTANT OF INTEGRATION**

To find the constant of integration, we need information such as a set of values that satisfy the function. A point through which the curve passes would provide the necessary information. This is illustrated in the following examples.

**EXAMPLE 8** Evaluating the constant of integration

Find  $y$  in terms of  $x$ , given that  $dy/dx = 3x - 1$  and the curve passes through  $(1, 4)$ .

The solution is as follows.

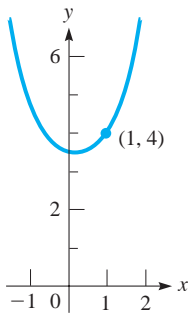


Fig. 25.2

$$dy = (3x - 1) dx \quad \text{rewrite equation—solve for } dy \text{ in terms of } dx$$

$$\int dy = \int (3x - 1) dx \quad \text{set up integration}$$

$$y = \frac{3}{2}x^2 - x + C \quad \text{integrate}$$

$$4 = \frac{3}{2} - 1 + C \quad \text{or} \quad C = \frac{7}{2} \quad \text{evaluate } C; \text{ point } (1, 4) \text{ satisfies equation}$$

$$y = \frac{3}{2}x^2 - x + \frac{7}{2} \quad \text{or} \quad 2y = 3x^2 - 2x + 7 \quad \text{see Fig. 25.2} \quad \blacksquare$$

**EXAMPLE 9** Integral with a negative exponent

The time rate of change of the displacement (velocity) of a robot arm is  $ds/dt = 8t/(t^2 + 4)^2$ . Find the expression for the displacement as a function of time if  $s = -1$  m when  $t = 0$  s.

First, we write  $ds = \frac{8t dt}{(t^2 + 4)^2}$  and then integrate. To integrate the expression on the right, we note that  $n = -2$ ,  $u = t^2 + 4$ , and  $du = 2t dt$ . This means we need a 2 with  $t dt$  to form the proper  $du$ . In turn, this means we place a  $\frac{1}{2}$  before the integral sign. Also, we place the 8 in front of the integral sign. Therefore,

$$\int ds = \int \frac{8t dt}{(t^2 + 4)^2} = \left(\frac{1}{2}\right)(8) \int (t^2 + 4)^{-2}(2t dt) \quad \text{set up integration}$$

$$s = 4 \left(\frac{1}{-1}\right) (t^2 + 4)^{-1} + C = \frac{-4}{t^2 + 4} + C \quad \text{integrate; } -2 + 1 = -1$$

$$-1 = \frac{-4}{0 + 4} + C \quad \text{or} \quad C = 0 \quad \text{evaluate } C \text{ given } s = -1 \text{ m when } t = 0 \text{ s}$$

$$s = \frac{-4}{t^2 + 4} \quad \text{expression for displacement} \quad \blacksquare$$

■ See the chapter introduction.

## EXERCISES 25.2

In Exercises 1–4, make the given changes in the indicated examples of this section, and then solve the resulting problems.

- In Example 3, change the coefficient 6 to 8.
- In Example 5, change  $\sqrt{r}$  to  $\sqrt[3]{r}$ .
- In Example 6, change the power 3 to 4.
- In Example 8, change  $(1, 4)$  to  $(2, 3)$ .

In Exercises 5–36, integrate each of the given expressions.

- |  |   |                            |
|--|---|----------------------------|
| 5. $\int 2x dx$  | 6. $\int 5x^4 dx$   | 7. $\int x^7 dx$           |
| 8. $\int 0.6y^5 dy$                                      | 9. $\int 8x^{3/2} dx$                                       | 10. $\int 6\sqrt[3]{x} dx$ |
| 11. $\int 9R^{-4} dR$                                    | 12. $\int \frac{4}{\sqrt{x}} dx$                            | 13. $\int (x^2 - x^5) dx$  |
| 14. $\int (1 - 3x) dx$                                   | 15. $\int (9x^2 + x + 3) dx$                                | 16. $\int x(x - 2)^2 dx$   |
| 17. $\int \left(\frac{t^2}{2} - \frac{2}{t^2}\right) dt$ | 18. $\int \frac{3x^2 - 4}{x^2} dx$                          |                            |
| 19. $\int \sqrt{x}(x^2 - x) dx$                          | 20. $\int (3R\sqrt{R} - 5R^2) dR$                           |                            |
| 21. $\int (2x^{-2/3} + 3^{-2}) dx$                       | 22. $\int (x^{1/3} + x^{1/5} + x^{-1/7}) dx$                |                            |
| 23. $\int (1 + 12x^2)^2 dx$                              | 24. $\int (x^2 + 4x + 4)^{1/3} dx$                          |                            |
| 25. $\int (x^2 - 1)^5 (2x dx)$                           | 26. $\int (t^3 - 2)^6 (3t^2 dt)$                            |                            |
| 27. $\int (x^4 + 3)^4 (4x^3 dx)$                         | 28. $\int (1 - 2x)^{1/3} (-2 dx)$                           |                            |
| 29. $\int (2\theta^5 + 5)^7 \theta^4 d\theta$            | 30. $\int 6x^2 (1 - x^3)^{4/3} dx$                          |                            |
| 31. $\int \sqrt{8x + 1} dx$                              | 32. $\int \frac{dV}{(0.3 + 2V)^3}$                          |                            |
| 33. $\int \frac{x dx}{\sqrt{6x^2 + 1}}$                  | 34. $\int \frac{2x^2 dx}{\sqrt{2x^3 + 1}}$                  |                            |
| 35. $\int \frac{4z - 4}{\sqrt{z^2 - 2z}} dz$             | 36. $\int (x^2 - x) \left(x^3 - \frac{3}{2}x^2\right)^8 dx$ |                            |

In Exercises 37–40, find  $y$  in terms of  $x$ .

- $\frac{dy}{dx} = 6x^2$ , curve passes through  $(0, 2)$
- $\frac{dy}{dx} = 8x + 1$ , curve passes through  $(-1, 4)$
- $\frac{dy}{dx} = x^2(1 - x^3)^5$ , curve passes through  $(1, 5)$

40.  $\frac{dy}{dx} = 2x^3(x^4 - 6)^4$ , curve passes through  $(2, 10)$

In Exercises 41–60, solve the given problems. In Exercises 41–46, explain your answers.

- Is  $\int 3x^2 dx = x^3$ ?
- Can  $\int (x^2 - 1)^2 dx$  be integrated with  $u = x^2 - 1$ ?
- Can  $\int (4x^3 + 3)^5 x^4 dx$  be integrated with  $u = 4x^3 + 3$  and  $du = x^4 dx$ ?
- Is  $\int \sqrt{2x + 1} dx = \frac{2}{3}(2x + 1)^{3/2}$ ?
- Is  $\int 3(2x + 1)^2 dx = (2x + 1)^3 + C$ ?
- Is  $\int x^{-2} dx = -\frac{1}{3}x^{-3} + C$ ?
- Find the equation of the curve whose slope is  $-x\sqrt{1 - 4x^2}$  and that passes through  $(0, 7)$ .
- Find the equation of the curve whose slope is  $\sqrt{6x - 3}$  and that passes through  $(2, -1)$ .
- Find the general form of the function whose second derivative is  $\sqrt{x}$ .
- Find the general form of the expression for the displacement  $s$  of an object if its acceleration is  $9.8 \text{ m/s}^2$ .
- If the consumption of natural gas is  $0.14 + 0.00028t$  billion  $\text{m}^3/\text{year}$ , find the volume  $V$  that will be consumed in the next  $t$  years. (At the end of 2010, Canada had reserves of natural gas of about  $2.8 \times 10^{10} \text{ m}^3$ .)
- The radius  $r$  (in m) of a circular oil spill is increasing at the rate given by  $\frac{dr}{dt} = \frac{3}{\sqrt{4t + 1}}$ , where  $t$  is in minutes. Find the radius as a function of  $t$ , if  $t$  is measured from the time of the spill.
- The time rate of change of electric current in a circuit is given by  $di/dt = 4t - 0.6t^2$ . Find the expression for the current as a function of time if  $i = 2 \text{ A}$  when  $t = 0 \text{ s}$ .
- The rate of change of the frequency  $f$  of an electronic oscillator with respect to the inductance  $L$  is  $df/dL = 80(4 + L)^{-3/2}$ . Find  $f$  as a function of  $L$  if  $f = 80 \text{ Hz}$  for  $L = 0 \text{ H}$ .
- The rate of change of the temperature  $T$  (in  $^\circ\text{C}$ ) from the centre of a blast furnace to a distance  $r$  (in m) from the centre is given by  $dT/dr = -4500(r + 1)^{-3}$ . Express  $T$  as a function of  $r$  if  $T = 2500^\circ\text{C}$  for  $r = 0$ .
- The rate of change of current  $i$  (in mA) in a circuit with a variable inductance is given by  $di/dt = 300(5.0 - t)^{-2}$ , where  $t$  (in ms) is the time the circuit is closed. Find  $i$  as a function of  $t$  if  $i = 300 \text{ mA}$  for  $t = 2.0 \text{ ms}$ .



57. At a given site, the rate of change of the annual fraction  $f$  of energy supplied by solar energy with respect to the solar-collector area  $A$  (in  $\text{m}^2$ ) is  $\frac{df}{dA} = \frac{0.005}{\sqrt{0.01A + 1}}$ . Find  $f$  as a function of  $A$  if  $f = 0$  for  $A = 0 \text{ m}^2$ .
58. An analysis of a company's records shows that in a day the rate of change of profit  $p$  (in dollars) in producing  $x$  generators is  $\frac{dp}{dx} = \frac{600(30 - x)}{\sqrt{60x - x^2}}$ . Find the profit in producing  $x$  generators if a loss of \$5000 is incurred if none are produced.

59. Find the equation of the curve for which the second derivative is 6. The curve passes through  $(1, 2)$  with a slope of 8.
60. The second derivative of a function is  $12x^2$ . Explain how to find the function if its curve passes through the points  $(1, 6)$  and  $(2, 21)$ . Find the function.

#### Answers to Practice Exercises

1.  $2x^3 - 5x + C$     2.  $-\frac{4}{3}(1 - 2x^2)^{3/2} + C$

## 25.3 The Area Under a Curve

### Summing Areas of Inscribed Rectangles • Limit of Areas of Inscribed Rectangles • Area Under a Curve by Integration

In geometry, there are formulas and methods for finding the areas of regular figures. By means of integration, it is possible to find the area between curves for which we know the equations. The next example illustrates the basic idea behind the method.

#### EXAMPLE 1 Summing the areas of inscribed rectangles

Approximate the area in the first quadrant to the left of the line  $x = 4$  and under the parabola  $y = x^2 + 1$ . Here, “under” means between the curve and the  $x$ -axis. First, make this approximation by inscribing two rectangles of equal width under the parabola and finding the sum of the areas of these rectangles. Then, improve the approximation by repeating the process with eight inscribed rectangles.

The area to be approximated is shown in Fig. 25.3(a). The area with two rectangles inscribed under the curve is shown in Fig. 25.3(b). The first approximation, admittedly small, of the area can be found by adding the areas of the two rectangles. Both rectangles have a width of 2. The left rectangle is 1 unit high, and the right rectangle is 5 units high. Thus, the area of the two rectangles is

$$A = 2(1 + 5) = 12$$

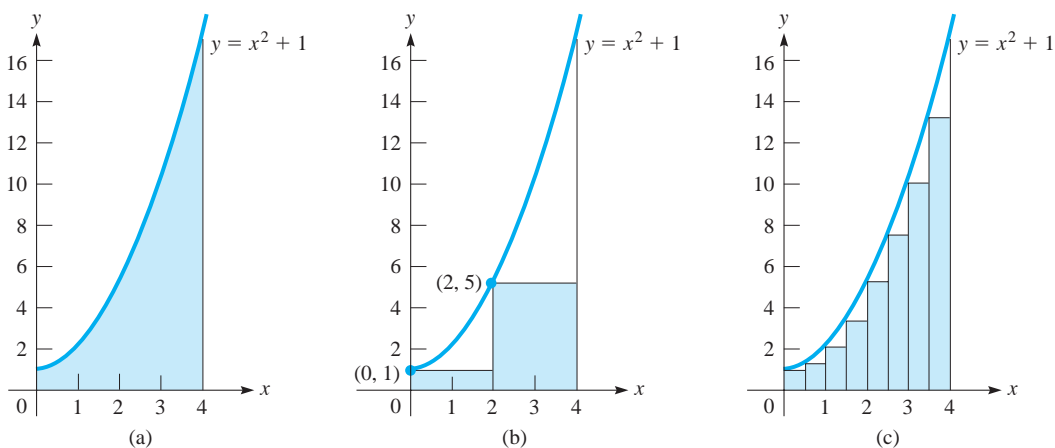


Fig. 25.3

A much better approximation is found by inscribing the eight rectangles as shown in Fig. 25.3(c). Each of these rectangles has a width of  $\frac{1}{2}$ . The leftmost rectangle has a height of 1. The next has a height of  $\frac{5}{4}$ , which is determined by finding  $y$  for  $x = \frac{1}{2}$ . The next rectangle has a height of 2, which is found by evaluating  $y$  for  $x = 1$ . Finding the heights of all rectangles and multiplying their sum by  $\frac{1}{2}$  gives the area of the eight rectangles as

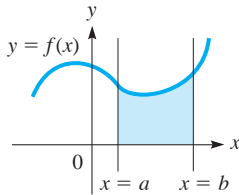
$$A = \frac{1}{2} \left( 1 + \frac{5}{4} + 2 + \frac{13}{4} + 5 + \frac{29}{4} + 10 + \frac{53}{4} \right) = \frac{43}{2} = 21.5$$

#### Practice Exercise

1. In Example 1, approximate the area by inscribing four rectangles.

**Table 25.1**

Number of Rectangles $n$	Total Area of Rectangles
8	21.5
100	25.0144
1000	25.301 344
10 000	25.330 134



**Fig. 25.4**

An even better approximation could be obtained by inscribing more rectangles under the curve. The greater the number of rectangles, the closer the sum of their areas is to the area under the curve. See Table 25.1. By using integration later in this section, we determine the *exact* area to be  $\frac{76}{3} = 25\frac{1}{3}$ . ■

We now develop the basic method used to find the area under a curve, which is the area bounded by the curve, the  $x$ -axis, and the lines  $x = a$  and  $x = b$ . See Fig. 25.4. We assume here that  $f(x)$  is never negative in the interval  $a < x < b$ . In Chapter 26, we will extend the method so that  $f(x)$  may be negative.

In finding the area under a curve, we consider the sum of the areas of inscribed rectangles, as the number of rectangles is assumed to increase without bound. The reason for this last condition is that, as we saw in Example 1, as the number of rectangles increases, the approximation of the area is better.

**EXAMPLE 2 Letting the number of rectangles approach infinity**

Find the area under the straight line  $y = 2x$ , above the  $x$ -axis, and to the left of the line  $x = 4$ .

Since this figure is a right triangle, the area can easily be found. However, the *method* we use here is the important concept. We first subdivide the interval from  $x = 0$  to  $x = 4$  into  $n$  inscribed rectangles of width  $\Delta x$ . The endpoints of the intervals are labelled  $a, x_1, x_2, \dots, b (= x_n)$ , as shown in Fig. 25.5, where  $a = 0, b = 4$  and

$$x_1 = \Delta x \quad x_2 = 2\Delta x, \dots \quad x_{n-1} = (n - 1)\Delta x \quad b = n\Delta x$$

The area of each of these  $n$  rectangles is as follows:

- First:  $f(a) \Delta x$ , where  $f(a) = f(0) = 2(0) = 0$  is the height.
- Second:  $f(x_1) \Delta x$ , where  $f(x_1) = 2(\Delta x) = 2 \Delta x$  is the height.
- Third:  $f(x_2) \Delta x$ , where  $f(x_2) = 2(2 \Delta x) = 4 \Delta x$  is the height.
- Fourth:  $f(x_3) \Delta x$ , where  $f(x_3) = 2(3 \Delta x) = 6 \Delta x$  is the height.
- ⋮
- Last:  $f(x_{n-1}) \Delta x$ , where  $f[(n - 1) \Delta x] = 2(n - 1) \Delta x$  is the height.

These areas are summed up as follows:

$$\begin{aligned}
 A_n &= f(a) \Delta x + f(x_1) \Delta x + f(x_2) \Delta x + \dots + f(x_{n-1}) \Delta x & (25.5) \\
 &= 0 + 2 \Delta x (\Delta x) + 4 \Delta x (\Delta x) + \dots + 2[(n - 1) \Delta x] \Delta x \\
 &= 2(\Delta x)^2 [1 + 2 + 3 + \dots + (n - 1)]
 \end{aligned}$$

Now,  $b = n \Delta x$ , or  $4 = n \Delta x$ , or  $\Delta x = 4/n$ . Thus,

$$A_n = 2 \left( \frac{4}{n} \right)^2 [1 + 2 + 3 + \dots + (n - 1)]$$

The sum of the arithmetic sequence  $1 + 2 + 3 + \dots + n - 1$  is

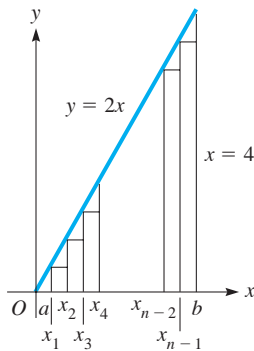
$$s = \frac{n - 1}{2} (1 + n - 1) = \frac{n(n - 1)}{2} = \frac{n^2 - n}{2}$$

Now, the expression for the sum of the areas can be written as

$$A_n = \frac{32}{n^2} \left( \frac{n^2 - n}{2} \right) = 16 \left( 1 - \frac{1}{n} \right)$$

This expression is an approximation of the actual area under consideration. The larger  $n$  becomes, the better the approximation. If we let  $n \rightarrow \infty$  (which is equivalent to letting  $\Delta x \rightarrow 0$ ), the limit of this sum will equal the area in question.

$$A = \lim_{n \rightarrow \infty} 16 \left( 1 - \frac{1}{n} \right) = 16 \quad 1/n \rightarrow 0 \text{ as } n \rightarrow \infty$$



**Fig. 25.5**

**LEARNING TIP**

The area under the curve is the limit of the sum of the areas of the inscribed rectangles, as the number of rectangles approaches infinity.

■ This checks with the geometric result.

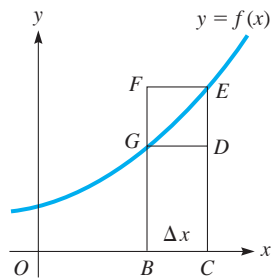


Fig. 25.6

The method indicated in Example 2 illustrates the interpretation of finding an area as a summation process, although it should not be considered as a proof. However, we will find that integration proves to be a much more useful method for finding an area. Let us now see how integration can be used directly.

Let  $\Delta A$  represent the area  $BCEG$  under the curve, as indicated in Fig. 25.6. We see that the following inequality is true for the indicated areas:

$$A_{BCDG} < \Delta A < A_{BCEF}$$

If the point  $G$  is now designated as  $(x, y)$  and  $E$  as  $(x + \Delta x, y + \Delta y)$ , we have  $y\Delta x < \Delta A < (y + \Delta y)\Delta x$ . Dividing through by  $\Delta x$ , we have

$$y < \frac{\Delta A}{\Delta x} < y + \Delta y$$

Now, we take the limit as  $\Delta x \rightarrow 0$  ( $\Delta y$  then approaches zero). This results in

$$\frac{dA}{dx} = y \quad (25.6)$$

This is true since the left member of the inequality is  $y$  and the right member approaches  $y$ . Also, in the definition of the derivative, Eq. (23.6),  $f(x + h) - f(x)$  is equivalent to  $\Delta A$ , and  $h$  is equivalent to  $\Delta x$ , which means

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta A}{\Delta x} = \frac{dA}{dx}$$

We shall now use Eq. (25.6) to show the method of finding the complete area under a curve. We now let  $x = a$  be the left boundary of the desired area and  $x = b$  be the right boundary (Fig. 25.7). The area under the curve to the right of  $x = a$  and bounded on the right by the line  $GB$  is now designated as  $A_{ax}$ . From Eq. (25.6), we have

$$dA_{ax} = [y dx]_a^x \quad \text{or} \quad A_{ax} = \left[ \int y dx \right]_a^x = \left[ \int f(x) dx \right]_a^x$$

where  $[ ]_a^x$  is the notation used to indicate the boundaries of the area. If the indefinite integral is given by  $F(x) + C$ , we have

$$A_{ax} = \left[ \int f(x) dx \right]_a^x = [F(x) + C]_a^x \quad (25.7)$$

But we know that if  $x = a$ , then  $A_{aa} = 0$ . Thus,  $0 = F(a) + C$ , or  $C = -F(a)$ . Therefore,

$$A_{ax} = \left[ \int f(x) dx \right]_a^x = F(x) - F(a) \quad (25.8)$$

Now, to find the area under the curve that reaches from  $a$  to  $b$ , we write

$$A_{ab} = F(b) - F(a) \quad (25.9)$$

Thus, the area under the curve that reaches from  $a$  to  $b$  is given by

$$A_{ab} = \left[ \int f(x) dx \right]_a^b = F(b) - F(a) \quad (25.10)$$

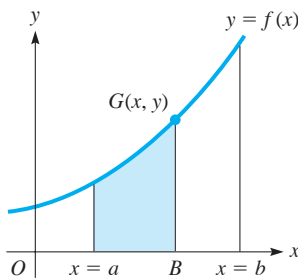


Fig. 25.7

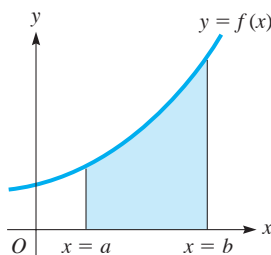


Fig. 25.8

### LEARNING TIP

Eq. (25.10) tells us that the area under the curve may be found by integrating the function  $f(x)$  to find the function  $F(x)$ , which is then evaluated at each boundary value. The area is the difference between these values of  $F(x)$ . See Fig. 25.8.

Note that we do not have to include the constant of integration when using  $F(x)$  in Eq. (25.10). Any constant added to  $F(x)$  cancels out when  $F(a)$  is subtracted from  $F(b)$ .

## Integration as Summation

In Example 2, we found an area under a curve by finding the limit of the sum of the areas of the inscribed rectangles as the number of rectangles approaches infinity. Eq. (25.10) expresses the area under a curve in terms of integration. We can now see that we have obtained an area by summation and also expressed it in terms of integration. Therefore, we conclude that

*summations can be evaluated by integration.*

Also, we have seen the connection between the problem of finding the slope of a tangent to a curve (differentiation) and the problem of finding an area under a curve (integration). We would not normally suspect that these two problems would have solutions that lead to reverse processes. We have also seen that the definition of integration has much more application than originally anticipated.

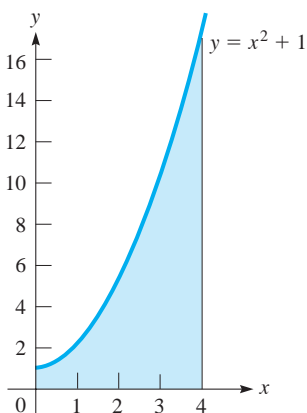


Fig. 25.3(a)

## Practice Exercise

2. In Example 3, change  $x^2 + 1$  to  $x^3 + 1$  and then find the area.

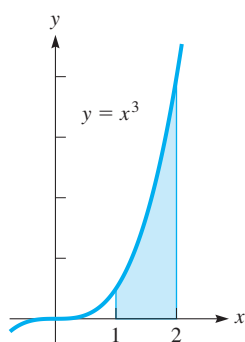


Fig. 25.9

## EXAMPLE 3 Area under a curve by integration

Find the area under the curve of  $y = x^2 + 1$  between the  $y$ -axis and the line  $x = 4$ . This is the same area that we illustrated in Example 1 and showed in Fig. 25.3(a). This figure is shown again here for reference.

Using Eq. (25.10), we note that  $f(x) = x^2 + 1$ . This means that

$$\int (x^2 + 1) dx = \frac{1}{3}x^3 + x + C$$

Therefore, with  $F(x) = \frac{1}{3}x^3 + x$ , the area is given by

$$\begin{aligned} A_{0,4} &= F(4) - F(0) && \text{using Eq. (25.10)} \\ &= \left[ \frac{1}{3}(4^3) + 4 \right] - \left[ \frac{1}{3}(0^3) + 0 \right] && \text{evaluating } F(x) \text{ at } x = 4 \text{ and } x = 0 \\ &= \frac{1}{3}(64) + 4 = \frac{76}{3} \end{aligned}$$

We note that  $76/3$  is a little more than 25 square units and is therefore about 4 square units more than the value obtained using eight inscribed rectangles in Example 1. Therefore, from this result, we know that the *exact* area under the curve is  $25\frac{1}{3}$ , as stated at the end of Example 1. ■

## EXAMPLE 4 Area under a curve by integration

Find the area under the curve  $y = x^3$  that is between the lines  $x = 1$  and  $x = 2$ , as shown in Fig. 25.9.

In Eq. (25.10),  $f(x) = x^3$ . Therefore,

$$\begin{aligned} \int x^3 dx &= \frac{1}{4}x^4 + C && \text{--- } F(x) \\ A_{1,2} &= F(2) - F(1) = \left[ \frac{1}{4}(2^4) \right] - \left[ \frac{1}{4}(1^4) \right] && \text{using Eq. (25.10) and evaluating} \\ &= 4 - \frac{1}{4} = \frac{15}{4} \end{aligned}$$

The calculated area of  $15/4$  is the exact area, not an approximation. ■

## EXERCISES 25.3

In Exercises 1–4, make the given changes in the indicated examples of this section and then solve the resulting problems.

- In Example 1, change  $x = 4$  to  $x = 2$  and find the area of (a) two inscribed rectangles and (b) four inscribed rectangles.
- In Example 3, change  $x = 4$  to  $x = 2$  and compare the results with those of Exercise 1.
- In Example 4, change  $x = 2$  to  $x = 3$ .
- In Example 4, change  $x = 1$  to  $x = 2$  and  $x = 2$  to  $x = 3$ . Note that the result added to the result of Example 4 is the same as the result for Exercise 3.

In Exercises 5–14, find the approximate area under the curves of the given equations by dividing the indicated intervals into  $n$  subintervals and then adding up the areas of the inscribed rectangles. There are two values of  $n$  for each exercise and therefore two approximations for each area. The height of each rectangle may be found by evaluating the function for the proper value of  $x$ . See Example 1. Round answers to 3 significant digits.

- $y = 3x$ , between  $x = 0$  and  $x = 3$ , for (a)  $n = 3$  ( $\Delta x = 1$ ), (b)  $n = 10$  ( $\Delta x = 0.3$ )
- $y = 2x + 1$ , between  $x = 0$  and  $x = 2$ , for (a)  $n = 4$  ( $\Delta x = 0.5$ ), (b)  $n = 10$  ( $\Delta x = 0.2$ )
- $y = x^2$ , between  $x = 0$  and  $x = 2$ , for (a)  $n = 5$  ( $\Delta x = 0.4$ ), (b)  $n = 10$  ( $\Delta x = 0.2$ )
- $y = 9 - x^2$ , between  $x = 2$  and  $x = 3$ , for (a)  $n = 5$  ( $\Delta x = 0.2$ ), (b)  $n = 10$  ( $\Delta x = 0.1$ )
- $y = 4x - x^2$ , between  $x = 1$  and  $x = 4$ , for (a)  $n = 6$ , (b)  $n = 10$
- $y = 1 - x^2$ , between  $x = 0.5$  and  $x = 1$ , for (a)  $n = 5$ , (b)  $n = 10$
- $y = \frac{1}{x^2}$ , between  $x = 1$  and  $x = 5$ , for (a)  $n = 4$ , (b)  $n = 8$
- $y = \sqrt{x}$ , between  $x = 1$  and  $x = 4$ , for (a)  $n = 3$ , (b)  $n = 12$
- $y = \frac{1}{\sqrt{x+1}}$ , between  $x = 3$  and  $x = 8$ , for (a)  $n = 5$ , (b)  $n = 10$
- $y = 2x\sqrt{x^2 + 1}$ , between  $x = 0$  and  $x = 6$ , for (a)  $n = 6$ , (b)  $n = 12$

In Exercises 15–24, find the exact area under the given curves between the indicated values of  $x$ . The functions are the same as those for which approximate areas were found in Exercises 5–14.

- $y = 3x$ , between  $x = 0$  and  $x = 3$
- $y = 2x + 1$ , between  $x = 0$  and  $x = 2$
- $y = x^2$ , between  $x = 0$  and  $x = 2$
- $y = 9 - x^2$ , between  $x = 2$  and  $x = 3$
- $y = 4x - x^2$ , between  $x = 1$  and  $x = 4$
- $y = 1 - x^2$ , between  $x = 0.5$  and  $x = 1$
- $y = \frac{1}{x^2}$ , between  $x = 1$  and  $x = 5$

22.  $y = \sqrt{x}$ , between  $x = 1$  and  $x = 4$

23.  $y = \frac{1}{\sqrt{x+1}}$ , between  $x = 3$  and  $x = 8$

24.  $y = 2x\sqrt{x^2 + 1}$ , between  $x = 0$  and  $x = 6$   
Explain the reason for the difference between this result and the two values found in Exercise 14.

In developing the concept of the area under a curve, we first (in Examples 1 and 2) considered rectangles **inscribed** under the curve. A more complete development also considers rectangles **circumscribed** above the curve and shows that the limiting area of the circumscribed rectangles equals the limiting area of the inscribed rectangles as the number of rectangles increases without bound. See Fig. 25.10 for an illustration of inscribed and circumscribed rectangles.

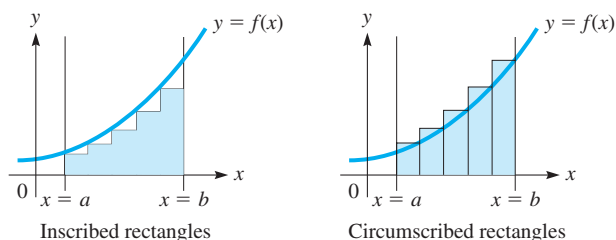


Fig. 25.10

In Exercises 25–28, find the sum of the areas of 10 circumscribed rectangles for each curve and show that the exact area (as shown in Exercises 15–18) is between the sum of the areas of the circumscribed rectangles and the inscribed rectangles (as found in Exercises 5(b)–8(b)). Also, note that the mean of the two sums is close to the exact value.

- $y = 3x$  between  $x = 0$  and  $x = 3$  (compare with Exercises 5(b) and 15). Why is the mean of the sums of the inscribed rectangles and circumscribed rectangles equal to the exact value?
- $y = 2x + 1$  between  $x = 0$  and  $x = 2$  (compare with Exercises 6(b) and 16). Why is the mean of the sums of the inscribed rectangles and circumscribed rectangles equal to the exact value?
- $y = x^2$  between  $x = 0$  and  $x = 2$  (compare with Exercises 7(b) and 17). Why is the mean of the sums of the inscribed rectangles and circumscribed rectangles greater than the exact value?
- $y = 9 - x^2$  between  $x = 2$  and  $x = 3$  (compare with Exercises 8(b) and 18). Why is the mean of the sums of the inscribed rectangles and circumscribed rectangles less than the exact value?

#### Answers to Practice Exercises

1.  $A = 18$     2.  $A = 68$

## 25.4 The Definite Integral

**Definite Integral - Result Is a Number • Lower Limit and Upper Limit**

Using reasoning similar to that in the preceding section, we define the **definite integral** of a function  $f(x)$  as

$$\int_a^b f(x) dx = F(b) - F(a) \quad (25.11)$$

### Limits of Integration

where  $F'(x) = f(x)$ . We call this a **definite integral because the final result of integrating and evaluating is a number**. (The *indefinite* integral had an arbitrary constant in the result.) The numbers  $a$  and  $b$  are called the **lower limit** and the **upper limit**, respectively. We can see that the value of a definite integral is found by evaluating the function (found by integration) at the upper limit and subtracting the value of this function at the lower limit.

■ That integration is equivalent to the limit of a sum is the reason Leibniz (see pages 660 and 765) used an elongated  $S$  for the integral sign. It stands for the Latin word for *sum*.

From Section 25.3, we know that this definite integral can be interpreted as the area under the curve of  $y = f(x)$  from  $x = a$  to  $x = b$ , and in general as a summation. This summation interpretation will be applied to many kinds of applied problems.

### EXAMPLE 1 Definite integral of a power of $x$

Evaluate the integral  $\int_0^2 x^4 dx$ .

$$\int_0^2 x^4 dx = \frac{x^5}{5} \Big|_0^2 = \frac{2^5}{5} - 0 = \frac{32}{5}$$

upper limit
lower limit
 $f(x)$ 
 $F(x)$

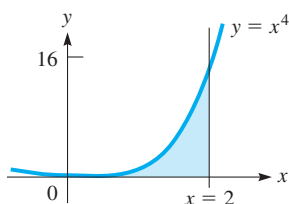


Fig. 25.11

Note that a vertical line—with the limits written at the top and the bottom—is the way the value is indicated after integration, but before evaluation.

The area that this definite integral can represent is shown in Fig. 25.11. ■

### EXAMPLE 2 Definite integral of a power with a negative exponent

Evaluate  $\int_1^3 (x^{-2} - 1) dx$ .

$$\begin{aligned} \int_1^3 (x^{-2} - 1) dx &= -\frac{1}{x} - x \Big|_1^3 = \left( -\frac{1}{3} - 3 \right) - \left( -1 - 1 \right) \\ &= -\frac{10}{3} + 2 = -\frac{4}{3} \end{aligned}$$

upper limit
subtract
lower limit

### EXAMPLE 3 Definite integral of a power of a function

Evaluate  $\int_0^1 5z(z^2 + 1)^5 dz$ .

For purposes of integration,  $n = 5$ ,  $u = z^2 + 1$ , and  $du = 2z dz$ . Hence,

$$\begin{aligned} \int_0^1 5z(z^2 + 1)^5 dz &= \frac{5}{2} \int_0^1 (z^2 + 1)^5 (2z dz) \\ &= \frac{5}{2} \left( \frac{1}{6} \right) (z^2 + 1)^6 \Big|_0^1 \\ &= \frac{5}{12} (2^6 - 1^6) = \frac{5(63)}{12} = \frac{105}{4} \end{aligned}$$

## Practice Exercise

2. Evaluate  $\int_0^6 \sqrt{4x+1} dx$ .

**EXAMPLE 4** Definite integral with a radical in the denominator

Evaluate  $\int_{0.1}^{2.7} \frac{dx}{\sqrt{4x+1}}$ .

In order to integrate, we have  $n = -\frac{1}{2}$ ,  $u = 4x + 1$ , and  $du = 4 dx$ . Therefore,

$$\begin{aligned} \int_{0.1}^{2.7} \frac{dx}{\sqrt{4x+1}} &= \int_{0.1}^{2.7} (4x+1)^{-1/2} dx = \frac{1}{4} \int_{0.1}^{2.7} (4x+1)^{-1/2} (4 dx) \\ &= \frac{1}{4} \left( \frac{1}{-\frac{1}{2}} \right) (4x+1)^{1/2} \Big|_{0.1}^{2.7} = \frac{1}{2} (4x+1)^{1/2} \Big|_{0.1}^{2.7} && \text{integrate} \\ &= \frac{1}{2} (\sqrt{11.8} - \sqrt{1.4}) = 1.13 && \text{evaluate} \quad \blacksquare \end{aligned}$$

**EXAMPLE 5** Integral involving a negative power of a function

Evaluate  $\int_0^4 \frac{x+1}{(x^2+2x+2)^3} dx$ .

For integrating,  $n = -3$ ,  $u = x^2 + 2x + 2$ , and  $du = (2x + 2) dx$ .

$$\begin{aligned} \int_0^4 (x^2+2x+2)^{-3} (x+1) dx &= \frac{1}{2} \int_0^4 (x^2+2x+2)^{-3} [2(x+1) dx] \\ &= \frac{1}{2} \left( \frac{1}{-2} \right) (x^2+2x+2)^{-2} \Big|_0^4 && \text{integrate} \\ &= -\frac{1}{4} (16+8+2)^{-2} + \frac{1}{4} (0+0+2)^{-2} && \text{evaluate} \\ &= \frac{1}{4} \left( -\frac{1}{26^2} + \frac{1}{2^2} \right) = \frac{1}{4} \left( \frac{1}{4} - \frac{1}{676} \right) \\ &= \frac{1}{4} \left( \frac{168}{676} \right) = \frac{21}{338} \quad \blacksquare \end{aligned}$$

The following example illustrates an application of the definite integral. In Chapter 26, we will see that the definite integral has many applications in science and technology.

**EXAMPLE 6** Definite integral—application

The rate of flow  $Q$  (in  $\text{m}^3/\text{s}$ ) of water over a certain dam is found by evaluating the definite integral in the equation  $Q = \int_0^{1.25} 240\sqrt{1.50-y} dy$ . See Fig. 25.12. Find  $Q$ .

The solution is as follows:

$$\begin{aligned} Q &= \int_0^{1.25} 240\sqrt{1.50-y} dy = -240 \int_0^{1.25} (1.50-y)^{1/2} (-dy) \\ &= -240 \left( \frac{2}{3} \right) (1.50-y)^{3/2} \Big|_0^{1.25} && \text{integrate} \\ &= -160 [ (1.50-1.25)^{3/2} - (1.50-0)^{3/2} ] && \text{evaluate} \\ &= -160 (0.25^{3/2} - 1.50^{3/2}) = 274 \text{ m}^3/\text{s} \quad \blacksquare \end{aligned}$$

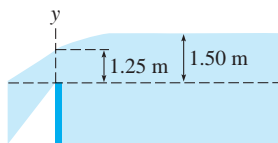


Fig. 25.12

Some graphing calculators are programmed to evaluate definite integrals and areas under curves. The manual should be consulted to determine how any particular model is used for integration.



## EXERCISES 25.4

In Exercises 1 and 2, make the given changes in the indicated examples of this section and then solve the resulting problems.

- In Example 2, change the upper limit from 3 to 4.
- In Example 4, change  $4x$  to  $2x$ .

In Exercises 3–34, evaluate the given definite integrals.

- $\int_0^1 2x \, dx$
- $\int_0^2 3x^2 \, dx$
- $\int_1^4 x^{5/2} \, dx$
- $\int_4^9 (p^{3/2} - 3) \, dp$
- $\int_3^6 \left( \frac{1}{\sqrt{x}} - 7 \right) dx$
- $\int_{1.2}^{1.6} \left( 5 + \frac{6}{x^4} \right) dx$
- $\int_{-1.6}^{0.7} (1 - x)^{1/3} \, dx$
- $\int_1^5 \sqrt{3v + 1} \, dv$
- $\int_{-2}^2 (T - 2)(T + 2) \, dT$
- $\int_1^2 (3x^5 - 2x^3) \, dx$
- $\int_{0.5}^{2.2} (\sqrt[3]{x} - 2) \, dx$
- $\int_{2.7}^{5.3} \left( \frac{1}{x\sqrt{x}} + 4 \right) dx$
- $\int_0^4 (1 - \sqrt{x})^2 \, dx$
- $\int_1^4 \frac{y + 4}{\sqrt{y}} \, dy$
- $\int_{-2}^{-1} 2x(4 - x^2)^3 \, dx$
- $\int_0^1 x(3x^2 - 1)^3 \, dx$
- $\int_0^4 \frac{x \, dx}{\sqrt{x^2 + 9}}$
- $\int_{0.2}^{0.7} x^2(x^3 + 2)^{3/2} \, dx$
- $\int_{2.75}^{3.25} \frac{dx}{\sqrt[3]{6x + 1}}$
- $\int_2^6 \frac{8 \, du}{\sqrt{4u + 1}}$
- $\int_1^3 \frac{12x \, dx}{(2x^2 + 1)^3}$
- $\int_{12.6}^{17.2} \frac{3 \, dx}{(6x - 1)^2}$
- $\int_3^7 \sqrt{16t^2 + 8t + 1} \, dt$
- $\int_{-5}^1 \sqrt{6 - 2x} \, dx$
- $\int_0^2 2x(9 - 2x^2)^2 \, dx$
- $\int_{-1}^2 V(V^3 + 1) \, dV$
- $\int_{-1}^2 \frac{8x - 2}{(2x^2 - x + 1)^3} \, dx$
- $\int_2^3 \frac{x^2 + 1}{(x^3 + 3x)^2} \, dx$
- $\int_0^1 (x^2 + 3)(x^3 + 9x + 6)^2 \, dx$
- $\int_{-3}^{-2} (3x^2 - 2)\sqrt[3]{2x^3 - 4x + 1} \, dx$
- $\int_{\sqrt{5}}^3 2z\sqrt[4]{z^4 + 8z^2 + 16} \, dz$
- $\int_{-2}^0 (\sqrt{2x + 4} - \sqrt[3]{3x + 8}) \, dx$

In Exercises 35–50, solve the given problems.

- Show that  $\int_0^1 x^3 \, dx + \int_1^2 x^3 \, dx = \int_0^2 x^3 \, dx$ . In terms of area, explain the result.

- Write  $\int_1^8 f(x) \, dx - \int_5^8 f(x) \, dx$  as a single definite integral.

- Evaluate  $\int_{x=1}^{x=4} y \, dx$ , when  $y^2 = 4x$  ( $y > 0$ ).

- Show that  $\int_1^3 4x \, dx = -\int_3^1 4x \, dx$ .

- Show that  $\int_0^1 x \, dx > \int_0^1 x^2 \, dx$  and  $\int_1^2 x \, dx < \int_1^2 x^2 \, dx$ .

In terms of area, explain the result.

- Given that  $\int_0^9 \sqrt{x} \, dx = 18$ , evaluate  $\int_0^9 2\sqrt{x} \, dx$ .

- Evaluate  $\int_{-1}^1 t^{2k} \, dt$ , where  $k$  is a positive integer.

- Evaluate the following integral, which arises in the study of electricity:  $\int_0^L \frac{1}{EI} \left( -\frac{1}{2}wx^2 \right) (-x) \, dx$ .

- Evaluate the following integral, which arises in the study of hydrodynamics:  $\int_H^h \frac{Ay^{-1/2} \, dy}{a\sqrt{2g}}$ .

- It is estimated that a newly discovered oil field will produce oil at the rate of  $\frac{dR}{dt} = \frac{400t^2}{(t^3 + 20)^2} + 10$  thousand barrels per year. How much oil can be expected from the field in the next ten years?

- The work  $W$  (in  $\text{N} \cdot \text{m}$ ) in winding up an 80-m cable is  $W = \int_0^{80} (1000 - 5x) \, dx$ . Evaluate  $W$ .

- The total volume  $V$  of liquid flowing through a certain pipe of radius  $R$  is  $V = k(R^2 \int_0^R r \, dr - \int_0^R r^3 \, dr)$ , where  $k$  is a constant. Evaluate  $V$  and explain why  $R$ , but not  $r$ , can be to the left of the integral sign.

- The surface area  $A$  (in  $\text{m}^2$ ) of a certain parabolic radio-wave reflector is  $A = 4\pi \int_0^2 \sqrt{3x + 9} \, dx$ . Evaluate  $A$ .

- The total force (in  $\text{N}$ ) on the circular end of a water tank is  $F = 19\,600 \int_0^5 y\sqrt{25 - y^2} \, dy$ . Evaluate  $F$ .

- In finding the average electron energy in a metal at very low temperatures, the integral  $\frac{3N}{2E_F^{3/2}} \int_0^{E_F} E^{3/2} \, dE$  is used. Evaluate this integral.

- In finding the electric field  $E$  caused by a surface electric charge on a disc, the equation  $E = k \int_0^R \frac{r \, dr}{(x^2 + r^2)^{3/2}}$  is used. Evaluate the integral.

## Answers to Practice Exercises

- 45/8
- 62/3

## 25.5 Numerical Integration: The Trapezoidal Rule

Numerical Integration •  
Trapezoidal Rule •  
Approximating Integrals

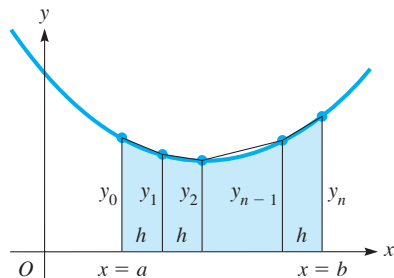


Fig. 25.13

For data and functions that cannot be directly integrated by available methods, it is possible to develop numerical methods of integration. These numerical methods are of greater importance today since they are readily adaptable for use on a calculator or computer. There are a great many such numerical techniques for approximating the value of an integral. In this section, we develop one of these, the *trapezoidal rule*. In the following section, another numerical method is discussed.

We know from Sections 25.3 and 25.4 that we can interpret a definite integral as the area under a curve. We will therefore show how to approximate the value of the integral by approximating the appropriate area by a set of inscribed trapezoids. The basic idea here is very similar to that used when rectangles were inscribed under a curve. However, the use of trapezoids reduces the error and provides a better approximation.

The area to be found is subdivided into  $n$  intervals of equal width. Perpendicular lines are then dropped from the curve (or points, if only a given set of numbers is available). If the points on the curve are joined by straight-line segments, the area of successive parts under the curve is approximated by finding the areas of the trapezoids formed. However, if these points are not too far apart, the approximation will be very good (see Fig. 25.13). From geometry, recall that the area of a trapezoid equals one-half the product of the sum of the bases times the altitude. For these trapezoids, the bases are the  $y$ -coordinates, and the altitudes are  $h$ . Therefore, when we indicate the sum of these trapezoidal areas, we have

$$A_T = \frac{1}{2}(y_0 + y_1)h + \frac{1}{2}(y_1 + y_2)h + \frac{1}{2}(y_2 + y_3)h + \cdots \\ + \frac{1}{2}(y_{n-2} + y_{n-1})h + \frac{1}{2}(y_{n-1} + y_n)h$$

We note, when this addition is performed, that the result is

$$A_T = h\left(\frac{1}{2}y_0 + y_1 + y_2 + \cdots + y_{n-1} + \frac{1}{2}y_n\right) \quad (25.12)$$

The  $y$ -values to be used either are derived from the function  $y = f(x)$  or are the  $y$ -coordinates of a set of data.

Since  $A_T$  approximates the area under the curve, it also approximates the value of the definite integral, or

$$\int_a^b f(x) dx \approx \frac{h}{2}(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n) \quad (25.13)$$

Eq. (25.13) is known as the **trapezoidal rule**. This is essentially the same rule as (Eq. 2.13) that we used in Chapter 2 to measure irregular geometric areas (see page 74). Since the definite integral can be interpreted as the area under a curve, we can now use the trapezoidal rule to find the approximate value of a definite integral.

Wherever the curve of the function being integrated is concave up, the approximating segments are above the curve and each trapezoid has slightly more area than the corresponding area under the curve. If the curve is concave down, the approximating segments are below the curve, and each trapezoid has slightly less area than the corresponding area under the curve. For a straight-line segment, the trapezoidal rule gives an exact value.

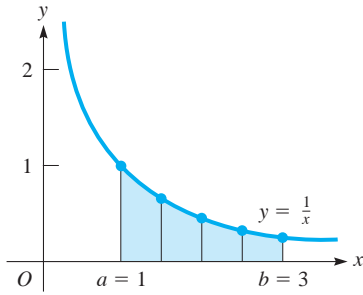


Fig. 25.14

**Practice Exercise**

- In Example 1, use the trapezoidal rule with  $n = 2$ .

**Table 25.2**

Number of Trapezoids $n$	Total Area of Trapezoids
4	1.116 666 7
100	1.098 641 9
1000	1.098 612 6
10 000	1.098 612 3

**EXAMPLE 1** Approximating an integral by the trapezoidal rule

Approximate the value of  $\int_1^3 \frac{1}{x} dx$  by the trapezoidal rule. Let  $n = 4$ .

We are to approximate the area under  $y = 1/x$  from  $x = 1$  to  $x = 3$  by dividing the area into four trapezoids. This area is found by applying Eq. (25.12), which is the approximate value of the integral, as shown in Eq. (25.13). Figure 25.14 shows the graph. In this example,  $f(x) = 1/x$ , and

$$\begin{aligned}
 h &= \frac{3 - 1}{4} = \frac{1}{2} & y_0 &= f(a) = f(1) = 1 \\
 y_1 &= f\left(\frac{3}{2}\right) = \frac{2}{3} & y_2 &= f(2) = \frac{1}{2} \\
 y_3 &= f\left(\frac{5}{2}\right) = \frac{2}{5} & y_n &= y_4 = f(b) = f(3) = \frac{1}{3} \\
 A_T &= \frac{1/2}{2} \left[ 1 + 2\left(\frac{2}{3}\right) + 2\left(\frac{1}{2}\right) + 2\left(\frac{2}{5}\right) + \frac{1}{3} \right] \\
 &= \frac{1}{4} \left( \frac{15 + 20 + 15 + 12 + 5}{15} \right) = \frac{1}{4} \left( \frac{67}{15} \right) = \frac{67}{60}
 \end{aligned}$$

Therefore,

$$\int_1^3 \frac{1}{x} dx \approx \frac{67}{60}$$

We cannot perform this integration directly by methods developed up to this point. As we increase the number of trapezoids, the value becomes more accurate. See Table 25.2. The actual value to seven decimal places is 1.098 612 3. ■

**EXAMPLE 2** Approximating an integral by the trapezoidal rule

Approximate the value of  $\int_0^1 \sqrt{x^2 + 1} dx$  by the trapezoidal rule. Let  $n = 5$ .

Figure 25.15 shows the graph. In this example,

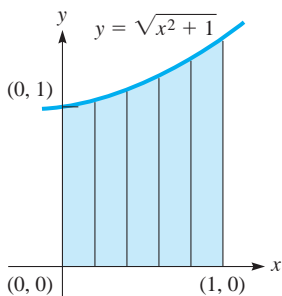


Fig. 25.15

$$h = \frac{1 - 0}{5} = 0.2$$

$$\begin{aligned}
 y_0 &= f(0) = 1 & y_1 &= f(0.2) = \sqrt{1.04} = 1.019\ 803\ 9 \\
 y_2 &= f(0.4) = \sqrt{1.16} = 1.077\ 033\ 0 & y_3 &= f(0.6) = \sqrt{1.36} = 1.166\ 190\ 4 \\
 y_4 &= f(0.8) = \sqrt{1.64} = 1.280\ 624\ 8 & y_5 &= f(1) = \sqrt{2.00} = 1.414\ 213\ 6
 \end{aligned}$$

Hence, we have

$$\begin{aligned}
 A_T &= \frac{0.2}{2} [1 + 2(1.109\ 803\ 9) + 2(1.077\ 033\ 0) + 2(1.166\ 190\ 4) \\
 &\quad + 2(1.280\ 624\ 8) + 1.414\ 213\ 6] \\
 &= 1.15 \quad (\text{rounded off to 3 significant digits})
 \end{aligned}$$

This means that

$$\int_0^1 \sqrt{x^2 + 1} dx \approx 1.15 \quad \text{the actual value is 1.148 to three decimal places}$$

We note that the entire calculation can be done on a calculator without tabulating values by entering the formula directly in terms of square roots. ■

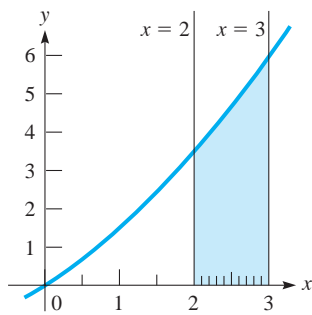


Fig. 25.16

**EXAMPLE 3** Approximating an integral by the trapezoidal rule

Approximate the value of the integral  $\int_2^3 x\sqrt{x+1} dx$  by using the trapezoidal rule.

Use  $n = 10$ .

In Fig. 25.16, the graph of the function and the area used in the trapezoidal rule are shown. From the given values, we have  $h = \frac{3-2}{10} = 0.1$ . Therefore,

$$y_0 = f(2) = 2\sqrt{3} = 3.464\ 101\ 6 \quad y_1 = f(2.1) = 2.1\sqrt{3.1} = 3.697\ 431\ 5$$

$$y_2 = f(2.2) = 2.2\sqrt{3.2} = 3.935\ 479\ 6 \quad y_3 = f(2.3) = 2.3\sqrt{3.3} = 4.178\ 157\ 5$$

$$y_4 = f(2.4) = 2.4\sqrt{3.4} = 4.425\ 381\ 3 \quad y_5 = f(2.5) = 2.5\sqrt{3.5} = 4.677\ 071\ 7$$

$$y_6 = f(2.6) = 2.6\sqrt{3.6} = 4.933\ 153\ 2 \quad y_7 = f(2.7) = 2.7\sqrt{3.7} = 5.193\ 553\ 7$$

$$y_8 = f(2.8) = 2.8\sqrt{3.8} = 5.458\ 204\ 8 \quad y_9 = f(2.9) = 2.9\sqrt{3.9} = 5.727\ 041\ 1$$

$$y_{10} = f(3) = 3\sqrt{4} = 6.000\ 000\ 0$$

$$\begin{aligned} A_T &= \frac{0.1}{2} [3.464\ 101\ 6 + 2(3.697\ 431\ 5) + \cdots + 2(5.727\ 041\ 1) + 6.000\ 000\ 0] \\ &= 4.6958 \end{aligned}$$

Therefore,  $\int_2^3 x\sqrt{x+1} dx \approx 4.6958$ . The actual value of the integral is 4.6954 to four decimal places. ■

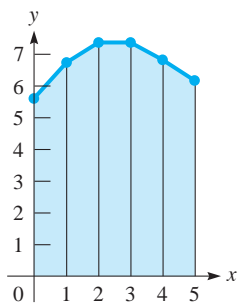


Fig. 25.17

**EXAMPLE 4** The trapezoidal rule for empirical data

The following points were found empirically:

$x$	0	1	2	3	4	5
$y$	5.68	6.75	7.32	7.35	6.88	6.24

Approximate the value of the integral of the function defined by these points between  $x = 0$  and  $x = 5$  by the trapezoidal rule.

In order to find  $A_T$ , we use the values of  $y_0, y_1$ , and so on, directly from the table. We also note that  $h = 1$ . The graph is shown in Fig. 25.17. Therefore, we have

$$A_T = \frac{1}{2} [5.68 + 2(6.75) + 2(7.32) + 2(7.35) + 2(6.88) + 6.24] = 34.26$$

Although we do not know the algebraic form of the function, we can state that

$$\int_0^5 f(x) dx \approx 34.26 \quad \blacksquare$$

**EXERCISES 25.5**

In Exercises 1 and 2, make the given changes in the indicated examples of this section and then solve the resulting problems.

- In Example 1, change  $n$  from 4 to 2.
- In Example 3, change  $n$  from 10 to 5.

In Exercises 3–6, (a) approximate the value of each of the given integrals by use of the trapezoidal rule, using the given value of  $n$ , and (b) check by direct integration.

- $\int_0^2 2x^2 dx, n = 4$
- $\int_0^1 (1 - x^2) dx, n = 3$
- $\int_1^4 (1 + \sqrt{x}) dx, n = 6$
- $\int_3^8 \sqrt{1+x} dx, n = 5$

In Exercises 7–14, approximate the value of each of the given integrals by use of the trapezoidal rule, using the given value of  $n$ . Round to 3 significant digits.

- $\int_2^3 \frac{1}{2x} dx, n = 2$
- $\int_2^6 \frac{dx}{x+3}, n = 4$
- $\int_0^5 \sqrt{25-x^2} dx, n = 5$
- $\int_0^2 \sqrt{x^3+1} dx, n = 4$
- $\int_1^5 \frac{1}{x^2+x} dx, n = 10$
- $\int_2^4 \frac{1}{x^2+1} dx, n = 10$
- $\int_0^5 2^x dx, n = 12$
- $\int_0^{1.5} 10^x dx, n = 15$

In Exercises 15 and 16, approximate the values of the integrals defined by the given sets of points.

$$15. \int_2^{14} y dx \quad \begin{array}{c|cccccccc} x & 2 & 4 & 6 & 8 & 10 & 12 & 14 \\ \hline y & 0.670 & 2.34 & 4.56 & 3.67 & 3.56 & 4.78 & 6.87 \end{array}$$

$$16. \int_{1.4}^{3.2} y dx \quad \begin{array}{c|cccccccc} x & 1.4 & 1.7 & 2.0 & 2.3 & 2.6 & 2.9 & 3.2 \\ \hline y & 0.180 & 7.87 & 18.23 & 23.53 & 24.62 & 20.93 & 20.76 \end{array}$$

In Exercises 17–22, solve each given problem by using the trapezoidal rule. Round to 3 significant digits unless otherwise stated.

17. Explain why the approximate value of the integral in Exercise 5 is less than the exact value.

18.  $\int_0^2 \sqrt{4-x^2} dx = \pi$  (see Exercise 27 of Section 28.8). Approximate the value of the integral with  $n = 8$ . Compare with  $\pi$ .

19.  $\int_0^3 \frac{dx}{2x+2} = \ln 2$  (see Exercise 38 of Section 28.2). Approximate the value of the integral with  $n = 6$ . Compare with  $\ln 2$ .

20. A force  $F$  that a distributed electric charge has on a point charge is  $F = k \int_0^2 \frac{dx}{(4+x^2)^{3/2}}$ , where  $x$  is the distance along the distributed charge and  $k$  is a constant. With  $n = 8$ , evaluate  $F$  in terms of  $k$ .

21. The length  $L$  (in m) of telephone wire needed (considering the sag) between two poles exactly 100 m apart is  $L = 2 \int_0^{50} \sqrt{6.4 \times 10^{-7}x^2 + 1} dx$ . With  $n = 10$ , evaluate  $L$  (to 6 significant digits).

22. The amount  $A$  (in standard pollution index) of a pollutant in the air in a city is measured to be  $A = \frac{150}{1 + 0.25(t - 4.0)^2} + 25$ , here  $t$  is the time (in h) after 6 A.M. With  $n = 6$ , find the total value of  $A$  between 6 A.M. and noon.

### Answer to Practice Exercise

1.  $7/6 = 1.1667$

## 25.6 Simpson's Rule

### Parabolic Arcs • Simpson's Rule • Number of Intervals Must Be Even

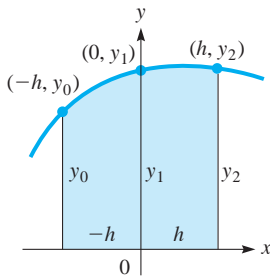


Fig. 25.18

The numerical method of integration developed in this section is also readily programmable for use on a computer or easily usable with the necessary calculations done on a calculator. It is obtained by interpreting the definite integral as the area under a curve, as we did in developing the trapezoidal rule, and by approximating the curve by a set of parabolic arcs. The use of parabolic arcs, rather than chords as with the trapezoidal rule, usually gives a better approximation.

Since we will be using parabolic arcs, we first derive a formula for the area that is under a parabolic arc. The curve shown in Fig. 25.18 represents the parabola  $y = ax^2 + bx + c$ . The points shown on this curve are  $(-h, y_0)$ ,  $(0, y_1)$ , and  $(h, y_2)$ . The area under the parabola is given by

$$\begin{aligned} A &= \int_{-h}^h y dx = \int_{-h}^h (ax^2 + bx + c) dx = \frac{ax^3}{3} + \frac{bx^2}{2} + cx \Big|_{-h}^h \\ &= \frac{2}{3}ah^3 + 2ch \end{aligned}$$

$$A = \frac{h}{3}(2ah^2 + 6c) \quad (25.14)$$

The coordinates of the three points also satisfy the equation  $y = ax^2 + bx + c$ . This means that

$$\begin{aligned} y_0 &= ah^2 - bh + c \\ y_1 &= c \\ y_2 &= ah^2 + bh + c \end{aligned}$$

By finding the sum of  $y_0 + 4y_1 + y_2$ , we have

$$y_0 + 4y_1 + y_2 = 2ah^2 + 6c \quad (25.15)$$

Substituting Eq. (25.15) into Eq. (25.14), we have

$$A = \frac{h}{3}(y_0 + 4y_1 + y_2) \quad (25.16)$$

### LEARNING TIP

Note that the area under the parabolic arc depends only on the distance  $h$  and the three  $y$ -coordinates. The coefficients of the equation of the parabola are not required, so they are not calculated.

Now, let us consider the area under the curve in Fig. 25.19. If a parabolic arc is passed through the points  $(x_0, y_0)$ ,  $(x_1, y_1)$ , and  $(x_2, y_2)$ , we may use Eq. (25.16) to approximate the area under the curve between  $x_0$  and  $x_2$ . We again note that the distance  $h$  is the difference in the  $x$ -coordinates. Therefore, the area under the curve between  $x_0$  and  $x_2$  is

$$A_1 = \frac{h}{3}(y_0 + 4y_1 + y_2)$$

Similarly, if a parabolic arc is passed through the three points starting with  $(x_2, y_2)$ , the area between  $x_2$  and  $x_4$  is

$$A_2 = \frac{h}{3}(y_2 + 4y_3 + y_4)$$

The sum of these areas is

$$A_1 + A_2 = \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4) \quad (25.17)$$

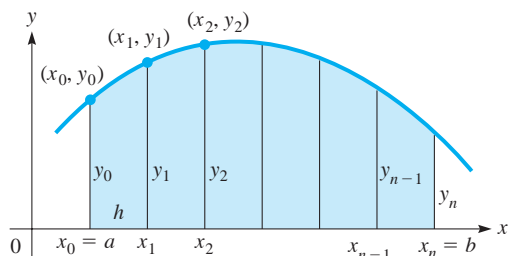


Fig. 25.19

We can continue this procedure until the approximate value of the entire area has been found. We must note, however, that **the number of intervals  $n$  of width  $h$  must be even**. Therefore, generalizing Eq. (25.17) and recalling again that the value of the definite integral is the area under the curve, we have

$$\int_a^b f(x) dx \approx \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 4y_{n-1} + y_n), n \text{ even} \quad (25.18)$$

■ Although Simpson's rule is named for the English mathematician Thomas Simpson (1710–1761), he did not discover the rule. It was well known when he included it in some of his many books on mathematics.

Eq. (25.18) is known as **Simpson's rule**. As with the trapezoidal rule, we used Simpson's rule in Chapter 2 to measure irregular areas (see page 75). We now see how it is derived. Simpson's rule is also used in many calculator models for the evaluation of definite integrals.

### EXAMPLE 1 Approximating an integral by using Simpson's rule

Approximate the value of the integral  $\int_0^1 \frac{dx}{x+1}$  by Simpson's rule. Let  $n = 2$ .

In Fig. 25.20, the graph of the function and the area used are shown. We are to approximate the integral by using Eq. (25.18). We therefore note that  $f(x) = 1/(x+1)$ . Also,  $x_0 = a = 0$ ,  $x_1 = 0.5$ , and  $x_2 = b = 1$ . This is due to the fact that  $n = 2$  and  $h = 0.5$  since the total interval is 1 unit (from  $x = 0$  to  $x = 1$ ). Therefore,

$$y_0 = \frac{1}{0+1} = 1.0000 \quad y_1 = \frac{1}{0.5+1} = 0.6667 \quad y_2 = \frac{1}{1+1} = 0.5000$$

Substituting, we have

$$\begin{aligned} \int_0^1 \frac{dx}{x+1} &= \frac{0.5}{3} [1.0000 + 4(0.6667) + 0.5000] \\ &= 0.694 \end{aligned}$$

To three decimal places, the actual value of the integral is 0.693. We will consider the method of integrating this function in a later chapter. ■

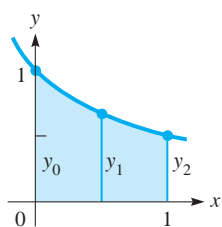


Fig. 25.20

#### Practice Exercise

- In Example 1, use Simpson's rule with  $n = 4$ .

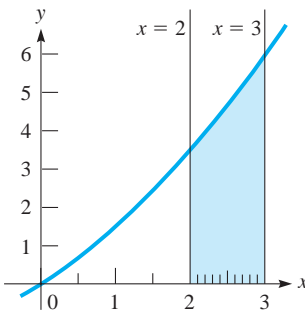


Fig. 25.21

**EXAMPLE 2** Approximating an integral by using Simpson's rule

Approximate the value of  $\int_2^3 x\sqrt{x+1} dx$  by Simpson's rule. Use  $n = 10$ .

Since the necessary values for this function are shown in Example 3 of Section 25.5, we shall simply tabulate them here ( $h = 0.1$ ). See Fig. 25.21.

$$\begin{aligned} y_0 &= 3.464\ 101\ 6 & y_1 &= 3.697\ 431\ 5 & y_2 &= 3.935\ 479\ 6 & y_3 &= 4.178\ 157\ 5 \\ y_4 &= 4.425\ 381\ 3 & y_5 &= 4.677\ 071\ 7 & y_6 &= 4.933\ 153\ 2 & y_7 &= 5.193\ 553\ 7 \\ y_8 &= 5.458\ 204\ 8 & y_9 &= 5.727\ 041\ 1 & y_{10} &= 6.000\ 000\ 0 \end{aligned}$$

Therefore, we evaluate the integral as follows:

$$\begin{aligned} \int_2^3 x\sqrt{x+1} dx &= \frac{0.1}{3} [3.464\ 101\ 6 + 4(3.697\ 431\ 5) + 2(3.935\ 479\ 6) \\ &\quad + 4(4.178\ 157\ 5) + 2(4.425\ 381\ 3) + 4(4.677\ 071\ 7) \\ &\quad + 2(4.933\ 153\ 2) + 4(5.193\ 553\ 7) + 2(5.458\ 204\ 8) \\ &\quad + 4(5.727\ 041\ 1) + 6.000\ 000\ 0] \\ &= \frac{0.1}{3} (140.861\ 56) = 4.695\ 385\ 4 \end{aligned}$$

This result agrees with the actual value to the 8 significant digits shown. The value we obtained earlier with the trapezoidal rule was 4.6958. ■

**EXAMPLE 3** Simpson's rule—application

The rear stabilizer of a certain aircraft is shown in Fig. 25.22. The area  $A$  (in  $\text{m}^2$ ) of one side of the stabilizer is  $A = \int_0^3 (3x^2 - x^3)^{0.6} dx$ . Find this area, using Simpson's rule with  $n = 6$ .

Here, we note that  $f(x) = (3x^2 - x^3)^{0.6}$ ,  $a = 0$ ,  $b = 3$ , and  $h = \frac{3-0}{6} = 0.5$ . Therefore,

$$\begin{aligned} y_0 &= f(0) = [3(0)^2 - 0^3]^{0.6} = 0 \\ y_1 &= f(0.5) = [3(0.5)^2 - 0.5^3]^{0.6} = 0.754\ 272\ 0 \\ y_2 &= f(1) = [3(1)^2 - 1^3]^{0.6} = 1.515\ 716\ 6 \\ y_3 &= f(1.5) = [3(1.5)^2 - 1.5^3]^{0.6} = 2.074\ 742\ 8 \\ y_4 &= f(2) = [3(2)^2 - 2^3]^{0.6} = 2.297\ 396\ 7 \\ y_5 &= f(2.5) = [3(2.5)^2 - 2.5^3]^{0.6} = 1.981\ 116\ 5 \\ y_6 &= f(3) = [3(3)^2 - 3^3]^{0.6} = 0 \end{aligned}$$

$$\begin{aligned} A &= \int_0^3 (3x^2 - x^3)^{0.6} dx = \frac{0.5}{3} [0 + 4(0.754\ 272\ 0) + 2(1.515\ 716\ 6) \\ &\quad + 4(2.074\ 742\ 8) + 2(2.297\ 396\ 7) + 4(1.981\ 116\ 5) + 0] \\ &= 4.477\ 792\ 0\ \text{m}^2 \end{aligned}$$

Thus, the area of one side of the stabilizer is  $4.8\ \text{m}^2$  (rounded off). As with the trapezoidal rule, the calculation can be done completely on a calculator, without recording the above values. ■

Just like the trapezoidal rule, Simpson's rule is especially useful when approximating integrals of functions for which only a given set of points is available. This is often the case for functions obtained from empirical data or from experimental measurements. Examples can be found in the exercises.

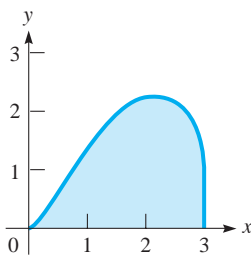


Fig. 25.22



## EXERCISES 25.6

In Exercises 1 and 2, make the given changes in the indicated examples of this section, and then solve the indicated problems.

- In Example 1, change the denominator  $x + 1$  to  $x + 2$  and then find the approximate value of the integral.
- In Example 2, change  $n$  such that  $h = 0.2$  and explain why Simpson's rule cannot be used.

In Exercises 3–6, (a) approximate the value of each of the given integrals by use of Simpson's rule, using the given value of  $n$ , and (b) check by direct integration. Round to 3 significant digits.

- $\int_0^2 (1 + x^3) dx, n = 2$
- $\int_0^8 x^{1/3} dx, n = 2$
- $\int_1^4 (2x + \sqrt{x}) dx, n = 6$
- $\int_0^2 x\sqrt{x^2 + 1} dx, n = 4$

In Exercises 7–12, approximate the value of each of the given integrals by use of Simpson's rule, using the given values of  $n$ . Exercises 8–10 are the same as Exercises 10–12 of Section 25.5. Round to 3 significant digits.

- $\int_0^5 \sqrt{25 - x^2} dx, n = 4$
- $\int_0^2 \sqrt{x^3 + 1} dx, n = 4$
- $\int_1^5 \frac{1}{x^2 + x} dx, n = 10$
- $\int_2^4 \frac{1}{x^2 + 1} dx, n = 10$
- $\int_{-4}^5 (2x^4 + 1)^{0.1} dx, n = 6$
- $\int_0^{2.4} \frac{dx}{(4 + \sqrt{x})^{3/2}}, n = 8$

In Exercises 13 and 14, approximate the values of the integrals defined by the given sets of points by using Simpson's rule. These are the same as Exercises 15 and 16 of Section 25.5.

$$13. \int_2^{14} y dx \quad \begin{array}{c|cccccccc} x & 2 & 4 & 6 & 8 & 10 & 12 & 14 \\ \hline y & 0.670 & 2.34 & 4.56 & 3.67 & 3.56 & 4.78 & 6.87 \end{array}$$

$$14. \int_{1.4}^{3.2} y dx \quad \begin{array}{c|cccccccc} x & 1.4 & 1.7 & 2.0 & 2.3 & 2.6 & 2.9 & 3.2 \\ \hline y & 0.180 & 7.87 & 18.23 & 23.53 & 24.62 & 20.93 & 20.76 \end{array}$$

In Exercises 15–18, solve the given problems using Simpson's rule. Exercise 15 and 16 are the same as Exercises 18 and 19 of Section 25.5.

- $\int_0^2 \sqrt{4 - x^2} dx = \pi$  (see Exercise 27 of Section 28.8). Approximate the value of the integral with  $n = 8$ . Compare with  $\pi$ .
- $\int_0^3 \frac{dx}{2x + 2} = \ln 2$  (see Exercise 38 of Section 28.2). Approximate the value of the integral with  $n = 6$ . Compare with  $\ln 2$ .
- The distance  $\bar{x}$  (in cm) from one end of a barrel plug (with vertical cross section) to its centre of mass, as shown in Fig. 25.23, is  $\bar{x} = 0.9129 \int_0^3 x\sqrt{0.3 - 0.1x} dx$ . Find  $\bar{x}$  with  $n = 12$ .

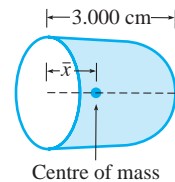


Fig. 25.23 Centre of mass

- The average value of the electric current  $i_{av}$  (in A) in a circuit for the first 4 s is  $i_{av} = \frac{1}{4} \int_0^4 (4t - t^2)^{0.2} dt$ . Find  $i_{av}$  with  $n = 10$ .

## Answer to Practice Exercise

- 0.693

## CHAPTER 25 EQUATIONS

## Indefinite integral

$$\int f(x) dx = F(x) + C \quad (25.1)$$

## Integrals

$$\int c du = c \int du = cu + C \quad (25.2)$$

$$\int (du + dv) = u + v + C \quad (25.3)$$

## Power formula

$$\int u^n du = \frac{u^{n+1}}{n+1} + C \quad (n \neq -1) \quad (25.4)$$

## Area under a curve

$$A_{ab} = \left[ \int f(x) dx \right]_a^b = F(b) - F(a) \quad (25.10)$$

## Definite integral

$$\int_a^b f(x) dx = F(b) - F(a) \quad (25.11)$$

## Trapezoidal rule

$$\int_a^b f(x) dx \approx \frac{h}{2} (y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n) \quad (25.13)$$

## Simpson's rule

$$\int_a^b f(x) dx \approx \frac{h}{3} (y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 4y_{n-1} + y_n), n \text{ even} \quad (25.18)$$

## CHAPTER 25 REVIEW EXERCISES

In Exercises 1–24, evaluate the given integrals.

1.  $\int (4x^3 - x) dx$
2.  $\int (5 + 3t^2) dt$
3.  $\int \sqrt{u}(u^2 + 8) du$
4.  $\int x(x - 3x^4) dx$
5.  $\int_1^4 \left( \frac{\sqrt{x}}{2} + \frac{2}{\sqrt{x}} \right) dx$
6.  $\int_1^2 \left( x + \frac{1}{x^2} \right) dx$
7.  $\int_0^2 x(4 - x) dx$
8.  $\int_0^1 2t(2t + 1)^2 dt$
9.  $\int \left( 5 + \frac{6}{x^3} \right) dx$
10.  $\int \left( 3\sqrt{x} + \frac{1}{2\sqrt{x}} - \frac{1}{4} \right) dx$
11.  $\int_{-2}^5 \frac{dx}{\sqrt[3]{x^2 + 6x + 9}}$
12.  $\int_{0.35}^{0.85} x(\sqrt{1 - x^2} + 1) dx$
13.  $\int \frac{dn}{(9 - 5n)^3}$
14.  $\int \frac{1}{x^2} \sqrt{1 + \frac{1}{x}} dx$
15.  $\int 3(7 - 2x)^{3/4} dx$
16.  $\int (y^3 + 3y^2 + 3y + 1)^{2/3} dy$
17.  $\int_0^2 \frac{3x dx}{\sqrt[3]{1 + 2x^2}}$
18.  $\int_1^6 \frac{12 dx}{(3x - 2)^{3/4}}$
19.  $\int x^2(1 - 2x^3)^4 dx$
20.  $\int 3R^3(1 - 5R^4) dR$
21.  $\int \frac{(2 - 3x^2) dx}{(2x - x^3)^2}$
22.  $\int \frac{x^2 - 3}{\sqrt{6 + 9x - x^3}} dx$
23.  $\int_1^3 (x^2 + x + 2)(2x^3 + 3x^2 + 12x) dx$
24.  $\int_0^2 (4x + 18x^2)(x^2 + 3x^3)^2 dx$

In Exercises 25–38, solve the given problems.

25. Find the equation of the curve that passes through  $(-1, 3)$  for which the slope is given by  $3 - x^2$ .
26. Find the equation of the curve that passes through  $(1, -2)$  for which the slope is  $x(x^2 + 1)^2$ .
27. Perform the integration  $\int (1 - 2x) dx$  (a) term by term, labeling the constant of integration as  $C_1$ , and then (b) by letting  $u = 1 - 2x$ , using the general power rule and labelling the constant of integration as  $C_2$ . Is  $C_1 = C_2$ ? Explain.
28. Following the methods (a) and (b) in Exercise 27, perform the integration  $\int (3x + 2) dx$ . In (b) let  $u = 3x + 2$ . Is  $C_1 = C_2$ ? Explain.
29. Write  $\int_3^8 F(v) dv - \int_4^8 F(v) dv$  as a single definite integral.
30. Show that  $\int_{-3}^0 x^3 dx = -\int_0^3 x^3 dx$ .
31. Find the general form of the function whose second derivative is  $1/\sqrt{6x + 5}$ .

32. Find the equation of the curve for which the second derivative is  $-6x$  if the curve passes through  $(-1, 3)$  with a slope of  $-3$ .

33. Show that  $\int_0^1 x^3 dx = \int_1^2 (x - 1)^3 dx$ . In terms of area, explain this result.
34. If  $\int_0^1 [f(x) - g(x)] dx = 3$  and  $\int_0^1 g(x) dx = -1$ , find the value of  $\int_0^1 2f(x) dx$ .
35. Use Eq. (25.10) to find the area under  $y = 6x - 1$  between  $x = 1$  and  $x = 3$ .
36. Use Eq. (25.10) to find the first-quadrant area under  $y = 8x - x^4$ .
37. Given that  $f(x)$  is continuous,  $f(x) > 0$ , and  $f''(x) < 0$  for  $a \leq x \leq b$ , explain why the exact value of  $\int_a^b f(x) dx$  is greater than the approximate value found by use of the trapezoidal rule.
38. It is shown in more advanced works that when evaluating  $\int_a^b f(x) dx$ , the maximum error in using Simpson's rule is  $\frac{M(b - a)^5}{180n^4}$ , where  $M$  is the greatest absolute value of the fourth derivative of  $f(x)$  for  $a \leq x \leq b$ . Evaluate the maximum error for the integral  $\int_2^3 \frac{dx}{2x}$  with  $n = 4$ .

In Exercises 39 and 40, solve the given problems by using the trapezoidal rule. In Exercises 41 and 42, solve the given problems using Simpson's rule.

39. Approximate  $\int_1^3 \frac{dx}{2x - 1}$  with  $n = 4$ .

40. The streamflow  $F$  (in  $\text{m}^3/\text{s}$ ) passing through a cross section of a stream is found by evaluating  $F = \int_0^L g(x) dx$ , where  $L$  is the width of the stream cross section and  $g(x)$  is the product of the depth and velocity of the stream  $x$  m from the bank. From the following table, estimate  $F$  with  $L = 24$  m.

$x$ (m)	0.0	3.0	6.0	9.0	12.0	15.0	18.0	21.0	24.0
$g(x)$	0.00	0.59	0.13	0.34	0.76	0.65	0.29	0.07	0.00

41. Approximate  $\int_1^3 \frac{dx}{2x - 1}$  with  $n = 4$  (see Exercise 39).
42. The velocity  $v$  (in  $\text{km/h}$ ) of a car was recorded at 1-min intervals as shown. Estimate the distance traveled by the car.

$t$ (min)	0	1	2	3	4	5	6	7	8	9	10
$v$ (km/h)	60	62	65	69	72	74	76	77	77	75	76

In Exercises 43–48, use the function  $y = x\sqrt[3]{2x^2 + 1}$  and approximate the area under the curve between  $x = 1$  and  $x = 4$  by the indicated method. See Fig. 25.24.

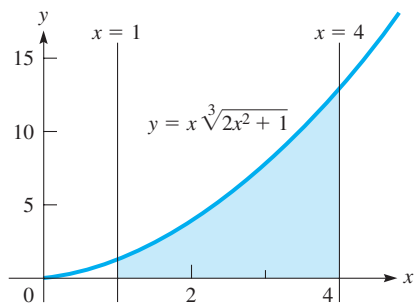


Fig. 25.24

43. Find the sum of the areas of three inscribed rectangles.
44. Find the sum of the areas of six inscribed rectangles.
45. Use the trapezoidal rule with  $n = 3$ .
46. Use the trapezoidal rule with  $n = 6$ .
47. Use Simpson's rule with  $n = 6$ .
48. Use integration (for the exact area).

In Exercises 49–51, find the area of the archway, as shown in Fig. 25.25, by the indicated method. The archway can be described as the area bounded by the elliptical arc  $y = 4 + \sqrt{1 + 8x - 2x^2}$ ,  $x = 0$ ,  $x = 4$ , and  $y = 0$ , where dimensions are in metres.

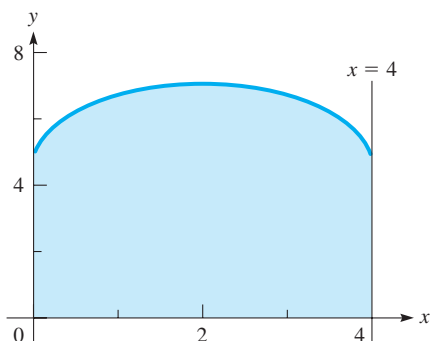


Fig. 25.25

49. Find the sum of the areas of eight inscribed rectangles.
50. Use the trapezoidal rule with  $n = 8$ .
51. Use Simpson's rule with  $n = 8$ .

In Exercises 52–58, solve the given problems by integration.

52. The charge  $Q$  (in C) transmitted in a certain electric circuit in 3 s is given by  $Q = \int_0^3 6t^2 dt$ . Evaluate  $Q$ .
53. The velocity  $ds/dt$  (in m/s) of a projectile is  $ds/dt = -9.8t + 16$ . Find the displacement  $s$  of the object after 4.0 s if the initial displacement is 48 m.
54. The deflection  $y$  of a certain beam at a distance  $x$  from one end is given by  $dy/dx = k(2L^3 - 12Lx + 2x^4)$ , where  $k$  is a constant and  $L$  is the length of the beam. Find  $y$  as a function of  $x$  if  $y = 0$  for  $x = 0$ .
55. The total electric charge  $Q$  on a charged sphere is given by  $Q = k \int (r^2 - \frac{r^3}{R}) dr$ , where  $k$  is a constant,  $r$  is the distance from the centre of the sphere, and  $R$  is the radius of the sphere. Find  $Q$  as a function of  $r$  if  $Q = Q_0$  for  $r = R$ .
56. Part of the deck of a boat is the parabolic area shown in Fig. 25.26. The area  $A$  (in  $\text{m}^2$ ) is  $A = 2 \int_0^5 \sqrt{5-y} dy$ . Evaluate  $A$ .

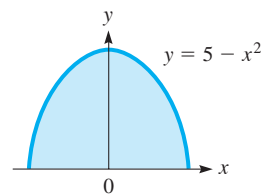


Fig. 25.26

57. The distance  $s$  (in cm) through which a cam follower moves in 4 s is  $s = \int_0^4 t\sqrt{4 + 9t^2} dt$ . Evaluate  $s$ .
58. The probability that the time of failure (in years) of a certain hydraulic component is between 3 and 6 years is given by  $p = \int_3^6 \frac{32dx}{(x+4)^3}$ . Evaluate  $p$ .

### Writing Exercise

59. A computer science student is writing a program to find a good approximation for the value of  $\pi$  by using the formula  $A = \pi r^2$  for a circle. The value of  $\pi$  is to be found by approximating the area of a circle with a given radius. Write two or three paragraphs explaining how the value of  $\pi$  can be approximated in this way. Include any equations and values that may be used, but do not actually make the calculations.

## CHAPTER 25 PRACTICE TEST

1. Find an antiderivative of  $f(x) = 2x - (1-x)^4$ .
2. Integrate  $\int x\sqrt{1-2x^2} dx$ .
3. Find  $y$  in terms of  $x$  if  $dy/dx = (6-x)^4$  and the curve passes through  $(5, 2)$ .
4. Approximate the area under  $y = \frac{1}{x+2}$  between  $x = 1$  and  $x = 4$  (above the  $x$ -axis) by inscribing six rectangles and finding the sum of their areas.
5. Evaluate  $\int_1^4 \frac{dx}{x+2}$  by using the trapezoidal rule with  $n = 6$ .
6. Evaluate the definite integral of Problem 5 by using Simpson's rule with  $n = 6$ .
7. The total electric current  $i$  (in A) to pass a point in the circuit between  $t = 1$  s and  $t = 3$  s is  $i = \int_1^3 (t^2 + \frac{1}{t^2}) dt$ . Evaluate  $i$ .

# Applications of Integration

**W**ith the development of the calculus, many problems being studied in the 1600s and later were much more easily solved. As we saw in Chapters 23 and 24, differential calculus led to the solution of problems such as finding velocities, maximum and minimum values, and various types of rates of change.

Integral calculus also led to the solution of many types of problems that were being studied. As Newton and Leibniz developed the methods of integral calculus, they were interested in finding areas and the problems that could be solved by finding these areas. They were also very interested in applications in which the rate of change was known and therefore led to the relation between the variables being studied.

One of the problems being studied in the 1600s by the French mathematician and physicist Blaise Pascal was that of the pressure within a liquid and the force on the walls of the container due to this pressure. Since pressure is a measure of the force on an area, finding the force became essentially solving an area problem. Also at the time, many mathematicians and physicists were studying various kinds of motion, such as motion along a curved path and the motion of a rotating object. When information about the velocity is known, the solution is found by integrating. Later, in the 1800s, since electric current is the time rate change of electric charge, the current could be found by integrating known expressions for the charge.

As it turns out, integration is useful in many areas of science, engineering, and technology. It has important applications in areas such as electricity, mechanics, architecture, machine design, statistics, and business, as well as other areas of physics and geometry.

In the first section of this chapter, we present some important applications of the indefinite integral, with emphasis on the motion of an object and the voltage across a capacitor. In the remaining sections, we show uses of the definite integral related to geometry, mechanics, work by a variable force, and force due to liquid pressure.

## 26

### LEARNING OUTCOMES

**After completion of this chapter, the student should be able to:**

- Solve application problems involving indefinite integrals, including velocity and displacement and voltage across a capacitor
- Find the area between curves
- Obtain the volume of a solid of revolution
- Find the centre of mass, the moment of inertia, and the radius of gyration of a flat plate and of a solid of revolution
- Solve application problems involving definite integrals, including work, liquid pressure, and average value of a function



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◀ In Section 26.6, integration is used to find the force water exerts on the floodgate of a dam.

## 26.1 Applications of the Indefinite Integral

### Velocity and Displacement • Voltage Across a Capacitor

■ Velocity as a first derivative and acceleration as a second derivative were introduced in Chapter 23.

### VELOCITY AND DISPLACEMENT

We first apply integration to the problem of finding the displacement and velocity as functions of time, when we know the relationship between acceleration and time, and certain values of displacement and velocity. As shown in Section 25.2, these values are needed for finding the constants of integration that are introduced.

Recalling that the acceleration  $a$  of an object is given by  $a = dv/dt$ , we can find the expression for the velocity  $v$  in terms of  $a$ , the time  $t$ , and the constant of integration. Therefore, we write  $dv = a dt$ , or

$$v = \int a dt \quad (26.1)$$

If the acceleration is constant, we have

$$v = at + C_1 \quad (26.2)$$

In general, Eq. (26.1) is used to find the velocity as a function of time when we know the acceleration as a function of time. Since the case of constant acceleration is often encountered, Eq. (26.2) can often be used.

#### EXAMPLE 1 Find the velocity, given the acceleration

Find the expression for the velocity if  $a = 12t$ , given that  $v = 8$  when  $t = 1$ .

Using Eq. (26.1), we have

$$v = \int (12t) dt = 6t^2 + C_1$$

Substituting the known values, we have  $8 = 6 + C_1$ , or  $C_1 = 2$ . This means that

$$v = 6t^2 + 2 \quad \blacksquare$$

#### EXAMPLE 2 Find the velocity for a constant acceleration

For an object falling under the influence of gravity, the acceleration due to gravity is essentially constant. Its value is  $-9.8 \text{ m/s}^2$ . (The negative sign is chosen so that *all quantities directed up are positive and all quantities directed down are negative*.) Find the expression for the velocity of an object under the influence of gravity if  $v = v_0$  when  $t = 0$ .

We write

$$\begin{aligned} v &= \int (-9.8) dt && \text{substitute } a = -9.8 \text{ into Eq. (26.1)} \\ &= -9.8t + C_1 && \text{integrate} \\ v_0 &= -9.8(0) + C_1 && \text{substitute given values} \\ C_1 &= v_0 && \text{solve for } C_1 \\ v &= v_0 - 9.8t && \text{substitute} \end{aligned}$$

The velocity  $v_0$  is called the *initial velocity*. If the object is given an initial upward velocity of  $40.0 \text{ m/s}$ ,  $v_0 = 40.0 \text{ m/s}$ . If the object is dropped,  $v_0 = 0$ . If the object is given an initial downward velocity of  $40.0 \text{ m/s}$ ,  $v_0 = -40.0 \text{ m/s}$ .  $\blacksquare$

Once we have the expression for velocity, we can then integrate to find the expression for displacement  $s$  in terms of time. Since  $v = ds/dt$ , we can write  $ds = vdt$ , or

$$s = \int v dt \quad (26.3)$$

### EXAMPLE 3 Find the initial velocity

A ball is thrown vertically from the top of a building 24.5 m high and hits the ground 5.0 s later. What initial velocity was the ball given?

Measuring vertical distances from the ground, we know that  $s = 24.5$  m when  $t = 0$  and that  $v = v_0 - 9.8t$ . Thus,

$$s = \int (v_0 - 9.8t) dt = v_0 t - 4.9t^2 + C \quad \text{integrate}$$

$$24.5 = v_0(0) - 4.9(0) + C, \quad C = 24.5 \quad \text{evaluate } C$$

$$s = v_0 t - 4.9t^2 + 24.5$$

We also know that  $s = 0$  when  $t = 5.0$  s. Thus,

$$0 = v_0(5.0) - 4.9(5.0)^2 + 24.5 \quad \text{substitute given values}$$

$$5.0v_0 = 98.0$$

$$v_0 = 19.6 \text{ m/s}$$

This means that the initial velocity was 19.6 m/s upward. See Fig. 26.1. ■

### EXAMPLE 4 Find the displacement, given the acceleration

During the initial stage of launching a spacecraft vertically, the acceleration  $a$  (in  $\text{m/s}^2$ ) of the spacecraft is  $a = 6t^2$ . Find the height  $s$  of the spacecraft after 6.0 s if  $s = 12$  m for  $t = 0.0$  s and  $v = 16$  m/s for  $t = 2.0$  s.

First, we use Eq. (26.1) to get an expression for the velocity:

$$v = \int 6t^2 dt = 2t^3 + C_1 \quad \text{integrate}$$

$$16 = 2(2.0)^3 + C_1, \quad C_1 = 0 \quad \text{evaluate } C_1$$

$$v = 2t^3$$

We now use Eq. (26.3) to get an expression for the displacement:

$$s = \int 2t^3 dt = \frac{1}{2}t^4 + C_2 \quad \text{integrate}$$

$$12 = \frac{1}{2}(0.0)^4 + C_2, \quad C_2 = 12 \quad \text{evaluate}$$

$$s = \frac{1}{2}t^4 + 12$$

Now, finding  $s$  for  $t = 6.0$  s, we have  $s = \frac{1}{2}(6.0)^4 + 12 = 660$  m ■

## VOLTAGE ACROSS A CAPACITOR

The second basic application of the indefinite integral we will discuss comes from the field of electricity. By definition, the current  $i$  in an electric circuit equals the time rate of change of the charge  $q$  (in coulombs) that passes a given point in the circuit, or

$$i = \frac{dq}{dt} \quad (26.4)$$

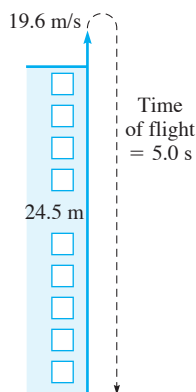


Fig. 26.1

### Practice Exercise

- In Example 4, change the acceleration to  $a = 4 \text{ m/s}^2$  and then find the height under the same given conditions.



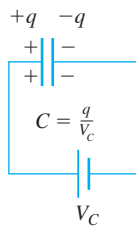


Fig. 26.2

**LEARNING TIP**

When using Eq. (26.6), all quantities must be in the units listed. If the quantities arising in a given problem involve different unit prefixes, we must use the proper power of 10 that corresponds to each prefix.

**Practice Exercise**

2. In Example 5, change the current to  $i = 12t + 6$  and then find the charge under the same given conditions.

■ Here,  $C$  represents coulombs and is not the  $C$  for capacitance or the constant of integration.

Rewriting this expression  $dq = i dt$  and integrating both sides, we have

$$q = \int i dt \quad (26.5)$$

Now, the voltage  $V_C$  across a capacitor with capacitance  $C$  (see Fig. 26.2) is given by  $V_C = q/C$ . By combining equations, the voltage  $V_C$  is given by

$$V_C = \frac{1}{C} \int i dt \quad (26.6)$$

Here,  $V_C$  is measured in volts,  $C$  in farads,  $i$  in amperes, and  $t$  in seconds.

**EXAMPLE 5 Find the electric charge, given the current**

The current  $i$  (in A) in an electric circuit as a function of time  $t$  (in s) is given by  $i = 6t^2 + 4$ . Find an expression for the electric charge  $q$  (in C) that passes a point in the circuit as a function of  $t$ . If  $q = 0$  C when  $t = 0$  s, determine the total charge that passes the point in 2.0 s.

Since  $q = \int i dt$ , we have

$$\begin{aligned} q &= \int (6t^2 + 4) dt && \text{substitute into Eq. (26.5)} \\ &= 2t^3 + 4t + C_1 && \text{integrate} \end{aligned}$$

This is the required expression of charge as a function of time. When  $t = 0$ ,  $q = C_1$ , which means the constant of integration represents the initial charge, or the charge that passed a given point before the timing started. Using  $q_0$  to represent this charge, we have

$$q = 2t^3 + 4t + q_0$$

Returning to the second part of the problem, we note that  $q_0 = 0$ . Evaluating  $q$  for  $t = 2.0$  s, we have  $q = 2(2.0)^3 + 4(2.0) = 24$  C, which is the charge that passes any point in 2.0 s. ■

**EXAMPLE 6 Find the voltage across a capacitor**

The voltage across a  $5.0\text{-}\mu\text{F}$  capacitor is zero. What is the voltage after 20 ms if a current of 75 mA charges the capacitor?

Since the current is 75 mA, we know that  $i = 0.075$  A  $= 7.5 \times 10^{-2}$  A. Since  $5.0 \mu\text{F} = 5.0 \times 10^{-6}$  F, we have

$$\begin{aligned} V_C &= \frac{1}{5.0 \times 10^{-6}} \int 7.5 \times 10^{-2} dt && \text{substituting into Eq. (26.6)} \\ &= (1.5 \times 10^4) \int dt = (1.5 \times 10^4)t + C_1 && \text{integrate} \end{aligned}$$

From the given information, we know that  $V_C = 0$  when  $t = 0$ . Thus,

$$0 = (1.5 \times 10^4)(0) + C_1 \quad \text{or} \quad C_1 = 0 \quad \text{evaluate } C_1$$

This means that

$$V_C = (1.5 \times 10^4)t$$

Evaluating this expression for  $t = 20 \times 10^{-3}$  s, we have

$$\begin{aligned} V_C &= (1.5 \times 10^4)(20 \times 10^{-3}) \\ &= 30 \times 10 = 300 \text{ V} \end{aligned} \quad \blacksquare$$



**EXAMPLE 7** Find the capacitance of a capacitor

A certain capacitor has 100 V across it. At this instant, a current  $i = 0.06t^{1/2}$  is sent through the circuit. After 0.25 s, the voltage across the capacitor is 140 V. What is the capacitance?

Substituting  $i = 0.06t^{1/2}$ , we find that

$$\begin{aligned} V_C &= \frac{1}{C} \int (0.06t^{1/2} dt) = \frac{0.06}{C} \int t^{1/2} dt && \text{using Eq. (26.6)} \\ &= \frac{0.04}{C} t^{3/2} + C_1 && \text{integrate} \end{aligned}$$

From the given information, we know that  $V_C = 100$  V when  $t = 0$ . Thus,

$$\begin{aligned} 100 &= \frac{0.04}{C}(0) + C_1 \quad \text{or} \quad C_1 = 100 \text{ V} && \text{evaluate} \\ V_C &= \frac{0.04}{C} t^{3/2} + 100 && \text{substituting 100 for } C_1 \end{aligned}$$

We also know that  $V_C = 140$  V when  $t = 0.25$  s. Therefore,

$$\begin{aligned} 140 &= \frac{0.04}{C}(0.25)^{3/2} + 100 \\ 40 &= \frac{0.04}{C}(0.125) \\ C &= 1.25 \times 10^{-4} \text{ F} = 125 \mu\text{F} \end{aligned}$$

**EXERCISES 26.1**

- In Example 3, change 5.0 s to 1.0 s and then solve the resulting problem.
- In Example 7, change  $0.06\sqrt{t}$  to  $0.06t$  and then solve the resulting problem.
- What is the velocity (in m/s) of a sandbag 1.5 s after it is released from a hot-air balloon that is stationary in the air?
- A beach ball is rolled up a shallow slope with an initial velocity of 7.2 m/s. If the acceleration of the ball is  $1.2 \text{ m/s}^2$  down the slope, find the velocity of the ball after 8.0 s.
- A conveyor belt 8.00 m long moves at 0.25 m/s. If a package is placed at one end, find its displacement from the other end as a function of time.
- During each cycle, the velocity  $v$  (in mm/s) of a piston is  $v = 6t - 6t^2$ , where  $t$  is the time (in s). Find the displacement  $s$  of the piston after 0.75 s if the initial displacement is zero.
- While in the barrel of a tennis ball machine, the acceleration  $a$  (in  $\text{m/s}^2$ ) of a ball is  $a = 30\sqrt{1 - 4t}$ , where  $t$  is the time (in s). If  $v = 0$  for  $t = 0$ , find the velocity of the ball as it leaves the barrel at  $t = 0.25$  s.
- A person skis down a slope with an acceleration (in  $\text{m/s}^2$ ) given by  $a = \frac{600t}{(60 + 0.5t^2)^2}$ , where  $t$  is the time (in s). Find the skier's velocity as a function of time if  $v = 0$  when  $t = 0$ .
- If a car decelerates at  $250 \text{ m/s}^2$  (about the maximum a human body can survive) during an accident, and the car was going at 96 km/h at impact, over what distance must an airbag stop a person in order to survive the crash?
- The engine of a lunar lander is cut off when the lander is 5.0 m above the surface of the moon and descending at 2.0 m/s. If the acceleration due to gravity on the moon is  $1.6 \text{ m/s}^2$ , what is the speed of the lander just before it touches the surface?
- If an aircraft is to attain a take-off velocity of 75 m/s after traveling 240 m along the flight deck of an aircraft carrier, find the aircraft's acceleration (assumed constant).
- A stone is thrown straight up from the edge of a 45.0-m-high cliff. A loose stone at the edge of the cliff falls off 1.50 s later. What is the vertical velocity of the first stone, if the two stones reach the ground below at the same time?
- What must be the nozzle velocity of the water from a fire hose if it is to reach a point 30 m directly above the nozzle?
- An arrow is shot from the edge of a cliff with a vertical upward velocity of 40.0 m/s. If it strikes the plain below after 9.0 s, how high is the cliff?
- In coming to a stop, the acceleration of a motorcycle is  $-4.0t$ . If it is travelling at 32 m/s when the brakes are applied, how far does it travel while stopping?
- A hoist mechanism raises a crate with an acceleration (in  $\text{m/s}^2$ )  $a = \sqrt{1 + 0.2t}$ , where  $t$  is the time in seconds. Find the displacement of the crate as a function of time if  $v = 0 \text{ m/s}$  and  $s = 2 \text{ m}$  for  $t = 0$  s.
- The electric current in a microprocessor circuit is  $0.230 \mu\text{A}$ . How many coulombs pass a given point in the circuit in 1.50 ms?
- A capacitor in a microwave oven circuit has a charge of  $0.10 \mu\text{F}$ . If it is further charged by a current of  $0.25 \mu\text{A}$ , what is the charge on the capacitor 3.0  $\mu\text{s}$  later?

19. In an amplifier circuit, the current  $i$  (in A) changes with time  $t$  (in s) according to  $i = 0.06t\sqrt{1 + t^2}$ . If 0.015 C of charge has passed a point in the circuit at  $t = 0$ , find the total charge to have passed the point at  $t = 0.25$  s.
20. The current  $i$  (in  $\mu\text{A}$ ) in a DVD player circuit is given by  $i = 6.0 - 0.50t$ , where  $t$  is the time (in  $\mu\text{s}$ ) and  $0 \leq t \leq 30 \mu\text{s}$ . If  $q_0 = 0$  C, for what value of  $t$  is  $q = 0$  C? What interpretation can be given to this result?
21. The voltage across a  $2.5\text{-}\mu\text{F}$  capacitor in a copying machine is zero. What is the voltage after 12 ms if a current of 25 mA charges the capacitor?
22. The voltage across an  $8.50\text{-nF}$  capacitor in an FM receiver circuit is zero. Find the voltage after  $2.00 \mu\text{s}$  if a current (in mA)  $i = 0.042t$  charges the capacitor.
23. The voltage across a  $3.75\text{-}\mu\text{F}$  capacitor in a television circuit is 4.50 mV. Find the voltage after 0.565 ms if a current (in  $\mu\text{A}$ )  $i = \sqrt[3]{1 + 6t}$  further charges the capacitor.
24. A current  $i = t/\sqrt{t^2 + 1}$  (in A) is sent through an electric dryer circuit containing a previously uncharged  $2.0\text{-}\mu\text{F}$  capacitor. How long does it take for the capacitor voltage to reach 120 V?
25. The angular velocity  $\omega$  is the time rate of change of the angular displacement  $\theta$  of a rotating object. See Fig. 26.3. In testing the shaft of an engine, its angular velocity is  $\omega = 16t + 0.50t^2$ , where  $t$  is the time (in s) of rotation. Find the angular displacement through which the shaft goes in 10.0 s.

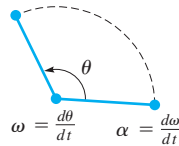


Fig. 26.3

26. The angular acceleration  $\alpha$  is the time rate of change of angular velocity  $\omega$  of a rotating object. See Fig. 26.3. When starting up, the angular acceleration of a helicopter blade is  $\alpha = \sqrt{8t + 1}$ . Find the expression for  $\theta$  if  $\omega = 0$  and  $\theta = 0$  for  $t = 0$ .
27. An inductor in an electric circuit is essentially a coil of wire in which the voltage is affected by a changing current. By definition,

the voltage caused by the changing current is given by  $V_L = L(di/dt)$ , where  $L$  is the inductance (in H). If  $V_L = 12.0 - 0.2t$  for a 3.0-H inductor, find the current in the circuit after 20 s if the initial current was zero.

28. If the inner and outer walls of a container are at different temperatures, the rate of change of temperature with respect to the distance from one wall is a function of the distance from the wall. Symbolically, this is stated as  $dT/dx = f(x)$ , where  $T$  is the temperature. If  $x$  is measured from the outer wall, at  $20^\circ\text{C}$ , and  $f(x) = 72x^2$ , find the temperature at the inner wall if the container walls are 0.5 cm thick.
29. Surrounding an electrically charged particle is an electric field. The rate of change of electric potential with respect to the distance from the particle creating the field equals the negative of the value of the electric field. That is,  $dV/dx = -E$ , where  $E$  is the electric field. If  $E = k/x^2$ , where  $k$  is a constant, find the electric potential at a distance  $x_1$  from the particle, if  $V \rightarrow 0$  as  $x \rightarrow \infty$ .
30. The rate of change of the vertical deflection  $y$  with respect to the horizontal distance  $x$  from one end of a beam is a function of  $x$ . For a particular beam, the function is  $k(x^5 + 1350x^3 - 7000x^2)$ , where  $k$  is a constant. Find  $y$  as a function of  $x$ .
31. Freshwater is flowing into a brine solution, with an equal volume of mixed solution flowing out. The amount of salt in the solution decreases, but more slowly as time increases. Under certain conditions, the time rate of change of mass of salt (in g/min) is given by  $-1/\sqrt{t + 1}$ . Find the mass  $m$  of salt as a function of time if 1000 g were originally present. Under these conditions, how long would it take for all the salt to be removed?
32. A holograph of a circle is formed. The rate of change of the radius  $r$  of the circle with respect to the wavelength  $\lambda$  of the light used is inversely proportional to the square root of  $\lambda$ . If  $dr/d\lambda = 3.55 \times 10^4$  and  $r = 4.08$  cm for  $\lambda = 574$  nm, find  $r$  as a function of  $\lambda$ .

Answers to Practice Exercises

1. 132 m    2. 36 C

## 26.2 Areas by Integration

Summing Elements of Area •  
Vertical Elements •  
Horizontal Elements

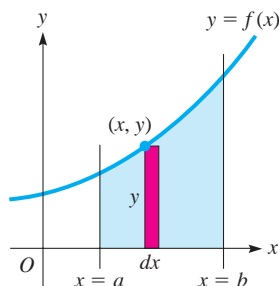


Fig. 26.4

In Section 25.3, we introduced the method of finding the area under a curve by integration. We also showed that the area can be found by a summation process on the rectangles inscribed under the curve, which means that integration can be interpreted as a summation process. *The applications of the definite integral use this summation interpretation of the integral.* We now develop a general procedure for finding the area for which the bounding curves are known by summing the areas of inscribed rectangles and using integration for the summation.

The first step is to make a sketch of the area. Next, a representative **element of area**  $dA$  (a typical rectangle) is drawn. In Fig. 26.4, the width of the element is  $dx$ . The length of the element is determined by the  $y$ -coordinate (of the vertex of the element) of the point on the curve. Thus, the length is  $y$ . The area of this element is  $y dx$ , which in turn means that  $dA = y dx$ , or

$$A = \int_a^b y dx = \int_a^b f(x) dx \tag{26.7}$$

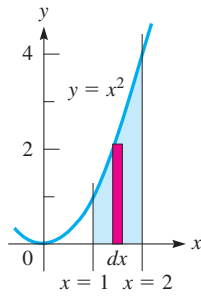


Fig. 26.5

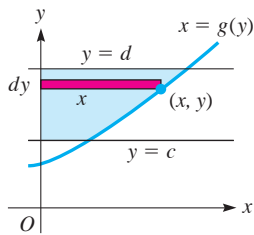


Fig. 26.6

This equation states that the elements are to be summed (this is the meaning of the integral sign) from  $a$  (the left boundary) to  $b$  (the right boundary).

**EXAMPLE 1 Find an area using vertical elements**

Find the area bounded by  $y = x^2$ ,  $x = 1$ , and  $x = 2$ .

This area is shown in Fig. 26.5. The rectangle shown is the representative element, and its area is  $y dx$ . The elements are to be summed from  $x = 1$  to  $x = 2$ .

$$\begin{aligned}
 A &= \int_1^2 y dx && \text{sum} && \text{right boundary} && A = \int_1^2 y dx = \int_1^2 x^2 dx && \text{substitute } x^2 \text{ for } y \\
 &= \int_1^2 y dx && \text{left boundary} && \text{area of element} && = \frac{1}{3} x^3 \Big|_1^2 = \frac{1}{3} (8) - \frac{1}{3} (1) && \text{integrate and evaluate} \\
 & && && && = \frac{7}{3} &&
 \end{aligned}$$

In Figs. 26.4 and 26.5, the elements of area are vertical. Some problems are simplified using horizontal elements. With horizontal elements, the length (longest dimension) is measured in terms of the  $x$ -coordinate of the point on the curve, and the width is  $dy$ . In Fig. 26.6, the area of the element is  $x dy$ , which means  $dA = x dy$ , or

$$A = \int_c^d x dy = \int_c^d g(y) dy \tag{26.8}$$

**LEARNING TIP**

Note that the limits of integration in Eqs. (26.7) and (26.8) are chosen so that summation is done in the positive direction. Vertical elements are summed from left to right (with boundaries represented by  $x$  values), and horizontal elements are summed from bottom to top (with boundaries represented by  $y$  values).

In using Eq. (26.8), the elements are summed from  $c$  (the lower boundary) to  $d$  (the upper boundary).

**EXAMPLE 2 Vertical and horizontal elements of an area**

Find the area in the first quadrant bounded by  $y = 9 - x^2$ . See Fig. 26.7.

$$\begin{aligned}
 A &= \int_0^3 y dx && \text{vertical element of length } y \text{ and width } dx && \text{sum of areas of elements} && A = \int_0^9 x dy && \text{horizontal element of length } x \text{ and width } dy \\
 &= \int_0^3 (9 - x^2) dx && \text{substitute for } y; \text{ substitute for } x && && = \int_0^9 \sqrt{9 - y} dy = - \int_0^9 (9 - y)^{1/2} (-dy) \\
 &= \left( 9x - \frac{x^3}{3} \right) \Big|_0^3 && \text{integrate} && && = -\frac{2}{3} (9 - y)^{3/2} \Big|_0^9 \\
 &= (27 - 9) - 0 = 18 && \text{evaluate} && && = -\frac{2}{3} (9 - 9)^{3/2} + \frac{2}{3} (9 - 0)^{3/2} = 18
 \end{aligned}$$

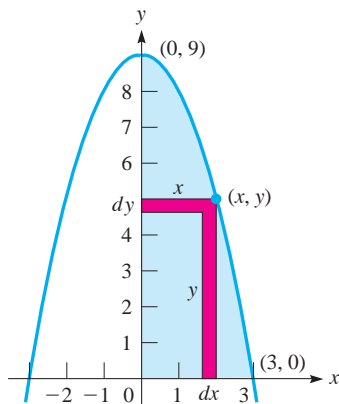


Fig. 26.7

Note that the limits are 0 to 3 for the vertical elements, and 0 to 9 for the horizontal elements.

The choice of vertical or horizontal elements is determined by (1) which one leads to the simplest solution or (2) the form of the resulting integral. In some problems, it makes little difference which is chosen. However, our present methods of integration do not include many types of integrals.

**AREA BETWEEN TWO CURVES**

It is also possible to find the area between two curves when one of the curves is not an axis. In such a case, the length of the element of area becomes the difference in the  $y$ - or  $x$ -coordinates, depending on whether a vertical element or a horizontal element is used.

**Practice Exercise**

1. Find the area in the first quadrant bounded by  $y = 4 - x^2$ .

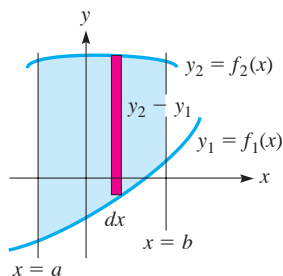


Fig. 26.8

$$A = \int_a^b (y_2 - y_1) dx \tag{26.9}$$

In Fig. 26.9, by using horizontal elements, the element of area is bounded on the left by  $x_1 = g_1(y)$  and on the right by  $x_2 = g_2(y)$ . The length of the element is  $x_2 - x_1$ , and its width is  $dy$ . Thus, the area is

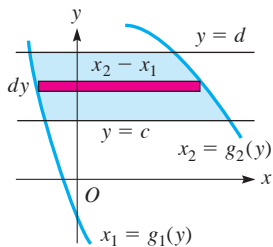


Fig. 26.9

$$A = \int_c^d (x_2 - x_1) dy \tag{26.10}$$

The following examples show the use of Eqs. (26.9) and (26.10) to find the indicated areas.

**EXAMPLE 3** Area between curves using vertical elements

Find the area bounded by  $y = x^2$  and  $y = x + 2$ .

First, by sketching each curve, we see that the area to be found is that shown in Fig. 26.10. The points of intersection of these curves are found by solving the equations simultaneously. The solution for the  $x$ -values is shown at the left. We then find the  $y$ -coordinates by substituting into either equation. The substitution shows the points of intersection to be  $(-1, 1)$  and  $(2, 4)$ .

Here, we choose vertical elements, since they are all bounded at the top by the line  $y = x + 2$  and at the bottom by the parabola  $y = x^2$ . If we were to choose horizontal elements, the bounding curves would be different above  $(-1, 1)$  from below this point. Choosing horizontal elements would then require two separate integrals. Therefore, using vertical elements, we have

■  $x^2 = x + 2$   
 $x^2 - x - 2 = 0$   
 $(x + 1)(x - 2) = 0$   
 $x = -1, 2$

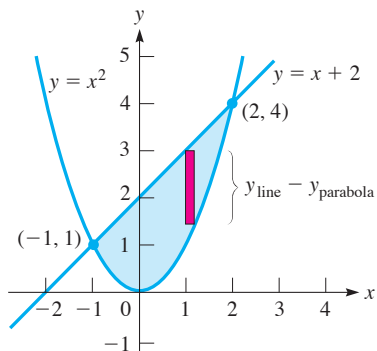


Fig. 26.10

$$\begin{aligned} A &= \int_{-1}^2 (y_{\text{line}} - y_{\text{parabola}}) dx && \text{using Eq. (26.9)} \\ &= \int_{-1}^2 (x + 2 - x^2) dx = \left( \frac{x^2}{2} + 2x - \frac{x^3}{3} \right) \Big|_{-1}^2 \\ &= \left( 2 + 4 - \frac{8}{3} \right) - \left( \frac{1}{2} - 2 + \frac{1}{3} \right) \\ &= \frac{10}{3} + \frac{7}{6} = \frac{27}{6} \\ &= \frac{9}{2} \end{aligned}$$

**EXAMPLE 4** Area between curves using horizontal elements

Find the area bounded by the curve  $y = x^3 - 3$  and the lines  $x = 2$ ,  $y = -1$ , and  $y = 3$ .

Sketching the curve and lines, we show the area in Fig. 26.11. Horizontal elements are better, since they avoid having to evaluate the area in two parts. Therefore, we have

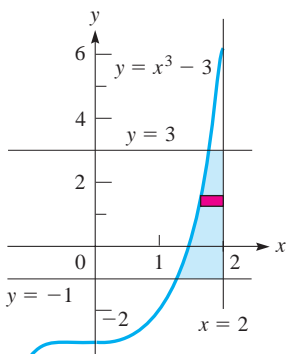


Fig. 26.11

$$\begin{aligned}
 A &= \int_{-1}^3 (x_{\text{line}} - x_{\text{cubic}}) dy = \int_{-1}^3 (2 - \sqrt[3]{y+3}) dy \quad \text{using Eq. (26.10)} \\
 &= 2y - \frac{3}{4}(y+3)^{4/3} \Big|_{-1}^3 = \left[ 6 - \frac{3}{4}(6^{4/3}) \right] - \left[ -2 - \frac{3}{4}(2^{4/3}) \right] \\
 &= 8 - \frac{9}{2}\sqrt[3]{6} + \frac{3}{2}\sqrt[3]{2} = 1.71
 \end{aligned}$$

As we see, the choice of horizontal elements leads to limits of  $-1$  and  $3$ . If we had chosen vertical elements, the limits would have been  $\sqrt[3]{2}$  and  $\sqrt[3]{6}$  for the area to the left of  $(\sqrt[3]{6}, 3)$ , and  $\sqrt[3]{6}$  and  $2$  to the right of this point. ■

### COMMON ERROR

It is important to set up the element of area so that its length is positive. A common error is to take the difference incorrectly, resulting in a negative area.

- To ensure positive lengths for vertical elements, **subtract  $y$  of the lower curve from  $y$  of the upper curve.**
- To ensure positive lengths for horizontal elements, **subtract  $x$  of the left curve from  $x$  of the right curve.**

Example 5 illustrates this important point.

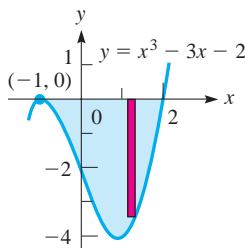


Fig. 26.12

### EXAMPLE 5 Area below the $x$ -axis

Find the area bounded by  $x^3 - 3x - 2$  and the  $x$ -axis.

Sketching the graph, we find that  $y = x^3 - 3x - 2$  has a maximum point at  $(-1, 0)$ , a minimum point at  $(1, -4)$ , and an intercept at  $(2, 0)$ . The curve is shown in Fig. 26.12, and we see that the area is *below* the  $x$ -axis. Using vertical elements, we see that the top is the  $x$ -axis ( $y = 0$ ) and the bottom is the curve  $y = x^3 - 3x - 2$ . Therefore, we have

$$\begin{aligned}
 A &= \int_{-1}^2 [0 - (x^3 - 3x - 2)] dx = \int_{-1}^2 (-x^3 + 3x + 2) dx \\
 &= -\frac{1}{4}x^4 + \frac{3}{2}x^2 + 2x \Big|_{-1}^2 \\
 &= \left[ -\frac{1}{4}(2^4) + \frac{3}{2}(2^2) + 2(2) \right] - \left[ -\frac{1}{4}(-1)^4 + \frac{3}{2}(-1)^2 + 2(-1) \right] \\
 &= \frac{27}{4} = 6.75
 \end{aligned}$$

If we had simply set up the area as  $A = \int_{-1}^2 (x^3 - 3x - 2) dx$ , we would have found  $A = -6.75$ . The negative sign shows that the area is below the  $x$ -axis. Again, we avoid any complications with negative areas by making the length of the element positive.

Also, note that since

$$0 - (x^3 - 3x - 2) = -(x^3 - 3x - 2)$$

an area bounded on top by the  $x$ -axis can be found by setting up the area as being “under” the curve and using the negative of the function. ■

### Practice Exercise

- Find the area bounded by  $y = x^2 - 4$  and the  $x$ -axis.

### LEARNING TIP

We must be very careful if the bounding curves of an area cross. In such a case, for part of the area one curve is above the area, and for a different part of the area this same curve is below the area. When this happens, two integrals must be used to find the area. Example 6 illustrates the necessity of using this procedure.

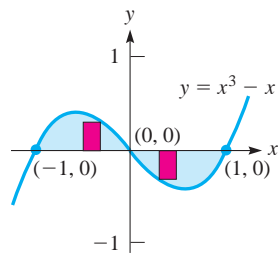


Fig. 26.13

**EXAMPLE 6** Area above and below the  $x$ -axis

Find the area between  $y = x^3 - x$  and the  $x$ -axis.

We note from Fig. 26.13 that the area to the left of the origin is above the axis and the area to the right is below. If we find the area from

$$\begin{aligned} A &= \int_{-1}^1 (x^3 - x) dx = \left. \frac{x^4}{4} - \frac{x^2}{2} \right|_{-1}^1 \\ &= \left( \frac{1}{4} - \frac{1}{2} \right) - \left( \frac{1}{4} - \frac{1}{2} \right) = 0 \end{aligned}$$

we see that the apparent area is zero. From the figure, we know this is not correct. Noting that the  $y$ -values (of the area) are negative to the right of the origin, we set up the integrals

$$\begin{aligned} A &= \int_{-1}^0 (x^3 - x) dx + \int_0^1 [0 - (x^3 - x)] dx \\ &= \left( \frac{x^4}{4} - \frac{x^2}{2} \right) \Big|_{-1}^0 - \left( \frac{x^4}{4} - \frac{x^2}{2} \right) \Big|_0^1 \\ &= 0 - \left( \frac{1}{4} - \frac{1}{2} \right) - \left( \frac{1}{4} - \frac{1}{2} \right) + 0 = \frac{1}{2} \end{aligned}$$

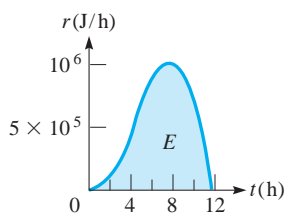


Fig. 26.14

**EXAMPLE 7** Area under a curve—application

Measurements of solar radiation on a particular surface indicate that the rate  $r$  (in J/h) at which solar energy is received during the day is given by  $r = 3600(12t^2 - t^3)$ , where  $t$  is the time (in h). Since  $r$  is a rate, we write  $r = dE/dt$ , where  $E$  is the energy (in J) received at the surface. Thus,  $dE = 3600(12t^2 - t^3) dt$ , and we find the total energy by evaluating the integral.

$$E = 3600 \int_0^{12} (12t^2 - t^3) dt$$

This integral can be interpreted as being the area under  $f(t) = 3600(12t^2 - t^3)$  from  $t = 0$  to  $t = 12$ , as shown in Fig. 26.14 (the limits of integration are found from the  $t$ -intercepts of the curve). Evaluating this integral, we have

$$\begin{aligned} E &= 3600 \int_0^{12} (12t^2 - t^3) dt = 3600 \left( 4t^3 - \frac{1}{4}t^4 \right) \Big|_0^{12} \\ &= 3600 \left[ 4(12^3) - \frac{1}{4}(12^4) - 0 \right] \\ &= 6.22 \times 10^6 \text{ J} \end{aligned}$$

Therefore, 6.22 MJ of energy were received in 12 h.

**EXERCISES 26.2**

In Exercises 1 and 2, make the given changes in the indicated examples of this section and then find the resulting areas.

- In Example 1, change  $x = 2$  to  $x = 3$ .
- In Example 3, change  $y = x + 2$  to  $y = 2x$ .

In Exercises 3–30, find the areas bounded by the indicated curves.

- $y = 4x, y = 0, x = 1$
- $y = 3x^2, y = 0, x = 3$
- $y = 6 - 4x, x = 0, y = 0, y = 3$
- $y = \frac{1}{2}x^2 + 2, x = 0, y = 4$  ( $x > 0$ )
- $y = x^2 - 4, y = 0, x = -4$  ( $y > 0$ )
- $y = x^2 - 5x, y = 0$
- $y = x^{-2}, y = 0, x = 2, x = 3$
- $y = 16 - x^2, y = 0, x = -2, x = 3$

11.  $y = \sqrt{x}, x = 0, y = 1, y = 3$
12.  $y = 3\sqrt{x+1}, x = 0, y = 6$
13.  $y = 2/\sqrt{x}, x = 0, y = 1, y = 4$
14.  $x = y^2 - y, x = 0$
15.  $y = 6 - 3x, x = 0, y = 0, y = 3$
16.  $y = x, y = 3 - x, x = 0$
17.  $y = x - 2\sqrt{x}, y = 0$
18.  $y = x^4 - 2x^3, y = 0$
19.  $y = x^2, y = 2 - x, x = 0 \quad (x \geq 0)$
20.  $y = x^2, y = 2 - x, y = 1$
21.  $y = x^4 - 8x^2 + 16, y = 16 - x^4$
22.  $y = \sqrt{x-1}, y = 3 - x, y = 0$
23.  $y = x^2 + 5x, y = 3 - x^2$
24.  $y = x^3, y = x^2 + 4, x = -1$
25.  $y = x^5, x = -1, x = 2, y = 0$
26.  $y = x^2 + 2x - 8, y = x + 4$
27.  $y = x^2, y = x + 6, y = -x + 2$
28.  $y = x^2, y = x^{1/3}, \text{ between } x = -1 \text{ and } x = 1$
29.  $y = 4 - x^2, y = 4x - x^2, x = 0, x = 2$
30.  $y = x - 1, y^2 = 2x + 6$

In Exercises 31–38, solve the given problems.

31. Describe a region for which the area is found by evaluating the integral  $\int_1^2 (2x^2 - x^3) dx$ .
32. Although the integral  $\int_{-2}^2 \sqrt{4 - x^2} dx$  cannot be integrated by methods we have developed to this point, by recognizing the region represented, it can be evaluated. Evaluate this integral.
33. Use integration to find the area of the triangle with vertices  $(0, 0)$ ,  $(4, 4)$ , and  $(10, 0)$ .
34. Show that the area bounded by the parabola  $y = x^2$  and the line  $y = b$  ( $b > 0$ ) is two-thirds of the area of the rectangle that circumscribes it.
35. Show that the curve  $y = x^n$  ( $n > 0$ ) divides the unit square bounded by  $x = 0, y = 0, x = 1$ , and  $y = 1$  into regions with areas in the ratio of  $n/1$ .
36. Why can the integral  $\int_a^2 (2 + x - x^2) dx$  be used to find the area bounded by  $x = a, y = 0$ , and  $y = 2 + x - x^2$  if  $a = -1$ , but not if  $a = -2$ ?
37. Find the value of  $c$  such that the region bounded by  $y = x^2$  and  $y = 4$  is divided by  $y = c$  into two regions of equal area.
38. Find the value(s) of  $c$  such that the region bounded by  $y = x^2 - c^2$  and  $y = -x^2 + c^2$  has an area of 576.

In Exercises 39–42, find the areas bounded by the indicated curves, using (a) vertical elements and (b) horizontal elements.

39.  $y = 8x, x = 0, y = 4$
40.  $y = x^3, x = 0, y = 3$
41.  $y = x^4, y = 8x$
42.  $y = 4x, y = x^3$

In Exercises 43–50, some applications of areas are shown.

43. Certain physical quantities are often represented as an area under a curve. By definition, power is the time rate of change of performing work. Thus,  $P = dw/dt$ , or  $dw = P dt$ . Therefore, if  $P = 12t - 4t^2$ , find the work (in J) performed in 3 s by finding the area under the curve of  $P$  vs.  $t$ . See Fig. 26.15.

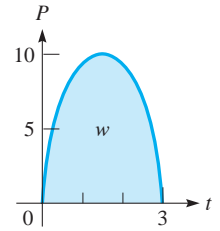


Fig. 26.15

44. The total electric charge  $Q$  (in C) to pass a point in the circuit from time  $t_1$  to  $t_2$  is  $Q = \int_{t_1}^{t_2} i dt$ , where  $i$  is the current (in A). Find  $Q$  if  $t_1 = 1$  s,  $t_2 = 4$  s, and  $i = 0.0032t\sqrt{t^2 + 1}$ .
45. Since the displacement  $s$ , velocity  $v$ , and time  $t$  of a moving object are related by  $s = \int v dt$ , it is possible to represent the change in displacement as an area. A rocket is launched such that its vertical velocity  $v$  (in km/s) as a function of time  $t$  (in s) is  $v = 1 - 0.01\sqrt{2t + 1}$ . Find the change in vertical displacement from  $t = 10$  s to  $t = 100$  s.
46. The total cost  $C$  (in dollars) of production can be interpreted as an area. If the cost per unit  $C'$  (in dollars per unit) of producing  $x$  units is given by  $100/(0.01x + 1)^2$ , find the total cost of producing 100 units by finding the area under the curve of  $C'$  vs.  $x$ .
47. A cam is designed such that one face of it is described as being the area between the curves  $y = x^3 - 2x^2 - x + 2$  and  $y = x^2 - 1$  (units in cm). Show that this description does not uniquely describe the face of the cam. Find the area of the face of the cam, if a complete description requires that  $x \leq 1$ .
48. Using CAD (computer-assisted design), an architect programs a computer to sketch the shape of a swimming pool designed between the curves  $y = \frac{800x}{(x^2 + 10)^2}$ ,  $y = 0.5x^2 - 4x$ ,  $x = 8$  (dimensions in m). Find the area of the surface of the pool.
49. A coffee-table top is designed to be the region between  $y = 0.25x^4$  and  $y = 12 - 0.25x^4$ . What is the area (in  $\text{dm}^2$ ) of the table top?
50. A window is designed to be the area between a parabolic section and a straight base, as shown in Fig. 26.16. What is the area of the window?

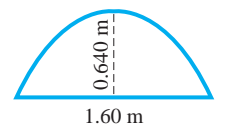


Fig. 26.16

### Answers to Practice Exercises

1. 16/3
2. 32/3



## 26.3 Volumes by Integration

Summing Elements of Volume •  
Disc • Cylindrical Shell

Consider a region in the  $xy$ -plane and its representative element of area, as shown in Fig. 26.17(a). When the region is revolved about the  $x$ -axis, it is said to generate a **solid of revolution**, which is also shown in the figure. We now show methods of finding volumes of solids that are generated in this way.

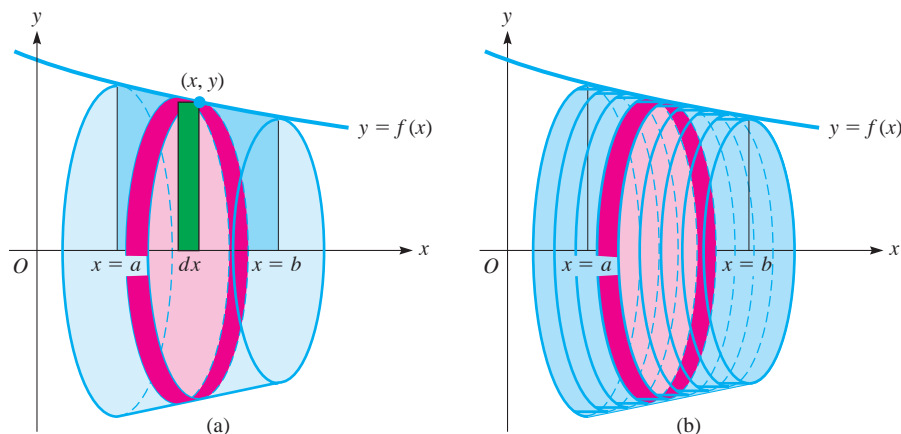


Fig. 26.17

As the region revolves about the  $x$ -axis, so does its representative element, which generates a solid for which the volume is known—an infinitesimally thin cylindrical **disc**. The volume of a right circular cylinder is  $\pi$  times its radius squared times its height (in this case, the thickness) of the cylinder. Since the element is revolved about the  $x$ -axis, the  $y$ -coordinate of the point on the curve that touches the element is the radius. Also, the thickness is  $dx$ . This disc, the representative **element of volume**, has a volume of  $dV = \pi y^2 dx$ . Summing these elements of volume from left to right, as shown in Fig. 26.17(b), we have for the total volume

■ This is the sum of the volumes of the discs whose thicknesses approach zero as the number of discs approaches infinity.

$$V = \pi \int_a^b y^2 dx = \pi \int_a^b [f(x)]^2 dx \quad (26.11)$$

The element of volume is a **disc**, and by use of Eq. (26.11), we can find the volume of the solid generated by a region bounded by the  $x$ -axis, which is revolved about the  $x$ -axis.

### EXAMPLE 1 Find a volume using vertical discs

Find the volume of the solid generated by revolving the region bounded by  $y = x^2$ ,  $x = 2$ , and  $y = 0$  about the  $x$ -axis. See Fig. 26.18.

From the figure, we see that the radius of the disc is  $y$  and its thickness is  $dx$ . The elements are summed from left ( $x = 0$ ) to right ( $x = 2$ ):

$$\begin{aligned} V &= \pi \int_0^2 y^2 dx && \text{using Eq. (26.11)} \\ &= \pi \int_0^2 (x^2)^2 dx = \pi \int_0^2 x^4 dx && \text{substitute } x^2 \text{ for } y \\ &= \frac{\pi}{5} x^5 \Big|_0^2 = \frac{32\pi}{5} && \text{integrate and evaluate} \end{aligned}$$

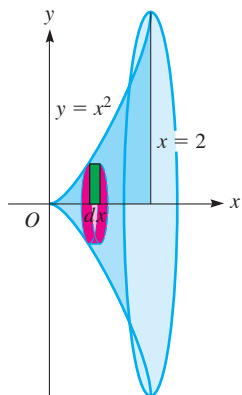


Fig. 26.18

Since  $\pi$  is used in Eq. (26.11), it is common to leave results in terms of  $\pi$ . In applied problems, a decimal result would normally be given. ■

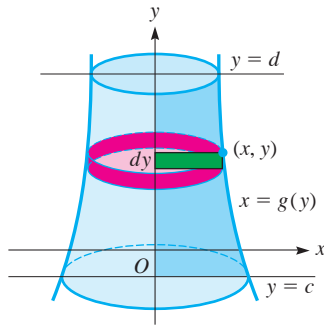


Fig. 26.19

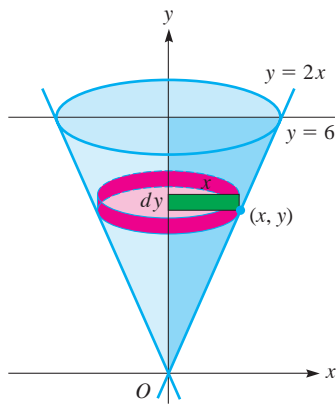


Fig. 26.20

If a region bounded by the y-axis is revolved about the y-axis, the volume of the solid generated is given by

$$V = \pi \int_c^d x^2 dy \tag{26.12}$$

In this case, the radius of the element of volume, a **disc**, is the x-coordinate of the point on the curve, and the thickness of the disc is  $dy$ , as shown in Fig. 26.19. One should always be careful to identify the radius and the thickness properly.

**EXAMPLE 2 Find a volume using horizontal discs**

Find the volume of the solid generated by revolving the region bounded by  $y = 2x$ ,  $y = 6$ , and  $x = 0$  about the y-axis.

Fig. 26.20 shows the volume to be found. Note that the radius of the disc is  $x$  and its thickness is  $dy$ .

$$\begin{aligned} V &= \pi \int_0^6 x^2 dy && \text{using Eq. (26.12)} \\ &= \pi \int_0^6 \left(\frac{y}{2}\right)^2 dy = \frac{\pi}{4} \int_0^6 y^2 dy && \text{substituting } \frac{y}{2} \text{ for } x \\ &= \frac{\pi}{12} y^3 \Big|_0^6 = 18\pi && \text{integrate and evaluate} \end{aligned}$$

Since this volume is a right circular cone, it is possible to check the result:

$$V = \frac{1}{3} \pi r^2 h = \frac{1}{3} \pi (3^2) (6) = 18\pi \quad \blacksquare$$

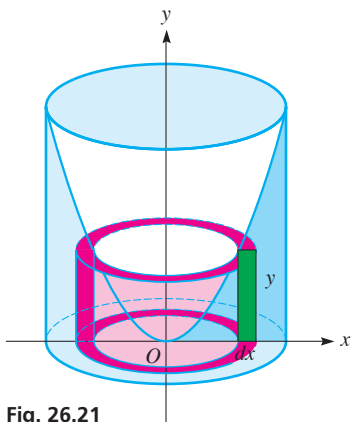


Fig. 26.21

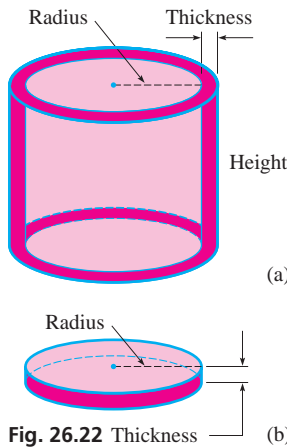


Fig. 26.22

If the region in Fig. 26.21 is revolved about the y-axis, the element of area  $y dx$  does not generate a disc, as when it was revolved about the x-axis. In Fig. 26.22, we see that the element of volume is a region bounded by two concentric cylinders of height  $y$ , or a **cylindrical shell** (note the hole through its centre). *The total volume is made up of an infinite number of concentric shells.* When the volumes of these shells are summed, we have the total volume generated. Thus, we must now find the approximate volume  $dV$  of the representative shell. By finding the circumference of the base and multiplying this by the height, we obtain an expression for the surface area of the shell. Then, by multiplying this by the thickness of the shell, we find its volume. The volume of the representative **shell** shown in Fig. 26.22(a) is

**Shell** 
$$dV = 2\pi (\text{radius}) \times (\text{height}) \times (\text{thickness}) \tag{26.13}$$

Similarly, when the element of volume is a disc (see Fig. 26.22(b)), its volume is given by

**Disc** 
$$dV = \pi (\text{radius})^2 \times (\text{thickness}) \tag{26.14}$$

## LEARNING TIP

It is generally better to remember the formulas for the elements of volume in the general forms given in Eqs. (26.13) and (26.14), and not in the specific forms such as Eqs. (26.11) and (26.12) (both of these use *discs*). If we remember the formulas in this way, we can readily apply these methods to finding the volume of any solid of revolution.

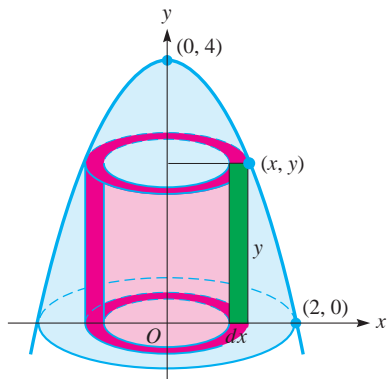


Fig. 26.23

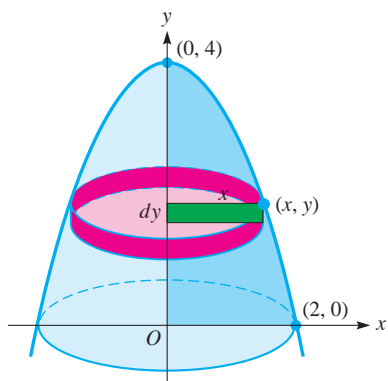


Fig. 26.24

## EXAMPLE 3 Find a volume using cylindrical shells

Use the method of cylindrical shells to find the volume of the solid generated by revolving the first-quadrant region bounded by  $y = 4 - x^2$ ,  $x = 0$ , and  $y = 0$  about the  $y$ -axis.

From Fig. 26.23, we identify the radius, the height, and the thickness of the shell:

$$\text{radius} = x \quad \text{height} = y \quad \text{thickness} = dx$$

*The fact that the elements of area that generate the shells go from  $x = 0$  to  $x = 2$  determines the limits of integration as 0 and 2.* Therefore,

$$\begin{aligned} V &= 2\pi \int_0^2 xy \, dx && \leftarrow \text{thickness} && \text{using Eq. (26.13)} \\ & \text{radius} \uparrow \quad \uparrow \text{height} \\ &= 2\pi \int_0^2 x(4 - x^2) \, dx = 2\pi \int_0^2 (4x - x^3) \, dx && \text{substitute } 4 - x^2 \text{ for } y \\ &= 2\pi \left( 2x^2 - \frac{1}{4}x^4 \right) \Big|_0^2 && \text{integrate} \\ &= 8\pi && \text{evaluate} \end{aligned}$$

We can find the volume shown in Example 3 by using discs, as we show in the following example.

## EXAMPLE 4 Same volume using horizontal discs

Use the method of discs to find the volume indicated in Example 3.

From Fig. 26.24, we identify the radius and the thickness of the disc:

$$\text{radius} = x \quad \text{thickness} = dy$$

*Since the elements of area that generate the discs go from  $y = 0$  to  $y = 4$ , the limits of integration are 0 and 4.* Thus,

$$\begin{aligned} V &= \pi \int_0^4 x^2 \, dy && \text{using Eq. (26.14)} \\ & \text{radius} \uparrow \quad \uparrow \text{thickness} \\ &= \pi \int_0^4 (4 - y) \, dy && \text{substitute } \sqrt{4 - y} \text{ for } x \\ &= \pi \left( 4y - \frac{1}{2}y^2 \right) \Big|_0^4 && \text{integrate} \\ &= 8\pi && \text{evaluate} \end{aligned}$$

We see that the volume of  $8\pi$  using discs agrees with the result we obtained using shells in Example 3.

## Practice Exercise

1. Find the volume of the solid generated by revolving the first-quadrant region bounded by  $y = 4 - 2x$  about the  $x$ -axis. Use discs.

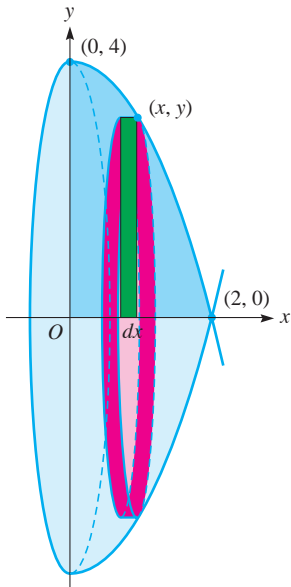


Fig. 26.25

**EXAMPLE 5 Find a volume using vertical discs**

Using discs, find the volume of the solid generated by revolving the first-quadrant region bounded by  $y = 4 - x^2$ ,  $x = 0$ , and  $y = 0$  about the  $x$ -axis. (This is the same region as used in Examples 3 and 4.)

For the disc in Fig. 26.25, we have

$$\text{radius} = y \quad \text{thickness} = dx$$

and the limits of integration are  $x = 0$  and  $x = 2$ . This gives us

$$\begin{aligned} V &= \pi \int_0^2 y^2 dx && \text{using Eq. (26.14)} \\ &= \pi \int_0^2 (4 - x^2)^2 dx && \text{substitute } 4 - x^2 \text{ for } y \\ &= \pi \int_0^2 (16 - 8x^2 + x^4) dx \\ &= \pi \left( 16x - \frac{8}{3}x^3 + \frac{1}{5}x^5 \right) \Big|_0^2 && \text{integrate} \\ &= \frac{256\pi}{15} && \text{evaluate} \end{aligned}$$

We now show how to set up the integral to find the volume of the solid shown in Example 5 by using cylindrical shells. As it turns out, we are not able at this point to integrate the expression that arises, but we are still able to set up the proper integral.

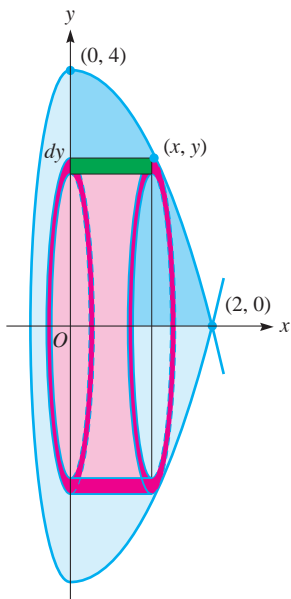


Fig. 26.26

**EXAMPLE 6 Same volume using cylindrical shells**

Use the method of cylindrical shells to find the volume indicated in Example 5.

From Fig. 26.26, we see that for the shell we have

$$\text{radius} = y \quad \text{height} = x \quad \text{thickness} = dy$$

Since the elements go from  $y = 0$  to  $y = 4$ , the limits of integration are 0 and 4. Hence,

$$\begin{aligned} V &= 2\pi \int_0^4 xy dy && \text{using Eq. (26.13)} \\ &= 2\pi \int_0^4 \sqrt{4-y}(y dy) && \text{substitute } \sqrt{4-y} \text{ for } x \\ &= \frac{256\pi}{15} \end{aligned}$$

The method of performing the integration  $\int \sqrt{4-y}(y dy)$  will be discussed in Section 28.7. We present the answer here for the reader's information to show that the volume found in this example is the same as that found in Example 5. ■

In the next example, we show how to find the volume of the solid generated if a region is revolved about a line other than one of the axes. We will see that a proper choice of the radius, height, and thickness for Eq. (26.13) leads to the result.

**Practice Exercise**

- Find the volume of the solid generated by revolving the first-quadrant region bounded by  $y = 4 - 2x$  about the  $x$ -axis. Use shells.

**EXAMPLE 7** Rotate a region about a line

Find the volume of the solid generated if the region in Examples 3–6 is revolved about the line  $x = 2$ .

Shells are convenient, since the volume of a shell can be expressed as a single integral. We can find the radius, height, and thickness of the shell from Fig. 26.27. Carefully note that *the radius is not  $x$  but is  $2 - x$* , since the region is revolved about the line  $x = 2$ . This means

$$\text{radius} = 2 - x \quad \text{height} = y \quad \text{thickness} = dx$$

Since the elements that generate the shells go from  $x = 0$  to  $x = 2$ , the limits of integration are 0 and 2. This means we have

$$\begin{aligned} V &= 2\pi \int_0^2 (2 - x)y \, dx \quad \leftarrow \text{thickness} \quad \text{using Eq. (25.13)} \\ &\quad \text{radius} \quad \uparrow \quad \uparrow \quad \text{height} \\ &= 2\pi \int_0^2 (2 - x)(4 - x^2) \, dx \quad \text{substitute } 4 - x^2 \text{ for } y \\ &= 2\pi \int_0^2 (8 - 2x^2 - 4x + x^3) \, dx \\ &= 2\pi \left( 8x - \frac{2}{3}x^3 - 2x^2 + \frac{1}{4}x^4 \right) \Big|_0^2 \quad \text{integrate} \\ &= \frac{40\pi}{3} \quad \text{evaluate} \end{aligned}$$

(If the region had been revolved about the line  $x = 3$ , the only difference in the integral would have been that  $r = 3 - x$ . Everything else, including the limits, would have remained the same.) ■

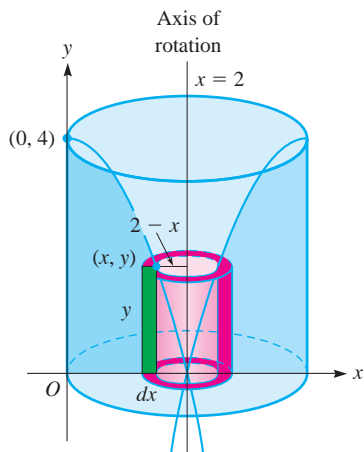


Fig. 26.27

**EXERCISES 26.3**

In Exercises 1 and 2, make the given changes in the indicated examples of this section and then find the indicated volumes.

- In Example 1, change  $y = x^2$  to  $y = x^3$ .
- In Example 3, change  $y = 4 - x^2$  to  $y = 4 - x$ .

In Exercises 3–6, find the volume generated by revolving the region bounded by  $y = 2 - x$ ,  $x = 0$ , and  $y = 0$  about the indicated axis, using the indicated element of volume.

- |                       |                       |
|-----------------------|-----------------------|
| 3. $x$ -axis (discs)  | 4. $y$ -axis (discs)  |
| 5. $y$ -axis (shells) | 6. $x$ -axis (shells) |

In Exercises 7–16, find the volume generated by revolving the regions bounded by the given curves about the  $x$ -axis. Use the indicated method in each case.

- $y = x$ ,  $y = 0$ ,  $x = 3$  (discs)
- $y = \sqrt{x}$ ,  $x = 0$ ,  $y = 2$  (shells)
- $y = 3\sqrt{x}$ ,  $y = 0$ ,  $x = 4$  (discs)
- $y = 4x - x^2$ ,  $y = 0$  (discs)

- $y = x^3$ ,  $y = 8$ ,  $x = 0$  (shells)
- $y = x^2$ ,  $y = x$  (shells)
- $y = x^2 + 1$ ,  $x = 0$ ,  $x = 3$ ,  $y = 0$  (discs)
- $y = 6 - x - x^2$ ,  $x = 0$ ,  $y = 0$  (quadrant I), (discs)
- $x = 4y - y^2 - 3$ ,  $x = 0$  (shells)
- $y = x^4$ ,  $x = 0$ ,  $y = 1$ ,  $y = 8$  (shells)

In Exercises 17–26, find the volume generated by revolving the regions bounded by the given curves about the  $y$ -axis. Use the indicated method in each case.

- $y = x^{1/3}$ ,  $x = 0$ ,  $y = 2$  (discs)
- $y = \sqrt{x^2 - 1}$ ,  $y = 0$ ,  $x = 3$  (shells)
- $y = 2\sqrt{x}$ ,  $x = 0$ ,  $y = 3$  (discs)
- $y^2 = x$ ,  $y = 4$ ,  $x = 0$  (discs)
- $x^2 - 4y^2 = 4$ ,  $x = 3$  (shells)
- $y = 3x^2 - x^3$ ,  $y = 0$  (shells)

23.  $x = 6y - y^2, x = 0$  (discs)  
 24.  $x^2 + 4y^2 = 4$  (quadrant I), (discs)  
 25.  $y = \sqrt{4 - x^2}$  (quadrant I), (shells)  
 26.  $y = 8 - x^3, x = 0, y = 0$  (shells)

In Exercises 27–38, find the indicated volumes by integration.

27. Describe a region that is revolved about the  $x$ -axis to generate a volume found by evaluating the integral  $\pi \int_1^2 x^3 dx$ .
28. Describe a region that is revolved about the  $y$ -axis to generate a volume found by the integral in Exercise 27.
29. Find the volume generated if the region of Exercise 10 is revolved about the line  $x = 4$ .
30. Find the volume generated if the region bounded by  $y = \sqrt{x}$  and  $y = x/2$  is revolved about the line  $y = 4$ .
31. Derive the formula for the volume of a right circular cone of radius  $r$  and height  $h$  by revolving the area bounded by  $y = (r/h)x, y = 0$ , and  $x = h$  about the  $x$ -axis.
32. Explain how to derive the formula for the volume of a sphere by using the disc method.
33. The base of a solid is the region bounded by  $y = x^2, y = 0$ , and  $x = 1$ . All vertical cross sections are *squares* with an edge in the  $xy$ -plane. Find the volume of the solid. (Note that this is not a solid generated by revolving a region about an axis.)
34. The oil in a spherical tank 20.0 m in diameter is 7.5 m deep. How much oil is in the tank?
35. A *drumlin* is an oval hill composed of relatively soft soil that was deposited beneath glacial ice. (The Halifax Citadel National Historic Site in Halifax, Nova Scotia, is built on a drumlin.) Computer analysis showed that the surface of a certain drumlin can be approximated by  $y = 10(1 - 0.0001x^2)$  revolved  $180^\circ$

about the  $x$ -axis from  $x = -100$  to  $x = 100$  (see Fig. 26.28). Find the volume (in  $\text{m}^3$ ) of this drumlin.

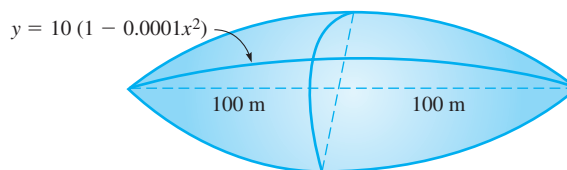


Fig. 26.28

36. A commercial dirigible used for outdoor advertising has a helium-filled balloon in the shape of an ellipse revolved about its major axis. If the balloon is 41.3 m long and 12.0 m in diameter, what volume of helium is required to fill it? See Fig. 26.29.

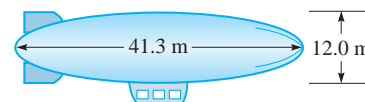


Fig. 26.29

37. A hole 2.00 cm in diameter is drilled through the centre of a spherical lead weight 6.00 cm in diameter. How much lead is removed? See Fig. 26.30.

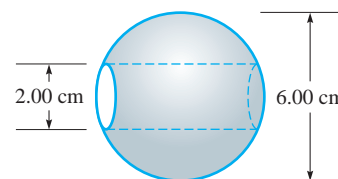


Fig. 26.30

38. All horizontal cross sections of a keg 1.20 m tall are circular, and the sides of the keg are parabolic. The diameter at the top and bottom is 0.80 m, and the diameter in the middle is 1.0 m. Find the volume that the keg holds.

### Answers to Practice Exercises

1.  $32\pi/3$     2.  $32\pi/3$

## 26.4 Centroids

Centre of Mass • Centroid •  
 Centroid of a Thin Flat Plate •  
 Centroid of a Solid of Revolution

In the study of mechanics, a very important property of an object is its centre of mass. In this section, we explain the meaning of centre of mass and then show how integration is used to determine the centre of mass for regions and solids of revolution.

If a mass  $m$  is at a distance  $d$  from a specified point  $O$ , the **moment** of the mass about  $O$  is defined as  $md$ . If several masses  $m_1, m_2, \dots, m_n$  are at distances  $d_1, d_2, \dots, d_n$ , respectively, from point  $O$ , the total moment (as a group) about  $O$  is defined as  $m_1d_1 + m_2d_2 + \dots + m_nd_n$ . The **centre of mass** is that point  $\bar{d}$  units from  $O$  at which all the masses could be concentrated to get the same total moment. Therefore,  $\bar{d}$  is defined by the equation

$$m_1d_1 + m_2d_2 + \dots + m_nd_n = (m_1 + m_2 + \dots + m_n)\bar{d} \quad (26.15)$$

The moment of a mass is a measure of its tendency to rotate about a point. A weight far from the point of balance of a long rod is more likely to make the rod turn than if the same weight were placed near the point of balance. It is easier to open a door if you push near the doorknob than if you push near the hinges. This is the type of physical property that the moment of mass measures.

### LEARNING TIP

When finding moments, we must always use **directed distances**.

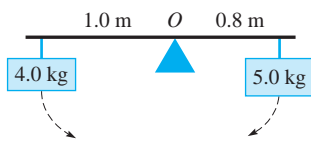


Fig. 26.31

**EXAMPLE 1** Balancing masses

One of the simplest and most basic illustrations of moments and centre of mass is seen in balancing a long rod (of negligible mass) with masses of different sizes, one on either side of the balance point.

In Fig. 26.31, a mass of 5.0 kg is hung from the rod 0.8 m to the right of point  $O$ . We see that this 5.0-kg mass tends to turn the rod clockwise. A mass placed on the opposite side of  $O$  will tend to turn the rod counterclockwise. Neglecting the mass of the rod, in order to balance the rod at  $O$ , the moments must be equal in magnitude but opposite in sign.

For instance, a 4-kg mass would have to be placed at a distance  $d_2$ , satisfying

$$5.0(0.8) + 4.0(d_2) = (5.0 + 4.0)(0) \quad \text{Eq. [26.15] with } \bar{d} = 0 \\ d_2 = -1.0$$

In other words, the centre of mass of the combination of the 5.0-kg mass and the 4.0-kg mass is at  $O$  (with  $\bar{d} = 0$ ) if the 4.0-kg mass is placed 1.0 m to the left of  $O$ . ■

**EXAMPLE 2** Centre of mass of three masses

A mass of 3.0 g is placed at  $(2.0, 0)$  on the  $x$ -axis (distances in cm). Another mass of 6.0 g is placed at  $(5.0, 0)$ , and a third mass of 7.0 g is placed at  $(6.0, 0)$ . See Fig. 26.32. Find the centre of mass of these three masses.

Taking the reference point as the origin, we find  $d_1 = 2.0$  cm,  $d_2 = 5.0$  cm, and  $d_3 = 6.0$  cm. Thus,  $m_1d_1 + m_2d_2 + m_3d_3 = (m_1 + m_2 + m_3)\bar{d}$  becomes

$$3.0(2.0) + 6.0(5.0) + 7.0(6.0) = (3.0 + 6.0 + 7.0)\bar{d} \quad \text{or } \bar{d} = 4.9 \text{ cm}$$

This means that the centre of mass of the three masses is at  $(4.9, 0)$ . Therefore, a mass of 16.0 g placed at this point has the same moment as the three masses as a unit. ■

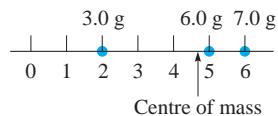


Fig. 26.32

**EXAMPLE 3** Centre of mass of a metal plate

Find the centre of mass of the flat metal plate that is shown in Fig. 26.33.

We first note that the centre of mass is not *on* either axis. This can be seen from the fact that the major portion of the area is in the first quadrant. *We will therefore measure the moments with respect to each axis to find the point that is the centre of mass. This point is also called the centroid of the plate.*

The easiest method of finding this centroid is to divide the plate into rectangles, as indicated by the dashed line in Fig. 26.33, and assume that we may consider the mass of each rectangle to be concentrated at its centre. In this way, the centre of the left rectangle is at  $(-1.0, 1.0)$  (distances in cm), and the centre of the right rectangle is at  $(2.5, 2.0)$ . The mass of each rectangle area, assumed uniform, is proportional to its area. The area of the left rectangle is  $8.0 \text{ cm}^2$ , and that of the right rectangle is  $12.0 \text{ cm}^2$ . Thus, taking moments with respect to the  $y$ -axis, we have

$$8.0(-1.0) + 12.0(2.5) = (8.0 + 12.0)\bar{x}$$

where  $\bar{x}$  is the  $x$ -coordinate of the centroid. Solving for  $\bar{x}$ , we have  $\bar{x} = 1.1$  cm.

Now, taking moments with respect to the  $x$ -axis, we have

$$8.0(1.0) + 12.0(2.0) = (8.0 + 12.0)\bar{y}$$

where  $\bar{y}$  is the  $y$ -coordinate of the centroid. Thus,  $\bar{y} = 1.6$  cm. This means that the coordinates of the centroid, the centre of mass, are  $(1.1, 1.6)$ . This may be interpreted as meaning that a plate of this shape would balance on a single support under this point. As an approximate check, we note from the figure that this point appears to be a reasonable balance point for the plate. ■

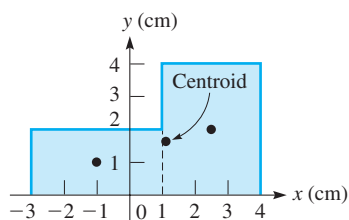


Fig. 26.33

■ Since the centre of mass does not depend on the density of the metal, we have assumed the constant of proportionality to be 1.



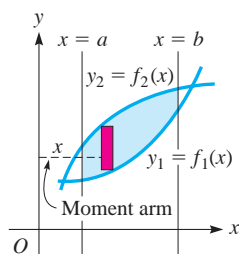


Fig. 26.34

### CENTROID OF A THIN, FLAT PLATE BY INTEGRATION

If a thin, flat plate covers the region bounded by  $y_1 = f_1(x)$ ,  $y_2 = f_2(x)$ ,  $x = a$ , and  $x = b$ , as shown in Fig. 26.34, the moment of the mass of the element of area about the  $y$ -axis is given by  $(\rho dA)x$ , where  $\rho$  is the mass per unit area. In this expression,  $\rho dA$  is the mass of the element, and  $x$  is its distance (moment arm) from the  $y$ -axis. The element  $dA$  may be written as  $(y_2 - y_1) dx$ , which means that the moment may be written as  $\rho x(y_2 - y_1) dx$ . If we then sum up the moments of all the elements and express this as an integral (which, of course, means sum), we have  $\rho \int_a^b x(y_2 - y_1) dx$ . If we consider all the mass of the plate to be concentrated at one point  $\bar{x}$  units from the  $y$ -axis, the moment would be  $(\rho A)\bar{x}$ , where  $\rho A$  is the mass of the entire plate and  $\bar{x}$  is the distance the centre of mass is from the  $y$ -axis. By the previous discussion, these two expressions should be equal. This means  $\rho \int_a^b x(y_2 - y_1) dx = \rho A\bar{x}$ . Since  $\rho$  appears on each side of the equation, we divide it out (we are assuming that the mass per unit area is constant). The area  $A$  is found by the integral  $\int_a^b (y_2 - y_1) dx$ . Therefore, the  $x$ -coordinate of the centroid of the plate is given by

$$\bar{x} = \frac{\int_a^b x(y_2 - y_1) dx}{\int_a^b (y_2 - y_1) dx} \quad (26.16)$$

Eq. (26.16) gives us the  $x$ -coordinate of the centroid of the plate if vertical elements are used.

#### COMMON ERROR

**The two integrals in Eq. (26.16) must be evaluated separately.** We cannot cancel out the apparent common factor  $y_2 - y_1$ , and we cannot combine quantities and perform only one integration. The two integrals must be evaluated separately first. Then any possible cancellations of factors common to the numerator and the denominator may be made.

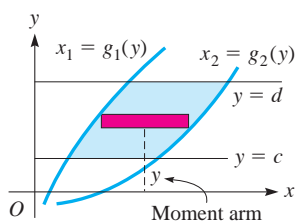


Fig. 26.35

Following the same reasoning that we used in developing Eq. (26.16), for a thin plate covering the region bounded by the functions  $x_1 = g_1(y)$ ,  $x_2 = g_2(y)$ ,  $y = c$ , and  $y = d$ , as shown in Fig. 26.35, the  $y$ -coordinate of the centroid of the plate is given by the equation

$$\bar{y} = \frac{\int_c^d y(x_2 - x_1) dy}{\int_c^d (x_2 - x_1) dy} \quad (26.17)$$

In this equation, horizontal elements are used.

#### LEARNING TIP

In applying Eqs. (26.16) and (26.17), we should keep in mind that each denominator on the right-hand sides gives the area of the plate and that, once we have found this area, we may use it for both  $\bar{x}$  and  $\bar{y}$ . In this way, we can avoid having to set up and perform one of the indicated integrations. Also, in finding the coordinates of the centroid, we should look for and utilize any symmetry the region may have.

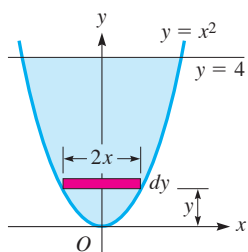


Fig. 26.36

**EXAMPLE 4** Centroid of a thin plate by integration

Find the coordinates of the centroid of a thin plate covering the region bounded by the parabola  $y = x^2$  and the line  $y = 4$ .

We sketch a graph indicating the region and an element of area (see Fig. 26.36). The curve is a parabola whose axis is the  $y$ -axis. Since the region is symmetric to the  $y$ -axis, the centroid must be on this axis. This means that the  $x$ -coordinate of the centroid is zero, or  $\bar{x} = 0$ . To find the  $y$ -coordinate of the centroid, we have

$$\begin{aligned} \bar{y} &= \frac{\int_0^4 y(2x) dy}{\int_0^4 2x dy} && \begin{array}{l} \text{moment arm of element} \\ \text{using Eq. (26.17)} \\ \text{area} \end{array} \\ &= \frac{\int_0^4 y(2\sqrt{y}) dy}{\int_0^4 2\sqrt{y} dy} = \frac{2 \int_0^4 y^{3/2} dy}{2 \int_0^4 y^{1/2} dy} = \frac{2(\frac{2}{5})y^{5/2}|_0^4}{2(\frac{2}{3})y^{3/2}|_0^4} && \text{integrate and evaluate numerator and denominator separately} \\ &= \frac{\frac{4}{5}(32)}{\frac{4}{3}(8)} = \frac{128}{5} \times \frac{3}{32} = \frac{12}{5} \end{aligned}$$

The coordinates of the centroid are  $(0, \frac{12}{5})$ . This plate would balance if a single pointed support were to be put under this point. ■

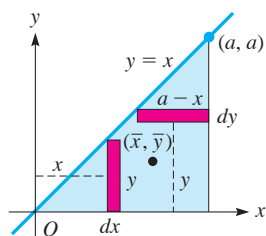


Fig. 26.37

**EXAMPLE 5** Centroid of a triangular plate by integration

Find the coordinates of the centroid of an isosceles right triangular plate with side  $a$ . See Fig. 26.37.

We must first set up the region in the  $xy$ -plane. The choice shown in Fig. 26.37 is to place the triangle with one vertex at the origin and the right angle on the  $x$ -axis. Since each side is  $a$ , the hypotenuse passes through the point  $(a, a)$ . The equation of the hypotenuse is  $y = x$ . The  $x$ -coordinate of the centroid is found by using Eq. (26.16):

$$\bar{x} = \frac{\int_0^a xy dx}{\int_0^a y dx} = \frac{\int_0^a x(x) dx}{\int_0^a x dx} = \frac{\int_0^a x^2 dx}{\frac{1}{2}x^2|_0^a} = \frac{\frac{1}{3}x^3|_0^a}{\frac{a^2}{2}} = \frac{\frac{a^3}{3}}{\frac{a^2}{2}} = \frac{2a}{3}$$

The  $y$ -coordinate of the centroid is found by using Eq. (26.17):

$$\begin{aligned} \bar{y} &= \frac{\int_0^a y(a-x) dy}{\frac{a^2}{2}} = \frac{\int_0^a y(a-y) dy}{\frac{a^2}{2}} = \frac{\int_0^a (ay - y^2) dy}{\frac{a^2}{2}} \\ &= \frac{\frac{ay^2}{2} - \frac{y^3}{3}|_0^a}{\frac{a^2}{2}} = \frac{\frac{a^3}{6}}{\frac{a^2}{2}} = \frac{a}{3} \end{aligned}$$

Thus, the coordinates of the centroid are  $(\frac{2}{3}a, \frac{1}{3}a)$ . The results indicate that the centre of mass is  $\frac{1}{3}a$  units from each of the equal sides. ■

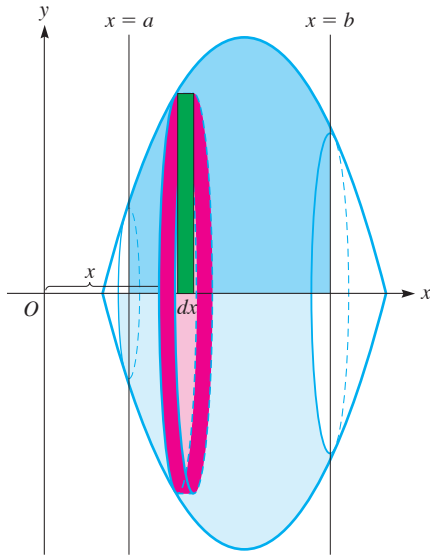


Fig. 26.38

**CENTROID OF A SOLID OF REVOLUTION**

Another figure for which we wish to find the centroid is a solid of revolution. If the density of the solid is constant, the centroid is on the axis of revolution. The problem that remains is to find just where on the axis the centroid is located.

If a region bounded by the  $x$ -axis, as shown in Fig. 26.38, is revolved about the  $x$ -axis, a vertical element of area generates a disc element of volume. The centre of mass of the disc is at its centre, and we may consider its mass concentrated there. The moment about the  $y$ -axis of a typical element is  $x(\rho)(\pi y^2 dx)$ , where  $x$  is the moment arm,  $\rho$  is the density, and  $\pi y^2 dx$  is the volume. The sum of the moments of the elements can be expressed as an integral; it equals the volume times the density times the  $x$ -coordinate of the centroid of the volume. Since  $\pi$  and the density  $\rho$  would appear on each side of the equation, they cancel and need not be written. Therefore,

$$\bar{x} = \frac{\int_a^b xy^2 dx}{\int_a^b y^2 dx} \tag{26.18}$$

is the equation for the  $x$ -coordinate of the centroid of a solid of revolution about the  $x$ -axis.

In the same manner, we may find that the  $y$ -coordinate of the centroid of a solid of revolution about the  $y$ -axis is

$$\bar{y} = \frac{\int_c^d yx^2 dy}{\int_c^d x^2 dy} \tag{26.19}$$

**EXAMPLE 6 Centroid of a solid by integration**

Find the coordinates of the centroid of the volume generated by revolving the first-quadrant region under the curve  $y = 4 - x^2$  about the  $y$ -axis as shown in Fig. 26.39.

Since the curve is rotated about the  $y$ -axis,  $\bar{x} = 0$ . The  $y$ -coordinate is

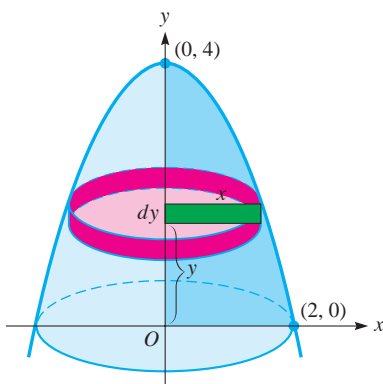


Fig. 26.39

$$\begin{aligned} \bar{y} &= \frac{\int_0^4 yx^2 dy}{\int_0^4 x^2 dy} \quad \begin{array}{l} \text{moment arm} \\ \text{using Eq. (26.19)} \end{array} \\ &= \frac{\int_0^4 y(4 - y) dy}{\int_0^4 (4 - y) dy} = \frac{\int_0^4 (4y - y^2) dy}{\int_0^4 (4 - y) dy} = \frac{2y^2 - \frac{1}{3}y^3 \Big|_0^4}{4y - \frac{1}{2}y^2 \Big|_0^4} \\ &= \frac{32 - \frac{64}{3}}{16 - 8} = \frac{4}{3} \end{aligned}$$

The coordinates of the centroid are  $(0, \frac{4}{3})$ . ■

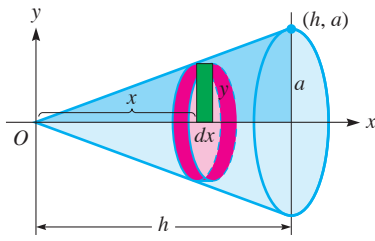


Fig. 26.40

**EXAMPLE 7 Centroid of a right circular cone**

Find the centroid of a solid right circular cone of radius  $r$  and height  $h$ .

To generate a right circular cone, we may revolve a right triangle about one of its legs (Fig. 26.40). Placing a leg of length  $h$  along the  $x$ -axis, we rotate the right triangle whose hypotenuse is given by  $y = (a/h)x$  about the  $x$ -axis. Therefore,

$$\bar{x} = \frac{\int_0^h xy^2 dx}{\int_0^h y^2 dx}$$

moment arm using Eq. (26.18)

$$= \frac{\int_0^h x \left[ \left( \frac{a}{h} \right) x \right]^2 dx}{\int_0^h \left[ \left( \frac{a}{h} \right) x \right]^2 dx} = \frac{\left( \frac{a^2}{h^2} \right) \left( \frac{1}{4} x^4 \right) \Big|_0^h}{\left( \frac{a^2}{h^2} \right) \left( \frac{1}{3} x^3 \right) \Big|_0^h} = \frac{3}{4} h$$

The centroid is located along the height  $\frac{3}{4}$  of the way from the vertex to the base. ■

**EXERCISES 26.4**

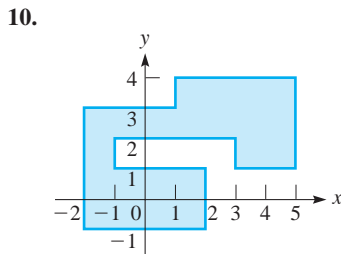
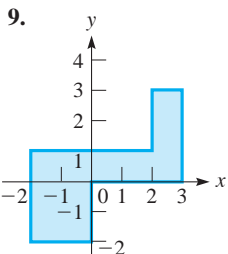
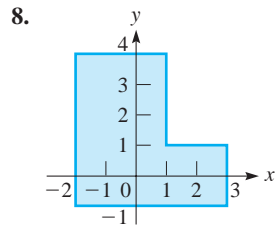
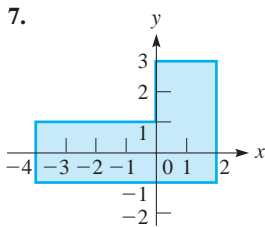
In Exercises 1 and 2, make the given changes in the indicated examples of this section and then find the coordinates of the centroid.

1. In Example 4, change  $y = x^2$  to  $y = |x|$  ( $y = x$  for  $x \geq 0$ , and  $y = -x$  for  $x < 0$ ).
2. In Example 6, change  $y = 4 - x^2$  to  $y = 4 - x$ .

In Exercises 3–6, find the centre of mass (in cm) of the particles with the given masses located at the given points on the  $x$ -axis.

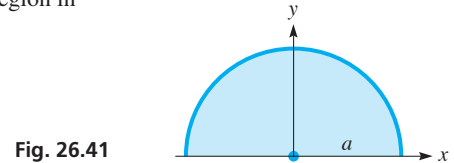
3. 5.0 g at (1.0, 0), 8.5 g at (4.2, 0), 3.6 g at (2.5, 0)
4. 2.3 g at (1.3, 0), 6.5 g at (5.8, 0), 1.2 g at (9.5, 0)
5. 42 g at (-3.5, 0), 24 g at (0, 0), 15 g at (2.6, 0), 84 g at (3.7, 0)
6. 550 g at (-42, 0), 230 g at (-27, 0), 470 g at (16, 0), 120 g at (22, 0)

In Exercises 7–10, find the coordinates (to 0.01 cm) of the centroids of the uniform flat-plate machine parts shown.



In Exercises 11–34, find the coordinates of the centroids of the given figures. In Exercises 11–22, each region is covered by a thin, flat plate.

11. The region bounded by  $y = x^2$  and  $y = 2$
12. The semicircular region in Fig. 26.41



13. The region bounded by  $y = 4 - x$  and the axes
14. The region bounded by  $y = x^3$ ,  $x = 2$ , and the  $x$ -axis
15. The region bounded by  $y = 4x^2$  and  $y = 2x^3$
16. The region bounded by  $y^2 = x$ ,  $y = 2$ , and  $x = 0$
17. The region bounded by  $y = 2x$ ,  $y = 3x$ , and  $y = 6$
18. The region bounded by  $y = x^{2/3}$ ,  $x = 8$ , and  $y = 0$
19. The region bounded by  $y = 4 - 2x$ ,  $x = 2$ , and  $y = 4$
20. The region bounded by  $y = \sqrt{x}$ ,  $y = 0$ , and  $x = 9$
21. The region bounded by  $x^2 = 4py$  and  $y = a$  if  $p > 0$  and  $a > 0$
22. The region above the  $x$ -axis, bounded by the ellipse with vertices  $(a, 0)$  and  $(-a, 0)$ , and minor axis  $2b$  (The area of an ellipse is  $\pi ab$ .)
23. The solid generated by revolving the region bounded by  $y = x^3$ ,  $y = 0$ , and  $x = 1$  about the  $x$ -axis
24. The solid generated by revolving the region bounded by  $y = 2 - 2x$ ,  $x = 0$ , and  $y = 0$  about the  $y$ -axis
25. The solid generated by revolving the region in the first quadrant bounded by  $y^2 = 4x$ ,  $y = 0$ , and  $x = 1$  about the  $y$ -axis
26. The solid generated by revolving the region bounded by  $y = x^2$ ,  $x = 2$ , and the  $x$ -axis about the  $x$ -axis

- 27. The solid generated by revolving the region bounded by  $y^2 = 4x$  and  $x = 1$  about the  $x$ -axis
- 28. The solid generated by revolving the region bounded by  $x^2 - y^2 = 9$ ,  $y = 4$ , and the  $x$ -axis about the  $y$ -axis
- 29. Explain how to find the centroid of a right triangular plate with legs  $a$  and  $b$ . Find the location of the centroid.
- 30. Find the location of the centroid of a hemisphere of radius  $a$ .
- 31. A lens with semielliptical vertical cross sections and circular horizontal cross sections is shown in Fig. 26.42. For proper installation in an optical device, its centroid must be known. Locate its centroid.
- 32. A sanding machine disc can be described as the solid generated by rotating the region bounded by  $y^2 = 4/x$ ,  $y = 1$ ,  $y = 2$ , and the  $y$ -axis about the  $y$ -axis (measurements in cm). Locate the centroid of the disc.
- 33. A highway marking pylon has the shape of a frustum of a cone. Find its centroid if the radii of its bases are 5.00 cm and 20.0 cm and the height between bases is 60.0 cm.
- 34. A floodgate is in the shape of an isosceles trapezoid. Find the location of the centroid of the floodgate if the upper base is 20 m, the lower base is 12 m, and the height between bases is 6.0 m. See Fig. 26.43.

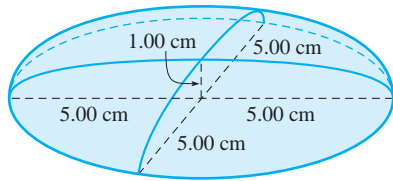


Fig. 26.42

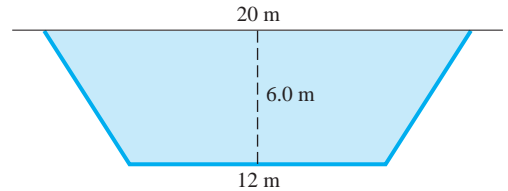


Fig. 26.43

## 26.5 Moments of Inertia

Radius of Gyration • Moment of Inertia of a Thin, Flat Plate • Moment of Inertia of a Solid

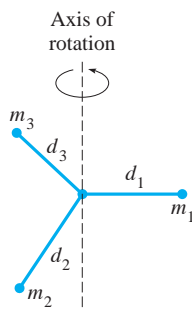


Fig. 26.44

Important in the rotational motion of an object is its **moment of inertia**, which is analogous to the mass of a moving object. In each case, *the moment of inertia or mass is the measure of the tendency of the object to resist a change in motion.*

Suppose that a particle of mass  $m$  is rotating about some point: We define its moment of inertia as  $md^2$ , where  $d$  is the distance from the particle to the point. If a group of particles of masses  $m_1, m_2, \dots, m_n$  are rotating about an axis, as shown in Fig. 26.44, the moment of inertia  $I$  with respect to the axis of the group is

$$I = m_1d_1^2 + m_2d_2^2 + \dots + m_nd_n^2$$

where the  $d$ 's are the respective distances of the particles from the axis. If all the masses were at the same distance  $R$  from the axis of rotation, so that the total moment of inertia were the same, we would have

$$m_1d_1^2 + m_2d_2^2 + \dots + m_nd_n^2 = (m_1 + m_2 + \dots + m_n)R^2 \quad (26.20)$$

**Radius of Gyration** where  $R$  is called the **radius of gyration**.

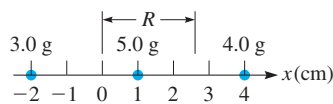


Fig. 26.45

### EXAMPLE 1 Moment of inertia and radius of gyration

Find the moment of inertia and the radius of gyration of the array of three masses, one of 3.0 g at  $(-2.0, 0)$ , another of 5.0 g at  $(1.0, 0)$ , and the third of 4.0 g at  $(4.0, 0)$ , with respect to the origin (distances in cm). See Fig. 26.45.

The moment of inertia of the array is

$$I = 3.0(-2.0)^2 + 5.0(1.0)^2 + 4.0(4.0)^2 = 81 \text{ g} \cdot \text{cm}^2$$

The radius of gyration is found from  $I = (m_1 + m_2 + m_3)R^2$ . Thus,

$$81 = (3.0 + 5.0 + 4.0)R^2, \quad R^2 = \frac{81}{12}, \quad R = 2.6 \text{ cm}$$

Therefore, a mass of 12.0 g placed at  $(2.6, 0)$  (or  $(-2.6, 0)$ ) has the same rotational inertia about the origin as the array of masses as a unit. ■

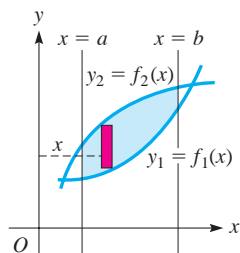


Fig. 26.46

### MOMENT OF INERTIA OF A THIN, FLAT PLATE

If a thin, flat plate covering the region is bounded by the curves of the functions  $y_1 = f_1(x)$ ,  $y_2 = f_2(x)$ , and the lines  $x = a$  and  $x = b$ , as shown in Fig. 26.46, the moment of inertia of this plate with respect to the  $y$ -axis,  $I_y$ , is given by the sum of the moments of inertia of the individual elements. The mass of each element is  $\rho(y_2 - y_1)dx$ , where  $\rho$  is the mass per unit area and  $(y_2 - y_1)dx$  is the area of the element. The distance of the element from the  $y$ -axis is  $x$ . Representing this sum as an integral, we have

$$I_y = \rho \int_a^b x^2 (y_2 - y_1) dx \tag{26.21}$$

To find the radius of gyration of the plate with respect to the  $y$ -axis,  $R_y$ , first find the moment of inertia, divide this by the mass of the plate, and take the square root of this result.

In the same manner, the moment of inertia of a thin plate, with respect to the  $x$ -axis, bounded by  $x_1 = g_1(y)$  and  $x_2 = g_2(y)$  is given by

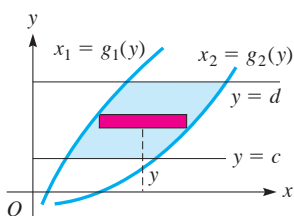


Fig. 26.47

$$I_x = \rho \int_c^d y^2 (x_2 - x_1) dy \tag{26.22}$$

We find the radius of gyration of the plate with respect to the  $x$ -axis,  $R_x$ , in the same manner as we find it with respect to the  $y$ -axis (see Fig. 26.47).

### EXAMPLE 2 Moment of inertia of a plate

Find the moment of inertia and radius of gyration of the plate covering the region bounded by  $y = 4x^2$ ,  $x = 1$ , and the  $x$ -axis with respect to the  $y$ -axis.

We find the moment of inertia of this plate (see Fig. 26.48) as follows:

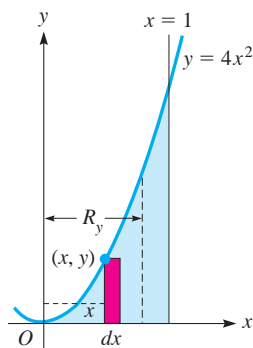


Fig. 26.48

$$\begin{aligned} I_y &= \rho \int_0^1 x^2 y dx && \text{distance from element to axis} \\ &= \rho \int_0^1 x^2 (4x^2) dx && \text{using Eq. (26.21)} \\ &= 4\rho \int_0^1 x^4 dx \\ &= 4\rho \left( \frac{1}{5} x^5 \right) \Big|_0^1 = \frac{4\rho}{5} \end{aligned}$$

To find the radius of gyration, we first determine the mass of the plate:

$$\begin{aligned} m &= \rho \int_0^1 y dx = \rho \int_0^1 (4x^2) dx && m = \rho A \\ &= 4\rho \left( \frac{1}{3} x^3 \right) \Big|_0^1 = \frac{4\rho}{3} \\ R_y^2 &= \frac{I_y}{m} = \frac{4\rho}{5} \times \frac{3}{4\rho} = \frac{3}{5} && R_y^2 = I_y/m \\ R_y &= \sqrt{\frac{3}{5}} = \frac{\sqrt{15}}{5} = 0.775 \end{aligned}$$

Therefore, if all of the mass of the plate were at a distance of 0.775 from the  $y$ -axis, the moment of inertia about the  $y$ -axis is the same as the moment of inertia of the plate itself. ■

**EXAMPLE 3** Moment of inertia of a triangular plate

Find the moment of inertia of a right triangular plate with sides  $a$  and  $b$  with respect to side  $b$ . Assume that  $\rho = 1$ .

Placing the triangle as shown in Fig. 26.49, we see that the equation of the hypotenuse is  $y = (a/b)x$ . The moment of inertia is

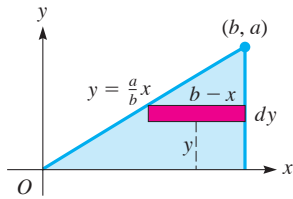


Fig. 26.49

$$\begin{aligned}
 I_x &= \int_0^a y^2 (b-x) dy \quad \text{using Eq. (26.22)} \\
 &= \int_0^a y^2 \left( b - \frac{b}{a}y \right) dy = b \int_0^a \left( y^2 - \frac{1}{a}y^3 \right) dy \\
 &= b \left( \frac{1}{3}y^3 - \frac{1}{4a}y^4 \right) \Big|_0^a = b \left( \frac{a^3}{3} - \frac{a^3}{4} \right) = \frac{ba^3}{12}
 \end{aligned}$$

**MOMENT OF INERTIA OF A SOLID**

In applications, among the most important moments of inertia are those of solids of revolution. Since *all parts of an element of mass should be at the same distance from the axis*, the most convenient element of volume to use is the cylindrical shell. In Fig. 26.50, if the region bounded by the curves  $y_1 = f_1(x)$ ,  $y_2 = f_2(x)$ ,  $x = a$ , and  $x = b$  is revolved about the  $y$ -axis, the moment of inertia of the element of volume is  $\rho [2\pi x (y_2 - y_1) dx] (x^2)$ , where  $\rho$  is the density,  $2\pi x (y_2 - y_1) dx$  is the volume of the element, and  $x^2$  is the square of the distance from the  $y$ -axis. Expressing the sum of the elements as an integral, the moment of inertia of the solid with respect to the  $y$ -axis,  $I_y$ , is

■ Note carefully that Eq. (26.23) gives the moment of inertia with respect to the  $y$ -axis and that  $(y_2 - y_1)$  is the height of the shell (see Fig. 26.21).

$$I_y = 2\pi\rho \int_a^b (y_2 - y_1) x^3 dx \tag{26.23}$$

The radius of gyration  $R_y$  is found by determining (1) the moment of inertia, (2) the mass of the solid, and (3) the square root of the quotient of the moment of inertia divided by the mass.

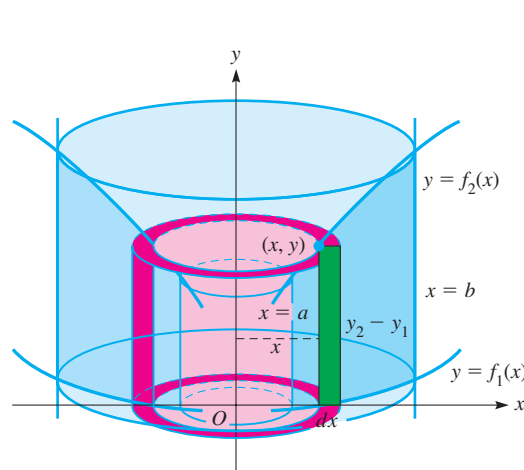


Fig. 26.50

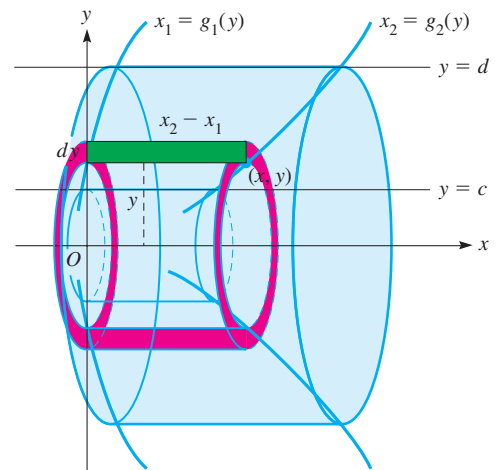


Fig. 26.51

The moment of inertia of the solid (see Fig. 26.51) generated by revolving the region bounded by  $x_1 = g_1(y)$ ,  $x_2 = g_2(y)$ ,  $y = c$ , and  $y = d$  about the  $x$ -axis,  $I_x$ , is given by



■ Note carefully that Eq. (26.24) gives the moment of inertia with respect to the  $x$ -axis and that  $(x_2 - x_1)$  is the height of the shell (see Fig. 26.21).

$$I_x = 2\pi\rho \int_c^d (x_2 - x_1)y^3 dy \tag{26.24}$$

The radius of gyration with respect to the  $x$ -axis,  $R_x$ , is found in the same manner as  $R_y$ .

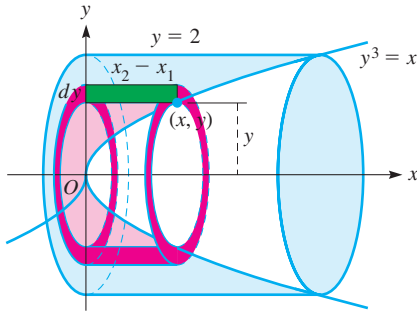


Fig. 26.52

**EXAMPLE 4** Moment of inertia of a solid

Find the moment of inertia and radius of gyration with respect to the  $x$ -axis of the solid generated by revolving the region bounded by the curves of  $y^3 = x$ ,  $y = 2$ , and the  $y$ -axis about the  $x$ -axis. See Fig. 26.52.

$$\begin{aligned}
 I_x &= 2\pi\rho \int_0^2 (x_2 - x_1)y^3 dy && \text{using Eq. (26.24)} \\
 &= 2\pi\rho \int_0^2 (y^3)y^3 dy && x_2 - x_1 = y^3 - 0 = y^3 \\
 &= 2\pi\rho \left(\frac{1}{7}y^7\right)\Big|_0^2 = \frac{256\pi\rho}{7} \\
 m &= 2\pi\rho \int_0^2 xy dy && \text{mass} = \rho \times \text{volume} \\
 &= 2\pi\rho \int_0^2 y^3 y dy = 2\pi\rho \left(\frac{1}{5}y^5\right)\Big|_0^2 = \frac{64\pi\rho}{5} \\
 R_x^2 &= \frac{256\pi\rho}{7} \times \frac{5}{64\pi\rho} = \frac{20}{7} && R_x^2 = I_x/m \\
 R_x &= \sqrt{\frac{20}{7}} = \frac{2}{7}\sqrt{35} = 1.69
 \end{aligned}$$

**EXAMPLE 5** Moment of inertia of a solid disc

As noted at the beginning of this section, the moment of inertia is important when studying the rotational motion of an object. For this reason, the moments of inertia of various objects are calculated, and the formulas tabulated. Such formulas are usually expressed in terms of the mass of the object.

Among the objects for which the moment of inertia is important is a solid disc. Find the moment of inertia of a disc with respect to its axis and rewrite it in terms of its mass.

To generate a disc (see Fig. 26.53), we rotate the region bounded by the axes,  $x = r$  and  $y = b$ , about the  $y$ -axis. We then have

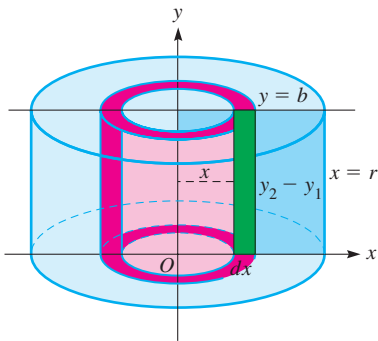


Fig. 26.53

$$\begin{aligned}
 I_y &= 2\pi\rho \int_0^r (y_2 - y_1)x^3 dx && \text{using Eq. (26.23)} \\
 &= 2\pi\rho \int_0^r (b)x^3 dx = 2\pi\rho b \int_0^r x^3 dx && y_2 - y_1 = b - 0 = b \\
 &= 2\pi\rho b \left(\frac{1}{4}x^4\right)\Big|_0^r = \frac{\pi\rho br^4}{2}
 \end{aligned}$$

The mass of the disc is  $\rho(\pi r^2)b$ . Rewriting the expression for  $I_y$ , we have

$$I_y = \frac{(\pi\rho br^2)r^2}{2} = \frac{mr^2}{2}$$

Due to the limited methods of integration available at this point, we cannot integrate the expressions for the moments of inertia of circular areas or of a sphere. These will be introduced in Section 28.8 in the exercises, by which point the proper method of integration will have been developed.

## EXERCISES 26.5

In Exercises 1 and 2, make the given changes in the indicated examples of this section, and then solve the resulting problems.

1. In Example 2, change  $y = 4x^2$  to  $y = 4x$ .
2. In Example 4, change  $y^3 = x$  to  $y^2 = x$ .

In Exercises 3–6, find the moment of inertia (in  $\text{g} \cdot \text{cm}^2$ ) and the radius of gyration (in cm) with respect to the origin of each of the given arrays of masses located at the given points on the  $x$ -axis.

3. 5.0 g at  $(2.4, 0)$ , 3.2 g at  $(3.5, 0)$
4. 3.4 g at  $(-1.5, 0)$ , 6.0 g at  $(2.1, 0)$ , 2.6 g at  $(3.8, 0)$
5. 45.0 g at  $(-3.80, 0)$ , 90.0 g at  $(0.00, 0)$ , 62.0 g at  $(5.50, 0)$
6. 564 g at  $(-45.0, 0)$ , 326 g at  $(-22.5, 0)$ , 720 g at  $(15.4, 0)$ , 205 g at  $(64.0, 0)$

In Exercises 7–28, find the indicated moment of inertia or radius of gyration.

7. Find the moment of inertia of a plate covering the region bounded by  $x = -1$ ,  $x = 1$ ,  $y = 0$ , and  $y = 1$  with respect to the  $x$ -axis.
8. Find the radius of gyration of a plate covering the region bounded by  $x = 2$ ,  $x = 4$ ,  $y = 0$ , and  $y = 4$  with respect to the  $y$ -axis.
9. Find the moment of inertia of a plate covering the first-quadrant region bounded by  $y^2 = x$ ,  $x = 9$ , and the  $x$ -axis with respect to the  $x$ -axis.
10. Find the moment of inertia of a plate covering the region bounded by  $y = 2x$ ,  $x = 1$ ,  $x = 2$ , and the  $x$ -axis with respect to the  $y$ -axis.
11. Find the radius of gyration of a plate covering the region bounded by  $y = x^3$ ,  $x = 3$ , and the  $x$ -axis with respect to the  $y$ -axis.
12. Find the radius of gyration of a plate covering the first-quadrant region bounded by  $y^2 = 1 - x$  with respect to the  $x$ -axis.
13. Find the moment of inertia of a right triangular plate with sides  $a$  and  $b$  with respect to side  $a$  in terms of the mass of the plate.
14. Find the moment of inertia of a rectangular plate of sides  $a$  and  $b$  with respect to side  $a$ . Express the result in terms of the mass of the plate.
15. Find the radius of gyration of a plate covering the region bounded by  $y = x^2$ ,  $x = 3$ , and the  $x$ -axis with respect to the  $x$ -axis.
16. Find the radius of gyration of a plate covering the region bounded by  $y^2 = x^3$ ,  $y = 8$ , and the  $y$ -axis with respect to the  $y$ -axis.
17. Find the radius of gyration of the plate of Exercise 16 with respect to the  $x$ -axis.
18. Find the radius of gyration of a plate covering the first-quadrant region bounded by  $x = 1$ ,  $y = 2 - x$ , and the  $y$ -axis with respect to the  $y$ -axis.

19. Find the moment of inertia with respect to its axis of the solid generated by revolving the region bounded by  $y^2 = 4x$ ,  $y = 2$ , and the  $y$ -axis about the  $x$ -axis.
20. Find the radius of gyration with respect to its axis of the solid generated by revolving the first-quadrant region under the curve  $y = 4 - x^2$  about the  $y$ -axis.
21. Find the radius of gyration with respect to its axis of the solid generated by revolving the region bounded by  $y = 4x - x^2$  and the  $x$ -axis about the  $y$ -axis.
22. Find the radius of gyration with respect to its axis of the solid generated by revolving the region bounded by  $y = 2x$  and  $y = x^2$  about the  $y$ -axis.
23. Find the moment of inertia in terms of its mass of a right circular cone of radius  $r$  and height  $h$  with respect to its axis. See Fig. 26.54.

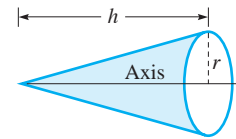


Fig. 26.54

24. Find the moment of inertia in terms of its mass of a circular hoop of radius  $r$  and of negligible thickness with respect to its centre.
25. A rotating drill head is in the shape of a right circular cone. Find the moment of inertia of the drill head with respect to its axis if its radius is 0.600 cm, its height is 0.800 cm, and its mass is 3.00 g. (See Exercise 23.)
26. Find the moment of inertia (in  $\text{kg} \cdot \text{m}^2$ ) of a rectangular door 2 m high and 1 m wide with respect to its hinges if  $\rho = 3 \text{ kg/m}^2$ . (See Exercise 14.)
27. Find the moment of inertia of a flywheel with respect to its axis if its inner radius is 4.0 cm, its outer radius is 6.0 cm, and its mass is 1.2 kg. See Fig. 26.55.

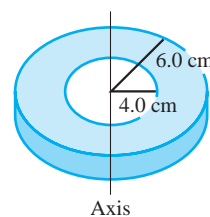


Fig. 26.55

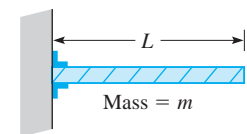


Fig. 26.56

28. A cantilever beam is supported only at its left end, as shown in Fig. 26.56. Explain how to find the formula for the moment of inertia of this beam with respect to a vertical axis through its left end if its length is  $L$  and its mass is  $m$ . (Consider the mass to be distributed evenly along the beam. This is not an area or volume type of problem.) Find the formula for the moment of inertia.

## 26.6 Other Applications

Work Done by a Variable Force •  
Force Due to Liquid Pressure •  
Average Value of a Function

We have seen that the definite integral is used to find the exact measure of the sum of products in which one factor is an increment that approaches a limit of zero. This makes the definite integral a powerful mathematical tool in that a great many applications can be expressed in this form. The following examples show three more applications of the definite integral, and others are shown in the exercises.

### WORK BY A VARIABLE FORCE

In physics, **work** is defined as the product of a constant force times the distance through which it acts. When we consider the work done in stretching a spring, the first thing we recognize is that the more the spring is stretched, the greater is the force necessary to stretch it. Thus, the force varies. However, if we are stretching the spring a distance  $\Delta x$ , where we are considering the limit as  $\Delta x \rightarrow 0$ , the force can be considered as constant over  $\Delta x$ . Adding the product of force<sub>1</sub> times  $\Delta x_1$ , force<sub>2</sub> times  $\Delta x_2$ , and so forth, we see that the total is the sum of these products. Thus, the work can be expressed as a definite integral in the form

$$W = \int_a^b f(x) dx \quad (26.25)$$

where  $f(x)$  is the force as a function of the distance the spring is stretched. The **limits  $a$  and  $b$**  refer to the initial and final distances the spring is stretched from its normal length.

One problem remains: We must find the function  $f(x)$ . From physics, we learn that the force required to stretch a spring is proportional to the amount it is stretched (Hooke's law). If a spring is stretched  $x$  units from its normal length, then  $f(x) = kx$ . From conditions stated for a particular spring, the value of  $k$  may be determined. Thus,  $W = \int_a^b kx dx$  is the formula for finding the total work done in stretching a spring. Here,  $a$  and  $b$  are the initial and final amounts the spring is stretched from its natural length.

■ Hooke's law is named for the English physicist Robert Hooke (1635–1703).

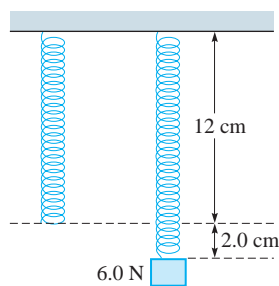


Fig. 26.57

### EXAMPLE 1 Work done stretching a spring

A spring of natural length 12 cm requires a force of 6.0 N to stretch it 2.0 cm. See Fig. 26.57. Find the work done in stretching it 6.0 cm.

From Hooke's law, we find the constant  $k$  for the spring, and therefore  $f(x)$  as

$$\begin{aligned} f(x) &= kx, & 6.0 &= k(2.0), & k &= 3.0 \text{ N/cm} \\ &= 3.0x \end{aligned}$$

Since the spring is to be stretched 6.0 cm,  $a = 0$  (it starts unstretched) and  $b = 6.0$  (it is 6.0 cm longer than its normal length). Therefore, the work done in stretching it is

$$\begin{aligned} W &= \int_0^{6.0} 3.0x dx = 1.5x^2 \Big|_0^{6.0} \quad \text{using Eq. (26.25)} \\ &= 54 \text{ N} \cdot \text{cm} \end{aligned}$$

Problems involving work by a variable force arise in many fields of technology. On the following page is an illustration from electricity that deals with the motion of an electric charge through an electric field created by another electric charge.

Electric charges are of two types, designated as positive and negative. A basic law is that charges of the same sign repel each other and charges of opposite signs attract each other. *The force between charges is proportional to the product of their charges, and inversely proportional to the square of the distance between them.*

The force  $f(x)$  between electric charges is therefore given by

$$f(x) = \frac{kq_1q_2}{x^2} \quad (26.26)$$

when  $q_1$  and  $q_2$  are the charges (in coulombs),  $x$  is the distance (in metres), the force is in newtons, and  $k = 9.0 \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2$ . For other systems of units, the numerical value of  $k$  is different. We can find the work done when electric charges move toward each other or when they separate by use of Eq. (26.26) in Eq. (26.25).

### EXAMPLE 2 Work done in moving $\alpha$ -particles

Find the work done when two  $\alpha$ -particles,  $q = 0.32 \text{ aC}$  each, move until they are  $10 \text{ nm}$  apart, if they were originally separated by  $1.0 \text{ m}$ .

From the given information, we have for each  $\alpha$ -particle

$$q = 0.32 \text{ aC} = 0.32 \times 10^{-18} \text{ C} = 3.2 \times 10^{-19} \text{ C}$$

Since the particles start  $1.0 \text{ m}$  apart and are moved to  $10 \text{ nm}$  apart,  $a = 1.0 \text{ m}$  and  $b = 10 \times 10^{-9} \text{ m} = 10^{-8} \text{ m}$ . The work done is

$$\begin{aligned} W &= \int_{1.0}^{10^{-8}} \frac{9.0 \times 10^9 (3.2 \times 10^{-19})^2}{x^2} dx && \text{using Eq. (26.25)} \\ &= 9.2 \times 10^{-28} \int_{1.0}^{10^{-8}} \frac{dx}{x^2} = 9.2 \times 10^{-28} \left( -\frac{1}{x} \right) \Big|_{1.0}^{10^{-8}} \\ &= -9.2 \times 10^{-28} (10^8 - 1) = -9.2 \times 10^{-20} \text{ J} \end{aligned}$$

$f(x)$  from Eq. (26.26)

Since  $10^8 \gg 1$ , where  $\gg$  means “much greater than,” the 1 may be neglected in the calculation. The minus sign in the result means that work must be done *on* the system to move the particles toward each other. If free to move, they tend to separate. ■

The following is another type of problem involving work by a variable force.

### EXAMPLE 3 Work done winding up a cable

Find the work done in winding up  $60.0 \text{ m}$  of a  $100\text{-m}$  cable that weighs  $4.00 \text{ N/m}$ . See Fig. 26.58.

First, we let  $x$  denote the length of cable that has been wound up at any time. Then the force required to raise the remaining cable equals the weight of the cable that has not yet been wound up. This weight is the product of the unwound cable length,  $100 - x$ , and its weight per unit length,  $4.00 \text{ N/m}$ , or

$$f(x) = 4.00(100 - x)$$

Since  $60.0 \text{ m}$  of cable are to be wound up,  $a = 0$  (none is initially wound up) and  $b = 60.0 \text{ m}$ . The work done is

$$\begin{aligned} W &= \int_0^{60.0} 4.00(100 - x) dx && \text{using Eq. (26.25)} \\ &= \int_0^{60.0} (400 - 4.00x) dx = 400x - 2.00x^2 \Big|_0^{60.0} = 16\,800 \text{ N} \cdot \text{m} \quad \blacksquare \end{aligned}$$

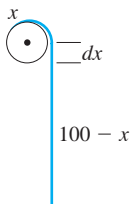


Fig. 26.58

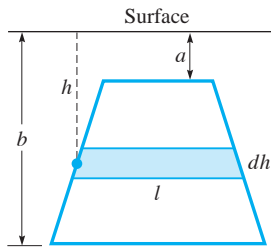


Fig. 26.59

**FORCE DUE TO LIQUID PRESSURE**

The second application of integration in this section deals with the force due to liquid pressure. The force  $F$  on an area  $A$  at the depth  $h$  in a liquid of density  $\gamma$  is  $F = \gamma hA$ . Let us assume that the plate shown in Fig. 26.59 is submerged vertically in the liquid. Using integration to sum the forces on the elements of area, the total force on the plate is given by

$$F = \gamma \int_a^b lh \, dh \tag{26.27}$$

Here,  $l$  is the length of the element of area,  $h$  is the depth of the element of area,  $\gamma$  is the weight per unit volume of the liquid,  $a$  is the depth of the top, and  $b$  is the depth of the bottom of the area on which the force is exerted. When the liquid is water, we use  $\gamma = 9800 \text{ N/m}^3$ .

**LEARNING TIP**

Note that integration in Eq. (26.27) is done from the top to the bottom of the submerged plate. If a problem is set up using coordinate axes, it is convenient to have the  $y$ -axis directed downward so that integration is done in the positive direction.

■ See the chapter introduction.

**EXAMPLE 4 Force on a floodgate**

A vertical floodgate of a dam is 3.00 m wide and 2.00 m high. Find the force on the floodgate if its upper edge is 1.00 m below the surface of the water. See Fig. 26.60.

Each element of area of the floodgate has a length of 3.00 m, which means that  $l = 3.00 \text{ m}$ . Since the top of the gate is 1.00 m below the surface,  $a = 1.00 \text{ m}$ , and since the gate is 2.00 m high,  $b = 3.00 \text{ m}$ . The force on the gate is

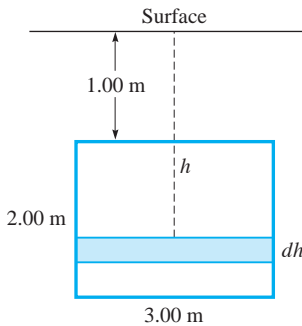


Fig. 26.60

$$\begin{aligned} F &= 9800 \int_{1.00}^{3.00} 3.00h \, dh \quad \text{using Eq. (26.27)} \\ &= 29\,400 \int_{1.00}^{3.00} h \, dh \\ &= 14\,700h^2 \Big|_{1.00}^{3.00} = 14\,700(9.00 - 1.00) \\ &= 118\,000 \text{ N} = 118 \text{ kN} \end{aligned}$$

**EXAMPLE 5 Force on the end of a tank of water**

The vertical end of a tank full of water is in the shape of a right triangle as shown in Fig. 26.61. (Note the  $y$ -axis directed downward.) What is the force on the end of the tank?

The equation of the line  $OA$  is  $y = \frac{1}{2}x$ . Thus, we see that the length of an element of area of the end of the tank is  $4.0 - x$ , the depth of the element of area is  $y$ , the top of the tank is  $y = 0$ , and the bottom is  $y = 2.0 \text{ m}$ . Therefore, the force on the end of the tank is

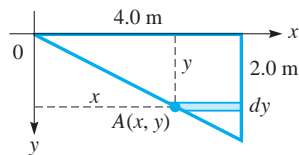


Fig. 26.61

$$\begin{aligned} F &= 9800 \int_0^{2.0} (4.0 - x)(y)(dy) \quad \text{using Eq. (26.27)} \\ &= 9800 \int_0^{2.0} (4.0 - 2y)(y) \, dy \\ &= 19\,600 \int_0^{2.0} (2.0y - y^2) \, dy = 19\,600 \left( 1.0y^2 - \frac{1}{3}y^3 \right) \Big|_0^{2.0} \\ &= 26\,100 \text{ N} \end{aligned}$$

### AVERAGE VALUE OF A FUNCTION

The third application of integration shown in this section is that of the *average value* of a function. In general, *an average is found by summing up the quantities to be averaged and then dividing by the total number of them.* Generalizing on this and using integration for the summation, the **average value** of a function  $y$  with respect to  $x$  from  $x = a$  to  $x = b$  is given by

$$y_{\text{av}} = \frac{\int_a^b y \, dx}{b - a} \quad (26.28)$$

The following examples illustrate applications of the average value of a function.

#### EXAMPLE 6 Average value of velocity

The velocity  $v$  (in m/s) of an object falling under the influence of gravity as a function of time  $t$  (in s) is given by  $v = 9.80t$ . What is the average velocity of the object with respect to time for the first 3.0 s?

In this case, we want the average value of the function  $v = 9.80t$  from  $t = 0$  s to  $t = 3.0$  s. This gives us

$$\begin{aligned} v_{\text{av}} &= \frac{\int_0^{3.0} v \, dt}{3.0 - 0} \quad \text{using Eq. (26.28)} \\ &= \frac{\int_0^{3.0} 9.80t \, dt}{3.0} = \frac{4.90t^2}{3.0} \Big|_0^{3.0} \\ &= 14.7 \text{ m/s} \end{aligned}$$

This result can be interpreted as meaning that an average velocity of 14.7 m/s for 3.0 s would result in the same distance, 44.1 m, being travelled by the object as that with the variable velocity. Since  $s = \int v \, dt$ , the numerator represents the distance travelled. ■

#### EXAMPLE 7 Average value of electric power

The power  $P$  (in W) developed in a certain resistor as a function of the current  $i$  (in A) is  $P = 6.0i^2$ . What is the average power with respect to the current as the current changes from 2.0 A to 5.0 A?

In this case, we are to find the average value of the function  $P$  from  $i = 2.0$  A to  $i = 5.0$  A. This average value of  $P$  is

$$\begin{aligned} P_{\text{av}} &= \frac{\int_{2.0}^{5.0} P \, di}{5.0 - 2.0} \quad \text{using Eq. (26.28)} \\ &= \frac{6.0 \int_{2.0}^{5.0} i^2 \, di}{3.0} = \frac{2.0i^3}{3.0} \Big|_{2.0}^{5.0} = \frac{2.0(125 - 8.0)}{3.0} = 78 \text{ W} \quad \blacksquare \end{aligned}$$

In general, it might be noted that the average value of  $y$  with respect to  $x$  is that value of  $y$  which, when multiplied by the length of the interval for  $x$ , gives the same area as that under the curve of  $y$  as a function of  $x$ .

## EXERCISES 26.6

- In Example 1, find the work done in stretching the spring from a length of 15.0 cm to a length of 18.0 cm.
- In Example 3, find the work done in winding up all the cable.
- In Example 4, find the force on the floodgate if the upper edge is 2.00 m below the surface.
- In Example 6, find the average velocity of the object with respect to time between 3.00 s and 6.00 s.
- The spring of a spring balance is 8.0 cm long when there is no weight on the balance, and it is 9.5 cm long with 6.0 N hung from the balance. How much work is done in stretching it from 8.0 cm to a length of 10.0 cm?
- How much work is done in stretching the spring of Exercise 5 from a length of 10.0 cm to 12.0 cm?
- A 640-N person compresses a bathroom scale 0.080 cm. If the scale obeys Hooke's law, how much work is done compressing the scale if a 720-N person stands on it?
- A force  $F$  of 25 N on the spring in the lever-spring mechanism shown in Fig. 26.62 stretches the spring by 16 mm. How much work is done by the 25-N force in stretching the spring?
- An electron has a  $1.6 \times 10^{-19}$  C negative charge. How much work is done in separating two electrons from 1.0 pm to 4.0 pm?

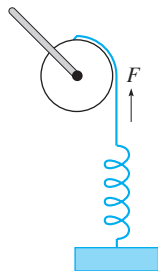


Fig. 26.62

- How much work is done in separating an electron (see Exercise 9) and an oxygen nucleus, which has a positive charge of  $1.3 \times 10^{-18}$  C, from a distance of  $2.0 \mu\text{m}$  to a distance of 1.0 m?
- The gravitational force (in N) of attraction between two objects is given by  $F = k/x^2$ , where  $x$  is the distance between the objects. If the objects are 10 m apart, find the work required to separate them until they are 100 m apart. Express the result in terms of  $k$ .
- Find the work done by winding up 20 m of a 25-m rope on which the force of gravity is 6.0 N/m.
- A 6000-N elevator is suspended on cables that together weigh 48 N/m. How much work is done in raising the elevator from the basement to the second floor, a distance of 8.0 m?
- A chain is being unwound from a winch. The force of gravity on it is 12.0 N/m. When 20 m have been unwound, how much work is done by gravity in unwinding another 30 m?
- At liftoff, a rocket weighs 320 kN, including the weight of its fuel. During the first (vertical) stage of ascent, fuel is consumed at the rate of 12 kN per 1000 m of ascent. How much work is done in lifting the rocket to an altitude of 12 000 m? (Disregard the decrease in weight due to increasing elevation.)
- While descending, a 550-N weather balloon enters a zone of freezing rain in which ice forms on the balloon at the rate of 7.50 N per 100 m of descent. Find the work done on the balloon during the first 1000 m of descent through the freezing rain.
- A meteorite is 75 000 km from the centre of Earth and falls to the surface of Earth. From Newton's law of gravity (see page 501), the force of gravity varies inversely as the square of the distance

between the meteorite and the centre of Earth. Find the work done by gravity if the meteorite weighs 160 N at the surface, and the radius of Earth is 6400 km.

- A rectangular swimming pool full of water is 5.50 m wide, 13.5 m long, and 1.75 m deep. Find the work done in pumping the water from the pool to a level 1.25 m above the top of the pool.
- Find the work done in pumping the water out of the top of a cylindrical tank 2.0 m in radius and 4.0 m high, given that the tank is initially full and water weighs  $9.8 \text{ kN/m}^3$ . (Hint: If horizontal slices  $dx$  m thick are used, each element weighs  $9800(\pi)(2.00^2) dx$  N, and each element must be raised  $4.0 - x$  m, if  $x$  is the distance from the base to the element (see Fig. 26.63). In this way, the force, which is the weight of the slice, and the distance through which the force acts are determined. Thus, the products of force and distance are summed by integration.)

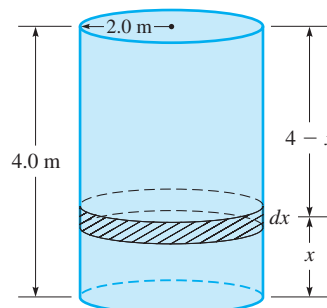


Fig. 26.63

- A hemispherical tank of radius 3.0 m is full of water. Find the work done in pumping the water out of the top of the tank. (See Exercise 19. This problem is similar, except that the weight of each element is  $9800\pi(\text{radius})^2$  (thickness), where the radius of each element is different. If we let  $x$  be the radius of an element and  $y$  be the distance the element must be raised, we have  $9800\pi x^2 dy$  with  $x^2 + y^2 = 9.0$ .)
- One end of a spa is a vertical rectangular wall 4.00 m wide. What is the force exerted on this wall by the water if it is 0.80 m deep?
- Find the force on one side of a cubical container 6.0 cm on an edge if the container is filled with mercury. The density of mercury is  $133 \text{ kN/m}^3$ .
- A rectangular sea aquarium observation window is 3.00 m wide and 2.00 m high. What is the force on this window if the upper edge is 1.50 m below the surface of the water? The density of seawater is  $10.1 \text{ kN/m}^3$ .
- A horizontal tank has vertical circular ends, each with a radius of 2.00 m. It is filled to a depth of 2.00 m with oil of density  $9400 \text{ N/m}^3$ . Find the force on one end of the tank.
- A swimming pool is 6.00 m wide and 15.0 m long. The bottom has a constant slope such that the water is 1.00 m deep at one end and 2.00 m deep at the other end. Find the force of the water on one of the sides of the pool.
- Find the force on the lower half of the wall at the deep end of the swimming pool in Exercise 25.
- A small dam is in the shape of the area bounded by  $y = x^2$  and  $y = 20$  (distances in m). Find the force on the area below  $y = 4$  if the surface of the water is at the top of the dam.



28. The tank on a tanker truck has vertical elliptical ends with the major axis horizontal. The major axis is 2.0 m and the minor axis 1.3 m. Find the force on one end of the tank when it is half-filled with fuel oil of density  $7.8 \text{ kN/m}^3$ .
29. A watertight cubical box with an edge of 2.00 m is suspended in water such that the top surface is 1.00 m below water level. Find the total force on the top of the box and the total force on the bottom of the box. What meaning can you give to the difference of these two forces?
30. Find the force on the region bounded by  $x = 2y - y^2$  and the  $y$ -axis if the upper point of the area is at the surface of the water. All distances are in metres.
31. The electric current  $i$  (in A) as a function of the time  $t$  (in s) for a certain circuit is given by  $i = 4t - t^2$ . Find the average value of the current with respect to time for the first 4.0 s.
32. The temperature  $T$  (in  $^\circ\text{C}$ ) recorded in a city during a given day approximately followed the curve of  $T = 0.00100t^4 - 0.280t^2 + 25.0$ , where  $t$  is the number of hours from noon ( $-12 \text{ h} \leq t \leq 12 \text{ h}$ ). What was the average temperature during the day?
33. The efficiency  $\eta$  (in %) of an automobile engine is given by  $\eta = 0.768s - 0.00004s^3$ , where  $s$  is the speed (in km/h) of the car. Find the average efficiency with respect to the speed for  $s = 30.0 \text{ km/h}$  to  $s = 90.0 \text{ km/h}$ . (See Example 1 of Section 24.7.)
34. Find the average value of the volume of a sphere with respect to the radius. Explain the meaning of the result.

35. The length of arc  $s$  of a curve from  $x = a$  to  $x = b$  is

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \quad (26.29)$$

The cable of a bridge can be described by the equation  $y = 0.04x^{3/2}$  from  $x = 0$  to  $x = 100 \text{ m}$ . Find the length of the cable. See Fig. 26.64.

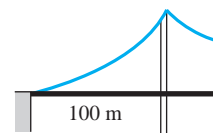


Fig. 26.64

36. A rocket takes off in a path described by the equation  $y = \frac{2}{3}(x^2 - 1)^{3/2}$ . Find the distance travelled by the rocket for  $x = 1.0 \text{ km}$  to  $x = 3.0 \text{ km}$ . (See Exercise 35.)
37. The area of a surface of revolution from  $x = a$  to  $x = b$  is
- $$S = 2\pi \int_a^b y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$
- Find the formula for the lateral surface area of a right circular cone of radius  $r$  and height  $h$ .
38. The grinding surface of a grinding machine can be described as the surface generated by rotating the curve  $y = 0.2x^3$  from  $x = 0$  to  $x = 2.0 \text{ cm}$  about the  $x$  axis. Find the grinding surface area. (See Exercise 37.)

## CHAPTER 26 EQUATIONS

### Velocity

$$v = \int a dt \quad (26.1)$$

$$v = at + C_1 \quad (26.2)$$

### Displacement

$$s = \int v dt \quad (26.3)$$

### Electric current

$$i = \frac{dq}{dt} \quad (26.4)$$

### Electric charge

$$q = \int i dt \quad (26.5)$$

### Voltage across a capacitor

$$V_C = \frac{1}{C} \int i dt \quad (26.6)$$

### Area

$$A = \int_a^b y dx = \int_a^b f(x) dx \quad (26.7)$$

$$A = \int_c^d x dy = \int_c^d g(y) dy \quad (26.8)$$

$$A = \int_a^b (y_2 - y_1) dx \quad (26.9)$$

$$A = \int_c^d (x_2 - x_1) dy \quad (26.10)$$

$$\text{Volume} \quad V = \pi \int_a^b y^2 dx = \pi \int_a^b [f(x)]^2 dx \quad (26.11)$$

$$V = \pi \int_c^d x^2 dy \quad (26.12)$$

$$\text{Shell} \quad dV = 2\pi(\text{radius}) \times (\text{height}) \times (\text{thickness}) \quad (26.13)$$

$$\text{Disc} \quad dV = \pi(\text{radius})^2 \times (\text{thickness}) \quad (26.14)$$

$$\text{Centre of mass} \quad m_1 d_1 + m_2 d_2 + \cdots + m_n d_n = (m_1 + m_2 + \cdots + m_n) \bar{d} \quad (26.15)$$

$$\text{Centroid of a flat plate} \quad \bar{x} = \frac{\int_a^b x(y_2 - y_1) dx}{\int_a^b (y_2 - y_1) dx} \quad (26.16) \quad \bar{y} = \frac{\int_c^d y(x_2 - x_1) dy}{\int_c^d (x_2 - x_1) dy} \quad (26.17)$$

$$\text{Centroid of a solid of revolution} \quad \bar{x} = \frac{\int_a^b xy^2 dx}{\int_a^b y^2 dx} \quad (26.18) \quad \bar{y} = \frac{\int_c^d yx^2 dy}{\int_c^d x^2 dy} \quad (26.19)$$

$$\text{Radius of gyration} \quad m_1 d_1^2 + m_2 d_2^2 + \cdots + m_n d_n^2 = (m_1 + m_2 + \cdots + m_n) R^2 \quad (26.20)$$

$$\text{Moment of inertia of a flat plate} \quad I_y = \rho \int_a^b x^2 (y_2 - y_1) dx \quad (26.21)$$

$$I_x = \rho \int_c^d y^2 (x_2 - x_1) dy \quad (26.22)$$

$$\text{Moment of inertia of a solid of revolution} \quad I_y = 2\pi\rho \int_a^b (y_2 - y_1)x^3 dx \quad (26.23)$$

$$I_x = 2\pi\rho \int_c^d (x_2 - x_1)y^3 dy \quad (26.24)$$

$$\text{Work} \quad W = \int_a^b f(x) dx \quad (26.25)$$

$$\text{Force between electric charges} \quad f(x) = \frac{kq_1 q_2}{x^2} \quad (26.26)$$

$$\text{Force due to liquid pressure} \quad F = \gamma \int_a^b lh dh \quad (26.27)$$

$$\text{Average value} \quad y_{\text{av}} = \frac{\int_a^b y dx}{b - a} \quad (26.28)$$

$$\text{Length of arc} \quad s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \quad (26.29)$$

## CHAPTER 26 REVIEW EXERCISES

- A pitcher releases a baseball horizontally at 45.0 m/s. How far does it drop while travelling 17.1 m to home plate?
- If the velocity  $v$  (m/s) of a subway train after the brakes are applied can be expressed as  $v = \sqrt{400 - 20t}$ , where  $t$  is the time in seconds, how far does it travel in coming to a stop?
- A weather balloon is rising at the rate of 10.0 m/s when a small metal part drops off. If the balloon is 60.0 m high at this instant, when will the part hit the ground?
- A float is dropped into a river at a point where it is flowing at 1.5 m/s. How far does the float travel in 30 s if it accelerates downstream at  $0.010 \text{ m/s}^2$ ?
- A golf ball is putted straight for the hole with an initial velocity of 2.50 m/s and acceleration of  $-0.750 \text{ m/s}^2$ . Will the ball make it to the hole, which is 4.20 m away?
- What is the initial vertical velocity of a baseball that just reaches the ceiling of an indoor stadium that is 65 m high?
- The electric current  $i$  (in A) in a circuit as a function of the time  $t$  (in s) is  $i = 0.25(2\sqrt{t} - t)$ . Find the total charge to pass a point in the circuit in 2.0 s.
- The current  $i$  (in A) in a certain electric circuit is given by  $i = \sqrt{1 + 4t}$ , where  $t$  is the time (in s). Find the charge that passes a given point from  $t = 1.0$  s to  $t = 3.0$  s if  $q_0 = 0$ .
- The voltage across a 5.5-nF capacitor in an FM radio receiver is zero. What is the voltage after  $25 \mu\text{s}$  if a current of 12 mA charges the capacitor?
- The initial voltage across a capacitor is zero, and  $V_C = 2.50 \text{ V}$  after 8.00 ms. If a current  $i = t/\sqrt{t^2 + 1}$ , where  $i$  is the current (in A) and  $t$  is the time (in s), charges the capacitor, find the capacitance  $C$  of the capacitor.
- The distribution of weight on a cable is not uniform. If the slope of the cable at any point is given by  $dy/dx = 20 + 0.025x^2$  and if the origin of the coordinate system is at the lowest point, find the equation that gives the curve described by the cable.
- The time rate of change of the reliability  $R$  (in %) of a computer system is  $dR/dt = -2.5(0.05t + 1)^{-1.5}$ , where  $t$  is the time (in h). If  $R = 100$  for  $t = 0$ , find  $R$  for  $t = 100$  h.
- Find the area between  $y = \sqrt{9 - x}$  and the coordinate axes.
- Find the area bounded by  $y = 3x^2 - x^3$  and the  $x$ -axis.
- Find the area bounded by  $y^2 = 2x$  and  $y = x - 4$ .
- Find the area bounded by  $y = 6/(x + 3)^2$ ,  $y = 0$ ,  $x = -1$ , and  $x = 3$ .
- Find the area between  $y = x^2$  and  $y = x^3 - 2x^2$ .
- Find the area between  $y = x^2 + 8$  and  $y = 3x^2$ .
- Show that the curve  $y = x^{2n}$  ( $n > 0$ ) divides the square bounded by  $x = 0$ ,  $y = 0$ ,  $x = 1$ , and  $y = 1$  into two regions, the areas of which are in the ratio  $2n/1$ .
- Find the value of  $a$  such that the line  $x = a$  bisects the area under the curve  $y = 1/x^2$  between  $x = 1$  and  $x = 4$ .
- Find the volume generated by revolving the region bounded by  $y = 3 + x^2$  and the line  $y = 4$  about the  $x$ -axis.
- Find the volume generated by revolving the region bounded by  $y = 8x - x^4$  and the  $x$ -axis about the  $x$ -axis.
- Find the volume generated by revolving the region bounded by  $y = x^3 - 4x^2$  and the  $x$ -axis about the  $y$ -axis.
- Find the volume generated by revolving the region bounded by  $y = x$  and  $y = 3x - x^2$  about the  $y$ -axis.
- Find the volume generated by revolving an ellipse about its major axis.
- A hole of radius 1.00 cm is bored along the diameter of a sphere of radius 4.00 cm. Find the volume of the material that is removed from the sphere.
- Find the centre of mass of the following array of four masses in the  $xy$ -plane (distances in cm): 60 g at  $(4, 4)$ , 160 g at  $(-3, 6)$ , 70 g at  $(-5, -4)$ , and 130 g at  $(3, -5)$ .
- Find the centroid of the flat-plate machine part shown in Fig. 26.65. Each section is uniform, and the mass of the section to the right of the  $y$ -axis is twice that of the section to the left.

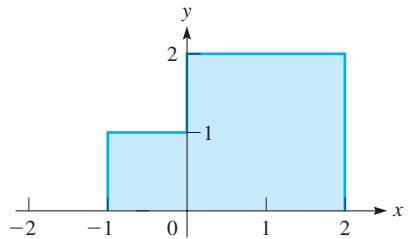


Fig. 26.65

- Find the centroid of a flat plate covering the region bounded by  $y^2 = x^3$  and  $y = 3x$ .
- Find the centroid of a flat plate covering the region bounded by  $y = 2x - 4$ ,  $x = 1$ , and  $y = 0$ .
- Find the centroid of the volume generated by revolving the region bounded by  $y = \sqrt{x}$ ,  $x = 1$ ,  $x = 4$ , and  $y = 0$  about the  $x$ -axis.
- Find the centroid of the volume generated by revolving the region bounded by  $yx^4 = 1$ ,  $y = 1$ , and  $y = 4$  about the  $y$ -axis.
- Find the moment of inertia of a flat plate covering the region bounded by  $y = 3x - x^2$  and  $y = x$  with respect to the  $y$ -axis.
- Find the radius of gyration of a flat plate covering the region bounded by  $y = 8 - x^3$ , and the axes, with respect to the  $y$ -axis.
- Find the moment of inertia with respect to its axis of a lead bullet that is defined by revolving the region bounded by  $y = 3.00x^{0.10}$ ,  $x = 0$ ,  $x = 20.0$ , and  $y = 0$  about the  $x$ -axis (all measurements in mm). The density of lead is  $0.0114 \text{ g/mm}^3$ .
- Find the radius of gyration with respect to its axis of a rotating machine part that can be defined by revolving the region bounded by  $y = 1/x$ ,  $x = 1.00$ ,  $x = 4.00$ , and  $y = 0.25$  (all measurements in cm) about the  $x$ -axis.
- A pail and its contents weigh 80 N. The pail is attached to the end of a 30-m rope that weighs 20 N and is hanging vertically. How much work is done in winding up the rope with the pail attached?
- The gravitational force (in N) of the earth on a satellite (the weight of the satellite) is given by  $f(x) = 10^{11}/x^2$ , where  $x$  is the vertical distance (in km) from the centre of the earth to the satellite. How much work is done in moving the satellite from the earth's surface to an altitude of 3000 km? The radius of the earth is 6370 km.

39. A decorative glass table-top is designed to be the region between the curves of  $y = 0.0001x^4$  and  $y = 110 - 0.0001x^4$ . Find the area (in  $\text{cm}^2$ ) of the table top.
40. A level putting green at a golf course can be approximated as the area bounded by  $y = 0.003x^3 - 2x$  and  $y = 1.5x - 0.001x^3$ . Find the area (in  $\text{m}^2$ ) of the green.
41. In a video game, a large rock is propelled up a slope at  $12 \text{ m/s}$ , but is accelerating down the slope at  $6.0 \text{ m/s}^2$ . What is the velocity of the rock after  $5.0 \text{ s}$ ?
42. The vertical end of a trough is made of a right triangular section of concrete and a similar section of wood, as shown in Fig. 26.66. Find the force on each section if the trough is full of water.

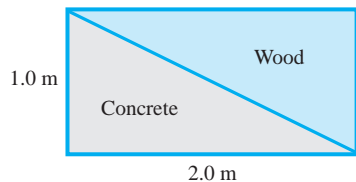


Fig. 26.66

43. The rear stabilizer of a certain aircraft can be described as the region under the curve  $y = 3x^2 - x^3$ , as shown in Fig. 26.67. Find the  $x$ -coordinate (in  $\text{m}$ ) of the centroid of the stabilizer.

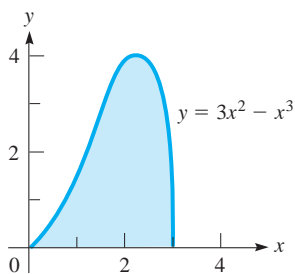


Fig. 26.67

44. The diameter of a circular swimming pool is  $12 \text{ m}$ , and the sides are  $2.0 \text{ m}$  high. If the depth of the water is  $1.5 \text{ m}$ , how much work is done in pumping all of the water out over the side?
45. The nose cone of a rocket has the shape of a semiellipse revolved about its major axis, as shown in Fig. 26.68. What is the volume of the nose cone?

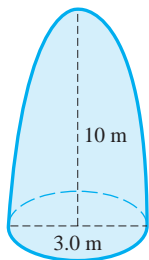


Fig. 26.68

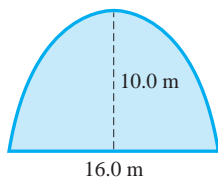


Fig. 26.69

46. The deck area of a boat is a parabolic section as shown in Fig. 26.69. What is the area of the deck?

47. The vertical ends of a fuel storage tank have a parabolic bottom section and a triangular top section, as shown in Fig. 26.70. What volume does the tank hold?

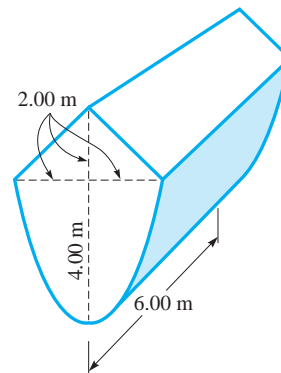


Fig. 26.70

48. The capillary tube shown in Fig. 26.71 has circular horizontal cross sections of inner radius  $1.1 \text{ mm}$ . What is the volume of the liquid in the tube above the level of liquid outside the tube if the top of the liquid in the centre vertical cross section is described by the equation  $y = x^4 + 1.5$ , as shown?

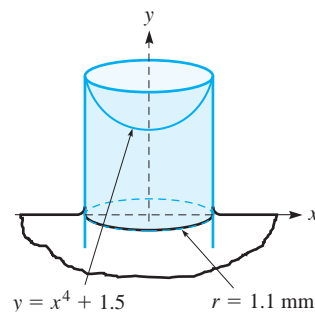


Fig. 26.71

49. A cylindrical chemical waste-holding tank  $4.50 \text{ m}$  in radius has a depth of  $3.25 \text{ m}$ . Find the total force on the circular side of the tank when it is filled with liquid with a density of  $10.6 \text{ kN/m}^3$ .
50. A section of a dam is in the shape of a right triangle. The base of the triangle is  $6.00 \text{ m}$  and is in the surface of the water. If the triangular section goes to a depth of  $4.00 \text{ m}$ , find the force on it. See Fig. 26.72.

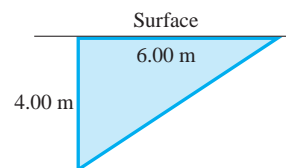


Fig. 26.72

51. The electric resistance of a wire is inversely proportional to the square of its radius. If a certain wire has a resistance of  $0.30 \Omega$  when its radius is  $2.0 \text{ mm}$ , find the average value of the resistance with respect to the radius as the radius changes from  $2.0 \text{ mm}$  to  $2.1 \text{ mm}$ .
52. The mass of earth is  $5.98 \times 10^{24} \text{ kg}$ , and the mass of the moon is  $7.36 \times 10^{22} \text{ kg}$ . Assuming all of the mass of each is at its centre, find the centre of mass of the earth-moon system, if their centres are  $3.82 \times 10^8 \text{ m}$  apart. Compare this position with the radius of earth, which is  $6.37 \times 10^6 \text{ m}$ .

53. In the television tube of an older TV set, electrons are accelerated from rest with an acceleration of  $5 \times 10^{14} \text{ m/s}^2$ . What is their velocity after travelling 2.5 cm?
54. A horizontal straight section of pipe is supported at its centre by a vertical wire as shown in Fig. 26.73. Find the formula for the moment of inertia of the pipe with respect to an axis along the wire if the pipe is of length  $L$  and mass  $m$ .

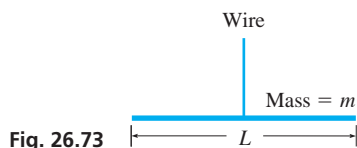


Fig. 26.73

55. Table 22.1 in Section 22.4 gives areas under the curve from 0 to  $z$  for the standard normal distribution, given by  $y = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ . Use the information given in the table to find  $\frac{1}{\sqrt{2\pi}} \int_1^2 e^{-x^2/2} dx$ .

56. Use the information given in Table 22.1 and the symmetry of the standard normal distribution to find  $\frac{1}{\sqrt{2\pi}} \int_{-2}^3 e^{-x^2/2} dx$  (see Exercise 55).

### Writing Exercise

57. The float for a certain valve control has a circular top of radius  $a$ . All cross sections of the float that are perpendicular to a fixed diameter of the top are squares. Write one or two paragraphs explaining how to derive the formula that gives the volume of the float. What is the formula?

## CHAPTER 26 PRACTICE TEST

In Problems 1–3, use the region bounded by  $y = \frac{1}{4}x^2$ ,  $y = 0$ , and  $x = 2$ .

- Find the area.
- Find the coordinates of the centroid of a flat plate that covers the region.
- Find the volume if the given region is revolved about the  $x$ -axis.

In Problems 4 and 5, use the first-quadrant region bounded by  $y = x^2$ ,  $x = 0$ , and  $y = 9$ .

- Find the volume if the given region is revolved about the  $x$ -axis.
- Find the moment of inertia of a flat plate that covers the region, with respect to the  $y$ -axis.

- The velocity  $v$  of an object as a function of the time  $t$  is  $v = 60 - 4t$ . Find the expression for the displacement  $s$  if  $s = 10$  for  $t = 0$ .
- The natural length of a spring is 8.0 cm. A force of 12 N stretches it to a length of 10.0 cm. How much work is done in stretching it from a length of 10.0 cm to a length of 14.0 cm?
- A vertical rectangular floodgate is 6.00 m wide and 2.00 m high. Find the force on the gate if its upper edge is 1.00 m below the surface of the water ( $\gamma = 9.80 \text{ kN/m}^3$ ).

# 27

## LEARNING OUTCOMES

After completion of this chapter, the student should be able to:

- Find the derivative of expressions involving trigonometric and inverse trigonometric functions
- Find the derivative of expressions involving logarithmic and exponential functions
- Find limits of indeterminate forms using L'Hospital's rule
- Solve application problems involving derivatives of transcendental functions

# Differentiation of Transcendental Functions

While studying vibrations in a rod in the mid-1700s, the Swiss mathematician Euler noted that the trigonometric functions arose naturally as solutions to equations in which derivatives appeared. This was the first treatment of the trigonometric functions as functions of numbers essentially as we do today. Later, in 1755, Euler wrote a textbook on differential calculus in which he included differentiation of the trigonometric, inverse trigonometric, logarithmic, and exponential functions. Euler called these functions “transcendental,” as “they transcend the power of algebraic methods.” These functions are not algebraic in that they cannot be expressed using algebraic operations (addition, subtraction, multiplication, division, and taking roots).

As noted on page 370, logarithms were developed as a tool for calculation. In establishing the calculus, Newton and Leibniz did some formulation of the logarithmic and exponential functions. However, the calculus of the transcendental functions was formulated mostly in the 1700s by Euler and a number of other mathematicians. This led to the rapid progress made later in many technical and scientific areas. For example, transcendental functions, along with their derivatives and integrals, have been of great importance in the development of the fields of electricity and electronics in the 1800s, 1900s, and 2000s, particularly with respect to alternating current.

In this chapter, we develop formulas for the derivatives of these transcendental functions, and in the next chapter, we will take up integration that involves these functions. Other areas in which we will show important applications include harmonic motion, rocket motion, monetary interest calculations, population growth, acoustics, and optics.

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► In Sections 27.4 and 27.8, we use derivatives of transcendental functions in analysing the motion of a rocket.



## 27.1 Derivatives of the Sine and Cosine Functions

- Limit of  $(\sin)\theta/\theta$  as  $\theta \rightarrow 0$  •
- Derivative of  $\sin u$  •
- Derivative of  $\cos u$

■ For reference, Eq. (20.18) is  $\sin x - \sin y = 2 \sin \frac{1}{2}(x - y) \cos \frac{1}{2}(x + y)$ .

We now find the derivative of the sine function. We will then be able to use it in finding the derivatives of the other trigonometric and inverse trigonometric functions.

Let  $y = \sin x$ , where  $x$  is expressed in radians. If  $x$  changes by an amount  $h$ , from the definition of the derivative, we have

$$\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{\sin(x + h) - \sin x}{h}$$

Referring now to Eq. (20.18), we have

$$\begin{aligned} \frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{2 \sin \frac{1}{2}(x + h - x) \cos \frac{1}{2}(x + h + x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sin(h/2) \cos(x + h/2)}{h/2} \end{aligned}$$

Looking ahead to the next step of letting  $h \rightarrow 0$ , we see that the numerator and denominator both approach zero. This situation is precisely the same as that in which we were finding the derivatives of the algebraic functions. To find the limit, we must find

$$\lim_{h \rightarrow 0} \frac{\sin(h/2)}{h/2}$$

since these are the factors that cause the numerator and denominator to approach zero.

In finding this limit, we let  $\theta = h/2$  for convenience of notation. This means that we are to determine  $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta}$ . Of course, it would be convenient to know before proceeding if this limit does actually exist. Therefore, by using a calculator, we can develop a table of values of  $\frac{\sin \theta}{\theta}$  as  $\theta$  becomes very small:

$\theta$ (radians)	0.5	0.1	0.05	0.01	0.001
$\frac{\sin \theta}{\theta}$	0.958 851 1	0.998 334 2	0.999 583 4	0.999 983 3	0.999 999 8

We see from this table that the limit of  $\frac{\sin \theta}{\theta}$ , as  $\theta \rightarrow 0$ , appears to be 1.

In order to prove that  $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$ , we use a geometric approach. Considering

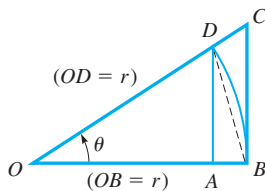


Fig. 27.1

Fig. 27.1, we see that the following inequality is true:

$$\text{Area triangle } OBD < \text{area sector } OBD < \text{area triangle } OBC$$

$$\frac{1}{2}r(r \sin \theta) < \frac{1}{2}r^2\theta < \frac{1}{2}r(r \tan \theta) \quad \text{or} \quad \sin \theta < \theta < \tan \theta$$

Remembering that we want to find the limit of  $(\sin \theta)/\theta$ , we next divide through by  $\sin \theta$  and then take reciprocals:

$$1 < \frac{\theta}{\sin \theta} < \frac{1}{\cos \theta} \quad \text{or} \quad 1 > \frac{\sin \theta}{\theta} > \cos \theta$$



When we consider the limit as  $\theta \rightarrow 0$ , we see that the left member remains 1 and the right member approaches 1. Thus,  $(\sin \theta)/\theta$  must approach 1. This means

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = \lim_{h \rightarrow 0} \frac{\sin (h/2)}{h/2} = 1 \quad (27.1)$$

Using the result in Eq. (27.1) in the expression for  $dy/dx$ , we have

$$\begin{aligned} \lim_{h \rightarrow 0} \left[ \cos (x + h/2) \frac{\sin (h/2)}{h/2} \right] &= \cos x \\ \frac{dy}{dx} &= \cos x \end{aligned} \quad (27.2)$$

To find the derivative of  $y = \sin u$ , where  $u$  is a function of  $x$ , we use the chain rule, Eq. (23.14), which we repeat here for reference:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \quad (27.3)$$

Therefore, for  $y = \sin u$ ,  $dy/du = \cos u$ , we have

$$\frac{d(\sin u)}{dx} = \cos u \frac{du}{dx} \quad (27.4)$$

### EXAMPLE 1 Derivative of $\sin u$

(a) Find the derivative of  $y = \sin 2x$ .

In this example,  $u = 2x$ . Therefore, using Eq. (27.4),

$$\begin{aligned} \frac{dy}{dx} &= \frac{d(\sin 2x)}{dx} = \cos 2x \frac{d(2x)}{dx} = (\cos 2x) \left( 2 \frac{du}{dx} \right) \\ &= 2 \cos 2x \end{aligned}$$

(b) Find the derivative of  $y = 2 \sin(x^2)$ .

In this example,  $u = x^2$ , which means that  $du/dx = 2x$ . This means

$$\begin{aligned} \frac{dy}{dx} &= 2[\cos(x^2)](2x) \quad \text{using Eq. (27.4)} \\ &= 4x \cos(x^2) \end{aligned}$$

#### Practice Exercise

1. Find the derivative of  
 $y = 3 \sin(4x + 1)$ .

#### LEARNING TIP

It is important here, just as it is in finding the derivatives of powers of all functions, to remember to include the factor  $du/dx$ .

■ For reference, Eq. (23.15) is

$$\frac{du^n}{dx} = nu^{n-1} \left( \frac{du}{dx} \right).$$

### EXAMPLE 2 Derivative of a power of $\sin u$

Find the derivative of  $r = \sin^2 \theta$ .

This example is a combination of the use of the power rule, Eq. (23.15) and the derivative of the sine function Eq. (27.4). Since  $\sin^2 \theta$  means  $(\sin \theta)^2$ , in using the power rule we have  $u = \sin \theta$ . Thus,

$$\begin{aligned} \frac{dr}{d\theta} &= 2(\sin \theta) \frac{d \sin \theta}{d\theta} \quad \text{using Eq. (23.15)} \\ &= 2 \sin \theta \cos \theta \quad \text{using Eq. (27.4)} \\ &= \sin 2\theta \quad \text{using identity (Eq. 20.21)} \end{aligned}$$

In order to find the derivative of the cosine function, we write it in the form  $\cos u = \sin\left(\frac{\pi}{2} - u\right)$ . Thus, if  $y = \sin\left(\frac{\pi}{2} - u\right)$ , we have

$$\begin{aligned}\frac{dy}{dx} &= \cos\left(\frac{\pi}{2} - u\right) \frac{d\left(\frac{\pi}{2} - u\right)}{dx} = \cos\left(\frac{\pi}{2} - u\right) \left(-\frac{du}{dx}\right) \\ &= -\cos\left(\frac{\pi}{2} - u\right) \frac{du}{dx}\end{aligned}$$

Since  $\cos\left(\frac{\pi}{2} - u\right) = \sin u$ , we have

$$\frac{d(\cos u)}{dx} = -\sin u \frac{du}{dx} \quad (27.5)$$

### EXAMPLE 3 Derivative of $\cos u$ —application

The electric power  $P$  developed in a resistor of an amplifier circuit is  $P = 25 \cos^2 120\pi t$ , where  $t$  is the time. Find the expression for the time rate of change of power.

From Chapter 23, we know that we are to find the derivative  $dP/dt$ . Therefore,

$$\begin{aligned}P &= 25 \cos^2 120\pi t \\ \frac{dP}{dt} &= 25(2 \cos 120\pi t) \frac{d \cos 120\pi t}{dt} && \text{using Eq. (23.15)} \\ &= 50 \cos 120\pi t (-\sin 120\pi t) \frac{d(120\pi t)}{dt} && \text{using Eq. (27.5)} \\ &= (-50 \cos 120\pi t \sin 120\pi t) (120\pi) \\ &= -6000\pi \cos 120\pi t \sin 120\pi t \\ &= -3000\pi \sin 240\pi t && \text{using Eq. (20.21)}\end{aligned}$$

■ For reference, Eq. (20.21) is  $\sin 2\alpha = 2 \sin \alpha \cos \alpha$ .

### EXAMPLE 4 Derivative of a root containing $\cos u$

Find the derivative of  $y = \sqrt{1 + \cos 2x}$ .

$$\begin{aligned}y &= (1 + \cos 2x)^{1/2} \\ \frac{dy}{dx} &= \frac{1}{2} (1 + \cos 2x)^{-1/2} \frac{d(1 + \cos 2x)}{dx} && \text{using Eq. (23.15)} \\ &= \frac{1}{2} (1 + \cos 2x)^{-1/2} (-\sin 2x) (2) && \text{using Eq. (27.5)} \\ &= -\frac{\sin 2x}{\sqrt{1 + \cos 2x}}\end{aligned}$$

#### Practice Exercise

2. Find the derivative of  $y = 5(1 + \cos x^2)^2$ .

### EXAMPLE 5 Derivative of the product $\sin u \cos u$

Find the differential of  $y = \sin 2x \cos x^2$ .

From Section 24.8, recall that the differential of a function  $y = f(x)$  is  $dy = f'(x) dx$ . Thus, using the derivative product rule and the derivatives of the sine and cosine functions, we arrive at the following result:

$$\begin{aligned}y &= \sin 2x \cos x^2 && y = (\sin 2x) \cdot (\cos x^2) \\ dy &= [\sin 2x (-\sin x^2) (2x) + \cos x^2 (\cos 2x) (2)] dx \\ &= (-2x \sin 2x \sin x^2 + 2 \cos 2x \cos x^2) dx\end{aligned}$$

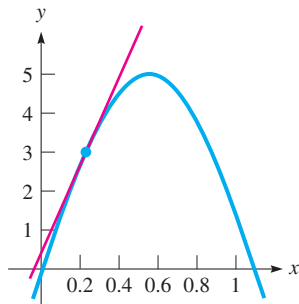


Fig. 27.2

**EXAMPLE 6** Slope of a tangent

Find the slope of a line tangent to the curve of  $y = 5 \sin 3x$  at  $x = 0.2$ .

Here, we are to find the derivative of  $y = 5 \sin 3x$  and then evaluate the derivative for  $x = 0.2$ . Therefore, we have the following:

$$y = 5 \sin 3x$$

$$\frac{dy}{dx} = 5(\cos 3x)(3) = 15 \cos 3x \quad \text{find derivative}$$

$$\begin{aligned} \left. \frac{dy}{dx} \right|_{x=0.2} &= 15 \cos 3(0.2) = 15 \cos 0.6 && \text{evaluate} \\ &= 12.38 \end{aligned}$$

In evaluating the slope, we must remember that  $x = 0.2$  means the *values are in radians*. Therefore, the slope is 12.38.

The curve and the tangent line at  $x = 0.2$  are shown in Fig. 27.2. The tangent line and its slope can be checked on some graphing calculators using the *tangent* and *numerical derivative* features. ■

**EXERCISES 27.1**

In Exercises 1 and 2, make the given changes in the indicated examples of this section and then find the derivatives.

- In Example 2, in the given function, change  $\theta$  to  $2\theta^2$ .
- In Example 4, in the given function, change  $2x$  to  $x^2$ .

In Exercises 3–34, find the derivatives of the given functions.

- |   |   |
|---|---|
| 3. $y = \sin(x + 2)$                          | 4. $y = 3 \sin 4x$                          |
| 5. $y = 2 \sin(2x^3 - 1)$                     | 6. $s = 5 \sin(7 - 3t)$                     |
| 7. $y = 6 \cos \frac{1}{2}x$                  | 8. $y = \cos(1 - x)$                        |
| 9. $y = 2 \cos(3x - \pi)$                     | 10. $y = 4 \cos(6x^2 + 5)$                  |
| 11. $r = \sin^2 3\pi\theta$                   | 12. $y = 3 \sin^3(2x^4 + 1)$                |
| 13. $y = 3 \cos^3(5x + 2)$                    | 14. $y = 4 \cos^2 \sqrt{x}$                 |
| 15. $y = x \sin 3x$                           | 16. $v = 6t^2 \sin 3\pi t$                  |
| 17. $y = 3x^3 \cos 5x$                        | 18. $y = 0.5\theta \cos(2\theta + \pi/4)$   |
| 19. $u = 3 \sin v^2 \cos 5v$                  | 20. $y = 6 \sin x \cos 4x$                  |
| 21. $y = \sqrt{1 + \sin 4x}$                  | 22. $y = (x - \cos^2 x)^4$                  |
| 23. $r = \frac{\sin(3t - \pi/3)}{2t}$         | 24. $T = \frac{4z + 3}{\sin \pi z}$         |
| 25. $y = \frac{2 \cos x^2}{3x - 1}$           | 26. $y = \frac{\cos^2 3x}{1 + 2 \sin^2 2x}$ |
| 27. $y = 2 \sin^2 3x \cos 2x$                 | 28. $y = \cos^3 4x \sin^2 2x$               |
| 29. $s = \sin(\sin 2t)$                       | 30. $z = 0.2 \cos(\sin 3\phi)$              |
| 31. $y = \sin^3 x - \cos 2x$                  | 32. $y = x \sin x + \cos x$                 |
| 33. $p = \frac{1}{\sin s} + \frac{1}{\cos s}$ |   |
| 34. $y = 2x \sin x + 2 \cos x - x^2 \cos x$   |   |

In Exercises 35–56, solve the given problems.

- Using a graphing calculator: (a) display the graph of  $y = (\sin x)/x$  to verify that  $(\sin \theta)/\theta \rightarrow 1$  as  $\theta \rightarrow 0$ , and (b) verify the values for  $(\sin \theta)/\theta$  in the table on page 815.
- Evaluate  $\lim_{\theta \rightarrow 0} (\tan \theta)/\theta$ . (Use the fact that  $\lim_{\theta \rightarrow 0} (\sin \theta)/\theta = 1$ .)
- On a calculator, find the values of (a)  $\cos 1.0000$  and (b)  $(\sin 1.0001 - \sin 1.0000)/0.0001$ . Compare the values and give the meaning of each in relation to the derivative of the sine function where  $x = 1$ .
- On a calculator, find the values of (a)  $-\sin 1.0000$  and (b)  $(\cos 1.0001 - \cos 1.0000)/0.0001$ . Compare the values and give the meaning of each in relation to the derivative of the cosine function where  $x = 1$ .
- On the graph of  $y = \sin x$  in Fig. 27.3, draw tangent lines at the indicated points and determine the slopes of these tangent lines. Then plot the values of these slopes for the same values of  $x$  and join the points with a smooth curve. Compare the resulting curve with  $y = \cos x$ . (Recall the meaning of the derivative as the slope of a tangent line.)

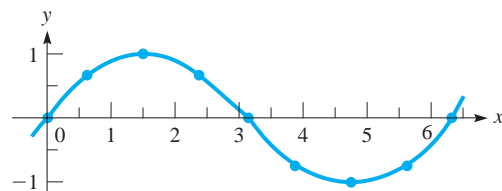


Fig. 27.3

40. Repeat the instructions given in Exercise 39 for the graph of  $y = \cos x$  in Fig. 27.4. Compare the resulting curve with  $y = \sin x$ . (Be careful in this comparison and remember the difference between  $y = \sin x$  and the derivative of  $y = \cos x$ . As in Exercise 39, note the meaning of the derivative as the slope of a tangent line.)

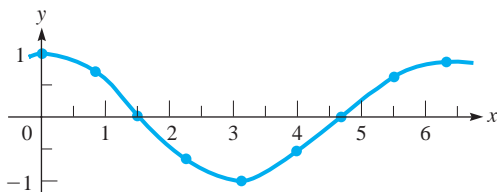


Fig. 27.4

41. Find the derivative of the implicit function  $\sin(xy) + \cos 2y = x^2$ .
42. Find the derivative of the implicit function  $x \cos 2y + \sin x \cos y = 1$ .
43. Show that  $\frac{d^4 \sin x}{dx^4} = \sin x$ .
44. Show that  $y = A \sin kx + B \cos kx$  satisfies the equation  $y'' + k^2 y = 0$ .
45. Find the derivative of each member of the identity  $\cos 2x = 2 \cos^2 x - 1$  and thereby obtain another trigonometric identity.
46. Find values of  $x$  for which the following curves have horizontal tangents: (a)  $y = x + \sin x$  and (b)  $y = 4x + \cos \pi x$ .
47. Use differentials to estimate the value of  $\sin 31^\circ$ .
48. Find the linearization  $L(x)$  of the function  $f(x) = \sin(\cos x)$  for  $a = \pi/2$ .
49. Find the slope of a line tangent to the curve  $y = \frac{2 \sin 3x}{x}$ , where  $x = 0.15$ .
50. An object is oscillating vertically on the end of a spring such that its displacement  $d$  (in cm) is  $d = 2.5 \cos 16t$ , where  $t$  is the time (in s). What is the acceleration of the object after 1.5 s?
51. The voltage  $V$  in a certain electric circuit as a function of the time  $t$  (in s) is given by  $V = 3.00 \sin 188t \cos 188t$ . How fast is the voltage changing when  $t = 2.00$  ms?

52. A water slide at an amusement park follows the curve ( $y$  in m)  $y = 2.0 + 2.0 \cos(0.53x + 0.40)$  for  $0 \leq x \leq 5.0$  m. Find the angle with the horizontal of the slide for  $x = 2.5$  m.
53. The blade of a sabre saw moves vertically up and down, and its displacement  $y$  (in cm) is given by  $y = 1.85 \sin 36\pi t$ , where  $t$  is the time (in s). Find the velocity of the blade for  $t = 0.0250$  s.
54. The current  $i$  (in A) in an amplifier circuit as a function of the time  $t$  (in s) is given by  $i = 0.10 \cos(120\pi t + \pi/6)$ . Find the expression for the voltage across a 2.0-mH inductor in the circuit. (See Exercise 27 of Section 26.1 on page 782.)
55. In testing a heat-seeking rocket, it is found to be always moving directly toward a remote-controlled aircraft. At a certain instant, the distance  $r$  (in km) from the rocket to the aircraft is  $r = \frac{100}{1 - \cos \theta}$ , where  $\theta$  is the angle between their directions of flight. Find  $dr/d\theta$  for  $\theta = 120^\circ$ . See Fig. 27.5.

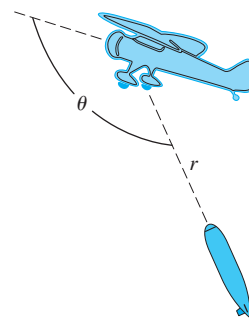


Fig. 27.5

56. The number  $N$  of reflections of a light ray passing through an optic fibre of length  $L$  and diameter  $d$  is  $N = \frac{L \sin \theta}{d \sqrt{n^2 - \sin^2 \theta}}$ . Here,  $n$  is the index of refraction of the fibre, and  $\theta$  is the angle between the light ray and the fibre's axis. Find  $dN/d\theta$ .

### Answers to Practice Exercises

1.  $y' = 12 \cos(4x + 1)$     2.  $y' = -20x \sin x^2(1 + \cos x^2)$

## 27.2 Derivatives of the Other Trigonometric Functions

### Derivatives of $\tan u$ , $\cot u$ , $\sec u$ , $\csc u$

We obtain the derivative of  $\tan u$  by expressing  $\tan u$  as  $\sin u/\cos u$ . Therefore, letting  $y = \sin u/\cos u$ , by employing the quotient rule, we have

$$\begin{aligned} \frac{dy}{dx} &= \frac{\cos u[\cos u(du/dx)] - \sin u[-\sin u(du/dx)]}{\cos^2 u} \\ &= \frac{\cos^2 u + \sin^2 u}{\cos^2 u} \frac{du}{dx} = \frac{1}{\cos^2 u} \frac{du}{dx} = \sec^2 u \frac{du}{dx} \end{aligned}$$

$$\frac{d(\tan u)}{dx} = \sec^2 u \frac{du}{dx} \quad (24.6)$$

We find the derivative of  $\cot u$  by letting  $y = \cos u/\sin u$ , again using the quotient rule.

$$\begin{aligned}\frac{dy}{dx} &= \frac{\sin u[-\sin u(du/dx)] - \cos u[\cos u(du/dx)]}{\sin^2 u} \\ &= \frac{-\sin^2 u - \cos^2 u}{\sin^2 u} \frac{du}{dx}\end{aligned}$$

$$\frac{d(\cot u)}{dx} = -\csc^2 u \frac{du}{dx} \quad (27.7)$$

To obtain the derivative of  $\sec u$ , we let  $y = 1/\cos u$ . Then,

$$\frac{dy}{dx} = -(\cos u)^{-2} \left[ (-\sin u) \left( \frac{du}{dx} \right) \right] = \frac{1}{\cos u} \frac{\sin u}{\cos u} \frac{du}{dx}$$

$$\frac{d(\sec u)}{dx} = \sec u \tan u \frac{du}{dx} \quad (27.8)$$

We obtain the derivative of  $\csc u$  by letting  $y = 1/\sin u$ . And so,

$$\frac{dy}{dx} = -(\sin u)^{-2} \left( \cos u \frac{du}{dx} \right) = -\frac{1}{\sin u} \frac{\cos u}{\sin u} \frac{du}{dx}$$

$$\frac{d(\csc u)}{dx} = -\csc u \cot u \frac{du}{dx} \quad (27.9)$$

### EXAMPLE 1 Derivative of a power of $\sec u$

Find the derivative of  $y = 3 \sec^2 4x$ .

Using the power rule and Eq. (27.8), we have

$$\begin{aligned}\frac{dy}{dx} &= 3(2)(\sec 4x) \frac{d(\sec 4x)}{dx} && \text{using } \frac{du^n}{dx} = nu^{n-1} \frac{du}{dx} \\ &= 6(\sec 4x)(\sec 4x \tan 4x)(4) && \text{using } \frac{d(\sec u)}{dx} = \sec u \tan u \frac{du}{dx} \\ &= 24 \sec^2 4x \tan 4x\end{aligned}$$

### EXAMPLE 2 Derivative of a product with $\csc u$

Find the derivative of  $y = t \csc^3 2t$ .

Using the power rule, the product rule, and Eq. (27.9), we have

$$\begin{aligned}\frac{dy}{dt} &= t(3 \csc^2 2t)(-\csc 2t \cot 2t)(2) + (\csc^3 2t)(1) \\ &= \csc^3 2t(-6t \cot 2t + 1)\end{aligned}$$

### EXAMPLE 3 Derivative of a power with $\tan u$ and $\sec u$

Find the derivative of  $y = (\tan 2x + \sec 2x)^3$ .

Using the power rule and Eqs. (27.6) and (27.8), we have

$$\begin{aligned}\frac{dy}{dx} &= 3(\tan 2x + \sec 2x)^2 [\sec^2 2x(2) + \sec 2x \tan 2x(2)] \\ &= 3(\tan 2x + \sec 2x)^2 (2 \sec 2x)(\sec 2x + \tan 2x) \\ &= 6 \sec 2x (\tan 2x + \sec 2x)^3\end{aligned}$$

#### Practice Exercise

- Find the derivative of  $y = 3 \tan 8x$ .

#### Practice Exercise

- Find the derivative of  $y = 5(\cot 3x + \csc 3x)^2$ .

**EXAMPLE 4** Differential with  $\sin u$  and  $\tan u$ 

Find the differential of  $r = \sin 2\theta \tan \theta^2$ .

Here, we are to find the derivative of the given function and multiply by  $d\theta$ . Therefore, using the product rule along with Eqs. (27.4) and (27.6), we have

$$\begin{aligned} dr &= [(\sin 2\theta)(\sec^2 \theta^2)(2\theta) + (\tan \theta^2)(\cos 2\theta)(2)] d\theta \\ &= (2\theta \sin 2\theta \sec^2 \theta^2 + 2 \cos 2\theta \tan \theta^2) d\theta \end{aligned}$$

don't forget the  $d\theta$  ■

**EXAMPLE 5** Derivative of an implicit function

Find  $dy/dx$  if  $\cot 2x - 3 \csc xy = y^2$ .

In finding the derivative of this implicit function, we must be careful not to forget the factor  $dy/dx$  when it occurs. The derivative is found as follows:

$$\begin{aligned} \cot 2x - 3 \csc xy &= y^2 \\ (-\csc^2 2x)(2) - 3(-\csc xy \cot xy) \left( x \frac{dy}{dx} + y \right) &= 2y \frac{dy}{dx} \\ 3x \csc xy \cot xy \frac{dy}{dx} - 2y \frac{dy}{dx} &= 2 \csc^2 2x - 3y \csc xy \cot xy \\ \frac{dy}{dx} &= \frac{2 \csc^2 2x - 3y \csc xy \cot xy}{3x \csc xy \cot xy - 2y} \end{aligned}$$

**EXAMPLE 6** Evaluation of a derivative

Evaluate the derivative of  $y = \frac{2x}{1 - \cot 3x}$ , where  $x = 0.25$ .

Finding the derivative, we have

$$\begin{aligned} \frac{dy}{dx} &= \frac{(1 - \cot 3x)(2) - 2x(\csc^2 3x)(3)}{(1 - \cot 3x)^2} \\ &= \frac{2 - 2 \cot 3x - 6x \csc^2 3x}{(1 - \cot 3x)^2} \end{aligned}$$

Now, substituting  $x = 0.25$ , we have

$$\begin{aligned} \left. \frac{dy}{dx} \right|_{x=0.25} &= \frac{2 - 2 \cot 0.75 - 6(0.25) \csc^2 0.75}{(1 - \cot 0.75)^2} \\ &= -626.0 \end{aligned}$$

In the above calculation, we have used the calculator in *radian mode*. Moreover, we have calculated  $\cot 0.75$  and  $\csc 0.75$  as the reciprocals of  $\tan 0.75$  and  $\sin 0.75$ , respectively. ■

**EXERCISES 27.2**

In Exercises 1 and 2, make the given changes in the indicated examples of this section and then find the derivatives.

- In Example 1, in the given function, change  $4x$  to  $x^2$ .
- In Example 4, in the given function, change  $\theta^2$  to  $3\theta$ .

In Exercises 3–34, find the derivatives of the given functions.

- $y = \tan 5x$
- $y = 5 \cot(0.25\pi - \theta)$
- $u = 3 \sec 5v$
- $y = 3 \tan(3x + 2)$
- $y = 3 \cot 6x$
- $y = \sec \sqrt{1 - x}$

- $y = -3 \csc \sqrt{2x + 3}$
- $R = 5 \tan^2 3\pi t$
- $y = 2 \cot^4 \frac{1}{2}x$
- $y = \sqrt{\sec 4x}$
- $y = 3 \csc^4 7x$
- $r = t^2 \tan 0.5t$
- $y = 4 \cos x \csc x^2$
- $y = \frac{\csc x}{x}$
- $h = 0.5 \csc(1 - 2\pi t)$
- $y = 2 \tan^2(x^2)$
- $p = 3 \cot^2(4 - 3r^2)$
- $y = 0.8 \sec^3 5u$
- $y = 7 \csc^2(9x^3)$
- $y = 3x \sec 2\pi x$
- $y = \frac{1}{2} \sin 2x \sec x$
- $u = \frac{\cot 0.25z}{2z}$

25.  $y = \frac{2 \cos 4x}{1 + \cot 3x}$       26.  $y = \frac{\tan^2 3x}{2 + \sin x^2}$   
 27.  $y = \frac{1}{3} \tan^3 x - \tan x$       28.  $y = 4 \csc 4x - 2 \cot 4x$   
 29.  $r = \tan(\sin 2\pi\theta)$       30.  $y = x \tan x + \sec^2 2x$   
 31.  $y = \sqrt{2x + \tan 4x}$       32.  $V = (1 - \csc^2 3r)^3$   
 33.  $x \sec y - 2y = \sin 2x$       34.  $3 \cot(x + y) = \cos y^2$

In Exercises 35–38, find the differentials of the given functions.

35.  $y = 4 \tan^2 3x$       36.  $y = 2.5 \sec^3 2t$   
 37.  $y = \tan 4x \sec 4x$       38.  $y = 2x \cot 3x$

In Exercises 39–52, solve the given problems.

39. On a calculator, find the values of (a)  $\sec^2 1.0000$  and (b)  $(\tan 1.0001 - \tan 1.0000)/0.0001$ . Compare the values and give the meaning of each in relation to the derivative of  $\tan x$  where  $x = 1$ .  
 40. On a calculator, find the values of (a)  $\sec 1.0000 \tan 1.0000$  and (b)  $(\sec 1.0001 - \sec 1.0000)/0.0001$ . Compare the values and give the meaning of each in relation to the derivative of  $\sec x$  where  $x = 1$ .  
 41. (a) Display the graph of  $y = \tan x$  on a graphing calculator, and using the *derivative* feature, evaluate  $dy/dx$  for  $x = 1$ . (b) Display the graph of  $y = \sec^2 x$  and evaluate  $y'$  for  $x = 1$ . (Compare the values in parts (a) and (b).)  
 42. Follow the instructions in Exercise 41, using the graphs of  $y = \sec x$  and  $y = \sec x \tan x$ .  
 43. Find the derivative of each member of the identity  $1 + \tan^2 x = \sec^2 x$  and show that the results are equal.  
 44. Find the points where a tangent to the curve of  $y = \tan x$  is parallel to the line  $y = 2x$  if  $0 < x < 2\pi$ .  
 45. Find the slope of a line tangent to the curve of  $y = 2 \cot 3x$  where  $x = \pi/12$ .  
 46. Find the slope of a line normal to the curve of  $y = \csc \sqrt{2x + 1}$  where  $x = 0.45$ .  
 47. Show that  $y = 2 \tan x - \sec x$  satisfies  $\frac{dy}{dx} = \frac{2 - \sin x}{\cos^2 x}$ .

48. A helicopter takes off such that its height  $h$  (in m) above the ground is  $h = 25 \sec 0.16t$  for the first 8.0 s of flight. What is its vertical velocity after 6.0 s?  
 49. The vertical displacement  $y$  (in cm) of the end of an industrial robot arm for each cycle is  $y = 2.0t^{1.5} - \tan 0.10t$ , where  $t$  is the time (in s). Find its vertical velocity for  $t = 15$  s.  
 50. The electric charge  $q$  (in C) passing a given point in a circuit is given by  $q = t \sec \sqrt{0.20t^2 + 1.0}$ , where  $t$  is the time (in s). Find the current  $i$  (in A) for  $t = 0.80$  s. ( $i = dq/dt$ )  
 51. An observer to a rocket launch was 1000 m from the takeoff position. The observer found the angle of elevation of the rocket as a function of time to be  $\theta = 3t/(2t + 10)$ . Therefore, the height  $h$  (in m) of the rocket was  $h = 1000 \tan \frac{3t}{2t + 10}$ . Find the time rate of change of height after 5.0 s. See Fig. 27.6.

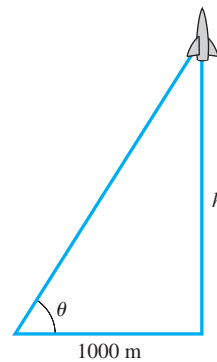


Fig. 27.6

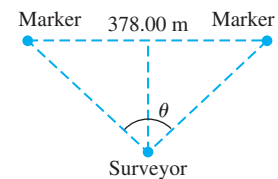


Fig. 27.7

52. A surveyor measures the distance between two markers to be 378.00 m. Then, moving along a line equidistant from the markers, the distance  $d$  from the surveyor to each marker is  $d = 189.00 \csc \frac{1}{2}\theta$ , where  $\theta$  is the angle between the lines of sight to the markers. See Fig. 27.7. By using differentials, find the change in  $d$  if  $\theta$  changes from  $98.20^\circ$  to  $98.45^\circ$ .

### Answers to Practice Exercises

1.  $y' = 24 \sec^2 8x$       2.  $y' = -30 \csc 3x (\cot 3x + \csc 3x)^2$

## 27.3 Derivatives of the Inverse Trigonometric Functions

### Derivatives of $\sin^{-1} u$ , $\cos^{-1} u$ , $\tan^{-1} u$

To obtain the derivative of  $y = \sin^{-1} u$ , we first solve for  $u$  in the form  $u = \sin y$ , and then take the derivative with respect to  $x$ . This results in  $\frac{du}{dx} = \cos y \frac{dy}{dx}$ . Solving this for  $dy/dx$ , we have

$$\frac{dy}{dx} = \frac{1}{\cos y} \frac{du}{dx} = \frac{1}{\sqrt{1 - \sin^2 y}} \frac{du}{dx} = \frac{1}{\sqrt{1 - u^2}} \frac{du}{dx}$$

We choose the positive square root since  $\cos y > 0$  for  $-\frac{\pi}{2} < y < \frac{\pi}{2}$ , which is the range of the defined values of  $\sin^{-1} u$ . Therefore, we obtain the following result:

$$\frac{d(\sin^{-1} u)}{dx} = \frac{1}{\sqrt{1 - u^2}} \frac{du}{dx} \quad (27.10)$$

■ Note that the derivative of the inverse sine function is an algebraic function.



**EXAMPLE 1** Derivative of  $\sin^{-1}u$ 

Find the derivative of  $y = \sin^{-1} 4x$ .

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - (4x)^2}} (4) = \frac{4}{\sqrt{1 - 16x^2}} \quad \text{using Eq. (27.10)}$$

We find the derivative of the inverse cosine function by letting  $y = \cos^{-1} u$  and by following the same procedure as that used in finding the derivative of  $\sin^{-1} u$ :

$$u = \cos y, \quad \frac{du}{dx} = -\sin y \frac{dy}{dx}$$

$$\frac{dy}{dx} = -\frac{1}{\sin y} \frac{du}{dx} = -\frac{1}{\sqrt{1 - \cos^2 y}} \frac{du}{dx}$$

$$\frac{d(\cos^{-1} u)}{dx} = -\frac{1}{\sqrt{1 - u^2}} \frac{du}{dx} \quad (27.11)$$

**Practice Exercise**

1. Find the derivative of  $y = 5 \sin^{-1} x^2$ .

The positive square root is chosen here since  $\sin y > 0$  for  $0 < y < \pi$ , which is the range of the defined values of  $\cos^{-1} u$ . We note that the derivative of the inverse cosine is the negative of the derivative of the inverse sine.

By letting  $y = \tan^{-1} u$ , solving for  $u$ , and taking derivatives, we find the derivative of the inverse tangent function:

$$u = \tan y, \quad \frac{du}{dx} = \sec^2 y \frac{dy}{dx}, \quad \frac{dy}{dx} = \frac{1}{\sec^2 y} \frac{du}{dx} = \frac{1}{1 + \tan^2 y} \frac{du}{dx}$$

$$\frac{d(\tan^{-1} u)}{dx} = \frac{1}{1 + u^2} \frac{du}{dx} \quad (27.12)$$

We can see that the derivative of the inverse tangent is also an algebraic function.

The inverse sine, inverse cosine, and inverse tangent prove to be the inverse trigonometric functions of greatest importance in applications and in further development of mathematics. Therefore, the formulas for the derivatives of the other inverse functions are not presented here, although they are included in the exercises.

**EXAMPLE 2** Derivative of  $\cos^{-1}u$ —application

A 20-N force acts on a sign as shown in Fig. 27.8. Express the angle  $\theta$  as a function of the  $x$ -component,  $F_x$ , of the force, and then find the expression for the instantaneous rate of change of  $\theta$  with respect to  $F_x$ .

From the figure, we see that  $F_x = 20 \cos \theta$ . Solving for  $\theta$ , we have  $\theta = \cos^{-1}(F_x/20)$ . To find the instantaneous rate of change of  $\theta$  with respect to  $F_x$ , we are to take the derivative  $d\theta/dF_x$ :

$$\theta = \cos^{-1} \frac{F_x}{20} = \cos^{-1} 0.05F_x$$

$$\frac{d\theta}{dF_x} = -\frac{1}{\sqrt{1 - (0.05F_x)^2}} (0.05) \quad \text{using Eq. (27.11)}$$

$$= \frac{-0.05}{\sqrt{1 - 0.0025F_x^2}}$$

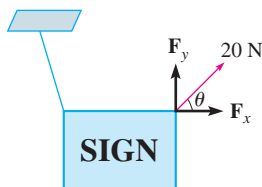


Fig. 27.8

**EXAMPLE 3** Derivative of  $\tan^{-1} u$ 

Find the derivative of  $y = (x^2 + 1)\tan^{-1} x - x$ .

Using the product rule along with Eq. (27.12) on the first term, we have

$$\begin{aligned}\frac{dy}{dx} &= (x^2 + 1) \underbrace{\left(\frac{1}{1+x^2}\right)}_{\text{using Eq. (27.12)}} (1) + (\tan^{-1} x)(2x) - 1 \\ &= 2x \tan^{-1} x\end{aligned}$$

**Practice Exercise**

2. Find the derivative of  $y = (\tan^{-1} 3x)^2$ .

**EXAMPLE 4** Differential of  $\sin^{-1} u$ 

Find the derivative of  $y = x \sin^{-1} 2x + \frac{1}{2}\sqrt{1-4x^2}$ .

using Eq. (27.10)

using Eq. (23.15)

$$\begin{aligned}\frac{dy}{dx} &= x \left(\frac{2}{\sqrt{1-4x^2}}\right) + \sin^{-1} 2x + \frac{1}{2} \left(\frac{1}{2}\right) (1-4x^2)^{-1/2} (-8x) \\ &= \frac{2x}{\sqrt{1-4x^2}} + \sin^{-1} 2x - \frac{2x}{\sqrt{1-4x^2}} \\ &= \sin^{-1} 2x\end{aligned}$$

■ For reference, Eq. (23.15) is

$$\frac{du^n}{dx} = nu^{n-1} \left(\frac{du}{dx}\right).$$

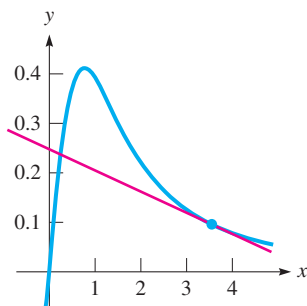


Fig. 27.9

**EXAMPLE 5** Tangent line for a quotient with  $\tan^{-1} u$ 

Find the slope of a tangent to the curve of  $y = \frac{\tan^{-1} x}{x^2 + 1}$ , where  $x = 3.60$ . In Fig. 27.9, the function and the tangent line are shown.

Here, we are to find the derivative and then evaluate it for  $x = 3.60$ .

$$\begin{aligned}\frac{dy}{dx} &= \frac{(x^2 + 1) \left(\frac{1}{1+x^2}\right) (1) - (\tan^{-1} x)(2x)}{(x^2 + 1)^2} && \text{take derivative} \\ &= \frac{1 - 2x \tan^{-1} x}{(x^2 + 1)^2} \\ \frac{dy}{dx} \Big|_{x=3.60} &= \frac{1 - 2(3.60)(\tan^{-1} 3.60)}{(3.60^2 + 1)^2} = -0.0429 && \text{evaluate}\end{aligned}$$

**EXERCISES 27.3**

In Exercises 1 and 2, make the given changes in the indicated examples of this section and then find the derivatives.

- In Example 1, in the given function, change  $4x$  to  $x^2$ .
- In Example 3, in the given function, change  $(x^2 + 1)\tan^{-1} x$  to  $(4x^2 + 1)\tan^{-1} 2x$ .

In Exercises 3–34, find the derivatives of the given functions.

- $y = \sin^{-1} 7x$
- $R = 3 \sin^{-1}(4 - t^2)$
- $y = 2 \sin^{-1} 3x^3$
- $y = \sin^{-1} \sqrt{1 - 2x}$
- $y = 3.6 \cos^{-1} 0.5s$
- $\theta = 0.2 \cos^{-1} 5t$
- $y = 2 \cos^{-1} \sqrt{2 - x}$
- $y = 3 \cos^{-1}(x^2 + 0.5)$
- $V = 8 \tan^{-1} \sqrt{s}$
- $y = \tan^{-1}(1 - x)$
- $y = 6 \tan^{-1}(1/x)$
- $w = 4 \tan^{-1} \pi u^4$
- $y = 5x \sin^{-1} x$
- $y = x^2 \cos^{-1} x$
- $v = 0.4u \tan^{-1} 2u$
- $T = \frac{3R - 1}{\sin^{-1} 2R}$
- $y = \frac{\sin^{-1} 2x}{\cos^{-1} 2x}$
- $y = 2(\cos^{-1} 4x)^3$
- $u = [\sin^{-1}(4t + 3)]^2$
- $y = \tan^{-1}\left(\frac{1-t}{1+t}\right)$
- $y = \frac{1}{1+4x^2} - \tan^{-1} 2x$
- $y = 3(4 - \cos^{-1} 2x)^3$
- $2 \tan^{-1} xy + x = 3$
- $y = (x^2 + 1) \sin^{-1} 4x$
- $\theta = \frac{\tan^{-1} 2r}{\pi r}$
- $y = \frac{x^2 + 1}{\tan^{-1} x}$
- $r = 0.5(\sin^{-1} 3t)^4$
- $y = \sqrt{\sin^{-1}(x - 1)}$
- $p = \frac{3}{\cos^{-1} 2w}$
- $y = \sin^{-1} x - \sqrt{1 - x^2}$
- $\sin^{-1}(x + y) + y = x^2$
- $y = \sqrt{2\pi - \sin^{-1} 4x}$

In Exercises 35–54, solve the given problems.

35. On a calculator, find the values of (a)  $1/\sqrt{1-0.5^2}$  and (b)  $(\sin^{-1} 0.5001 - \sin^{-1} 0.5000)/0.0001$ . Compare the values and give the meaning of each in relation to the derivative of  $\sin^{-1} x$  where  $x = 0.5$ .
36. On a calculator, find the values of (a)  $1/(1+0.5^2)$  and (b)  $(\tan^{-1} 0.5001 - \tan^{-1} 0.5000)/0.0001$ . Compare the values and give the meaning of each in relation to the derivative of  $\tan^{-1} x$  where  $x = 0.5$ .
37. Find the differential of the function  $y = (\sin^{-1} x)^3$ .
38. Find the linearization  $L(x)$  of the function  $f(x) = 2x \cos^{-1} x$  for  $a = 0$ .
39. Find the slope of a line tangent to the curve of  $y = x/\tan^{-1} x$  at  $x = 0.80$ .
40. Explain what is wrong with a problem that requires finding the derivative of  $y = \sin^{-1}(x^2 + 1)$ .
41. Find the second derivative of  $y = x \tan^{-1} x$ .
42. Find the point(s) at which the line normal to  $y = 2 \sin^{-1} 0.5x$  is parallel to the line  $y = 1 - x$ .
43. Use a graphing calculator to display the graphs of  $y = \sin^{-1} x$  and  $y = 1/\sqrt{1-x^2}$ . By roughly estimating slopes of tangent lines of  $y = \sin^{-1} x$ , note that  $y = 1/\sqrt{1-x^2}$  gives reasonable values for the derivative of  $y = \sin^{-1} x$ .
44. Use a graphing calculator to display the graphs of  $y = \tan^{-1} x$  and  $y = 1/(1+x^2)$ . By roughly estimating slopes of tangent lines of  $y = \tan^{-1} x$ , note that  $y = 1/(1+x^2)$  gives reasonable values for the derivative of  $y = \tan^{-1} x$ .
45. Find the second derivative of the function  $y = \tan^{-1} 2x$ .
46. Show that  $\frac{d(\cot^{-1} u)}{dx} = -\frac{1}{1+u^2} \frac{du}{dx}$ .
47. Show that  $\frac{d(\sec^{-1} u)}{dx} = \frac{1}{\sqrt{u^2(u^2-1)}} \frac{du}{dx}$ .
48. Show that  $\frac{d(\csc^{-1} u)}{dx} = -\frac{1}{\sqrt{u^2(u^2-1)}} \frac{du}{dx}$ .

## 27.4 Applications

Tangents and Normals • Newton's Method • Time Rates of Change • Curve Sketching • Maximum and Minimum Problems • Differentials

With our development of the formulas for the derivatives of the trigonometric and inverse trigonometric functions, it is now possible to use these derivatives in the same manner as we applied the derivatives of algebraic functions. We can now use these functions in the types of applications listed at the left as shown in the examples and exercises of this section.

### EXAMPLE 1 Sketching a curve

Sketch the curve  $y = \sin^2 x - \frac{x}{2}$  ( $0 \leq x \leq 2\pi$ ).

First, by setting  $x = 0$ , we see that the only easily obtainable intercept is  $(0, 0)$ . Replacing  $x$  by  $-x$  and  $y$  by  $-y$ , we find that the curve is not symmetric to either axis or to the origin. Also, since  $x$  does not appear in a denominator, there are no vertical asymptotes. We are considering only the restricted domain  $0 \leq x \leq 2\pi$ . (Without this restriction, the domain is all  $x$  and the range is all  $y$ .)

49. In the analysis of the waveform of an AM radio wave, the equation  $t = \frac{1}{\omega} \sin^{-1} \frac{A-E}{mE}$  arises. Find  $dt/dm$ , assuming that the other quantities are constant.
50. An equation that arises in the theory of solar collectors is  $\alpha = \cos^{-1} \frac{2f-r}{r}$ . Find the expression for  $d\alpha/dr$  if  $f$  is constant.
51. When an alternating current passes through a series  $RLC$  circuit, the voltage and current are out of phase by angle  $\theta$  (see Section 12.7). Here  $\theta = \tan^{-1} [(X_L - X_C)/R]$ , where  $X_L$  and  $X_C$  are the reactances of the inductor and capacitor, respectively, and  $R$  is the resistance. Find  $d\theta/dX_C$  for constant  $X_L$  and  $R$ .
52. When passing through glass, a light ray is refracted (bent) such that the angle of refraction  $r$  is given by  $r = \sin^{-1}[(\sin i)/\mu]$ . Here,  $i$  is the angle of incidence, and  $\mu$  is the index of refraction of the glass (see Fig. 27.10). For different types of glass,  $\mu$  differs. Find the expression for  $dr$  for a constant value of  $i$ .

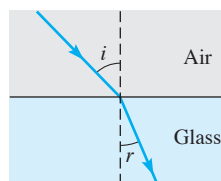


Fig. 27.10

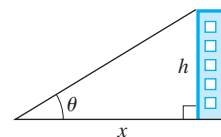


Fig. 27.11

53. As a person approaches a building of height  $h$ , the angle of elevation of the top of the building is a function of the person's distance from the building. Express the angle of elevation  $\theta$  in terms of  $h$  and the distance  $x$  from the building and then find  $d\theta/dx$ . Assume the person's height is negligible to that of the building. See Fig. 27.11.
54. A triangular metal frame is designed as shown in Fig. 27.12. Express angle  $A$  as a function of  $x$  and evaluate  $dA/dx$  for  $x = 6$  cm.

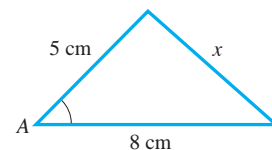


Fig. 27.12

### Answers to Practice Exercises

1.  $y' = 10x/\sqrt{1-x^4}$     2.  $y' = (6 \tan^{-1} 3x)/(1+9x^2)$

In order to find the information from the derivatives, we write

$$\frac{dy}{dx} = 2 \sin x \cos x - \frac{1}{2} = \sin 2x - \frac{1}{2}$$

$$\frac{d^2y}{dx^2} = 2 \cos 2x$$

Relative maximum and minimum points will occur for  $\sin 2x = 1/2$ . Thus, we have possible relative maximum and minimum points for

$$2x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \quad \text{or} \quad x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$$

Now, using the second derivative, we find that  $d^2y/dx^2$  is positive for  $x = \frac{\pi}{12}$  and  $x = \frac{13\pi}{12}$  and is negative for  $x = \frac{5\pi}{12}$  and  $x = \frac{17\pi}{12}$ . Thus, the maximum points are  $(\frac{5\pi}{12}, 0.279)$  and  $(\frac{17\pi}{12}, -1.29)$ . Minimum points are  $(\frac{\pi}{12}, -0.064)$  and  $(\frac{13\pi}{12}, -1.63)$ . Inflection points occur for  $\cos 2x = 0$ , or

$$2x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \quad \text{or} \quad x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

Therefore, the points of inflection are  $(\frac{\pi}{4}, 0.11)$ ,  $(\frac{3\pi}{4}, -0.68)$ ,  $(\frac{5\pi}{4}, -1.46)$ , and  $(\frac{7\pi}{4}, -2.25)$ . Using this information, we sketch the curve in Fig. 27.13. ■

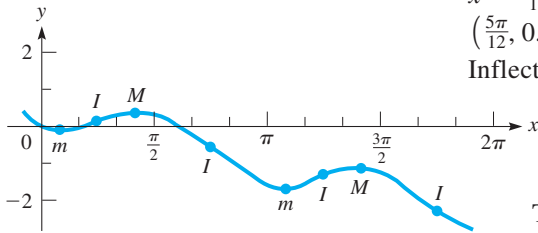


Fig. 27.13

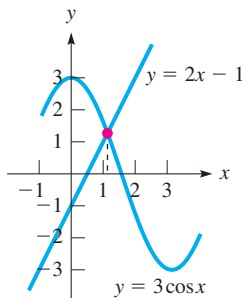


Fig. 27.14

**EXAMPLE 2** Solve an equation using Newton’s method

By using Newton’s method, solve the equation  $2x - 1 = 3 \cos x$ .

First, we locate the required root approximately by sketching  $y_1 = 2x - 1$  and  $y_2 = 3 \cos x$ . As we can see in Fig. 27.14, they intersect between  $x = 1$  and  $x = 2$ , near  $x = 1.2$ . Therefore, using  $x_1 = 1.2$ , with

$$f(x) = 2x - 1 - 3 \cos x$$

$$f'(x) = 2 + 3 \sin x$$

we use Eq. (24.1) with  $n = 2$ , which is

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

To find  $x_2$ , we have

$$f(1.2) = 2(1.2) - 1 - 3 \cos 1.2 = 0.312\ 926\ 7$$

$$f'(1.2) = 2 + 3 \sin 1.2 = 4.796\ 117\ 3$$

$$x_2 = 1.2 - \frac{0.312\ 926\ 7}{4.796\ 117\ 3} = 1.134\ 754\ 2$$

Finding the next approximation, we find  $x_3 = 1.134\ 236\ 6$ , which is accurate to the value shown. Again, when using the calculator, it is not necessary to list the values of  $f(x_1)$  and  $f'(x_1)$ , as the complete calculation can be done directly on the calculator. ■

**EXAMPLE 3** Finding a maximum value

Logs with a circular cross section 1.20 m in diameter are cut in half lengthwise. Find the largest rectangular cross-sectional area that can then be cut from one of the halves.

From Fig. 27.15, we see that  $x = 0.60 \cos \theta$  and  $y = 0.60 \sin \theta$ , which means the area of the rectangle inscribed within the semicircular area is

$$A = (2x)y = 2(0.60 \cos \theta)(0.60 \sin \theta) = 0.72 \cos \theta \sin \theta$$

$$= 0.36 \sin 2\theta \quad \text{using trigonometric identity (Eq. 20.21)}$$

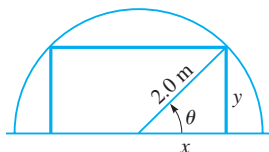


Fig. 27.15

Now, taking the derivative and setting it equal to zero, we have

$$\begin{aligned}\frac{dA}{d\theta} &= (0.36 \cos 2\theta)(2) = 0.72 \cos 2\theta \\ 0.72 \cos 2\theta &= 0, \quad 2\theta = \frac{\pi}{2}, \quad \theta = \frac{\pi}{4} \quad \cos \frac{\pi}{2} = 0\end{aligned}$$

(Using  $2\theta = \frac{3\pi}{2}$ ,  $\theta = \frac{3\pi}{4}$  leads to the same solution.) Since the minimum area is zero, we have the maximum area when  $\theta = \frac{\pi}{4}$ , and this maximum area is

$$A = 0.36 \sin 2\left(\frac{\pi}{4}\right) = 0.36 \sin \frac{\pi}{2} = 0.36 \text{ m}^2 \quad \sin \frac{\pi}{2} = 1$$

Therefore, the largest rectangular cross-sectional area is  $0.36 \text{ m}^2$ . ■

#### EXAMPLE 4 Finding the time rate of change

A rocket is taking off vertically at a distance of 6500 m from an observer. If, when the angle of elevation is  $38.4^\circ$ , it is changing at  $5.00^\circ/\text{s}$ , how fast is the rocket ascending?

From Fig. 27.16, letting  $x$  = the height of the rocket, we see that  $\tan \theta = x/6500$ , or  $\theta = \tan^{-1}(x/6500)$ . Taking derivatives with respect to time, we have

$$\frac{d\theta}{dt} = \frac{1}{1 + (x/6500)^2} \frac{dx/dt}{6500} = \frac{6500 dx/dt}{6500^2 + x^2}$$

We must remember to express angles in radians. This means  $d\theta/dt = 5.00^\circ/\text{s} = 0.0873 \text{ rad/s}$ . Substituting this value and  $x = 6500 \tan 38.4^\circ = 5150 \text{ m}$ , we have

$$0.0873 = \frac{6500 dx/dt}{6500^2 + 5150^2}$$

$$\frac{dx}{dt} = 924 \text{ m/s} \quad \blacksquare$$

#### EXAMPLE 5 Application of the differential

On level ground, 180 m from the base of a building, the angle of elevation of the top of the building is  $30.00^\circ$ . What error in calculating the height  $h$  of the building would be caused by an error of  $0.25^\circ$  in the angle?

From Fig. 27.17,  $h = 180 \tan \theta$ . To find the error in  $h$ , we must find the differential  $dh$ .

$$dh = 180 \sec^2 \theta d\theta$$

The possible error in  $\theta$  is  $0.25^\circ$ , which in *radian measure* is  $0.25\pi/180$ , which is the value we should use in the calculation of  $dh$ . Calculating  $dh$ , we have

$$dh = \frac{180(0.25\pi/180)}{\cos^2 \frac{\pi}{6}} = 1.05 \text{ m}$$

An error of  $0.25^\circ$  in the angle results in an error of over 1 m in the calculated value of the height. In using the calculator, we divide by  $\cos^2 \frac{\pi}{6}$  since  $\sec \theta = 1/\cos \theta$ . ■

■ See the chapter introduction.

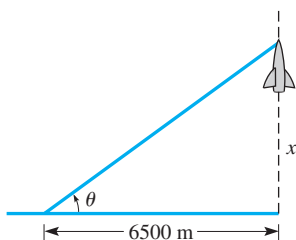


Fig. 27.16

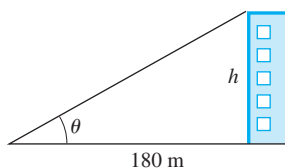


Fig. 27.17

**EXAMPLE 6** Curvilinear motion—parametric equations

A particle is rotating so that its  $x$ - and  $y$ -coordinates are given by  $x = \cos 2t$  and  $y = \sin 2t$ . Find the magnitude and direction of its velocity when  $t = \pi/8$ .

Taking derivatives with respect to time, we have

$$v_x = \frac{dx}{dt} = -2 \sin 2t \quad v_y = \frac{dy}{dt} = 2 \cos 2t$$

$$v_x|_{t=\pi/8} = -2 \sin 2\left(\frac{\pi}{8}\right) = -2\left(\frac{\sqrt{2}}{2}\right) = -\sqrt{2} \quad \text{evaluating}$$

$$v_y|_{t=\pi/8} = 2 \cos 2\left(\frac{\pi}{8}\right) = 2\left(\frac{\sqrt{2}}{2}\right) = \sqrt{2}$$

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{2 + 2} = 2 \quad \text{magnitude}$$

$$\tan \theta_{\text{ref}} = \left| \frac{v_y}{v_x} \right| = \frac{\sqrt{2}}{\sqrt{2}} = 1$$

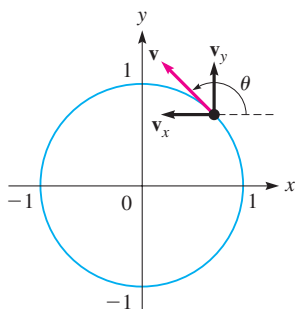


Fig. 27.18

Since  $v_x$  is negative and  $v_y$  is positive,  $\theta$  is in the second quadrant, so  $\theta = \pi - \pi/4 = 3\pi/4$  (or  $\theta = 135^\circ$ ).

We can plot the curve by squaring  $x$  and  $y$  and adding, thus removing the parameter  $t$ . This gives  $x^2 + y^2 = \cos^2 2t + \sin^2 2t = 1$ , so the curve is a circle of radius 1 (see Fig. 27.18). The direction of the resultant velocity tells us that the particle is moving counterclockwise around the circle. ■

**EXERCISES 27.4**

In Exercises 1 and 2, make the given changes in the indicated examples of this section and then solve the resulting problems.

- In Example 1, change  $\sin^2 x$  to  $\sin x$ .
- In Example 6, change  $\cos 2t$  to  $3 \cos 2t$ , and  $\sin 2t$  to  $2 \sin 2t$ .

In Exercises 3–38, solve the given problems. Where necessary, round answers to 3 significant digits.

- Show that the slopes of the sine and cosine curves are negatives of each other at the points of intersection.
- Show that the graph of the tangent function is always increasing (when the tangent is defined).
- Show that the curve of  $y = \tan^{-1} x$  is always increasing.
- Sketch the graph of  $y = \sin x + \cos x$  ( $0 \leq x \leq 2\pi$ ).
- Sketch the graph of  $y = x - \tan x$  ( $-\frac{\pi}{2} < x < \frac{\pi}{2}$ ).
- Sketch the graph of  $y = 2 \sin x + \sin 2x$  ( $0 \leq x \leq 2\pi$ ).
- Find the equation of the line tangent to the curve of  $y = x \sin^{-1} x$  at  $x = 0.50$ .
- Find the equation of the line normal to the curve of  $y = 3 \tan x^2$  at  $x = 0.25$ .
- By Newton's method, find the positive root of the equation  $x^2 - 4 \sin x = 0$  to at least four decimal places.
- By Newton's method, find the smallest positive root of the equation  $\tan x = 2x$  to at least four decimal places.
- Find the minimum value of the function  $y = 6 \cos x - 8 \sin x$ .
- Find the maximum value of the function  $y = \tan^{-1}(1+x) + \tan^{-1}(1-x)$ .
- Power  $P$  is the time rate of change of work  $W$ . Find the equation for the power in a circuit for which  $W = 8 \sin^2 2t$ .
- The phase shift  $\phi$  in a certain electric circuit with a resistance  $R$  and variable capacitance  $C$  is  $\phi = \tan^{-1} \omega RC$ . Find the equation for the instantaneous rate of change of  $\phi$  with respect to  $C$ .
- In studying water waves, the vertical displacement  $y$  (in m) of a wave was determined to be  $y = 0.50 \sin 2t + 0.30 \cos t$ , where  $t$  is the time (in s). Find the velocity and the acceleration for  $t = 0.40$  s.
- At  $45^\circ\text{N}$  latitude, the number of hours  $h$  of daylight each day during the year is given approximately by the equation  $h = 12.2 + 3.5 \sin \left[ \frac{2\pi}{365}(t - 81) \right]$ , where  $t$  is measured in days ( $t = 37$  is Feb. 6, etc.). Find the date of the longest day and the date of the shortest day. (Cities at  $45^\circ\text{N}$  are Ottawa, Ontario, and Venice, Italy.)
- Find the time rate of change of the horizontal component  $T_x$  of the constant 46.6-N tension shown in Fig. 27.19 if  $d\theta/dt = 0.36^\circ/\text{s}$  for  $\theta = 14.2^\circ$ .

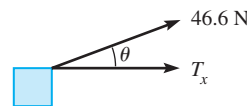


Fig. 27.19

21. A point on the outer edge of a 38.0-cm wheel can be described by the equations  $x = 19.0 \cos 6\pi t$  and  $y = 19.0 \sin 6\pi t$ . Find the velocity of the point for  $t = 0.600$  s.
22. A machine is programmed to move an etching tool such that the position (in cm) of the tool is given by  $x = 2 \cos 3t$  and  $y = \cos 2t$ , where  $t$  is the time (in s). Find the velocity of the tool for  $t = 4.1$  s.
23. Find the acceleration of the tool of Exercise 22 for  $t = 4.1$  s.
24. The volume  $V$  (in  $\text{m}^3$ ) of water used each day by a community during the summer is found to be  $V = 2500 + 480 \sin(\pi t/90)$ , where  $t$  is the number of the summer day, and  $t = 0$  is the first day of summer. On what summer day is the water usage the greatest?
25. A person observes an object dropped from the top of a building 40.0 m away. If the top of the building is 60.0 m above the person's eye level, how fast is the angle of elevation of the object changing after 1.0 s? (The distance the object drops is given by  $s = 4.9t^2$ .) See Fig. 27.20.

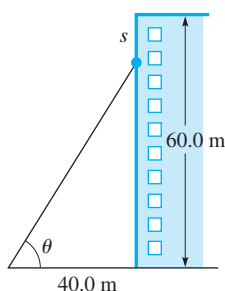


Fig. 27.20

26. A car passes directly under a police helicopter 150 m above a straight and level highway. After the car has travelled another 20.0 m, the angle of depression of the car from the helicopter is decreasing at the rate of 0.215 rad/s. What is the speed of the car?
27. A searchlight is 225 m from a straight wall. As the beam moves along the wall, the angle between the beam and the perpendicular to the wall is increasing at the rate of  $1.5^\circ/\text{s}$ . How fast is the length of the beam increasing when it is 315 m long? See Fig. 27.21.

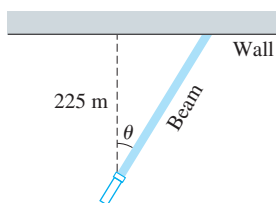


Fig. 27.21

28. A person standing 35.0 m from the base of the Skylon Tower in Niagara Falls observes one of the exterior elevators rising at a rate

of 2.60 m/s. How fast is the angle of elevation of the line of sight to the elevator increasing when the elevator is 40.0 m above the ground?

29. A crate of weight  $w$  is being pulled along a level floor by a force  $F$  that is at an angle  $\theta$  with the floor. The force is given by  $F = \frac{0.25w}{0.25 \sin \theta + \cos \theta}$ . Find  $\theta$  for the minimum value of  $F$ .
30. The electric power  $P$  (in W) developed in a resistor in an FM receiver circuit is  $P = 0.0307 \cos^2 120\pi t$ , where  $t$  is the time (in s). Linearize  $P$  for  $t = 0.0010$  s.
31. When an astronaut views the horizon of earth from a spacecraft at an altitude of 610 km, the angle  $\theta$  in Fig. 27.22 is found to be  $65.8^\circ \pm 0.5^\circ$ . Use differentials to approximate the possible error in the astronaut's calculation of earth's radius.

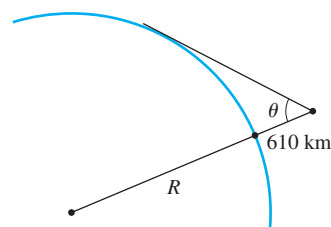


Fig. 27.22

32. A surveyor measures two sides and the included angle of a triangular parcel of land to be 82.04 m, 75.37 m, and  $38.38^\circ$ . What error is caused in the calculation of the third side by an error of  $0.15^\circ$  in the angle?
33. The volume  $V$  (in L) of air in a person's lungs during one normal cycle of inhaling and exhaling at any time  $t$  is  $V = 0.48(1.2 - \cos 1.26t)$ . What is the maximum flow rate (in L/s) of air?
34. To connect the four vertices of a square with the minimum amount of electric wire requires using the wiring pattern shown in Fig. 27.23. Find  $\theta$  for the total length of wire ( $L = 4x + y$ ) to be a minimum.

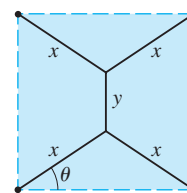


Fig. 27.23

35. The strength  $S$  of a rectangular beam is directly proportional to the product of its width  $w$  and the square of its depth  $d$ . Use trigonometric functions to find the dimensions of the strongest beam that can be cut from a circular log 16.0 cm in diameter. (See Example 4 on page 739.)
36. An architect is designing a window in the shape of an isosceles triangle with a perimeter of 150 cm. What is the vertex angle of the window of greatest area?



37. A wall is 1.8 m high and 1.2 m from a building. What is the length of the shortest pole that can touch the building and the ground beyond the wall? (*Hint:* From Fig. 27.24, show that  $y = 1.8 \csc \theta + 1.2 \sec \theta$ .)
38. The television screen at a sports arena is vertical and 2.4 m high. The lower edge is 8.5 m above an observer's eye level. If the best view of the screen is obtained when the angle subtended by the screen at eye level is a maximum, how far from directly below the screen must the observer's eye be? See Fig. 27.25.

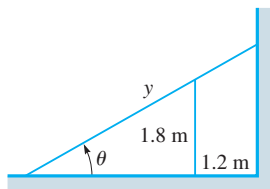


Fig. 27.24

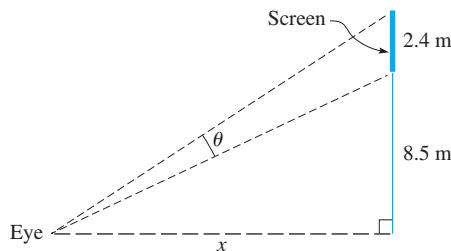


Fig. 27.25

## 27.5 Derivative of the Logarithmic Function

Limit of  $(1 + h/x)^{x/h}$  as  $h \rightarrow 0$  • Derivative of  $\log_b u$  • Derivative of  $\ln u$  • Using Properties of Logarithms

Using the definition of a derivative, we next find the derivative of the logarithmic function. Therefore, letting  $y = \log_b x$ , we have

$$\begin{aligned} \frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{\log_b(x+h) - \log_b x}{h} = \lim_{h \rightarrow 0} \frac{\log_b \frac{x+h}{x}}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{x} \frac{x}{h} \log_b \left(1 + \frac{h}{x}\right) && \text{multiply and divide by } x \\ &= \frac{1}{x} \lim_{h \rightarrow 0} \log_b \left(1 + \frac{h}{x}\right)^{x/h} && \text{using Eq. (13.9)} \\ &= \frac{1}{x} \log_b \left( \lim_{h \rightarrow 0} \left(1 + \frac{h}{x}\right)^{x/h} \right) \end{aligned}$$

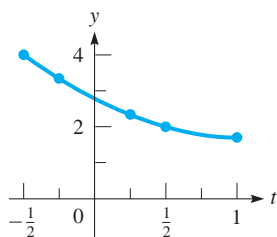


Fig. 27.26

In the last line, we have interchanged the limit and the logarithm because the logarithm function is continuous. We can see that the exponent becomes unbounded, but the number being raised to this exponent approaches 1. Therefore, we will investigate this limiting value.

To approximate the value, we graph the function  $y = (1+t)^{1/t}$  (for purposes of graphing, we let  $h/x = t$ ). Constructing a table of values, we then graph this function in Fig. 27.26.

$t$	-0.5	-0.25	+0.25	+0.50	+1.00
$y$	4.00	3.16	2.44	2.25	2.00

Only these values are shown, since we are interested in the  $y$ -value corresponding to  $t = 0$ . We see from the graph that this value is approximately 2.7. Choosing very small values of  $t$ , we may obtain these values:

$t$	0.1	0.01	0.001	0.0001
$y$	2.5937	2.7048	2.7169	2.718 15

By methods developed in Chapter 29, it can be shown that this value is about 2.718 281 8. **The limiting value is the irrational number  $e$ .** This is the same number used in the exponential form of a complex number in Chapter 12 and as the base of natural logarithms in Chapter 13.

Returning to the derivative of the logarithmic function, we have

$$\frac{dy}{dx} = \frac{1}{x} \log_b \left( \lim_{h \rightarrow 0} \left( 1 + \frac{h}{x} \right)^{x/h} \right) = \frac{1}{x} \log_b e$$

Now, for  $y = \log_b u$ , where  $u$  is a function of  $x$ , using Eq. (27.3), we have

$$\frac{d(\log_b u)}{dx} = \frac{1}{u} \log_b e \frac{du}{dx} \quad (27.13)$$

At this point, we see that if we choose  $e$  as the base of a system of logarithms, the above formula becomes

$$\frac{d(\ln u)}{dx} = \frac{1}{u} \frac{du}{dx} \quad (27.14)$$

■ For reference, Eq. (27.3) is

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

### LEARNING TIP

The choice of  $e$  as the base  $b$  makes  $\log_e e = 1$ ; thus, this factor does not appear in Eq. (27.14). We now see why the number  $e$  is chosen as the base for a system of logarithms, the natural logarithms. The notation  $\ln u$  is the same as that used in Chapter 13 for natural logarithms.

### EXAMPLE 1 Derivative of $\log u$

Find the derivative of  $y = \log 4x$ .

Using Eq. (27.13), we find

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{4x} (\log e) (4) \\ &= \frac{1}{x} \log e \end{aligned} \quad \begin{array}{l} \frac{du}{dx} \\ \log e = 0.4343 \end{array}$$

### EXAMPLE 2 Derivative of $\ln u$

Find the derivative of  $s = \ln 3t^4$ .

Using Eq. (27.14), we have (with  $u = 3t^4$ )

$$\begin{aligned} \frac{ds}{dt} &= \frac{1}{3t^4} (12t^3) \\ &= \frac{4}{t} \end{aligned} \quad \frac{du}{dt}$$

### Practice Exercise

1. Find the derivative of  $y = 2 \ln 5x^2$ .

### EXAMPLE 3 Derivative of $\ln \tan u$

Find the derivative of  $y = \ln \tan 4x$ .

Using Eq. (27.14), along with the derivative of the tangent, we have

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{\tan 4x} (\sec^2 4x) (4) \\ &= \frac{\cos 4x}{\sin 4x} \frac{4}{\cos^2 4x} \quad \text{using trigonometric relations} \\ &= \frac{1}{\sin 4x} \frac{4}{\cos 4x} = 4 \csc 4x \sec 4x \end{aligned}$$

**LEARNING TIP**

Often, finding the derivative of a logarithmic function is simplified by **using the properties of logarithms to simplify the logarithmic expression before taking the derivative.**

■ For reference, Eqs. (13.7), (13.8), and (13.9) are  
 $\log_b xy = \log_b x + \log_b y$   
 $\log_b \left(\frac{x}{y}\right) = \log_b x - \log_b y$   
 $\log_b x^n = n \log_b x$

**Practice Exercise**

2. Find the derivative of  $y = \ln \frac{4x}{x+4}$ .

**EXAMPLE 4** Derivative using properties of logarithms

Find the derivative of  $y = \ln \frac{x-1}{x+1}$ .

In this example, it is easier to find the derivative if we write  $y$  in the form

$$y = \ln(x-1) - \ln(x+1)$$

by using the properties of logarithms (Eq. 13.8). Hence,

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{x-1} - \frac{1}{x+1} = \frac{x+1-x+1}{(x-1)(x+1)} \\ &= \frac{2}{x^2-1} \end{aligned}$$

**EXAMPLE 5** Derivative of  $\ln u^3$  and  $\ln^3 u$ 

(a) Find the derivative of  $y = \ln(1-2x)^3$ .

First, using Eq. (13.9), we rewrite the equation as  $y = 3 \ln(1-2x)$ . Then we have

$$\frac{dy}{dx} = 3 \left( \frac{1}{1-2x} \right) (-2) = \frac{-6}{1-2x}$$

(b) Find the derivative of  $y = \ln^3(1-2x)$ .

First, we note that

$$y = \ln^3(1-2x) = [\ln(1-2x)]^3$$

where  $\ln^3(1-2x)$  is usually the preferred notation.

Next, we must be careful to distinguish this function from that in part (a). For  $y = \ln^3(1-2x)$ , it is the logarithm of  $1-2x$  that is being cubed, whereas for  $y = \ln(1-2x)^3$ , it is  $1-2x$  that is being cubed.

Now, finding the derivative of  $y = \ln^3(1-2x)$ , we have

$$\begin{aligned} \frac{dy}{dx} &= 3[\ln^2(1-2x)] \left( \frac{1}{1-2x} \right) (-2) \\ &= -\frac{6 \ln^2(1-2x)}{1-2x} \quad \begin{array}{l} \uparrow \\ \frac{d \ln(1-2x)}{dx} \end{array} \end{aligned}$$

**EXAMPLE 6** Evaluation of the derivative of  $\ln u$ 

Evaluate the derivative of  $y = \ln[(\sin 2x)(\sqrt{x^2+1})]$  for  $x = 0.375$ .

First, using the properties of logarithms, we rewrite the function as

$$y = \ln \sin 2x + \frac{1}{2} \ln(x^2+1)$$

Now, we have

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{\sin 2x} (\cos 2x) (2) + \frac{1}{2} \left( \frac{1}{x^2+1} \right) (2x) && \text{take the derivative} \\ &= 2 \cot 2x + \frac{x}{x^2+1} \end{aligned}$$

$$\left. \frac{dy}{dx} \right|_{x=0.375} = 2 \cot 0.750 + \frac{0.375}{0.375^2+1} = 2.48 \quad \text{evaluate}$$

## EXERCISES 27.5

In Exercises 1 and 2, make the given changes in the indicated examples of this section and then find the derivatives.

- In Example 3, in the given function, change  $\tan$  to  $\cos$ .
- In Example 4, in the given function, change  $x - 1$  to  $x^2$ .

In Exercises 3–34, find the derivatives of the given functions.

- $y = \log x^2$
- $y = \log_2 6x$
- $y = 4 \log_5 (3 - x)$
- $y = \log_7 (x^2 + 1)$
- $u = 8 \ln(3 - x)$
- $y = 2 \ln(3x^2 - 1)$
- $y = 2 \ln \tan 2x$
- $s = \ln \sin^2 t$
- $R = \ln \sqrt{T}$
- $y = \ln(4x - 3)^3$
- $y = \ln(x - x^2)^3$
- $s = 3 \ln^2(7t^3 - 1)$
- $v = 3(t + \ln t^2)^2$
- $y = 6x^2 \ln 5x$
- $y = 3x \ln(6 - x)$
- $y = \frac{8 \ln x}{x}$
- $y = \ln(\ln x)$
- $y = \ln \frac{2x}{1 + x}$
- $r = 0.5 \ln \cos(\pi\theta^2)$
- $y = \ln(x\sqrt{x + 1})$
- $y = \sin \ln x$
- $y = \tan^{-1} \ln 2x$
- $u = 3v \ln^2 2v$
- $h = 0.1s \ln^4 s$
- $y = \ln(x \tan x)$
- $y = \ln(x + \sqrt{x^2 - 1})$
- $r = \ln \frac{v^2}{v + 2}$
- $y = \sqrt{x + \ln 3x}$
- $y = \sqrt{x^2 + 1} - \ln \frac{1 + \sqrt{x^2 + 1}}{x}$
- $3 \ln xy + \sin y = x^2$
- $y = x - \ln^2(x + y)$
- $y = \ln(x + \ln x)$

In Exercises 35–56, solve the given problems.

- On a calculator, find the value of  $(\ln 2.0001 - \ln 2.0000)/0.0001$  and compare it with 0.5. Give the meanings of the value found and 0.5 in relation to the derivative of  $\ln x$ , where  $x = 2$ .
- On a calculator, find the value of  $(\ln 0.5001 - \ln 0.5000)/0.0001$  and compare it with 2. Give the meanings of the value found and 2 in relation to the derivative of  $\ln x$ , where  $x = 0.5$ .
- Using a graphing calculator, (a) display the graph of  $y = (1 + x)^{1/x}$  to verify that  $(1 + x)^{1/x} \rightarrow 2.718$  as  $x \rightarrow 0$  and (b) verify the values for  $(1 + x)^{1/x}$  in the tables on page 830.
- (a) Display the graph of  $y = \ln x$  on a graphing calculator, and using the derivative feature, evaluate  $dy/dx$  for  $x = 2$ . (b) Display the graph of  $y = 1/x$ , and evaluate  $y$  for  $x = 2$ . (c) Compare the values in parts (a) and (b).
- Given that  $\ln \sin 45^\circ = -0.3466$ , use differentials to approximate  $\ln \sin 44^\circ$ .
- Find the second derivative of the function  $y = x^2 \ln x$ .
- Evaluate the derivative of  $y = \sin^{-1} 2x + \ln \sqrt{1 - 4x^2}$ , where  $x = 0.250$ .
- Evaluate the derivative of  $y = \ln \frac{\sqrt{2x + 1}}{3x + 1}$ , where  $x = 2.75$ .

- Find the linearization  $L(x)$  for the function  $f(x) = 2 \ln \tan x$  for  $a = \pi/4$ .
- Find the differential of the function  $y = 6 \log_x 2$ .
- Find the slope of a line tangent to the curve of  $y = \tan^{-1} 2x + \ln(4x^2 + 1)$ , where  $x = 0.625$ .
- Find the slope of a line tangent to the curve of  $y = x \ln 3x$  at  $x = 4$ .
- Find the derivative of  $y = x^x$  by first taking logarithms of each side of the equation. Explain why Eq. (23.15) cannot be used to find the derivative of this function.
- Find the derivative of  $y = (\sin x)^x$  by first taking logarithms of each side of the equation. Explain why Eq. (23.15) cannot be used to find the derivative of this function.
- Find the derivatives of  $y_1 = \ln(x^2)$  and  $y_2 = 2 \ln x$ , and evaluate these derivatives for  $x = -1$ . Explain your results.
- The inductance  $L$  (in  $\mu\text{H}$ ) of a coaxial cable is given by  $L = 0.032 + 0.15 \log(a/x)$ , where  $a$  and  $x$  are the radii of the outer and inner conductors, respectively. For constant  $a$ , find  $dL/dx$ .
- If the loudness  $b$  (in decibels) of a sound of intensity  $I$  is given by  $b = 10 \log(I/I_0)$ , where  $I_0$  is a constant, find the expression for  $db/dt$  in terms of  $dI/dt$ .
- The time  $t$  for a particular computer system to process  $N$  bits of data is directly proportional to  $N \ln N$ . Find the expression for  $dt/dN$ .
- When a tractor-trailer turns a right-angle corner, the rear wheels follow a curve known as a *tractrix*, the equation for which is  $y = \ln \left( \frac{1 + \sqrt{1 + x^2}}{x} \right) - \sqrt{1 + x^2}$ . Find  $dy/dx$ .
- When designing a computer to sort files on a hard disk, the equation  $y = xA \log_x A$  arises. If  $A$  is constant, find  $dy/dx$ .
- When air friction is considered, the time  $t$  (in s) it takes a certain falling object to attain a velocity  $v$  (in m/s) is given by  $t = 5 \ln \frac{5}{5 - 0.1v}$ . Find  $dt/dv$  for  $v = 10.0$  m/s.
- The electric potential  $V$  at a point  $P$  at a distance  $x$  from an electric charge distributed along a wire of length  $2a$  (see Fig. 27.27) is  $V = k \ln \frac{\sqrt{a^2 + x^2} + a}{\sqrt{a^2 + x^2} - a}$ , where  $k$  is a constant. Find the expression for the electric field  $E$ , where  $E = -dV/dx$ .

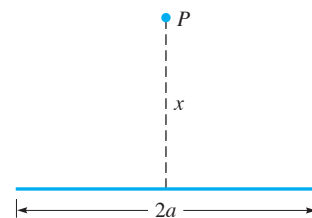


Fig. 27.27

Answers to Practice Exercises

- $y' = 4/x$
- $y' = 4/(x^2 + 4x)$

## 27.6 Derivative of the Exponential Function

Derivative of  $b^x$  • Derivative of  $e^x$  • Using Laws of Exponents

To obtain the derivative of the exponential function, we let  $y = b^u$ , then take natural logarithms of both sides, and then take derivatives of both sides:

$$\begin{aligned}\ln y &= \ln b^u = u \ln b \\ \frac{1}{y} \frac{dy}{dx} &= \ln b \frac{du}{dx} \\ \frac{dy}{dx} &= y \ln b \frac{du}{dx}\end{aligned}$$

$$\frac{d(b^u)}{dx} = b^u \ln b \left( \frac{du}{dx} \right) \quad (27.15)$$

If we let  $b = e$ , Eq. (27.15) becomes

$$\frac{d(e^u)}{dx} = e^u \left( \frac{du}{dx} \right) \quad (27.16)$$

The simplicity of Eq. (27.16) compared with Eq. (27.15) again shows the advantage of choosing  $e$  as the base of natural logarithms. It is for this reason that  $e$  appears so often in applications of calculus.

### EXAMPLE 1 Derivative of $e^x$

Find the derivative of  $y = e^x$ .

Using Eq. (27.16), we have

$$\frac{dy}{dx} = e^x \left( \frac{du}{dx} \right) = e^x (1) = e^x$$

We see that the derivative of the function  $e^x$  equals itself. This exponential function is widely used in applications of calculus. ■

■ For reference, Eq. (23.15) is  $\frac{du^n}{dx} = nu^{n-1} \left( \frac{du}{dx} \right)$ .

### LEARNING TIP

Note carefully that Eq. (23.15) is used with a variable raised to a constant exponent, whereas Eqs. (27.15) and (27.16) are used with a constant raised to a variable exponent.

$$\begin{array}{cc} \begin{array}{c} \text{variable} \\ \downarrow \\ \frac{du^n}{dx} = nu^{n-1} \left( \frac{du}{dx} \right) \end{array} & \begin{array}{c} \text{constant} \\ \swarrow \\ \frac{db^u}{dx} = b^u \ln b \left( \frac{du}{dx} \right) \end{array} \end{array}$$

In the following example, we must note carefully this difference in the type of function that leads us to use either Eq. (23.15) or Eq. (27.15) in order to find the derivative.

### EXAMPLE 2 Derivative of $u^2$ and $2^u$

Find the derivatives of  $y = (4x)^2$  and  $y = 2^{4x}$ .

Using Eq. (23.15), we have

$$\begin{aligned}y &= (4x)^2 \\ \frac{dy}{dx} &= 2(4x)^1 (4) \\ &= 32x\end{aligned}$$

Using Eq. (27.15), we have

$$\begin{aligned}y &= 2^{4x} \\ \frac{dy}{dx} &= 2^{4x} (\ln 2) (4) \\ &= (4 \ln 2) (2^{4x})\end{aligned}$$

**EXAMPLE 3 Derivative of  $\ln \cos e^{2x}$**

Find the derivative of  $y = \ln \cos e^{2x}$ .

Here, we use Eq. (27.16) and the derivatives of the logarithmic and cosine functions.

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{\cos e^{2x}} \frac{d \cos e^{2x}}{dx} && \text{using } \frac{d \ln u}{dx} = \frac{1}{u} \frac{du}{dx} \\ &= \frac{1}{\cos e^{2x}} (-\sin e^{2x}) \frac{de^{2x}}{dx} && \text{using } \frac{d \cos u}{dx} = -\sin u \frac{du}{dx} \\ &= \frac{-\sin e^{2x}}{\cos e^{2x}} (e^{2x}) (2) && \text{using } \frac{de^u}{dx} = e^u \frac{du}{dx} \\ &= -2e^{2x} \tan e^{2x} && \text{using } \frac{\sin \theta}{\cos \theta} = \tan \theta \end{aligned}$$

**Practice Exercise**

- Find the derivative of  $y = \ln (e^{3x} + 1)$ .

**EXAMPLE 4 Derivative of a product**

Find the derivative of  $r = \theta e^{\tan \theta}$ .

Here, we use Eq. (27.16) with the derivatives of a product and the tangent:

$$\begin{aligned} \frac{dr}{d\theta} &= \theta e^{\tan \theta} (\sec^2 \theta) + e^{\tan \theta} (1) \\ &= e^{\tan \theta} (\theta \sec^2 \theta + 1) \end{aligned}$$

**Practice Exercise**

- Find the derivative of  $y = 8e^{4x} \cos x$ .

**EXAMPLE 5 Using the laws of exponents**

Find the derivative of  $y = (e^{1/x})^2$ .

In this example, we use Eqs. (23.15) and (27.16):

$$\begin{aligned} \frac{dy}{dx} &= 2(e^{1/x}) (e^{1/x}) \left( -\frac{1}{x^2} \right) \\ &= \frac{-2(e^{1/x})^2}{x^2} = \frac{-2e^{2/x}}{x^2} \end{aligned}$$

using Eqs. (27.16) and (23.15) to find  $\frac{du}{dx}$  of Eq. (23.15)

We can also solve this problem by using the laws of exponents and writing the function as  $y = e^{2/x}$ . This change in form simplifies the steps for finding the derivative:

$$\frac{dy}{dx} = e^{2/x} \left( -\frac{2}{x^2} \right) = \frac{-2e^{2/x}}{x^2}$$

using Eq. (23.15) to find  $\frac{du}{dx}$  of Eq. (27.16)

**EXAMPLE 6 Evaluation of the slope of a tangent**

Find the slope of a line tangent to the curve of  $y = \frac{3e^{2x}}{x^2 + 1}$  at  $x = 1.275$ .

Here, we are to evaluate the derivative for  $x = 1.275$ . The solution is as follows:

$$\begin{aligned} \frac{dy}{dx} &= \frac{(x^2 + 1)(3e^{2x})(2) - 3e^{2x}(2x)}{(x^2 + 1)^2} && \text{take the derivative} \\ &= \frac{6e^{2x}(x^2 - x + 1)}{(x^2 + 1)^2} \\ \frac{dy}{dx} \Big|_{x=1.275} &= \frac{6e^{2(1.275)}(1.275^2 - 1.275 + 1)}{(1.275^2 + 1)^2} = 15.05 && \text{evaluate} \end{aligned}$$

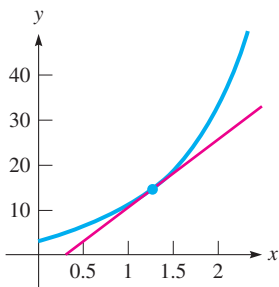


Fig. 27.28

The function and the tangent line are shown in Fig. 27.28.

**EXAMPLE 7** Derivative of a power

Find the derivative of  $y = (3e^{4x} + 4x^2 \ln x)^3$ .

Using the general power rule (Eq. 23.15) for the derivatives, the derivative of the exponential function (Eq. 27.16), the derivative of a product (Eq. 23.12), and the derivative of a logarithm (Eq. 27.14), we have

$$\begin{aligned}\frac{dy}{dx} &= 3(3e^{4x} + 4x^2 \ln x)^2 \left[ 12e^{4x} + 4x^2 \left( \frac{1}{x} \right) + (\ln x)(8x) \right] \\ &= 3(3e^{4x} + 4x^2 \ln x)^2 (12e^{4x} + 4x + 8x \ln x)\end{aligned}$$

**EXERCISES 27.6**

In Exercises 1 and 2, make the given changes in the indicated examples of this section and then solve the resulting problems.

- In Example 3, in the given function, change  $\cos$  to  $\sin$ .
- In Example 6, in the given function, change  $x^2 + 1$  to  $x + 1$ .

In Exercises 3–32, find the derivatives of the given functions.

- $y = 4^{6x}$
- $y = 6e^{\sqrt{x}}$
- $y = 4e^t(e^{2t} - e^t)$
- $R = Te^{-T}$
- $y = xe^{\sin x}$
- $r = \frac{2(e^{2s} - e^{-2s})}{e^{2s}}$
- $y = e^{-3x} \sin 4x$
- $y = \frac{2e^{3x}}{4x + 3}$
- $y = 0.5 \ln(e^{t^2} + 4)$
- $y = (2e^{2x})^3 \sin x^2$
- $u = 4\sqrt{\ln 2t + e^{2t}}$
- $y = xe^{xy} + \sin y$
- $y = 3e^{2x} \ln x$
- $I = \ln \sin 2e^{6t}$
- $y = 2 \sin^{-1} e^{2x}$
- $y = 10^{x^2}$
- $r = 0.3e^{\theta^2}$
- $y = 0.6 \ln(e^{5x} + 3)$
- $y = 5x^2 e^{2x}$
- $y = 4e^x \sin \frac{1}{2}x$
- $u = \frac{e^{0.5v}}{2v}$
- $y = (\cos 2x)(e^{x^2-1})$
- $y = \frac{7 \ln 3x}{e^{2x} + 8}$
- $p = (3e^{2n} + e^2)^3$
- $y = (e^{3/x} \cos x)^2$
- $y = (2e^{x^2} + x^2)^3$
- $y = 4e^{-2/x} \ln y + 1$
- $r = 0.4e^{2\theta} \ln \cos \theta$
- $y = 6 \tan e^{x+1}$
- $W = \tan^{-1} e^{3s}$

In Exercises 33–54, solve the given problems. Where necessary, round answers to 3 significant digits.

- On a calculator, find the values of (a)  $e$  and (b)  $(e^{1.0001} - e^{1.0000})/0.0001$ . Compare the values and give the meaning of each in relation to the derivative of  $e^x$ , where  $x = 1$ .
- On a calculator, find the values of (a)  $e^2$  and (b)  $(e^{2.0001} - e^{2.0000})/0.0001$ . Compare the values and give the meaning of each in relation to the derivative of  $e^x$ , where  $x = 2$ .
- Display the graph of  $y = e^x$  on a graphing calculator. Using the *derivative* feature, evaluate  $dy/dx$  for  $x = 2$  and compare with the value of  $y$  for  $x = 2$ .
- Find a formula for the  $n$ th derivative of  $y = ae^{bx}$ .
- Find the slope of a line tangent to the curve of  $y = e^{-x/2} \cos 4x$  for  $x = 0.625$ .
- Find the slope of a line tangent to the curve of  $y = \frac{e^{-x}}{1 + \ln 4x}$  for  $x = 1.842$ .
- Find the differential of the function  $y = \frac{12e^{4x}}{x + 6}$ .
- Find the linearization of the function  $f(x) = \frac{6e^{4x}}{2x + 3}$  for  $a = 0$ .
- Use a graphing calculator to display the graph of  $y = e^x$ . By roughly estimating slopes of tangent lines, note that it is reasonable that these values are equal to the  $y$ -coordinates of the points at which these estimates are made. (Remember: For  $y = e^x$ ,  $dy/dx = e^x$  also.)
- Use a graphing calculator to display the graphs of  $y = e^{-x}$  and  $y = -e^{-x}$ . By roughly estimating slopes of tangent lines of  $y = e^{-x}$ , note that  $y = -e^{-x}$  gives reasonable values for the derivative of  $y = e^{-x}$ .
- Show that  $y = xe^{-x}$  satisfies the equation  $(dy/dx) + y = e^{-x}$ .
- Show that  $y = e^{-x} \sin x$  satisfies the equation  $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$ .
- For  $y = \frac{e^{2x} - 1}{e^{2x} + 1}$ , show that  $\frac{dy}{dx} = 1 - y^2$ .
- If  $e^x + e^y = e^{x+y}$ , show that  $dy/dx = -e^{y-x}$ .
- For what values of  $m$  does the function  $y = ae^{mx}$  satisfy the equation  $y'' + y' - 6y = 0$ ?
- For what values of  $m$  does the function  $y = ae^{mx}$  satisfy the equation  $y'' + 4y = 0$ ?
- If  $y = Ae^{kx} + Be^{-kx}$ , show that  $y'' = k^2y$ .
- The average energy consumption  $C$  (in MJ/year) of a certain model of refrigerator-freezer is approximately  $C = 5350e^{-0.0748t} + 1800$ , where  $t$  is measured in years, with  $t = 0$  corresponding to 1990, and a newer model is produced each year. Assuming the function is continuous, use differentials to estimate the reduction in energy consumption of the 2012 model from that of the 2011 model.
- The reliability  $R$  ( $0 \leq R \leq 1$ ) of a certain computer system is given by  $R = e^{-0.002t}$ , where  $t$  is the time of operation (in h). Find  $dR/dt$  for  $t = 100$  h.



52. A thermometer is taken from a freezer at  $-16^\circ\text{C}$  and placed in a room at  $24^\circ\text{C}$ . The temperature  $T$  of the thermometer as a function of the time  $t$  (in min) after removal is given by  $T = 8.0(3.0 - 5.0e^{-0.50t})$ . How fast is the temperature changing when  $t = 6.0$  min?
53. For the electric circuit shown in Fig. 27.29, the current  $i$  (in A) is given by  $i = 4.42e^{-66.7t} \sin 226t$ , where  $t$  is the time (in s). Find the expression for  $di/dt$ .

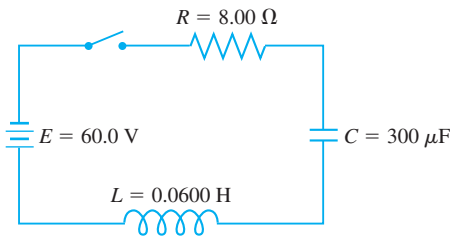


Fig. 27.29

54. Under certain assumptions of limitations to population growth, the population  $P$  (in billions) of the world is given by the logistic equation  $P = \frac{10}{1 + 0.65e^{-0.060t}}$ , where  $t$  is the number of years after the year 2000. Find the expression for  $dP/dt$ .

In Exercises 55–59, use the following information.

The **hyperbolic sine** of  $u$  is defined as

$$\sinh u = \frac{1}{2}(e^u - e^{-u}).$$

Figure 27.30 shows the graph of  $y = \sinh x$ .

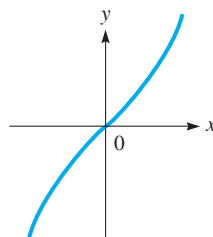


Fig. 27.30

The **hyperbolic cosine** of  $u$  is defined as

$$\cosh u = \frac{1}{2}(e^u + e^{-u}).$$

Figure 27.31 shows the graph of  $y = \cosh x$ .

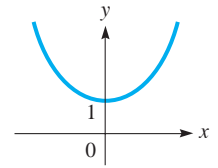


Fig. 27.31

These functions are called hyperbolic functions since, if  $x = \cosh u$  and  $y = \sinh u$ ,  $x$  and  $y$  satisfy the equation of the hyperbola  $x^2 - y^2 = 1$ .

55. Verify the fact that the exponential expressions for the hyperbolic sine and hyperbolic cosine given above satisfy the equation of the hyperbola.
56. Show that  $\sinh u$  and  $\cosh u$  satisfy the identity  $\cosh^2 u - \sinh^2 u = 1$ .
57. Show that  $\frac{d}{dx} \sinh u = \cosh u \frac{du}{dx}$  and  $\frac{d}{dx} \cosh u = \sinh u \frac{du}{dx}$  where  $u$  is a function of  $x$ .
58. Show that  $\frac{d^2 \sinh x}{dx^2} = \sinh x$  and  $\frac{d^2 \cosh x}{dx^2} = \cosh x$ .
59. A telephone wire hangs so that its shape is described by  $y = 50 \cosh \frac{x}{50}$ . Find the expression for  $dy/dx$ . (See Exercise 57.)

### Answers to Practice Exercises

1.  $y' = 3e^{3x}/(e^{3x} + 1)$     2.  $y' = 8e^{4x}(4 \cos x - \sin x)$

## 27.7 L'Hospital's Rule

Indeterminate Forms  $0/0$  and  $\infty/\infty$  •  
L'Hospital's Rule • Numerical Verification •  
Indeterminate Form  $0 \cdot \infty$

■ The method is named after the French mathematician the Marquis de l'Hospital (1661–1704), who incorporated it in the first textbook on differential calculus to appear in print (in 1696). However, the result was derived two years earlier by his calculus tutor, Johann Bernoulli (1667–1748).

We dedicate this section to finding limits of quotients where both numerator and denominator approach zero or where both numerator and denominator approach infinity. We have already encountered this type of expression in Section 23.1, where we evaluated the limit using algebraic techniques, and in Section 27.1, where we found that  $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$  using a geometric approach. Now we study a more general method to evaluate such limits by using derivatives.

For the limit of  $u(x)/v(x)$  as  $x \rightarrow a$ , if both  $u(x)$  and  $v(x)$  approach zero, the limit is called an **indeterminate form of the type  $0/0$** . If  $u(x)$  and  $v(x)$  approach infinity, the limit is called an **indeterminate form of the type  $\infty/\infty$** .

To find these limits, we use **L'Hospital's rule**. The proof of L'Hospital's rule can be found in texts covering advanced methods in calculus.

### L'Hospital's Rule

If  $u(x)$  and  $v(x)$  are differentiable such that  $v'(x)$  is not zero, and if  $\lim_{x \rightarrow a} u(x) = 0$  and  $\lim_{x \rightarrow a} v(x) = 0$  or  $\lim_{x \rightarrow a} u(x) = \infty$  and  $\lim_{x \rightarrow a} v(x) = \infty$ , then

$$\lim_{x \rightarrow a} \frac{u(x)}{v(x)} = \lim_{x \rightarrow a} \frac{u'(x)}{v'(x)}$$

**EXAMPLE 1** L'Hospital's rule—indeterminate form 0/0

Using L'Hospital's rule, find  $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta}$ .

This, of course, is the same limit we found in Section 27.1 when deriving the derivative of the sine function. Noting again that

$$\lim_{\theta \rightarrow 0} \sin \theta = 0 \quad \text{and} \quad \lim_{\theta \rightarrow 0} \theta = 0$$

we see that this is of the indeterminate form 0/0. Applying L'Hospital's rule, we have

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = \lim_{\theta \rightarrow 0} \frac{\frac{d}{d\theta} \sin \theta}{\frac{d}{d\theta} \theta} = \lim_{\theta \rightarrow 0} \frac{\cos \theta}{1} = \frac{1}{1} = 1$$

This agrees with the result in Section 27.1. ■

**EXAMPLE 2** L'Hospital's rule—indeterminate form  $\infty/\infty$ 

Evaluate  $\lim_{x \rightarrow \infty} \frac{3e^{2x}}{5x}$ .

Here, we see that  $3e^{2x} \rightarrow \infty$  and  $5x \rightarrow \infty$  as  $x \rightarrow \infty$ , which means the limit is of the indeterminate form  $\infty/\infty$ . Therefore, using L'Hospital's rule, we have

$$\lim_{x \rightarrow \infty} \frac{3e^{2x}}{5x} = \lim_{x \rightarrow \infty} \frac{\frac{d}{dx} 3e^{2x}}{\frac{d}{dx} 5x} = \lim_{x \rightarrow \infty} \frac{6e^{2x}}{5} = \infty$$

This means that the limit does not exist. ■

**EXAMPLE 3** L'Hospital's rule—limit as  $t \rightarrow \pi/2$ 

Evaluate  $\lim_{t \rightarrow \pi/2} \frac{1 - \sin t}{\cos t}$ .

Here,  $\lim_{t \rightarrow \pi/2} (1 - \sin t) = 0$  and  $\lim_{t \rightarrow \pi/2} \cos t = 0$ , which means the limit is of the indeterminate form 0/0. Using L'Hospital's rule, we have

$$\lim_{t \rightarrow \pi/2} \frac{1 - \sin t}{\cos t} = \lim_{t \rightarrow \pi/2} \frac{\frac{d}{dt} (1 - \sin t)}{\frac{d}{dt} \cos t} = \lim_{t \rightarrow \pi/2} \frac{-\cos t}{-\sin t} = \frac{0}{-1} = 0 \quad \blacksquare$$

**EXAMPLE 4** Using L'Hospital's rule twice

Using L'Hospital's rule, find  $\lim_{x \rightarrow \infty} \frac{x^2 + 1}{2x^2 + 3}$ .

This is the same limit we found algebraically in Example 11 on page 664.

Noting that  $(x^2 + 1) \rightarrow \infty$  and  $(2x^2 + 3) \rightarrow \infty$  as  $x \rightarrow \infty$ , this limit is of the indeterminate form  $\infty/\infty$ . Therefore, applying L'Hospital's rule, we have

$$\lim_{x \rightarrow \infty} \frac{x^2 + 1}{2x^2 + 3} = \lim_{x \rightarrow \infty} \frac{\frac{d}{dx} (x^2 + 1)}{\frac{d}{dx} (2x^2 + 3)} = \lim_{x \rightarrow \infty} \frac{2x}{4x} = \lim_{x \rightarrow \infty} \frac{1}{2} = \frac{1}{2}$$

After applying L'Hospital's rule, we were able to algebraically simplify the expression. However, because  $2x \rightarrow \infty$  and  $4x \rightarrow \infty$  as  $x \rightarrow \infty$ , this last limit also is of the indeterminate form  $\infty/\infty$  and we can use L'Hospital's rule again. This gives us

$$\lim_{x \rightarrow \infty} \frac{2x}{4x} = \lim_{x \rightarrow \infty} \frac{\frac{d}{dx} 2x}{\frac{d}{dx} 4x} = \lim_{x \rightarrow \infty} \frac{2}{4} = \frac{1}{2}$$

We note that this result agrees with the result of Example 11 on page 664. ■

**Practice Exercise**

1. Find  $\lim_{x \rightarrow 0} \frac{1 - e^{4x}}{2x}$ .

**Practice Exercise**

2. Find  $\lim_{x \rightarrow \infty} \frac{\ln x}{x}$ .

**COMMON ERROR**

When using L'Hospital's rule, *always verify that the limit fits the indeterminate form of  $0/0$  or  $\infty/\infty$  before you apply the rule.* If you use the rule when it does not apply, it will lead to incorrect calculations, as can be seen in Example 5.

**EXAMPLE 5** Be careful to check the indeterminate form

Find  $\lim_{x \rightarrow \pi^-} \frac{2 \sin x}{1 - \cos x}$ .

Applying L'Hospital's rule, we have

$$\lim_{x \rightarrow \pi^-} \frac{2 \sin x}{1 - \cos x} = \lim_{x \rightarrow \pi^-} \frac{\frac{d}{dx} 2 \sin x}{\frac{d}{dx} (1 - \cos x)} = \lim_{x \rightarrow \pi^-} \frac{2 \cos x}{\sin x} = -\infty$$

However, *this result is INCORRECT.* Checking back to the original expression, we note that *it does not fit the indeterminate form  $0/0$ .* As  $x \rightarrow \pi^-$  ( $x$  approaches  $\pi$  from values below  $\pi$ ),  $2 \sin x$  approaches 0, but  $1 - \cos x$  approaches 2. Therefore, we cannot apply L'Hospital's rule to this limit.

Since this function is continuous as  $x \rightarrow \pi^-$ , we have

$$\lim_{x \rightarrow \pi^-} \frac{2 \sin x}{1 - \cos x} = \frac{2 \sin \pi}{1 - \cos \pi} = \frac{0}{1 - (-1)} = 0$$

We can verify the value of a limit found using L'Hospital's rule by successive evaluation. The table derived in Example 6 can be obtained using the *table* feature of some graphing calculators.

**EXAMPLE 6** Numerical verification of a limit

Find  $\lim_{x \rightarrow \pi} \frac{1 + \cos x}{(\pi - x)^2}$  and verify the value of the limit numerically using values of  $x$  that approach  $\pi$ .

Because  $\cos \pi = -1$ , we see that both the numerator and the denominator approach zero as  $x$  approaches  $\pi$ . Therefore, we have

$$\begin{aligned} \lim_{x \rightarrow \pi} \frac{1 + \cos x}{(\pi - x)^2} &= \lim_{x \rightarrow \pi} \frac{\frac{d}{dx} (1 + \cos x)}{\frac{d}{dx} (\pi - x)^2} = \lim_{x \rightarrow \pi} \frac{-\sin x}{-2(\pi - x)} && \text{indeterminate form } 0/0; \\ &&& \text{use L'Hospital's rule} \\ &&& \text{again} \\ &= \lim_{x \rightarrow \pi} \frac{\frac{d}{dx} (-\sin x)}{\frac{d}{dx} [-2(\pi - x)]} = \lim_{x \rightarrow \pi} \frac{-\cos x}{2} = \frac{-(-1)}{2} = \frac{1}{2} \end{aligned}$$

$x$	$\frac{1 + \cos x}{(\pi - x)^2}$
3.1376	0.499 999 336
3.1386	0.499 999 627
3.1396	0.499 999 835
3.1406	0.499 999 954
$\pi$	Error
3.1426	0.499 999 956
3.1436	0.499 999 832
3.1446	0.499 999 624
3.1456	0.499 999 331

Fig. 27.32

To verify the limit numerically, we form a table of values of  $(1 + \cos x)/(\pi - x)^2$  using values of  $x$  that approach  $\pi$ . Such a table is shown in Fig. 27.32. We see that the limit  $1/2$  is verified.

There are indeterminate forms other than  $0/0$  and  $\infty/\infty$ . One of these is illustrated in the following example, and others are covered in the exercises.

**EXAMPLE 7** Indeterminate form  $0 \cdot \infty$ 

Find  $\lim_{x \rightarrow 0^+} x^2 \ln x$ .

As  $x \rightarrow 0^+$ , we note that  $x^2 \rightarrow 0$  and  $\ln x \rightarrow -\infty$ . This is the indeterminate form  $0 \cdot \infty$ . By writing  $x^2$  as  $1/(1/x^2)$  we can change this indeterminate form to  $\infty/\infty$ . Making this change, and then applying L'Hospital's rule, we have

$$\lim_{x \rightarrow 0^+} x^2 \ln x = \lim_{x \rightarrow 0^+} \frac{\ln x}{\frac{1}{x^2}} = \lim_{x \rightarrow 0^+} \frac{\frac{d}{dx} \ln x}{\frac{d}{dx} x^{-2}} = \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-\frac{2}{x^3}} = \lim_{x \rightarrow 0^+} \left( -\frac{x^2}{2} \right) = 0$$

## EXERCISES 27.7

In Exercises 1 and 2, make the given changes in the indicated examples of this section and then solve the resulting problems.

- In Example 2, change the denominator  $5x$  to  $5 \ln x$ .
- In Example 5, change the denominator  $1 - \cos x$  to  $1 + \cos x$ .

In Exercises 3–30, evaluate each limit (if it exists). Use L'Hospital's rule (if appropriate).

- $\lim_{\theta \rightarrow 0} \frac{\tan \theta}{\theta}$
- $\lim_{x \rightarrow \infty} \frac{x \ln x}{x + \ln x}$
- $\lim_{t \rightarrow \pi/4} \frac{1 - \sin 2t}{\frac{\pi}{4} - t}$
- $\lim_{x \rightarrow 0} \frac{\ln x}{x^{-1}}$
- $\lim_{x \rightarrow 0} \frac{\sin x - x}{x^3}$
- $\lim_{x \rightarrow 1} \frac{\sin \pi x}{x - 1}$
- $\lim_{x \rightarrow 0} x \cot x$
- $\lim_{x \rightarrow 0^+} x \ln \sin x$
- $\lim_{x \rightarrow 0} \frac{2 \sin x}{5e^x}$
- $\lim_{x \rightarrow +\infty} \frac{1 + e^{2x}}{2 + \ln x}$
- $\lim_{x \rightarrow 0} \frac{\ln \sin x}{\ln \tan x}$
- $\lim_{x \rightarrow 0^+} (\sin x)(\ln x)$
- $\lim_{x \rightarrow 2} \frac{2x^3 - 7x^2 + 11x - 10}{x^3 - 5x^2 + 7x - 2}$
- $\lim_{x \rightarrow 1} \frac{x^4 - x^3 - 3x^2 + 5x - 2}{x^4 - 5x^3 + 9x^2 - 7x + 2}$
- $\lim_{x \rightarrow 0} \frac{\sqrt{1-x} - \sqrt{1+x}}{x}$
- $\lim_{x \rightarrow 0} \frac{e^x + e^{-x} - 2}{1 - \cos 2x}$
- $\lim_{x \rightarrow \infty} \frac{e^x}{x^2}$
- $\lim_{x \rightarrow 1} \frac{\ln x}{x - 1}$
- $\lim_{x \rightarrow 0} \frac{\ln \cos x}{x}$
- $\lim_{\theta \rightarrow \pi/2} \frac{1 + \sec \theta}{\tan \theta}$
- $\lim_{x \rightarrow \infty} \frac{x^2 + x}{e^x + 1}$
- $\lim_{x \rightarrow 0} \frac{\tan^{-1} x}{x}$
- $\lim_{z \rightarrow +\infty} z e^{-z}$
- $\lim_{x \rightarrow \pi/2} (1 - \sin x) \tan x$
- $\lim_{x \rightarrow \infty} \frac{e^{2x} - 1}{4x + 1}$
- $\lim_{x \rightarrow \pi/2} \frac{\frac{\pi}{2} - x}{1 + \sin x}$
- $\lim_{t \rightarrow +\infty} \frac{\ln \ln t}{\ln t}$
- $\lim_{x \rightarrow 0^+} (\sin^{-1} x)(\ln x)$

In Exercises 31–40, solve the given problems.

- Another indeterminate form is  $\infty - \infty$ . Often, it is possible to make an algebraic or trigonometric change in the function so that it will take on the form of a  $0/0$  or  $\infty/\infty$  indeterminate form. Find  $\lim_{\theta \rightarrow \frac{\pi}{2}} (\sec \theta - \tan \theta)$ .
- Find  $\lim_{x \rightarrow 0} \left( \frac{1}{\sin x} - \frac{1}{x} \right)$ . (See Exercise 31.)
- Three other indeterminate forms are  $0^0$ ,  $\infty^0$ , and  $1^\infty$ . For the function  $y = [f(x)]^{g(x)}$  and the following limits, we have the indicated indeterminate form:
 
$$\lim_{x \rightarrow a} f(x) = 0 \text{ and } \lim_{x \rightarrow a} g(x) = 0 \quad (\text{indet. form } 0^0)$$

$$\lim_{x \rightarrow a} f(x) = \infty \text{ and } \lim_{x \rightarrow a} g(x) = 0 \quad (\text{indet. form } \infty^0)$$

$$\lim_{x \rightarrow a} f(x) = 1 \text{ and } \lim_{x \rightarrow a} g(x) = \infty \quad (\text{indet. form } 1^\infty)$$
 By taking the logarithm of  $y = [f(x)]^{g(x)}$ , we have  $\ln y = g(x) \ln f(x)$  and in each case the right-hand member is a type that can be solved by L'Hospital's rule. By knowing the limit of  $\ln y$ , we can find the limit of  $y$ . Find  $\lim_{x \rightarrow 0} x^x$ .
- Find  $\lim_{x \rightarrow 0} \left( \frac{1}{x} \right)^{\sin x}$ . (See Exercise 33.)
- Find  $\lim_{x \rightarrow +\infty} (1 + x^2)^{1/x}$ . (See Exercise 33.)
- Find  $\lim_{\theta \rightarrow \pi/2} (\sin \theta)^{\tan \theta}$ . (See Exercise 33.)
- Explain why L'Hospital's rule cannot be applied to  $\lim_{x \rightarrow \infty} x \sin x$ .
- By inspection, find  $\lim_{x \rightarrow \frac{\pi}{2}^-} (\cos x)^{\tan x}$ . What is the form of this limit? (Note that it is not one of the indeterminate forms noted in this section.)
- If the force resisting the fall of an object of mass  $m$  through the atmosphere is directly proportional to the velocity  $v$ , then the velocity at time  $t$  is  $v = \frac{mg}{k} (1 - e^{-kt/m})$ , where  $g$  is the acceleration due to gravity and  $k$  is a positive constant. Find  $\lim_{k \rightarrow 0^+} v$ .
- In Exercise 39, find  $\lim_{m \rightarrow \infty} v$ .

## Answers to Practice Exercises

1.  $-2$  2. 0

## 27.8 Applications

Curve Sketching • Newton's Method • Time-Rate-of-Change Problems

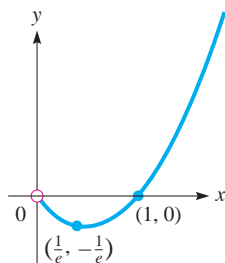


Fig. 27.33

The following examples show applications of the logarithmic and exponential functions to the types of applications shown at the left. These and other applications are included in the exercises.

### EXAMPLE 1 Sketching a graph

Sketch the graph of the function  $y = x \ln x$ .

First, we note that  $x$  cannot be zero since  $\ln x$  is not defined at  $x = 0$ . Since  $\ln 1 = 0$ , we have an intercept at  $(1, 0)$ . There is no symmetry to the axes or origin, and there are no vertical asymptotes. Also, because  $\ln x$  is defined only for  $x > 0$ , the domain is  $x > 0$ .

Finding the first two derivatives, we have

$$\frac{dy}{dx} = x\left(\frac{1}{x}\right) + \ln x = 1 + \ln x \quad \frac{d^2y}{dx^2} = \frac{1}{x}$$

The first derivative is zero if  $\ln x = -1$ , or  $x = e^{-1}$ . The second derivative is positive for this value of  $x$ . Thus, there is a minimum point at  $(1/e, -1/e)$ . Since the domain is  $x > 0$ , the second derivative indicates that the curve is always concave up. In turn, we now see that the range of the function is  $y \geq -1/e$ . The graph is shown in Fig. 27.33.

Using L'Hospital's rule, we find  $\lim_{x \rightarrow 0} x \ln x = 0$ . Therefore, although the curve approaches the origin as  $x$  approaches zero, the origin is not included on the graph of the function. ■

### EXAMPLE 2 Sketching a graph

Sketch the graph of the function  $y = e^{-x} \cos x$  ( $0 \leq x \leq 2\pi$ ).

This curve has intercepts for all values for which  $\cos x$  is zero. Those values in the domain  $0 \leq x \leq 2\pi$  for which  $\cos x = 0$  are  $x = \frac{\pi}{2}$  and  $x = \frac{3\pi}{2}$ . The factor  $e^{-x}$  is always positive, and  $e^{-x} = 1$  for  $x = 0$ , which means  $(0, 1)$  is also an intercept. There is no symmetry to the axes or the origin, and there are no vertical asymptotes.

Next, finding the first derivative, we have

$$\frac{dy}{dx} = -e^{-x} \sin x - e^{-x} \cos x = -e^{-x} (\sin x + \cos x)$$

Setting the derivative equal to zero, since  $e^{-x}$  is always positive, we have

$$\sin x + \cos x = 0, \quad \tan x = -1, \quad x = \frac{3\pi}{4}, \frac{7\pi}{4}$$

Now, finding the second derivative, we have

$$\frac{d^2y}{dx^2} = -e^{-x} (\cos x - \sin x) - e^{-x} (-1) (\sin x + \cos x) = 2e^{-x} \sin x$$

The sign of the second derivative depends only on  $\sin x$ . Therefore,

$$\frac{d^2y}{dx^2} > 0 \quad \text{for } x = \frac{3\pi}{4} \quad \text{and} \quad \frac{d^2y}{dx^2} < 0 \quad \text{for } x = \frac{7\pi}{4}$$

This means that  $(\frac{3\pi}{4}, -0.067)$  is a minimum and  $(\frac{7\pi}{4}, 0.003)$  is a maximum.

Also, from the second derivative, points of inflection occur for  $x = 0, \pi$ , and  $2\pi$  since  $\sin x = 0$  for these values. The graph is shown in Fig. 27.34. ■

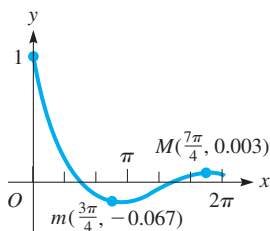


Fig. 27.34

**EXAMPLE 3** Solving an equation using Newton's method

Find the root of the equation  $e^{2x} - 4 \cos x = 0$  that lies between 0 and 1, by using Newton's method.

Here,

$$\begin{aligned} f(x) &= e^{2x} - 4 \cos x \\ f'(x) &= 2e^{2x} + 4 \sin x \end{aligned}$$

**Practice Exercise**

1. In Example 3, choose  $x_1 = 0.7$  and then find  $x_2$ .

This means that  $f(0) = -3$  and  $f(1) = 5.2$ . Therefore, we choose  $x_1 = 0.5$ . Using Eq. (24.1), which is

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

we have these values:

$$\begin{aligned} f(x_1) &= e^{2(0.5)} - 4 \cos 0.5 = -0.792\,048\,4 \\ f'(x_1) &= 2e^{2(0.5)} + 4 \sin 0.5 = 7.354\,265\,8 \\ x_2 &= 0.5 - \frac{-0.792\,048\,4}{7.354\,265\,8} = 0.607\,699\,2 \end{aligned}$$

Using the method again, we find  $x_3 = 0.597\,975\,1$ , which is correct to three decimal places. ■

**EXAMPLE 4** Time rate of change—population growth

One model for population growth is that the population  $P$  at time  $t$  is given by  $P = P_0 e^{kt}$ , where  $P_0$  is the *initial* population ( $t = 0$ , when timing starts for the population being considered) and  $k$  is a constant. Show that the instantaneous time rate of change of population is directly proportional to the population present at time  $t$ .

To find the time rate of change, we find the derivative  $dP/dt$ :

$$\begin{aligned} \frac{dP}{dt} &= (P_0 e^{kt})(k) = kP_0 e^{kt} \\ &= kP \quad \text{since } P = P_0 e^{kt} \end{aligned}$$

Thus, we see that population growth increases as population increases. ■

**EXAMPLE 5** Acceleration of a rocket

A rocket is moving such that the only force acting on it is due to gravity and its mass is decreasing (because of the use of fuel) at a constant rate  $r$ . If it moves vertically, its velocity  $v$  as a function of time  $t$  is given by

$$v = v_0 - gt - k \ln \left( 1 - \frac{rt}{m_0} \right)$$

where  $v_0$  is the initial velocity,  $g$  is the acceleration due to gravity,  $t$  is the time,  $m_0$  is the initial mass, and  $k$  is a constant. Determine the expression for the acceleration.

Since acceleration is the time rate of change of the velocity, we must find  $dv/dt$ . Therefore,

$$\begin{aligned} \frac{dv}{dt} &= -g - k \frac{1}{1 - \frac{rt}{m_0}} \left( \frac{-r}{m_0} \right) = -g + \frac{km_0}{m_0 - rt} \left( \frac{r}{m_0} \right) \\ &= \frac{kr}{m_0 - rt} - g \end{aligned}$$

■ See the chapter introduction.

## EXERCISES 27.8

In Exercises 1 and 2, make the given changes in the indicated examples of this section and then solve the resulting problems.

- In Example 2, in the given function, change  $\cos$  to  $\sin$ .
- In Example 3, in the given function, change  $e^{2x}$  to  $2e^x$ .

In Exercises 3–14, sketch the graphs of the given functions.

- $y = \ln \cos x$
- $y = \frac{2 \ln x}{x}$
- $y = 3xe^{-x}$
- $y = \frac{e^x}{x}$
- $y = \ln \frac{1}{x^2 + 1}$
- $y = 5 \ln \frac{e}{x}$
- $y = 4e^{-x^2}$
- $y = 4x - e^x$
- $y = 8 \ln x - 2x$
- $y = e^{-x} \sin x$
- $y = \frac{1}{2}(e^x - e^{-x})$  (See Exercise 55 of Section 27.6.)
- $y = \frac{1}{2}(e^x + e^{-x})$  (See Exercise 55 of Section 27.6.)

In Exercises 15–44, solve the given problems by finding the appropriate derivative.

- Find the values of  $x$  for which the graphs of  $y_1 = e^{2x^2}$  and  $y_2 = e^{-2x^2}$  are increasing and decreasing. Explain why they differ as they do.
- Find the values of  $x$  for which the graphs of  $y_1 = \ln(x^2 + 1)$  and  $y_2 = \ln(x^2 - 1)$  are concave up and concave down. Explain why they differ as they do.
- Find the equation of the line tangent to the curve of  $y = x^2 \ln x$  at the point  $(1, 0)$ .
- Find the equation of the line tangent to the curve of  $y = \tan^{-1} 2x$ , where  $x = 1$ .
- Find the equation of the line normal to the curve of  $y = 2 \sin \frac{1}{2}x$ , where  $x = 3\pi/2$ .
- Find the equation of the line normal to the curve of  $y = e^{2x}/x$ , at  $x = 1$ .
- By Newton's method, solve the equation  $x^2 - 3 + \ln 4x = 0$  to at least four decimal places.
- By Newton's method, find the value of  $x$  to at least four decimal places for which  $y = e^{\cos x}$  is minimum ( $0 < x < 2\pi$ ).
- The electric current  $i$  (in A) through an inductor of 0.50 H as a function of time  $t$  (in s) is  $i = e^{-5.0t} \sin 120\pi t$ . Find the voltage across the inductor for  $t = 1.0$  ms. (See Exercise 27 on page 782.)
- A computer analysis showed that the population density  $D$  (in persons/km<sup>2</sup>) at a distance  $r$  (in km) from the centre of a city is approximately  $D = 200(1 + 5e^{-0.01r^2})$  if  $r < 20$  km. At what distance from the city centre does the decrease in population density ( $dD/dr$ ) itself start to decrease?
- The power supply  $P$  (in W) in a satellite is  $P = 100e^{-0.005t}$ , where  $t$  is measured in days. Find the time rate of change of power after 100 days.
- The number  $N$  of atoms of radium at any time  $t$  is given in terms of the number at  $t = 0$ ,  $N_0$ , by  $N = N_0e^{-kt}$ . Show that the time rate of change of  $N$  is proportional to  $N$ .
- A metal bar is heated, and then allowed to cool. Its temperature  $T$  (in °C) is found to be  $T = 15 + 75e^{-0.25t}$ , where  $t$  (in min) is the time of cooling. Find the time rate of change of temperature after 5.0 min.
- The insulation resistance  $R$  (in  $\Omega/m$ ) of a shielded cable is given by  $R = k \ln(r_2/r_1)$ . Here  $r_1$  and  $r_2$  are the inner and outer radii of the insulation. Find the expression for  $dR/dr_2$  if  $k$  and  $r_1$  are constant.
- The vapour pressure  $p$  and thermodynamic temperature  $T$  of a gas are related by the equation  $\ln p = \frac{a}{T} + b \ln T + c$ , where  $a$ ,  $b$ , and  $c$  are constants. Find the expression for  $dp/dT$ .
- The charge  $q$  on a capacitor in a circuit containing a capacitor of capacitance  $C$ , a resistance  $R$ , and a source of voltage  $E$  is given by  $q = CE(1 - e^{-t/RC})$ . Show that this equation satisfies the equation  $R \frac{dq}{dt} + \frac{q}{C} = E$ .
- Assuming that force is proportional to acceleration, show that a particle moving along the  $x$ -axis, so that its displacement is  $x = Ae^{kt} + Be^{-kt}$ , has a force acting on it which is proportional to its displacement.
- The radius of curvature at a point on a curve is given by 
$$R = \frac{[1 + (dy/dx)^2]^{3/2}}{d^2y/dx^2}.$$
 A roller mechanism moves along the path defined by  $y = \ln \sec x$  ( $-1.5 \text{ dm} \leq x \leq 1.5 \text{ dm}$ ). Find the radius of curvature of this path for  $x = 0.85 \text{ dm}$ .
- Sketch the graph of  $y = \ln \sec x$ , marking that part which is the path of the roller mechanism of Exercise 32.
- In an electronic device, the maximum current density  $i_m$  as a function of the temperature  $T$  is given by  $i_m = AT^2e^{k/T}$ , where  $A$  and  $k$  are constants. Find the expression for a small change in  $i_m$  for a small change in  $T$ .
- The energy  $E$  (in J) dissipated by a certain resistor after  $t$  seconds is given by  $E = \ln(t + 1) - 0.25t$ . At what time is the energy dissipated the greatest?
- In a study of traffic control, the number  $n$  of vehicles on a certain section of a highway from 2 P.M. to 8 P.M. was found to be  $n = 200(1 + t^3e^{-t})$ , where  $t$  is the number of hours after 2 P.M. At what time is the number of vehicles the greatest?
- A meteorologist sketched the path of the jet stream on a map of the northern United States and southern Canada on which all latitudes were parallel and all longitudes were parallel and equally spaced. A computer analysis showed this path to be  $y = 6.0e^{-0.020x} \sin 0.20x$  ( $0 \leq x \leq 60$ ), where the origin is 125.0°W, 45.0°N and  $(60, 0)$  is 65.0°W, 45.0°N. Find the locations of the maximum and minimum latitudes of the jet stream between 65°W and 125°W for that day.
- The reliability  $R$  ( $0 \leq R \leq 1$ ) of a certain computer system after  $t$  hours of operation is found from  $R = 3e^{-0.004t} - 2e^{-0.006t}$ . Use Newton's method to find how long the system operates to have a reliability of 0.8 (80% probability that there will be no system failure).



39. An object on the end of a spring is moving so that its displacement (in cm) from the equilibrium position is given by  $y = e^{-0.5t}(0.4 \cos 6t - 0.2 \sin 6t)$ . Find the expression for the velocity of the object. What is the velocity when  $t = 0.26$  s? The motion described by this equation is called *damped harmonic motion*.
40. A package of weather instruments is propelled into the air to an altitude of about 7 km. A parachute then opens, and the package returns to the surface. The altitude  $y$  of the package as a function of the time  $t$  (in min) is given by  $y = \frac{10t}{e^{0.4t} + 1}$ . Find the vertical velocity of the package for  $t = 8.0$  min.
41. The speed  $s$  of signaling by use of a certain communications cable is directly proportional to  $x^2 \ln x^{-1}$ , where  $x$  is the ratio of the radius of the core of the cable to the thickness of the surrounding insulation. For what value of  $x$  is  $s$  a maximum?
42. A computer is programmed to inscribe a series of rectangles in the first quadrant under the curve of  $y = e^{-x}$ . What is the area of the largest rectangle that can be inscribed?
43. The relative number  $N$  of gas molecules in a container that are moving at a velocity  $v$  can be shown to be  $N = av^2e^{-bv^2}$ , where  $a$  and  $b$  are constants. Find  $v$  for the maximum  $N$ .
44. A missile is launched and travels along a path that can be represented by  $y = \sqrt{x}$ . A radar tracking station is located 2.00 km directly behind the launch pad. Placing the launch pad at the origin and the radar station at  $(-2.00, 0)$ , find the largest angle of elevation required of the radar to track the missile.

**Answer to Practice Exercise**

1.  $x_2 = 0.6068$

**CHAPTER 27 EQUATIONS**

Limit of  $\frac{\sin \theta}{\theta}$  as  $\theta \rightarrow 0$

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = \lim_{h \rightarrow 0} \frac{\sin(h/2)}{h/2} = 1 \quad (27.1)$$

Chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \quad (27.3)$$

Derivatives

$$\frac{d(\sin u)}{dx} = \cos u \frac{du}{dx} \quad (27.4)$$

$$\frac{d(\cos u)}{dx} = -\sin u \frac{du}{dx} \quad (27.5)$$

$$\frac{d(\tan u)}{dx} = \sec^2 u \frac{du}{dx} \quad (27.6)$$

$$\frac{d(\cot u)}{dx} = -\csc^2 u \frac{du}{dx} \quad (27.7)$$

$$\frac{d(\sec u)}{dx} = \sec u \tan u \frac{du}{dx} \quad (27.8)$$

$$\frac{d(\csc u)}{dx} = -\csc u \cot u \frac{du}{dx} \quad (27.9)$$

$$\frac{d(\sin^{-1} u)}{dx} = \frac{1}{\sqrt{1-u^2}} \frac{du}{dx} \quad (27.10)$$

$$\frac{d(\cos^{-1} u)}{dx} = -\frac{1}{\sqrt{1-u^2}} \frac{du}{dx} \quad (27.11)$$

$$\frac{d(\tan^{-1} u)}{dx} = \frac{1}{1+u^2} \frac{du}{dx} \quad (27.12)$$

$$\frac{d(\log_b u)}{dx} = \frac{1}{u} \log_b e \frac{du}{dx} \quad (27.13)$$

$$\frac{d(\ln u)}{dx} = \frac{1}{u} \frac{du}{dx} \quad (27.14)$$

$$\frac{d(b^u)}{dx} = b^u \ln b \frac{du}{dx} \quad (27.15)$$

$$\frac{d(e^u)}{dx} = e^u \frac{du}{dx} \quad (27.16)$$

## CHAPTER 27 REVIEW EXERCISES

In Exercises 1–40, find the derivative of the given functions.

1.  $y = 3 \cos(4x - 1)$
2.  $y = 4 \sec(1 - x^3)$
3.  $u = 0.2 \tan \sqrt{3 - 2v}$
4.  $y = 5 \sin(1 - 6x)$
5.  $y = \csc^2(3x + 2)$
6.  $r = \cot^2 5\pi\theta$
7.  $y = 3 \cos^4 x^2$
8.  $y = 2 \sin^3 \sqrt{x}$
9.  $y = (e^{x-3})^2$
10.  $y = 0.5e^{\sin 2x}$
11.  $y = 3 \ln(x^2 + 1)$
12.  $R = \ln(3 + \sin T^2)$
13.  $y = 10 \tan^{-1}(x/5)$
14.  $y = 0.4 \cos^{-1}(2\pi t + 1)$
15.  $\theta = \ln \sin^{-1} 0.1t$
16.  $y = \sin(\tan^{-1} x)$
17.  $y = \sqrt{\csc 4x + \cot 4x}$
18.  $y = 3 \cos^2(\tan 3x)$
19.  $y = 7 \ln(x - e^{-x})^2$
20.  $h = \ln \sqrt[3]{\sin 6\theta}$
21.  $y = \frac{\cos^2 x}{e^{3x} + \pi^2}$
22.  $y = \sqrt{\frac{1 + \cos 2x}{2}}$
23.  $v = \frac{u^2}{\tan^{-1} 2u}$
24.  $y = \frac{\sin^{-1} x}{4x}$
25.  $y = \ln(\csc x^2)$
26.  $u = 0.5 \ln \tan e^x$
27.  $y = \ln^2(3 + \sin x)$
28.  $y = \ln(3 + \sin \pi t)^2$
29.  $L = 0.1e^{-2t} \sec \pi t$
30.  $y = 5e^{3x} \ln x$
31.  $y = \sqrt{\sin 2x + e^{4x}}$
32.  $x + y \ln 2x = y^2$
33.  $\tan^{-1} \frac{y}{x} = x^2 e^y$
34.  $3y + \ln xy = 2 + x^2$
35.  $r = 0.5t(e^{2t} + 1)(e^{-2t} - 1)$
36.  $y = (\ln 4x - \tan 4x)^3$
37.  $\ln xy + ye^{-x} = 1$
38.  $y = x(\sin^{-1} x)^2 + 2\sqrt{1 - x^2} \sin^{-1} x - 2x$
39.  $y = x \cos^{-1} x - \sqrt{1 - x^2}$
40.  $W = \ln(4s^2 + 1) + \tan^{-1} 2s$

In Exercises 41–44, sketch the graphs of the given functions.

41.  $y = x - \cos 0.5x$
42.  $y = 4 \sin x + \cos 2x$
43.  $y = x(\ln x)^2$
44.  $y = 2 \ln(3 + x)$

In Exercises 45–48, find the equations of the indicated tangent or normal lines.

45. Find the equation of the line tangent to the curve of  $y = 4 \cos^2(x^2)$  at  $x = 1$ .
46. Find the equation of the line tangent to the curve of  $y = \ln \cos x$  at  $x = \frac{\pi}{6}$ .
47. Find the equation of the line normal to the curve of  $y = e^{x^2}$  at  $x = \frac{1}{2}$ .
48. Find the equation of the line normal to the curve of  $y = \tan^{-1} 4x$  at  $x = 1/2$ .

In Exercises 49–54, find the indicated limits by use of L'Hospital's rule.

49.  $\lim_{x \rightarrow 0} \frac{\sin 2x}{\sin 3x}$
50.  $\lim_{x \rightarrow 0} \frac{xe^x}{1 - e^x}$
51.  $\lim_{x \rightarrow \pi^+} \frac{\sin x}{x - \pi}$
52.  $\lim_{t \rightarrow +\infty} \frac{t^4 + 5t^2 + 1}{3t^4 + 4}$
53.  $\lim_{x \rightarrow \infty} \frac{\ln x}{\sqrt[3]{x}}$
54.  $\lim_{x \rightarrow \pi} (x - \pi) \cot x$

In Exercises 55–98, solve the given problems.

55. Find the derivative of each member of the identity  $\sin^2 x + \cos^2 x = 1$  and show that the results are equal.
56. Find the derivative of each member of the identity  $\sin(x + 1) = \sin x \cos 1 + \cos x \sin 1$  and show that the results are equal.
57. If  $y = \sin 3x$ , show that  $\frac{d^2 y}{dx^2} = -9y$ .
58. If  $y = e^{5x}(a + bx)$ , show that  $\frac{d^2 y}{dx^2} - 10 \frac{dy}{dx} + 25y = 0$ .
59. By Newton's method, solve the equation  $e^x - x^2 = 0$ .
60. By Newton's method, find the nonzero solution to the equation  $x^2 = \tan^{-1} x$ .
61. Find the values of  $x$  for which the graph of  $y = e^x - 2e^{-x}$  is concave up.
62. Find the values of  $x$  for which the graph of  $y = 2e^{\sin(x/2)}$  has maximum or minimum points.
63. If a block is placed on a plane inclined with the horizontal at an angle  $\theta$  such that the block stays at the same position, the coefficient of friction  $\mu$  is given by  $\mu = \tan \theta$ . Use differentials to find the change in  $\mu$  if  $\theta$  changes from  $18^\circ$  to  $20^\circ$ .
64. The tensile strength  $S$  (in N) of a plastic is tested and found to change with the temperature  $T$  (in  $^\circ\text{C}$ ) according to the equation  $S = 1800 \ln(T + 25) - 40T + 8600$ . For what temperature is the tensile strength the greatest?
65. If a 200-N crate is dragged along a horizontal floor by a force  $F$  (in N) acting along a rope at an angle  $\theta$  with the floor, the magnitude of  $F$  is given by  $F = \frac{200 \mu}{\mu \sin \theta + \cos \theta}$ , where  $\mu$  is the coefficient of friction. Evaluate the instantaneous rate of change of  $F$  with respect to  $\theta$  when  $\theta = 15^\circ$  if  $\mu = 0.20$ .
66. Find the date of the maximum number of hours of daylight in Toronto, Ontario. See Exercise 73 on page 319.

67. Periodically, a robot moves a part vertically  $y$  cm in an automobile assembly line. If  $y$  as a function of the time  $t$  (in s) is  $y = 0.75(\sec \sqrt{0.15t} - 1)$ , find the velocity at which the part is moved after 5.0 s of each period.
68. The length  $L$  (in m) of the shadow of a tree 15 m tall is  $L = 15 \cot \theta$ , where  $\theta$  is the angle of elevation of the sun. Approximate the change in  $L$  if  $\theta$  changes from  $50^\circ$  to  $52^\circ$ .
69. An analysis of temperature records for Sydney, Australia, indicates that the average daily temperature (in  $^\circ\text{C}$ ) during the year is given approximately by  $T = 17.2 + 5.2 \cos \left[ \frac{\pi}{6}(x - 0.50) \right]$ , where  $x$  is measured in months ( $x = 0.5$  is Jan. 15, etc.). What is the *daily* time rate of change of temperature on March 1? (Hint: 12 months/365 days = 0.033 month/day =  $dx/dt$ .)
70. An earth-orbiting satellite is launched such that its altitude (in km) is given by  $y = 240(1 - e^{-0.05t})$ , where  $t$  is the time (in min). Find the vertical velocity of the satellite for  $t = 10.0$  min.
71. Power  $P$  is the time rate of change of doing work  $W$ . If work is being done in an electric circuit according to  $W = 25 \sin^2 2t$ , find  $P$  as a function of  $t$ .
72. The value  $V$  of a bank account in which \$1000 is deposited and then earns 6% annual interest, compounded continuously (daily compounding approximates this, and some banks actually use continuous compounding), is  $V = 1000e^{0.06t}$  after  $t$  years. How fast is the account growing after exactly 2 years?
73. In determining how to divide files on the hard disk of a computer, we can use the equation  $n = xN \log_x N$ . Sketch the graph of  $n$  as a function of  $x$  for  $1 < x \leq 10$  if  $N = 8$ .
74. Under certain conditions, the potential  $V$  (in V) due to a magnet is given by  $V = -k \ln \left( 1 + \frac{L}{x} \right)$ , where  $L$  is the length of the magnet and  $x$  is the distance from the point where the potential is measured. Find the expression for  $dV/dx$ .
75. In the theory of making images by holography, an expression used for the light-intensity distribution is  $I = kE_0^2 \cos^2 \frac{1}{2}\theta$ , where  $k$  and  $E_0$  are constant and  $\theta$  is the phase angle between two light waves. Find the expression for  $dI/d\theta$ .
76. Neglecting air resistance, the range  $R$  of a bullet fired at an angle  $\theta$  with the horizontal is  $R = \frac{v_0^2}{g} \sin 2\theta$ , where  $v_0$  is the initial velocity and  $g$  is the acceleration due to gravity. Find  $\theta$  for the maximum range. See Fig. 27.35.
77. In the design of a cone-type clutch, an equation that relates the cone angle  $\theta$  and the applied force  $F$  is  $\theta = \sin^{-1}(Ff/R)$ , where  $R$  is the frictional resistance and  $f$  is the coefficient of friction. For constant  $R$  and  $f$ , find  $d\theta/dF$ .
78. If inflation makes the dollar worth 5% less each year, then the value of \$100 in  $t$  years will be  $V = 100(0.95)^t$ . What is the approximate change in the value during the fourth year?
79. An object attached to a cord of length  $l$ , as shown in Fig. 27.36, moves in a circular path. The angular velocity  $\omega$  is given by  $\omega = \sqrt{g/(l \cos \theta)}$ . By use of differentials, find the approximate change in  $\omega$  if  $\theta$  changes from  $32.50^\circ$  to  $32.75^\circ$ , given that  $g = 9.800 \text{ m/s}^2$  and  $l = 0.6375 \text{ m}$ .

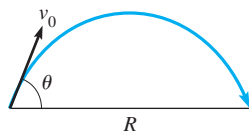


Fig. 27.35

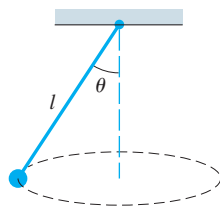


Fig. 27.36

80. An analysis of samples of air for a city showed that the number of parts per million  $p$  of sulfur dioxide on a certain day was  $p = 0.05 \ln(2 + 24t - t^2)$ , where  $t$  is the hour of the day. Using differentials, find the approximate change in the amount of sulfur dioxide between 10 A.M. and noon.
81. According to Newton's law of cooling (Isaac Newton, again), the rate at which a body cools is proportional to the difference in temperature between it and the surrounding medium. By use of this law, the temperature  $T$  (in  $^\circ\text{C}$ ) of an engine coolant as a function of the time  $t$  (in min) is  $T = 30 + 60(0.5)^{0.200t}$ . The coolant was initially at  $90^\circ\text{C}$  and the air temperature was  $30^\circ\text{C}$ . Linearize this function for  $t = 5.00$  min.
82. The charge  $q$  on a certain capacitor in an amplifier circuit as a function of time  $t$  is given by  $q = e^{-0.1t}(0.2 \sin 120\pi t + 0.8 \cos 120\pi t)$ . The current  $i$  in the circuit is the instantaneous time rate of change of the charge. Find the expression for  $i$  as a function of  $t$ .
83. A projection of the number  $n$  (in millions) of users of the Internet is  $n = 160 - 140e^{-0.30t}$ , where  $t$  is the number of years after 2000. What is the projected annual rate of increase in 2015?
84. A football is thrown horizontally (very little arc) at 18 m/s parallel to the sideline. A TV camera is 31 m from the path of the football. Find  $d\theta/dt$ , the rate at which the camera must turn to follow the ball when  $\theta = 15^\circ$ . See Fig. 27.37.

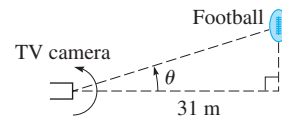


Fig. 27.37

85. An architect designs an arch of height  $y$  (in m) over a walkway by the curve of the equation  $y = 3.00e^{-0.500x^2}$ . What are the dimensions of the largest rectangular passage area under the arch?
86. A force  $P$  (in N) at an angle  $\theta$  above the horizontal drags a 50-N box across a level floor. The coefficient of friction between the floor and the box is constant and equals 0.20. The magnitude of the force  $P$  is given by  $P = \frac{(0.20)(50)}{0.20 \sin \theta + \cos \theta}$ . Find  $\theta$  such that  $P$  is a minimum.
87. A jet is flying at 220 m/s directly away from the control tower of an airport. If the jet is at a constant altitude of 1700 m, how fast is the angle of elevation of the jet from the control tower changing when it is  $13.0^\circ$ ?
88. The current  $i$  in an electric circuit with a resistance  $R$  and an inductance  $L$  is  $i = i_0 e^{-Rt/L}$ , where  $i_0$  is the initial current. Show that the time rate of change of the current is directly proportional to the current.
89. A silo constructed as shown in Fig. 27.38 is to hold  $2880 \text{ m}^3$  of silage when completely full. It can be shown (can you?) that the surface area  $S$  (in  $\text{m}^2$ ) (not including the base) is  $S = 640 + 81\pi(\csc \theta - \frac{2}{3} \cot \theta)$ . Find  $\theta$  such that  $S$  is a minimum.

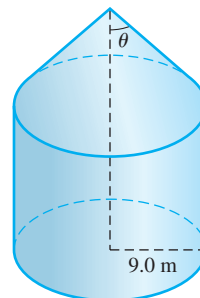


Fig. 27.38

90. Light passing through a narrow slit forms patterns of light and dark (see Fig. 27.39). The intensity  $I$  of the light at an angle  $\theta$  is given by  $I = I_0 \left[ \frac{\sin(k \sin \theta)}{k \sin \theta} \right]^2$ , where  $k$  and  $I_0$  are constants. Show that the maximum and minimum values of  $I$  occur for  $k \sin \theta = \tan(k \sin \theta)$ .

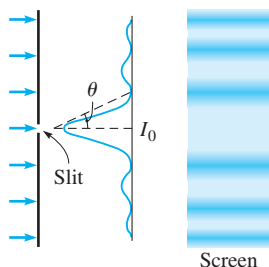


Fig. 27.39

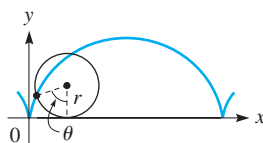


Fig. 27.40

91. When a wheel rolls along a straight line, a point  $P$  on the circumference traces a curve called a *cycloid*. See Fig. 27.40. The parametric equations of a cycloid are  $x = r(\theta - \sin \theta)$  and  $y = r(1 - \cos \theta)$ . Find the velocity of the point on the rim of a wheel for which  $r = 5.50$  cm and  $d\theta/dt = 0.120$  rad/s for  $\theta = 35.0^\circ$ . (An inverted cycloid is the path of least time of descent (the *brachistochrone*) of an object acted on only by gravity.)
92. In the study of atomic spectra, it is necessary to solve the equation  $x = 5(1 - e^{-x})$  for  $x$ . Use Newton's method to find the solution.
93. The illuminance from a point source of light varies directly as the cosine of the angle of incidence (measured from the perpendicular) and inversely as the square of the distance  $r$  from the source. How high above the centre of a circle of radius 10.0 cm should a light be placed so that illuminance at the circumference will be a maximum? See Fig. 27.41.

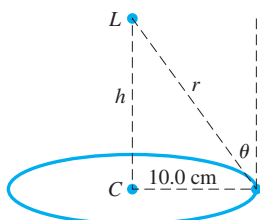


Fig. 27.41

94. A Y-shaped metal bracket is to be made such that its height is 10.0 cm and its width across the top is 6.00 cm. What shape will require the least amount of material? See Fig. 27.42.

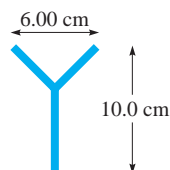


Fig. 27.42

95. A gutter is to be made from a sheet of metal 30.0 cm wide by turning up strips of width 10.0 cm along each side to make equal angles  $\theta$  with the vertical. Sketch a graph of the cross-sectional area  $A$  as a function of  $\theta$ . See Fig. 27.43.

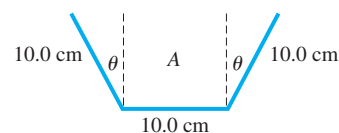


Fig. 27.43

96. The displacement  $y$  (in cm) of a weight on a spring in water is given by  $y = 3.0te^{-0.20t}$ , where  $t$  is the time (in s). What is the maximum displacement? (For this type of displacement, the motion is called *critically damped*, as the weight returns to its equilibrium position as quickly as possible without oscillating.)
97. Show that the equation of the hyperbolic cosine function  $y = \frac{H}{w} \cosh \frac{wx}{H}$  ( $w$  and  $H$  are constants) satisfies the equation  $\frac{d^2y}{dx^2} = \frac{w}{H} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$  (see Exercise 55 on page 837). A *catenary* (see Exercise 59 on page 392) is the curve of a uniform cable hanging under its own weight and is in the shape of a hyperbolic cosine curve. This shape (inverted) was chosen for the St. Louis Gateway Arch (shown in Fig. 27.44) and makes the arch self-supporting. Also, an igloo has the shape of a rotated inverted catenary (called a *catenoid*).



Fig. 27.44

98. A conical filter is made from a circular piece of wire mesh of radius 24.0 cm by cutting out a sector with central angle  $\theta$  and then taping the cut edges of the remaining piece together (see Fig. 27.45). What is the maximum possible volume the resulting filter can hold?

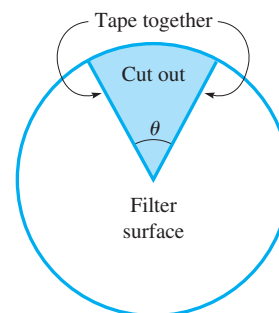


Fig. 27.45

### Writing Exercise

99. To find the area of the largest rectangular microprocessor chip with a perimeter of 40 mm, it is possible to use either an algebraic function or a trigonometric function. Write two or three paragraphs to explain how each type of function can be used to find the required area.

## CHAPTER 27 PRACTICE TEST

In Problems 1–3, find the derivative of each of the functions.

1.  $y = \tan^3 2x + \tan^{-1} 2x$

2.  $y = 2(3 + \cot 4x)^3$

3.  $y \sec 2x = \sin^{-1} 3y$

4. Find the differential of the function  $y = \frac{\cos^2(3x + 1)}{x}$ .

5. Find the slope of a tangent to the curve of  $y = \ln \frac{2x - 1}{1 + x^2}$  for  $x = 2$ .

6. Find the expression for the time rate of change of electric current that is given by the equation  $i = 8e^{-t} \sin 10t$ , where  $t$  is the time.7. Sketch the graph of the function  $y = xe^x$ .

8. A balloon leaves the ground 250 m from an observer and rises at the rate of 5.0 m/s. How fast is the angle of elevation of the balloon increasing after 8.0 s?

9. Find  $\lim_{x \rightarrow 0} \frac{\tan^{-1} x}{2 \sin x}$  using L'Hospital's rule.

# Methods of Integration

In developing calculus, mathematicians saw that integration and differentiation were inverse processes and that many integrals could be formed by finding the antiderivative. However, many of the integrals that arose mathematically and from the study of mechanical systems did not fit a form from which the antiderivative could be found directly. This led to the creation of numerous methods to integrate various types of functions.

Some of these methods of integration had been developed by the early 1700s, and were used by many mathematicians, including Newton and Leibniz. By the mid-1700s, many of these methods were included in textbooks. One of these texts was written by the Italian mathematician Maria Agnesi. Her text included analytic geometry, differential calculus, integral calculus, and some advanced topics and was noted for clear and organized explanations with many examples. Another important set of textbooks was written by Euler, who wrote separate texts on precalculus topics, differential calculus, and integral calculus. These texts were also noted for clear and organized presentations and were widely used until the early 1800s. Euler's integral calculus text included most of the methods of integration presented in this chapter.

Being able to integrate functions by using special methods, as well as by directly using antiderivatives, made integral calculus much more useful in developing many areas of geometry, science, and technology in the 1800s and 1900s. In earlier chapters, we have noted a number of applications of integration, and additional examples are found in the examples and exercises of this chapter.

In this chapter, we expand the use of the general power formula for integration for use with integrands that include transcendental functions and several special methods of integration for integrands that do not directly fit standard forms. In using all forms of integration, *recognition of the integral form* is of great importance.

# 28

## LEARNING OUTCOMES

After completion of this chapter, the student should be able to:

- Perform integrals by any of the following methods:
  - the general power formula
  - integration of basic logarithmic and exponential forms
  - integration of basic trigonometric and inverse trigonometric forms
  - integration by parts
  - trigonometric substitution
  - integration by partial fractions
  - using tables
- Solve application problems involving integration



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◀ In Section 28.5, we show an application of integration that is important in the design of computers, electronic equipment, and electric appliances.



## 28.1 The General Power Formula

**Integration Using the Power Formula** •  
**Recognizing  $u$ ,  $n$ , and  $du$**  •  
**Special Care with  $du$**

### LEARNING TIP

In applying Eq. (28.1) to transcendental integrands, as well as with algebraic integrands, **we must properly recognize the quantities  $u$ ,  $n$ , and  $du$** . This requires familiarity with the differential forms of Chapters 23 and 27.

The first formula for integration that we will discuss is the general power formula, and we will expand its use to include transcendental integrands. It was first introduced with the integration of basic algebraic forms in Chapter 25 and is repeated here for reference.

$$\int u^n du = \frac{u^{n+1}}{n+1} + C \quad (n \neq -1) \quad (28.1)$$

### EXAMPLE 1 Trigonometric integrand

Integrate  $\int \sin^3 x \cos x dx$ .

Since  $d(\sin x) = \cos x dx$ , we note that this integral fits the form of Eq. (28.1) for  $u = \sin x$ . Thus, with  $u = \sin x$ , we have  $du = \cos x dx$ , which means that this integral is of the form  $\int u^3 du$ . Therefore, the integration can now be completed:

$$\begin{aligned} \int \sin^3 x \cos x dx &= \int \sin^3 x (\cos x dx) \\ &= \frac{1}{4} \sin^4 x + C \end{aligned}$$

do not forget the constant of integration

We note here that **the factor  $\cos x$  is a necessary part of the  $du$**  in order to have the proper form of integration **and therefore does not appear in the final result**.

We check our result by finding the derivative of  $\frac{1}{4} \sin^4 x + C$ , which is

$$\begin{aligned} \frac{d}{dx} \left( \frac{1}{4} \sin^4 x + C \right) &= \frac{1}{4} (4) \sin^3 x \cos x \\ &= \sin^3 x \cos x \end{aligned}$$

### EXAMPLE 2 Trigonometric integrand

Integrate  $\int 2\sqrt{1 + \tan \theta} \sec^2 \theta d\theta$ .

Here, we note that  $d(\tan \theta) = \sec^2 \theta d\theta$ , which means that the integral fits the form of Eq. (28.1) with

$$u = 1 + \tan \theta \quad du = \sec^2 \theta d\theta \quad n = \frac{1}{2}$$

The integral is of the form  $\int u^{1/2} du$ . Thus,

$$\begin{aligned} \int 2\sqrt{1 + \tan \theta} (\sec^2 \theta d\theta) &= 2 \int (1 + \tan \theta)^{1/2} (\sec^2 \theta d\theta) \\ &= 2 \left( \frac{2}{3} \right) (1 + \tan \theta)^{3/2} + C \\ &= \frac{4}{3} (1 + \tan \theta)^{3/2} + C \end{aligned}$$

### EXAMPLE 3 Logarithmic integral

Integrate  $\int \ln x \left( \frac{dx}{x} \right)$ .

By noting that  $d(\ln x) = \frac{dx}{x}$ , we have:  $u = \ln x \quad du = \frac{dx}{x} \quad n = 1$

This means that the integral is of the form  $\int u du$ . Thus,

$$\int \ln x \left( \frac{dx}{x} \right) = \frac{1}{2} (\ln x)^2 + C = \frac{1}{2} \ln^2 x + C$$

### Practice Exercise

1. Integrate  $\int \cos^2 x \sin x dx$ .

### Practice Exercise

2. Integrate  $\int \frac{2 \ln 2x dx}{x}$ .



**EXAMPLE 4** Inverse trigonometric integrand

Find the value of  $\int_0^{0.5} \frac{\sin^{-1} x}{\sqrt{1-x^2}} dx$ .

For purposes of integrating:  $u = \sin^{-1} x$   $du = \frac{dx}{\sqrt{1-x^2}}$   $n = 1$

$$\begin{aligned} \int_0^{0.5} \frac{\sin^{-1} x}{\sqrt{1-x^2}} dx &= \int_0^{0.5} \sin^{-1} x \left( \frac{dx}{\sqrt{1-x^2}} \right) && \int u du \\ &= \frac{(\sin^{-1} x)^2}{2} \Big|_0^{0.5} && \text{integrate} \\ &= \frac{(\frac{\pi}{6})^2}{2} - 0 = \frac{\pi^2}{72} && \text{evaluate} \end{aligned}$$

**EXAMPLE 5** Inverse trigonometric integrand

Find the first-quadrant area bounded by  $y = \frac{e^{2x}}{\sqrt{e^{2x} + 1}}$  and  $x = 1.5$ .

The area is shown in Fig. 28.1. Using a representative element of area  $y dx$ , the area is found by evaluating the integral

$$\int_0^{1.5} \frac{e^{2x} dx}{\sqrt{e^{2x} + 1}}$$

For the purpose of integration,  $n = -\frac{1}{2}$ ,  $u = e^{2x} + 1$ , and  $du = 2e^{2x} dx$ . Therefore,

$$\begin{aligned} \int_0^{1.5} (e^{2x} + 1)^{-1/2} e^{2x} dx &= \frac{1}{2} \int_0^{1.5} (e^{2x} + 1)^{-1/2} (2e^{2x} dx) \\ &= \frac{1}{2} (2) (e^{2x} + 1)^{1/2} \Big|_0^{1.5} = (e^{2x} + 1)^{1/2} \Big|_0^{1.5} \\ &= \sqrt{e^3 + 1} - \sqrt{2} = 3.178 \end{aligned}$$

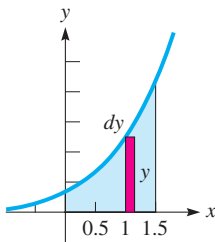


Fig. 28.1

**EXERCISES 28.1**

In Exercises 1 and 2, make the given changes in the indicated examples of this section and then solve the given problems.

- In Example 1, change  $\sin^3 x$  to  $\cos^3 x$ . What other change must be made in the integrand to have a result of  $\frac{1}{4} \cos^4 x + C$ ?
- In Example 3, change  $\ln x$  to  $\ln x^2$  and then integrate.

In Exercises 3–26, integrate each of the functions.

3.  $\int \sin^4 x \cos x dx$

4.  $\int \cos^5 x (-\sin x dx)$

5.  $\int 0.4 \sqrt{\cos \theta} \sin \theta d\theta$

6.  $\int 8 \sin^{1/3} x \cos x dx$

7.  $\int 4 \tan^2 x \sec^2 x dx$

8.  $\int \sec^3 x (\sec x \tan x) dx$

9.  $\int_0^{\pi/8} \cos 2x \sin 2x dx$

10.  $\int_{\pi/6}^{\pi/4} 3 \sqrt{\cot x} \csc^2 x dx$

11.  $\int (\sin^{-1} x)^3 \left( \frac{dx}{\sqrt{1-x^2}} \right)$

12.  $\int \frac{20(\cos^{-1} 2t)^4 dt}{\sqrt{1-4t^2}}$

13.  $\int \frac{5 \tan^{-1} 5x}{1+25x^2} dx$

14.  $\int \frac{\sin^{-1} 4x dx}{\sqrt{1-16x^2}}$

15.  $\int [\ln(x+1)]^2 \frac{dx}{x+1}$

16.  $\int 0.8(3+2 \ln u)^3 \frac{du}{u}$

17.  $\int_0^{1/2} \frac{\ln(2x+3)}{2x+3} dx$

18.  $\int_1^e \frac{(1-2 \ln x) dx}{x}$

19.  $\int (4+e^x)^3 e^x dx$

20.  $\int 2\sqrt{1-e^{-x}} (e^{-x} dx)$

21.  $\int \frac{e^{2t} dt}{(1-e^{2t})^3}$

22.  $\int \frac{(1+3e^{-2x})^4 dx}{e^{2x}}$

23.  $\int (1+\sec^2 x)^4 (\sec^2 x \tan x dx)$

24.  $\int (e^x + e^{-x})^{1/4} (e^x - e^{-x}) dx$

25.  $\int_{\pi/6}^{\pi/4} (1 + \cot x)^2 \csc^2 x dx$     26.  $\int_{\pi/3}^{\pi/2} \frac{\sin \theta d\theta}{\sqrt{1 + \cos \theta}}$

In Exercises 27–30, rewrite the given integrals so that they fit the form  $\int u^n du$ , and identify  $u$ ,  $n$ , and  $du$ .

27.  $\int \sec^5 x \sin x dx$

28.  $\int \frac{\tan^3 x dx}{\cos^2 x}$

29.  $\int \frac{dx}{x \ln^2 x}$

30.  $\int \frac{\sqrt{(1 + e^{-r})(1 - e^{-r})}}{e^{2r}} dr$

In Exercises 31–40, solve the given problems by integration.

31. Find the first-quadrant area under the curve of  $y = \ln^2 x/x$  from  $x = 1$  to  $x = 4$ .

32. Find the first-quadrant area under the curve of  $y = \sin^3 x \cos x$  from  $x = 0$  to  $x = \frac{\pi}{2}$ .

33. Find the area under the curve  $y = \frac{1 + \tan^{-1} 2x}{1 + 4x^2}$  from  $x = 0$  to  $x = 2$ .

34. Find the first-quadrant area bounded by  $y = \frac{\ln(4x + 1)}{4x + 1}$  and  $x = 5$ .

35. The general expression for the slope of a given curve is  $(\ln x)^2/x$ . If the curve passes through  $(1, 2)$ , find its equation.

36. Find the equation of the curve for which  $dy/dx = (1 + \tan 2x)^2 \sec^2 2x$  if the curve passes through  $(2, 1)$ .

37. In the development of the expression for the total pressure  $p$  on a wall due to molecules with mass  $m$  and velocity  $v$  striking the wall, the equation  $p = mv^2 \int_0^{\pi/2} \sin \theta \cos^2 \theta d\theta$  is found. The symbol  $n$  represents the number of molecules per unit volume, and  $\theta$  represents the angle between a perpendicular to the wall and the direction of the molecule. Find the expression for  $p$ .

38. The solar energy  $E$  passing through a hemispherical surface per unit time, per unit area, is  $E = 2\pi I \int_0^{\pi/2} \cos \theta \sin \theta d\theta$ , where  $I$  is the solar intensity and  $\theta$  is the angle at which it is directed (from the perpendicular). Evaluate this integral.

39. After an electric power interruption, the current  $i$  in a circuit is given by  $i = 3(1 - e^{-t})^2 (e^{-t})$ , where  $t$  is the time. Find the expression for the total electric charge  $q$  to pass a point in the circuit if  $q = 0$  for  $t = 0$ .

40. A space vehicle is launched vertically from the ground such that its velocity  $v$  (in km/s) is given by  $v = [\ln^2(t^3 + 1)] \frac{t^2}{t^3 + 1}$ , where  $t$  is the time (in s). Find the altitude of the vehicle after 10.0 s.

### Answers to Practice Exercises

1.  $-\frac{1}{3} \cos^3 x + C$     2.  $\ln^2 2x + C$

## 28.2 The Basic Logarithmic Form

Integration of  $du/u$  •  
Result Is Absolute Value of  $u$

The general power formula for integration, Eq. (28.1), is valid for all values of  $n$  except  $n = -1$ . If  $n$  were set equal to  $-1$ , this would cause the result to be undefined. When we obtained the derivative of the logarithmic function, we found

$$\frac{d(\ln u)}{dx} = \frac{1}{u} \frac{du}{dx}$$

This means the differential of the logarithmic form is  $d(\ln u) = du/u$ . Reversing the process, we then determine that  $\int du/u = \ln u + C$ . In other words, when the exponent of the expression being integrated is  $-1$ , the expression is a logarithmic form.

Logarithms are defined only for positive numbers. Thus,  $\int du/u = \ln u + C$  is valid if  $u > 0$ . If  $u < 0$ , then  $-u > 0$ . In this case,  $d(-u) = -du$ , or  $\int (-du)/(-u) = \ln(-u) + C$ . However,  $\int du/u = \int (-du)/(-u)$ . These results can be combined into a single form using the absolute value of  $u$ . Therefore,

$$\int \frac{du}{u} = \ln |u| + C \quad (28.2)$$

**EXAMPLE 1 Algebraic integrand**

Integrate  $\int \frac{dx}{x+1}$ .

Since  $d(x+1) = dx$ , this integral fits the form of Eq. (28.2) with  $u = x+1$  and  $du = dx$ . Therefore, we have

$$\int \frac{dx}{x+1} = \ln|x+1| + C$$

**EXAMPLE 2 Algebraic integrand—application**

Newton's law of cooling states that the rate at which an object cools is directly proportional to the difference in its temperature  $T$  and the temperature of the surrounding medium. By use of this law, the time  $t$  (in min) a certain object takes to cool from  $80^\circ\text{C}$  to  $50^\circ\text{C}$  in air at  $20^\circ\text{C}$  is found to be

$$t = -9.8 \int_{80}^{50} \frac{dT}{T-20}$$

Find the value of  $t$ .

We see that the integral fits Eq. (28.2) with  $u = T-20$  and  $du = dT$ . Thus,

$$\begin{aligned} t &= -9.8 \int_{80}^{50} \frac{dT}{T-20} \quad \leftarrow \begin{array}{l} du \\ u \end{array} \\ &= -9.8 \ln|T-20| \Big|_{80}^{50} && \text{integrate} \\ &= -9.8 (\ln 30 - \ln 60) && \text{evaluate} \\ &= -9.8 \ln \frac{30}{60} = -9.8 \ln(0.50) && \ln x - \ln y = \ln \frac{x}{y} \\ &= 6.8 \text{ min} \end{aligned}$$

**Practice Exercise**

1. Integrate  $\int \frac{3 dx}{5-2x}$ .

**EXAMPLE 3 Trigonometric integrand**

Integrate  $\int \frac{\cos x}{\sin x} dx$ .

We note that  $d(\sin x) = \cos x dx$ . This means that this integral fits the form of Eq. (28.2) with  $u = \sin x$  and  $du = \cos x dx$ . Thus,

$$\begin{aligned} \int \frac{\cos x}{\sin x} dx &= \int \frac{\cos x dx}{\sin x} \quad \leftarrow \begin{array}{l} du \\ u \end{array} \\ &= \ln|\sin x| + C \end{aligned}$$

**EXAMPLE 4 Comparing a logarithmic form with a power formula form**

Integrate  $\int \frac{x dx}{4-x^2}$ .

This integral fits the form of Eq. (28.2) with  $u = 4-x^2$  and  $du = -2x dx$ . This means that we must introduce a factor of  $-2$  into the numerator and a factor of  $-\frac{1}{2}$  before the integral. Therefore,

$$\begin{aligned} \int \frac{x dx}{4-x^2} &= -\frac{1}{2} \int \frac{-2x dx}{4-x^2} \quad \leftarrow \begin{array}{l} du \\ u \end{array} \\ &= -\frac{1}{2} \ln|4-x^2| + C \end{aligned}$$

We should note that if the quantity  $4 - x^2$  were raised to any power other than  $-1$ , as in the example, we would have to employ the general power formula for integration. For example,

$$\begin{aligned}\int \frac{x \, dx}{(4 - x^2)^2} &= -\frac{1}{2} \int \frac{-2x \, dx}{(4 - x^2)^2} \quad \leftarrow du \\ &= -\frac{1}{2} \frac{(4 - x^2)^{-1}}{-1} + C = \frac{1}{2(4 - x^2)} + C\end{aligned}$$

### EXAMPLE 5 Exponential integrand

Integrate  $\int \frac{e^{4x} \, dx}{1 + 3e^{4x}}$ .

Since  $d(1 + 3e^{4x})/dx = 12e^{4x}$ , we see that we can use Eq. (28.2) with  $u = 1 + 3e^{4x}$  and  $du = 12e^{4x} \, dx$ . Therefore, we write

$$\begin{aligned}\int \frac{e^{4x} \, dx}{1 + 3e^{4x}} &= \frac{1}{12} \int \frac{12e^{4x} \, dx}{1 + 3e^{4x}} && \text{multiply and divide by 12} \\ &= \frac{1}{12} \ln|1 + 3e^{4x}| + C && \text{integrate} \\ &= \frac{1}{12} \ln(1 + 3e^{4x}) + C && 1 + 3e^{4x} > 0 \text{ for all } x\end{aligned}$$

### Practice Exercise

2. Integrate  $\int \frac{\sin x \, dx}{1 - \cos x}$ .

### EXAMPLE 6 Definite integral with a trigonometric integrand

Evaluate  $\int_0^{\pi/8} \frac{\sec^2 2\theta}{1 + \tan 2\theta} \, d\theta$ .

Since  $d(1 + \tan 2\theta) = 2 \sec^2 2\theta \, d\theta$ , we see that we can use Eq. (28.2) with  $u = 1 + \tan 2\theta$  and  $du = 2 \sec^2 2\theta \, d\theta$ . Therefore, we have

$$\begin{aligned}\int_0^{\pi/8} \frac{\sec^2 2\theta}{1 + \tan 2\theta} \, d\theta &= \frac{1}{2} \int_0^{\pi/8} \frac{2 \sec^2 2\theta \, d\theta}{1 + \tan 2\theta} && \text{multiply and divide by 2} \\ &= \frac{1}{2} \ln|1 + \tan 2\theta|_0^{\pi/8} && \text{integrate} \\ &= \frac{1}{2} (\ln|1 + 1| - \ln|1 + 0|) && \text{evaluate} \\ &= \frac{1}{2} (\ln 2 - \ln 1) = \frac{1}{2} (\ln 2 - 0) \\ &= \frac{1}{2} \ln 2\end{aligned}$$

### EXAMPLE 7 Find a volume using a logarithmic form

Find the volume within the piece of tapered tubing shown in Fig. 28.2, which can be described as the volume generated by revolving the region bounded by the curve of  $y = \frac{3}{\sqrt{4x + 3}}$ ,  $x = 2.50$  cm, and the axes about the  $x$ -axis.

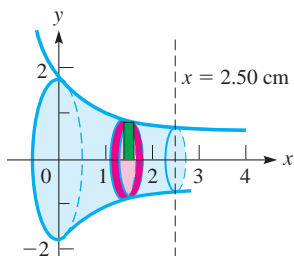


Fig. 28.2

The volume can be found by setting up only one integral by using a disc element of volume, as shown. The volume is found as follows:

$$\begin{aligned}
 V &= \pi \int_0^{2.50} y^2 dx = \pi \int_0^{2.50} \left( \frac{3}{\sqrt{4x+3}} \right)^2 dx \\
 &= \pi \int_0^{2.50} \frac{9 dx}{4x+3} = \frac{9\pi}{4} \int_0^{2.50} \frac{4 dx}{4x+3} \\
 &= \frac{9\pi}{4} \ln(4x+3) \Big|_0^{2.50} = \frac{9\pi}{4} (\ln 13.0 - \ln 3) \\
 &= \frac{9\pi}{4} \ln \frac{13.0}{3} = 10.4 \text{ cm}^3
 \end{aligned}$$

## EXERCISES 28.2

In Exercises 1 and 2, make the given changes in the indicated examples of this section and then solve the given problems.

- In Example 1, change  $x + 1$  to  $x^2 + 1$ . What other change must be made in the integrand to have a result of  $\ln|x^2 + 1| + C$ ?
- In Example 3, interchange  $\cos x$  and  $\sin x$  and then integrate.

In Exercises 3–30, integrate each of the given functions.

- $\int \frac{dx}{1+4x}$
- $\int \frac{2x dx}{4-3x^2}$
- $\int_0^2 \frac{dx}{8-3x}$
- $\int \frac{0.4 \csc^2 2\theta d\theta}{\cot 2\theta}$
- $\int_0^{\pi/2} \frac{\cos x dx}{1+\sin x}$
- $\int \frac{e^{-x}}{1-e^{-x}} dx$
- $\int \frac{1+e^x}{x+e^x} dx$
- $\int \frac{\sec x \tan x dx}{1+4 \sec x}$
- $\int_1^3 \frac{1+x}{4x+2x^2} dx$
- $\int \frac{0.5 dr}{r \ln r}$
- $\int \frac{2+\sec^2 x}{2x+\tan x} dx$
- $\int \frac{6 dx}{\sqrt{1-2x}}$
- $\int \frac{dx}{1-4x}$
- $\int \frac{4\sqrt{u} du}{1+u\sqrt{u}}$
- $\int_{-1}^3 \frac{8x^3 dx}{x^4+1}$
- $\int \frac{\sin 3x}{\cos 3x} dx$
- $\int_0^{\pi/4} \frac{\sec^2 x dx}{4+\tan x}$
- $\int \frac{5e^{3x}}{1-e^{3x}} dx$
- $\int \frac{3e^t dt}{\sqrt{e^{2t}+4e^t+4}}$
- $\int \frac{\sin 2x}{1-\cos^2 x} dx$
- $\int_1^2 \frac{4x+6x^2}{x^2+x^3} dx$
- $\int \frac{dx}{x(1+2 \ln x)}$
- $\int \frac{x+\cos 2x}{x^2+\sin 2x} dx$
- $\int \frac{4x dx}{(1+x^2)^2}$

- $\int \frac{x+2}{x^2} dx$
- $\int \frac{3v^2-2v}{v^2} dv$
- $\int_0^{\pi/12} \frac{\sec^2 3x}{4+\tan 3x} dx$
- $\int_1^2 \frac{x^2+1}{x^3+3x} dx$

In Exercises 31–48, solve the given problems by integration.

- Find the area bounded by  $y(x+1) = 1$ ,  $x = 0$ ,  $y = 0$ , and  $x = 2$ . See Fig. 28.3.

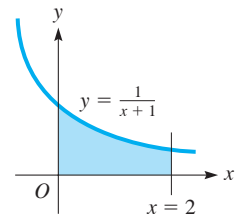


Fig. 28.3

- Evaluate  $\int_1^2 x^{-1} dx$  and  $\int_2^4 x^{-1} dx$ . Give a geometric interpretation of these two results.
- Integrate  $\int \frac{x-4}{x+4} dx$  by first using algebraic division to change the form of the integrand.
- Integrate  $\int \sec x dx$  by first multiplying the integrand by  $\frac{\sec x + \tan x}{\sec x + \tan x}$  (a special form of 1).
- Find the volume generated by revolving the region bounded by  $y = 1/(x^2+1)$ ,  $x = 0$ ,  $x = 1$ , and  $y = 0$  about the  $y$ -axis. Use shells.
- Find the volume of the solid generated by revolving the region bounded by  $y = \frac{2}{\sqrt{3x+1}}$ ,  $x = 0$ ,  $x = 3.5$ , and  $y = 0$  about the  $x$ -axis.
- The general expression for the slope of a curve is  $\frac{\sin x}{3+\cos x}$ . If the curve passes through the point  $(\pi/3, 2)$ , find its equation.
- Show that  $\int_0^3 \frac{dx}{2x+2} = \ln 2$ .
- If  $a > 0$  and  $b > 0$ , show that  $\int_1^a \frac{du}{u} + \int_1^b \frac{du}{u} = \int_1^{ab} \frac{du}{u}$ .

40. If  $x > 0$ , find  $f(x)$  if  $f''(x) = x^{-2}$ ,  $f(1) = 0$ , and  $f(2) = 0$ .
41. The acceleration  $a$  (in  $\text{m/s}^2$ ) of an object is  $a = (t + 4)^{-1}$ . Find the velocity for  $t = 4$  s, if the initial velocity is zero.
42. The pressure  $p$  (in kPa) and volume  $V$  (in  $\text{cm}^3$ ) of a gas are related by  $pV = 8600$ . Find the average value of  $p$  from  $V = 75 \text{ cm}^3$  to  $V = 95 \text{ cm}^3$ .
43. Under ideal conditions, the natural law of population growth is that the population increases at a rate proportional to the population  $P$  present at any time  $t$ . This leads to the equation  $t = \frac{1}{k} \int \frac{dP}{P}$ . Assuming ideal conditions for Canada, if  $P = 30.7$  million in 2000 ( $t = 0$ ) and  $P = 34.1$  million in 2010 ( $t = 10$  years), find the population that is projected for 2030 ( $t = 30$  years).
44. In determining the temperature that is absolute zero (0 K, or about  $-273^\circ\text{C}$ ), the equation  $\ln T = -\int \frac{dr}{r-1}$  is used. Here,  $T$  is the thermodynamic temperature and  $r$  is the ratio between certain specific vapour pressures. If  $T = 273.16$  K for  $r = 1.3361$ , find  $T$  as a function of  $r$  (if  $r > 1$  for all  $T$ ).
45. The time  $t$  and electric current  $i$  for a certain circuit with a voltage  $E$ , a resistance  $R$ , and an inductance  $L$  is given by  $t = L \int \frac{di}{E - iR}$ . If  $t = 0$  for  $i = 0$ , integrate and express  $i$  as a function of  $t$ .
46. Conditions are often such that a force proportional to the velocity tends to retard the motion of an object moving through a resisting medium. Under such conditions, the acceleration of a certain object moving down an inclined plane is given by  $20 - v$ . This leads to the equation  $t = \int \frac{dv}{20 - v}$ . If the object starts from rest, find the expression for the velocity as a function of time.
47. An architect designs a wall panel that can be described as the first-quadrant area bounded by  $y = \frac{50}{x^2 + 20}$  and  $x = 3.00$ . If the area of the panel is  $6.61 \text{ m}^2$ , find the  $x$ -coordinate (in m) of the centroid of the panel.
48. The electric power  $P$  developed in a certain resistor is given by  $P = 3 \int \frac{\sin \pi t}{2 + \cos \pi t} dt$ , where  $t$  is the time. Express  $P$  as a function of  $t$ .

Answers to Practice Exercises

1.  $-\frac{3}{2} \ln |5 - 2x| + C$     2.  $\ln |1 - \cos x| + C$

## 28.3 The Exponential Form

### Integration of $e^u du$

In deriving the derivative for the exponential function, we obtained the result  $de^u/dx = e^u (du/dx)$ . This means that the differential of the exponential form is  $d(e^u) = e^u du$ . Reversing this form to find the proper form of the integral for the exponential function, we have

$$\int e^u du = e^u + C \tag{28.3}$$

#### EXAMPLE 1 Algebraic $u$

Integrate  $\int xe^{x^2} dx$ .

Since  $d(x^2) = 2x dx$ , we can write this integral in the form of Eq. (28.3) with  $u = x^2$  and  $du = 2x dx$ . Thus,

$$\begin{aligned} \int xe^{x^2} dx &= \frac{1}{2} \int e^{x^2} (2x dx) \\ &= \frac{1}{2} e^{x^2} + C \end{aligned}$$

#### Practice Exercise

1. Integrate  $\int 6e^{-3x} dx$ .

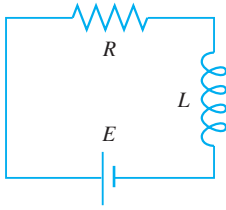


Fig. 28.4

**EXAMPLE 2 Algebraic  $u$ —application**

For an electric circuit containing a direct voltage source  $E$ , a resistance  $R$ , and an inductance  $L$ , the current  $i$  and time  $t$  are related by  $ie^{Rt/L} = \frac{E}{L} \int e^{Rt/L} dt$ . See Fig. 28.4. If  $i = 0$  for  $t = 0$ , perform the integration and then solve for  $i$  as a function of  $t$ .

For this integral, we see that  $u = \frac{Rt}{L}$ , which means that  $du = \frac{R}{L} dt$ . The solution is then as follows:

$$ie^{Rt/L} = \frac{E}{L} \int e^{Rt/L} dt = \frac{E}{L} \left( \frac{L}{R} \right) \int e^{Rt/L} \left( \frac{R}{L} dt \right) \quad \text{introduce factor } \frac{R}{L}$$

$$= \frac{E}{R} e^{Rt/L} + C \quad \text{integrate}$$

$$0(e^0) = \frac{E}{R} e^0 + C, \quad C = -\frac{E}{R} \quad i = 0 \text{ for } t = 0; \text{ evaluate } C$$

$$ie^{Rt/L} = \frac{E}{R} e^{Rt/L} - \frac{E}{R} \quad \text{substitute for } C$$

$$i = \frac{E}{R} - \frac{E}{R} e^{-Rt/L} = \frac{E}{R} (1 - e^{-Rt/L}) \quad \text{solve for } i$$

**EXAMPLE 3  $e^u$  in the denominator**

Integrate  $\int \frac{dx}{e^{3x}}$ .

This integral can be put in proper form by writing it as  $\int e^{-3x} dx$ . In this form,  $u = -3x$  and  $du = -3 dx$ . Thus,

$$\begin{aligned} \int \frac{dx}{e^{3x}} &= \int e^{-3x} dx = -\frac{1}{3} \int e^{-3x} (-3 dx) \\ &= -\frac{1}{3} e^{-3x} + C \end{aligned}$$

**EXAMPLE 4 Proper form using laws of exponents**

Integrate  $\int \frac{4e^{3x} - 3e^x}{e^{x+1}} dx$ .

By using the laws of exponents, this integral can be put in proper form for integration, and then integrated as follows:

$$\begin{aligned} \int \frac{4e^{3x} - 3e^x}{e^{x+1}} dx &= \int \frac{4e^{3x}}{e^{x+1}} dx - \int \frac{3e^x}{e^{x+1}} dx \\ &= 4 \int e^{3x-(x+1)} dx - 3 \int e^{x-(x+1)} dx \quad \text{using Eq. (11.2)} \end{aligned}$$

$$= 4 \int e^{2x-1} dx - 3 \int e^{-1} dx$$

$$= \frac{4}{2} \int e^{2x-1} (2 dx) - \frac{3}{e} \int dx$$

$$= 2e^{2x-1} - \frac{3}{e} x + C$$

■ For reference, Eq. (11.2) is  $\frac{a^m}{a^n} = a^{m-n}$ .



**EXAMPLE 5** Definite integral with a trigonometric  $u$ Evaluate:  $\int_0^{\pi/2} (\sin 2\theta) (e^{\cos 2\theta}) d\theta$ .With  $u = \cos 2\theta$ ,  $du = -2 \sin 2\theta d\theta$ , we have

$$\begin{aligned} \int_0^{\pi/2} (\sin 2\theta) (e^{\cos 2\theta}) d\theta &= -\frac{1}{2} \int_0^{\pi/2} (e^{\cos 2\theta}) (-2 \sin 2\theta d\theta) \\ &= -\frac{1}{2} e^{\cos 2\theta} \Big|_0^{\pi/2} && \text{integrate} \\ &= -\frac{1}{2} \left( \frac{1}{e} - e \right) = 1.175 && \text{evaluate} \quad \blacksquare \end{aligned}$$

**EXAMPLE 6** Algebraic  $u$ —Finding the equation of a curveFind the equation of the curve for which  $\frac{dy}{dx} = \frac{e^{\sqrt{x+1}}}{\sqrt{x+1}}$  if the curve passes through  $(0, 1)$ .

The solution of this problem requires that we integrate the given function and then evaluate the constant of integration. Hence,

$$dy = \frac{e^{\sqrt{x+1}}}{\sqrt{x+1}} dx \quad \int dy = \int \frac{e^{\sqrt{x+1}}}{\sqrt{x+1}} dx$$

For purposes of integrating the right-hand side,

$$u = \sqrt{x+1} \quad \text{and} \quad du = \frac{1}{2\sqrt{x+1}} dx$$

$$\begin{aligned} y &= 2 \int e^{\sqrt{x+1}} \left( \frac{1}{2\sqrt{x+1}} dx \right) \\ &= 2e^{\sqrt{x+1}} + C \end{aligned}$$

Letting  $x = 0$  and  $y = 1$ , we have  $1 = 2e + C$ , or  $C = 1 - 2e$ . This means that the equation is

$$y = 2e^{\sqrt{x+1}} + 1 - 2e \quad \text{see Fig. 28.5.} \quad \blacksquare$$

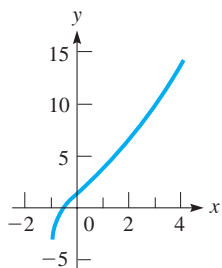
**Practice Exercise**2. Integrate  $\int \frac{e^{\sin x}}{\sec x} dx$ .

Fig. 28.5

**EXERCISES 28.3**

In Exercises 1 and 2, make the given changes in the indicated examples of this section and then solve the given problems.

- In Example 1, change  $e^{x^2}$  to  $e^{x^3}$ . What other change must be made in the integrand to have a result of  $e^{x^3} + C$ ?
- In Example 4, change  $e^{x+1}$  to  $e^{x-1}$  and then integrate.

In Exercises 3–28, integrate each of the given functions.

3.  $\int e^{7x} (7 dx)$

4.  $\int e^{x^4} (4x^3 dx)$

5.  $\int e^{2x+5} dx$

6.  $\int 2e^{-4x} dx$

7.  $\int_{-2}^2 6e^{s/2} ds$

8.  $\int_1^2 3e^{4x} dx$

9.  $\int 6x^2 e^{x^3} dx$

10.  $\int xe^{-x^2} dx$

11.  $\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$

12.  $\int_0^1 4(\ln e^u) e^{-2u^2} du$

13.  $\int 4(\sec \theta \tan \theta) e^{2 \sec \theta} d\theta$

14.  $\int (\sec^2 x) e^{\tan x} dx$

15.  $\int \sqrt{e^{2y} + e^{3y}} dy$

16.  $\int \frac{4 dx}{x^2 e^{1/x}}$

17.  $\int_1^3 3e^{2x} (e^{-2x} - 1) dx$

18.  $\int_0^{0.5} \frac{3e^{3x+1}}{e^x} dx$

19.  $\int \frac{2 dx}{\sqrt{x} e^{\sqrt{x}}}$

20.  $\int \frac{4 dx}{e^{\sin x} \sec x}$

21.  $\int \frac{e^{\tan^{-1} 2x}}{4x^2 + 1} dx$

22.  $\int \frac{e^{\sin^{-1} 2x} dx}{\sqrt{1 - 4x^2}}$

23.  $\int \frac{e^{\cos 3x} dx}{\csc 3x}$

24.  $\int (e^x - e^{-x})^2 dx$

25.  $\int_0^\pi (\sin 2x) e^{\cos^2 x} dx$       26.  $\int_0^2 \frac{e^{2t} dt}{\sqrt{e^{2t} + 4}}$   
 27.  $\int \frac{6e^x dx}{e^x + 1}$       28.  $\int \frac{4e^x dx}{(1 - e^x)^2}$

In Exercises 29–45, solve the given problems by integration.

29. Find the area bounded by  $y = 3e^x$ ,  $x = 0$ ,  $y = 0$ , and  $x = 2$ .  
 30. Find the area bounded by  $x = a$ ,  $x = b$ ,  $y = 0$ , and  $y = e^x$ . Explain the meaning of the result.  
 31. Integrate  $\int 2e^{x^2 + \ln x} dx$  by first showing that  $e^{\ln x} = x$ . (Hint: Let  $y = e^{\ln x}$  and then take natural logarithms of both sides.)  
 32. Integrate  $\int e^{x + \ln x} \frac{dx}{x}$ . See Exercise 31.  
 33. Integrate  $\int \frac{dx}{1 + e^x}$  by first changing the form of the integrand by algebraic division.  
 34. Integrate  $\int \frac{e^{2x} dx}{1 + e^x}$  by first changing the form of the integrand by algebraic division.  
 35. Find the volume generated by revolving the region bounded by  $y = e^{x^2}$ ,  $x = 1$ ,  $y = 0$ , and  $x = 2$  about the  $y$ -axis. See Fig. 28.6.

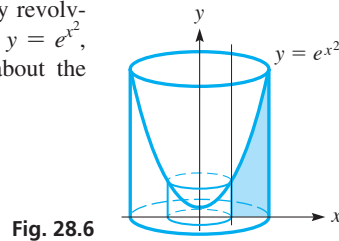


Fig. 28.6

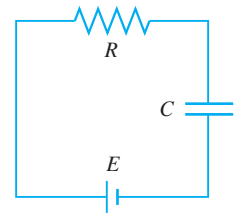
36. Find the equation of the curve for which  $dy/dx = \sqrt{e^{x+3}}$  if the curve passes through  $(1, 0)$ .  
 37. Find the average value of the function  $y = 4e^{x/2}$  from  $x = 0$  to  $x = 4$ .

38. Find the moment of inertia with respect to the  $y$ -axis of a flat plate that covers the first-quadrant region bounded by  $y = e^{x^3}$ ,  $x = 1$ , and the axes.

39. Using Eq. (27.15), show that  $\int b^u du = \frac{b^u}{\ln b} + C$  ( $b > 0$ ,  $b \neq 1$ ).

40. Find the first-quadrant area bounded by  $y = 2^x$  and  $x = 3$ . See Exercise 39.

41. For an electric circuit containing a voltage source  $E$ , a resistance  $R$ , and a capacitance  $C$ , an equation relating the charge  $q$  on the capacitor and the time



$t$  is  $q e^{t/RC} = \frac{E}{R} \int e^{t/RC} dt$ . See

Fig. 28.7. If  $q = 0$  for  $t = 0$ , perform the integration and then solve for  $q$  as a function of  $t$ .

Fig. 28.7

42. In the theory dealing with energy propagation of lasers, the equation  $E = a \int_0^{I_0} e^{-Tx} dx$  is used. Here,  $a$ ,  $I_0$ , and  $T$  are constants. Evaluate this integral.

43. The St. Louis Gateway Arch (see Fig. 27.44) has a shape that is given approximately by (measurements in m)  $y = -19.46(e^{x/38.92} + e^{-x/38.92}) + 230.9$ . Find the total open area under the arch. (Hint: The limits of integration are the  $x$ -intercepts. To find them, set  $y = 0$  and solve as an equation in quadratic form.)

44. The force  $F$  (in N) exerted by a robot programmed to staple carton sections together is given by  $F = 6 \int e^{\sin \pi t} \cos \pi t dt$ , where  $t$  is the time (in s). Find  $F$  as a function of  $t$  if  $F = 0$  for  $t = 1.5$  s.

45. The average lifetime of a certain kind of emergency backup battery is 200 hours. The proportion of those batteries that are expected to last less than 100 hours is given by  $p = \int_0^{100} (0.05)x^{-0.5} e^{-0.1x^{0.5}} dx$ . Find  $p$ .

Answers to Practice Exercises

1.  $-2e^{-3x} + C$     2.  $e^{\sin x} + C$

## 28.4 Basic Trigonometric Forms

### Integration of the Six Trigonometric Functions and Four Trigonometric Integrals That Directly Give Trigonometric Functions

By noting the formulas for differentiating the six trigonometric functions, and the anti-derivatives that are found using them, we have the following six integration formulas:

$$\int \sin u \, du = -\cos u + C \tag{28.4}$$

$$\int \cos u \, du = \sin u + C \tag{28.5}$$

$$\int \sec^2 u \, du = \tan u + C \tag{28.6}$$

$$\int \csc^2 u \, du = -\cot u + C \tag{28.7}$$

$$\int \sec u \tan u \, du = \sec u + C \tag{28.8}$$

$$\int \csc u \cot u \, du = -\csc u + C \tag{28.9}$$

**EXAMPLE 1** Integration of  $\sec^2 u \, du$ Integrate  $\int x \sec^2 x^2 \, dx$ .With  $u = x^2$ ,  $du = 2x \, dx$ , we have**Practice Exercise**1. Integrate  $\int \sin 5x \, dx$ .

$$\begin{aligned} \int x \sec^2 x^2 \, dx &= \frac{1}{2} \int (\sec^2 x^2) (2x \, dx) \\ &= \frac{1}{2} \tan x^2 + C \end{aligned}$$

using Eq. (28.6) ■

**EXAMPLE 2** Integration of  $\tan u \sec u \, du$ Integrate  $\int \frac{\tan 2x}{\cos 2x} \, dx$ .By using the basic identity  $\sec \theta = 1/\cos \theta$ , we can transform this integral into the form  $\int \sec 2x \tan 2x \, dx$ . In this form,  $u = 2x$  and  $du = 2 \, dx$ . Therefore,**Practice Exercise**2. Integrate  $\int 6 \csc^2 3x \, dx$ .

$$\begin{aligned} \int \frac{\tan 2x}{\cos 2x} \, dx &= \int \sec 2x \tan 2x \, dx = \frac{1}{2} \int \sec 2x \tan 2x (2 \, dx) \\ &= \frac{1}{2} \sec 2x + C \end{aligned}$$

using Eq. (28.8) ■

**EXAMPLE 3** Integration of  $\cos u \, du$ —applicationThe vertical velocity  $v$  (in cm/s) of the end of a vibrating rod is given by  $v = 80 \cos 20\pi t$ , where  $t$  is the time in seconds. Find the vertical displacement  $y$  (in cm) as a function of  $t$  if  $y = 0$  for  $t = 0$ .Since  $v = dy/dt$ , we have the following solution:

$$\begin{aligned} \frac{dy}{dt} &= 80 \cos 20\pi t \\ \int dy &= \int 80 \cos 20\pi t \, dt = \frac{80}{20\pi} \int (\cos 20\pi t) (20\pi \, dt) && \text{set up integration} \\ y &= \frac{4}{\pi} \sin 20\pi t + C && \text{using Eq. (28.5)} \\ 0 &= 1.27 \sin 0 + C, \quad C = 0 && \text{evaluate } C \\ y &= 1.27 \sin 20\pi t && \text{solution} \end{aligned}$$

■

To find the integrals for the other trigonometric functions, we must change them to a form for which the integral can be determined by methods previously discussed. We can accomplish this by using the basic trigonometric relations.

The formula for  $\int \tan u \, du$  is found by expressing the integral in the form  $\int (\sin u / \cos u) \, du$ . We recognize this as being a logarithmic form, where the  $u$  of the logarithmic form is  $\cos u$  in this integral. The differential of  $\cos u$  is  $-\sin u \, du$ . Therefore, we have

$$\int \tan u \, du = \int \frac{\sin u}{\cos u} \, du = - \int \frac{-\sin u \, du}{\cos u} = -\ln|\cos u| + C$$

The formula for  $\int \cot u \, du$  is found by writing it in the form  $\int (\cos u / \sin u) \, du$ . In this manner, we obtain the result

$$\int \cot u \, du = \int \frac{\cos u}{\sin u} \, du = \int \frac{\cos u \, du}{\sin u} = \ln|\sin u| + C$$

The formula for  $\int \sec u \, du$  is found by writing it in the form

$$\int \frac{\sec u (\sec u + \tan u)}{\sec u + \tan u} \, du$$

We see that this form is also a logarithmic form, since

$$d(\sec u + \tan u) = (\sec u \tan u + \sec^2 u) \, du$$

The right side of this equation is the expression appearing in the numerator of the integral. Thus,

$$\begin{aligned} \int \sec u \, du &= \int \frac{\sec u (\sec u + \tan u) \, du}{\sec u + \tan u} = \int \frac{\sec u \tan u + \sec^2 u}{\sec u + \tan u} \, du \\ &= \ln|\sec u + \tan u| + C \end{aligned}$$

To obtain the formula for  $\int \csc u \, du$ , we write it in the form

$$\int \frac{\csc u (\csc u - \cot u) \, du}{\csc u - \cot u}$$

Thus, we have

$$\begin{aligned} \int \csc u \, du &= \int \frac{\csc u (\csc u - \cot u) \, du}{\csc u - \cot u} = \int \frac{(-\csc u \cot u + \csc^2 u) \, du}{\csc u - \cot u} \\ &= \ln|\csc u - \cot u| + C \end{aligned}$$

Summarizing these results, we have the following integrals:

$$\int \tan u \, du = -\ln|\cos u| + C \quad (28.10)$$

$$\int \cot u \, du = \ln|\sin u| + C \quad (28.11)$$

$$\int \sec u \, du = \ln|\sec u + \tan u| + C \quad (28.12)$$

$$\int \csc u \, du = \ln|\csc u - \cot u| + C \quad (28.13)$$

#### EXAMPLE 4 Integration of $\tan u \, du$

Integrate  $\int \tan 4\theta \, d\theta$ .

Noting that  $u = 4\theta$ ,  $du = 4 \, d\theta$ , we have

$$\begin{aligned} \int \tan 4\theta \, d\theta &= \frac{1}{4} \int \tan 4\theta (4 \, d\theta) && \text{multiply and divide by 4} \\ &= -\frac{1}{4} \ln|\cos 4\theta| + C && \text{using Eq. (28.10)} \end{aligned}$$

#### Practice Exercise

3. Integrate  $\int 4x \cot x^2 \, dx$ .

#### EXAMPLE 5 Integration of $\sec u \, du$

Integrate  $\int \frac{\sec e^{-x} \, dx}{e^x}$ .

In this integral,  $u = e^{-x}$ ,  $du = -e^{-x} \, dx$ . Therefore,

$$\begin{aligned} \int \frac{\sec e^{-x} \, dx}{e^x} &= - \int (\sec e^{-x}) (-e^{-x} \, dx) && \text{introducing - sign} \\ &= -\ln|\sec e^{-x} + \tan e^{-x}| + C && \text{using Eq. (28.12)} \end{aligned}$$

**EXAMPLE 6** Integration of  $\csc u du$  and  $\cot u du$ Evaluate  $\int_{\pi/6}^{\pi/4} \frac{1 + \cos x}{\sin x} dx$ .

$$\begin{aligned} \int_{\pi/6}^{\pi/4} \frac{1 + \cos x}{\sin x} dx &= \int_{\pi/6}^{\pi/4} \csc x dx + \int_{\pi/6}^{\pi/4} \cot x dx && \text{using Eqs. (20.1) and (20.5)} \\ &= \ln|\csc x - \cot x| \Big|_{\pi/6}^{\pi/4} + \ln|\sin x| \Big|_{\pi/6}^{\pi/4} && \text{integrating} \\ &= \ln|\sqrt{2} - 1| - \ln|2 - \sqrt{3}| + \ln\left|\frac{1}{2}\sqrt{2}\right| - \ln\left|\frac{1}{2}\right| && \text{evaluating} \\ &= \ln\frac{(\frac{1}{2}\sqrt{2})(\sqrt{2} - 1)}{(\frac{1}{2})(2 - \sqrt{3})} = \ln\frac{2 - \sqrt{2}}{2 - \sqrt{3}} = 0.782 \end{aligned}$$

**EXERCISES 28.4**

In Exercises 1 and 2, make the given changes in the indicated examples of this section and then solve the given problems.

- In Example 1, change  $\sec^2 x^2$  to  $\sec^2 x^3$ . What other change must be made in the integrand to have a result of  $\tan x^3 + C$ ?
- In Example 4, change  $\tan$  to  $\cot$  and then integrate.

In Exercises 3–26, integrate each of the given functions. You may need to use trigonometric identities to change some of the integrals into a basic trigonometric form.

- $\int \cos 2x dx$
- $\int 4 \sin(2 - x) dx$
- $\int 0.3 \sec^2 3\theta d\theta$
- $\int \csc 8x \cot 8x dx$
- $\int \sec \frac{1}{2}x \tan \frac{1}{2}x dx$
- $\int e^x \csc^2(e^x) dx$
- $\int_{0.5}^1 x^2 \cot x^3 dx$
- $\int_0^1 6 \sin \frac{1}{2}t \sec \frac{1}{2}t dt$
- $\int 3\phi \sec^2 \phi^2 \cos \phi^2 d\phi$
- $\int (\cos^2 4x - \sin^2 4x) dx$
- $\int \frac{\sin(1/x)}{x^2} dx$
- $\int \frac{3 dx}{\sin 4x}$
- $\int_0^{\pi/6} \frac{dx}{\cos^2 2x}$
- $\int_0^1 \frac{2e^s ds}{\sec e^s}$
- $\int \sqrt{\frac{1 - \cos x}{2}} dx$  ( $0 \leq x < \frac{\pi}{2}$ )
- $\int \frac{\sin 2x}{\cos^2 x} dx$
- $\int \sqrt{\tan^2 2x + 1} dx$
- $\int 5(\tan u)(\ln \cos u) du$
- $\int \frac{2 \tan T}{1 - \tan^2 T} dT$
- $\int \frac{1 - \cot^2 x}{\cos^2 x} dx$

- $\int \frac{1 - \sin x}{1 + \cos x} dx$
- $\int \frac{1 + \sec^2 x}{x + \tan x} dx$
- $\int_0^{\pi/9} \sin 3x(\csc 3x + \sec 3x) dx$
- $\int_{\pi/4}^{\pi/3} (1 + \sec x)^2 dx$

In Exercises 27–40, solve the given problems by integration.

- Find the area bounded by  $y = 2 \tan x$ ,  $x = \frac{\pi}{4}$ , and  $y = 0$ .
- Find the area under the curve  $y = \sin x$  from  $x = 0$  to  $x = \pi$ .
- Although  $\int \frac{dx}{1 + \sin x}$  does not appear to fit a form for integration, show that it can be integrated by multiplying the numerator and the denominator by  $1 - \sin x$ .
- Change the integrand of  $\int \sin^2 2x \cos x dx$  to a form that can be integrated by methods of this section.
- Integrate  $\int \sec^2 x \tan x dx$  (a) with  $u = \tan x$ , and (b) with  $u = \sec x$ . Explain why the answers appear to be different.
- Evaluate  $\int_a^{a+2\pi} \sin x dx$  for any real value of  $a$ . Show an interpretation of the result in terms of the area under a curve.
- Find the volume generated by revolving the region bounded by  $y = \sec x$ ,  $x = 0$ ,  $x = \frac{\pi}{3}$ , and  $y = 0$  about the  $x$ -axis.
- Find the volume generated by revolving the region bounded by  $y = \cos x^2$ ,  $x = 0$ ,  $y = 0$ , and  $x = 1$  about the  $y$ -axis.
- The angular velocity  $\omega$  (in rad/s) of a pendulum is  $\omega = -0.25 \sin 2.5t$ . Find the angular displacement  $\theta$  as a function of  $t$  if  $\theta = 0.10$  for  $t = 0$ .
- If the current  $i$  (in A) in a certain electric circuit is given by  $i = 110 \cos 377t$ , find the expression for the voltage across a  $500\text{-}\mu\text{F}$  capacitor as a function of time. The initial voltage is zero. Show that the voltage across the capacitor is  $90^\circ$  out of phase with the current.

37. A fin on a wind-direction indicator has a shape that can be described as the region bounded by  $y = \tan x^2$ ,  $y = 0$ , and  $x = 1$ . Find the  $x$ -coordinate (in m) of the centroid of the fin if its area is  $0.3984 \text{ m}^2$ .
38. A force is given as a function of the distance from the origin as  $F = \frac{2 + \tan x}{\cos x}$ . Express the work done by this force as a function of  $x$  if  $W = 0$  for  $x = 0$ .
39. The probability that the phase error in a tracking device is between 0 and  $\pi/4$  is given by  $p = \int_0^{\pi/4} \cos x \, dx$ . Find  $p$ .
40. The number of hours  $h$  of daylight each day during the year in Toronto, Ontario, can be approximated by  $h = 12.2 + 3.3 \sin\left(\frac{2\pi}{365}t - \frac{160\pi}{365}\right)$ , where  $t$  is measured in days ( $t = 15$  is Jan. 15, etc.). Find the average number of daylight hours during the month of January.

#### Answers to Practice Exercises

1.  $-\frac{1}{5} \cos 5x + C$    2.  $-2 \cot 3x + C$    3.  $2 \ln|\sin x^2| + C$

## 28.5 Other Trigonometric Forms

Odd Power of  $\sin u$  or  $\cos u$  •  
 Even Power of  $\sin u$  or  $\cos u$  •  
 Power of  $\tan u$ ,  $\cot u$ ,  $\sec u$ ,  $\csc u$  •  
 Root-Mean-Square Value of Function

By use of trigonometric relations developed in Chapter 20, it is possible to transform many integrals involving powers of the trigonometric functions into integrable form. Let us first state the relationships that are useful for these integrals:

$$\cos^2 x + \sin^2 x = 1 \quad (28.14)$$

$$1 + \tan^2 x = \sec^2 x \quad (28.15)$$

$$1 + \cot^2 x = \csc^2 x \quad (28.16)$$

$$2 \cos^2 x = 1 + \cos 2x \quad (28.17)$$

$$2 \sin^2 x = 1 - \cos 2x \quad (28.18)$$

We now summarize how each of these identities can be used for integrating powers of trigonometric functions. Some examples are given after the summary.

### Integrals Involving Powers of Trigonometric Functions

- **Products of sines and cosines, with at least one of the powers odd:** Use Eq. (28.14) to write the integrand in powers of either the sine or the cosine, with only a first power of the other. (This first power will be the required factor in  $du$ .)
- **Products of even powers of sines and cosines:** Use Eqs. (28.17) and (28.18) to halve the even powers. The most commonly met integrands are  $\cos^2 u$  and  $\sin^2 u$ , which can be integrated directly after one substitution.
- **Products of powers of tangents and secants or even powers of secants:** Use Eq. (28.15) to write the integrand as powers of  $\tan u$  multiplied by  $\sec^2 u$  (which becomes part of  $du$ ) or as powers of  $\sec u$  multiplied by  $\sec u \tan u$  (which becomes part of  $du$ ).
- **Products of powers of cotangents and cosecants or even powers of cosecants:** Use Eq. (28.16) to write the integrand as powers of  $\cot u$  multiplied by  $\csc^2 u$  (which becomes part of  $du$ ) or as powers of  $\csc u$  multiplied by  $\csc u \cot u$  (which becomes part of  $du$ ).

■ Odd powers of secants and odd powers of cosecants are integrated by parts (see Section 28.7).

**EXAMPLE 1** Integration with an odd power of  $\sin u$ 

Integrate  $\int \sin^3 x \cos^2 x dx$ .

Since  $\sin^3 x = \sin^2 x \sin x = (1 - \cos^2 x) \sin x$ , we can write this integral with powers of  $\cos x$  along with  $-\sin x$ , which is the necessary  $du$  for this integral. Therefore,

$$\begin{aligned} \int \sin^3 x \cos^2 x dx &= \int (1 - \cos^2 x) (\sin x) (\cos^2 x) dx && \text{using Eq. (28.14)} \\ &= \int (\cos^2 x - \cos^4 x) (\sin x dx) \\ &= - \int \cos^2 x (-\sin x dx) + \int \cos^4 x (-\sin x dx) \\ &= -\frac{1}{3} \cos^3 x + \frac{1}{5} \cos^5 x + C \end{aligned}$$

**Practice Exercise**

1. Integrate  $\int \sin^3 x dx$ .

**EXAMPLE 2** Integration with an odd power of  $\cos u$ 

Integrate  $\int \cos^5 2x dx$ .

Since  $\cos^5 2x = \cos^4 2x \cos 2x = (1 - \sin^2 2x)^2 \cos 2x$ , it is possible to write this integral with powers of  $\sin 2x$  along with  $\cos 2x dx$ . Thus, with the introduction of a factor of 2,  $(\cos 2x) (2 dx)$  is the necessary  $du$  for this integral. Thus,

$$\begin{aligned} \int \cos^5 2x dx &= \int (1 - \sin^2 2x)^2 \cos 2x dx && \text{using Eq. (28.14)} \\ &= \int (1 - 2 \sin^2 2x + \sin^4 2x) \cos 2x dx \\ &= \int \cos 2x dx - \int 2 \sin^2 2x \cos 2x dx + \int \sin^4 2x \cos 2x dx \\ &= \frac{1}{2} \int \cos 2x (2 dx) - \int \sin^2 2x (2 \cos 2x dx) + \frac{1}{2} \int \sin^4 2x (2 \cos 2x dx) \\ &= \frac{1}{2} \sin 2x - \frac{1}{3} \sin^3 2x + \frac{1}{10} \sin^5 2x + C \end{aligned}$$

**EXAMPLE 3** Integration with an even power of  $\sin u$ 

Integrate  $\int \sin^2 2x dx$ .

Using Eq. (28.18) in the form  $\sin^2 2x = \frac{1}{2} (1 - \cos 4x)$ , this integral can be transformed into a form that can be integrated. (Here, we note *the  $x$  of Eq. 28.18 is treated as  $2x$*  for this integral.) Therefore, we write

$$\begin{aligned} \int \sin^2 2x dx &= \int \left[ \frac{1}{2} (1 - \cos 4x) \right] dx && \text{using Eq. (28.18)} \\ &= \frac{1}{2} \int dx - \frac{1}{8} \int \cos 4x (4 dx) \\ &= \frac{x}{2} - \frac{1}{8} \sin 4x + C \end{aligned}$$



**EXAMPLE 4** Integration with a power of  $\sec u$ 

Integrate  $\int \sec^3 t \tan t \, dt$ .

By writing  $\sec^3 t \tan t$  as  $\sec^2 t (\sec t \tan t)$ , we can use the  $\sec t \tan t \, dt$  as the  $du$  of the integral. Thus,

$$\begin{aligned} \int \sec^3 t \tan t \, dt &= \int (\sec^2 t) (\sec t \tan t \, dt) \\ &= \frac{1}{3} \sec^3 t + C \end{aligned}$$

**Practice Exercise**

2. Integrate  $\int \sec^4 x \, dx$ .

**EXAMPLE 5** Integration with a power of  $\tan u$ 

Integrate  $\int \tan^5 x \, dx$ .

Since  $\tan^5 x = \tan^3 x \tan^2 x = \tan^3 x (\sec^2 x - 1)$ , we can write this integral with powers of  $\tan x$  along with  $\sec^2 x \, dx$ . Thus,  $\sec^2 x \, dx$  becomes the necessary  $du$  of the integral. It is necessary to replace  $\tan^2 x$  with  $\sec^2 x - 1$  twice during the integration. Therefore,

$$\begin{aligned} \int \tan^5 x \, dx &= \int \tan^3 x (\sec^2 x - 1) \, dx && \text{using Eq. (28.15)} \\ &= \int \tan^3 x (\sec^2 x \, dx) - \int \tan^3 x \, dx \\ &= \frac{1}{4} \tan^4 x - \int \tan x (\sec^2 x - 1) \, dx && \text{using Eq. (28.15) again} \\ &= \frac{1}{4} \tan^4 x - \int \tan x (\sec^2 x \, dx) + \int \tan x \, dx \\ &= \frac{1}{4} \tan^4 x - \frac{1}{2} \tan^2 x - \ln|\cos x| + C \end{aligned}$$

**EXAMPLE 6** Integration of another trigonometric form

Integrate  $\int_0^{\pi/4} \frac{\tan^3 x}{\sec^3 x} \, dx$ .

Of several possible ways in which the integrand can be transformed into an integrable form, among the easiest is the following:

$$\begin{aligned} \int_0^{\pi/4} \frac{\tan^3 x}{\sec^3 x} \, dx &= \int_0^{\pi/4} \frac{\sin^3 x}{\cos^3 x} \frac{1}{\sec^3 x} \, dx && \tan x = \sin x / \cos x \\ &= \int_0^{\pi/4} \sin^3 x \, dx = \int_0^{\pi/4} (1 - \cos^2 x) \sin x \, dx && \cos x \sec x = 1 \\ &= \int_0^{\pi/4} \sin x \, dx - \int_0^{\pi/4} \cos^2 x \sin x \, dx \\ &= -\cos x + \frac{1}{3} \cos^3 x \Big|_0^{\pi/4} && \text{integrate} \\ &= -\frac{\sqrt{2}}{2} + \frac{1}{3} \left( \frac{\sqrt{2}}{2} \right)^3 - \left( -1 + \frac{1}{3} \right) && \text{evaluate} \\ &= \frac{8 - 5\sqrt{2}}{12} = 0.0774 \end{aligned}$$

**EXAMPLE 7** Root-mean-square value—application

The *root-mean-square value* of a function with respect to  $x$  is defined by the following equation:

$$y_{\text{rms}} = \sqrt{\frac{1}{T} \int_0^T y^2 dx} \quad (28.19)$$

■ See the chapter introduction.

Usually, the value of  $T$  that is of importance is the period of the function. Find the root-mean-square value of the electric current  $i$  (in A) used in a desktop computer, for which  $i = 3.82 \cos 120\pi t$ , for one period.

The period is  $\frac{2\pi}{120\pi} = \frac{1}{60.0}$  s. Thus, we must find the square root of the integral

$$\frac{1}{1/60.0} \int_0^{1/60.0} (3.82 \cos 120\pi t)^2 dt = 876 \int_0^{1/60.0} \cos^2 120\pi t dt$$

■ In most countries, the rms voltage is 240 V. In Canada and the United States, it is 120 V.

Evaluating this integral, we have

$$\begin{aligned} 876 \int_0^{1/60.0} \cos^2 120\pi t dt &= 438 \int_0^{1/60.0} (1 + \cos 240\pi t) dt \\ &= 438t \Big|_0^{1/60.0} + \frac{438}{240\pi} \int_0^{1/60.0} \cos 240\pi t (240\pi dt) \\ &= 7.3 + \frac{438}{240\pi} \sin 240\pi t \Big|_0^{1/60.0} = 7.3 \end{aligned}$$

This means the root-mean-square current is

$$i_{\text{rms}} = \sqrt{7.3} = 2.70 \text{ A}$$

This value of the current, often referred to as the *effective current*, is the value of direct current that would produce the same quantity of heat energy in the same time. It is important in the design of electronic equipment and electric appliances. ■

**EXERCISES 28.5**

In Exercises 1 and 2, answer the given questions related to the indicated examples of this section.

- In Example 3, what change must be made in the integrand in order to have a result of  $\frac{1}{6} \sin^3 2x + C$ ?
- In Example 5, what change must be made in the integrand in order to have a result of  $\frac{1}{6} \tan^6 x + C$ ?

In Exercises 3–34, integrate each of the given functions.

3.  $\int \sin^2 x \cos x dx$

4.  $\int \sin x \cos^5 x dx$

5.  $\int \sin^3 2x dx$

6.  $\int 3 \cos^3 T dT$

7.  $\int 4(\cos^4 \theta - \sin^4 \theta) d\theta$

8.  $\int \sin^3 x \cos^6 x dx$

9.  $\int_0^{\pi/4} 5 \sin^5 x dx$

10.  $\int_{\pi/3}^{\pi/2} 10 \sin t (1 - \cos 2t)^2 dt$

11.  $\int \sin^2 x dx$

13.  $\int 2(1 + \cos 3\phi)^2 d\phi$

15.  $\int \tan^3 x dx$

17.  $\int_0^{\pi/4} \tan x \sec^4 x dx$

19.  $\int \tan^4 2x dx$

21.  $\int 0.5 \sin s \sin 2s ds$

23.  $\int (\sin x + \cos x)^2 dx$

12.  $\int \cos^2 2x dx$

14.  $\int_0^1 \sin^2 4x dx$

16.  $\int \frac{6 \cot^2 y}{\tan y} dy$

18.  $\int \cot 4x \csc^4 4x dx$

20.  $\int 4 \cot^4 x dx$

22.  $\int \sqrt{\tan x} \sec^4 x dx$

24.  $\int (\tan 2x + \cot 2x)^2 dx$

25.  $\int \frac{1 - \cot x}{\sin^4 x} dx$

27.  $\int_{\pi/6}^{\pi/4} \cot^5 p dp$

29.  $\int \sec^6 x dx$

31.  $\int \frac{\sin 2x}{\cos^3 x} dx$

33.  $\int \frac{\sec e^{-x}}{e^x} dx$

26.  $\int \frac{(\sin u + \sin^2 u)^2}{\sec u} du$

28.  $\int_{\pi/6}^{\pi/3} \frac{2 dx}{1 + \sin x}$

30.  $\int \tan^7 x dx$

32.  $\int \frac{\sec^2 t \tan t}{4 + \sec^2 t} dt$

34.  $\int_0^{\pi/4} \sqrt{1 + \cos 4x} dx$

43. Show that  $\int_0^{\pi} \sin^2 nx dx = \frac{1}{2}\pi$ , where  $n$  is any positive integer.

44. The velocity  $v$  (in cm/s) of an object is  $v = \cos^2 \pi t$ . How far does the object move in 4.0 s?

45. The acceleration  $a$  (in  $\text{m/s}^2$ ) of an object is  $a = \sin^2 t \cos t$ . If the object starts at the origin with a velocity of 6 m/s, what is its position at time  $t$ ?

46. Find the root-mean-square current in a circuit from  $t = 0$  s to  $t = 0.50$  s if  $i = i_0 \sin t \sqrt{\cos t}$ .

47. In the study of the rate of radiation by an accelerated charge, the following integral must be evaluated:  $\int_0^{\pi} \sin^3 \theta d\theta$ . Find the value of the integral.

48. In finding the volume of a special O-ring for a space vehicle, the integral  $\int \frac{\sin^2 \theta}{\cos^2 \theta} d\theta$  must be evaluated. Perform this integration.

49. For a voltage  $V = 340 \sin 120\pi t$ , show that the root-mean-square voltage for one period is 240 V.

50. For a current  $i = i_0 \sin \omega t$ , show that the root-mean-square current for one period is  $i_0/\sqrt{2}$ .

51. In the analysis of the intensity of light from a certain source, the equation  $I = A \int_{-a/2}^{a/2} \cos^2 [b\pi(c-x)] dx$  is used. Here,  $A$ ,  $a$ ,  $b$ , and  $c$  are constants. Evaluate this integral. (The simplification is quite lengthy.)

52. In the study of the lifting force  $L$  due to a stream of fluid passing around a cylinder, the equation  $L = k \int_0^{2\pi} (a \sin \theta + b \sin^2 \theta - b \sin^3 \theta) d\theta$  is used. Here,  $k$ ,  $a$ , and  $b$  are constants and  $\theta$  is the angle from the direction of flow. Evaluate the integral.

In Exercises 35–52, solve the given problems by integration.

35. Using Eq. (20.14) on page 541, integrate  $\int \sin 4x \cos 5x dx$ .

36. Using Eq. (20.15) on page 541, integrate  $\int \cos 3x \cos 4x dx$ .

37. Find the volume generated by revolving the region bounded by  $y = \sin x$  and  $y = 0$ , from  $x = 0$  to  $x = \pi$ , about the  $x$ -axis.

38. Find the volume generated by revolving the region bounded by  $y = \tan^3(x^2)$ ,  $y = 0$ , and  $x = \frac{\pi}{4}$  about the  $y$ -axis.

39. Find the area bounded by  $y = \sin x$ ,  $y = \cos x$ , and  $x = 0$  in the first quadrant.

40. Find the length of the curve  $y = \ln \cos x$  from  $x = 0$  to  $x = \frac{\pi}{3}$ . (Use Eq. 26.29.)

41. Show that  $\int \sin x \cos x dx$  can be integrated in two ways. Explain the difference in the answers.

42. For  $n > 0$ , show that  $\int \tan x \sec^n x dx = \frac{1}{n} \sec^n x + C$ .

### Answers to Practice Exercises

1.  $-\cos x + \frac{1}{3}\cos^3 x + C$  2.  $\tan x + \frac{1}{3}\tan^3 x + C$

## 28.6 Inverse Trigonometric Forms

**Inverse Sine Form •  
Inverse Tangent Form •  
Comparison of Inverse Sine,  
Inverse Tangent, Power, and  
Logarithmic Forms**

■ For reference, Eq. (27.10) is

$$\frac{d(\sin^{-1} u)}{dx} = \frac{1}{\sqrt{1-u^2}} \frac{du}{dx}$$

Using Eq. (27.10), we can find the differential of  $\sin^{-1}(u/a)$ , where  $a$  is constant:

$$d\left(\sin^{-1} \frac{u}{a}\right) = \frac{1}{\sqrt{1-(u/a)^2}} \frac{du}{a} = \frac{a}{\sqrt{a^2-u^2}} \frac{du}{a} = \frac{du}{\sqrt{a^2-u^2}}$$

Noting this differentiation formula, and the antiderivative of the result, we have the important integration formula

$$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1} \frac{u}{a} + C \quad (28.20)$$

By finding the differential of  $\tan^{-1}(u/a)$ , we have

$$d\left(\tan^{-1} \frac{u}{a}\right) = \frac{1}{1+(u/a)^2} \frac{du}{a} = \frac{a^2}{a^2+u^2} \frac{du}{a} = \frac{a du}{a^2+u^2}$$

Again, noting the antiderivative, we have another important integration formula:

$$\int \frac{du}{a^2 + u^2} = \frac{1}{a} \tan^{-1} \frac{u}{a} + C \quad (28.21)$$

This shows one of the principal uses of the inverse trigonometric functions: They provide a solution to the integration of important algebraic functions.

### EXAMPLE 1 Inverse sine form

Integrate  $\int \frac{dx}{\sqrt{9 - x^2}}$ .

This integral fits the form of Eq. (28.20) with  $u = x$ ,  $du = dx$ , and  $a = 3$ . Thus,

$$\begin{aligned} \int \frac{dx}{\sqrt{9 - x^2}} &= \int \frac{dx}{\sqrt{3^2 - x^2}} \\ &= \sin^{-1} \frac{x}{3} + C \end{aligned}$$

#### Practice Exercise

1. Integrate  $\int \frac{dx}{\sqrt{9 - 4x^2}}$ .

### EXAMPLE 2 Inverse tangent form—application

The volume flow rate  $Q$  (in  $\text{m}^3/\text{s}$ ) of a constantly flowing liquid is given by  $Q = 24 \int_0^2 \frac{dx}{6 + x^2}$ , where  $x$  is the distance from the centre of flow. Find the value of  $Q$ .

For the integral, we see that it fits Eq. (28.21) with  $u = x$ ,  $du = dx$ , and  $a = \sqrt{6}$ .

$$\begin{aligned} Q &= 24 \int_0^2 \frac{dx}{6 + x^2} = 24 \int_0^2 \frac{dx}{(\sqrt{6})^2 + x^2} \\ &= \frac{24}{\sqrt{6}} \tan^{-1} \left| \frac{x}{\sqrt{6}} \right|_0^2 \\ &= \frac{24}{\sqrt{6}} \left( \tan^{-1} \frac{2}{\sqrt{6}} - \tan^{-1} 0 \right) \\ &= 6.71 \text{ m}^3/\text{s} \end{aligned}$$

### EXAMPLE 3 Inverse sine form

Integrate  $\int \frac{dx}{\sqrt{25 - 4x^2}}$ .

This integral fits the form of Eq. (28.20) with  $u = 2x$ ,  $du = 2 dx$ , and  $a = 5$ . Thus, in order to have the proper  $du$ , we must include a factor of 2 in the numerator, and therefore we also place a  $\frac{1}{2}$  before the integral. This leads to

$$\begin{aligned} \int \frac{dx}{\sqrt{25 - 4x^2}} &= \frac{1}{2} \int \frac{2 dx}{\sqrt{5^2 - (2x)^2}} \\ &= \frac{1}{2} \sin^{-1} \frac{2x}{5} + C \end{aligned}$$

**EXAMPLE 4** Completing the square to fit the  $\tan^{-1}u$  form

Integrate  $\int_{-1}^3 \frac{dx}{x^2 + 6x + 13}$ .

At first glance, it does not appear that this integral fits any of the forms presented up to this point. However, by writing the denominator in the form  $(x^2 + 6x + 9) + 4 = (x + 3)^2 + 2^2$  (see Example 4 on page 227), **we recognize that**  $u = x + 3$ ,  $du = dx$ , and  $a = 2$ . Thus,

$$\begin{aligned} \int_{-1}^3 \frac{dx}{x^2 + 6x + 13} &= \int_{-1}^3 \frac{dx}{(x + 3)^2 + 2^2} \\ &= \frac{1}{2} \tan^{-1} \frac{x + 3}{2} \Big|_{-1}^3 && \text{integrate} \\ &= \frac{1}{2} (\tan^{-1} 3 - \tan^{-1} 1) && \text{evaluate} \\ &= 0.2318 \end{aligned}$$

**Practice Exercise**

2. Integrate  $\int \frac{dx}{4 + 9x^2}$ .

Now, we can see the use of completing the square when we are transforming integrals into proper form. ■

**EXAMPLE 5** Inverse tangent and logarithmic forms

Integrate  $\int \frac{2r + 5}{r^2 + 9} dr$ .

By writing this integral as the sum of two integrals, we may integrate each of these separately:

$$\int \frac{2r + 5}{r^2 + 9} dr = \int \frac{2r dr}{r^2 + 9} + \int \frac{5 dr}{r^2 + 9}$$

The first integral is a logarithmic form, and the second is an inverse tangent form. For the first,  $u = r^2 + 9$ ,  $du = 2r dr$ . For the second,  $u = r$ ,  $du = dr$ ,  $a = 3$ .

$$\int \frac{2r dr}{r^2 + 9} + 5 \int \frac{dr}{r^2 + 9} = \ln|r^2 + 9| + \frac{5}{3} \tan^{-1} \frac{r}{3} + C \quad \blacksquare$$

**LEARNING TIP**

The inverse trigonometric integral forms show very well the importance of **proper recognition of the form of the integral**. It is important that these forms are not confused with those of the general power rule or the logarithmic form.

**EXAMPLE 6** Comparing the inverse sine, power, and logarithmic forms

The integral  $\int \frac{dx}{\sqrt{1 - x^2}}$  is of the inverse sine form with  $u = x$ ,  $du = dx$ , and  $a = 1$ . Thus,

$$\int \frac{dx}{\sqrt{1 - x^2}} = \sin^{-1} x + C$$

The integral  $\int \frac{x dx}{\sqrt{1 - x^2}}$  is not of the inverse sine form due to the factor of  $x$  in the numerator. It is integrated by use of the general power rule, with  $u = 1 - x^2$ ,  $du = -2x dx$ , and  $n = -\frac{1}{2}$ . Thus,

$$\int \frac{x dx}{\sqrt{1 - x^2}} = -\sqrt{1 - x^2} + C$$

The integral  $\int \frac{x dx}{1 - x^2}$  is of the basic logarithmic form with  $u = 1 - x^2$  and  $du = -2x dx$ . If  $1 - x^2$  is raised to any power other than 1 in the denominator, we would use the general power rule. To be of the inverse sine form, we would have the square root of  $1 - x^2$  and no factor of  $x$ , as in the first illustration. Thus,

$$\int \frac{x dx}{1 - x^2} = -\frac{1}{2} \ln|1 - x^2| + C \quad \blacksquare$$

### EXAMPLE 7 Recognizing different forms

The following integrals are of the form indicated:

$\int \frac{dx}{1 + x^2}$	Inverse tangent form	$u = x, du = dx$
$\int \frac{x dx}{1 + x^2}$	Logarithmic form	$u = 1 + x^2, du = 2x dx$
$\int \frac{x dx}{\sqrt{1 + x^2}}$	General power form	$u = 1 + x^2, du = 2x dx$
$\int \frac{dx}{1 + x}$	Logarithmic form	$u = 1 + x, du = dx$
$\int \frac{x dx}{\sqrt{1 - x^4}}$	Inverse sine form	$u = x^2, du = 2x dx$
$\int \frac{x dx}{1 + x^4}$	Inverse tangent form	$u = x^2, du = 2x dx$

There are a number of integrals whose forms appear to be similar to those in Examples 6 and 7, but which do not fit the forms we have discussed. They include

$$\int \frac{dx}{\sqrt{x^2 - 1}} \quad \int \frac{dx}{\sqrt{1 + x^2}} \quad \int \frac{dx}{1 - x^2} \quad \int \frac{dx}{x\sqrt{1 + x^2}}$$

We will develop methods to integrate some of these forms, and all of them can be integrated by the tables discussed in Section 28.11.

## EXERCISES 28.6

In Exercises 1 and 2, answer the given questions related to the indicated examples of this section.

- In Example 1, what change must be made in the integrand in order to have a result of  $\sqrt{9 - x^2} + C$ ?
- In Example 2, what change must be made in the integrand in order that the integration would lead to a result of  $\ln(6 + x^2) + C$ ?

In Exercises 3–30, integrate each of the given functions.

3.  $\int \frac{dx}{\sqrt{4 - x^2}}$

4.  $\int \frac{dx}{\sqrt{49 - x^2}}$

5.  $\int \frac{dx}{64 + x^2}$

6.  $\int \frac{6p^2 dp}{4 + p^6}$

7.  $\int \frac{8x dx}{\sqrt{1 - 16x^4}}$

8.  $\int_0^1 \frac{2 dx}{\sqrt{9 - 4x^2}}$

9.  $\int_0^2 \frac{3e^{-t} dt}{1 + 9e^{-2t}}$

10.  $\int_1^3 \frac{4 dx}{49 + 4x^2}$

11.  $\int_0^{0.4} \frac{2 dx}{\sqrt{4 - 5x^2}}$

13.  $\int \frac{8x dx}{9x^2 + 16}$

15.  $\int_1^e \frac{3 du}{u[1 + (\ln u)^2]}$

17.  $\int \frac{e^x dx}{\sqrt{1 - e^{2x}}}$

19.  $\int \frac{dT}{T^2 + 2T + 2}$

21.  $\int \frac{4 dx}{\sqrt{-4x - x^2}}$

23.  $\int_{\pi/6}^{\pi/2} \frac{2 \cos 2\theta d\theta}{1 + \sin^2 2\theta}$

12.  $\int \frac{dx}{2\sqrt{x}\sqrt{1 - x}}$

14.  $\int \frac{4y dy}{\sqrt{25 - 16y^2}}$

16.  $\int_0^1 \frac{4x dx}{1 + x^4}$

18.  $\int \frac{\sec^2 x dx}{\sqrt{1 - \tan^2 x}}$

20.  $\int \frac{2 dx}{x^2 + 8x + 17}$

22.  $\int \frac{0.3 ds}{\sqrt{2s - s^2}}$

24.  $\int_{-4}^0 \frac{dx}{x^2 + 4x + 5}$

25.  $\int \frac{2-x}{\sqrt{4-x^2}} dx$

26.  $\int \frac{3-2x}{1+4x^2} dx$

27.  $\int \frac{\sin^{-1} x}{\sqrt{1-x^2}} dx$

28.  $\int \frac{dx}{e^x + e^{-x}}$

29.  $\int \frac{x^3 + 3x^5}{1+x^6} dx$

30.  $\int \frac{x \tan^{-1} x^2}{1+x^4} dx$

In Exercises 31–34, identify the form of each integral as being inverse sine, inverse tangent, logarithmic, or general power, as in Examples 6 and 7. Do not integrate. In each part (a), explain how the choice was made.

31. (a)  $\int \frac{2 dx}{4+9x^2}$  (b)  $\int \frac{2 dx}{4+9x}$  (c)  $\int \frac{2x dx}{\sqrt{4+9x^2}}$

32. (a)  $\int \frac{2x dx}{4-9x^2}$  (b)  $\int \frac{2 dx}{\sqrt{4-9x}}$  (c)  $\int \frac{2x dx}{4+9x^2}$

33. (a)  $\int \frac{2x dx}{\sqrt{4-9x^2}}$  (b)  $\int \frac{2 dx}{\sqrt{4-9x^2}}$  (c)  $\int \frac{2 dx}{4-9x}$

34. (a)  $\int \frac{2 dx}{9x^2+4}$  (b)  $\int \frac{2x dx}{\sqrt{9x^2-4}}$  (c)  $\int \frac{2x dx}{9x^2-4}$

In Exercises 35–44, solve the given problems by integration.

35. Explain how to integrate  $\int \frac{dx}{\sqrt{x}(1+x)}$ . What is the result?

36. Integrate  $\int \sqrt{\frac{1+x}{1-x}} dx$  by first multiplying the numerator and denominator of the fraction under the radical by  $1+x$ .

37. Find the area bounded by  $y(1+x^2) = 1$ ,  $x = 0$ ,  $y = 0$ , and  $x = 2$ . See Fig. 28.8.

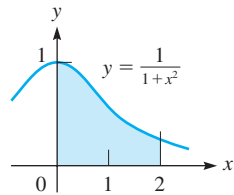


Fig. 28.8

## 28.7 Integration by Parts

Integration by Parts •  
Identifying  $u$  and  $dv$  •  
Repeated Use of the Method

There are many methods of transforming integrals into forms that can be integrated by one of the basic formulas. In the preceding sections, we saw that completing the square and trigonometric identities can be used for this purpose. In this section and the following one, we develop two general methods. The method of *integration by parts* is discussed in this section.

Since the derivative of a product of functions is found by use of the formula

$$\frac{d(uv)}{dx} = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}$$

38. Find the circumference of the circle  $x^2 + y^2 = r^2$  by integration. (Use Eq. 26.29.)
39. To find the electric field  $E$  from an electric charge distributed uniformly over the entire  $xy$ -plane at a distance  $d$  from the plane, it is necessary to evaluate the integral  $kd \int \frac{dx}{d^2 + x^2}$ . Here,  $x$  is the distance from the origin to the element of charge. Perform the indicated integration.
40. An oil-storage tank can be described as the volume generated by revolving the region bounded by  $y = 24/\sqrt{16+x^2}$ ,  $x = 0$ ,  $y = 0$ , and  $x = 3$  about the  $x$ -axis. Find the volume (in  $m^3$ ) of the tank.
41. In dealing with the theory for simple harmonic motion, it is necessary to solve the equation  $\frac{dx}{\sqrt{A^2 - x^2}} = \sqrt{\frac{k}{m}} dt$  ( $k$ ,  $m$ , and  $A$  are constants). Determine the solution if  $x = x_0$  when  $t = 0$ .
42. During each cycle, the velocity  $v$  (in m/s) of a robotic welding device is given by  $v = 2t - \frac{12}{2+t^2}$ , where  $t$  is the time (in s). Find the expression for the displacement  $s$  (in m) as a function of  $t$  if  $s = 0$  for  $t = 0$ .
43. Find the moment of inertia with respect to the  $y$ -axis for a flat plate covering the region bounded by  $y = 1/(1+x^6)$ , the  $x$ -axis,  $x = 1$ , and  $x = 2$ .
44. Find the length of arc along the curve  $y = \sqrt{1-x^2}$  between  $x = 0$  and  $x = 1$ . (Use Eq. 26.29.)

### Answers to Practice Exercises

1.  $\frac{1}{2} \sin^{-1} \frac{2x}{3} + C$     2.  $\frac{1}{6} \tan^{-1} \frac{3x}{2} + C$



the differential of a product of functions is given by  $d(uv) = u dv + v du$ . Integrating both sides of this equation, we have  $uv = \int u dv + \int v du$ . Solving for  $\int u dv$ , we obtain

$$\int u dv = uv - \int v du \quad (28.22)$$

Integration by use of Eq. (28.22) is called **integration by parts**.

### EXAMPLE 1 Algebraic-trigonometric integrand

Integrate  $\int x \sin x dx$ .

This integral does not fit any of the previous forms we have discussed, since neither  $x$  nor  $\sin x$  can be made a factor of a proper  $du$ . However, by choosing  $u = x$  and  $dv = \sin x dx$ , integration by parts may be used. Thus,

$$u = x \quad dv = \sin x dx$$

By finding the differential of  $u$  and integrating  $dv$ , we find  $du$  and  $v$ . This gives us

$$du = dx \quad v = -\cos x + C_1$$

Now, substituting in Eq. (28.22), we have

$$\begin{aligned} \int \overset{u}{\downarrow} \overset{dv}{\downarrow} &= \overset{u}{\downarrow} \overset{v}{\downarrow} - \int \overset{v}{\downarrow} \overset{du}{\downarrow} \\ \int (x)(\sin x dx) &= (x)(-\cos x + C_1) - \int (-\cos x + C_1)(dx) \\ &= -x \cos x + C_1 x + \int \cos x dx - \int C_1 dx \\ &= -x \cos x + C_1 x + \sin x - C_1 x + C \\ &= -x \cos x + \sin x + C \end{aligned}$$

Other choices of  $u$  and  $dv$  may be made, but they are not useful. For example, if we choose  $u = \sin x$  and  $dv = x dx$ , then  $du = \cos x dx$  and  $v = \frac{1}{2}x^2 + C_2$ . This makes  $\int v du = \int (\frac{1}{2}x^2 + C_2)(\cos x dx)$ , which is more complex than the integrand of the original problem. ■

#### Practice Exercise

1. Integrate  $\int x \sec^2 x dx$ .

#### LEARNING TIP

We note that the constant  $C_1$  that was introduced when we integrated  $dv$  does not appear in the final result. This constant will always cancel out, and therefore we do not need a constant of integration when finding  $v$ .

As in Example 1, there is often more than one choice as to the part of the integrand that is selected to be  $u$  and the part that is selected to be  $dv$ . There are no set rules that may be stated for the best choice of  $u$  and  $dv$ , but two guidelines may be stated.

#### Guidelines for Choosing $u$ and $dv$

1. The quantity  $u$  is normally chosen such that  $du/dx$  is of simpler form than  $u$ .
2. The differential  $dv$  is normally chosen such that  $\int dv$  is easily integrated.

Working examples, and thereby gaining experience in methods of integration, is the best way to determine when this method should be used and how to use it.

**EXAMPLE 2 Algebraic integrand**

Integrate  $\int x\sqrt{1-x} dx$ .

We see that this form does not fit the general power rule, for  $x dx$  is not a factor of the differential of  $1-x$ . By choosing  $u = x$  and  $dv = \sqrt{1-x} dx$ , we have  $du/dx = 1$ , and  $v$  can readily be determined. Thus,

$$\begin{aligned}u &= x & dv &= \sqrt{1-x} dx = (1-x)^{1/2} dx \\ du &= dx & v &= -\frac{2}{3}(1-x)^{3/2}\end{aligned}$$

Substituting in Eq. (28.22), we have

$$\int \overset{u}{\downarrow} x \left[ \overset{dv}{\downarrow} (1-x)^{1/2} dx \right] = x \left[ \overset{u}{\downarrow} -\frac{2}{3} (1-x)^{3/2} \right] - \int \left[ \overset{v}{\downarrow} -\frac{2}{3} (1-x)^{3/2} \right] \overset{du}{\downarrow} dx$$

At this point, we see that we can complete the integration. Thus,

$$\begin{aligned}\int x(1-x)^{1/2} dx &= -\frac{2x}{3}(1-x)^{3/2} + \frac{2}{3} \int (1-x)^{3/2} dx \\ &= -\frac{2x}{3}(1-x)^{3/2} + \frac{2}{3} \left( -\frac{2}{5} \right) (1-x)^{5/2} + C \\ &= -\frac{2}{3}(1-x)^{3/2} \left[ x + \frac{2}{5}(1-x) \right] + C \\ &= -\frac{2}{15}(1-x)^{3/2}(2+3x) + C\end{aligned}$$

**EXAMPLE 3 Algebraic-logarithmic integrand**

Integrate  $\int \sqrt{x} \ln x dx$ .

For this integral, we have

$$\begin{aligned}u &= \ln x & dv &= x^{1/2} dx \\ du &= \frac{1}{x} dx & v &= \frac{2}{3}x^{3/2}\end{aligned}$$

$$\begin{aligned}\int \sqrt{x} \ln x dx &= \frac{2}{3}x^{3/2} \ln x - \frac{2}{3} \int x^{1/2} dx \\ &= \frac{2}{3}x^{3/2} \ln x - \frac{4}{9}x^{3/2} + C\end{aligned}$$

**EXAMPLE 4 Inverse sine integrand**

Integrate  $\int \sin^{-1} x dx$ .

We write

$$\begin{aligned}u &= \sin^{-1} x & dv &= dx \\ du &= \frac{dx}{\sqrt{1-x^2}} & v &= x\end{aligned}$$

$$\begin{aligned}\int \sin^{-1} x dx &= x \sin^{-1} x - \int \frac{x dx}{\sqrt{1-x^2}} = x \sin^{-1} x + \frac{1}{2} \int \frac{-2x dx}{\sqrt{1-x^2}} \\ &= x \sin^{-1} x + \sqrt{1-x^2} + C\end{aligned}$$

**EXAMPLE 5** Application with an algebraic-exponential integrand

In a certain electric circuit, the current  $i$  (in A) is given by the equation  $i = te^{-t}$ , where  $t$  is the time (in s). Find the charge  $q$  to pass a point in the circuit between  $t = 0$  s and  $t = 1.0$  s.

Since  $i = \frac{dq}{dt}$ , we have  $\frac{dq}{dt} = te^{-t}$ . Thus, with  $q = \int_0^{1.0} te^{-t} dt$ , we are to solve for  $q$ .

For this integral,

$$\begin{aligned} u &= t & dv &= e^{-t} dt \\ du &= dt & v &= -e^{-t} \end{aligned}$$

$$\begin{aligned} q &= \int_0^{1.0} te^{-t} dt = -te^{-t} \Big|_0^{1.0} + \int_0^{1.0} e^{-t} dt = -te^{-t} - e^{-t} \Big|_0^{1.0} \\ &= -e^{-1.0} - e^{-1.0} + 1.0 = 0.26 \text{ C} \end{aligned}$$

Therefore, 0.26 C pass a given point in the circuit. ■

There are instances when integration by parts must be used more than once in order to complete an integral. Consider the following examples.

**EXAMPLE 6** Integration by parts twice

Integrate  $\int x^2 \cos x dx$ .

Let  $u = x^2$ ,  $dv = \cos x dx$ ,  $du = 2x dx$ ,  $v = \sin x$ . Then

$$\int x^2 \cos x dx = x^2 \sin x - 2 \int x \sin x dx$$

We apply integration by parts a second time to the right-hand integral, with  $u = x$ ,  $dv = \sin x dx$ ,  $du = dx$ ,  $v = -\cos x$ . Therefore,

$$\int x \sin x dx = -x \cos x + \int \cos x dx = -x \cos x + \sin x$$

Substituting, we get

$$\int x^2 \cos x dx = x^2 \sin x + 2x \cos x - 2 \sin x \quad \blacksquare$$

**EXAMPLE 7** Integration leading to an equation

Integrate  $\int e^x \sin x dx$ .

Let  $u = \sin x$ ,  $dv = e^x dx$ ,  $du = \cos x dx$ ,  $v = e^x$ .

$$\int e^x \sin x dx = e^x \sin x - \int e^x \cos x dx$$

At first glance, it appears that we have made no progress. Nevertheless, we carry on and apply integration by parts to the integral  $\int e^x \cos x dx$ :

$$u = \cos x \quad dv = e^x dx \quad du = -\sin x dx \quad v = e^x$$

And so  $\int e^x \cos x dx = e^x \cos x + \int e^x \sin x dx$ . Substituting this expression into the expression for  $\int e^x \sin x dx$ , we obtain

$$\begin{aligned} \int e^x \sin x dx &= e^x \sin x - \left( e^x \cos x + \int e^x \sin x dx \right) \\ &= e^x \sin x - e^x \cos x - \int e^x \sin x dx \end{aligned}$$

**LEARNING TIP**

There are some integrals for which using integration by parts twice leads to an integral of the same form as the original integral but with a different coefficient. Although this may seem as if the integration took us back where we started, what we now have is an equation, which can be solved by combining the integrals of like form. See Example 7.

Adding the right-hand integral to both sides gives

$$2 \int e^x \sin x \, dx = e^x (\sin x - \cos x) + 2C$$

$$\int e^x \sin x \, dx = \frac{e^x}{2} (\sin x - \cos x) + C$$

Thus, by combining integrals of like form, we obtain the desired result. ■

## EXERCISES 28.7

In Exercises 1 and 2, answer the given questions related to the indicated examples of this section.

- In Example 2, do the choices  $u = \sqrt{1-x}$  and  $dv = x \, dx$  work for this integral? Explain.
- In Example 7, do the choices  $u = e^x$  and  $dv = \sin x \, dx$  work for this integral? Explain.

In Exercises 3–22, integrate each of the given functions.

- |   |   |
|---|---|
| 3. $\int \theta \cos \theta \, d\theta$         | 4. $\int x \sin 2x \, dx$                 |
| 5. $\int 4xe^{2x} \, dx$                        | 6. $\int 3xe^x \, dx$                     |
| 7. $\int 3x \csc^2 x \, dx$                     | 8. $\int_0^{\pi/4} x \sec x \tan x \, dx$ |
| 9. $\int 2 \tan^{-1} x \, dx$                   | 10. $\int \ln s \, ds$                    |
| 11. $\int_{-3}^0 \frac{4t \, dt}{\sqrt{1-t}}$   | 12. $\int x\sqrt{x+1} \, dx$              |
| 13. $\int x \ln x \, dx$                        | 14. $\int x^2 \ln 4x \, dx$               |
| 15. $\int 2\phi^2 \sin \phi \cos \phi \, d\phi$ | 16. $\int_0^1 r^2 e^{2r} \, dr$           |
| 17. $\int_0^{\pi/2} e^x \cos x \, dx$           | 18. $\int e^{-x} \sin 2x \, dx$           |
| 19. $\int (x+4)^8 (x+5) \, dx$                  | 20. $\int \cos x \ln(\sin x) \, dx$       |
| 21. $\int \cos(\ln x) \, dx$                    | 22. $\int \sec^3 x \, dx$                 |

In Exercises 23–36, solve the given problems by integration.

- To integrate  $\int e^{-\sqrt{x}} \, dx$ , the only choices possible of  $u = e^{-\sqrt{x}}$  and  $dv = dx$  do not work. However, if we first let  $t = \sqrt{x}$ ,  $dt = dx/2\sqrt{x}$ , the integration can be done by parts. Perform this integration.
- To integrate  $\int x \ln(x+1) \, dx$ , the substitution  $t = x+1$ ,  $dt = dx$  leads to an integral that can be done readily by parts. Perform this integration in this way.

- Find the area bounded by  $y = xe^{-x}$ ,  $y = 0$ , and  $x = 2$ . See Fig. 28.9.

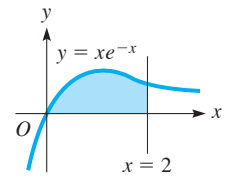


Fig. 28.9

- Find the area bounded by  $y = 2(\ln x)/x^2$ ,  $y = 0$ , and  $x = 3$ .
- Find the volume generated by revolving the region bounded by  $y = \tan^2 x$ ,  $y = 0$ , and  $x = 0.5$  about the  $y$ -axis.
- Find the volume generated by revolving the region bounded by  $y = \sin x$  and  $y = 0$  from  $x = 0$  to  $x = \pi$  about the  $y$ -axis.
- Find the  $x$ -coordinate of the centroid of a flat plate covering the region bounded by  $y = \cos x$  and  $y = 0$  for  $0 \leq x \leq \pi/2$ .
- Find the moment of inertia with respect to its axis of the solid generated by revolving the region bounded by  $y = e^x$ ,  $x = 1$ , and the coordinate axes about the  $y$ -axis.
- Find the root-mean-square value of the function  $y = \sqrt{\sin^{-1} x}$  between  $x = 0$  and  $x = 1$ . (See Example 7 of Section 28.5.)
- The general expression for the slope of a curve is  $dy/dx = x^3\sqrt{1+x^2}$ . Find the equation of the curve if it passes through the origin.
- Computer simulation shows that the velocity  $v$  (in m/s) of a test car is  $v = t^3/\sqrt{t^2+1}$  from  $t = 0$  to  $t = 8.0$  s. Find the expression for the distance travelled by the car in  $t$  seconds.
- The nose cone of a rocket has the shape of the solid that is generated by revolving the region bounded by  $y = \ln x$ ,  $y = 0$ , and  $x = 9.5$  about the  $x$ -axis. Find the volume (in  $\text{m}^3$ ) of the nose cone. See Fig. 28.10.

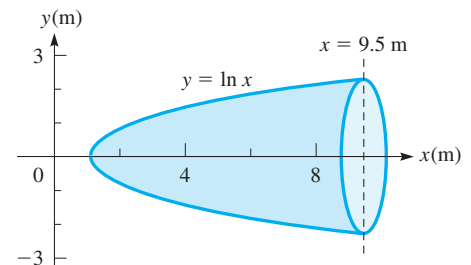


Fig. 28.10

- The current in a given circuit is given by  $i = e^{-2t} \cos t$ . Find an expression for the amount of charge that passes a given point in the circuit as a function of the time, if  $q_0 = 0$ .
- In finding the average length  $\bar{x}$  (in nm) of a certain type of large molecule, we use the equation  $\bar{x} = \lim_{b \rightarrow \infty} [0.1 \int_0^b x^3 e^{-x^2/8} \, dx]$ . Evaluate the integral and then use L'Hospital's rule (or a calculator) to show that  $\bar{x} \rightarrow 3.2$  nm as  $b \rightarrow \infty$ .

### Answer to Practice Exercise

- $x \tan x + \ln|\cos x| + C$

## 28.8 Integration by Trigonometric Substitution

Transform Algebraic Integrals into Trigonometric Integrals

In this section, we show how trigonometric relations are useful in integrating algebraic integrals involving the expressions  $\sqrt{a^2 - x^2}$ ,  $\sqrt{a^2 + x^2}$ , or  $\sqrt{x^2 - a^2}$ . We begin by summarizing the method, which is known as **trigonometric substitution**, and then consider some examples.

### Trigonometric Substitution

1. Select the appropriate trigonometric substitution according to the following table and rewrite the radical as shown:

If the Integral Involves	Then Substitute	Use the Identity	To Rewrite the Radical as
$\sqrt{a^2 - x^2}$	$x = a \sin \theta$	$1 - \sin^2 \theta = \cos^2 \theta$	$a \cos \theta$
$\sqrt{a^2 + x^2}$	$x = a \tan \theta$	$1 + \tan^2 \theta = \sec^2 \theta$	$a \sec \theta$
$\sqrt{x^2 - a^2}$	$x = a \sec \theta$	$\sec^2 \theta - 1 = \tan^2 \theta$	$a \tan \theta$

2. Replace all factors of the integral with expressions in terms of  $\theta$ , thus obtaining a trigonometric integral.
3. Integrate the trigonometric integral.
4. Express the answer in terms of the original variable  $x$ . The values of the different trigonometric functions may be read from a suitable reference triangle.

### EXAMPLE 1 For $\sqrt{a^2 - x^2}$ , let $x = a \sin \theta$

Integrate  $\int \frac{dx}{x^2 \sqrt{1 - x^2}}$ .

If we let  $x = \sin \theta$ , then  $\sqrt{1 - \sin^2 \theta} = \cos \theta$ , and the integral can be transformed into a trigonometric integral. Carefully **replacing all factors of the integral with expressions in terms of  $\theta$** , we have  $x = \sin \theta$ ,  $\sqrt{1 - x^2} = \cos \theta$ , and  $dx = \cos \theta d\theta$ . Therefore,

$$\begin{aligned} \int \frac{dx}{x^2 \sqrt{1 - x^2}} &= \int \frac{\cos \theta d\theta}{\sin^2 \theta \sqrt{1 - \sin^2 \theta}} && \text{substituting} \\ &= \int \frac{\cos \theta d\theta}{\sin^2 \theta \cos \theta} = \int \csc^2 \theta d\theta && \text{using trigonometric relations} \\ &= -\cot \theta + C && \text{using Eq. (28.7)} \end{aligned}$$

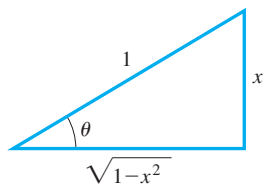


Fig. 28.11

Making a triangle with an angle  $\theta$  such that  $\sin \theta = x/1$  (see Fig. 28.11), we may express any of the trigonometric functions in terms of  $x$ . (This is the method used with inverse trigonometric functions.) Thus,

$$\cot \theta = \frac{\sqrt{1 - x^2}}{x}$$

Therefore, the result of the integration becomes

$$\int \frac{dx}{x^2 \sqrt{1 - x^2}} = -\cot \theta + C = -\frac{\sqrt{1 - x^2}}{x} + C \quad \blacksquare$$

**EXAMPLE 2** For  $\sqrt{a^2 + x^2}$ , let  $x = a \tan \theta$ 

Integrate  $\int \frac{dx}{\sqrt{x^2 + 4}}$ .

If we let  $x = 2 \tan \theta$ , the radical in this integral becomes

$$\sqrt{x^2 + 4} = \sqrt{4 \tan^2 \theta + 4} = 2\sqrt{\tan^2 \theta + 1} = 2\sqrt{\sec^2 \theta} = 2 \sec \theta$$

Therefore, with  $x = 2 \tan \theta$  and  $dx = 2 \sec^2 \theta d\theta$ , we have

$$\begin{aligned} \int \frac{dx}{\sqrt{x^2 + 4}} &= \int \frac{2 \sec^2 \theta d\theta}{\sqrt{4 \tan^2 \theta + 4}} = \int \frac{2 \sec^2 \theta d\theta}{2 \sec \theta} && \text{substituting} \\ &= \int \sec \theta d\theta = \ln |\sec \theta + \tan \theta| + C && \text{using Eq. (28.12)} \\ &= \ln \left| \frac{\sqrt{x^2 + 4}}{2} + \frac{x}{2} \right| + C = \ln \left| \frac{\sqrt{x^2 + 4} + x}{2} \right| + C && \text{see Fig. (28.12)} \end{aligned}$$

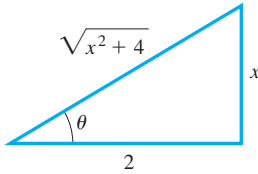


Fig. 28.12

This answer is acceptable, but by using the properties of logarithms, we have

$$\begin{aligned} \ln \left| \frac{\sqrt{x^2 + 4} + x}{2} \right| + C &= \ln |\sqrt{x^2 + 4} + x| + (C - \ln 2) \\ &= \ln |\sqrt{x^2 + 4} + x| + C' \end{aligned}$$

■ Combining constants, as in  $C' = C - \ln 2$ , is a common practice in integration problems.

**EXAMPLE 3** For  $\sqrt{x^2 - a^2}$ , let  $x = a \sec \theta$ 

Integrate  $\int \frac{2 dx}{x\sqrt{x^2 - 9}}$ .

If we let  $x = 3 \sec \theta$ , the radical in this integral becomes

$$\sqrt{x^2 - 9} = \sqrt{9 \sec^2 \theta - 9} = 3\sqrt{\sec^2 \theta - 1} = 3\sqrt{\tan^2 \theta} = 3 \tan \theta$$

Therefore, with  $x = 3 \sec \theta$  and  $dx = 3 \sec \theta \tan \theta d\theta$ , we have

$$\begin{aligned} \int \frac{2 dx}{x\sqrt{x^2 - 9}} &= 2 \int \frac{3 \sec \theta \tan \theta d\theta}{3 \sec \theta \sqrt{9 \sec^2 \theta - 9}} = 2 \int \frac{\tan \theta d\theta}{3 \tan \theta} \\ &= \frac{2}{3} \int d\theta = \frac{2}{3} \theta + C = \frac{2}{3} \sec^{-1} \frac{x}{3} + C \end{aligned}$$

It is not necessary to refer to a triangle to express the result in terms of  $x$ . The solution is found by solving  $x = 3 \sec \theta$  for  $\theta$ , as indicated. ■

**EXAMPLE 4** Trigonometric substitution—application

The joint between two links of a robot arm moves back and forth along the curve defined by  $y = 3.0 \ln x$  from  $x = 1.0$  cm to  $x = 4.0$  cm. Find the distance the joint moves in one cycle.

The required distance is found by use of the equation for the length of arc, Eq. (26.29). Therefore, we find the derivative as  $dy/dx = 3.0/x$ , which means the total distance  $s$  moved in one cycle is

$$s = 2 \int_{1.0}^{4.0} \sqrt{1 + \left(\frac{3.0}{x}\right)^2} dx = 2 \int_{1.0}^{4.0} \frac{\sqrt{x^2 + 9.0}}{x} dx$$

To integrate, we make the substitution  $x = 3.0 \tan \theta$  and  $dx = 3.0 \sec^2 \theta d\theta$ . Thus,

For reference, Eq. (26.29) is

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$\begin{aligned} \int \frac{\sqrt{x^2 + 9.0}}{x} dx &= \int \frac{\sqrt{9.0 \tan^2 \theta + 9.0}}{3.0 \tan \theta} (3.0 \sec^2 \theta d\theta) = 3.0 \int \frac{\sec \theta}{\tan \theta} (1 + \tan^2 \theta) d\theta \\ &= 3.0 \left( \int \frac{\sec \theta}{\tan \theta} d\theta + \int \tan \theta \sec \theta d\theta \right) = 3.0 \left( \int \csc \theta d\theta + \int \tan \theta \sec \theta d\theta \right) \quad \text{see Fig. 28.13} \\ &= 3.0 (\ln |\csc \theta - \cot \theta| + \sec \theta) + C = 3.0 \left[ \ln \left| \frac{\sqrt{x^2 + 9.0}}{x} - \frac{3.0}{x} \right| + \frac{\sqrt{x^2 + 9.0}}{3.0} \right] + C \end{aligned}$$

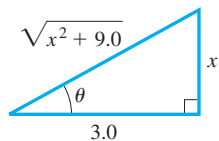


Fig. 28.13

Limits have not been included, due to the change in variables. Evaluating, we have

$$\begin{aligned} s &= 2 \int_{1.0}^{4.0} \frac{\sqrt{x^2 + 9.0}}{x} dx = 6.0 \left[ \ln \left| \frac{\sqrt{x^2 + 9.0}}{x} - \frac{3.0}{x} \right| + \frac{\sqrt{x^2 + 9.0}}{3.0} \right] \Bigg|_{1.0}^{4.0} \\ &= 6.0 \left[ \left( \ln 0.50 - \ln \frac{\sqrt{10.0} - 3.0}{1.0} \right) + \frac{5.0}{3.0} - \frac{\sqrt{10.0}}{3.0} \right] = 10.4 \text{ cm} \quad \blacksquare \end{aligned}$$

For easy reference, we rewrite the radical forms and their appropriate trigonometric substitutions here:

### Basic Trigonometric Substitutions

For $\sqrt{a^2 - x^2}$ ,	use $x = a \sin \theta$	(28.23)
For $\sqrt{a^2 + x^2}$ ,	use $x = a \tan \theta$	
For $\sqrt{x^2 - a^2}$ ,	use $x = a \sec \theta$	

## EXERCISES 28.8

In Exercises 1 and 2, answer the given equations related to the indicated examples of this section.

- In Example 1, how must the integrand be changed in order to have a result of  $\sin^{-1} x + C$ ?
- In Example 2, how must the integrand be changed in order to have a result of  $\tan^{-1}(x/2) + C$ ?

In Exercises 3–8, give the proper trigonometric substitution and find the transformed integral, but do not integrate.

- $\int \frac{\sqrt{9 - x^2}}{x^2} dx$
- $\int \frac{dx}{\sqrt{x^2 - 16}}$
- $\int \frac{dx}{x^2 \sqrt{x^2 + 1}}$
- $\int \frac{dx}{(1 - x^2)^{3/2}}$
- $\int \frac{dx}{x \sqrt{x^2 - 1}}$
- $\int \sqrt{25 + x^2} dx$

In Exercises 9–24, integrate each of the given functions.

- $\int \frac{\sqrt{1 - x^2}}{x^2} dx$
- $\int_0^4 \frac{dt}{(t^2 + 9)^{3/2}}$
- $\int \frac{2 dx}{\sqrt{x^2 - 4}}$
- $\int \frac{\sqrt{x^2 - 25}}{x} dx$
- $\int \frac{6 dz}{z^2 \sqrt{z^2 + 9}}$
- $\int \frac{3 dx}{x \sqrt{4 - x^2}}$
- $\int \frac{4 dx}{(4 - x^2)^{3/2}}$
- $\int \frac{6p^3 dp}{\sqrt{9 + p^2}}$
- $\int_0^{0.5} \frac{x^3 dx}{\sqrt{1 - x^2}}$
- $\int_4^5 \frac{\sqrt{x^2 - 16}}{x^2} dx$

- $\int \frac{5 dx}{\sqrt{x^2 + 2x + 2}}$
- $\int_{2.5}^3 \frac{dy}{y \sqrt{4y^2 - 9}}$
- $\int \frac{2 dx}{\sqrt{e^{2x} - 1}}$
- $\int \frac{12 \sec^2 u du}{(4 - \tan^2 u)^{3/2}}$
- $\int \frac{dx}{\sqrt{x^2 + 2x}}$
- $\int \sqrt{16 - x^2} dx$

In Exercises 25–36, solve the given problems by integration.

- Perform the integration  $\int x \sqrt{1 - x^2} dx$  (a) by using the power formula, and (b) by trigonometric substitution. Compare results.
- Perform the integration  $\int \frac{x dx}{x^2 + 1}$  (a) by using the logarithmic formula, and (b) by trigonometric substitution. Compare results.
- Find the area of a quarter circle of radius 2. See Exercise 18 of Section 25.5.
- Find the area bounded by  $y = \frac{1}{x^2 \sqrt{x^2 - 1}}$ ,  $x = \sqrt{2}$ ,  $x = \sqrt{5}$ , and  $y = 0$ . See Fig. 28.14.

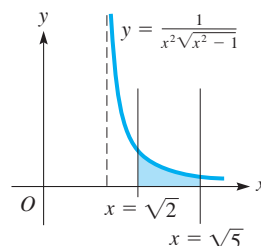


Fig. 28.14



29. Find the moment of inertia with respect to the  $y$ -axis of a flat plate covering the first-quadrant region under the circle  $x^2 + y^2 = a^2$  in terms of its mass.
30. Find the moment of inertia of a sphere of radius  $a$  with respect to its axis in terms of its mass.
31. Find the volume generated by revolving the region bounded by  $y = \frac{\sqrt{x^2 - 16}}{x^2}$ ,  $y = 0$ , and  $x = 5$  about the  $y$ -axis.
32. The perimeter of the rudder of a boat can be described as the region bound by  $y = -0.5x^2\sqrt{4 - x^2}$  and the  $x$ -axis. Find the area of one side of the rudder.
33. Find the  $x$ -coordinate of the centroid of the boat rudder in Exercise 32.
34. The vertical cross section of a highway culvert is defined by the region within the ellipse  $1.00x^2 + 9.00y^2 = 9.00$ , where dimensions are in metres. Find the area of the cross section of the culvert.
35. If an electric charge  $Q$  is distributed along a straight wire of length  $2a$ , the electric potential  $V$  at a point  $P$ , which is at a distance  $b$  from the centre of the wire, is  $V = kQ \int_{-a}^a \frac{dx}{\sqrt{b^2 + x^2}}$ .

Here,  $k$  is a constant and  $x$  is the distance along the wire. Evaluate the integral.

36. An electric insulating ring for a machine part can be described as the volume generated by revolving the region bounded by  $y = x^2\sqrt{x^2 - 4}$ ,  $y = 0$ , and  $x = 2.5$  cm about the  $y$ -axis. Find the volume (in  $\text{cm}^3$ ) of material in the ring.

*Certain algebraic integrals can be transformed into integrable form with the appropriate algebraic substitution. For an expression of the form  $(ax + b)^{p/q}$ , a substitution of the form  $u = (ax + b)^{1/q}$  may put it into an integrable form. In Exercises 37–40, use this type of substitution for the given integrals.*

37.  $\int x\sqrt{x+1} dx$                       38.  $\int x\sqrt[3]{8-x} dx$
39.  $\int x(x-4)^{2/3} dx$                       40.  $\int \frac{x^2 dx}{(4x+1)^{5/2}}$

## 28.9 Integration by Partial Fractions: Nonrepeated Linear Factors

**Method of Partial Fractions •**  
**Nonrepeated Linear Factors •**  
**Solution by Substitution •**  
**Solution by Equating Coefficients**

We have seen how the *derivative of a product* and *trigonometric identities* are used to write integrals into a form that can be integrated. In this section and the next, we show an algebraic method by which integrands that are fractions can also be changed into an integrable form.

In algebra, we combine fractions into a single fraction by means of addition. However, if we wish to integrate an expression that contains rational fractions, in which both numerator and denominator are polynomials, it is often advantageous to reverse the process of addition and express the rational fraction as the sum of simpler fractions.

### EXAMPLE 1 Illustrating partial fractions

In attempting to integrate  $\int \frac{7-x}{x^2+x-2} dx$ , we find that it does not fit any of the standard forms in this chapter. However, we can show that

$$\frac{7-x}{x^2+x-2} = \frac{2}{x-1} - \frac{3}{x+2}$$

This means that

$$\begin{aligned} \int \frac{7-x}{x^2+x-2} dx &= \int \frac{2 dx}{x-1} - \int \frac{3 dx}{x+2} \\ &= 2 \ln|x-1| - 3 \ln|x+2| + C \end{aligned}$$

We see that by writing the fraction in the original integrand as the sum of the simpler fractions, each resulting integrand can be integrated. ■

In Example 1, we saw that the integral is readily determined once the rational fraction  $(7-x)/(x^2+x-2)$  is replaced by the simpler fractions. In this section and the next, we describe how certain rational fractions can be expressed in terms of simpler fractions and thereby be integrated. *This technique is called the method of partial fractions.*

**LEARNING TIP**

In order to express the rational fraction  $f(x)/g(x)$  in terms of simpler partial fractions, the degree of the numerator  $f(x)$  must be less than that of the denominator  $g(x)$ . If this is not the case, we divide numerator by denominator until the remainder is of the proper form. Then the denominator  $g(x)$  is factored into a product of linear and quadratic factors. The method of determining the partial fractions depends on the factors that are obtained.

■ The method of partial fractions essentially reverses the process of combining fractions over a common denominator. By determining that

$$\frac{7-x}{x^2+x-2} = \frac{2}{x-1} - \frac{3}{x+2}$$

reverses the process by which

$$\begin{aligned} \frac{2}{x-1} - \frac{3}{x+2} &= \frac{2(x+2) - 3(x-1)}{(x-1)(x+2)} \\ &= \frac{7-x}{x^2+x-2} \end{aligned}$$

**Practice Exercise**

1. Find the partial fractions for

$$\frac{3x+11}{x^2-2x-3}$$

There are four cases for the types of factors of the denominator. They are (1) *nonrepeated linear factors*, (2) *repeated linear factors*, (3) *nonrepeated quadratic factors*, and (4) *repeated quadratic factors*. In this section, we consider the case of nonrepeated linear factors, and the other cases are discussed in the next section.

**NONREPEATED LINEAR FACTORS**

For the case of nonrepeated linear factors, we use the fact that *corresponding to each linear factor  $ax + b$ , occurring once in the denominator, there will be a partial fraction of the form*

$$\frac{A}{ax+b}$$

where  $A$  is a constant to be determined. The following examples illustrate the method.

**EXAMPLE 2 Integration by partial fractions**

Integrate  $\int \frac{7-x}{x^2+x-2} dx$ . (This is the same integral as in Example 1. Here, we see how the partial fractions are found.)

First, we note that the degree of the numerator is 1 (the highest-power term is  $x$ ) and that of the denominator is 2 (the highest-power term is  $x^2$ ). Since the degree of the denominator is higher, we may proceed to factoring it. Thus,

$$\frac{7-x}{x^2+x-2} = \frac{7-x}{(x-1)(x+2)}$$

There are two linear factors,  $(x-1)$  and  $(x+2)$ , in the denominator, and they are different. This means that there are two partial fractions. Therefore, we write

$$\frac{7-x}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2} \quad (1)$$

We are to determine constants  $A$  and  $B$  so that Eq. (1) is an identity. In finding  $A$  and  $B$ , we clear Eq. (1) of fractions by multiplying both sides by  $(x-1)(x+2)$ .

$$7-x = A(x+2) + B(x-1) \quad (2)$$

Eq. (2) is also an identity, which means that there are two ways of determining the values of  $A$  and  $B$ .

*Solution by substitution:* Since Eq. (2) is an identity, *it is true for any value of  $x$* . Thus, in turn we pick  $x = -2$  and  $x = 1$ , for each of these values makes a factor on the right equal to zero, and the values of  $B$  and  $A$  are easily found. Therefore,

$$\text{For } x = -2: \quad 7 - (-2) = A(-2 + 2) + B(-2 - 1)$$

$$9 = -3B \quad B = -3$$

$$\text{For } x = 1: \quad 7 - 1 = A(1 + 2) + B(1 - 1)$$

$$6 = 3A, \quad A = 2$$

*Solution by equating coefficients:* Since Eq. (2) is an identity, another way of finding the constants  $A$  and  $B$  is to *equate coefficients of like powers of  $x$  from each side*. Thus, writing Eq. (2) as

$$7 - x = (2A - B) + (A + B)x$$

we have

$$2A - B = 7 \quad \text{equating constants: } x^0 \text{ terms}$$

$$A + B = -1 \quad \text{equating coefficients of } x$$

$$\begin{array}{r} \blacksquare \quad 2A - B = 7 \\ \quad \quad A + B = -1 \\ \hline \quad \quad 3A = 6 \\ \quad \quad A = 2 \\ 2(2) - B = 7 \\ \quad \quad -B = 3 \\ \quad \quad B = -3 \end{array}$$

Now, using the values  $A = 2$  and  $B = -3$  (as found at the left), we have

$$\frac{7-x}{(x-1)(x+2)} = \frac{2}{x-1} - \frac{3}{x+2}$$

Therefore, the integral is found as in Example 1.

$$\begin{aligned} \int \frac{7-x}{x^2+x-2} dx &= \int \frac{7-x}{(x-1)(x+2)} dx = \int \frac{2 dx}{x-1} - \int \frac{3 dx}{x+2} \\ &= 2 \ln|x-1| - 3 \ln|x+2| + C \end{aligned}$$

Using the properties of logarithms, we may write this as

$$\int \frac{7-x}{x^2+x-2} dx = \ln \left| \frac{(x-1)^2}{(x+2)^3} \right| + C \quad \blacksquare$$

### EXAMPLE 3 Integration by partial fractions

Integrate  $\int \frac{6x^2 - 14x - 11}{(x+1)(x-2)(2x+1)} dx$ .

The denominator is factored and is of degree 3 (when multiplied out, the highest-power term of  $x$  is  $x^3$ ). This means we have three nonrepeated linear factors.

$$\frac{6x^2 - 14x - 11}{(x+1)(x-2)(2x+1)} = \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{2x+1}$$

Multiplying through by  $(x+1)(x-2)(2x+1)$ , we have

$$6x^2 - 14x - 11 = A(x-2)(2x+1) + B(x+1)(2x+1) + C(x+1)(x-2)$$

We now substitute the values of 2,  $-\frac{1}{2}$ , and  $-1$  for  $x$ . Again, these are chosen because they make factors of the coefficients of  $A$ ,  $B$ , or  $C$  equal to zero, although any values may be chosen. Therefore,

$$\text{For } x = 2: \quad 6(4) - 14(2) - 11 = A(0)(5) + B(3)(5) + C(3)(0), \quad B = -1$$

$$\text{For } x = -\frac{1}{2}: \quad 6\left(\frac{1}{4}\right) - 14\left(-\frac{1}{2}\right) - 11 = A\left(-\frac{5}{2}\right)(0) + B\left(\frac{1}{2}\right)(0) + C\left(\frac{1}{2}\right)\left(-\frac{5}{2}\right), \quad C = 2$$

$$\text{For } x = -1: \quad 6(1) - 14(-1) - 11 = A(-3)(-1) + B(0)(-1) + C(0)(-3), \quad A = 3$$

Therefore,

$$\begin{aligned} \int \frac{6x^2 - 14x - 11}{(x+1)(x-2)(2x+1)} dx &= \int \frac{3 dx}{x+1} - \int \frac{dx}{x-2} + \int \frac{2 dx}{2x+1} \\ &= 3 \ln|x+1| - \ln|x-2| + \ln|2x+1| + C_1 = \ln \left| \frac{(2x+1)(x+1)^3}{x-2} \right| + C_1 \end{aligned}$$

Here, we have let the constant of integration be  $C_1$  since we used  $C$  as the numerator of the third partial fraction.  $\blacksquare$

### EXAMPLE 4 Integration by partial fractions

Integrate  $\int \frac{2x^4 - x^3 - 9x^2 + x - 12}{x^3 - x^2 - 6x} dx$ .

Since the numerator is of a higher degree than the denominator, we must first divide the numerator by the denominator. This gives

$$\frac{2x^4 - x^3 - 9x^2 + x - 12}{x^3 - x^2 - 6x} = 2x + 1 + \frac{4x^2 + 7x - 12}{x^3 - x^2 - 6x}$$

$\blacksquare$  The partial fractions corresponding to a quotient of polynomials can be found using the *expand* feature on some graphing calculators.

We must now express this rational fraction in terms of its partial fractions.

$$\frac{4x^2 + 7x - 12}{x^3 - x^2 - 6x} = \frac{4x^2 + 7x - 12}{x(x+2)(x-3)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-3}$$

Clearing fractions, we have

$$4x^2 + 7x - 12 = A(x+2)(x-3) + Bx(x-3) + Cx(x+2)$$

Now, using values of  $x$  of  $-2$ ,  $3$ , and  $0$  for substitution, we obtain the values of  $B = -1$ ,  $C = 3$ , and  $A = 2$ , respectively. Therefore,

$$\begin{aligned} \int \frac{2x^4 - x^3 - 9x^2 + x - 12}{x^3 - x^2 - 6x} dx &= \int \left( 2x + 1 + \frac{2}{x} - \frac{1}{x+2} + \frac{3}{x-3} \right) dx \\ &= x^2 + x + 2 \ln|x| - \ln|x+2| + 3 \ln|x-3| + C_1 \\ &= x^2 + x + \ln \left| \frac{x^2(x-3)^3}{x+2} \right| + C_1 \quad \blacksquare \end{aligned}$$

In Example 2, we showed two ways of finding the values of  $A$  and  $B$ . The method of substitution is generally easier to use with linear factors, and we used it in Examples 3 and 4. However, as we will see in the next section, the method of equating coefficients can be very useful for other cases with partial fractions.

## EXERCISES 28.9

In Exercises 1 and 2, make the given changes in the integrands of the indicated examples of this section, and then find the resulting fractions to be used in the integration. Do not integrate.

- In Example 2, change the numerator to  $10 - x$ .
- In Example 3, change the numerator to  $x^2 - 12x - 10$ .

In Exercises 3–6, write out the form of the partial fractions, similar to that shown in Eq. (1) of Example 2, that would be used to perform the indicated integrations. Do not evaluate the constants.

- $\int \frac{3x+2}{x^2+x} dx$
- $\int \frac{9-x}{x^2+2x-3} dx$
- $\int \frac{x^2-6x-8}{x^3-4x} dx$
- $\int \frac{2x^2-5x-7}{x^3+2x^2-x-2} dx$

In Exercises 7–24, integrate each of the given functions.

- $\int \frac{x+3}{(x+1)(x+2)} dx$
- $\int \frac{x+2}{x(x+1)} dx$
- $\int \frac{dx}{x^2-4}$
- $\int \frac{p-9}{2p^2-3p+1} dp$
- $\int \frac{x^2+3}{x^2+3x} dx$
- $\int \frac{x^3}{x^2+3x+2} dx$
- $\int_0^1 \frac{2t+4}{3t^2+5t+2} dt$
- $\int_1^3 \frac{x-1}{4x^2+x} dx$
- $\int \frac{4x^2-10}{x(x+1)(x-5)} dx$
- $\int \frac{4x^2+21x+6}{(x+2)(x-3)(x+4)} dx$
- $\int \frac{6x^2-2x-1}{4x^3-x} dx$
- $\int_2^3 \frac{dR}{R^3-R}$

$$19. \int_1^2 \frac{x^3+7x^2+9x+2}{x(x^2+3x+2)} dx$$

$$20. \int \frac{2x^3+x-1}{x^3+x^2-4x-4} dx$$

$$21. \int \frac{dV}{(V^2-4)(V^2-9)}$$

$$22. \int \frac{5x^3-2x^2-15x+24}{x^4-2x^3-11x^2+12x} dx$$

$$23. \int \frac{2x dx}{x^4-3x^3+2x^2}$$

$$24. \int \frac{e^x dx}{e^{2x}+3e^x+2}$$

In Exercises 25–34, solve the given problems by integration.

$$25. \text{ Derive the general formula } \int \frac{du}{u(a+bu)} = -\frac{1}{a} \ln \frac{a+bu}{u} + C.$$

$$26. \text{ Derive the general formula } \int \frac{du}{u^2-a^2} = \frac{1}{2a} \ln \frac{u-a}{u+a} + C.$$

$$27. \text{ Integrate } \int \frac{\cos \theta}{\sin^2 \theta + 2 \sin \theta - 3} d\theta. \text{ (Hint: First, find the partial fractions for } 1/(\sin^2 \theta + 2 \sin \theta - 3)\text{.)}$$

$$28. \text{ Find the first-quadrant area bounded by } y = 1/(x^3 + 3x^2 + 2x), x = 1, \text{ and } x = 3.$$

$$29. \text{ Find the volume generated if the region of Exercise 28 is revolved about the } y\text{-axis.}$$

$$30. \text{ Find the } x\text{-coordinate of the centroid of a flat plate that covers the region bounded by } y(x^2 - 1) = 1, y = 0, x = 2, \text{ and } x = 4.$$

$$31. \text{ The general expression for the slope of a curve is } (3x+5)/(x^2+5x). \text{ Find the equation of the curve if it passes through } (1, 0).$$

32. The current  $i$  (in A) as a function of the time  $t$  (in s) in a certain electric circuit is given by  $i = (4t + 3)/(2t^2 + 3t + 1)$ . Find the total charge that passes a given point in the circuit during the first second.
33. The force  $F$  (in N) applied by a stamping machine in making a certain computer part is  $F = 4x/(x^2 + 3x + 2)$ , where  $x$  is the distance (in cm) through which the force acts. Find the work done by the force from  $x = 0$  to  $x = 0.500$  cm.
34. Under specified conditions, the time  $t$  (in min) required to form  $x$  grams of a substance during a chemical reaction is given by  $t = \int dx/[ (4 - x)(2 - x) ]$ . Find the equation relating  $t$  and  $x$  if  $x = 0$  g when  $t = 0$  min.

#### Answer to Practice Exercise

$$1. \frac{3x+11}{x^2-2x-3} = \frac{5}{x-3} - \frac{2}{x+1}$$

## 28.10 Integration by Partial Fractions: Other Cases

Repeated Linear Factors •  
The Only Factor Is Repeated •  
Nonrepeated Quadratic Factors •  
Repeated Quadratic Factors

In the previous section, we introduced the method of partial fractions and considered the case of nonrepeated linear factors. In this section, we develop the use of partial fractions for the cases of repeated linear factors and nonrepeated quadratic factors. We also briefly discuss the case of repeated quadratic factors.

### REPEATED LINEAR FACTORS

For repeated linear factors, we use the fact that *corresponding to each linear factor  $ax + b$  that occurs  $n$  times in the denominator there will be  $n$  partial fractions*

$$\frac{A_1}{ax + b} + \frac{A_2}{(ax + b)^2} + \cdots + \frac{A_n}{(ax + b)^n}$$

where  $A_1, A_2, \dots, A_n$  are constants to be determined.

#### EXAMPLE 1 Two factors—one repeated

Integrate  $\int \frac{dx}{x(x+3)^2}$ .

Here, we see that the denominator has a factor of  $x$  and two factors of  $x + 3$ . For the factor of  $x$ , we use a partial fraction as in the previous section, for it is a nonrepeated factor. For the factor  $x + 3$ , we need two partial fractions, one with a denominator of  $x + 3$  and the other with a denominator of  $(x + 3)^2$ . Thus, we write

$$\frac{1}{x(x+3)^2} = \frac{A}{x} + \frac{B}{x+3} + \frac{C}{(x+3)^2}$$

Multiplying each side by  $x(x+3)^2$ , we have

$$1 = A(x+3)^2 + Bx(x+3) + Cx \quad (1)$$

Using the values of  $x$  of  $-3$  and  $0$ , we have

$$\text{For } x = -3: \quad 1 = A(0^2) + B(-3)(0) + (-3)C, \quad C = -\frac{1}{3}$$

$$\text{For } x = 0: \quad 1 = A(3^2) + B(0)(3) + C(0), \quad A = \frac{1}{9}$$

Since no other numbers make a factor in Eq. (1) equal to zero, we either choose some other value of  $x$  or equate coefficients of some power of  $x$  in Eq. (1). Since Eq. (1) is an identity, we may choose any value of  $x$ . With  $x = 1$ , we have

$$\begin{aligned} 1 &= A(4^2) + B(1)(4) + C(1) \\ 1 &= 16A + 4B + C \end{aligned}$$

#### Practice Exercise

1. Find the partial fractions for  $\frac{2}{x(x-1)^2}$ .

Using the known values of  $A$  and  $C$ , we have

$$1 = 16\left(\frac{1}{9}\right) + 4B - \frac{1}{3}, \quad B = -\frac{1}{9}$$

This means that

$$\frac{1}{x(x+3)^2} = \frac{\frac{1}{9}}{x} + \frac{-\frac{1}{9}}{x+3} + \frac{-\frac{1}{3}}{(x+3)^2}$$

or

$$\begin{aligned} \int \frac{dx}{x(x+3)^2} &= \frac{1}{9} \int \frac{dx}{x} - \frac{1}{9} \int \frac{dx}{x+3} - \frac{1}{3} \int \frac{dx}{(x+3)^2} \\ &= \frac{1}{9} \ln|x| - \frac{1}{9} \ln|x+3| - \frac{1}{3} \left(\frac{1}{-1}\right) (x+3)^{-1} + C_1 \\ &= \frac{1}{9} \ln \left| \frac{x}{x+3} \right| + \frac{1}{3(x+3)} + C_1 \quad \blacksquare \end{aligned}$$

### EXAMPLE 2 Two factors—one repeated

Integrate  $\int \frac{3x^3 + 15x^2 + 21x + 15}{(x-1)(x+2)^3} dx$ .

First, we set up the partial fractions as

$$\frac{3x^3 + 15x^2 + 21x + 15}{(x-1)(x+2)^3} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2} + \frac{D}{(x+2)^3}$$

Next, we clear fractions:

$$3x^3 + 15x^2 + 21x + 15 = A(x+2)^3 + B(x-1)(x+2)^2 + C(x-1)(x+2) + D(x-1) \quad (1)$$

$$\text{For } x = 1: \quad 3 + 15 + 21 + 15 = 27A, \quad 54 = 27A, \quad A = 2$$

$$\text{For } x = -2: \quad 3(-8) + 15(4) + 21(-2) + 15 = -3D, \quad 9 = -3D, \quad D = -3$$

To find  $B$  and  $C$ , we equate coefficients of powers of  $x$ . Therefore, we write Eq. (1) as

$$3x^3 + 15x^2 + 21x + 15 = (A+B)x^3 + (6A+3B+C)x^2 + (12A+C+D)x + (8A-4B-2C-D)$$

$$\text{Coefficients of } x^3: \quad 3 = A + B, \quad 3 = 2 + B, \quad B = 1$$

$$\text{Coefficients of } x^2: \quad 15 = 6A + 3B + C, \quad 15 = 12 + 3 + C, \quad C = 0$$

$$\frac{3x^3 + 15x^2 + 21x + 15}{(x-1)(x+2)^3} = \frac{2}{x-1} + \frac{1}{x+2} + \frac{0}{(x+2)^2} + \frac{-3}{(x+2)^3}$$

$$\begin{aligned} \int \frac{3x^3 + 15x^2 + 21x + 15}{(x-1)(x+2)^3} dx &= 2 \int \frac{dx}{x-1} + \int \frac{dx}{x+2} - 3 \int \frac{dx}{(x+2)^3} \\ &= 2 \ln|x-1| + \ln|x+2| - 3 \left(\frac{1}{-2}\right) (x+2)^{-2} + C_1 \\ &= \ln|(x-1)^2(x+2)| + \frac{3}{2(x+2)^2} + C_1 \quad \blacksquare \end{aligned}$$

If there is one repeated factor in the denominator and it is the only factor present in the denominator, a substitution is easier and more convenient than using partial fractions. This is illustrated in the following example.

**EXAMPLE 3** The only factor is repeated

Integrate  $\int \frac{x \, dx}{(x-2)^3}$ .

This could be integrated by first setting up the appropriate partial fractions. However, the solution is more easily found by using the substitution  $u = x - 2$ . Using this, we have

$$\begin{aligned} u = x - 2 \quad x = u + 2 \quad dx = du \\ \int \frac{x \, dx}{(x-2)^3} &= \int \frac{(u+2)(du)}{u^3} = \int \frac{du}{u^2} + 2 \int \frac{du}{u^3} \\ &= \int u^{-2} \, du + 2 \int u^{-3} \, du = \frac{1}{-u} + \frac{2}{-2} u^{-2} + C \\ &= -\frac{1}{u} - \frac{1}{u^2} + C = -\frac{u+1}{u^2} + C \\ &= -\frac{x-2+1}{(x-2)^2} + C = \frac{1-x}{(x-2)^2} + C \end{aligned}$$

**NONREPEATED QUADRATIC FACTORS**

For the case of nonrepeated quadratic factors, we use the fact that *corresponding to each irreducible quadratic factor  $ax^2 + bx + c$  that occurs once in the denominator there is a partial fraction of the form*

$$\frac{Ax + B}{ax^2 + bx + c}$$

where  $A$  and  $B$  are constants to be determined. (Here, an *irreducible quadratic factor* is one that cannot be further factored into linear factors involving only real numbers.) This case is illustrated in the following examples.

**EXAMPLE 4** Two factors—one quadratic

Integrate  $\int \frac{4x + 4}{x^3 + 4x} \, dx$ .

In setting up the partial fractions, we note that the denominator factors as  $x^3 + 4x = x(x^2 + 4)$ . Here, the factor  $x^2 + 4$  cannot be further factored. This means we have

$$\frac{4x + 4}{x^3 + 4x} = \frac{4x + 4}{x(x^2 + 4)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4}$$

Clearing fractions, we have

$$\begin{aligned} 4x + 4 &= A(x^2 + 4) + Bx^2 + Cx \\ &= (A + B)x^2 + Cx + 4A \end{aligned}$$

Equating coefficients of powers of  $x$  gives us

$$\text{For } x^2: \quad 0 = A + B$$

$$\text{For } x: \quad 4 = C$$

$$\text{For constants: } 4 = 4A, \quad A = 1$$

**Practice Exercise**

2. Find the partial fractions for  $\frac{x^2 - 2}{x^3 + 2x}$ .



Therefore, we easily find that  $B = -1$  from the first equation. This means that

$$\frac{4x + 4}{x^3 + 4x} = \frac{1}{x} + \frac{-x + 4}{x^2 + 4}$$

and

$$\begin{aligned} \int \frac{4x + 4}{x^3 + 4x} dx &= \int \frac{1}{x} dx + \int \frac{-x + 4}{x^2 + 4} dx \\ &= \int \frac{1}{x} dx - \int \frac{x dx}{x^2 + 4} + \int \frac{4 dx}{x^2 + 4} \\ &= \ln|x| - \frac{1}{2} \ln|x^2 + 4| + 2 \tan^{-1} \frac{x}{2} + C_1 \end{aligned}$$

We could use the properties of logarithms to combine the first two terms of the answer. Doing so, we have the following result:

$$\int \frac{4x + 4}{x^3 + 4x} dx = \ln \frac{|x|}{\sqrt{x^2 + 4}} + 2 \tan^{-1} \frac{x}{2} + C_1 \quad \blacksquare$$

### EXAMPLE 5 One quadratic factor and a repeated linear factor

Integrate  $\int \frac{x^3 + 3x^2 + 2x + 4}{x^2(x^2 + 2x + 2)} dx$ .

In the denominator, we have a repeated linear factor,  $x^2$ , and a quadratic factor. Therefore,

$$\frac{x^3 + 3x^2 + 2x + 4}{x^2(x^2 + 2x + 2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{x^2 + 2x + 2}$$

$$\begin{aligned} x^3 + 3x^2 + 2x + 4 &= Ax(x^2 + 2x + 2) + B(x^2 + 2x + 2) + Cx^3 + Dx^2 \\ &= (A + C)x^3 + (2A + B + D)x^2 + (2A + 2B)x + 2B \end{aligned}$$

Equating coefficients, we find that

For constants:	$2B = 4,$	$B = 2$	
For $x$ :	$2A + 2B = 2,$	$A + B = 1,$	$A = -1$
For $x^2$ :	$2A + B + D = 3,$	$-2 + 2 + D = 3,$	$D = 3$
For $x^3$ :	$A + C = 1,$	$-1 + C = 1,$	$C = 2$

$$\begin{aligned} \frac{x^3 + 3x^2 + 2x + 4}{x^2(x^2 + 2x + 2)} &= -\frac{1}{x} + \frac{2}{x^2} + \frac{2x + 3}{x^2 + 2x + 2} \\ \int \frac{x^3 + 3x^2 + 2x + 4}{x^2(x^2 + 2x + 2)} dx &= -\int \frac{dx}{x} + 2 \int \frac{dx}{x^2} + \int \frac{2x + 3}{x^2 + 2x + 2} dx \\ &= -\ln|x| - 2\left(\frac{1}{x}\right) + \int \frac{2x + 2 + 1}{x^2 + 2x + 2} dx \\ &= -\ln|x| - \frac{2}{x} + \int \frac{2x + 2}{x^2 + 2x + 2} dx + \int \frac{dx}{(x^2 + 2x + 1) + 1} \\ &= -\ln|x| - \frac{2}{x} + \ln|x^2 + 2x + 2| + \tan^{-1}(x + 1) + C_1 \end{aligned}$$

*Note the manner in which the integral with the quadratic denominator was handled for the purpose of integration.* First, the numerator,  $2x + 3$ , was written in the form  $(2x + 2) + 1$  so that we could fit the logarithmic form with the  $2x + 2$ . Then we completed the square in the denominator of the final integral so that it then fit an inverse tangent form. ■

### REPEATED QUADRATIC FACTORS

Finally, considering the case of repeated quadratic factors, we use the fact that *corresponding to each irreducible quadratic factor  $ax^2 + bx + c$  that occurs  $n$  times in the denominator there will be  $n$  partial fractions*

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_nx + B_n}{(ax^2 + bx + c)^n}$$

where  $A_1, A_2, \dots, A_n, B_1, B_2, \dots, B_n$  are constants to be determined. The procedures that lead to the solution are the same as those for the other cases. Exercises 19 and 20 in this section are solved by using these partial fractions for repeated quadratic factors.

We now summarize the method of partial fractions for all four cases discussed in the last two sections.

#### Method of Partial Fractions

To integrate a rational expression of the form  $f(x)/g(x)$  by the method of partial fractions:

1. Make sure that the degree of the numerator  $f(x)$  is less than the degree of the denominator  $g(x)$ . If it is not, divide the numerator by the denominator until the remainder is of the proper form.
2. Factor the denominator as completely as possible.
3. Select the term(s) that correspond to each factor in the denominator according to the following table. Keep in mind that if there is only one factor present, integration by substitution may be more convenient.

Factor in Denominator	Term(s) in Partial Fractions Decomposition
$ax + b$	$\frac{A}{ax + b}$
$(ax + b)^n$	$\frac{A_1}{ax + b} + \frac{A_2}{(ax + b)^2} + \cdots + \frac{A_n}{(ax + b)^n}$
$ax^2 + bx + c$	$\frac{Ax + B}{ax^2 + bx + c}$
$(ax^2 + bx + c)^n$	$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_nx + B_n}{(ax^2 + bx + c)^n}$

4. Determine the values of the coefficients either by substitution of convenient values of  $x$ , or by equating coefficients of like powers of  $x$  from each side.
5. Integrate.

## EXERCISES 28.10

In Exercises 1–4, make the given changes in the integrands of the indicated examples of this section. Then write out the equation that shows the partial fractions that would be used for the integration. Do not evaluate the constants.

- In Example 1, change the numerator from 1 to 2.
- In Example 2, change the denominator to  $(x - 1)^2(x + 2)^2$ .
- In Example 4, change the denominator to  $x^3 - 9x^2$ .
- In Example 5, change the denominator to  $x^2(x^2 + 3x + 2)$ .

In Exercises 5–20, integrate each of the given functions.

- $\int \frac{x - 8}{x^3 - 4x^2 + 4x} dx$
- $\int \frac{dT}{T^3 - T^2}$
- $\int \frac{2 dx}{x^2(x^2 - 1)}$
- $\int_1^3 \frac{3x^3 + 8x^2 + 10x + 2}{x(x + 1)^3} dx$
- $\int_1^2 \frac{2s ds}{(s - 3)^3}$
- $\int \frac{x dx}{(x + 2)^4}$
- $\int \frac{x^3 - 2x^2 - 7x + 28}{(x + 1)^2(x - 3)^2} dx$
- $\int \frac{4 dx}{(x + 1)^2(x - 1)^2}$
- $\int_0^2 \frac{x^2 + x + 5}{(x + 1)(x^2 + 4)} dx$
- $\int \frac{v^2 + v - 1}{(v^2 + 1)(v - 2)} dv$
- $\int \frac{5x^2 + 8x + 16}{x^2(x^2 + 4x + 8)} dx$
- $\int \frac{2x^2 + x + 3}{(x^2 + 2)(x - 1)} dx$
- $\int \frac{10x^3 + 40x^2 + 22x + 7}{(4x^2 + 1)(x^2 + 6x + 10)} dx$
- $\int_3^4 \frac{5x^3 - 4x}{x^4 - 16} dx$
- $\int \frac{-x^3 + x^2 + x + 3}{(x + 1)(x^2 + 1)^2} dx$
- $\int \frac{2r^3}{(r^2 + 1)^2} dr$

In Exercises 21–30, solve the given problems by integration.

- For the integral of Example 3, set up the integration by partial fractions, and then integrate. Compare results with Example 3.

- Integrate  $\int \frac{\cos x dx}{\sin x + \sin^3 x}$ . (Hint: The numerator for the quadratic factor is  $B \sin x + C$ .)

- Find the area bounded by  $y = \frac{x - 3}{x^3 + x^2}$ ,  $y = 0$ , and  $x = 1$ .

- Find the first-quadrant area bounded by  $y = \frac{3x^2 + 2x + 9}{(x^2 + 9)(x + 1)}$  and  $x = 2$ .

- Find the volume generated by revolving the first-quadrant region bounded by  $y = 4/(x^4 + 6x^2 + 5)$  and  $x = 2$  about the  $y$ -axis.

- Find the volume generated by revolving the first-quadrant region bounded by  $y = x/(x + 3)^2$  and  $x = 3$  about the  $x$ -axis.

- Under certain conditions, the velocity  $v$  (in m/s) of an object moving along a straight line as a function of the time  $t$  (in s) is given by  $v = \frac{t^2 + 14t + 27}{(2t + 1)(t + 5)^2}$ . Find the distance travelled by the object during the first 2.00 s.

- By a computer analysis, the electric current  $i$  (in A) in a certain circuit is given by  $i = \frac{0.0010(7t^2 + 16t + 48)}{(t + 4)(t^2 + 16)}$ , where  $t$  is the time (in s). Find the total charge that passes a point in the circuit in the first 0.250 s.

- Find the  $x$ -coordinate of the centroid of a flat plate covering the region bounded by  $y = 4/(x^3 + x)$ ,  $x = 1$ ,  $x = 2$ , and  $y = 0$ .

- The slope of a curve is given by  $\frac{dy}{dx} = \frac{29x^2 + 36}{4x^4 + 9x^2}$ . Find the equation of the curve if it passes through  $(1, 5)$ .

## Answers to Practice Exercises

- $\frac{2}{x(x - 1)^2} = \frac{2}{x} - \frac{2}{x - 1} + \frac{2}{(x - 1)^2}$
- $\frac{x^2 - 2}{x^3 + 2x} = -\frac{1}{x} + \frac{2x}{x^2 + 2}$

## 28.11 Integration by Use of Tables

Proper Recognition of the Form, Variables, and Constants • Proper Identification of  $u$  and  $du$

## LEARNING TIP

The use of the tables depends on proper recognition of the form and the variables and constants of the integral.

In this chapter, we have introduced certain basic integrals and have also brought in some methods of reducing other integrals to these basic forms. Often, this transformation and integration requires a number of steps to be performed, and therefore integrals are tabulated for reference. The integrals found in tables have been derived by using the methods introduced thus far, as well as many other methods that can be used. Therefore, an understanding of the basic forms and some of the basic methods is very useful in finding integrals from tables. Such an understanding forms a basis for proper recognition of the forms that are used in the tables, as well as the types of results that may be expected. The following examples illustrate the use of the table of integrals found in Appendix B. More extensive tables are available in other sources.

**EXAMPLE 1** Integral fits the form of formula 6

Integrate  $\int \frac{x dx}{\sqrt{2 + 3x}}$ .

We first note that this integral fits the form of formula 6 of Appendix B, with  $u = x$ ,  $a = 2$ , and  $b = 3$ . Therefore,

$$\int \frac{x dx}{\sqrt{2 + 3x}} = -\frac{2(4 - 3x)\sqrt{2 + 3x}}{27} + C$$

**EXAMPLE 2** Integral fits the form of formula 18

Integrate  $\int \frac{\sqrt{4 - 9x^2}}{x} dx$ .

This fits the form of formula 18, with proper identification of constants;  $u = 3x$ ,  $du = 3 dx$ ,  $a = 2$ . Hence,

$$\begin{aligned} \int \frac{\sqrt{4 - 9x^2}}{x} dx &= \int \frac{\sqrt{4 - 9x^2}}{3x} 3 dx \\ &= \sqrt{4 - 9x^2} - 2 \ln\left(\frac{2 + \sqrt{4 - 9x^2}}{3x}\right) + C \end{aligned}$$

**EXAMPLE 3** Integral fits the form of formula 37

Integrate  $\int 5 \sec^3 2x dx$ .

This fits the form of formula 37;  $n = 3$ ,  $u = 2x$ ,  $du = 2 dx$ . And so,

$$\begin{aligned} \int 5 \sec^3 2x dx &= 5\left(\frac{1}{2}\right) \int \sec^3 2x (2 dx) \\ &= \frac{5}{2} \frac{\sec 2x \tan 2x}{2} + \frac{5}{2}\left(\frac{1}{2}\right) \int \sec 2x (2 dx) \end{aligned}$$

To complete this integral, we must use the basic form of Eq. (28.12). Thus, we complete it by

$$\int 5 \sec^3 2x dx = \frac{5 \sec 2x \tan 2x}{4} + \frac{5}{4} \ln|\sec 2x + \tan 2x| + C$$

**EXAMPLE 4** Area—integral fits the form of formula 46

Find the area bounded by  $y = x^2 \ln 2x$ ,  $y = 0$ , and  $x = e$ .

From Fig. 28.15, we see that the area is

$$A = \int_{0.5}^e x^2 \ln 2x dx$$

This integral fits the form of formula 46 if  $u = 2x$ . Thus, we have

$$\begin{aligned} A &= \frac{1}{8} \int_{0.5}^e (2x)^2 \ln 2x (2 dx) = \frac{1}{8} (2x)^3 \left[ \frac{\ln 2x}{3} - \frac{1}{9} \right]_{0.5}^e \\ &= e^3 \left( \frac{\ln 2e}{3} - \frac{1}{9} \right) - \frac{1}{8} \left( \frac{\ln 1}{3} - \frac{1}{9} \right) = e^3 \left( \frac{3 \ln 2e - 1}{9} \right) + \frac{1}{72} \\ &= 9.118 \end{aligned}$$

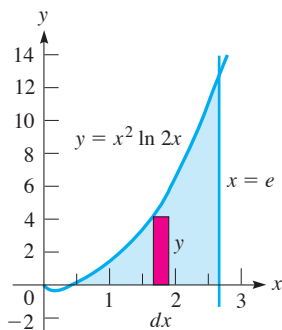


Fig. 28.15

The proper identification of  $u$  and  $du$  is the key step in the use of tables. Therefore, for the integrals in the following example, the proper  $u$  and  $du$ , along with the appropriate formula from the table, are identified, but the integrations are not performed.

■ For reference, formula 18 is

$$\int \frac{\sqrt{a^2 - u^2}}{u} du = \sqrt{a^2 - u^2} - a \ln\left(\frac{a + \sqrt{a^2 - u^2}}{u}\right)$$

■ For reference, formula 37 is

$$\int \sec^n u du = \frac{\sec^{n-2} u \tan u}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} u du$$

■ For reference, Eq. (28.12) is

$$\int \sec u du = \ln|\sec u + \tan u| + C$$

**EXAMPLE 5** Identify the formula,  $u$ , and  $du$ 

$$(a) \int x\sqrt{1-x^4} dx \quad u = x^2, \quad du = 2x dx \quad \text{formula 15}$$

$$(b) \int \frac{(4x^6 - 9)^{3/2}}{x} dx \quad u = 2x^3, \quad du = 6x^2 dx \quad \text{formula 22}$$

introduce a factor of  $x^2$  into numerator and denominator

$$(c) \int x^3 \sin x^2 dx \quad u = x^2, \quad du = 2x dx \quad \text{formula 47} \quad \blacksquare$$

Following is a brief summary of the approach to integration we have used to obtain the exact result. Also, definite integrals may be approximated by methods such as the trapezoidal rule or Simpson's rule.

**Basic Approach to Integrating a Function**

1. Write the integral so that it fits an integral form. Either a basic form as developed in this chapter or a form from a table of integrals may be used.
2. Use a method of transforming the integral so that an integral form may be used. Appropriate methods are covered in this chapter or in other sources.

**EXERCISES 28.11**

In Exercises 1 and 2, make the given changes in the indicated examples of this section, and then state which formula from Appendix B would be used to complete the integration.

1. In Example 1, change the denominator to  $(2 + 3x)^2$ .
2. In Example 2, in the numerator, change  $-$  to  $+$ .

In Exercises 3–8, identify  $u$ ,  $du$ , and the formula from Appendix B that would be used to complete the integration. Do not integrate.

- |                                      |  |
|--------------------------------------|--|
| $3. \int \frac{4 dy}{3y\sqrt{1+2y}}$ | $4. \int \frac{x dx}{\sqrt{x^4 - 16}}$ |
| $5. \int \frac{x dx}{(4-x^4)^{3/2}}$ | $6. \int x^5 \ln x^3 dx$               |
| $7. \int x \cos^2(x^2) dx$           | $8. \int \frac{ds}{s(s^4 - 1)^{3/2}}$  |

In Exercises 9–52, integrate each function by using the table in Appendix B.

- |   |   |
|---|---|
| $9. \int \frac{3x dx}{2+5x}$            | $10. \int \frac{4x dx}{(1+x)^2}$          |
| $11. \int_2^7 4x\sqrt{2+x} dx$          | $12. \int \frac{dx}{x^2 - 4}$             |
| $13. \int \frac{dy}{(y^2 + 4)^{3/2}}$   | $14. \int_0^{\pi/3} \sin^3 x dx$          |
| $15. \int \sin 2x \sin 3x dx$           | $16. \int 6 \sin^{-1} 3x dx$              |
| $17. \int \frac{\sqrt{4x^2 - 9}}{x} dx$ | $18. \int \frac{(9x^2 + 16)^{3/2}}{x} dx$ |

$$19. \int \cos^5 4x dx$$

$$21. \int 6r \tan^{-1} r^2 dr$$

$$23. \int_1^2 (4-x^2)^{3/2} dx$$

$$25. \int \frac{dx}{x\sqrt{4x^2+1}}$$

$$27. \int \frac{8 dx}{x\sqrt{1-4x^2}}$$

$$29. \int_0^{\pi/12} \sin \theta \cos 5\theta d\theta$$

$$31. \int x^5 \cos x^3 dx$$

$$33. \int \frac{2x dx}{(1-x^4)^{3/2}}$$

$$35. \int_1^3 \frac{\sqrt{3+5x^2} dx}{x}$$

$$37. \int x^3 \ln x^2 dx$$

$$39. \int \frac{9x^2 dx}{(x^6 - 1)^{3/2}}$$

$$41. \int t^2 (t^6 + 1)^{3/2} dt$$

$$43. \int \sin^3 4x \cos^3 4x dx$$

$$20. \int 0.2 \tan^2 2\phi d\phi$$

$$22. \int 5xe^{4x} dx$$

$$24. \int \frac{3 dx}{9 - 16x^2}$$

$$26. \int \frac{\sqrt{4+x^2}}{x} dx$$

$$28. \int \frac{dx}{x(1+4x)^2}$$

$$30. \int_0^2 x^2 e^{3x} dx$$

$$32. \int 5 \sin^3 t \cos^2 t dt$$

$$34. \int \frac{dx}{x(1-4x)}$$

$$36. \int_{1/2}^1 \frac{\sqrt{9-4x^2}}{x} dx$$

$$38. \int \frac{1.2u du}{u^2 \sqrt{u^4 - 9}}$$

$$40. \int x^7 \sqrt{x^4 + 4} dx$$

$$42. \int \frac{\sqrt{3+4x^2} dx}{x}$$

$$44. \int 6 \cot^4 2x dx$$

45. Find the length of arc of the curve  $y = x^2$  from  $x = 0$  to  $x = 1$ . (Use Eq. 26.29.)
46. Find the moment of inertia with respect to its axis of the solid generated by revolving the region bounded by  $y = 3 \ln x$ ,  $x = e$ , and the  $x$ -axis about the  $y$ -axis.
47. Find the area of an ellipse with a major axis  $2a$  and a minor axis  $2b$ .
48. The voltage across a  $5.0\text{-}\mu\text{F}$  capacitor in an electric circuit is zero. What is the voltage after  $5.00\ \mu\text{s}$  if a current  $i$  (in mA) as a function of the time  $t$  (in s) given by  $i = \tan^{-1} 2t$  charges the capacitor?
49. Find the force (in N) on the region bounded by  $x = 1/\sqrt{1+y}$ ,  $y = 0$ ,  $y = 3$ , and the  $y$ -axis, if the surface of the water is at the upper edge of the area.
50. If 6.00 g of a chemical are placed in water, the time  $t$  (in min) it takes to dissolve half of the chemical is given by  $t = 560 \int_3^6 \frac{dx}{x(x+4)}$ , where  $x$  is the amount of undissolved chemical at any time. Evaluate  $t$ .
51. The dome of a sports arena is the surface generated by revolving  $y = 20.0 \cos 0.0196x$  ( $0 \leq x \leq 80.0$  m) about the  $y$ -axis. Find the volume within the dome.
52. If an electric charge  $Q$  is distributed along a wire of length  $2a$ , the force  $F$  exerted on an electric charge  $q$  placed at point  $P$  is  $F = kqQ \int \frac{b dx}{(b^2 + x^2)^{3/2}}$ . Integrate to find  $F$  as a function of  $x$ .

## CHAPTER 28 EQUATIONS

### Integrals

$$\int u^n du = \frac{u^{n+1}}{n+1} + C \quad (n \neq -1) \quad (28.1)$$

$$\int \frac{du}{u} = \ln|u| + C \quad (28.2)$$

$$\int e^u du = e^u + C \quad (28.3)$$

$$\int \sin u du = -\cos u + C \quad (28.4)$$

$$\int \cos u du = \sin u + C \quad (28.5)$$

$$\int \sec^2 u du = \tan u + C \quad (28.6)$$

$$\int \csc^2 u du = -\cot u + C \quad (28.7)$$

$$\int \sec u \tan u du = \sec u + C \quad (28.8)$$

$$\int \csc u \cot u du = -\csc u + C \quad (28.9)$$

$$\int \tan u du = -\ln|\cos u| + C \quad (28.10)$$

$$\int \cot u du = \ln|\sin u| + C \quad (28.11)$$

$$\int \sec u du = \ln|\sec u + \tan u| + C \quad (28.12)$$

$$\int \csc u du = \ln|\csc u - \cot u| + C \quad (28.13)$$

## Trigonometric relations

$$\cos^2 x + \sin^2 x = 1 \quad (28.14)$$

$$1 + \tan^2 x = \sec^2 x \quad (28.15)$$

$$1 + \cot^2 x = \csc^2 x \quad (28.16)$$

$$2 \cos^2 x = 1 + \cos 2x \quad (28.17)$$

$$2 \sin^2 x = 1 - \cos 2x \quad (28.18)$$

## Root-mean-square value

$$y_{\text{rms}} = \sqrt{\frac{1}{T} \int_0^T y^2 dx} \quad (28.19)$$

## Integrals

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1} \frac{u}{a} + C \quad (28.20)$$

$$\int \frac{du}{a^2 + u^2} = \frac{1}{a} \tan^{-1} \frac{u}{a} + C \quad (28.21)$$

$$\int u dv = uv - \int v du \quad (28.22)$$

## Trigonometric substitutions

For  $\sqrt{a^2 - x^2}$  use  $x = a \sin \theta$

For  $\sqrt{a^2 + x^2}$  use  $x = a \tan \theta$

For  $\sqrt{x^2 - a^2}$  use  $x = a \sec \theta$

(28.23)

## CHAPTER 28 REVIEW EXERCISES

In Exercises 1–42, integrate the given functions without using a table of integrals.

- |   |   |  |  |
|---|---|--|--|
| 1. $\int e^{-8x} dx$  | 2. $\int e^{\cos 2x} \sin x \cos x dx$                          | 21. $\int \frac{3x dx}{4 + x^4}$                   | 22. $\int_1^3 \frac{12 dR}{\sqrt{R}(1 + R)}$       |
| 3. $\int \frac{dx}{x(\ln 2x)^2}$                                  | 4. $\int_1^8 y^{1/3} \sqrt{y^{4/3} + 9} dy$                     | 23. $\int \frac{4 dx}{\sqrt{4x^2 - 9}}$            | 24. $\int \frac{x^2 dx}{\sqrt{9 - x^2}}$           |
| 5. $\int_0^{\pi/2} \frac{4 \cos \theta d\theta}{1 + \sin \theta}$ | 6. $\int \frac{\sec^2 x dx}{2 + \tan x}$                        | 25. $\int \frac{e^{2x} dx}{\sqrt{e^{2x} + 1}}$     | 26. $\int \frac{x^2 - 2x + 3}{(x - 1)^3} dx$       |
| 7. $\int \frac{2 dx}{25 + 49x^2}$                                 | 8. $\int \frac{dx}{\sqrt{1 - 4x^2}}$                            | 27. $\int \frac{2x^2 + 3x + 18}{x^3 + 9x} dx$      | 28. $\int_{1/2}^{e/2} \frac{(4 + \ln 2u)^3 du}{u}$ |
| 9. $\int_0^{\pi/2} \cos^3 2\theta d\theta$                        | 10. $\int_0^{\pi/18} \sec^3 6x \tan 6x dx$                      | 29. $\int_0^{\pi/6} 3 \sin^2 3\phi d\phi$          | 30. $\int \sin^4 x dx$                             |
| 11. $\int_0^2 \frac{x dx}{4 + x^2}$                               | 12. $\int_1^e \frac{\ln v^2 dv}{\ln e^v}$                       | 31. $\int x \csc^2 2x dx$                          | 32. $\int x \tan^{-1} x dx$                        |
| 13. $\int (\sin t + \cos t)^2 \sin t dt$                          | 14. $\int \frac{\sin^3 x dx}{\sqrt{\cos x}}$                    | 33. $\int \frac{3u^2 - 6u - 2}{3u^3 + u^2} du$     | 34. $\int \frac{R^2 + 3}{R^4 + 3R^2 + 2} dR$       |
| 15. $\int \frac{e^x dx}{1 + e^{2x}}$                              | 16. $\int \frac{p + 25}{p^2 - 25} dp$                           | 35. $\int e^{2x} \cos e^{2x} dx$                   | 36. $\int \frac{3 dx}{x^2 + 6x + 10}$              |
| 17. $\int \sec^4 3x dx$   | 18. $\int \frac{(1 - \cos^2 \theta) d\theta}{1 + \cos 2\theta}$ | 37. $\int_1^e \frac{3 \cos(\ln x) dx}{x}$          | 38. $\int_1^3 \frac{2 dx}{x^2 - 2x + 5}$           |
| 19. $\int \frac{2x^2 + 6x + 1}{2x^3 - x^2 - x} dx$                | 20. $\int \frac{4 - e^{\sqrt{x}}}{\sqrt{x} e^{\sqrt{x}}} dx$    | 39. $\int \frac{u^2 - 8}{u + 3} du$                | 40. $\int \frac{\log_x 2 dx}{x \ln x}$             |
|   |   | 41. $\int \frac{\sin x \cos^2 x}{5 + \cos^2 x} dx$ | 42. $\int \frac{e^{2x} dx}{16 + e^{4x}}$           |



In Exercises 43–84, solve the given problems by integration.

43. For the integral  $\int \frac{dx}{x\sqrt{2-x^2}}$ , using a trigonometric substitution, find the transformed integral but do not integrate.
44. For the integral  $\int \frac{dx}{\sqrt{x^2+4x+3}}$ , using a trigonometric substitution, find the transformed integral but do not integrate.
45. Perform the integrations  $\int e^{\ln 4x} dx$  and  $\int \ln e^{4x} dx$ . Compare results.
46. Perform the integrations  $\int \sin 6x \sin 5x dx$  using Eq. (20.16) on page 541.
47. For the integral  $\int \frac{\sqrt{16+x^8}}{x} dx$ , identify the formula found in Appendix B,  $u$ , and  $du$  that would be used for the integration. Do not integrate.
48. For the integral  $\int x^9 \sqrt{x^5+8} dx$ , identify the formula found in Appendix B,  $u$ , and  $du$ , that would be used for the integration. Do not integrate.
49. Show that  $\int e^x (e^x + 1)^2 dx$  can be integrated in two ways. Explain the difference in the answers.
50. Show that  $\int \frac{1}{x} (1 + \ln x) dx$  can be integrated in two ways. Explain the difference in the answers.
51. Integrate  $\int \frac{\sin x dx}{1 + \sin x}$  by first rewriting the integrand. (*Hint*: First divide and then note Exercise 29 on page 862.)
52. Integrate  $\int \frac{dx}{1 + e^x}$  by first rewriting the integrand. (*Hint*: It is possible to multiply the numerator and the denominator by an appropriate expression.)
53. The integral  $\int \frac{x}{\sqrt{x^2+4}} dx$  can be integrated in more than one way. Explain what methods can be used and which is simpler.
54. Find the equation of the curve for which  $dy/dx = e^x(2 - e^x)^2$ , if the curve passes through  $(0, 4)$ .
55. Find the equation of the curve for which  $dy/dx = \sec^4 x$ , if the curve passes through the origin.
56. Find the equation of the curve for which  $\frac{dy}{dx} = \frac{\sqrt{4+x^2}}{x^4}$ , if the curve passes through  $(2, 1)$ .
57. Find the area bounded by  $y = 4e^{2x}$ ,  $x = 1.5$ , and the axes.
58. Find the area bounded by  $y = x/(1+x)^2$ , the  $x$ -axis, and the line  $x = 4$ .
59. Find the area inside the circle  $x^2 + y^2 = 25$  and to the right of the line  $x = 3$ .
60. Find the area bounded by  $y = x\sqrt{x+4}$ ,  $y = 0$ , and  $x = 5$ .
61. Find the area bounded by  $y = \tan^{-1} 2x$ ,  $x = 2$ , and the  $x$ -axis.
62. In polar coordinates, the area  $A$  bounded by the curve  $r = f(\theta)$ ,  $\theta = \alpha$ , and  $\theta = \beta$  is found by evaluating the integral  $A = \frac{1}{2} \int_{\alpha}^{\beta} r^2 d\theta$ . Find the area bounded by  $\alpha = 0$ ,  $\beta = \pi/2$ , and  $r = e^{\theta}$ .
63. Find the volume generated by revolving the region bounded by  $y = xe^x$ ,  $y = 0$ , and  $x = 2$  about the  $y$ -axis.
64. Find the volume generated by revolving about the  $y$ -axis the region bounded by  $y = x + \sqrt{x+1}$ ,  $x = 3$ , and the axes.
65. Find the volume of the solid generated by revolving the region bounded by  $y = e^x \sin x$  and the  $x$ -axis between  $x = 0$  and  $x = \pi$  about the  $x$ -axis.
66. Find the centroid of a flat plate that covers the region bounded by  $y = \ln x$ ,  $x = 2$ , and the  $x$ -axis.
67. Find the length of arc along the curve of  $y = \ln \sin x$  from  $x = \pi/3$  to  $x = 2\pi/3$ . (Use Eq. 26.29.)
68. Find the area of the surface generated by revolving the curve of  $y = \sqrt{4-x^2}$  from  $x = -2$  to  $x = 2$  about the  $x$ -axis. See Exercise 37 of Section 26.6.
69. The force  $F$  (in N) on a nail by a hammer is  $F = 5/(1+2t)$ , where  $t$  is the time (in s). The impulse  $I$  of the force from  $t = 0$  s to  $t = 0.25$  s is  $I = \int_0^{0.25} F dt$ . Find the impulse.
70. The acceleration of a parachutist is given by  $dv/dt = g - kv$ , where  $k$  is a constant depending on the resisting force due to air friction. Find  $v$  as a function of the time  $t$ .
71. The change in the thermodynamic entity of entropy  $\Delta S$  may be expressed as  $\Delta S = \int (c_v/T) dT$ , where  $c_v$  is the heat capacity at constant volume and  $T$  is the temperature. For increased accuracy,  $c_v$  is often given by the equation  $c_v = a + bT + cT^2$ , where  $a$ ,  $b$ , and  $c$  are constants. Express  $\Delta S$  as a function of temperature.
72. A certain type of chemical reaction leads to the equation  $dt = \frac{dx}{k(a-x)(b-x)}$ , where  $a$ ,  $b$ , and  $k$  are constants. Solve for  $t$  as a function of  $x$ .
73. An electric transmission line between two towers has a shape given by  $y = 16.0(e^{x/32} + e^{-x/32})$ . Find the length of transmission line if the towers are 50.0 m apart (from  $x = -25.0$  m to  $x = 25.0$  m). (Use Eq. 26.29.)
74. An object at the end of a spring is immersed in liquid. Its velocity (in cm/s) is then described by the equation  $v = 2e^{-2t} + 3e^{-5t}$ , where  $t$  is the time (in s). Such motion is called *overdamped*. Find the displacement  $s$  as a function of  $t$  if  $s = -1.6$  cm for  $t = 0$ .
75. When we consider the resisting force of the air, the velocity  $v$  (in m/s) of a falling brick in terms of the time  $t$  (in s) is given by  $dv/(9.8 - 0.1v) = dt$ . If  $v = 0$  when  $t = 0$ , find  $v$  as a function of  $t$ .
76. The power delivered to an electric circuit is given by  $P = Ei$ , where  $E$  and  $i$  are the instantaneous voltage and the instantaneous current in the circuit, respectively. The mean power, averaged over a period  $2\pi/\omega$ , is given by  $P_{av} = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} Ei dt$ . If  $E = 20 \cos 2t$  and  $i = 3 \sin 2t$ , find the average power over a period of  $\pi/4$ .

77. Find the root-mean-square value for one period of the electric current  $i$  if  $i = 2 \sin t$ .
78. In atomic theory, when finding the number  $n$  of atoms per unit volume of a substance, we use the equation  $n = A \int_0^\pi e^{a \cos \theta} \sin \theta \, d\theta$ . Perform the indicated integration.
79. In the study of the effects of an electric field on molecular orientation, the integral  $\int_0^\pi (1 + k \cos \theta) \cos \theta \sin \theta \, d\theta$  is used. Evaluate this integral.
80. In finding the lift of the air flowing around an airplane wing, we use the integral  $\int_{-\pi/2}^{\pi/2} \theta^2 \cos \theta \, d\theta$ . Evaluate this integral.
81. Find the volume within the piece of tubing in an oil distribution line shown in Fig. 28.16. All cross sections are circular.

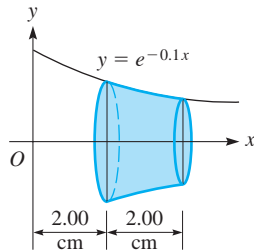


Fig. 28.16

82. A metal plate has a shape shown in Fig. 28.17. Find the  $x$ -coordinate of the centroid of the plate.

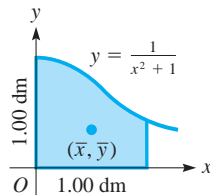


Fig. 28.17

83. The nose cone of a space vehicle is to be covered with a heat shield. The cone is designed such that a cross section  $x$  metres from the tip and perpendicular to its axis is a circle of radius  $1.5x^{2/3}$  metres. Find the surface area of the heat shield if the nose cone is 4.00 m long. See Fig. 28.18. (See Exercise 37 of Section 26.6.)

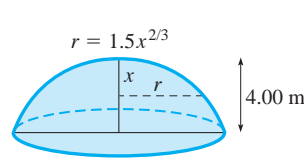


Fig. 28.18

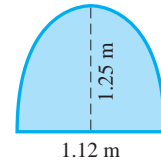


Fig. 28.19

84. A window has a shape of a semiellipse, as shown in Fig. 28.19. What is the area of the window?

### Writing Exercise

85. The side of a blade designed to cut leather at a shoe factory can be described as the region bounded by  $y = 4 \cos^2 x$  and  $y = 4$  from  $x = 0$  to  $x = 3.14$ . Write two or three paragraphs explaining how the area of the blade may be found by integrating the appropriate integral by either of two methods of this chapter.

## CHAPTER 28 PRACTICE TEST

In Problems 1–7, evaluate the given integrals.

- $\int (\sec x - \sec^3 x \tan x) \, dx$
- $\int \sin^3 x \, dx$
- $\int \tan^3 2x \, dx$
- $\int \cos^2 4\theta \, d\theta$
- $\int \frac{dx}{x^2 \sqrt{4 - x^2}}$
- $\int x e^{-2x} \, dx$
- $\int \frac{x^3 + 5x^2 + x + 2}{x^4 + x^2} \, dx$

8. The electric current in a certain circuit is given by  $i = \int \frac{6t + 1}{4t^2 + 9} \, dt$ , where  $t$  is the time. Integrate and find the resulting function if  $i = 0$  for  $t = 0$ .
9. Find the first-quadrant area bounded by  $y = \frac{1}{\sqrt{16 - x^2}}$  and  $x = 3$ .

# Partial Derivatives and Double Integrals

# 29

**T**o this point, we have been dealing with functions that have a single independent variable. There are, however, numerous applications in which functions with more than one independent variable are used. Although a number of these functions involve three or more independent variables, many involve only two, and we shall be concerned primarily with these.

In the first two sections of this chapter, we establish the meaning of a function of two independent variables, and discuss how the graph of this type of function is shown. In the last two sections, we develop some of the basic concepts of the calculus of these functions, primarily *partial derivatives* and *double integrals*. In finding the partial derivative, we take the derivative of the function with respect to one independent variable, holding the other constant. Similarly, when integrating an expression with two differentials, we integrate with respect to one, holding the other constant, and then integrate with respect to the other.

While studying problems in motion and optics in the early 1700s, mathematicians including Leibniz developed the meaning and use of partial derivatives. By the middle of the eighteenth century, double integrals seem to have been known by the leading mathematicians of the time. However, their wide dissemination is due to Euler who, in 1769, published the first detailed explanation of them.

Partial derivatives and double integrals have applications in many areas of science and technology, including acoustics, electricity, electronics, mechanics, product design, and wave motion. Some of these applications are shown in the examples and exercises of this chapter.

## LEARNING OUTCOMES

**After completion of this chapter, the student should be able to:**

- Set up and evaluate a function of two variables
- Use traces and sections to sketch the graph of a surface in the rectangular coordinate system in three dimensions
- Convert points and equations from rectangular to cylindrical coordinates and vice versa
- Find the first- and second-order partial derivatives of a function of several variables
- Evaluate double integrals
- Understand the geometric interpretation of the partial derivatives and the double integral of a function of two variables
- Solve application problems involving functions of two variables, their partial derivatives, and their integrals



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◀ In Section 29.2 we see how the shape of the roof of the Saddledome arena in Calgary, Alberta, is determined by a function of two variables.

## 29.1 Functions of Two Variables

Functions of Two Independent Variables •  $z = f(x, y)$  Notation • Restrictions on  $x$  and  $y$

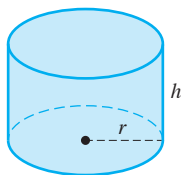


Fig. 29.1

Many familiar formulas express one variable in terms of two or more other variables. The following example illustrates one from geometry.

### EXAMPLE 1 A function of two variables

The total surface area  $A$  of a right circular cylinder is a function of the radius  $r$  and the height  $h$  of the cylinder. That is, the area will change if either or both of these change. The formula for the total surface area is

$$A = 2\pi r^2 + 2\pi rh$$

We say that  $A$  is a function of  $r$  and  $h$ . See Fig. 29.1. ■

We define a function of two variables as follows: *If  $z$  is uniquely determined for given values of  $x$  and  $y$ , then  $z$  is a function of  $x$  and  $y$ .* The notation used is similar to that used for one independent variable. It is  $z = f(x, y)$ , where both  $x$  and  $y$  are independent variables. Therefore, it follows that  $f(a, b)$  means “the value of the function when  $x = a$  and  $y = b$ .”

### EXAMPLE 2 Illustrating the notation $z = f(x, y)$

If  $f(x, y) = 3x^2 + 2xy - y^3$ , find  $f(-1, 2)$ .

Substituting  $-1$  for  $x$  and  $2$  for  $y$ , we have

$$\begin{aligned} f(-1, 2) &= 3(-1)^2 + 2(-1)(2) - (2)^3 \\ &= 3 - 4 - 8 = -9 \end{aligned}$$

### EXAMPLE 3 Application of a function of two variables

For a certain electric circuit, the current  $i$  (in A), in terms of the voltage  $E$  and resistance  $R$  (in  $\Omega$ ) is given by

$$i = \frac{E}{R + 0.25}$$

Find the current for  $E = 1.50$  V and  $R = 1.20$   $\Omega$  and for  $E = 1.60$  V and  $R = 1.05$   $\Omega$ .

Substituting the first values, we have

$$i = \frac{1.50}{1.20 + 0.25} = 1.03 \text{ A}$$

For the second pair of values, we have

$$i = \frac{1.60}{1.05 + 0.25} = 1.23 \text{ A}$$

For this circuit, the current generally changes if either or both of  $E$  and  $R$  change. ■

### EXAMPLE 4 Using $f(x, y)$ notation

If  $f(x, y) = 2xy^2 - y$ , find  $f(x, 2x) - f(x, x^2)$ .

We note that in each evaluation, the  $x$  factor remains as  $x$ , but that we are to substitute  $2x$  for  $y$  and subtract the function for which  $x^2$  is substituted for  $y$ .

$$\begin{aligned} f(x, 2x) - f(x, x^2) &= [2x(2x)^2 - (2x)] - [2x(x^2)^2 - x^2] \\ &= [8x^3 - 2x] - [2x^5 - x^2] \\ &= 8x^3 - 2x - 2x^5 + x^2 \\ &= -2x^5 + 8x^3 + x^2 - 2x \end{aligned}$$

This type of difference of functions is important in Section 29.4. ■

#### Practice Exercise

1. If  $f(x, y) = 4xy^2 - 3x^2y$ , find  $f(1, 4) - f(x, 2x)$ .

Restricting values of the function to real numbers means that *values of either  $x$  or  $y$  or both that lead to division by zero or to imaginary values for  $z$  are not permissible.*

**EXAMPLE 5** Restrictions on independent variables

If  $f(x, y) = \frac{3xy}{(x - y)(x + 3)}$ , all values of  $x$  and  $y$  are permissible except those for which  $x = y$  and  $x = -3$ . Each of these would indicate division by zero.

If  $f(x, y) = \sqrt{4 - x^2 - y^2}$ , neither  $x$  nor  $y$  may be greater than 2 in absolute value, and the sum of their squares may not exceed 4. Otherwise, imaginary values of the function would result. ■

Following is an example of setting up a function of two variables from stated conditions. As with word problems with one unknown, although no general rules can be given for this procedure, a careful analysis of the statement should lead to the required function.

**EXAMPLE 6** Setting up a function of two variables

An open rectangular fish tank is to hold  $0.25 \text{ m}^3$  of water when completely full. Express the total surface area of glass required to make the tank as a function of the length and width of the tank.

An “open” tank is one that has no top. Therefore, the surface area  $S$  of the tank is

$$S = lw + 2lh + 2hw$$

where  $l$  is the length,  $w$  the width, and  $h$  the height of the tank (see Fig. 29.2). However, this equation contains three independent variables. Using the condition that the volume of water is  $0.25 \text{ m}^3$ , we have  $lwh = 0.25$ . Since we wish to have only  $l$  and  $w$ , we solve this equation for  $h$  and find that  $h = 0.25/lw$ . Substituting for  $h$  in the equation for the surface area, we have

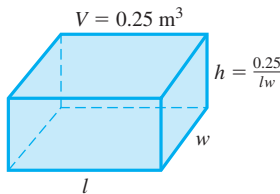


Fig. 29.2

$$\begin{aligned} S &= lw + 2l\left(\frac{0.25}{lw}\right) + 2\left(\frac{0.25}{lw}\right)w \\ &= lw + \frac{0.50}{w} + \frac{0.50}{l} \end{aligned}$$

the required function ■

**EXERCISES 29.1**

In Exercises 1 and 2, make the given changes in the indicated examples of this section and then solve the resulting problems.

- In Example 2, change  $f(-1, 2)$  to  $f(-2, 1)$ .
- In Example 4, change  $f(x, 2x) - f(x, x^2)$  to  $f(2x, x) - f(x^2, x)$ .

In Exercises 3–8, determine the indicated function.

- Express the volume  $V$  of a right circular cylinder as a function of the radius  $r$  and height  $h$ .
- Express the length of a diagonal of a rectangle as a function of the length and the width.
- A cylindrical can is to be made to contain a volume  $V$ . Express the total surface area (including the top) of the can as a function of  $V$  and the radius of the can.
- The angle between two forces  $\mathbf{F}_1$  and  $\mathbf{F}_2$  is  $30.0^\circ$ . Express the magnitude of the resultant  $\mathbf{R}$  in terms of  $F_1$  and  $F_2$ . See Fig. 29.3.

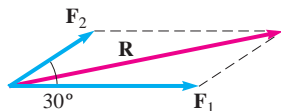


Fig. 29.3

- A right circular cylinder is to be inscribed in a sphere of radius  $r$ . Express the volume of the cylinder as a function of the height  $h$  of the cylinder and  $r$ . See Fig. 29.4.

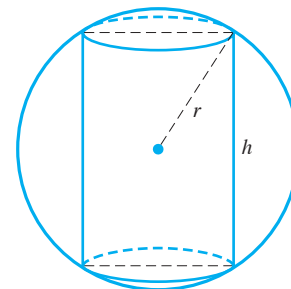


Fig. 29.4

- An office-furniture-leasing firm charges a monthly fee  $F$  based on the length of time a corporation has used the service, plus \$100 for each week the furniture is leased. Express the total monthly charge  $T$  as a function of  $F$  and the number of weeks  $w$  the furniture is leased.



In Exercises 9–24, evaluate the given functions.

9.  $f(x, y) = 2x - 6y$ ; find  $f(0, -4)$ .
10.  $F(x, y) = x^2 - 5y + y^2$ ; find  $F(2, -2)$ .
11.  $g(r, s) = r - 2rs - r^2s$ ; find  $g(-2, 1)$ .
12.  $f(r, \theta) = 2r(r \tan \theta - \sin 2\theta)$ ; find  $f(3, \pi/4)$ .
13.  $Y(y, t) = \frac{2 - 3y}{t - 1} + 2y^2t$ ; find  $Y(y, 2)$ .
14.  $f(r, t) = r^3 - 3r^2t + 3rt^2 - t^3$ ; find  $f(3, t)$ .
15.  $X(x, t) = -6xt + xt^2 - t^3$ ; find  $X(x, -t)$ .
16.  $g(y, z) = 2yz^2 - 6y^2z - y^2z^2$ ; find  $g(y, 2y)$ .
17.  $H(p, q) = p - \frac{p - 2q^2 - 5q}{p + q}$ ; find  $H(p, q + k)$ .
18.  $g(x, z) = z \tan^{-1}(x^2 + xz)$ ; find  $g(-x, z)$ .
19.  $f(x, y) = x^2 - 2xy - 4x$ ; find  $f(x + h, y + k) - f(x, y)$ .
20.  $g(y, z) = 4yz - z^3 + 4y$ ; find  $g(y + 1, z + 2) - g(y, z)$ .
21.  $f(x, y) = xy + x^2 - y^2$ ; find  $f(x, x) - f(x, 0)$ .
22.  $f(x, y) = 4x^2 - xy - 2y$ ; find  $f(x, x^2) - f(x, 1)$ .
23.  $g(y, z) = 3y^3 - y^2z + 5z^2$ ; find  $g(3z^2, z) - g(z, z)$ .
24.  $X(x, t) = 2x - \frac{t^2 - 2x^2}{x}$ ; find  $X(2t, t) - X(2t^2, t)$ .

In Exercises 25–28, determine which values of  $x$  and  $y$ , if any, are not permissible. In Exercise 27, explain your answer.

25.  $f(x, y) = \frac{\sqrt{y}}{2x}$
26.  $f(x, y) = \frac{x^2 - 4y^2}{x^2 + 9}$
27.  $f(x, y) = \sqrt{x^2 + y^2 - x^2y - y^3}$
28.  $f(x, y) = \frac{1}{xy - y}$

In Exercises 29–40, solve the given problems. When necessary, round answers to 3 significant digits.

29. The voltage  $V$  across a resistor  $R$  in an electric circuit is given by  $V = iR$ , where  $i$  is the current. What is the voltage if the current is 3 A and the resistance is 6  $\Omega$ ?
30. The centripetal acceleration  $a$  of an object moving in a circular path is  $a = v^2/R$ , where  $v$  is the velocity of the object and  $R$  is the radius of the circle. What is the centripetal acceleration of an object moving at 6 m/s in a circular path of radius 4 m?
31. The pressure  $p$  (in Pa) of a gas as a function of its volume  $V$  and temperature  $T$  is  $p = nRT/V$ . If  $n = 3.00$  mol and  $R = 8.31$  J/mol  $\cdot$  K, find  $p$  for  $T = 300$  K and  $V = 50.0$  m<sup>3</sup>.
32. The power  $P$  (in W) supplied to a resistance  $R$  in an electric circuit by a current  $i$  is  $P = i^2R$ . Find the power supplied by a current of 4.0 A to a resistance of 2.4  $\Omega$ .
33. The atmospheric temperature  $T$  near ground level in a certain region is  $T = ax^2 + by^2$ , where  $a$  and  $b$  are constants. What type of curve is each isotherm (along which the temperature is constant) in this region?

34. The pressure  $p$  exerted by a force  $F$  on an area  $A$  is  $p = F/A$ . If a given force is doubled on an area that is  $2/3$  of a given area, what is the ratio of the initial pressure to the final pressure?

35. The current  $i$  (in A) in a certain electric circuit is a function of the time  $t$  (in s) and a variable resistor  $R$  (in  $\Omega$ ), given by

$$i = \frac{6.0 \sin 0.01t}{R + 0.12}. \text{ Find } i \text{ for } t = 0.75 \text{ s and } R = 1.50 \Omega.$$

36. The reciprocal of the image distance  $q$  from a lens as a function of the object distance  $p$  and the focal length  $f$  of the lens is

$$\frac{1}{q} = \frac{1}{f} - \frac{1}{p}.$$

Find the image distance of an object 20 cm from a lens whose focal length is 5 cm.

37. A rectangular solar cell panel has a perimeter  $p$  and a width  $w$ . Express the area  $A$  of the panel in terms of  $p$  and  $w$  and evaluate the area for  $p = 250$  cm and  $w = 55$  cm. See Fig. 29.5.

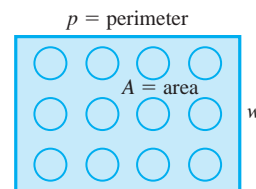


Fig. 29.5

38. A gasoline storage tank is in the shape of a right circular cylinder with a hemisphere at each end, as shown in Fig. 29.6. Express the volume  $V$  of the tank in terms of  $r$  and  $h$  and then evaluate the volume for  $r = 1.25$  m and  $h = 4.17$  m.

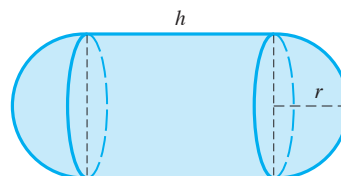


Fig. 29.6

39. The crushing load  $L$  of a pillar varies as the fourth power of its radius  $r$  and inversely as the square of its length  $l$ . Express  $L$  as a function of  $r$  and  $l$  for a pillar 6.0 m tall and 1.0 m in diameter that is crushed by a load of 20 Mg.
40. The resonant frequency  $f$  (in Hz) of an electric circuit containing an inductance  $L$  and capacitance  $C$  is inversely proportional to the square root of the product of the inductance and the capacitance. If the resonant frequency of a circuit containing a 4-H inductor and a 64- $\mu$ F capacitor is 10 Hz, express  $f$  as a function of  $L$  and  $C$ .

#### Answer to Practice Exercise

1.  $f(1, 4) - f(x, 2x) = 52 - 10x^3$

## 29.2 Curves and Surfaces in Three Dimensions

Rectangular Coordinates • Octant • Plane • Surfaces and Curves • Trace • Section • Cylindrical Coordinates

We will now undertake a brief description of the graphical representation of a function of two variables. We shall show first a method of representation in the rectangular coordinate system in two dimensions. The following example illustrates the method.

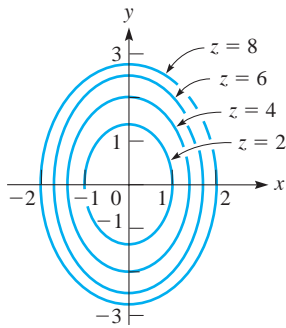


Fig. 29.7

### EXAMPLE 1 Representing $f(x, y)$ in two dimensions

In order to represent  $z = 2x^2 + y^2$ , we will assume various values of  $z$  and sketch the curve of the resulting equation in the  $xy$ -plane. For example, if  $z = 2$  we have

$$2x^2 + y^2 = 2$$

We recognize this as an ellipse with its major axis along the  $y$ -axis and vertices at  $(0, \sqrt{2})$  and  $(0, -\sqrt{2})$ . The ends of the minor axis are at  $(1, 0)$  and  $(-1, 0)$ . However, the ellipse  $2x^2 + y^2 = 2$  represents the function  $z = 2x^2 + y^2$  only for the value of  $z = 2$ . If  $z = 4$ , we have  $2x^2 + y^2 = 4$ , which is another ellipse. In fact, for all positive values of  $z$ , an ellipse is the resulting curve. Negative values of  $z$  are not possible since neither  $x^2$  nor  $y^2$  may be negative. Fig. 29.7 shows the ellipses that are obtained by using the indicated values of  $z$ .

The method of representation illustrated in Example 1 is useful if only a few specific values of  $z$  are to be used, or at least if the various curves do not intersect in such a way that they cannot be distinguished. If a general representation of  $z$  as a function of  $x$  and  $y$  is desired, it is necessary to use three coordinate axes, one each for  $x$ ,  $y$ , and  $z$ . The most widely applicable system of this kind is to place a third coordinate axis at right angles to each of the  $x$ - and  $y$ -axes. In this way, we employ three dimensions for the representation.

The three mutually perpendicular coordinate axes—the  $x$ -axis, the  $y$ -axis, and the  $z$ -axis—are the basis of the **rectangular coordinate system in three dimensions**. Together they form three mutually perpendicular planes in space: the  $xy$ -plane, the  $yz$ -plane, and the  $xz$ -plane. To every point in space of the coordinate system is associated the set of numbers  $(x, y, z)$ . The point at which the axes meet is the *origin*. The positive directions of the axes are indicated in Fig. 29.8. That part of space in which all values of the coordinates are positive is called the **first octant**. Numbers are not assigned to the other octants.

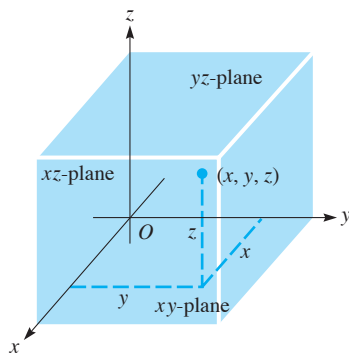


Fig. 29.8

### EXAMPLE 2 Representing $(x, y, z)$ in rectangular coordinates

Represent the point  $(2, 4, 3)$  in rectangular coordinates.

We first note that when representing a point in three dimensions, a certain distortion is necessary to represent values of  $x$  reasonably, since the  $x$ -axis “comes out” of the plane of the page. Units that are  $\sqrt{2}/2$  ( $\approx 0.7$ ) as long as those used on the other axes give a good representation. With this in mind, we draw a line 4 units long from the point  $(2, 0, 0)$  on the  $x$ -axis in the  $xy$ -plane, parallel to the  $y$ -axis. This locates the point  $(2, 4, 0)$ . From this point, a line 3 units long is drawn vertically upward, and this locates the point  $(2, 4, 3)$ . See the points and dashed lines in Fig. 29.9.

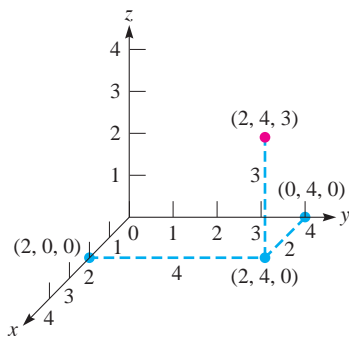


Fig. 29.9

This point  $(2, 4, 3)$  may also be located by starting from  $(0, 4, 0)$  on the  $y$ -axis, then proceeding 2 units parallel to the  $x$ -axis to  $(2, 4, 0)$ , and then proceeding vertically 3 units upward to  $(2, 4, 3)$ .

It may also be located by starting from the point  $(0, 0, 3)$  on the  $z$ -axis, although it is generally preferred to start on the  $x$ -axis, or possibly the  $y$ -axis.

We now show certain basic techniques by which three-dimensional figures may be drawn. We start by showing the general equation of a plane.



Equation of a Plane

$$Ax + By + Cz + D = 0 \tag{29.1}$$

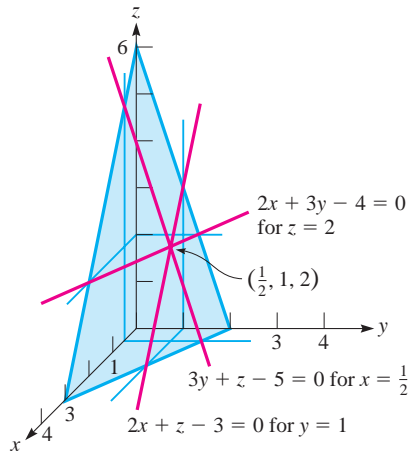


Fig. 29.10

is a **plane** in three dimensions.

**EXAMPLE 3** Illustrating the graph of a plane

Show that the graph of the linear equation  $2x + 3y + z - 6 = 0$  in the rectangular coordinate system of three dimensions is a plane.

If we let any of the three variables take on a specific value, we obtain a linear equation in the other two variables. For example, the point  $(\frac{1}{2}, 1, 2)$  satisfies the equation and therefore lies on the graph of the equation. For  $x = \frac{1}{2}$ , we have

$$3y + z - 5 = 0$$

which is the equation of a straight line. This means that all pairs of values of  $y$  and  $z$  that satisfy this equation, along with  $x = \frac{1}{2}$ , satisfy the given equation. Thus, for  $x = \frac{1}{2}$ , the straight line  $3y + z - 5 = 0$  lies on the graph of the equation.

For  $z = 2$ , we have

$$2x + 3y - 4 = 0$$

which is also a straight line. By similar reasoning, this line lies on the graph of the equation. Since two lines through a point define a plane, these lines through  $(\frac{1}{2}, 1, 2)$  define a plane. This plane is the graph of the equation (see Fig. 29.10).

For any point on the graph of this equation, there is a straight line on the graph that is parallel to one of the coordinate planes. Therefore, there are intersecting straight lines through the point. Thus, the graph is a plane. A similar analysis can be made for any equation of the same linear form. ■

**EXAMPLE 4** Sketching the graph of a plane

Sketch the graph of  $3x - y + 2z - 4 = 0$ .

As in the case of two-dimensional graphs, the intercepts of a graph in three dimensions are those points where it crosses the respective axes. Therefore, by letting two of the variables at a time equal zero, we obtain the intercepts. For the graph of the given equation, the intercepts are  $(\frac{4}{3}, 0, 0)$ ,  $(0, -4, 0)$ , and  $(0, 0, 2)$ . These points are located (see Fig. 29.11), and lines drawn between them to represent the plane. ■

The intersection of two surfaces is a **curve** in space. This has been seen in Examples 3 and 4, since the intersections of the given planes and the coordinate planes are lines (which in the general sense are curves). We define the **traces** of a surface to be the curves of intersection of the surface and the coordinate planes. The traces of a plane are those lines drawn between the intercepts to represent the plane. Many surfaces may be sketched by finding their traces and intercepts.

**EXAMPLE 5** Using intercepts and traces to sketch a graph

Find the intercepts and traces for the graph of the equation  $z = 4 - x^2 - y^2$  and then sketch the graph.

The intercepts of the graph of the equation are  $(2, 0, 0)$ ,  $(-2, 0, 0)$ ,  $(0, 2, 0)$ ,  $(0, -2, 0)$ , and  $(0, 0, 4)$ .

Since the traces of a surface lie within the coordinate planes, for each trace one of the variables is zero. Thus, by **letting each variable in turn be zero, we find the trace of the surface in the plane of the other two variables**. Therefore, the traces of this surface are

**LEARNING TIP**

Since we know that the graph of an equation of the form of Eq. (29.1) is a plane, its graph can be found by determining its three intercepts, and the plane can then be represented by drawing in the lines between these intercepts. If the plane passes through the origin, by letting two of the variables in turn be zero, two straight lines that define the plane are found.

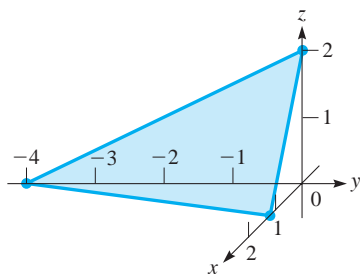


Fig. 29.11

Traces

**LEARNING TIP**

The graphs of planes and of functions with two independent variables are examples of **surfaces** in space. More generally, the graph of an equation relating three variables is also a surface.

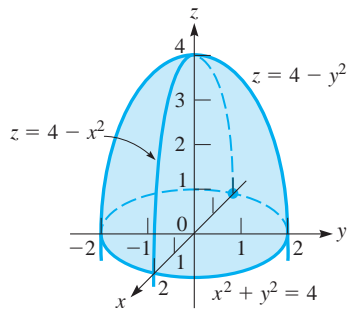


Fig. 29.12

**Section**

- in the  $yz$ -plane:  $z = 4 - y^2$  (a parabola)
- in the  $xz$ -plane:  $z = 4 - x^2$  (a parabola)
- in the  $xy$ -plane:  $x^2 + y^2 = 4$  (a circle)

Using the intercepts and sketching the traces, we obtain the surface represented by the equation as shown in Fig. 29.12. This figure is called a **circular paraboloid**. ■

There are numerous techniques for analysing the equation of a surface in order to obtain its graph. Another that we shall discuss here, which is closely associated with a trace, is that of a **section**. By assuming a specific value of one of the variables, we obtain an equation in two variables, the graph of which lies in a plane parallel to the coordinate plane of the two variables. The following examples illustrate sketching a surface by use of intercepts, traces, and sections.

**EXAMPLE 6 Using traces and sections to sketch a graph**

Sketch the graph of  $4x^2 + y^2 - z^2 = 4$ .

The intercepts are  $(1, 0, 0)$ ,  $(-1, 0, 0)$ ,  $(0, 2, 0)$ , and  $(0, -2, 0)$ . We note that there are no intercepts on the  $z$ -axis, for this would necessitate  $z^2 = -4$ .

The traces are

- in the  $yz$ -plane:  $y^2 - z^2 = 4$  (a hyperbola)
- in the  $xz$ -plane:  $4x^2 - z^2 = 4$  (a hyperbola)
- in the  $xy$ -plane:  $4x^2 + y^2 = 4$  (an ellipse)

The surface is reasonably defined by these curves, but by assuming suitable values of  $z$ , we may indicate its shape better. For example, if  $z = 3$ , we have  $4x^2 + y^2 = 13$ , which is an ellipse. In using it, we must remember that it is valid for  $z = 3$  and therefore should be drawn 3 units above the  $xy$ -plane. If  $z = -2$ , we have  $4x^2 + y^2 = 8$ , which is also an ellipse. Thus, we have the following sections:

- for  $z = 3$ :  $4x^2 + y^2 = 13$  (an ellipse)
- for  $z = -2$ :  $4x^2 + y^2 = 8$  (an ellipse)

Other sections could be found, but these are sufficient to obtain a good sketch of the graph (see Fig. 29.13). The figure is called an **elliptic hyperboloid**. ■

**EXAMPLE 7 Sketching a graph—application**

Sketch the graph of  $z = y^2 - x^2$  (a hyperbolic paraboloid).

There is only one intercept at the origin. The traces are

- in the  $yz$ -plane:  $z = y^2$  (a parabola opening upward)
- in the  $xz$ -plane:  $z = -x^2$  (a parabola opening downward)
- in the  $xy$ -plane:  $y = \pm x$  (two intersecting lines)

Sections in the  $yz$ -plane are all parabolas opening upward, and sections in the  $xz$ -plane are all parabolas opening downward. Finally, sections in the horizontal  $xy$ -plane are hyperbolas of the form  $y^2 - x^2 = k$ . For example, if  $z = -1$ , we have the hyperbola  $x^2 - y^2 = 1$ .

This surface is not easy to draw, but using the information above, we obtain a sketch of the surface, as shown in Fig. 29.14. Note that the origin looks like a minimum from one direction but like a maximum from another. This behaviour at the origin is what gives this surface its characteristic saddle shape.

The shape of a hyperbolic paraboloid has intrinsic structural strength that is often exploited in the building of large roofs. For example, the cable net roof of the Saddledome arena in Calgary, Alberta, is a hyperbolic paraboloid that can be approximated by the equation

$$z = \frac{1}{327}y^2 - \frac{1}{763}x^2$$

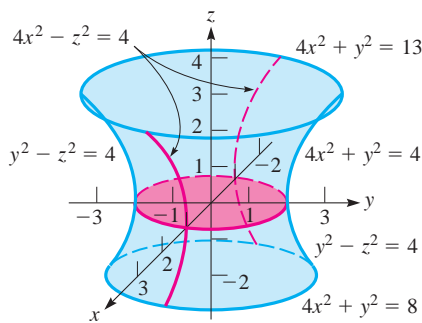


Fig. 29.13

**Practice Exercise**

1. What is the trace of  $x^2 + 4y^2 - 4z^2 = 4$  in the (a)  $xy$ -plane? (b)  $xz$ -plane?

■ See the chapter introduction.

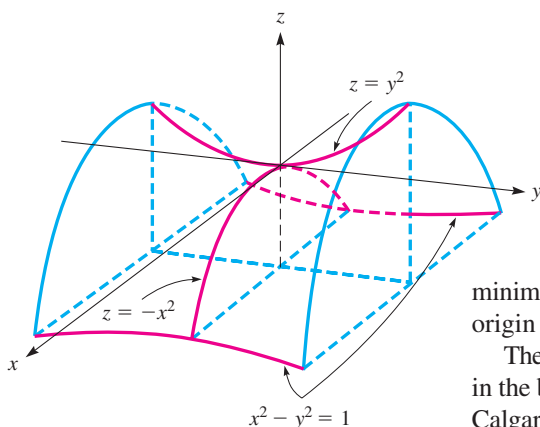


Fig. 29.14

(placing the origin at the centre of the roof, with all measurements in metres). Other structures with saddle-shaped cable net roofs include the Stadium of Peace and Friendship in Athens, Greece, and the velodrome in London, England (host of the London 2012 track cycling events). ■

Having developed the rectangular coordinate system in three dimensions, we can compare the graph of a function using two dimensions and three dimensions. The next example shows the surface for the function of Example 1.

**EXAMPLE 8** Comparing a 2D graph and a 3D graph

Sketch the graph of  $z = 2x^2 + y^2$ .

The only intercept is  $(0, 0, 0)$ . The traces are

in the  $yz$ -plane:  $z = y^2$  (a parabola)

in the  $xz$ -plane:  $z = 2x^2$  (a parabola)

in the  $xy$ -plane: the origin

The trace in the  $xy$ -plane is only the point of origin, since  $2x^2 + y^2 = 0$  may be written as  $y^2 = -2x^2$ , which is true only for  $x = 0$  and  $y = 0$ .

To get a better graph, we should use some positive values for  $z$ . As we noted in Example 1, negative values of  $z$  cannot be used. Since we used  $z = 2, z = 4, z = 6$ , and  $z = 8$  in Example 1, we shall use these values here. Therefore,

for  $z = 2$ :  $2x^2 + y^2 = 2$       for  $z = 4$ :  $2x^2 + y^2 = 4$

for  $z = 6$ :  $2x^2 + y^2 = 6$       for  $z = 8$ :  $2x^2 + y^2 = 8$

Each of these sections is an ellipse. The surface, called an **elliptic paraboloid**, is shown in Fig. 29.15. Compare with Fig. 29.7. ■

Example 8 illustrates how *topographic maps* may be drawn. These maps represent three-dimensional terrain in two dimensions. For example, if Fig. 29.15 represents an excavation in the surface of the earth, then Fig. 29.7 represents the curves of constant elevation, or *contours*, with equally spaced elevations measured from the bottom of the excavation.

An equation with only two variables may represent a surface in space. Since only two variables are included in the equation, the surface is independent of the other variable. Another interpretation is that all sections, for all values of the variable not included, are the same. That is, *all sections parallel to the coordinate plane of the included variables are the same as the trace in that plane.*

**EXAMPLE 9** Comparing a linear equation in 2D and 3D

Sketch the graph of  $x + y = 2$  in the rectangular coordinate system in three dimensions and in two dimensions.

Since  $z$  does not appear in the equation, we can consider the equation to be  $x + y + 0z = 2$ . Therefore, we see that *for any value of  $z$* , the section is the straight line  $x + y = 2$ . Thus, the graph is a vertical plane as shown in Fig. 29.16(a). The graph as a straight line in two dimensions is shown in Fig. 29.16(b).

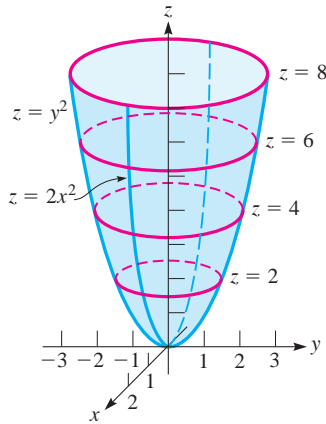


Fig. 29.15

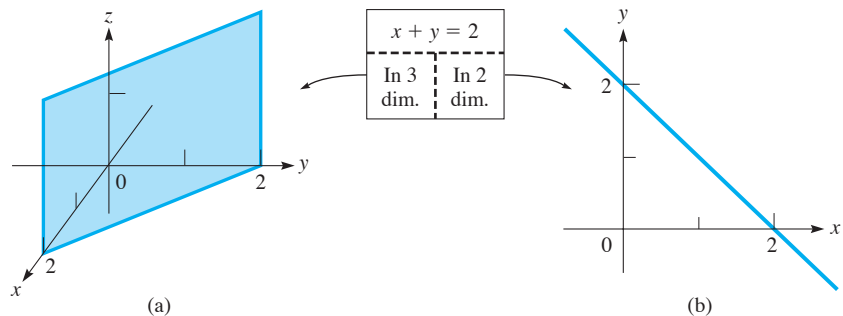


Fig. 29.16

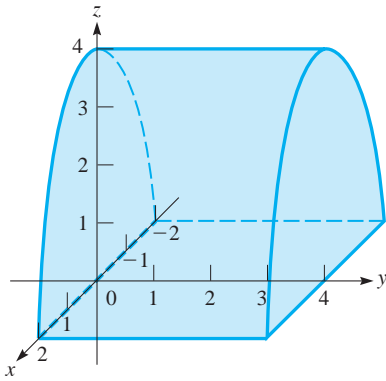


Fig. 29.17

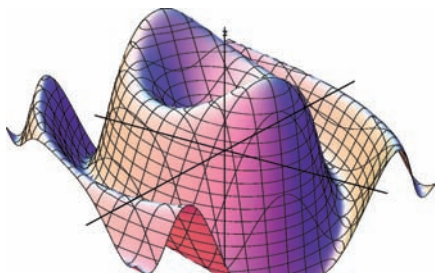


Fig. 29.18

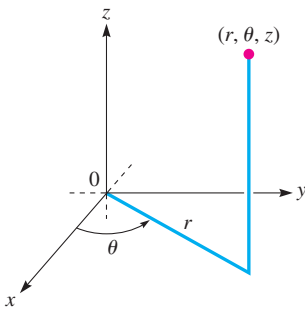


Fig. 29.19

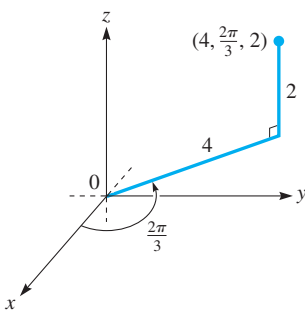


Fig. 29.20

**EXAMPLE 10 Sketching a cylindrical surface**

The graph of the equation  $z = 4 - x^2$  in three dimensions is a surface whose sections, for all values of  $y$ , are given by the parabola  $z = 4 - x^2$ . The surface is shown in Fig. 29.17.

The surface in Example 10 is known as a **cylindrical surface**. In general, a cylindrical surface is one that can be generated by a line moving parallel to a fixed line while passing through a plane curve.

It must be realized that most of the figures shown extend beyond the ranges indicated by the traces and sections. However, these traces and sections are convenient for representing and visualizing these surfaces.

There are various computer programs, and some graphing calculators, that can be used to display three-dimensional surfaces. They generally use sections in planes that are perpendicular to both the  $x$ - and  $y$ -axes. Such computer-drawn surfaces are of great value for visualizing all types of surfaces, especially those of a complex nature. Fig. 29.18 shows the graph of

$$z = \frac{\sin(2x^2 + y^2)}{x^2 + 1}$$

drawn by a computer program.

**CYLINDRICAL COORDINATES**

Another set of coordinates that can be used to display a three-dimensional figure are **cylindrical coordinates**, which are polar coordinates combined with the  $z$ -axis. In using cylindrical coordinates, every point in space is designated by the coordinates  $(r, \theta, z)$ , as shown in Fig. 29.19. The equations relating the rectangular coordinates  $(x, y, z)$  and the cylindrical coordinates  $(r, \theta, z)$  of a point are

$$\begin{aligned} x &= r \cos \theta & y &= r \sin \theta & z &= z \\ r^2 &= x^2 + y^2 & \tan \theta &= \frac{y}{x} \end{aligned} \tag{29.2}$$

The following examples illustrate the use of cylindrical coordinates.

**EXAMPLE 11 Plotting points in cylindrical coordinates**

- (a) Plot the point with cylindrical coordinates  $(4, 2\pi/3, 2)$  and find the corresponding rectangular coordinates.

This point is shown in Fig. 29.20. From Eq. (29.2), we have

$$x = 4 \cos \frac{2\pi}{3} = 4 \left( -\frac{1}{2} \right) = -2 \quad y = 4 \sin \frac{2\pi}{3} = 4 \left( \frac{\sqrt{3}}{2} \right) = 2\sqrt{3} \quad z = 2$$

Therefore, the rectangular coordinates of the point are  $(-2, 2\sqrt{3}, 2)$ .

- (b) Find the cylindrical coordinates of the point with rectangular coordinates  $(2, -2, 6)$ .

Using Eq. (29.2), we have

$$r = \sqrt{2^2 + (-2)^2} = 2\sqrt{2} \quad z = 6$$

$$\tan \theta = \frac{-2}{2} = -1 \quad (\text{quadrant IV}) \quad \theta = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$$

The cylindrical coordinates are  $(2\sqrt{2}, 7\pi/4, 6)$ . As with polar coordinates, the value of  $\theta$  can be  $7\pi/4 + 2n\pi$ , where  $n$  is an integer.

**EXAMPLE 12** Sketching a surface in cylindrical coordinates

Find the equation for the surface  $z = 4x^2 + 4y^2$  in cylindrical coordinates and sketch the surface.

Factoring the 4 from the terms on the right, we have  $z = 4(x^2 + y^2)$ . We can now use the fact that  $x^2 + y^2 = r^2$  to write this equation in cylindrical coordinates  $(r, \theta, z)$  as  $z = 4r^2$ . From this equation, we see that  $z \geq 0$  and that as  $r$  increases, we have circular sections (parallel to the  $xy$ -plane) of increasing radius. Therefore, we see that it is a **circular paraboloid**. See Fig. 29.21. ■

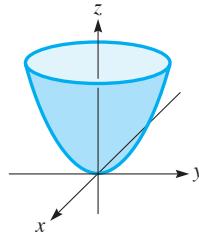


Fig. 29.21

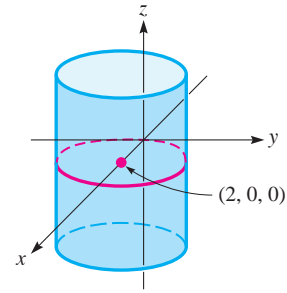


Fig. 29.22

**EXAMPLE 13** Sketching a surface in cylindrical coordinates

Find the equation for the surface  $r = 4 \cos \theta$  in rectangular coordinates and sketch the surface.

If we multiply each side of the equation by  $r$ , we obtain  $r^2 = 4r \cos \theta$ . Now, from Eq. (29.2), we have  $r^2 = x^2 + y^2$  and  $r \cos \theta = x$ , which means that  $x^2 + y^2 = 4x$ , or

$$\begin{aligned}x^2 - 4x + 4 + y^2 &= 4 \\(x - 2)^2 + y^2 &= 4\end{aligned}$$

We recognize this as a cylinder that has as its trace in the  $xy$ -plane a circle with centre at  $(2, 0, 0)$ , as shown in Fig. 29.22. ■

**LEARNING TIP**

If the centre of the trace of a circular cylinder in the  $xy$ -plane is at the origin, the equation of the cylinder in cylindrical coordinates is simply  $r = a$ . This is why these coordinates are called “cylindrical” coordinates.

**EXERCISES 29.2**

In Exercises 1 and 2, make the given changes in the indicated examples of this section and then solve the resulting problems.

- In Example 4, change the  $-4$  to  $+6$ .
- In Example 6, change the  $-$  sign before  $z^2$  to  $+$ .

In Exercises 3 and 4, use the method of Example 1 and show the graphs of the given equations for the given values of  $z$ .

- $z = x^2 + y^2$ ,  $z = 1$ ,  $z = 4$ ,  $z = 9$
- $z = y - x^2$ ,  $z = 0$ ,  $z = 2$ ,  $z = 4$

In Exercises 5–24, sketch the graphs of the given equations in the rectangular coordinate system in three dimensions.

- $x + y + 2z - 4 = 0$
- $4x - 2y + z - 8 = 0$
- $z = y - 2x - 2$
- $x + 2y = 4$
- $x^2 + y^2 + z^2 = 4$
- $z = 4 - 4x^2 - y^2$
- $2x - y - z + 6 = 0$
- $3x + 3y - 2z - 6 = 0$
- $z = x - 4y$
- $2x - 3z = 6$
- $2x^2 + 2y^2 + z^2 = 8$
- $z = x^2 + y^2$

- $z = 2x^2 + y^2 + 2$
- $x^2 - y^2 - z^2 = 9$
- $x^2 + y^2 = 16$
- $y^2 + 9z^2 = 9$
- $x^2 + y^2 - 4z^2 = 4$
- $z^2 = 9x^2 + 4y^2$
- $4z = x^2$
- $xy = 2$

In Exercises 25–30, perform the indicated operations involving cylindrical coordinates.

- Find the rectangular coordinates of the points whose cylindrical coordinates are (a)  $(3, \pi/4, 5)$ , (b)  $(2, \pi/2, 3)$ , (c)  $(4, \pi/3, 2)$ .
- Find the cylindrical coordinates of the points whose rectangular coordinates are (a)  $(3, 4, 5)$ , (b)  $(8, 15, -6)$ , (c)  $(\sqrt{3}, -2, 1)$ .
- Describe the surface for which the cylindrical coordinate equation is (a)  $r = 2$ , (b)  $\theta = 2$ , (c)  $z = 2$ .
- Write the equation  $x^2 + y^2 + 4z^2 = 4$  in cylindrical coordinates and sketch the surface.
- Write the equation  $r^2 = 4z$  in rectangular coordinates and sketch the surface.
- Write the equation  $r = 2 \sin \theta$  in rectangular coordinates and sketch the surface.

In Exercises 31–37, sketch the indicated curves and surfaces.

31. Curves that represent a constant temperature are called *isotherms*. The temperature at a point  $(x, y)$  of a flat plate is  $t$  ( $^{\circ}\text{C}$ ), where  $t = 4x - y^2$ . In two dimensions, draw the isotherms for  $t = -4, 0, 8$ .
32. At a point  $(x, y)$  in the  $xy$ -plane, the electric potential  $V$  (in volts) is given by  $V = y^2 - x^2$ . Draw the lines of equal potential for  $V = -9, 0, 9$ .
33. An electric charge is so distributed that the electric potential at all points on an imaginary surface is the same. Such a surface is called an *equipotential surface*. Sketch the graph of the equipotential surface whose equation is  $2x^2 + 2y^2 + 3z^2 = 6$ .
34. The surface of a small hill can be roughly approximated by the equation  $z(2x^2 + y^2 + 100) = 1500$ , where the units are metres. Draw the surface of the hill and the contours for  $z = 3$  m,  $z = 6$  m,  $z = 9$  m,  $z = 12$  m, and  $z = 15$  m.
35. The pressure  $p$  (in kPa), volume  $V$  (in  $\text{m}^3$ ), and temperature  $T$  (in K) for a certain gas are related by the equation  $p = T/2V$ . Sketch the  $p$ - $V$ - $T$  surface by using the  $z$ -axis for  $p$ , the  $x$ -axis for  $V$ , and the  $y$ -axis for  $T$ . Use units of 100 K for  $T$  and 10 m for  $V$ . Sections must be used for this surface, a *thermodynamic surface*, since none of the variables may equal zero.
36. Sketch the line in space defined by the intersection of the planes  $x + 2y + 3z - 6 = 0$  and  $2x + y + z - 4 = 0$ .
37. Sketch the graph of  $x^2 + y^2 - 2y = 0$  in three dimensions and in two dimensions.

#### Answer to Practice Exercise

1. (a)  $x^2 + 4y^2 = 4$  (ellipse) (b)  $x^2 - 4z^2 = 4$  (hyperbola)

## 29.3 Partial Derivatives

Meaning of the Partial Derivative •  
Notation • Geometric Interpretation •  
Partial Derivatives of Higher Order

In Chapter 23, when showing the derivative to be the instantaneous rate of change of one variable with respect to another, only one independent variable was present. To extend the derivative to functions of two (or more) variables, we find the derivative of the function with respect to one variable, while holding the other variable(s) constant.

If  $z = f(x, y)$  and  $y$  is held constant,  $z$  becomes a function of only  $x$ . The derivative of  $f(x, y)$  with respect to  $x$  is termed the **partial derivative** of  $z$  with respect to  $x$ . Similarly, if  $x$  is held constant, the derivative of  $f(x, y)$  with respect to  $y$  is the **partial derivative** of  $z$  with respect to  $y$ . The notations for the partial derivative of  $z = f(x, y)$  with respect to  $x$  include

$$\frac{\partial z}{\partial x} \quad \frac{\partial f}{\partial x} \quad f_x \quad \frac{\partial}{\partial x} f(x, y) \quad f_x(x, y)$$

■ The symbol  $\partial$  was introduced by the German mathematician Carl Jacobi (1804–1851).

Similarly,  $\partial z/\partial y$  denotes the partial derivative of  $z$  with respect to  $y$ . In speaking, this is often shortened to “the partial of  $z$  with respect to  $y$ .”

### EXAMPLE 1 Finding partial derivatives

If  $z = 4x^2 + xy - y^2$ , find  $\partial z/\partial x$  and  $\partial z/\partial y$ .

Finding the partial derivatives of  $z$ , we have

$$\begin{aligned} z &= 4x^2 + xy - y^2 \\ \frac{\partial z}{\partial x} &= 8x + y && \text{treat } y \text{ as a constant} \\ \frac{\partial z}{\partial y} &= x - 2y && \text{treat } x \text{ as a constant} \end{aligned}$$

### EXAMPLE 2 Finding partial derivatives

If  $z = \frac{x \ln y}{x^2 + 1}$ , find  $\partial z/\partial x$  and  $\partial z/\partial y$ .

$$\begin{aligned} \frac{\partial z}{\partial x} &= \frac{(x^2 + 1)(\ln y) - (x \ln y)(2x)}{(x^2 + 1)^2} = \frac{(1 - x^2) \ln y}{(1 + x^2)^2} \\ \frac{\partial z}{\partial y} &= \left( \frac{x}{x^2 + 1} \right) \left( \frac{1}{y} \right) = \frac{x}{y(x^2 + 1)} \end{aligned}$$

#### Practice Exercise

1. If  $z = 4x^2 + x \sin y$ , find  $\partial z/\partial x$  and  $\partial z/\partial y$ .



We note that in finding  $\partial z/\partial x$ , it is necessary to use the quotient rule, since  $x$  appears in both numerator and denominator. However, when finding  $\partial z/\partial y$ , the only derivative needed is that of  $\ln y$ . ■

**EXAMPLE 3** Evaluating a partial derivative

For the function  $f(x, y) = x^2y\sqrt{2 + xy^2}$ , find  $f_y(2, 1)$ .

The notation  $f_y(2, 1)$  means the partial derivative of  $f$  with respect to  $y$ , evaluated for  $x = 2$  and  $y = 1$ . Thus, first finding  $f_y(x, y)$ , we have

$$\begin{aligned} f(x, y) &= x^2y(2 + xy^2)^{1/2} \\ f_y(x, y) &= x^2y\left(\frac{1}{2}\right)(2 + xy^2)^{-1/2}(2xy) + (2 + xy^2)^{1/2}(x^2) \\ &= \frac{x^3y^2}{(2 + xy^2)^{1/2}} + x^2(2 + xy^2)^{1/2} \\ &= \frac{x^3y^2 + x^2(2 + xy^2)}{(2 + xy^2)^{1/2}} = \frac{2x^2 + 2x^3y^2}{(2 + xy^2)^{1/2}} \\ f_y(2, 1) &= \frac{2(4) + 2(8)(1)}{(2 + 2)^{1/2}} = 12 \end{aligned}$$

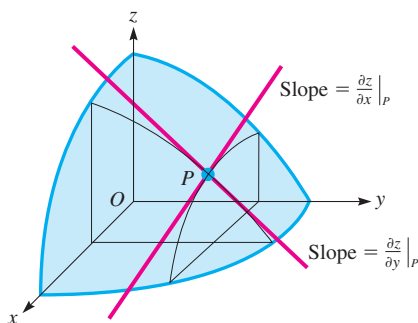


Fig. 29.23

To determine the geometric interpretation of a partial derivative, assume that  $z = f(x, y)$  is the surface shown in Fig. 29.23. Choosing a point  $P$  on the surface, we then draw a plane through  $P$  parallel to the  $xz$ -plane. On this plane through  $P$ , the value of  $y$  is constant. The intersection of this plane and the surface is the curve as indicated. *The partial derivative of  $z$  with respect to  $x$  represents the slope of a line tangent to this curve.* When the values of the coordinates of point  $P$  are substituted into the expression for this partial derivative, it gives the slope of the tangent line at that point. In the same way, the partial derivative of  $z$  with respect to  $y$ , evaluated at  $P$ , gives the slope of the line tangent to the curve that is found from the intersection of the surface and the plane parallel to the  $yz$ -plane through  $P$ .

**EXAMPLE 4** Slopes of lines tangent to a surface

Find the slope of a line tangent to the surface  $2z = x^2 + 2y^2$  and parallel to the  $xz$ -plane at the point  $(2, 1, 3)$ . Also, find the slope of the line tangent to this surface and parallel to the  $yz$ -plane at the same point.

Finding the partial derivative with respect to  $x$ , we have

$$2 \frac{\partial z}{\partial x} = 2x \quad \text{or} \quad \frac{\partial z}{\partial x} = x$$

*This derivative, evaluated at the point  $(2, 1, 3)$ , will give us the slope of the line tangent that is also parallel to the  $xz$ -plane.* Therefore, the first required slope is

$$\left. \frac{\partial z}{\partial x} \right|_{(2, 1, 3)} = 2$$

The partial derivative of  $z$  with respect to  $y$ , evaluated at  $(2, 1, 3)$ , will give us the second required slope. Thus,

$$\frac{\partial z}{\partial y} = 2y, \quad \left. \frac{\partial z}{\partial y} \right|_{(2, 1, 3)} = 2$$

Therefore, both slopes are 2. See Fig. 29.24. ■

Applications of partial derivatives are found in many fields of technology. We show here an application from electricity, and others are found in the exercises.

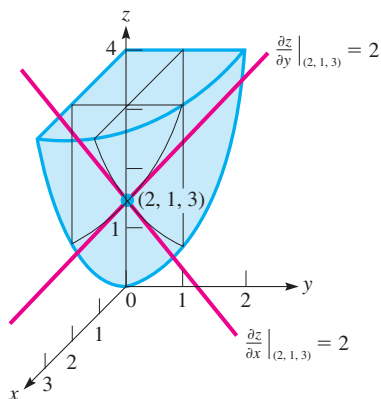


Fig. 29.24

**LEARNING TIP**

The general interpretation of the partial derivative follows that of a derivative of a function with one independent variable. *The partial derivative  $f_x(x_0, y_0)$  is the instantaneous rate of change of the function  $f(x, y)$  with respect to  $x$ , with  $y$  held constant at the value of  $y_0$ .* This holds regardless of what the variables represent.



**EXAMPLE 5** Partial derivative—application

An electric circuit in a microwave transmitter has parallel resistances  $r$  and  $R$ . The current through  $r$  can be found from

$$i = \frac{IR}{r + R}$$

where  $I$  is the total current for the two branches. Assuming that  $I$  is constant at 85.4 mA, find  $\partial i/\partial r$  and evaluate it for  $R = 0.150 \Omega$  and  $r = 0.032 \Omega$ .

Substituting for  $I$  and finding the partial derivative, we have the following:

$$\begin{aligned} i &= \frac{85.4R}{r + R} = 85.4R(r + R)^{-1} \\ \frac{\partial i}{\partial r} &= (-1)(85.4R)(r + R)^{-2}(1) = \frac{-85.4R}{(r + R)^2} \\ \frac{\partial i}{\partial r} \Big|_{\substack{r=0.032 \\ R=0.150}} &= \frac{-85.4(0.150)}{(0.032 + 0.150)^2} = -387 \text{ mA}/\Omega \end{aligned}$$

This result tells us that the current is decreasing at the rate of 387 mA per ohm of change in the smaller resistor at the instant when  $r = 0.032 \Omega$ , for a fixed value of  $R = 0.150 \Omega$ . ■

Since the partial derivatives  $\partial f/\partial x$  and  $\partial f/\partial y$  are functions of  $x$  and  $y$ , we can take partial derivatives of each of them. This gives rise to **partial derivatives of higher order**, in a manner similar to the higher derivatives of a function of one independent variable. *The possible second-order partial derivatives of a function  $f(x, y)$  are*

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) & \frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial y} \right) \\ \frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) & \frac{\partial^2 f}{\partial y \partial x} &= \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) \end{aligned}$$

**EXAMPLE 6** Second-order partial derivatives

Find the second-order partial derivatives of  $z = x^3y^2 - 3xy^3$ .

First, we find  $\partial z/\partial x$  and  $\partial z/\partial y$ :

$$\frac{\partial z}{\partial x} = 3x^2y^2 - 3y^3 \quad \text{or} \quad \frac{\partial z}{\partial y} = 2x^3y - 9xy^2$$

Therefore, we have the following second-order partial derivatives:

$$\begin{aligned} \frac{\partial^2 z}{\partial x^2} &= \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial x} \right) = 6xy^2 & \frac{\partial^2 z}{\partial y^2} &= \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial y} \right) = 2x^3 - 18xy \\ \frac{\partial^2 z}{\partial x \partial y} &= \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial y} \right) = 6x^2y - 9y^2 & \frac{\partial^2 z}{\partial y \partial x} &= \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial x} \right) = 6x^2y - 9y^2 \end{aligned}$$

In Example 6, we note that

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} \quad (29.3)$$

In general, this is true if the function and partial derivatives are continuous.

**Practice Exercise**

2. If  $z = 2x^2y - 5x^2y^4$ , find  $\frac{\partial^2 z}{\partial x \partial y}$ .

**EXAMPLE 7** Equality of second-order partial derivatives

For  $f(x, y) = \tan^{-1} \frac{y}{x^2}$ , show that  $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ .

Finding  $\partial f / \partial x$  and  $\partial f / \partial y$ , we have

$$\frac{\partial f}{\partial x} = \frac{1}{1 + \left(\frac{y}{x^2}\right)^2} \left(\frac{-2y}{x^3}\right) = \frac{x^4}{x^4 + y^2} \left(\frac{-2y}{x^3}\right) = \frac{-2xy}{x^4 + y^2}$$

$$\frac{\partial f}{\partial y} = \frac{1}{1 + \left(\frac{y}{x^2}\right)^2} \left(\frac{1}{x^2}\right) = \frac{x^4}{x^4 + y^2} \left(\frac{1}{x^2}\right) = \frac{x^2}{x^4 + y^2}$$

Now, finding  $\partial^2 f / \partial x \partial y$  and  $\partial^2 f / \partial y \partial x$ , we have

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{(x^4 + y^2)(2x) - x^2(4x^3)}{(x^4 + y^2)^2} = \frac{-2x^5 + 2xy^2}{(x^4 + y^2)^2}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{(x^4 + y^2)(-2x) - (-2xy)(2y)}{(x^4 + y^2)^2} = \frac{-2x^5 + 2xy^2}{(x^4 + y^2)^2}$$

We see that they are equal. ■

**EXERCISES 29.3**

In Exercises 1 and 2, make the given changes in the indicated examples of this section and then solve the resulting problems.

- In Example 2, change  $x^2$  to  $y^2$ .
- In Example 4, change the equation to  $2z = 2x^2 + y^2$  and the point  $(2, 1, 3)$  to  $(1, 2, 3)$ .

In Exercises 3–24, find the partial derivative of the dependent variable or function with respect to each of the independent variables.

- $z = 5x + 4x^2y$
- $z = 3x^2y^3 - 3x + 4y$
- $f(x, y) = xe^{2y}$
- $z = 3y \cos 2x$
- $f(x, y) = \frac{2 + \cos x}{1 - \sec 3y}$
- $f(x, y) = \frac{\tan^{-1} y}{2 + x^2}$
- $\phi = r\sqrt{1 + 2rs}$
- $w = uv^2\sqrt{1 - u}$
- $z = (x^2 + xy^3)^4$
- $f(x, y) = (2xy - x^2)^5$
- $z = \sin xy$
- $y = \tan^{-1}\left(\frac{x}{t}\right)$
- $y = \ln(r^2 + s)$
- $u = e^{x+2y}$
- $f(x, y) = \frac{2 \sin^3 2x}{1 - 3y}$
- $f(x, y) = \frac{3x + \ln y}{x^2 + y^2}$
- $z = \frac{\sin^{-1} xy}{3 + x^2}$
- $z = \frac{\sqrt{1 - \tan xy}}{xy + y^2}$
- $z = \sin x + \cos xy - \cos y$
- $t = 2re^{rs^2} - \tan(2r + s)$

$$23. f(x, y) = e^x \cos xy + e^{-2x} \tan y$$

$$24. u = \ln \frac{y^2}{x - y} + e^{-x} (\sin y - \cos 2y)$$

In Exercises 25–28, evaluate the indicated partial derivatives at the given points.

$$25. z = 3xy - x^2, \frac{\partial z}{\partial x} \Big|_{(1, -2, -7)} \quad 26. z = x^2 \cos 4y, \frac{\partial z}{\partial y} \Big|_{(2, \frac{\pi}{2}, 4)}$$

$$27. z = x\sqrt{x^2 - y^2}, \frac{\partial z}{\partial x} \Big|_{(5, 3, 20)} \quad 28. z = e^y \ln xy, \frac{\partial z}{\partial y} \Big|_{(e, 1, e)}$$

In Exercises 29–32, find all second partial derivatives.

- $z = 2xy^3 - 3x^2y$
- $F(x, y) = y \ln(x + 2y)$
- $z = \frac{x}{y} + e^x \sin y$
- $f(x, y) = \frac{2 + \cos y}{1 + x^2}$

In Exercises 33–45, solve the given problems.

- Find the slope of a line tangent to the surface  $z = 9 - x^2 - y^2$  and parallel to the  $yz$ -plane that passes through  $(1, 2, 4)$ . Repeat the instructions for the line through  $(2, 2, 1)$ . Draw an appropriate figure.
- A metal plate in the shape of a circular segment of radius  $r$  expands by being heated. Express the width  $w$  (straight dimension) as a function of  $r$  and the height  $h$ . Then find both  $\partial w / \partial r$  and  $\partial w / \partial h$ .
- Two resistors  $R_1$  and  $R_2$ , placed in parallel, have a combined resistance  $R_T$  given by  $\frac{1}{R_T} = \frac{1}{R_1} + \frac{1}{R_2}$ . Find  $\frac{\partial R_T}{\partial R_1}$ .

36. Find  $\partial z/\partial y$  for the function  $z = 4x^2 - 8$ . Explain your result. Draw an appropriate figure.
37. A metallic machine part contracts while cooling. It is in the shape of a hemisphere attached to a cylinder, as shown in Fig. 29.25. Find the rate of change of volume with respect to  $r$  when  $r = 2.65$  cm and  $h = 4.20$  cm.

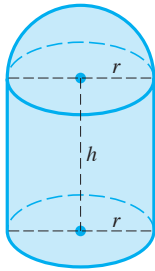


Fig. 29.25

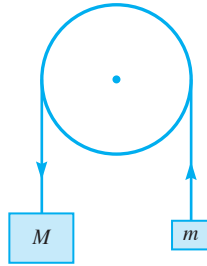


Fig. 29.26

38. Two masses  $M$  and  $m$  are attached as shown in Fig. 29.26. If  $M > m$ , the downward acceleration  $a$  of mass  $M$  is given by  $a = \frac{M - m}{M + m}g$ , where  $g$  is the acceleration due to gravity. Show that  $M \frac{\partial a}{\partial M} + m \frac{\partial a}{\partial m} = 0$ .
39. In quality testing, a rectangular sheet of vinyl is stretched. Set up the length of the diagonal  $d$  of the sheet as a function of the sides  $x$  and  $y$ . Find the rate of change of  $d$  with respect to  $x$  for  $x = 2.20$  m if  $y$  remains constant at 1.55 m.
40. If an observer and a source of sound are moving toward or away from each other, the observed frequency of sound is different from that emitted. This is known as the *Doppler effect*. The equation relating the frequency  $f_o$  the observer hears and the frequency  $f_s$  emitted by the source (a constant) is  $f_o = f_s \left( \frac{v + v_o}{v - v_s} \right)$ , where  $v$  is the velocity of sound in air (a

constant),  $v_o$  is the velocity of the observer, and  $v_s$  is the velocity of the source. Show that  $f_s \frac{\partial f_o}{\partial v_s} = f_o \frac{\partial f_o}{\partial v_o}$ . Explain the meaning of  $\partial f_o/\partial v_s$ .

41. The *mutual conductance* (in  $1/\Omega$ ) of a certain electronic device is defined as  $g_m = \partial i_b/\partial V_c$ . Under certain circumstances, the current  $i_b$  (in  $\mu\text{A}$ ) is given by  $i_b = 50(V_b + 5V_c)^{1.5}$ . Find  $g_m$  when  $V_b = 200$  V and  $V_c = -20$  V.
42. The *amplification factor* of the electronic device of Exercise 41 is defined as  $\mu = -\partial V_b/\partial V_c$ . For the device of Exercise 41, under the given conditions, find the amplification factor.
43. The temperature  $u$  in a metal bar depends on the distance  $x$  from one end and the time  $t$ . Show that  $u(x, t) = 5e^{-t} \sin 4x$  satisfies the *one-dimensional heat-conduction equation*  $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$ , where  $k$  is called the *diffusivity*. Here  $k = 1/16$ .
44. The displacement  $y$  at any point in a taut, flexible string depends on the distance  $x$  from one end of the string and the time  $t$ . Show that  $y(x, t) = 2 \sin 2x \cos 4t$  satisfies the *wave equation*  $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$  with  $a = 2$ .
45. The steady-state temperature  $u$  of a thin, flat plate satisfies *Laplace's equation*  $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ . Show that the function  $u(x, y) = e^{-x} \sin y$  satisfies Laplace's equation.

Answers to Practice Exercises

1.  $\partial z/\partial x = 8x + \sin y, \partial z/\partial y = x \cos y$
2.  $\frac{\partial^2 z}{\partial x \partial y} = 4x - 40xy^3$

## 29.4 Double Integrals

Meaning of the Double Integral • Notation • Volume Under a Surface

We now turn our attention to integration in the case of a function of two variables. The analysis has similarities to that of partial differentiation, in that an operation is performed while holding one of the independent variables constant.

If  $z = f(x, y)$  and we wish to integrate with respect to  $x$  and  $y$ , we first consider either  $x$  or  $y$  constant and integrate with respect to the other. After this integral is evaluated, we then integrate with respect to the variable first held constant. We shall now define this type of integral and then give an appropriate geometric interpretation.

If  $z = f(x, y)$  the **double integral** of the function over  $x$  and  $y$  is defined as

$$\int_a^b \left[ \int_{g(x)}^{G(x)} f(x, y) dy \right] dx$$

Since it is customary not to include the brackets in writing a double integral, we write

$$\int_a^b \left[ \int_{g(x)}^{G(x)} f(x, y) dy \right] dx = \int_a^b \int_{g(x)}^{G(x)} f(x, y) dy dx \tag{29.4}$$

**LEARNING TIP**

- When the outer differential of a double integral is  $dx$ , the limits on the inner integral are functions of  $x$ , and those on the outer integral are explicit values of  $x$ . The inner integral is performed first, integrating with respect to  $y$  while keeping  $x$  constant. This results in an integral with  $x$  only, which can then be evaluated.
- When the outer differential of a double integral is  $dy$ , the limits on the inner integral are functions of  $y$ , and those on the outer integral are explicit values of  $y$ . The inner integral is performed first, integrating with respect to  $x$  while keeping  $y$  constant. This results in an integral with  $y$  only, which can then be evaluated.

**Practice Exercise**

1. Evaluate  $\int_0^1 \int_0^x 2xy \, dy \, dx$ .

**EXAMPLE 1** Evaluating a double integral

Evaluate  $\int_0^1 \int_{x^2}^x xy \, dy \, dx$ .

First, we integrate the inner integral with  $y$  as the variable and  $x$  as a constant.

$$\int_{x^2}^x xy \, dy = \left( x \frac{y^2}{2} \right) \Big|_{x^2}^x = x \left( \frac{x^2}{2} - \frac{x^4}{2} \right) = \frac{1}{2} (x^3 - x^5)$$

This means

$$\begin{aligned} \int_0^1 \int_{x^2}^x xy \, dy \, dx &= \int_0^1 \frac{1}{2} (x^3 - x^5) \, dx = \frac{1}{2} \left( \frac{x^4}{4} - \frac{x^6}{6} \right) \Big|_0^1 \\ &= \frac{1}{2} \left( \frac{1}{4} - \frac{1}{6} \right) - \frac{1}{2} (0) = \frac{1}{24} \end{aligned}$$

**EXAMPLE 2** Evaluating a double integral

Evaluate  $\int_0^{\pi/2} \int_0^{\sin y} e^{2x} \cos y \, dx \, dy$ .

Since the inner differential is  $dx$ , we first integrate with  $x$  as the variable and  $y$  as a constant. The second integration is with  $y$  as the variable.

$$\begin{aligned} \int_0^{\pi/2} \int_0^{\sin y} e^{2x} \cos y \, dx \, dy &= \int_0^{\pi/2} \left[ \frac{1}{2} e^{2x} \cos y \right]_0^{\sin y} \, dy \\ &= \frac{1}{2} \int_0^{\pi/2} (e^{2 \sin y} \cos y - \cos y) \, dy \\ &= \frac{1}{2} \left[ \frac{1}{2} e^{2 \sin y} - \sin y \right]_0^{\pi/2} = \frac{1}{2} \left( \frac{1}{2} e^2 - 1 \right) - \frac{1}{2} \left( \frac{1}{2} - 0 \right) \\ &= \frac{1}{4} e^2 - \frac{1}{2} - \frac{1}{4} = \frac{1}{4} (e^2 - 3) = 1.097 \end{aligned}$$

**LEARNING TIP**

We interpret a double integral as the **volume under a surface**, in the same way as the integral was interpreted as the area of a plane figure.

For the geometric interpretation of a double integral, consider the surface shown in Fig. 29.27(a). An **element of volume** (dimensions of  $dx$ ,  $dy$ , and  $z$ ) extends from the  $xy$ -plane to the surface. With  $x$  a constant, sum (integrate) these elements of volume from the left boundary,  $y = g(x)$ , to the right boundary,  $y = G(x)$ . Now, the volume of the vertical slice is a function of  $x$ , as shown in Fig. 29.27(b). By summing (integrating) the volumes of these slices from  $x = a$  ( $x = 0$  in the figure) to  $x = b$ , we have the complete volume, as shown in Fig. 29.27(c).

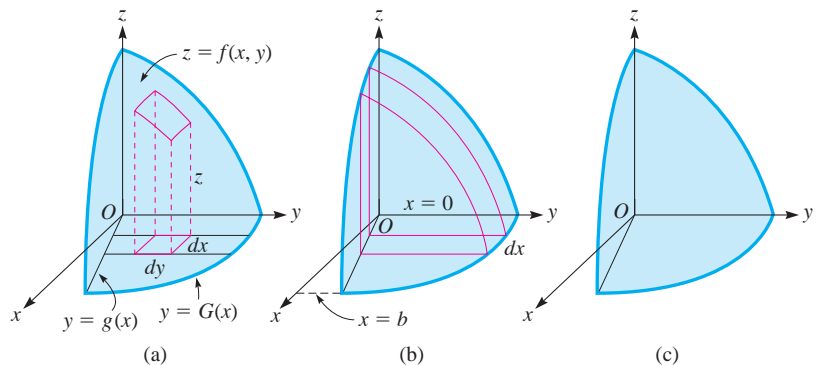


Fig. 29.27

**EXAMPLE 3** Volume under a plane

Find the volume that is in the first octant and under the plane  $x + 2y + 4z - 8 = 0$ . See Fig. 29.28.

This figure is a tetrahedron, for which  $V = \frac{1}{3}Bh$ . Assuming the base is in the  $xy$ -plane,  $B = \frac{1}{2}(4)(8) = 16$ , and  $h = 2$ . Therefore,  $V = \frac{1}{3}(16)(2) = \frac{32}{3}$  cubic units. We shall use this value to check the one we find by double integration.

To find  $z = f(x, y)$ , we solve the given equation for  $z$ . Thus,

$$z = \frac{8 - x - 2y}{4}$$

Next, we must find the limits on  $y$  and  $x$ . Choosing to integrate over  $y$  first, we see that  $y$  goes from  $y = 0$  to  $y = (8 - x)/2$ . **This last limit is the trace of the surface in the  $xy$ -plane.** Next, we note that  $x$  goes from  $x = 0$  to  $x = 8$ . Therefore, we set up and evaluate the integral:

$$\begin{aligned} V &= \int_0^8 \int_0^{(8-x)/2} \left( \frac{8-x-2y}{4} \right) dy dx \\ &= \int_0^8 \left[ \frac{1}{4} (8y - xy - y^2) \Big|_0^{(8-x)/2} \right] dx \\ &= \frac{1}{4} \int_0^8 \left[ 8 \left( \frac{8-x}{2} \right) - x \left( \frac{8-x}{2} \right) - \left( \frac{8-x}{2} \right)^2 \right] dx \\ &= \frac{1}{4} \int_0^8 \left( 32 - 4x - 4x + \frac{x^2}{2} - 16 + 4x - \frac{x^2}{4} \right) dx \\ &= \frac{1}{4} \int_0^8 \left( 16 - 4x + \frac{x^2}{4} \right) dx \\ &= \frac{1}{4} \left( 16x - 2x^2 + \frac{x^3}{12} \right) \Big|_0^8 = \frac{1}{4} \left( 128 - 128 + \frac{512}{12} \right) \\ &= \frac{1}{4} \left( \frac{128}{3} \right) = \frac{32}{3} \text{ cubic units} \end{aligned}$$

We see that the values obtained by the two different methods agree. ■

**EXAMPLE 4** Volume under a surface

Find the volume above the  $xy$ -plane, below the surface  $z = xy$ , and enclosed by the cylinder  $y = x^2$  and the plane  $y = x$ .

Because the cylinders are perpendicular to the  $xy$ -plane, the volume of interest lies under the function  $z = xy$  and above the region on the  $xy$ -plane bounded by the parabola  $y = x^2$  and the line  $y = x$  (see Fig. 29.29). The limits of integration can be found directly from the graph of this region without having to plot the surface. Integrating over  $y$  first, the limits on  $y$  are  $y = x^2$  to  $y = x$ . The corresponding limits on  $x$  are  $x = 0$  to  $x = 1$ . Therefore, the double integral to be evaluated is

$$V = \int_0^1 \int_{x^2}^x xy \, dy \, dx$$

This integral has already been evaluated in Example 1 of this section, and we can now see the geometric interpretation of that integral. Using the result from Example 1, we see that the required volume is  $1/24$  cubic unit. ■

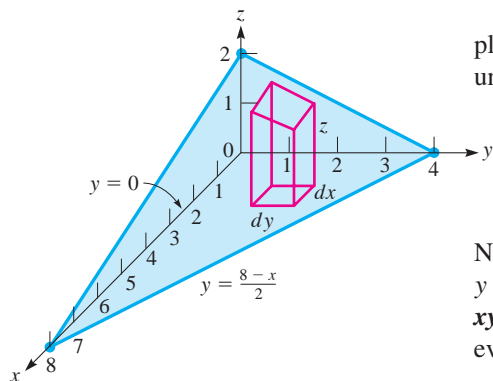


Fig. 29.28

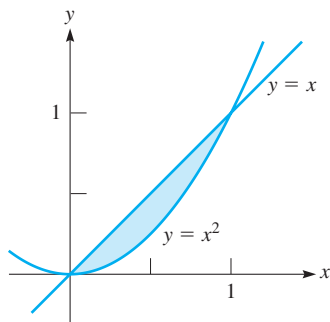


Fig. 29.29

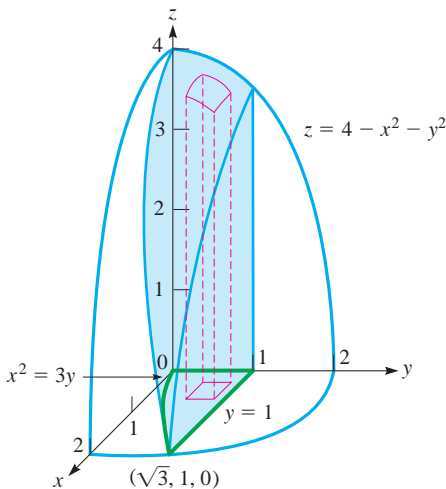


Fig. 29.30

**EXAMPLE 5** Volume under a surface

Find the volume in the first octant that is under the surface  $z = 4 - x^2 - y^2$ , and is between the cylinder  $x^2 = 3y$  and the plane  $y = 1$ . See Fig. 29.30.

We find the limits of integration from the shaded region on the  $xy$ -plane. Integrating over  $x$  first, we have

$$\begin{aligned} V &= \int_0^1 \int_0^{\sqrt{3y}} (4 - x^2 - y^2) dx dy \\ &= \int_0^1 \left[ 4x - \frac{x^3}{3} - y^2x \right]_0^{\sqrt{3y}} dy \\ &= \int_0^1 (4\sqrt{3y} - \sqrt{3}y^{3/2} - \sqrt{3}y^{5/2}) dy \\ &= \sqrt{3} \left[ 4\left(\frac{2}{3}\right)y^{3/2} - \frac{2}{5}y^{5/2} - \frac{2}{7}y^{7/2} \right] \Big|_0^1 \\ &= \sqrt{3} \left( \frac{8}{3} - \frac{2}{5} - \frac{2}{7} \right) = \frac{208\sqrt{3}}{105} \approx 3.43 \text{ cubic units} \end{aligned}$$

If we integrate over  $y$  first, we arrive at the same result evaluating the double integral

$$V = \int_0^{\sqrt{3}} \int_{x^2/3}^1 (4 - x^2 - y^2) dy dx$$

**EXERCISES 29.4**

In Exercises 1 and 2, make the given changes in the indicated examples of this section and then solve the resulting problems.

- In Example 1, change the integrand  $xy$  to  $(x + y)$ .
- In Example 2, delete the  $e^{2x}$ .

In Exercises 3–16, evaluate the given double integrals.

- $\int_2^4 \int_0^1 xy^2 dx dy$
- $\int_0^2 \int_0^1 \frac{y}{(xy + 1)^2} dx dy$
- $\int_1^2 \int_0^{y^2} xy^2 dx dy$
- $\int_0^4 \int_1^{\sqrt{y}} (x - y) dx dy$
- $\int_0^1 \int_0^{\sqrt{1-x^2}} y dy dx$
- $\int_4^9 \int_0^x \sqrt{x-y} dy dx$
- $\int_0^{\pi/6} \int_{\pi/3}^y \sin x dx dy$
- $\int_0^{\sqrt{3}} \int_{x^2/3}^1 (4 - x^2) dy dx$
- $\int_1^e \int_1^y \frac{1}{x} dx dy$
- $\int_{-1}^1 \int_1^{e^x} \frac{1}{xy} dy dx$
- $\int_1^2 \int_0^x yx^3 e^{xy^2} dy dx$
- $\int_0^{\pi/6} \int_0^1 y \sin x dy dx$
- $\int_0^{\ln 3} \int_0^x e^{2x+3y} dy dx$
- $\int_0^{1/2} \int_y^{y^2} \frac{dx dy}{\sqrt{y^2 - x^2}}$

In Exercises 17–26, find the indicated volumes by double integration.

- The first-octant volume under the plane  $x + y + z - 4 = 0$
- The first-octant volume under the surface  $z = y^2$  and bounded by the planes  $x = 2$  and  $y = 3$

- The volume above the  $xy$ -plane and under the surface  $z = 4 - x^2 - y^2$
- The volume above the  $xy$ -plane, below the surface  $z = x^2 + y^2$ , and inside the cylinder  $x^2 + y^2 = 4$
- The first-octant volume bounded by the  $xy$ -plane, the planes  $x = y$ ,  $y = 2$ , and  $z = 2 + x^2 + y^2$
- The volume bounded by the planes  $x + 3y + 2z - 6 = 0$ ,  $2x = y$ ,  $x = 0$ , and  $z = 0$
- The first-octant volume under the plane  $z = x + y$  and inside the cylinder  $x^2 + y^2 = 9$
- The volume above the  $xy$ -plane and bounded by the cylinders  $x = y^2$ ,  $y = 8x^2$ , and  $z = x^2 + 1$ . Integrate over  $y$  first and then check by integrating over  $x$  first.
- A wedge is to be made in the shape shown in Fig. 29.31. (All vertical cross sections are equal right triangles.) By double integration, find the volume of the wedge.

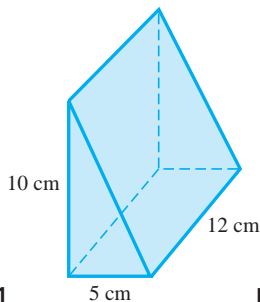


Fig. 29.31

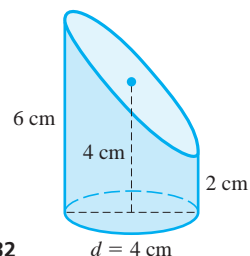


Fig. 29.32

- A circular piece of pipe is cut as shown in Fig. 29.32. Find the volume within the pipe. Describe how to set up the coordinate system in order to determine the required volume.

In Exercises 27 and 28, draw the appropriate figure.

27. Draw the appropriate figure indicating a volume that is found from the integral

$$\int_0^1 \int_{x^2}^1 (4 - x - 2y) dy dx.$$

28. Repeat Exercise 27 for the integral

$$\int_1^2 \int_0^{2-y} \sqrt{1 + x^2 + y^2} dx dy.$$

**Answer to Practice Exercise**

1. 1/4

## CHAPTER 29 EQUATIONS

### Equation of a plane

$$Ax + By + Cz + D = 0 \quad (29.1)$$

### Cylindrical coordinates

$$x = r \cos \theta \quad y = r \sin \theta \quad z = z$$

$$r^2 = x^2 + y^2 \quad \tan \theta = \frac{y}{x} \quad (29.2)$$

### Partial derivatives

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} \quad (29.3)$$

### Double integral

$$\int_a^b \left[ \int_{g(x)}^{G(x)} f(x, y) dy \right] dx = \int_a^b \int_{g(x)}^{G(x)} f(x, y) dy dx \quad (29.4)$$

## CHAPTER 29 REVIEW EXERCISES

In Exercises 1–4, evaluate the given functions.

- $f(x, y) = 3x^2y - y^3$ , find  $f(-1, 4)$
- $f(r, \theta) = r^2 \cos 2\theta - r \sin \theta$ , find  $f(3, \pi)$
- $f(s, t) = \frac{2st^2 - t}{s}$ , find  $f(4, s^2)$
- $f(x, y) = \frac{4x}{xy - 2}$ , find  $f(x^2, 2x)$

In Exercise 5–8, sketch the graphs of the given equations in the rectangular coordinate system in three dimensions.

- $x - y + 2z - 4 = 0$
- $2y + 3z = 6$
- $z = x^2 + 4y^2$
- $x^2 + y^2 - 4z^2 - 4 = 0$

In Exercises 9–18, find the partial derivatives of the given functions with respect to each of the independent variables.

$$9. z = 5x^3y^2 - 2xy^4 \quad 10. z = 2x\sqrt{y} - x^2y$$

$$11. z = \sqrt{x^2 - 3y^2} \quad 12. u = \frac{r}{(r - 3s)^2}$$

$$13. z = \frac{2x - 3y}{x^2y + 1} \quad 14. z = x(y^2 + xy + 2)^4$$

$$15. u = y \ln \sin(x^2 + 2y) \quad 16. q = p \ln(r + 1) - \frac{rp}{r + 1}$$

$$17. z = \sin^{-1} \sqrt{x + y}$$

$$18. z = ye^{xy} \sin(2x - y)$$

In Exercises 19 and 20, find all of the second partial derivatives of the given functions.

$$19. z = 3x^2y - y^3 + 2xy$$

$$20. z = x\sqrt{2y + 1} + y^2(x - 2)^3$$

In Exercises 21–28, evaluate each of the given double integrals.

$$21. \int_0^2 \int_1^2 (3y + 2xy) dx dy \quad 22. \int_2^7 \int_0^1 x\sqrt{2 + x^2y} dx dy$$

$$23. \int_0^3 \int_1^x (x + 2y) dy dx \quad 24. \int_1^2 \int_0^{\pi/4} r \sec^2 \theta d\theta dr$$

$$25. \int_0^1 \int_0^{2x} x^2 e^{xy} dy dx \quad 26. \int_{\pi/4}^{\pi/2} \int_1^{\sqrt{\cos \theta}} r \sin \theta dr d\theta$$

$$27. \int_1^e \int_1^x \frac{\ln y}{xy} dy dx \quad 28. \int_1^3 \int_0^x \frac{2}{x^2 + y^2} dy dx$$

In Exercises 29–46, solve the given problems.

29. Sketch the surface representing  $z = \sqrt{x^2 + 4y^2}$ .

30. For the function of Exercise 29, find the equation of a line tangent to the surface at  $(2, 1, 2\sqrt{2})$  that is parallel to the  $yz$ -plane.

31. Sketch the surface representing  $z = e^{x+y}$ .



32. For the function of Exercise 31, find the volume in the first octant under the surface and inside the planes  $x = 1$  and  $y = x$ .
33. Describe the surface for which the cylindrical coordinate equation is (a)  $\theta = 3$ , (b)  $z = r^2$ .
34. Write the cylindrical coordinate equation  $r = 2(\sin \theta + \cos \theta)$  in rectangular coordinates and sketch the surface.
35. In a simple series electric circuit, with two resistors  $r$  and  $R$  connected across a voltage source  $E$ , the voltage  $V$  across  $r$  is  $V = rE/(r + R)$ . Assuming  $E$  to be constant, find  $\partial V/\partial r$  and  $\partial V/\partial R$ .
36. For a gas, the volume expansivity is defined as  $\beta = \frac{1}{V} \frac{\partial V}{\partial T}$ , where  $V$  is the volume and  $T$  is the temperature of the gas. If  $V$  as a function of  $T$  and the pressure  $p$  is given by  $V = a + bT/p - c/T^2$ , where  $a$ ,  $b$ , and  $c$  are constants, find  $\beta$ .
37. In the theory dealing with transistors, the current gain  $\alpha$  of a transistor is defined as  $\alpha = \partial i_c/\partial i_e$ , where  $i_c$  is the collector current and  $i_e$  is the emitter current. If  $i_c$  is a function of  $i_e$  and the collector voltage  $V_c$  given by  $i_c = i_e(1 - e^{-2V_c})$ , find  $\alpha$  if  $V_c$  is 2 V.
38. Young's modulus, which measures the ratio of the stress to strain in a stretched wire, is defined as  $Y = \frac{L}{A} \frac{\partial F}{\partial L}$ , where  $L$  is the length of the wire,  $A$  is its cross-sectional area, and  $F$  is the tension in the wire. Find  $Y$  (in Pa) if  $F = -0.0100 T/L^2$  for  $L = 1.10$  m,  $T = 300$  K, and  $A = 1.00 \times 10^{-6}$  m<sup>2</sup>.
39. The period  $T$  of the pendulum as a function of its length  $l$  and the acceleration due to gravity  $g$  is given by  $T = 2\pi\sqrt{l/g}$ . Show that  $\partial T/\partial l = T/2l$ .
40. The volume of a right circular cone of radius  $r$  and height  $h$  is given by  $V = \frac{1}{3}\pi r^2 h$ . Show that  $\partial V/\partial r = 2V/r$ .
41. The coefficient of linear expansion  $\alpha$  of a wire whose length  $L$  is a function of the tension and temperature  $T$  is given by  $\alpha = \frac{1}{L} \left( \frac{\partial L}{\partial T} \right)$ . If  $L$  is a function of  $T$  and the tension  $F$  given by  $L = L_0 + k_1 F + k_2 T + k_3 F T^2$ , where  $k_1$ ,  $k_2$ , and  $k_3$  are constants, find the expression for  $\alpha$ .
42. An isothermal process is one during which the temperature does not change. If the volume  $V$ , pressure  $p$ , and temperature  $T$  of an ideal gas are related by the equation  $pV = nRT$ , where  $n$  and  $R$  are constants, find the expression for  $\partial p/\partial V$ , which is the rate of change of pressure with respect to volume for an isothermal process.
43. For the ideal gas of Exercise 42, show that  $\left( \frac{\partial V}{\partial T} \right) \left( \frac{\partial T}{\partial p} \right) \left( \frac{\partial p}{\partial V} \right) = -1$ .
44. Find the volume in the first octant below the plane  $x + y + z - 6 = 0$  and inside the cylinder  $y = 4 - x^2$ .
45. Find the first-octant volume bounded by  $x^2 + y^2 = 16$  and  $x + z = 8$ . Describe each of the bounding surfaces.
46. Find the first-octant volume below the surface  $z = 4 - y^2$  and inside the plane  $x + y = 2$ .

### Writing Exercise

47. An architectural design student determined the area of a patio could be described by the double integral  $\int_0^4 \int_0^{\sqrt{y}} f(x, y) dx dy$ . Write the integration with the order of integration interchanged. Write one or two paragraphs to explain your method on interchanging the order of integration.

## CHAPTER 29 PRACTICE TEST

- Given  $f(x, y) = \frac{2y}{x^2 - y^2}$ , find  $f(-1, 3)$ .
- Sketch the surface representing the function  $z = 4 - x^2 - 4y^2$ .
- Given  $z = xe^{2xy}$ , find  $\frac{\partial z}{\partial x}$  and  $\frac{\partial z}{\partial y}$ .
- Given  $z = 3x^3y + 2x^2y^4$ , find  $\frac{\partial^2 z}{\partial x \partial y}$ .
- Evaluate  $\int_0^2 \int_{x^2}^{2x} (x^3 + 4y) dy dx$ .
- Evaluate  $\int_1^{\ln 8} \int_0^{\ln y} e^{x+y} dx dy$ .
- Find the volume in the first octant bounded by the coordinate planes and the cylinders  $x^2 + y^2 = 9$  and  $y^2 + z^2 = 9$ .
- The fundamental frequency of vibration  $f$  of a string varies directly as the square root of the tension  $T$  and inversely as the length  $L$ . If a string 60 cm long is under a tension of 65 N and has a fundamental frequency of 30 Hz, find the partial derivative of  $f$  with respect to  $T$  and evaluate it for the given values.

# Expansion of Functions in Series

In the mid-1600s, mathematicians found that transcendental functions can be represented by polynomials and that by using these polynomials it was possible to calculate the values of these functions more easily. In this chapter, we show how a given function may be expressed in terms of a polynomial and how this polynomial is used to evaluate the function.

The polynomials that we will develop are known as *power series*, and they can be expressed with an unlimited number of terms. Although first noted for their usefulness in calculating values of transcendental functions, many mathematicians, including Newton, used power series extensively in their contributions to various areas of mathematics. In fact, the French mathematician Joseph-Louis Lagrange (1736–1813) attempted to make power series the basis for the development of all methods in calculus.

Another type of series was developed by the French physicist and mathematician Jean Baptiste Joseph Fourier (1768–1830) in the study of heat conduction. In 1822, he showed that a function can be expressed in a series of sine and cosine terms. Today, these series are very important in the study of electricity and electronics. They are also useful in the study of mechanical vibrations and other applications that are periodic in nature. We study these series in the last two sections of this chapter.

We see again that a concept first used to ease calculation became very important in the later development of mathematics. Also, a concept developed for the study of heat, long before the advent of electronics, has become important in electronics.

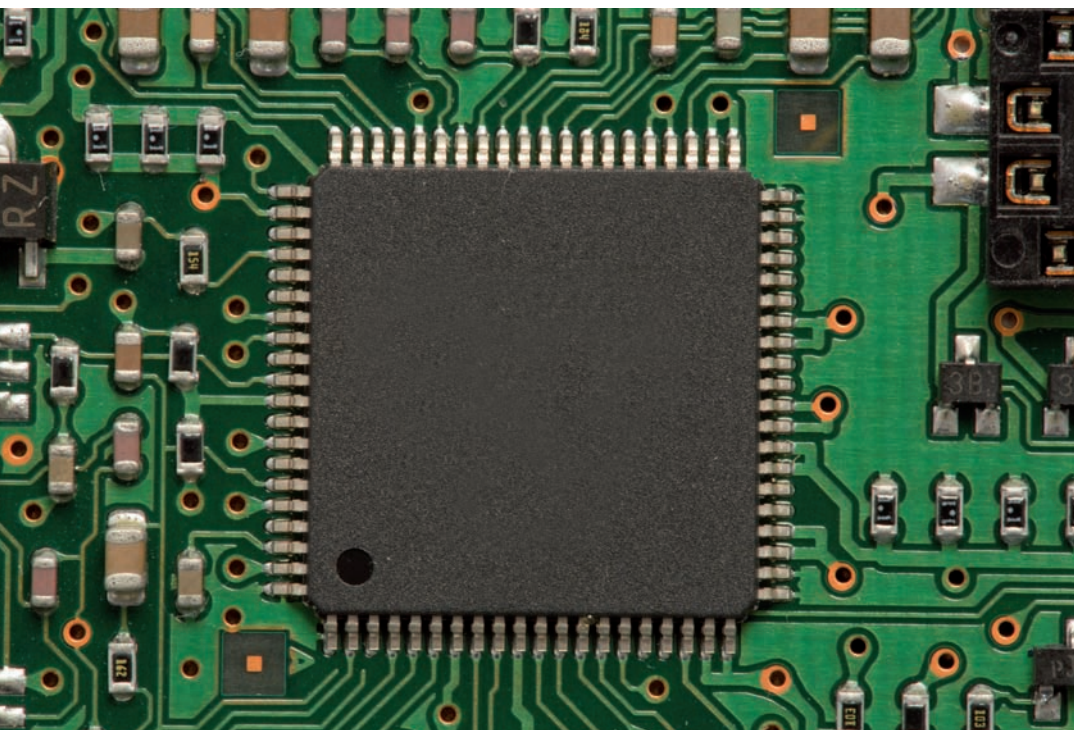
In Chapter 19, we discussed arithmetic and geometric sequences and the concept of an infinite geometric sequence. In the first section of this chapter, we develop these topics further for use in the sections that follow.

# 30

## LEARNING OUTCOMES

After completion of this chapter, the student should be able to:

- Find the terms of sequences and series
- Decide whether a geometric series is convergent or divergent and, if convergent, find its sum
- Find the Maclaurin series expansion of a function
- Use algebraic, trigonometric, or calculus procedures on known series to obtain other series expansions
- Find the Taylor series expansion of a function
- Approximate the value of functions by using series
- Find the Fourier series expansion of a periodic function
- Find the half-range Fourier series of a function
- Solve application problems using series



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◀ Many types of electronic devices are used to control the current in a circuit. In Section 30.6, we see how series are used to analyse one such device.

## 30.1 Infinite Series

Sequence • Term • Partial Sum •  
Convergent • Divergent • Geometric Series

Aside from arithmetic sequences and geometric sequences, there are many other ways of generating sequences of numbers. The squares of the integers 1, 4, 9, 16, 25 . . . form a sequence. Also, the successive approximations  $x_1, x_2, x_3, \dots$  found by using Newton's method in solving a particular equation form a sequence.

In general, a **sequence** (or **infinite sequence**) is an infinite succession of numbers. Each of the numbers is a **term** of the sequence. Each term of the sequence is associated with a positive integer, although at times it is convenient to associate the first term with zero (or some specified positive integer). We shall use  $a_n$  to designate the term of the sequence corresponding to the integer  $n$ .

### EXAMPLE 1 Find the terms of a sequence, given the general term

Find the first three terms of the sequence for which the general term is  $a_n = 2n + 1$ ,  $n = 1, 2, 3, \dots$

Substituting the values of  $n$ , we obtain the values

$$a_1 = 2(1) + 1 = 3 \quad a_2 = 2(2) + 1 = 5 \quad a_3 = 2(3) + 1 = 7, \dots$$

Therefore, we have the sequence 3, 5, 7, . . . .

Given  $a_n = 2n + 1$  for  $n = 0, 1, 2, \dots$ , the sequence is 1, 3, 5, . . . . ■

As we stated in Chapter 19, the sum of the terms of a sequence is called an **infinite series**. Thus, for the sequence

$$a_1, a_2, a_3, \dots, a_n, \dots$$

the associated infinite series is

$$a_1 + a_2 + a_3 + \dots + a_n + \dots$$

Using the summation sign  $\Sigma$  (see Section 22.2) to indicate the sum, we have

Infinite Series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots + a_n + \dots \quad (30.1)$$

Since it is not possible to find the sum of infinitely many terms, we define the sum for an infinite series in terms of a limit. For the infinite series of Eq. (30.1), we let  $S_n$  represent the sum of the first  $n$  terms. Therefore,

$$\begin{aligned} S_1 &= a_1 \\ S_2 &= a_1 + a_2 \\ S_3 &= a_1 + a_2 + a_3 \\ S_n &= a_1 + a_2 + a_3 + \dots + a_n \end{aligned}$$

Partial Sum

The numbers  $S_1, S_2, S_3, \dots, S_n, \dots$  form a sequence. Each term of this sequence is called a **partial sum**. We say that the infinite series, Eq. (30.1), is **convergent** and has the sum  $S$  given by

$$S = \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \sum_{i=1}^n a_i \quad (30.2)$$

if this limit exists. If the limit does not exist, the series is **divergent**.

■ The *sequence* feature on some graphic calculators can be used to find the terms of a sequence.

### Practice Exercise

- Find the first three terms of the sequence for which  $a_n = \frac{n+3}{n^2+1}$ ,  $n = 1, 2, 3, \dots$

**EXAMPLE 2** Convergence of partial sums

For the infinite series

$$\sum_{n=0}^{\infty} \frac{1}{5^n} = \frac{1}{5^0} + \frac{1}{5^1} + \frac{1}{5^2} + \cdots + \frac{1}{5^n} + \cdots$$

the first six partial sums are

$$\begin{aligned} S_0 &= 1 && \text{first term} \\ S_1 &= 1 + \frac{1}{5} = 1.2 && \text{sum of first two terms} \\ S_2 &= 1 + \frac{1}{5} + \frac{1}{25} = 1.24 && \text{sum of first three terms} \\ S_3 &= 1 + \frac{1}{5} + \frac{1}{25} + \frac{1}{125} = 1.248 \\ S_4 &= 1 + \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \frac{1}{625} = 1.2496 \\ S_5 &= 1 + \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \frac{1}{625} + \frac{1}{3125} = 1.24992 \end{aligned}$$

These values can be found using the standard calculational features of a calculator, or by the *cumulative sum* (cumSum) feature. Here, it appears that the sequence of partial sums approaches the value 1.25. We therefore conclude that this infinite series converges and that its sum is approximately 1.25. (In Example 4 of this section, we show that this infinite series does in fact converge and that its sum is 1.25.) ■

**EXAMPLE 3** Divergent series

(a) The infinite series

$$\sum_{n=1}^{\infty} 5^n = 5 + 5^2 + 5^3 + \cdots + 5^n + \cdots$$

is a divergent series. The first four partial sums are

$$S_1 = 5 \quad S_2 = 30 \quad S_3 = 155 \quad S_4 = 780$$

Obviously, they are increasing without bound.

(b) The infinite series

$$\sum_{n=0}^{\infty} (-1)^n = 1 + (-1) + 1 + (-1) + \cdots + (-1)^n + \cdots$$

has as its first five partial sums

$$S_0 = 1 \quad S_1 = 0 \quad S_2 = 1 \quad S_3 = 0 \quad S_4 = 1$$

The values of these partial sums do not approach a limiting value, and therefore, the series diverges. ■

Since convergent series are those that have a value associated with them, they are the ones that are of primary use to us. However, generally, it is not easy to determine whether a given series is convergent, and many types of tests have been developed for this purpose. These tests for convergence may be found in most textbooks that include the more advanced topics in calculus.

One important series for which we are able to determine the convergence, and its sum if convergent, is the geometric series. For this series, the  $n$ th partial sum is

$$S_n = a_1 + a_1r + a_1r^2 + \cdots + a_1r^{n-1}$$

**LEARNING TIP**

If  $r = 1$ , we see that the series is  $a_1 + a_1 + a_1 + \dots + a_1 + \dots$  and is, therefore, divergent. If  $r = -1$ , the series is  $a_1 - a_1 + a_1 - a_1 + \dots$  and is also divergent. If  $|r| > 1$ ,  $\lim_{n \rightarrow \infty} r^n$  is unbounded. Therefore, **the geometric series is convergent only if  $|r| < 1$  and has the value given by Eq. (30.3).**

**Practice Exercise**

2. Show that the infinite series  $\sum_{n=0}^{\infty} \frac{2}{3^n}$  is convergent and find its sum.

■ See Example 2.

where  $r$  is the fixed number by which we multiply a given term to get the next term. In Chapter 19, we determined that if  $|r| < 1$ , the sum  $S$  of the infinite geometric series is

$$S = \lim_{n \rightarrow \infty} S_n = \frac{a_1}{1 - r} \tag{30.3}$$

**EXAMPLE 4 Geometric series**

Show that the infinite series

$$\sum_{n=0}^{\infty} \frac{1}{5^n} = \frac{1}{5^0} + \frac{1}{5^1} + \frac{1}{5^2} + \dots + \frac{1}{5^n} + \dots$$

is convergent and find its sum. This is the same series as in Example 2.

This is a geometric series with  $r = \frac{1}{5}$ . Since  $|r| < 1$ , the series converges. The sum is

$$S = \frac{1}{1 - \frac{1}{5}} = \frac{1}{\frac{4}{5}} = \frac{5}{4} = 1.25 \quad \text{using Eq. (30.3)} \quad \blacksquare$$

**EXERCISES 30.1**

In Exercises 1 and 2, make the given changes in the indicated examples of this section and then solve the given problems.

- 1. In Example 3(a), change  $5^n$  to  $0.5^n$ . What other changes occur?
- 2. In Example 4, change  $n = 0$  to  $n = 1$ . What is the value of  $S$ ?

In Exercises 3–6, give the first four terms of the sequences for which  $a_n$  is given.

- 3.  $a_n = n^2, n = 1, 2, 3, \dots$
- 4.  $a_n = \frac{2^{n+1}}{n!}, n = 1, 2, 3, \dots$
- 5.  $a_n = \frac{1}{n+2}, n = 0, 1, 2, \dots$
- 6.  $a_n = \frac{n^2 + 1}{2n + 1}, n = 0, 1, 2, \dots$

In Exercises 7–10, give (a) the first four terms of the sequence for which  $a_n$  is given and (b) the first four terms of the infinite series associated with the sequence.

- 7.  $a_n = \left(-\frac{2}{5}\right)^n, n = 1, 2, 3, \dots$
- 8.  $a_n = \frac{1}{n} + \frac{1}{n+1}, n = 1, 2, 3, \dots$
- 9.  $a_n = \cos \frac{n\pi}{2}, n = 0, 1, 2, \dots$
- 10.  $a_n = \frac{n^n}{n!}, n = 2, 3, 4, \dots$

In Exercises 11–14, find the  $n$ th term of the given infinite series for which  $n = 1, 2, 3, \dots$

- 11.  $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots$
- 12.  $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$

13.  $\frac{1}{2 \times 3} - \frac{1}{3 \times 4} + \frac{1}{4 \times 5} - \frac{1}{5 \times 6} + \dots$

14.  $\frac{1}{\sqrt{2}} - \frac{1}{2} + \frac{\sqrt{2}}{4} - \frac{1}{4} + \dots$

In Exercises 15–24, find the first five partial sums of the given series and determine whether the series appears to be convergent or divergent. If it is convergent, find its approximate sum.

15.  $1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \frac{1}{125} + \dots$

16.  $1 + 2 + 5 + 10 + 17 + \dots$

17.  $1 + \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \dots$

18.  $\frac{1}{3} - \frac{1}{9} + \frac{1}{27} - \frac{1}{81} + \frac{1}{243} - \dots$

19.  $\sum_{n=0}^{\infty} \sqrt{n}$       20.  $\sum_{n=1}^{\infty} \frac{2}{n(n+1)}$       21.  $\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2}$

22.  $\sum_{n=1}^{\infty} \frac{n}{2n+1}$       23.  $\sum_{n=1}^{\infty} \frac{\sin n}{4^n}$       24.  $\sum_{n=3}^{\infty} \frac{\ln n}{e^n}$

In Exercises 25–32, test each of the given geometric series for convergence or divergence. Find the sum of each series that is convergent.

25.  $1 + 2 + 4 + \dots + 2^n + \dots$

26.  $1 + \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^n} + \dots$

27.  $1 - \frac{1}{3} + \frac{1}{9} - \dots + \left(-\frac{1}{3}\right)^n + \dots$

28.  $1 - \frac{3}{2} + \frac{9}{4} - \dots + \left(-\frac{3}{2}\right)^n + \dots$



29.  $10 + 9 + 8.1 + 7.29 + 6.561 + \dots$

30.  $4 + 1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \dots$

31.  $512 - 64 + 8 - 1 + \frac{1}{8} - \dots$

32.  $16 + 12 + 9 + \frac{27}{4} + \frac{81}{16} + \dots$

In Exercises 33 and 34, find the values of  $x$  for which the given series converge.

33.  $\sum_{n=0}^{\infty} (x - 4)^n$

34.  $\sum_{n=2}^{\infty} \frac{x^n}{5^n}$

In Exercises 35–44, solve the given problems as indicated.

35. Using a calculator, take successive square roots of 2 and find at least 20 approximate values for the terms of the sequence  $2^{1/2}, 2^{1/4}, 2^{1/8}, 2^{1/16}, \dots$ . From the values that are obtained, (a) what do you observe about the value of  $\lim_{n \rightarrow \infty} 2^{1/2^n}$ ? (b) Determine whether the infinite series for this sequence converges or diverges.

36. Using a calculator, (a) take successive square roots of 0.01 and then (b) take successive square roots of 100. From these sequences of square roots, state any general conclusions that might be drawn.

37. Referring to Chapter 19, we see that the sum of the first  $n$  terms of a geometric sequence is

$$S_n = \frac{a_1(1 - r^n)}{1 - r} \quad (r \neq 1) \quad \text{Eq. (19.6)}$$

where  $a_1$  is the first term and  $r$  is the common ratio. We can visualize the corresponding infinite series by graphing the function  $f(x) = a_1(1 - r^x)/(1 - r)$  ( $r \neq 1$ ) (or using a calculator that can graph a sequence). The graph represents the sequence of partial sums for values where  $x = n$ , since  $f(n) = S_n$ .

Use a graphing calculator to visualize the first five partial sums of the series

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$$

What value does the infinite series approach? (Remember: Only points for which  $x$  is an integer have real meaning.)

38. Following Exercise 37, use a graphing calculator to show that the sum of the infinite series of Example 4 is 1.25. (Be careful: Because of the definition of the series,  $x = 1$  corresponds to  $n = 0$ .)

39. The value  $V$  (in dollars) of a certain investment after  $n$  years can be expressed as

$$V = 100(1.05 + 1.05^2 + 1.05^3 + \dots + 1.05^n)$$

(a) By finding partial sums, determine whether this series converges or diverges. (b) Following Exercise 37, use a graphing calculator to visualize the first 10 partial sums. (See Example 7 of Section 19.2.)

40. If an electric discharge is passed through hydrogen gas, a spectrum of isolated parallel lines, called the Balmer series, is formed. See Fig. 30.1. The wavelengths  $\lambda$  (in nm) of the light for these lines is given by the formula

$$\frac{1}{\lambda} = 1.097 \times 10^{-2} \left( \frac{1}{2^2} - \frac{1}{n^2} \right) \quad (n = 3, 4, 5, \dots)$$

Find the wavelengths of the first three lines and the shortest wavelength of all the lines of the series.

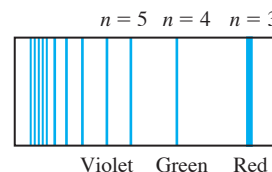


Fig. 30.1

41. Use geometric series to show that  $\sum_{n=0}^{\infty} x^n = \frac{1}{1 - x}$  for  $|x| < 1$ .

42. Use geometric series to show that  $\sum_{n=0}^{\infty} (-1)^n x^n = \frac{1}{1 + x}$  for  $|x| < 1$ .

43. If term  $a_1$  is given along with a rule to find term  $a_{n+1}$  from term  $a_n$ , the sequence is said to be defined *recursively*. If  $a_1 = 2$  and  $a_{n+1} = (n + 1)a_n$ , find the first five terms of the sequence.

44. A sequence is defined recursively (see Exercise 43) by  $x_1 = \frac{N}{2}$ ,  $x_{n+1} = \frac{1}{2} \left( x_n + \frac{N}{x_n} \right)$ . With  $N = 10$ , find  $x_6$  and compare the value with  $\sqrt{10}$ . It can be seen that  $\sqrt{N}$  can be approximated using this recursion sequence.

Answers to Practice Exercises

1. 2, 1, 3/5, ...    2.  $r = 1/3 < 1, S = 3$

## 30.2 Maclaurin Series

### Power-Series Expansion • Interval of Convergence • Maclaurin Series Expansion

$$\begin{array}{r} \blacksquare \quad 1 + \frac{x}{2} \\ 2 - x \overline{) 2} \\ \underline{2 - x} \phantom{0} \\ x \phantom{00} \\ x - \frac{x^2}{2} \\ \underline{\phantom{x} - \frac{x^2}{2}} \\ \phantom{x} \frac{x^2}{2} \phantom{00} \end{array}$$

In this section, we develop a very important basic polynomial form of a function. Before developing the method using calculus, we will review how this can be done for some functions algebraically.

#### EXAMPLE 1 An algebraic function represented by a series

By using long division (as started at the left), we have

$$\frac{2}{2 - x} = 1 + \frac{1}{2}x + \frac{1}{4}x^2 + \dots + \left(\frac{1}{2}\right)^{n-1}x^{n-1} + \dots \quad (1)$$

where  $n$  is the number of the term of the expression on the right. Since  $x$  represents a number, the right-hand side of Eq. (1) becomes a geometric series.

From Eq. (30.3), we know that the sum of a geometric series with first term  $a_1$  and common ratio  $r$  is

$$S = \frac{a_1}{1 - r}$$

when  $|r| < 1$  and the series converges.

If  $x = 1$ , the right-hand side of Eq. (1) is

$$1 + \frac{1}{2} + \frac{1}{4} + \cdots + \left(\frac{1}{2}\right)^{n-1} + \cdots$$

For this series,  $r = \frac{1}{2}$  and  $a_1 = 1$ , which means that the series converges and  $S = 2$ . Moreover, the left side of Eq. (1) is also 2 when  $x = 1$ , so the function may be represented by the series for this value of  $x$ .

If  $x = 3$ , the right-hand side of Eq. (1) is

$$1 + \frac{3}{2} + \frac{9}{4} + \cdots + \left(\frac{3}{2}\right)^{n-1} + \cdots$$

which diverges since  $r > 1$ . Therefore, the left-hand side of Eq. (1) cannot be represented by the series when  $x = 3$ .

As we can see, the function and the series agree when the series converges. Since the series converges as long as  $|x| < 2$ , we conclude that the series on the right properly represents the function on the left, as long as  $|x| < 2$ . ■

From Example 1, we see that an algebraic function may be properly represented by a function of the form

#### Power Series

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots \quad (30.4)$$

Eq. (30.4) is known as a **power-series expansion** of the function  $f(x)$ . The problem now arises as to whether or not functions in general may be represented in this form. If such a representation were possible, it would provide a means of evaluating the transcendental functions for the purpose of making tables of values. Also, since a power-series expansion is in the form of a polynomial, it makes algebraic operations much simpler due to the properties of polynomials. A further study of calculus shows many other uses of power series.

In Example 1, we saw that the function could be represented by a power series as long as  $|x| < 2$ . That is, if we substitute any value of  $x$  in this interval into the series and also into the function, the series will converge to the value of the function. *This interval of values for which the series converges is called the **interval of convergence**.*

#### EXAMPLE 2 Interval of convergence

In Example 1, the interval of convergence for the series

$$1 + \frac{1}{2}x + \frac{1}{4}x^2 + \cdots + \left(\frac{1}{2}x\right)^{n-1} + \cdots$$

is  $|x| < 2$ . We saw that the series converges for  $x = 1$ , with  $S = 2$ , and that the value of the function is 2 for  $x = 1$ . This verifies that  $x = 1$  is in the interval of convergence.

Also, we saw that the series diverges for  $x = 3$ , which verifies that  $x = 3$  is not in the interval of convergence. ■



At this point, we will assume that unless otherwise noted, the functions with which we will be dealing may be properly represented by a power-series expansion (it takes more advanced methods to prove that this is generally possible), for appropriate intervals of convergence. We will find that the methods of calculus are very useful in developing the method of general representation. Thus, writing a general power series, along with the first few derivatives, we have

$$\begin{aligned} f(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \dots + a_nx^n + \dots \\ f'(x) &= a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \dots + na_nx^{n-1} + \dots \\ f''(x) &= 2a_2 + 2(3)a_3x + 3(4)a_4x^2 + 4(5)a_5x^3 + \dots + (n-1)na_nx^{n-2} + \dots \\ f'''(x) &= 2(3)a_3 + 2(3)(4)a_4x + 3(4)(5)a_5x^2 + \dots + (n-2)(n-1)na_nx^{n-3} + \dots \\ f^{iv}(x) &= 2(3)(4)a_4 + 2(3)(4)(5)a_5x + \dots + (n-3)(n-2)(n-1)na_nx^{n-4} + \dots \end{aligned}$$

Regardless of the values of the constants  $a_n$  for any power series, **if  $x = 0$ , the left and right sides must be equal**, and all the terms on the right are zero except the first. Thus, setting  $x = 0$  in each of the above equations, we have

$$\begin{aligned} f(0) &= a_0 & f'(0) &= a_1 & f''(0) &= 2a_2 \\ f'''(0) &= 2(3)a_3 & f^{iv}(0) &= 2(3)(4)a_4 \end{aligned}$$

Solving each of these for the constants  $a_n$ , we have

$$a_0 = f(0) \quad a_1 = f'(0) \quad a_2 = \frac{f''(0)}{2!} \quad a_3 = \frac{f'''(0)}{3!} \quad a_4 = \frac{f^{iv}(0)}{4!}$$

Substituting these into the expression for  $f(x)$ , we have

**Maclaurin Series**

$$f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \dots + \frac{f^n(0)x^n}{n!} + \dots \tag{30.5}$$

■ The Maclaurin series is named for the Scottish mathematician Colin Maclaurin (1698–1746).

Eq. (30.5) is known as the **Maclaurin series expansion** of a function. For a function to be represented by a Maclaurin expansion, the function and all of its derivatives must exist at  $x = 0$ . Also, we note that the factorial notation introduced in Section 19.4 is used in writing the Maclaurin series expansion.

As we mentioned earlier, one of the uses we will make of series expansions is that of determining the values of functions for particular values of  $x$ . If  $x$  is sufficiently small, successive terms become smaller and smaller and the series will converge rapidly. This is considered in the sections that follow.

The following examples illustrate Maclaurin expansions for algebraic, exponential, and trigonometric functions.

**EXAMPLE 3** Maclaurin series for an algebraic function

Find the first four terms of the Maclaurin series expansion of  $f(x) = \frac{2}{2-x}$ .

$$\begin{aligned} f(x) &= \frac{2}{2-x} & f(0) &= 1 & f''(x) &= \frac{4}{(2-x)^3} & f''(0) &= \frac{1}{2} & \text{find derivatives} \\ f'(x) &= \frac{2}{(2-x)^2} & f'(0) &= \frac{1}{2} & f'''(x) &= \frac{12}{(2-x)^4} & f'''(0) &= \frac{3}{4} & \text{and evaluate} \\ & & & & & & & & \text{each at } x = 0 \end{aligned}$$

**Practice Exercise**

- Find the first four terms of the Maclaurin series expansion for

$$f(x) = \frac{1}{1+x}.$$

$$f(x) = 1 + \frac{1}{2}x + \frac{1}{2}\left(\frac{x^2}{2!}\right) + \frac{3}{4}\left(\frac{x^3}{3!}\right) + \dots \quad \text{using Eq. (30.5)}$$

$$\frac{2}{2-x} = 1 + \frac{1}{2}x + \frac{1}{4}x^2 + \frac{1}{8}x^3 + \dots$$

■ Compare with Example 1.

**EXAMPLE 4** Maclaurin series for an exponential function

Find the first four terms of the Maclaurin series expansion of  $f(x) = e^{-x}$ .

$$\begin{array}{llll} f(x) = e^{-x} & f(0) = 1 & f''(x) = e^{-x} & f''(0) = 1 \\ f'(x) = -e^{-x} & f'(0) = -1 & f'''(x) = -e^{-x} & f'''(0) = -1 \end{array} \quad \begin{array}{l} \text{find derivatives} \\ \text{and evaluate} \\ \text{each at } x = 0 \end{array}$$

$$f(x) = 1 + (-1)x + 1\left(\frac{x^2}{2!}\right) + (-1)\left(\frac{x^3}{3!}\right) + \dots \quad \text{using Eq. (30.5)}$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots$$

**EXAMPLE 5** Maclaurin series for a trigonometric function

Find the first three nonzero terms of the Maclaurin series expansion of  $f(x) = \sin 2x$ .

$$\begin{array}{llll} f(x) = \sin 2x & f(0) = 0 & f'''(x) = -8 \cos 2x & f'''(0) = -8 \\ f'(x) = 2 \cos 2x & f'(0) = 2 & f^{iv}(x) = 16 \sin 2x & f^{iv}(0) = 0 \\ f''(x) = -4 \sin 2x & f''(0) = 0 & f^v(x) = 32 \cos 2x & f^v(0) = 32 \end{array}$$

$$f(x) = 0 + 2x + 0 + (-8)\frac{x^3}{3!} + 0 + 32\frac{x^5}{5!} + \dots$$

$$\sin 2x = 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 - \dots \quad \text{This series is called an alternating series since every other term is negative.}$$

**EXAMPLE 6** Maclaurin series—application

Frictional forces in the spring shown in Fig. 30.2 are just sufficient so that the lever does not oscillate after being depressed. Such motion is called *critically damped*. The displacement  $y$  as a function of the time  $t$  for one case is  $y = (1 + t)e^{-t}$ . To study the motion for small values of  $t$ , a Maclaurin expansion of  $y = f(t)$  is to be used. Find the first four terms of the expansion.

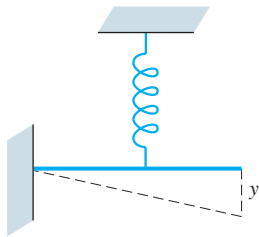


Fig. 30.2

$$\begin{array}{ll} f(t) = (1 + t)e^{-t} & f(0) = 1 \\ f'(t) = (1 + t)e^{-t}(-1) + e^{-t} = -te^{-t} & f'(0) = 0 \\ f''(t) = te^{-t} - e^{-t} & f''(0) = -1 \\ f'''(t) = -te^{-t} + e^{-t} + e^{-t} = 2e^{-t} - te^{-t} & f'''(0) = 2 \\ f^{iv}(t) = -2e^{-t} + te^{-t} - e^{-t} = te^{-t} - 3e^{-t} & f^{iv}(0) = -3 \end{array}$$

$$f(t) = 1 + 0 + (-1)\frac{t^2}{2!} + 2\frac{t^3}{3!} + (-3)\frac{t^4}{4!} + \dots$$

$$(1 + t)e^{-t} = 1 - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{8} + \dots$$

**EXERCISES 30.2**

In Exercises 1 and 2, make the given changes in the indicated examples of this section and then find the resulting series.

1. In Example 3, in  $f(x)$ , change the denominator to  $2 + x$ .
2. In Example 5, in  $f(x)$ , change  $2x$  to  $(-2x)$ .

In Exercises 3–20, find the first three nonzero terms of the Maclaurin expansion of the given functions.

3.  $f(x) = e^x$
4.  $f(x) = \sin x$
5.  $f(x) = \cos x$
6.  $f(x) = \ln(1 + x)$
7.  $f(x) = \sqrt{1 + x}$
8.  $f(x) = \sqrt[3]{1 + x}$

9.  $f(x) = e^{-2x}$
10.  $f(x) = \frac{1}{\sqrt{1 + x}}$
11.  $f(x) = \cos 4\pi x$
12.  $f(x) = e^x \sin x$
13.  $f(x) = \frac{1}{1 - x}$
14.  $f(x) = \frac{1}{(1 + x)^2}$
15.  $f(x) = \ln(1 - 2x)$
16.  $f(x) = (1 + x)^{3/2}$
17.  $f(x) = \cos^2 x$
18.  $f(x) = \ln(1 + 4x)$
19.  $f(x) = \sin(x + \frac{\pi}{4})$
20.  $f(x) = (2x - 1)^2$

In Exercises 21–28, find the first two nonzero terms of the Maclaurin expansion of the given functions.

- 21.  $f(x) = \tan^{-1} x$
- 22.  $f(x) = \cos x^2$
- 23.  $f(x) = \tan x$
- 24.  $f(x) = \sec x$
- 25.  $f(x) = \ln \cos x$
- 26.  $f(x) = xe^{\sin x}$
- 27.  $f(x) = \sqrt{1 + \sin x}$
- 28.  $f(x) = xe^{-x^2}$

In Exercises 29–42, solve the given problems.

- 29. Is it possible to find a Maclaurin expansion for (a)  $f(x) = \csc x$  or (b)  $f(x) = \ln x$ ? Explain.
- 30. Is it possible to find a Maclaurin expansion for (a)  $f(x) = \sqrt{x}$  or (b)  $f(x) = \sqrt{1+x}$ ? Explain.
- 31. Find the first three nonzero terms of the Maclaurin expansion for (a)  $f(x) = e^x$  and (b)  $f(x) = e^{x^2}$ . Compare these expansions.
- 32. By finding the Maclaurin expansion of  $f(x) = (1+x)^n$ , derive the first four terms of the binomial series, which is Eq. (19.10). Its interval of convergence is  $|x| < 1$  for all values of  $n$ .
- 33. If  $f(x) = e^{3x}$ , compare the Maclaurin expansion with the linearization for  $a = 0$ .
- 34. Find the Maclaurin series for  $y = \sinh x$ . (See Exercises 55–59 on page 837.)
- 35. Find the Maclaurin series for  $y = \cosh x$ . (See Exercises 55–59 on page 837.)
- 36. Find the Maclaurin series for  $f(x) = \cos^2 x$ , by using the identity  $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$ . Compare the result with that of Exercise 17.
- 37. If  $f(x) = x^2$ , show that this function is obtained when a Maclaurin expansion is found.
- 38. If  $f(x) = x^4 + 2x^2$ , show that this function is obtained when a Maclaurin expansion is found.
- 39. The displacement  $y$  (in cm) of an object hung vertically from a spring and allowed to oscillate is given by the equation  $y = 4e^{-0.2t} \cos t$ , where  $t$  is the time (in s). Find the first three terms of the Maclaurin expansion of this function.

- 40. For the circuit shown in Fig. 30.3, after the switch is closed, the transient current  $i$  (in A) is given by  $i = 2.5(1 + e^{-0.1t})$ . Find the first three terms of the Maclaurin expansion of this function.

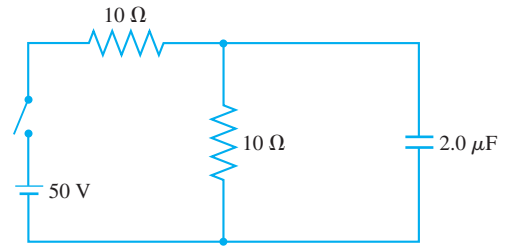


Fig. 30.3

- 41. The reliability  $R$  ( $0 \leq R \leq 1$ ) of a certain computer system is  $R = e^{-0.001t}$ , where  $t$  is the time of operation (in min). Express  $R = f(t)$  in polynomial form by using the first three terms of the Maclaurin expansion.
- 42. In the analysis of the optical paths of light from a narrow slit  $S$  to a point  $P$ , as shown in Fig. 30.4, the law of cosines is used to obtain the equation

$$c^2 = a^2 + (a + b)^2 - 2a(a + b) \cos \frac{s}{a}$$

where  $s$  is part of the circular arc  $\widehat{AB}$ . By using two nonzero terms of the Maclaurin expansion of  $\cos \frac{s}{a}$ , simplify the right side of the equation. (In finding the expansion, let  $x = \frac{s}{a}$  and then substitute back into the expansion.)

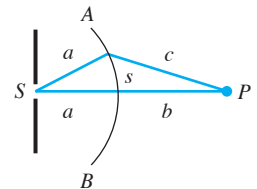


Fig. 30.4

**Answer to Practice Exercise**

1.  $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots$

### 30.3 Operations with Series

Form New Series from Known Series • Using Functional Notation • Using Algebraic Operations • Differentiating and Integrating • Accuracy of Series

The series found in Exercises 3 to 6 and 32 (the binomial series) of Section 30.2 are of particular importance. They are used to evaluate exponential functions, trigonometric functions, logarithms, powers, and roots, as well as develop other series. For reference, we give them here with their intervals of convergence.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \quad (\text{all } x) \quad (30.6)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \quad (\text{all } x) \quad (30.7)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \quad (\text{all } x) \quad (30.8)$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \quad (|x| < 1) \quad (30.9)$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \dots \quad (|x| < 1) \quad (30.10)$$

**LEARNING TIP**

When we discussed functions in Chapter 3, we mentioned functions such as  $f(2x)$  and  $f(-x)$ . By using functional notation and the preceding series, we can find the series expansions of many other series without using direct expansion. This can often save time in finding a desired series.

In the next section, we will see how to use these series in finding values of functions. In this section, we see how new series are developed by using the above basic series, and we also show other uses of series.

**EXAMPLE 1 Series formed using functional notation**

Find the Maclaurin expansion of  $e^{2x}$ .

From Eq. (30.6), we know the expansion of  $e^x$ . Hence,

$$f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

Since  $e^{2x} = f(2x)$ , we have

$$\begin{aligned} f(2x) &= 1 + (2x) + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \dots \quad \text{in } f(x), \text{ replace } x \text{ by } 2x \\ e^{2x} &= 1 + 2x + 2x^2 + \frac{4x^3}{3} + \dots \end{aligned}$$

**EXAMPLE 2 Series formed using functional notation**

Find the Maclaurin expansion of  $\sin x^2$ .

From Eq. (30.7), we know the expansion of  $\sin x$ . Therefore,

$$f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$f(x^2) = (x^2) - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \dots \quad \text{in } f(x), \text{ replace } x \text{ by } x^2$$

$$\sin x^2 = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \dots$$

**Practice Exercise**

- Using the Maclaurin series for  $\ln(1+x)$ , find the first four terms of the Maclaurin expansion of  $\ln(1-2x)$ .

Direct expansion of this series is quite lengthy.

The basic algebraic operations may be applied to series in the same manner they are applied to polynomials. That is, we may add, subtract, multiply, or divide series in order to obtain other series. The interval of convergence for the resulting series is that which is common to those of the series being used. The multiplication of series is illustrated in the following example.

**EXAMPLE 3 Series formed by multiplication**

Multiply the series expansion for  $e^x$  by the series expansion for  $\cos x$  to obtain the series expansion for  $e^x \cos x$ .

Using the series expansion for  $e^x$  and  $\cos x$  as shown in Eqs. (30.6) and (30.8), we have the following indicated multiplication:

$$e^x \cos x = \left( 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \right) \left( 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \right)$$

By multiplying the series on the right, we have the following result, considering through the  $x^4$  terms in the product.

$$\begin{aligned} e^x \cos x &= 1 - \frac{x^2}{2} + \frac{x^4}{24} + x - \frac{x^3}{2} + \frac{x^2}{2} - \frac{x^4}{4} + \frac{x^3}{6} + \frac{x^4}{24} + \dots \\ &= 1 + x - \frac{1}{3}x^3 - \frac{1}{6}x^4 + \dots \end{aligned}$$

It is also possible to use the operations of differentiation and integration to obtain series expansions, although the proof of this is found in more advanced texts. Consider the following example.

#### EXAMPLE 4 Series formed by differentiation

Show that by differentiating the series for  $\ln(1+x)$  term by term, the result is the same as the series for  $\frac{1}{1+x}$ .

The series for  $\ln(1+x)$  is shown in Eq. (30.9) as

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

Differentiating, we have

$$\begin{aligned} \frac{1}{1+x} &= 1 - \frac{2x}{2} + \frac{3x^2}{3} - \frac{4x^3}{4} + \cdots \\ &= 1 - x + x^2 - x^3 + \cdots \end{aligned}$$

Using the binomial expansion for  $\frac{1}{1+x} = (1+x)^{-1}$ , we have

$$\begin{aligned} (1+x)^{-1} &= 1 + (-1)x + \frac{(-1)(-2)}{2!}x^2 + \frac{(-1)(-2)(-3)}{3!}x^3 + \cdots \\ &= 1 - x + x^2 - x^3 + \cdots \end{aligned} \quad \begin{array}{l} \text{using} \\ \text{Eq. (30.10)} \\ \text{with } n = -1 \end{array}$$

We see that the results are the same. ■

We can use algebraic operations on series to verify that the definition of the exponential form of a complex number, as shown in Eq. (12.11), is consistent with other definitions. The only assumption required here is that the Maclaurin expansions for  $e^x$ ,  $\sin x$ , and  $\cos x$  are also valid for complex numbers. This is shown in advanced calculus. Thus,

$$e^{j\theta} = 1 + j\theta + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \cdots = 1 + j\theta - \frac{\theta^2}{2!} - j\frac{\theta^3}{3!} + \cdots \quad (30.11)$$

$$j \sin \theta = j\theta - j\frac{\theta^3}{3!} + \cdots \quad (30.12)$$

$$\cos \theta = 1 - \frac{\theta^2}{2!} + \cdots \quad (30.13)$$

When we add the terms of Eq. (30.12) to those of Eq. (30.13), the result is the series given in Eq. (30.11). Thus,

$$e^{j\theta} = \cos \theta + j \sin \theta \quad (30.14)$$

A comparison of Eqs. (12.11) and (30.14) indicates the reason for the choice of the definition of the exponential form of a complex number.

An additional use of power series is now shown. Many integrals that occur in practice cannot be integrated by methods given in the preceding chapters. However, power series can be very useful in giving excellent approximations to some definite integrals.

■ For reference, Eq. (12.11) is  $re^{j\theta} = r(\cos \theta + j \sin \theta)$ .

■ Eq. (30.14) is known as Euler's Formula. If  $\theta = \pi$ , we have  $e^{j\pi} = -1$ , which can be written as

$$e^{j\pi} + 1 = 0$$

This equation connects the five fundamental numbers  $e, j, \pi, 1$ , and  $0$ , and it has been called a "beautiful" equation by mathematicians. (In nontechnical sources,  $i$  would appear in place of  $j$  (see page 342).)

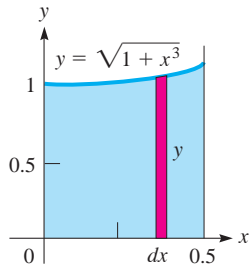


Fig. 30.5

**EXAMPLE 5** Using series for integration

Find the first-quadrant area bounded by  $y = \sqrt{1 + x^3}$  and  $x = 0.5$ .

From Fig. 30.5, we see that the area is

$$A = \int_0^{0.5} \sqrt{1 + x^3} dx$$

This integral does not fit any form we have used. However, its value can be closely approximated by using the binomial expansion for  $\sqrt{1 + x^3}$  and then integrating.

Using the binomial expansion to find the first three terms of the expansion for  $\sqrt{1 + x^3}$ , we have

$$\begin{aligned} \sqrt{1 + x^3} &= (1 + x^3)^{0.5} = 1 + 0.5x^3 + \frac{0.5(-0.5)}{2}(x^3)^2 + \dots \\ &= 1 + 0.5x^3 - 0.125x^6 + \dots \end{aligned}$$

Substituting in the integral, we have

$$\begin{aligned} A &= \int_0^{0.5} (1 + 0.5x^3 - 0.125x^6 + \dots) dx \\ &= x + \frac{0.5}{4}x^4 - \frac{0.125}{7}x^7 + \dots \Big|_0^{0.5} \\ &= 0.5 + 0.007\,812\,5 - 0.000\,139\,5 + \dots = 0.507\,673 + \dots \end{aligned}$$

We can see that each of the terms omitted was very small. The result shown is correct to four decimal places, or  $A = 0.5077$ . Additional accuracy can be obtained by using more terms of the expansion. ■

**EXAMPLE 6** Using series for integration

Evaluate  $\int_0^{0.1} e^{-x^2} dx$ .

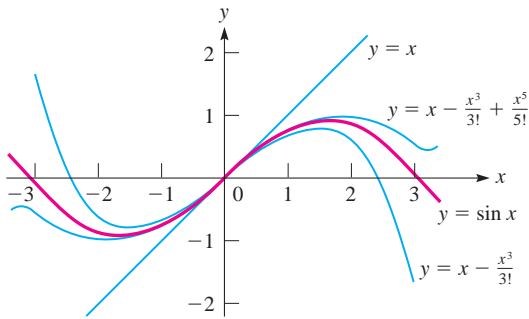


Fig. 30.6

$$\begin{aligned} e^{-x^2} &= 1 + (-x^2) + \frac{(-x^2)^2}{2!} + \dots && \text{using Eq. (30.6)} \\ \int_0^{0.1} e^{-x^2} dx &= \int_0^{0.1} \left(1 - x^2 + \frac{x^4}{2} - \dots\right) dx && \text{substitute} \\ &= \left(x - \frac{x^3}{3} + \frac{x^5}{10} - \dots\right) \Big|_0^{0.1} && \text{integrate} \\ &= 0.1 - \frac{0.001}{3} + \frac{0.000\,01}{10} = 0.099\,667\,7 && \text{evaluate} \end{aligned}$$

This answer is correct to the indicated accuracy. ■

**LEARNING TIP**

For small values of  $x$ , a Maclaurin series gives good accuracy with a very few terms. In this case, the series converges rapidly. For this reason, a Maclaurin series is of particular use for small values of  $x$ . For larger values of  $x$ , a function is usually expanded in a Taylor series (see Section 30.5). Of course, if we omit any term in a series, there is some error in the calculation.

The question of accuracy now arises. The integrals just evaluated indicate that the more terms used, the greater the accuracy of the result. To show the accuracy involved graphically, Fig. 30.6 depicts the graphs of  $y = \sin x$  and the graphs of

$$y = x \quad y = x - \frac{x^3}{3!} \quad y = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

which are the first three approximations of  $y = \sin x$ . We can see that each term added gives a better fit to the curve of  $y = \sin x$ . Also, this gives a graphical representation of the meaning of a series expansion.

## EXERCISES 30.3

In Exercises 1 and 2, make the given changes in the indicated examples of this section, and then find the resulting series.

- In Example 1, change  $e^{2x}$  to  $e^{2x^2}$ .
- In Example 3, change  $e^x$  to  $e^{-x}$ .

In Exercises 3–10, find the first four nonzero terms of the Maclaurin expansions of the given functions by using Eqs. (29.6) to (29.10).

- $f(x) = e^{3x}$
- $f(x) = \sin \frac{1}{2}x$
- $f(x) = x \cos 4x$
- $f(x) = \ln(1 + x^2)$
- $f(x) = e^{-2x}$
- $f(x) = \sin x^4$
- $f(x) = \sqrt{1 - x^4}$
- $f(x) = x^2 \ln(1 - x)$

In Exercises 11–14, evaluate the given integrals by using three terms of the appropriate series. Round answers to 3 significant digits.

- $\int_0^1 \sin x^2 dx$
- $\int_0^{0.4} \sqrt[4]{1 - 2x^2} dx$
- $\int_0^{0.2} \cos \sqrt{x} dx$
- $\int_{0.1}^{0.2} \frac{\cos x - 1}{x} dx$

In Exercises 15–28, find the indicated series by the given operation.

- Find the first four terms of the Maclaurin expansion of the function  $f(x) = \frac{2}{1 - x^2}$  by adding the terms of the series for the functions  $\frac{1}{1 - x}$  and  $\frac{1}{1 + x}$ .
- Find the first four nonzero terms of the expansion of the function  $f(x) = \frac{1}{2}(e^x - e^{-x})$  by subtracting the terms of the appropriate series. The result is the series for  $\sinh x$ . (See Exercise 55 of Section 27.6.)
- Find the first three terms of the expansion for  $e^x \sin x$  by multiplying the proper expansions together, term by term.
- Find the first three nonzero terms of the expansion for  $f(x) = \tan x$  by dividing the series for  $\sin x$  by that for  $\cos x$ .
- By using the properties of logarithms and the series for  $\ln(1 + x)$ , find the series for  $x^2 \ln(1 - x)^2$ .
- By using the properties of logarithms and the series for  $\ln(1 + x)$ , find the series for  $\ln \frac{1 + x}{1 - x}$ .
- Find the first three terms of the expansion for  $\ln(1 + \sin x)$  by using the expansions for  $\ln(1 + x)$  and  $\sin x$ .
- Show that by differentiating term by term the expansion for  $\sin x$ , the result is the expansion for  $\cos x$ .
- Show that by differentiating term by term the expansion for  $e^x$ , the result is also the expansion for  $e^x$ .
- Find the expansion for  $\sin x + x \cos x$  by differentiating term by term the expansion for  $x \sin x$ .
- Show that by integrating term by term the expansion for  $\cos x$ , the result is the expansion for  $\sin x$ .
- Show that by integrating term by term the expansion for  $-1/(1 - x)$  (see Exercise 13 of Section 30.2), the result is the expansion for  $\ln(1 - x)$ .

27. By multiplication of series, find the first three terms of the expansion for the displacement of the oscillating object of Exercise 39 on page 923.

28. By using the series for  $e^x$ , find the first three terms of the expansion of the electric current given in Exercise 40 on page 923.

In Exercises 29–40, solve the given problems. Where necessary, round answers to 3 significant digits.

- Evaluate  $\int_0^1 e^x dx$  directly and compare the result obtained by using four terms of the series for  $e^x$  and then integrating.
- Evaluate  $\lim_{x \rightarrow 0} \frac{\sin x}{x}$  by using the series expansion for  $\sin x$ . Compare the result with Eq. (27.1).
- Evaluate  $\lim_{x \rightarrow 0} \frac{\sin x - x}{x^3}$  by using the expansion for  $\sin x$ .
- Find the approximate area bounded by  $y = \sin x$ ,  $y = 0$ , and  $x = \pi/6$  by using two terms of the expansion for  $\sin x$ . Compare the result with that found by direct integration.
- Find the approximate value of the area bounded by  $y = x^2 e^x$ ,  $x = 0.2$ , and the  $x$ -axis by using three terms of the appropriate Maclaurin series.
- Find the approximate area under the graph of  $y = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$  from  $x = -1$  to  $x = 1$  by using three terms of the appropriate series. See Fig. 30.7. Compare with the value obtained from Table 22.1.

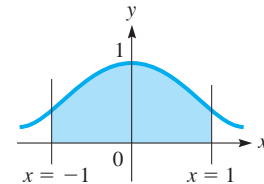


Fig. 30.7

- The Fresnel integral  $\int_0^x \cos t^2 dt$  is used in the analysis of beam displacements (and in optics). Evaluate this integral for  $x = 0.2$  by using two terms of the appropriate series.
- The dome of a sports arena is designed as the surface generated by revolving the curve of  $y = 20.0 \cos 0.0196x$  ( $0 \leq x \leq 80.0$  m) about the  $y$ -axis. Find the volume within the dome by using three terms of the appropriate series. (This is the same dome as in Exercise 51, page 891. Compare the results.)
- In the theory of relativity, when studying the kinetic (moving) energy of an object, the equation  $K = \left[ \left( 1 - \frac{v^2}{c^2} \right)^{-1/2} - 1 \right] mc^2$  is used. Here, for a given object,  $K$  is the kinetic energy,  $v$  is its velocity, and  $c$  is the velocity of light. If  $v$  is much smaller than  $c$ , show that  $K = \frac{1}{2}mv^2$ , which is the classical expression for  $K$ .
- The charge  $q$  on a capacitor in a certain electric circuit is given by  $q = ce^{-at} \sin 6at$ , where  $t$  is the time. By multiplication of series, find the first four nonzero terms of the expansion for  $q$ .
- By differentiating the expansion for  $\frac{1}{1 - x}$ , show that  $\sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1 - x)^2}$  for  $|x| < 1$ .





**EXAMPLE 3** Trigonometric value

Calculate the value of  $\cos 0.5429$ .

Since the angle is expressed in radians, we have

$$\begin{aligned}\cos 0.5429 &= 1 - \frac{0.5429^2}{2} + \frac{0.5429^4}{4!} - \cdots \quad \text{using Eq. (30.8)} \\ &= 0.856\,249\,5 \quad \text{using 3 terms}\end{aligned}$$

**Practice Exercise**

1. Using two terms of the appropriate series, calculate the value of  $\cos 2^\circ$ .

A calculator shows that  $\cos 0.5429 = 0.856\,214\,082\,4$ . Since the angle is not small, additional terms are needed to obtain this accuracy. With one more term, the value  $0.856\,213\,9$  is obtained. ■

**EXAMPLE 4** Logarithmic value

Calculate the value of  $\ln 1.2$ .

$$\begin{aligned}\ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \quad \text{Eq. (30.9)} \\ \ln 1.2 &= \ln(1+0.2) \\ &= 0.2 - \frac{(0.2)^2}{2} + \frac{(0.2)^3}{3} - \cdots = 0.1827\end{aligned}$$

To 4 significant digits,  $\ln(1.2) = 0.1823$ . One more term is required to obtain this accuracy. ■

We now illustrate the use of series in error calculations and measurement approximations. We also discussed these as applications of differentials. A series solution allows as close a value of the calculated error or approximation as needed, whereas with differentials, the calculation is limited to one term.

**EXAMPLE 5** Approximation of the value of a velocity

The velocity  $v$  of an object that has fallen  $h$  m is  $v = 4.43\sqrt{h}$ . Find the approximate error in calculating the velocity of an object that has fallen 100.0 m with a possible error of 2.0 m.

If we let  $v = 4.43\sqrt{100.0 + x}$ , where  $x$  is the error in  $h$ , we may express  $v$  as a Maclaurin expansion in  $x$ :

$$\begin{aligned}f(x) &= 4.43(100.0 + x)^{1/2} & f(0) &= 44.3 \\ f'(x) &= 2.22(100.0 + x)^{-1/2} & f'(0) &= 0.222 \\ f''(x) &= -1.11(100.0 + x)^{-3/2} & f''(0) &= -0.001\,11\end{aligned}$$

Therefore,

$$v = 4.43\sqrt{100.0 + x} = 44.3 + 0.222x - 0.000\,56x^2 + \cdots$$

Since the calculated value of  $v$  for  $x = 0$  is 44.3, the error  $E$  in the value of  $v$  is

$$E = 0.222x - 0.000\,56x^2 + \cdots$$

Calculating, the error for  $x = 2.0$  is

$$E = 0.222(2.0) - 0.000\,56(4.0) = 0.444 - 0.002 = 0.442 \text{ m/s}$$

The value 0.444 is that which is found using differentials. The additional terms are corrections to this term. The additional term in this case shows that the first term is a good approximation to the error. Although this problem can be done numerically, a series solution allows us to find the error for any value of  $x$ . ■

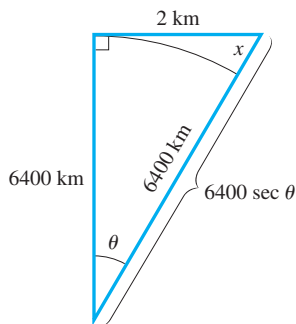


Fig. 30.8

**EXAMPLE 6** Approximation of a tangent to earth's surface

From a point on the surface of the earth, a laser beam is aimed tangentially toward a vertical rod 2 km distant. How far up on the rod does the beam touch? (Assume earth is a perfect sphere of radius 6400 km.)

From Fig. 30.8, we see that

$$x = 6400 \sec \theta - 6400$$

Finding the series for  $\sec \theta$ , we have

$$\begin{aligned} f(\theta) &= \sec \theta & f(0) &= 1 \\ f'(\theta) &= \sec \theta \tan \theta & f'(0) &= 0 \\ f''(\theta) &= \sec^3 \theta + \sec \theta \tan^2 \theta & f''(0) &= 1 \end{aligned}$$

Thus, the first two nonzero terms are  $\sec \theta = 1 + (\theta^2/2)$ . Therefore,

$$\begin{aligned} x &= 6400(\sec \theta - 1) \\ &= 6400\left(1 + \frac{\theta^2}{2} - 1\right) = 3200 \theta^2 \end{aligned}$$

The first two terms of the expansion for  $\tan \theta$  are  $\theta + \theta^3/3$ , which means that  $\tan \theta \approx \theta$ , since  $\theta$  is small (see Section 8.4). From Fig. 30.8,  $\tan \theta = 2/6400$ , and therefore  $\theta = 1/3200$ . Therefore, we have

$$x = 3200\left(\frac{1}{3200}\right)^2 = \frac{1}{3200} = 0.0003 \text{ km}$$

This means the 2-km-long beam touches the rod only 30 cm above the surface! ■

**EXERCISES 30.4**

In Exercises 1 and 2, make the given changes in the indicated examples of this section, and then solve the resulting problems.

- In Example 1, change  $e^{0.1}$  to  $e^{-0.1}$ .
- In Example 4, change  $\ln 1.2$  to  $\ln 0.8$ .

In Exercises 3–20, calculate the value of each of the given functions. Use the indicated number of terms of the appropriate series. Compare with the value found directly on a calculator.

- $e^{0.2}$  (3)
- $\sin 0.1$  (2)
- $e$  (7)
- $\cos \pi^\circ$  (2)
- $\ln 1.4$  (4)
- $\sin 0.3625$  (3)
- $\ln 0.8461$  (5)
- $1.032^6$  (3)
- $1.1^{-0.2}$  (3)
- $1.01^{-1}$  (4)
- $\cos 0.05$  (3)
- $e^{-0.5}$  (5)
- $\sin 8^\circ$  (3)
- $\ln 0.95$  (4)
- $\cos 1$  (4)
- $\ln 1.0534$  (3)
- $0.9982^8$  (3)
- $0.96^{-1}$  (3)

In Exercises 21–24, calculate the value of each of the given functions. In Exercises 21 and 22, use the expansion for  $\sqrt{1+x}$ , and in Exercises 23 and 24, use the expansion for  $\sqrt[3]{1+x}$ . Use three terms of the appropriate series.

- $\sqrt{1.1076}$
- $\sqrt{0.7915}$
- $\sqrt[3]{0.9628}$
- $\sqrt[3]{1.1392}$

In Exercises 25–28, calculate the maximum error of the values calculated in the indicated exercises. If a series is alternating (every other term is negative), the maximum possible error in the calculated value is the value of the first term omitted.

- Exercise 5
- Exercise 9
- Exercise 4
- Exercise 11

In Exercises 29–40, solve the given problems by using series expansions.

- Evaluate  $\sqrt{3.92}$  by noting that  $\sqrt{3.92} = \sqrt{4 - 0.08} = 2\sqrt{1 - 0.02}$ .
- Evaluate  $\sin 32^\circ$  by first finding the expansion for  $\sin(x + \pi/6)$ .
- We can evaluate  $\pi$  by use of  $\frac{1}{4}\pi = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}$  (see Exercise 62 of Section 20.6), along with the series for  $\tan^{-1}x$ . The first three terms are  $\tan^{-1}x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5$ . Using these terms, expand  $\tan^{-1}\frac{1}{2}$  and  $\tan^{-1}\frac{1}{3}$  and approximate the value of  $\pi$ .
- Use the fact that  $\frac{1}{4}\pi = \tan^{-1}\frac{1}{7} + 2\tan^{-1}\frac{1}{3}$  to approximate the value of  $\pi$ . (See Exercise 31.)
- Explain why  $e^x > 1 + x + \frac{1}{2}x^2$  for  $x > 0$ .
- Using a calculator, determine how many terms of the expansion for  $\ln(1+x)$  are needed to give the value of  $\ln 1.3$  accurate to five decimal places.
- The time  $t$  (in years) for an investment to increase by 10% when the interest rate is 6% is given by  $t = \frac{\ln 1.1}{0.06}$ . Evaluate this expression by using the first four terms of the appropriate series.

36. The period  $T$  of a pendulum of length  $L$  is given by

$$T = 2\pi\sqrt{\frac{L}{g}}\left(1 + \frac{1}{4}\sin^2\frac{\theta}{2} + \frac{9}{64}\sin^4\frac{\theta}{2} + \dots\right)$$

where  $g$  is the acceleration due to gravity and  $\theta$  is the maximum angular displacement. If  $L = 1.000$  m and  $g = 9.800$  m/s<sup>2</sup>, calculate  $T$  for  $\theta = 10.0^\circ$  (a) if only one term (the 1) of the series is used and (b) if two terms of the indicated series are used. In the second term, substitute one term of the series for  $\sin^2(\theta/2)$ .

37. The current in a circuit containing a resistance  $R$ , an inductance  $L$ , and a battery whose voltage is  $E$  is given by the equation  $i = \frac{E}{R}(1 - e^{-Rt/L})$ , where  $t$  is the time. Approximate this expression by using the first three terms of the appropriate exponential series. Under what conditions will this approximation be valid?
38. The image distance  $q$  from a certain lens as a function of the object distance  $p$  is given by  $q = 20p/(p - 20)$ . Find the first

three nonzero terms of the expansion of the right side. From this expression, calculate  $q$  for  $p = 2.00$  cm and compare it with the value found by substituting 2.00 in the original expression.

39. At what height above the shoreline of Lake Ontario must an observer be in order to see a point 15 km distant on the surface of the lake? (The radius of the earth is 6400 km.)
40. The efficiency  $\eta$  (in %) of an internal combustion engine in terms of its compression ratio  $c$  is given by  $\eta = 100(1 - c^{-0.40})$ . Determine the possible approximate error in the efficiency for a compression ratio measured to be 6.00 with a possible error of 0.50. (Hint: Set up a series for  $(6 + x)^{-0.40}$ .)

#### Answer to Practice Exercise

1.  $\cos 2^\circ = 0.999\ 390\ 8$

## 30.5 Taylor Series

Taylor Series Expansion • More General Than Maclaurin Series • Choice of the Value of  $a$

To obtain accurate values of a function for values of  $x$  that are not close to zero, it is usually necessary to use many terms of a Maclaurin expansion. However, we can use another type of series, called a **Taylor series**, which is a more general expansion than a Maclaurin expansion. Also, functions for which a Maclaurin series may not be found may have a Taylor series.

The basic assumption in formulating a Taylor expansion is that a function may be expanded in a polynomial of the form

$$f(x) = c_0 + c_1(x - a) + c_2(x - a)^2 + \dots \quad (30.15)$$

Following the same line of reasoning as in deriving the Maclaurin expansion, we may find the constants  $c_0, c_1, c_2, \dots$ . That is, derivatives of Eq. (30.15) are taken, and the function and its derivatives are evaluated at  $x = a$ . This leads to

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)(x - a)^2}{2!} + \dots \quad (30.16)$$

■ The Taylor series is named for the English mathematician Brook Taylor (1685–1731).

Equation (30.16) is the **Taylor series expansion** of a function. It converges rapidly for values of  $x$  that are close to  $a$ , and this is illustrated in Examples 3 and 4.

### EXAMPLE 1 Taylor series for $e^x$

Expand  $f(x) = e^x$  in a Taylor series with  $a = 1$ .

$$f(x) = e^x \quad f(1) = e \quad \text{find derivatives and evaluate each at } x = 1$$

$$f'(x) = e^x \quad f'(1) = e$$

$$f''(x) = e^x \quad f''(1) = e$$

$$f'''(x) = e^x \quad f'''(1) = e$$

$$f(x) = e + e(x - 1) + e\frac{(x - 1)^2}{2!} + e\frac{(x - 1)^3}{3!} + \dots \quad \text{using Eq. (30.16)}$$

$$e^x = e\left[1 + (x - 1) + \frac{(x - 1)^2}{2} + \frac{(x - 1)^3}{6} + \dots\right]$$

This series can be used in evaluating  $e^x$  for values of  $x$  near 1. ■

#### Practice Exercise

1. Expand  $f(x) = e^x$  in a Taylor series with  $a = 3$ .

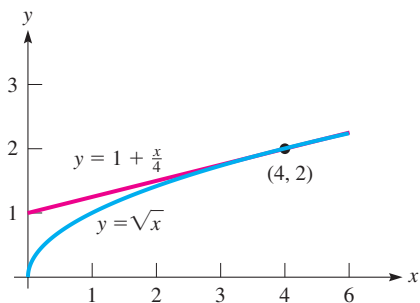


Fig. 30.9

**EXAMPLE 2** Taylor series for  $\sqrt{x}$ 

Expand  $f(x) = \sqrt{x}$  in powers of  $(x - 4)$ .

Another way of stating this is to find the Taylor series for  $f(x) = \sqrt{x}$ , with  $a = 4$ . Thus,

$$f(x) = x^{1/2} \quad f(4) = 2 \quad \text{find derivatives and evaluate each at } x = 4$$

$$f'(x) = \frac{1}{2x^{1/2}} \quad f'(4) = \frac{1}{4}$$

$$f''(x) = -\frac{1}{4x^{3/2}} \quad f''(4) = -\frac{1}{32}$$

$$f'''(x) = \frac{3}{8x^{5/2}} \quad f'''(4) = \frac{3}{256}$$

$$f(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{32} \frac{(x-4)^2}{2!} + \frac{3}{256} \frac{(x-4)^3}{3!} - \dots \quad \text{using Eq. (30.16)}$$

$$\sqrt{x} = 2 + \frac{(x-4)}{4} - \frac{(x-4)^2}{64} + \frac{(x-4)^3}{512} - \dots$$

This series would be used to evaluate square roots of numbers near 4.

Fig. 30.9 shows the graphs of  $y = \sqrt{x}$  and  $y = 1 + x/4$  (the first two terms of the series, and the linearization of the function at  $x = 4$ ). We see that each curve passes through  $(4, 2)$ , and they have nearly equal values of  $y$  for values of  $x$  near 4. ■

In the last section, we evaluated functions by using Maclaurin series. In the following examples, we use Taylor series to evaluate functions.

**EXAMPLE 3** Evaluating a square root using Taylor series

By using Taylor series, evaluate  $\sqrt{4.5}$ .

Using the four terms of the series found in Example 2, we have

$$\begin{aligned} \sqrt{4.5} &= 2 + \frac{(4.5 - 4)}{4} - \frac{(4.5 - 4)^2}{64} + \frac{(4.5 - 4)^3}{512} \quad \text{substitute 4.5 for } x \\ &= 2 + \frac{(0.5)}{4} - \frac{(0.5)^2}{64} + \frac{(0.5)^3}{512} \\ &= 2.121\,337\,891 \end{aligned}$$

The value found directly on a calculator is 2.121 320 344. Therefore, the value found by these terms of the series expansion is correct to four decimal places. ■

**EXAMPLE 4** Evaluating a sine value using Taylor series

Calculate the approximate value of  $\sin 29^\circ$  by using three terms of the appropriate Taylor expansion.

Since the value of  $\sin 30^\circ$  is known to be  $\frac{1}{2}$ , we let  $a = \frac{\pi}{6}$  (remember, we must use values expressed in radians) when we evaluate the expansion for  $x = 29^\circ$ . When expressed in radians, the quantity  $(x - a)$  is  $-\frac{\pi}{180}$  (equivalent to  $-1^\circ$ ). This means that its numerical values are small and become smaller when it is raised to higher powers. Therefore,

**LEARNING TIP**

In Example 3, we see that successive terms become small rapidly. If a value of  $x$  is chosen such that  $x - a$  is larger, the successive terms may not become small rapidly, and many terms may be required. Therefore, **we should choose the value of  $a$  as conveniently close as possible to the  $x$ -values that will be used.** Also, we should note that a Maclaurin expansion for  $\sqrt{x}$  cannot be used since the derivatives of  $\sqrt{x}$  are not defined for  $x = 0$ .

■ The *taylor* feature on some graphic calculators can be used to find the Taylor expansion of a function.

$$f(x) = \sin x \quad f\left(\frac{\pi}{6}\right) = \frac{1}{2} \quad \text{find derivatives and evaluate each at } x = \frac{\pi}{6}$$

$$f'(x) = \cos x \quad f'\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

$$f''(x) = -\sin x \quad f''\left(\frac{\pi}{6}\right) = -\frac{1}{2}$$

$$f(x) = \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right) - \frac{1}{4}\left(x - \frac{\pi}{6}\right)^2 - \dots \quad \text{using Eq. (30.16)}$$

$$\sin x = \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right) - \frac{1}{4}\left(x - \frac{\pi}{6}\right)^2 - \dots \quad f(x) = \sin x$$

$$\sin 29^\circ = \sin\left(\frac{\pi}{6} - \frac{\pi}{180}\right) \quad 29^\circ = 30^\circ - 1^\circ = \frac{\pi}{6} - \frac{\pi}{180}$$

$$= \frac{1}{2} + \frac{\sqrt{3}}{2}\left(\frac{\pi}{6} - \frac{\pi}{180} - \frac{\pi}{6}\right) - \frac{1}{4}\left(\frac{\pi}{6} - \frac{\pi}{180} - \frac{\pi}{6}\right)^2 - \dots \quad \text{substitute } \frac{\pi}{6} - \frac{\pi}{180} \text{ for } x$$

$$= \frac{1}{2} + \frac{\sqrt{3}}{2}\left(-\frac{\pi}{180}\right) - \frac{1}{4}\left(-\frac{\pi}{180}\right)^2 - \dots$$

$$= 0.484\,808\,850\,9$$

The value found directly on a calculator is 0.484 809 620 2. ■

## EXERCISES 30.5

In Exercises 1 and 2, make the given changes in the indicated examples of this section, and then solve the resulting problems.

- In Example 2, change  $(x - 4)$  to  $(x - 1)$ .
- In Example 4, change  $\sin 29^\circ$  to  $\sin 31^\circ$ .

In Exercises 3–10, evaluate the given functions by using the series developed in the examples of this section.

- $e^{1.2}$
- $\sqrt[3]{4.2}$
- $\sin 28^\circ$
- $\sin 29.53^\circ$
- $e^{0.7}$
- $\sqrt{3.5}$
- $\sin 28^\circ$
- $\sqrt{3.8527}$

In Exercises 11–22, find the first three nonzero terms of the Taylor expansion for the given function and given value of  $a$ .

- $e^{-x}$  ( $a = 2$ )
- $\cos x$  ( $a = \frac{\pi}{4}$ )
- $\sin x$  ( $a = \frac{\pi}{3}$ )
- $\ln x$  ( $a = 3$ )
- $\sqrt[3]{x}$  ( $a = 8$ )
- $\frac{1}{x}$  ( $a = 2$ )
- $\tan x$  ( $a = \frac{\pi}{4}$ )
- $\ln \sin x$  ( $a = \frac{\pi}{2}$ )
- $e^x \sin x$  ( $a = \frac{\pi}{2}$ )
- $xe^{-x}$  ( $a = -1$ )
- $\frac{1}{x+2}$  ( $a = 3$ )
- $\frac{1}{(1+x)^2}$  ( $a = -2$ )

In Exercises 23–30, evaluate the given functions by using three terms of the appropriate Taylor series.

- $e^\pi$
- $\ln 3.1$
- $\sqrt{9.3}$
- $2.056^{-1}$

- $\sqrt[3]{8.3}$
- $\tan 46^\circ$
- $\sin 61^\circ$
- $\cos 42^\circ$

In Exercises 31–38, solve the given problems.

- By completing the steps indicated before Eq. (30.16) in the text, complete the derivation of Eq. (30.16).
- Find the first three terms of the Taylor expansion of  $f(x) = \ln x$  with  $a = 1$ . Compare this Taylor expansion with the linearization  $L(x)$  of  $f(x)$  with  $a = 1$ . Compare the graphs of  $f(x)$ ,  $L(x)$ , and the Taylor expansion on a graphing calculator.
- Show that the polynomial  $2x^3 + x^2 - 3x + 5$  can be written as  $2(x-1)^3 + 7(x-1)^2 + 5(x-1) + 5$ .
- Calculate  $\sqrt{3}$  using the series in Example 2 and compare with the value using the series in Exercise 1. Which is the better approximation?
- Calculate  $\sin 31^\circ$  by using three terms of the Maclaurin expansion for  $\sin x$ . Also, calculate  $\sin 31^\circ$  by using three terms of the Taylor expansion in Example 4 (see Exercise 2). Compare the accuracy of the values obtained with that found directly on a calculator.
- Referring to Eq. (30.16), show that a Taylor series can be expressed in the form

$$f(a+h) = f(a) + f'(a)h + \frac{f''(a)}{2!}h^2 + \dots$$

- The current  $i$  in a certain electric circuit is  $i = 6 \sin \pi t$ . Write the first three terms of the Taylor series of this function about  $t = \pi/2$ .

38. In the analysis of the electric potential of an electric charge distributed along a straight wire of length  $L$ , the expression  $\ln \frac{x+L}{x}$  is used. Find three terms of the Taylor expansion of this expression in powers of  $(x-L)$ .

In Exercises 39–42, use a graphing calculator to display (a) the function in the indicated exercise of this set and (b) the first two terms of the Taylor series found for that exercise in the same display. Describe how closely the graph in part (b) fits the graph in part (a). Use the given values of  $x$  for  $X_{\min}$  and  $X_{\max}$ .

- 39. Exercise 13 ( $\sin x$ ),  $x = 0$  to  $x = 2$
- 40. Exercise 15 ( $\sqrt[3]{x}$ ),  $x = 0$  to  $x = 16$
- 41. Exercise 16 ( $1/x$ ),  $x = 0$  to  $x = 4$
- 42. Exercise 17 ( $\tan x$ ),  $x = 0$  to  $x = 1.5$

Answer to Practice Exercise

$$1. f(x) = e^3 \left[ 1 + (x-3) + \frac{(x-3)^2}{2} + \frac{(x-3)^3}{6} + \dots \right]$$

## 30.6 Introduction to Fourier Series

Periodic Functions • Fourier Series Is a Series of Sines and Cosines • Formulas for Coefficients

Many problems encountered in the various fields of science and technology involve functions that are periodic. A *periodic function* is one for which  $F(x + P) = F(x)$ , where  $P$  is the period. We noted that the trigonometric functions are periodic when we discussed their graphs in Chapter 10. Illustrations of applied problems that involve periodic functions are alternating-current voltages and mechanical oscillations.

In this section, we use a series made of terms of sines and cosines. This allows us to represent complicated periodic functions in terms of the simpler sines and cosines. It also provides us a good approximation over a greater interval than Maclaurin and Taylor series, which give good approximations with a few terms only near a specific value. Illustrations of applications of this type of series are given in Example 3 and in the exercises.

We will assume that a function  $f(x)$  may be represented by the series of sines and cosines as indicated:

$$f(x) = a_0 + a_1 \cos x + a_2 \cos 2x + \dots + a_n \cos nx + \dots + b_1 \sin x + b_2 \sin 2x + \dots + b_n \sin nx + \dots \tag{30.17}$$

Note that although the period of individual terms in the series may be less than  $2\pi$  (for instance,  $\sin 2x$  has period  $\pi$ ), all terms repeat every  $2\pi$  units. Therefore, the series expansion indicated in Eq. (30.17) has period  $2\pi$ . This series is called a **Fourier series**.

The principal problem to be solved is that of finding the coefficients  $a_n$  and  $b_n$ . Derivatives proved to be useful in finding the coefficients for a Maclaurin expansion. In contrast, we use the properties of certain integrals to find the coefficients of a Fourier series. To utilize these properties, we multiply all terms of Eq. (30.17) by  $\cos mx$  and then evaluate from  $-\pi$  to  $\pi$  (in this way we take advantage of the period  $2\pi$ ). Thus, we have

$$\int_{-\pi}^{\pi} f(x) \cos mx \, dx = \int_{-\pi}^{\pi} (a_0 + a_1 \cos x + a_2 \cos 2x + \dots) (\cos mx) \, dx + \int_{-\pi}^{\pi} (b_1 \sin x + b_2 \sin 2x + \dots) (\cos mx) \, dx \tag{30.18}$$

Using the methods of integration of Chapter 28, we now find the values of the coefficients  $a_n$  and  $b_n$ . For the coefficients  $a_n$  we find that the values differ depending on whether or not  $n = m$ . Therefore, first considering the case for which  $n \neq m$ , we have

$$\int_{-\pi}^{\pi} a_0 \cos mx \, dx = \frac{a_0}{m} \sin mx \Big|_{-\pi}^{\pi} = \frac{a_0}{m} (0 - 0) = 0 \tag{30.19}$$

$$\int_{-\pi}^{\pi} a_n \cos nx \cos mx \, dx = a_n \left[ \frac{\sin(n-m)x}{2(n-m)} + \frac{\sin(n+m)x}{2(n+m)} \right] \Big|_{-\pi}^{\pi} = 0 \quad (n \neq m) \tag{30.20}$$

The Fourier series is named for the French mathematician and physicist Jean Baptiste Joseph Fourier (1768–1830).



These values are all equal to zero since the sine of any multiple of  $\pi$  is zero.

Now, considering the case for which  $n = m$ , we have

$$\begin{aligned} \int_{-\pi}^{\pi} a_n \cos nx \cos nx \, dx &= \int_{-\pi}^{\pi} a_n \cos^2 nx \, dx \\ &= \left( \frac{a_n x}{2} + \frac{a_n}{2n} \sin nx \cos nx \right) \Big|_{-\pi}^{\pi} \\ &= \frac{a_n x}{2} \Big|_{-\pi}^{\pi} = \pi a_n \end{aligned} \quad (30.21)$$

Now, finding the values of the coefficient  $b_n$ , we have

$$\begin{aligned} \int_{-\pi}^{\pi} b_n \sin nx \cos mx \, dx &= b_n \left[ -\frac{\cos(n-m)x}{2(n-m)} - \frac{\cos(n+m)x}{2(n+m)} \right] \Big|_{-\pi}^{\pi} \\ &= b_n \left[ -\frac{\cos(n-m)\pi}{2(n-m)} - \frac{\cos(n+m)\pi}{2(n+m)} \right] \\ &\quad + \frac{\cos(n-m)(-\pi)}{2(n-m)} + \frac{\cos(n+m)(-\pi)}{2(n+m)} \Big] \\ &= 0 \text{ [since } \cos \theta = \cos(-\theta) \text{]} \quad (n \neq m) \end{aligned} \quad (30.22)$$

$$\int_{-\pi}^{\pi} b_n \sin nx \cos nx \, dx = \frac{b_n}{2n} \sin^2 nx \Big|_{-\pi}^{\pi} = 0 \quad (30.23)$$

These integrals are seen to be zero, except for the one specific case of  $\int_{-\pi}^{\pi} a_n \cos nx \cos mx \, dx$  when  $n = m$ , for which the result is indicated in Eq. (30.21). Using these results in Eq. (30.18), we have

$$\begin{aligned} \int_{-\pi}^{\pi} f(x) \cos nx \, dx &= a_n \int_{-\pi}^{\pi} \cos^2 nx \, dx = \pi a_n \\ a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \end{aligned} \quad (30.24)$$

This equation allows us to find the coefficients  $a_n$ , except  $a_0$ . We find the term  $a_0$  by direct integration of Eq. (30.17) from  $-\pi$  to  $\pi$ . When we perform this integration, all the sine and cosine terms integrate to zero, thereby giving the result

$$\begin{aligned} \int_{-\pi}^{\pi} f(x) \, dx &= \int_{-\pi}^{\pi} a_0 \, dx = a_0 x \Big|_{-\pi}^{\pi} = 2\pi a_0 \\ a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx \end{aligned} \quad (30.25)$$

By multiplying all terms of Eq. (30.17) by  $\sin mx$  and then integrating from  $-\pi$  to  $\pi$ , we find the coefficients  $b_n$ . We obtain the result

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \quad (30.26)$$

We can restate our equations for the Fourier series of a function  $f(x)$ :

$$\begin{aligned} f(x) &= a_0 + a_1 \cos x + a_2 \cos 2x + \cdots + a_n \cos nx + \cdots \\ &\quad + b_1 \sin x + b_2 \sin 2x + \cdots + b_n \sin nx + \cdots \end{aligned} \quad (30.17)$$

where the coefficients are found by

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx \tag{30.25}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx \tag{30.24}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx \tag{30.26}$$

**EXAMPLE 1** Fourier series for a square wave function

Find the Fourier series for the square wave function

$$f(x) = \begin{cases} -1 & -\pi \leq x < 0 \\ 1 & 0 \leq x < \pi \end{cases}$$

(Many of the functions we shall expand in Fourier series are discontinuous (not continuous) like this one. See Section 23.1 for a discussion of continuity.)

**LEARNING TIP**

When  $f(x)$  is defined differently for the intervals of  $x$  indicated, **it requires two integrals for each coefficient.**

using Eq. (30.25)

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^0 (-1) dx + \frac{1}{2\pi} \int_0^{\pi} (1) dx = -\frac{x}{2\pi} \Big|_{-\pi}^0 + \frac{x}{2\pi} \Big|_0^{\pi} = -\frac{1}{2} + \frac{1}{2} = 0$$

using Eq. (30.24)

$$a_n = \frac{1}{\pi} \int_{-\pi}^0 (-1) \cos nx dx + \frac{1}{\pi} \int_0^{\pi} (1) \cos nx dx = -\frac{1}{n\pi} \sin nx \Big|_{-\pi}^0 + \frac{1}{n\pi} \sin nx \Big|_0^{\pi} = 0 + 0 = 0$$

for all values of  $n$ , since  $\sin n\pi = 0$ ;

using Eq. (30.26)  
with  $n = 1$

$$b_1 = \frac{1}{\pi} \int_{-\pi}^0 (-1) \sin x dx + \frac{1}{\pi} \int_0^{\pi} (1) \sin x dx = \frac{1}{\pi} \cos x \Big|_{-\pi}^0 - \frac{1}{\pi} \cos x \Big|_0^{\pi} \\ = \frac{1}{\pi} (1 + 1) - \frac{1}{\pi} (-1 - 1) = \frac{4}{\pi}$$

using Eq. (30.26)  
with  $n = 2$

$$b_2 = \frac{1}{\pi} \int_{-\pi}^0 (-1) \sin 2x dx + \frac{1}{\pi} \int_0^{\pi} (1) \sin 2x dx = \frac{1}{2\pi} \cos 2x \Big|_{-\pi}^0 - \frac{1}{2\pi} \cos 2x \Big|_0^{\pi} \\ = \frac{1}{2\pi} (1 - 1) - \frac{1}{2\pi} (1 - 1) = 0$$

using Eq. (30.26)  
with  $n = 3$

$$b_3 = \frac{1}{\pi} \int_{-\pi}^0 (-1) \sin 3x dx + \frac{1}{\pi} \int_0^{\pi} (1) \sin 3x dx = \frac{1}{3\pi} \cos 3x \Big|_{-\pi}^0 - \frac{1}{3\pi} \cos 3x \Big|_0^{\pi} \\ = \frac{1}{3\pi} (1 + 1) - \frac{1}{3\pi} (-1 - 1) = \frac{4}{3\pi}$$

In general, if  $n$  is even,  $b_n = 0$ , and if  $n$  is odd, then  $b_n = 4/n\pi$ . Therefore,

$$f(x) = \frac{4}{\pi} \sin x + \frac{4}{3\pi} \sin 3x + \frac{4}{5\pi} \sin 5x + \cdots = \frac{4}{\pi} \left( \sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \cdots \right)$$

A graph of the function as defined, and the curve found by using the first three terms of the Fourier series, are shown in Fig. 30.10.

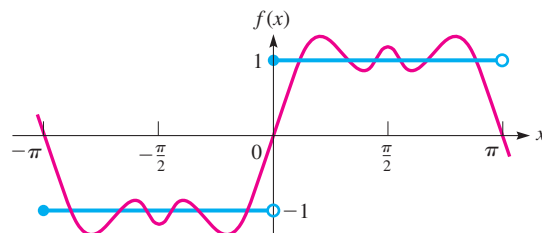


Fig. 30.10

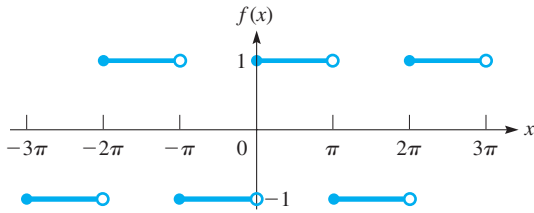


Fig. 30.11

Since functions found by Fourier series have a period of  $2\pi$ , they can represent functions with this period. If the function  $f(x)$  were defined to be periodic with period  $2\pi$ , with the same definitions as originally indicated, we would graph the function as shown in Fig. 30.11. The Fourier series representation would follow it as in Fig. 30.10. If more terms were used, the fit would be closer. ■

**EXAMPLE 2 Finding a Fourier series**

Find the Fourier series for the function

$$f(x) = \begin{cases} 1 & -\pi \leq x < 0 \\ x & 0 \leq x < \pi \end{cases}$$

For the periodic function, let  $f(x + 2\pi) = f(x)$  for all  $x$ .

A graph of three periods of this function is shown in Fig. 30.12.

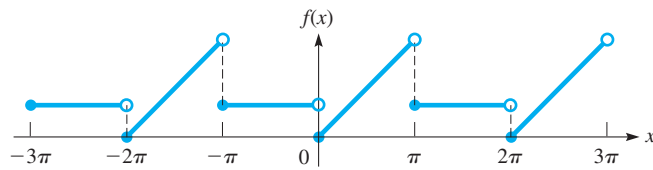


Fig. 30.12

Now, finding the coefficients, we have

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \int_{-\pi}^0 dx + \frac{1}{2\pi} \int_0^{\pi} x dx = \frac{x}{2\pi} \Big|_{-\pi}^0 + \frac{x^2}{4\pi} \Big|_0^{\pi} && \text{using Eq. (30.25)} \\ &= \frac{1}{2} + \frac{\pi}{4} = \frac{2 + \pi}{4} \end{aligned}$$

$$\begin{aligned} a_1 &= \frac{1}{\pi} \int_{-\pi}^0 \cos x dx + \frac{1}{\pi} \int_0^{\pi} x \cos x dx && \text{using Eq. (30.24)} \\ &= \frac{1}{\pi} \sin x \Big|_{-\pi}^0 + \frac{1}{\pi} (\cos x + x \sin x) \Big|_0^{\pi} = -\frac{2}{\pi} && \text{with } n = 1 \end{aligned}$$

$$\begin{aligned} a_2 &= \frac{1}{\pi} \int_{-\pi}^0 \cos 2x dx + \frac{1}{\pi} \int_0^{\pi} x \cos 2x dx && \text{using Eq. (30.24)} \\ &= \frac{1}{2\pi} \sin 2x \Big|_{-\pi}^0 + \frac{1}{4\pi} (\cos 2x + 2x \sin 2x) \Big|_0^{\pi} = 0 && \text{with } n = 2 \end{aligned}$$

$$\begin{aligned} a_3 &= \frac{1}{\pi} \int_{-\pi}^0 \cos 3x dx + \frac{1}{\pi} \int_0^{\pi} x \cos 3x dx && \text{using Eq. (30.24)} \\ &= \frac{1}{3\pi} \sin 3x \Big|_{-\pi}^0 + \frac{1}{9\pi} (\cos 3x + 3x \sin 3x) \Big|_0^{\pi} = -\frac{2}{9\pi} && \text{with } n = 3 \end{aligned}$$

$$\begin{aligned} b_1 &= \frac{1}{\pi} \int_{-\pi}^0 \sin x dx + \frac{1}{\pi} \int_0^{\pi} x \sin x dx && \text{using Eq. (30.26)} \\ &= -\frac{1}{\pi} \cos x \Big|_{-\pi}^0 + \frac{1}{\pi} (\sin x - x \cos x) \Big|_0^{\pi} = \frac{\pi - 2}{\pi} && \text{with } n = 1 \end{aligned}$$

$$\begin{aligned} b_2 &= \frac{1}{\pi} \int_{-\pi}^0 \sin 2x dx + \frac{1}{\pi} \int_0^{\pi} x \sin 2x dx && \text{using Eq. (30.26)} \\ &= -\frac{\cos 2x}{2\pi} \Big|_{-\pi}^0 + \frac{\sin 2x - 2x \cos 2x}{4\pi} \Big|_0^{\pi} = -\frac{1}{2} && \text{with } n = 2 \end{aligned}$$

Therefore, the first few terms of the Fourier series are

$$f(x) = \frac{2 + \pi}{4} - \frac{2}{\pi} \cos x - \frac{2}{9\pi} \cos 3x - \dots + \left(\frac{\pi - 2}{\pi}\right) \sin x - \frac{1}{2} \sin 2x + \dots$$

■ See the chapter introduction.

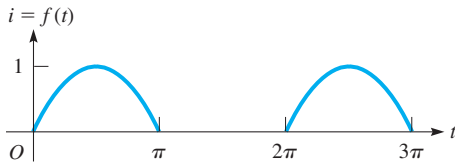


Fig. 30.13

**EXAMPLE 3** Fourier series for a half-wave rectifier

Certain electronic devices allow an electric current to pass through in only one direction. When an alternating current is applied to the circuit, the current exists for only half the cycle. Figure 30.13 is a representation of such a current as a function of time. This type of electronic device is called a *half-wave rectifier*. Derive the Fourier series for a rectified wave for which half is defined by  $f(t) = \sin t$  ( $0 \leq t \leq \pi$ ) and for which the other half is defined by  $f(t) = 0$ .

In finding the Fourier coefficients, we first find  $a_0$  as

$$a_0 = \frac{1}{2\pi} \int_0^\pi \sin t \, dt = \frac{1}{2\pi} (-\cos t) \Big|_0^\pi = \frac{1}{2\pi} (1 + 1) = \frac{1}{\pi}$$

In the previous example, we evaluated each of the coefficients individually. Here, we show how to set up a general expression for  $a_n$  and another for  $b_n$ . Once we have determined these, we can substitute values of  $n$  in the formula to obtain the individual coefficients:

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_0^\pi \sin t \cos nt \, dt = -\frac{1}{2\pi} \left[ \frac{\cos(1-n)t}{1-n} + \frac{\cos(1+n)t}{1+n} \right]_0^\pi \\ &= -\frac{1}{2\pi} \left[ \frac{\cos(1-n)\pi}{1-n} + \frac{\cos(1+n)\pi}{1+n} - \frac{1}{1-n} - \frac{1}{1+n} \right] \end{aligned}$$

See formula 40 in the table of integrals in Appendix B. It is valid for all values of  $n$  except  $n = 1$ . Now, we write

$$\begin{aligned} a_1 &= \frac{1}{\pi} \int_0^\pi \sin t \cos t \, dt = \frac{1}{2\pi} \sin^2 t \Big|_0^\pi = 0 \\ a_2 &= -\frac{1}{2\pi} \left( \frac{-1}{-1} + \frac{-1}{3} - \frac{1}{-1} - \frac{1}{3} \right) = -\frac{2}{3\pi} \\ a_3 &= -\frac{1}{2\pi} \left( \frac{1}{-2} + \frac{1}{4} - \frac{1}{-2} - \frac{1}{4} \right) = 0 \\ a_4 &= -\frac{1}{2\pi} \left( \frac{-1}{-3} + \frac{-1}{5} - \frac{1}{-3} - \frac{1}{5} \right) = -\frac{2}{15\pi} \\ b_n &= \frac{1}{\pi} \int_0^\pi \sin t \sin nt \, dt = \frac{1}{2\pi} \left[ \frac{\sin(1-n)t}{1-n} - \frac{\sin(1+n)t}{1+n} \right]_0^\pi \\ &= -\frac{1}{2\pi} \left[ \frac{\sin(1-n)\pi}{1-n} - \frac{\sin(1+n)\pi}{1+n} \right] \end{aligned}$$

See formula 39 in Appendix B. It is valid for all values of  $n$  except  $n = 1$ .

Therefore, we have

$$b_1 = \frac{1}{\pi} \int_0^\pi \sin t \sin t \, dt = \frac{1}{\pi} \int_0^\pi \sin^2 t \, dt = \frac{1}{2\pi} (t - \sin t \cos t) \Big|_0^\pi = \frac{1}{2}$$

We see that  $b_n = 0$  if  $n > 1$ , since each is evaluated in terms of the sine of a multiple of  $\pi$ .

Therefore, the Fourier series for the rectified wave is

$$f(t) = \frac{1}{\pi} + \frac{1}{2} \sin t - \frac{2}{\pi} \left( \frac{1}{3} \cos 2t + \frac{1}{15} \cos 4t + \dots \right)$$

The graph of these terms of the Fourier series and the original function are shown in Fig. 30.14. ■

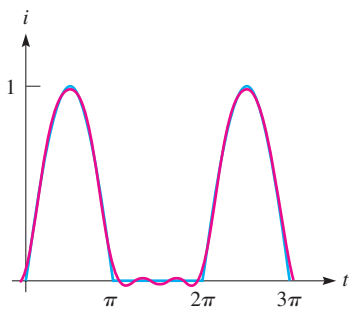


Fig. 30.14

All the types of periodic functions included in this section (as well as many others) may actually be seen on an oscilloscope when the proper signal is sent into it. In this way, the oscilloscope may be used to analyse the periodic nature of such phenomena as sound waves and electric currents.

### EXERCISES 30.6

In Exercises 1 and 2, make the given changes in Example 1 of this section and then find the resulting Fourier series.

1. Change the  $-1$  to  $-2$ , and the  $1$  to  $2$ .
2. Change the  $-1$  to  $0$ .

In Exercises 3–14, find at least three nonzero terms (including  $a_0$  and at least two cosine terms and two sine terms if they are not all zero) of the Fourier series for the given periodic functions and sketch at least three periods of the function.

3.  $f(x) = \begin{cases} 1 & -\pi \leq x < 0 \\ 0 & 0 \leq x < \pi \end{cases}$
4.  $f(x) = \begin{cases} 0 & -\pi \leq x < -\frac{\pi}{2}, \frac{\pi}{2} \leq x < \pi \\ 2 & -\frac{\pi}{2} \leq x < \frac{\pi}{2} \end{cases}$
5.  $f(x) = \begin{cases} 1 & -\pi \leq x < 0 \\ 2 & 0 \leq x < \pi \end{cases}$
6.  $f(x) = \begin{cases} 0 & -\pi \leq x < 0, \frac{\pi}{2} < x < \pi \\ 1 & 0 \leq x \leq \frac{\pi}{2} \end{cases}$
7.  $f(x) = \begin{cases} 0 & -\pi \leq x < 0 \\ x & 0 \leq x < \pi \end{cases}$
8.  $f(x) = x \quad -\pi \leq x < \pi$
9.  $f(x) = \begin{cases} -1 & -\pi \leq x < 0 \\ 0 & 0 \leq x < \frac{\pi}{2} \\ 1 & \frac{\pi}{2} \leq x < \pi \end{cases}$
10.  $f(x) = x^2 \quad -\pi \leq x < \pi$
11.  $f(x) = \begin{cases} -x & -\pi \leq x < 0 \\ x & 0 \leq x < \pi \end{cases}$
12.  $f(x) = \begin{cases} 0 & -\pi \leq x < 0 \\ x^2 & 0 \leq x < \pi \end{cases}$
13.  $f(x) = e^x \quad -\pi \leq x < \pi$
14.  $f(x) = \begin{cases} \pi + x & -\pi \leq x < 0 \\ \pi - x & 0 < x < \pi \end{cases}$

In Exercises 15–20, use a graphing calculator to display the terms of the Fourier series given in the indicated example or answer for the indicated exercise. Compare with the sketch of the function. For each calculator display, use  $X_{\min} = -8$  and  $X_{\max} = 8$ .

- |                |                 |                 |
|----------------|-----------------|-----------------|
| 15. Example 1  | 16. Example 2   | 17. Exercise 5  |
| 18. Exercise 7 | 19. Exercise 11 | 20. Exercise 10 |

In Exercises 21–24, solve the given problems.

21. The periodic force  $F$  (in N) applied in testing a spring system can be represented by  $F = 0$  for  $-\pi \leq t < 0$  and  $F = t^2 + t$  for  $0 < t < \pi$ , where the time  $t$  is in seconds. Find the Fourier series that represents this force.
22. Another representation for a half-wave rectifier (see Example 3) is  $f(t) = \cos t$  ( $-\pi/2 \leq t < \pi/2$ ),  $f(t) = 0$  ( $-\pi < t < -\pi/2, \pi/2 < t < \pi$ ). Find the Fourier series for this half-wave rectifier.
23. Find the Fourier expansion of the electronic device known as a full-wave rectifier. This is found by using as the function for the current  $f(t) = -\sin t$  for  $-\pi \leq t \leq 0$  and  $f(t) = \sin t$  for  $0 < t \leq \pi$ . The graph of this function is shown in Fig. 30.15. The portion of the curve to the left of the origin is dashed because from a physical point of view we can give no significance to this part of the wave, although mathematically we can derive the proper form of the Fourier expansion by using it.

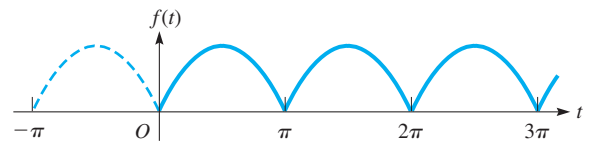


Fig. 30.15

24. The loudness  $L$  (in decibels) of a certain siren as a function of time  $t$  (in s) can be described by the function
 
$$L = 0 \quad -\pi \leq t < 0$$

$$L = 100t \quad 0 \leq t < \pi/2$$

$$L = 100(\pi - t) \quad \pi/2 \leq t < \pi$$
 with a period of  $2\pi$  seconds (where only positive values of  $t$  have physical significance). Find  $a_0$ , the first nonzero cosine term, and the first two nonzero sine terms of the Fourier expansion for the loudness of the siren. See Fig. 30.16.

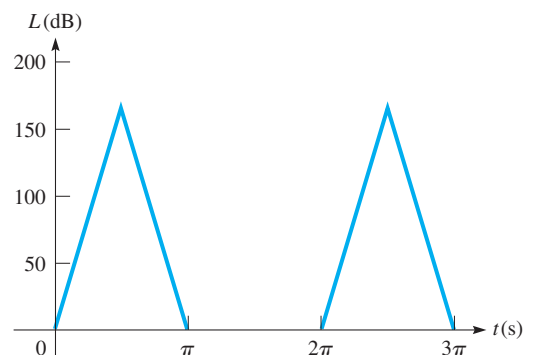


Fig. 30.16

## 30.7 More About Fourier Series

Even and Odd Functions • Constant Added to Fourier Series • Fourier Series with Period  $2L$  • Half-Range Expansions

When finding the Fourier expansion of some functions, it may turn out that all the sine terms evaluate to be zero or that all the cosine terms evaluate to be zero. In fact, in Example 1 on page 936, we see that all of the cosine terms were zero and that the expansion contained only sine terms. We now show how to quickly determine if an expansion will contain only sine terms, or only cosine terms.

### EVEN FUNCTIONS AND ODD FUNCTIONS

In Chapter 21 (page 574), we showed that when  $-x$  replaces  $x$  in a function  $f(x)$ , and the function does not change, the curve of the function is symmetric to the  $y$ -axis. *Such a function is called an even function.*

#### EXAMPLE 1 $\cos x$ is an even function

We can show that the function  $y = \cos x$  is an even function by using the Maclaurin expansions for  $\cos x$  and  $\cos(-x)$ . These are

$$\begin{aligned}\cos x &= 1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots \\ \cos(-x) &= 1 - \frac{(-x)^2}{2} + \frac{(-x)^4}{24} - \cdots = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots\end{aligned}$$

Since the expansions are the same,  $\cos x$  is an even function (and so is  $\cos nx$  for all  $n$ ). ■

Again referring to Chapter 21 (page 574), we recall that if  $-x$  replaces  $x$  and  $-y$  replaces  $y$  at the same time, and the function does not change, then the function is symmetric to the origin. *Such a function is called an odd function.*

#### EXAMPLE 2 $\sin x$ is an odd function

We can show that the function  $y = \sin x$  is an odd function by using the Maclaurin expansions for  $\sin x$  and  $-\sin(-x)$  (the  $-$  sign before  $\sin(-x)$  is equivalent to making  $y$  negative). These are

$$\begin{aligned}\sin x &= x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots \\ -\sin(-x) &= -\left[(-x) - \frac{(-x)^3}{6} + \frac{(-x)^5}{120} - \cdots\right] = x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\end{aligned}$$

Since  $\sin x = -\sin(-x)$ ,  $\sin x$  is an odd function (and so is  $\sin nx$  for all  $n$ ). ■

It follows from the definitions that the product of two odd functions is even and that the product of an even function and an odd function is also odd. Moreover, if  $f(x)$  is odd,  $\int_{-\pi}^{\pi} f(x) = 0$  (the area from  $-\pi$  to 0 cancels out with the area from 0 to  $\pi$ ).

These properties combined with the results of Examples 1 and 2 imply that when  $f(x)$  is even, the integrand in Eq. (30.26) is odd, and all  $b_n$  terms are 0. Similarly, when  $f(x)$  is odd, the integrands in Eqs. (30.25) and (30.24) are odd, and all  $a_n$  terms are 0. We summarize this in the Learning Tip to the left.

#### Practice Exercises

Determine whether the following functions are even or odd or neither:

1.  $f(x) = x^{1/3}$     2.  $f(x) = \sin^2 x$

#### LEARNING TIP

- The Fourier series expansion of an even function contains only cosine terms (and possibly a constant term). When finding the Fourier series for such a function, we do not have to find any sine terms.
- The Fourier series expansion of an odd function contains only sine terms (and no constant term). When finding the Fourier series for such a function, we do not have to find any cosine terms.

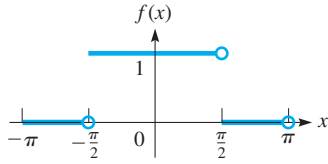


Fig. 30.17

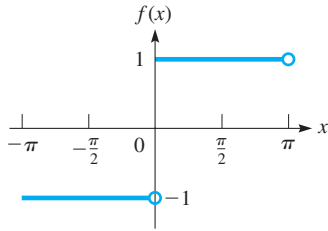


Fig. 30.18

**LEARNING TIP**

If a constant  $k$  is added to a function  $f_1(x)$ , the resulting function  $f(x)$  is

$$f(x) = k + f_1(x)$$

Therefore, if we know the Fourier series expansion for  $f_1(x)$ , the Fourier series expansion of  $f(x)$  is found by adding  $k$  to the Fourier series expansion of  $f_1(x)$ .

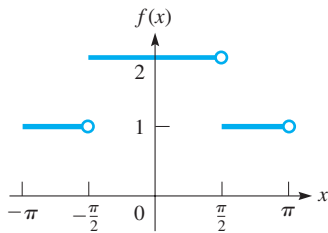


Fig. 30.19

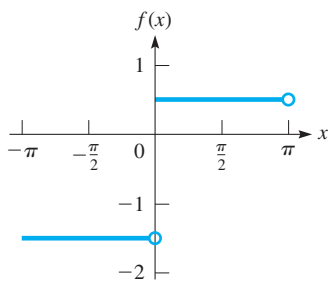


Fig. 30.20

**EXAMPLE 3** Fourier series of an even function

The function

$$f(x) = \begin{cases} 0 & -\pi \leq x < -\pi/2, \pi/2 \leq x < \pi \\ 1 & -\pi/2 \leq x < \pi/2 \end{cases}$$

is even (the symmetry to the  $y$ -axis can be seen in Fig. 30.17). Therefore, its Fourier series expansion contains only cosine terms (and a constant). We find the series to be

$$f(x) = \frac{1}{2} + \frac{2}{\pi} \left( \cos x - \frac{1}{3} \cos 3x + \frac{1}{5} \cos 5x - \cdots \right).$$

**EXAMPLE 4** Fourier series of an odd function

The function

$$f(x) = \begin{cases} -1 & -\pi \leq x < 0 \\ 1 & 0 \leq x < \pi \end{cases}$$

is odd (the symmetry to the origin can be seen in Fig. 30.18). Therefore, its Fourier series expansion contains only sine terms and no constant. As we showed in Example 1,

$$\text{the series is } f(x) = \frac{4}{\pi} \left( \sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \cdots \right).$$

**EXAMPLE 5** Constant added to a Fourier series

The values of the function

$$f(x) = \begin{cases} 1 & -\pi \leq x < -\pi/2, \pi/2 \leq x < \pi \\ 2 & -\pi/2 \leq x < \pi/2 \end{cases}$$

are all 1 greater than those of the function of Example 3. Therefore, denoting the function of Example 3 as  $f_1(x)$ , we have  $f(x) = 1 + f_1(x)$ . This means that the Fourier series for  $f(x)$  is

$$\begin{aligned} f(x) &= 1 + \left[ \frac{1}{2} + \frac{2}{\pi} \left( \cos x - \frac{1}{3} \cos 3x + \frac{1}{5} \cos 5x - \cdots \right) \right] \\ &= \frac{3}{2} + \frac{2}{\pi} \left( \cos x - \frac{1}{3} \cos 3x + \frac{1}{5} \cos 5x - \cdots \right) \end{aligned}$$

In Fig. 30.19, we see that the graph of  $f(x)$  is shifted up vertically by 1 unit from the graph of  $f_1(x)$  in Fig. 30.17. This is equivalent to a vertical translation of axes. We also note that  $f(x)$  is an even function.

**EXAMPLE 6** Constant subtracted from a Fourier series

The values of the function

$$f(x) = \begin{cases} -\frac{3}{2} & -\pi \leq x < 0 \\ \frac{1}{2} & 0 \leq x < \pi \end{cases}$$

are all  $\frac{1}{2}$  less than those of the function of Example 4. Therefore, denoting the function of Example 4 as  $f_1(x)$ , we have  $f(x) = -\frac{1}{2} + f_1(x)$ . This means that the Fourier series for  $f(x)$  is

$$f(x) = -\frac{1}{2} + \frac{4}{\pi} \left( \sin x - \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x - \cdots \right)$$

In Fig. 30.20, we see that the graph of  $f(x)$  is shifted vertically down by  $\frac{1}{2}$  unit from the graph of  $f_1(x)$  in Fig. 30.18. Because of the presence of the constant term,  $f(x)$  is not an odd function. However, since  $f(x)$  is obtained as a shift of an odd function, its series expansion contains only sine terms.



**LEARNING TIP**

A function that is obtained as a vertical shift of an odd function will no longer be odd. However, its series expansion will still contain only sine terms, as seen in Example 6. It is therefore worth identifying when a function is a shift of an odd function, since in that case no cosine terms need to be computed.

**FOURIER SERIES WITH PERIOD 2L**

The standard form of a Fourier series we have considered to this point is defined over the interval from  $x = -\pi$  to  $x = \pi$ . At times, it is preferable to have a series that is defined over a different interval.

Noting that

$$\sin \frac{n\pi}{L}(x + 2L) = \sin n\left(\frac{\pi x}{L} + 2\pi\right) = \sin \frac{n\pi x}{L}$$

we see that  $\sin(n\pi x/L)$  has a period of  $2L$ . Thus, by using  $\sin(n\pi x/L)$  and  $\cos(n\pi x/L)$  and the same method of derivation, the following equations are found for the coefficients for the Fourier series for the interval from  $x = -L$  to  $x = L$ .

$$a_0 = \frac{1}{2L} \int_{-L}^L f(x) dx \tag{30.27}$$

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx \tag{30.28}$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx \tag{30.29}$$

Note that the conclusions regarding the Fourier series of odd and even functions also apply when the period is  $2L$ .

**EXAMPLE 7** Fourier series with period of 8

In checking electric circuit elements, a square wave representing the current  $i$  (in mA) as a function of the time  $t$  (in ms) is given by

$$f(t) = \begin{cases} 0 & -4 \leq t < 0 \\ 2 & 0 \leq t < 4 \end{cases}$$

and for which the period is 8 ms. See Fig. 30.21.

Since the period is 8 ms,  $L = 4$  ms. Next, we note that  $f(t) = 1 + f_1(t)$ , where  $f_1(t)$  is an odd function (from the definition of  $f(t)$ , and from Fig. 30.21 we can see the symmetry to the point  $(0, 1)$ ). Therefore, *the constant is 1 and there are no cosine terms* in the Fourier series for  $f(t)$ . Now, finding the sine terms, we have

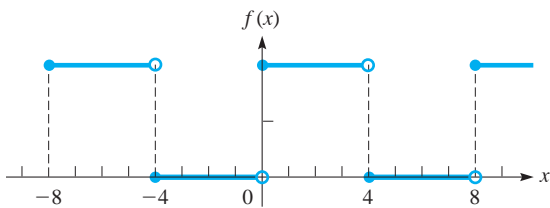


Fig. 30.21

$$b_n = \frac{1}{4} \int_{-4}^0 0 \sin \frac{n\pi t}{4} dt + \frac{1}{4} \int_0^4 2 \sin \frac{n\pi t}{4} dt \tag{using Eq. (30.29)}$$

$$= \frac{1}{2} \left( \frac{4}{n\pi} \right) \int_0^4 \sin \frac{n\pi t}{4} \left( \frac{n\pi dt}{4} \right) = -\frac{2}{n\pi} \cos \frac{n\pi t}{4} \Big|_0^4$$

$$= -\frac{2}{n\pi} (\cos n\pi - \cos 0) = \frac{2}{n\pi} (1 - \cos n\pi)$$

$$b_1 = \frac{2}{\pi} [1 - (-1)] = \frac{4}{\pi} \qquad b_2 = \frac{2}{2\pi} (1 - 1) = 0$$

$$b_3 = \frac{2}{3\pi} [1 - (-1)] = \frac{4}{3\pi} \qquad b_4 = \frac{2}{4\pi} (1 - 1) = 0$$

Therefore, the Fourier series is

$$f(t) = 1 + \frac{4}{\pi} \sin \frac{\pi t}{4} + \frac{4}{3\pi} \sin \frac{3\pi t}{4} + \dots$$

See Exercise 43 on page 316. ■

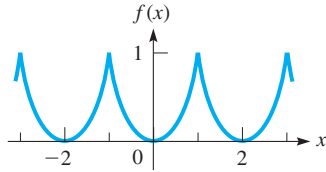


Fig. 30.22

**EXAMPLE 8** Fourier series with period of 2

Find the Fourier series for the function

$$f(x) = x^2 \quad -1 \leq x < 1$$

for which the period is 2. See Fig. 30.22.

Since the period is 2,  $L = 1$ . Next, we note that  $f(x) = f(-x)$ , which means it is an even function. Therefore, there are no sine terms in the Fourier series. Finding the constant and the cosine terms, we have

$$\begin{aligned}
 a_0 &= \frac{1}{2(1)} \int_{-1}^1 x^2 dx = \frac{1}{6} x^3 \Big|_{-1}^1 = \frac{1}{6} (1 + 1) = \frac{1}{3} \\
 a_n &= \frac{1}{1} \int_{-1}^1 x^2 \cos \frac{n\pi x}{1} dx = \int_{-1}^1 x^2 \cos n\pi x dx && \text{integrating by parts: } u = x^2, du = 2x dx, \\
 &= x^2 \left( \frac{1}{n\pi} \sin n\pi x \right) \Big|_{-1}^1 - \frac{2}{n\pi} \int_{-1}^1 x \sin n\pi x dx && \text{integrating by parts: } u = x, du = dx, \\
 &= \frac{1}{n\pi} \sin n\pi - \frac{1}{n\pi} \sin(-n\pi) - \frac{2}{n\pi} \left[ x \left( -\frac{1}{n\pi} \cos n\pi x \right) \Big|_{-1}^1 - \left( -\frac{1}{n\pi} \int_{-1}^1 \cos n\pi x dx \right) \right] && dv = \cos n\pi x dx, v = (1/n\pi) \sin n\pi x \\
 &= \frac{2}{n^2 \pi^2} [\cos n\pi + \cos(-n\pi)] + \frac{1}{n^2 \pi^2} \sin n\pi x \Big|_{-1}^1 = \frac{2}{n^2 \pi^2} (2 \cos n\pi) = \frac{4}{n^2 \pi^2} \cos n\pi && dv = \sin n\pi x dx, v = (-1/n\pi) \cos n\pi x \\
 a_1 &= \frac{4}{\pi^2} \cos \pi = -\frac{4}{\pi^2} \quad a_2 = \frac{4}{4\pi^2} \cos 2\pi = \frac{4}{4\pi^2} \quad a_3 = \frac{4}{9\pi^2} \cos 3\pi = -\frac{4}{9\pi^2}
 \end{aligned}$$

Therefore, the Fourier series is

$$f(x) = \frac{1}{3} - \frac{4}{\pi^2} \left( \cos \pi x - \frac{1}{4} \cos 2\pi x + \frac{1}{9} \cos 3\pi x - \dots \right)$$

**Practice Exercise**

3. Find the Fourier series for the function  $f(x) = x^2 + 2 \quad -1 \leq x < 1$ .

**HALF-RANGE EXPANSIONS**

We have seen that the Fourier series expansion for an even function contains only cosine terms (and possibly a constant), and the expansion of an odd function contains only sine terms. It is also possible to force a function to be even or odd, so that the expansion will contain only cosine terms or only sine terms.

Considering the symmetry of an even function, the area under the curve from  $-L$  to 0 is the same as the area under the curve from 0 to  $L$  (see Fig. 30.23). This means the value of the integral from  $-L$  to 0 equals the value of the integral from 0 to  $L$ . Therefore, the value of the integral from  $-L$  to  $L$  equals twice the value of the integral from 0 to  $L$ , or

$$\int_{-L}^L f(x) dx = 2 \int_0^L f(x) dx \quad f(x) \text{ even}$$

Therefore, to obtain the Fourier coefficients for an expression from  $-L$  to  $L$  for an even function, we can multiply the coefficients obtained using Eqs. (30.27) and (30.28) from 0 to  $L$  by 2. Similar reasoning shows that the Fourier coefficients for an expansion from  $-L$  to  $L$  for an odd function may be found by multiplying the coefficients obtained using Eq. (30.29) from 0 to  $L$  by 2.

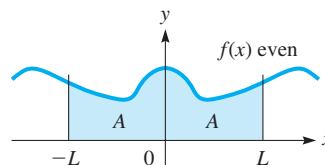


Fig. 30.23

A **half-range Fourier cosine series** is a series that contains only cosine terms, and a **half-range Fourier sine series** is a series that contains only sine terms. To find the half-range expansion for a function  $f(x)$ , it is defined for interval 0 to  $L$  (half of the interval from  $-L$  to  $L$ ) and then specified as odd or even, thereby clearly defining the function in the interval from  $-L$  to 0. This means that the *Fourier coefficients for a half-range cosine series are given by*

$$a_0 = \frac{1}{L} \int_0^L f(x) dx \quad \text{and} \quad a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx \quad (n = 1, 2, \dots) \quad (30.30)$$

Similarly, the *Fourier coefficients for a half-range sine series are given by*

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx \quad (n = 1, 2, \dots) \quad (30.31)$$

**EXAMPLE 9 Half-range cosine series**

Find  $f(x) = x$  in a half-range cosine series for  $0 \leq x < 2$ .

Since we are to have a cosine series, we extend the function to be an even function with its graph as shown in Fig. 30.24. The red portion between  $x = 0$  and  $x = L$  shows the given function as defined, and blue portions show the extension that makes it an even function. Now, by use of Eqs. (30.30), we find the Fourier expansion coefficients, with  $L = 2$ .

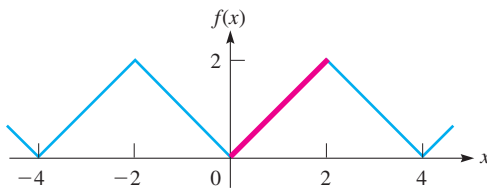


Fig. 30.24

$$a_0 = \frac{1}{2} \int_0^2 x dx = \frac{1}{4} x^2 \Big|_0^2 = 1$$

$$a_n = \frac{2}{2} \int_0^2 x \cos \frac{n\pi x}{2} dx = x \left[ \left( \frac{2}{n\pi} \sin \frac{n\pi x}{2} \right) - \left( \frac{-4}{n^2 \pi^2} \cos \frac{n\pi x}{2} \right) \right] \Big|_0^2$$

$$= \frac{4}{n^2 \pi^2} (\cos n\pi - 1) \quad (n \neq 0)$$

If  $n$  is even,  $\cos n\pi - 1 = 0$ . Therefore, we evaluate  $a_n$  for the odd values of  $n$ , and find the expansion is

$$f(x) = 1 - \frac{8}{\pi^2} \left( \cos \frac{\pi x}{2} + \frac{1}{9} \cos \frac{3\pi x}{2} + \frac{1}{25} \cos \frac{5\pi x}{2} + \dots \right) \quad \blacksquare$$

■ See Exercise 44 on page 316.

**EXAMPLE 10 Half-range sine series**

Expand  $f(x) = x$  in a half-range sine series for  $0 \leq x < 2$ .

Since we are to have a sine series, we extend the function to be an odd function with its graph as shown in Fig. 30.25. Again, the red portion shows the given function as defined, and blue portions show the extension that makes it an odd function. By using Eq. (30.31), we find the Fourier expansion coefficients, with  $L = 2$ .

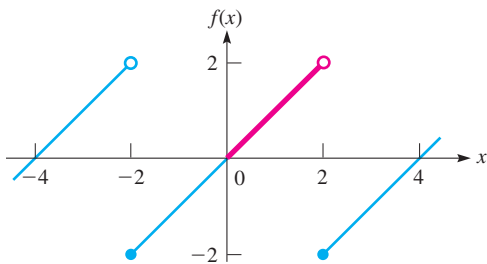


Fig. 30.25

$$b_n = \frac{2}{2} \left( \int_0^2 x \sin \frac{n\pi x}{2} \right) dx$$

$$= x \left[ \left( \frac{-2}{n\pi} \cos \frac{n\pi x}{2} \right) - \left( \frac{-4}{n^2 \pi^2} \sin \frac{n\pi x}{2} \right) \right] \Big|_0^2 = -\frac{4}{n\pi} \cos n\pi$$

$$f(x) = \frac{4}{\pi} \left( \sin \frac{\pi x}{2} - \frac{1}{2} \sin \pi x + \frac{1}{3} \sin \frac{3\pi x}{2} - \dots \right) \quad \blacksquare$$

## EXERCISES 30.7

In Exercises 1–4, write the Fourier series for each function by comparing it to an appropriate function given in an example of this section. Do not use any of the formulas for  $a_0$ ,  $a_n$ , or  $b_n$ .

$$1. f(x) = \begin{cases} 2 & -\pi \leq x < -\pi/2, \pi/2 \leq x < \pi \\ 3 & -\pi/2 \leq x < \pi/2 \end{cases}$$

$$2. f(x) = \begin{cases} -\frac{1}{2} & -\pi \leq x < 0 \\ \frac{3}{2} & 0 \leq x < \pi \end{cases}$$

$$3. f(x) = \begin{cases} -2 & -4 \leq x < 0 \\ 0 & 0 \leq x < 4 \end{cases}$$

$$4. f(x) = \begin{cases} -\frac{1}{3} & -\pi \leq x < -\pi/2, \pi/2 \leq x < \pi \\ \frac{2}{3} & -\pi/2 \leq x < \pi/2 \end{cases}$$

In Exercises 5–12, determine whether the given function is even, or odd, or neither. One period is defined for each function.

$$5. f(x) = \begin{cases} 5 & -3 \leq x < 0 \\ 0 & 0 \leq x < 3 \end{cases} \quad 6. f(x) = \begin{cases} -1 & -2 \leq x < 0 \\ 1 & 0 \leq x < 2 \end{cases}$$

$$7. f(x) = \begin{cases} 2 & -1 \leq x < 1 \\ 0 & -2 \leq x < -1, 1 \leq x < 2 \end{cases}$$

$$8. f(x) = \begin{cases} 0 & -2 \leq x < 0, 1 \leq x < 2 \\ 1 & 0 \leq x < 1 \end{cases}$$

$$9. f(x) = |x| \quad -4 \leq x < 4 \quad 10. f(x) = \begin{cases} 0 & -1 \leq x < 0 \\ e^x & 0 \leq x < 1 \end{cases}$$

$$11. f(x) = -x \cos 3x \quad -3 \leq x < 3$$

$$12. f(x) = x \sin 2x \cos x \quad -4 \leq x < 4$$

In Exercises 13–16, determine whether the Fourier series of the given functions will include only sine terms, only cosine terms, or both sine terms and cosine terms.

$$13. f(x) = 2 - x \quad -4 \leq x < 4$$

$$14. f(x) = \cos(\sin x) \quad -\pi \leq x < \pi$$

$$15. f(x) = \begin{cases} 0 & -\pi \leq x < 0 \\ \cos x & 0 \leq x < \pi \end{cases}$$

$$16. f(x) = \begin{cases} -3 & -3 \leq x < 0 \\ 0 & 0 \leq x < 3 \end{cases}$$

In Exercises 17–22, find at least three nonzero terms (including  $a_0$  and at least two cosine terms and two sine terms if they are not all zero) of the Fourier series for the function from the indicated exercise of this section.

17. Exercise 5

18. Exercise 6

19. Exercise 7

20. Exercise 8

21. Exercise 9

22. Exercise 10

In Exercises 23–28, solve the given problems.

23. Expand  $f(x) = 1$  in a half-range sine series for  $0 \leq x < 4$ .

24. Expand  $f(x) = 1$  ( $0 \leq x < 2$ ),  $f(x) = 0$  ( $2 \leq x < 4$ ) in a half-range cosine series for  $0 \leq x < 4$ .

25. Expand  $f(x) = x^2$  in a half-range cosine series for  $0 \leq x < 2$ .

26. Expand  $f(x) = x^2$  in a half-range sine series for  $0 \leq x < 2$ .

27. Each pulse of a pulsating force  $F$  of a pressing machine is 8 N. The force lasts for 1 s, followed by a 3-s pause. Thus, it can be represented by  $F = 0$  for  $-2 \leq t < 0$  and  $1 \leq t < 2$ , and  $F = 8$  for  $0 \leq t < 1$ , with a period of 4 s (only positive values of  $t$  have physical significance). Find the Fourier series for the force.

28. A pulsating electric current  $i$  (in mA) with a period of 2 s can be described by  $i = e^{-t}$  for  $-1 \leq t < 1$  s for one period (only positive values of  $t$  have physical significance). Find the Fourier series that represents this current.

## Answers to Practice Exercises

$$1. \text{ odd} \quad 2. \text{ even} \quad 3. f(x) = \frac{7}{3} - \frac{4}{\pi^2} (\cos \pi x - \frac{1}{4} \cos 2\pi x + \dots)$$

## CHAPTER 30 EQUATIONS

Infinite series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots + a_n + \dots \quad (30.1)$$

Sum of series

$$S = \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \sum_{i=1}^n a_i \quad (30.2)$$

Sum of geometric series

$$S = \lim_{n \rightarrow \infty} S_n = \frac{a_1}{1-r} \quad \text{for } |r| < 1 \quad (30.3)$$

Power series

$$f(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots \quad (30.4)$$

Maclaurin series

$$f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \dots + \frac{f^n(0)x^n}{n!} + \dots \quad (30.5)$$

**Special series** 
$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \quad (\text{all } x) \quad (30.6)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \quad (\text{all } x) \quad (30.7)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \quad (\text{all } x) \quad (30.8)$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad (|x| < 1) \quad (30.9)$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots \quad (|x| < 1) \quad (30.10)$$

**Taylor series** 
$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \cdots \quad (30.16)$$

**Fourier series** 
$$f(x) = a_0 + a_1 \cos x + a_2 \cos 2x + \cdots + a_n \cos nx + \cdots \\ + b_1 \sin x + b_2 \sin 2x + \cdots + b_n \sin nx + \cdots \quad (30.17)$$

**Period =  $2\pi$**  
$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx \quad (30.25)$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx \quad (30.24)$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx \quad (30.26)$$

**Period =  $2L$**  
$$a_0 = \frac{1}{2L} \int_{-L}^L f(x) dx \quad (30.27)$$

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx \quad (30.28)$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx \quad (30.29)$$

**Half-range expansions** 
$$a_0 = \frac{1}{L} \int_0^L f(x) dx \quad \text{and} \quad a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx \quad (n = 1, 2, \dots) \quad (30.30)$$

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx \quad (n = 1, 2, \dots) \quad (30.31)$$

## CHAPTER 30 REVIEW EXERCISES

In Exercises 1–10, find the first three nonzero terms of the Maclaurin expansion of the given functions.

1.  $f(x) = \frac{1}{1+e^x}$

2.  $f(x) = e^{\cos x}$

5.  $f(x) = (x+1)^{1/3}$

6.  $f(x) = \frac{x^2}{1+x^2}$

3.  $f(x) = \sin 2x^2$

4.  $f(x) = \frac{1}{(1-x)^2}$

7.  $f(x) = \sin^{-1} x$

8.  $f(x) = \frac{1}{1-\sin x}$

9.  $f(x) = \cos(a+x)$

10.  $f(x) = \ln(a+x)$

In Exercises 11–22, calculate the value of each of the given functions. Use three terms of the appropriate series. Round answers to 3 significant digits.

- |                      |                   |                        |
|----------------------|-------------------|------------------------|
| 11. $e^{-0.2}$       | 12. $\ln(1.10)$   | 13. $\sqrt[3]{1.3}$    |
| 14. $\sin 3.5^\circ$ | 15. $1.086^{-1}$  | 16. $0.9839^{10}$      |
| 17. $\ln 0.8172$     | 18. $\cos 0.1376$ | 19. $\tan 43.62^\circ$ |
| 20. $\sqrt[4]{260}$  | 21. $\sqrt{148}$  | 22. $\cos 47^\circ$    |

In Exercises 23 and 24, evaluate the given integrals by using three terms of the appropriate series.

$$23. \int_{0.1}^{0.2} \frac{\cos x}{\sqrt{x}} dx \qquad 24. \int_0^{0.1} \sqrt[3]{1+x^2} dx$$

In Exercises 25 and 26, find the first three terms of the Taylor expansion for the given function and value of  $a$ .

25.  $\cos x$  ( $a = \pi/3$ )      26.  $\ln \cos x$  ( $a = \pi/4$ )

In Exercises 27–30, write the Fourier series for each function by comparing it to an appropriate function in an example of either Section 30.6 or 30.7. One period is given for each function. Do not use any formulas for  $a_0$ ,  $a_n$ , or  $b_n$ .

27.  $f(x) = \begin{cases} 0 & -\pi \leq x < 0 \\ x-1 & 0 \leq x < \pi \end{cases}$
28.  $f(x) = x^2 - 1$   $-1 \leq x < 1$
29.  $f(x) = \begin{cases} \pi - 1 & -4 \leq x < 0 \\ \pi + 1 & 0 \leq x < 4 \end{cases}$
30.  $f(x) = \begin{cases} 1 & -\pi \leq x < 0 \\ 1 + \sin x & 0 \leq x < \pi \end{cases}$

In Exercises 31–34, find at least three nonzero terms (including  $a_0$  and at least two cosine terms and two sine terms if they are not all zero) of the Fourier series for the given function. One period is given for each function.

31.  $f(x) = \begin{cases} 0 & -\pi \leq x < -\pi/2, \pi/2 \leq x < \pi \\ 1 & -\pi/2 \leq x < \pi/2 \end{cases}$  (See Example 3, page 941.)
32.  $f(x) = \begin{cases} -x & -\pi \leq x < 0 \\ 0 & 0 \leq x < \pi \end{cases}$
33.  $f(x) = x$   $-2 \leq x < 2$
34.  $f(x) = \begin{cases} -2 & -3 \leq x < 0 \\ 2 & 0 \leq x < 3 \end{cases}$

In Exercises 35–74, solve the given problems.

35. Test the series  $1000 + 800 + 640 + 512 + \dots$  for convergence or divergence. If convergent, find its sum.
36. Test the series  $1 + 1.1 + 1.21 + 1.331 + \dots$  for convergence or divergence. If convergent, find its sum.
37. Find the sum of the series  $64 + 48 + 36 + 27 + \dots$ .
38. Find the first five partial sums of the series  $\sum_{n=1}^{\infty} \frac{n}{3n+1}$  and determine whether it appears to be convergent or divergent.
39. Express the integration of the indefinite integral  $\int \sin(x^2) dx$  as an infinite series.

40. Express the integration of the indefinite integral  $\int (1+x^4)^{-1} dx$  as an infinite series.
41. Find the first three terms of the Taylor series for  $f(x) = \tan x$  with  $a = \pi/4$ .
42. Integrate the series found in Exercise 41 to find the Taylor expansion for  $\ln \cos x$  with  $a = \pi/4$ .
43. Differentiate the series found in Exercise 41 to find the Taylor expansion for  $\sec^2 x$  with  $a = \pi/4$ .
44. Use the series for  $(1-x^2)^{-1/2}$  to find the Maclaurin series for  $\sin^{-1} x$ .
45. If  $h$  is small, show that  $\sin(x+h) - \sin(x-h) = 2h \cos x$ .
46. Find the first three nonzero terms of the Maclaurin expansion of the function  $\sin x + x \cos x$  by differentiating the expansion term by term for  $x \sin x$ .
47. Using the properties of logarithms and Eq. (30.9), find four terms of the Maclaurin expansion of  $\ln(1+x)^4$ .
48. By multiplication of series, show that the first two terms of the Maclaurin series for  $2 \sin x \cos x$  are the same as those of the series for  $\sin 2x$ .
49. Find the first four nonzero terms of the expansion for  $\sin^2 x$  by using the identity  $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$  and the series for  $\cos x$ .
50. Evaluate the integral  $\int_0^1 x \sin x dx$  (a) by methods of Chapter 28 and (b) using three terms of the series for  $\sin x$ . Compare results.
51. Find the first three terms of the Maclaurin expansion for  $\sec x$  by finding the reciprocal of the series for  $\cos x$ .
52. By simplifying the sum of the squares for the Maclaurin series for  $\sin x$  and  $\cos x$ , verify (to this extent) that  $\sin^2 x + \cos^2 x = 1$ .
53. From the Maclaurin series for  $f(x) = 1/(1-x)$  (see Exercise 13 of Section 30.2), find the series for  $1/(1+x)$ .
54. Show that the Maclaurin expansions for  $\cos x$  and  $\cos(-x)$  are the same.
55. Expand  $f(x) = x^2$  in a half-range cosine series for  $0 < x \leq 1$ .
56. Expand  $f(x) = 2-x$  in a half-range sine series for  $0 < x \leq 2$ .
57. Calculate  $e^{0.9}$  by using four terms of the Maclaurin expansion for  $e^x$ . Also, calculate  $e^{0.9}$  by using the first three terms of the Taylor expansion in Example 1 on page 931. Compare the accuracy of the values obtained with that found directly on a calculator.
58. Find the volume generated by revolving the region bounded by  $y = e^{-x}$ ,  $y = 0$ ,  $x = 0$ , and  $x = 0.1$  about the  $y$ -axis by using three terms of the appropriate series.
59. Find the approximate area between the curve of  $y = \frac{x - \sin x}{x^2}$  and the  $x$ -axis between  $x = 0.1$  and  $x = 0.2$ .
60. Find the approximate value of the moment of inertia with respect to its axis of the solid generated by revolving the smaller region bounded by  $y = \sin x$ ,  $x = 0.3$ , and the  $x$ -axis about the  $y$ -axis. Use two terms of the appropriate series.
61. Find three terms of the Maclaurin series for  $\tan^{-1} x$  by integrating the series for  $1/(1+x^2)$ , term by term.

62. The current  $i$  in an electric circuit containing a resistance  $R$  and an inductance  $L$  is given by  $i = Ie^{-Rt/L}$ , where  $I$  is the current at  $t = 0$ . Express  $i$  as an infinite series.
63. A piano wire vibrates with a displacement  $y$  (in mm) given by  $y = 3.2 \cos 880\pi t$ , where  $t$  is the time (in s). Express  $y$  as an infinite series.
64. The displacement  $y$  (in m) of a water wave as a function of the time  $t$  (in s) is  $y = 0.5 \sin 0.5t - 0.2 \sin 0.4t$ . Find the first three terms of the Maclaurin series for the displacement.
65. The number  $N$  of radioactive nuclei in a radioactive sample is  $N = N_0 e^{-\lambda t}$ . Here,  $t$  is the time,  $N_0$  is the number at  $t = 0$ , and  $\lambda$  is the decay constant. By using four terms of the appropriate series, express the right side of this equation as a polynomial.
66. The length of Lake Erie is a great circle arc of 390 km. If the lake is assumed to be flat, use series to find the error in calculating the distance from the centre of the lake to the centre of the earth. The radius of the earth is 6400 km.
67. From what height can a person see a point 10 km distant on earth's surface?
68. The vertical displacement  $y$  of a mass at the end of a spring is given by  $y = \sin 3t - \cos 2t$ , where  $t$  is the time. By subtraction of series, find the first four nonzero terms of the series for  $y$ .
69. The electric potential  $V$  at a distance  $x$  along a certain surface is given by  $V = \ln \frac{1+x}{1-x}$ . Find the first four terms of the Maclaurin series for  $V$ .
70. If a mass  $M$  is hung from a spring of mass  $m$ , the ratio of the masses is  $m/M = k\omega \tan k\omega$ , where  $k$  is a constant and  $\omega$  is a measure of the frequency of vibration. By using two terms of the appropriate series, express  $m/M$  as a polynomial in terms of  $\omega$ .
71. In the study of electromagnetic radiation, the expression  $\frac{N_0}{1 - e^{-k/T}}$  is used. Here,  $T$  is the thermodynamic temperature, and

$N_0$  and  $k$  are constants. Show that this expression can be written as  $N_0(1 + e^{-k/T} + e^{-2k/T} + \dots)$ . (Hint: Let  $x = e^{-k/T}$ .)

72. In the analysis of reflection from a spherical mirror, it is necessary to express the  $x$ -coordinate on the surface shown in Fig. 30.26 in terms of the  $y$ -coordinate and the radius  $R$ . Using the equation of the semicircle shown, solve for  $x$  (note that  $x \leq R$ ). Then express the result as a series. (Note that the first approximation gives a parabolic surface.)

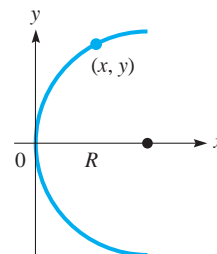


Fig. 30.26

73. A certain electric current is pulsating so that the current as a function of time is given by  $f(t) = 0$  if  $-\pi \leq t < 0$  and  $\pi/2 < t < \pi$ . If  $0 < t < \pi/2$ ,  $f(t) = \sin t$ . Find the Fourier expansion for this pulsating current and sketch three periods.
74. The force  $F$  applied to a spring system as a function of the time  $t$  is given by  $F = t/\pi$  if  $0 \leq t \leq \pi$  and  $F = 0$  if  $\pi < t < 2\pi$ . If the period of the force is  $2\pi$ , find the first few terms of the Fourier series that represents the force.

### Writing Exercise

75. A computer science class is assigned to write a program to find the Maclaurin series for  $\sin^2 x$ , using only the series for  $\sin x$  and/or  $\cos x$ , and any algebraic, trigonometric, or calculus procedures. Write a paragraph or two explaining how this can be done in at least four different ways.

## CHAPTER 30 PRACTICE TEST

- By direct expansion, find the first four nonzero terms of the Maclaurin expansion for  $f(x) = (1 + e^x)^2$ .
- Find the first three nonzero terms of the Taylor expansion for  $f(x) = \cos x$ , with  $a = \pi/3$ .
- Evaluate  $\ln 0.96$  by using four terms of the expansion for  $\ln(1 + x)$ .
- Find the first three nonzero terms of the expansion for  $f(x) = \frac{1}{\sqrt{1 - 2x}}$  by using the binomial series.
- Evaluate  $\int_0^1 x \cos x \, dx$  by using three terms of the appropriate series.
- An electric current is pulsating such that it is a function of the time with a period of  $2\pi$ . If  $f(t) = 2$  for  $0 \leq t < \pi$  and  $f(t) = 0$  for the other half-cycle, find the first three nonzero terms of the Fourier series for this current.
- $f(x) = x^2 + 2$  for  $-2 \leq x < 2$  (period = 4). Is  $f(x)$  an even function, an odd function, or neither? Expressing the Fourier series as  $F(x)$ , what is the Fourier series for the function  $g(x) = x^2 - 1$  for  $-2 \leq x < 2$  (period = 4)? (Do not use integration to derive specific terms for either series.)



# Differential Equations

Many of the physical problems being studied in the 1700s, such as velocity and light, led to equations that involved derivatives or differentials. These equations are called *differential equations*. Therefore, solving these differential equations became a very important topic of mathematical development during the eighteenth century.

In this chapter, we study some of the basic methods of solving differential equations. Many of these methods were first developed by the famous Swiss mathematician Leonhard Euler (1707–1783). He is undoubtedly the most prolific mathematician of all time, in that his work in mathematics and other fields filled over 70 large volumes.

The final topic covered in this chapter is the solution of differential equations by *Laplace transforms*. They are named for the French mathematician Pierre Laplace (1749–1827). Actually, Laplace had devised a mathematical method in the late 1700s that the English electric engineer Oliver Heaviside (1850–1925) refined and developed into its present useful form in the late 1800s. The Laplace transform is particularly useful for solving problems involving electrical circuits and mechanical systems.

Here, we see again that a field of mathematics was developed in response to the need for solving real-life physical problems. Also, we see that a method developed for purely mathematical reasons was then very usefully applied 100 years later in electricity, an area of study that did not exist when the method was first devised.

We actually solved a few simple differential equations in earlier chapters when we started a solution with the expression for the slope of a tangent line or the velocity of an object. Also, we have noted the applications of differential equations in electrical circuits, mechanical systems, and the study of light. Other areas of application include chemical reactions, interest calculations, changes in pressure and temperature, population growth, forces on beams and structures, and nuclear energy.



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# 31

## LEARNING OUTCOMES

After completion of this chapter, the student should be able to:

- Show that a function is a solution of a differential equation
- Solve first-order differential equations by separation of variables or by recognizing integrating combinations
- Solve first-order linear differential equations using an integrating factor
- Solve first-order differential equations numerically by Euler's method or by the Runge–Kutta method
- Solve homogeneous linear differential equations of higher order
- Solve higher-order nonhomogeneous linear differential equations by the method of undetermined coefficients
- Find the Laplace transform and the inverse Laplace transform of a function
- Solve differential equations using Laplace transforms
- Solve application problems involving differential equations

◀ In Section 31.10, we show how differential equations are used in the dating of geological samples (such as those from the Dome Glacier in Alberta) by using carbon dating.

## 31.1 Solutions of Differential Equations

Differential Equation • Order • Degree • General Solution • Particular Solution • Show Equation Is a Solution

A **differential equation** is an equation that contains derivatives or differentials. Most differential equations we shall consider contain first and/or second derivatives, although some will have higher derivatives. An equation that contains only first derivatives is called a **first-order differential equation**. An equation that contains second derivatives, and possibly first derivatives, is called a **second-order differential equation**. In general, the **order** of the differential equation is that of the highest derivative in the equation, and the **degree** is the highest power of that derivative.

### EXAMPLE 1 Illustrations of differential equations

- (a) The equation  $dy/dx + x = y$  is a first-order differential equation since it contains only a first derivative.
- (b) The equations  $\frac{d^2y}{dx^2} + y = 3x^2$  and  $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = x$  are second-order differential equations since each contains a second derivative and no higher derivatives. The  $dy/dx$  in the second equation does not affect the order.
- (c) The equation  $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^4 - y = 6$  is a differential equation of the second order and first degree. That is, the highest derivative that appears is the second, and it is raised to the first power. Since the second derivative appears, the fourth power of the first derivative does not affect the degree. ■

A **solution** of a differential equation is a relation between the variables that satisfies the differential equation. That is, when this relation is substituted into the differential equation, an algebraic identity results. A solution containing a number of independent arbitrary constants equal to the order of the differential equation is called the **general solution** of the equation. When specific values are given to at least one of these constants, the solution is called a **particular solution**.

General Solution

Particular Solution

### EXAMPLE 2 Independent arbitrary constants

Any coefficients that are not specified numerically after like terms have been combined are independent arbitrary constants. In the expression  $c_1x + c_2 + c_3x$ , there are only two arbitrary constants, since the  $x$ -terms may be combined;  $c_2 + c_4x$  is an equivalent expression with  $c_4 = c_1 + c_3$ . ■

### EXAMPLE 3 Illustration of the general solution

$y = c_1e^{-x} + c_2e^{2x}$  is the general solution of the differential equation

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} = 2y$$

The order of this differential equation is 2, and there are two independent arbitrary constants in the solution. The equation  $y = 4e^{-x}$  is a particular solution. It can be derived from the general solution by letting  $c_1 = 4$  and  $c_2 = 0$ . Each of these solutions can be shown to satisfy the differential equation by taking two derivatives and substituting. ■

To solve a differential equation, we have to find some method of transforming the equation so that each term may be integrated. Some of these methods will be considered after this section.

**LEARNING TIP**

To show that a given equation is a solution of the differential equation, we take the required derivatives and show that an identity results after substitution.

**Practice Exercise**

1. Show that  $y = 2 + ce^{2x}$  is a solution of  $y'' - 2y' = 0$ . Is it the general solution?

**EXAMPLE 4** Showing that an equation is the general solution

Show that  $y = c_1 \sin x + c_2 \cos x$  is the general solution of the differential equation  $y'' + y' = 0$ .

The function and its first two derivatives are

$$\begin{aligned}y &= c_1 \sin x + c_2 \cos x \\y' &= c_1 \cos x - c_2 \sin x \\y'' &= -c_1 \sin x - c_2 \cos x\end{aligned}$$

Substituting these into the differential equation, we have

$$\begin{array}{ccc}y'' & + & y & = & 0 \\ \downarrow & & \downarrow & & \\ (-c_1 \sin x - c_2 \cos x) & + & (c_1 \sin x + c_2 \cos x) & = & 0 \quad \text{or} \quad 0 = 0\end{array}$$

We know that this must be the general solution, since there are two independent arbitrary constants and the order of the differential equation is 2. ■

**EXAMPLE 5** Showing that an equation is a particular solution

Show that  $y = 3x + x^2$  is a solution of the differential equation  $xy' - y = x^2$ .

Taking one derivative of the function and substituting in the differential equation, we have

$$\begin{array}{l}y = 3x + x^2 \xrightarrow{\text{particular solution—no arbitrary constants}} \\ y' = 3 + 2x \xrightarrow{\text{particular solution—no arbitrary constants}} \\ xy' - y = x^2 \\ x(3 + 2x) - (3x + x^2) = x^2 \quad \text{or} \quad x^2 = x^2\end{array}$$

**EXERCISES 31.1**

In Exercises 1 and 2, show that the indicated solutions are, in fact, solutions of the differential equations in the indicated examples.

- In Example 3, two solutions are shown for the given differential equation. Show that each is a solution.
- In Example 4, show that  $y_1 = c \sin x + 5 \cos x$  and  $y_2 = 2 \sin x - 3 \cos x$  are solutions of the given differential equation.

In Exercises 3–6, determine whether the given equation is the general solution or a particular solution of the given differential equation.

- $\frac{dy}{dx} + 2xy = 0$ ,  $y = e^{-x^2}$
- $y' \ln x - \frac{y}{x} = 0$ ,  $y = c \ln x$
- $y'' + 3y' - 4y = 3e^x$ ,  $y = c_1 e^x + c_2 e^{-4x} + \frac{3}{5} x e^x$
- $\frac{d^2 y}{dx^2} + 4y = 8$ ,  $y = c \sin 2x + 3 \cos 2x + 2$

In Exercises 7–10, show that each function  $y = f(x)$  is a solution of the given differential equation.

- $\frac{dy}{dx} - y = 1$ ;  $y = e^x - 1$ ,  $y = 5e^x - 1$
- $\frac{dy}{dx} = 2xy^2$ ;  $y = -\frac{1}{x^2}$ ,  $y = -\frac{1}{x^2 + c}$

- $y'' + 4y = 0$ ;  $y = 3 \cos 2x$ ,  $y = c_1 \sin 2x + c_2 \cos 2x$
- $y'' = 2y'$ ;  $y = 3e^{2x}$ ,  $y = 2e^{2x} - 5$

In Exercises 11–32, show that the given equation is a solution of the given differential equation.

- $\frac{dy}{dx} = 2x$ ,  $y = x^2 + 1$
- $xy' = 2y$ ,  $y = cx^2$
- $\frac{dy}{dx} = 1 - 3x^2$ ,  $y = 2 + x - x^3$
- $\frac{dy}{dx} = 3y + 2x$ ,  $y = ce^{3x} - \frac{2}{3}x - \frac{2}{9}$
- $y' + 2y = 2x$ ,  $y = ce^{-2x} + x - \frac{1}{2}$
- $y'' = 6x + 2$ ,  $y = x^3 + x^2 + c$
- $y'' + 9y = 4 \cos x$ ,  $2y = \cos x$
- $y'' - 4y' + 4y = e^{2x}$ ,  $y = e^{2x} \left( c_1 + c_2 x + \frac{x^2}{2} \right)$
- $x^2 y' + y^2 = 0$ ,  $xy = cx + cy$
- $xy' - 3y = x^2$ ,  $y = cx^3 - x^2$
- $x \frac{d^2 y}{dx^2} + \frac{dy}{dx} = 0$ ,  $y = c_1 \ln x + c_2$
- $y'' + 4y = 10e^x$ ,  $y = c_1 \sin 2x + c_2 \cos 2x + 2e^x$
- $y' + y = 2 \cos x$ ,  $y = \sin x + \cos x - e^{-x}$

24.  $(x + y) - xy' = 0$ ,  $y = x \ln x - cx$   
 25.  $y'' - 3y' + 2y = 3$ ,  $y = c_1e^x + c_2e^{2x} + 3/2$   
 26.  $xy'' + y' = 16x^3$ ,  $y = x^4 + c_1 + c_2 \ln x$   
 27.  $\cos x \frac{dy}{dx} + \sin x = 1 - y$ ,  $y = \frac{x + c}{\sec x + \tan x}$   
 28.  $2xyy' + x^2 = y^2$ ,  $x^2 + y^2 = cx$   
 29.  $(y')^2 + xy' = y$ ,  $y = cx + c^2$   
 30.  $x^4(y')^2 - xy' = y$ ,  $y = c^2 + \frac{c}{x}$   
 31.  $\frac{d^3y}{dx^3} = \frac{d^2y}{dx^2}$ ,  $y = c_1 + c_2x + c_3e^x$   
 32.  $\frac{d^3y}{dx^3} + 4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} = 0$ ,  $y = c_1 + c_2e^{-2x} + xe^{-2x}$

In Exercises 33 and 34, solve the given problems.

33. The general solution of the differential equation  $y'' - y'/x = 3x$  is  $y = x^3 + c_1x^2 + c_2$ . Find the particular solution if the graph of the solution passes through the point  $(0, -4)$ .  
 34. For a given electric circuit, the charge  $q$  in the circuit as a function of the time  $t$  is  $q = 0.01(1 - \cos 316t)$ . Show that this satisfies the differential equation  $0.1 \frac{d^2q}{dt^2} + 10^4q = 100$ .

#### Answer to Practice Exercise

1.  $(4ce^{2x}) - 2(2ce^{2x}) = 0$ , No (two constants required).

## 31.2 Separation of Variables

### Separation of Variables • Solutions with Logarithms • Finding Particular Solutions

We will now solve differential equations of the first order and first degree. Of the many methods for solving such equations, a few are presented in this and the next two sections. The first of these is *the method of separation of variables*.

A differential equation of the first order and first degree contains the first derivative to the first power. That is, it may be written as  $dy/dx = f(x, y)$ . This type of equation is more commonly expressed in its differential form,

$$M(x, y) dx + N(x, y) dy = 0 \quad (31.1)$$

where  $M(x, y)$  and  $N(x, y)$  may represent constants, functions of either  $x$  or  $y$ , or functions of  $x$  and  $y$ .

To solve an equation of the form of Eq. (31.1), we must integrate. However, if  $M(x, y)$  contains  $y$ , the first term cannot be integrated. Also, if  $N(x, y)$  contains  $x$ , the second term cannot be integrated. If it is possible to rewrite Eq. (31.1) as

$$A(x) dx + B(y) dy = 0 \quad (31.2)$$

where  $A(x)$  does not contain  $y$  and  $B(y)$  does not contain  $x$ , then *we may find the solution by integrating each term and adding the constant of integration*. Many differential equations can be solved in this way.

#### EXAMPLE 1 Separate variables—divide by $x$

Solve the differential equation  $dx - 4xy^3dy = 0$ .

We can write this equation as

$$(1) dx + (-4xy^3) dy = 0$$

which means that  $M(x, y) = 1$  and  $N(x, y) = -4xy^3$ .

We remove the  $x$  from the coefficient of  $dy$  by dividing each term by  $x$ , which gives us

$$dx/x - 4y^3dy = 0$$

It is now possible to integrate each term. Performing this integration, we have

$$\ln|x| - y^4 = c$$

The constant of integration  $c$  becomes the arbitrary constant of the solution. ■

#### LEARNING TIP

Eq. (31.1) can be rewritten in the form of Eq. (31.2) when the functions  $M(x, y)$  and  $N(x, y)$  are both products of a function involving only  $x$  and a function involving only  $y$ . Therefore, the  $x$  and  $y$  terms can be separated by multiplying or dividing the equation by an appropriate expression.

It is important to note that, if division is used, the solution is not valid for values that make the divisor zero.

**LEARNING TIP**

In Example 1, we showed the integration of  $dx/x$  as  $\ln|x|$ , which follows our discussion in Section 28.2. We know  $\ln|x| = \ln x$  if  $x > 0$  and  $\ln|x| = \ln(-x)$  if  $x < 0$ . Since we know the values being used when we find a particular solution, we generally will not use the absolute value notation when integrating logarithmic forms. We would show the integration of  $dx/x$  as  $\ln x$ , with the understanding that we know  $x > 0$ . When using negative values of  $x$ , we would express it as  $\ln(-x)$ .

**Practice Exercise**

1. Find the general solution of the differential equation  $dx + 2y \sec x dy = 0$ .

■ For reference, Eq. (13.9) is  $\log_b x^n = n \log_b x$  and Eq. (13.7) is  $\log_b xy = \log_b x + \log_b y$ .

**LEARNING TIP**

Any expression that represents a constant may be chosen as the constant of integration and leads to a correct solution. In checking answers, we must remember that a different choice of constant will lead to a different form of the solution. Thus, two different-looking answers may both be correct. Often there is more than one reasonable choice of a constant, and different forms of the solution may be expected.

**EXAMPLE 2** Separate variables—divide by  $y(x^2 + 1)$ 

Solve the differential equation  $xy dx + (x^2 + 1) dy = 0$ .

In order to integrate each term, it is necessary to divide each term by  $y(x^2 + 1)$ . When this is done, we have

$$\frac{x dx}{x^2 + 1} + \frac{dy}{y} = 0$$

Integrating, we obtain the solution

$$\frac{1}{2} \ln(x^2 + 1) + \ln y = c$$

It is possible to make use of the properties of logarithms to make the form of this solution neater. If we write the constant of integration as  $\ln c_1$ , rather than  $c$ , we have  $\frac{1}{2} \ln(x^2 + 1) + \ln y = \ln c_1$ . Multiplying through by 2 and using the property of logarithms given by Eq. (13.9), we have  $\ln(x^2 + 1) + \ln y^2 = \ln c_1^2$ . Next, using the property of logarithms given by Eq. (13.7), we then have  $\ln(x^2 + 1)y^2 = \ln c_1^2$ , which means

$$(x^2 + 1)y^2 = c_1^2$$

This form of the solution is more compact and generally would be preferred. ■

**EXAMPLE 3** Divide by  $\theta$  and multiply by  $dt$ 

Solve the differential equation  $\frac{d\theta}{dt} = \frac{\theta}{t^2 + 4}$ .

The solution proceeds as follows:

$$\begin{aligned} \frac{d\theta}{\theta} &= \frac{dt}{t^2 + 4} && \text{separate variables by multiplying} \\ &&& \text{by } dt \text{ and dividing by } \theta \\ \ln \theta &= \frac{1}{2} \tan^{-1} \frac{t}{2} + \frac{c}{2} && \text{integrate} \\ 2 \ln \theta &= \tan^{-1} \frac{t}{2} + c \\ \ln \theta^2 &= \tan^{-1} \frac{t}{2} + c \end{aligned}$$

Note the different forms of the result using  $c/2$  as the constant of integration. These forms would differ somewhat had we chosen  $c$  as the constant.

The choice of  $\ln c$  as the constant of integration is also reasonable. On the left, it would lead to the result  $2 \ln c\theta = \tan^{-1}(t/2)$ . On the right, after solving explicitly for  $\theta$  using the exponential function, it would lead to the expression  $\theta = ce^{\frac{1}{2}\tan^{-1}(\frac{t}{2})}$ . ■

**EXAMPLE 4** Divide by  $e^x \sin y$ 

Solve the differential equation  $2e^{3x} \sin y dx + e^x \csc y dy = 0$ .

To separate variables, we divide by  $e^x \sin y$  to remove  $\sin y$  from the first term and  $e^x$  from the second term. Using properties of trigonometric and exponential functions, we have

$$\begin{aligned} \frac{2e^{3x} \sin y dx}{e^x \sin y} + \frac{e^x \csc y dy}{e^x \sin y} &= 0 && \text{divide by } e^x \sin y \\ 2e^{2x} dx + \csc^2 y dy &= 0 && \text{variables separated} \\ e^{2x} (2 dx) + \csc^2 y dy &= 0 && \text{form for integrating} \\ e^{2x} - \cot y &= c && \text{integrate} \end{aligned}$$



## FINDING PARTICULAR SOLUTIONS

To find a particular solution, we need information that allows us to evaluate the constant of integration. We now show how to find particular solutions, and graphically show the difference between the general solution and a particular solution.

## EXAMPLE 5 Finding a particular solution

Solve the equation  $(x^2 + 1)^2 dy + 4x dx = 0$ , subject to the condition that  $x = 1$  when  $y = 3$ .

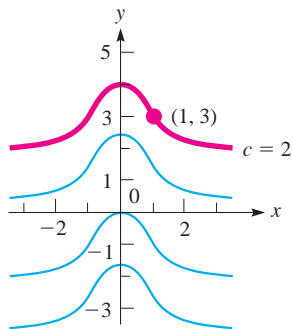


Fig. 31.1

$$\begin{aligned} dy + \frac{4x dx}{(x^2 + 1)^2} &= 0 && \text{dividing by } (x^2 + 1)^2 \\ y - \frac{2}{x^2 + 1} &= c && \text{integrating} \\ y &= \frac{2}{x^2 + 1} + c && \text{general solution} \\ 3 &= \frac{2}{1 + 1} + c, \quad c = 2 && \text{use } x = 1, y = 3 \text{ to evaluate } c \\ y &= \frac{2}{x^2 + 1} + 2 && \text{particular solution} \end{aligned}$$

The general solution defines a *family* of curves, one member of the family for each value of  $c$ . A few of these curves are shown in Fig. 31.1. When  $c$  is specified as in the particular solution, we have the specific curve shown. ■

## EXAMPLE 6 The constant does not affect the result

Find the particular solution in Example 2, given that  $x = 0$  when  $y = e$ .

Using the solution  $\frac{1}{2} \ln(x^2 + 1) + \ln y = c$ , we have

$$\begin{aligned} \frac{1}{2} \ln(0 + 1) + \ln e &= c, && \frac{1}{2} \ln 1 + 1 = c, \quad c = 1 \\ \frac{1}{2} \ln(x^2 + 1) + \ln y &= 1 && \text{substitute } c = 1 \\ \ln(x^2 + 1) + 2 \ln y &= 2 \\ \ln y^2(x^2 + 1) &= 2 && \text{using properties of logarithms} \\ y^2(x^2 + 1) &= e^2 && \text{exponential form of particular solution} \end{aligned}$$

Using the general solution  $(x^2 + 1)y^2 = c_1^2$ , we have

$$(0 + 1)e^2 = c_1^2, \quad c_1^2 = e^2 \\ y^2(x^2 + 1) = e^2$$

which is precisely the same solution as above. This shows that *the choice of the form of the constant does not affect the final result, and the constant is truly arbitrary.* ■

## Practice Exercise

2. In Example 4, find the particular solution if  $x = 0$  when  $y = \pi/4$ .

## EXERCISES 31.2

In Exercises 1 and 2, make the given changes in the indicated examples of this section and then solve the resulting differential equations.

- In Example 2, change the first term to  $2xy dx$ .
- In Example 4, change the second term to  $e^{2x} \csc y dy$ .

In Exercises 3–32, solve the given differential equations. Explain your method of solution for Exercise 15.

- $2x dx + dy = 0$
- $y^2 dy + x^3 dx = 0$
- $y^2 dx + dy = 0$
- $y dt + t dy = 3ty dt$

- $\frac{dV}{dP} = -\frac{V}{P^2}$
- $x^2 + (x^3 + 5)y' = 0$
- $dy + \ln xy dx = (4x + \ln y) dx$
- $r\sqrt{1 - \theta^2} \frac{dr}{d\theta} = \theta + 4$
- $e^{x^2} dy = x\sqrt{1 - y} dx$
- $\sqrt{1 + 4x^2} dy = y^3 x dx$
- $e^{2x} dy + e^x dx = 4 dx$
- $\frac{2dy}{dx} = \frac{y(x+1)}{x}$
- $xyy' + \sqrt{1 + y^2} = 0$
- $e^{x+y} dx + dy = 0$
- $y' - y = 4$

18.  $ds - s^2 dt = 9 dt$

19.  $x \frac{dy}{dx} = y^2 + y^2 \ln x$

20.  $(yx^2 + y) \frac{dy}{dx} = \tan^{-1} x$

21.  $y \tan x dx + \cos^2 x dy = 0$

22.  $\sin x \sec y dx = dy$

23.  $yx^2 dx = y dx - x^2 dy$

24.  $y\sqrt{1-x^2} dy + 2 dx = 0$

25.  $e^{\cos \theta} \tan \theta d\theta + \sec \theta dy = 0$

26.  $(x^3 + x^2) dx + (xy + x + y + 1) dy = 0$

27.  $xe^{x^2-y} dx = dy$

28.  $\cos^2 \phi + y \csc \phi \frac{dy}{d\phi} = 0$

29.  $2 \ln t dt + t di = 0$

30.  $y^2 e^t + (e^t + 4)y' = 0$

31.  $2y(x^3 + 1) dy + 3x^2(y^2 - 1) dx = 0$

32.  $V + 1 + (1 + \sin T) \sec T \frac{dV}{dT} = 0$

In Exercises 33–38, find the particular solution of the given differential equation for the indicated values.

33.  $\frac{dy}{dx} + yx^2 = 0$ ;  $x = 0$  when  $y = 1$

34.  $\frac{ds}{dt} = \sec s$ ;  $t = 0$  when  $s = 0$

35.  $y' = (1 - y)\cos x$ ;  $x = \pi/6$  when  $y = 0$

36.  $x dy = y \ln y dx$ ;  $x = 2$  when  $y = e$

37.  $y^2 e^x dx + e^{-x} dy = y^2 dx$ ;  $x = 0$  when  $y = 2$

38.  $2y \cos y dy - \sin y dy = y \sin y dx$ ;  $x = 0$  when  $y = \pi/2$

In Exercise 39–44, solve the given problems.

39. In Example 6, show that the particular solution is the same if the constant of integration is  $\ln c$ , rather than  $c$ .

40. Find  $f(x)$  if  $f'(x) = x^2 f(x)$  and  $f(0) = 1$ .

41. The temperature reading  $T$  (in  $^\circ\text{C}$ ) at time  $t$  (in s) of a thermometer initially reading  $40^\circ\text{C}$  and then placed in water at  $10^\circ\text{C}$  is found by solving the equation  $dT + 0.15(T - 10)dt = 0$ . Solve for  $T$  as a function of  $t$ .

42. The current  $i$  (in A) in an electric circuit changes with time  $t$  (in s) according to the equation  $di + 10i dt = 6 dt$ . Find  $i$  as a function of  $t$  if the initial current is zero.

43. In the study of fluid mechanics, streamlines are often drawn to visualize the flow field. (A streamline is tangent to the velocity vector.) Find an expression for the streamlines of a fluid flow if these satisfy the equation  $\frac{dy}{dx} = -\frac{x}{y}$ . What shape do these lines have?

44. Find an expression for the streamlines of a fluid flow if these satisfy the equation  $\frac{dy}{dx} = -\frac{y}{x}$ . What shape do these lines have? (See Exercise 43.)

#### Answers to Practice Exercises

1.  $\sin x + y^2 = c$     2.  $\cot y = e^{2x}$

## 31.3 Integrating Combinations

### Combinations of Differentials Integrated as a Unit • Basic Combinations

For different equations that cannot be solved by separation of variables, other methods have been developed. One is based on the fact that *certain combinations of differentials can be integrated as a unit*. The following differentials show some of the possible combinations:

$$d(xy) = xdy + ydx \quad (31.3)$$

$$d(x^2 + y^2) = 2(xdx + ydy) \quad (31.4)$$

$$d\left(\frac{y}{x}\right) = \frac{xdy - ydx}{x^2} \quad (31.5)$$

$$d\left(\frac{x}{y}\right) = \frac{ydx - xdy}{y^2} \quad (31.6)$$

Eq. (31.3) suggests that if the combination  $xdy + ydx$  occurs in a differential equation, we should look for a function of  $xy$  as a solution. Eq. (31.4) suggests that if the combination  $x dx + y dy$  occurs, we should look for a function of  $x^2 + y^2$ . Eqs. (31.5) and (31.6) suggest that if either of the combinations  $xdy - ydx$  or  $ydx - xdy$  occurs, we should look for a function of  $y/x$  or  $x/y$ .



**EXAMPLE 1** Recognizing the differential of  $xy$ 

Solve the differential equation  $x dy + y dx + xy dy = 0$ .

By dividing through by  $xy$ , we have

$$\frac{x dy + y dx}{xy} + dy = 0$$

The left term is the differential of  $xy$  divided by  $xy$ . Thus, it integrates to  $\ln xy$ .

$$\frac{d(xy)}{xy} + dy = 0$$

for which the solution is

$$\ln xy + y = c$$

**EXAMPLE 2** Recognizing the differential of  $y/x$ 

Solve the differential equation  $y dx - x dy + x dx = 0$ .

The combination of  $y dx - x dy$  suggests that this equation might make use of either Eq. (31.5) or (31.6). This would require dividing through by  $x^2$  or  $y^2$ . If we divide by  $y^2$ , the last term cannot be integrated, but **division by  $x^2$  still allows integration of the last term**. Performing this division, we obtain

$$\frac{y dx - x dy}{x^2} + \frac{dx}{x} = 0$$

This left combination is the negative of Eq. (31.5). Thus, we have

$$-d\left(\frac{y}{x}\right) + \frac{dx}{x} = 0$$

for which the solution is  $-\frac{y}{x} + \ln x = c$ .

**EXAMPLE 3** Two combinations in the same equation

Find the general solution of the differential equation

$$(x^3 + xy^2 + 2y)dx + (y^3 + x^2y + 2x)dy = 0$$

which satisfies the condition that  $x = 1$  when  $y = 0$ .

Regrouping the terms of the equation, we have

$$x(x^2 + y^2)dx + y(x^2 + y^2)dy + 2(ydx + xdy) = 0$$

Factoring  $x^2 + y^2$  from each of the first two terms gives

$$(x^2 + y^2)(x dx + y dy) + 2(y dx + x dy) = 0$$

$$\frac{1}{2}(x^2 + y^2) \overset{d(x^2 + y^2)}{(2x dx + 2y dy)} + 2 \overset{d(xy)}{(y dx + x dy)} = 0$$

$$\frac{1}{2}\left(\frac{1}{2}\right)(x^2 + y^2)^2 + 2xy + \frac{c}{4} = 0 \quad \text{integrating}$$

$$(x^2 + y^2)^2 + 8xy + c = 0$$

**Practice Exercise**

1. Find the general solution of the differential equation  $(4x - y)dx = xdy$ .

**LEARNING TIP**

The use of integrating combinations depends on proper recognition of the forms. It may take two or three arrangements to find the combination that leads to the solution. Of course, many equations cannot be arranged so as to give integrable combinations in all terms.

**EXAMPLE 4** Particular solution—differential of  $x^2 + y^2$ 

Find the particular solution of the differential equation  $(x^2 + y^2 + x) dx + y dy = 0$  that satisfies the condition  $x = 1$  when  $y = 0$ .

Regrouping the terms of this equation, we have

$$(x^2 + y^2) dx + (x dx + y dy) = 0 \quad \text{divide each term by } x^2 + y^2$$

$$dx + \frac{x dx + y dy}{x^2 + y^2} = 0$$

The right term now can be put in the form of  $du/u$  (with  $u = x^2 + y^2$ ) by multiplying each of the terms of the numerator by 2. This leads to

$$dx + \left(\frac{1}{2}\right) \frac{2x dx + 2y dy}{x^2 + y^2} = 0 \quad d(x^2 + y^2) = 2x dx + 2y dy$$

$$x + \frac{1}{2} \ln(x^2 + y^2) = \frac{c}{2} \quad \text{or} \quad 2x + \ln(x^2 + y^2) = c$$

Using the condition that  $x = 1$  when  $y = 0$ ,  $2(1) + \ln(1^2 + 0^2) = c$ , or  $c = 2$ . The particular solution is then  $2x + \ln(x^2 + y^2) = 2$ . ■

**EXERCISES 31.3**

In Exercises 1 and 2, make the given changes in the indicated examples of this section and then solve the resulting differential equations.

- In Example 1, change the third term to  $2xy^2 dy$ .
- In Example 2, change the third term to  $2 dx$ .

In Exercises 3–18, solve the given differential equations.

- $x dy + y dx + x dx = 0$
- $(2y + t) dy + y dt = 0$
- $y dx - x dy + x^3 dx = 2 dx$
- $x dy - y dx + y^2 dx = 0$
- $A^3 dr + A^2 r dA + r dA = A dr$
- $\sec(xy) dx + (x dy + y dx) = 0$
- $\sin x dy = (1 - y \cos x) dx$
- $x(x + e^{xy}) dy + y(2x + e^{xy}) dx = 0$
- $\sqrt{x^2 + y^2} dx - 2y dy = 2x dx$
- $R dR + (R^2 + T^2 + T) dT = 0$
- $\tan(x^2 + y^2) dy + x dx + y dy = 0$
- $(x^2 + y^3)^2 dy + 2x dx + 3y^2 dy = 0$
- $y dy + (y^2 - x^2) dx = x dx$  (Explain your solution.)
- $e^{x+y} (dx + dy) + 4x dx = 0$

- $10x dy + 5y dx + 3y dy = 0$
- $2(u dv + v du) \ln uv + 3u^3 v du = 0$

In Exercises 19–24, find the particular solutions to the given differential equations that satisfy the given conditions.

- $2(x dy + y dx) + 3x^2 dx = 0$ ;  $x = 1$  when  $y = 2$
- $t dt + s ds = 2(t^2 + s^2) dt$ ;  $t = 1$  when  $s = 0$
- $y dx - x dy = y^3 dx + y^2 x dy$ ;  $x = 2$  when  $y = 4$
- $e^{x/y} (x dy - y dx) = y^4 dy$ ;  $x = 0$  when  $y = 2$
- $2 \csc(xy) dx + x dy + y dx = 0$ ;  $x = 0$  when  $y = \pi/2$
- $\sqrt[3]{x^2 + y^2} dy = 3(x dx + y dy)$ ;  $x = 0$  when  $y = 8$

In Exercises 25 and 26, rewrite each equation such that each resulting term or combination is in an integrable form.

- $e^{-x} dy - 2y dy = ye^{-x} dx$
- $\cos(x + 2y) dx + 2 \cos(x + 2y) dy - x dy = y dx$

**Answer to Practice Exercise**

- $2x^2 - xy = c$

**31.4 The Linear Differential Equation of the First Order**

**Linear Differential Equation • P and Q Functions of x only • Identifying P, Q, and  $e^{\int P dx}$**

There is one type of differential equation of the first order and first degree for which an integrable combination can always be found. It is the **linear differential equation of the first order** and is of the form

$$dy + Py dx = Q dx \quad (31.7)$$

where  $P$  and  $Q$  are functions of  $x$  only. This type of equation occurs widely in applications.

**LEARNING TIP**

In finding the factor  $e^{\int P dx}$ , we often obtain an expression of the form  $e^{\ln u}$ . Using the properties of logarithms, we recall how  $e^{\ln u} = u$ :

$$\text{Let } y = e^{\ln u}.$$

$$\ln y = \ln e^{\ln u} = \ln u (\ln e) = \ln u$$

$$y = u \quad \text{or} \quad e^{\ln u} = u$$

**LEARNING TIP**

In finding  $e^{\int P dx}$ , the constant of integration in the exponent  $\int P dx$  can always be taken as zero, as we did in Example 1. To show why this is so, let  $P = 2/x$  as in Example 1:

$$e^{\int (2/x) dx} = e^{\ln x^2 + c} = (e^{\ln x^2}) (e^c) = x^2 e^c$$

The solution to the differential equation, as given in Eq. (31.8), is then

$$y(x^2)(e^c) = \int 4x(x^2)(e^c) dx + c_1 e^c$$

Regardless of the value of  $c$ , the factor  $e^c$  can be divided out. Therefore, it is convenient to let  $c = 0$  and have  $e^c = 1$ .

If each side of Eq. (31.7) is multiplied by  $e^{\int P dx}$ , it becomes integrable, since the left side becomes of the form  $du$  with  $u = ye^{\int P dx}$  and the right side is a function of  $x$  only. This is shown by finding the differential of  $ye^{\int P dx}$ . Thus,

$$d(ye^{\int P dx}) = e^{\int P dx}(dy + Py dx)$$

In finding the differential of  $\int P dx$ , we use the fact that, by definition, the integral and the differential are reverse processes. Thus,  $d(\int P dx) = P dx$ . Therefore, if each side is multiplied by  $e^{\int P dx}$ , the left side may be immediately integrated to  $ye^{\int P dx}$ , and the right-side integration may be indicated. The solution becomes

$$ye^{\int P dx} = \int Qe^{\int P dx} dx + c \tag{31.8}$$

**EXAMPLE 1 Quadratic factor**

Solve the differential equation  $dy + \left(\frac{2}{x}\right)y dx = 4x dx$ .

This equation fits the form of Eq. (31.7) with  $P = 2/x$  and  $Q = 4x$ . The first expression to find is  $e^{\int P dx}$ . In this case, this is

$$e^{\int (2/x) dx} = e^{2 \ln x} = e^{\ln x^2} = x^2 \quad \text{see text comments following example}$$

The left side integrates to  $yx^2$ , while the right side becomes  $\int 4x(x^2) dx$ . Thus,

$$\begin{aligned} ye^{\int P dx} &= \int Qe^{\int P dx} dx + c \\ \downarrow \downarrow & \quad \quad \quad \downarrow \downarrow \\ y(x^2) &= \int (4x)(x^2) dx + c \quad \text{using Eq. (31.8)} \end{aligned}$$

$$yx^2 = \int 4x^3 dx + c = x^4 + c \quad \text{integrating}$$

$$y = x^2 + cx^{-2}$$

**EXAMPLE 2 Power function factor**

Solve the differential equation  $xdy - 3y dx = x^3 dx$ .

Putting this equation in the form of Eq. (31.7) by dividing through by  $x$  gives  $dy - (3/x)y dx = x^2 dx$ . Here,  $P = -3/x$ ,  $Q = x^2$ , and the factor  $e^{\int P dx}$  becomes

$$e^{\int (-3/x) dx} = e^{-3 \ln x} = e^{\ln x^{-3}} = x^{-3}$$

Therefore,

$$\begin{aligned} ye^{\int P dx} &= \int Qe^{\int P dx} dx + c \\ \downarrow \downarrow & \quad \quad \quad \downarrow \downarrow \\ yx^{-3} &= \int x^2(x^{-3}) dx + c \quad \text{using Eq. (31.8)} \end{aligned}$$

$$= \int x^{-1} dx + c = \ln x + c$$

$$y = x^3(\ln x + c)$$

**Practice Exercise**

1. Find the general solution of the differential equation  $xdy - 2y dx = 2x^4 dx$ .

**EXAMPLE 3 Exponential factor**

Solve the differential equation  $dy + ydx = xdx$ .

Here,  $P = 1$ ,  $Q = x$ , and  $e^{\int P dx} = e^{\int (1) dx} = e^x$ . Therefore,

$$ye^x = \int xe^x dx + c = e^x(x - 1) + c$$

$$y = x - 1 + ce^{-x}$$

using Eq. (31.8) and integrating  
by parts or tables

**EXAMPLE 4 Trigonometric factor**

Solve the differential equation  $\cos x \frac{dy}{dx} = 1 - y \sin x$ .

Writing this in the form of Eq. (31.7), we have

$$dy + y \tan x dx = \sec x dx \quad \text{dividing by } \cos x$$

Thus, with  $P = \tan x$ , we have

$$e^{\int P dx} = e^{\int \tan x dx} = e^{-\ln \cos x} = \sec x \quad \text{using } e^{\ln u} = u$$

$$y \sec x = \int \sec^2 x dx = \tan x + c \quad \text{using Eq. (31.8)}$$

$$y = \sin x + c \cos x$$

**EXAMPLE 5 Finding a particular solution**

For the differential equation  $dy = (1 - 2y)x dx$ , find the particular solution such that  $x = 0$  when  $y = 2$ .

The solution proceeds as follows:

$$dy + 2xy dx = x dx \quad \text{form of Eq. (31.7)}$$

$$e^{\int P dx} = e^{\int 2x dx} = e^{x^2} \quad \text{find } e^{\int P dx}$$

$$ye^{x^2} = \int xe^{x^2} dx \quad \text{using Eq. (31.8)}$$

$$= \frac{1}{2} e^{x^2} + c \quad \text{general solution}$$

$$(2)(e^0) = \frac{1}{2}(e^0) + c, \quad 2 = \frac{1}{2} + c, \quad c = \frac{3}{2} \quad \text{x = 0, y = 2; evaluate c}$$

$$ye^{x^2} = \frac{1}{2} e^{x^2} + \frac{3}{2} \quad \text{substitute } c = \frac{3}{2}$$

$$y = \frac{1}{2}(1 + 3e^{-x^2}) \quad \text{particular solution}$$

**EXERCISES 31.4**

In Exercises 1 and 2, make the given changes in the indicated examples of this section and then solve the resulting differential equations.

1. In Example 1, change the right side to  $3 dx$ .
2. In Example 3, change the right side to  $2 dx$ .

In Exercises 3–28, solve the given differential equations.

3.  $dy + ydx = e^{-x} dx$
4.  $dy + 3ydx = e^{-3x} dx$
5.  $dy + 2ydx = e^{-4x} dx$

6.  $di + i dt = e^{-t} \cos t dt$

7.  $\frac{dy}{dx} - 2y = 4$

8.  $2 \frac{dy}{dx} = 5 - 6y$

9.  $dy = 3x^2(2 - y)dx$

10.  $x dy + 3y dx = dx$

11.  $2x dy + y dx = 8x^3 dx$

12.  $3x dy - y dx = 9x dx$

13.  $dr + r \cot \theta d\theta = d\theta$

14.  $y' = x^2 y + 3x^2$

15.  $\sin x \frac{dy}{dx} = 1 - y \cos x$

16.  $\frac{dv}{dt} - \frac{v}{t} = \ln t$

17.  $y' + y = x + e^x$

18.  $y' + 2y = \sin x$

19.  $ds = (te^{4t} + 4s) dt$

20.  $y' - 2y = 2e^{2x}$

21.  $y' = x^3(1 - 4y)$

22.  $y' + y \tan x = -\sin x$

23.  $x \frac{dy}{dx} = y + (x^2 - 1)^2$

24.  $dy = dt - \frac{y dt}{(1 + t^2) \tan^{-1} t}$

25.  $\sqrt{1 + x^2} dy + x(1 + y) dx = 0$

26.  $(1 + x^2) dy + xy dx = x dx$

27.  $\tan \theta \frac{dr}{d\theta} - r = \tan^2 \theta$

28.  $y' + y = y^2$  (Solve by letting  $y = 1/u$  and solving the resulting linear equation for  $u$ .)

In Exercises 29 and 30, solve the given differential equations. Explain how each can be solved using either of two different methods.

29.  $y' = 2(1 - y)$

30.  $x dy = (2x - y) dx$

In Exercises 31–36, find the indicated particular solutions of the given differential equations.

31.  $\frac{dy}{dx} + 2y = e^{-x}$ ;  $x = 0$  when  $y = 1$

32.  $dq - 4q du = 2 du$ ;  $q = 2$  when  $u = 0$

33.  $y' + 2y \cot x = 4 \cos x$ ;  $x = \pi/2$  when  $y = 1/3$

34.  $y' \sqrt{x} + \frac{1}{2}y = e^{\sqrt{x}}$ ;  $x = 1$  when  $y = 3$

35.  $(\sin x)y' + y = \tan x$ ;  $x = \pi/4$  when  $y = 0$

36.  $f(x)dy + 2yf'(x)dx = f(x)f'(x)dx$ ;  $f(x) = -1$  when  $y = 3$

In Exercises 37–40, solve the given problems.

37. The differential equation  $y' + P(x)y = Q(x)y^2$  is not linear. Show that the substitution  $u = y^{-1}$  will transform it into a linear equation.

38. An equation used in the analysis of rocket motion is  $m dv + kv dt = 0$ , where  $m$  and  $k$  are positive constants. Solve this equation for  $v$  as a function of  $t$  in two ways.

39. The electric current  $i$  in a circuit with a voltage  $V$ , resistance  $R$ , and inductance  $L$ , is a function of the time  $t$  given by  $\frac{di}{dt} + \frac{R}{L}i = \frac{V}{L}$ . Solve for  $i$  as a function of  $t$  if the initial current is zero.

40. If  $A$  dollars are placed in an account that pays 5% interest, compounded *continuously*, and  $A$  dollars are added to the account each year, the number of dollars  $n$  in the account after  $t$  years is given by  $dn/dt = A + 0.05n$ . Solve for  $n$  as a function of  $t$ .

#### Answer to Practice Exercise

1.  $y = x^4 + cx^2$

## 31.5 Numerical Solutions of First-Order Equations

### Euler's Method • Runge–Kutta Method

Many differential equations do not have exact solutions. Therefore, in this section, we show one basic method and one more advanced method of solving such equations numerically.

#### EULER'S METHOD

To find an approximate solution to a differential equation of the form  $dy/dx = f(x, y)$ , that passes through a known point  $(x_0, y_0)$ , we write the equation as  $dy = f(x, y) dx$  and then approximate  $dy$  as  $y_1 - y_0$ , and replace  $dx$  with  $\Delta x$ . From Section 24.8, we recall that  $\Delta y$  closely approximates  $dy$  for a small  $dx$  and that  $dx = \Delta x$ . This gives us

$$y_1 = y_0 + f(x_0, y_0) \Delta x \quad \text{and} \quad x_1 = x_0 + \Delta x$$

Therefore, we now know another point  $(x_1, y_1)$  that is on (or very nearly on) the curve of the solution. We can now repeat this process using  $(x_1, y_1)$  as a known point to obtain a next point  $(x_2, y_2)$ . Continuing this process, we can get a series of points that are approximately on the solution curve. The method is called *Euler's method*.

■ Euler's method is named for the Swiss mathematician Leonhard Euler (1707–1783).

**EXAMPLE 1 Euler's method**

For the differential equation  $dy/dx = x + y$ , use Euler's method to find the  $y$ -values of the solution for  $x = 0$  to  $x = 0.5$  with  $\Delta x = 0.1$ , if the curve of the solution passes through  $(0, 1)$ .

Using the method outlined above, we have  $x_0 = 0$ ,  $y_0 = 1$ , and

$$y_1 = 1 + (0 + 1)(0.1) = 1.1 \quad \text{and} \quad x_1 = 0 + 0.1 = 0.1$$

This tells us that the curve passes (or nearly passes) through the point  $(0.1, 1.1)$ . Assuming this point is correct, we use it to find the next point on the curve.

$$y_2 = 1.1 + (0.1 + 1.1)(0.1) = 1.22 \quad \text{and} \quad x_2 = 0.1 + 0.1 = 0.2$$

Therefore, the next approximate point is  $(0.2, 1.22)$ . Continuing this process, we find a set of points that would approximately satisfy the function that is the solution of the differential equation. Tabulating results, we have the following table:

$x$	$y$	Correct Value of $y$
0.0	1.0000	1.0000
0.1	1.1000	1.1103
0.2	1.2200	1.2428
0.3	1.3620	1.3997
0.4	1.5282	1.5836
0.5	1.7210	1.7974

the values shown have been rounded off, although more digits were carried in the calculations

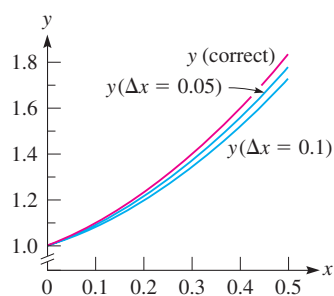


Fig. 31.2

In this case, we are able to find the correct values since the equation can be written as  $dy/dx - y = x$ , and the solution is  $y = 2e^x - x - 1$ . Although numerical methods are generally used with equations that cannot be solved exactly, we chose this equation so that we could compare values obtained with known values.

We can see that as  $x$  increases, the error in  $y$  increases. More accurate values can be found by using smaller values of  $\Delta x$ . In Fig. 31.2 the solution curves using  $\Delta x = 0.1$  and  $\Delta x = 0.05$  are shown along with the correct values of  $y$ .

Euler's method is easy to use and understand, but it is less accurate than other methods. We will show one of the more accurate methods in the next example. ■

**RUNGE-KUTTA METHOD**

For more accurate numerical solutions of a differential equation, the **Runge-Kutta method** is often used. Starting at a first point  $(x_0, y_0)$ , the coordinates of the second point  $(x_1, y_1)$  are found by using a weighted average of the slopes calculated at the points where  $x = x_0$ ,  $x = x_0 + \frac{1}{2}\Delta x$ , and  $x = x_0 + \Delta x$ . The formulas for  $y_1$  and  $x_1$  are

$$y_1 = y_0 + \frac{1}{6}H(J + 2K + 2L + M) \quad \text{and} \quad x_1 = x_0 + H \quad (\text{for convenience, } H = \Delta x)$$

where

$$J = f(x_0, y_0)$$

$$K = f(x_0 + 0.5H, y_0 + 0.5HJ)$$

$$L = f(x_0 + 0.5H, y_0 + 0.5HK)$$

$$M = f(x_0 + H, y_0 + HL)$$

We have used uppercase letters to correspond to calculator use. Traditional sources normally use  $h$  for  $H$  and a lowercase letter (such as  $k$ ) with subscripts for  $J, K, L$ , and  $M$ , and express 0.5 as  $1/2$ .

As with Euler's method, once  $(x_1, y_1)$  is determined, we use the formulas again to find  $(x_2, y_2)$  by replacing  $(x_0, y_0)$  with  $(x_1, y_1)$ . The following example illustrates the use of the Runge-Kutta method.

■ The Runge-Kutta method is named for the German mathematicians Carl Runge (1856–1927) and Martin Kutta (1867–1944).

x	y
0.0	0.0
0.1	0.0050125208
0.2	0.0202013395
0.3	0.0460278455
0.4	0.0832868181
0.5	0.1331460062

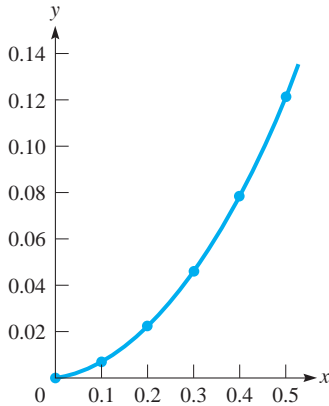


Fig. 31.3

**EXAMPLE 2** Runge–Kutta method

For the differential equation  $dy/dx = x + \sin xy$ , use the Runge–Kutta method to find  $y$ -values of the solution for  $x = 0$  to  $x = 0.5$  with  $\Delta x = 0.1$ , if the curve of the solution passes through  $(0, 0)$ .

Using the formulas and method outlined above, we have the following solution, with calculator notes to the right of the equations. Also, calculator symbols are used on the right sides of the equations to indicate the way in which they should be entered.

$$\begin{aligned}
 x_0 &= 0 && \text{store as } X \\
 y_0 &= 0 && \text{store as } Y \\
 H &= 0.1 && \text{store as } H \\
 J &= X + \sin XY = 0 && \text{store as } J \\
 K &= X + 0.5H + \sin[(X + 0.5H)(Y + 0.5HJ)] = 0.05 && \text{store as } K \\
 L &= X + 0.5H + \sin[(X + 0.5H)(Y + 0.5HK)] = 0.050125 && \text{store as } L \\
 M &= X + H + \sin[(X + H)(Y + HL)] = 0.10050125 && \text{store as } M \\
 y_1 &= Y + (H/6)(J + 2K + 2L + M) = 0.0050125208 && \text{store as } Y \\
 x_1 &= X + H = 0.1 && \text{store as } X
 \end{aligned}$$

We now use  $(x_1, y_1)$  as we just used  $(x_0, y_0)$  to get the next point  $(x_2, y_2)$ , which is then used to find  $(x_3, y_3)$ , and so on. A table showing calculator values and a graph of these values is shown in Fig. 31.3.

**EXERCISES 31.5**

In Exercises 1–8, use Euler’s method to find  $y$ -values of the solution for the given values of  $x$  and  $\Delta x$ , if the curve of the solution passes through the given point. Check the results against known values by solving the differential equations exactly.

- $\frac{dy}{dx} = x + 1$ ;  $x = 0$  to  $x = 1$ ;  $\Delta x = 0.2$ ;  $(0, 1)$
- $\frac{dy}{dx} = \sqrt{2x + 1}$ ;  $x = 0$  to  $x = 1.2$ ;  $\Delta x = 0.3$ ;  $(0, 2)$
- $\frac{dy}{dx} = y(0.4x + 1)$ ;  $x = -0.2$  to  $x = 0.3$ ;  $\Delta x = 0.1$ ;  $(-0.2, 2)$
- $\frac{dy}{dx} = y + e^x$ ;  $x = 0$  to  $x = 0.5$ ;  $\Delta x = 0.1$ ;  $(0, 0)$
- The differential equation of Exercise 1 with  $\Delta x = 0.1$
- The differential equation of Exercise 2 with  $\Delta x = 0.1$
- The differential equation of Exercise 3 with  $\Delta x = 0.05$
- The differential equation of Exercise 4 with  $\Delta x = 0.05$

In Exercises 9–14, use the Runge–Kutta method to find  $y$ -values of the solution for the given values of  $x$  and  $\Delta x$ , if the curve of the solution passes through the given point.

- $\frac{dy}{dx} = xy + 1$ ;  $x = 0$  to  $x = 0.4$ ;  $\Delta x = 0.1$ ;  $(0, 0)$
- $\frac{dy}{dx} = x^2 + y^2$ ;  $x = 0$  to  $x = 0.4$ ;  $\Delta x = 0.1$ ;  $(0, 1)$

- $\frac{dy}{dx} = e^{xy}$ ;  $x = 0$  to  $x = 1$ ;  $\Delta x = 0.2$ ;  $(0, 0)$
- $\frac{dy}{dx} = \sqrt{1 + xy}$ ;  $x = 0$  to  $x = 0.2$ ;  $\Delta x = 0.05$ ;  $(0, 1)$
- $\frac{dy}{dx} = \cos(x + y)$ ;  $x = 0$  to  $x = 0.6$ ;  $\Delta x = 0.1$ ;  $(0, \pi/2)$
- $\frac{dy}{dx} = y + \sin x$ ;  $x = 0.5$  to  $x = 1.0$ ;  $\Delta x = 0.1$ ;  $(0.5, 0)$

In Exercises 15–18, solve the given problems.

- In Example 1, use Euler’s method to find the  $y$ -values from  $x = 0$  to  $x = 3$  with  $\Delta x = 1$ . Compare with the value found using the exact solution. Comment on the use of Euler’s method in finding the value of  $y$  for  $x = 3$ .
- For the differential equation  $dy/dx = x + 1$ , if the curve of the solution passes through  $(0, 0)$ , calculate the  $y$ -value for  $x = 0.04$  with  $\Delta x = 0.01$ . Find the exact solution, and compare the result using three terms of the Maclaurin series that represents the solution.
- An electric circuit contains a 1-H inductor, a 2- $\Omega$  resistor, and a voltage source of  $\sin t$ . The resulting differential equation relating the current  $i$  and the time  $t$  is  $di/dt + 2i = \sin t$ . Find  $i$  after 0.5 s by Euler’s method with  $\Delta t = 0.1$  s if the initial current is zero. Solve the equation exactly and compare the values.
- An object is being heated such that the rate of change of the temperature  $T$  (in  $^\circ\text{C}$ ) with respect to time  $t$  (in min) is  $dT/dt = \sqrt[3]{1 + t^3}$ . Find  $T$  for  $t = 5$  min by using the Runge–Kutta method with  $\Delta t = 1$  min, if the initial temperature is  $0^\circ\text{C}$ .



## 31.6 Elementary Applications

Geometric Applications •  
Electrical, Mechanical, and  
Other Technical Applications

The differential equations of the first order and first degree we have discussed thus far have numerous applications in geometry and the various fields of technology. In this section, we illustrate some of these applications.

### EXAMPLE 1 Find a curve, given its slope

The slope of a curve is given by the expression  $6xy$ . Find the equation of the curve if it passes through the point  $(2, 1)$ .

Since the slope is  $6xy$ , the differential equation for the curve is

$$\frac{dy}{dx} = 6xy$$

We now want to find the particular solution of this equation for which  $x = 2$  when  $y = 1$ . The solution follows:

$$\begin{aligned} \frac{dy}{y} &= 6x \, dx && \text{separate variables} \\ \ln y &= 3x^2 + c && \text{general solution} \\ \ln 1 &= 3(2^2) + c && \text{evaluate } c \\ 0 &= 12 + c, \quad c = -12 \\ \ln y &= 3x^2 - 12 && \text{particular solution} \end{aligned}$$

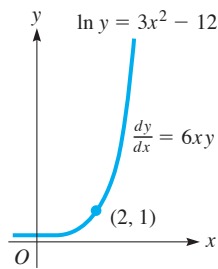


Fig. 31.4

The graph of this solution is shown in Fig. 31.4. ■

### EXAMPLE 2 Orthogonal trajectories

A curve that intersects all members of a family of curves at right angles is called an **orthogonal trajectory** of the family. Find the equations of the orthogonal trajectories of the parabolas  $x^2 = ky$ . As before, each value of  $k$  gives us a particular member of the family.

The derivative of the given equation is  $dy/dx = 2x/k$ . This equation contains the constant  $k$ , which depends on the point  $(x, y)$  on the parabola. **Eliminating this constant between the equations of the parabolas and the derivative**, we have

$$k = \frac{x^2}{y} \quad \frac{dy}{dx} = \frac{2x}{k} = \frac{2x}{x^2/y} \quad \text{or} \quad \frac{dy}{dx} = \frac{2y}{x}$$

This equation gives a general expression for the slope of any of the members of the family. For a curve to be perpendicular, its slope must equal the negative reciprocal of this expression, or the slope of the orthogonal trajectories must be

$$\left. \frac{dy}{dx} \right|_{\text{OT}} = -\frac{x}{2y} \quad \text{this equation must not contain the constant } k$$

Solving this differential equation gives the family of orthogonal trajectories.

$$\begin{aligned} 2y \, dy &= -x \, dx \\ y^2 &= -\frac{x^2}{2} + \frac{c}{2} \\ 2y^2 + x^2 &= c \quad \text{orthogonal trajectories} \end{aligned}$$

Thus, the orthogonal trajectories are ellipses. Note in Fig. 31.5 that each parabola intersects each ellipse at right angles. ■

■ Some calculators have a feature by which a family of curves can be graphed.

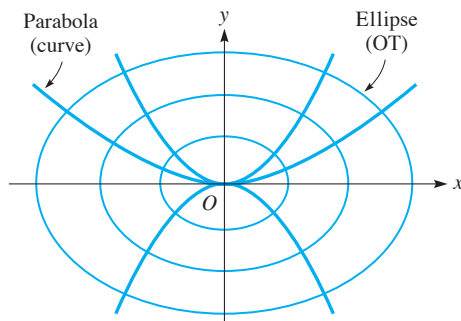


Fig. 31.5

**EXAMPLE 3** Radioactivity

Radioactive elements decay at rates that are proportional to the amount of the element present. Carbon-14 decays such that one-half of an original amount decays into other forms in about 5730 years. By measuring the proportion of carbon-14 in remains at a given site, the approximate age of the remains can be determined. This method is called *carbon dating*.

The analysis of some detrital wood washed from the Dome Glacier in Alberta showed that the concentration of carbon-14 was 47.7% of the concentration that new wood would have. Determine the equation relating the amount of carbon-14 present with the time and then determine the age of the wood from the Dome Glacier. (Scientists use paleoecological samples of this kind to study past climate change.)

Let  $N_0$  be the original amount and  $N$  be the amount present at any time  $t$  (in years). The rate of decay can be expressed as a derivative. Therefore, since the rate of change is proportional to  $N$ , we have the equation

$$\frac{dN}{dt} = kN$$

Solving this differential equation, we have

$$\begin{aligned} \frac{dN}{N} &= k dt && \text{separate variables} \\ \ln N &= kt + \ln c && \text{general solution} \\ \ln N_0 &= k(0) + \ln c && N = N_0 \text{ for } t = 0 \\ c &= N_0 && \text{solve for } c \\ \ln N &= kt + \ln N_0 && \text{substitute } N_0 \text{ for } c \\ \ln N - \ln N_0 &= kt && \text{use properties of logarithms} \\ \ln \frac{N}{N_0} &= kt \\ N &= N_0 e^{kt} && \text{exponential form} \end{aligned}$$

Now, using the condition that one-half of carbon-14 decays in 5730 years, we have  $N = N_0/2$  when  $t = 5730$  years. This gives

$$\frac{N_0}{2} = N_0 e^{5730k} \quad \text{or} \quad \frac{1}{2} = (e^k)^{5730} \quad \text{and}$$

$$e^k = 0.5^{1/5730} \quad \text{both sides to the power } 1/5730$$

Therefore, the equation relating  $N$  and  $t$  is

$$N = N_0 (0.5)^{t/5730} \quad \text{substitute in the exponential form}$$

Since the present concentration is  $N = 0.477N_0$ , we can solve for  $t$ :

$$0.477N_0 = N_0 (0.5)^{t/5730} \quad \text{or} \quad 0.477 = 0.5^{t/5730}$$

$$\ln 0.477 = \ln 0.5^{t/5730}, \quad \ln 0.477 = \frac{t}{5730} \ln 0.5 \quad \text{solving an exponential equation}$$

$$t = 5730 \frac{\ln 0.477}{\ln 0.5} = 6120 \text{ years}$$

Therefore, the wood from the Dome Glacier is about 6120 years old. ■

■ See the chapter introduction.

■ Radioactivity was discovered in 1898 by the French physicist Henri Becquerel (1852–1908). Carbon dating was developed in 1947 by the U.S. chemist Willard Libby (1908–1980).

**EXAMPLE 4** Electric circuits

The general equation relating the current  $i$ , voltage  $E$ , inductance  $L$ , capacitance  $C$ , and resistance  $R$  in a simple electric circuit (see Fig. 31.6) is

$$L \frac{di}{dt} + Ri + \frac{q}{C} = E \quad (31.9)$$

where  $q$  is the charge on the capacitor. Find the general expression for the current in a circuit containing an inductance, a resistance, and a voltage source if  $i = 0$  when  $t = 0$ .

The differential equation for this circuit is

$$L \frac{di}{dt} + Ri = E$$

Using the method of the linear differential equation of the first order, we have the equation

$$di + \frac{R}{L} i dt = \frac{E}{L} dt$$

The factor  $e^{\int P dt}$  is  $e^{\int (R/L) dt} = e^{(R/L)t}$ . This gives

$$ie^{(R/L)t} = \frac{E}{L} \int e^{(R/L)t} dt = \frac{E}{R} e^{(R/L)t} + c$$

Letting the current be zero for  $t = 0$ , we have  $c = -E/R$ . The result is

$$ie^{(R/L)t} = \frac{E}{R} e^{(R/L)t} - \frac{E}{R}$$

$$i = \frac{E}{R} (1 - e^{-(R/L)t})$$

(We can see that  $i \rightarrow E/R$  as  $t \rightarrow \infty$ . In practice, the exponential term becomes negligible very quickly.)

**EXAMPLE 5** Salt solution

Fifty litres of brine originally containing 3.00 kg of salt are in a tank into which 2.00 L of water run each minute with the same amount of mixture running out each minute. How much salt is in the tank after 10.0 min?

Let  $x$  = the number of kilograms of salt in the tank after  $t$  minutes. Each litre of brine contains  $x/50$  kg of salt, and in time  $dt$ ,  $2 dt$  L of mixture leave the tank **with  $(x/50)(2 dt)$  kg of salt**. The amount of salt that is leaving may also be written as  $-dx$  (the minus sign is included to show that  $x$  is decreasing). Thus,

$$-dx = \frac{2x dt}{50} \quad \text{or} \quad \frac{dx}{x} = -\frac{dt}{25}$$

This leads to  $\ln x = -(t/25) + \ln c$ . Using the fact that  $x = 3.00$  kg when  $t = 0$ , we find that  $\ln 3.00 = \ln c$ , or  $c = 3.00$ . Therefore,

$$x = 3.00e^{-t/25}$$

is the general expression for the amount of salt in the tank at time  $t$ . Therefore, when  $t = 10.0$  min, we have

$$x = 3.00e^{-10/25} = 3.00e^{-0.4} = 3.00(0.670) = 2.01 \text{ kg}$$

There are 2.01 kg of salt in the tank after 10.0 min. (Although the data were given with 3 significant digits, we did not use all significant digits in writing the equations that were used.)

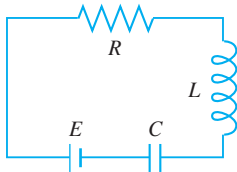


Fig. 31.6

**EXAMPLE 6** Motion in a resisting medium

An object moving through (or across) a resisting medium often experiences a retarding force approximately proportional to the velocity as well as the force that causes the motion. An example is a ball falling due to the force of gravity, with air resistance producing a retarding force. Applying Newton's laws of motion (from physics) to the ball leads to the equation

$$m \frac{dv}{dt} = F - kv \quad (31.10)$$

where  $m$  is the mass of the object,  $v$  is the velocity of the object,  $t$  is the time,  $F$  is the force causing the motion, and  $k$  ( $k > 0$ ) is a constant. The quantity  $kv$  is the retarding force.

We assume that these conditions hold for a falling object whose mass is 5.00 kg and experiences a force (its own weight) of 49.0 N. The object starts from rest, and the air causing a retarding force numerically equal to 0.200 times the velocity.

Substituting in Eq. (31.10) and then solving the differential equation, we have

$$\begin{aligned} 5 \frac{dv}{dt} &= 49 - 0.2v \\ \frac{5dv}{49 - 0.2v} &= dt && \text{separate variables} \\ -25 \ln(49 - 0.2v) &= t - 25 \ln c && \text{integrate} \\ \ln(49 - 0.2v) &= -\frac{t}{25} + \ln c && \text{solve for } v \\ \ln \frac{49 - 0.2v}{c} &= -\frac{t}{25} \\ 49 - 0.2v &= ce^{-t/25} \\ 0.2v &= 49 - ce^{-t/25} \\ v &= 5(49 - ce^{-t/25}) && \text{general solution} \end{aligned}$$

Since the object started from rest,  $v = 0$  when  $t = 0$ . Thus,

$$\begin{aligned} 0 &= 5(49 - c) \quad \text{or} \quad c = 49 && \text{evaluate } c \\ v &= 245(1 - e^{-t/5}) && \text{particular solution} \\ &= 245(1 - e^{-0.200}) = 245(1 - 0.819) && \text{evaluating } v \text{ for } t = 5.00 \text{ s} \\ &= 44.4 \text{ m/s} \end{aligned}$$

After 5.00 s, the velocity is 44.4 m/s. Without the air resistance, the velocity would be about 49.0 m/s. ■

■ There is at least some resistance to the motion of any object moving through a medium (not a vacuum). The exact nature of the resistance is not usually known.

■ Slug is the unit of mass when force is expressed in pounds.

■ The data were given to 3 significant digits, but not all significant digits were used in writing the equations.

**EXERCISES 31.6**

In Exercises 1–4, make the given changes in the indicated examples of this section, and then solve the resulting problems.

- In Example 2, change  $x^2 = ky$  to  $y^2 = kx$ .
- In Example 4, if the term  $L \frac{di}{dt}$  is deleted (no inductance) in Eq. (31.9), find the expression for the charge in the circuit. (There is a capacitance  $C$ , and  $i = dq/dt$ .) The initial charge is zero.
- In Example 5, change 2.00 L to 1.00 L.
- In Example 6, change 0.200 to 0.100 (for the retarding force).

In Exercises 5–8, find the equation of the curve for the given slope and point through which it passes. Use a graphing calculator to display the curve.

- Slope given by  $2x/y$ ; passes through  $(2, 3)$
- Slope given by  $-y/(x + y)$ ; passes through  $(-1, 3)$
- Slope given by  $y + x$ ; passes through  $(0, 1)$
- Slope given by  $-2y + e^{-x}$ ; passes through  $(0, 2)$

In Exercises 9–12, find the equation of the orthogonal trajectories of the curves for the given equations. Use a graphing calculator to display at least two members of the family of curves and at least two of the orthogonal trajectories.

9. The exponential curves  $y = ke^x$
10. The hyperbolas  $x^2 - y^2 = a^2$
11. The curves  $y = k(\sec x + \tan x)$
12. The family of circles, all with centres at the origin

In Exercises 13–50, solve the given problems by solving the appropriate differential equation.

13. The isotope cobalt-60, with half-life of 5.27 years, is used in treating cancerous tumors. What percent of an initial amount remains after 2.00 years?
14. Radium-226 decays such that 10% of the original amount disintegrates in 246 years. Find the half-life (the time for one-half of the original amount to disintegrate) of radium-226.
15. A possible health hazard in the home is radon gas. It is radioactive and about 90.0% of an original amount disintegrates in 12.7 days. Find the half-life of radon gas. (The problem with radon is that it is a gas and is being continually produced by the radioactive decay of minute amounts of radioactive radium found in the soil and rocks of an area.)
16. Noting Example 3, another element used to date more recent events is helium-3, which has a half-life of 12.3 years. If a building has 5.0% of helium-3 that a new building would have, about how old is the building?
17. A radioactive element leaks from a nuclear power plant at a constant rate  $r$ , and it decays at a rate that is proportional to the amount present. Find the relation between the amount  $N$  present in the environment in which it leaks and the time  $t$ , if  $N = 0$  when  $t = 0$ .
18. Most use of the pesticide DDT was banned in Canada in 1970. It has been found that 19% of an initial amount of DDT is degraded into harmless products in 10 years. If no DDT has been used in an area since 1970, in what year will the concentration become only 20% of the 1970 amount?
19. The growth of the population  $P$  of a nation with a constant immigration rate  $I$  may be expressed as  $\frac{dP}{dt} = kP + I$ , where  $t$  is in years. If the population of Canada in 2012 was 34.9 million and about 0.20 million immigrants enter Canada each year, what will the population of Canada be in 2025, given that the growth rate  $k$  is about 1.0% (0.010) annually?
20. Assuming that the natural environment of the earth is limited and that the maximum population it can sustain is  $M$ , the rate of growth of the population  $P$  is given by the *logistic* differential equation  $\frac{dP}{dt} = kP(M - P)$ . Using this equation for the earth, if  $P = 7.0$  billion in 2012,  $k = 0.00030$ ,  $M = 50$  billion, what will be the population of the earth in 2025?
21. The rate of change of the radial stress  $S$  on the walls of a pipe with respect to the distance  $r$  from the axis of the pipe is given by  $r \frac{dS}{dr} = 2(a - S)$ , where  $a$  is a constant. Solve for  $S$  as a function of  $r$ .
22. The velocity  $v$  of a meteor approaching the earth is given by  $v \frac{dv}{dr} = -\frac{GM}{r^2}$ , where  $r$  is the distance from the centre of the earth,  $M$  is the mass of the earth, and  $G$  is a universal gravitational constant. If  $v = 0$  for  $r = r_0$ , solve for  $v$  as a function of  $r$ .
23. Assume that the rate at which highway construction increases is directly proportional to the total length  $M$  of all highways already completed at time  $t$  (in years). Solve for  $M$  as a function of  $t$  if  $M = 5250$  km for a certain county when  $t = 0$  and  $M = 5460$  km for  $t = 2.00$  years.
24. The marginal profit function gives the change in the total profit  $P$  of a business due to a change in the business, such as adding new machinery or reducing the size of the sales staff. A company determines that the marginal profit  $dP/dx$  is  $e^{-x^2} - 2Px$ , where  $x$  is the amount invested in new machinery. Determine the total profit (in thousands of dollars) as a function of  $x$ , if  $P = 0$  for  $x = 0$ .
25. According to Newton's law of cooling, the rate at which a body cools is proportional to the difference in temperature between it and the surrounding medium. Assuming Newton's law holds, how long will it take a cup of hot water, initially at 90°C, to cool to 40°C if the room temperature is 25°C, if it cools to 60°C in 5.0 min?
26. An object whose temperature is 100°C is placed in a medium whose temperature is 20°C. The temperature of the object falls to 50°C in 10 min. Express the temperature  $T$  of the object as a function of time  $t$  (in min). (Assume it cools according to Newton's law of cooling as stated in Exercise 25.)
27. If interest in a bank account is compounded continuously, the amount grows at a rate that is proportional to the amount present in the account. Interest that is compounded daily very closely approximates this situation. Determine the amount in an account after one year if \$1000 is placed in the account and it pays 4% interest per year, compounded continuously.
28. The rate of change in the intensity  $I$  of light below the surface of the ocean with respect to the depth  $y$  is proportional to  $I$ . If the intensity at 5.0 m is 50% of the intensity  $I_0$  at the surface, at what depth is the intensity 15% of  $I_0$ ?
29. For a DNA sample in a liquid containing a solute of constant concentration  $c_0$ , the rate at which the concentration  $c(t)$  of solute in the sample changes is proportional to  $c_0 - c(t)$ . Find  $c(t)$  if  $c(0) = 0$ .
30. In a town of  $N$  persons, during a flu epidemic, it was determined that the rate  $dS/dt$  at which persons were being infected was proportional to the product of the number  $S$  of infected persons and the number  $N - S$  of healthy persons. Find  $S$  as a function of  $t$ .

31. If the current in an  $RL$  circuit with a voltage source  $E$  is zero when  $t = 0$  (see Example 4), show that  $\lim_{t \rightarrow \infty} i = E/R$ . See Fig. 31.7.

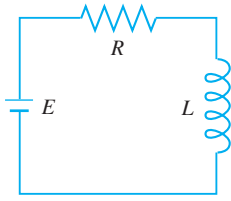


Fig. 31.7

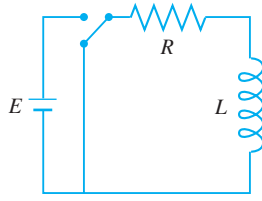


Fig. 31.8

32. If a circuit contains only an inductance and a resistance, with  $L = 2.0$  H and  $R = 30 \Omega$ , find the current  $i$  as a function of time  $t$  if  $i = 0.020$  A when  $t = 0$ . See Fig. 31.8.
33. An amplifier circuit contains a resistance  $R$ , an inductance  $L$ , and a voltage source  $E \sin \omega t$ . Express the current in the circuit as a function of the time  $t$  if the initial current is zero.
34. A radio transmitter circuit contains a resistance of  $2.0 \Omega$ , a variable inductor of  $100 - t$  henrys, and a voltage source of  $4.0$  V. Find the current  $i$  in the circuit as a function of the time  $t$  for  $0 \leq t \leq 100$  s if the initial current is zero.

35. If a circuit contains only a resistance and a capacitance  $C$ , find the equation relating the charge  $q$  on the capacitor in terms of the time  $t$  if  $i = dq/dt$  and  $q = q_0$  when  $t = 0$ . See Fig. 31.9.

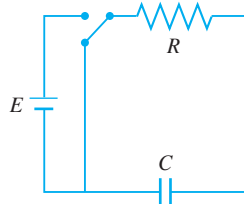


Fig. 31.9

36. One hundred litres of brine originally containing  $4.0$  kg of salt are in a tank into which  $5.0$  L of water run each minute. The same amount of mixture from the tank leaves each minute. How much salt is in the tank after  $20$  min?
37. An object falling under the influence of gravity has a variable acceleration given by  $9.8 - v$ , where  $v$  represents the velocity. If the object starts from rest, find an expression for the velocity in terms of the time. Also, find the limiting value of the velocity (find  $\lim_{t \rightarrow \infty} v$ ).
38. In a ballistics test, a bullet is fired into a sandbag. The acceleration of the bullet within the sandbag is  $-15\sqrt{v}$ , where  $v$  is the velocity (in m/s). When will the bullet stop if it enters the sandbag at  $300$  m/s?
39. A boat with a mass of  $150$  kg is being towed at  $8.0$  km/h. The tow rope is then cut, and a motor that exerts a force of  $80$  N on the boat is started. If the water exerts a retarding force that numerically equals twice the velocity, what is the velocity of the boat  $3.0$  min later?
40. A parachutist is falling at a rate of  $60.0$  m/s when her parachute opens. If the air resists the fall with a force equal to  $5v^2$ , find the velocity as a function of time. The person and equipment have a combined mass of  $100$  kg (weight is  $980$  N).

41. For each cycle, a roller mechanism follows a path described by  $y = 2x - x^2$ ,  $y \geq 0$ , such that  $dx/dt = 6t - 3t^2$ . Find  $x$  and  $y$  (in cm) in terms of the time  $t$  (in s) if  $x$  and  $y$  are zero for  $t = 0$ .
42. In studying the flow of water in a stream, it is found that an object follows the hyperbolic path  $y(x + 1) = 10$  such that  $(t + 1) dx = (x - 2) dt$ . Find  $x$  and  $y$  (in m) in terms of time  $t$  (in s) if  $x = 4$  m and  $y = 2$  m for  $t = 0$ .
43. The rate of change of air pressure  $p$  (in kPa) with respect to height  $h$  (in m) is approximately proportional to the pressure. If the pressure is  $100$  kPa when  $h = 0$  and  $p = 80$  kPa when  $h = 2000$  m, find the expression relating pressure and height.
44. Water flows from a vertical cylindrical storage tank through a hole of area  $A$  at the bottom of the tank. The rate of flow is  $2.6 A \sqrt{h}$ , where  $h$  is the distance (in m) from the surface of the water to the hole. If  $h$  changes from  $9.0$  m to  $8.0$  m in  $16$  min, how long will it take the tank to empty? See Fig. 31.10.

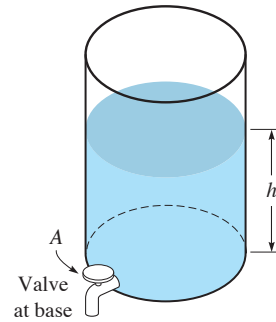


Fig. 31.10

45. Assume that the rate of depreciation of an object is proportional to its value at any time  $t$ . If a car costs  $\$33\,000$  new and its value  $3$  years later is  $\$19\,700$ , what is its value  $11$  years after it was purchased?
46. Assume that sugar dissolves at a rate proportional to the undissolved amount. If there are initially  $525$  g of sugar and  $225$  g remain after  $4.00$  min, how long does it take to dissolve  $375$  g?
47. Fresh air is being circulated into a room whose volume is  $120$  m<sup>3</sup>. Under specified conditions the number of cubic metres  $x$  of carbon dioxide present at any time  $t$  (in min) is found by solving the differential equation  $dx/dt = 1 - 5.0x$ . Find  $x$  as a function of  $t$  if  $x = 0.35$  m<sup>3</sup> when  $t = 0$ .
48. Moisture evaporates from a surface at a rate proportional to the amount of moisture present at any time. If  $75\%$  of the moisture evaporates from a certain surface in  $1.00$  h, how long did it take for  $50\%$  to evaporate?
49. On a certain weather map, the *isobars* (curves of equal barometric pressure) are given by  $y = e^{x/2} + k$ . Find the equation of the orthogonal trajectories (curves that show the wind direction), and display a few of each on a calculator.
50. The lines of equal potential in a field of force are all at right angles to the lines of force. In an electric field of force caused by charged particles, the lines of force are given by  $x^2 + y^2 = kx$ . Find the equation of the lines of equal potential. Use a graphing calculator to view a few members of the lines of force and those of equal potential.

## 31.7 Higher-Order Homogeneous Equations

*n*th Order Linear Differential Equation • Operator  $D$  • Homogeneous and Nonhomogeneous Equations • Second-Order Homogeneous Equations with Constant Coefficients • Auxiliary Equation

Another important type of differential equation is the linear differential equation of higher order with constant coefficients. First, we shall briefly describe the general higher-order equation and the notation we shall use with this type of equation.

The general **linear differential equation of the  $n$ th order** is of the form

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = b \quad (31.11)$$

where the  $a$ 's and  $b$  are either functions of  $x$  or constants.

For convenience of notation, the  $n$ th derivative with respect to the independent variable will be denoted by  $D^n$ . Here,  $D$  is called the **operator**, since it denotes the *operation* of differentiation. Using this notation with  $x$  as the independent variable, Eq. (31.11) becomes

$$a_0 D^n y + a_1 D^{n-1} y + \cdots + a_{n-1} D y + a_n y = b \quad (31.12)$$

If  $b = 0$ , the general linear equation is called **homogeneous**, and if  $b \neq 0$ , it is called **nonhomogeneous**. Both types of equations have important applications.

### EXAMPLE 1 Illustrating the operator $D$

Using the operator form of Eq. (31.12), the differential equation

$$\frac{d^3 y}{dx^3} - 3 \frac{d^2 y}{dx^2} + 4 \frac{dy}{dx} - 2y = e^x \sec x$$

is written as

$$D^3 y - 3D^2 y + 4D y - 2y = e^x \sec x$$

This equation is nonhomogeneous since  $b = e^x \sec x$ . ■

Although the  $a$ 's may be functions of  $x$ , we shall restrict our attention to the cases in which they are constants. We shall, however, consider both homogeneous equations and nonhomogeneous equations. Also, since second-order linear equations are the most commonly found in elementary applications, we shall devote most of our attention to them. The methods used to solve second-order equations may be applied to equations of higher order, and we shall consider certain of these higher-order equations.

### SECOND-ORDER HOMOGENEOUS EQUATIONS WITH CONSTANT COEFFICIENTS

Using the operator notation, a *second-order, linear, homogeneous differential equation with constant coefficients* is one of the form

$$a_0 D^2 y + a_1 D y + a_2 y = 0 \quad (31.13)$$

where the  $a$ 's are constants. The following example indicates the kind of solution we should expect for this type of equation.



**EXAMPLE 2** Solution for a second-order equation

Solve the differential equation  $D^2y - Dy - 2y = 0$ .

First, we put this equation in the form  $(D^2 - D - 2)y = 0$ . This is another way of saying that we are to take the second derivative of  $y$ , subtract the first derivative, and finally subtract twice the function. This expression may now be factored as  $(D - 2)(D + 1)y = 0$ . (We will not develop the algebra of the operator  $D$ . However, most such algebraic operations can be shown to be valid.) This formula tells us to find the first derivative of the function and add this to the function. Then twice this result is to be subtracted from the derivative of this result. If we let  $z = (D + 1)y$ , which is valid since  $(D + 1)y$  is a function of  $x$ , we have  $(D - 2)z = 0$ . This equation is easily solved by separation of variables. Thus,

$$\begin{aligned} \frac{dz}{dx} - 2z &= 0 & \frac{dz}{z} - 2dx &= 0 & \ln z - 2x &= \ln c_1 \\ \ln \frac{z}{c_1} &= 2x & \text{or } z &= c_1 e^{2x} \end{aligned}$$

Replacing  $z$  by  $(D + 1)y$ , we have

$$(D + 1)y = c_1 e^{2x}$$

This is a linear equation of the first order. Then,

$$dy + ydx = c_1 e^{2x} dx$$

The factor  $e^{\int P dx}$  is  $e^{\int dx} = e^x$ . And so,

$$\begin{aligned} ye^x &= \int c_1 e^{3x} dx = \frac{c_1}{3} e^{3x} + c_2 \quad \text{using Eq. (31.8)} \\ y &= c_1' e^{2x} + c_2 e^{-x} \end{aligned}$$

where  $c_1' = \frac{1}{3}c_1$ . This example indicates that solutions of the form  $e^{mx}$  result for this equation. ■

Based on the result of Example 2, assume that an equation of the form of Eq. (31.13) has a particular solution  $ce^{mx}$ . Substituting this into Eq. (31.13) gives

$$a_0 cm^2 e^{mx} + a_1 cm e^{mx} + a_2 ce^{mx} = 0$$

Since the exponential function  $e^{mx} > 0$  for all real  $x$ , this equation will be satisfied if  $m$  is a root of the equation

**Auxiliary Equation**

$$a_0 m^2 + a_1 m + a_2 = 0 \quad (31.14)$$

Eq. (31.14) is called the **auxiliary equation** of Eq. (31.13). Note that it may be formed directly by inspection from Eq. (31.13).

There are two roots of the auxiliary Eq. (31.14), and there are two arbitrary constants in the solution of Eq. (31.13). These factors lead us to the *general solution* of Eq. (31.13), which is

**General Solution**

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} \quad (31.15)$$

where  $m_1$  and  $m_2$  are the two real and distinct solutions of Eq. (31.14). We see that this is in agreement with the results of Example 2.

**EXAMPLE 3** Using the auxiliary equation

Solve the differential equation  $D^2y - 5Dy + 6y = 0$ .

From this operator form of the differential equation, we write the auxiliary equation

$$\begin{aligned} m^2 - 5m + 6 &= 0 \\ (m - 3)(m - 2) &= 0 && \text{solving auxiliary equation} \\ m_1 = 3 \quad m_2 = 2 & \\ y = c_1e^{3x} + c_2e^{2x} & && \text{using Eq. (31.15)} \end{aligned}$$

It makes no difference which constant is written with each exponential term.

**EXAMPLE 4** The auxiliary equation has zero as a root

Solve the differential equation  $y'' = 6y'$ .

We first rewrite this equation using the  $D$  notation for derivatives. Also, we want to write it in the proper form of a homogeneous equation. This gives us

$$D^2y - 6Dy = 0$$

Proceeding with the solution, we have

$$\begin{aligned} m^2 - 6m &= 0 && \text{auxiliary equation} \\ m(m - 6) &= 0 && \text{solve for } m \\ m_1 = 0 \quad m_2 = 6 & \\ y = c_1e^{0x} + c_2e^{6x} & && \text{using Eq. (31.15)} \end{aligned}$$

Since  $e^{0x} = 1$ , we have

$$y = c_1 + c_2e^{6x} \quad \text{general solution}$$

**Practice Exercise**

1. Solve the differential equation  $2D^2y - Dy - 3y = 0$ .

**EXAMPLE 5** A third-order equation

Solve the differential equation  $2\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} - 7\frac{dy}{dx} = 0$ .

Although this is a third-order equation, the *method* of solution is the same as in the previous examples. Using the  $D$ -notation for derivatives, we have

$$\begin{aligned} 2D^3y + D^2y - 7Dy &= 0 \\ 2m^3 + m^2 - 7m &= 0 && \text{auxiliary equation} \\ m(2m^2 + m - 7) &= 0 \end{aligned}$$

We can see that one root is  $m = 0$ . The quadratic factor is not factorable, but we can find the roots from it by using the quadratic formula. This gives

$$m = \frac{-1 \pm \sqrt{1 + 56}}{4} = \frac{-1 \pm \sqrt{57}}{4}$$

Since there are *three* roots, there are *three* arbitrary constants. Following the solution indicated by Eq. (31.15), we have

$$y = c_1e^{0x} + c_2e^{(-1+\sqrt{57})x/4} + c_3e^{(-1-\sqrt{57})x/4}$$

Again,  $e^{0x} = 1$ . Also, factoring  $e^{-x/4}$  from the second and third terms, we have

$$y = c_1 + e^{-x/4} (c_2e^{x\sqrt{57}/4} + c_3e^{-x\sqrt{57}/4})$$

In all examples and exercises of this section, all the roots of the auxiliary equation are different, and they do not include complex numbers. For such roots, the solutions have a different form. They are the topic of the following section.

**EXAMPLE 6** A particular solution

Solve the differential equation  $D^2y - 2Dy - 15y = 0$  and find the particular solution that satisfies the conditions  $Dy = 2$  and  $y = -1$ , when  $x = 0$ . (It is necessary to give two conditions since there are two constants to evaluate.)

We have

$$\begin{aligned} m^2 - 2m - 15 &= 0, & (m - 5)(m + 3) &= 0 \\ m_1 &= 5 & m_2 &= -3 \\ y &= c_1e^{5x} + c_2e^{-3x} \end{aligned}$$

This equation is the general solution. In order to evaluate the constants  $c_1$  and  $c_2$ , we use the given conditions to find two simultaneous equations in  $c_1$  and  $c_2$ . These are then solved to determine the particular solution. Thus,

$$y' = 5c_1e^{5x} - 3c_2e^{-3x}$$

Using the given conditions in the general solution and its derivative, we have

$$\begin{aligned} c_1 + c_2 &= -1 & y = -1 \text{ when } x = 0 \\ 5c_1 - 3c_2 &= 2 & Dy = 2 \text{ when } x = 0 \end{aligned}$$

The solution to this system of equations is  $c_1 = -\frac{1}{8}$  and  $c_2 = -\frac{7}{8}$ . The particular solution becomes

$$y = -\frac{1}{8}e^{5x} - \frac{7}{8}e^{-3x} \quad \text{or} \quad 8y + e^{5x} + 7e^{-3x} = 0 \quad \blacksquare$$

**EXERCISES 31.7**

In Exercises 1 and 2, make the given changes in the indicated examples of this section and then solve the resulting differential equations.

- In Example 3, delete the  $+6y$  term.
- In Example 4, add  $16y$  to the right side.

In Exercises 3–26, solve the given differential equations.

- $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0$
- $3\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + y = 0$
- $D^2y - 3Dy = 0$
- $2D^2y - 3y = Dy$
- $3D^2y + 12y = 20Dy$
- $3y'' + 8y' - 3y = 0$
- $3y'' + 2y' - y = 0$
- $2\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + y = 0$
- $4D^2y - 3Dy = 2y$
- $y'' = 3y' + y$
- $y'' + y' = 8y$
- $2D^2y + 5aDy - 12a^2y = 0 \quad (a > 0)$
- $3k^4D^2y + 14k^2Dy - 5y = 0$
- $\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$
- $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 8y = 0$
- $D^2y = 25y$
- $D^2y + 7Dy + 6y = 0$
- $4D^2y + 12Dy = 7y$
- $8y'' + 6y' = 9y$
- $2y'' - 7y' + 6y = 0$
- $\frac{d^2y}{dx^2} + \frac{dy}{dx} = 5y$
- $2D^2y - 3Dy - y = 0$
- $5y'' - y' = 3y$
- $8y'' = y' + y$

In Exercises 27–30, find the particular solutions of the given differential equations that satisfy the given conditions.

- $D^2y - 4Dy - 21y = 0$ ;  $Dy = 0$  and  $y = 2$  when  $x = 0$
- $4D^2y - Dy = 0$ ;  $Dy = 2$  and  $y = 4$  when  $x = 0$
- $D^2y - Dy = 12y$ ;  $y = 0$  when  $x = 0$ , and  $y = 1$  when  $x = 1$
- $2D^2y + 5Dy = 0$ ;  $y = 0$  when  $x = 0$ , and  $y = 2$  when  $x = 1$

In Exercises 31–34, solve the given third- and fourth-order differential equations.

- $y''' - 2y'' - 3y' = 0$
- $D^3y - 6D^2y + 11Dy - 6y = 0$
- $D^4y - 5D^2y + 4y = 0$
- $D^4y - D^3y - 9D^2y + 9Dy = 0$

In Exercises 35 and 36, solve the given problems.

- The voltage  $v$  at a distance  $s$  along a transmission line is given by  $d^2v/ds^2 = a^2v$ , where  $a$  is called the *attenuation constant*. Solve for  $v$  as a function of  $s$ .
- Following the method of Example 2, solve the differential equation  $D^2y - 3Dy + 2y = 0$ . Do not use Eqs. (31.14) and (31.15).

**Answer to Practice Exercise**

- $y = c_1e^{-x} + c_2e^{3x/2}$

## 31.8 Auxiliary Equation with Repeated or Complex Roots

Repeated Roots • Complex Roots •  
 Roots Repeated More Than Once •  
 Repeated Complex Roots

In solving higher-order homogeneous differential equations in the previous section, we purposely avoided repeated or complex roots of the auxiliary equation. In this section, we develop the solutions for such equations. The following example indicates the type of solution that results from the case of repeated roots.

### EXAMPLE 1 Solution for a double root

Solve the differential equation  $D^2y - 4Dy + 4y = 0$ .

Using the method of Example 2 of the previous section, we have the following steps:

$$(D^2 - 4D + 4)y = 0, \quad (D - 2)(D - 2)y = 0, \quad (D - 2)z = 0$$

where  $z = (D - 2)y$ . The solution to  $(D - 2)z = 0$  is found by separation of variables. And so,

$$\begin{aligned} \frac{dz}{dx} - 2z &= 0 & \frac{dz}{z} - 2dx &= 0 \\ \ln z - 2x &= \ln c_1 & \text{or } z &= c_1 e^{2x} \end{aligned}$$

Substituting back, we have  $(D - 2)y = c_1 e^{2x}$ , which is a linear equation of the first order. Then

$$dy - 2y dx = c_1 e^{2x} dx \quad e^{\int -2 dx} = e^{-2x}$$

This leads to

$$y e^{-2x} = c_1 \int dx = c_1 x + c_2 \quad \text{or } y = c_1 x e^{2x} + c_2 e^{2x}$$

This example indicates the type of solution that results when the auxiliary equation has repeated roots. If the method of the previous section were to be used, the solution of the above example would be  $y = c_1 e^{2x} + c_2 e^{2x}$ . This would not be the general solution, since both terms are similar, which means that there is only one independent constant. The constants can be combined to give a solution of the form  $y = c e^{2x}$ , where  $c = c_1 + c_2$ . This solution would contain only one constant for a second-order equation. ■

Based on the above example, *the solution to Eq. (31.13) when the auxiliary Eq. (31.14) has repeated roots is*

**Solution with Repeated Roots**

$$y = e^{mx} (c_1 + c_2 x) \quad (31.16)$$

where  $m$  is the double root. (In Example 1, this double root is 2.)

### EXAMPLE 2 Solving an equation with a double root

Solve the differential equation  $(D + 2)^2 y = 0$ .

The solution is as follows:

$$\begin{aligned} (m + 2)^2 &= 0 && \text{auxiliary equation} \\ m &= -2, -2 \\ y &= e^{-2x} (c_1 + c_2 x) && \text{using Eq. (31.16)} \end{aligned}$$

■ For reference, Eq. (31.13) is  $a_0 D^2 y + a_1 D y + a_2 y = 0$  and Eq. (31.14) is  $a_0 m^2 + a_1 m + a_2 = 0$ .

**EXAMPLE 3 Solving an equation with a double root**

Solve the differential equation  $\frac{d^2y}{dx^2} + 25y = 10\frac{dy}{dx}$ .

$$\begin{aligned}
 D^2y + 25y &= 10Dy && \text{using operator } D \text{ notation} \\
 D^2y - 10Dy + 25y &= 0 && \text{put in proper form with 0 on right} \\
 m^2 - 10m + 25 &= 0 && \text{auxiliary equation} \\
 (m - 5)^2 &= 0 && \text{solve for } m \\
 m &= 5, 5 && \text{double root} \\
 y &= e^{5x}(c_1 + c_2x) && \text{using Eq. (31.16)}
 \end{aligned}$$

**Practice Exercise**

1. Solve the differential equation  $D^2y + 8Dy + 16y = 0$ .

When the auxiliary equation has complex roots, it can be solved by the method of the previous section and the solution can be put in a more useful form. For complex roots of the auxiliary equation  $m = \alpha \pm j\beta$ , the solution is of the form

$$y = c_1e^{(\alpha + j\beta)x} + c_2e^{(\alpha - j\beta)x} = e^{\alpha x}(c_1e^{j\beta x} + c_2e^{-j\beta x})$$

Using the exponential form of a complex number, Eq. (12.11), we have

$$\begin{aligned}
 y &= e^{\alpha x}[c_1 \cos \beta x + jc_1 \sin \beta x + c_2 \cos(-\beta x) + jc_2 \sin(-\beta x)] \\
 &= e^{\alpha x}(c_1 \cos \beta x + c_2 \cos \beta x + jc_1 \sin \beta x - jc_2 \sin \beta x) \\
 &= e^{\alpha x}(c_3 \cos \beta x + c_4 \sin \beta x)
 \end{aligned}$$

where  $c_3 = c_1 + c_2$  and  $c_4 = jc_1 - jc_2$ .

Therefore, if the auxiliary equation has complex roots of the form  $\alpha \pm j\beta$ ,

**Solution with Complex Roots**

$$y = e^{\alpha x}(c_1 \sin \beta x + c_2 \cos \beta x) \tag{31.17}$$

is the solution to Eq. (31.13). The  $c_1$  and  $c_2$  here are not the same as those above. They are simply the two arbitrary constants of the solution.

**EXAMPLE 4 Solving an equation with complex roots**

Solve the differential equation  $D^2y - Dy + y = 0$ .

We have the following solution:

$$\begin{aligned}
 m^2 - m + 1 &= 0 && \text{auxiliary equation} \\
 m &= \frac{1 \pm j\sqrt{3}}{2} && \text{complex roots} \\
 \alpha &= \frac{1}{2} \quad \beta = \frac{\sqrt{3}}{2} && \text{identify } \alpha \text{ and } \beta \\
 y &= e^{x/2} \left( c_1 \sin \frac{\sqrt{3}}{2}x + c_2 \cos \frac{\sqrt{3}}{2}x \right) && \text{using Eq. (31.17)}
 \end{aligned}$$

**EXAMPLE 5 Solving a third-order equation**

Solve the differential equation  $D^3y + 4Dy = 0$ .

This is a third-order equation, which means there are three arbitrary constants in the general solution.

$$\begin{aligned}
 m^3 + 4m &= 0, \quad m(m^2 + 4) = 0 && \text{auxiliary equation} \\
 m_1 = 0 \quad m_2 = 2j \quad m_3 = -2j &&& \text{three roots, two of them complex} \\
 \alpha = 0 \quad \beta = 2 &&& \text{identify } \alpha \text{ and } \beta \text{ for complex roots} \\
 y &= c_1e^{0x} + e^{0x}(c_2 \sin 2x + c_3 \cos 2x) && \text{using Eqs. (31.15) and (31.17)} \\
 &= c_1 + c_2 \sin 2x + c_3 \cos 2x && e^0 = 1
 \end{aligned}$$

**Practice Exercise**

2. Solve the differential equation  $D^2y + 2Dy + 5y = 0$ .

**EXAMPLE 6** Complex roots—a particular solution

Solve the differential equation  $y'' - 2y' + 12y = 0$ , if  $y' = 2$  and  $y = 1$  when  $x = 0$ .

$$\begin{aligned} D^2y - 2Dy + 12y &= 0 && \text{using operator } D \text{ notation} \\ m^2 - 2m + 12 &= 0 && \text{auxiliary equation} \\ m &= \frac{2 \pm \sqrt{4 - 48}}{2} = 1 \pm j\sqrt{11} && \text{complex roots: } \alpha = 1, \beta = \sqrt{11} \\ y &= e^x (c_1 \cos \sqrt{11}x + c_2 \sin \sqrt{11}x) && \text{general solution} \end{aligned}$$

Using the condition that  $y = 1$  when  $x = 0$ , we have

$$1 = e^0(c_1 \cos 0 + c_2 \sin 0) \quad \text{or} \quad c_1 = 1$$

Since  $y' = 2$  when  $x = 0$ , we find the derivative and then evaluate  $c_2$ .

$$\begin{aligned} y' &= e^x(c_1 \cos \sqrt{11}x + c_2 \sin \sqrt{11}x - \sqrt{11}c_1 \sin \sqrt{11}x + \sqrt{11}c_2 \cos \sqrt{11}x) \\ 2 &= e^0(\cos 0 + c_2 \sin 0 - \sqrt{11} \sin 0 + \sqrt{11}c_2 \cos 0) && y' = 2 \text{ when } x = 0 \\ 2 &= 1 + \sqrt{11}c_2, \quad c_2 = \frac{1}{11}\sqrt{11} && \text{solve for } c_2 \\ y &= e^x \left( \cos \sqrt{11}x + \frac{1}{11}\sqrt{11} \sin \sqrt{11}x \right) && \text{particular solution} \quad \blacksquare \end{aligned}$$

If the root of the auxiliary equation is repeated more than once—for example, a *triple root*—an additional term with another arbitrary constant and the next higher power of  $x$  is added to the solution for each additional root. Also, if a pair of complex roots is repeated, an additional term with a factor of  $x$  and another arbitrary constant is added for each root of the pair. These are illustrated in the following example.

**EXAMPLE 7** Root repeated more than once

(a) For the differential equation  $D^3y + 3D^2y + 3Dy + y = 0$ , the auxiliary equation is

$$m^3 + 3m^2 + 3m + 1 = 0, \quad (m + 1)^3 = 0$$

Each of the three roots is  $m = -1$ . The equation is a third-order equation, which means there are three arbitrary constants. Therefore, the general solution is

$$y = e^{-x}(c_1 + c_2x + c_3x^2)$$

**Repeated Complex Roots**

(b) For the differential equation  $D^4y + 8D^2y + 16y = 0$ , the auxiliary equation is

$$m^4 + 8m^2 + 16 = 0, \quad (m^2 + 4)^2 = 0$$

With two factors of  $m^2 + 4$ , the roots are  $2j$ ,  $2j$ ,  $-2j$ , and  $-2j$ . The fourth-order equation, and four roots, indicate four arbitrary constants. Since  $e^{0x} = 1$ , the general solution is

$$y = (c_1 + c_2x)\sin 2x + (c_3 + c_4x)\cos 2x \quad \blacksquare$$

Knowing the various types of possible solutions, it is possible to determine the differential equation if the solution is known. Consider the following example.

**EXAMPLE 8** Determine the equation from the solution

- (a) A solution of  $y = c_1e^x + c_2e^{2x}$  indicates an auxiliary equation with roots of  $m_1 = 1$  and  $m_2 = 2$ . Thus, the auxiliary equation is  $(m - 1)(m - 2) = 0$ , and the simplest form of the differential equation is  $D^2y - 3Dy + 2y = 0$ .
- (b) A solution of  $y = e^{2x}(c_1 + c_2x)$  indicates repeated roots  $m_1 = m_2 = 2$  of the auxiliary equation  $(m - 2)^2 = 0$ , and a differential equation is  $D^2y - 4Dy + 4y = 0$ . \blacksquare

Let us summarize the type of solution that results from each of the cases discussed in the last two sections.

### The Homogeneous Linear Differential Equation

The terms of the general solution to a homogeneous linear differential equation of constant coefficients are related to the roots of the auxiliary equation as follows.

Root of the auxiliary equation	Terms in the general solution
$m$ a single real root	$c_1 e^{mx}$
$m$ a double real root	$e^{mx}(c_1 + c_2 x)$
$m$ an $n$ -fold real root	$e^{mx}(c_1 + c_2 x + \cdots + c_n x^{n-1})$
$m = \alpha \pm \beta j$ complex conjugate roots	$e^{\alpha x}(c_1 \sin \beta x + c_2 \cos \beta x)$

## EXERCISES 31.8

In Exercises 1–4, make the given changes in the indicated examples of this section, and then solve the resulting differential equations.

- In Example 3, change the sign of the term  $10 \frac{dy}{dx}$ .
- In Example 3, delete the term  $10 \frac{dy}{dx}$ .
- In Example 3, delete the term  $25y$ .
- In Example 4, change the  $-$  sign to  $+$ .

In Exercises 5–32, solve the given differential equations.

- $\frac{d^2 y}{dx^2} - 2 \frac{dy}{dx} + y = 0$
- $\frac{d^2 y}{dx^2} - 6 \frac{dy}{dx} + 9y = 0$
- $D^2 y + 12Dy + 36y = 0$
- $16D^2 y + 8Dy + y = 0$
- $\frac{d^2 y}{dx^2} + 9y = 0$
- $\frac{d^2 y}{dx^2} + y = 0$
- $D^2 y + Dy + 2y = 0$
- $D^2 y + 4y = 2Dy$
- $D^4 y - y = 0$
- $4D^2 y = 12Dy - 9y$
- $4D^2 y + y = 0$
- $9D^2 y + 4y = 0$
- $16y'' - 24y' + 9y = 0$
- $9y'' - 24y' + 16y = 0$
- $25y'' + 4y = 0$
- $y'' + 5y = 4y'$
- $2D^2 y + 5y = 4Dy$
- $D^2 y + 4Dy + 6y = 0$
- $25y'' + 16y = 40y'$
- $9y''' + 0.6y'' + 0.01y' = 0$
- $2D^2 y - 3Dy - y = 0$
- $D^2 y - 5Dy = 14y$
- $3D^2 y + 12Dy = 2y$

- $36D^2 y = 25y$
- $D^3 y - 6D^2 y + 12Dy - 8y = 0$
- $D^4 y - 2D^3 y + 2D^2 y - 2Dy + y = 0$
- $D^4 y + 2D^2 y + y = 0$
- $16D^4 y - y = 0$

In Exercises 33–36, find the particular solutions of the given differential equations that satisfy the given conditions.

- $y'' + 2y' + 10y = 0$ ;  $y = 0$  when  $x = 0$  and  $y = e^{-\pi/6}$  when  $x = \pi/6$
- $9D^2 y + 16y = 0$ ;  $Dy = 0$  and  $y = 2$  when  $x = \pi/2$
- $D^2 y + 16y = 8Dy$ ;  $Dy = 2$  and  $y = 4$  when  $x = 0$
- $D^4 y + 3D^3 y + 2D^2 y = 0$ ;  $y = 0$  and  $Dy = 4$  and  $D^2 y = -8$  and  $D^3 y = 16$  when  $x = 0$

In Exercises 37–40, find the simplest form of the second-order homogeneous linear differential equation that has the given solution. In Exercises 38 and 39, explain how the equation is found.

- $y = c_1 e^{3x} + c_2 e^{-3x}$
- $y = c_1 e^{3x} + c_2 x e^{3x}$
- $y = c_1 \cos 3x + c_2 \sin 3x$
- $y = c_1 e^{2x} \cos x + c_2 e^{2x} \sin x$

In Exercises 41 and 42, solve the given problems.

- Find the solution of the equation  $D^2 y + ay = 0$ , if  $y = 0$  for  $x = 0$  and  $x = 1$ , (a) if  $a = 0$ , and (b)  $a < 0$ .
- What is the solution of the equation in Exercise 41, with the same conditions, if  $0 < a < \pi^2$ ?

### Answers to Practice Exercises

- $y = e^{-4x}(c_1 + c_2 x)$
- $y = e^{-x}(c_1 \sin 2x + c_2 \cos 2x)$



## 31.9 Solutions of Nonhomogeneous Equations

Complementary Solution •  
Particular Solution • Method of  
Undetermined Coefficients •  
Special Case

We now consider the solution of a nonhomogeneous linear equation of the form

$$a_0 D^2 y + a_1 D y + a_2 y = b \quad (31.18)$$

where  $b$  is a function of  $x$  or is a constant. When the solution is substituted into the left side, we must obtain  $b$ . Solutions found from the methods of Sections 31.7 and 31.8 give zero when substituted into the left side, but they do contain the arbitrary constants necessary in the solution. If we could find a particular solution that when substituted into the left side produced  $b$ , it could be added to the solution containing the arbitrary constants. Therefore, *the solution is of the form*

$$y = y_c + y_p \quad (31.19)$$

where  $y_c$ , called the **complementary solution**, is obtained by solving the corresponding homogeneous equation and where  $y_p$  is the **particular solution** necessary to produce the expression  $b$  of Eq. (31.18). It should be noted that  $y_p$  satisfies the differential equation, but it has no arbitrary constants and therefore cannot be the general solution. The arbitrary constants are part of  $y_c$ .

### EXAMPLE 1 Complementary and particular solutions

The differential equation  $D^2 y - Dy - 6y = e^x$  has the solution

$$y = c_1 e^{3x} + c_2 e^{-2x} - \frac{1}{6} e^x$$

where the complementary solution  $y_c$  and particular solution  $y_p$  are

$$y_c = c_1 e^{3x} + c_2 e^{-2x} \quad y_p = -\frac{1}{6} e^x$$

The complementary solution  $y_c$  is obtained by solving the corresponding homogeneous equation  $D^2 y - Dy - 6y = 0$ , and we shall discuss below the method of finding  $y_p$ . Again, we note that  $y_c$  contains the arbitrary constants,  $y_p$  contains the expression needed to produce the  $e^x$  on the right, and therefore both are needed to have the complete general solution. ■

The method that is used to find a particular solution  $y_p$  is called the **method of undetermined coefficients**.

### Solving Nonhomogeneous Equations by Undetermined Coefficients

1. Obtain the complementary solution  $y_c$  by solving the corresponding homogeneous equation.
2. **Make a guess as to the form** of a particular solution  $y_p$ . Since a combination of the particular solution and its derivatives must form the function  $b$  on the right side of the equation,  $y_p$  **should contain all possible forms of  $b$  and its derivatives**. Leave the coefficients of  $y_p$  undetermined.
3. Substitute the chosen form of  $y_p$  into the differential equation and equate the coefficients of like terms. Find the values of the coefficients by solving the resulting equation(s).
4. Complete the solution by adding  $y_p$  to  $y_c$ . Note that some adjustments will be necessary in step 3 if the proposed  $y_p$  and  $y_c$  have similar terms (see Example 9).

**EXAMPLE 2** Forms of particular solutions

- (a) If the function  $b$  is  $4x$ , we choose the particular solution  $y_p$  to be of the form  $y_p = A + Bx$ . The  $Bx$ -term is included to account for the  $4x$ . Since the derivative of  $Bx$  is a constant, the  $A$ -term is included to account for any first derivative of the  $Bx$ -term that may be present. Since the derivative of  $A$  is zero, no other terms are needed to account for higher-derivative terms of the  $Bx$ -term.
- (b) If the function  $b$  is  $e^{2x}$ , we choose the form of the particular solution to be  $y_p = Ce^{2x}$ . Since all derivatives of  $Ce^{2x}$  are a constant times  $e^{2x}$ , no other forms appear in the derivatives, and no other forms are needed in  $y_p$ .
- (c) If the function  $b$  is  $4x + e^{2x}$ , we choose the form of the particular solution to be  $y_p = A + Bx + Ce^{2x}$ . ■

**EXAMPLE 3** Forms of particular solutions

- (a) If  $b$  is of the form  $x^2 + e^{-x}$ , we choose the particular solution to be of the form  $y_p = A + Bx + Cx^2 + Ee^{-x}$ .
- (b) If  $b$  is of the form  $xe^{-2x} - 5$ , we choose the form of the particular solution to be  $y_p = Ae^{-2x} + Bxe^{-2x} + C$ .
- (c) If  $b$  is of the form  $x \sin x$ , we choose the form of the particular solution to be  $y_p = A \sin x + B \cos x + Cx \sin x + Ex \cos x$ . All these types of terms occur in the derivatives of  $x \sin x$ . ■

**Practice Exercise**

1. Find the form of  $y_p$  if  $b$  is of the form  $x + \cos x$ .

**EXAMPLE 4** Forms of particular solutions

- (a) If  $b$  is of the form  $e^x + xe^x$ , we should then choose  $y_p$  to be of the form  $y_p = Ae^x + Bxe^x$ . These terms occur for  $xe^x$  and its derivatives. Since the form of the  $e^x$ -term of  $b$  is already included in  $Ae^x$ , we do not include another  $e^x$ -term in  $y_p$ .
- (b) In the same way, if  $b$  is of the form  $2x + 4x^2$ , we choose the form of  $y_p$  to be  $y_p = A + Bx + Cx^2$ . There are only forms that occur in either  $2x$  or  $4x^2$  and their derivatives. ■

**EXAMPLE 5** Solving a nonhomogeneous equation

Solve the differential equation  $D^2y - Dy - 6y = e^x$ .

In this case, the solution of the auxiliary equation  $m^2 - m - 6 = 0$  gives us the roots  $m_1 = 3$  and  $m_2 = -2$ . Thus,

$$y_c = c_1e^{3x} + c_2e^{-2x}$$

The proper form of the particular solution is  $y_p = Ae^x$ . This means that  $Dy_p = Ae^x$  and  $D^2y_p = Ae^x$ . Substituting  $y_p$  and its derivatives into the differential equation, we have

$$Ae^x - Ae^x - 6Ae^x = e^x$$

To produce equality, the coefficients of  $e^x$  must be the same on each side of the equation. Thus,

$$-6A = 1 \quad \text{or} \quad A = -1/6$$

Therefore,  $y_p = -\frac{1}{6}e^x$ . This gives the complete solution  $y = y_c + y_p$ ,

$$y = c_1e^{3x} + c_2e^{-2x} - \frac{1}{6}e^x \quad \text{see Example 1}$$

This solution checks when substituted into the original differential equation. ■

**EXAMPLE 6** Solving a nonhomogeneous equation

Solve the differential equation  $D^2y + 4y = x - 4e^{-x}$ .

In this case, we have  $m^2 + 4 = 0$ , which give us  $m_1 = 2j$  and  $m_2 = -2j$ . Therefore,  $y_c = c_1 \sin 2x + c_2 \cos 2x$ .

The proper form of the particular solution is  $y_p = A + Bx + Ce^{-x}$ . Finding two derivatives and then substituting into the differential equation gives

$$\begin{array}{rcl}
 y_p = A + Bx + Ce^{-x} & Dy_p = B - Ce^{-x} & D^2y_p = Ce^{-x} \\
 \downarrow & \swarrow & \\
 D^2y + 4y = x - 4e^{-x} & & \text{differential equation} \\
 (Ce^{-x}) + 4(A + Bx + Ce^{-x}) = x - 4e^{-x} & & \text{substituting} \\
 Ce^{-x} + 4A + 4Bx + 4Ce^{-x} = x - 4e^{-x} & & \\
 (4A) + (4B)x + (5C)e^{-x} = 0 + (1)x + (-4)e^{-x} & & \text{note coefficients}
 \end{array}$$

Equating the constants and the coefficients of  $x$  and  $e^{-x}$  on either side gives

$$\begin{array}{rcl}
 4A = 0 & 4B = 1 & 5C = -4 \\
 A = 0 & B = 1/4 & C = -4/5
 \end{array}$$

This means that the particular solution is

$$y_p = \frac{1}{4}x - \frac{4}{5}e^{-x}$$

In turn, this tells us that the complete solution is

$$y = c_1 \sin 2x + c_2 \cos 2x + \frac{1}{4}x - \frac{4}{5}e^{-x}$$

Substitution into the original differential equation verifies this solution. ■

**EXAMPLE 7** Solving a nonhomogeneous equation

Solve the differential equation  $D^3y - 3D^2y + 2Dy = 10 \sin x$ .

$$\begin{array}{rcl}
 m^3 - 3m^2 + 2m = 0 & m(m-1)(m-2) = 0 & m_1 = 0 \quad m_2 = 1 \quad m_3 = 2 \quad \text{auxiliary equation} \\
 y_c = c_1 + c_2e^x + c_3e^{2x} & & \text{complementary solution}
 \end{array}$$

We now find the particular solution:

$$\begin{array}{rcl}
 y_p = A \sin x + B \cos x & & \text{particular solution form} \\
 Dy_p = A \cos x - B \sin x & D^2y_p = -A \sin x - B \cos x & D^3y_p = -A \cos x + B \sin x \quad \text{find three derivatives} \\
 (-A \cos x + B \sin x) - 3(-A \sin x - B \cos x) + 2(A \cos x - B \sin x) = 10 \sin x & & \text{substitute into differential equation} \\
 (3A - B) \sin x + (A + 3B) \cos x = 10 \sin x & & \\
 3A - B = 10 & A + 3B = 0 & \text{equate coefficients}
 \end{array}$$

The solution of this system is  $A = 3$ ,  $B = -1$ .

$$\begin{array}{rcl}
 y_p = 3 \sin x - \cos x & & \text{particular solution} \\
 y = c_1 + c_2e^x + c_3e^{2x} + 3 \sin x - \cos x & & \text{complete general solution}
 \end{array}$$

Check this solution in the differential equation. ■

**Practice Exercise**

2. Find  $y_p$  for the differential equation  $D^2y + 4y = 8xe^{2x}$ .

**EXAMPLE 8 Particular solution**

Find the particular solution of  $y'' + 16y = 2e^{-x}$  if  $Dy = -2$  and  $y = 1$  when  $x = 0$ .  
 In this case, we must not only find  $y_c$  and  $y_p$ , but we must also evaluate the constants of  $y_c$  from the given conditions. The solution is as follows:

$$\begin{aligned}
 D^2y + 16y &= 2e^{-x} && \text{operator } D \text{ form} \\
 m^2 + 16 &= 0, \quad m = \pm 4j && \text{auxiliary equation} \\
 y_c &= c_1 \sin 4x + c_2 \cos 4x && \text{complementary solution} \\
 y_p &= Ae^{-x} && \text{particular solution form} \\
 Dy_p &= -Ae^{-x} & D^2y_p &= Ae^{-x} \\
 Ae^{-x} + 16Ae^{-x} &= 2e^{-x} && \text{substituting} \\
 17Ae^{-x} &= 2e^{-x}, \quad A = \frac{2}{17} && \text{equate coefficients} \\
 y_p &= \frac{2}{17}e^{-x} \\
 y &= c_1 \sin 4x + c_2 \cos 4x + \frac{2}{17}e^{-x} && \text{complete general solution}
 \end{aligned}$$

We now evaluate  $c_1$  and  $c_2$  from the given conditions:

$$\begin{aligned}
 Dy &= 4c_1 \cos 4x - 4c_2 \sin 4x - \frac{2}{17}e^{-x} \\
 1 &= c_1(0) + c_2(1) + \frac{2}{17}(1), \quad c_2 = \frac{15}{17} && y = 1 \text{ when } x = 0 \\
 -2 &= 4c_1(1) - 4c_2(0) - \frac{2}{17}(1), \quad c_1 = -\frac{8}{17} && Dy = -2 \text{ when } x = 0 \\
 y &= -\frac{8}{17} \sin 4x + \frac{15}{17} \cos 4x + \frac{2}{17}e^{-x} && \text{required particular solution}
 \end{aligned}$$

This solution checks when substituted into the differential equation. ■

**FOURIER SERIES SOLUTIONS**

The particular solution of a non-homogeneous differential equation can also be obtained when the function  $b$  in Eq. (31.18) is periodic. By expanding  $b$  in a Fourier series, the particular solution can be found by undetermined coefficients, as we see in the next example.

**EXAMPLE 9 A solution using Fourier series**

Solve the differential equation  $D^2y + 4y = f(x)$  where  
 $f(x) = \begin{cases} x & 0 \leq x < 2 \\ 4 - x & 2 \leq x < 4 \end{cases}$ ,  $f(x + 4) = f(x)$ , and  $f(x)$  is extended to be even.  
 See Fig. 31.11.

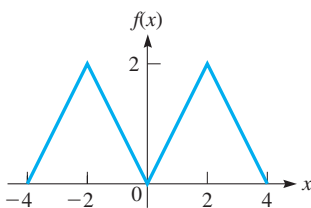


Fig. 31.11

We have seen in Example 6 that the complementary solution is of the form  $y_c = c_1 \sin 2x + c_2 \cos 2x$ . We now find a particular solution.

We begin by writing  $f(x)$  in a half-range cosine series ( $f(x)$  is even). We have already done that in Example 9 of Section 30.7. We have

$$f(x) = 1 - \frac{8}{\pi^2} \left( \cos \frac{\pi x}{2} + \frac{1}{9} \cos \frac{3\pi x}{2} + \frac{1}{25} \cos \frac{5\pi x}{2} + \dots \right)$$

Since the differential equation contains only  $y$  and its second derivatives, a particular solution involving only cosines is appropriate. Therefore we choose a particular solution of the form

$$y_p = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{2}$$

Therefore,

$$D^2y_p = \sum_{n=1}^{\infty} \left( -\frac{n^2\pi^2}{4} \right) A_n \cos \frac{n\pi x}{2}$$

We substitute  $y_p$  and  $D^2y_p$  into the differential equation and equate the coefficients of like terms. If  $n = 0$  we have  $4A_0 = 1$ , so  $A_0 = 1/4$ . If  $n$  is even,  $A_n = 0$ , and if  $n$  is odd,

$$\left( -\frac{n^2\pi^2}{4} + 4 \right) A_n = -\frac{8}{\pi^2 n^2} \quad \text{or} \quad A_n = \frac{32}{\pi^2 (\pi^2 n^4 - 16n^2)}$$

Therefore, the particular solution is

$$y_p = \frac{1}{4} + \frac{32}{\pi^2} \left( \frac{1}{\pi^2 - 16} \cos \frac{\pi x}{2} + \frac{1}{81\pi^2 - 144} \cos \frac{3\pi x}{2} + \dots \right)$$

The complete solution is

$$y = c_1 \sin 2x + c_2 \cos 2x + \frac{1}{4} + \frac{32}{\pi^2} \left( \frac{1}{\pi^2 - 16} \cos \frac{\pi x}{2} + \frac{1}{81\pi^2 - 144} \cos \frac{3\pi x}{2} + \dots \right) \blacksquare$$

■ From Eq. (31.18), we note that the function  $b$  is the function on the right side of the differential equation.

#### LEARNING TIP

If a term of the proposed  $y_p$  is similar to a term of  $y_c$ , any term of the proposed  $y_p$  included to account for the similar term of the function  $b$  must be multiplied by the smallest possible integral power of  $x$  such that any resulting term  $y_p$  is not similar to the term of  $y_c$ .

■ Note that the  $A + Bx$  are not multiplied by  $x^2$  since they are not included in  $y_p$  to account for the  $e^x$  to the right.

#### A SPECIAL CASE

It may happen that a term of the proposed  $y_p$  is similar to a term of  $y_c$ . Since any term of  $y_c$  gives zero when substituted in the differential equation, so will that term of the proposed  $y_p$ . This means the proposed  $y_p$  must be modified. The following example shows us how.

#### EXAMPLE 10 $y_p$ and $y_c$ have similar terms

Solve the differential equation  $D^2y - 2Dy + y = x + e^x$ .

We find that the auxiliary equation and complementary solution are

$$m^2 - 2m + 1 = 0, \quad (m - 1)^2 = 0 \quad m_1 = 1 \quad m_2 = 1$$

$$y_c = e^x (c_1 + c_2x)$$

Based on the function  $b$  on the right side, the proposed form of  $y_p$  is

$$y_p = A + Bx + Ce^x \quad \text{proposed form}$$

We now note that the term  $Ce^x$  is similar to the term  $c_1e^x$  of  $y_c$ . Therefore, we must multiply the term  $Ce^x$  by the smallest power of  $x$  such that it is not similar to any term of  $y_c$ . If we multiply by  $x$ , the term becomes similar to  $c_2xe^x$ . Therefore, we must multiply  $Ce^x$  by  $x^2$  such that  $y_p$  is

$$y_p = A + Bx + Cx^2e^x \quad \text{correct modified form}$$

Using this form of  $y_p$ , we now complete the solution.

$$Dy_p = B + Cx^2e^x + 2Cxe^x \quad D^2y_p = Cx^2e^x + 2Cxe^x + 2Ce^x + 2Cxe^x = 2Ce^x + 4Cxe^x + Cx^2e^x$$

$$(2Ce^x + 4Cxe^x + Cx^2e^x) - 2(B + Cx^2e^x + 2Cxe^x) + (A + Bx + Cx^2e^x) = x + e^x$$

$$(A - 2B) + Bx + 2Ce^x = x + e^x$$

$$A - 2B = 0 \quad B = 1 \quad 2C = 1 \quad A = 2 \quad B = 1 \quad C = 1/2$$

$$y_p = 2 + x + \frac{1}{2}x^2e^x$$

$$y = e^x (c_1 + c_2x) + 2 + x + \frac{1}{2}x^2e^x \quad \blacksquare$$

## EXERCISES 31.9

In Exercises 1–4, make the given changes in the indicated examples of this section and then solve the given problems.

- In Example 3(a), add  $2x$  to the form  $b$  and then determine the form of  $y_p$ .
- In Example 5, change  $e^x$  to  $e^{2x}$  and then find the solution.
- In Example 6, on the right side, change  $x$  to  $\sin x$ , and then find the solution.
- In Example 8, change the right side to  $x^2 + xe^x$  and then find the proper form for  $y_p$ .

In Exercises 5–16, solve the given differential equations. The form of  $y_p$  is given.

- $D^2y - Dy - 2y = 4$  (Let  $y_p = A$ .)
- $D^2y - Dy - 6y = 4x$  (Let  $y_p = A + Bx$ .)
- $D^2y - y = 2 + x^2$  (Let  $y_p = A + Bx + Cx^2$ .)
- $D^2y + 4Dy + 3y = 2 + e^x$  (Let  $y_p = A + Be^x$ .)
- $y'' - 3y' = 2e^x + xe^x$  (Let  $y_p = Ae^x + Bxe^x$ .)
- $y'' + y' - 2y = 8 + 4x + 2xe^{2x}$   
(Let  $y_p = A + Bx + Ce^{2x} + Exe^{2x}$ .)
- $9D^2y - y = \sin x$  (Let  $y_p = A \sin x + B \cos x$ .)
- $D^2y + 4y = \sin x + 4$  (Let  $y_p = A + B \sin x + C \cos x$ .)
- $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 2x + x^2 + \sin 3x$   
(Let  $y_p = A + Bx + Cx^2 + E \sin 3x + F \cos 3x$ .)
- $D^2y - y = e^{-x}$  (Let  $y_p = Axe^{-x}$ .)
- $D^2y + 4y = -12 \sin 2x$  (Let  $y_p = A \sin 2x + B \cos 2x$ .)
- $y'' - 2y' + y = 3 + e^x$  (Let  $y_p = A + Bx^2e^x$ .)

In Exercises 17–32, solve the given differential equations.

- $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 30y = 10$
- $2\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 8x$
- $3\frac{d^2y}{dx^2} - \frac{dy}{dx} - 4y = 5e^{3x}$
- $\frac{d^2y}{dx^2} + 4y = 2 \sin 3x$
- $D^2y - 4y = \sin x + 2 \cos x$

- $6D^2y + Dy - y = e^x - e^{-x}$
- $D^2y + y = 4 + \sin 2x$
- $D^2y - Dy + y = x + \sin x$
- $D^2y + 5Dy + 4y = xe^x + 4$
- $3D^2y + Dy - 2y = 4 + 2x + e^x$
- $y''' - y' = \sin 2x$
- $D^4y - y = x$
- $D^2y + y = \cos x$
- $4y'' - 4y' + y = 4e^{x/2}$
- $D^2y + 2Dy = 8x + e^{-2x}$
- $D^3y - Dy = 4e^{-x} + 3e^{2x}$

In Exercises 33–36, find the particular solution of each differential equation for the given conditions.

- $D^2y - Dy - 6y = 5 - e^x$ ;  $Dy = 4$  and  $y = 2$  when  $x = 0$
- $3y'' - 10y' + 3y = xe^{-2x}$ ;  $y' = -\frac{9}{35}$  and  $y = -\frac{13}{35}$  when  $x = 0$
- $y'' + y = x + \sin 2x$ ;  $y' = 1$  and  $y = 0$  when  $x = \pi$
- $D^2y - 2Dy + y = xe^{2x} - e^{2x}$ ;  $Dy = 4$  and  $y = -2$  when  $x = 0$

In Exercises 37 and 38, solve the given problems.

- Solve the first-order equation  $Dy - y = x^2$  by the method of undetermined coefficients. For this equation, why is this method easier than the method of Section 31.4?
- Show that the equation  $(D^2 + 1)y = x^3$ , subject to the conditions that  $y = 0$  for  $x = 0$  and  $x = \pi$ , has no solution.

In Exercises 39–40, find a particular solution of the equation  $D^2y + 4y = f(x)$  for the given function. (One period is defined for each function.) See Example 9.

- $f(x) = \begin{cases} 1 & 0 \leq x < 2 \\ -1 & 2 \leq x < 4 \end{cases}$  and  $f(x)$  is extended to be odd.
- $f(x) = \begin{cases} 1 & 0 \leq x < 4 \\ 8 - x & 4 \leq x < 8 \end{cases}$  and  $f(x)$  is extended to be even.

## Answers to Practice Exercises

- $y_p = A + Bx + C \sin x + D \cos x$
- $y_p = -\frac{1}{2}e^{2x} + xe^{2x}$

## 31.10 Applications of Higher-Order Equations

Simple Harmonic Motion •  
Damped Simple Harmonic Motion •  
Electric Circuits • Deflection of Beams

We now show important applications of second-order differential equations to simple harmonic motion and simple electric circuits. Also, we will show an application of a fourth-order differential equation to the deflection of a beam.

## EXAMPLE 1 Simple harmonic motion

Simple harmonic motion may be defined as motion in a straight line for which the acceleration is proportional to the displacement and in the opposite direction. Examples of this type of motion are a weight on a spring, a simple pendulum, and an object bobbing in water. If  $x$  represents the displacement,  $d^2x/dt^2$  is the acceleration.

Using the definition of simple harmonic motion, we have

$$\frac{d^2x}{dt^2} = -k^2x$$

(We chose  $k^2$  for convenience of notation in the solution.) We write this equation in the form

$$D^2x + k^2x = 0 \quad \text{here, } D = d/dt$$

The roots of the auxiliary equation are  $kj$  and  $-kj$ , and the solution is

$$x = c_1 \sin kt + c_2 \cos kt$$

This solution indicates an oscillating motion, which is known to be the case. If, for example,  $k = 4$  and we know that  $x = 2$  and  $Dx = 0$  (which means the velocity is zero) for  $t = 0$ , we have

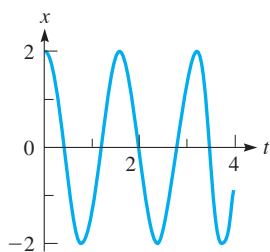


Fig. 31.12

$$\begin{aligned} Dx &= 4c_1 \cos 4t - 4c_2 \sin 4t \\ 2 &= c_1(0) + c_2(1) & x = 2 \text{ for } t = 0 \\ 0 &= 4c_1(1) - 4c_2(0) & Dx = 0 \text{ for } t = 0 \end{aligned}$$

which gives  $c_1 = 0$  and  $c_2 = 2$ . Therefore,

$$x = 2 \cos 4t$$

is the equation relating the displacement and time;  $Dx$  is the velocity and  $D^2x$  is the acceleration. See Fig. 31.12. ■

■ Newton's second law states that the net force acting on an object is equal to its mass times its acceleration. (This is one of Newton's best-known contributions to physics.)

### EXAMPLE 2 Damped simple harmonic motion

In practice, an object moving with simple harmonic motion will in time cease to move due to unavoidable frictional forces. A “freely” oscillating object has a retarding force that is approximately proportional to the velocity. The differential equation for this case is  $D^2x = -k^2x - bDx$ . This results from applying (from physics) Newton's second law of motion (see the margin note at the left). Again, using the operator  $D^2x = d^2x/dt^2$ , the term  $D^2x$  represents the acceleration of the object, the term  $-k^2x$  is a measure of the restoring force (of the spring, for example), and the term  $-bDx$  represents the retarding (damping) force. This equation can be written as

$$D^2x + bDx + k^2x = 0$$

The auxiliary equation is  $m^2 + bm + k^2 = 0$ , for which the roots are

$$m = \frac{-b \pm \sqrt{b^2 - 4k^2}}{2}$$

If  $k = 3$  and  $b = 4$ ,  $m = -2 \pm j\sqrt{5}$ , which means the solution is

$$x = e^{-2t} (c_1 \sin \sqrt{5}t + c_2 \cos \sqrt{5}t) \quad (1)$$

Here,  $4k^2 > b^2$ , and this case is called **underdamped**. In this case, the object oscillates as the amplitude becomes smaller.

If  $k = 2$  and  $b = 5$ ,  $m = -1, -4$ , which means the solution is

$$x = c_1 e^{-t} + c_2 e^{-4t} \quad (2)$$

Here,  $4k^2 < b^2$ , and the motion is called **overdamped**. Note that the motion is not oscillatory, since no sine or cosine terms appear. In this case, the object returns slowly to equilibrium without oscillating.

If  $k = 2$  and  $b = 4$ ,  $m = -2, -2$ , which means the solution is

$$x = e^{-2t} (c_1 + c_2 t) \quad (3)$$

Here,  $4k^2 = b^2$ , and the motion is called **critically damped**. Again the motion is not oscillatory. In this case, there is just enough damping to prevent any oscillations. The object returns to equilibrium in the minimum time.

See Fig. 31.13, in which Eqs. (1), (2), and (3) are represented in general. The actual values depend on  $c_1$  and  $c_2$ , which in turn depend on the conditions imposed on the motion. ■

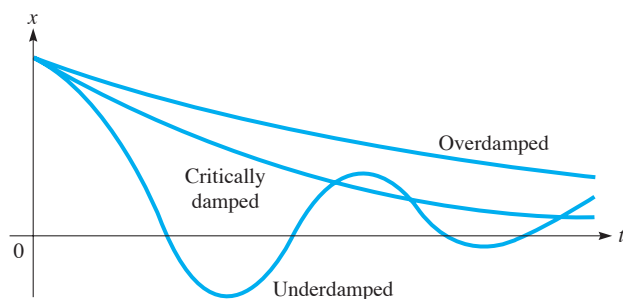


Fig. 31.13



**EXAMPLE 3 Underdamped harmonic motion**

In testing the characteristics of a particular type of spring, it is found that a weight of 4.90 N stretches the spring 0.490 m when the weight and spring are placed in a fluid that resists the motion with a force equal to twice the velocity. If the weight is brought to rest and then given a velocity of 12.0 m/s, find the equation of motion. See Fig. 31.14.

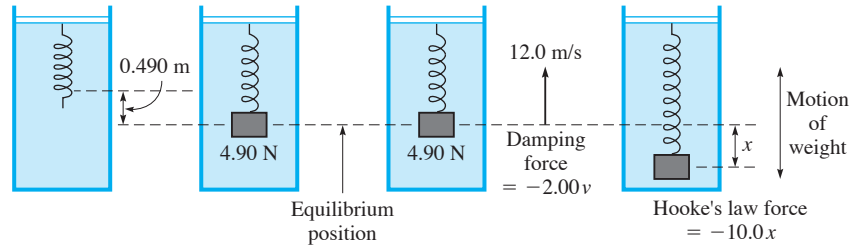


Fig. 31.14

In order to find the equation of motion, we use Newton's second law of motion (see Example 2). The weight (one force) at the end of the spring is offset by the equilibrium position force exerted by the spring, in accordance with Hooke's law (see Section 26.6). Therefore, the net force acting on the weight is the sum of the Hooke's law force due to the displacement from the equilibrium position and the resisting force. Using Newton's second law, we have

$$\begin{aligned} \text{mass} \times \text{acceleration} &= \text{resisting force} + \text{Hooke's law force} \\ mD^2x &= -2.00 Dx - kx \end{aligned}$$

The mass of an object is its weight divided by the acceleration due to gravity. The weight is 4.90 N, and the acceleration due to gravity is  $9.80 \text{ m/s}^2$ . Thus the mass  $m$  is

$$m = \frac{4.90 \text{ N}}{9.80 \text{ m/s}^2} = 0.500 \text{ kg}$$

where the kilogram is the unit of mass.

The constant  $k$  for the Hooke's law force is found from the fact that the spring stretches 0.490 m for a force of 4.90 N. Thus, using Hooke's law,

$$4.90 = k(0.490), \quad k = 10.0 \text{ N/m}$$

This means that the differential equation to be solved is

$$0.500D^2x + 2.00Dx + 10.0x = 0$$

or

$$1.00D^2x + 4.00Dx + 20.0x = 0$$

Solving this equation, we have

$$1.00m^2 + 4.00m + 20.0 = 0 \quad \text{auxiliary equation}$$

$$m = \frac{-4.00 \pm \sqrt{16.0 - 4(20.0)(1.00)}}{2.00}$$

$$= -2.00 \pm 4.00j \quad \text{complex roots}$$

$$x = e^{-2.00t} (c_1 \cos 4.00t + c_2 \sin 4.00t) \quad \text{general solution}$$

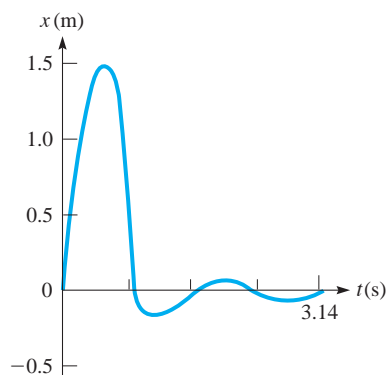


Fig. 31.15

Since the weight started from the equilibrium position with a velocity of 12.0 m/s, we know that  $x = 0$  and  $Dx = 12.0$  for  $t = 0$ . Thus,

$$0 = e^0(c_1 + 0c_2) \quad \text{or} \quad c_1 = 0 \quad x = 0 \text{ for } t = 0$$

Thus, since  $c_1 = 0$ , we have

$$\begin{aligned} x &= c_2 e^{-2.00t} \sin 4.00t \\ Dx &= c_2 e^{-2.00t} (\cos 4.00t) (4.00) + c_2 \sin 4.00t (e^{-2.00t}) (-2.00) \\ 12.0 &= c_2 e^0 (1) (4.00) + c_2 (0) (e^0) (-2.00) \quad Dx = 12.0 \text{ for } t = 0 \\ c_2 &= 3.00 \end{aligned}$$

This means that the equation of motion is

$$x = 3.00e^{-2.00t} \sin 4.00t$$

The motion is underdamped; the graph is shown in Fig. 31.15. ■

It is possible to have an additional force acting on a weight such as the one in Example 3. For example, a vibratory force may be applied to the support of the spring. In such a case, called **forced vibrations**, this additional external force is added to the other net force. This means that the added force  $F(t)$  becomes a nonzero function on the right side of the differential equation, and we must then solve a nonhomogeneous equation.

#### EXAMPLE 4 Electric circuits

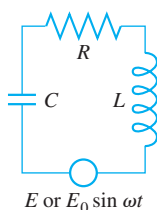


Fig. 31.16

The impressed voltage in the electric circuit shown in Fig. 31.16 equals the sum of the voltages across the components of the circuit. For this circuit with a resistance  $R$ , an inductance  $L$ , a capacitance  $C$ , and a voltage source  $E$ , we have

$$L \frac{d^2q}{dt^2} + R \frac{dq}{dt} + \frac{q}{C} = E \quad (31.20)$$

By definition,  $q$  represents the electric charge,  $dq/dt = i$  is the current, and  $d^2q/dt^2$  is the time rate of change of current. This equation may be written as

$$LD^2q + RDq + q/C = E$$

The auxiliary equation is  $Lm^2 + Rm + 1/C = 0$ . The roots are

$$\begin{aligned} m &= \frac{-R \pm \sqrt{R^2 - 4L/C}}{2L} \\ &= -\frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} \end{aligned}$$

If we let  $a = R/2L$  and  $\omega = \sqrt{1/LC - R^2/4L^2}$ , we have (assuming complex roots, which corresponds to realistic values of  $R$ ,  $L$ , and  $C$ )

$$q_c = e^{-at} (c_1 \sin \omega t + c_2 \cos \omega t)$$

This indicates an oscillating charge, or an alternating current. However, the exponential term usually is such that the current dies out rapidly unless there is a source of voltage in the circuit. ■

**LEARNING TIP**

To find the steady-state solution for Eq. (31.20) we need find only the particular solution.

If there is no source of voltage in the circuit of Example 4, we have a homogeneous differential equation to solve. If we have a constant voltage source, the particular solution is of the form  $q_p = A$ . If there is an alternating voltage source, the particular solution is of the form  $q_p = A \sin \omega_1 t + B \cos \omega_1 t$ , where  $\omega_1$  is the angular velocity of the source. After a very short time, the exponential factor in the complementary solution makes it negligible. For this reason, it is referred to as the **transient** term, and *the particular solution is the steady-state solution*.

It should be noted that the complementary solutions of the mechanical and electric cases are of identical form. There is also an equivalent mechanical case to that of an impressed sinusoidal voltage source in the electric case. This arises in the case of forced vibrations, when an external force affecting the vibrations is applied to the system. Thus, we may have transient and steady-state solutions to mechanical and other nonelectric situations.

**EXAMPLE 5 Electric circuit**

Find the steady-state solution for the current in a circuit containing the following elements:  $C = 400 \mu\text{F}$ ,  $L = 1.00 \text{ H}$ ,  $R = 10.0 \Omega$ , and a voltage source of  $500 \sin 100t$ . See Fig. 31.17.

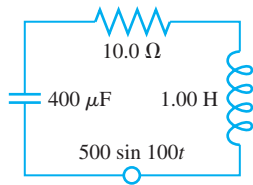


Fig. 31.17

This means the differential equation to be solved is

$$\frac{d^2q}{dt^2} + 10 \frac{dq}{dt} + \frac{10^4}{4}q = 500 \sin 100t$$

Since we wish to find the steady-state solution, we must find  $q_p$ , from which we may find  $i_p$  by finding a derivative. The solution now follows:

$$q_p = A \sin 100t + B \cos 100t \quad \text{particular solution form}$$

$$\frac{dq_p}{dt} = 100A \cos 100t - 100B \sin 100t$$

$$\frac{d^2q_p}{dt^2} = -10^4A \sin 100t - 10^4B \cos 100t$$

$$-10^4A \sin 100t - 10^4B \cos 100t + 10^3A \cos 100t - 10^3B \sin 100t \quad \text{substitute into differential equation}$$

$$+ \frac{10^4}{4}A \sin 100t + \frac{10^4}{4}B \cos 100t = 500 \sin 100t$$

$$(-0.75 \times 10^4A - 10^3B) \sin 100t + (-0.75 \times 10^4B + 10^3A) \cos 100t = 500 \sin 100t$$

$$-7.5 \times 10^3A - 10^3B = 500 \quad \text{equate coefficients of } \sin 100t$$

$$10^3A - 7.5 \times 10^3B = 0 \quad \text{equate coefficients of } \cos 100t$$

Solving these equations, we obtain

$$B = -8.73 \times 10^{-3} \quad \text{and} \quad A = -65.5 \times 10^{-3}$$

Therefore,

$$q_p = -65.5 \times 10^{-3} \sin 100t - 8.73 \times 10^{-3} \cos 100t$$

$$i_p = \frac{dq_p}{dt} = -6.55 \cos 100t + 0.87 \sin 100t$$

which is the required solution. (We assumed 3 significant digits for the data but did not use all of them in most equations of the solution.) ■

**Deflection of Beams**

The solutions to the second-order differential equations for the applications of simple harmonic motion and electric circuits generally include sines and cosines, because of the oscillatory nature of these applications. We now consider problems involving the deflections of beams, which involve fourth-order differential equations and algebraic functions.

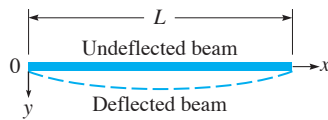


Fig. 31.18

In the study of the strength of materials and elasticity, it is shown that the deflection  $y$  of a beam of length  $L$  satisfies the differential equation  $EI d^4y/dx^4 = w(x)$ , where  $EI$  is a measure of the stiffness of the beam and  $w(x)$  is the weight distribution along the beam. See Fig. 31.18. Since this is a fourth-order equation, it is necessary to specify four conditions to obtain a solution. These conditions are determined by the way in which the ends, where  $x = 0$  and where  $x = L$ , are held. For an end held in the specified manner, these conditions are *clamped*:  $y = 0$  and  $y' = 0$ ; *hinged*:  $y = 0$  and  $y'' = 0$ ; *free*:  $y'' = 0$  and  $y''' = 0$  ( $y' = 0$  indicates no change in alignment;  $y'' = 0$  indicates no curvature;  $y''' = 0$  indicates no shearing force). Since the conditions are given for specific positions, this kind of problem is called a *boundary value problem*. Consider the following example.

### EXAMPLE 6 Deflection of a beam

A uniform beam of length  $L$  is hinged at both ends and has a constant load distribution of  $w$  due to its own weight. Find the deflection  $y$  of the beam in terms of the distance  $x$  from one end of the beam.

Using the differential equation given above, we have  $EI d^4y/dx^4 = w$ . For convenience in the solution, let  $k = w/EI$ . Thus, the solution is as follows:

$$\begin{aligned} D^4y &= k \\ m^4 &= 0 \quad m_1 = m_2 = m_3 = m_4 = 0 \end{aligned}$$

Since the four roots of the auxiliary equation are equal,

$$y_c = c_1 + c_2x + c_3x^2 + c_4x^3$$

The form of  $y_p$  indicates that we must multiply the *proposed*  $y_p = A$  by  $x^4$  so that it is not similar to any of the terms of  $y_c$ . This gives us  $y_p = x^4$ . Therefore, we now find the general solution as follows:

$$\begin{aligned} y_p &= Ax^4 & Dy_p &= 4Ax^3 & D^2y_p &= 12Ax^2 & D^3y_p &= 24Ax & D^4y_p &= 24A \\ 24A &= k, & A &= k/24 \end{aligned}$$

$$y = c_1 + c_2x + c_3x^2 + c_4x^3 + \frac{k}{24}x^4 \quad \text{general solution}$$

From the discussion about beams before this example, we now find four boundary conditions in order to evaluate the four constants in  $y_c$ . For a beam hinged at both ends, we know that  $y = 0$  and  $D^2y = 0$  for both  $x = 0$  and  $x = L$ . Therefore, we now find two derivatives, use these conditions, and thereby find  $y_c$ .

$$Dy = c_2 + 2c_3x + 3c_4x^2 + \frac{k}{6}x^3 \quad \text{find derivatives}$$

$$D^2y = 2c_3 + 6c_4x + \frac{k}{2}x^2$$

$$\text{At } x = 0, y = 0: \quad c_1 = 0; \quad \text{At } x = 0, D^2y = 0: \quad c_3 = 0 \quad \text{use conditions to evaluate constants}$$

$$\text{At } x = L, D^2y = 0: \quad 0 = 6c_4L + \frac{kL^2}{2}; \quad c_4 = -\frac{kL}{12}$$

$$\text{At } x = L, y = 0: \quad 0 = c_2L + c_4L^3 + \frac{kL^4}{24}$$

$$0 = c_2L + \left(-\frac{kL}{12}\right)L^3 + \frac{kL^4}{24}; \quad c_2 = \frac{kL^3}{24}$$

Therefore, substituting the values of the four constants in the general solution above, the particular solution that satisfies these conditions is

$$\begin{aligned} y &= \frac{kL^3}{24}x - \frac{kL}{12}x^3 + \frac{kx^4}{24} = \frac{k}{24}(L^3x - 2Lx^3 + x^4) \\ &= \frac{w}{24EI}(L^3x - 2Lx^3 + x^4) \quad k = w/EI \end{aligned}$$

EXERCISES 31.10

In Exercises 1 and 2, make the given changes in the indicated examples of this section and then solve the resulting problems.

1. In Example 1, change the conditions that  $x = 2$  and  $Dx = 0$  for  $t = 0$  to  $x = 0$  and  $Dx = 2$  for  $t = 0$ .
2. In Example 5, change the voltage source to  $500 \cos 100t$ .

In Exercises 3–27, solve the given problems.

3. An object moves with simple harmonic motion according to  $D^2x + 0.2Dx + 100x = 0$ ,  $D = d/dt$ . Find the displacement as a function of time, subject to the conditions  $x = 4$  and  $Dx = 0$  when  $t = 0$ .
4. What must be the value of  $b$  so that the motion of an object given by the equation  $D^2x + bDx + 100x = 0$  is critically damped?
5. When the angular displacement  $\theta$  of a pendulum is small (less than about  $6^\circ$ ), the pendulum moves with simple harmonic motion closely approximated by  $D^2\theta + \frac{g}{l}\theta = 0$ . Here,  $D = d/dt$ ,  $g$  is the acceleration due to gravity, and  $l$  is the length of the pendulum. Find  $\theta$  as a function of time (in s) if  $g = 9.8 \text{ m/s}^2$ ,  $l = 1.0 \text{ m}$ ,  $\theta = 0.1$ , and  $D\theta = 0$  when  $t = 0$ . Sketch the curve.
6. A block of wood floating in oil is depressed from its equilibrium position such that its equation of motion is  $D^2y + 8Dy + 3y = 0$ , where  $y$  is the displacement (in cm) and  $D = d/dt$ . Find its displacement after 12 s if  $y = 6.0 \text{ cm}$  and  $Dy = 0$  when  $t = 0$ .
7. A car suspension is depressed from its equilibrium position such that its equation of motion is  $D^2y + bDy + 25y = 0$ , where  $y$  is the displacement and  $D = d/dt$ . What must be the value of  $b$  if the motion is critically damped?
8. In an electric circuit, if a capacitor discharges through a negligible resistance, the current  $i$  is related to the time  $t$  by the equation  $d^2i/dt^2 = -a^2i$ , where  $a$  is a constant. Find the frequency of the current if  $a = 1000$ .
9. For an elastic band that is stretched vertically, with one end fixed and a mass  $m$  at the other end, the displacement  $s$  of the mass is given by  $m \frac{d^2s}{dt^2} = -\frac{mg}{e}(s - L)$ , where  $L$  is the natural length of the band and  $e$  is the elongation due to the weight  $mg$ . Find  $s$  if  $s = s_0$  and  $ds/dt = 0$  when  $t = 0$ .
10. A mass of 0.820 kg stretches a given spring by 0.250 m. The mass is pulled down 0.150 m below the equilibrium position and released. Find the equation of motion of the mass if there is no damping.
11. A 4.00-N weight stretches a certain spring 5.00 cm. With this weight attached, the spring is pulled 10.0 cm longer than its equilibrium length and released. Find the equation of the resulting motion, assuming no damping.
12. Find the solution for the spring of Exercise 11 if a damping force numerically equal to the velocity is present.
13. Find the solution for the spring of Exercise 11 if no damping is present but an external force of  $4 \sin 2t$  is acting on the spring.
14. Find the solution for the spring of Exercise 11 if the damping force of Exercise 12 and the impressed force of Exercise 13 are both acting.

15. Find the equation relating the charge and the time in an electric circuit with the following elements:  $L = 0.200 \text{ H}$ ,  $R = 8.00 \Omega$ ,  $C = 1.00 \mu\text{F}$ , and  $E = 0$ . In this circuit,  $q = 0$  and  $i = 0.500 \text{ A}$  when  $t = 0$ .
16. For a given electric circuit,  $L = 2 \text{ mH}$ ,  $R = 0$ ,  $C = 50 \text{ nF}$ , and  $E = 0$ . Find the equation relating the charge and the time if  $q = 10^5 \text{ C}$  and  $i = 0$  when  $t = 0$ .
17. For a given circuit,  $L = 0.100 \text{ H}$ ,  $R = 0$ ,  $C = 100 \mu\text{F}$ , and  $E = 100 \text{ V}$ . Find the equation relating the charge and the time if  $q = 0$  and  $i = 0$  when  $t = 0$ .
18. Find the relation between the current and the time for the circuit of Exercise 17.
19. For a radio tuning circuit,  $L = 0.500 \text{ H}$ ,  $R = 10.0 \Omega$ ,  $C = 200 \mu\text{F}$ , and  $E = 120 \sin 120\pi t$ . Find the equation relating the charge and time.
20. Find the steady-state current for the circuit of Exercise 19.
21. In a given electric circuit  $L = 8.00 \text{ mH}$ ,  $R = 0$ ,  $C = 0.500 \mu\text{F}$ , and  $E = 20.0e^{-200t} \text{ mV}$ . Find the relation between the current and the time if  $q = 0$  and  $i = 0$  for  $t = 0$ .
22. Find the current as a function of time for a circuit in which  $L = 0.400 \text{ H}$ ,  $R = 60.0 \Omega$ ,  $C = 0.200 \mu\text{F}$ , and  $E = 0.800e^{-100t} \text{ V}$ , if  $q = 0$  and  $i = 5.00 \text{ mA}$  for  $t = 0$ .
23. Find the steady-state current for a circuit with  $L = 1.00 \text{ H}$ ,  $R = 5.00 \Omega$ ,  $C = 150 \mu\text{F}$ , and  $E = 120 \sin 100t \text{ V}$ .
24. Find the steady-state solution for the current in an electric circuit containing the following elements:  $C = 20.0 \mu\text{F}$ ,  $L = 2.00 \text{ H}$ ,  $R = 20.0 \Omega$ , and  $E = 200 \sin 10t \text{ V}$ .
25. A cantilever beam is clamped at the end  $x = 0$  and is free at the end  $x = L$ . Find the equation for the deflection  $y$  of the beam in terms of the distance  $x$  from one end if it has a constant load distribution of  $w$  due to its own weight. See Fig. 31.19.

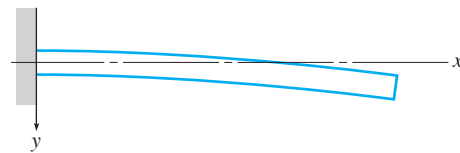


Fig. 31.19

26. A beam 10 m in length is hinged at both ends and has a variable load distribution of  $w = kEIx$ , where  $k = 7.2 \times 10^{-4}/\text{m}$  and  $x$  is the distance from one end. Find the equation of the deflection  $y$  in terms of  $x$ .
27. A mass of  $\frac{1}{16} \text{ kg}$  stretches a spring for which  $k = 4 \text{ N/m}$ . A periodic external force  $f(x)$  is applied to the spring, with one period defined by  $f(t) = \begin{cases} \pi t & 0 \leq t < 1 \\ \pi t - 2\pi & 1 \leq t < 2 \end{cases}$ , and  $f(t)$  is extended to be odd. Express the displacement  $y$  of the object as a function of time. (See Example 9 in Section 31.9.)

## 31.11 Laplace Transforms

Definition of Laplace Transform • Improper Integral • Table of Laplace Transforms • Linearity Property • Inverse Transforms

■ The Laplace transform is named for the French mathematician and astronomer Pierre Laplace (1749–1827).

**Laplace transforms** provide an algebraic method of obtaining a *particular* solution of a differential equation from stated initial conditions. Since this is frequently what is wanted, Laplace transforms are often used in engineering and electronics. The treatment in this text is intended only as an introduction to Laplace transforms.

The Laplace transform of a function  $f(t)$  is defined as the function  $F(s)$  as

$$F(s) = \int_0^{\infty} e^{-st} f(t) dt \quad (31.21)$$

By writing the transform as  $F(s)$ , we show that the result of integrating and evaluating is a function of  $s$ . To denote that we are dealing with “the Laplace transform of the function  $f(t)$ ,” the notation  $\mathcal{L}(f)$  is used. Thus,

$$F(s) = \mathcal{L}(f) = \int_0^{\infty} e^{-st} f(t) dt \quad (31.22)$$

We shall see that both notations are quite useful.

In Eqs. (31.21) and (31.22), we note that the upper limit is  $\infty$ , which means it is unbounded. This integral is one type of what is known as an **improper integral**. In evaluating at the upper limit, it is necessary to find the limit of the resulting function as the upper limit approaches infinity. This may be shown as

$$\lim_{c \rightarrow \infty} \int_0^c e^{-st} f(t) dt$$

where we substitute  $c$  for  $t$  in the resulting function and determine the limit as  $c \rightarrow \infty$  to determine the result for the upper limit. This also means that the Laplace transform,  $F(s)$ , is defined only for those values of  $s$  for which the limit is defined.

### EXAMPLE 1 Finding a transform from the definition

Find the Laplace transform of the function  $f(t) = t, t > 0$ .

By the definition of the Laplace transform.

$$\mathcal{L}(f) = \mathcal{L}(t) = \int_0^{\infty} e^{-st} t dt$$

This may be integrated by parts or by formula 44 in Appendix B. Using the formula, we have

$$\begin{aligned} \mathcal{L}(t) &= \int_0^{\infty} t e^{-st} dt = \lim_{c \rightarrow \infty} \int_0^c t e^{-st} dt = \lim_{c \rightarrow \infty} \left. \frac{e^{-st}(-st - 1)}{s^2} \right|_0^c \\ &= \lim_{c \rightarrow \infty} \left[ \frac{e^{-sc}(-sc - 1)}{s^2} \right] + \frac{1}{s^2} \end{aligned}$$

For  $s > 0$ , we can rewrite the quotient so as to have an  $\infty/\infty$  indeterminate form and use L'Hospital's rule (see Section 27.7) to evaluate the limit. We have

$$\begin{aligned} \mathcal{L}(t) &= -\lim_{c \rightarrow \infty} \left[ \frac{sc + 1}{s^2 e^{sc}} \right] + \frac{1}{s^2} && \text{rewriting to have } \infty/\infty \\ &= -\lim_{c \rightarrow \infty} \left[ \frac{s}{s^3 e^{sc}} \right] + \frac{1}{s^2} && \text{differentiate numerator and denominator with respect to } c \\ &= 0 + \frac{1}{s^2} = \frac{1}{s^2} && \text{defined for } s > 0 \quad \blacksquare \end{aligned}$$

■ For reference, if  $\lim_{c \rightarrow \infty} u(c) \lim_{c \rightarrow \infty} v(c) = \infty$ , then by L'Hospital's rules

$$\lim_{c \rightarrow \infty} \frac{u(c)}{v(c)} = \lim_{c \rightarrow \infty} \frac{u'(c)}{v'(c)}$$

**EXAMPLE 2** Finding a transform from the definition

Find the Laplace transform of the function  $f(t) = \cos at$ .

By definition,

$$\mathcal{L}(f) = \mathcal{L}(\cos at) = \int_0^{\infty} e^{-st} \cos at \, dt$$

Using formula 50 in Appendix B, we have

$$\begin{aligned} \mathcal{L}(\cos at) &= \int_0^{\infty} e^{-st} \cos at \, dt = \lim_{c \rightarrow \infty} \int_0^c e^{-st} \cos at \, dt \\ &= \lim_{c \rightarrow \infty} \left. \frac{e^{-st}(-s \cos at + a \sin at)}{s^2 + a^2} \right|_0^c \\ &= \lim_{c \rightarrow \infty} \frac{e^{-sc}(-s \cos ac + a \sin ac)}{s^2 + a^2} - \left( -\frac{s}{s^2 + a^2} \right) \\ &= 0 + \frac{s}{s^2 + a^2} = \frac{s}{s^2 + a^2} \quad (s > 0) \end{aligned}$$

Therefore, the Laplace transform of the function  $\cos at$  is

$$\mathcal{L}(\cos at) = \frac{s}{s^2 + a^2}$$

In both examples, the resulting transform was an algebraic function of  $s$ . ■

We now present a short table of Laplace transforms. They are sufficient for our work in this chapter. More complete tables are available in many references.

**Table of Laplace Transforms**

	$f(t) = \mathcal{L}^{-1}(F)$	$\mathcal{L}(f) = F(s)$		$f(t) = \mathcal{L}^{-1}(F)$	$\mathcal{L}(f) = F(s)$
1.	1	$\frac{1}{s}$	11.	$te^{-at}$	$\frac{1}{(s+a)^2}$
2.	$\frac{t^{n-1}}{(n-1)!}$	$\frac{1}{s^n} (n = 1, 2, 3, \dots)$	12.	$t^{n-1}e^{-at}$	$\frac{(n-1)!}{(s+a)^n}$
3.	$e^{-at}$	$\frac{1}{s+a}$	13.	$e^{-at}(1-at)$	$\frac{s}{(s+a)^2}$
4.	$1 - e^{-at}$	$\frac{a}{s(s+a)}$	14.	$[(b-a)t + 1]e^{-at}$	$\frac{s+b}{(s+a)^2}$
5.	$\cos at$	$\frac{s}{s^2 + a^2}$	15.	$\sin at - at \cos at$	$\frac{2a^3}{(s^2 + a^2)^2}$
6.	$\sin at$	$\frac{a}{s^2 + a^2}$	16.	$t \sin at$	$\frac{2as}{(s^2 + a^2)^2}$
7.	$1 - \cos at$	$\frac{a^2}{s(s^2 + a^2)}$	17.	$\sin at + at \cos at$	$\frac{2as^2}{(s^2 + a^2)^2}$
8.	$at - \sin at$	$\frac{a^3}{s^2(s^2 + a^2)}$	18.	$t \cos at$	$\frac{s^2 - a^2}{(s^2 + a^2)^2}$
9.	$e^{-at} - e^{-bt}$	$\frac{b-a}{(s+a)(s+b)}$	19.	$e^{-at} \sin bt$	$\frac{b}{(s+a)^2 + b^2}$
10.	$ae^{-at} - be^{-bt}$	$\frac{s(a-b)}{(s+a)(s+b)}$	20.	$e^{-at} \cos bt$	$\frac{s+a}{(s+a)^2 + b^2}$



An important property of transforms is the **linearity property**,

$$\mathcal{L}[af + bg] = a\mathcal{L}(f) + b\mathcal{L}(g) \quad (31.23)$$

We state this property here since it determines that the transform of a sum of functions is the sum of the transforms. This is of definite importance when dealing with a sum of functions. This property is a direct result of the definition of the Laplace transform.

Another Laplace transform important to the solution of a differential equation is the transform of the derivative of a function. Let us first find the Laplace transform of the first derivative of a function.

By definition,

$$\mathcal{L}(f') = \int_0^{\infty} e^{-st} f'(t) dt$$

To integrate by parts, let  $u = e^{-st}$  and  $dv = f'(t) dt$ , so  $du = -se^{-st} dt$  and  $v = f(t)$  (the integral of the derivative of a function is the function). Therefore,

$$\begin{aligned} \mathcal{L}(f') &= \lim_{c \rightarrow \infty} e^{-st} f(t) \Big|_0^c + s \int_0^{\infty} e^{-st} f(t) dt \\ &= 0 - f(0) + s\mathcal{L}(f) \end{aligned}$$

It is noted that the integral in the second term on the right is the Laplace transform of  $f(t)$  by definition. Therefore, *the Laplace transform of the first derivative of a function is*

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0) \quad (31.24)$$

Applying the same analysis, we may find *the Laplace transform of the second derivative of a function. It is*

$$\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0) \quad (31.25)$$

Here, it is necessary to integrate by parts twice to derive the result. The transforms of higher derivatives are found in a similar manner.

Eqs. (31.24) and (31.25) allow us to express the transform of each derivative in terms of  $s$  and the transform itself. This is illustrated in the following example.

### EXAMPLE 3 Linearity property—transforms of derivatives

Given that  $f(0) = 0$  and  $f'(0) = 1$ , express the transform of  $f''(t) - 2f'(t)$  in terms of  $s$  and the transform of  $f(t)$ .

By using the linearity property and the transforms of the derivatives, we have

$$\begin{aligned} \mathcal{L}[f'' - 2f'] &= \mathcal{L}(f'') - 2\mathcal{L}(f') && \text{using Eq. (31.23)} \\ &= [s^2\mathcal{L}(f) - sf(0) - f'(0)] - 2[s\mathcal{L}(f) - f(0)] && \text{using Eqs. (31.25) and (31.24)} \\ &= [s^2\mathcal{L}(f) - s \cdot 0 - 1] - 2[s\mathcal{L}(f) - 0] && \text{substitute given values} \\ &= (s^2 - 2s)\mathcal{L}(f) - 1 \end{aligned}$$

#### Practice Exercise

- Given that  $f(0) = 1$  and  $f'(0) = 0$ , express the transform of  $f''(t) - f'(t)$  in terms of  $s$  and the transform of  $f(t)$ .

### INVERSE TRANSFORMS

If the Laplace transform of a function is known, it is then possible to find the function by finding the **inverse transform**,

$$\mathcal{L}^{-1}(F) = f(t) \quad (31.26)$$

where  $\mathcal{L}^{-1}$  denotes the inverse transform.

#### EXAMPLE 4 Inverse transform from the table

If  $F(s) = \frac{s}{s^2 + a^2}$ , from Transform 5 of the table, we see that

$$\begin{aligned} \mathcal{L}^{-1}(F) &= \mathcal{L}^{-1}\left(\frac{s}{s^2 + a^2}\right) = \cos at \\ f(t) &= \cos at \end{aligned}$$

#### EXAMPLE 5 Inverse transform from the table

If  $(s^2 - 2s)\mathcal{L}(f) - 1 = 0$ , then

$$\mathcal{L}(f) = \frac{1}{s^2 - 2s} \quad \text{or} \quad F(s) = \frac{1}{s(s - 2)}$$

Therefore, we have

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}(F) = \mathcal{L}^{-1}\left[\frac{1}{s(s - 2)}\right] && \text{inverse transform} \\ &= -\frac{1}{2}\mathcal{L}^{-1}\left[\frac{-2}{s(s - 2)}\right] && \text{fit form of Transform 4} \\ &= -\frac{1}{2}(1 - e^{2t}) && \text{use Transform 4} \end{aligned}$$

#### Practice Exercise

2. Find  $f(t)$  if  $F(s) = \frac{6}{s^2 + 9}$ .

The introduction of the factor  $-2$  in Example 5 illustrates that it often takes some algebra steps to get  $F(s)$  to match the proper form in the table. Another algebraic step that may be useful is *completing the square*. For a review of this algebraic method, see Section 7.2 (in particular, see Example 4 on page 227). The following example illustrates its use in finding an inverse transform.

#### EXAMPLE 6 Inverse transform by completing the square

If  $F(s) = \frac{s + 5}{s^2 + 6s + 10}$ , then

$$\mathcal{L}^{-1}(F) = \mathcal{L}^{-1}\left[\frac{s + 5}{s^2 + 6s + 10}\right]$$

It appears that this function does not fit any of the forms given. However,

$$s^2 + 6s + 10 = (s^2 + 6s + 9) + 1 = (s + 3)^2 + 1$$

By writing  $F(s)$  as

$$F(s) = \frac{(s + 3) + 2}{(s + 3)^2 + 1} = \frac{s + 3}{(s + 3)^2 + 1} + \frac{2}{(s + 3)^2 + 1}$$

we can find the inverse of each term. Therefore,

$$\begin{aligned} \mathcal{L}^{-1}(F) &= e^{-3t} \cos t + 2e^{-3t} \sin t && \text{using Transforms 20 and 19} \\ f(t) &= e^{-3t}(\cos t + 2 \sin t) \end{aligned}$$

The following example shows how *partial fractions* can be used to find the inverse transform of  $F(s)$ . For a review of the method of expressing a given algebraic fraction in terms of partial fractions, refer to Examples 2–4 on pages 880 and 881 and Examples 1–4 on pages 883–887.

### EXAMPLE 7 Inverse transform by partial fractions

$$\text{If } F(s) = \frac{5s^2 - 17s + 32}{s^3 - 8s^2 + 16s}, \text{ then}$$

$$\mathcal{L}^{-1}(F) = \mathcal{L}^{-1}\left[\frac{5s^2 - 17s + 32}{s^3 - 8s^2 + 16s}\right]$$

To fit forms in the table, we will now use partial fractions.

$$\begin{aligned} \frac{5s^2 - 17s + 32}{s^3 - 8s^2 + 16s} &= \frac{5s^2 - 17s + 32}{s(s-4)^2} = \frac{A}{s} + \frac{B}{s-4} + \frac{C}{(s-4)^2} && \text{factor of } s, \\ &&& \text{repeated factor } s-4 \\ 5s^2 - 17s + 32 &= A(s-4)^2 + Bs(s-4) + Cs && \text{multiply each side by } s(s-4)^2 \\ s=0: \quad 32 &= 16A, \quad A=2 \\ s=4: \quad 5(4^2) - 17(4) + 32 &= 4C, \quad C=11 \\ s^2 \text{ terms:} \quad 5 &= A+B, \quad 5=2+B, \quad B=3 \\ \mathcal{L}^{-1}(F) &= \mathcal{L}^{-1}\left[\frac{2}{s} + \frac{3}{s-4} + \frac{11}{(s-4)^2}\right] && \text{substitute in } F(s) \\ \mathcal{L}^{-1}(F) &= f(t) = 2 + 3e^{4t} + 11te^{4t} && \text{using Transforms 1, 3, and 11} \quad \blacksquare \end{aligned}$$

## EXERCISES 31.11

In Exercises 1–4, make the given changes in the indicated examples of this section, and then solve the resulting problems.

- In Example 1, change the function  $f(t)$ . Let  $f(t) = 1$ .
- In Example 2, change the function  $f(t)$ . Let  $f(t) = \sin at$ .
- In Example 3, interchange the values of  $f(0)$  and  $f'(0)$ .
- In Example 4, in the function  $F(s)$ , change the numerator to  $a$ .

In Exercises 5–12, find the transforms of the given functions by use of the table.

- $f(t) = e^{3t}$
- $f(t) = 1 - \cos 2t$
- $f(t) = 5t^3e^{-2t}$
- $f(t) = 8e^{-3t} \sin 4t$
- $f(t) = \cos 2t - \sin 2t$
- $f(t) = 2t \sin 3t + e^{-3t} \cos t$
- $f(t) = 3 + 2t \cos 3t$
- $f(t) = t^3 - 3te^{-t}$

In Exercises 13–16, express the transforms of the given expressions in terms of  $s$  and  $\mathcal{L}(f)$ .

- $f'' + f', f(0) = 0, f'(0) = 0$
- $f'' - 3f', f(0) = 2, f'(0) = -1$
- $2f'' - f' + f, f(0) = 1, f'(0) = 0$
- $f'' - 3f' + 2f, f(0) = -1, f'(0) = 2$

In Exercises 17–28, find the inverse transforms of the given functions of  $s$ .

- $F(s) = \frac{2}{s^3}$
- $F(s) = \frac{6}{s^2 + 4}$

- $F(s) = \frac{15}{2s + 6}$
- $F(s) = \frac{1}{s^3 + 3s^2 + 3s + 1}$
- $F(s) = \frac{s + 2}{(s^2 + 9)^2}$
- $F(s) = \frac{4s^2 - 8}{(s + 1)(s - 2)(s - 3)}$
- $F(s) = \frac{3s + 1}{(s - 1)(s^2 + 1)}$
- $F(s) = \frac{2s + 3}{s^2 - 2s + 5}$
- $F(s) = \frac{3s^4 + 3s^3 + 6s^2 + s + 1}{s^5 + s^3}$  (Explain your method of solution.)

In Exercises 29 and 30, find the indicated Laplace transforms.

- For the Laplace transform  $F(s)$  of the function  $f(t)$ , it can be shown that  $\mathcal{L}\{tf(t)\} = -\frac{d}{ds}F(s)$ . Verify this relationship by deriving Transform 11 from Transform 3.
- Using the equation given in Exercise 29, derive the transform for  $t^2 \cos at$  from Transform 18.

### Answers to Practice Exercises

- $(s^2 - s)\mathcal{L}(f) - s + 1$
- $2 \sin 3t$

## 31.12 Solving Differential Equations by Laplace Transforms

Solutions are Particular Solutions • Change Differential Equation into Algebraic Form

We will now show how certain differential equations can be solved by using Laplace transforms. *It must be remembered that these solutions are the **particular** solutions of the equations subject to the given conditions.* The necessary operations were developed in the preceding section. The following examples illustrate the method.

### EXAMPLE 1 Solution of a first-order equation

Using Laplace transforms, solve the differential equation  $2y' - y = 0$ , if  $y(0) = 1$ . (Note that we are using  $y$  to denote the function.)

Taking transforms of each term in the equation, we have

$$\mathcal{L}(2y') - \mathcal{L}(y) = \mathcal{L}(0)$$

$$2\mathcal{L}(y') - \mathcal{L}(y) = 0$$

$\mathcal{L}(0) = 0$  by direct use of the definition of the transform. Now, using Eq. (31.24),  $\mathcal{L}(y') = s\mathcal{L}(y) - y(0)$ , we have

$$2[s\mathcal{L}(y) - 1] - \mathcal{L}(y) = 0 \quad y(0) = 1$$

Solving for  $\mathcal{L}(y)$ , we obtain

$$2s\mathcal{L}(y) - \mathcal{L}(y) = 2$$

$$\mathcal{L}(y) = \frac{2}{2s - 1} = \frac{1}{s - \frac{1}{2}}$$

Finding the inverse transform, we have

$$y = e^{t/2} \quad \text{using Transform 3}$$

You should check this solution with that obtained by methods developed earlier. ■

### EXAMPLE 2 Solution of a second-order equation

Using Laplace transforms, solve the differential equation  $y'' + 2y' + 2y = 0$ , if  $y(0) = 0$  and  $y'(0) = 1$ .

Using the same steps as outlined in Example 1, we have

$$\mathcal{L}(y'') + 2\mathcal{L}(y') + 2\mathcal{L}(y) = 0 \quad \text{take transforms}$$

$$[s^2\mathcal{L}(y) - sy(0) - y'(0)] + 2[s\mathcal{L}(y) - y(0)] + 2\mathcal{L}(y) = 0 \quad \text{using Eqs. (31.25) and (31.24)}$$

$$[s^2\mathcal{L}(y) - s(0) - 1] + 2[s\mathcal{L}(y) - 0] + 2\mathcal{L}(y) = 0 \quad \text{substitute given values}$$

$$s^2\mathcal{L}(y) - 1 + 2s\mathcal{L}(y) + 2\mathcal{L}(y) = 0$$

$$(s^2 + 2s + 2)\mathcal{L}(y) = 1 \quad \text{solve for } \mathcal{L}(y)$$

$$\mathcal{L}(y) = \frac{1}{s^2 + 2s + 2} = \frac{1}{(s + 1)^2 + 1} \quad \text{take inverse transform}$$

$$y = e^{-t} \sin t \quad \text{using Transform 19} \quad \blacksquare$$

#### LEARNING TIP

It should be noted that the solution in Example 1 was essentially an algebraic one. This points out the power and usefulness of Laplace transforms. **We are able to change a differential equation into an algebraic form**, which can in turn be translated into the solution of the differential equation. Thus, we can solve a differential equation by using algebra and specific algebraic forms.

**EXAMPLE 3** Solution of a second-order equation

Solve the differential equation  $y'' + y = \cos t$ , if  $y(0) = 1$  and  $y'(0) = 2$ .

$$\mathcal{L}(y'') + \mathcal{L}(y) = \mathcal{L}(\cos t) \quad \text{take transforms}$$

$$[s^2\mathcal{L}(y) - s(1) - 2] + \mathcal{L}(y) = \frac{s}{s^2 + 1} \quad \text{using Eq. (31.25) and Transform 5}$$

$$(s^2 + 1)\mathcal{L}(y) = \frac{s}{s^2 + 1} + s + 2$$

$$\mathcal{L}(y) = \frac{s}{(s^2 + 1)^2} + \frac{s}{s^2 + 1} + \frac{2}{s^2 + 1} \quad \text{do not combine the fractions}$$

$$y = \frac{t}{2} \sin t + \cos t + 2 \sin t \quad \text{using Transforms 16, 5, and 6}$$

**EXAMPLE 4** Application—simple harmonic motion

A spring is stretched 0.31 m by a weight of 4.9 N (mass of 0.5 kg). The medium resists the motion with a force of  $4v$ , where  $v$  is the velocity of the motion. The differential equation describing the displacement  $y$  of the weight is

$$\frac{1}{2} \frac{d^2y}{dt^2} + 4 \frac{dy}{dt} + 16y = 0 \quad \text{see Example 3, page 984}$$

Find  $y$  as a function of time  $t$ , if  $y(0) = 1$  and  $dy/dx = 0$  for  $t = 0$ .

Clearing fractions and denoting derivatives by  $y''$  and  $y'$ , we have the following differential equation and solution.

$$y'' + 8y' + 32y = 0$$

$$\mathcal{L}(y'') + 8\mathcal{L}(y') + 32\mathcal{L}(y) = 0 \quad \text{take transforms}$$

$$[s^2\mathcal{L}(y) - s \cdot 1 - 0] + 8[s\mathcal{L}(y) - 1] + 32\mathcal{L}(y) = 0 \quad \text{substitute given values}$$

$$(s^2 + 8s + 32)\mathcal{L}(y) = s + 8 \quad \text{solve for } \mathcal{L}(y)$$

$$\mathcal{L}(y) = \frac{s + 8}{(s + 4)^2 + 4^2} = \frac{s + 4}{(s + 4)^2 + 4^2} + \frac{4}{(s + 4)^2 + 4^2} \quad \text{fit transform forms}$$

$$y = e^{-4t} \cos 4t + e^{-4t} \sin 4t = e^{-4t}(\cos 4t + \sin 4t) \quad \text{take inverse transforms}$$

The graph of this solution is shown in Fig. 31.20.

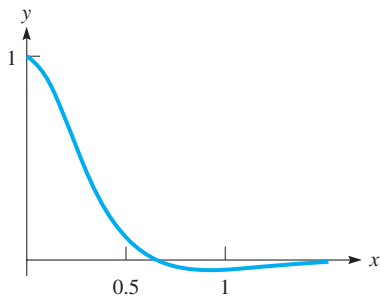


Fig. 31.20

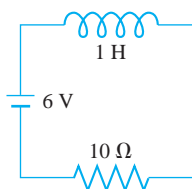


Fig. 31.21

**EXAMPLE 5** Application—electric current

The initial current in the circuit shown in Fig. 31.21 is zero. Find the current as a function of the time  $t$ .

Setting up the differential equation, and then solving it, we have

$$\frac{di}{dt} + 10i = 6 \quad \text{using Eq. (31.20)}$$

$$\mathcal{L}\left(\frac{di}{dt}\right) + 10\mathcal{L}(i) = \mathcal{L}(6) \quad \text{take transforms}$$

$$[s\mathcal{L}(i) - 0] + 10\mathcal{L}(i) = \frac{6}{s} \quad \text{substitute given values and find transform on right}$$

$$\mathcal{L}(i) = \frac{6}{s(s + 10)} \quad \text{solve for } \mathcal{L}(i)$$

$$i = 0.6(1 - e^{-10t}) \quad \text{take inverse transform}$$

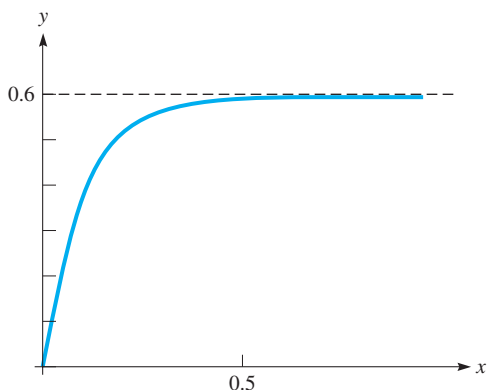


Fig. 31.22

The graph of this solution is shown in Fig. 31.22.

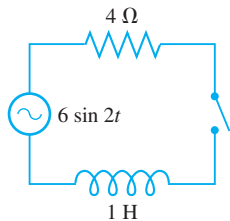


Fig. 31.23

■  $s = 0: 12 = 4A + 4C, 3 = A + C$   
 $s$  terms:  $0 = 4B + C$   
 $s^2$  terms:  $0 = A + B$

**EXAMPLE 6 Application—electric current**

An electric circuit in an FM radio transmitter contains a 1-H inductor and a 4-Ω resistor. It is being tested using a voltage source of  $6 \sin 2t$ . If the initial current is zero, find the current  $i$  as a function of time  $t$ . See Fig. 31.23.

The solution is as follows:

$(1)Di + 4i = 6 \sin 2t$  differential equation,  $D = d/dt$

$\mathcal{L}(Di) + 4\mathcal{L}(i) = 6\mathcal{L}(\sin 2t)$  take transforms

$[s\mathcal{L}(i) - 0] + 4\mathcal{L}(i) = \frac{6(2)}{s^2 + 4}$   $i(0) = 0$

$\mathcal{L}(i) = \frac{12}{(s + 4)(s^2 + 4)} = \frac{A}{s + 4} + \frac{Bs + C}{s^2 + 4}$  use partial fractions

$12 = A(s^2 + 4) + B(s^2 + 4s) + C(s + 4)$

The equations that give us the following values are shown in the margin at the left.

$A = 0.6 \quad B = -0.6 \quad C = 2.4$

$\mathcal{L}(i) = 0.6\left(\frac{1}{s + 4}\right) - 0.6\left(\frac{s}{s^2 + 4}\right) + 1.2\left(\frac{2}{s^2 + 4}\right)$  take inverse transforms

$\mathcal{L} = 0.6e^{-4t} - 0.6 \cos 2t + 1.2 \sin 2t$

This is checked by showing that  $i(0) = 0$  and that it satisfies the original equation. ■

**EXAMPLE 7 Application—electric current**

An electric circuit contains a 0.1-H inductor, a 250-μF capacitor, a voltage source of  $10 \sin 4t$ , and negligible resistance (assume  $R = 0$ ). See Fig. 31.24. If the initial charge on the capacitor is zero, and the initial current is also zero, find the current in the circuit as a function of the time  $t$ .

The solution is as follows:

$0.1D^2q + \frac{1}{250 \times 10^{-6}}q = 10 \sin 4t$  differential equation,  $D = d/dt$

$D^2q + 40,000q = 100 \sin 4t$

$\mathcal{L}(D^2q) + 40,000\mathcal{L}(q) = 100\mathcal{L}(\sin 4t)$  take transforms

$[s^2\mathcal{L}(q) - sq(0) - Dq(0)] + 40,000\mathcal{L}(q) = \frac{400}{s^2 + 16}$   $q(0) = 0, D(q) = 0$

$\mathcal{L}(q) = \frac{400}{(s^2 + 200^2)(s^2 + 16)} = \frac{As + B}{s^2 + 200^2} + \frac{Cs + E}{s^2 + 16}$  use partial fractions

$400 = (As + B)(s^2 + 16) + (Cs + E)(s^2 + 200^2)$

The equations that give us the following values are shown in the margin at the left.

$A = 0 \quad B = -0.010 \quad C = 0 \quad E = 0.010$

$\mathcal{L}(q) = \frac{0.010}{s^2 + 16} - \frac{0.010}{s^2 + 200^2} = \frac{0.010}{4} \left(\frac{4}{s^2 + 16}\right) - \frac{0.010}{200} \left(\frac{200}{s^2 + 200^2}\right)$

$q = 0.0025 \sin 4t - 5.0 \times 10^{-5} \sin 200t$  take inverse transforms

$i = 0.010 \cos 4t - 0.010 \cos 200t$  ■

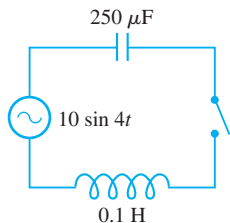


Fig. 31.24

■  $s = 0: 400 = 16B + 200^2E$   
 $s$  terms:  $0 = 16A + 200^2C$   
 $s^2$  terms:  $0 = B + E$   
 $s^3$  terms:  $0 = A + C$

## EXERCISES 31.12

In Exercises 1–4, make the given changes in the indicated examples of this section, and then solve the resulting problems.

- In Example 1, change the function  $y(0)$  from 1 to 2.
- In Example 2, interchange the values of  $y(0)$  and  $y'(0)$ .
- In Example 3, interchange the values of  $y(0)$  and  $y'(0)$ .
- In Example 5, change the initial current to 1 A.

In Exercises 5–37, solve the given differential equations by Laplace transforms. The function is subject to the given conditions.

- $y' + y = 0, y(0) = 1$
- $y' - 2y = 0, y(0) = 2$
- $2y' - 3y = 0, y(0) = -1$
- $y' + 2y = 1, y(0) = 0$
- $y' + 3y = e^{-3t}, y(0) = 1$
- $y' + 2y = te^{-2t}, y(0) = 0$
- $y'' + 4y = 0, y(0) = 0, y'(0) = 1$
- $9y'' - 4y = 0, y(0) = 1, y'(0) = 0$
- $4y'' + 4y' + 5y = 0, y(0) = 1, y'(0) = -1/2$
- $y'' + 2y' + y = 0, y(0) = 0, y'(0) = -2$
- $y'' - 4y' + 5y = 0, y(0) = 1, y'(0) = 2$
- $4y'' + 4y' + y = 0, y(0) = 1, y'(0) = 0$
- $y'' + y = 1, y(0) = 1, y'(0) = 1$
- $9y'' + 4y = 2t, y(0) = 0, y'(0) = 0$
- $y'' + 2y' + y = e^{-t}, y(0) = 1, y'(0) = 2$
- $2y'' + 8y = 3 \sin 2t, y(0) = 0, y'(0) = 0$
- $y'' - 4y = 10e^{3t}, y(0) = 5, y'(0) = 0$
- $y'' - 2y' + y = e^{2t}, y(0) = 1, y'(0) = 3$
- $y'' - y = 5 \sin 2t, y(0) = 0, y'(0) = 1$
- $2y'' + y' - y = \sin 3t, y(0) = 0, y'(0) = 0$
- A constant force of 6 N moves a 2-kg mass through a medium that resists the motion with a force equal to the velocity  $v$ . The equation relating the velocity and the time is  $2 \frac{dv}{dt} = 6 - v$ . Find  $v$  as a function of  $t$  if the object starts from rest.
- A pendulum moves with simple harmonic motion according to the differential equation  $D^2\theta + 20\theta = 0$ , where  $\theta$  is the angular displacement and  $D = d/dt$ . Find  $\theta$  as a function of  $t$  if  $\theta = 0$  and  $D\theta = 0.40$  rad/s when  $t = 0$ .
- The end of a certain vibrating metal rod oscillates according to  $D^2y + 6400y = 0$  (assuming no damping), where  $D^2 = d^2/dt^2$ . If  $y = 4$  mm and  $Dy = 0$  when  $t = 0$ , find the equation of motion.
- If there is a retarding force of  $0.2Dy$  to the motion of the rod in Exercise 27, find the equation of motion.
- A 50- $\Omega$  resistor, a 4.0- $\mu\text{F}$  capacitor, and a 40-V battery are connected in series. Find the charge on the capacitor as a function of time  $t$  if the initial charge is zero.
- A 2-H inductor, an 80- $\Omega$  resistor, and an 8-V battery are connected in series. Find the current in the circuit as a function of time if the initial current is zero.

- A 10-H inductor, a 40- $\mu\text{F}$  capacitor, and a voltage supply whose voltage is given by  $100 \sin 50t$  are connected in series in an electric circuit. Find the current as a function of the time if the initial charge on the capacitor is zero and the initial current is zero.
- A 20-mH inductor, a 40- $\Omega$  resistor, a 50- $\mu\text{F}$  capacitor, and a voltage source of  $100e^{-1000t}$  are connected in series in an electric circuit. Find the charge on the capacitor as a function of time  $t$ , if  $q = 0$  and  $i = 0$  when  $t = 0$ .
- The weight on a spring undergoes forced vibrations according to the equation  $D^2y + 9y = 18 \sin 3t$ . Find its displacement  $y$  as a function of the time  $t$ , if  $y = 0$  and  $Dy = 0$  when  $t = 0$ .
- A spring is stretched 1 m by a 20-N weight. The spring is stretched 0.5 m below the equilibrium position with the weight attached and then released. If it is in a medium that resists the motion with a force equal to  $12v$ , where  $v$  is the velocity, find the displacement  $y$  of the weight as a function of the time.
- For the electric circuit shown in Fig. 31.25, find the current as a function of the time  $t$  if the initial current is zero.

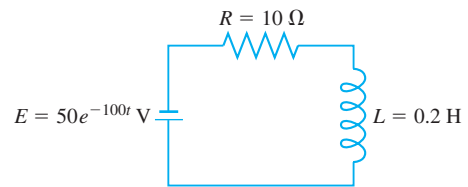


Fig. 31.25

- For the electric circuit shown in Fig. 31.26, find the current as a function of time  $t$  if the initial charge on the capacitor is zero and the initial current is zero.

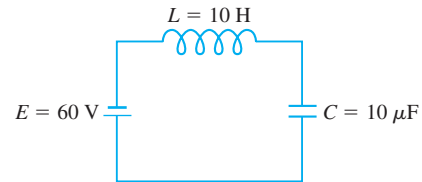


Fig. 31.26

- For the beam in Example 6 on page 987, find the deflection  $y$  as a function of  $x$  using Laplace transforms. The Laplace transform of the fourth derivative  $y^{(4)}$  is given by

$$\mathcal{L}(y^{(4)}) = s^4 \mathcal{L}(y) - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0)$$

Also, since  $y'(0)$  and  $y'''(0)$  are not given, but are constants, assume  $y'(0) = a$ , and  $y'''(0) = b$ . It is then possible to evaluate  $a$  and  $b$  to obtain the solution.



## CHAPTER 31 EQUATIONS

Separation of variables	$M(x, y)dx + N(x, y)dy = 0$	(31.1)
	$A(x)dx + B(y)dy = 0$	(31.2)
Integrating combinations	$d(xy) = x dy + y dx$	(31.3)
	$d(x^2 + y^2) = 2(x dx + y dy)$	(31.4)
	$d\left(\frac{y}{x}\right) = \frac{x dy - y dx}{x^2}$	(31.5)
	$d\left(\frac{x}{y}\right) = \frac{y dx - x dy}{y^2}$	(31.6)
Linear differential equation of first order	$dy + Py dx = Q dx$	(31.7)
	$ye^{\int P dx} = \int Qe^{\int P dx} dx + c$	(31.8)
Electric circuit	$L \frac{di}{dt} + Ri + \frac{q}{C} = E$	(31.9)
Motion in resisting medium	$m \frac{dv}{dt} = F - kv$	(31.10)
General linear differential equation	$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = b$	(31.11)
	$a_0 D^n y + a_1 D^{n-1} y + \cdots + a_{n-1} Dy + a_n y = b$	(31.12)
Homogeneous linear differential equation	$a_0 D^2 y + a_1 Dy + a_2 y = 0$	(31.13)
Auxiliary equation	$a_0 m^2 + a_1 m + a_2 = 0$	(31.14)
Distinct roots	$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$	(31.15)
Repeated roots	$y = e^{mx}(c_1 + c_2 x)$	(31.16)
Complex roots	$y = e^{\alpha x}(c_1 \sin \beta x + c_2 \cos \beta x)$	(31.17)
Nonhomogeneous linear differential equation	$a_0 D^2 y + a_1 Dy + a_2 y = b$	(31.18)
	$y = y_c + y_p$	(31.19)
Electric circuit	$L \frac{d^2 q}{dt^2} + R \frac{dq}{dt} + \frac{q}{C} = E$	(31.20)
Laplace transforms	$F(s) = \int_0^{\infty} e^{-st} f(t) dt$	(31.21)
	$F(s) = \mathcal{L}(f) = \int_0^{\infty} e^{-st} f(t) dt$	(31.22)
	$\mathcal{L}[af + bg] = a\mathcal{L}(f) + b\mathcal{L}(g)$	(31.23)
	$\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$	(31.24)
	$\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0)$	(31.25)
Inverse transform	$\mathcal{L}^{-1}(F) = f(t)$	(31.26)

## CHAPTER 31 REVIEW EXERCISES

In Exercises 1–32, find the general solution to the given differential equations.

1.  $4xy^3 dx + (x^2 + 1)dy = 0$
2.  $\frac{dy}{dx} = e^{x-y}$
3.  $\sin 2x dx + y \sin x dy = \sin x dx$
4.  $x dy + y dx = y dy$
5.  $2D^2y + Dy = 0$
6.  $2D^2y - 5Dy + 2y = 0$
7.  $16y'' - 8y' + y = 0$
8.  $y'' + 2y' + 2y = 0$
9.  $(x + y)dx + (x + y^3)dy = 0$
10.  $R \ln L dL = L dR$
11.  $V \frac{dP}{dV} - 5P = V^2$
12.  $dy - 2y dx = (x - 2)e^x dx$
13.  $dy = 2y dx + y^2 dx$
14.  $x^2 y dy = (1 + x) \csc y dx$
15.  $D^2y + 2Dy + 6y = 0$
16.  $4D^2y - 4Dy + y = 0$
17.  $y' + 4y = 2e^{-2x}$
18.  $2uv du = (2v - \ln v) dv$
19.  $\sin x \frac{dy}{dx} + y \cos x + x = 0$
20.  $y dy = (x^2 + y^2 - x) dx$
21.  $2 \frac{d^2s}{dt^2} + \frac{ds}{dt} - 3s = 6$
22.  $\frac{d^2y}{dx^2} + 6 \frac{dy}{dx} + 9y = 3x$
23.  $y'' + y' - y = 2e^x$
24.  $4D^3y + 9Dy = xe^x$
25.  $9D^2y - 18Dy + 8y = 16 + 4x$
26.  $9y'' + 4y = 4 \cos 2x$
27.  $D^3y - D^2y + 9Dy - 9y = \sin x$
28.  $y'' + y' = e^x + \cos 2x$
29.  $y'' - 7y' - 8y = 2e^{-x}$
30.  $3y'' - 6y' = 4 + xe^x$
31.  $D^2y + 25y = 50 \cos 5x$
32.  $D^2y + 4y = 8x \sin 2x$

In Exercises 33–40, find the indicated particular solution of the given differential equations.

33.  $3y' = 2y \cot x$ ;  $x = \frac{\pi}{2}$  when  $y = 2$
34.  $TdV - VdT = V^3 dV$ ;  $T = 1$  when  $V = 3$
35.  $y' = 4x - 2y$ ;  $x = 0$  when  $y = -2$
36.  $xy^2 dx + e^x dy = 0$ ;  $x = 0$  when  $y = 2$
37.  $\frac{d^2v}{dt^2} + \frac{dv}{dt} + 4v = 0$ ;  $\frac{dv}{dt} = \sqrt{15}$ ,  $v = 0$  when  $t = 0$
38.  $5y'' + 7y' - 6y = 0$ ;  $y' = 10$ ,  $y = 2$  when  $x = 0$
39.  $D^2y + 4Dy + 4y = 4 \cos x$ ;  $Dy = 1$ ,  $y = 0$  when  $x = 0$
40.  $y'' - 2y' + y = e^x + x$ ;  $y = 0$ ,  $y' = 0$  when  $x = 0$

In Exercises 41–48, solve the given differential equations by using Laplace transforms, where the function is subject to the given conditions.

41.  $4y' - y = 0$ ,  $y(0) = 1$
42.  $2y' - y = 4$ ,  $y(0) = 1$
43.  $y' - 3y = e^t$ ,  $y(0) = 0$
44.  $y' + 2y = e^{-2t}$ ,  $y(0) = 2$

45.  $y'' + y = 0$ ,  $y(0) = 0$ ,  $y'(0) = -4$
46.  $y'' + 4y' + 5y = 0$ ,  $y(0) = 1$ ,  $y'(0) = 1$
47.  $16y'' + 9y = 3e^x$ ,  $y(0) = 0$ ,  $y'(0) = 0$
48.  $y'' - 2y' + y = e^x + x$ ,  $y(0) = 0$ ,  $y'(0) = 1$

In Exercises 49–91, solve the given problems.

49. Use Euler's method to find the  $y$ -values of the solution of the equation  $dy/dx = 1 + y^2$  from  $x = 0$  to  $x = 0.4$ , with  $\Delta x = 0.1$ , if the curve passes through  $(0, 0)$ .
50. Solve the equation of Exercise 49, subject to the same conditions, using the Runge–Kutta method.
51. Find the particular solution of the equation  $dy/dx - 2y = e^{3x}$ , if  $y = 1$  when  $x = 0$ , (a) as a first-order linear equation, and (b) using Laplace transforms.
52. Find the particular solution of the equation  $D^2y - 4y = 2 - 8x$ , if  $y = 0$  and  $y' = 0$  when  $x = 0$ , (a) by the method of undetermined coefficients, and (b) by using Laplace transforms.
53. An object moves along a hyperbolic path described by  $xy = 1$ , such that  $dx/dt = 2t$ . Express  $x$  and  $y$  in terms of  $t$  if  $x = 1$ ,  $y = 1$  when  $t = 0$ .
54. An object moves along a parabolic path described by  $y = x^2 + x$ , such that  $dx/dt = 4t + 1$ . Express  $x$  and  $y$  in terms of  $t$ , if both  $x$  and  $y$  are zero when  $t = 0$ .
55. The time rate of change of volume of an evaporating substance is proportional to the surface area. Express the radius of an evaporating sphere of ice as a function of time. Let  $r = r_0$  when  $t = 0$ . (*Hint:* Express both  $V$  and  $A$  in terms of the radius  $r$ .)
56. An insulated tank is filled with a solution containing radioactive cobalt. Due to the radioactivity, energy is released and the temperature  $T$  (in  $^\circ\text{C}$ ) of the solution rises with the time  $t$  (in h). The following equation expresses the relation between temperature and time for a specific case:

$$56\,600 = 262(T - 70) + 20\,200 \frac{dT}{dt}$$

If the initial temperature is  $70^\circ\text{C}$ , what is the temperature 24 h later?

57. In a certain chemical reaction, the velocity of the reaction is proportional to the mass  $m$  of the chemical that remains unchanged. If  $m_0$  is the initial mass and  $dm/dt$  is the velocity of the reaction, find  $m$  as a function of the time  $t$ .
58. Under proper conditions, bacteria grow at a rate proportional to the number present. In a certain culture, there were  $10^4$  bacteria present at a given time, and there were  $3.0 \times 10^5$  bacteria present after 10 h. How many were present after 5.0 h?
59. An object with a mass of 1.00 kg slides down a long inclined plane. The effective force of gravity is 4.00 N, and the motion is retarded by a force numerically equal to the velocity. If the object starts from rest, what is the velocity (in m/s) 4.00 s later?

60. A 760-N object falls from rest under the influence of gravity. Find the equation for the velocity at any time  $t$  (in s) if the air resists the motion with a force numerically equal to twice the velocity.
61. A particle is moving along a path  $y = f(x)$  such that the slope of the path is  $y/(y - x)$ , and the path passes through the point  $(-1, 2)$ . Find the equation of the path ( $y > 0$ ).
62. On a certain weather map, the lines indicating equal temperature (*isotherms*) are given by  $y = x^3 + k$ . Find the equation of the orthogonal trajectories of the isotherms, the curves that show the direction of heat flow.
63. After 10.0 s, it is noted that 15.9% of the radioactive isotope neon-23 has decayed. Find the half-life of neon-23.
64. The isotope iodine-131, with a half-life of 8.04 days, is used in nuclear medicine to study the thyroid gland. Of an original amount, what percent of iodine-131 remains after 21 days?
65. Radioactive potassium-40 with a half-life of  $1.28 \times 10^9$  years is used for dating rock samples. If a given rock sample has 75% of its original amount of potassium-40, how old is the rock?
66. When a gas undergoes an adiabatic change (no gain or loss of heat), the rate of change of pressure with respect to volume is directly proportional to the pressure and inversely proportional to the volume. Express the pressure in terms of the volume.
67. Under ideal conditions, the natural law of population change is that the population increases at a rate proportional to the population at any time. Under these conditions, project the population of the world in 2020 if it reached 6.0 billion in 1999 and 6.9 billion in 2010.
68. A spherical balloon is being blown up such that its volume  $V$  increases at a rate proportional to its surface area. Show that this leads to the differential equation  $dV/dt = kV^{2/3}$  and solve for  $V$  as a function of  $t$ .
69. Find the orthogonal trajectories of the family of curves  $y = kx^5$ .
70. Find the equation of the curves for which their normals at all points are in the direction of the lines connecting the points and the origin.
71. Find the temperature after 1.0 h of an object originally  $100^\circ\text{C}$ , if it cools to  $90^\circ$  in 5.0 min in air that is at  $20^\circ\text{C}$ . (See Exercise 25 on page 967.)
72. If a circuit contains a resistance  $R$ , a capacitance  $C$ , and a source of voltage  $E$ , express the charge  $q$  on the capacitor as a function of time.
73. A 2-H inductor, a  $40\text{-}\Omega$  resistor, and a 20-V battery are connected in series. Find the current in the circuit as a function of time if the initial current is zero.
74. A hollow cylinder moves vertically up and down in water according to the equation  $D^2y + 6.5y = 0$ , where  $D = d/dt$ . Find the displacement  $y$  as a function of the time  $t$ , if  $y = 8.0$  cm and  $y' = 0$  cm/s when  $t = 0$  s.
75. A certain spring stretches 0.5 m by a 40-N weight. With this weight suspended on it, the spring is stretched 0.5 m beyond the equilibrium position and released. Find the equation of the resulting motion if the medium in which the weight is suspended retards the motion with a force equal to 16 times the velocity. Classify the motion as underdamped, critically damped, or overdamped. Explain your choice.
76. The end of a vibrating rod moves according to the equation  $D^2y + 0.2Dy + 4000y = 0$ , where  $y$  is the displacement and  $D = d/dt$ . Find  $y$  as a function of  $t$  if  $y = 3.00$  cm and  $Dy = -0.300$  cm/s when  $t = 0$ .
77. A 0.5-H inductor, a  $6\text{-}\Omega$  resistor, and a 20-mF capacitor are connected in series with a generator for which  $E = 24 \sin 10t$ . Find the charge on the capacitor as a function of time if the initial charge and initial current are zero.
78. A 5.00-mH inductor and a  $10.0\text{-}\mu\text{F}$  capacitor are connected in series with a voltage source of  $0.200e^{-200t}$  V. Find the charge on the capacitor as a function of time if  $q = 0$  and  $i = 4.00$  mA when  $t = 0$ .
79. Find the equation for the current as a function of time if a resistor of  $20\ \Omega$ , an inductor of 4 H, a capacitor of  $100\ \mu\text{F}$ , and a battery of 100 V are in series. The initial charge on the capacitor is 10 mC, and the initial current is zero.
80. If an electric circuit contains an inductance  $L$ , a capacitor with a capacitance  $C$ , and a sinusoidal source of voltage  $E_0 \sin \omega t$ , express the charge  $q$  on the capacitor as a function of the time. Assume  $q = 0$ ,  $i = 0$  when  $t = 0$ .
81. The differential equation relating the current and time for a certain electric circuit is  $2 di/dt + i = 12$ . Solve this equation by use of Laplace transforms, given that the initial current is zero. Evaluate the current for  $t = 0.300$  s.
82. A 6-H inductor and a  $30\text{-}\Omega$  resistor are connected in series with a voltage source of  $10 \sin 20t$ . Find the current as a function of time if the initial current is zero. Use Laplace transforms.
83. A 0.25-H inductor, a  $4.0\text{-}\Omega$  resistor, and a  $100\text{-}\mu\text{F}$  capacitor are connected in series. If the initial charge on the capacitor is  $400\ \mu\text{C}$  and the initial current is zero, find the charge on the capacitor as a function of time. Use Laplace transforms.
84. An inductor of 0.5 H, a resistor of  $6\ \Omega$ , and a capacitor of  $200\ \mu\text{F}$  are connected in series. If the initial charge on the capacitor is 10 mC and the initial current is zero, find the charge on the capacitor as a function of time after the switch is closed. Use Laplace transforms.
85. A mass of 0.25 kg stretches a spring for which  $k = 16$  N/m. An external force of  $\cos 8t$  is applied to the spring. Express the displacement  $y$  of the object as a function of time if the initial displacement and velocity are zero. Use Laplace transforms.
86. A spring is stretched 1.00 m by a mass of 5.00 kg (assume the weight to be 50.0 N). Find the displacement  $y$  of the object as a function of time if  $y(0) = 1$  m and  $dy/dt = 0$  when  $t = 0$ . Use Laplace transforms.

87. Air containing 20% oxygen passes into a 5.00-L container initially filled with 100% oxygen. A uniform mixture of the air and oxygen then passes from the container at the same rate. What volume of oxygen is in the container after 5.00 L of air have passed into it?
88. When a circular disc of mass  $m$  and radius  $r$  is suspended by a wire at the centre of one of its flat faces and the disc is twisted through an angle  $\theta$ , torsion in the wire tends to turn the disc back in the opposite direction. The differential equation for this case is  $\frac{1}{2}mr^2\frac{d^2\theta}{dt^2} = -k\theta$ , where  $k$  is a constant. Determine the equation of motion if  $\theta = \theta_0$  and  $d\theta/dt = \omega_0$  when  $t = 0$ . See Fig. 31.27.

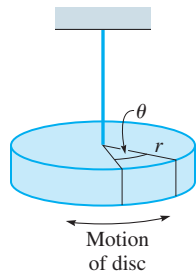


Fig. 31.27

89. The approximate differential equation relating the displacement  $y$  of a beam at a horizontal distance  $x$  from one end is  $EI\frac{d^2y}{dx^2} = M$ , where  $E$  is the modulus of elasticity,  $I$  is the moment of inertia of the cross section of the beam perpendicular to its axis, and  $M$  is the bending moment at the cross section. If  $M = 2000x - 40x^2$  for a particular beam of length  $L$  for which  $y = 0$  when  $x = 0$  and when  $x = L$ , express  $y$  in terms of  $x$ . Consider  $E$  and  $I$  as constants.

90. The gravitational acceleration of an object is inversely proportional to the square of its distance  $r$  from the centre of the earth. Use the chain rule, Eq. (23.14), to show that the acceleration is  $\frac{dv}{dt} = v\frac{dv}{dr}$ , where  $v = \frac{dr}{dt}$  is the velocity of the object. Then solve for  $v$  as a function of  $r$  if  $dv/dt = -g$  and  $v = v_0$  for  $r = R$ , where  $R$  is the radius of the earth. Finally, show that a spacecraft must have a velocity of at least  $v_0 = \sqrt{2gR}$  (11.2 km/s) in order to escape from the earth's gravitation. (Note the expression for  $v^2$  as  $r \rightarrow \infty$ .)
91. For the electric circuit shown in Fig. 31.28, find the current as a function of the time  $t$ , if the initial charge on the capacitor is zero and the initial current is zero.

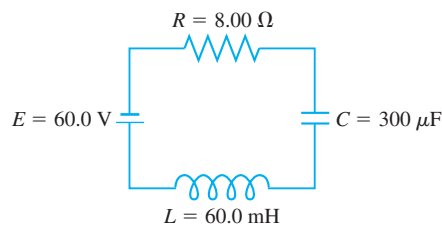


Fig. 31.28

### Writing Exercise

92. An electric circuit contains an inductor  $L$ , a resistor  $R$ , and a battery of voltage  $E$ . The initial current in the circuit is zero. Write three or four paragraphs explaining how the differential equation for the current in the circuit is solved using (a) separation of variables, (b) the linear differential equation of the first order, and (c) Laplace transforms.

## CHAPTER 31 PRACTICE TEST

In Problems 1–6, find the general solution of each of the given differential equations.

- $x\frac{dy}{dx} + 2y = 4$
- $y'' + 2y' + 5y = 0$
- $x dx + y dy = x^2 dx + y^2 dx$
- $2D^2y - Dy = 2 \cos x$
- $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 3x$
- $D^2y - 2Dy - 8y = 4e^{-2x}$
- Find the particular solution of the differential equation  $(xy + y)\frac{dy}{dx} = 2$ , if  $y = 2$  when  $x = 0$ .

- If interest in a bank account is compounded continuously, the amount grows at a rate that is proportional to the amount present. Derive the equation for the amount  $A$  in an account with continuous compounding in which the initial amount is  $A_0$  and the interest rate is  $r$  as a function of the time  $t$  after  $A_0$  is deposited.
- Using Laplace transforms, solve the differential equation  $y'' + 9y = 9$ , if  $y(0) = 0$  and  $y'(0) = 1$ .
- Using Laplace transforms, solve the differential equation  $D^2y - Dy - 2y = 12$ , if  $y(0) = 0$  and  $y'(0) = 0$ .
- Find the equation for the current as a function of the time (in s) in a circuit containing a 2-H inductance, an 8- $\Omega$  resistor, and a 6-V battery in series, if  $i = 0$  when  $t = 0$ .
- A mass of 0.5 kg stretches a spring for which  $k = 32$  N/m. With this weight attached, the spring is pulled 0.3 m longer than its equilibrium length and released. Find the equation of the resulting motion, assuming no damping. (The acceleration due to gravity is  $9.8$  m/s<sup>2</sup>.)

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# Solving Word Problems

Drill-type problems require a working knowledge of the methods presented, and some algebraic steps to change the algebraic form may be required to complete the solution. *Word problems*, however, require a proper interpretation of the statement of the problem before they can be put in a form for solution.

We have to put word problems in symbolic form in order to solve them, and it is this procedure that most students find difficult. Because such problems require more than going through a certain routine, they demand more analysis and appear to be more difficult. Among the reasons for the student's difficulty at solving word problems are (1) unsuccessful previous attempts at solving word problems, leading the student to believe that all word problems are "impossible," (2) a poorly organized approach to the solution, and (3) failure to read the problem carefully, thereby having an improper and incomplete interpretation of the statement given. These can be overcome with proper attitude and care.

A specific procedure for solving word problems is shown on page 48, when word problems are first covered in our study of algebra. There are over 120 completely worked examples of word problems (as well as numerous other examples that show a similar analysis) throughout this text, illustrating proper interpretations and approaches to these problems.

## RISERS

The procedure shown on page 48 is similar to that used by most instructors and texts. One of the variations that a number of instructors use is called *RISERS*. This is a word formed from the first letters (an *acronym*) of the words that outline the procedure. These are *Read*, *Imagine*, *Sketch*, *Equate*, *Relate*, and *Solve*. We now briefly outline this procedure here.

**Read** the statement of the problem carefully.

**Imagine.** Take time to get a mental image of the situation described.

**Sketch** a figure.

**Equate**, *on the sketch*, the known and unknown quantities.

**Relate** the known and unknown quantities with an equation.

**Solve** the equation.

There are problems where a *sketch* may simply be words and numbers placed so that we may properly *equate* the known and unknown quantities. For example, in Example 2 on page 48, the *sketch*, *equate*, and *relate* steps might look like this:

<i>sketch</i>	34 resistors	56 $\Omega$
	1.5- $\Omega$ resistors	2.0- $\Omega$ resistors
<i>equate</i>	$x$	$34 - x$
resistance	$1.5x$	$2.0(34 - x)$
<i>relate</i>	$1.5x$	$+ 2.0(34 - x) = 56$

If you follow the method on page 48, or this *RISERS* variation, or any appropriate step-by-step method, and *write out the solution neatly*, you will find that word problems lend themselves to solution more readily than you have previously found.

# A Table of Integrals

The basic forms of Chapter 28 are not included. The constant of integration is omitted.

Forms containing  $a + bu$  and  $\sqrt{a + bu}$

$$1. \int \frac{u \, du}{a + bu} = \frac{1}{b^2} [ (a + bu) - a \ln(a + bu) ]$$

$$2. \int \frac{du}{u(a + bu)} = -\frac{1}{a} \ln \frac{a + bu}{u}$$

$$3. \int \frac{u \, du}{(a + bu)^2} = \frac{1}{b^2} \left( \frac{a}{a + bu} + \ln(a + bu) \right)$$

$$4. \int \frac{du}{u(a + bu)^2} = \frac{1}{a(a + bu)} - \frac{1}{a^2} \ln \frac{a + bu}{u}$$

$$5. \int u\sqrt{a + bu} \, du = -\frac{2(2a - 3bu)(a + bu)^{3/2}}{15b^2}$$

$$6. \int \frac{u \, du}{\sqrt{a + bu}} = -\frac{2(2a - bu)\sqrt{a + bu}}{3b^2}$$

$$7. \int \frac{du}{u\sqrt{a + bu}} = \frac{1}{\sqrt{a}} \ln \left( \frac{\sqrt{a + bu} - \sqrt{a}}{\sqrt{a + bu} + \sqrt{a}} \right), \quad a > 0$$

$$8. \int \frac{\sqrt{a + bu}}{u} \, du = 2\sqrt{a + bu} + a \int \frac{du}{u\sqrt{a + bu}}$$

Forms containing  $\sqrt{u^2 \pm a^2}$  and  $\sqrt{a^2 - u^2}$

$$9. \int \frac{du}{u^2 - a^2} = \frac{1}{2a} \ln \frac{u - a}{u + a}$$

$$10. \int \frac{du}{\sqrt{u^2 \pm a^2}} = \ln(u + \sqrt{u^2 \pm a^2})$$

$$11. \int \frac{du}{u\sqrt{u^2 + a^2}} = -\frac{1}{a} \ln \left( \frac{a + \sqrt{u^2 + a^2}}{u} \right)$$

$$12. \int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \sec^{-1} \frac{u}{a}$$

$$13. \int \frac{du}{u\sqrt{a^2 - u^2}} = -\frac{1}{a} \ln \left( \frac{a + \sqrt{a^2 - u^2}}{u} \right)$$

$$14. \int \sqrt{u^2 \pm a^2} \, du = \frac{u}{2} \sqrt{u^2 \pm a^2} \pm \frac{a^2}{2} \ln(u + \sqrt{u^2 \pm a^2})$$

$$15. \int \sqrt{a^2 - u^2} \, du = \frac{u}{2} \sqrt{a^2 - u^2} + \frac{a^2}{2} \sin^{-1} \frac{u}{a}$$



$$\begin{aligned}
 16. \int \frac{\sqrt{u^2 + a^2}}{u} du &= \sqrt{u^2 + a^2} - a \ln \left( \frac{a + \sqrt{u^2 + a^2}}{u} \right) \\
 17. \int \frac{\sqrt{u^2 - a^2}}{u} du &= \sqrt{u^2 - a^2} - a \sec^{-1} \frac{u}{a} \\
 18. \int \frac{\sqrt{a^2 - u^2}}{u} du &= \sqrt{a^2 - u^2} - a \ln \left( \frac{a + \sqrt{a^2 - u^2}}{u} \right) \\
 19. \int (u^2 \pm a^2)^{3/2} du &= \frac{u}{4} (u^2 \pm a^2)^{3/2} \pm \frac{3a^2 u}{8} \sqrt{u^2 \pm a^2} + \frac{3a^4}{8} \ln (u + \sqrt{u^2 \pm a^2}) \\
 20. \int (a^2 - u^2)^{3/2} du &= \frac{u}{4} (a^2 - u^2)^{3/2} + \frac{3a^2 u}{8} \sqrt{a^2 - u^2} + \frac{3a^4}{8} \sin^{-1} \frac{u}{a} \\
 21. \int \frac{(u^2 + a^2)^{3/2}}{u} du &= \frac{1}{3} (u^2 + a^2)^{3/2} + a^2 \sqrt{u^2 + a^2} - a^3 \ln \left( \frac{a + \sqrt{u^2 + a^2}}{u} \right) \\
 22. \int \frac{(u^2 - a^2)^{3/2}}{u} du &= \frac{1}{3} (u^2 - a^2)^{3/2} - a^2 \sqrt{u^2 - a^2} + a^3 \sec^{-1} \frac{u}{a} \\
 23. \int \frac{(a^2 - u^2)^{3/2}}{u} du &= \frac{1}{3} (a^2 - u^2)^{3/2} - a^2 \sqrt{a^2 - u^2} + a^3 \ln \left( \frac{a + \sqrt{a^2 - u^2}}{u} \right) \\
 24. \int \frac{du}{(u^2 \pm a^2)^{3/2}} &= \pm \frac{u}{a^2 \sqrt{u^2 \pm a^2}} \\
 25. \int \frac{du}{(a^2 - u^2)^{3/2}} &= \frac{u}{a^2 \sqrt{a^2 - u^2}} \\
 26. \int \frac{du}{u(u^2 + a^2)^{3/2}} &= \frac{1}{a^2 \sqrt{u^2 + a^2}} - \frac{1}{a^3} \ln \left( \frac{a + \sqrt{u^2 + a^2}}{u} \right) \\
 27. \int \frac{du}{u(u^2 - a^2)^{3/2}} &= -\frac{1}{a^2 \sqrt{u^2 - a^2}} - \frac{1}{a^3} \sec^{-1} \frac{u}{a} \\
 28. \int \frac{du}{u(a^2 - u^2)^{3/2}} &= \frac{1}{a^2 \sqrt{a^2 - u^2}} - \frac{1}{a^3} \ln \left( \frac{a + \sqrt{a^2 - u^2}}{u} \right)
 \end{aligned}$$

Trigonometric forms

$$\begin{aligned}
 29. \int \sin^2 u du &= \frac{u}{2} - \frac{1}{2} \sin u \cos u \\
 30. \int \sin^3 u du &= -\cos u + \frac{1}{3} \cos^3 u \\
 31. \int \sin^n u du &= -\frac{1}{n} \sin^{n-1} u \cos u + \frac{n-1}{n} \int \sin^{n-2} u du \\
 32. \int \cos^2 u du &= \frac{u}{2} + \frac{1}{2} \sin u \cos u \\
 33. \int \cos^3 u du &= \sin u - \frac{1}{3} \sin^3 u \\
 34. \int \cos^n u du &= \frac{1}{n} \cos^{n-1} u \sin u + \frac{n-1}{n} \int \cos^{n-2} u du \\
 35. \int \tan^n u du &= \frac{\tan^{n-1} u}{n-1} - \int \tan^{n-2} u du
 \end{aligned}$$

$$36. \int \cot^n u \, du = -\frac{\cot^{n-1} u}{n-1} - \int \cot^{n-2} u \, du$$

$$37. \int \sec^n u \, du = \frac{\sec^{n-2} u \tan u}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} u \, du$$

$$38. \int \csc^n u \, du = \frac{\csc^{n-2} u \cot u}{n-1} + \frac{n-2}{n-1} \int \csc^{n-2} u \, du$$

$$39. \int \sin au \sin bu \, du = \frac{\sin(a-b)u}{2(a-b)} - \frac{\sin(a+b)u}{2(a+b)}$$

$$40. \int \sin au \cos bu \, du = -\frac{\cos(a-b)u}{2(a-b)} - \frac{\cos(a+b)u}{2(a+b)}$$

$$41. \int \cos au \cos bu \, du = \frac{\sin(a-b)u}{2(a-b)} + \frac{\sin(a+b)u}{2(a+b)}$$

$$42. \int \sin^m u \cos^n u \, du = \frac{\sin^{m+1} u \cos^{n-1} u}{m+n} + \frac{n-1}{m+n} \int \sin^m u \cos^{n-2} u \, du$$

$$43. \int \sin^m u \cos^n u \, du = -\frac{\sin^{m-1} u \cos^{n+1} u}{m+n} + \frac{m-1}{m+n} \int \sin^{m-2} u \cos^n u \, du$$

Other forms

$$44. \int u e^{au} \, du = \frac{e^{au}(au-1)}{a^2}$$

$$45. \int u^2 e^{au} \, du = \frac{e^{au}}{a^3} (a^2 u^2 - 2au + 2)$$

$$46. \int u^n \ln u \, du = u^{n+1} \left( \frac{\ln u}{n+1} - \frac{1}{(n+1)^2} \right)$$

$$47. \int u \sin u \, du = \sin u - u \cos u$$

$$48. \int u \cos u \, du = \cos u + u \sin u$$

$$49. \int e^{au} \sin bu \, du = \frac{e^{au}(a \sin bu - b \cos bu)}{a^2 + b^2}$$

$$50. \int e^{au} \cos bu \, du = \frac{e^{au}(a \cos bu + b \sin bu)}{a^2 + b^2}$$

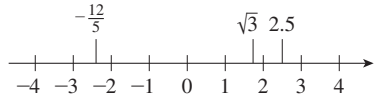
$$51. \int \sin^{-1} u \, du = u \sin^{-1} u + \sqrt{1-u^2}$$

$$52. \int \tan^{-1} u \, du = u \tan^{-1} u - \frac{1}{2} \ln(1+u^2)$$

# Answers to Odd-Numbered Exercises

Since statements will vary for writing exercises, answers here are in abbreviated form. Answers are not included for end-of-chapter writing exercises.

## Exercises 1.1, page 5

1. Change  $\frac{5}{1}$  to  $\frac{-3}{1}$  and  $\frac{-19}{1}$  to  $\frac{14}{1}$ .    3.  $-6 < -4$ ;  $-6$  is to the left of  $-4$ . The figure must be changed to show  $-6$  to the left of  $-4$ .
5. 3: integer, rational, real.  $\sqrt{-4}$ : imag.;  $-\frac{\pi}{6}$ : irrational, real.
7. 3, 4,  $\frac{\pi}{2}$     9.  $6 < 8$     11.  $\pi > -3.2$
13.  $-4 < -|-3|$     15.  $-\frac{1}{3} > -\frac{1}{2}$     17.  $\frac{1}{3}, -\frac{\sqrt{3}}{4}, \frac{b}{y}$
19. 
21. No,  $|0| = 0$     23. The number itself
25.  $\frac{3}{23}$     27.  $-3.1, -|-3|, -1, \sqrt{5}, \pi, |-8|, 9$
29. (a) Positive integer (b) Negative integer  
(c) Positive rational number less than 1
31. (a) Yes (b) Yes
33. (a) To the right of zero (b) To the left of  $-4$
35. Positive number less than 1
37. All real values of  $a$ ;  $b = 0$ .
39. 0.00080 F    41.  $N = 1000$  *an* bits
43. Yes;  $-30^\circ\text{C}$  is to the left of  $-20^\circ\text{C}$

## Exercises 1.2, page 10

1. 22    3.  $-4$     5. 4    7. 6    9.  $-3$     11.  $-24$
13. 35    15. 20    17. 40    19.  $-1$     21.  $-17$
23. Undefined    25. 20    27. 16    29.  $-9$     31. 24
33.  $-6$     35. 3    37. Commutative law of multiplication
39. Distributive law    41. Associative law of addition
43. Associative law of multiplication
45. d    47. b    49. (a) Positive (b) Negative
51. Correct. (For  $x > 0$ ,  $x$  is positive; for  $x < 0$ ,  $-x$  is positive.)
53. (a) Negative reciprocals of each other (b) All values of  $x$  and  $y$  with  $x \neq y$     55.  $-2.4 \text{ kW} \cdot \text{h}$     57.  $2.0^\circ\text{C}$     59. 10 V
61.  $100 \text{ m} + 200 \text{ m} = 200 \text{ m} + 100 \text{ m}$ ; commutative law of addition
63.  $4(8 \text{ min} + 6 \text{ min})$  OR  $4(8 \text{ min}) + 4(6 \text{ min})$ ; distributive law

## Exercises 1.3, page 19

1. 0.390 has three significant digits; the zero is not necessary as a placeholder and should not be written unless it is significant.
3. 75.7    5. 1 000 000 Hz    7. 0.001 m    9. 1000 volts
11. 0.001 amperes    13. 100 000 cm    15. 0.000 02 Ms
17. 0.000 25  $\text{m}^2$     19. 80 000 L (with 3 significant digits)
21. 4500 cm/s (with 3 significant digits)

23. 3 530 000  $\text{cm}/\text{min}^2$
25. 90 000 000 ms (with 2 significant digits)
27. 0.0150 mC/V    29. 110 000 km/h    31. 56 000  $\text{cm}^3$
33. 24 000 km/h    35. 0.0112  $\text{m}^2$     37. 1000 g/L
39. 1200 km/h (with 3 significant digits)
41. 0.135 J/(s  $\cdot$   $\text{cm}^2$ )    43. 120 000 mA/ $\text{cm}^2$
45. 8 cylinders: exact; 55 km/h: approximate
47. Both are exact.
49. 107 has 3 significant digits; 3004 has 4 significant digits.
51. 6.80 has 3 significant digits; 6.08 has 3 significant digits.
53. 3000 has 1 significant digit; 3000.1 has 5 significant digits.
55. (a) 0.01 (b) 30.8    57. (a) Same (b) 78.0
59. (a) 0.004 (b) Same    61. (a) 4.94. (b) 4.9
63. (a) 50 900 (b) 51 000    65. (a) 9540 (b) 9500
67. (a) 0.945 (b) 0.94    69. (a) 12 (b) 12.20    71. (a) 0.015  
(b) 0.011 4    73. (a) 0.2 (b) 0.1356 to 4 significant digits
75. (a) 6.5 (b) 6.085 610 98 (6.086 to 4 significant digits)
77. 15.8788    79. 204.2
81. Measurements can be more than 2.745 MHz and less than 2.755.
83. Too many sig. digits; time has only 2 sig. digits.
85. (a) 19.3 (b) 27
87. (a) 2 (b) 2 (c)  $-2$  (d) 0 (e) Undefined
89. Example:  $(231\,465 - 164\,352) \div 9 = 7457$ ; integer
91. (a)  $\pi < 3.1416$  (b)  $\pi < (22 \div 7)$
93. (a)  $1 \div 3 = 0.333\,333\dots$  (b)  $5 \div 11 = 0.454\,545\dots$   
(c)  $2 \div 5 = 0.400\,000\dots$  (0 repeats)    95. 95.3 MJ
97. 262 144 bytes    99. 59.14%

## Exercises 1.4, page 25

1.  $x^6$     3.  $\frac{a^6 x^2}{b^4 t^2}$     5.  $x^7$     7.  $2b^6$     9.  $m^2$     11.  $\frac{1}{7n^4}$
13.  $P^8$     15.  $8\pi^3$     17.  $a^{30}T^{60}$     19.  $\frac{8}{b^3}$     21.  $\frac{x^8}{16}$
23. 1    25.  $-3$     27.  $\frac{1}{6}$     29.  $R^2$     31.  $-t^{14}$
33.  $\frac{1}{64v^{12}}$     35.  $-L^2$     37.  $\frac{1}{8}$     39. 1    41.  $\frac{a}{x^2}$
43.  $\frac{64s^6}{g^2}$     45.  $\frac{x^3}{64a^3}$     47.  $\frac{5n^3}{T}$     49.  $-53$     51. 253
53.  $-0.421$     55. 9990    57. Yes    59. 625
61. 1, provided that  $x \neq 0$     63.  $\frac{G^2 k^5 T^5}{h}$
65.  $\frac{r}{6}$     67. \$3212.27

**B.2 ANSWERS TO ODD-NUMBERED EXERCISES**

**Exercises 1.5, page 29**

1. 8060    3. 45 000    5. 0.002 01    7. 3.23    9. 18.6  
 11.  $4 \times 10^3$     13.  $8.7 \times 10^{-3}$     15.  $6.09 \times 10^0$   
 17.  $6.3 \times 10^{-2}$     19.  $1 \times 10^0$     21.  $5.6 \times 10^{13}$   
 23.  $2.2 \times 10^8$     25.  $3.2 \times 10^{-34}$     27.  $1.728 \times 10^{87}$   
 29.  $4.85 \times 10^{10}$     31.  $1.59 \times 10^7$     33.  $9.965 \times 10^{-3}$   
 35.  $3.38 \times 10^{16}$     37.  $2 \times 10^6$  kW    39.  $3 \times 10^{-6}$  W  
 41.  $2 \times 10^9$  Hz    43. 0.000 000 000 001 6 W  
 45. 2 GW    47.  $3 \mu$ W    49. 2 GHz  
 51. (a)  $8.09 \times 10^6$  (b)  $809 \times 10^3$  (c)  $80.9 \times 10^{-3}$   
 53.  $10^{21}$     55.  $3.3 \times 10^{-11}$  mg  
 57. (a)  $8.64 \times 10^4$  s    (b)  $3.155\ 760\ 0 \times 10^9$  s  
 59.  $4.8 \times 10^2$  W    61.  $2.998 \times 10^5$  km/s

**Exercises 1.6, page 31**

1. -4    3. 12    5. 9    7. -11    9. -7    11. 0.3  
 13. 5    15. -6    17. 5    19. 47    21. 53    23.  $20\sqrt{3}$   
 25.  $4\sqrt{21}$     27.  $2\sqrt{5}$     29. 4    31. 7    33. 10  
 35.  $3\sqrt{10}$     37. 9.24    39. 0.6877  
 41. (a) 60.00 (b) 84.00    43. (a) 0.0388 (b) 0.0246  
 45. 98 km/h    47. 1450 m/s    49. 107 cm  
 51. No, not true if  $a < 0$     53. (a) 12.9 (b) -0.598  
 55. (a) Imaginary (b) Real

**Exercises 1.7, page 35**

1.  $3x - 3y$     3.  $4ax + 5s$     5.  $8x$     7.  $y + 4x$   
 9.  $5F - 3T - 2$     11.  $-a^2b - a^2b^2$     13.  $3s - 4$   
 15.  $-v + 5x - 4$     17.  $5a - 5$     19.  $-5a + 2$   
 21.  $-2t + 5u$     23.  $7r + 8s$     25.  $19j - 50$   
 27.  $3n - 9$     29.  $-2t^2 + 18$     31. 6a    33.  $2aZ + 1$   
 35.  $4c - 6$     37.  $8p - 5q$     39.  $-4x^2 + 22$     41.  $7V^2 - 3$   
 43.  $-6t + 13$     45.  $-24R + 4Z$     47.  $2D + d$   
 49.  $3B - 2a$     51.  $40x + 250$   
 53. (a)  $x^2 + 2y + 2a - b$  (b)  $3x^2 - 4y + 2a + b$     55.  $|(b - a)|$

**Exercises 1.8, page 38**

1.  $-8s^8t^{13}$     3.  $x^2 - 5x + 6$     5.  $a^3x$     7.  $-a^2c^3x^3$   
 9.  $-8a^3x^5$     11.  $i^2R + 2i^2r$     13.  $-3s^3 + 15st$   
 15.  $5m^3n + 15m^2n$     17.  $-3M^2 - 3MN + 6M$   
 19.  $acx^4 + acx^3y^3$     21.  $x^2 + 2x - 15$     23.  $2x^2 + 9x - 5$   
 25.  $6a^2 - 7ab + 2b^2$     27.  $6s^2 + 11st - 35t^2$   
 29.  $2x^3 + 5x^2 - 2x - 5$     31.  $x^2 + 4y^2 - 4xy - 16$   
 33.  $2a^2 - 16a - 18$     35.  $18T^2 - 15T - 18$   
 37.  $-2L^3 + 6L^2 + 8L$     39.  $4x^2 - 20x + 25$   
 41.  $x_1^2 + 6x_1x_2 + 9x_2^2$     43.  $x^2y^2z^2 - 4xyz + 4$   
 45.  $2x^2 + 32x + 128$     47.  $-x^3 + 2x^2 + 5x - 6$   
 49.  $6T^3 + 9T^2 - 6T$     51. (a)  $49 \neq 9 + 16$  (b)  $1 \neq 9 - 16$   
 53. If  $1 < x < 9$ , then  $x^2 - 1$  can be factored to  $(x - 1)(x + 1)$ .  
 55.  $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

57.  $0.0001r^2P + 0.02rP + P$     59.  $3R^2 - 4RX$   
 61.  $n^2 + 200n + 10\ 000$     63.  $R_1^2 - R_2^2$

**Exercises 1.9, page 40**

1.  $\frac{3}{y^3}$     3.  $3x - 2$     5.  $-4x^2y$     7.  $\frac{4t^4}{r^2}$     9.  $4x^2$   
 11.  $-6a$     13.  $a^2 + 2y$     15.  $-2rt^2 + t$   
 17.  $-4q^3 + 2p + q$     19.  $\frac{2L}{R} - R$     21.  $\frac{1}{3a} - \frac{2b}{3a} + 1$   
 23.  $3y^n - 2ay$     25.  $2x + 1$     27.  $x - 1$   
 29.  $4x^2 - x - 1 - \frac{3}{2x - 3}$     31.  $Z - 2 - \frac{1}{4Z + 3}$   
 33.  $x^2 + x - 6$     35.  $2a^2 + 8$   
 37.  $x^2 - 2x + 4$     39.  $x - y$     41.  $x - y + z$     43. -5  
 45.  $\frac{x^4 + 1}{x + 1} = x^3 - x^2 + x - 1 + \frac{2}{x + 1} \neq x^3$   
 47.  $A + \frac{\mu^2E^2}{2A} - \frac{\mu^4E^4}{8A^3}$     49.  $\frac{GMm}{R}$   
 51.  $s^2 + 2s + 6 + \frac{16s + 16}{s^2 - 2s - 2}$

**Exercises 1.10, page 44**

1. (a) -9 (b) -15 (c) -36 (d) -4    3.  $\frac{1}{8}$     5. 9  
 7. -1    9. -10    11. -5    13. -3    15. 4    17.  $-\frac{7}{2}$   
 19. 8    21.  $\frac{10}{3}$     23. 2    25.  $-\frac{5}{2}$     27. 0    29. 8  
 31. 9 or -9    33. 9.5    35. -2.1    37. 5.7  
 39. 0.85    41. (a) Identity (b) Conditional    43. 3.75  
 45. 55 km/h    47. 120 °C    49. 750 L    51. 820 km

**Exercises 1.11, page 46**

1.  $\frac{v - v_0}{t}$     3.  $\frac{V_0 + bTV_0 - V}{bV_0}$     5.  $\frac{E}{I}$     7.  $g_2 - rL$   
 9.  $\frac{Q}{Sd^2}$     11.  $\frac{p - p_a}{dg}$     13.  $\frac{APV}{R}$     15.  $\frac{-ct^2 + 0.3t}{c}$   
 17.  $Tv - c$     19.  $\frac{K_1m_1 - K_2m_1}{K_2}$     21.  $\frac{2gm - 2am}{a}$   
 23.  $\frac{C_0^2 - C_1^2}{2C_1^2}$     25.  $\frac{Ar - N}{r}$     27.  $100T_1 - 100T_2$   
 29.  $\frac{Q_1 + PQ_1}{P}$     31.  $\frac{N + N_2 - N_2T}{T}$     33.  $\frac{L - \pi r_2 - 2x_1 - 2x_2}{\pi}$   
 35.  $\frac{V_1^2 + gJP}{V_1}$     37.  $\frac{Cd(k_1 + k_2)}{2Ak_1k_2}$     39.  $\frac{CN - NV}{C}$   
 41. 1070 K    43. 32.3 °C    45. 3.22  $\Omega$   
 47.  $\frac{d - v_2(4\text{ h}) - v_1(2\text{ h})}{v_1}$

**Exercises 1.12, page 50**

1. 29 1.5  $\Omega$  resistors and 5 2.5  $\Omega$  resistors  
 3. 3.000 h    5. \$22 000 6 years ago, \$27 000 today  
 7.  $1.9 \times 10^5$  the first year,  $2.6 \times 10^5$  the second year  
 9. 50 hectares at \$200 per hectare and 90 hectares at \$300 per hectare  
 11. 60 ppm/h    13. 20 girders    15.  $-2.3 \mu$ A,  $-4.6 \mu$ A, 6.9  $\mu$ A

17. 6.9 km, 9.5 km    19. 32 CDs and 22 DVDs    21. 900 m  
 23. 84.2 km/h, 92.2 km/h    25. 395 s, first car  
 27. 146 km from A    29. 4.0 L    31. 79 km/h

**Review Exercises for Chapter 1, page 52**

1. -10    3. -20    5. -22    7. -25    9. -4    11. 5  
 13.  $4r^2t^4$     15.  $-\frac{24t}{m^2n}$     17.  $\frac{8T^3}{N}$     19.  $3\sqrt{5}$     21. (a) 3 (b) 8800  
 23. (a) 4 (b) 9.0    25. 18.0    27.  $1.3 \times 10^{-4}$     29.  $-2ab - a$   
 31.  $7LC - 3$     33.  $2x^2 + 9x - 5$     35.  $x^2 + 16x + 64$   
 37.  $-3h^2k^4 + hk$     39.  $7R - 6r$     41.  $13xy - 10z$   
 43.  $2x^3 - x^2 - 7x - 3$     45.  $-3x^2y + 24xy^2 - 48y^3$   
 47.  $18p^2q - 9p^2 + 3pq$     49.  $\frac{3q^4}{p^3} + \frac{6q}{p} - 2$   
 51.  $2x - 5$     53.  $x^2 - 2x + 3$     55.  $4x^3 - 2x^2 + 6x, R = -1$   
 57.  $15r - 3s - 3t$     59.  $y^2 + 5y - 1, R = 4$     61.  $-\frac{9}{2}$   
 63.  $\frac{21}{10}$     65.  $-\frac{7}{3}$     67. 3    69.  $-\frac{19}{5}$     71. 1.0  
 73. (a)  $6 \times 10^{10}$  bytes (b) 60 gigabytes  
 75. (a)  $1.92 \times 10^8$  km (b) 192 Gm  
 77. (a) 40 500 000 000 000 km (b) 40.5 Pm  
 79. (a) 0.000 000 000 001 W/m<sup>2</sup> (b) 1 pW/m<sup>2</sup>  
 81. (a) 0.15 Bq/L (b)  $150 \times 10^{-3}$  mBq/L    83.  $\frac{R}{n^2}$   
 85.  $\frac{L^2P}{\pi^2I}$     87.  $\frac{Rr - Pp}{Q}$     89.  $\frac{d + A}{A}$     91.  $\frac{N_1 - N_3 + N_3T}{T}$   
 93.  $\frac{HR + AT_1}{A}$     95.  $\frac{d - 3bkx^2 + kx^3}{3kx^2}$     97.  $8.2 \times 10^5$   
 99. 1.20    101. 0.0188  $\Omega$     103.  $101x + 198a$  cm  
 105.  $-2t^2 - 2h^2 - 4ht + 4t + 4h$     107. Yes, (18 to 0)  
 109. Identity    111.  $-(y - x)^3$     113.  $4 \times 10^{-7}$   
 115. \$59, \$131    117. 160 cm<sup>3</sup>, 80 cm<sup>3</sup>, 320 cm<sup>3</sup>  
 119. 1900  $\Omega$ , 3100  $\Omega$     121. 6.7 h    123. 34.5 h  
 125. 400 L of 0.50% grade oil and 600 L of 0.75% grade oil  
 127. 27 m<sup>2</sup>    129. 5.500%

**Exercises 2.1, page 58**

1. 90°    3. 4 pairs    5.  $\angle EBD$  and  $\angle DBC$     7.  $\angle ABC$   
 9. 25°    11.  $BD$  and  $BC$     13. 140°    15. 40°    17. 35°  
 19. 62°    21. 28°    23. 134°    25. 44°    27. 46°  
 29. 4.53 m    31. 3.40 m    33. 133°    35. 882 m  
 37. 180°    39. Sum of angles is 180°

**Exercises 2.2, page 64**

1. 65°    3. 7.02 m    5. 56°    7. 48°    9. 8.4 m<sup>2</sup>  
 11. 32 300 cm<sup>2</sup>    13. 4.41 cm<sup>2</sup>    15. 0.390 m<sup>2</sup>    17. 942 cm  
 19. 64.5 cm    21. 26.6 mm    23. 522 cm    25. 67°  
 27. 227.2 cm    29. 45°    31. An equilateral triangle  
 33. The smaller triangles are similar.  
 35.  $\angle LMK = \angle OMN; \angle KLM = \angle MON; \Delta MKL \sim \Delta MNO$   
 37. 8    39. 65°    41. 1150 cm<sup>2</sup>    43. 9.6 m<sup>2</sup>    45. 5.7 m  
 47. 23.1 m    49. 7.5 m, 9.0 m    51. 3.4 m    53. 20.0 km

**Exercises 2.3, page 68**

1. Trapezoid    3. 8900 m<sup>2</sup>    5. 260 m    7. 3.324 mm  
 9. 12.8 m    11. 214.4 dm    13. 7.3 mm<sup>2</sup>    15. 0.683 km<sup>2</sup>  
 17. 9.2 m<sup>2</sup>    19.  $2.00 \times 10^3$  dm<sup>2</sup>    21.  $p = 2b + 4a$   
 23.  $A = bh + a^2$     25. Rectangle    27. 288 cm<sup>2</sup>  
 29. The diagonal always divides the rhombus into two congruent triangles. All outer sides are always equal.  
 31. 348 m    33. 2000 mm, 8000 mm    35. 2.4 L  
 37. 3.04 km<sup>2</sup>  
 39. 360°. A diagonal divides a quadrilateral into two triangles, and the sum of the interior angles of each triangle is 180°.

**Exercises 2.4, page 72**

1. 18°    3.  $p = 11.6$  cm,  $A = 8.30$  cm<sup>2</sup>    5. (a)  $AD$  (b)  $AF$   
 7. (a)  $AF \perp OE$  (b)  $\Delta OCE$     9. 1730 cm    11. 72.6 mm  
 13. 0.0285 km<sup>2</sup>    15. 4.26 m<sup>2</sup>    17. 25°    19. 25°  
 21. 120°    23. 40°    25. 0.393 rad    27. 2.185 rad  
 29.  $\frac{\pi r}{2} + 2r$     31.  $\frac{1}{4}\pi r^2 - \frac{1}{2}r^2$   
 33. All are on the same diameter.    35. 10.3 cm<sup>2</sup>  
 37. 40 060 km    39.  $\frac{0.445}{1}$     41. 35.7 cm    43. 9500 cm<sup>2</sup>  
 45.  $9.7 \times 10^7$  mm<sup>2</sup>  
 47. Horizontally and opposite to original direction    49. 630 km

**Exercises 2.5, page 76**

1. Simpson's rule. The use of smaller intervals improves the approximation.  
 3. Simpson's rule. It accounts better for the arcs between points on the curve.  
 5. 84 m<sup>2</sup>    7. 0.45 m<sup>2</sup>    9. 9.8 km<sup>2</sup>    11. 19 000 km<sup>2</sup>  
 13. 11 800 m<sup>2</sup>    15.  $8.0 \times 10^3$  m<sup>2</sup>  
 17. 2.73 cm<sup>2</sup>. All of the trapezoids are inscribed.  
 19. 2.98 cm<sup>2</sup>. The ends of the areas are curved so they can get closer to the boundary.

**Exercises 2.6, page 80**

1. The volume increases by a factor of 4.  
 3. 771 cm<sup>3</sup>    5. 366 cm<sup>3</sup>    7.  $3.99 \times 10^6$  m<sup>2</sup>    9. 2.83 m<sup>3</sup>  
 11. 20 500 cm<sup>2</sup>    13.  $2.5 \times 10^5$  cm<sup>3</sup>    15. 3.358 m<sup>2</sup>  
 17. 0.15 cm<sup>3</sup>    19. 72.3 cm<sup>2</sup>    21.  $\frac{1}{6}\pi d^3$     23.  $\frac{6}{1}$     25.  $\frac{4}{1}$   
 27. 604 cm<sup>2</sup>    29.  $1.4 \times 10^6$  m<sup>3</sup>    31.  $2.6 \times 10^6$  m<sup>3</sup>  
 33. 66 600 m<sup>3</sup>    35. 1560 mm<sup>2</sup>    37. 7330 cm<sup>3</sup>    39. 1.10 cm<sup>3</sup>

**Review Exercises for Chapter 2, page 82**

1. 32°    3. 32°    5. 41    7. 700    9. 7.36    11. 21.1  
 13. 25.5 mm    15. 3.06 m<sup>2</sup>    17. 309 mm    19. 3320 cm<sup>2</sup>  
 21. 6190 cm<sup>3</sup>    23.  $1.60 \times 10^5$  m<sup>3</sup>    25. 1.62 m<sup>2</sup>  
 27. 66.6 mm<sup>2</sup>    29. 25°    31. 65°    33. 53°    35. 2.4  
 37.  $p = b + \sqrt{b^2 + 4a^2} + \pi a$     39.  $A = ab + \frac{1}{2}\pi a^2$     41. Yes

**B.4 ANSWERS TO ODD-NUMBERED EXERCISES**

43.  $n^2$ ;  $A = \pi(nr)^2 = n^2(\pi r^2)$     45.  $\frac{a}{d} = \frac{b}{c}$     47.  $71^\circ$   
 49. 7.89 m    51. 12 cm    53. 30 m    55. 6.0 m  
 57. 5.91 km    59. 42 100 km    61.  $2.7 \times 10^6 \text{ mm}^2$   
 63.  $1.1 \times 10^6 \text{ m}^2$     65.  $193 \text{ m}^3$     67. 10.000 m    69. 873 L  
 71.  $h = 52.5 \text{ cm}$ ,  $w = 93.3 \text{ cm}$     73.  $136\,000 \text{ m}^2$

**Exercises 3.1, page 90**

1. -13    3.  $f(T - 10) = 9.1 + 0.08T + 0.001T^2$   
 5. (a)  $A(r) = \pi r^2$  (b)  $A(d) = \frac{\pi d^2}{4}$   
 7.  $d(V) = \sqrt[3]{\frac{6V}{\pi}}$     9.  $A(s) = s^2$ ,  $s(A) = \sqrt{A}$   
 11.  $A(r) = 4r^2 - \pi r^2$     13. 3, -1    15. 5, 5  
 17. -2.66,  $-\frac{1}{2}$     19.  $\frac{a}{4} + \frac{a^2}{2}$ , 0  
 21.  $3s^2 + s + 6$ ,  $12s^2 - 2s + 6$     23. -8  
 25. 62.9,  $2.60 \times 10^2$     27. -299.67  
 29. Square the value of the independent variable and add 2 to the result.  
 31. Multiply the value of the independent variable by 6, cube the value of the independent variable, and subtract the second result from the first.  
 33. Multiply the value of the independent variable by 2 and then add 5. Multiply this result by 3, then subtract 1.  
 35. Multiply the value of the independent variable by 2 and then add 3. Divide this result by the sum of the independent variable and 2.  
 37.  $A = 5s^2$   
 $f(s) = 5s^2$   
 39.  $A = 6.00 - 0.25t$   
 $f(t) = 6.00 - 0.25t$   
 41. 10.4 m    43. 13.2m,  $0.4v + 0.32v^2$ , 40.8 m, 40.8 m    45.  $d(t) = 55t$   
 47. (a)  $f[f(x)]$  means "function of the function of  $x$ ." (b)  $8x^4$

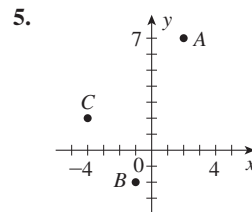
**Exercises 3.2, page 94**

1.  $f(x) = -x^2 + 2$  is defined for all real values of  $x$ .  
 Since  $x^2$  cannot be negative, the maximum value of  $f(x)$  is 2.  
 Range: all real numbers  $f(x) \leq 2$   
 3. 4 mA, 0 mA  
 5. Domain: all real numbers  
 Range: all real numbers  
 7. Domain: all real numbers  $R \neq 0$ , or  $(-\infty, 0)$  and  $(0, \infty)$   
 Range: all real numbers  $G(R) \neq 0$ , or  $(-\infty, 0)$  and  $(0, \infty)$   
 9. Domain: all real numbers  $s \neq 0$ , or  $(-\infty, 0)$  and  $(0, \infty)$   
 Range: all real numbers  $f(s) > 0$ , or  $(0, \infty)$   
 11. Domain: all real numbers  $h \geq 0$ , or  $[0, \infty)$   
 Range: all real numbers  $H(h) \geq 1$ , or  $[1, \infty)$   
 13. Domain: all real numbers  
 Range: all real numbers  $y \geq 0$ , or  $[0, \infty)$   
 15. Domain: all real numbers  $y > 2$ , or  $(2, \infty)$   
 17. Domain: all real numbers  $D \neq 2, -4$ , or  $(-\infty, -4)$ ,  $(-4, 2)$ , and  $(2, \infty)$

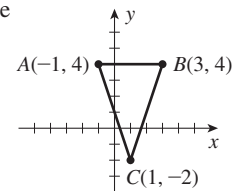
19. 2, does not exist    21. 2, 0.75    23.  $d(t) = 120 + 80t$   
 25.  $w(t) = 5500 - 2t$   
 27.  $m(h) = 0.5h - 390$  for  $h > 1000$  m  
 29.  $C(l) = 5l + 250$   
 31. (a)  $y(x) = \frac{1200 - 0.1x}{0.4}$  (b) 2900 L  
 33.  $y(x) = \frac{2750 - 15x}{25}$     35.  $A(p) = \frac{p^2}{16} + \frac{(60 - p)^2}{4\pi}$   
 37.  $d(h) = \sqrt{h^2 + 14\,400}$   
 Domain: all real values  $h \geq 0$ , or  $[0, \infty)$   
 Range: all real values  $d \geq 120$  m, or  $[120, \infty)$   
 39.  $v(t) = \frac{300}{t}$   
 Domain: all real values  $t > 0$ , or  $(0, \infty)$   
 Range: all real numbers  $0 < v < c$ , or  $(0, c)$   
 41. Domain: all real values  $C > 0$ , or  $(0, \infty)$   
 43.  $m(h) = \begin{cases} 110 & \text{for } 0 \leq h \leq 1000 \text{ m} \\ 0.5h - 390 & \text{for } h > 1000 \text{ m} \end{cases}$   
 45. (a)  $V(w) = 10w^2 - 150w + 500$   
 (b) Domain:  $w > 10$  cm, or  $(10 \text{ cm}, \infty)$   
 47. 1    49. Range: all real values  $f(x) \geq 2$ , or  $[2, \infty)$

**Exercises 3.3, page 96**

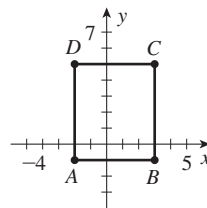
1. (-1, 1)    3.  $A(2, 1)$ ,  $B(-1, 2)$ ,  $C(-2, -3)$



7. Isosceles triangle

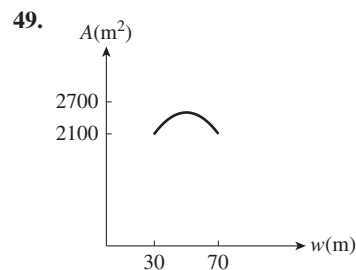
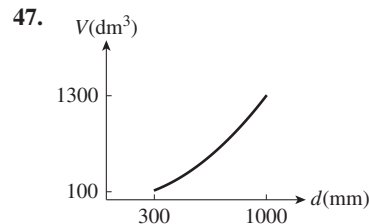
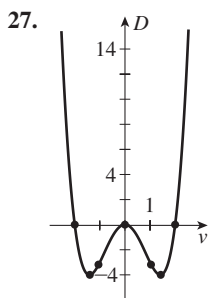
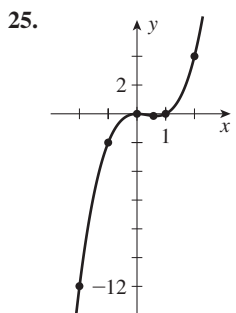
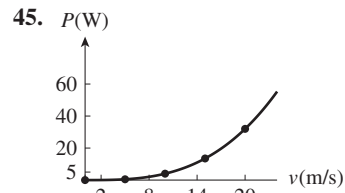
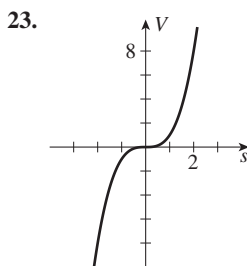
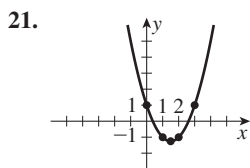
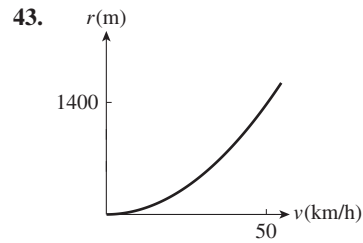
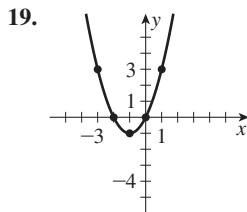
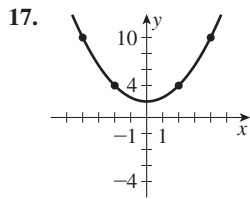
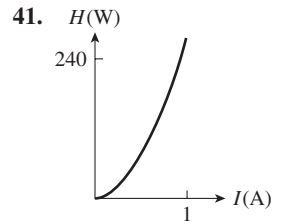
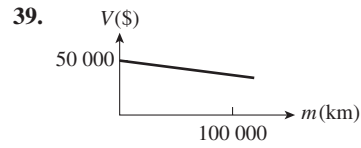
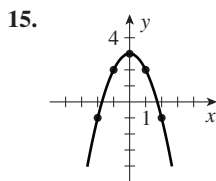
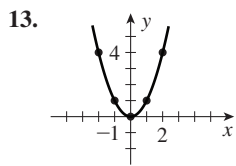
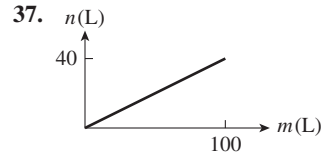
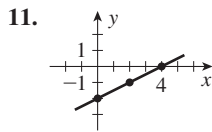
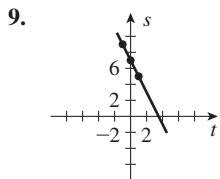
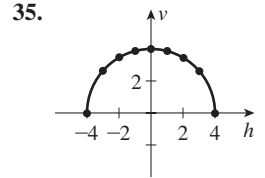
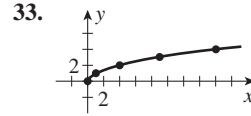
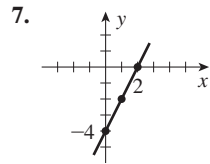
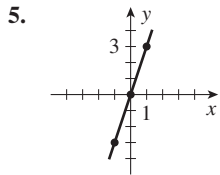
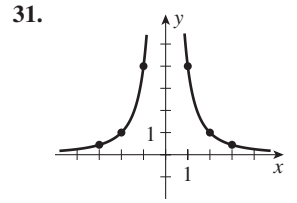
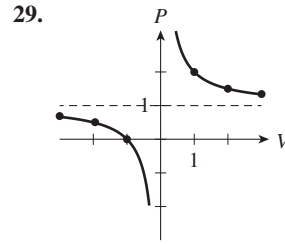
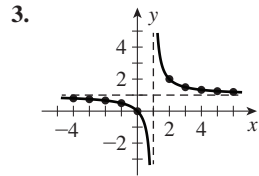
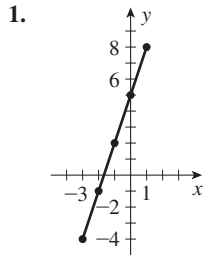


9. Rectangle



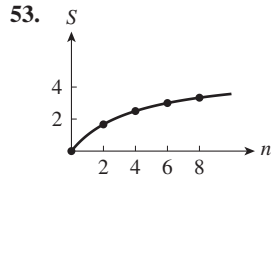
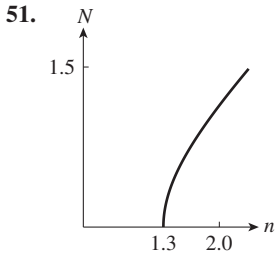
11. (5, 4)    13. (3, -2)  
 15. On a vertical line one unit to the right of the  $y$ -axis  
 17. On a horizontal line 3 units above the  $x$ -axis  
 19. On a  $45^\circ$  line through the origin    21. 0  
 23. To the right of the  $y$ -axis  
 25. To the left of a line parallel to the  $y$ -axis, one unit to its left  
 27. Quadrant I or Quadrant III    29. On the  $x$ - or  $y$ -axis  
 31. In Quadrant III    33. (a) 8 (b) 6

Exercises 3.4, page 102



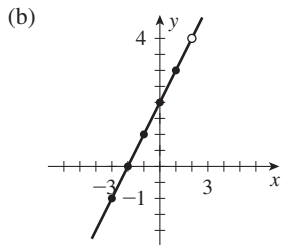
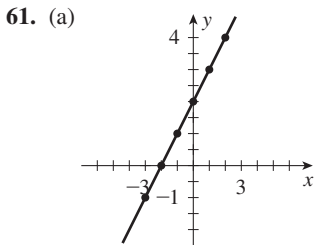
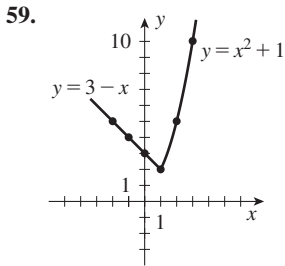
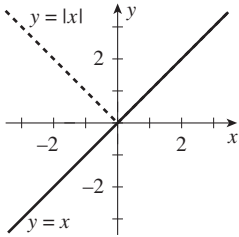


**B.6 ANSWERS TO ODD-NUMBERED EXERCISES**



55. No.  $f(1) = 2$  says nothing about  $f(2)$ .

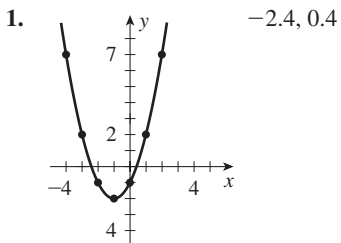
57.  $y = x$  is the same as  $y = |x|$  for  $x \geq 0$ .  
 $y = |x|$  is the same as  $y = -x$  for  $x < 0$ .



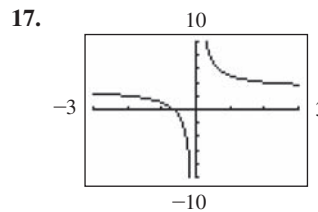
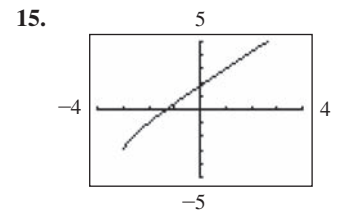
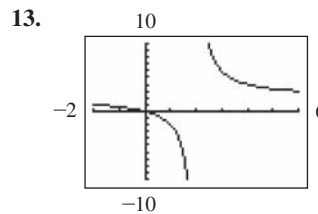
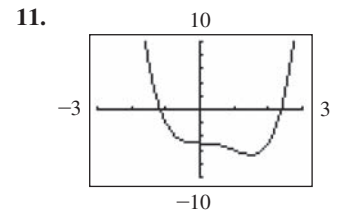
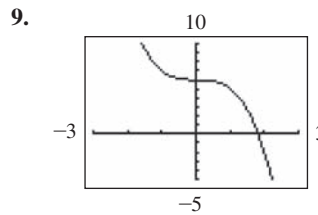
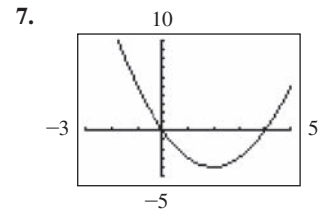
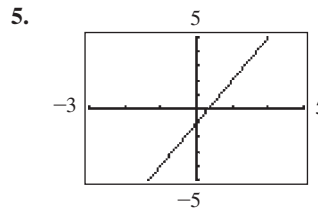
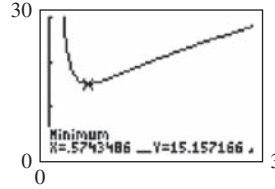
The graphs are identical except the second curve is undefined at a single point  $x = 2$ .

63. Yes    65. No

**Exercises 3.5, page 107**



3. Range: all real values  $y \geq 15.2$ , or  $[15.2, \infty)$



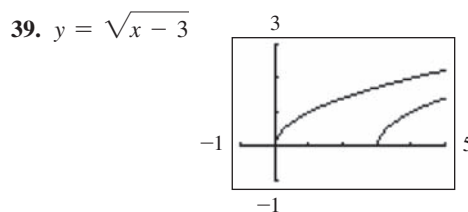
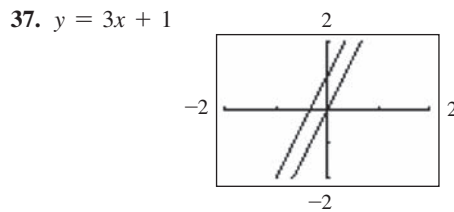
19. -2.791, 1.791    21. 2.104    23. 4.321    25. 1.400

27. No solution    29. Range: all real values  $y > 0$  or  $y \leq -1$

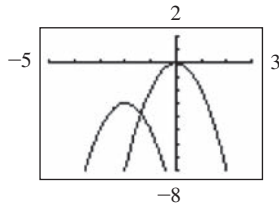
31. Range: all real values  $y \leq -4$  or  $y \geq 0$

33. Range: all real values  $Y(y) \geq 3.464$  (approx.)

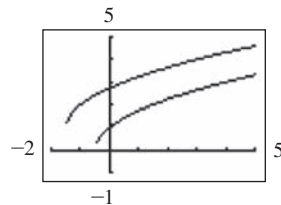
35. Range: all real numbers



41.  $y = -2(x + 2)^2 - 3$



43.  $y = \sqrt{2x + 3} + 1$

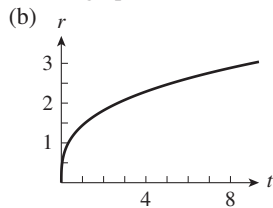


45. 6.00 V      47. 24.4 cm, 29.4 cm

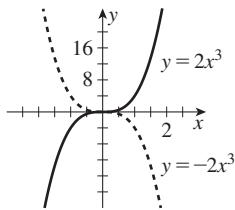
49.  $w = 17.6$  cm,  $l = 29.6$  cm.      51. 66.7 m

53. 3.9362 cm, 3.9362 cm, and 1.9362 cm      55. 0.250 cm/min

57. (a) The graph will look similar to the graph in (b).

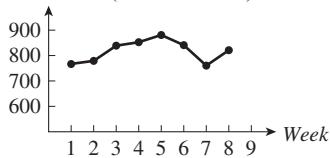


59. Curves are reflected about the  $x$ -axis.

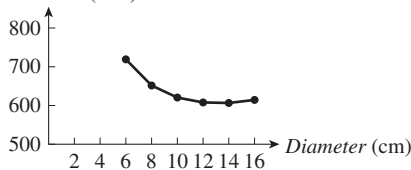


**Exercises 3.6, page 111**

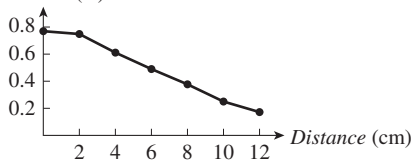
1. Production (1000's of litres)



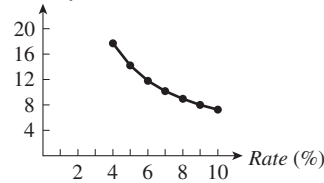
3. Material (cm<sup>2</sup>)



5.  $M$ . ind. (H)



7. Time (yrs)

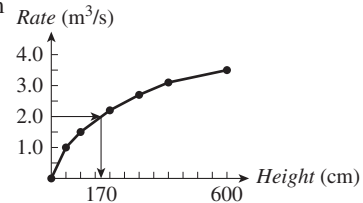


9. (a) 132°C (b) 0.7 min

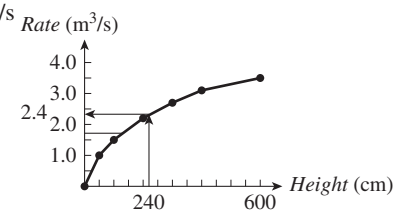
11. (a) 1.5 mm (b) 3.2 mm

13. 0.30 H      15. 7.2%

17. (a) 170 cm



(b) 2.4 m<sup>3</sup>/s



19. 1.3 m<sup>3</sup>/s

21. 0.34

23. 76 m<sup>2</sup>

25. 130.3°C

27. 3.7 m<sup>3</sup>/s

**Review Exercises for Chapter 3, page 112**

1.  $A(t) = 4\pi t^2$       3.  $y(x) = \frac{2000 - 0.05x}{0.04}$

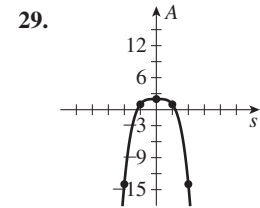
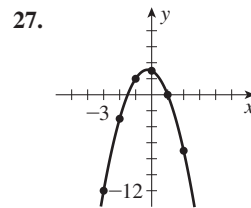
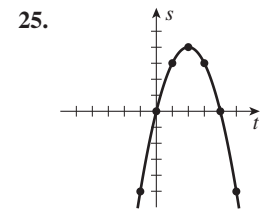
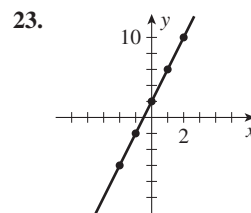
5. 16, -47      7.  $3, \sqrt{1 - 4h}$       9.  $h^3 + 11h^2 + 36h$

11. -3      13. -3.67, 16.6      15. 0.165 03, -0.214 76

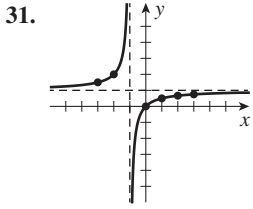
17. Domain: all real numbers, or  $(-\infty, \infty)$   
Range: all real numbers  $f(x) \geq 1$ , or  $[1, \infty)$

19. Domain: all real numbers  $t > -4$ , or  $(-4, \infty)$   
Range: all real numbers  $g(t) > 0$ , or  $(0, \infty)$

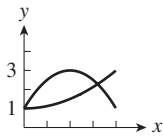
21. Domain: all real numbers  $n \neq 5$ , or  $(-\infty, 5)$  and  $(5, \infty)$   
Range: all real numbers  $f(n) > 1$ , or  $(1, \infty)$



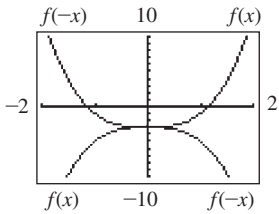
**B.8 ANSWERS TO ODD-NUMBERED EXERCISES**



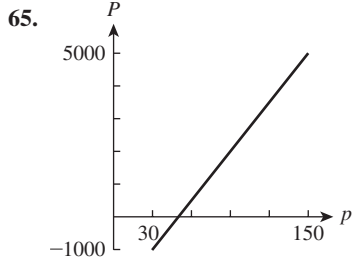
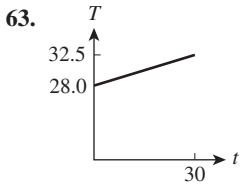
33. 0.43    35. 0.17, 5.83    37. 1.35    39. -0.71, 0.71  
 41. Range: all real values  $y \geq -6.25$  or  $[-6.25, \infty)$   
 43. Range: all real values  $A \leq -2.83$  or  $A \geq 2.83$ , or  $(-\infty, -2.83]$  or  $[2.83, \infty)$   
 45.  $a$  and  $b$  have opposite signs.    47.  $(1, \pm\sqrt{3})$   
 49. All values of  $(x, y)$  that are not on the  $x$ -axis or  $y$ -axis.  
 51. There are many possibilities. Two are shown.



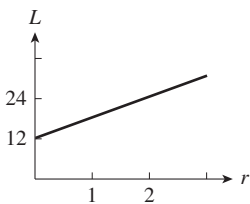
53.  $y = \sqrt{x+1} + 1$   
 55. The graphs are reflections of each other across the  $y$ -axis.



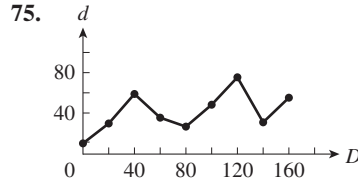
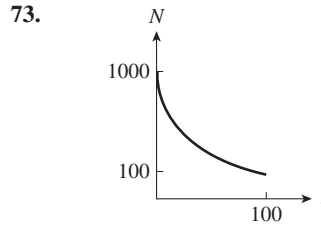
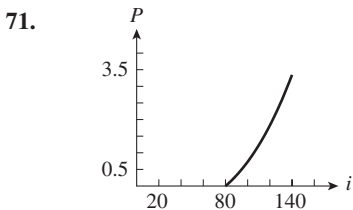
57. Yes; this passes the vertical line test.    59. 13.4    61.  $72^\circ$



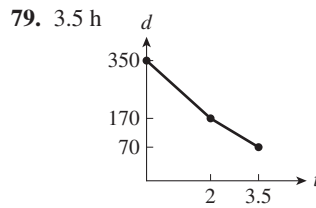
67.  $L = f(r) = 2\pi r + 12$



69. 2.63%



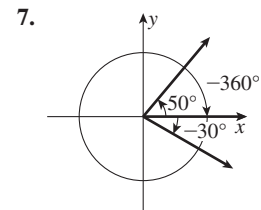
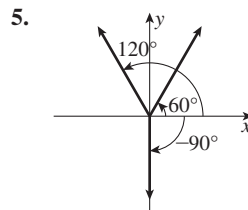
77. 3.46 m



81.  $32.8^\circ\text{C}$     83. 0.389 m    85. 6.53 hours

**Exercises 4.1, page 118**

1.  $865.6^\circ$ , or  $-574.4^\circ$     3.  $-135^\circ$

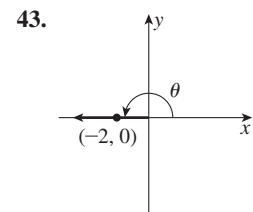
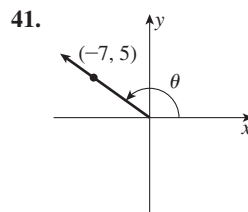
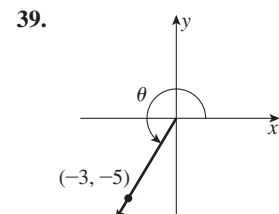
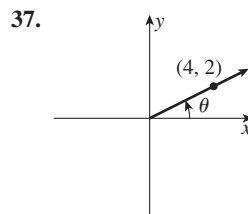


9.  $405^\circ, -315^\circ$     11.  $210^\circ, -510^\circ$     13.  $430^\circ30', -289^\circ30'$

15.  $638.1^\circ, -81.9^\circ$     17.  $15.18^\circ$     19.  $82.91^\circ$     21.  $18.85^\circ$

23.  $-235.49^\circ$     25. 0.977 rad    27. 6.72 rad    29.  $47^\circ30'$

31.  $-5^\circ37'$     33.  $15.20^\circ$     35.  $301.27^\circ$



45. First-quadrant angle, fourth-quadrant angle

47. First-quadrant angle, quadrantal angle

49. First-quadrant angle, second-quadrant angle

51. Third-quadrant angle, first-quadrant angle  
55.  $86^{\circ}16'26''$

**Exercises 4.2, page 122**

$$\begin{array}{ll} 1. \sin \theta = \frac{3}{5} & \csc \theta = \frac{5}{3} \\ \cos \theta = \frac{4}{5} & \sec \theta = \frac{5}{4} \\ \tan \theta = \frac{3}{4} & \cot \theta = \frac{4}{3} \end{array} \quad \begin{array}{ll} 3. \sin \theta = \frac{4}{5} & \csc \theta = \frac{5}{4} \\ \cos \theta = \frac{3}{5} & \sec \theta = \frac{5}{3} \\ \tan \theta = \frac{4}{3} & \cot \theta = \frac{3}{4} \end{array}$$

$$\begin{array}{ll} 5. \sin \theta = \frac{8}{17} & \csc \theta = \frac{17}{8} \\ \cos \theta = \frac{15}{17} & \sec \theta = \frac{17}{15} \\ \tan \theta = \frac{8}{15} & \cot \theta = \frac{15}{8} \end{array} \quad \begin{array}{ll} 7. \sin \theta = \frac{40}{41} & \csc \theta = \frac{41}{40} \\ \cos \theta = \frac{9}{41} & \sec \theta = \frac{41}{9} \\ \tan \theta = \frac{40}{9} & \cot \theta = \frac{9}{40} \end{array}$$

$$9. \sin \theta = \frac{\sqrt{15}}{4} \quad \csc \theta = \frac{4}{\sqrt{15}}$$

$$\cos \theta = \frac{1}{4} \quad \sec \theta = 4$$

$$\tan \theta = \sqrt{15} \quad \cot \theta = \frac{1}{\sqrt{15}}$$

$$11. \sin \theta = \frac{1}{\sqrt{2}} \quad \csc \theta = \sqrt{2}$$

$$\cos \theta = \frac{1}{\sqrt{2}} \quad \sec \theta = \sqrt{2}$$

$$\tan \theta = 1 \quad \cot \theta = 1$$

$$13. \sin \theta = \frac{2}{\sqrt{29}} \quad \csc \theta = \frac{\sqrt{29}}{2}$$

$$\cos \theta = \frac{5}{\sqrt{29}} \quad \sec \theta = \frac{\sqrt{29}}{5}$$

$$\tan \theta = \frac{2}{5} \quad \cot \theta = \frac{5}{2}$$

$$15. \sin \theta = 0.808 \quad \csc \theta = 1.24$$

$$\cos \theta = 0.589 \quad \sec \theta = 1.70$$

$$\tan \theta = 1.37 \quad \cot \theta = 0.729$$

$$17. \frac{5}{13}, \frac{12}{5} \quad 19. \frac{2}{\sqrt{5}}, \sqrt{5} \quad 21. 0.882, 1.33$$

$$23. 0.246, 3.94 \quad 25. \sin \theta = \frac{4}{5}, \tan \theta = \frac{4}{3}$$

$$27. \tan \theta = \frac{1}{3}, \sec \theta = \frac{\sqrt{10}}{3} \quad 29. 1 \quad 31. \cos \theta = \sqrt{1-y^2}$$

$$33. -\frac{7}{3} \quad 35. \sec \theta$$

**Exercises 4.3, page 125**

$$1. 20.65^{\circ} \quad 3. 71.0^{\circ}$$

$$5. \sin 40^{\circ} = 0.65 \quad \csc 40^{\circ} = 1.5$$

$$\cos 40^{\circ} = 0.76 \quad \sec 40^{\circ} = 1.3$$

$$\tan 40^{\circ} = 0.86 \quad \cot 40^{\circ} = 1.2$$

$$7. \sin 15^{\circ} = 0.26 \quad \csc 15^{\circ} = 3.8$$

$$\cos 15^{\circ} = 0.97 \quad \sec 15^{\circ} = 1.0$$

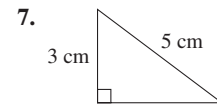
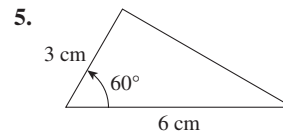
$$\tan 15^{\circ} = 0.27 \quad \cot 15^{\circ} = 3.7$$

9. 0.381    11. 1.58    13. 0.9626    15. 1.00    17. 0.4085
19. 1.57    21. 116.9    23. 0.07063    25.  $70.97^{\circ}$
27.  $65.70^{\circ}$     29.  $11.7^{\circ}$     31.  $49.453^{\circ}$     33.  $53.44^{\circ}$
35.  $86.53^{\circ}$     37.  $74.1^{\circ}$     39.  $6.95^{\circ}$     41.  $0.956 = 0.956$
43.  $2.7 = 2.7$
45.  $y$  is always less than or equal to  $r$ , and the minimum value of  $y$  is 0.
47. As  $\theta$  increases from  $0^{\circ}$  to  $90^{\circ}$ , the radius vector rotates counter-clockwise and  $x$  decreases.
49. 0.8885    51. 0.93614    53.  $71.6^{\circ}$     55. 87.5 dB
57.  $48.6^{\circ}$

**Exercises 4.4, page 130**

1.  $\sin A = 0.868$ ,  $\cos A = 0.496$ ,  $\tan A = 1.75$ ,  $\sin B = 0.496$   
 $\cos B = 0.868$ ,  $\tan B = 0.571$

3.  $a = 44.68$   
 $A = 34.17^{\circ}$   
 $B = 55.83^{\circ}$



9.  $c = 6850$     11.  $b = 309$     13.  $b = 12.6$   
 $b = 1450$      $A = 26.3^{\circ}$      $a = 20.2$   
 $B = 12.2^{\circ}$      $B = 63.7^{\circ}$      $A = 57.9^{\circ}$
15.  $a = 32$     17.  $a = 30.21$     19.  $c = 71.85$   
 $A = 21^{\circ}$      $b = 48.16$      $A = 52.15^{\circ}$   
 $B = 69^{\circ}$      $B = 57.90^{\circ}$      $B = 37.85^{\circ}$
21.  $c = 1.09$     23.  $c = 1.922$     25.  $c = 648.46$   
 $b = 0.661$      $a = 0.5239$      $A = 65.883^{\circ}$   
 $A = 52.5^{\circ}$      $A = 15.82^{\circ}$      $B = 24.117^{\circ}$
27.  $c = 14.61$     29.  $b = 0.0162$   
 $a = 0.7584$      $a = 0.0959$   
 $B = 87.025^{\circ}$      $A = 80.44^{\circ}$

31. The given information does not determine a unique triangle.

33. 4.45    35.  $40.24^{\circ}$     37.  $43.1^{\circ}$     39. 788

41.  $a = c \sin A$     43.  $c = \frac{a}{\cos B}$   
 $b = c \cos A$      $b = a \tan B$   
 $B = 90^{\circ} - A$      $A = 90^{\circ} - B$

**Exercises 4.5, page 133**

1. 639 m    3. 98.7 m    5. 44.0 m    7.  $0.390^{\circ}$
9. 850.1 cm    11. 7610 mm    13. 0.34 km    15.  $26.6^{\circ}, 63.4^{\circ}$
17.  $3.4^{\circ}$     19.  $23.5^{\circ}$     21.  $8.12^{\circ}$     23. 3.07 cm    25. 651 m
27.  $30.2^{\circ}$     29. 6.28 cm    31. 47.3 m    33. 282 m
35.  $d = 2x \tan(\frac{\theta}{2})$     37.  $A = a \sin \theta (b + a \cos \theta)$
39.  $35.3^{\circ}$

**Review Exercises for Chapter 4, page 137**

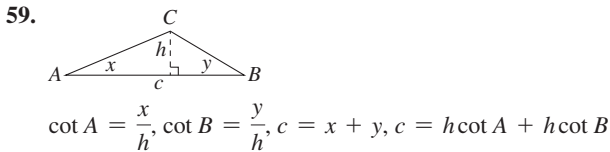
1.  $377.0^{\circ}, -343.0^{\circ}$     3.  $142.5^{\circ}, -577.5^{\circ}$     5.  $31.90^{\circ}$
7.  $-38.10^{\circ}$     9.  $17^{\circ}30'$     11.  $749^{\circ}42'$

**B.10 ANSWERS TO ODD-NUMBERED EXERCISES**

13.  $\sin \theta = \frac{7}{25}$      $\csc \theta = \frac{25}{7}$   
 $\cos \theta = \frac{24}{25}$      $\sec \theta = \frac{25}{24}$   
 $\tan \theta = \frac{7}{24}$      $\cot \theta = \frac{24}{7}$

15.  $\sin \theta = \frac{1}{\sqrt{2}}$      $\csc \theta = \sqrt{2}$   
 $\cos \theta = \frac{1}{\sqrt{2}}$      $\sec \theta = \sqrt{2}$   
 $\tan \theta = 1$      $\cot \theta = 1$

17. 0.923, 2.40    19. 0.447, 1.12    21. 0.952    23. 42.12  
 25. 1.05    27. 0.00    29. 18.2°    31. 57.57°    33. 12.25°  
 35. 87.7°    37. 7.998°    39. 88.85°  
 41.  $B = 73.0^\circ$     43.  $c = 104$     45.  $B = 52.5^\circ$   
 $a = 1.83$      $A = 51.5^\circ$      $b = 15.6$   
 $c = 6.27$      $B = 38.5^\circ$      $c = 19.7$   
 47.  $a = 4.006$     49.  $B = 40.33^\circ$     51.  $b = 10.196$   
 $A = 31.61^\circ$      $a = 0.6292$      $A = 48.813^\circ$   
 $B = 58.39^\circ$      $b = 0.5341$      $B = 41.187^\circ$   
 53. 4.6    55. 61.2    57. 10.5



61.  $\frac{\sqrt{x^2 + 1}}{x}$     63. 16.7°    65. 80.3°    67. 12.0°

69. (a)  $A = \frac{1}{2}bh = \frac{1}{2}b(a \sin C) = \frac{1}{2}ab \sin C$   
 (b) 679.2 m<sup>2</sup>  
 71. 4.92 km    73. 0.977 m<sup>2</sup>    75. 15.3 m<sup>2</sup>    77. 4.43 m  
 79. 34 m    81. 56%    83. 10.2 cm  
 85. End: 30.2 km  
 Middle: 30.2 km  
 87. 1.83 km  
 89. (a) 147 000 mm<sup>2</sup> (b) 155 000 mm<sup>2</sup>  
 (c) The flat surface approximation does not account for the curved surface of the soccer ball.  
 91. 464 m    93. 73.3 cm    95. 35 m<sup>2</sup>

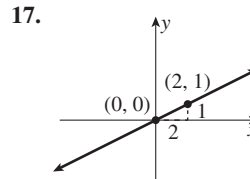
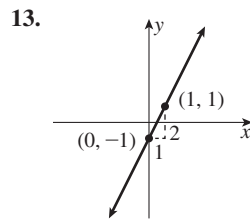
**Exercises 5.1, page 145**

1.  $x - \frac{1}{6}y + z - 4w = 7$  is linear  
 3.  $x = 4, y = 4$     5. No; yes    7. Yes; no    9.  $\frac{5}{3}, \frac{50}{3}$   
 11.  $-\frac{7}{2}, -9$     13. -2.0, 2.5    15. Yes    17. No    19. No  
 21. No    23. Yes    25. 6    27. Yes    29. No

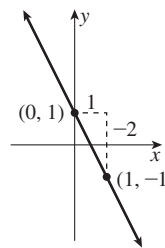
**Exercises 5.2, page 149**

1. The line rises 1 unit for each 4 units in going from left to right.  
 3. The slope is  $\frac{2}{3}$  and the y-intercept is the point  $(0, -\frac{4}{3})$ .

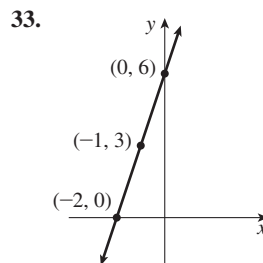
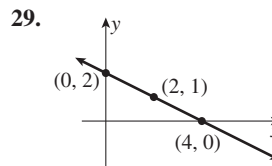
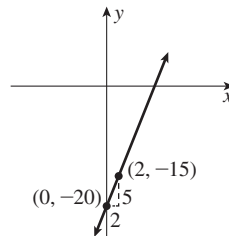
5. 4    7. -5    9.  $\frac{2}{7}$     11. 0.5



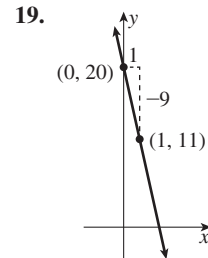
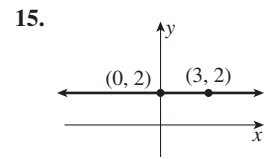
21.  $m = -2, b = 1$



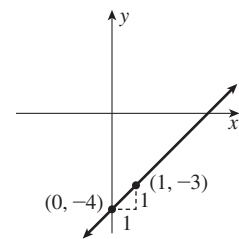
25.  $m = \frac{5}{2}, b = -20$



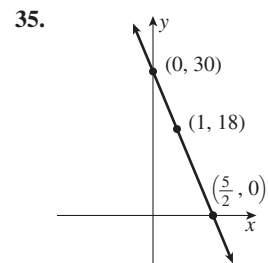
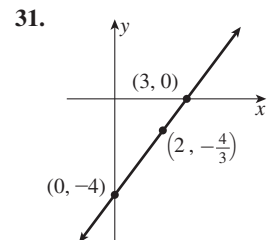
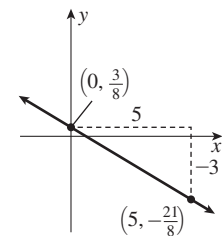
37. 6

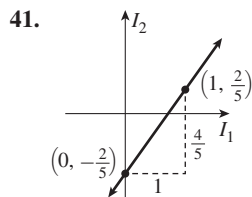
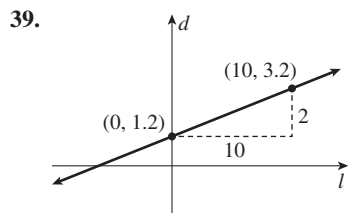


23.  $m = 1, b = -4$



27.  $m = -\frac{3}{5}, b = \frac{3}{8}$





**Exercises 5.3, page 152**

1.  $x = 1.5, y = 1.4$     3.  $x = 3.0, y = 1.0$   
 5.  $x = 3.0, y = 0.0$     7.  $x = 2.2, y = -0.3$   
 9.  $x = -0.9, y = -2.3$     11.  $t = -1.7, s = 1.1$   
 13.  $x = 0, y = 3$     15.  $x = -14.0, y = -5.0$   
 17.  $r_1 = 4.0, r_2 = 7.5$     19.  $x = 2.1, y = -0.4$   
 21.  $x = -3.600, y = -1.400$   
 23.  $x = 1.111, y = 0.841$     25. Dependent  
 27.  $t = -1.887, v = -3.179$   
 29.  $x = 1.500, y = 4.500$     31. Inconsistent  
 33.  $T_1 = 50\text{ N}, T_2 = 47\text{ N}$     35. 2.6 m, 3.4 m  
 37. 40 kg, 80 kg

**Exercises 5.4, page 158**

1.  $x = -3, y = -3$     3.  $x = \frac{16}{13}, y = -\frac{2}{13}$   
 5.  $x = 1, y = -2$     7.  $p = 3, V = 7$     9.  $x = -1, y = -4$   
 11.  $x = \frac{1}{2}, y = 2$     13.  $x = 1, y = \frac{1}{2}$     15.  $x = 3, y = 1$   
 17.  $x = -1, y = -2$     19.  $t = -\frac{1}{3}, y = 2$     21. Inconsistent  
 23.  $x = -\frac{14}{5}, y = -\frac{16}{5}$     25.  $x = \frac{1}{2}, y = -4$   
 27.  $x = -\frac{2}{3}, y = 0$     29.  $x = \frac{3}{5}, y = \frac{1}{5}$     31.  $V = -2, C = -1$   
 33.  $A = -1, B = 3$     35.  $a = \frac{1}{3}, b = -7$   
 37.  $x = \frac{69}{29}, y = \frac{13}{29}$     39.  $s = \frac{39}{10}, t = \frac{16}{5}$   
 41.  $V_1 = 9\text{ V}, V_2 = 6\text{ V}$     43.  $x = 6250\text{ L}, y = 3750\text{ L}$   
 45. 54    47.  $W_f = 9580\text{ N}, W_r = 8120\text{ N}$   
 49.  $t_1 = 32\text{ s}, t_2 = 20\text{ s}$   
 51. 34 at \$900 per month, 20 at \$1250 per month  
 53. 7.00 kW, 9.39 kW  
 55. This is an inconsistent system. There could be an error in the sales figures, and/or the conclusion is in error.  
 57.  $a \neq b$     59.  $(-3, -4), (9, 0)$

**Exercises 5.5, page 164**

1. 86    3.  $x = -13, y = -27$     5. -10    7. 29  
 9. 32    11. 9300    13. 1.083    15. 96    17.  $x = 3, y = 1$   
 19.  $x = -1, y = -2$     21.  $t = -\frac{1}{3}, y = 2$     23. Inconsistent  
 25.  $x = -\frac{14}{5}, y = -\frac{16}{5}$     27.  $x = \frac{69}{29}, y = \frac{13}{29}$

29.  $s = \frac{39}{10}, t = \frac{16}{5}$     31.  $x = -11.2, y = -9.26$   
 33.  $x = -1.0, y = -2.0$     35. 0    37. 0  
 39.  $F_1 = 15\text{ N}, F_2 = 6.0\text{ N}$   
 41.  $x = 73.6\text{ L}, y = 70.4\text{ L}$   
 43. 160 3-bedroom homes, 80 4-bedroom homes  
 45. 210 phones, 110 detectors    47. \$2000, 6.0%  
 49. 2.5 h, 2.1 h    51.  $V(i) = 4.5i - 3.2$

**Exercises 5.6, page 168**

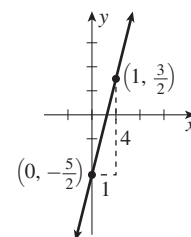
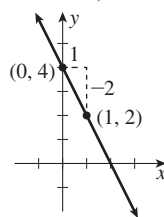
1.  $x = 1, y = -4, z = \frac{1}{3}$     3.  $x = 2, y = -1, z = 1$   
 5.  $x = 4, y = -3, z = 3$     7.  $i = \frac{1}{2}, w = \frac{2}{3}, h = \frac{1}{6}$   
 9.  $x = \frac{2}{3}, y = -\frac{1}{3}, z = 1$     11.  $x = \frac{4}{15}, y = -\frac{3}{5}, z = \frac{1}{3}$   
 13.  $x = \frac{3}{4}, y = 1, z = -\frac{1}{2}$     15.  $r = 0, s = 0, t = 0, u = -1$   
 17.  $A = 2, B = 1, C = 1$     19.  $P = 800\text{ h}, M = 125\text{ h}, I = 225\text{ h}$   
 21.  $F_1 = 9.43\text{ N}, F_2 = 8.33\text{ N}, F_3 = 1.67\text{ N}$   
 23.  $A = 22.5^\circ, B = 45.0^\circ, C = 112.5^\circ$   
 25.  $a = 0.295, b = -5.53, c = 24.2$   
 27.  $x = 70.0\text{ kg}, y = 100.0\text{ kg}, z = 30.0\text{ kg}$   
 29. Infinite number of solutions. A possible solution is  $x = -10, y = -6, z = 0$   
 31. Inconsistent

**Exercises 5.7, page 174**

1. -38    3. 122    5. 651    7. -439    9. 202  
 11. 29 440    13. 0.128    15.  $x = -1, y = 2, z = 0$   
 17.  $x = 2, y = -1, z = 1$     19.  $x = 4, y = -3, z = 3$   
 21.  $l = \frac{1}{2}, w = \frac{2}{3}, h = \frac{1}{6}$     23.  $x = \frac{2}{3}, y = -\frac{1}{3}, z = 1$   
 25.  $x = \frac{4}{15}, y = -\frac{3}{5}, z = \frac{1}{3}$     27.  $p = -2, q = \frac{2}{3}, r = \frac{1}{3}$   
 29.  $x = \frac{3}{4}, y = 1, z = -\frac{1}{2}$     31. Value changes from 19 to -19.  
 33. 19 (no change)    35.  $A = 125\text{ N}, B = 60.0\text{ N}, C = 75.0\text{ N}$   
 37.  $s_0 = 2.00\text{ m}, v_0 = 5.00\text{ m/s}, a = 4.00\text{ m/s}^2$   
 39.  $V = f(T) = 5.00 + 0.500T + 0.100T^2$   
 41. 79% nickel, 16% iron, 5% molybdenum  
 43.  $v_c = 45.9\text{ km/h}, v_j = 551\text{ km/h}, v_t = 30.9\text{ km/h}$   
 45.  $F_1 = 27.9\text{ N}, F_2 = 44.5\text{ N}, F_3 = 52.6\text{ N}$

**Review Exercises for Chapter 5, page 176**

1. -17    3. -1485    5. -4    7.  $\frac{2}{7}$   
 9.  $m = -2, b = 4$     11.  $m = 4, b = -\frac{5}{2}$



**B.12 ANSWERS TO ODD-NUMBERED EXERCISES**

13.  $x = 2, y = 0$     15.  $A = 2.2, B = 2.7$   
 17.  $x = 1.5, y = -1.9$     19.  $M = 1.5, N = 0.4$   
 21.  $x = 1, y = 2$     23.  $x = \frac{1}{2}, y = -2$     25.  $i = 2, v = -\frac{1}{3}$   
 27.  $x = -\frac{6}{19}, y = \frac{36}{19}$     29.  $x = \frac{43}{39}, y = \frac{7}{13}$     31.  $x = 1, y = 2$   
 33.  $x = \frac{1}{2}, y = -2$     35.  $i = 2, v = -\frac{1}{3}$   
 37.  $x = -\frac{6}{19}, y = \frac{36}{19}$     39.  $x = \frac{43}{39}, y = \frac{7}{13}$   
 41. 33 (the second equation is already solved for  $y$ )  
 43. 40 (the coefficients don't indicate another method)  
 45. -115    47. 230.08    49.  $x = 2, y = -1, z = 1$   
 51.  $r = 3, s = -1, t = \frac{3}{2}$     53.  $x = -0.168, y = 0.156, z = 2.41$   
 55.  $x = 2, y = -1, z = 1$     57.  $r = 3, s = -1, t = \frac{3}{2}$   
 59.  $x = -0.168, y = 0.156, z = 2.41$     61. 4    63.  $-\frac{15}{7}$   
 65.  $x = \frac{8}{3}, y = -8$   
 67.  $x = 1, y = 3$     69. -6    71.  $-\frac{4}{3}$   
 73.  $F_1 = 21\,000\text{ N}, F_2 = 2400\text{ N}, F_3 = 18\,000\text{ N}$   
 75.  $p_1 = 42\%, p_2 = 58\%$     77. \$40\,000, \$4000  
 79. 27.6 tonnes of 6.0% copper, 14.4 tonnes of 2.4% copper  
 81.  $a = 444\text{ m}\cdot^\circ\text{C}, b = 9.56\text{ }^\circ\text{C}$     83. 22\,800 km/h, 1400 km/h  
 85.  $R_1 = 0.500\ \Omega, R_2 = 1.50\ \Omega$     87.  $L = 10.0\text{ N}, w = 40.0\text{ N}$   
 89.  $A = 75.0^\circ, B = 65.0^\circ, C = 40.0^\circ$   
 91.  $A = 68.0\text{ MB}, B = 24.0\text{ MB}, C = 48.0\text{ MB}$   
 93. 4.26 N, 1.74 N    95. 8.84 m<sup>3</sup>/h, 4.65 m<sup>3</sup>/h

**Exercises 6.1, page 184**

1.  $9r^2 - 4s^2$     3.  $x^2 + 2xy + y^2 - 4x - 4y + 4$   
 5.  $40x - 40y$     7.  $2x^3 - 8x^2$     9.  $T^2 - 36$   
 11.  $9v^2 - 4$     13.  $16x^2 - 25y^2$     15.  $144 - 25a^2b^2$   
 17.  $25f^2 + 40f + 16$     19.  $4x^2 + 68x + 289$   
 21.  $L^4 - 2L^2 + 1$     23.  $16a^2 + 56axy + 49x^2y^2$   
 25.  $0.36s^2 - 1.2st + t^2$     27.  $x^2 + 6x + 5$   
 29.  $18 + 9C^2 + C^4$     31.  $20x^2 - 21x - 5$   
 33.  $40v^2 + 138v - 45$     35.  $600x^2 - 130xy - 63y^2$   
 37.  $2x^2 - 8$     39.  $8a^3 - 2a$     41.  $6ax^2 + 24abx + 24ab^2$   
 43.  $20n^4 + 100n^3 + 125n^2$     45.  $16R^4 - 72R^2r^2 + 81r^4$   
 47.  $x^2 + 2xy + y^2 + 2x + 2y + 1$   
 49.  $9 - 6x - 6y + x^2 + 2xy + y^2$     51.  $-t^3 + 15t^2 - 75t + 125$   
 53.  $27L^3 + 189L^2R + 441LR^2 + 343R^3$     55.  $w^2 + 2wh + h^2 - 1$   
 57.  $x^3 + 8$     59.  $64 - 27x^3$     61.  $x^4 - 2x^2y^2 + y^4$   
 63.  $P_1P_0C + P_1G$     65.  $4p^2 + 8pDA + 4D^2A^2$   
 67.  $\frac{1}{2}\pi R^2 - \frac{1}{2}\pi r^2$     69.  $\frac{L}{6}x^3 - \frac{L}{2}ax^2 + \frac{L}{2}a^2x - \frac{L}{6}a^3$   
 71.  $L_0 + aL_0T - aL_0T_0$     73. 2499    75.  $4x^2 - 9$   
 77. (a)  $(x + y)^2$     (b)  $x^2 + 2xy + y^2$  (Eq. 6.3)  
 79.  $4x^2 - y^2 - 4y - 4$

**Exercises 6.2, page 189**

1.  $2ax(2x - 1)$     3.  $5(x + 3)(x - 3)$   
 5.  $6(x + y)$     7.  $5(a - 1)$     9.  $3x(x - 3)$   
 11.  $7b(bh - 4)$     13.  $24n(12n + 1)$

15.  $2(x + 2y - 4z)$     17.  $3ab(b - 2 + 4b^2)$   
 19.  $4pq(3q - 2 - 7q^2)$     21.  $2(a^2 - b^2 + 2c^2 - 3d^2)$   
 23.  $(x + 2)(x - 2)$     25.  $(10 + 3A)(10 - 3A)$   
 27.  $36a^4 + 1$     29.  $2(9s + 5t)(9s - 5t)$   
 31.  $(12n + 13p^2)(12n - 13p^2)$   
 33.  $(x + y + 3)(x + y - 3)$     35.  $2(x + 2)(x - 2)$   
 37.  $300(x + 3z)(x - 3z)$     39.  $2(I - 1)(I - 5)$   
 41.  $(x^2 + 4)(x + 2)(x - 2)$   
 43.  $(x^4 + 1)(x^2 + 1)(x + 1)(x - 1)$   
 45.  $\frac{b + 3}{2 - b}$     47.  $\frac{3}{2(t - 1)}$     49.  $\frac{5}{2}$     51.  $(x - y)(b + 3)$   
 53.  $(a + x)(a - b)$     55.  $(x + 3)(x + 2)(x - 2)$   
 57.  $(x - y)(x + y + 1)$     59. 16\,777\,216  
 61.  $n^2 + n = n(n + 1)$ ; the product is even.    63.  $2\pi r(h + r)$   
 65.  $Rv(1 + v + v^2)$     67.  $r(R + r)(R - r)$   
 69.  $r^2(\pi - 2)$     71.  $\frac{i_2R_2}{R_1 + R_2}$     73.  $\frac{5Y}{3(3S - Y)}$   
 75.  $\frac{ER}{A(T_0 - T_1)}$

**Exercises 6.3, page 195**

1.  $(x + 3)(x + 1)$     3.  $(2x - 1)(x - 5)$   
 5.  $2(x + 6)(x - 3)$     7.  $(x + 1)(x + 4)$   
 9.  $(s - 7)(s + 6)$     11.  $(t + 8)(t - 3)$     13.  $(x + 1)^2$   
 15.  $(L - 2K)^2$     17.  $(3x + 1)(x - 2)$   
 19.  $4(3y + 1)(y - 3)$     21.  $(2s + 11)(s + 1)$   
 23.  $(3f^2 - 1)(f^2 - 5)$     25.  $(2t - 3)(t + 5)$   
 27.  $(3t - 4u)(t - u)$     29.  $(4x - 7)(x + 1)$   
 31.  $(x + y)(9x - 2y)$     33.  $(2m + 5)^2$     35.  $2(2x - 3)^2$   
 37.  $(3t - 4)(3t - 1)$     39.  $(2b - 1)(4b^2 + 2b + 1)(b^3 + 4)$   
 41.  $(4p - q)(p - 6q)$     43.  $(12x - y)(x + 4y)$   
 45.  $2(x - 1)(x - 6)$     47.  $2x(2x^2 - 1)(x^2 + 4)$   
 49.  $ax(x + 6a)(x - 2a)$     51.  $(a + b + 2)(a + b - 2)$   
 53.  $(5a + 5x + y)(5a - 5x - y)$     55.  $(4x^n - 3)(x^n + 4)$   
 57.  $16(t - 4)(t - 1)$     59.  $4(s + 3)(s + 1)$   
 61.  $100(2n + 3)(n - 12)$     63.  $(V - nB)^2$   
 65.  $wx^2(x - 3L)(x - 2L)$     67.  $Ad(3u - v)(u - v)$   
 69.  $k = -1, (2x + 1)^2$     71.  $(x^2 + 2x + 2)(x^2 - 2x + 2)$   
 73.  $9(4x^2 + 1)$     75. (a)  $(2n^2 + 3)(6n^2 - 5)$   
       (b)  $(p - 6q)(4p - q)$     (c)  $(6x + 5y)(2x - y)$   
       (d)  $(x + 4y)(12x - y)$

**Exercises 6.4, page 197**

1.  $(x - 2)(x^2 + 2x + 4)$     3.  $(x + 1)(x^2 - x + 1)$   
 5.  $(2 - t)(4 + 2t + t^2)$     7.  $(3x - 2a^2)(9x^2 + 6a^2x + 4a^4)$   
 9.  $4(x + 2)(x^2 - 2x + 4)$     11.  $6A^3(A - 1)(A^2 + A + 1)$   
 13.  $6x^3y(3 - y)(9 + 3y + y^2)$   
 15.  $x^3y^3(x + y)(x^2 - xy + y^2)$   
 17.  $3a^3(a - 1)(a^2 + 2a + 1)$   
 19.  $0.001(R - 4r)(R^2 + 4Rr + 16r^2)$   
 21.  $27L^3(L + 2)(L^2 - 2L + 4)$



23.  $(a + b + 4)(a^2 + 2ab + b^2 - 4a - 4b + 16)$   
 25.  $(2 + x)(2 - x)(16 + 4x^2 + x^4)$   
 27.  $2(x + 5)(x^2 - 5x + 25)$   
 29.  $D(D - d)(D^2 + Dd + d^2)$   
 31.  $QH(H + Q)(H^2 - HQ + Q^2)$   
 33.  $x^5 - y^5 = (x - y)(x^4 + x^3y + x^2y^2 + xy^3 + y^4)$ ,  
 $x^7 - y^7 = (x - y)(x^6 + x^5y + x^4y^2 + x^3y^3 + x^2y^4 + xy^5 + y^6)$   
 35.  $(x + y)(x - y)(x^4 + x^2y^2 + y^4)$   
 37.  $n^3 + 1 = (n + 1)(n^2 - n + 1)$ ;  $(n + 1)$  is a factor of  $n^3 + 1$  so  $n^3 + 1$  is not prime.

**Exercises 6.5, page 201**

1.  $\frac{3c}{4b}$ ,  $a, b, c \neq 0$     3.  $\frac{x + 2}{x - 2}$ ,  $x \neq -2$     5.  $\frac{14}{21}$     7.  $\frac{2ax^2}{2xy}$   
 9.  $\frac{2x - 4}{x^2 + x - 6}$     11.  $\frac{ax^2 - ay^2}{x^2 - xy - 2y^2}$     13.  $\frac{7}{11}$   
 15.  $\frac{2xy}{4y^2}$ ,  $x \neq 0$     17.  $\frac{2}{R + 1}$ ,  $R \neq 1$     19.  $\frac{s - 5}{2s - 1}$ ,  $s \neq -2$   
 21.  $9xy$     23.  $a^2 - 25$     25.  $2x$     27.  $1, x \neq b$   
 29.  $\frac{1}{4}$ ,  $a \neq 0$     31.  $\frac{3x}{4}$ ,  $x, y \neq 0$     33.  $\frac{1}{5a}$ ,  $a + b \neq 0$   
 35.  $\frac{2(a - b)}{2a - b}$     37.  $\frac{4x^2 + 1}{(2x + 1)(2x - 1)}$     39.  $3x, x \neq 2$   
 41.  $\frac{1}{2y^2}$ ,  $y \neq -\frac{3}{2}$     43.  $\frac{x - 4}{x + 4}$ ,  $x \neq 4$     45.  $\frac{2w^2 - 1}{w^2 + 8}$   
 47.  $\frac{5x + 4}{x(x + 3)}$ ,  $x \neq 2$     49.  $\frac{(N^2 + 4)(N + 2)}{8}$ ,  $N \neq 2$   
 51.  $\frac{1}{2t + 1}$ ,  $t \neq -4$     53.  $\frac{x + 3}{x - 3}$ ,  $x \neq 1$     55.  $-\frac{y + x}{2}$ ,  $x \neq y$   
 57.  $\frac{x + 1}{x - 1}$ ,  $x \neq \pm 1$     59.  $\frac{(x + 5)(x - 3)}{(5 - x)(x + 3)}$ ,  $x \neq \pm 2$   
 61.  $\frac{x^2 - xy + y^2}{2}$ ,  $x \neq -y$     63.  $\frac{2x}{9x^2 - 3x + 1}$ ,  $x \neq -\frac{1}{3}$   
 65. (a)  $\frac{x^2(x + 2)}{x^2 + 4}$  ( $x^2 + 4$  does not factor)  
 (b)  $\frac{x^2}{(x + 2)(x - 2)}$  (there are no more common factors)  
 67. (a)  $\frac{(x - 2)(x + 1)}{x(x - 1)}$  (no common factor)  
 (b)  $\frac{x - 2}{x}$ ,  $x \neq -1$  (no common factor)  
 69.  $u + v, u \neq v$     71.  $\frac{E^2(R - r)}{(R + r)^3}$

**Exercises 6.6, page 205**

1.  $\frac{2(x + y)}{3(x - y)}$ ,  $x \neq y, -\frac{3y}{2}$     3.  $3(x + 2y)$ ,  $x \neq -y$     5.  $\frac{3}{28}$   
 7.  $6xy, y \neq 0$     9.  $\frac{7}{18}$     11.  $\frac{xy^2}{bz^2}$ ,  $a \neq 0$     13.  $4t, x \neq -3$   
 15.  $3(u + v)(u - v), u \neq -2v$     17.  $\frac{50}{3(a + 4)}$ ,  $a \neq -4$   
 19.  $\frac{x^2 - 3}{x^2(x^2 + 3)}$     21.  $\frac{3x}{5a}$ ,  $x \neq 0, 3, -\frac{1}{2}$  and  $a \neq 0$

23.  $\frac{(x + 1)(x - 1)(x - 4)}{4(x + 2)}$ ,  $x \neq 0$   
 25.  $\frac{x^2}{a + x}$ ,  $x \neq -a, b \neq \frac{cx}{2}$     27.  $\frac{15}{4}, a \neq -\frac{5}{7}, -\frac{11}{4}$   
 29.  $\frac{3}{4x + 3}$ ,  $x \neq 5, 1, -\frac{7}{2}$   
 31.  $\frac{2(3T + N)(5V - 6)}{(V - 7)(4T + N)}$ ,  $T \neq \frac{N}{2}, V \neq -\frac{5}{2}$     33.  $\frac{7x^4}{3a^4}$ ,  $x \neq 0$   
 35.  $\frac{4t(2t - 1)(t + 5)}{(2t + 1)^2}$ ,  $t \neq 5, -\frac{1}{2}$     37.  $\frac{x + y}{2}$   
 39.  $(x + y)(3p + 7q)$ ,  $a \neq -b, p \neq q$   
 41.  $\frac{x - 2}{6x}$ ,  $x \neq -2, 0$     43.  $-\frac{2(2x - 1)}{5(x - 2)}$ ,  $x \neq -\frac{1}{2}, -2$   
 45.  $\frac{2v_1v_2}{v_1 + v_2}$ ,  $d \neq 0$     47.  $\frac{\pi}{2}$ ,  $a, b, \lambda \neq 0, a \neq -b$

**Exercises 6.7, page 210**

1.  $12a^2b^3$     3.  $\frac{x^3 - 10x^2 + 36x - 90}{2(x - 4)^2(x + 3)}$     5.  $\frac{9}{5}$     7.  $\frac{8}{x}$     9.  $\frac{5}{4}$   
 11.  $\frac{3 + 7ax + 8x}{4x}$     13.  $\frac{ax - b}{x^2}$     15.  $\frac{30 + ax^2}{25x^3}$   
 17.  $\frac{14 - a^2}{10a}$     19.  $\frac{xy + 2y + 7x - 8}{4xy}$     21.  $\frac{7}{2(2x - 1)}$   
 23.  $\frac{5 - 3x}{2x(x + 1)}$     25.  $\frac{-3}{4(s - 3)}$     27.  $\frac{7R + 6}{3(R + 3)(R - 3)}$   
 29.  $\frac{2x - 5}{(x - 4)^2}$     31.  $\frac{-4}{(v + 1)(v - 3)}$     33.  $\frac{9x^2 + x - 2}{(3x - 1)(x - 4)}$   
 35.  $\frac{-t(2t^2 - 3t - 15)}{(t + 3)^2(t - 3)(t + 2)}$     37.  $\frac{-w(2w^2 - w + 1)}{(w + 1)(w^2 - w + 1)}$   
 39.  $\frac{1}{x - 1}$ ,  $x \neq 0$     41.  $\frac{x - y}{y}$ ,  $x \neq 0, -y$   
 43.  $-\frac{(3x + 4)(x - 1)}{2x}$ ,  $x \neq 0, 1, -1$   
 45.  $\frac{h}{(x + 1)(x + h + 1)}$     47.  $\frac{-h(2x + h)}{x^2(x + h)^2}$   
 49.  $\frac{r^2 + y^2 - rx}{r^2}$     51.  $\frac{2a - 1}{a^2}$     53.  $\frac{a^2 + 2a - 1}{a + 1}$   
 55.  $ab, a \neq -b$     57.  $\frac{2mn}{m^2 + n^2}$     59.  $\frac{3(H - H_0)}{4\pi H}$   
 61.  $\frac{n(2n + 1)}{2(n - 1)(n + 2)}$     63.  $\frac{P^2(9x^2 + L^2)}{4L^4}$   
 65.  $\frac{Ls + R}{CLs^2 + CRs + 1}$     67.  $\frac{\omega^2L^2 + \omega^4L^2c^2R^2 - 2\omega^2LcR^2 + R^2}{R^2\omega^2L^2}$

**Exercises 6.8, page 215**

1.  $\frac{2}{b - 1}$     3.  $\frac{arv^2}{M(2a - r)}$     5.  $4$     7.  $-3$     9.  $\frac{13}{2}$   
 11.  $\frac{16}{21}$     13.  $-9$     15.  $-\frac{2}{13}$     17.  $\frac{5}{3}$     19.  $-2$   
 21.  $-\frac{1}{2}$     23.  $\frac{37}{6}$     25.  $7$     27.  $\frac{63}{8}$     29. No solution  
 31.  $\frac{2}{3}$     33.  $\frac{3b}{1 - 2b}$     35.  $\frac{(2b - 1)(b + 6)}{2(b - 1)}$

**B.14 ANSWERS TO ODD-NUMBERED EXERCISES**

37.  $\frac{2(s - s_0) - tv_0}{t}$     39.  $\frac{8.0(V - 6.0)}{15.6 - V}$     41.  $\frac{jX}{1 - 8mz}$   
 43.  $\frac{pV^3 - pV^2b + aV - ab}{RV^2}$     45.  $\frac{CC_3 + CC_2 - C_2C_3}{C_2 - C}$   
 47.  $\frac{f(1 - n)(R_2)}{nf - f - R_2}$     49. 3.08 h    51. 3.24 min    53. 220 m  
 55. 80.1 km/h    57. 2.23 V    59.  $A = 3, B = -2$

**Review Exercises for Chapter 6, page 217**

1.  $12ax + 15a^2$     3.  $4a^2 - 49b^2$     5.  $4a^2 + 4a + 1$   
 7.  $b^2 + 3b - 28$     9.  $2x^2 - 13x - 45$     11.  $4c^2a - d^2$   
 13.  $3(s + 3t)$     15.  $a^2(x^2 + 1)$     17.  $b^x(Wb + 12)(Wb - 12)$   
 19.  $(4x + 8 + t^2)(4x + 8 - t^2)$     21.  $4(3t - 1)^2$   
 23.  $(5t + 1)^2$     25.  $(x + 8)(x - 7)$   
 27.  $(t + 3)(t - 3)(t^2 + 4)$     29.  $(2k - 9)(k + 4)$   
 31.  $2(2x + 5)(2x - 7)$     33.  $(5b - 1)(2b + 5)$   
 35.  $4(x + 4y)(x - 4y)$     37.  $2(5 - 2y^2)(25 + 10y^2 + 4y^4)$   
 39.  $(2x + 3)(4x^2 - 6x + 9)$     41.  $(a - 3)(b^2 + 1)$   
 43.  $(x + 5)(n - x + 5)$     45.  $\frac{16x^2}{3a^2}, a, x, y \neq 0$   
 47.  $\frac{3x + 1}{2x - 1}, x \neq \frac{3}{2}$     49.  $\frac{16}{5x(x - y)}, x \neq 0, -y$   
 51.  $\frac{-6}{L - 5}, L \neq \pm 3$     53.  $\frac{x + 2}{2x(7x - 1)}, x \neq 0, -2$   
 55.  $\frac{1}{x - 1}, x \neq 0$     57.  $\frac{16x - 15}{36x^2}$     59.  $\frac{5y + 6}{2xy}$   
 61.  $\frac{-2(2a + 3)}{a(a + 2)}$     63.  $\frac{4x^2 - x + 1}{2x(x + 3)(x - 1)}$   
 65.  $\frac{12x^2 - 7x - 4}{2(4x - 1)(x + 1)(x - 1)}$     67.  $\frac{x^3 + 6x^2 - 2x + 2}{x(x + 3)(x - 1)}$   
 69.  $(x + \sqrt{5})(x - \sqrt{5})$     71.  $(\sqrt{x} + 1)(\sqrt{x} - 1)$   
 73.  $x\left(1 + \frac{y}{x}\right)$     75.  $2\left(\frac{x}{2} - y\right)$     77. 2    79.  $-\frac{(a - 1)^2}{2a}$   
 81.  $\frac{10}{3}$     83. -6    85. (a) Changes the sign of the fraction.  
 (b) Leaves the sign of the fraction unchanged.  
 87.  $\frac{1}{4}[(x + y)^2 - (x - y)^2]$   
 $= \frac{1}{4}[x^2 + 2xy + y^2 - (x^2 - 2xy + y^2)]$   
 $= \frac{1}{4}[x^2 + 2xy + y^2 - x^2 + 2xy - y^2]$   
 $= \frac{1}{4}[4xy]$   
 $= xy$   
 89.  $2zS^2 + 2zS$     91.  $4b^2 + 4bn\lambda - 4b\lambda + n^2\lambda^2 - 2n\lambda^2 + \lambda^2$   
 93.  $(T_2 - T_1)(c + R)$     95.  $R(3R - 4r)$   
 97.  $8n^6 + 36n^5 + 66n^4 + 63n^3 + 33n^2 + 9n + 1$   
 99.  $10aT - 10at + aT^2 - 2aTt + at^2$

101.  $4(3x^2 + 12x + 16)$     103.  $\frac{12wv^2\pi^2D}{gn^2t}$   
 105.  $\frac{R^2 + r^2}{2}, R \neq r, -r$     107.  $\frac{120 - 60d^2 + 5d^4 - d^6}{120}$   
 109.  $\frac{4k^2 + k - 2}{4k(k - 1)}$     111.  $\frac{4r^3 - 3ar^2 - a^3}{4r^3}$   
 113.  $\frac{c^2(u^2 - 2gx)}{1 - u^2c^2 + 2gc^2x}$     115.  $\frac{W}{g(h_2 - h_1)}$     117.  $\frac{RHw}{w - RH}$   
 119.  $\frac{p^2}{2E - 2V_0 - (m + M)V^2}$     121.  $-\frac{(s^2b^2m + kL^2)}{sb^2}$   
 123.  $\frac{i}{sV - V_0}$     125. 3.43 h    127. 600 Hz    129. 11.3  
 131. 18.0  $\Omega$

**Exercises 7.1, page 224**

1.  $-\frac{1}{2}, 4$     3.  $a = 1, b = -2, c = -4$   
 5. No  $x^2$  term so it is not quadratic.    7.  $a = 1, b = -1, c = 0$   
 9. -2, 2    11.  $-\frac{3}{2}, \frac{3}{2}$     13. 9, -1    15. 4, 3    17.  $0, \frac{5}{2}$   
 19.  $\frac{1}{3}, -\frac{1}{3}$     21.  $4, \frac{1}{3}$     23.  $-1, \frac{4}{7}$     25.  $\frac{2}{3}, \frac{3}{2}$     27.  $\frac{1}{2}, -\frac{3}{2}$   
 29.  $\frac{b}{2}, -\frac{2b}{3}$     31.  $\frac{5}{2}, -\frac{9}{2}$     33. 0, -2    35.  $b - a, -b - a$   
 37.  $-a - b, b - a$     39.  $\frac{1}{2} + 3 = \frac{7}{2} = -\frac{-7}{2} = -\frac{b}{a}$   
 41. -6.00 A or 2.00 A    43. 2:00 a.m., 10:00 a.m.  
 45. -1, 0, 1    47.  $4, \frac{3}{2}$     49.  $-2, \frac{1}{2}$   
 51. 3 N/cm, 6 N/cm    53. 30 km/h going, 40 km/h returning

**Exercises 7.2, page 228**

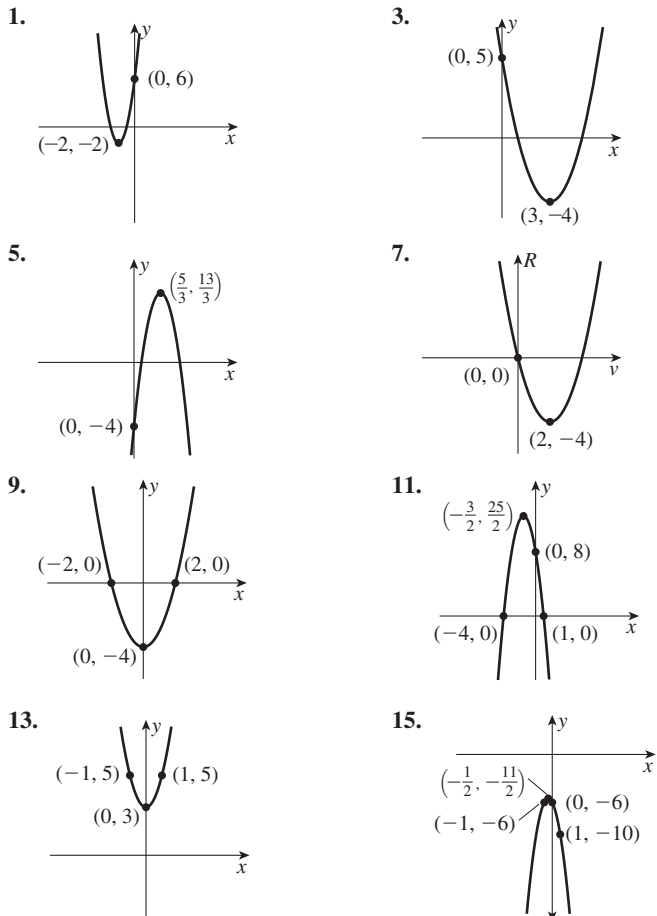
1.  $-3 \pm \sqrt{17}$     3.  $\pm 5$     5.  $\pm \sqrt{7}$     7. -3, 7  
 9.  $-3 \pm \sqrt{7}$     11. 2, -4    13. -2, -1    15.  $2 \pm \sqrt{2}$   
 17. -5, 3    19.  $-3, \frac{1}{2}$     21.  $\frac{1}{2} \pm \frac{\sqrt{33}}{6}$     23.  $\frac{1}{4} \pm \frac{\sqrt{17}}{4}$   
 25.  $1 \pm \frac{\sqrt{5}}{5}$     27.  $-\frac{1}{3}$  (double root)    29.  $-b \pm \sqrt{b^2 - c}$   
 31.  $(x + 3)^2 + 2^2$     33. 5.0°C, 15°C    35. 5.61 m

**Exercises 7.3, page 231**

1. -2, -3    3.  $\frac{4}{3}$  (double root)    5. 2, -4    7. -2, -1  
 9.  $2 \pm \sqrt{2}$     11. -5, 3    13.  $-3, \frac{1}{2}$   
 15.  $\frac{1}{6}(3 \pm \sqrt{33})$     17.  $\frac{1 \pm \sqrt{17}}{4}$     19.  $-\frac{8}{5}, \frac{5}{6}$

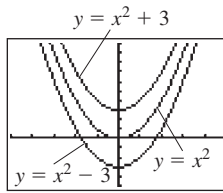
21.  $\frac{-61 \pm \sqrt{-119}}{16}$  (imaginary roots)  
 23.  $\frac{1}{6}(1 \pm \sqrt{109})$     25.  $\frac{11}{5}, -\frac{11}{5}$     27.  $\frac{3}{4}, -\frac{5}{8}$   
 29. -0.540, 0.740    31. -0.256, 2.43  
 33.  $-c \pm \sqrt{c^2 + 1}$     35.  $\frac{b + 1 \pm \sqrt{4ab^2 - 3b^2 + 2b + 1}}{2b^2}$   
 37. Unequal, imaginary    39. Real, irrational, and unequal  
 41. 4    43.  $\pm 2, \pm 1$     45. 4.38 cm    47. 1.618  
 49.  $\frac{-R \pm \sqrt{R^2 - 4L/C}}{2L}$     51. 1.78 cm  
 53.  $l = 23.8$  m,  $w = 11.0$  m  
 55. 1.11 m    57. 80.0 km/h, 100.0 km/h

**Exercises 7.4, page 235**

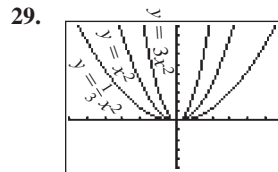
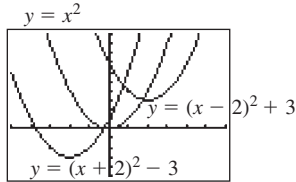


17. -1.22, 1.22    19. 0.532, 3.14    21. No real solutions  
 23. -2.41, 1.24  
 25. The parabola  $y = x^2 + 3$  is shifted up +3 units (minimum point  $(0, 3)$ ).  
 The parabola  $y = x^2 - 3$  is shifted down -3 units

(minimum point  $(0, -3)$ ).

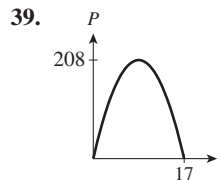
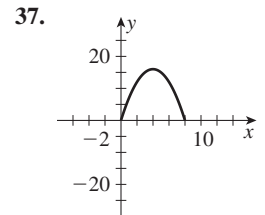
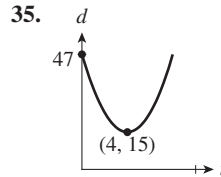


27.  $y = (x - 2)^2 + 3$  is  $y = x^2$  shifted right 2 and up 3.  
 $y = (x + 2)^2 - 3$  is  $y = x^2$  shifted left 2 and down 3.



29. The graph of  $y = 3x^2$  is the graph of  $y = x^2$  narrowed. The graph of  $y = \frac{1}{3}x^2$  is the graph of  $y = x^2$  broadened.

31. 7    33. -2



37. (a) 18.9 s    (b) 463 m    (c) 2.6 s, 15.8 s  
 41. 4.53 cm    43. 21.6 m by 92.6 m, or 32.4 m by 61.7 m

**Review Exercises for Chapter 7, page 237**

1. -4, 1    3. 2, 8    5.  $-4, \frac{1}{3}$     7.  $\frac{1}{2}, \frac{5}{3}$   
 9.  $0, \frac{25}{6}$     11.  $-\frac{3}{2}, \frac{7}{2}$     13. -10.0, 11.0  
 15.  $-1 \pm \sqrt{7}$     17.  $\frac{9}{2}, -4$     19.  $\frac{3 \pm \sqrt{41}}{8}$   
 21.  $\frac{-2.3 \pm \sqrt{-40.91}}{4.2}$  (imaginary roots)  
 23.  $\frac{-2 \pm \sqrt{58}}{6}$     25.  $-2 \pm 2\sqrt{2}$     27.  $\frac{-4 \pm \sqrt{103}}{8}$

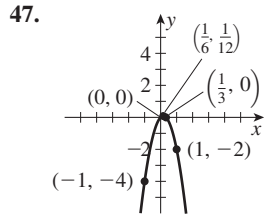
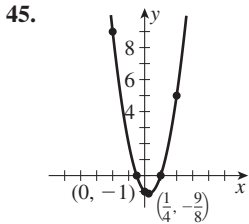
**B.16 ANSWERS TO ODD-NUMBERED EXERCISES**

29.  $-1, \frac{5}{4}$     31.  $\frac{-3 \pm \sqrt{-47}}{4}$  (imaginary roots)

33.  $\frac{-1 \pm \sqrt{-1}}{a}$  ( $a \neq 0$ , imaginary roots)

35.  $\frac{-3 \pm \sqrt{9 + 4a^2}}{2a}$     37.  $-5, 6$     39.  $\frac{1 \pm \sqrt{33}}{4}$

41.  $3 \pm \sqrt{7}$     43. 0

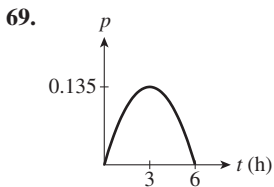


49.  $-1.69, 1.19$     51. No real roots.    53.  $-4$

55. 0,  $L$     57. 17    59.  $95.8^\circ\text{C}$     61. 0.55 s, 2.22s

63. 8000    65.  $\frac{-\pi h + \sqrt{\pi^2 h^2 + 2\pi A}}{2\pi}$

67.  $\frac{r + 1 \pm \sqrt{(r + 1)^2 - 4rp_2}}{2r}$



71. 81.4 m and 61.4 m    73. 9.88 cm

75. 570 m by 570 m, 670 m by 970 m.

77.  $40.7 \text{ cm} \times 55.2 \text{ cm}$     79. 25

81.

83. 3.56

**Exercises 8.1, page 243**

1. (a)  $-$ ,  $+$ ,  $-$ ,  $-$ ,  $+$ ,  $-$ , (b)  $+$ ,  $-$ ,  $+$ ,  $+$ ,  $+$ ,  $-$     3.  $+$ ,  $-$   
 5.  $+$ ,  $+$     7.  $-$ ,  $+$     9.  $+$ ,  $-$     11.  $-$ ,  $-$     13.  $-$ ,  $+$

15.  $\sin \theta = \frac{1}{\sqrt{5}}$ ,  $\cos \theta = \frac{2}{\sqrt{5}}$ ,  $\tan \theta = \frac{1}{2}$ ,  
 $\csc \theta = \sqrt{5}$ ,  $\sec \theta = \frac{\sqrt{5}}{2}$ ,  $\cot \theta = 2$

17.  $\sin \theta = \frac{-3}{\sqrt{13}}$ ,  $\cos \theta = \frac{-2}{\sqrt{13}}$ ,  $\tan \theta = \frac{3}{2}$ ,  
 $\csc \theta = -\frac{\sqrt{13}}{3}$ ,  $\sec \theta = -\frac{\sqrt{13}}{2}$ ,  $\cot \theta = \frac{2}{3}$

19.  $\sin \theta = \frac{12}{13}$ ,  $\cos \theta = -\frac{5}{13}$ ,  $\tan \theta = -\frac{12}{5}$ ,  
 $\csc \theta = \frac{13}{12}$ ,  $\sec \theta = -\frac{13}{5}$ ,  $\cot \theta = -\frac{5}{12}$

21.  $\sin \theta = \frac{-2}{\sqrt{29}}$ ,  $\cos \theta = \frac{5}{\sqrt{29}}$ ,  $\tan \theta = -\frac{2}{5}$ ,  
 $\csc \theta = -\frac{\sqrt{29}}{2}$ ,  $\sec \theta = \frac{\sqrt{29}}{5}$ ,  $\cot \theta = -\frac{5}{2}$

23. I, II    25. I, III    27. II, III    29. II    31. II  
 33. IV    35. III    37. II    39.  $-$     41.  $-$

**Exercises 8.2, page 248**

1.  $\sin 200^\circ = -\sin 20^\circ = -0.342$ ,  
 $\cos 265^\circ = -\cos 85^\circ = -0.0872$ ,  
 $\tan 150^\circ = -\tan 30^\circ = -0.577$ ,

$\cot 300^\circ = -\cot 60^\circ = -\frac{1}{\tan 60^\circ} = -0.577$ ,

$\sec 344^\circ = \sec 16^\circ = \frac{1}{\cos 16^\circ} = 1.04$ ,

$\sin 397^\circ = \sin 37^\circ = 0.602$

3.  $296.0^\circ$     5.  $\sin 20^\circ$ ,  $-\cos 40^\circ$     7.  $-\tan 75^\circ$ ,  $-\csc 58^\circ$

9.  $\cos 40^\circ$ ,  $-\tan 40^\circ$     11.  $-\sin 15^\circ = -0.259$

13.  $-\cos 73.7^\circ = -0.2807$     15.  $\sec 31.67^\circ = 1.1750$

17.  $-\tan 31.5^\circ = -0.613$     19.  $-0.5228$     21.  $-0.76199$

23.  $-3.9096$     25. 0.2989    27.  $237.99^\circ$ ,  $302.01^\circ$

29.  $66.40^\circ$ ,  $293.60^\circ$     31.  $90.7^\circ$ ,  $270.7^\circ$     33.  $119.5^\circ$

35.  $263^\circ$     37.  $306.21^\circ$     39.  $299.24^\circ$     41.  $-0.7002$

43.  $-0.777$     45.  $<$     47.  $=$     49.  $-0.9659$     51.  $\cot \theta$

53. 0.0183 A    55. 12.6 cm

**Exercises 8.3, page 252**

1.  $160^\circ$     3. 1.082, 2.060    5.  $\frac{\pi}{12}$ ,  $\frac{5\pi}{6}$     7.  $\frac{5\pi}{12}$ ,  $\frac{11\pi}{6}$

9.  $\frac{7\pi}{6}$ ,  $\frac{3\pi}{20}$     11.  $4\pi$ ,  $-\frac{\pi}{20}$     13.  $72^\circ$ ,  $270^\circ$

15.  $10^\circ$ ,  $315^\circ$     17.  $70^\circ$ ,  $300^\circ$     19.  $-20^\circ$ ,  $27^\circ$

21. 0.401 rad    23. 4.40 rad    25. 5.821 rad

27. 8.351 rad    29.  $43.0^\circ$     31.  $195.2^\circ$     33.  $722^\circ$

35.  $-940.8^\circ$     37. 0.7071    39. 3.732

41.  $-0.8660$     43.  $-8.3265$     45. 0.9056    47.  $-0.890$

49.  $-2.09$     51. 0.149    53. 0.3141, 2.827

55. 2.932, 6.074    57. 0.8309, 5.452    59. 2.442, 3.841

61.  $-0.3827$     63.  $-\cot \theta$     65. 612 mil    67. 11.0 rad

69. 0.030 J    71. 2900 m

**Exercises 8.4, page 257**

1. 2.36 cm    3. 14.4 m/s    5. 3.46 cm    7. 426 mm

9. 0.1849 km<sup>2</sup>    11. 67 cm<sup>2</sup>    13. 0.0647 m    15. 8.21 m

17. 43.1 m    19. 1.81 rad, 14.5 cm<sup>2</sup>    21. 12:32:44

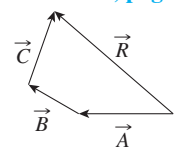
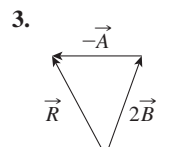
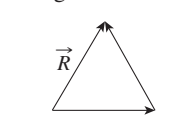
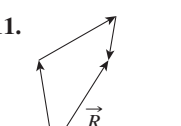
23. 627 m<sup>2</sup>    25. 0.52 rad/s    27. 34.73 m<sup>2</sup>    29. 0.704 m

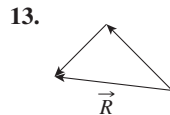
31.  $22.6 \text{ m}^2$     33.  $369 \text{ m}^3$     35.  $0.4 \text{ km}$     37.  $5.42 \text{ m/s}$   
 39.  $188 \text{ rad/s}^2$     41.  $509\,000 \text{ km/h}$     43.  $75.8 \text{ r/min}$   
 45.  $9.41 \text{ m}$     47.  $1260 \text{ mm/s}$     49.  $0.433 \text{ rad/s}$   
 51.  $1070 \text{ km/h}$     53.  $11.0 \times 10^3 \text{ m}^2$     55.  $14.9 \text{ m}^3$   
 57. The sequences both converge to 1.    59.  $1.15 \times 10^8 \text{ km}$

**Review Exercises for Chapter 8, page 261**

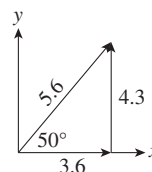
1.  $\sin \theta = \frac{4}{5}$ ,  $\cos \theta = \frac{3}{5}$ ,  $\tan \theta = \frac{4}{3}$ ,  
 $\csc \theta = \frac{5}{4}$ ,  $\sec \theta = \frac{5}{3}$ ,  $\cot \theta = \frac{3}{4}$   
 3.  $\sin \theta = -\frac{2}{\sqrt{53}}$ ,  $\cos \theta = \frac{7}{\sqrt{53}}$ ,  $\tan \theta = -\frac{2}{7}$ ,  
 $\csc \theta = -\frac{\sqrt{53}}{2}$ ,  $\sec \theta = \frac{\sqrt{53}}{7}$ ,  $\cot \theta = -\frac{7}{2}$   
 5.  $-\cos 48^\circ$ ,  $\tan 14^\circ$     7.  $-\sin 71^\circ$ ,  $\sec 15^\circ$     9.  $\frac{2\pi}{9}$ ,  $\frac{17\pi}{20}$   
 11.  $\frac{34\pi}{15}$ ,  $\frac{9\pi}{8}$     13.  $252^\circ$ ,  $130^\circ$     15.  $12^\circ$ ,  $330^\circ$     17.  $32.1^\circ$   
 19.  $2067^\circ$     21.  $1.78 \text{ rad}$     23.  $0.3534 \text{ rad}$     25.  $4.5736 \text{ rad}$   
 27.  $-11.10 \text{ rad}$     29.  $-0.4147$     31.  $-0.466$     33.  $-1.0799$   
 35.  $-0.42641$     37.  $-1.638$     39.  $4.1398$     41.  $-0.5878$   
 43.  $-0.8660$     45.  $0.5569$     47.  $1.197$     49.  $10.30^\circ$ ,  $190.30^\circ$   
 51.  $118.23^\circ$ ,  $241.77^\circ$     53.  $0.5759$ ,  $5.707$   
 55.  $4.187$ ,  $5.238$     57.  $223.76^\circ$     59.  $246.78^\circ$     61.  $10.8 \text{ cm}$   
 63.  $3.23 \text{ rad}$     65.  $14.2 \text{ m}$     67.  $2100 \text{ cm}^2$   
 69.  $A = \frac{1}{2}\theta r^2 - \frac{1}{2}r(r \sin \theta) = \frac{1}{2}r^2(\theta - \sin \theta)$   
 71.  $1040 \text{ m}^2$     73.  $0.0562 \text{ W}$     75.  $45.8^\circ$     77.  $137 \text{ r/min}$   
 79.  $3600 \text{ km/h}$     81. (a)  $6670 \text{ km}$  (b)  $10\,000 \text{ km}$ ; The distance over the north pole is shorter.  
 83.  $47.1 \text{ m/s}$     85.  $2.70 \text{ m}^2$     87.  $302 \text{ m/s}$     89.  $138 \text{ m}^2$   
 91.  $3.58 \times 10^5 \text{ km}$

**Exercises 9.1, page 261**

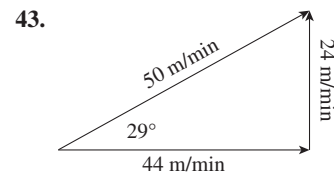
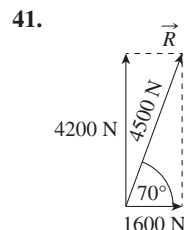
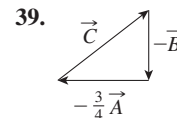
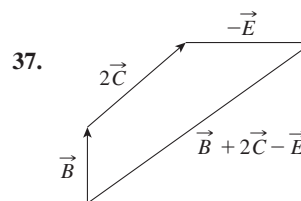
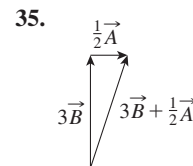
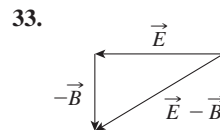
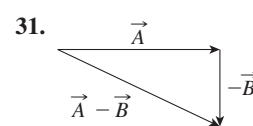
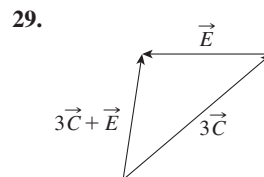
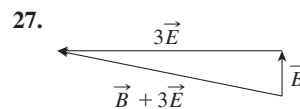
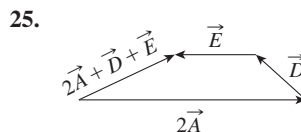
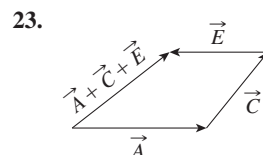
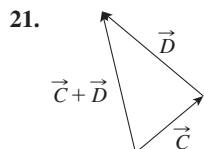
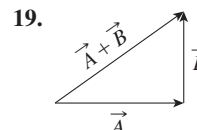
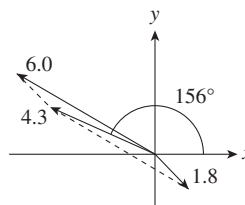
1.     3.   
 5. (a) Scalar, no direction given  
 (b) Vector, magnitude and direction both given  
 7. (a) A vector; it has magnitude and direction (b) Scalar; it has magnitude but not direction  
 9.     11. 



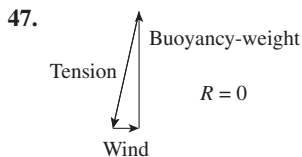
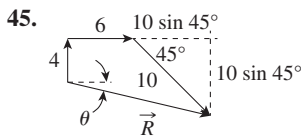
15.  $5.6 \text{ cm}$ ,  $50^\circ$



17.  $4.3 \text{ cm}$ ,  $156^\circ$



**B.18 ANSWERS TO ODD-NUMBERED EXERCISES**



**Exercises 9.2, page 271**

1.  $V_x = -11.6, V_y = -8.46$     3.  $A_x = 0, A_y = -375.4$   
 5. 662, 352    7. -349, -664    9. -750, 0  
 11. 3.22 N, 7.97 N    13. -62.9 m/s, 44.1 m/s  
 15.  $0 \text{ mm/s}^2, -9040 \text{ mm/s}^2$     17. -2.53 mN, -0.788 mN  
 19. -0.8088 dm, 0.3296 dm    21. 23.9 km/h, 7.43 km/h  
 23. 12 400 N, 760 N    25. 116 km/h    27. 0.237 m, 1.18 m  
 29. 89 N, -190 N    31. 290 N  
 33. 0.57 (km/h)/m, 0.48 (km/h)/m

**Exercises 9.3, page 276**

1.  $R = 1650, \theta = 73.8^\circ$     3.  $R = 24.2, \theta = 52.6^\circ$   
 5.  $R = 7.781, \theta = 66.63^\circ$  (with  $\vec{A}$ )    7.  $R = 10.0, \theta = 58.8^\circ$   
 9.  $R = 2.74, \theta = 111.0^\circ$     11.  $R = 2130, \theta = 107.7^\circ$   
 13.  $R = 7052, \theta = 349.82^\circ$     15.  $R = 29.2, \theta = 10.8^\circ$   
 17.  $R = 5920, \theta = 88.4^\circ$     19.  $R = 27.27, \theta = 33.14^\circ$   
 21.  $R = 50.2, \theta = 50.3^\circ$     23.  $R = 0.242, \theta = 285.9^\circ$   
 25.  $R = 532, \theta = 95.7^\circ$     27.  $R = 235, \theta = 121.6^\circ$   
 29.  $R = 68\ 000 \text{ N}, \theta = 345.1^\circ$   
 31. 40 N    33. 24.3 kg · m/s

**Exercises 9.4, page 280**

1. 47.08 km,  $10.27^\circ$  N of E  
 3. 39.6 N,  $29.5^\circ$  from 34.5-N force    5. 11 000 N,  $44^\circ$  above horizontal  
 7. 3070 m,  $17.8^\circ$  S of W    9. 229.4 m,  $72.82^\circ$  N of E  
 11. 3750 N,  $56^\circ$  from 2080-N force  
 13. 25.3 km/h,  $29.6^\circ$  S of E    15.  $9.6 \text{ m/s}^2, 7.0^\circ$  from vertical  
 17. 781 N,  $9.3^\circ$  above horizontal  
 19. 540 km/h,  $6^\circ$  from direction of plane  
 21. 29 180 km/h,  $0.03^\circ$  from direction of shuttle  
 23. 27.0 km at a bearing of  $192.4^\circ$     25. 9.3 km/h    27. 910 N  
 29.  $184\ 000 \text{ cm/min}^2, 89.6^\circ$     31. 138 km,  $65.0^\circ$  N of E  
 33. 79.0 m/s,  $75.6^\circ$  from vertical  
 35. 4.06 A/m,  $11.6^\circ$  with magnet

**Exercises 9.5, page 286**

1.  $b = 76.01, C = 40.77^\circ, c = 50.01$   
 3.  $b = 38.1, C = 66.0^\circ, c = 46.1$   
 5.  $b = 2620, C = 108.0^\circ, c = 2800$   
 7.  $B = 12.20^\circ, C = 149.57^\circ, c = 7.448$   
 9.  $A = 149.7^\circ, a = 11\ 050, C = 9.574^\circ$   
 11.  $A = 3.40^\circ, a = 0.0776, c = .005\ 66$   
 13.  $A = 99.4^\circ, b = 55.1, c = 24.4$

15.  $A = 68.01^\circ, a = 5520, c = 5376$   
 17.  $A_1 = 61.36^\circ, c_1 = 5.628, C_1 = 70.51^\circ;$   
 $A_2 = 118.64^\circ, c_2 = 1.366, C_2 = 13.23^\circ$   
 19.  $A_1 = 107.3^\circ, a_1 = 5280, C_1 = 41.3^\circ;$   
 $A_2 = 9.9^\circ, a_2 = 950, C_2 = 138.7^\circ$   
 21. No solution    23. 25.3 cm    25.  $9.4^\circ$     27. 6.0 m  
 29. 880 N    31. 1200 m    33. 13.94 cm    35. 27 300 km  
 37.  $77.4^\circ$  with bank downstream  
 39. 62 m downhill at  $62^\circ$  with the hillside slope    41. 105 m

**Exercises 9.6, page 291**

1.  $A = 14.0^\circ, B = 21.0^\circ, c = 107$   
 3.  $A = 50.3^\circ, B = 75.7^\circ, c = 6.31$   
 5.  $A = 70.9^\circ, B = 11.1^\circ, c = 4750$   
 7.  $A = 34.73^\circ, B = 40.67^\circ, C = 104.6^\circ$   
 9.  $A = 18.21^\circ, B = 22.28^\circ, C = 139.51^\circ$   
 11.  $A = 6.0^\circ, B = 16.0^\circ, c = 1150$   
 13.  $A = 82.3^\circ, b = 2160, C = 11.4^\circ,$   
 15.  $A = 36.24^\circ, a = 97.22, B = 39.09^\circ$   
 17.  $A = 46.94^\circ, B = 61.52^\circ, C = 71.24^\circ$   
 19.  $A = 138^\circ, B = 33.7^\circ, C = 8.3^\circ$   
 21.  $b = 3700, C = 25^\circ, c = 2400$   
 23. (Intermediate steps)  

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$1 + \cos A = \frac{b^2 + 2bc + c^2 - a^2}{2bc}$$
  
 25. Case 3: no; Case 4: yes, if sum of lengths of two sides is smaller than third side.    27. 69.4 km    29.  $47^\circ$   
 31.  $57.3^\circ, 141.7^\circ$     33. 16.5 mm  
 35. 5.09 km/h,  $91.6^\circ$  with respect to the land  
 37. 17.8 km    39. 2.05 km, 2.48 km

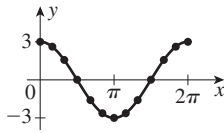
**Review Exercises for Chapter 9, page 293**

1.  $A_x = 57.4, A_y = 30.5$     3.  $A_x = -0.7485, A_y = -0.5357$   
 5.  $R = 602, \theta = 32.9^\circ$  with B    7.  $R = 5960, \theta = 33.60^\circ$  with A  
 9.  $R = 965, \theta = 8.58^\circ$     11.  $R = 26.12, \theta = 146.03^\circ$   
 13.  $R = 71.94, \theta = 336.5^\circ$     15.  $R = 0.994, \theta = 359.6^\circ$   
 17.  $b = 181, C = 64.0^\circ, c = 175$   
 19.  $A = 21.2^\circ, b = 34.8, c = 51.5$   
 21.  $a = 17\ 340, b = 24\ 660, C = 7.99^\circ$   
 23.  $A = 39.89^\circ, a = 5195, C = 30.02^\circ$   
 25.  $A_1 = 54.8^\circ, a_1 = 12.7, A_2 = 12.0^\circ, a_2 = 3.24,$   
 $B_1 = 68.6^\circ, B_2 = 111.4^\circ$   
 27.  $A = 32.3^\circ, b = 267, C = 17.7^\circ$   
 29.  $A = 176.4^\circ, B = 1.1^\circ, c = 5.41$   
 31.  $a = 1782, b = 1920, C = 16.00^\circ$   
 33.  $A = 37^\circ, B = 25^\circ, C = 118^\circ$   
 35.  $A = 20.6^\circ, B = 35.4^\circ, C = 124^\circ$

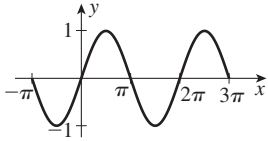
37. Add the 3 forms of the law of cosines together; simplify; divide by  $2abc$ .
39. 130 cm    41.  $A_r = \frac{1}{2}ab$ ;  $b = \frac{a \sin B}{\sin A}$ ; substitute
43. -155.7 N, 81.14 N    45. 630 m/s    47. 9.8 km
49.  $R = 2700$  N,  $\theta = 107^\circ$     51. 6.1 m/s,  $35^\circ$  with horizontal
53. 0.11 N    55. 1.5 pm    57. 2.30 m, 2.49 m
59. 0.039 km    61. 52 900 km    63. 2.65 km    65. 293 km
67. 1270 m or 1680 m (ambiguous)
69. 810 N,  $36^\circ$  N of E

**Exercises 10.1, page 299**

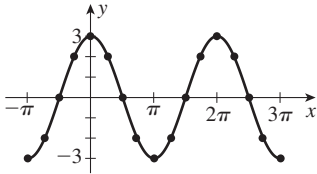
1.  $y = 3 \cos x$



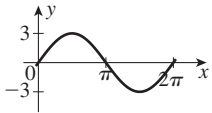
3. 0, -0.7, -1, -0.7, 0, 0.7, 1, 0.7, 0, -0.7, -1, -0.7, 0, 0.7, 1, 0.7, 0



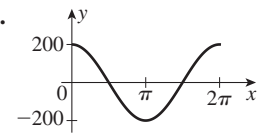
5. -3, -2.1, 0, 2.1, 3, 2.1, 0, -2.1, -3, -2.1, 0, 2.1, 3, 2.1, 0, -2.1, -3



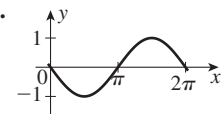
7.



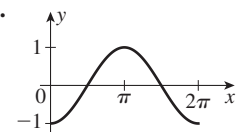
11.



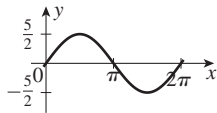
15.



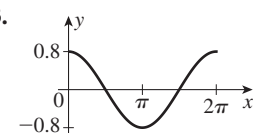
19.



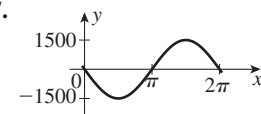
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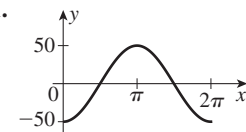
13.



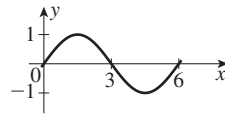
17.



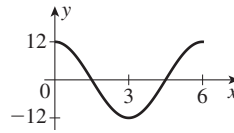
21.



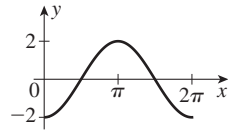
23. 0, 0.84, 0.91, 0.14, -0.76, -0.96, -0.28, 0.66



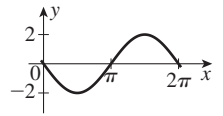
25. 12, 6.48, -4.99, -11.9, -7.84, 3.40, 11.5, 9.05



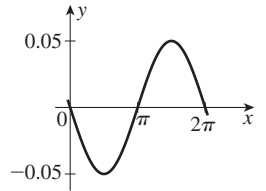
29.  $y = -2 \cos x$



27.



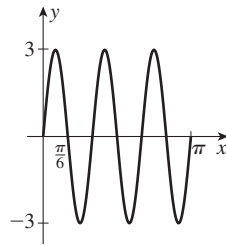
31.



33.  $y = 4 \sin x$     35.  $y = -1.5 \cos x$
37.  $y = -2.50 \sin x$     39.  $y = -2.50 \cos x$

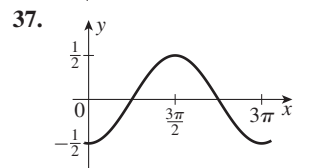
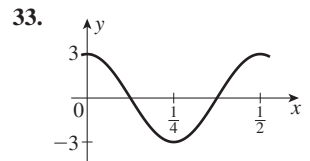
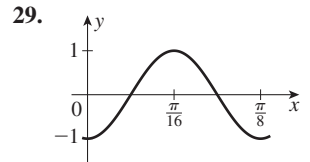
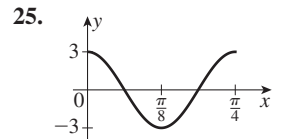
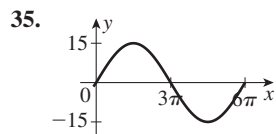
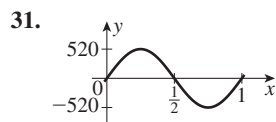
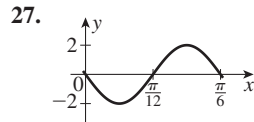
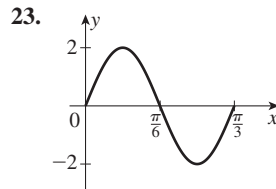
**Exercises 10.2, page 302**

1.  $y = 3 \sin 6x$



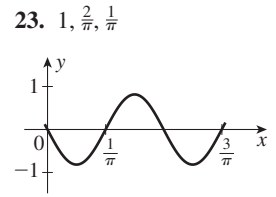
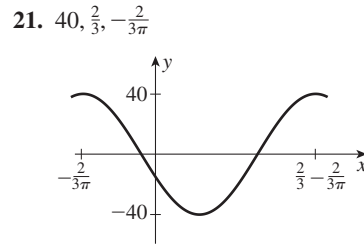
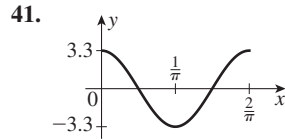
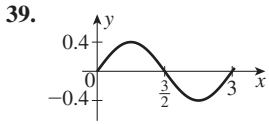
3.  $\frac{\pi}{3}$     5.  $\frac{\pi}{4}$     7.  $\frac{\pi}{6}$     9.  $\frac{\pi}{8}$     11. 1    13.  $\frac{1}{2}$

15.  $6\pi$     17.  $3\pi$     19. 3    21.  $\frac{2}{\pi}$

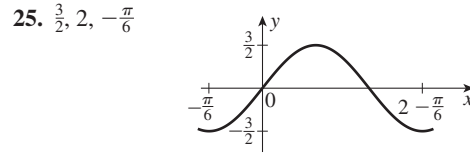
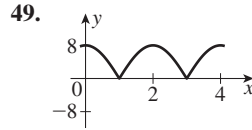
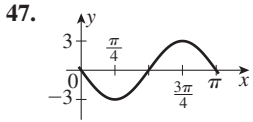




**B.20 ANSWERS TO ODD-NUMBERED EXERCISES**

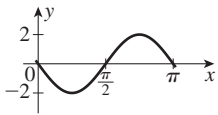


43.  $y = \sin 6x$     45.  $y = \sin 6\pi x$

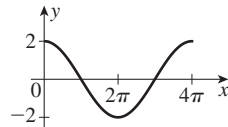


51.  $2\pi$

53.  $y = -2 \sin 2x$

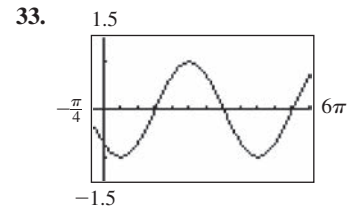
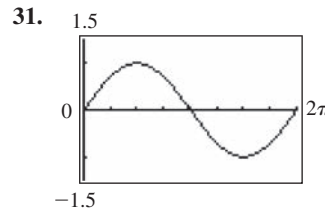
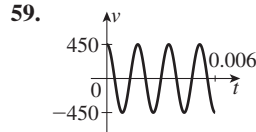
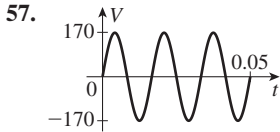


55.  $y = 2 \cos \frac{1}{2}x$



27.  $y = 4 \sin(\frac{2}{3}x + \frac{\pi}{6})$

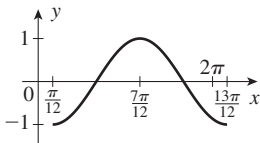
29.  $y = 12 \cos(4\pi x - \frac{\pi}{2})$



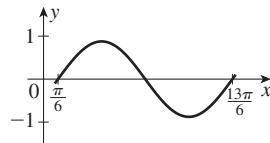
61.  $y = \frac{1}{2} \cos 2x$     63.  $y = -4 \sin \pi x$

**Exercises 10.3, page 306**

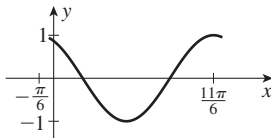
1.  $y = -\cos(2x - \frac{\pi}{6})$



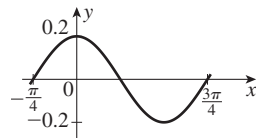
3.  $1, 2\pi, \frac{\pi}{6}$



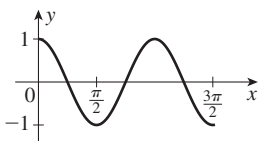
5.  $1, 2\pi, -\frac{\pi}{6}$



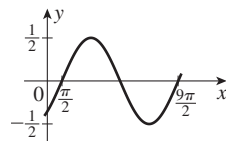
7.  $0.2, \pi, -\frac{\pi}{4}$



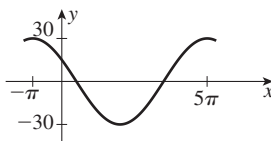
9.  $1, \pi, \frac{\pi}{2}$



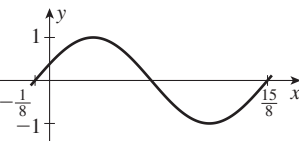
11.  $\frac{1}{2}, 4\pi, \frac{\pi}{2}$



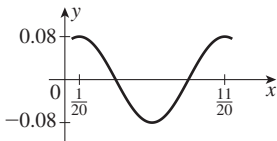
13.  $30, 6\pi, -\pi$



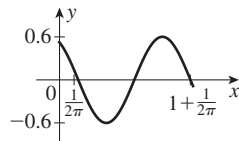
15.  $1, 2, -\frac{1}{8}$



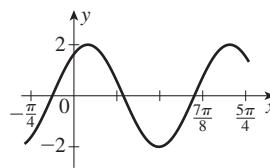
17.  $0.08, \frac{1}{2}, \frac{1}{20}$



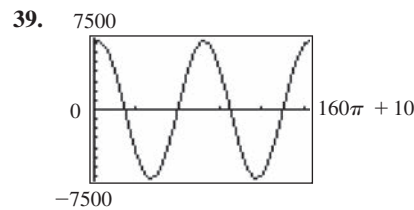
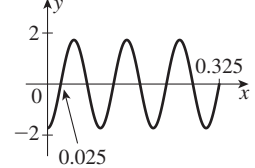
19.  $0.6, 1, \frac{1}{2\pi}$



35.  $y = 2 \sin(2x + \frac{\pi}{4})$



37.

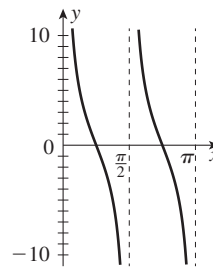


41.  $y = 5 \sin(\frac{\pi}{8}x + \frac{\pi}{8})$ . As shown: amplitude = 5; period =  $\frac{2\pi}{b} = 16$ ,  $b = \frac{\pi}{8}$ ; displacement =  $-\frac{c}{b} = -1$ ,  $c = \frac{\pi}{8}$

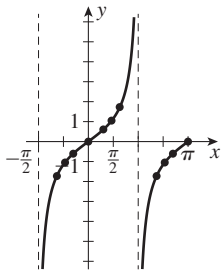
43.  $y = -0.8 \cos 2x$ . As shown: amplitude =  $|-0.8| = 0.8$ ; period =  $\frac{2\pi}{b} = \pi$ ,  $b = 2$ ; displacement =  $-\frac{c}{b} = 0$ ,  $c = 0$

**Exercises 10.4, page 309**

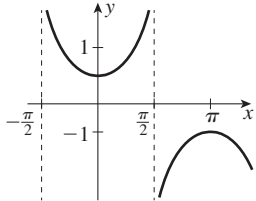
1.  $y = 5 \cot 2x$



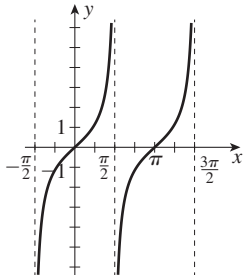
3. Undef., -1.7, -1, -0.58, 0, 0.58, 1, 1.7, undef., -1.7, -1, -0.58, 0



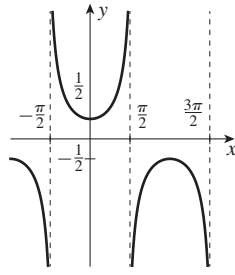
5. Undef., 2, 1.4, 1.2, 1, 1.2, 1.4, 2, undef., -2, -1.4, -1.2, -1



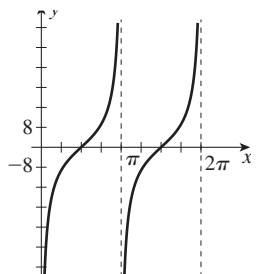
7.



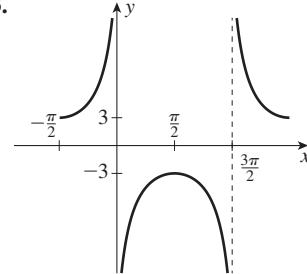
9.



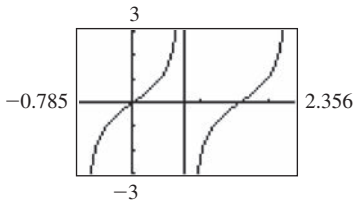
11.



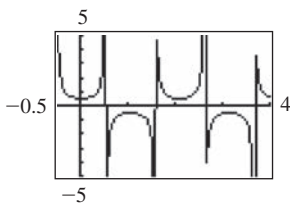
13.



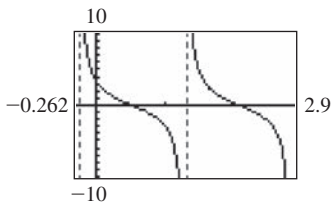
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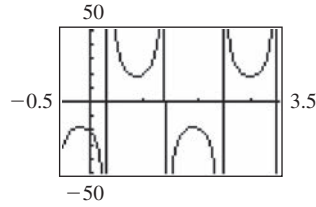
17.



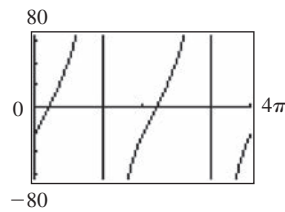
19.



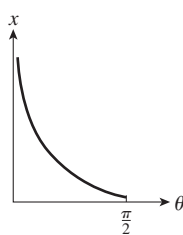
21.



23.

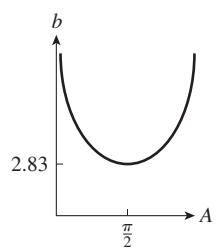


27.



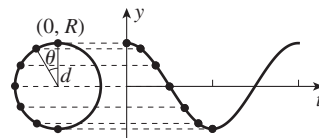
25.  $y = -3 \sec(x/2)$

29.

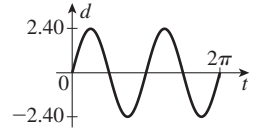


Exercises 10.5, page 312

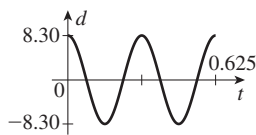
1.  $d = R \cos \omega t$



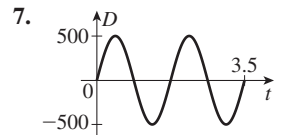
3.



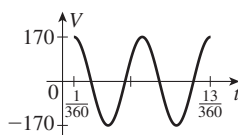
5.



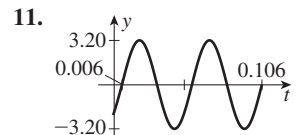
7.



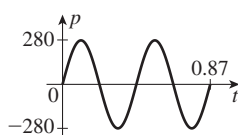
9.



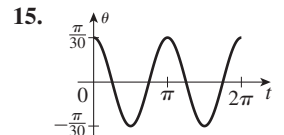
11.



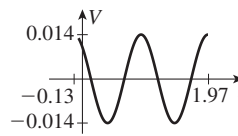
13.



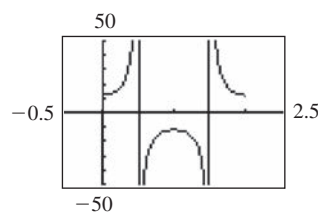
15.



17.



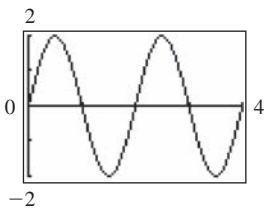
19. (a)



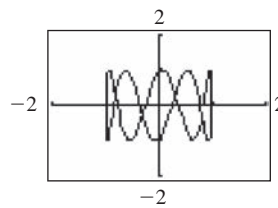
(b)  $L > 0$ , above  $x$ -axis

**B.22** ANSWERS TO ODD-NUMBERED EXERCISES

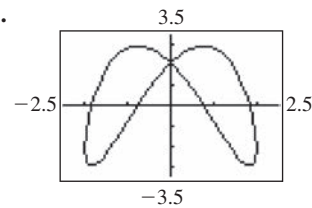
21.  $d = 1.98 \sin(\pi t)$



29.

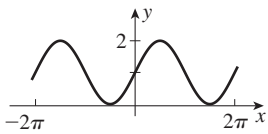


31.

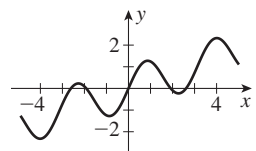


**Exercises 10.6, page 316**

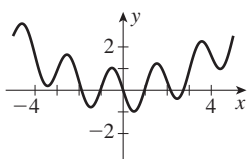
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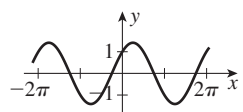
3.



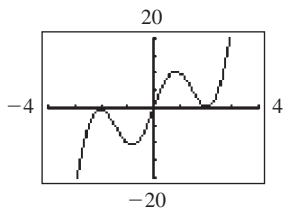
5.



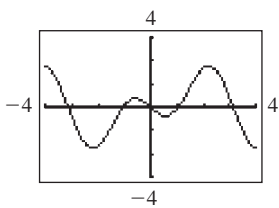
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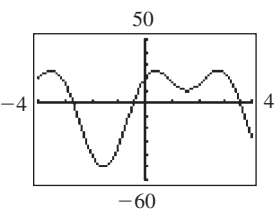
9.



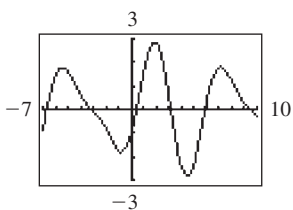
11.



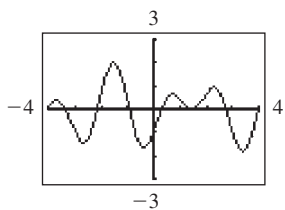
13.



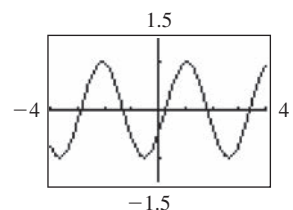
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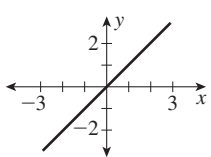
17.



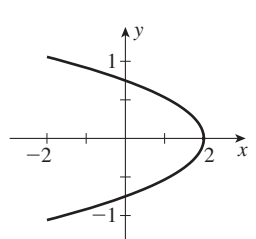
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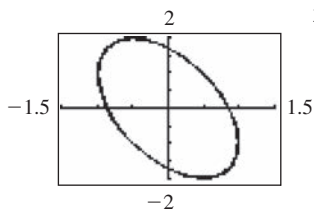
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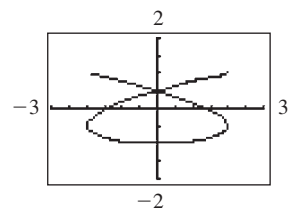
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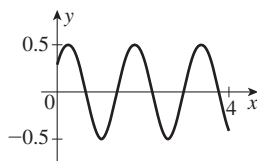
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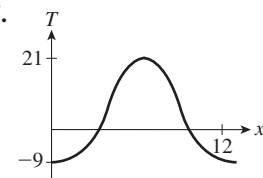
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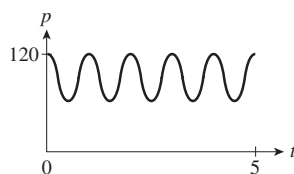
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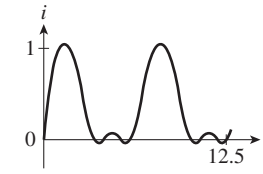
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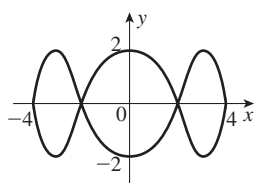
37.  $p = 100 + 20 \cos 2\pi t$



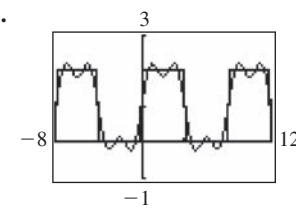
39.



41.

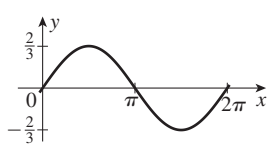


43.

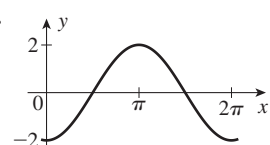


**Review Exercises for Chapter 10, page 317**

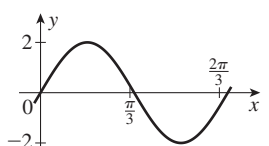
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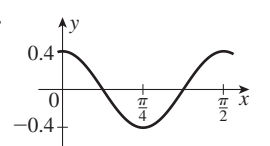
3.



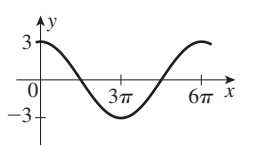
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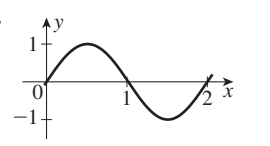
7.



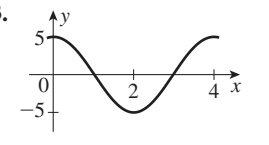
9.



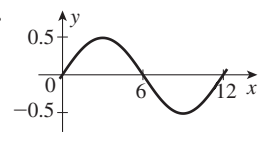
11.



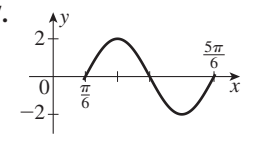
13.



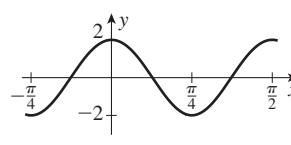
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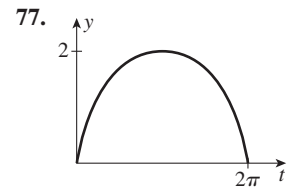
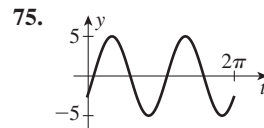
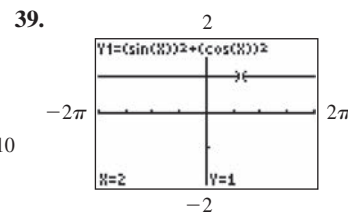
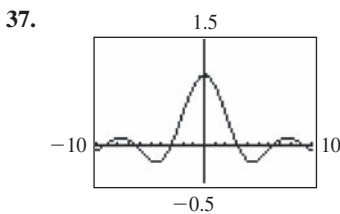
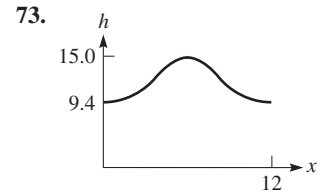
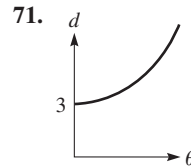
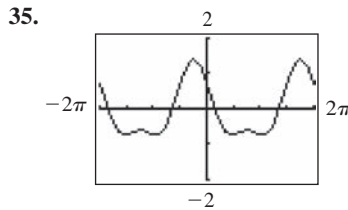
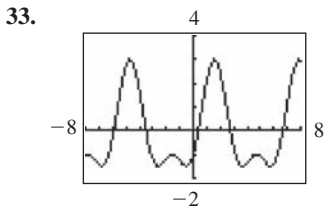
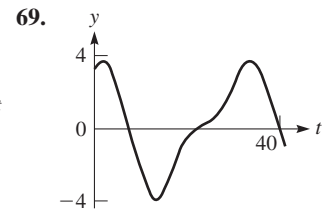
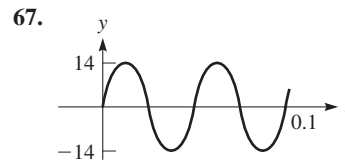
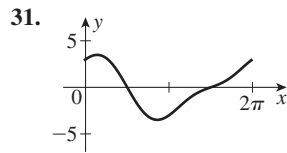
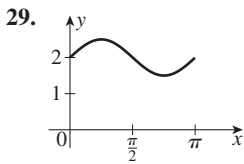
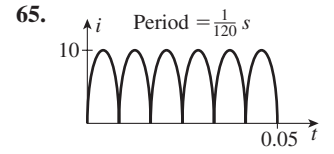
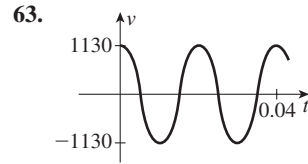
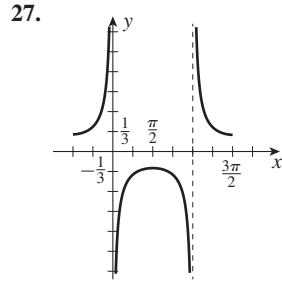
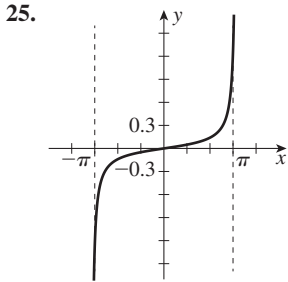
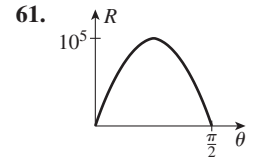
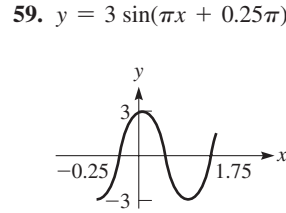
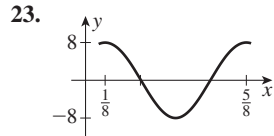
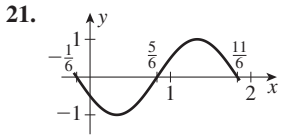


17.



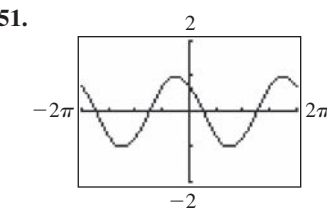
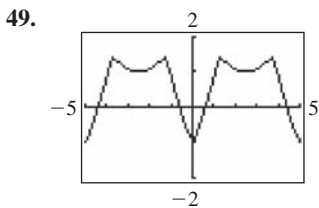
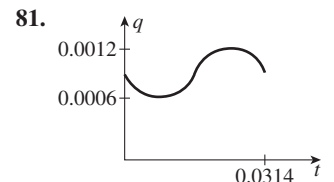
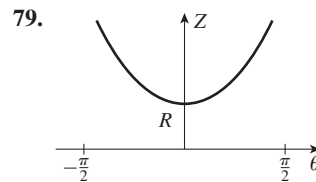
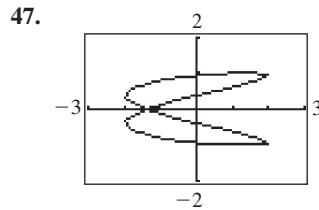
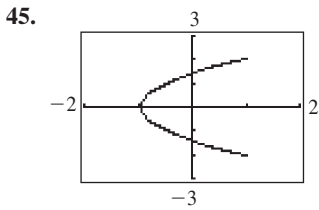
19.





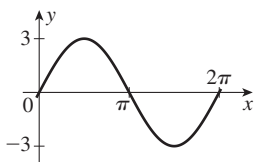
41.  $y = 2 \sin(2x + \frac{\pi}{2})$

43.  $y = \cos(\frac{\pi}{4}x - \frac{3\pi}{4})$

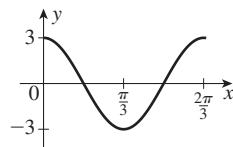


53.  $4\pi$

55.  $y = 3 \sin x$



57.  $y = 3 \cos 3x$



**Exercises 11.1, page 324**

1.  $\frac{y^2}{4x^2}$     3.  $\frac{3}{416}$     5.  $x^3$     7.  $\frac{2}{a^4}$     9.  $\frac{1}{125}$     11.  $\frac{4\pi^2}{x^2}$   
 13.  $\frac{2n^2}{5a}$     15. 1    17. -7    19.  $\frac{3}{x^2}$     21.  $\frac{a^3}{343x^3}$   
 23.  $\frac{n^9}{8}$     25.  $\frac{3}{a^3b^6}$     27.  $\frac{1}{a+b}$     29.  $\frac{2x^2 + 3y^2}{x^2y^2}$   
 31.  $\frac{8}{3a^n}$     33.  $\frac{b^3}{432a}, a \neq 0$     35.  $\frac{4}{t^4V^4}, t, V \neq 0$   
 37.  $\frac{2a^6 + 16}{a^8}$     39.  $\frac{10}{9}$     41.  $\frac{R_1R_2}{R_1 + R_2}$   
 43.  $\frac{4n^2 - 4n + 1}{n^4}$     45.  $\frac{8}{99}$     47.  $\frac{x+y}{xy}, x, y \neq 0, x \neq y$   
 49.  $\frac{t^2 + t + 2}{t^2}$     51.  $\frac{2D}{D^2 - 1}$     53. No  
 55. (a)  $4^5$  (b)  $2^{10}$

**B.24 ANSWERS TO ODD-NUMBERED EXERCISES**

57. (a)  $\left(\frac{a}{b}\right)^{-n} = \frac{1}{\left(\frac{a}{b}\right)^n} = \frac{1}{\frac{a^n}{b^n}} = \frac{b^n}{a^n} = \left(\frac{b}{a}\right)^n$

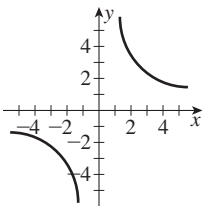
(b) 303.551 82 = 303.551 82

59.  $n = 3$     61. 7    63. N · m    65. J/s<sup>3</sup>    67. 1

69.  $\frac{p[(1+i)^n - 1]}{i(1+i)^n}$

**Exercises 11.2, page 328**

1. 16    3.    5. 5    7. 3



9. 10<sup>25</sup>    11.  $\frac{1}{2}$     13.  $\frac{1}{16}$     15. 25    17. 81    19. -200

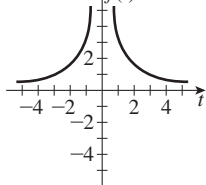
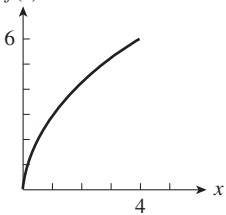
21.  $\frac{3}{5}$     23. -2    25.  $\frac{39}{1000}$     27.  $\frac{3}{5}$     29. 2.059

31. 0.538 91    33. B<sup>7/6</sup>    35.  $\frac{1}{-y^{9/10}}$     37.  $\frac{1}{x^{3/2}}$     39. 2ab<sup>2</sup>

41.  $\frac{1}{8a^3b^{9/4}}$     43. a<sup>1/12</sup>    45.  $\frac{4x}{(4x^2 + 1)^{1/2}}$     47. -y, y ≠ 1

49.  $\frac{T}{(T+2)^{1/2}}$     51.  $\frac{a^2 + 1}{a^4}$     53.  $\frac{a + 1}{a^{1/2}}$     55.  $\frac{x(5x - 2)}{(2x - 1)^{1/2}}$

57. f(x)    59.    61.  $\sqrt[3]{x^2}$



63. If  $(A/S)^{-1/4} = 0.5 = 1/2$ , then  $(A/S)^{1/4} = 2$ . Raise each to the fourth power and get A/S = 16.

65.  $R = \frac{T^{2/3}}{k^{1/3}} - d$     67. 1.91 mA

**Exercises 11.3, page 332**

1. ab<sup>2</sup>√a    3.  $\frac{\sqrt[4]{54}}{3}$     5. 2√6    7. 6√3    9. xy<sup>2</sup>√y

11. xy<sup>2</sup>z√z    13. 3R<sup>2</sup>V<sup>2</sup>√2RT    15. 2√[3]{2}    17. 2√[5]{3}

19. 2√[3]{a<sup>2</sup>}    21. 2st√[4]{4r<sup>3</sup>t}    23. 2    25. P√[3]{V}

27.  $\frac{1}{2}\sqrt{6}$     29.  $\frac{1}{2}\sqrt[3]{6}$     31.  $\frac{1}{3}\sqrt[5]{27}$     33. 2√5    35. 2

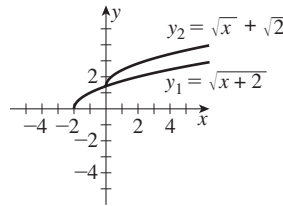
37. 200    39. 2000    41. √2a    43.  $\frac{1}{2}\sqrt{2}$     45. √[3]{2}

47.  $\sqrt[8]{n}$     49.  $\frac{2u\sqrt{7uv}}{v^3}$     51. 4√13    53.  $\frac{\sqrt{6x}}{3c^2}$

55.  $\frac{3}{4}\sqrt{2}$     57.  $\frac{\sqrt{xy(x^2 + y^2)}}{xy}$     59.  $\frac{\sqrt{(C-2)(C+2)}}{(C+2)}$

61.  $\sqrt{a^2 + b^2}$     63. 3x - 1    65.  $\sqrt[6]{a^3}$ ;  $\sqrt[6]{b^2}$ ;  $\sqrt[6]{c}$

67.    69.  $\sqrt{2ag}$



71.  $\frac{8Af\sqrt{f^2 + f_0^2}}{\pi^2(f^2 + f_0^2)}$

**Exercises 11.4, page 334**

1. 16√5    3. 7√3    5. √5 - √7    7. 3√5

9. -4r√3    11. -2√2a    13. 19√7    15. -20√2

17.  $\sqrt{R}(23\sqrt{3} - 6\sqrt{2})$     19.  $\frac{7}{3}\sqrt{15}$     21. -9√2

23. 13√[3]{3}    25. √[4]{2}    27. (5a - 2b<sup>2</sup>)√ab

29. (3 - 2a)√10    31. (2b - a)√[3]{3a<sup>2</sup>b}

33.  $\frac{(a^2 - c^3)\sqrt{ac}}{a^2c^3}$     35.  $\frac{(a - 2b)\sqrt[3]{ab^2}}{ab}$

37.  $\frac{-2V\sqrt{T^2 - V^2}}{T^2 - V^2}$     39. 15√3 - 11√5 = 1.384 014 4

41.  $\frac{1}{6}\sqrt{6} = 0.408 248 3$     43.  $\frac{1}{2}\sqrt[3]{2} = 4.409 723 7$

45. 3√3    47. Positive,  $\sqrt{1000} = 10\sqrt{10}$ ,  $\sqrt{11} > \sqrt{10}$

49. 6√2 + 2√6 units    51. 5400 + 900√2 = 6670 mm

**Exercises 11.5, page 338**

1. 3√10 - 16    3. √3 - √2    5. √30    7. 2√3

9. 2    11. 50    13. 2√5    15. √6 - √15    17. -1

19. 498 + 27√10    21. 66 + 13√11x - 5x

23. a√b + c√ac    25.  $\frac{2 - \sqrt{6}}{2}$     27. 2a - 3b + 2√2ab

29. √[6]{72}    31.  $\frac{1}{4}(\sqrt{7} - \sqrt{3})$

33.  $\frac{1}{11}(\sqrt{7} + 3\sqrt{2} - 6 - \sqrt{14})$     35.  $\frac{1}{17}(-56 + 9\sqrt{15})$

37.  $\frac{2(x + \sqrt{5x})}{x - 5}$     39. 1    41.  $-\frac{(R+2)(R-1)}{R}$

43.  $-\frac{\sqrt{x^2 - y^2} + \sqrt{x^2 + xy}}{y}$     45. a - 1 + √a(a - 2)

47. -1 - √66 = -9.124 038 4

49.  $-\frac{16 + 5\sqrt{30}}{26} = -1.668 697 2$     51.  $\frac{2x + 1}{\sqrt{x}}$

53.  $\frac{x(5x + 2)}{\sqrt{2x + 1}}$     55.  $\frac{17}{15\sqrt{10} - 12\sqrt{5}}$     57.  $\frac{1}{\sqrt{x+h} + \sqrt{x}}$

59. (1 - √2)<sup>2</sup> - 2(1 - √2) - 1 = 1 - 2√2 + 2 - 2 + 2√2 - 1 = 0

61. a = c    63.  $\frac{\sqrt[3]{x} - 1}{x - 1}$

$$65. \left[ \frac{1}{2}(\sqrt{b^2 - 4k^2} - b) \right]^2 + b \left[ \frac{1}{2}(\sqrt{b^2 - 4k^2} - b) \right] + k^2$$

$$= \frac{1}{4}(b^2 - 4k^2) - \frac{b}{2}\sqrt{b^2 - 4k^2} + \frac{1}{4}b^2$$

$$+ \frac{b}{2}\sqrt{b^2 - 4k^2} - \frac{1}{2}b^2 + k^2 = 0$$

67.  $\frac{2500 - 50\sqrt{V}}{2500 - V}$     69.  $2Q\sqrt{\sqrt{2} + 1}$     71.  $\frac{\sqrt{C(L - R^2C)}}{LC}$

**Review Exercises for Chapter 11, page 339**

1.  $\frac{2}{a^2}$     3.  $\frac{9d^3}{c}$     5. 375    7.  $\frac{1}{8000}$     9.  $\frac{t^4}{9}$     11. -28

13.  $64a^2b^5$     15.  $-8m^9n^6$     17.  $\frac{2(C - 2L^2)}{CL^2}$

19.  $\frac{2y}{x + 2y}, x \neq 0$     21.  $\frac{b}{ab - 3}$     23.  $\frac{(x^3y^3 - 1)^{1/3}}{y}$

25.  $\frac{1}{W + H}$     27.  $\frac{-2(x + 1)}{(x - 1)^3}$     29.  $2\sqrt{17}$     31.  $b^2c\sqrt{ab}$

33.  $3ab^2\sqrt{a}$     35.  $\frac{2t\sqrt{21st}}{u}$     37.  $\frac{5\sqrt{2s}}{2s}$     39.  $\frac{1}{9}\sqrt{33}$

41.  $mn^2\sqrt{m}\sqrt[4]{8n}$     43.  $\sqrt{2}$     45. 0    47.  $-7\sqrt{7}$

49.  $3ax\sqrt{2x}$     51.  $(2a + b)\sqrt[3]{a}$     53.  $25(6 - \sqrt{7})$

55.  $4(\sqrt{3} - \sqrt{5})$     57.  $6 - 51B - 7\sqrt{17B}$

59.  $42 - 7\sqrt{7a} - 3a$     61.  $\frac{6x + \sqrt{3xy}}{12x - y}$     63.  $-\frac{8 + \sqrt{6}}{29}$

65.  $\frac{13 - 2\sqrt{35}}{29}$     67.  $\frac{6x - 13a\sqrt{x} + 5a^2}{9x - 25a^2}$     69.  $\sqrt{4b^2 + 1}$

71.  $x(9 - 4x)$     73.  $1 + 6^{1/2}$     75.  $\frac{15 - 2\sqrt{15}}{4}$

77.  $\frac{3 + n + \sqrt{n(n + 3)}}{3}$     79.  $51 - 7\sqrt{105}$

81.  $\sqrt{\sqrt{2} - 1}(\sqrt{2} + 1) = \sqrt{(\sqrt{2} - 1)(\sqrt{2} + 1)^2}$   
 $= \sqrt{(\sqrt{2} - 1)(3 + 2\sqrt{2})} = \sqrt{\sqrt{2} + 1} = 1.5537740$

83.  $12\sqrt{2}$  units    85.  $\frac{33}{4} - 2\sqrt{3}$     87. 2.528%

89. (a)  $v = k(P/w)^{1/3}$  (b)  $v = \frac{k\sqrt[3]{Pw^2}}{w}$     91.  $\frac{100(1 + 2\sqrt{x})}{x}$

93.  $\frac{c}{(c^2 - v^2)^{1/2}}$     95.  $\frac{1}{\sqrt{A + h} + \sqrt{A}}$     97.  $6\sqrt{2}$  cm

99.  $\frac{\sqrt{LC_1C_2(C_1 + C_2)}}{2\pi LC_1C_2}$

**Exercises 12.1, page 344**

1.  $-\sqrt{6}$     3. -1    5.  $9j$     7.  $-2j$     9.  $0.6j$     11.  $8j\sqrt{2}$

13.  $\frac{1}{2}j\sqrt{7}$     15.  $-2ej$     17. (a) -7 (b) 7    19. (a) 4 (b) -4

21.  $-\frac{3}{5}$     23.  $10j$     25. (a) 1 (b) -1    27. 0    29.  $-2j$

31.  $-j$     33.  $2 + 3j$     35.  $-7j$     37.  $2 + 2j$

39.  $-2 + 3j$     41.  $3\sqrt{2} - 2j\sqrt{2}$     43. -1

45. (a)  $6 + 7j$  (b)  $8 - j$     47. (a)  $-2j$  (b) -4

49.  $x = 2, y = -2$     51.  $x = 10, y = -6$

53.  $x = -2, y = 3$     55. (a) No change (b) Sign changes

57. Yes    59. 0    61. Yes; imag. part is zero.

63. Each is  $x$ .

**Exercises 12.2, page 347**

1.  $1 - 5j$     3.  $\frac{1}{25}(29 + 22j)$     5.  $5 - 8j$     7.  $-25 + 10j$

9.  $-0.23 + 0.86j$     11.  $-36 + 21j$     13.  $7 + 49j$

15.  $22 + 3j$     17.  $-18j\sqrt{2}$     19.  $-28j$     21.  $3\sqrt{7} + 3j$

23.  $-40 - 42j$     25.  $-2 - 2j$     27.  $\frac{1}{29}(-30 + 12j)$

29.  $-\frac{1}{3}(1 + j)$     31.  $\frac{1}{11}(-13 + 8j\sqrt{2})$     33.  $\frac{-1 + 3j}{5}$

35.  $\frac{-45}{13} + \frac{48j}{13}$     37.  $-8j$     39.  $80j$     41.  $-\frac{7}{8} + 4j$

43.  $(-1 - j)^2 + 2(-1 - j) + 2 = 1 + 2j - 1 - 2 - 2j + 2 = 0$

45. 10    47.  $\frac{3}{10} + \frac{1}{10}j$     49.  $-1 + j$     51. -4

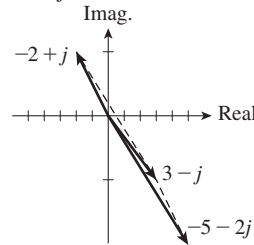
53.  $\frac{1}{10}(11 + 27j)$     55.  $281 + 35.2j$  volts

57.  $0.016 + 0.037j$  amperes    59. (a) Real (b) Pure imaginary

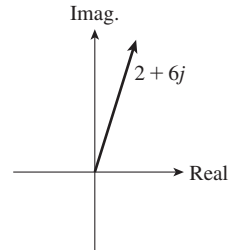
61. Product is the sum of squares of two real numbers  
 $(a + bj)(a - bj) = a^2 + b^2$

**Exercises 12.3, page 349**

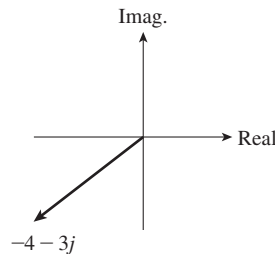
1.  $3 - j$



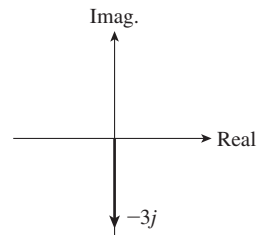
3.



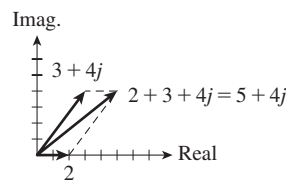
5.



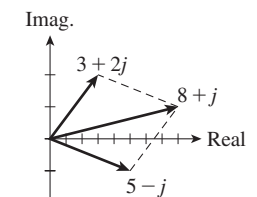
7.  $-3j$



9.  $5 + 4j$

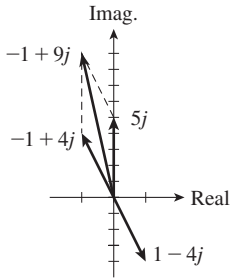


11.  $8 + j$

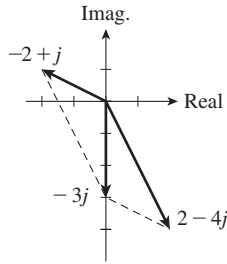


**B.26 ANSWERS TO ODD-NUMBERED EXERCISES**

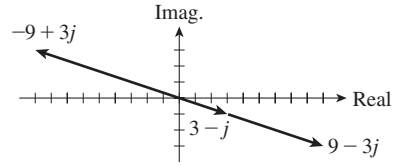
13.  $-1 + 9j$



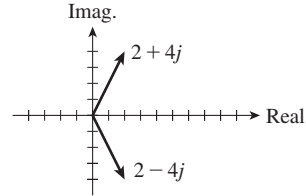
15.  $-3j$



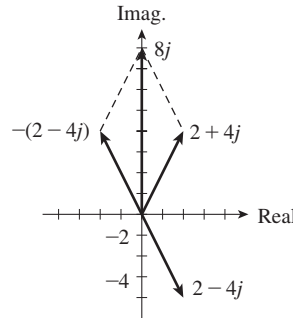
33.



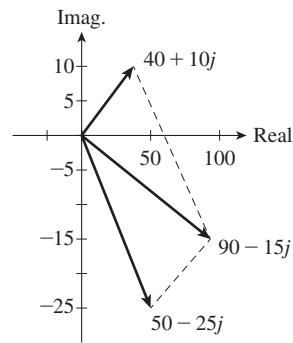
35. A complex number is on the opposite side of the real axis from its conjugate.



37. Subtracting the conjugate from the complex number results in an imaginary number.

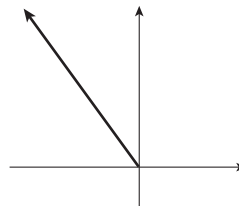


39.  $90 - 15jN$

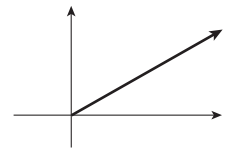


**Exercises 12.4, page 344**

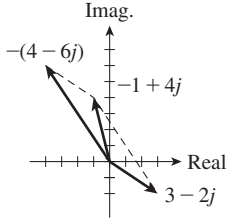
1.  $5(\cos 126.9^\circ + j \sin 126.9^\circ)$



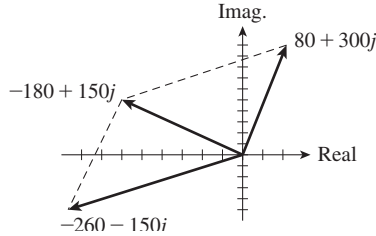
3.  $10(\cos 36.9^\circ + j \sin 36.9^\circ)$



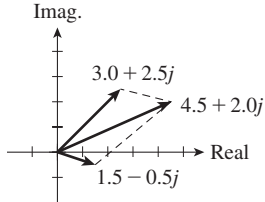
17.  $-1 + 4j$



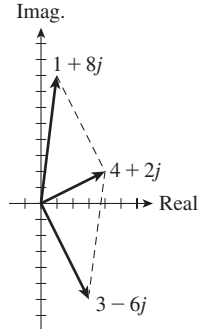
19.  $-180 + 150j$



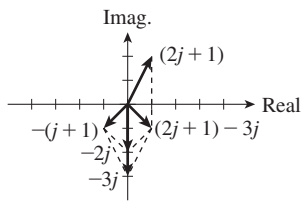
21.  $4.5 + 2.0j$



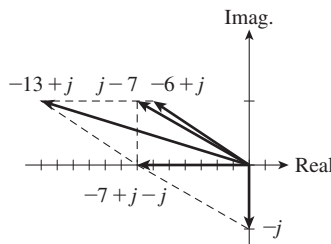
23.  $4 + 2j$



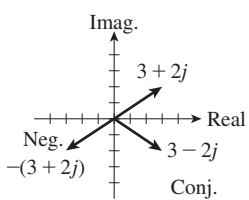
25.  $-2j$



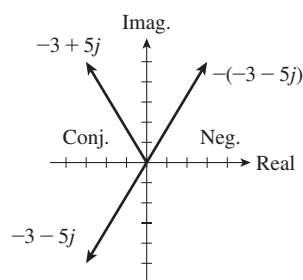
27.  $-13 + j$



29.

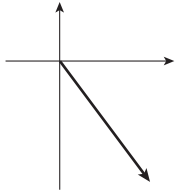


31.

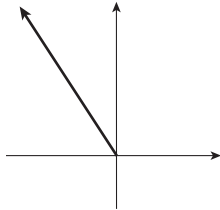




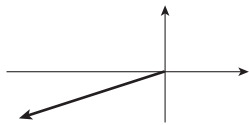
5.  $50(\cos 306.9^\circ + j\sin 306.9^\circ)$



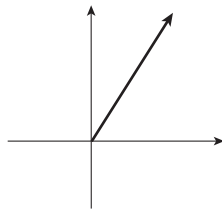
7.  $3.61(\cos 123.7^\circ + j\sin 123.7^\circ)$



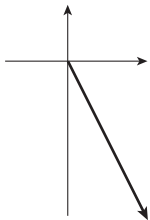
9.  $0.60(\cos 204^\circ + j\sin 204^\circ)$



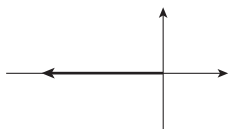
11.  $2(\cos 60^\circ + j\sin 60^\circ)$



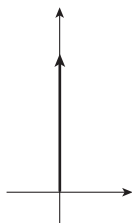
13.  $8.062(\cos 295.84^\circ + j\sin 295.84^\circ)$



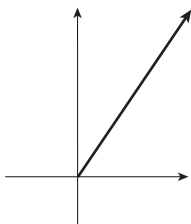
15.  $3(\cos 180^\circ + j\sin 180^\circ)$



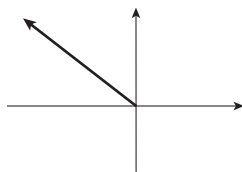
17.  $9(\cos 90^\circ + j\sin 90^\circ)$



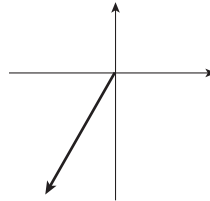
19.  $2.94 + 4.05j$



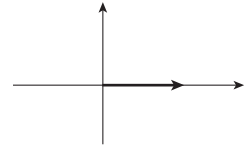
21.  $-140 + 80j$



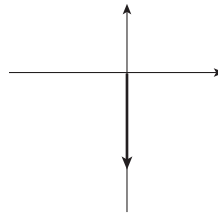
23.  $-1.85 - 2.36j$



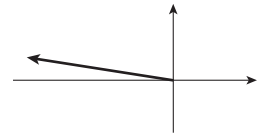
25.  $0.08$



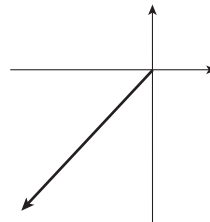
27.  $-120j$



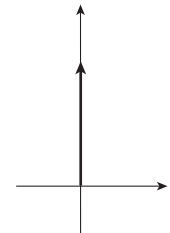
29.  $-4.71 + 0.595j$



31.  $-0.6052 - 0.7096j$



33.  $7.32j$



35.  $-6.961 + 86.14j$



37.  $180^\circ$

39. conj. of  $r(\cos \theta + j\sin \theta) = r(\cos \theta - j\sin \theta) = r(\cos(-\theta) + j\sin(-\theta))$

41.  $3.03/\underline{339.5^\circ}$  kV      43.  $2.51 + 12.1j$  V/m

**Exercises 12.5, page 354**

1.  $8.50e^{3.95j}$

3.  $2.00(\cos 217.7^\circ + j\sin 217.7^\circ) = -1.58 - 1.22j$

5.  $3.00e^{1.05j}$       7.  $0.450e^{4.93j}$       9.  $375.5e^{4.617j}$

11.  $0.515e^{3.461j}$       13.  $4.06e^{-1.07j} = 4.06e^{5.21j}$

15.  $9245e^{5.172j}$       17.  $5.00e^{5.36j}$       19.  $36.1e^{2.55j}$

21.  $6.37e^{0.386j}$       23.  $825.7e^{3.836j}$

25.  $3.00/\underline{28.6^\circ}; 2.63 + 1.44j$       27.  $464/\underline{106.0^\circ}, -128 + 446j$

29.  $3.20/\underline{50.0^\circ} = 2.06 + 2.45j$

31.  $0.1724/\underline{137.0^\circ}; -0.1261 + 0.1176j$

33.  $-18.2 + 9.95j; 20.7/\underline{151.3^\circ}$

35.  $-0.912 - 27.5j; 27.5/\underline{268^\circ}$       37.  $6.43e^{0.952j}$

39.  $391e^{0.285j}$  ohms;  $391\Omega$       41.  $3.17 \times 10^{-4}e^{0.478j}$  1/ $\Omega$

**B.28 ANSWERS TO ODD-NUMBERED EXERCISES**

**Exercises 12.6, page 360**

1.  $5.09(\cos 101.3^\circ + j\sin 101.3^\circ)$   
 3.  $613(\cos 281.5^\circ + j\sin 281.5^\circ)$     5.  $8(\cos 80^\circ + j\sin 80^\circ)$   
 7.  $3(\cos 250^\circ + j\sin 250^\circ)$     9.  $2(\cos 35^\circ + j\sin 35^\circ)$   
 11.  $2.4/\underline{170^\circ}$     13.  $0.008(\cos 105^\circ + j\sin 105^\circ)$   
 15.  $256(\cos 0^\circ + j\sin 0^\circ)$     17.  $0.8/\underline{273^\circ}$     19.  $5/\underline{87^\circ}$   
 21.  $1.73/\underline{79.8^\circ}$     23.  $11\,750/\underline{115.91^\circ}$   
 25.  $65.0(\cos 345.7^\circ + j\sin 345.7^\circ) = 63 - 16j$   
 27.  $61.4(\cos 343.9^\circ + j\sin 343.9^\circ); 59 - 17j$   
 29.  $2.21(\cos 71.6^\circ + j\sin 71.6^\circ); \frac{7}{10} + \frac{21}{10}j$   
 31.  $3.85(\cos 120.5^\circ + j\sin 120.5^\circ) = -1.95 + 3.31j$   
 33.  $625(\cos 212.5^\circ + j\sin 212.5^\circ) = -527 - 336j$   
 35.  $609.7(\cos 281.5^\circ + j\sin 281.5^\circ); 122 - 597j$   
 37.  $2(\cos 30^\circ + j\sin 30^\circ); 2(\cos 210^\circ + j\sin 210^\circ)$   
 39.  $-0.364 + 1.67j, -1.26 - 1.15j, 1.63 - 0.520j$   
 41.  $1.10 + 0.455j, -1.10 - 0.455j$     43.  $1, -1, j, -j$   
 45.  $3j, -\frac{3}{2}(\sqrt{3} + j), \frac{3}{2}(\sqrt{3} - j)$   
 47.  $1.62 + 1.18j, -0.618 + 1.90j, -2, -0.618 - 1.90j, 1.62 - 1.18j$   
 49.  $-5, \frac{5}{2} + j\frac{5\sqrt{3}}{2}, \frac{5}{2} - j\frac{5\sqrt{3}}{2}$   
 51.  $\left[\frac{1}{2}(1 - j\sqrt{3})\right]^3 = \frac{1}{8}[1 - 3(j\sqrt{3}) + 3(j\sqrt{3})^2 - (j\sqrt{3})^3] = \frac{1}{8}[1 - 3j\sqrt{3} - 9 + 3j\sqrt{3}] = \frac{1}{8}(-8) = -1$   
 53.  $-1, \frac{1}{2} + j\frac{\sqrt{3}}{2}, \frac{1}{2} - j\frac{\sqrt{3}}{2}$     55.  $p = 0.479/\underline{40.5^\circ}$  watts  
 57.  $43.3/\underline{165^\circ}, 43.3V$

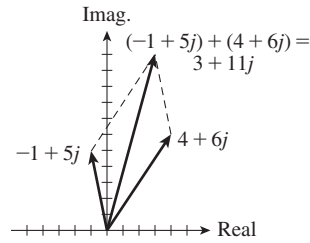
**Exercises 12.7, page 365**

1.  $V_R = 24.0V, V_L = 32.0V, V_{RL} = 40.0V, \theta = 53.1^\circ$   
 (voltage leads current)    3.  $12.9V$   
 5. (a)  $2850\Omega$  (b)  $37.9^\circ$  (c)  $16.4V$     7. (a)  $14.6\Omega$  (b)  $-90.0^\circ$   
 9. (a)  $47.8\Omega$  (b)  $19.8^\circ$   
 11.  $38.0V$     13.  $54.4\Omega, -62.3^\circ$     15.  $0.682H$   
 17.  $208kHz$     19.  $1.30 \times 10^{-11}F = 13pF$     21.  $1.02mW$   
 23.  $21.4 - 33.9j\Omega$

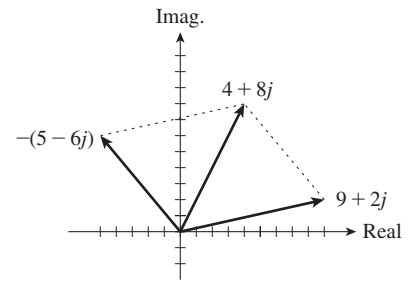
**Review Exercises for Chapter 12, page 367**

1.  $10 - j$     3.  $6 + 2j$     5.  $9 + 2j$     7.  $-12 + 66j$   
 9.  $\frac{1}{85}(21 + 18j)$     11.  $-2 - 3j$     13.  $\frac{1}{10}(-12 + 9j)$   
 15.  $\frac{1}{5}(13 + 11j)$     17.  $x = -3, y = -2$   
 19.  $x = \frac{10}{13}, y = \frac{11}{13}$

21.  $3 + 11j$



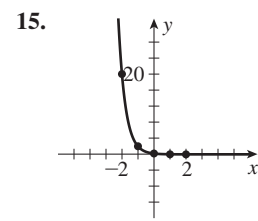
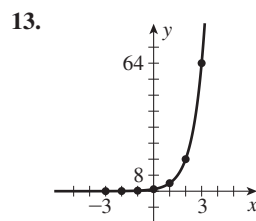
23.  $4 + 8j$

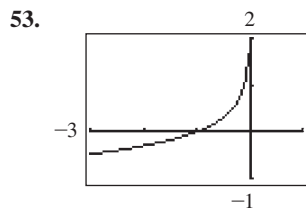
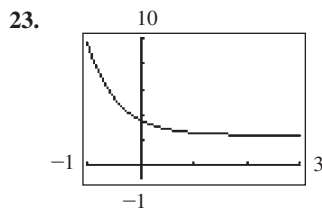
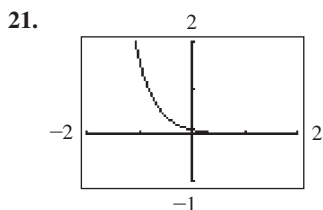
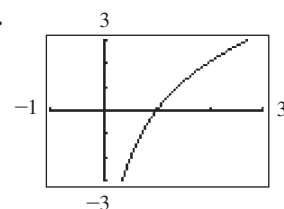
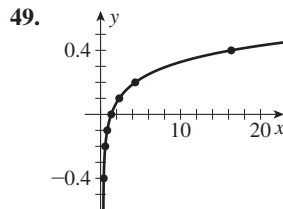
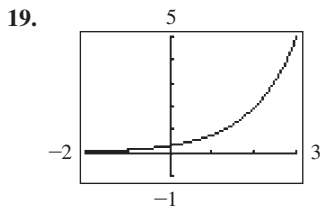
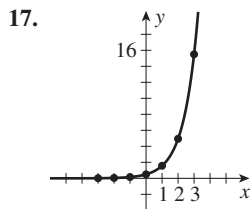


25.  $\sqrt{2}(\cos 315^\circ + j\sin 315^\circ) = \sqrt{2}e^{5.50j}$   
 27.  $80.1(\cos 254.1^\circ + j\sin 254.1^\circ) = 80.1e^{4.43j}$   
 29.  $4.67(\cos 76.8^\circ + j\sin 76.8^\circ); 4.67e^{1.34j}$   
 31.  $5000/\underline{0^\circ}, 5000e^{0j}$     33.  $-\sqrt{2} - j\sqrt{2}$   
 35.  $-2.789 + 4.163j$     37.  $0.19 - 0.59j$     39.  $26.31 - 6.427j$   
 41.  $1.94 + 0.495j$     43.  $-728.1 + 1017j$   
 45.  $15(\cos 84^\circ + j\sin 84^\circ)$     47.  $20/\underline{263^\circ}$   
 49.  $8(\cos 59^\circ + j\sin 59^\circ)$     51.  $14.29/\underline{133.61^\circ}$   
 53.  $1.26/\underline{59.7^\circ}$     55.  $9682/\underline{249.5^\circ}$   
 57.  $1024(\cos 160^\circ + j\sin 160^\circ)$     59.  $343/\underline{331.5^\circ}$   
 61.  $32(\cos 270^\circ + j\sin 270^\circ) = -32j$   
 63.  $\frac{625}{2}(\cos 270^\circ + j\sin 270^\circ) = -\frac{625}{2}j$   
 65.  $1 + j\sqrt{3}, -2, 1 - j\sqrt{3}$   
 67.  $0.383 + 0.924j, -0.924 + 0.383j, -0.383 - 0.924j, 0.924 - 0.383j$     69.  $40 + 9j, 41(\cos 12.7^\circ + j\sin 12.7^\circ)$   
 71.  $-15.0 - 10.9j, 18.5(\cos 216.0^\circ + j\sin 216.0^\circ)$   
 73.  $15 - 16j$     75.  $-j, -2j$     77.  $x^2 - 4x + 5 = 0$   
 79.  $1 - j$ , Yes;  $-1 - j$ , No    81.  $\frac{1}{2}$     83.  $2 + \frac{9}{2}j$   
 85.  $60V$     87.  $-21.6^\circ$     89.  $22.9Hz$   
 91.  $5500(\cos 53^\circ + j\sin 53^\circ)N$     93.  $\frac{\mu - j\omega n}{\mu^2 + \omega^2 n^2}$   
 95.  $e^{j\pi} = \cos \pi + j\sin \pi = -1$

**Exercises 13.1, page 372**

1.  $-\frac{1}{4}$     3. (a) Yes (b) Yes  
 5. (a) No (base cannot be negative) (b) Yes    7. 3  
 9.  $\frac{1}{81}$     11.  $\frac{1}{27}$

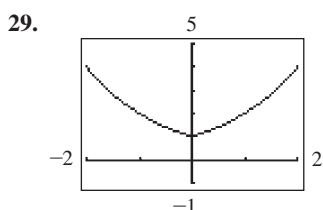




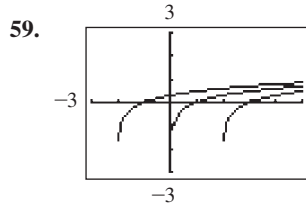
55. (a) -3  
(b) No value ( $x$  cannot be negative)

57. (a)  $\frac{1}{2}$   
(b) Not defined

25. 4      27.  $f(c + d) = b^{c+d} = b^c b^d = f(c) \cdot f(d)$

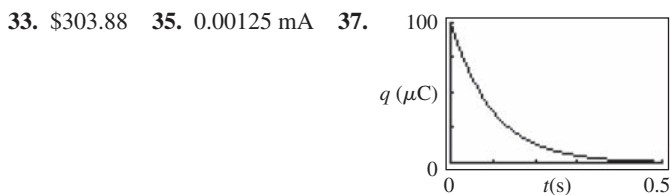


31. -0.767, 2, 4



$y_1 = \log(x+2)$   
 $y_2 = \log(x)$   
 $y_3 = \log(x-2)$

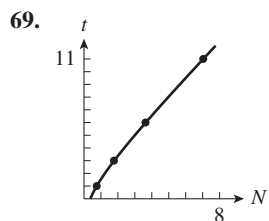
61. 0.0102, 2.376



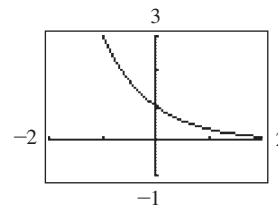
63.  $b_2 = b_1 10^{0.4(m_1 - m_2)}$

65.  $N = N_0 e^{-kt}$

67.  $V_1 = V_2 e^{-W/k}$



71. They are the same.



**Exercises 13.2, page 376**

1.  $32^{4/5} = 16$  in logarithmic form is  $\frac{4}{5} = \log_{32} 16$ .  
3. If  $x = 16$ ,  $y = \log_4 16$  means  $y = 2$ , since  $4^2 = 16$ . If  $x = \frac{1}{16}$ ,  
 $y = \log_4 \left(\frac{1}{16}\right)$  means  $y = -2$ , since  $4^{-2} = \frac{1}{16}$ .

5.  $\log_3 27 = 3$     7.  $\log_4 256 = 4$     9.  $\log_7 \left(\frac{1}{49}\right) = -2$

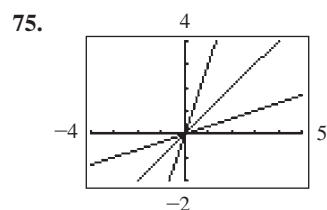
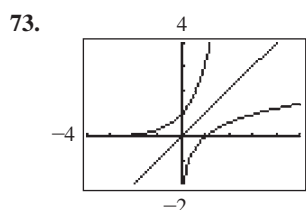
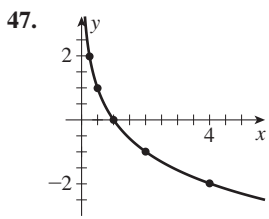
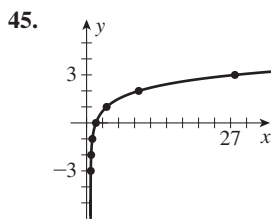
11.  $\log_2 \left(\frac{1}{64}\right) = -6$     13.  $\log_8 2 = \frac{1}{3}$

15.  $\log_{1/4} \left(\frac{1}{16}\right) = 2$     17.  $81 = 3^4$     19.  $9 = 3^2$

21.  $5 = 25^{1/2}$     23.  $3 = 243^{0.2}$     25.  $0.1 = 10^{-1}$

27.  $16 = (0.5)^{-4}$     29. 2    31. -2    33. 343

35.  $\frac{9}{4}$     37.  $\sqrt{5}$     39.  $\frac{1}{64}$     41. 0.2    43. -4

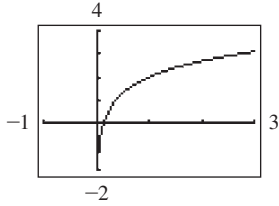


**Exercises 13.3, page 381**

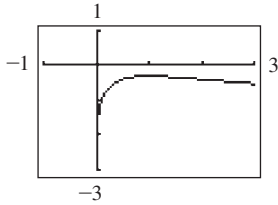
1.  $\log_4 3 + \log_4 7$     3.  $\log_4 \left(\frac{5}{x}\right)$     5. 3    7.  $\frac{1}{2}$   
9.  $\log_5 3 + \log_5 11$     11.  $\log_7 5 - \log_7 3$     13.  $3 \log_2 a$   
15.  $\log_6 a + \log_6 b + \log_6 c$     17.  $2 \log_5 y$   
19.  $\frac{1}{2} \log_2 x - 2 \log_2 a$     21.  $\log_b ac$     23.  $\log_5 3$   
25.  $\log_b x^{3/2}$     27.  $\log_e 4\pi^3$     29. -5    31. 2.5    33. 3  
35.  $\frac{3}{4}$     37.  $2 + \log_3 2$     39.  $-1 - \log_2 3$   
41.  $\frac{1}{2}(1 + \log_3 2)$     43.  $3 + \log_{10} 3$     45.  $y = 2x$   
47.  $y = \frac{3x}{5}$     49.  $y = \frac{49}{x^3}$     51.  $y = 2(2ax)^{1/5}$     53.  $y = \frac{2}{x}$   
55.  $y = \left(\frac{x^2}{9}\right)^{1/\log_5 3}$     57.  $\log_{10} x + \log_{10} 3 = \log_{10} 3x$

**B.30 ANSWERS TO ODD-NUMBERED EXERCISES**

59.  $y = \log_e(e^2x) = 2 \log_e e + \log_e x = 2 + \log_e x$  **61.** 8



63.  $y_1 = y_2$

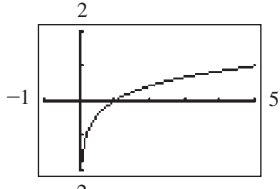


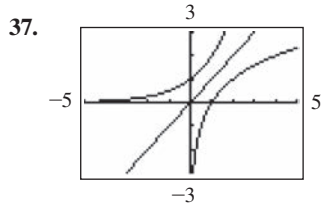
65.  $D = ae^{cr^2-br}$   
67.  $T = 90.0e^{-0.23t}$

**Exercises 13.4, page 384**

1. -0.4372    3. 2.754    5. 6.966    7. -0.2787    9. 1.219  
11. 27 400    13. 0.049 60    15. 2000.4    17. 0.005 788 2  
19. 85.5    21.  $7.37 \times 10^{101}$     23.  $1.1461 - 0.3010 = 0.8451$   
25.  $1.9085 = 1.9085$     27. 8.954    29. -13.886  
31. -30.040    33.  $2.65 \times 10^7$  m/s    35.  $3.2 \times 10^{-19}$  J  
37. 2    39. 15.2 dB    41. 7.9    43.  $2^{400} = 2.58 \times 10^{120}$

**Exercises 13.5, page 387**

1. 5.298    3. 3.258    5. 0.4460    7. -4.916 23  
9. 1.92    11. 4.806    13. 1.795    15. 3.940    17. 0.3322  
19. -0.008 335    21. -17.390 66    23. 1.6549  
25. -0.164 13    27. 8.94    29. 1.0085    31. 0.4757  
33.  $6.20 \times 10^{-11}$   
35. 

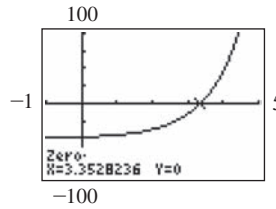


39.  $1.6094 + 2.0794 = 3.6889$     41.  $4.3944 = 4.3944$     43. 3  
45. 10    47.  $2x + y$     49.  $2.45 \times 10^9$  Hz  
51. 8.2%    53. 0.38 s    55. 21.7 s

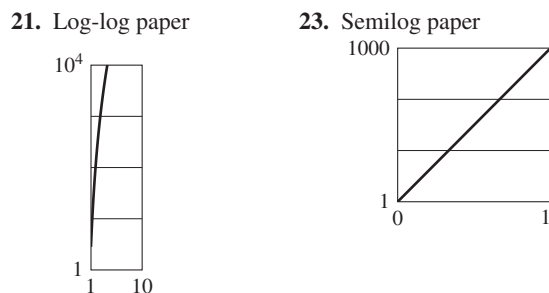
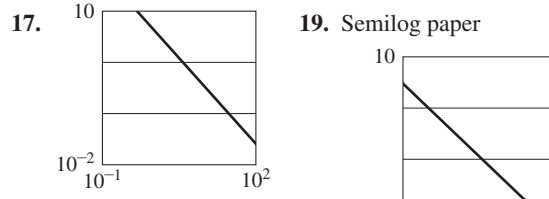
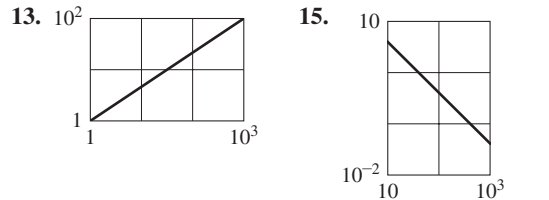
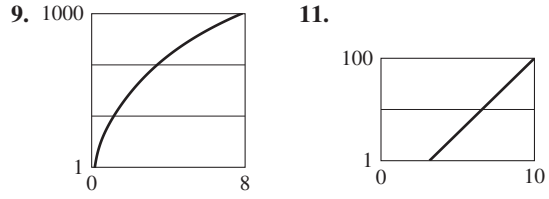
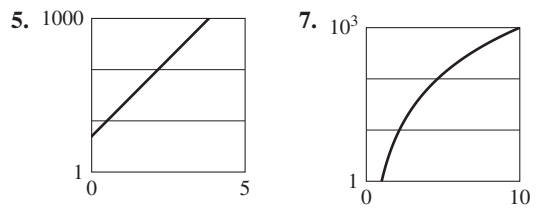
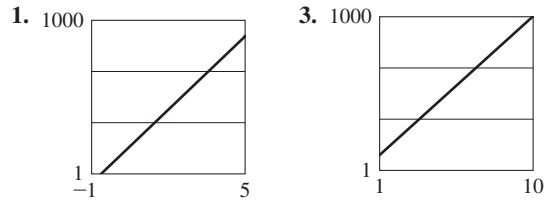
**Exercises 13.6, page 382**

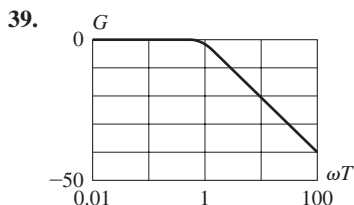
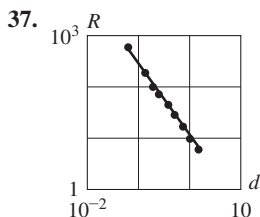
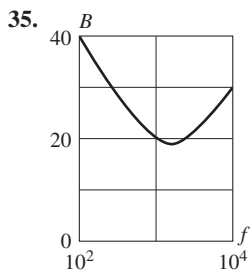
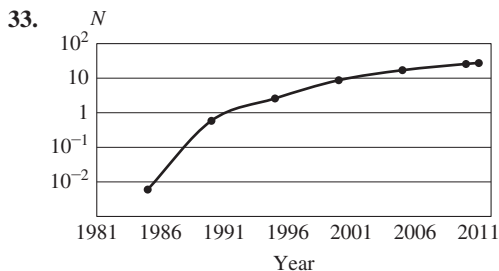
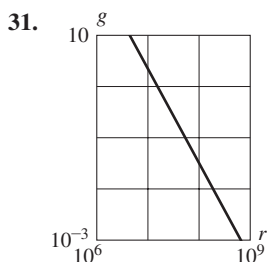
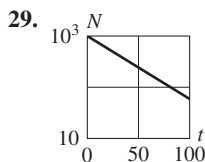
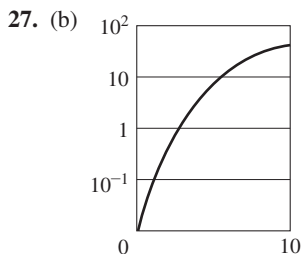
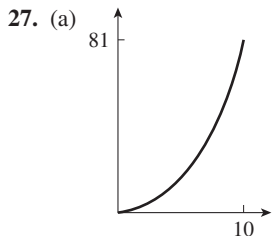
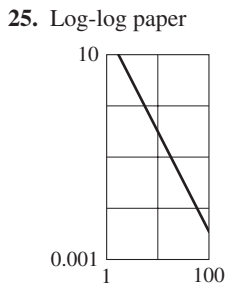
1. -0.535    3. 4    5. -0.7    7. 0.587    9. 1.43  
11. 1.854    13.  $0, \frac{\ln 0.6}{\ln 2} = -0.7$     15.  $\frac{1}{4}$     17. 1, 100  
19. 3    21. -0.162    23. 250    25. 4    27. 2  
29. 1.42    31. -0.104    33. 0.203    35. 0.974  
37. 10.9    39.  $\pm 0.9624$     41. 0.0025    43. (-1.21, 0)  
45. 28.0    47. Noon of previous day (36 h earlier)  
49.  $1.72 \times 10^{-5}$  mol/L    51.  $3.55 \times 10^{22}$   
53.  $c = 15e^{-0.20t}$     55.  $P = P_0(0.999)^t$

57.  $x = 3.35$     59.  $\pm 3.66$  m

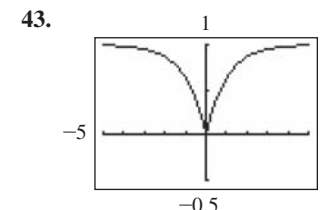
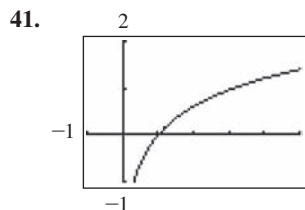
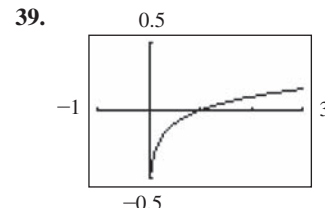
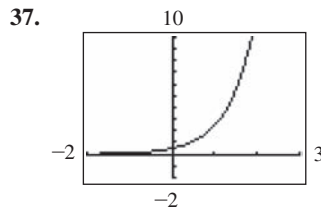


**Exercises 13.7, page 394**

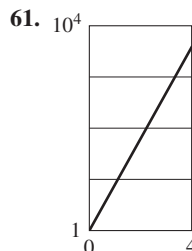




19.  $1 + \frac{1}{2} \log_4 3$     21.  $2 - \log_3 x$     23.  $3 + 4 \log_{10} x$   
 25.  $y = \frac{4}{x}$     27.  $y = e^{2/3}x$     29.  $y = \sqrt{7x}$     31.  $y = \frac{15}{x}$   
 33.  $y = 8x^3$     35.  $y = \frac{x}{\ln 2}$



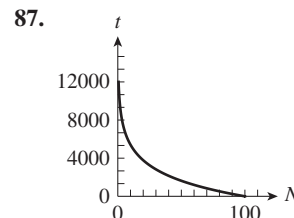
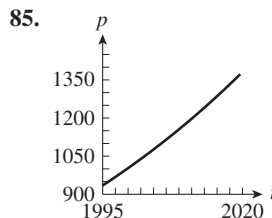
45. 2.182    47. -3.32    49. 1.8188    51. -2.7657  
 53. 0.805    55. 4.30    57. 2    59. -0.2



65. 4    67. 12    69.  $0.9542 - 0.7782 = 0.1761$     71. 0  
 73. 1    75. 0.706    77.  $1.28 \times 10^{-5}$  or 3.925

79.  $V = 1000(1.03)^{2t}$ ,  $t = \frac{\log(V/1000)}{2 \log 1.03}$     81.  $I = I_0 e^{-\beta t}$

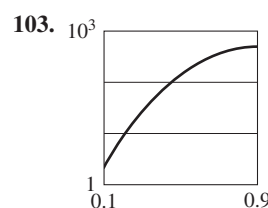
83.  $4.60 \times 10^6 \text{ N} \cdot \text{m}$



89.  $\sin \theta = \frac{l\omega^2}{3g}$     91.  $R = 2^{C/B} - 1$     93. 910 times brighter

95. 1.17 min    97. 0.813 cm    99.  $\frac{\ln \frac{P}{P_0}}{\ln \frac{PT_0}{P_0 T}}$

101.  $n = 20e^{-0.04t}$



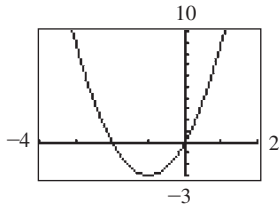
Review Exercises for Chapter 13, page 396

1. 10 000    3.  $\frac{1}{5}$     5. 6    7.  $\frac{5}{3}$     9. 6    11. 100  
 13.  $\log_3 2 + \log_3 x$     15.  $2 \log_3 t$     17.  $2 + \log_2 7$

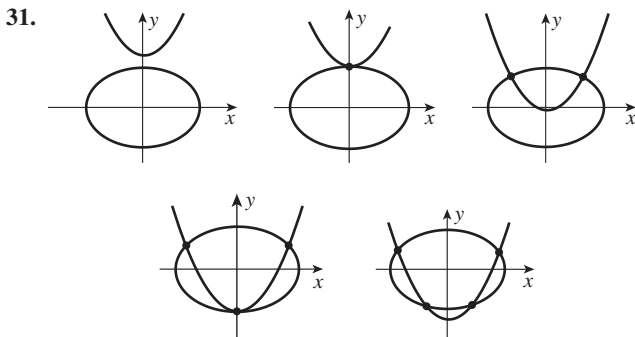
**B.32 ANSWERS TO ODD-NUMBERED EXERCISES**

**Exercises 14.1, page 402**

1.  $y = 3x^2 + 6x$     3.  $x = -1.2, y = 0.27; x = 0.76, y = 2.3$



5.  $x = 1.79, y = 3.58; x = -1.79, y = -3.58$   
 7.  $x = 0.00, y = -2.00; x = 2.67, y = -0.67$   
 9.  $x = 1.50, y = 0.25$     11.  $x = 2.79, y = 10.74$   
 13.  $x = 1.10, y = 2.79; x = -1.10, y = 2.79;$   
 $x = 2.41, y = -1.80; x = -2.41, y = -1.80$   
 15. No real solutions  
 17.  $x = -2.83, y = -1.00; x = 2.83, y = 1.00;$   
 $x = 2.83, y = -1.00; x = -2.83, y = 1.00$   
 19.  $x = -0.81, y = 2.52; x = 2.56, y = 0.66$   
 21.  $x = 0.00, y = 0.00; x = 0.88, y = 0.77$   
 23.  $x = -1.15, y = 3.15; x = 1.84, y = 0.16$   
 25.  $x = 16.34, y = 16.12$   
 27.  $x = 3.64, y = 0.97$   
 29.  $x = -2.06, y = 4.23; x = 1.06, y = 1.12$



33. 4.93 km N, 1.64 km E    35. 2.2 A, 0.9 A    37. No

**Exercises 14.2, page 405**

1.  $x = 2, y = 0; x = \frac{10}{3}, y = -\frac{8}{3}$   
 3.  $x = -\sqrt{6}, y = -\sqrt{3}; x = -\sqrt{6}, y = \sqrt{3};$   
 $x = \sqrt{6}, y = -\sqrt{3}; x = \sqrt{6}, y = \sqrt{3}$   
 5.  $x = 0, y = 1; x = 1, y = 2$   
 7.  $x = -\frac{19}{5}, y = \frac{17}{5}; x = 5, y = -1$     9.  $x = 1, y = 0$   
 11.  $x = \frac{2}{7}(3 + \sqrt{2}), y = \frac{2}{7}(-1 + 2\sqrt{2});$   
 $x = \frac{2}{7}(3 - \sqrt{2}), y = \frac{2}{7}(-1 - 2\sqrt{2})$   
 13.  $w = 1, h = 1$     15.  $x = -\frac{3}{2}, y = -\frac{8}{3}; x = 2, y = 2$   
 17.  $x = -5, y = 25; x = 5, y = 25$   
 19.  $x = 1, y = 2; x = -1, y = 2;$   
 $x = 2j, y = -3; x = -2j, y = -3$

21.  $D = 1, R = 0; D = -1, R = 0; D = \frac{1}{2}\sqrt{6}, R = \frac{1}{2};$

$D = -\frac{1}{2}\sqrt{6}, R = \frac{1}{2}$

23.  $x = \sqrt{19}, y = \sqrt{6}; x = \sqrt{19}, y = -\sqrt{6};$   
 $x = -\sqrt{19}, y = \sqrt{6}; x = -\sqrt{19}, y = -\sqrt{6}$   
 25.  $x = -5, y = -2; x = -5, y = 2;$   
 $x = 5, y = -2; x = 5, y = 2$   
 27.  $x = -3, y = 2; x = -1, y = -2$     29.  $x = a, y = b$   
 31.  $x = 50 \text{ km}, h = 25 \text{ km}$     33. 2.00 cm, 4.00 cm  
 35. 1.5 cm, 1.4 cm    37. 2.19 m, 0.21 m    39. 12, 10  
 41. 12.0 cm, 18.0 cm    43. 130 km/h

**Exercises 14.3, page 410**

1.  $\pm \frac{1}{2}j\sqrt{2}, \pm 2$     3. -3, -2, 2, 3    5.  $-\frac{1}{2}, \frac{1}{4}$   
 7.  $-\frac{1}{2}, \frac{1}{2}, \frac{1}{6}j\sqrt{6}, -\frac{1}{6}j\sqrt{6}$     9. 1,  $\frac{25}{4}$     11.  $\frac{64}{729}, 1$   
 13. -27, 125    15. 81    17. 26    19. -2, -1, 3, 4  
 21. 18    23. 1, -1,  $j\sqrt{2}, -j\sqrt{2}$     25. 0    27.  $\pm 2, \pm 4$   
 29. 1, 4    31. 10, 100    33. -2, -1, 1, 2  
 35.  $R_1 = 2.62 \Omega, R_2 = 1.62 \Omega$     37. 0.610  
 39. 40.4 cm, 55.4 cm

**Exercises 14.4, page 413**

1.  $\frac{13}{12}$     3. 10    5. 12    7. 2    9.  $\frac{2}{3}$     11. -1  
 13. 32    15. 4, 9    17.  $\pm 5$   
 19. 12 (Extraneous root introduced in squaring both sides of  $\sqrt{x+4} = x-8$ .)  
 21. 16    23.  $\frac{1}{2}$     25. 7, -1    27. 0    29. 5    31. 25  
 33. 258    35. 4    37. 4    39. 16  
 41.  $x = 2$ ; For  $x = 5$ , squaring  $3 - x$  is squaring a negative number. A different extraneous root is introduced.  
 43.  $-r \pm \sqrt{r^2 + R^2}$   
 45.  $r_1^2 = (kC + A - \sqrt{R_1^2 - R_2^2})^2 + r_2^2$   
 47.  $9.6 \text{ m}^2$     49. 9.2 km    51. 1.4 km

**Review Exercises for Chapter 14, page 414**

1.  $x = -0.93, y = 3.44; x = 0.81, y = 2.60$   
 3.  $x = 2.00, y = 0.00; x = 1.60, y = 0.60$   
 5.  $x = -0.56, y = 1.32; x = 0.56, y = 1.32$   
 7.  $x = -2, y = 7; x = 2, y = 7$   
 9.  $x = 0.00, y = 0.00; x = 2.38, y = 0.91$   
 11.  $x = 0, y = 0; x = 2, y = 16$   
 13.  $L = \sqrt{2}, R = 1; L = -\sqrt{2}, R = 1$   
 15.  $u = \frac{1}{12}(1 + \sqrt{97}), v = \frac{1}{18}(5 - \sqrt{97});$   
 $u = \frac{1}{12}(1 - \sqrt{97}), v = \frac{1}{18}(5 + \sqrt{97})$   
 17.  $x = 7, y = 5; x = 7, y = -5; x = -7, y = 5;$   
 $x = -7, y = -5$

19.  $x = -2, y = 2; x = \frac{2}{3}, y = \frac{10}{9}$   
 21.  $-4, -2, 2, 4$     23.  $1, 16$     25.  $\frac{1}{3}, -\frac{1}{7}$     27.  $1, 1.65$   
 29.  $\sqrt{3}, -\sqrt{3}, \frac{1}{2}j\sqrt{3}, -\frac{1}{2}j\sqrt{3}$     31.  $6$     33.  $8$     35.  $\frac{9}{16}$   
 37.  $\frac{1}{3}(11 - 4\sqrt{15})$     39.  $2$     41.  $4$     43.  $5$     45.  $-1, 2$   
 47.  $x = a + 1, y = a$     49.  $25$   
 51.  $x = -2, y = -3; x = 3, y = 2$   
 53.  $l = \frac{1}{2}(-1 + \sqrt{1 + 16\pi^2 L^2/h^2})$   
 55.  $m = \frac{1}{2}(-y \pm \sqrt{2s^2 - y^2})$     57.  $0.40 \text{ s}, 0.80 \text{ s}$     59.  $70 \text{ m}$   
 61.  $20 \text{ dm}, 26 \text{ dm}, 26 \text{ dm},$  or  $27.6 \text{ dm}, 22.2 \text{ dm}, 22.2 \text{ dm}$   
 63.  $3.0 \text{ mm}, 1.0 \text{ mm}$     65.  $34 \text{ mm}, 52 \text{ mm}$   
 67.  $0.353 \text{ cm}, 6.29 \text{ cm}$     69.  $27.0 \text{ km/h}, 23.8 \text{ km/h}$

**Exercises 15.1, page 422**

1.  $-25$     3. Coefficients  $1 \ 0 \ 0 \ -4 \ 11, R = -26$     5.  $0$   
 7.  $-40$     9.  $-43$     11.  $51$     13.  $-28$     15.  $14$   
 17. Yes    19. No    21. No    23.  $x^2 + 3x + 2, R = 0$   
 25.  $x^2 + x - 4, R = 8$   
 27.  $p^5 + 2p^4 + 4p^3 + 2p^2 + 2p + 4, R = 2$   
 29.  $x^6 + 2x^5 + 4x^4 + 8x^3 + 16x^2 + 32x + 64, R = 0$   
 31.  $x^3 + x^2 + 2x + 1, R = 0$     33. No    35. No    37. Yes  
 39. No    41. Yes    43. Yes    45.  $2x^2 - 5x + 1 - \frac{8}{x+4}$   
 47.  $(4x^3 + 8x^2 - x - 2) \div (2x - 1) = 2x^2 + 5x + 2; \text{no,}$   
 because the coefficient of  $x$  in  $2x - 1$  is  $2$ , not  $1$ .  
 49.  $-7$     51.  $x^2 - (3 + j)x + 3j, R = 0$   
 53. Yes: If  $r$  is a zero of  $f(x)$ ,  $f(r) = 0 = -g(r)$ , so  $r$  is a zero  
 of  $-g(x)$ .    55. (a) No (b) Yes    57. Yes

**Exercises 15.2, page 426**

(Note: Unknown roots listed)

1.  $1, 1, 1, -1, -1$     3.  $-3, 2, 3$     5.  $-3, -3, 2j, -2j$   
 7.  $-2, 3$     9.  $-2, -2$     11.  $-j, -\frac{2}{3}$     13.  $2j, -2j$   
 15.  $-1, \frac{1}{2}$     17.  $-2, 1$     19.  $1 - j, \frac{1}{2}, -2$     21.  $j, -j$   
 23.  $-j, -1, 1$     25.  $-2j, 3, -3$   
 27. Complex roots occur in conjugate pairs. Two real and one  
 complex is not a possibility for a third-degree polynomial.  
 29.  $x^3 - 2x^2 + 2x$

**Exercises 15.3, page 432**

1. No more than two positive roots and three negative roots  
 3.  $1, -1, -2$     5.  $2, -1, -3$   
 7.  $1, \frac{1}{6}(-3 + j\sqrt{15}), \frac{1}{6}(-3 - j\sqrt{15})$     9.  $\frac{1}{3}, -3, -1$   
 11.  $-2, -2, 2 \pm \sqrt{3}$     13.  $-2, \frac{2}{5}, 1 + j\sqrt{3}, 1 - j\sqrt{3}$

15.  $-3, -\frac{2}{3}, -\frac{1}{2}, \frac{1}{2}$     17.  $2, 2, -1, -1, -3$   
 19.  $2, 2, 2, \frac{1}{2}j, -\frac{1}{2}j$     21.  $-1.86, 0.68, 3.18$   
 23.  $-3.01, -1.49, -0.33, 0.33$     25.  $0.59$     27.  $-0.77$   
 29.  $x = -2, y = -28; x = 2\sqrt{3}, y = 20 - 12\sqrt{3};$   
 $x = 2\sqrt{3}, y = 20 + 12\sqrt{3}$     31.  $-\frac{3}{4}, \sqrt{5}, -\sqrt{5}$   
 33.  $1.73 \text{ s}, 4.51 \text{ s}$     35.  $0, L$     37.  $0.87 \text{ s}, 4.13 \text{ s}$   
 39.  $1.23 \text{ cm or } 2.14 \text{ cm}$     41.  $3.0 \text{ mm}, 4.0 \text{ mm}; 5.0 \text{ mm}, 6.0 \text{ mm}$   
 43.  $2800 \ \Omega, 4200 \ \Omega, 8400 \ \Omega$   
 45. Two positive, one negative; zero positive, one negative, two  
 complex

**Review Exercises for Chapter 15, page 433**

1.  $1$     3.  $-107$     5. Yes    7. No  
 9.  $x^2 + 4x + 10, R = 11$     11.  $2x^2 - 7x + 10, R = -17$   
 13.  $x^3 - 3x^2 - 2x + 10, R = -4$   
 15.  $2m^4 + 10m^3 + 4m^2 + 21m + 105, R = 516$     17. No  
 19. Yes    21. (unlisted roots)  $-2, 1$     23. (unlisted roots)  $3, 4$   
 25. (unlisted roots)  $-1 + j\sqrt{2}, -1 - j\sqrt{2}$   
 27. (unlisted roots)  $-j, \frac{1}{2}, -\frac{3}{2}$     29. (unlisted roots)  $2, -2$   
 31. (unlisted roots)  $-2 - j, \frac{1}{2}(-1 \pm j\sqrt{3})$     33.  $1, 2, -4$   
 35.  $-\frac{1}{2}, 1, 1$     37.  $-1, -\frac{1}{2}, \frac{5}{3}$   
 39.  $\frac{1}{2}, \frac{3}{2}, -1 + \sqrt{2}, -1 - \sqrt{2}$     41.  $1, 3, 5$     43.  $2 \text{ or } 4$   
 45. Use  $k$  as second coefficient in synthetic division, equate  
 remainder to  $0; k = 4$ .  
 47.  $x^3 - 5x^2 + x - 5$     49.  $x = -1, y = -2; x = 2, y = 1$   
 51.  $1.91$     53.  $11.22, 1.44$     55.  $0.75 \text{ cm}$     57.  $2 \text{ cm}$   
 59.  $5.1 \text{ m}, 8.3 \text{ m}$     61.  $h = 2.25 \text{ m}, w = 1.20 \text{ m}$

**Exercises 16.1, page 438**

1.  $\begin{bmatrix} 5 & 7 & -1 & 9 \\ 6 & 4 & 1 & 12 \end{bmatrix}$     3.  $a = 1, b = -3, c = 4, d = 7$   
 5.  $x = -2, y = 5, z = -9, r = 48, s = 4, t = -1$   
 7.  $C = 3, D = 2, E = -2$   
 9. Elements cannot be equated; different number of rows.  
 11.  $\begin{bmatrix} 1 & 10 \\ 0 & 2 \end{bmatrix}$     13.  $\begin{bmatrix} -5 & 0 \\ 11 & 71 \\ 11 & -5 \end{bmatrix}$     15.  $\begin{bmatrix} 6 & 13 & -13 \\ 6 & -7 & 3 \end{bmatrix}$   
 17. Cannot be added    19.  $\begin{bmatrix} 5 & 17 & -20 \\ 8 & -13 & 14 \end{bmatrix}$   
 21.  $\begin{bmatrix} -15 & -14 & 5 \\ -6 & -4 & 27 \end{bmatrix}$     23.  $\begin{bmatrix} 4 & -16 & 28 \\ -8 & 24 & -44 \end{bmatrix}$   
 25. Elements cannot be combined.  
 27.  $\begin{bmatrix} -3 & 12 & -21 \\ 6 & -18 & 33 \end{bmatrix}$     29.  $\begin{bmatrix} 20 & 31 & -25 \\ 14 & -9 & -13 \end{bmatrix}$   
 31. Cannot be subtracted    33.  $\begin{bmatrix} -43 & -50 & 29 \\ -22 & 0 & 59 \end{bmatrix}$



**B.34 ANSWERS TO ODD-NUMBERED EXERCISES**

35.  $A + B = B + A = \begin{bmatrix} 3 & 1 & 0 & 7 \\ 5 & -3 & -2 & 5 \\ 10 & 10 & 8 & 0 \end{bmatrix}$

37.  $-(A - B) = B - A = \begin{bmatrix} 5 & -3 & -6 & -7 \\ 5 & 3 & 0 & -3 \\ -8 & 12 & 8 & 4 \end{bmatrix}$

39.  $v_w = 30.9 \text{ km/h}, v_p = 249 \text{ km/h}$

41.  $\begin{bmatrix} 288 & 225 & 0 & 0 \\ 186 & 132 & 72 & 0 \\ 0 & 105 & 204 & 234 \end{bmatrix}$

43.  $B = \begin{bmatrix} 15 & 25 & 10 & 25 \\ 10 & 10 & 10 & 45 \end{bmatrix}, J = \begin{bmatrix} 15 & 30 & 3 & 3 \\ 0 & 100 & 2 & 2 \end{bmatrix}$

**Exercises 16.2, page 443**

1.  $\begin{bmatrix} 5 & 6 & 7 & -10 \\ -9 & 0 & -3 & 12 \\ 1 & 12 & 11 & -8 \end{bmatrix}$     3.  $[-8 \quad -12]$     5.  $\begin{bmatrix} 305 \\ -325 \end{bmatrix}$

7.  $\begin{bmatrix} -\frac{1}{2} & \frac{111}{4} \\ -\frac{121}{8} & -\frac{101}{2} \\ \frac{1}{10} & 22 \end{bmatrix}$     9.  $\begin{bmatrix} 33 & -22 \\ 31 & -12 \\ 15 & 13 \\ 50 & -41 \end{bmatrix}$     11.  $\begin{bmatrix} -15 & 15 & -26 \\ 8 & 5 & -13 \end{bmatrix}$

13.  $\begin{bmatrix} -49.43 & 55.20 \\ -53.02 & 79.16 \end{bmatrix}$

15.  $AB = [40], BA = \begin{bmatrix} -1 & 3 & -8 \\ 5 & -15 & 40 \\ 7 & -21 & 56 \end{bmatrix}$

17.  $AB = \begin{bmatrix} 45 \\ 327 \end{bmatrix}, BA$  not defined    19.  $AI = IA = A$

21.  $AI = IA = A$     23.  $B = A^{-1}$     25.  $B = A^{-1}$     27. Yes

29. No    31.  $A^2 = A$     33.  $B^3 = B$

35.  $\begin{bmatrix} ac + bd & ad + bc \\ bc + ad & bd + ac \end{bmatrix}$

37.  $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

39.  $A^2 - I = (A + I)(A - I) = \begin{bmatrix} 15 & 28 \\ 21 & 36 \end{bmatrix}$

41.  $\begin{bmatrix} 0 & -j \\ j & 0 \end{bmatrix} \begin{bmatrix} 0 & -j \\ j & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

43.  $V_2 = V_1, i_2 = -V_1/R + i_1$     45. Ellipse

**Exercises 16.3, page 448**

1.  $\begin{bmatrix} -\frac{5}{2} & \frac{3}{2} \\ -2 & 1 \end{bmatrix}$     3.  $\begin{bmatrix} -2 & -\frac{5}{2} \\ -1 & -1 \end{bmatrix}$     5.  $\begin{bmatrix} -\frac{1}{3} & \frac{1}{6} \\ \frac{2}{15} & \frac{1}{30} \end{bmatrix}$

7.  $\begin{bmatrix} \frac{3}{4} & \frac{1}{2} \\ -\frac{1}{4} & 0 \end{bmatrix}$     9.  $\begin{bmatrix} -\frac{8}{283} & -\frac{9}{566} \\ \frac{13}{1415} & \frac{5}{283} \end{bmatrix}$     11.  $\begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}$

13.  $\begin{bmatrix} -\frac{1}{2} & -2 \\ \frac{1}{2} & 1 \end{bmatrix}$     15.  $\begin{bmatrix} \frac{2}{9} & -\frac{5}{9} \\ \frac{1}{9} & \frac{2}{9} \end{bmatrix}$     17.  $\begin{bmatrix} \frac{7}{25} & \frac{3}{5} \\ -\frac{1}{5} & -\frac{1}{2} \end{bmatrix}$

19.  $\begin{bmatrix} -18 & -7 & 5 \\ -3 & -1 & 1 \\ -5 & -2 & 1 \end{bmatrix}$     21.  $\begin{bmatrix} 2 & 4 & \frac{7}{2} \\ -1 & -2 & -\frac{3}{2} \\ 1 & 1 & \frac{1}{2} \end{bmatrix}$

23.  $\begin{bmatrix} \frac{5}{2} & -2 & -2 \\ -1 & 1 & 1 \\ \frac{7}{4} & -\frac{3}{2} & -1 \end{bmatrix}$     25.  $\begin{bmatrix} 0.3 & -0.4 \\ 0.05 & 0.1 \end{bmatrix}$

27.  $\begin{bmatrix} 2 & 4 & 3.5 \\ -1 & -2 & -1.5 \\ 1 & 1 & 0.5 \end{bmatrix}$     29.  $\begin{bmatrix} 2.5 & -2 & -2 \\ -1 & 1 & 1 \\ 1.75 & -1.5 & -1 \end{bmatrix}$

31.  $\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 3 & 2 \\ 2 & 4 & 3 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

33.  $\begin{bmatrix} 2.537 & -0.950 & 0.159 & 0.470 \\ -0.213 & 0.687 & 0.290 & -0.272 \\ -1.006 & 0.870 & 0.496 & -0.532 \\ 0.113 & -0.123 & -0.033 & 0.385 \end{bmatrix}$

35.  $\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$  means the inverse does not exist.

37.  $\frac{1}{ad - bc} \begin{bmatrix} ad - bc & -ba + ab \\ cd - dc & -bc + ad \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

39.  $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{3}{2} & -\frac{5}{2} \\ \frac{1}{2} & -\frac{5}{2} & -\frac{7}{2} \end{bmatrix}$  which is symmetric    41.  $\begin{bmatrix} 0.8 & 0.0 & 0.6 \\ 0.0 & 1.0 & 0.0 \\ -0.6 & 0.0 & 0.8 \end{bmatrix}$

**Exercises 16.4, page 453**

1.  $x = 2, y = -3$     3.  $x = \frac{1}{2}, y = 3$     5.  $x = 1, y = 3$

7.  $x = 2, y = -2$     9.  $x = -1, y = 0, z = 3$

11.  $x = -\frac{3}{2}, y = -2$     13.  $x = 1.6, y = -2.5$

15.  $x = 2, y = -4, z = 1$     17.  $x = 2, y = -\frac{1}{2}, z = 3$

19.  $x = 1, y = -2, z = -3$     21.  $u = 2, v = -5, w = 4$

23.  $x = 2, y = 3, z = -2, t = 1$

25.  $v = 2, w = -1, x = \frac{1}{2}, y = \frac{3}{2}, z = -3$

27.  $x = \pm 2, y = -2$

29.  $x = 1, y = -2$ ; The three lines meet in the point  $(1, -2)$ .

31. 1180 N, 1860 N    33. 10 V, 8 V    35. 40 mL, 8 mL

37. 6.4 L, 1.6 L, 2.0 L

39.  $x = 2.5 \text{ ha}, y = 3.2 \text{ ha}, z = 1.5 \text{ ha}, t = 4 \text{ ha}$

**Exercises 16.5, page 457**

1.  $x = \frac{11}{7}, y = \frac{6}{7}$     3.  $x = 2, y = 1$     5.  $x = \frac{17}{14}, y = \frac{19}{14}$

7. Inconsistent    9.  $x = -2, y = -\frac{2}{3}, z = \frac{1}{3}$

11.  $w = 1, x = 0, y = -2, z = 3$

13. Unlimited:  $x = -3, y = -1, z = 1; x = 12, y = 0, z = -10$

15. Inconsistent    17.  $x = 4, y = \frac{2}{3}, z = -2$

19.  $x = \frac{1}{2}, y = -\frac{3}{2}$     21.  $x = \frac{1}{6}, y = -\frac{1}{2}$     23. Inconsistent

25. Inconsistent    27.  $r = 0, s = 0, t = 0, u = -1$

29.  $x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}, y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$

31.  $x = 57 \times 10^6 \text{ calc/s}, y = 68 \times 10^6 \text{ calc/s}$

33. 250 parts/h, 220 parts/h, 180 parts/h

**Exercises 16.6, page 461**

1.  $\begin{bmatrix} 3 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 3 \end{bmatrix} = 9, \begin{bmatrix} 0 & 0 & 3 \\ 0 & 1 & 1 \\ 3 & 1 & 2 \end{bmatrix} = -9$

3. -60    5. 0    7. -40    9. -40    11. 39    13. 57

15. -13    17. -72    19. 0    21. 39    23. 57

25. -13    27. -72    29. 0

31.  $x = -1, y = 0, z = 2, t = 1$

33.  $x = 1, y = 2, z = -1, t = 3$

35.  $x = 2, y = -1, z = -1, t = 3$

37.  $D = 1, E = 2, F = -1, G = -2$     39. 0    41. 8

43.  $\frac{33}{16} \text{ A}, \frac{11}{8} \text{ A}, -\frac{5}{8} \text{ A}, -\frac{15}{8} \text{ A}, -\frac{15}{16} \text{ A}$     45. 0.618, 1.618

47. ppm  $\text{SO}_2: 0.5, \text{NO}: 0.3, \text{NO}_2: 0.2, \text{CO}: 5.0$

**Review Exercises for Chapter 16, page 463**

1.  $a = 4, b = -1$

3.  $x = 2, y = -3, z = \frac{5}{2}, a = -1, b = -\frac{7}{2}, c = \frac{1}{2}$

5.  $x = -1, y = \frac{1}{2}, a = -\frac{1}{2}, b = -\frac{3}{2}$     7.  $\begin{bmatrix} 1 & -3 \\ 8 & -5 \\ -8 & -2 \\ 3 & -10 \end{bmatrix}$

9.  $\begin{bmatrix} -3 & 3 \\ 0 & -7 \\ 2 & -2 \\ -1 & -4 \end{bmatrix}$     11.  $\begin{bmatrix} 7 & -6 \\ -4 & 20 \\ -1 & 6 \\ 1 & 15 \end{bmatrix}$     13.  $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

15.  $\begin{bmatrix} 0.3 & 0.1 & 0.0 \\ 0.0 & -0.1 & 0.1 \\ 0.0 & -0.2 & 0.2 \end{bmatrix}$     17.  $\begin{bmatrix} -2 & \frac{5}{2} \\ -1 & 1 \end{bmatrix}$     19.  $\begin{bmatrix} \frac{40}{3} & \frac{5}{3} \\ -\frac{20}{3} & \frac{35}{3} \end{bmatrix}$

21.  $\begin{bmatrix} 11 & 10 & 3 \\ -4 & -4 & -1 \\ 3 & 3 & 1 \end{bmatrix}$     23.  $\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -1 \\ -3 & 2 & 1 \\ -4 & 3 & 2 \end{bmatrix}$

25.  $x = -3, y = 1$     27.  $x = 10, y = -15$

29.  $u = -1, v = -3, w = 0$     31.  $x = 1, y = \frac{1}{2}, z = -\frac{1}{3}$

33.  $x = -3, y = 1$     35.  $u = -1, v = -3, w = 0$

37.  $x = 1, y = \frac{1}{2}, z = -\frac{1}{3}$     39. Unlimited number of solutions

41.  $u = -1, v = -3, w = 0$     43.  $x = 1, y = \frac{1}{2}, z = -\frac{1}{3}$

45.  $x = 3, y = 1, z = -1$     47.  $x = 1, y = 2, z = -3, t = 1$

49.  $x = -\frac{1}{3}, y = 3, z = \frac{2}{3}, t = -4$

51.  $r = \frac{11}{10}, s = \frac{-92}{5}, t = \frac{3}{20}, u = \frac{64}{5}, v = \frac{3}{10}$

53.  $\begin{bmatrix} 1 & 0 \\ 15 & 16 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 63 & 64 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 255 & 256 \end{bmatrix}$

55.  $B^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$     57. 77    59. 323    61. 77

63. 323    65.  $N^{-1} = -N = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

67.  $\begin{bmatrix} n & 1+n \\ 1-n & -n \end{bmatrix} \begin{bmatrix} n & 1+n \\ 1-n & -n \end{bmatrix} = \begin{bmatrix} n^2 + (1-n)^2 & (n+n^2) - (n+n^2) \\ (n-n^2) - (n-n^2) & (1-n^2) + n^2 \end{bmatrix}$

69.  $(A+B)(A-B) = \begin{bmatrix} -6 & 2 \\ 4 & 2 \end{bmatrix}, A^2 - B^2 = \begin{bmatrix} -10 & -4 \\ 8 & 6 \end{bmatrix}$

71.  $\begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{1}{6} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & \frac{2}{3} \\ 0 & \frac{1}{3} \end{bmatrix}$     73.  $R_1 = 4 \Omega, R_2 = 6 \Omega$

75.  $F = 303 \text{ N}, T = 175 \text{ N}$     77.  $R_1 = 4 \Omega, R_2 = 6 \Omega$

79.  $F = 303 \text{ N}, T = 175 \text{ N}$

81. 0.20 h after police pass intersection    83. 30 g, 50 g, 20 g

85.  $x = 0.58, y = 0.22, z = 0.20$

87. 830 g, 880 g, 290 g    89. 1575 kJ, 581 kJ, 741 kJ

**Exercises 17.1, page 471**

1.  $x + 1 < 0$  is true for all values of  $x$  less than  $-1$ ;  $x < -1$ , or  $(-\infty, -1)$

3.  $-2 > -4$  multiplied by  $-3$ , gives  $6 < 12$ ; divided by  $-2$  gives  $1 < 2$ .

5.  $7 < 12$     7.  $20 < 45$     9.  $-4 > -9$     11.  $16 < 81$

13.  $x > -2$     15.  $x \leq 45$     17.  $1 < x < 7$

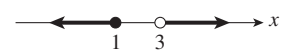
19.  $x < -9$  or  $x \geq -4$     21.  $x < 1$  or  $3 < x \leq 5$

23.  $-2 < x < 2$  or  $3 \leq x < 4$

25.  $x$  is greater than 0 and less than or equal to 2.

27.  $x$  is less than  $-10$ , or greater than or equal to 10 and less than 20.

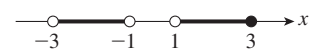
29.  $(-\infty, 3)$     31.  $(-\infty, 1]$  or  $(3, \infty)$



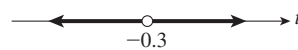
33.  $[0, 5)$     35.  $[-3, 5)$



37.  $(-\infty, -1)$  or  $[1, 4)$     39.  $(-3, -1)$  or  $(1, 3]$



41.  $(-\infty, -0.3)$  or  $(-0.3, \infty)$     43.  $(3, 5]$  or  $[8, 10)$



45. Absolute inequality    47. No,  $a - b < 0$

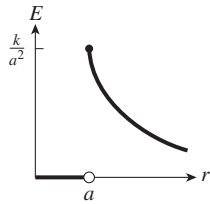
49.  $|x + y| < |x| + |y|$

51. From step (4) to step (5), both sides divided by  $\log 0.5$ , which is negative. Sign in step (5) should be  $>$ .

53.  $2000 \leq M \leq 1\,000\,000$     55.  $29\,000 < v < 40\,000 \text{ km/h}$

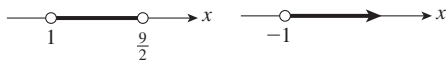
**B.36 ANSWERS TO ODD-NUMBERED EXERCISES**

57.  $0 < n \leq 2565$  steps    59.  $E = 0$  for  $0 < r < a$   
 $E = k/r^2$  for  $r \geq a$

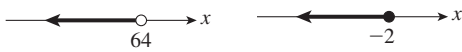


**Exercises 17.2, page 475**

1.  $x \leq 3, (-\infty, 3]$     3.  $1 < x < \frac{9}{2}, (1, \frac{9}{2})$     5.  $x > -1, (-1, \infty)$



7.  $x < 64, (-\infty, 64)$     9.  $x \leq -2, (-\infty, -2]$



11.  $y < -2, (-\infty, -2)$     13.  $x \leq \frac{5}{2}, (-\infty, \frac{5}{2}]$



15.  $T < -\frac{177}{20}, (-\infty, -\frac{177}{20})$     17.  $x > -1.80, (-1.80, \infty)$



19.  $L > -\frac{7}{9}, (-\frac{7}{9}, \infty)$     21.  $-1 < x < 1, (-1, 1)$



23.  $2 < x \leq 5, (2, 5]$     25.  $-3 \leq x < -1, [-3, -1)$



27. No values    29.  $x \geq 5, [5, \infty)$



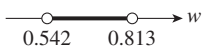
31.  $x \geq -5, [-5, \infty)$     33.  $-6 < k < 6, (-6, 6)$



35.  $n > 35 \text{ h}, (35, \infty)$     37.  $50^\circ < F < 68^\circ, (50, 68)$



39.  $0.542 \text{ m} < w < 0.813 \text{ m}, (0.542, 0.813)$



41.  $0.400 \text{ h} < t < 2.60 \text{ h}, (0.400, 2.60)$



43.  $0 \leq x \leq 500, 200 \leq y \leq 700, [200, 700]$

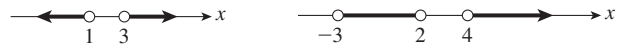


45.  $2 \text{ min} \leq \text{stop times} \leq 4 \text{ min}, [2, 4]$



**Exercises 17.3, page 481**

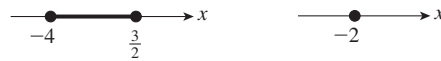
1.  $x < 1$  or  $x > 3, (-\infty, 1) \cup (3, \infty)$     3.  $-3 < x < 2$  or  $x > 4, (-3, 2) \cup (4, \infty)$



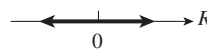
5.  $-4 < x < 4, (-4, 4)$     7.  $0 \leq x \leq 2, [0, 2]$



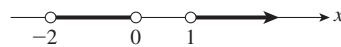
9.  $-4 \leq x \leq \frac{3}{2}, [-4, \frac{3}{2}]$     11.  $x = -2$



13. All  $R, (-\infty, \infty)$



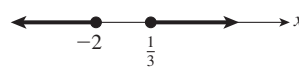
15.  $-2 < x < 0, x > 1, (-2, 0) \cup (1, \infty)$



17.  $-2 \leq s \leq -1, s \geq 1, [-2, -1] \cup [1, \infty)$



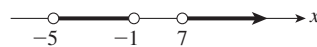
19.  $x \leq -2, x \geq \frac{1}{3}, (-\infty, -2] \cup [\frac{1}{3}, \infty)$



21.  $-6 < x \leq \frac{3}{2}, (-6, \frac{3}{2}]$



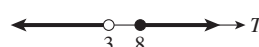
23.  $-5 < x < -1, x > 7, (-5, -1) \cup (7, \infty)$



25.  $x < -1, (-\infty, -1)$



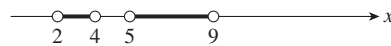
27.  $T < 3, T \geq 8, (-\infty, 3) \cup [8, \infty)$



29.  $-1 < x < \frac{3}{4}, x \geq 6, (-1, \frac{3}{4}) \cup [6, \infty)$



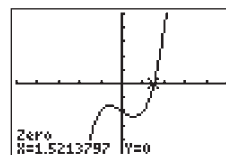
31.  $2 < x < 4, 5 < x < 9, (2, 4) \cup (5, 9)$



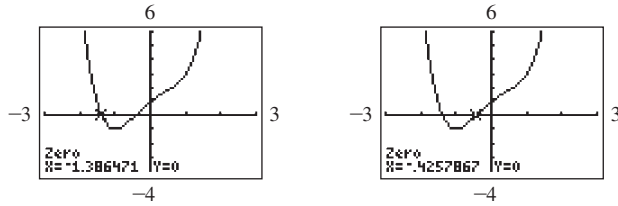
33.  $x \leq -2, x \geq 1, (-\infty, -2] \cup [1, \infty)$

35.  $-1 \leq x \leq 0, [-1, 0]$

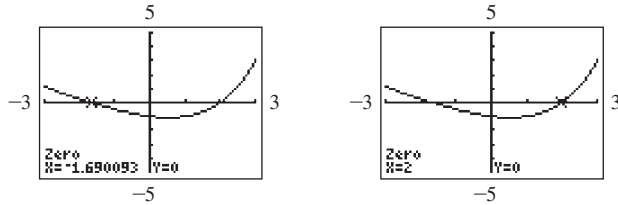
37.  $x > 1.52, (1.52, \infty)$



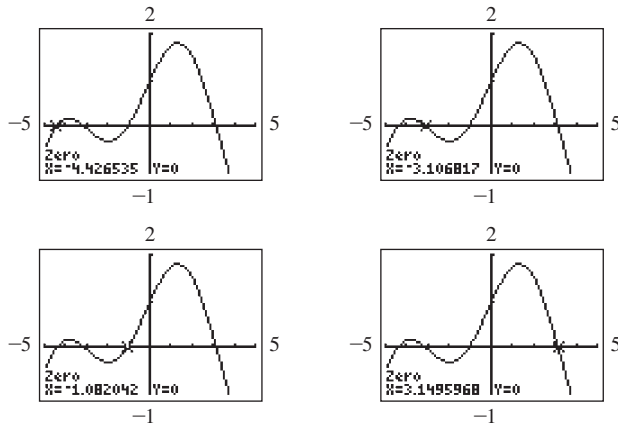
39.  $-1.39 < x < -0.43$ ,  $(-1.39, -0.43)$



41.  $x < -1.69$ ,  $x > 2.00$ ,  $(-\infty, -1.69)$  or  $(2.00, \infty)$



43.  $x < -4.43$ ,  $-3.11 < x < -1.08$ ,  $x > 3.15$ ,  
 $(-\infty, -4.43)$  or  $(-3.11, -1.08)$  or  $(3.15, \infty)$



45. No; not true if  $0 \leq x \leq 1$     47.  $x^2 - 3x - 4 < 0$

49.  $x < 3.11$ ,  $(-\infty, 3.11)$     51.  $a \neq b$

53.  $0.5 \text{ A} < i < 1 \text{ A}$ ,  $(0.5 \text{ A}, 1 \text{ A})$     55.  $4 \leq t \leq 16$ ,  $[4, 16]$

57.  $C_1 > \frac{4}{3} \mu\text{F}$ ,  $(\frac{4}{3} \mu\text{F}, \infty)$     59.  $h > 2640 \text{ km}$ ,  $(2640, \infty) \text{ km}$

61.  $3.0 \leq w < 5.0 \text{ mm}$ ,  $[3.0 \text{ mm}, 5.0 \text{ mm})$

63.  $0 \leq t < 0.92 \text{ h}$ ,  $[0, 0.92 \text{ h})$

**Exercises 17.4, page 484**

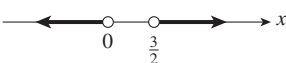
1.  $-2 < x < 3$ ,  $(-2, 3)$     3.  $3 < x < 5$ ,  $(3, 5)$



5.  $x < -2$ ,  $x > \frac{2}{5}$ ,  $(-\infty, -2)$  or  $(\frac{2}{5}, \infty)$     7.  $\frac{1}{6} \leq x \leq \frac{3}{2}$ ,  $[\frac{1}{6}, \frac{3}{2}]$



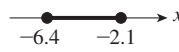
9.  $x < 0$ ,  $x > \frac{3}{2}$ ,  $(-\infty, 0)$  or  $(\frac{3}{2}, \infty)$



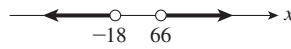
11.  $-26 < t < 24$ ,  $(-26, 24)$



13.  $-6.4 \leq x \leq -2.1$ ,  $[-6.4, -2.1]$



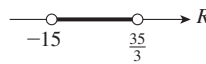
15.  $x < -18$ ,  $x > 66$ ,  $(-\infty, -18)$  or  $(66, \infty)$



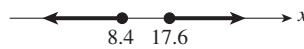
17.  $1 < x < 2$ ,  $(1, 2)$     19.  $x \leq \frac{1}{10}$ ,  $x \geq \frac{7}{10}$ ,  $(-\infty, \frac{1}{10}]$  or  $[\frac{7}{10}, \infty)$



21.  $-15 < R < \frac{35}{3}$ ,  $(-15, \frac{35}{3})$



23.  $x \leq 8.4$ ,  $x \geq 17.6$ ,  $(-\infty, 8.4]$  or  $[17.6, \infty)$



25.  $x < -3$ ,  $-2 < x < 1$ ,  $x > 2$   
 $(-\infty, -3)$  or  $(-2, 1)$  or  $(2, \infty)$

27.  $-3 < x < -2$ ,  $1 < x < 2$ ,  $(-3, -2)$  or  $(1, 2)$

29. No solutions;  $|x|$  is never less than 0.

31.  $x > \frac{a-c}{|b|}$     33. 4 km, 50 km

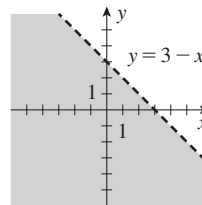
35.  $|p - 2\,000\,000| \leq 200\,000$ ; production is at least 1 800 000 barrels, but not greater than 2 200 000 barrels.

37.  $|d - 3.765| \leq 0.002$     39.  $|m - 1607.3| \leq 65.3$

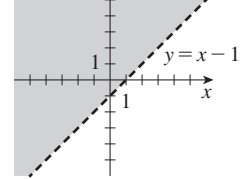
41.  $8.87 \text{ s} < t < 29.9 \text{ s}$ ,  $(8.87 \text{ s}, 29.9 \text{ s})$

**Exercises 17.5, page 487**

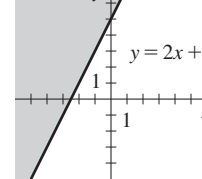
1.  $y < 3 - x$



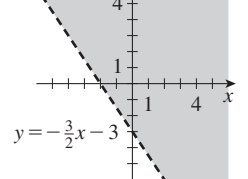
3.  $y > x - 1$



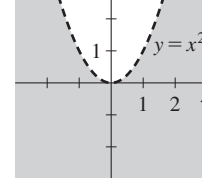
5.  $y > 2x + 5$



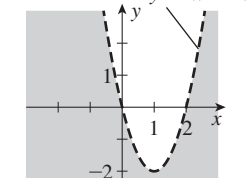
7.  $y < -\frac{3}{2}x - 3$

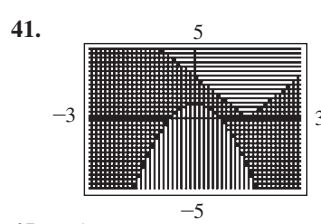
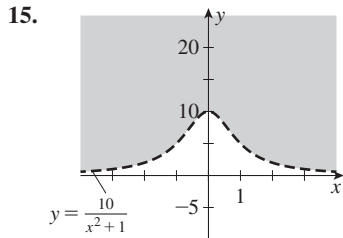
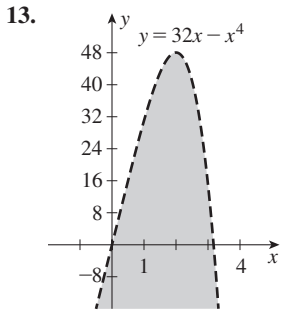


9.  $y > x^2$

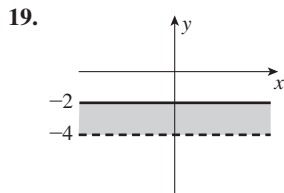
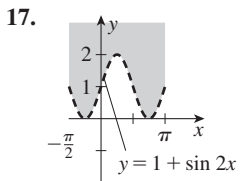


11.  $y < 2x^2 - 4x$

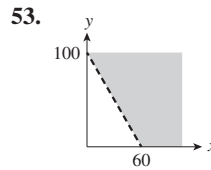
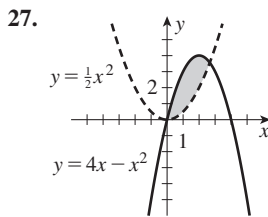
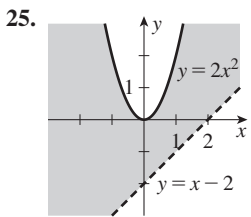
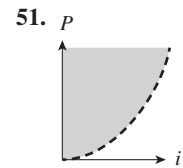
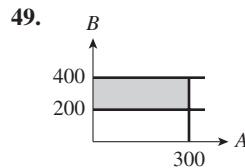
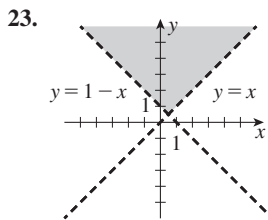
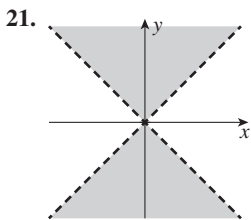
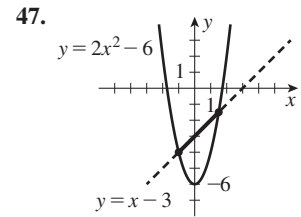




43.  $y < 2x + \frac{5}{2}$



45. Below



Exercises 17.6, page 491

1. Max  $F = 18$  at  $(0, 6)$

3. Max  $F = 58$  at  $(10, 7)$   
Min  $F = 0$  at  $(0, 0)$

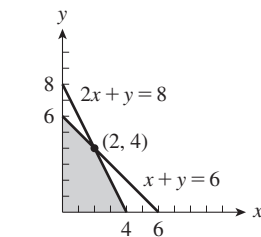
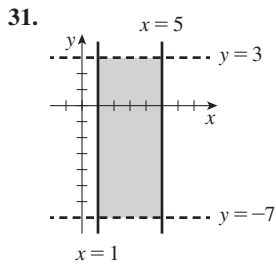
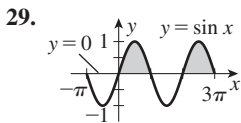
5. Max  $P = 30$  at  $(0, 6)$

7. Max  $P = 30$  at  $(6, 0)$

9. Min  $C = 24$  at  $(3, 2)$

11. Min  $F = 7$  at  $(1, 2)$

Max  $F = \frac{47}{3}$  at  $(\frac{5}{3}, \frac{14}{3})$



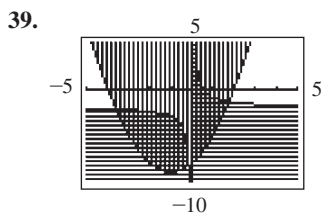
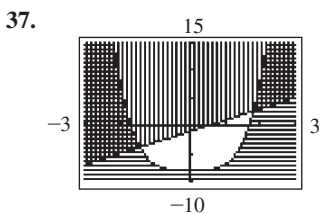
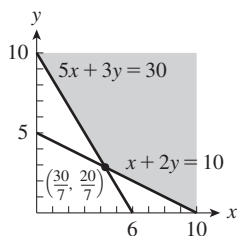
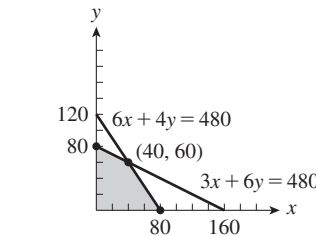
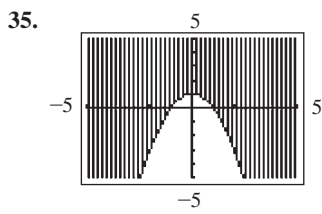
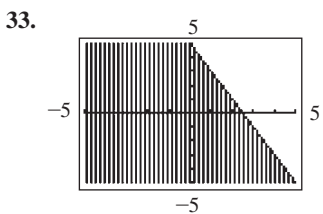
13. Max  $P = 27$  at  $(3, 0)$

15. Min  $C = 40$  at  $(4, 4)$

17. \$6000 at 6%, \$3000 at 5%

19. 40 business models  
60 graphing models

21.  $4\frac{2}{7}$  servings of  $A = \frac{900}{7}$  g of  $A$   
 $2\frac{6}{7}$  servings of  $B = \frac{600}{7}$  g of  $B$



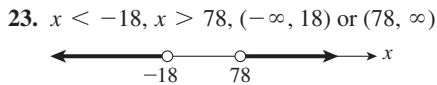
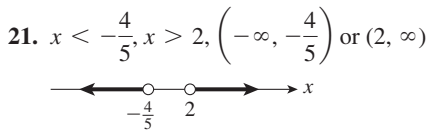
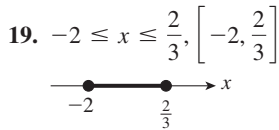
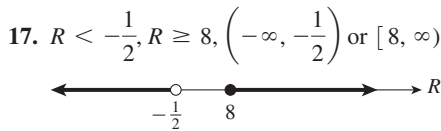
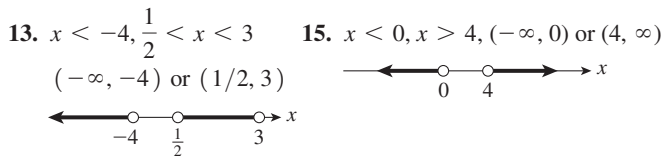
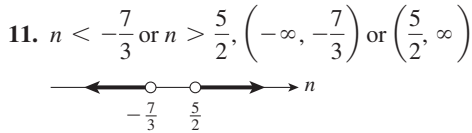
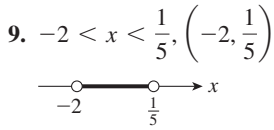
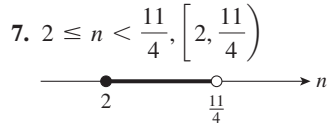
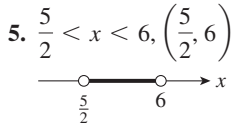
Review Exercises for Chapter 17, page 492

1.  $x > 6, (6, \infty)$

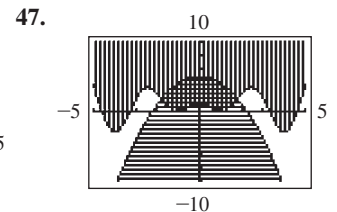
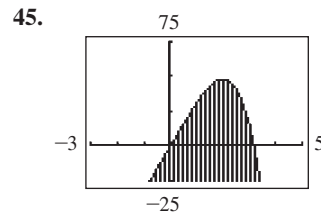
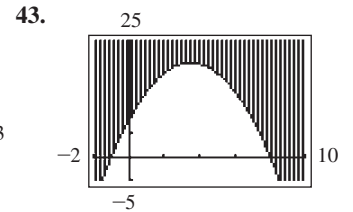
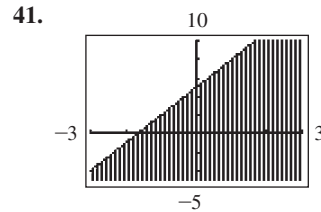
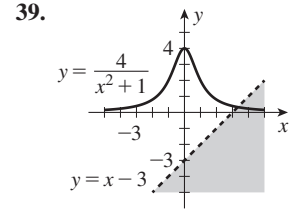
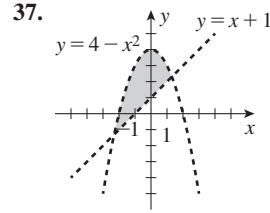
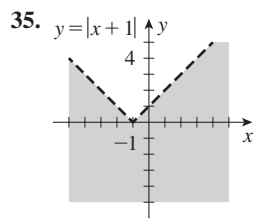
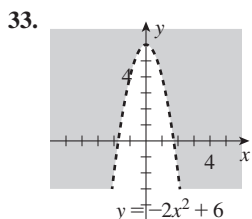
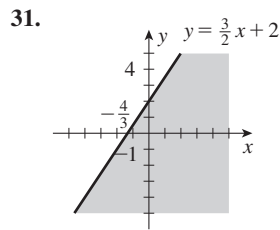
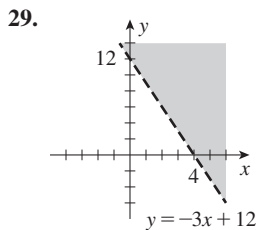
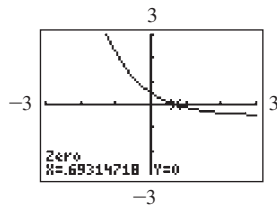
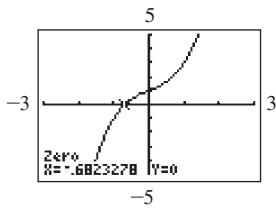


3.  $x > \frac{5}{3}, (\frac{5}{3}, \infty)$





25.  $x < -0.68, (-\infty, -0.68)$  27.  $x < 0.69, (-\infty, 0.69)$



49.  $x \leq 3, (-\infty, 3]$  51.  $x \leq -4, x \geq 0, (-\infty, 4] \cup [0, \infty)$

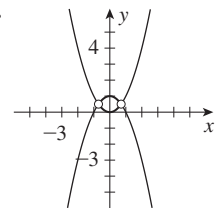
53. Max  $P = 29$  at  $(1, 3)$

55. Min  $C = \frac{65}{8}$  at  $\left(\frac{3}{8}, \frac{7}{4}\right)$

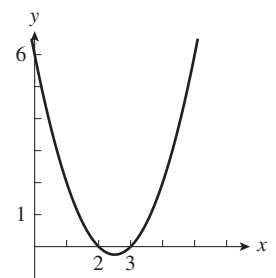
57.  $a$  and  $b$  must have different signs.

59.  $a + \frac{1}{a} - 2 > 0, a^2 + 1 - 2a > 0, (a - 1)^2 > 0$

61.  $x^2 - 3x - 10 < 0$  63.



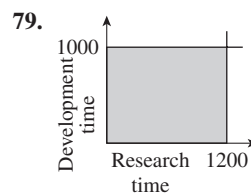
65.  $f(x) > 0$  for  $x < 2$  or  $x > 3$   
 $f(x) < 0$  for  $2 < x < 3$   
 $f(x) = 0$  for  $x = 2, x = 3$   
 $f(0) = 6, f(5) = 6$



67.  $0 < x \leq 8.0$  cm 69. Between \$5 and \$12.50

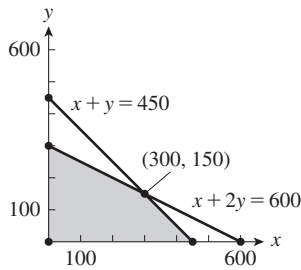
71.  $d > 39.5$  m 73.  $168 \leq B \leq 400$  MJ

75.  $0.456 < i < 0.816$  A 77.  $r > 5.7$



**B.40 ANSWERS TO ODD-NUMBERED EXERCISES**

81. 300 regular  
150 deluxe



**Exercises 18.1, page 498**

1. 56 km    3. 6    5.  $\frac{4}{3}$     7. 25    9. 40  
 11.  $0.27 = 27\%$     13. 0.41    15.  $0.025 \mu\text{F}$     17. 5.56  
 19. 0.931    21. 3.5%    23. 863 kg    25. 0.103  
 27.  $1.44 \Omega$     29. 0.45 ha    31. 237 600 s  
 33.  $2.50 \times 10^3 \text{ cm}^2$     35.  $0.0324 \text{ kL/h}$     37.  $23\,400 \text{ cm}^3$   
 39. 8.8 m    41. 3.0 m, 4.5 m    43. 19.67 kg  
 45. 17 500 chips    47. 112 cm

**Exercises 18.2, page 503**

1.  $C = kd, k = \pi$     3. 667 Hz    5.  $v = kr$   
 7.  $R = \frac{k}{d^2}$     9.  $p = \frac{k}{\sqrt{A}}$     11.  $S = kwd^3$   
 13. A varies directly as the square of r.  
 15. f varies directly as L and inversely as the square root of m.  
 17.  $V = \frac{H^2}{2048}$     19.  $p = \frac{16q}{r^3}$     21. 25    23. 0.033  
 25. 180    27.  $2.56 \times 10^5$   
 29.  $A = k_1x, B = k_2x; A + B = (k_1 + k_2)x$     31. 10.2 kB/s  
 33. 209 kJ    35. 2.40 cm    37. 560 kJ    39. 1.4 h  
 41. (a) Inverse (b)  $a = 60/\text{m}$   
 43. 7.65 MW    45.  $F = 2.32Av^2$     47. 0.536 N  
 49. 480 m/s    51.  $R = \frac{2.60 \times 10^{-5}l}{A}$     53. 80.0 W  
 55.  $G = \frac{5.9d^2}{\lambda^2}$     57.  $-6.58 \text{ cm/s}^2$     59.  $0.288 \text{ W/m}^2$

**Review Exercises for Chapter 18, page 506**

1. 200    3. 1.5    5. 3.1417    7. 5.6    9.  $7.39 \text{ N/cm}^2$   
 11.  $2260 \text{ J/g}$     13. 5.5%  
 15.  $\frac{a}{b} + \frac{b}{b} = \frac{c}{d} + \frac{d}{d} = \frac{a}{b} + \frac{c}{d}$     17. 630 km  
 19.  $2.66 \times 10^{-3} \text{ kJ}$     21. 240 pages    23. 140 mL  
 25. 71.9 m    27. 4500, 7500    29. 30.2 kg  
 31. 115 bolts, 207 bolts    33.  $y = 3x^2$     35.  $v = \frac{128x}{y^3}$   
 37. 11.7 cm    39. 10 cm    41.  $R = 8.5 \text{ A}$   
 43. 1500 bacteria/h    45.  $F = \frac{5500}{L}$     47. 2.22 s  
 49.  $4.3 \mu\text{C}$     51. 18.0 kW    53. 533 m    55. 1.4  
 57. 48.7 Hz    59.  $2.99 \times 10^8 \text{ m/s}$     61. 759 N  
 63. 3.25 cm    65.  $5.73 \times 10^4 \text{ m}$     67. 47 m    69. 150%  
 71. \$125.00, \$600.13    73. 4.80 MJ    75.  $0.023 \text{ W/m}^2$

**Exercises 19.1, page 514**

1. -3    3. 125 250    5. 4, 6, 8, 10, 12  
 7. 2.5, 1.5, 0.5, -0.5, -1.5    9. 22    11.  $-\frac{9\pi}{4}$     13. 30.9  
 15.  $49b$     17. 440    19.  $-\frac{85}{2}$     21.  $n = 6, S_6 = 150$   
 23.  $d = -\frac{2}{19}, a_{20} = -\frac{1}{3}$     25.  $a_1 = 19, a_{30} = 106$   
 27.  $n = 62, S_{62} = -486.7$   
 29.  $n = 13, a_{13} = 11k$     31.  $n = 8, d = \frac{1}{14}(b + 2c)$   
 33.  $a_1 = 360, d = 40, S_{10} = 5400$   
 35. Yes,  $d = \ln 2; a_5 = \ln 48$     37.  $a_n = 2n + 1$   
 39.  $d = \frac{b-a}{2}$     41.  $2b - c, b, c, 2c - b, 3c - 2b$   
 43. 5050    45. -8  
 47.  $2700 \text{ m}^2$     49. 195 logs    51. 10 rows  
 53. 13 years, \$11 700    55. 185 m  
 57.  $S_n = \frac{1}{2}n[2a_1 + (n-1)d]$     59.  $a_1 = 1, a_n = n$

**Exercises 19.2, page 518**

1. 243    3. 6400, 1600, 400, 100, 25    5.  $\frac{1}{6}, \frac{1}{2}, \frac{3}{2}, \frac{9}{2}, \frac{27}{2}$   
 7. 128    9.  $\frac{1}{125}$     11.  $\frac{100}{729}$     13. 1    15.  $\frac{341}{8}$     17. 762  
 19.  $\frac{3(2^{10} - k^{10})}{16(2+k)}$     21.  $a_6 = 64, S_6 = \frac{1365}{16}$   
 23.  $a_1 = 16, a_5 = 81$     25.  $a_1 = 1, r = 3$   
 27.  $n = 7, S_n = \frac{58\,593}{625}$     29. Yes;  $r = 3^x; a_{20} = 3^{19x+1}$   
 31.  $r = \sqrt{\frac{b}{a}}$     33. 5    35. (a)  $a_1$ , (b) 0    37. 2 or -3  
 39. 1.4%    41. 1.09 mA    43. \$443.96    45.  $5800 \text{ }^\circ\text{C}$   
 47. 43%    49.  $21.1 \text{ }^\circ\text{C}$     51. \$671 088.64  
 53.  $S_n = \frac{a_1 - ra_n}{1-r}$     55. Yes, the ratio is squared.  
 57. 8, 1, -6; 8, 16, 24; 8, 16, 24; 16, 24, 36

**Exercises 19.3, page 522**

1.  $\frac{32}{7}$     3. 8    5. 0.2    7.  $\frac{400}{21}$     9. 8    11.  $\frac{10\,000}{9999}$   
 13.  $\frac{1}{2}(5 + 3\sqrt{3})$     15.  $\frac{1}{3}$     17. 0.5    19.  $\frac{40}{99}$     21.  $\frac{2}{11}$   
 23.  $\frac{91}{3330}$     25.  $\frac{11}{30}$     27.  $\frac{100\,741}{999\,000}$     29.  $\frac{125}{2}, \frac{125}{3}$   
 31. 350 L    33. 346 g    35. 100 m    37.  $-\frac{1}{4}$     39. 0.368

**Exercises 19.4, page 527**

1.  $32x^5 + 240x^4 + 720x^3 + 1080x^2 + 810x + 243$   
 3.  $t^3 + 12t^2 + 48t + 64$     5.  $16x^4 - 32x^3 + 24x^2 - 8x + 1$   
 7. 8445.963 01  
 9.  $n^5 + 10\pi n^4 + 40\pi^2 n^3 + 80\pi^3 n^2 + 80\pi^4 n + 32\pi^5$



$$11. 64a^6 - 192a^5b^2 + 240a^4b^4 - 160a^3b^6 + 60a^2b^8 - 12ab^{10} + b^{12}$$

$$13. 625x^4 - 1500x^3 + 1350x^2 - 540x + 81$$

$$15. 64a^6 + 192a^5 + 240a^4 + 160a^3 + 60a^2 + 12a + 1$$

$$17. x^{10} + 20x^9 + 180x^8 + 960x^7 + \dots$$

$$19. 128a^7 - 448a^6 + 672a^5 - 560a^4 + \dots$$

$$21. x^6 - 48x^{11/2}y + 1056x^5y^2 - 14\,080x^9/2y^3 + \dots$$

$$23. b^{40} + 10b^{37} + \frac{95}{2}b^{34} + \frac{285}{2}b^{31} + \dots$$

$$25. 1.338 \quad 27. 1.015 \quad 29. 1 + 8x + 28x^2 + 56x^3 + \dots$$

$$31. 1 + 6x + 27x^2 + 108x^3 + \dots$$

$$33. 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \dots$$

$$35. \frac{1}{3} \left[ 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \dots \right]$$

$$37. (a) 3.557 \times 10^{14} \quad (b) 5.109 \times 10^{19} \quad (c) 8.536 \times 10^{15} \\ (d) 2.480 \times 10^{96}$$

$$39. n! = n(n-1)(n-2)(\dots)(2)(1) = n \times (n-1)!; \text{ for } n=1, 1! = 1 \times 0! \text{ Since } 1! = 1, 0! \text{ must} = 1.$$

$$41. 56a^3b^5 \quad 43. 10\,264\,320x^8b^4$$

$$45. n! \text{ contains factors 2 and 5.}$$

$$47. n=1, 1-1=0; n=2, 1-2+1=0; n=3, 1-3+3-1=0, \text{ etc.}$$

$$49. 2.45 \quad 51. V = A(1 - 5r + 10r^2 - 10r^3 + 5r^4 - r^5)$$

$$53. 1 - \frac{x}{a} + \frac{x^3}{2a^3} - \dots \quad 55. -\frac{2hk}{r^3} + \frac{3kh^2}{r^4}$$

### Review Exercises for Chapter 19, page 528

$$1. 81 \quad 3. 1.28 \times 10^{-3} \quad 5. 66.5 \quad 7. \frac{16}{243} \quad 9. \frac{195}{2}$$

$$11. \frac{5461}{512} \quad 13. 81 \quad 15. 32 \quad 17. -15 \quad 19. -0.25$$

$$21. 186 \quad 23. \frac{455}{2} \text{ (as), } 127 \text{ (gs), or } 43 \text{ (gs)} \quad 25. 2.7$$

$$27. 51 \quad 29. \frac{1}{33} \quad 31. \frac{4}{55}$$

$$33. x^4 - 8x^3 + 24x^2 - 32x + 16$$

$$35. x^{10} + 20x^8 + 160x^6 + 640x^4 + 1280x^2 + 1024$$

$$37. a^{10} + 20a^9e + 180a^8e^2 + 960a^7e^3 + \dots$$

$$39. p^{18} - \frac{3}{2}p^{16}q + p^{14}q^2 - \frac{7}{18}p^{12}q^3 + \dots$$

$$41. 1 + 12x + 66x^2 + 220x^3 + \dots$$

$$43. 1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \frac{1}{16}x^6 - \dots$$

$$45. 1 - \frac{1}{2}a^2 - \frac{1}{8}a^4 - \frac{1}{16}a^6 - \dots$$

$$47. \frac{1}{8} + \frac{3}{4}x + 3x^2 + 10x^3 + \dots \quad 49. 1\,001\,000$$

$$51. 4b - 3a \quad 53. \text{ No} \quad 55. a_n = -3n - 2 \quad 57. 0.716$$

$$59. 5.475 \quad 61. 11\text{th} \quad 63. 12.6 \text{ mm} \quad 65. 7690 \text{ mm}$$

$$67. \$4700 \quad 69. 1.65 \times 10^{10} \text{ cm} = 165\,000 \text{ km} \quad 71. 191 \text{ m}$$

$$73. 100 \text{ cm} \quad 75. \$47\,340.80 \quad 77. \$6.40 \quad 79. 4.0 \text{ }^\circ\text{C}$$

$$81. 1 + \frac{1}{2}am^2 + \frac{1}{8}am^4 \quad 83. 21 \text{ years} \quad 85. \text{ Five applications}$$

$$87. 22 \text{ years}$$

$$89. a_{\text{mid}} = a_{(n+1)/2} = a + \frac{n-1}{2}d \\ = \frac{1}{n} \left[ \frac{n}{2}(a + (a + (n-1)d)) \right] = \frac{S_n}{n}$$

$$91. \text{ Yes; term to term ratios are equal.}$$

### Exercises 20.1, page 531

(Note: "Answers" to trigonometric identities are intermediate steps of suggested reductions of the left member.)

$$1. \sin x = \frac{\tan x}{\sec x} = \frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x}} = \frac{\sin x}{\cos x} \cdot \frac{\cos x}{1} = \sin x$$

$$3. \frac{0.829}{0.559} = 1.48 \quad 5. \left(-\frac{1}{2}\sqrt{3}\right)^2 + \left(-\frac{1}{2}\right)^2 = \frac{3}{4} + \frac{1}{4} = 1$$

$$7. \sin x - 1 \quad 9. \cot \theta - 2 \cos \theta \quad 11. \sec^2 u$$

$$13. \tan x \sec x \quad 15. \sin t \cos t \quad 17. \cot^2 y (\csc^2 y + 1)$$

$$19. \frac{\sin x}{\sin x} = \frac{\sin x \left(\frac{\cos x}{\sin x}\right)}{1 \left(\frac{\cos x}{\sin x}\right)} \quad 21. \sin x \left(\frac{1}{\cos x}\right)$$

$$23. \csc^2 x (\sin^2 x)$$

$$25. \sin x (\csc^2 x) = \sin x \left(\frac{1}{\sin^2 x}\right)$$

$$27. \cos \theta \left(\frac{\cos \theta}{\sin \theta}\right) + \sin \theta = \frac{\cos^2 \theta + \sin^2 \theta}{\sin \theta} = \frac{1}{\sin \theta}$$

$$29. \cot \theta (\sec^2 \theta - 1) = \cot \theta \tan^2 \theta = (\cot \theta \tan \theta) \tan \theta$$

$$31. \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} = \frac{\sin^2 x + \cos^2 x}{\cos x \sin x} = \frac{1}{\cos x \sin x}$$

$$33. (1 - \sin^2 x) - \sin^2 x$$

$$35. \frac{\sin \theta}{1} + \frac{\cos \theta}{1} = \sin^2 \theta + \cos^2 \theta$$

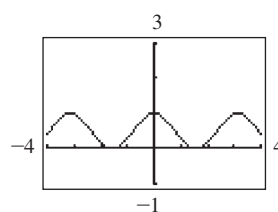
$$37. (2 \sin^2 x - 1)(\sin^2 x - 1)$$

$$39. \frac{\sin \pi t \left(\frac{\sin^2 \pi t + (1 - \cos \pi t)^2}{(1 - \cos \pi t) \sin \pi t}\right)}{2} \\ = \frac{\sin^2 \pi t + 1 - 2 \cos \pi t + \cos^2 \pi t}{2(1 - \cos \pi t)} = \frac{2(1 - \cos \pi t)}{2(1 - \cos \pi t)}$$

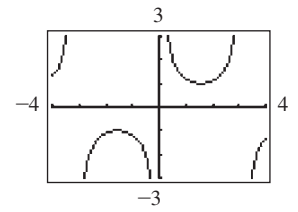
$$41. \text{ Geometric series with } a_1 = 1, r = \sin^2 x \quad 43. \cot x$$

$$45. \sin x \quad 47. \sec x \quad 49. \cos x$$

$$51.$$

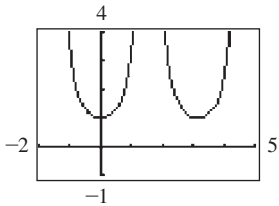


$$53.$$

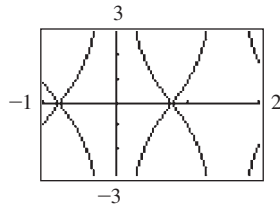


**B.42 ANSWERS TO ODD-NUMBERED EXERCISES**

55. Yes



57. No



59.  $0 = \cos A \cos B \cos C + \sin A \sin B$ ,

$$\cos C = -\frac{\sin A \sin B}{\cos A \cos B}$$

61.  $l = a \csc \theta + a \sec \theta = a\left(\frac{1}{\sin \theta} + \frac{\tan \theta}{\sin \theta}\right)$

63. Write  $\tan \theta$  in terms of  $\sin \theta$  and  $\cos \theta$ . Use  $\cos \theta$  as LCD and then Eq. (20.6) to simplify.

65.  $\sin^2 x - \sin^2 x \sec^2 x + \cos^2 x + \cos^2 x \sec^4 x$   
 $= \sin^2 x - \tan^2 x + \cos^2 x + \sec^2 x = 2$

67.  $\left(\frac{r}{x}\right)^2 + \left(\frac{r}{y}\right)^2 = \frac{r^2(x^2 + y^2)}{x^2 y^2}$

69.  $\sqrt{1 - \cos^2 \theta} = \sqrt{\sin^2 \theta}$

71.  $\sqrt{4 + 4 \tan^2 \theta} = 2\sqrt{1 + \tan^2 \theta}$

**Exercises 20.2, page 541**

1.  $\frac{16}{65}$

3.  $\sin 105^\circ = \sin 60^\circ \cos 45^\circ + \cos 60^\circ \sin 45^\circ$   
 $= \frac{\sqrt{3}}{2} \frac{\sqrt{2}}{2} + \frac{1}{2} \frac{\sqrt{2}}{2} = 0.966$

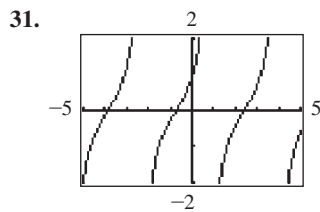
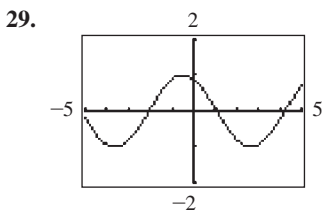
5.  $\cos 15^\circ = \cos(60^\circ - 45^\circ)$   
 $= \cos 60^\circ \cos 45^\circ + \sin 60^\circ \sin 45^\circ$   
 $= \left(\frac{1}{2}\right)\left(\frac{1}{2}\sqrt{2}\right) + \left(\frac{1}{2}\sqrt{3}\right)\left(\frac{1}{2}\sqrt{2}\right)$   
 $= \frac{1}{4}\sqrt{2} + \frac{1}{4}\sqrt{6} = \frac{1}{4}(\sqrt{2} + \sqrt{6}) = 0.966$

7.  $-\frac{33}{65}$     9.  $-\frac{56}{65}$     11.  $\sin 3x$     13.  $-\cos x$     15.  $-\sin x$

17.  $\tan x$     19. 0    21. 1    23. 0

25.  $(\sin x \cos y + \cos x \sin y)(\sin x \cos y - \cos x \sin y)$   
 $= \sin^2 x \cos^2 y - \cos^2 x \sin^2 y$   
 $= \sin^2 x(1 - \sin^2 y) - (1 - \sin^2 x)\sin^2 y$

27.  $(\cos \alpha \cos \beta - \sin \alpha \sin \beta)$   
 $+ (\cos \alpha \cos \beta + \sin \alpha \sin \beta)$



33, 35, 37, 39. Use the indicated method.

41.  $\frac{\sin(x+x)}{\sin x} = \frac{\sin x \cos x + \cos x \sin x}{\sin x}$   
 $= \frac{2 \sin x \cos x}{\sin x}$

43.  $\cos A \cos B \cos C - \sin A \sin B \cos C - \sin A \cos B \sin C$   
 $- \cos A \sin B \sin C$

45.  $20\sqrt{2}(\sin 120\pi t \cos \pi/4 + \sin \pi/4 \cos 120\pi t)$   
 $= 20\sqrt{2}\left(\frac{1}{\sqrt{2}}\right)(\sin 120\pi t + \cos 120\pi t)$

47.  $i_0 \sin(\omega t + \alpha) = i_0(\sin \omega t \cos \alpha + \cos \omega t \sin \alpha)$

49.  $\tan \alpha(R + \cos \beta) = \sin \beta, R = \frac{\sin \beta - \tan \alpha \cos \beta}{\tan \alpha}$   
 $= \frac{\sin \beta \cos \alpha - \cos \beta \sin \alpha}{\cos \alpha \tan \alpha}$

**Exercises 20.3, page 544**

1.  $-\sqrt{3}$     3.  $-\frac{24}{25}$

5.  $\sin 60^\circ = \sin 2(30^\circ) = 2 \sin 30^\circ \cos 30^\circ$   
 $= 2\left(\frac{1}{2}\right)\left(\frac{1}{2}\sqrt{3}\right) = \frac{1}{2}\sqrt{3}$

7.  $\tan 120^\circ = \frac{2 \tan 60^\circ}{1 - \tan^2 60^\circ} = \frac{2\sqrt{3}}{1 - (\sqrt{3})^2} = -\sqrt{3}$

9.  $\sin 258^\circ = 2 \sin 129^\circ \cos 129^\circ = -0.978$

11.  $\cos 96^\circ = \cos^2 48^\circ - \sin^2 48^\circ = -0.105$     13. 3.08

15.  $\frac{24}{25}$     17.  $-\sqrt{3}$     19.  $2 \sin 8x$     21.  $\cos 8x$     23.  $\cos x$

25.  $-2 \cos 4x$     27.  $2 \cos 2\theta$     29. 2

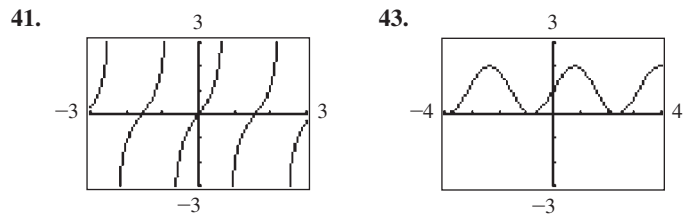
31.  $\cos^2 \alpha - (1 - \cos^2 \alpha)$

33.  $\frac{\cos x - (\sin x / \cos x) \sin x}{1 / \cos x} = \cos^2 x - \sin^2 x$

35.  $\frac{2 \sin \theta \cos \theta}{1 + 2 \cos^2 \theta - 1} = \frac{\sin \theta}{\cos \theta}$

37.  $1 - (1 - 2 \sin^2 2\theta) = \frac{2}{\csc^2 \theta}$

39.  $\ln \frac{1 - \cos 2x}{1 + \cos 2x} = \ln \frac{2 \sin^2 x}{2 \cos^2 x} = \ln \tan^2 x$

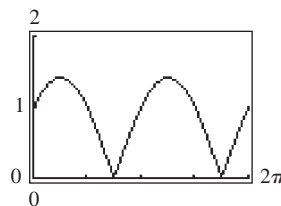


45.  $3 \sin x - 4 \sin^3 x$     47.  $8 \cos^4 x - 8 \cos^2 x + 1$

49.  $1 - 2 \sin^2 x + \frac{2 \sin x \cos x \sin x}{\cos x} = 1$

51. amp. = 2, per. =  $\pi$ ; write equation as  
 $y = 2(2 \sin x \cos x) = 2 \sin 2x$ .

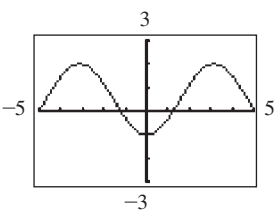
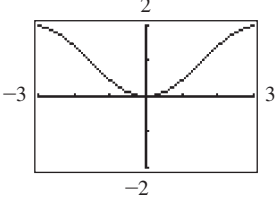
53.  $\sqrt{\sin^2 x + 2 \sin x \cos x + \cos^2 x} = \sqrt{1 + \sin 2x}$



55. 474 m    57.  $R = v \left( \frac{2v \sin \alpha}{g} \right) \cos \alpha = \frac{v^2 (2 \sin \alpha \cos \alpha)}{g}$

59.  $vi \sin \omega t \sin \left( \omega t - \frac{\pi}{2} \right)$   
 $= vi \sin \omega t \left( \sin \omega t \cos \frac{\pi}{2} - \cos \omega t \sin \frac{\pi}{2} \right)$   
 $= vi \sin \omega t [ -(\cos \omega t)(1) ] = -\frac{1}{2} vi (2 \sin \omega t \cos \omega t)$

**Exercises 20.4, page 547**

1.  $\cos 57^\circ$   
 3.  $\cos 15^\circ = \cos \frac{1}{2}(30^\circ) = \sqrt{\frac{1 + \cos 30^\circ}{2}} = \sqrt{\frac{1.8660}{2}} = 0.966$   
 5.  $\sin 105^\circ = \sin \frac{1}{2}(210^\circ) = \sqrt{\frac{1 - \cos 210^\circ}{2}} = \sqrt{\frac{1.8660}{2}} = 0.966$   
 7. -0.924    9.  $\sin 118^\circ = 0.882\,947\,6$   
 11.  $\sqrt{2 \left( \frac{1 + \cos 164^\circ}{2} \right)} = \sqrt{2} \cos 82^\circ = 0.196\,820\,5$   
 13.  $\sin 3x$     15.  $4 \cos 2x$     17.  $2\sqrt{2} \sin 5\theta$     19. 1  
 21.  $\frac{1}{26}\sqrt{26}$     23. 0.142    25.  $\pm \sqrt{\frac{2 \sec \alpha}{\sec \alpha - 1}}$   
 27.  $\tan \frac{1}{2} \alpha = \frac{1 - \cos \alpha}{\sin \alpha} = \frac{\sin \alpha}{1 + \cos \alpha}$   
 29.  $\frac{1 - \cos \alpha}{2 \sin \frac{1}{2} \alpha} = \frac{1 - \cos \alpha}{2\sqrt{\frac{1}{2}(1 - \cos \alpha)}} = \sqrt{\frac{1 - \cos \alpha}{2}}$   
 31.  $2\sqrt{\frac{(1 + \cos x)(1 + \cos x)}{2(2)(1 + \cos x)}} = \frac{2(1 + \cos x)}{2} \cdot \sqrt{\frac{2}{1 + \cos x}}$   
 33.     35. 

37.  $\pm \frac{24}{7}$     39.  $4 \sin \frac{\theta}{2}$   
 41.  $\sin^2 \omega t = \left( \sqrt{\frac{1 - \cos 2\omega t}{2}} \right)^2 = \frac{1 - \cos 2\omega t}{2}$   
 43.  $\frac{\sqrt{\frac{1 - \cos(A + \phi)}{2}}}{\sqrt{\frac{1 - \cos A}{2}}} = \sqrt{\frac{1 - \cos(A + \phi)}{1 - \cos A}}$

**Exercises 20.5, page 552**

1.  $\frac{\pi}{4}, \frac{5\pi}{4}$     3.  $0, \frac{\pi}{3}, \frac{5\pi}{3}$     5.  $\frac{\pi}{2}$     7.  $\frac{\pi}{2}, \frac{3\pi}{2}$   
 9.  $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$     11.  $0, \frac{\pi}{6}, \frac{5\pi}{6}, \pi$     13.  $\frac{\pi}{2}, \frac{3\pi}{2}$

15.  $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$     17. 0.262, 1.31, 3.40, 4.45

19.  $0, \pi$     21.  $\frac{3\pi}{4}, \frac{7\pi}{4}$   
 23. 1.98, 4.30    25.  $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$   
 27.  $\frac{\pi}{12}, \frac{\pi}{4}, \frac{5\pi}{12}, \frac{3\pi}{4}, \frac{13\pi}{12}, \frac{5\pi}{4}, \frac{17\pi}{12}, \frac{7\pi}{4}$     29.  $0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}$   
 31. 3.569, 5.856

33.  $\frac{\pi}{4}, \frac{5\pi}{4}, 1.25, \pi + 1.25$

35.  $\frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}, \frac{15\pi}{8}$

37.  $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$     39.  $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$

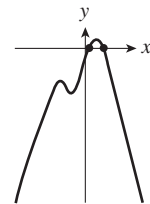
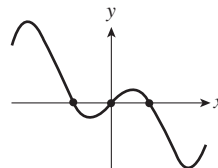
41. No;  $\cot \theta > 1, \sec \theta > 1, \csc \theta > 1$  for  $0 < \theta < \frac{\pi}{2}$

43.  $\theta = 0, r = 0; \theta = \frac{\pi}{3}, r = \frac{\sqrt{3}}{2}; \theta = \pi, r = 0; \theta = \frac{5\pi}{3}, r = -\frac{\sqrt{3}}{2}$

45.  $37.8^\circ$     47. 10.2 s, 15.7 s, 21.2 s, 47.1 s

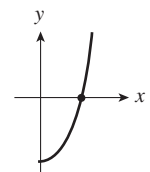
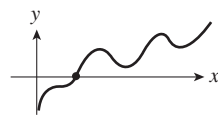
49.  $6.6 \times 10^{-4}$  rad    51. 300.0 N, 400.0 N

53. -2.28, 0.00, 2.28    55. 0.29, 0.95



57. 2.10

59. 1.08



**Exercises 20.6, page 557**

1.  $y$  is the angle whose tangent is  $3A$ .    3. 0  
 5.  $y$  is the angle whose cotangent is  $3x$ .  
 7.  $y$  is twice the angle whose sine is  $x$ .  
 9.  $y$  is five times the angle whose cosine is  $2x - 1$ .  
 11.  $\frac{\pi}{3}$     13.  $\frac{\pi}{4}$     15.  $-\frac{\pi}{3}$     17. No value    19.  $-\frac{\pi}{4}$   
 21.  $\frac{1}{2}\sqrt{3}$     23.  $\frac{\pi}{4}$     25.  $\frac{1}{\sqrt{26}}$     27. -1    29.  $2/\sqrt{21}$   
 31.  $2/\sqrt{3}$     33. -1.3090    35. 0.0219    37. -1.239  
 39. -0.2239    41.  $x = \frac{1}{3} \sin^{-1} y$     43.  $x = 4 \tan y$   
 45.  $x = 1 - \cos(1 - y)$     47.  $\frac{x}{\sqrt{1 - x^2}}$   
 49.  $xy + \sqrt{1 - x^2 - y^2 + x^2 y^2}$     51.  $\frac{3x}{\sqrt{9x^2 - 1}}$   
 53.  $2x\sqrt{1 - x^2}$

**B.44 ANSWERS TO ODD-NUMBERED EXERCISES**

55. No;  $\sin^{-1}(\sin x) = x$  for  $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ .

57.  $t = \frac{1}{2\omega} \cos^{-1} \frac{y}{A} - \frac{\phi}{\omega}$

59.  $t = \frac{1}{\omega} \left( \sin^{-1} \frac{i}{I_m} - \alpha - \phi \right)$

61.  $\sin \left( \sin^{-1} \frac{3}{5} + \sin^{-1} \frac{5}{13} \right) = \frac{3}{5} \cdot \frac{12}{13} + \frac{4}{5} \cdot \frac{5}{13} = \frac{56}{65}$

63.  $\frac{\pi}{2}$     65. 0    67.  $\sin^{-1} \left( \frac{a}{c} \right)$     69.  $\tan^{-1} \left( \frac{b \tan B}{a} \right)$

71. Let  $y =$  height to top of base;  $\tan \alpha = \frac{2.7 + y}{d}$ ,

$\tan \beta = \frac{y}{d}$ ;  $\tan \alpha = \frac{2.7 + d \tan \beta}{d}$

73.  $\theta = \tan^{-1} \left( \frac{y + 50}{x} \right) - \tan^{-1} \left( \frac{y}{x} \right)$

**Review Exercises for Chapter 20, page 559**

1.  $\sin(90^\circ + 30^\circ) = \sin 90^\circ \cos 30^\circ + \cos 90^\circ \sin 30^\circ$   
 $= (1) \left( \frac{1}{2} \sqrt{3} \right) + (0) \left( \frac{1}{2} \right) = \frac{1}{2} \sqrt{3}$

3.  $\sin(360^\circ - 45^\circ) = \sin 360^\circ \cos 45^\circ - \cos 360^\circ \sin 45^\circ$   
 $= 0 \left( \frac{1}{2} \sqrt{2} \right) - 1 \left( \frac{1}{2} \sqrt{2} \right) = -\frac{1}{2} \sqrt{2}$

5.  $\cos \left( 2 \frac{\pi}{2} \right) = \cos^2 \frac{\pi}{2} - \sin^2 \frac{\pi}{2} = 0 - 1^2 = -1$

7.  $\tan 2(30^\circ) = \frac{2 \tan 30^\circ}{1 - \tan^2 30^\circ} = \frac{2(\sqrt{3}/3)}{1 - (\sqrt{3}/3)^2} = \sqrt{3}$

9.  $\sin 52^\circ = 0.788$     11.  $\frac{1}{2}$     13.  $\cos 215^\circ = -0.819$

15.  $2 \tan 24^\circ = 0.890$     17.  $\sin 5x$     19.  $4 \sin 12x$

21.  $2 \cos 12x$     23.  $2 \cos x$     25.  $-\frac{\pi}{2}$     27. 0.2619

29.  $-\frac{1}{3} \sqrt{3}$     31.  $-\frac{\pi}{6}$     33.  $\sec^2 y - \tan^2 y = 1$

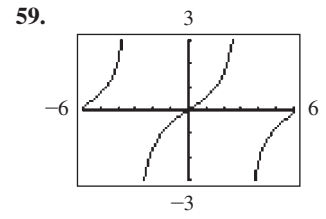
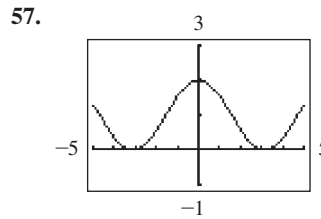
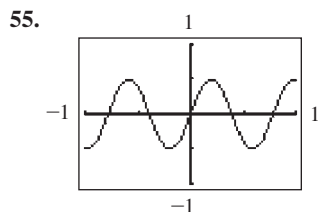
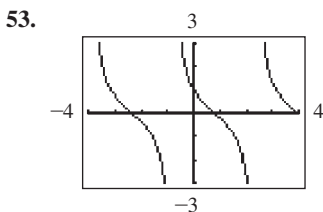
35.  $\sin x \csc x - \sin^2 x = 1 - \sin^2 x = \cos^2 x$

37.  $\frac{(\sec^2 x - 1)(\sec^2 x + 1)}{\tan^2 x} = \sec^2 x + 1$

39.  $2 \left( \frac{1}{\sin 2x} \right) \left( \frac{\cos x}{\sin x} \right) = 2 \left( \frac{1}{2 \sin x \cos x} \right) \left( \frac{\cos x}{\sin x} \right)$   
 $= \frac{1}{\sin^2 x}$

41.  $\frac{\cos^2 \theta}{\sin^2 \theta}$     43.  $\frac{1}{2} \left( 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} \right)$     45.  $\cot x$     47.  $\sec x$

49.  $\sin x$     51.  $\sin x$



61.  $x = \frac{1}{2} \cos^{-1} \frac{1}{2} y$     63.  $x = \frac{1}{5} \sin^{-1} \left( \frac{1}{4} \pi - y \right)$

65. 1.29, 4.43    67.  $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$     69.  $0, \frac{\pi}{3}, \frac{5\pi}{3}$

71.  $0, \frac{2\pi}{3}$     73.  $\frac{\pi}{10}, \frac{\pi}{2}, \frac{9\pi}{10}, \frac{13\pi}{10}, \frac{3\pi}{2}, \frac{17\pi}{10}$

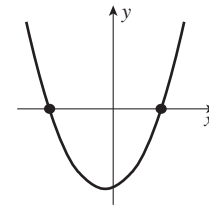
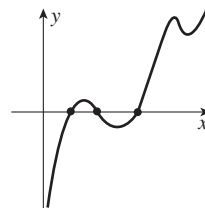
75. 0

77. Identity:

$$\tan x + \frac{1}{\tan x} = \frac{\tan^2 x + 1}{\tan x} = \sec^2 x (\cot x) = \frac{\sec^2 x \cos x}{\sin x}$$

79.  $\frac{\pi}{2}, \pi$

81. 1.56, 2.16, 3.46    83. -2.31, 1.14



85.  $\frac{1}{x}$     87.  $2x\sqrt{1-x^2}$     89.  $\frac{\sqrt{1-x^2}-xy}{\sqrt{1+y^2}}$

91.  $2\sqrt{1-\cos^2 \theta}$     93.  $\frac{\tan \theta}{\sqrt{1+\tan^2 \theta}} = \frac{\tan \theta}{\sec \theta}$

95.  $(\cos \theta + j \sin \theta)^2 = (\cos^2 \theta - \sin^2 \theta) + j(2 \sin \theta \cos \theta)$

97.  $\frac{\pi}{2} < x < \frac{3\pi}{2}$

99.  $C \left( \frac{A}{C} \sin 2t + \frac{B}{C} \cos 2t \right)$   
 $= C(\cos \alpha \sin 2t + \sin \alpha \cos 2t)$

101.  $R = \sqrt{(A \cos \theta - B \sin \theta)^2 + (A \sin \theta + B \cos \theta)^2}$   
 $= \sqrt{A^2(\cos^2 \theta + \sin^2 \theta) + B^2(\sin^2 \theta + \cos^2 \theta)}$

103.  $\frac{k}{2} \cdot \frac{1}{\sin^2 \frac{\theta}{2}} = \frac{k}{2} \cdot \frac{1}{\frac{1-\cos \theta}{2}}$     105.  $\theta = a + R \sin \omega t$

107.  $\frac{\cos 2\alpha}{2 \cos^2 \alpha}$     109.  $P = VI \cos \omega t [\cos(\phi + \omega t)]$

111. 6.8 m    113. 54.7°

**Exercises 21.1, page 566**

1.  $\sqrt{61}$     3.  $150^\circ$     5.  $2\sqrt{29}$     7. 3    9. 55

11.  $2\sqrt{53}$     13. 2.86    15.  $\frac{5}{2}$     17. Undefined    19.  $-\frac{3}{4}$

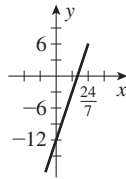
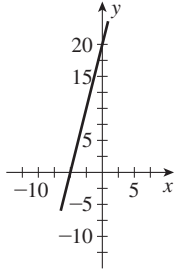
21.  $-\frac{5}{9}$     23. 0.747    25.  $\frac{1}{3} \sqrt{3}$     27. -0.311    29.  $20.0^\circ$

31.  $98.50^\circ$     33. Parallel    35. Perpendicular    37. 8, -2

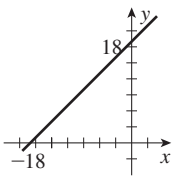
39. -3    41. Two sides equal  $2\sqrt{10}$ .    43.  $m_1 = \frac{5}{12}, m_2 = \frac{4}{3}$   
 45. 10    47.  $4\sqrt{10} + 4\sqrt{2} = 18.3$     49. (1, 5)  
 51. (-2.8, 4.2)    53.  $x^2 + y^2 = 9$     55.  $m_1 = 1, m_2 = -1$   
 57.  $(-\frac{11}{3}, 0)$     59.  $(0, -\frac{7}{9})$     61.  $0, \sqrt{3}, -\sqrt{3}$

**Exercises 21.2, page 571**

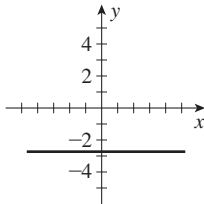
1.  $2y - x - 2 = 0$     3.  $2, (0, \frac{5}{2})$   
 5.  $4x - y + 20 = 0$     7.  $7x - 2y - 24 = 0$



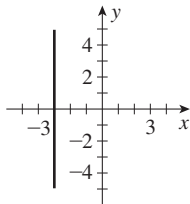
9.  $x - y + 19 = 0$



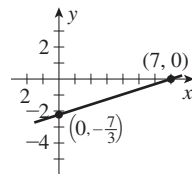
11.  $y = -2.7$



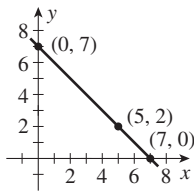
13.  $x = -3$



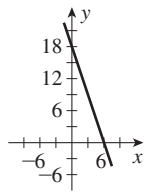
15.  $x - 3y - 7 = 0$



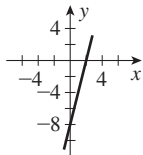
17.  $x + y - 7 = 0$



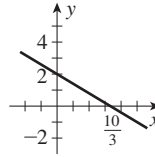
19.  $3x + y - 18 = 0$



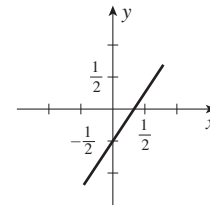
21.  $y = 4x - 8; m = 4, (0, -8)$



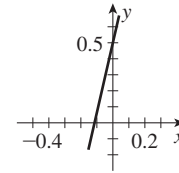
23.  $y = -\frac{3}{5}x + 2; m = -\frac{3}{5}, (0, 2)$



25.  $y = \frac{3}{2}x - \frac{1}{2}; m = \frac{3}{2}, (0, -\frac{1}{2})$



27.  $y = 3.5x + 0.5; m = 3.5, (0, 0.5)$     29. Parallel



31. Perpendicular    33. Neither    35. Perpendicular

37. -2

39. The slope of the first line is 3. A line perpendicular to it has a slope of  $-\frac{1}{3}$ . The slope of the second line is  $-\frac{k}{3}$ , so  $k = 1$ .

41.  $b - a$     43. 5

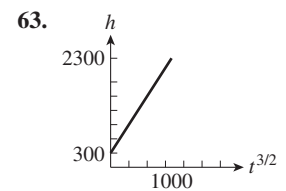
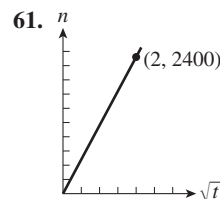
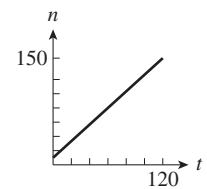
45.  $m_1 = -\frac{4}{5}, m_2 = \frac{2}{3}, m_3 = \frac{2}{3}, m_4 = -\frac{4}{5}, m_1 = m_4, m_2 = m_3$

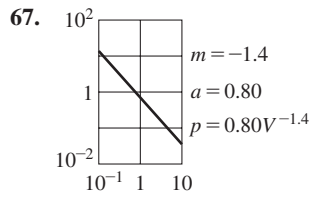
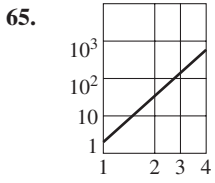
47.  $m_1 = -\frac{a}{b}, m_2 = -\frac{a}{b}, m_1 = m_2; m_3 = -\frac{a}{b}, m_4 = \frac{b}{a}, m_3 = -1/m_4$

49.  $C = \frac{5}{4}R$     51.  $v = 0.607T + 331$

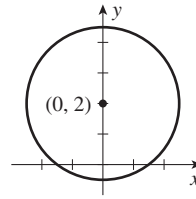
53.  $T = \frac{4}{3}x + 3$     55.  $P = 3.0 - 0.025T_5$

57.  $y = 10^{-5}(2.4 - 5.6x)$     59.  $n = \frac{7}{6}t + 10$ ; at 6:30,  $n = 10$ ; at 8:30,  $n = 150$

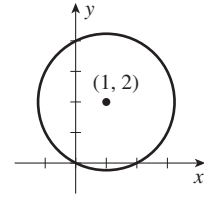




33.  $(0, 2), r = \frac{5}{2}$

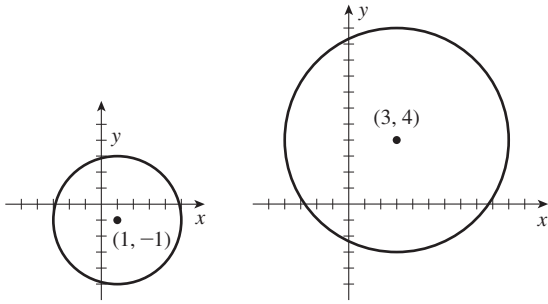


35.  $(1, 2), r = \frac{1}{2}\sqrt{22}$



Exercises 21.3, page 576

1.  $C(1, -1), r = 4$       3.  $C(3, 4), r = 7$



5.  $(2, 1), r = 5$       7.  $C(-1, 0), r = \frac{11}{2}$       9.  $x^2 + y^2 = 9$

11.  $(x - 2)^2 + (y - 2)^2 = 16$ , or  
 $x^2 + y^2 - 4x - 4y - 8 = 0$

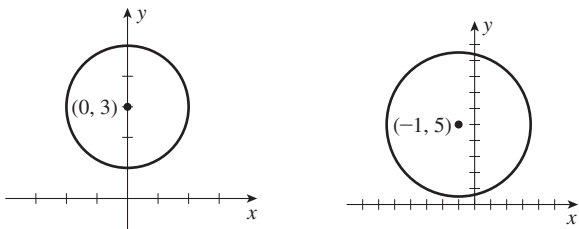
13.  $(x - 12)^2 + (y + 15)^2 = 324$ , or  
 $x^2 + y^2 - 24x + 30y + 45 = 0$

15.  $(x + 3)^2 + (y - 4)^2 = 25$       17.  $(x - 2)^2 + (y - 1)^2 = 8$

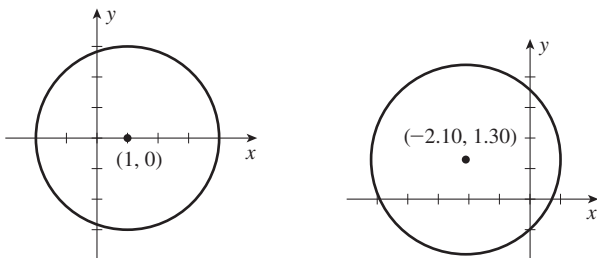
19.  $(x + 3)^2 + (y - 5)^2 = 25$ , or  
 $x^2 + y^2 + 6x - 10y + 9 = 0$

21.  $(x + 2)^2 + (y - 2)^2 = 4$  or      23.  $x^2 + y^2 = 2$   
 $x^2 + y^2 + 4x - 4y + 4 = 0$

25.  $(0, 3), r = 2$       27.  $(-1, 5), r = \frac{9}{2}$



29.  $(1, 0), r = 3$       31.  $(-2.1, 1.3), r = 3.1$

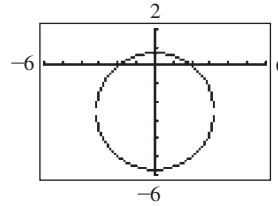


37. Symmetric to both axes and origin

39. Symmetric to y-axis      41.  $(7, 0), (-1, 0)$

43.  $3x^2 + 3y^2 + 4x + 8y - 20 = 0$ , circle

45. Graph  $y = -2.5 \pm \sqrt{10.25 - x^2}$



47. (a) Semicircle  
(b) Semicircle  
(c) Yes, there is only one value of  $y$  for each  $x$  in the domain.

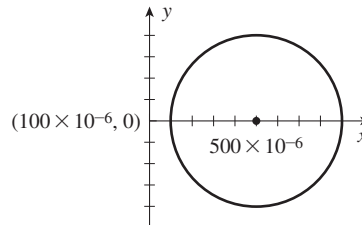
49. Outside

51.  $p > 0$ , circle;  $p = 0$ , point;  $p < 0$ , does not exist

53. 0.0912 cm      55. 2.82 cm      57.  $x^2 + y^2 = 0.0100$

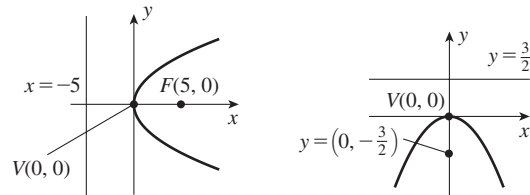
59.  $x^2 + (y + 2.4)^2 = 0.16$

61.  $(x - 500 \times 10^{-6})^2 + y^2 = 0.16 \times 10^{-6}$

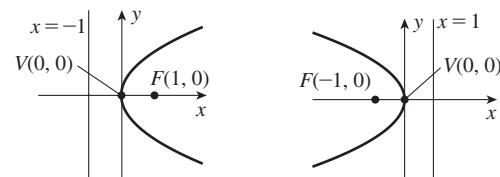


Exercises 21.4, page 581

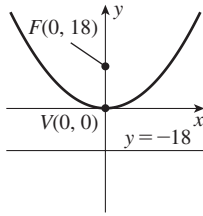
1.  $F(5, 0), x = -5$       3.  $F(0, -\frac{3}{2}), y = \frac{3}{2}$



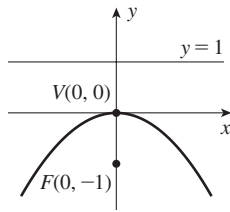
5.  $F(1, 0), x = -1$       7.  $F(-1, 0), x = 1$



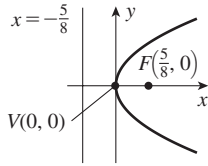
9.  $F(0, 18), y = -18$



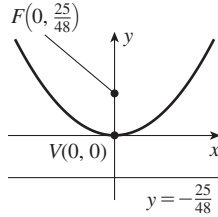
11.  $F(0, -1), y = 1$



13.  $F(\frac{5}{8}, 0), x = -\frac{5}{8}$



15.  $F(0, \frac{25}{48}), y = -\frac{25}{48}$



17.  $y^2 = 12x$

19.  $x^2 = -2y$

21.  $x^2 = 0.64y$

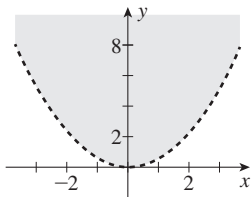
23.  $x^2 = 336y$

25.  $x^2 = \frac{1}{8}y$

27.  $y^2 = \frac{25}{3}x$

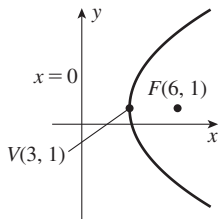
29.  $y^2 = 3x$

31.

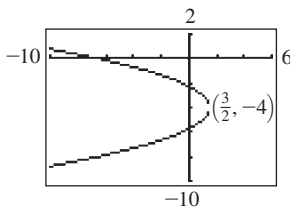


33.  $(0, 0), (8, -4)$

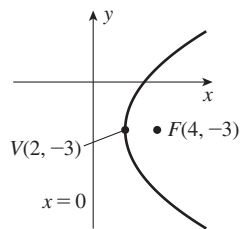
35.  $y^2 - 2y - 12x + 37 = 0$



37. Graph  $y = -4 \pm \sqrt{3 - 2x}$



39.



41.  $4|p|$

43. The parabola becomes broader as  $|p|$  increases.

45.  $x^2 = 1020y$

47.  $H$

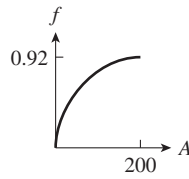


49. 57.6 m

51. No (a course correction is necessary)

53. 2.16 cm from vertex

55.

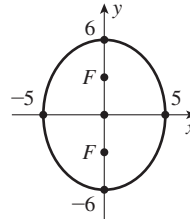


57.  $x^2 = 8y$  with vertex midway between island and shore

Exercises 21.5, page 586

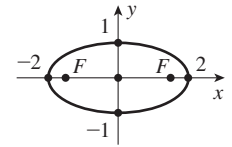
1.  $V(0, 6), V(0, -6),$

ends minor axis  $(5, 0), (-5, 0)$   
foci  $(0, \sqrt{11}), (0, -\sqrt{11})$



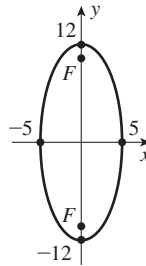
3.  $V(2, 0), V(-2, 0),$

$F(\sqrt{3}, 0), F(-\sqrt{3}, 0)$



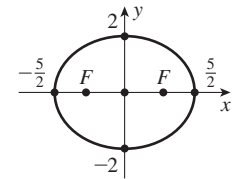
5.  $V(0, 12), V(0, -12),$

$F(0, \sqrt{119}), F(0, -\sqrt{119})$



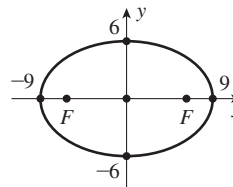
7.  $V(\frac{5}{2}, 0), V(-\frac{5}{2}, 0),$

$F(\frac{3}{2}, 0), F(-\frac{3}{2}, 0)$



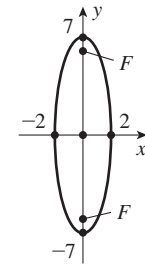
9.  $V(9, 0), V(-9, 0),$

$F(3\sqrt{5}, 0), F(-3\sqrt{5}, 0)$



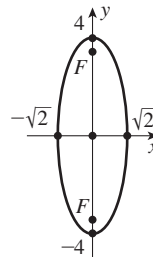
11.  $V(0, 7), V(0, -7),$

$F(0, \sqrt{45}), F(0, -\sqrt{45})$



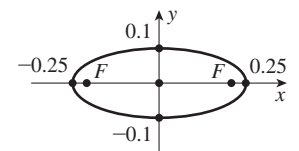
13.  $V(0, 4), V(0, -4),$

$F(0, \sqrt{14}), F(0, -\sqrt{14})$



15.  $V(0.25, 0), V(-0.25, 0),$

$F(0.23, 0), F(-0.23, 0)$





**B.48** ANSWERS TO ODD-NUMBERED EXERCISES

17.  $\frac{x^2}{225} + \frac{y^2}{144} = 1$ , or  $144x^2 + 225y^2 = 32\,400$

19.  $\frac{x^2}{225} + \frac{y^2}{289} = 1$ , or  $289x^2 + 225y^2 = 65\,025$

21.  $\frac{x^2}{208} + \frac{y^2}{144} = 1$ , or  $144x^2 + 208y^2 = 29\,952$

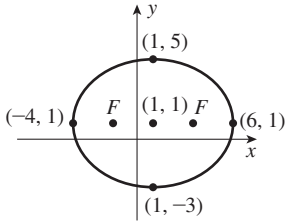
23.  $\frac{x^2}{64} + \frac{15y^2}{144} = 1$ , or  $3x^2 + 20y^2 = 192$

25.  $\frac{x^2}{5} + \frac{y^2}{20} = 1$ , or  $4x^2 + y^2 = 20$

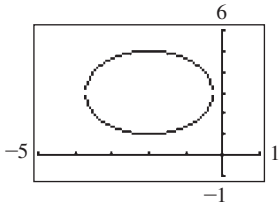
27.  $\frac{x^2}{100} + \frac{y^2}{64} = 1$ , or  $16x^2 + 25y^2 = 1600$

29.  $(1, -2), (1, 2)$

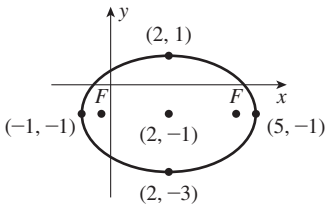
31.  $16x^2 + 25y^2 - 32x - 50y - 359 = 0$



33. Graph  $y = 3 \pm \frac{\sqrt{-12x^2 - 48x - 12}}{3}$



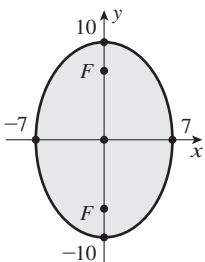
35.



37. Write equation as  $\frac{x^2}{1} + \frac{y^2}{1/k^2} = 1$ . Thus,  $\frac{1}{k^2} > 1$ , or  $|k| < 1$ .

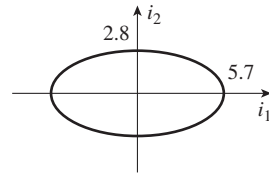
39.  $2x^2 + 3y^2 - 8x - 4 = 2x^2 + 3(-y)^2 - 8x - 4$

41.



43.  $\sin^2 t + \cos^2 t = \frac{x^2}{4} + \frac{y^2}{9} = 1$

45.  $i_1^2 + 4i_2^2 = 32$



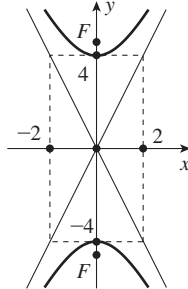
47. 0.44      49.  $7x^2 + 16y^2 = 112$

51. 27.5 m      53. 4.0 m

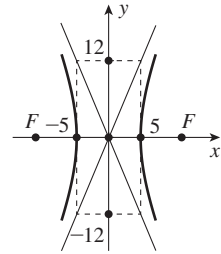
55. 843 m<sup>3</sup>

**Exercises 21.6, page 592**

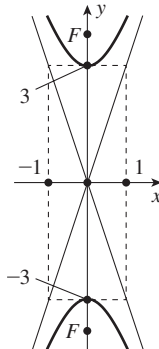
1.  $V(0, -4), V(0, 4)$ ,  
conj. axis  $(-2, 0), (2, 0)$   
 $F(0, -2\sqrt{5}), F(0, 2\sqrt{5})$



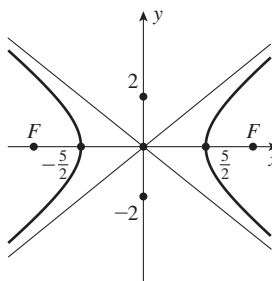
3.  $V(5, 0), V(-5, 0)$ ,  
 $F(13, 0), F(-13, 0)$



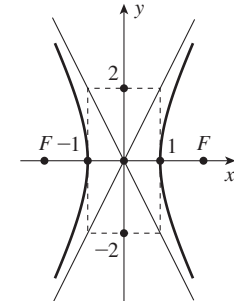
5.  $V(0, 3), V(0, -3)$ ,  
 $F(0, \sqrt{10}), F(0, -\sqrt{10})$



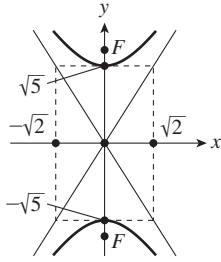
7.  $V(-\frac{5}{2}, 0), V(\frac{5}{2}, 0)$ ,  
 $F(-\frac{1}{2}\sqrt{41}, 0), F(\frac{1}{2}\sqrt{41}, 0)$



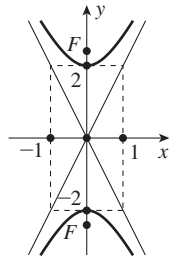
9.  $V(1, 0), V(-1, 0)$ ,  
 $F(\sqrt{5}, 0), F(-\sqrt{5}, 0)$



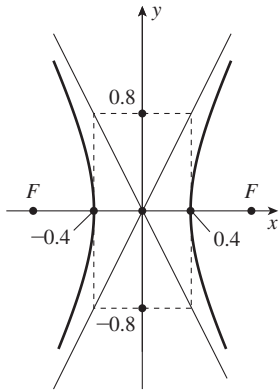
11.  $V(0, \sqrt{5}), V(0, -\sqrt{5}),$   
 $F(0, \sqrt{7}), F(0, -\sqrt{7})$



13.  $V(0, 2), V(0, -2),$   
 $F(0, \sqrt{5}), F(0, -\sqrt{5})$



15.  $V(0.4, 0), V(-0.4, 0),$   
 $F(0.9, 0), F(-0.9, 0)$



17.  $\frac{x^2}{9} - \frac{y^2}{16} = 1$ , or  $16x^2 - 9y^2 = 144$

19.  $\frac{y^2}{100} - \frac{x^2}{576} = 1$ , or  $144y^2 - 25x^2 = 14\,400$

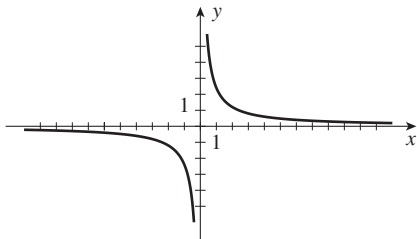
21.  $\frac{x^2}{1} - \frac{y^2}{3} = 1$ , or  $3x^2 - y^2 = 3$

23.  $\frac{x^2}{5} - \frac{y^2}{4} = 1$ , or  $4x^2 - 5y^2 = 20$

25.  $x^2 - \frac{y^2}{4} = 1$ , or  $4x^2 - y^2 = 4$

27.  $\frac{x^2}{36} - \frac{y^2}{64} = 1$ , or  $16x^2 - 9y^2 = 576$

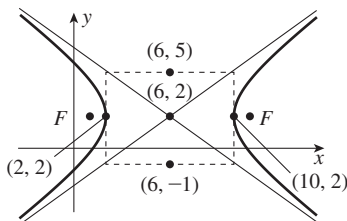
29.



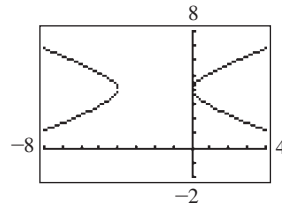
31.  $\sec^2 t - \tan^2 t = x^2 - y^2 = 1$

33.  $(-2, -3), (-2, 3), (2, -3), (2, 3)$

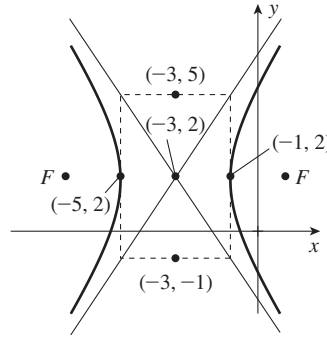
35.  $9x^2 - 16y^2 - 108x + 64y + 116 = 0$



37. Graph  $y = 4 \pm 0.5\sqrt{x^2 + 4x}$

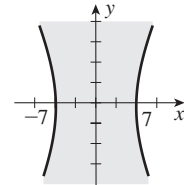


39.



41.  $x^2 - 2y^2 = 2$

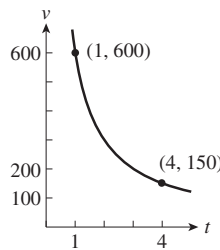
43.



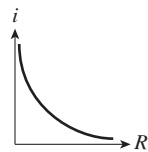
45.  $9x^2 - 16y^2 = 144$

47.  $3y^2 - x^2 = 27$

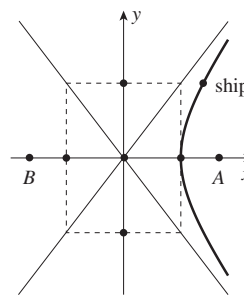
49.



51.  $i = 6.00/R$

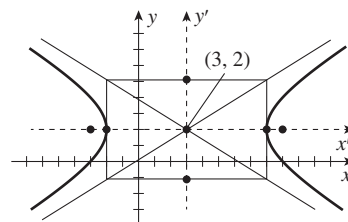


53.



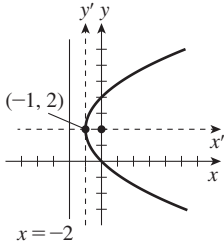
**Exercises 21.7, page 595**

1. Hyperbola,  $C(3, 2)$ , transverse axis parallel to  $x$ -axis  
 $a = 5, b = 3$

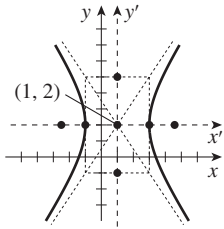


**B.50** ANSWERS TO ODD-NUMBERED EXERCISES

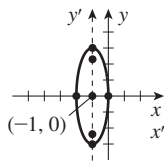
3. Parabola,  $(-1, 2)$



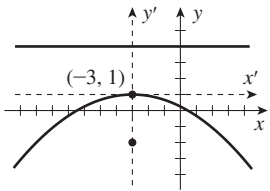
5. Hyperbola,  $(1, 2)$



7. Ellipse,  $(-1, 0)$



9. Parabola,  $(-3, 1)$



11.  $(y - 3)^2 = 16(x + 1)$ , or  $y^2 - 6y - 16x - 7 = 0$

13.  $y^2 = 24(x - 6)$

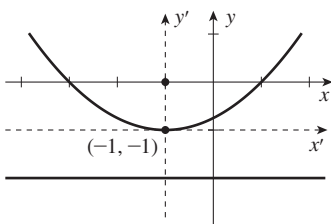
15.  $\frac{(x + 2)^2}{25} + \frac{(y - 2)^2}{16} = 1$ , or  
 $16x^2 + 25y^2 + 64x - 100y - 236 = 0$

17.  $\frac{(y - 1)^2}{16} + \frac{(x + 2)^2}{4} = 1$ , or  
 $4x^2 + y^2 + 16x - 2y + 1 = 0$

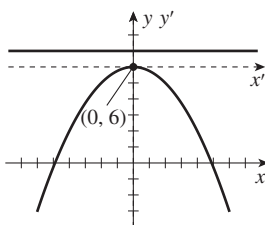
19.  $\frac{(y - 2)^2}{1} - \frac{(x + 1)^2}{3} = 1$ , or  
 $x^2 - 3y^2 + 2x + 12y - 8 = 0$

21.  $\frac{(x + 1)^2}{9} - \frac{(y - 1)^2}{16} = 1$ , or  
 $16x^2 - 9y^2 + 32x + 18y - 137 = 0$

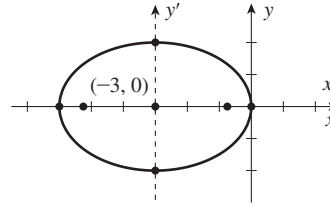
23. Parabola,  $(-1, -1)$



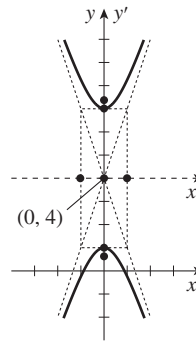
25. Parabola,  $(0, 6)$



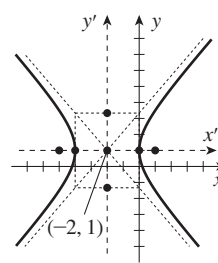
27. Ellipse,  $(-3, 0)$



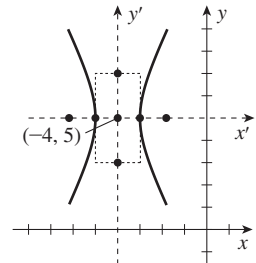
29. Hyperbola,  $(0, 4)$



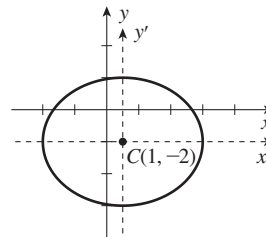
31. Hyperbola,  $(-2, 1)$



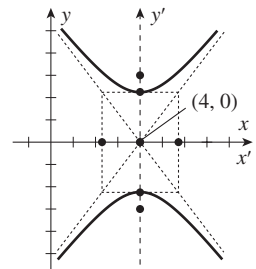
33. Hyperbola,  $(-4, 5)$



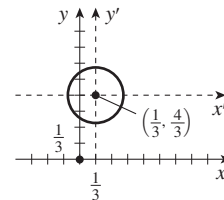
35. Ellipse,  $(1, -2)$



37. Hyperbola,  $(4, 0)$



39. Circle,  $(\frac{1}{3}, \frac{4}{3})$



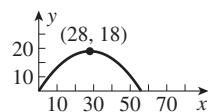
41.  $x^2 - y^2 + 4x - 2y - 22 = 0$

43.  $y^2 + 4x - 4 = 0$

45.  $y^2 = 4p(x - h)$

47.  $i' = \sin 2\pi t'$

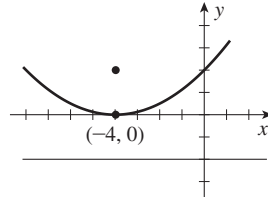
49.  $(x - 28)^2 = -\frac{28^2}{18}(y - 18)$



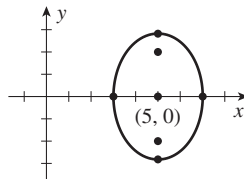
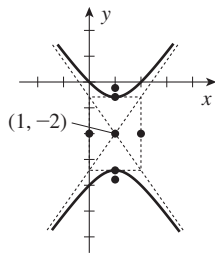
51.  $\frac{x^2}{9.0} + \frac{y^2}{16} = 1$ ,  $\frac{(x - 7.0)^2}{16} + \frac{y^2}{9.0} = 1$

**Exercises 21.8, page 598**

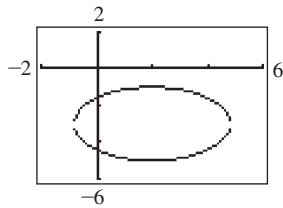
1. Hyperbola    3. Ellipse    5. Hyperbola    7. Circle  
 9. Parabola    11. Hyperbola    13. Circle  
 15. None (straight line)    17. Hyperbola    19. Ellipse  
 21. None (point at origin)    23. Ellipse  
 25. Parabola:  $V(-4, 0); F(-4, 2)$



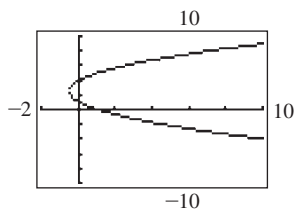
27. Hyperbola;  $C(1, -2); V(1, -2 \pm \sqrt{2})$     29. Ellipse;  $C(5, 0); V(5, \pm 2\sqrt{2})$



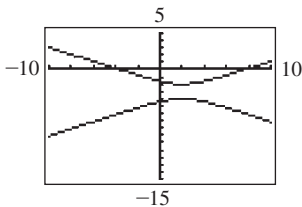
31. Ellipse; graph  $y = -3 \pm 0.5\sqrt{-2x^2 + 8x + 8}$



33. Parabola; graph  $y = \frac{5 \pm \sqrt{16x + 8}}{2}$



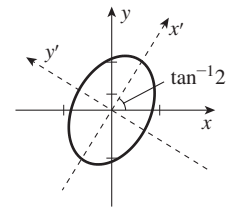
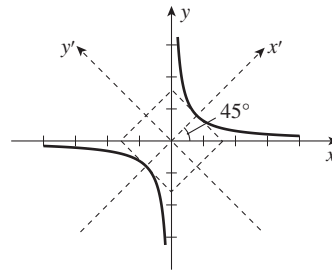
35. Hyperbola; graph  $y = -3 \pm \frac{\sqrt{x^2 - 4x + 8}}{2}$



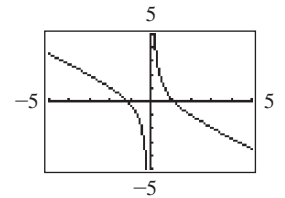
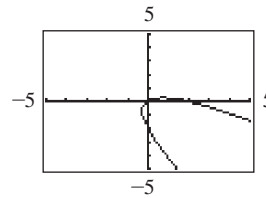
37. (a) Circle (b) Hyperbola (c) Ellipse    39. Point at the origin  
 41. Straight line    43. Parabola  
 45. Circle if light beam is perpendicular to floor; otherwise, an ellipse  
 47. One branch of a hyperbola

**Exercises 21.9, page 602**

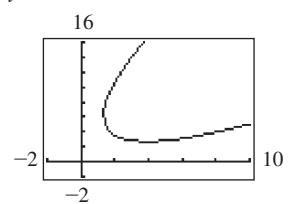
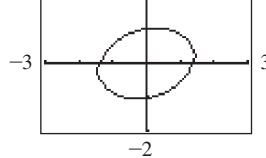
1. Hyperbola;  $2x'y' + 25 = 0$     3. Ellipse;  $4x'^2 + 9y'^2 = 36$



5. Hyperbola    7. Parabola    9. Ellipse  
 11. Parabola;  $x'^2 + \sqrt{2}y' = 0$     13. Hyperbola;  $4x'^2 - y'^2 = 4$



15. Ellipse;  $x'^2 + 2y'^2 = 2$     17. Parabola  
 $y'^2 - 4x' + 16 = 0$   
 $y''^2 = 4x''$

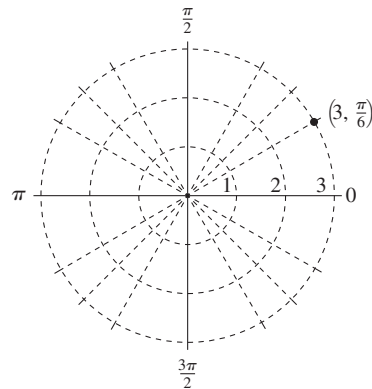


19. (a) Hyperbola (b) Hyperbola

**Exercises 21.10, page 605**

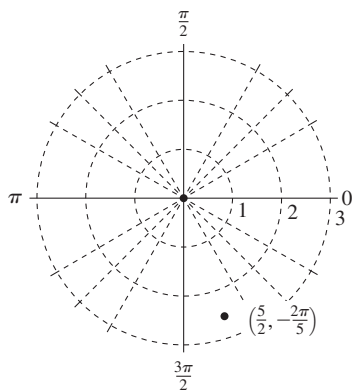
1.  $(3, \frac{-5\pi}{3}), (-3, \frac{\pi}{3})$     3. (5.83, 2.11)

5.

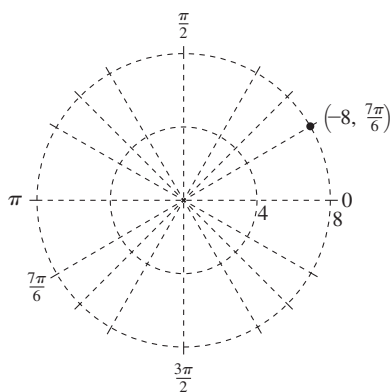


**B.52 ANSWERS TO ODD-NUMBERED EXERCISES**

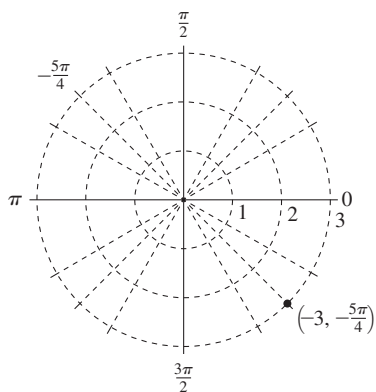
7.



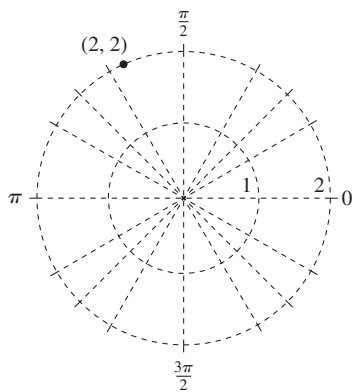
9.



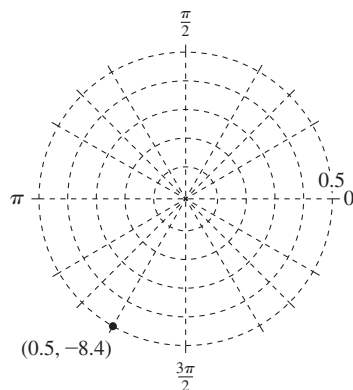
11.



13.



15.



17.  $(2, \frac{\pi}{6})$     19.  $(1, \frac{7\pi}{6})$     21.  $(-4, -4\sqrt{3})$

23.  $(2.76, -1.17)$     25.  $r = 3 \sec \theta$

27.  $r = \frac{-3}{\cos \theta + 2 \sin \theta}$     29.  $r = 4 \sin \theta$

31.  $r^2 = \frac{4}{1 + 3 \sin^2 \theta}$     33.  $r = 6 \sin \theta$

35.  $r = \frac{2 \sin 2\theta}{\cos^3 \theta + \sin^3 \theta}$     37.  $x^2 + y^2 - y = 0$ , circle

39.  $x = 4$ , straight line    41.  $x - 3y - 2 = 0$ , straight line

43.  $x^2 + y^2 - 4x - 2y = 0$ , circle

45.  $x^4 + y^4 - 4x^3 + 2x^2y^2 - 4xy^2 - 4y^2 = 0$

47.  $(x^2 + y^2)^2 = 2xy$     49. Yes  $\left[ \text{as } \left( 2, \frac{3\pi}{4} \right) \right]$

51.  $x^2 + y^2 - bx - ay = 0$

53.  $(2, 0), \left( 2, \frac{\pi}{3} \right), \left( 2, \frac{2\pi}{3} \right), \left( 2, \frac{4\pi}{3} \right), \left( 2, \frac{5\pi}{3} \right)$

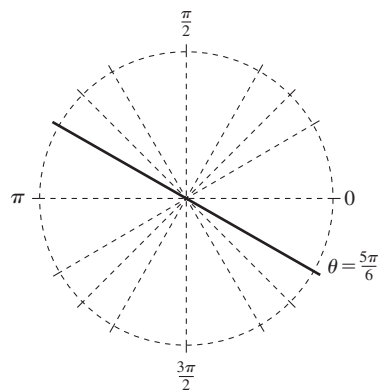
55.  $B_x = -\frac{k \sin \theta}{r}, B_y = \frac{k \cos \theta}{r}$

57.  $x^4 + y^4 + 2x^2y^2 + 2x^2y + 2y^3 - 9x^2 - 8y^2 = 0$

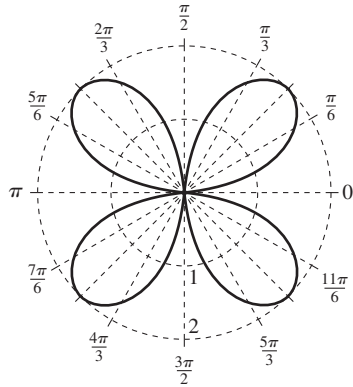
59. 13.8 km

**Exercises 21.11, page 609**

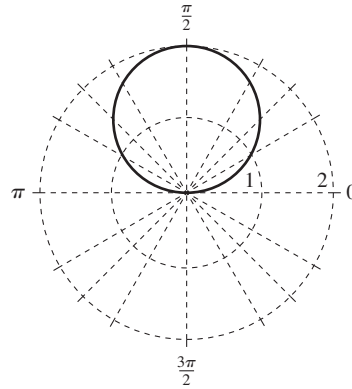
1.  $\theta = \frac{5\pi}{6}$



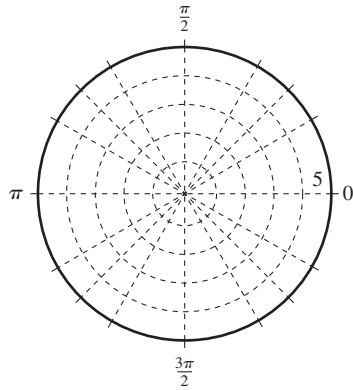
3.  $r = 2 \sin 2\theta$



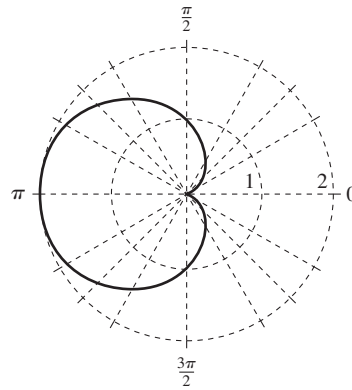
11.



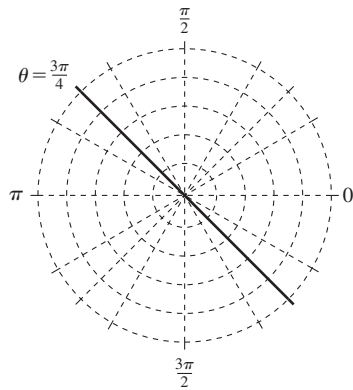
5.



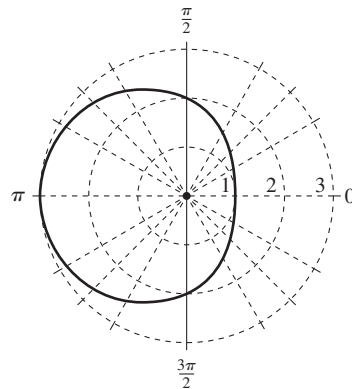
13.



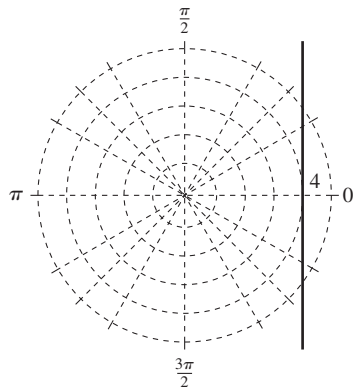
7.



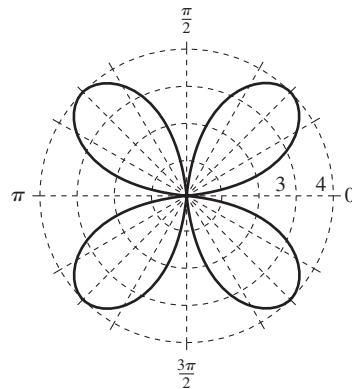
15.



9.

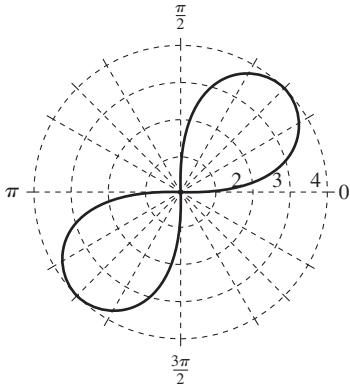


17.

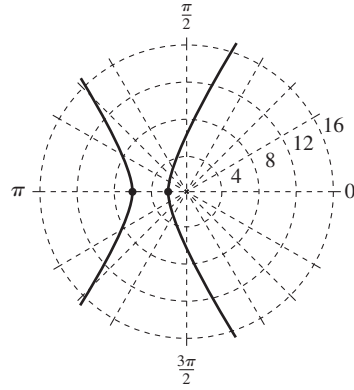


**B.54** ANSWERS TO ODD-NUMBERED EXERCISES

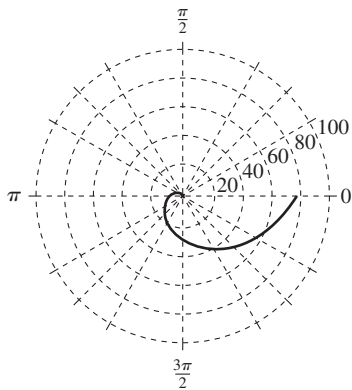
19.



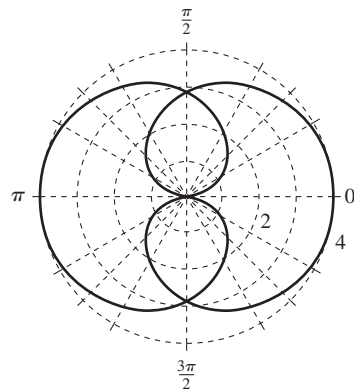
27.



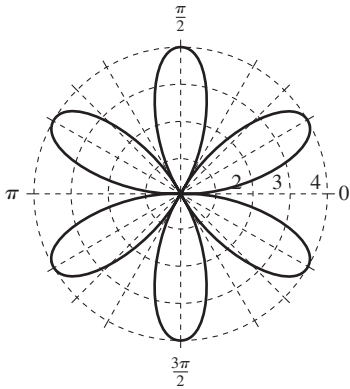
21.



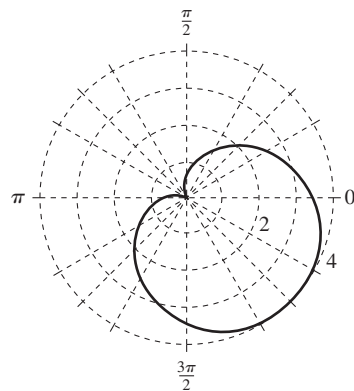
29.



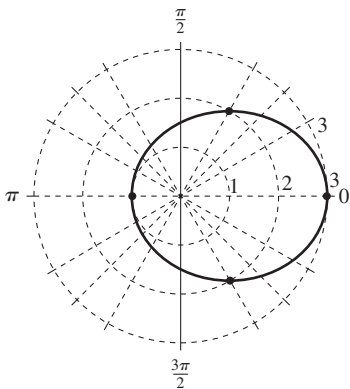
23.



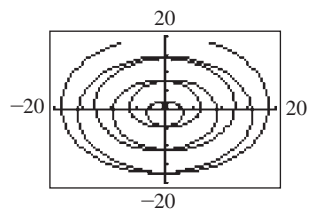
31.



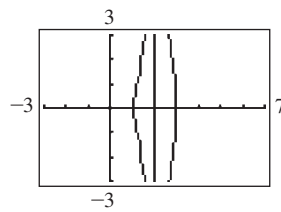
25.



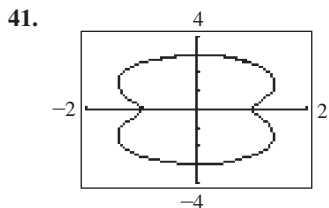
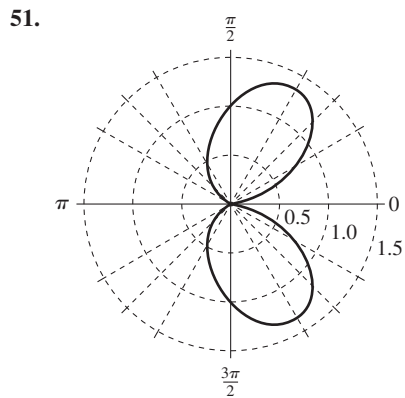
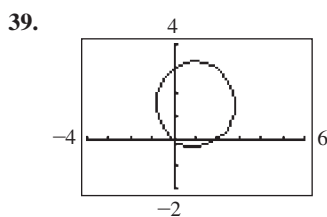
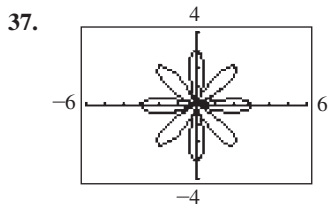
33.



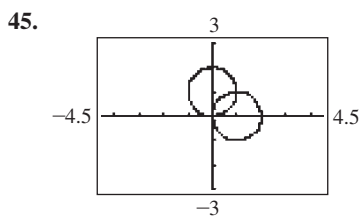
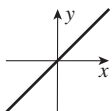
35.







43. Straight line

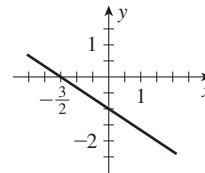
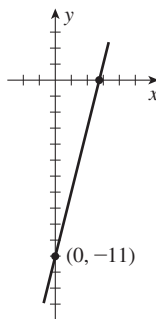


53. If  $n$  is odd, there are  $n$  loops. If  $n$  is even, there are  $2n$  loops.

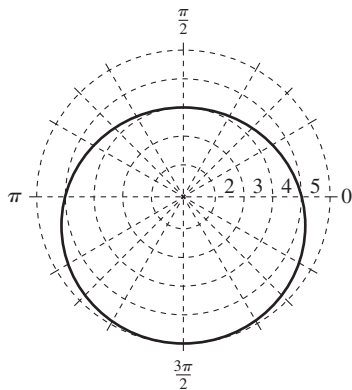
Review Exercises for Chapter 21, page 611

1.  $4x - y - 11 = 0$

3.  $2x + 3y + 3 = 0$

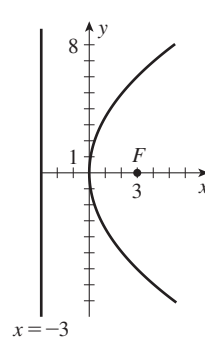
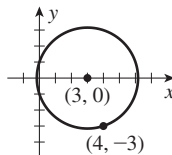


47.

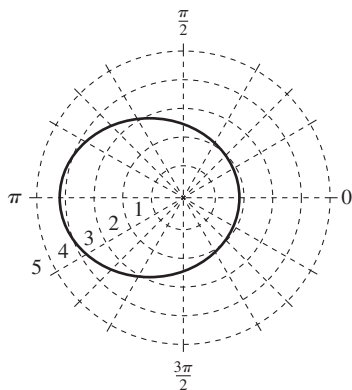


5.  $x^2 - 6x + y^2 - 1 = 0$

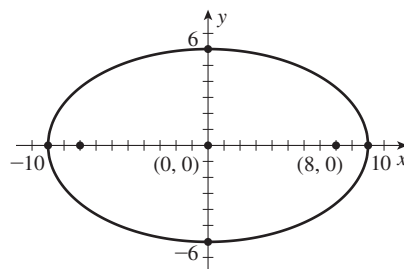
7.  $y^2 = 12x$



49.

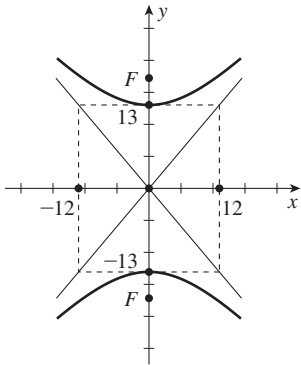


9.  $9x^2 + 25y^2 = 900$

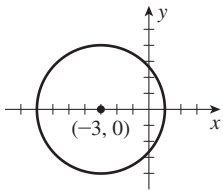


**B.56** ANSWERS TO ODD-NUMBERED EXERCISES

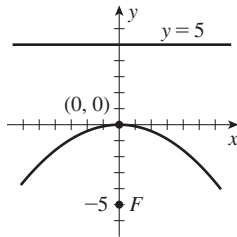
11.  $144y^2 - 169x^2 = 24\,336$



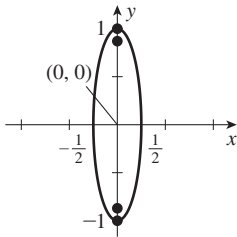
13.  $(-3, 0), r = 4$



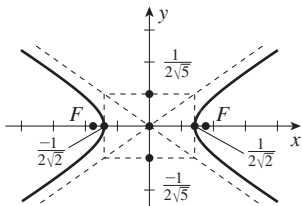
15.  $(0, -5), y = 5$



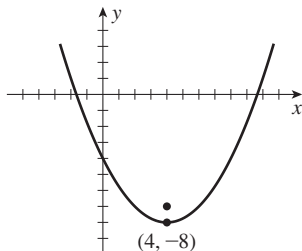
17.  $V(0, 1), V(0, -1), F\left(0, \frac{1}{2}\sqrt{3}\right), F\left(0, -\frac{1}{2}\sqrt{3}\right)$



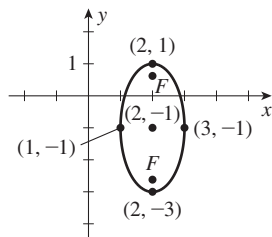
19.  $V\left(\frac{1}{2\sqrt{2}}, 0\right), V\left(-\frac{1}{2\sqrt{2}}, 0\right), F\left(\frac{\sqrt{70}}{20}, 0\right), F\left(-\frac{\sqrt{70}}{20}, 0\right)$



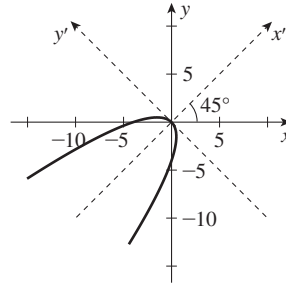
21.  $V(4, -8), F(4, -7)$



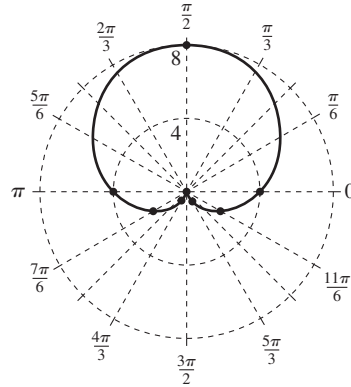
23.  $(2, -1)$



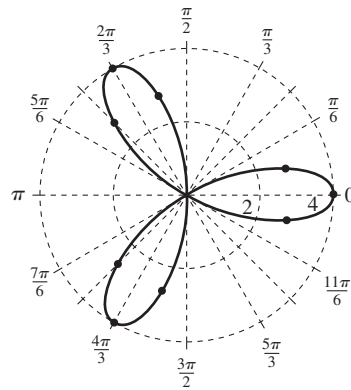
25.  $(0, 0)$



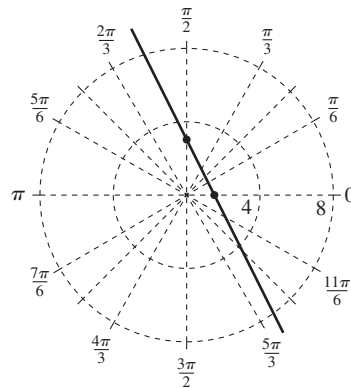
27.



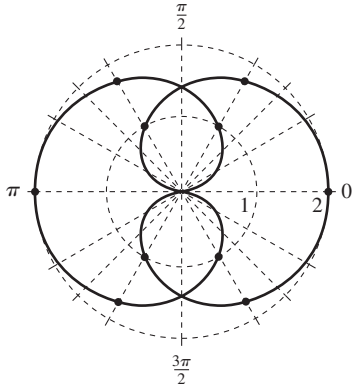
29.



31.

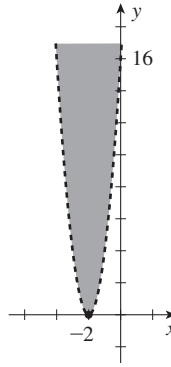


33.



35.  $\theta = \tan^{-1} 2 = 1.11$

71.



73. Hyperbola 75. 8

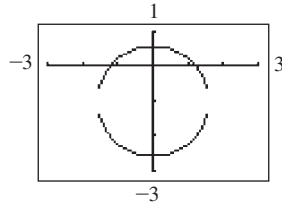
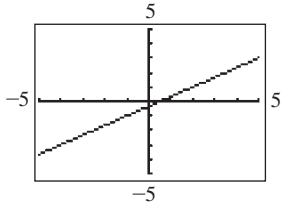
37.  $r^2 = \frac{2}{1 + \sin \theta \cos \theta}$

39.  $(x^2 + y^2)^3 = 16x^2y^2$

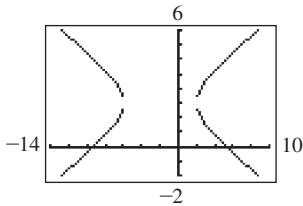
41.  $3x^2 + 4y^2 - 8x - 16 = 0$  43. 4 45. 2 47. 2

49. Graph  $y = \frac{2}{3}x - \frac{1}{3}$

51. Graph  $y = -1 \pm \frac{1}{2}\sqrt{10 - 4x^2}$

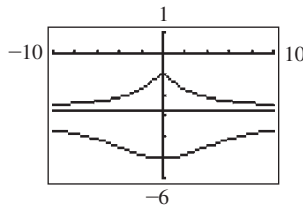
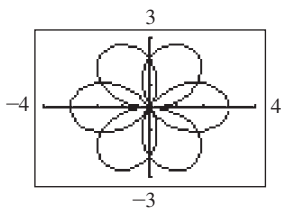


53. Graph  $y = \pm \sqrt{0.25x^2 + x - 3} + 3$



55. Graph  $r = 2 - \frac{3}{\sin \theta}$

57.



59.  $x^2 + y^2 - 6x + 8y + 9 = 0$

61.  $\frac{(x-4)^2}{16} + \frac{(y+3)^2}{7} = 1$  or

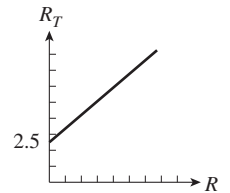
$7x^2 + 16y^2 - 56x + 96y + 144 = 0$

63. Circle 65. 1 67.  $\sqrt{45}$

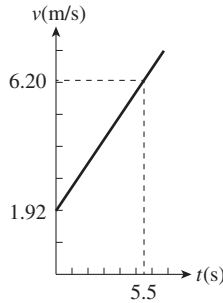
69.  $m_1 = -\frac{12}{5}, m_2 = \frac{5}{12}; 13^2 + 13^2 = 338$

77.  $x^2 - 6x - 8y + 1 = 0$

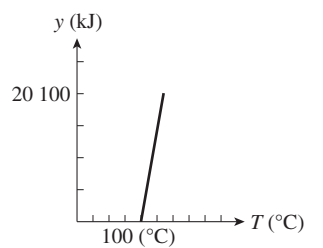
79.  $R_T = R + 2.5$



81.  $v = 1.92 + 0.778t$



83.  $y = 100.5T - 10\,050$

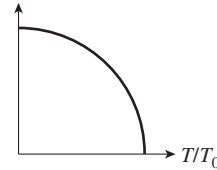


85.  $1400 \text{ m}^2$

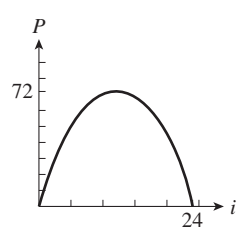
87.  $y = -\frac{1}{80}x^2$

89.  $y^2 = 32x$

91.  $H_T/H_0$



93.



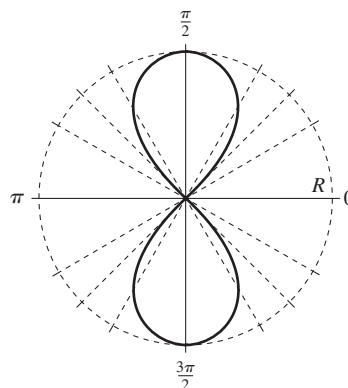
95.  $700 \text{ m}^2$

97. 18 cm, 8 cm

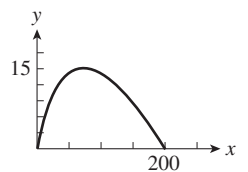
99. 11.3 m

101. Dist. from rifle to P - dist. from target to P = constant (related to dist. from rifle to target)

103.



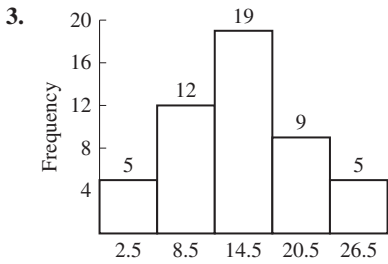
105.



**Exercises 22.1, page 619**

1.

<i>Est. hours</i>	0–5	6–11	12–17	18–23	24–29
<i>f</i>	5	12	19	9	5

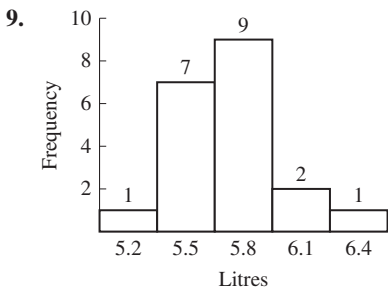


5.

<i>No.</i>	5.3	5.4	5.5	5.6	5.7	5.8	5.9	6.0	6.1	6.2	6.3
<i>f</i>	1	3	1	3	2	4	3	1	1	0	1

7.

<i>No.</i>	5.1–5.3	5.4–5.6	5.7–5.9	6.0–6.2	6.3–6.5
<i>f</i>	1	7	9	2	1

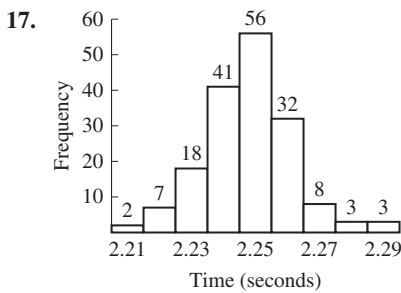
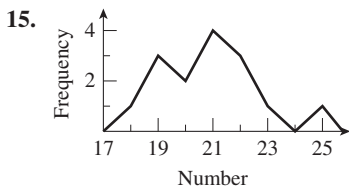


11.

<i>No.</i>	<5.4	<5.7	<6.0	<6.3	<6.6
<i>Cum. f</i>	1	8	17	19	20

13.

<i>No. inst.</i>	18	19	20	21	22	23	24	25
<i>f</i>	1	3	2	4	3	1	0	1

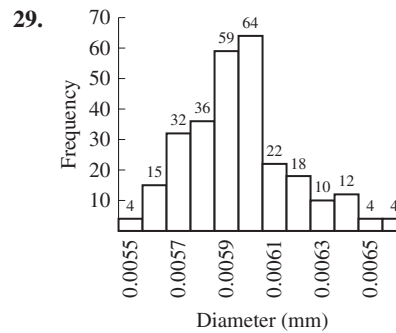
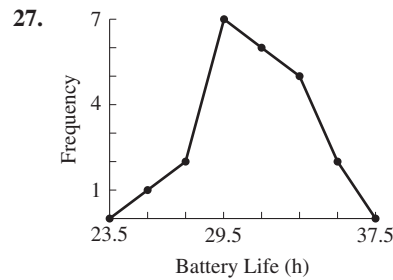
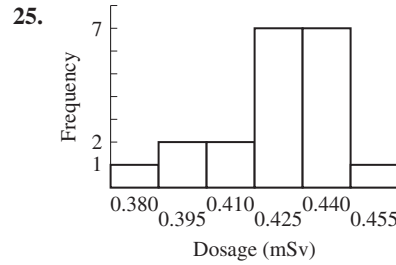
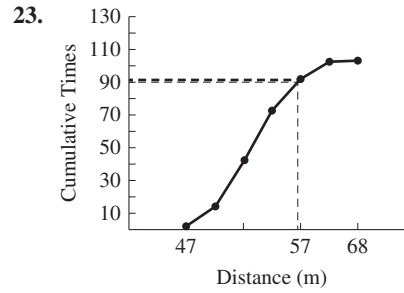


19.

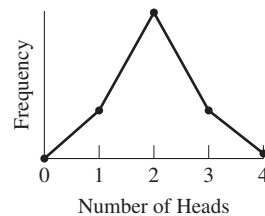
<i>Time (s)</i>	<2.22	<2.23	<2.24	<2.25	<2.26	<2.27	<2.28	<2.29	<2.30
<i>Cum. f</i>	2	9	27	68	124	156	164	167	170

21.

<i>Dist. (m)</i>	47–49	50–52	53–55	56–58	59–61	62–64	65–67
<i>Rel. Freq. (%)</i>	1.7	12.5	26.7	30.0	20.0	8.3	0.8



31. The greatest frequency should be at 2 and the least at 0 and 4. (Graphs will generally have approximately the shape shown.)



**Exercises 22.2, page 627**

1. 5    3. 5.5    5. 2.2    7. 94%    9. (a) 25.5 (b) 25 (c) 25  
 11. (a) 107.2 (b) 107.5 (c) 108    13. (a) 1.8 (b) 1.8

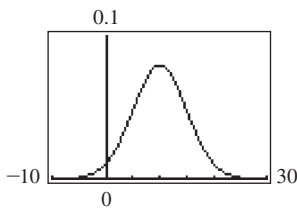
15. (a) 2.500 (b) 2.500    17. 0.257 L/100 km  
 19. Mean: 20.9 instructions, median: 21 instructions, mode: 21 instructions    21. Mean: 2.248 s, median: 2.25 s  
 23. Mean: 56.5 m, Median: 57 m    25. Mean: 0.4237 mSv, median: 0.4265 mSv, mode: 0.436 mSv    27. Mean: 31.2 h, median: 31 h, Mode: 30 h    29. Mean: 0.005 95 mm, median: 0.0059 mm    31. \$700  
 33. Range 275,  $s = \$84.62$     35. Median: 3600 h, mode: 3600 h  
 37. Median: 0.195 ppm, mode: 0.18 ppm  
 39. Range: 0.18 ppm,  $s = 0.0474$  ppm    41. \$662.50  
 43. Median: \$725, mean: \$748.2, mode: \$800. When adding a constant, the mean, the median, and the mode are all increased by the same constant.  
 45. \$883.90; The mean no longer represents the centre of the data.  
 47. 0.4254 mSv    49. At least 89%

**Exercises 22.3, page 633**

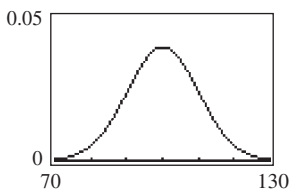
1.  $\mu = 10, \alpha = 5$  and  $\mu = 20, \alpha = 5$  result in the same curve, with the first centred at  $x = 10$  and the second centred at  $x = 20$ .

3. 0.2072

5.



7.



9. 136    11. 95    13. 341    15. 78.81%    17. 2166

19. 179    21. 141; about 68% of all samples of size 5000 have a mean lifetime from 99 859 km to 100 141 km.

23. 0.0145; 68% of all samples of size 500 will have a proportion of defectives that is within (0.1055, 0.1345).

25. -0.7    27. -0.6    29. 76% (normal dist. percentage is 68%)

31. 68% (normal dist. percentage is 68%)

**Exercises 22.4, page 640**

1. (32.4, 33.2)    3. 35    5. 1537    7. (243.2, 251.0)

9. (0.126, 0.228)    11. 277    13. (83.0, 84.4)

15. (83.0, 84.4). Margin of error has increased as well.

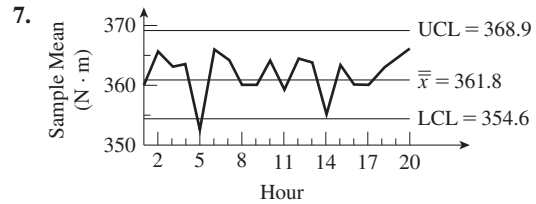
17. (104.9, 109.5)    19. 58    21. 601

**Exercises 22.5, page 645**

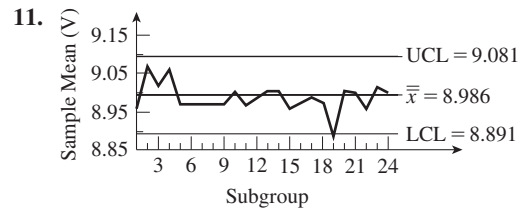
1.  $UCL(\bar{x}) = 504.7$  mg,  $LCL(\bar{x}) = 494.7$  mg

3. The first point would be below the  $\bar{x}$  line. The UCL and LCL lines would be 0.2 unit lower.

5.  $\bar{x} = 361.8$  N · m, UCL = 368.9 N · m, LCL = 354.6 N · m



9.  $\bar{x} = 8.986$  V, UCL = 9.081 V, LCL = 8.891 V



13.  $\mu = 2.725$  cm, UCL = 2.729 cm, LCL = 2.721 cm

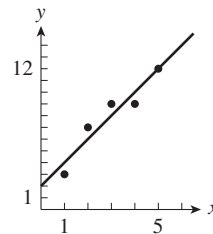
15.  $\mu = 5.57$  mL, UCL = 11.17 mL, LCL = 0.00 mL

17.  $\bar{p} = 0.0369$ , UCL = 0.0548, LCL = 0.0190

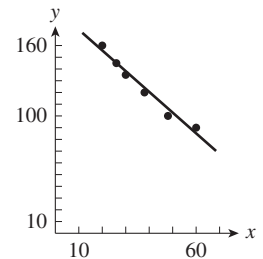
19.  $\bar{p} = 0.0580$ , UCL = 0.0894, LCL = 0.0266

**Exercises 22.6, page 650**

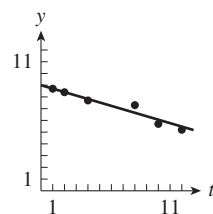
1.  $y = 2x + 2$



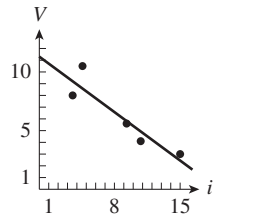
3.  $y = -1.77x + 191$



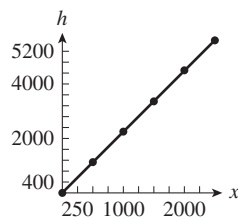
5.  $y = -0.308t + 9.07$



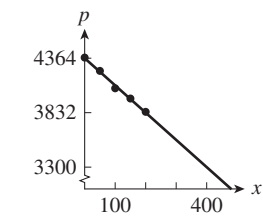
7.  $V = -0.590i + 11.3$



9.  $h = 2.24x + 5.2$

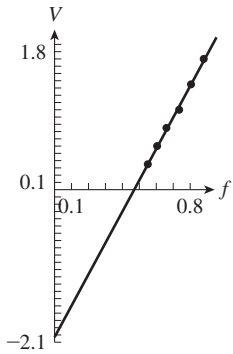


11.  $p = -2.66x + 4364$



**B.60 ANSWERS TO ODD-NUMBERED EXERCISES**

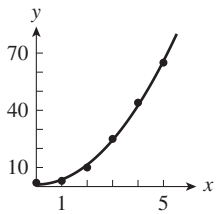
13.  $V = 4.32f - 2.03$ ,  
 $f_0 = 0.470$  PHz



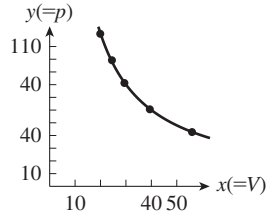
15. 0.953    17. -0.901

**Exercises 22.7, page 653**

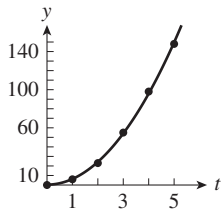
1.  $y = 2.59x^2 + 1.07$



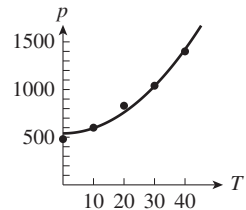
3.  $y = \frac{2410}{x} - 0.234$



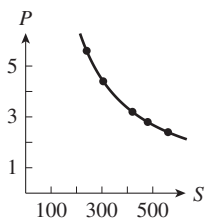
5.  $y = 5.97t^2 + 0.38$



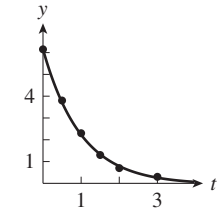
7.  $p = 0.551T^2 + 540$



9.  $P = \frac{1343}{S}$



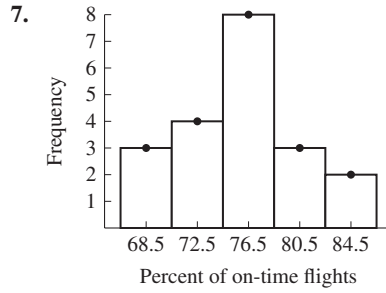
11.  $y = 6.20e^{-t} - 0.05$



**Review Exercises for Chapter 22, page 655**

1. 76.5    3. 76.2

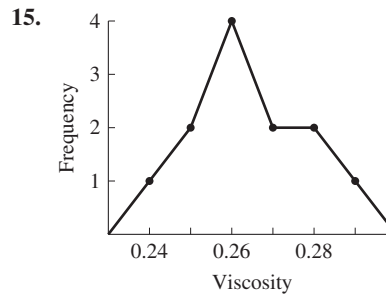
5.	No.	67-70	71-74	75-78	79-82	83-86
	f	3	4	8	3	2



9.

No.	<71	<75	<79	<83	<87
f	3	7	15	18	20

11.  $0.264 \text{ Pa} \cdot \text{s}$     13.  $0.014 \text{ Pa} \cdot \text{s}$



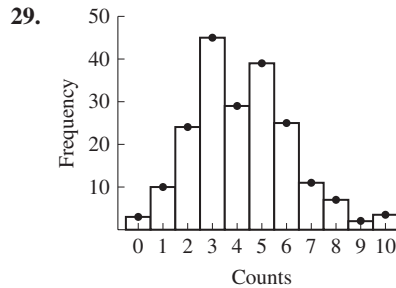
17. 700 W    19. 700 W    21. 17.3 W

23.

Power (W)	<660	<670	<680	<690
Cum. f	3	5	12	24

Power (W)	<700	<710	<720	<730	<740
Cum. f	51	85	100	116	121

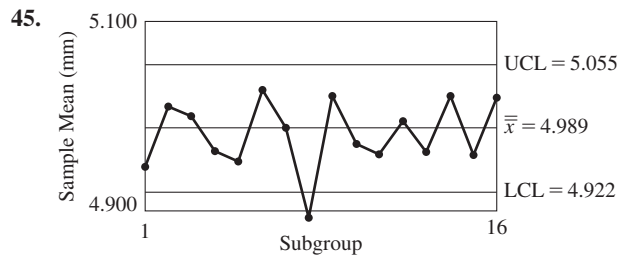
25. (692.8, 699.2)    27. 4



31. 66.2 km/h    33. 9.0 km/h

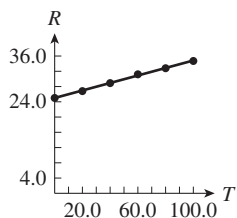
35. (64.6, 67.4)    37. At least 75%    39. (0.138, 0.252)

41. 415    43.  $\bar{p} = 0.0540$ , UCL = 0.0843, LCL = 0.0237

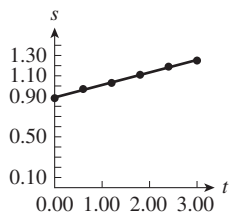


47. 322    49. 495

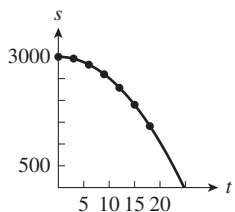
51.  $R = 0.0983T + 25.0$



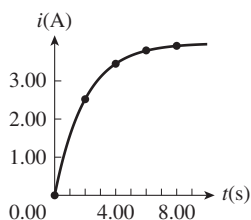
53.  $s = 0.123t + 0.887$



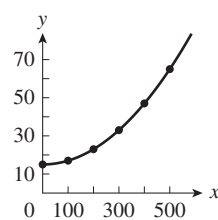
55.  $s = -4.90t^2 + 3000$



57.  $i = 3.99(1 - e^{-0.500t})$ , 3.99 A



59.  $y = 0.0002x^2 + 15$



61.  $y = 0.997x + 0.581$

63.  $y = 1.39x^{0.878}$

65. 2.7    67. 9.2 ppm

69. Substitute expressions for  $m$ ,  $b$ , and  $\bar{x}$ . Simplify to  $\bar{y} = \bar{y}$ .

**Exercises 23.1, page 667**

- 1. Not cont. at  $x = -2$     3.  $-4$     5. Cont. all  $x$
- 7. Not cont.  $x = 0$  and  $x = 1$ , div. by zero
- 9. Cont.  $x \leq 0, x > 2$ ; function not defined    11. Cont. all  $x$
- 13. Not cont.  $x = 2$ , jump in the graph
- 15. Not cont.  $x = 2, f(2) \neq \lim_{x \rightarrow 2} f(x)$
- 17. (a)  $-1$  (b)  $\lim_{x \rightarrow 2} f(x)$  does not exist
- 19. (a)  $0$  (b)  $\lim_{x \rightarrow 2} f(x) = -1$
- 21. Not cont.  $x = 2$ , jump in the graph    23. Cont. all  $x$

25. $x$	0.900	0.990	0.999	1.001
$f(x)$	1.7100	1.9701	1.9970	2.0030

$x$	1.010	1.100	$\lim_{x \rightarrow 1} f(x) = 2$
$f(x)$	2.0301	2.3100	

27. $x$	1.900	1.990	1.999	2.001
$f(x)$	-0.2516	-0.2502	-0.25002	-0.24998

$x$	2.010	2.100	$\lim_{x \rightarrow 2} f(x) = -0.25$
$f(x)$	-0.2498	-0.2485	

29. $x$	10	100	1000	$\lim_{x \rightarrow \infty} f(x) = 0.4$
$f(x)$	0.4468	0.4044	0.4004	

- 31. 7    33. 1    35.  $-\frac{2}{3}$     37. 27    39. 2

- 41. Does not exist    43.  $\frac{1}{2}$     45. 3    47. 1

49. $x$	-0.1	-0.01	-0.001	0.001
$f(x)$	-3.1	-3.01	-3.001	-2.999

$x$	0.01	0.1	$\lim_{x \rightarrow 0^+} f(x) = -3$ : use factoring
$f(x)$	-2.99	-2.9	

51. $x$	10	100	1000	$\lim_{x \rightarrow \infty} f(x) = 2$ ; divide numerator and denominator by $x^2$
$f(x)$	2.1649	2.0106	2.0010	

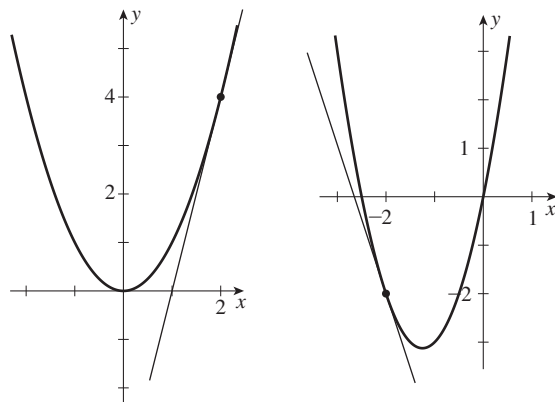
- 53. 34.9 °C, 0 °C    55. 3 cm/s    57. 2.77    59.  $e$

- 61. (a)  $-1$  (b) 2, (c) Does not exist    63. 0

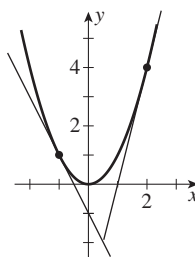
- 65.  $-1$ ;  $+1$ ; no,  $\lim_{x \rightarrow 0^+} f(x) \neq \lim_{x \rightarrow 0^-} f(x)$

**Exercises 23.2, page 673**

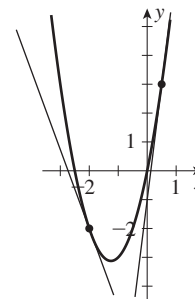
- 1. 9    3. (Slopes) 3.5, 3.9, 3.99    5. (Slopes)  $-2, -2.8, -2.98, -2.998$ ;  $m = -3$
- 3.999;  $m = 4$



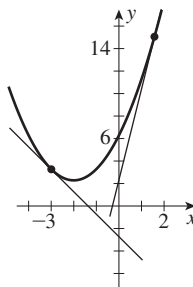
7.  $m_{\tan} = 2x; 4, -2$



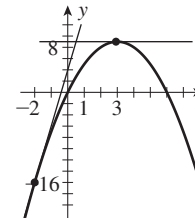
9.  $m_{\tan} = 4x + 5; -3, 7$



11.  $m_{\tan} = 2x + 4; -2, 8$



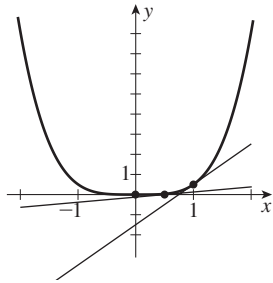
13.  $m_{\tan} = 6 - 2x; 10, 0$



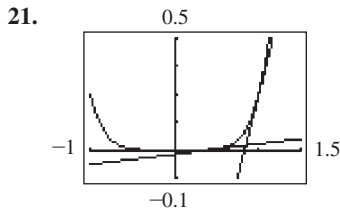
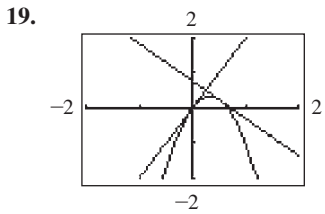
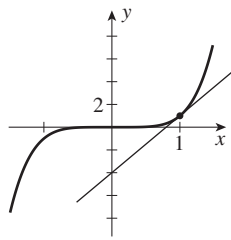


**B.62 ANSWERS TO ODD-NUMBERED EXERCISES**

15.  $m_{\tan} = 6x^3; 0, 0.75, 6$



17.  $m_{\tan} = 5x^4; 0, 0.31, 5$



23. av. ch. = 4.1,  $m_{\tan} = 4$     25. av. ch. = -12.6,  $m_{\tan} = -12.0$

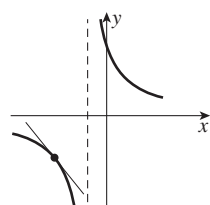
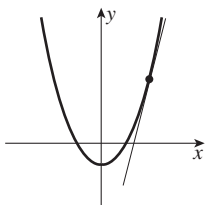
27.  $(-1, 2)$     29.  $-\frac{1}{6}$     31.  $9.46^\circ$     33. 3

**Exercises 23.3, page 677**

1.  $8x + 3$     3. 3    5. -2    7.  $2x$     9.  $2\pi x$
11.  $2x - 7$     13.  $4(2 - x)$     15.  $3x^2 + 4$     17.  $-\frac{\sqrt{3}}{(x+2)^2}$
19.  $1 - \frac{4}{3x^2}$     21.  $-\frac{4}{x^3}$     23.  $4x^3 + 3x^2 + 2x + 1$
25.  $4x^3 + \frac{2}{x^2}$     27.  $2(3x - 1); -8$     29.  $-\frac{33}{(3x+2)^2}; -\frac{3}{11}$
31.  $-\frac{2}{x^2}$ ; all real numbers except 0
33.  $\frac{-6x}{(x^2 - 1)^2}$ ; all real numbers except -1 and 1    35.  $(5, 5)$
37.  $(4, -32)$
39.  $\frac{1}{2\sqrt{x+1}}$ ; differentiable for  $x > -1$  since derivative and function are both defined for these values.
41.  $\frac{dy}{dx} = nx^{n-1}$     43.  $-11.3^\circ$

**Exercises 23.4, page 680**

1.  $\frac{ds}{dt} = 14.0 - 9.80t; -56.0 \text{ m/s}, -25.3 \text{ m/s}$
3.  $\left. \frac{dy}{dx} \right|_{x=2} = 4$     5.  $\left. \frac{dy}{dx} \right|_{x=-3} = -\frac{3}{4}$



7. 4.00, 4.00, 4.00, 4.00, 4.00;  $\lim_{t \rightarrow 3} v = 4.00 \text{ m/s}$

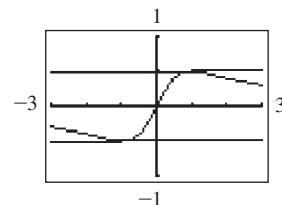
9. 5, 6.5, 7.7, 7.97, 7.997;  $\lim_{t \rightarrow 2} v = 8.00 \text{ m/s}$     11. 4; 4.00 m/s
13.  $6t - 4; 8.00 \text{ m/s}$     15. 48    17.  $3t(8 - t)$     19.  $3 + \frac{6}{5t^2}$
21.  $12t - 4$     23.  $6t$     25. 6.28 cm/cm    27. 4.49 s
29. -2    31.  $6w$     33. 0.460 kW    35.  $-83.1 \text{ W}/(\text{m}^2 \cdot \text{h})$
37.  $-\frac{48}{(t+3)^2}; -\$1330/\text{year}$     39.  $\pi d^2$     41.  $24.2/\sqrt{\lambda} \text{ m/m}$

**Exercises 23.5, page 685**

1.  $9r^8$     3.  $6x^2 - 12x + 8; 56$     5.  $5x^4$     7.  $-36x^8$
9.  $20x^3$     11.  $2(x+1)$     13.  $15r^2 - 2$
15.  $200x^7 - 170x^4 - \frac{1}{2\sqrt{x}}$     17.  $-42x^6 + 15x^2$
19.  $x^2 + x + \frac{1}{x^2}$     21. 16    23. 33    25. -4    27. -29
29.  $5(6t^4 - 1)$     31.  $-6(1 + t^2)$     33. 64.0 m/s
35. 0.00 m/s    37. 1    39.  $(2, 4)$     41.  $(1, -5)$
43.  $-\frac{1}{4}$     45.  $3\pi r^2$     47. 84.0 W/A    49.  $3.32 \Omega/^\circ\text{C}$
51.  $-0.00286 \text{ N}/^\circ\text{C}$     53.  $-98.0 \text{ m/km}$     55.  $391 \text{ mm}^3/\text{mm}$

**Exercises 23.6, page 689**

1.  $2(9x^2 - 9x - 5)$
3.  $6x(6x - 5) + (3x^2 - 5x)(6) = 6x(9x - 10)$
5.  $(3t + 2)(2) + (2t - 5)(3) = 12t - 11$
7.  $(x^4 - 3x^2 + 3)(-6x^2) + (1 - 2x^3)(4x^3 - 6x)$   
 $= 2x(-7x^5 + 15x^3 + 2x^2 - 9x - 3)$
9.  $(2x - 7)(-2) + (5 - 2x)(2) = -8(x - 3)$
11.  $(h^3 - 1)(4h - 1) + (2h^2 - h - 1)(3h^2)$   
 $= 10h^4 - 4h^3 - 3h^2 - 4h + 1$
13.  $\frac{3}{(2x+3)^2}$     15.  $-\frac{4\pi x}{(2x^2+1)^2}$     17.  $\frac{12x(3-x)}{(3-2x)^2}$
19.  $\frac{-6x^2 + 6x + 4}{(3x^2+2)^2}$     21.  $\frac{-3x^2 - 16x - 26}{(x^2+4x+2)^2}$
23.  $\frac{-2x^3 + 2x^2 + 5x + 4}{x^3(x+2)^2}$     25. -107    27. 75.0    29. 19
31. -5.64    33. Eq. (23.10)    35.  $(0, -1)$
37.  $x^2 f'(x) + 2xf(x)$
39. (1)  $\frac{-12x^3 + 45x^2 - 14x}{(3x-7)^2}$     (2)  $\frac{-12x^3 + 45x^2 - 14x}{(3x-7)^2}$
41. 12.0    43. 1, -1    45.  $-0.694 \text{ W/s}$

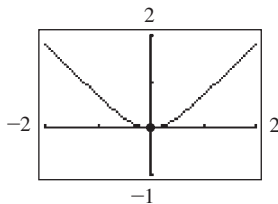


47.  $\frac{96}{(7R+12)^2}$     49.  $8t^3 - 45t^2 - 14t - 8$     51.  $1.18 \text{ }^\circ\text{C}/\text{h}$

53.  $\frac{2R(R+2r)}{3(R+r)^2}$     55.  $\frac{E^2(R-r)}{(R+r)^3}$     57.  $9.80 \times 10^3 \text{ Pa/m}$

**Exercises 23.7, page 697**

1.  $36x^2(2+3x^3)^3$     3.  $-\frac{3x}{(2-3x^2)^{1/2}}$     5.  $\frac{1}{2x^{1/2}}$     7.  $-\frac{6}{5t^3}$   
 9.  $-\frac{1}{x^{4/3}}$     11.  $\frac{3}{2}x^{1/2} + \frac{6}{x^2}$     13.  $10x(x^2+1)^4$   
 15.  $-216x^2(7-4x^3)^7$     17.  $\frac{2x^2}{(2x^3-3)^{2/3}}$     19.  $\frac{24y}{(4-y^2)^5}$   
 21.  $\frac{24x^3}{(2x^4-5)^{0.25}}$     23.  $\frac{-4x}{(1-8x^2)^{3/4}}$     25.  $\frac{12v+5}{(8v+5)^{1/2}}$   
 27.  $\frac{6(5x-1)}{x^4(1-6x)^{1/2}}$     29.  $\frac{x^2+12x+16}{(x+4)^2(x+2)^{1/2}}$   
 31.  $\frac{-1}{\sqrt{2R+1}(4R+1)^{3/2}}$     33.  $\frac{3}{10}$     35.  $\frac{5}{36}$   
 37.  $\frac{d}{dx}x(x^{1/2}) = x\left(\frac{1}{2}x^{-1/2}\right) + x^{1/2}(1) = \frac{3}{2}x^{1/2}$   
 39.  $\frac{x^3(0) - 1(3x^2)}{x^6} = -3x^{-4}$     41.  $x = 0$



43. Yes, at  $\left(-\frac{13}{9}, \frac{1}{3}\right)$     45. 1    47.  $\frac{0.485}{\sqrt{w}}$     49.  $-1.35 \text{ cm/s}$   
 51.  $\frac{-450\,000}{V^{5/2}}, -4.50 \text{ kPa/cm}^3$     53.  $l = a$   
 55.  $-45.2 \text{ W/(m}^2 \cdot \text{h)}$     57.  $\frac{8a^3}{(4a^2 - l^2)^{3/2}}$   
 59.  $\frac{\sqrt{2}(w+1)}{\sqrt{w^2+2w+2}}$

**Exercises 23.8, page 701**

1.  $-\frac{4x}{3y^2}$     3.  $x^2 \frac{dy}{dx} + 2xy$     5.  $-\frac{x \frac{dy}{dx} + y}{(xy)^2}$     7.  $-\frac{3}{2}$   
 9.  $\frac{6x+1}{4}$     11.  $\frac{x}{4y}$     13.  $\frac{2x}{5y^4}$     15.  $\frac{2x}{2y+1}$     17.  $\frac{-3y}{3x+1}$   
 19.  $\frac{y-x^3-2x^2y-xy^2}{x}$     21.  $\frac{3(y^2+1)(y^2-2x+1)}{(y^2+1)^2-6x^2y}$   
 23.  $\frac{4(2y-x)^3-2x}{8(2y-x)^3-1}$     25.  $\frac{-3x(x^2+1)^2}{y(y^2+1)}$     27. 3  
 29.  $-\frac{108}{157}$     31.  $\frac{1}{4}$     33.  $(2, 2), (2, -2)$   
 35. At  $(\sqrt{7}, 0)$  and  $(-\sqrt{7}, 0)$ ,  $m_{\tan} = -2$

37. 1    39.  $\frac{1}{2\omega L^3} \left(-\frac{L}{C} + 2R^2\right)$     41.  $\frac{nRT^2 + bnP}{VT^2 - aNT^2 + bnT}$   
 43.  $-\frac{x}{y}$     45.  $\frac{r-R+1}{r+1}$     47.  $\frac{2C^2r(12CSr-20Cr-3L)}{3(Cr+L)(Cr-L)}$

**Exercises 23.9, page 705**

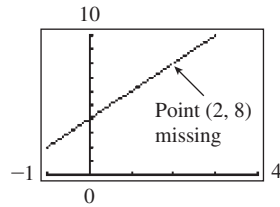
1.  $y' = 15x^2 - 4x, y'' = 30x - 4, y''' = 30, y^{(n)} = 0 (n \geq 4)$   
 3.  $y' = 3x^2 + 2x, y'' = 6x + 2, y''' = 6, y^{(n)} = 0 (n \geq 4)$   
 5.  $f'(x) = 3x^2 - 24x^3, f''(x) = 6x - 72x^2,$   
 $f'''(x) = 6 - 144x, f^{(4)}(x) = -144, f^{(n)}(x) = 0 (n \geq 5)$   
 7.  $y' = -8(1-2x)^3, y'' = 48(1-2x)^2,$   
 $y''' = -192(1-2x), y^{(4)} = 384, y^{(n)} = 0 (n \geq 5)$   
 9.  $f'(r) = (16r+9)(4r+9)^2,$   
 $f''(r) = 24(8r+9)(4r+9),$   
 $f'''(r) = 96(16r+27), f^{(4)}(r) = 1536,$   
 $f^{(n)}(r) = 0 (n \geq 5)$   
 11.  $84x^5 - 30x^4$     13.  $-\frac{1}{4x^{3/2}}$     15.  $-\frac{12}{(8x-3)^{7/4}}$   
 17.  $\frac{14.4\pi}{(1+2p)^{5/2}}$     19.  $600(2-5x)^2$   
 21.  $30(27x^2-1)(3x^2-1)^3$     23.  $\frac{4\pi^2}{(6-x)^3}$   
 25.  $\frac{450}{(v+15)^3}$     27.  $-\frac{9}{y^3}$     29.  $-\frac{6(x^2-xy+y^2)}{(2y-x)^3}$   
 31.  $\frac{9}{125}$     33.  $-\frac{13}{384}$     35.  $-4320$     37.  $-9.80 \text{ m/s}^2$   
 39.  $1.00 \text{ m/s}^2$   
 41.  $\frac{d}{dx}\left(u \frac{dv}{dx} + v \frac{du}{dx}\right) = u \frac{d^2v}{dx^2} + \frac{dv}{dx} \frac{du}{dx} + v \frac{d^2u}{dx^2} + \frac{du}{dx} \frac{dv}{dx}$   
 43. 48.0    45.  $\frac{d^2P}{dt^2} = 80.0$     47.  $-9.80 \text{ m/s}^2$   
 49.  $-\frac{1.60}{(2t+1)^{3/2}}$     51. 0.0490

**Review Exercises for Chapter 23, page 707**

1. -4    3. Does not exist    5. 1    7.  $\frac{1}{4}$     9.  $\frac{2}{3}$   
 11. -2    13. 5    15.  $-4x$     17.  $-\frac{4}{x^3}$     19.  $\frac{1}{2\sqrt{x+5}}$   
 21.  $2x(7x^5-3)$     23.  $\frac{1}{x^2}(2x^{3/2}+3)$     25.  $\frac{3}{(1-5y)^2}$   
 27.  $-28(2-7x)^3$     29.  $\frac{9\pi x}{(5-2x^2)^{7/4}}$   
 31.  $\frac{1}{\sqrt{1+\sqrt{1+\sqrt{1+8s}\sqrt{1+\sqrt{1+8s}\sqrt{1+8s}}}}}$   
 33.  $\frac{-2x-3}{2x^2(4x+3)^{1/2}}$     35.  $\frac{2x-6(2x-3y)^2}{1-9(2x-3y)^2}$     37.  $\frac{5}{48}$   
 39. 13.6    41.  $\frac{2(18x^5-1)}{x^3}$     43.  $\frac{56}{(1+4t)^3}$     45.  $\frac{1}{x^2}$   
 47.  $53.1^\circ$

**B.64 ANSWERS TO ODD-NUMBERED EXERCISES**

49. It appears to be 8, but using trace, there is no value shown for  $x = 2$ .



51. (a) 30.0 m/s (b) 6.00 m/s    53. -31.0    55.  $(0, \frac{1}{3}\sqrt{3})$

57.  $10\,000(1 + 0.250t)^7$     59. (a)  $v = \frac{4}{(1 + 8t)^{1/2}}$   
 (b)  $a = \frac{-16}{(1 + 8t)^{3/2}}$

61. 5    63.  $-k + k^2t - \frac{1}{2}k^3t^2$

65.  $-\frac{2k}{r^3}$     67.  $0.4(0.01t + 1)^2(0.04t + 1)$

69.  $\frac{2R(R + 2r)}{3(R + r)^2}$     71. 833 W    73.  $5k(x^4 + 270x^2 - 190)$

75.  $-\frac{1}{4\pi\sqrt{C}(L + 2)^{3/2}}$     77.  $\frac{-15}{(0.5t + 1)^2}$

79.  $y' = \frac{w}{6EI}(3L^2x - 3Lx^2 + x^3)$

$y'' = \frac{w}{2EI}(L - x)^2$

$y''' = \frac{w}{EI}(x - L)$

$y^{iv} = \frac{w}{EI}$

81.  $p = 2w + \frac{150}{w} \frac{dp}{dw} = 2 - \frac{150}{w^2}$

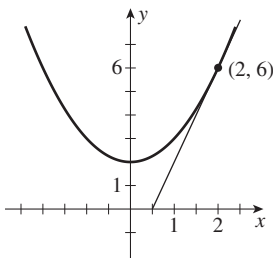
83.  $A = 4x - x^3, \frac{dA}{dx} = 4 - 3x^2$

85. 397 km/h

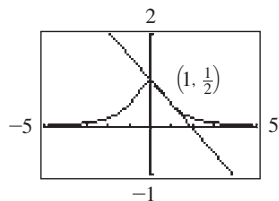
**Exercises 24.1, page 713**

1.  $x + 8y - 17 = 0$

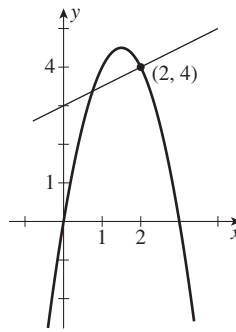
3.  $4x - y - 2 = 0$



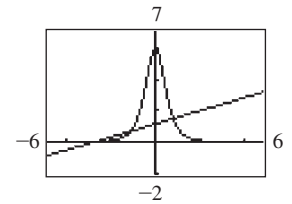
5.  $2y + x - 2 = 0$



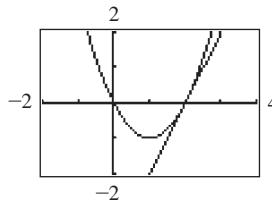
7.  $x - 2y + 6 = 0$



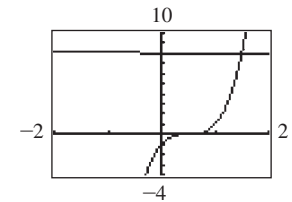
9.  $2x - 6y + 7 = 0$



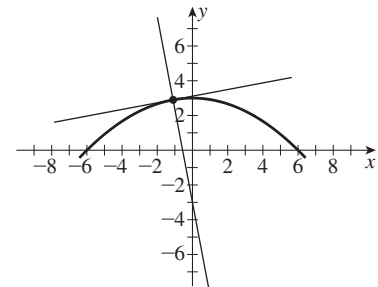
11.  $y = 2x - 4$



13.  $y - 8 = -\frac{1}{24}(x - \frac{3}{2})$ , or  $2x + 48y - 387 = 0$



15.  $2x - 12y + 37 = 0$   
 $72x + 12y + 37 = 0$



17. The line is  $y - x - 1 = 0$ .

19. Take derivatives; evaluate at  $(a, b)$ ; show that product  $m_1m_2 = -4a/b^2 = -1$ .

21.  $y - y_1 = -\frac{x_1}{y_1}(x - x_1); y_1y - y_1^2 = -x_1x + x_1^2;$

$x_1x + y_1y = x_1^2 + y_1^2 = a^2$

23.  $x - 4y + 2 = 0$     25.  $x - 2y - 20 = 0$

27.  $x + y - 6 = 0$

29.  $x + 2y - 3 = 0, x = 0, x - 2y + 3 = 0$

**Exercises 24.2, page 717**

1.  $x_3 = 0.2086$ ; calculator: 0.2087 (to three decimal places)

3. -0.180 460 4    5. 0.585 786 4    7. -3.7321

9. 2.561 552 8    11. -1.236 068    13. 0.917 543 3

15. 0.618 034 0    17. -1.855 772 5, 0.678 362 8, 3.177 409 7

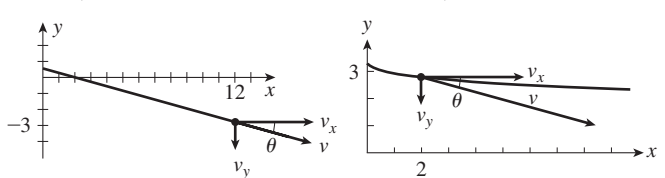
19. Find the real root of  $x^3 - 4 = 0$ ; 1.587 401 1

21.  $x_{n+1} = \frac{1}{2}x_n + \frac{a}{2x_n}$     23.  $x_n = 2x_{n-1} - ax_{n-1}^2, n \geq 2$

25. 4.21 cm    27. 2.214  $\mu$ F, 3.214  $\mu$ F, 4.214  $\mu$ F    29. 0.629 cm

31.  $1.00 \times 10^{-4}$

**Exercises 24.3, page 721**

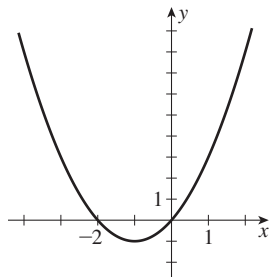
1. 16.5,  $-14.0^\circ$   
 3. 3.16,  $341.6^\circ$   
 5. 8.07,  $352.4^\circ$
- 
7.  $a = 0$     9. 20.0,  $3.68^\circ$     11. 6.42 m/s,  $321^\circ$   
 13. 1.3 m/min<sup>2</sup>,  $288^\circ$     15. 39 m/s,  $310^\circ$  and 7.4 m/s<sup>2</sup>,  $270^\circ$   
 17. 276 m/s,  $43.5^\circ$ ; 2090 m/s,  $16.7^\circ$     19. 1.32 cm/s,  $-24.9^\circ$   
 21. 22.1 m/s<sup>2</sup>,  $25.4^\circ$ ; 20.2 m/s<sup>2</sup>,  $8.5^\circ$     23. 21.2 km/min,  $296.6^\circ$   
 25.  $v_x = 731$  m/min,  $v_y = -690$  m/min    27. 370 m/s,  $19^\circ$

**Exercises 24.4, page 725**

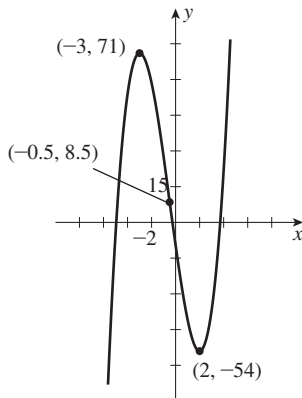
1. 5.20 V/min    3. 23    5. 3    7. 0.031 units/s  
 9. 0.36 m/s    11. 0.0900  $\Omega$ /s    13.  $-0.00251$  s/s  
 15. 330 km/h    17.  $4.1 \times 10^{-6}$  m/s  
 19.  $\frac{dB}{dt} = \frac{-3kr(dr/dt)}{[r^2 + (l/2)^2]^{5/2}}$     21. 0.0056 m/min  
 23. 0.15 mm<sup>2</sup>/month    25.  $-101$  mm<sup>3</sup>/min  
 27.  $\frac{dV}{dt} = -kA$ ;  $\frac{dr}{dt} = -k$     29.  $-4.6$  kPa/min  
 31.  $3.18 \times 10^6$  mm<sup>3</sup>/s    33.  $I = \frac{8k}{x^2}$ ;  $\frac{dI}{dt} = 0.000800k$  unit/s  
 35.  $-0.43$  N/s    37. 0.481 m/min    39. 820 km/h  
 41. 2.71 m/s    43. 2.50 m/s

**Exercises 24.5, page 731**

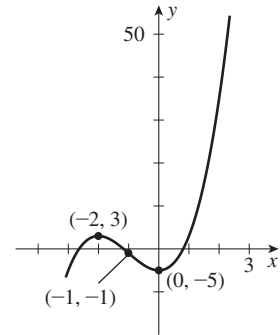
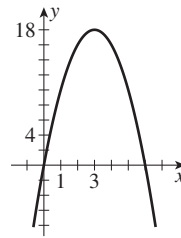
1. Inc.  $x < 0$ ,  $x > 4$ , dec.  $0 < x < 4$   
 3. Conc. down  $x < 0$ , conc. up  $x > 0$ , infl. (0, 0)  
 5. Inc.  $x > -1$ , dec.  $x < -1$   
 7. Inc.  $x < -3$ ,  $x > 2$ ; dec.  $-3 < x < 2$   
 9. Min. (-1, -1)    11. Max. (-3, 71); min. (2, -54)  
 13. Conc. up all  $x$   
 15. Conc. up,  $x > -\frac{1}{2}$ ; conc. down,  $x < -\frac{1}{2}$ ; infl.  $(-\frac{1}{2}, \frac{17}{2})$   
 17.



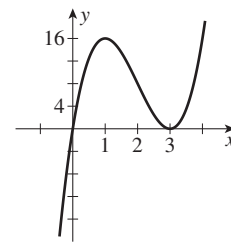
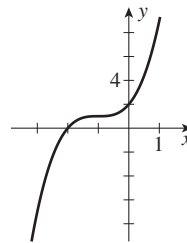
19.



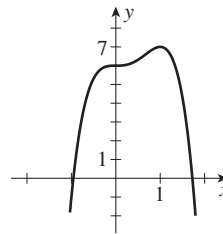
21. Max. (3, 18), conc. down all  $x$   
 23. Max. (-2, 3), min. (0, -5), infl. (-1, -1)



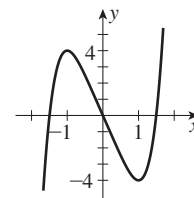
25. No max. or min., infl. (-1, 1)  
 27. Max. (1, 16), min. (3, 0), infl. (2, 8)



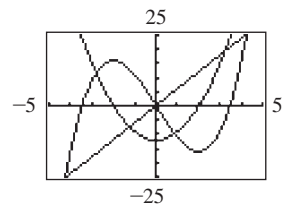
29. Max. (1, 7), infl. (0, 6),  $(\frac{2}{3}, \frac{178}{27})$



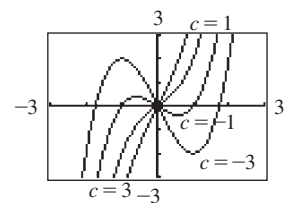
31. Max. (-1, 4), min. (1, -4), infl. (0, 0)



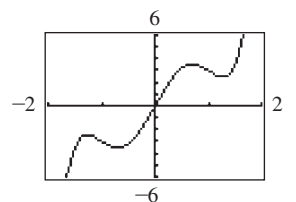
33. Where  $y' > 0$ ,  $y$  inc.  
 $y' = 0$ ,  $y$  has a max. or min.  
 $y' < 0$ ,  $y$  dec.  
 $y'' > 0$ ,  $y$  conc. up  
 $y'' = 0$ ,  $y$  has infl.  
 $y'' < 0$ ,  $y$  conc. down



35. Curve has a maximum and a minimum for  $c < 0$  but is always increasing for  $c > 0$ .

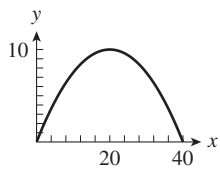


37. The left relative maximum point is above the left relative minimum point but below the right relative minimum point.

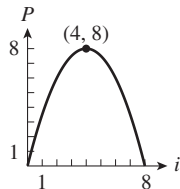


**B.66** ANSWERS TO ODD-NUMBERED EXERCISES

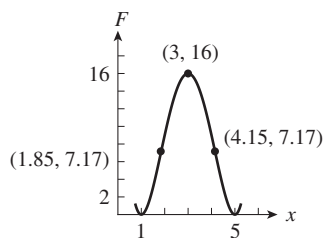
39. Max. (20, 10)



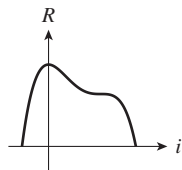
41. Max. (4, 8)



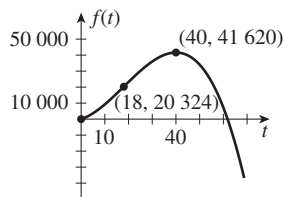
43. Max. (3, 16)



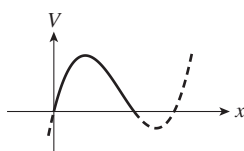
45. Max. (0, 75),  
infl. (1, 64), (3, 48)



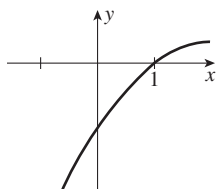
47. Max. (40, 41 620)



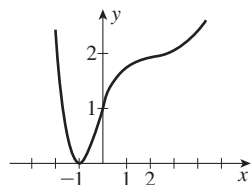
49.  $V = 4x^3 - 40x^2 + 96x$ ,  
max. (1.57, 67.6),  
infl.  $(\frac{10}{3}, 23.7)$



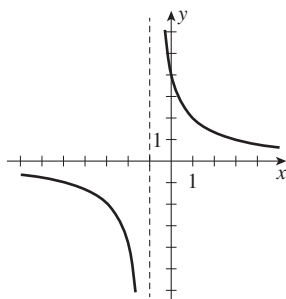
51.



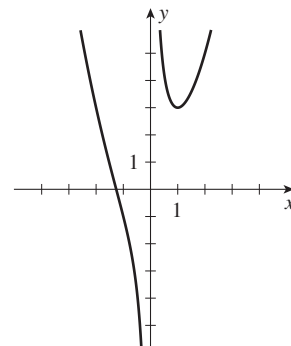
53.



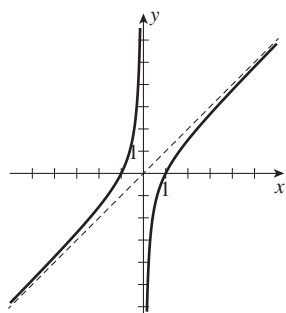
3. Dec.  $x < -1$ ,  $x > -1$ ,  
conc. up  $x > -1$ ,  
conc. down  $x < -1$ ,  
int. (0, 2),  
asym.  $x = -1$ ,  $y = 0$



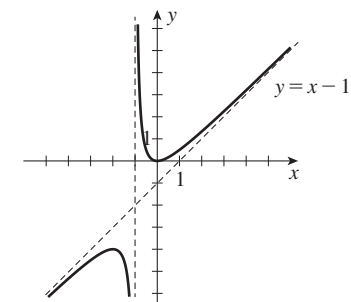
5. Int.  $(-\sqrt[3]{2}, 0)$ , min. (1, 3),  
infl.  $(-\sqrt[3]{2}, 0)$ , asym.  $x = 0$



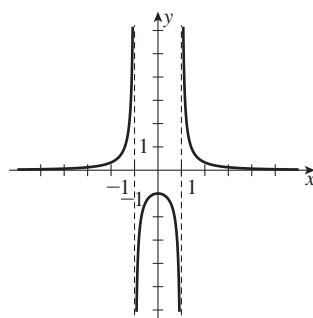
7. Int. (1, 0), (-1, 0),  
asym.  $x = 0$ ,  $y = x$ ,  
conc. up  $x < 0$ ,  
conc. down  $x > 0$



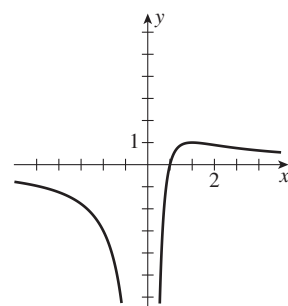
9. Int. (0, 0), max. (-2, -4),  
min. (0, 0), asym.  $x = -1$



11. Int. (0, -1),  
max. (0, -1),  
asym.  $x = 1$ ,  
 $x = -1$ ,  $y = 0$

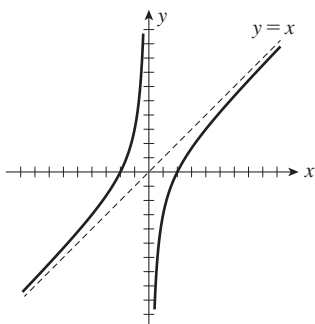


13. Int. (1, 0), max. (2, 1),  
infl.  $(3, \frac{8}{9})$ ,  
asym.  $x = 0$ ,  $y = 0$

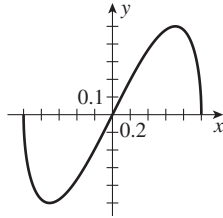


**Exercises 24.6, page 736**

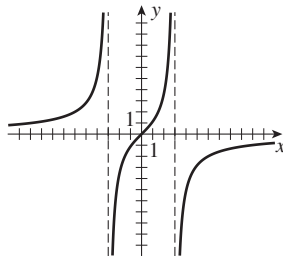
1.  $y = x - \frac{4}{x}$ ,  
int. (2, 0), (-2, 0);  
inc. all  $x$ , except  $x = 0$ ;  
conc. up  $x < 0$ ,  
conc. down  $x > 0$



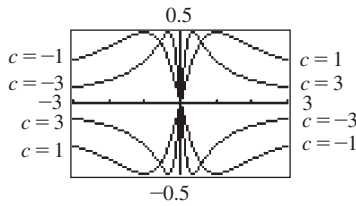
15. Int. (0, 0), (1, 0), (-1, 0),  
 max.  $(\frac{1}{2}\sqrt{2}, \frac{1}{2})$ ,  
 min.  $(-\frac{1}{2}\sqrt{2}, -\frac{1}{2})$



17. Int. (0, 0), infl. (0, 0),  
 asym.  $x = -3$ ,  
 $x = 3, y = 0$

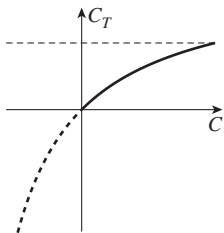


19. As  $c$  goes from  $-3$  to  $+3$ ,  
 the graph goes from second  
 and fourth quadrants to the  
 third and first quadrants.

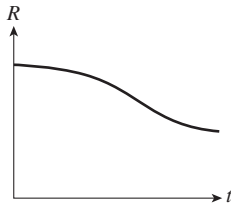


21. Dividing simplifies analysing  
 behaviour as  $x$  becomes large  
 and finding the range.

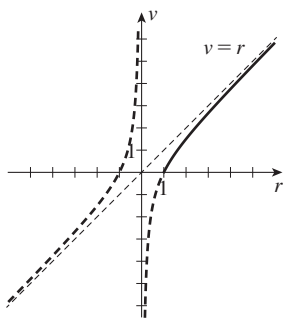
23. Int. (0, 0), asym.  $C_T = 6$ ,  
 inc.  $C \geq 0$ ,  
 conc. down  $C \geq 0$



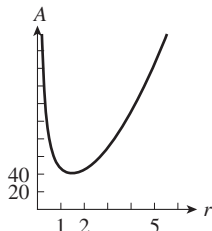
25. Int. (0, 1), max. (0, 1),  
 infl. (141, 0.82),  
 asym.  $R = 0$



27. Int. (1, 0), (-1, 0);  
 inc. all  $x$ , except  $x = 0$ ;  
 conc. up  $x < 0$ ,  
 conc. down  $x > 0$



29.  $A = 2\pi r^2 + \frac{40}{r}$ ,  
 min. (1.47, 40.8),  
 asym.  $r = 0$

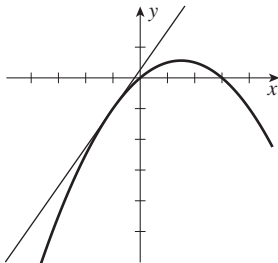
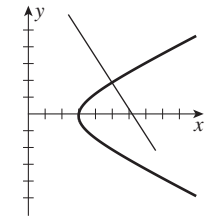


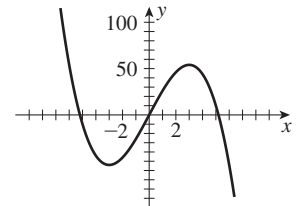
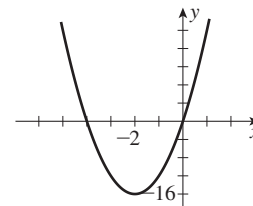
17. 12 cm by 12 cm    19. 5 mm, 5 mm  
 21.  $xy = A, p = 2x + 2y; p = 2x + \frac{2A}{x}; \frac{dp}{dx} = 2 - \frac{2A}{x^2}$ ,  
 $x = \sqrt{A}, y = \sqrt{A}; x = y$  (square)  
 23. 1.10 h    25. 8.49 cm, 8.49 cm  
 27.  $r = 0.5$  cm,  $l = 1$  cm,  $V = 0.8$  m<sup>3</sup>    29. 24.0 cm, 36.0 cm  
 31. 1.20 m    33. 2 cm    35. 12    37. 0.58L    39. 38.0°  
 41. 8/3 km from A    43. 3.3 cm    45. 100 m  
 47. 1250 units per week    49. 8.0 km from refinery  
 51. 59.2 m, 118 m

**Exercises 24.8, page 746**

1.  $\frac{-8(t^3 - 2)}{(t^3 + 4)^2} dt$   
 3.  $\frac{dx}{x} = 0.0065 = 0.65\%; \frac{dV}{V} = 0.019 = 1.9\%$   
 5.  $(5x^4 + 1)dx$     7.  $\frac{-10dr}{r^6}$     9.  $48t(3t^2 - 5)^3 dt$   
 11.  $\frac{-72xdx}{(3x^2 + 1)^2}$     13.  $x(1-x)^2(-5x+2)dx$     15.  $\frac{2dx}{(5x+2)^2}$   
 17. 12.28, 12    19. 0.626 49, 0.6257  
 21.  $L(x) = 2x$     23.  $L(x) = -2x - 3$     25. 1570 km  
 27. 0.246%    29. -31.3 nm    31.  $\frac{dr}{r} = \frac{1}{2} \cdot \frac{d\lambda}{\lambda}$   
 33.  $\frac{dA}{A} = \frac{2dr}{r}$     35. 2.0125  
 37.  $L(x) = 1^k + k(1^{k-1})x = 1 + kx$   
 39.  $L(x) = -\frac{1}{2}x + \frac{3}{2}; 1.45$     41.  $L(V) = 1.73 - 0.133V$

**Review Exercises for Chapter 24, page 748**

1.  $5x - y + 1 = 0$     3.  $8x + 5y - 50 = 0$
- 
- 
5.  $x - 2y + 3 = 0$     7. 4.19, 72.6°    9. 2.12  
 11. 2.00, 90.9°    13. 0.745 898 3    15. 1.4422  
 17. Min. (-2, -16),  
 conc. up all  $x$     19. Int. (0, 0),  $(\pm 3\sqrt{3}, 0)$ ;  
 max. (3, 54), min. (-3, -54);  
 infl. (0, 0)

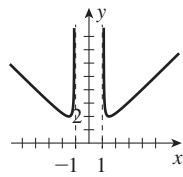
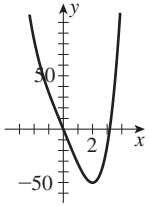


**Exercises 24.7, page 741**

1. 360 000 m<sup>2</sup>    3. 60 m    5.  $\frac{E}{2R}$     7. 0  
 9. 35 m<sup>2</sup>, \$8300    11.  $\frac{L}{R^2 + \omega^2 L^2}$     13. 1.94 units    15.  $a$

**B.68 ANSWERS TO ODD-NUMBERED EXERCISES**

21. Min.  $(2, -48)$ ;  
conc. up  $x < 0, x > 0$ ,  
int.  $(0, 0), (2\sqrt[3]{4}, 0)$
23. Min.  $(\pm\sqrt{2}, 2)$ ;  
asym.  $x = \pm 1$ ;  
conc. up  $x < 1, x > 1$



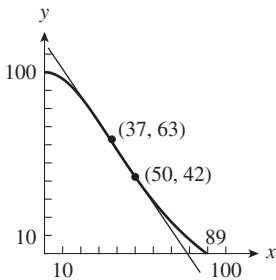
25.  $\left(12x^2 - \frac{1}{x^2}\right)dx$     27.  $\frac{(1-4x)dx}{(1-3x)^{2/3}}$     29. 0.244
31.  $L(x) = \frac{1}{3}(11x - 4)$     33.  $1.85 \text{ m}^3$     35.  $45.2 \text{ m}^3$

37.  $\frac{RdR}{R^2 + X^2}$     39. 251.1    41.  $2x - y + 1 = 0$

43. 0.0 m, 6.527 m    45. 8.8 m/s,  $336^\circ$     47. 0.23 m/s
49.  $-7.44 \text{ cm/s}$     51.    53.  $22 \text{ m}^3/\text{s}$

55.    57.  $-0.113 \text{ m/min}$

59.  $38\,000 \text{ m}^2/\text{min}$     61. 7.30 ft/s
63. Max.  $(0, 100)$ ; infl.  $(37, 63)$ ;    65. 1160 km/h  
int.  $(0, 100), (89, 0)$ ,  
 $L(x) = 113 - 1.42x$



67.  $6 \mu\text{F}, 6 \mu\text{F}$     69.  $13.5 \text{ dm}^3$     71. 3 km    73. 6.56 cm
75. 0.153 m/min    77.  $r = 2.09 \text{ cm}; h = 23.0 \text{ cm}$
79.  $l = 4.67 \text{ m}, w = 3.50 \text{ m}, h = 2.34 \text{ m}$

**Exercises 25.1, page 754**

1.  $F(x) = 3x^4$     3.  $F(x) = \frac{2}{3}x^{3/2} + \frac{1}{x^3}$     5. 1    7. 3
9. 6    11. -1    13.  $x^{5/2}$     15.  $\frac{3}{2}t^4 + 12t$     17.  $\frac{2}{3}x^3 - \frac{1}{2}x^2$
19.  $\frac{4}{3}x^{3/2} + 3x$     21.  $\frac{7}{5x^5}$     23.  $2v^2 + 3\pi^2v$
25.  $\frac{1}{3}x^3 - 4x - \frac{1}{x}$     27.  $(2x + 1)^6$     29.  $(p^2 - 1)^4$
31.  $\frac{1}{40}(2x^4 + 1)^5$     33.  $(6x + 1)^{3/2}$     35.  $\frac{1}{4}(3x + 1)^{4/3}$
37.  $\frac{1}{2x + 1}$
39.  $(2x + 5)^3$  is an antiderivative of  $6(2x + 5)^2$ . Factor of 2 from  $2x + 5$  is needed.

**Exercises 25.2, page 759**

1.  $4x^2 + C$     3.  $\frac{(x^2 + 1)^5}{5} + C$     5.  $x^2 + C$     7.  $\frac{1}{8}x^8 + C$
9.  $\frac{16}{5}x^{5/2} + C$     11.  $-\frac{3}{R^3} + C$     13.  $\frac{1}{3}x^3 - \frac{1}{6}x^6 + C$
15.  $3x^3 + \frac{1}{2}x^2 + 3x + C$     17.  $\frac{1}{6}t^3 + \frac{2}{t} + C$
19.  $\frac{2}{7}x^{7/2} - \frac{2}{5}x^{5/2} + C$     21.  $6x^{1/3} + \frac{1}{9}x + C$
23.  $x + 8x^3 + \frac{144}{5}x^5 + C$     25.  $\frac{1}{6}(x^2 - 1)^6 + C$
27.  $\frac{1}{5}(x^4 + 3)^5 + C$     29.  $\frac{1}{80}(2\theta^5 + 5)^8 + C$
31.  $\frac{1}{12}(8x + 1)^{3/2} + C$     33.  $\frac{1}{6}\sqrt{6x^2 + 1} + C$
35.  $4\sqrt{z^2 - 2z} + C$     37.  $y = 2x^3 + 2$
39.  $y = 5 - \frac{1}{18}(1 - x^3)^6$
41. No. The result should include the constant of integration.
43. No. With  $u = 4x^3 + 3$ ,  $du = 12x^2dx$ , and the  $x^4$  would have to be  $x^2$ .
45. No. There should be a factor of  $1/2$  in the result.
47.  $12y = 83 + (1 - 4x^2)^{3/2}$     49.  $\frac{4}{15}x^{5/2} + C_1x + C_2$
51.  $V = 0.14t + 0.00014t^2$     53.  $i = 2t^2 - 0.2t^3 + 2$
55.  $T = 2250(r + 1)^{-2} + 250$     57.  $f = \sqrt{0.01A + 1} - 1$
59.  $y = 3x^2 + 2x - 3$

**Exercises 25.3, page 764**

1. (a) 3    (b)  $\frac{15}{4}$     3. 20    5. 9, 12.2    7. 1.92, 2.28
9. 7.62, 8.21    11. 0.464, 0.599    13. 1.92, 1.96
15. 13.5    17.  $\frac{8}{3}$     19. 9    21. 0.8    23. 2
25. 14.85; the extra area above  $y = 3x$  using circumscribed rectangles is the same as the omitted area under  $y = 3x$  using inscribed rectangles.
27. 3.08; the extra area above  $y = x^2$  using circumscribed rectangles is greater than the omitted area under  $y = x^2$  using inscribed rectangles.

**Exercises 25.4, page 767**

1.  $-\frac{9}{4}$     3. 1    5.  $\frac{254}{7}$     7.  $2\sqrt{6} - 2\sqrt{3} - 21 = -19.57$
9. 2.53
11.  $-\frac{32}{3}$     13.  $\frac{33\sqrt{3}\sqrt{11}}{20} - \frac{3\sqrt{3}\sqrt{1}}{8} - \frac{17}{5} = -1.55$     15.  $\frac{4}{3}$
17.  $-\frac{81}{4}$     19. 2    21.  $\frac{1}{4}(20.5^{2/3} - 17.5^{2/3}) = 0.188$
23.  $\frac{176}{1083} = 0.162$     25. 84    27.  $\frac{364}{3}$     29.  $\frac{33}{784} = 0.0421$
31.  $\frac{3880}{9}$     33.  $\frac{2}{3}(13\sqrt{13} - 27) = 13.2$
35.  $\frac{1}{4} + \frac{15}{4} = 4$ ; under  $y = x^3$ , the area from  $x = 0$  to  $x = 1$  plus the area from  $x = 1$  to  $x = 2$  equals the area from  $x = 0$  to  $x = 2$ .
37.  $\frac{28}{3}$
39.  $\frac{1}{2} > \frac{1}{3}$  and  $\frac{3}{2} < \frac{7}{3}$ ; from  $x = 0$  to  $x = 1$  the area under  $y = x$  is greater than the area under  $y = x^2$ . Between  $x = 1$  and  $x = 2$ , the area under  $y = x$  is less than the area under  $y = x^2$ .
41.  $\frac{2}{2k + 1}$     43.  $\frac{A\sqrt{2}}{a\sqrt{g}}(\sqrt{h} - \sqrt{H})$
45.  $64\,000 \text{ N} \cdot \text{m}$     47.  $86.8 \text{ m}^2$     49.  $\frac{3NE_F}{5}$



**Exercises 25.5, page 770**

1.  $\frac{7}{6}$     3.  $\frac{11}{2} = 5.50, \frac{16}{3} = 5.33$     5.  $7.66, \frac{23}{3} = 7.67$   
 7. 0.204    9. 19.0    11. 0.520    13. 21.7    15. 45.4  
 17. The tops of all trapezoids are below  $y = 1 + \sqrt{x}$ .  
 19. 0.702,  $\ln 2 = 0.693$     21. 100.027 m

**Exercises 25.6, page 774**

1. 0.406    3. (a) 6 (b) 6    5. (a) 19.7 (b) 19.7  
 7. 19.3    9. 0.511    11. 13.1    13. 44.6    15. 3.12  
 17. 1.19 cm

**Review Exercises for Chapter 25, page 775**

1.  $x^4 - \frac{1}{2}x^2 + C$     3.  $\frac{2}{7}u^{7/2} + \frac{16}{3}u^{3/2} + C$     5.  $\frac{19}{3}$   
 7.  $\frac{16}{3}$     9.  $5x - \frac{3}{x^2} + C$     11. 3  
 13.  $\frac{1}{10(9-5n)^2} + C$     15.  $-\frac{6}{7}(7-2x)^{7/4} + C$   
 17.  $\frac{9}{8}(3\sqrt[3]{3}-1)$     19.  $-\frac{1}{30}(1-2x^3)^5 + C$   
 21.  $-\frac{1}{2x-x^3} + C$     23.  $\frac{3350}{3}$     25.  $y = 3x - \frac{1}{3}x^3 + \frac{17}{3}$

27. (a)  $x - x^2 + C_1$

(b)  $-\frac{1}{4}(1-2x)^2 + C_2 = x - x^2 + C_2 - \frac{1}{4}; C_1 = C_2 - \frac{1}{4}$

29.  $\int_3^4 F(v) dv$     31.  $\frac{1}{27}(6x+5)^{3/2} + C_1x + C_2$

33.  $0.25 = 0.25; y = x^3$  shifted 1 unit to the right is  $y = (x-1)^3$ .  
 Therefore, areas are the same.

35. 22    37. The graph of  $f(x)$  is concave down. This means the tops of the trapezoids are below  $y = f(x)$ .

39. 0.842    41. 0.811    43. 13.6    45. 19.30

47. 19.04    49. 25.7 m<sup>2</sup>    51. 25.8 m<sup>2</sup>

53. -14.4 m

55.  $Q = \frac{k}{12R}(4r^3R - 3r^4 - R^4) + Q_0$

57. 66.4 m

**Exercises 26.1, page 781**

1. -19.6 m/s    3. -15 m/s    5.  $s = 8.00 - 0.25t$     7. 5.0 m/s

9. 1.4 m    11. 12 m/s<sup>2</sup>    13. 24 m/s    15. 85 m

17. 0.345 nC    19. 0.015 C    21. 120 V    23. 4.68 mV

25. 970 rad    27. 67 A    29.  $\frac{k}{x_1}$

31.  $m = 1002 - 2\sqrt{t+1}, 2.51 \times 10^5$  min

**Exercises 26.2, page 786**

1.  $\frac{26}{3}$     3. 2    5.  $\frac{27}{8}$     7.  $\frac{32}{3}$     9.  $\frac{1}{6}$     11.  $\frac{26}{3}$

13. 3    15.  $\frac{9}{2}$     17.  $\frac{8}{3}$     19.  $\frac{7}{6}$     21.  $\frac{256}{15}$     23.  $\frac{343}{24}$

25.  $\frac{65}{6}$     27.  $\frac{49}{3}$     29. 4

31. The area bounded by  $x = 1, y = 2x^2$ , and  $y = x^3$  or the area bounded by  $x = 1, y = 0$ , and  $y = 2x^2 - x^3$ .

33. 20    35.  $\frac{A_1}{A_2} = \frac{1 - \frac{1}{n+1}}{\frac{1}{n+1}} = \frac{n}{n+1} \frac{n+1}{1} = \frac{n}{1}$

37.  $4^{2/3}$     39. 1    41.  $\frac{48}{5}$     43. 18 J    45. 80.8 km  
 47. 4 cm<sup>2</sup>    49. 42.5 dm<sup>2</sup>

**Exercises 26.3, page 792**

1.  $\frac{128\pi}{7}$     3.  $\frac{8}{3}\pi$     5.  $\frac{8}{3}\pi$     7.  $9\pi$     9.  $72\pi$     11.  $\frac{768}{7}\pi$

13.  $\frac{348}{5}\pi$     15.  $\frac{16}{3}\pi$     17.  $\frac{128}{7}\pi$     19.  $\frac{243}{80}\pi$     21.  $\frac{10\pi}{3}\sqrt{5}$

23.  $\frac{1296}{5}\pi$     25.  $\frac{16}{3}\pi$

27. The region bounded by  $y = x^{3/2}, y = 0, x = 1$ , and  $x = 2$

29.  $\frac{128}{3}\pi$     31.  $\frac{1}{3}\pi r^2 h$     33.  $\frac{1}{5}$     35. 16 800 m<sup>3</sup>

37. 18.3 cm<sup>3</sup>

**Exercises 26.4, page 798**

1.  $(0, \frac{8}{3})$     3. 2.9 cm    5. 1.2 cm    7. (-0.5 cm, 0.5 cm)

9.  $(\frac{6}{5}, \frac{192}{35})$     11.  $(0, \frac{6}{5})$     13.  $(\frac{4}{3}, \frac{4}{3})$

15. (1.20, 5.49)    17.  $(\frac{5}{3}, 4)$     19.  $(\frac{4}{3}, \frac{8}{3})$     21.  $(0, \frac{3}{5}a)$

23.  $(\frac{7}{8}, 0)$     25.  $(0, \frac{5}{6})$     27.  $(\frac{2}{3}, 0)$

29.  $(\frac{2}{3}b, \frac{1}{3}a)$ . Place triangle with a vertex at origin, side  $b$  and right angle on  $x$ -axis. Equation of hypotenuse is  $y = ax/b$ . Use Eqs. (26.16) and (26.17).

31. 0.375 cm above centre of base    33. 19.3 cm from larger base

**Exercises 26.5, page 803**

1.  $I_y = \rho, R_y = \frac{1}{2}\sqrt{2}$     3. 68 g · cm<sup>2</sup>, 2.9 cm

5. 2530 g · cm<sup>2</sup>, 3.58 cm    7.  $\frac{2}{3}\rho$     9.  $\frac{162}{5}\rho$     11.  $\sqrt{6}$

13.  $\frac{1}{6}mb^2$     15.  $\frac{9}{7}\sqrt{7}$     17.  $\frac{8}{11}\sqrt{55}$     19.  $\frac{16}{3}\pi\rho$     21.  $\frac{4}{5}\sqrt{10}$

23.  $\frac{3}{10}mr^2$     25. 0.324 g · cm<sup>2</sup>    27. 31.2 kg · cm<sup>2</sup>

**Exercises 26.6, page 808**

1. 41 N · cm    3. 176 kN    5. 8.0 N · cm    7. 32.4 N · m

9.  $1.7 \times 10^{-16}$  J    11. 0.09k N · m    13. 50 000 N · m

15.  $3.00 \times 10^6$  kN · m    17.  $9.37 \times 10^5$  N · km

19.  $9.8 \times 10^5$  N · m    21. 12.5 kN    23. 152 kN

25.  $1.72 \times 10^5$  N    27. 1.84 MN

29.  $3.92 \times 10^4$  N,  $1.18 \times 10^5$  N buoyant force    31. 2.7 A

33. 35.3%    35. 109 m    37.  $S = \pi r\sqrt{r^2 + h^2}$

**Review Exercises for Chapter 26, page 811**

1. 0.708 m    3. 4.7 s    5. No    7. 0.44 C    9. 55 V

11.  $y = 20x + \frac{1}{120}x^3$     13. 18    15. 18    17.  $\frac{27}{4}$

19.  $A_{\text{top}}/A_{\text{bot}} = \left(\frac{2n}{2n+1}\right) / \left(\frac{1}{2n+1}\right) = \frac{2n}{1}$

21.  $\frac{48}{5}\pi$     23.  $\frac{512}{5}\pi$     25.  $\frac{4}{3}\pi ab^2$     27. (-0.5 cm, 0.6 cm)

29.  $(\frac{30}{7}, \frac{45}{4})$     31.  $(\frac{14}{5}, 0)$     33.  $\frac{8}{5}\rho$     35. 68.7 g · mm<sup>2</sup>

37. 2700 N · m    39. 4790 cm<sup>2</sup>    41. -18 m/s    43. 1.8 m

45. 47 m<sup>3</sup>    47. 88 m<sup>3</sup>    49. 1580 kN    51. 0.29  $\Omega$

53.  $5 \times 10^6$  m/s    55. 0.1359

**B.70 ANSWERS TO ODD-NUMBERED EXERCISES**

**Exercises 27.1, page 818**

1.  $8\theta \sin 2\theta^2 \cos 2\theta^2 = 4\theta \sin 4\theta^2$     3.  $\cos(x + 2)$   
 5.  $12x^2 \cos(2x^3 - 1)$     7.  $-3 \sin \frac{1}{2}x$     9.  $-6 \sin(3x - \pi)$   
 11.  $6\pi \sin 3\pi\theta \cos 3\pi\theta = 3\pi \sin 6\pi\theta$   
 13.  $-45 \cos^2(5x + 2) \sin(5x + 2)$   
 15.  $\sin 3x + 3x \cos 3x$     17.  $9x^2 \cos 5x - 15x^3 \sin 5x$   
 19.  $6v \cos(5v) \cos v^2 - 15 \sin(5v) \sin v^2$   
 21.  $\frac{2 \cos 4x}{\sqrt{1 + \sin 4x}}$     23.  $\frac{3t \cos(3t - \pi/3) - \sin(3t - \pi/3)}{2t^2}$   
 25.  $\frac{4x(1 - 3x) \sin x^2 - 6 \cos x^2}{(3x - 1)^2}$   
 27.  $4 \sin 3x(3 \cos 3x \cos 2x - \sin 3x \sin 2x)$   
 29.  $2 \cos 2t \cos(\sin 2t)$     31.  $3 \sin^2 x \cos x + 2 \sin 2x$   
 33.  $-\frac{\cos s}{\sin^2 s} + \frac{\sin s}{\cos^2 s}$

35. (a)  (b) See the table.

37. (a) 0.540 302 3, value of derivative

- (b) 0.540 260 2, slope of secant line

39. Resulting curve is  $y = \cos x$ .

41.  $\frac{2x - y \cos xy}{x \cos xy - 2 \sin 2y}$

43.  $\frac{d \sin x}{dx} = \cos x, \frac{d^2 \sin x}{dx^2} = -\sin x,$

$\frac{d^3 \sin x}{dx^3} = -\cos x, \frac{d^4 \sin x}{dx^4} = \sin x$

45.  $\sin 2x = 2 \sin x \cos x$     47. 0.515    49. -2.65

51. 412 V/s    53. -199 cm/s    55. -38.5 km

**Exercises 27.2, page 821**

1.  $12x \sec^2 x^2 \tan x^2$     3.  $5 \sec^2 5x$     5.  $5 \csc^2(0.25\pi - \theta)$

7.  $15 \sec 5v \tan 5v$     9.  $\frac{3}{\sqrt{2x+3}} \csc \sqrt{2x+3} \cot \sqrt{2x+3}$

11.  $30\pi \tan 3\pi t \sec^2 3\pi t$     13.  $-4 \cot^3 \frac{1}{2}x \csc^2 \frac{1}{2}x$

15.  $2 \tan 4x \sqrt{\sec 4x}$     17.  $-84 \csc^4 7x \cot 7x$

19.  $0.5t^2 \sec^2 0.5t + 2t \tan 0.5t$

21.  $-4 \csc x^2(2x \cos x \cot x^2 + \sin x)$

23.  $\frac{\csc x(x \cot x + 1)}{x^2}$

25.  $\frac{2(-4 \sin 4x - 4 \sin 4x \cot 3x + 3 \cos 4x \csc^2 3x)}{(1 + \cot 3x)^2}$

27.  $\sec^2 x(\tan^2 x - 1)$     29.  $2\pi \cos 2\pi\theta \sec^2(\sin 2\pi\theta)$

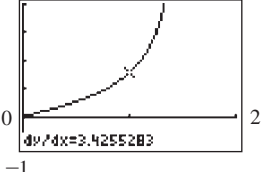
31.  $\frac{1 + 2 \sec^2 4x}{\sqrt{2x + \tan 4x}}$     33.  $\frac{2 \cos 2x - \sec y}{x \sec y \tan y - 2}$

35.  $24 \tan 3x \sec^2 3x dx$

37.  $4 \sec 4x(\tan^2 4x + \sec^2 4x) dx$

39. (a) 3.425 518 8, value of derivative

- (b) 3.426 052 4, slope of secant line

41. (a) 

- (b) 

43.  $2 \tan x \sec^2 x = 2 \sec x(\sec x \tan x)$     45. -12

47.  $2 \sec^2 x - \sec x \tan x = \frac{2}{\cos^2 x} - \frac{\sin x}{\cos^2 x}$

49. -8.4 cm/s    51. 140 m/s

**Exercises 27.3, page 824**

1.  $\frac{2x}{\sqrt{1-x^4}}$     3.  $\frac{7}{\sqrt{1-49x^2}}$     5.  $\frac{18x^2}{\sqrt{1-9x^6}}$

7.  $\frac{-1.8}{\sqrt{1-0.25s^2}}$     9.  $\frac{1}{\sqrt{(x-1)(2-x)}}$     11.  $\frac{4}{\sqrt{s(1+s)}}$

13.  $-\frac{6}{x^2+1}$     15.  $\frac{5x}{\sqrt{1-x^2}} + 5 \sin^{-1} x$

17.  $\frac{0.8u}{1+4u^2} + 0.4 \tan^{-1} 2u$

19.  $\frac{3\sqrt{1-4R^2} \sin^{-1} 2R - 6R + 2}{\sqrt{1-4R^2}(\sin^{-1} 2R)^2}$

21.  $\frac{2(\cos^{-1} 2x + \sin^{-1} 2x)}{\sqrt{1-4x^2}(\cos^{-1} 2x)^2}$     23.  $\frac{-24(\cos^{-1} 4x)^2}{\sqrt{1-16x^2}}$

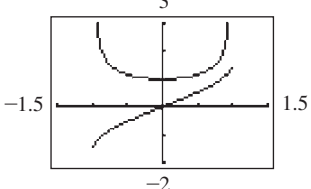
25.  $\frac{4 \sin^{-1}(4t+3)}{\sqrt{-(4t^2+6t+2)}}$     27.  $\frac{-1}{t^2+1}$     29.  $\frac{-2(2x+1)^2}{(1+4x^2)^2}$

31.  $\frac{18(4 - \cos^{-1} 2x)^2}{\sqrt{1-4x^2}}$     33.  $-\frac{x^2y^2 + 2y + 1}{2x}$

35. (a) 1.154 700 5, value of derivative

- (b) 1.154 739 0, slope of secant line

37.  $\frac{3(\sin^{-1} x)^2 dx}{\sqrt{1-x^2}}$     39. 0.41    41.  $\frac{2}{(1+x^2)^2}$

43.     45.  $\frac{-16x}{(1+4x^2)^2}$

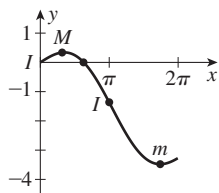
47. Let  $y = \sec^{-1} u$ ; solve for  $u$ ; take derivatives; substitute.

49.  $\frac{E-A}{\omega m \sqrt{m^2 E^2 - (A-E)^2}}$     51.  $-\frac{R}{R^2 + (X_L - X_C)^2}$

53.  $\theta = \tan^{-1} \frac{h}{x}; \frac{d\theta}{dx} = \frac{-h}{x^2 + h^2}$

**Exercises 27.4, page 828**

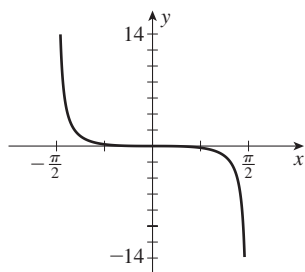
1. Max.  $(\frac{\pi}{3}, 0.34)$ ;  
 min.  $(\frac{5\pi}{3}, -3.48)$ ;  
 infl.  $(0, 0), (\pi, -\frac{\pi}{2})$



3.  $d \sin x/dx = \cos x$  and  $d \cos x/dx = -\sin x$ , and  $\sin x = \cos x$  at points of intersection.

5.  $\frac{1}{x^2 + 1}$  is always positive.

7.



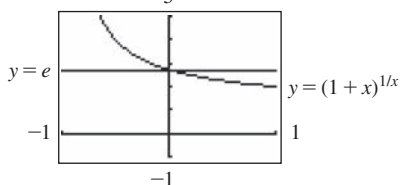
Dec.  $0 < x < \frac{\pi}{2}, -\frac{\pi}{2} < x < 0$ ;  
 infl.  $(0, 0)$ ;  
 asym.  $x = \frac{\pi}{2}, x = -\frac{\pi}{2}$

9.  $y = 1.10x - 0.29$     11. 1.933 753 8    13. -10  
 15.  $32 \sin 2t \cos 2t$     17. 0.58 m/s, -1.7 m/s<sup>2</sup>  
 19. -0.0718 N/s    21. 358 cm/s, 18.0°    23. 17.4 cm/s<sup>2</sup>, 176°  
 25. -0.085 rad/s    27. 8.08 m/s    29. 14.0°    31. 282 km  
 33. 1.06 L/s    35.  $w = 9.24$  cm,  $d = 13.1$  cm    37. 4.21 m

**Exercises 27.5, page 833**

1.  $-4 \tan 4x$     3.  $\frac{2 \log e}{x}$     5.  $\frac{4 \log_5 e}{x - 3}$     7.  $\frac{8}{x - 3}$   
 9.  $\frac{4 \sec^2 2x}{\tan 2x} = 4 \sec 2x \csc 2x$     11.  $\frac{1}{2T}$     13.  $\frac{3(1 - 2x)}{x - x^2}$   
 15.  $\frac{6(t + 2)(t + \ln t^2)}{t}$     17.  $3 \ln(6 - x) - \frac{3x}{6 - x}$   
 19.  $\frac{1}{x \ln x}$     21.  $-\pi\theta \tan(\pi\theta^2)$     23.  $\frac{\cos \ln x}{x}$   
 25.  $6 \ln 2v + 3 \ln^2 2v$     27.  $\frac{x \sec^2 x + \tan x}{x \tan x}$     29.  $\frac{v + 4}{v(v + 2)}$   
 31.  $\frac{\sqrt{x^2 + 1}}{x}$     33.  $\frac{x + y - 2 \ln(x + y)}{x + y + 2 \ln(x + y)}$

35. 0.5 is value of derivative; 0.499 987 5 is slope of secant line.  
 37. (a) 5 (b) See the table.



39. -0.3640    41. 0.976    43.  $L(x) = 4x - \pi$     45. 2.73  
 47.  $x^x(\ln x + 1)$ . In Eq. (23.15) the exponent is constant. For  $x^x$ , both the base and the exponent are variables.  
 49.  $\ln(x^2)$  is defined for all  $x, x \neq 0$ .

Therefore  $\frac{dy_1}{dx} \Big|_{x=-1} = \frac{2}{x} \Big|_{x=-1} = -2$ .

$\ln x$  is not defined for  $x \leq 0$ . Therefore  $\frac{dy_2}{dx} \Big|_{x=-1} = \frac{2}{x} \Big|_{x=-1}$  does not exist.

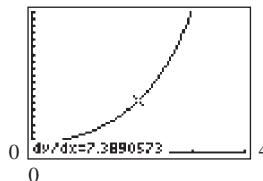
51.  $\frac{10 \log e}{I} \frac{dI}{dt}$     53.  $\frac{x}{x^2 + 1 + \sqrt{1 + x^2}} - \frac{1}{x} + \frac{x}{\sqrt{1 - x^2}}$   
 55.  $0.125 \text{ s}^2/\text{m}$

**Exercises 27.6, page 836**

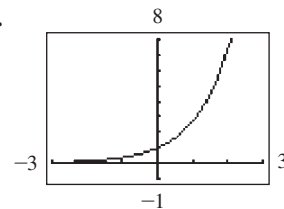
1.  $2e^{2x} \cot e^{2x}$     3.  $(6 \ln 4)4^{6x}$     5.  $\frac{3e^{\sqrt{x}}}{\sqrt{x}}$     7.  $4e^{2t}(3e^t - 2)$   
 9.  $e^{-T}(1 - T)$     11.  $e^{\sin x}(x \cos x + 1)$     13.  $8e^{-4s}$   
 15.  $e^{-3x}(4 \cos 4x - 3 \sin 4x)$     17.  $\frac{2e^{3x}(12x + 5)}{(4x + 3)^2}$   
 19.  $\frac{te^{t^2}}{e^{t^2} + 4}$     21.  $16e^{6x}(x \cos x^2 + 3 \sin x^2)$   
 23.  $\frac{2(1 + 2te^{2t})}{t\sqrt{\ln 2t + e^{2t}}}$     25.  $\frac{e^{xy}(xy + 1)}{1 - x^2e^{xy} - \cos y}$   
 27.  $\frac{3e^{2x}}{x} + 6e^{2x} \ln x$     29.  $12e^{6t} \cot 2e^{6t}$     31.  $\frac{4e^{2x}}{\sqrt{1 - e^{4x}}}$

33. (a) 2.718 281 8, value of derivative  
 (b) 2.718 417 7, slope of secant line

35. 20    37. -1.46



39.  $\frac{12e^{4x}(4x + 23) dx}{(x + 6)^2}$     41.



43.  $(-xe^{-x} + e^{-x}) + (xe^{-x}) = e^{-x}$   
 45.  $\frac{2e^{2x}(e^{2x} + 1) - 2e^{2x}(e^{2x} - 1)}{(e^{2x} + 1)^2} = \frac{4e^{2x}}{(e^{2x} + 1)^2}$   
 $= \frac{(e^{2x} + 1)^2 - (e^{2x} - 1)^2}{(e^{2x} + 1)^2}$   
 47. -3, 2    49.  $Ak^2e^{kx} + Bk^2e^{-kx} = k^2(Ae^{kx} + Be^{-kx})$   
 51. -0.001 64/h    53.  $e^{-66.7t}(999 \cos 226t - 295 \sin 226t)$

55. Substitute and simplify.

57.  $\frac{d}{dx} \left[ \frac{1}{2}(e^u - e^{-u}) \right] = \frac{1}{2}(e^u + e^{-u}) \frac{du}{dx}$ ;

$\frac{d}{dx} \left[ \frac{1}{2}(e^u + e^{-u}) \right] = \frac{1}{2}(e^u - e^{-u}) \frac{du}{dx}$

59.  $\sinh \frac{x}{50}$

**Exercises 27.7, page 840**

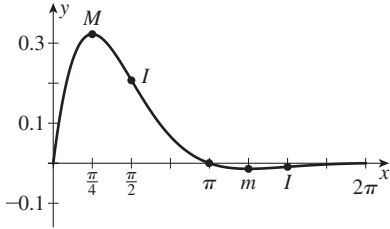
1.  $\infty$     3. 1    5.  $\infty$     7. 0    9. 0    11.  $-\frac{1}{6}$   
 13.  $-\pi$     15. 1    17. 0    19. 0    21.  $\infty$

**B.72 ANSWERS TO ODD-NUMBERED EXERCISES**

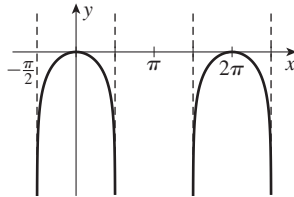
23. 1    25. 0    27. -7    29. -1    31. 0    33. 1  
 35. 1    37.  $\sin x$  varies between 1 and -1 as  $x \rightarrow \infty$ .    39.  $gt$

**Exercises 27.8, page 843**

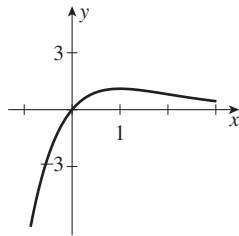
1. Int.  $(0, 0)$ ,  $(\pi, 0)$ ,  $(2\pi, 0)$ ;  
 max.  $(\frac{\pi}{4}, 0.322)$ ;  
 min.  $(\frac{5\pi}{4}, -0.014)$ ;  
 infl.  $(\frac{\pi}{2}, 0.208)$ ,  $(\frac{3\pi}{2}, -0.009)$



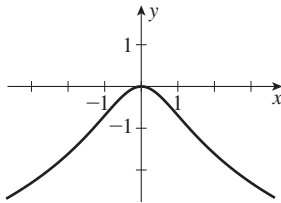
3. Int.  $(0, 0)$ , max.  $(0, 0)$ ,  
 not defined for  $\cos x < 0$ ,  
 asym.  $x = -\frac{1}{2}\pi, \frac{1}{2}\pi, \dots$



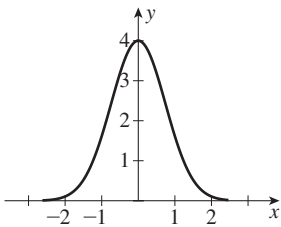
5. Int.  $(0, 0)$ , max.  $(1, \frac{1}{e})$ ,  
 infl.  $(2, \frac{6}{e^2})$ , asym.  $y = 0$



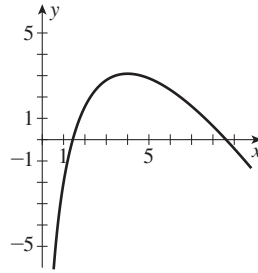
7. Int.  $(0, 0)$ , max.  $(0, 0)$ ,  
 infl.  $(-1, -\ln 2)$ ,  $(1, -\ln 2)$



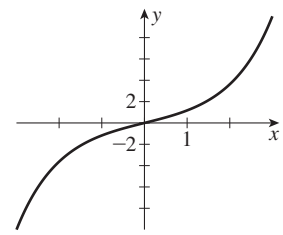
9. Int.  $(0, 4)$ ; max.  $(0, 4)$ ;  
 infl.  $(\frac{1}{2}\sqrt{2}, \frac{4}{e}\sqrt{e})$ ,  $(-\frac{1}{2}\sqrt{2}, \frac{4}{e}\sqrt{e})$ ;  
 asym.  $y = 0$



11. Max.  $(4, 3.09)$ ,  
 asym.  $x = 0$



13. Int.  $(0, 0)$ , infl.  $(0, 0)$ ,  
 inc. all  $x$



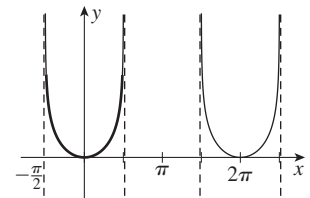
15.  $y_1$  inc.  $x < 0$ ,  $y_1$  dec.  $x > 0$ ;  
 $y_2$  inc.  $x > 0$ ,  $y_2$  dec.  $x < 0$ ;  
 $y_1$  and  $y_2$  are reciprocals.

17.  $y = x - 1$     19.  $2\sqrt{2}x - 2y + 2\sqrt{2} - 3\pi\sqrt{2} = 0$

21. 1.197 333 8    23. 174 V    25. -0.303 W/day

27.  $-5.4^\circ\text{C}/\text{min}$     29.  $\frac{p(-a + bT)}{T^2}$     31.  $a = k^2x$

33. Int.  $(0, 0)$ ;  
 min.  $(0, 0)$ ,  $(2\pi, 0)$ ,  $\dots$ ;  
 asym.  $x = -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \dots$



The path of the roller mechanism is the dark curve  $(-1.5 < x < 1.5)$

35. 3.0 s

37. Max.  $(117.6^\circ \text{ W}, 50.2^\circ \text{ N})$ ,  $(86.2^\circ \text{ W}, 47.7^\circ \text{ N})$ ;  
 min.  $(101.9^\circ \text{ W}, 41.2^\circ \text{ N})$ ,  $(70.5^\circ \text{ W}, 43.0^\circ \text{ N})$

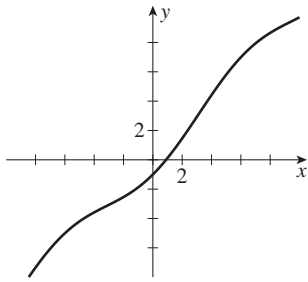
39.  $v = -e^{-0.5t}(1.4 \cos 6t + 2.3 \sin 6t)$ ,  $-2.03 \text{ cm/s}$

41.  $1/\sqrt{e} = 0.607$     43.  $\frac{1}{\sqrt{b}}$

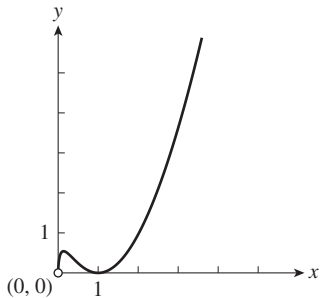
**Review Exercises for Chapter 27, page 845**

1.  $-12 \sin(4x - 1)$     3.  $\frac{-0.2 \sec^2 \sqrt{3 - 2v}}{\sqrt{3 - 2v}}$   
 5.  $-6 \csc^2(3x + 2) \cot(3x + 2)$     7.  $-24x \cos^3 x^2 \sin x^2$   
 9.  $2e^{2(x-3)}$     11.  $\frac{6x}{x^2 + 1}$     13.  $\frac{50}{25 + x^2}$   
 15.  $\frac{0.1}{\sin^{-1} 0.1t \sqrt{1 - 0.01t^2}}$     17.  $(-2 \csc 4x) \sqrt{\csc 4x + \cot 4x}$   
 19.  $\frac{14(1 + e^{-x})}{x - e^{-x}}$     21.  $-\frac{2 \sin x \cos x}{e^{3x} + \pi^2} - \frac{3e^{3x} \cos^2 x}{(e^{3x} + \pi^2)^2}$   
 23.  $\frac{2u(1 + 4u^2)(\tan^{-1} 2u) - 2u^2}{(1 + 4u^2)(\tan^{-1} 2u)^2}$     25.  $-2x \cot x^2$   
 27.  $\frac{2 \cos x \ln(3 + \sin x)}{3 + \sin x}$     29.  $0.1e^{-2t} \sec \pi t (-2 + \pi \tan \pi t)$   
 31.  $\frac{\cos 2x + 2e^{4x}}{\sqrt{\sin 2x + e^{4x}}}$     33.  $\frac{2x^3 e^y + 2xy^2 e^y + y}{x - x^4 e^y - x^2 y^2 e^y}$   
 35.  $0.5[e^{-2t}(1 - 2t) - e^{2t}(1 + 2t)]$     37.  $\frac{y(xye^{-x} - 1)}{x(ye^{-x} + 1)}$   
 39.  $\cos^{-1} x$

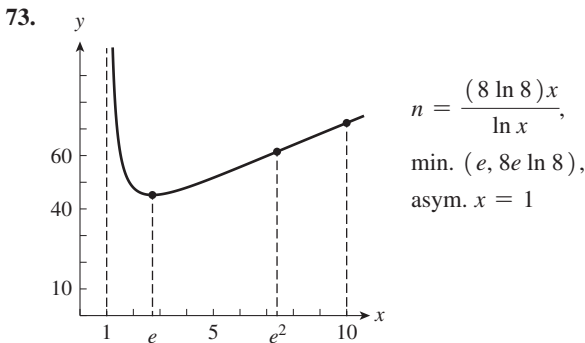
41. Inc. all  $x$ ,  
Infl.  $x = (2n + 1)\pi$



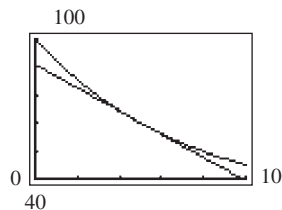
43. Max.  $(e^{-2}, 4e^{-2})$ , min.  $(1, 0)$ , infl.  $(e^{-1}, e^{-1})$       45.  $7.27x + y - 8.44 = 0$



47.  $2x + 2.57y - 4.30 = 0$     49.  $\frac{2}{3}$     51.  $-1$     53.  $0$   
55.  $2 \sin x \cos x - 2 \cos x \sin x = 0$   
57.  $-9 \sin 3x = -9(\sin 3x)$     59.  $-0.7034674$   
61.  $x > \frac{1}{2} \ln 2 (= 0.3466)$     63.  $0.039$     65.  $2.5 \text{ N}$   
67.  $0.12 \text{ cm/s}$     69.  $-0.064 \text{ }^\circ\text{C/day}$     71.  $50 \sin 4t$

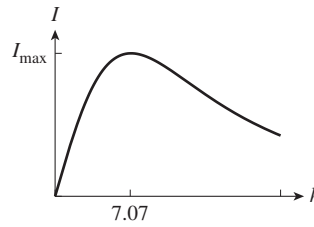


75.  $-kE_0^2 \cos \frac{1}{2} \theta \sin \frac{1}{2} \theta$     77.  $\frac{f}{\sqrt{R^2 - F^2 f^2}}$   
79.  $0.005934 \text{ rad/s}$     81.  $L(t) = 80.8 - 4.16t$



83. 470 000 users/year    85. 2.00 m, 1.82 m    87.  $-0.0065 \text{ rad/s}$   
89.  $48.2^\circ$     91.  $0.397 \text{ cm/s}, 72.6^\circ$

93. 7.07 cm      95.  $A = 50 \cos \theta (1 + 2 \sin \theta)$ ,  
max.  $(0.635, 88.0)$



97.  $\frac{w}{H} \cosh \frac{wx}{H} = \frac{w}{H} \sqrt{1 + \sinh^2 \frac{wx}{H}}$

**Exercises 28.1, page 851**

1. Change  $\cos x$  to  $-\sin x$ .    3.  $\frac{1}{5} \sin^5 x + C$   
5.  $-\frac{4}{15} (\cos \theta)^{3/2} + C$     7.  $\frac{4}{3} \tan^3 x + C$     9.  $\frac{1}{8}$   
11.  $\frac{1}{4} (\sin^{-1} x)^4 + C$     13.  $\frac{1}{2} (\tan^{-1} 5x)^2 + C$   
15.  $\frac{1}{3} [\ln(x + 1)]^3 + C$     17.  $0.179$     19.  $\frac{1}{4} (4 + e^x)^4 + C$   
21.  $\frac{1}{4(1 - e^{2t})^2} + C$     23.  $\frac{1}{10} (1 + \sec^2 x)^5 + C$     25.  $4.13$   
27.  $-\int \frac{-\sin x dx}{\cos^5 x}$ ;  $u = \cos x, du = -\sin x dx, n = -5$   
29.  $\int \frac{1}{\ln^2 x} \frac{dx}{x}$ ,  $u = \ln x, du = \frac{dx}{x}, n = -2$   
31.  $0.888$     33.  $1.10$     35.  $y = \frac{1}{3} (\ln x)^3 + 2$     37.  $\frac{1}{3} m v^2$   
39.  $q = (1 - e^{-t})^3$

**Exercises 28.2, page 855**

1. Change  $dx$  to  $2x dx$ .    3.  $\frac{1}{4} \ln|1 + 4x| + C$   
5.  $-\frac{1}{3} \ln|4 - 3x^2| + C$     7.  $\frac{1}{3} \ln 4 = 0.462$   
9.  $-0.2 \ln|\cot 2\theta| + C$     11.  $\ln 2 = 0.693$   
13.  $\ln|1 - e^{-x}| + C$     15.  $\ln|x + e^x| + C$   
17.  $\frac{1}{4} \ln|1 + 4 \sec x| + C$     19.  $\frac{1}{4} \ln 5 = 0.402$   
21.  $0.5 \ln|\ln r| + C$     23.  $\ln|2x + \tan x| + C$   
25.  $-6\sqrt{1 - 2x} + C$     27.  $\ln|x| - \frac{2}{x} + C$   
29.  $\frac{1}{3} \ln(\frac{5}{4}) = 0.0744$     31.  $1.10$   
33.  $\int \frac{x - 4}{x + 4} dx = \int (1 - \frac{8}{x + 4}) dx = x - 8 \ln|x + 4| + C$   
35.  $\pi \ln 2 = 2.18$     37.  $y = \ln(\frac{3.5}{3 + \cos x} + 2)$   
39.  $\ln a + \ln b = \ln ab$     41.  $\ln 2$     43. 42.1 million  
45.  $i = \frac{E}{R} (1 - e^{-Rt/L})$     47. 1.41 m

**Exercises 28.3, page 858**

1. Change  $x dx$  to  $3x^2 dx$ .    3.  $e^{7x} + C$     5.  $\frac{1}{2} e^{2x+5} + C$   
7. 28.2    9.  $2e^{x^3} + C$     11.  $2(e^2 - e) = 9.34$   
13.  $2e^{2 \sec \theta} + C$     15.  $\frac{2}{3} (1 + e^y)^{3/2} + C$   
17.  $6 - \frac{3(e^6 - e^2)}{2} = -588$     19.  $-\frac{4}{e^{\sqrt{x}}} + C$   
21.  $\frac{1}{2} e^{\tan^{-1} 2x} + C$     23.  $-\frac{1}{3} e^{\cos 3x} + C$     25.  $0$   
27.  $6 \ln(e^x + 1) + C$     29.  $3e^2 - 3 = 19.2$   
31.  $y = e^{\ln x}; \ln y = \ln e^{\ln x}, y = x; e^{x^2} + C$

**B.74 ANSWERS TO ODD-NUMBERED EXERCISES**

33.  $\int \left(1 - \frac{e^x}{e^x + 1}\right) dx = x - \ln(e^x + 1) + C$   
 35.  $\pi(e^4 - e) = 163$     37.  $2(e^2 - 1) = 12.8$   
 39.  $\ln b \int b^u du = b^u + C_1$     41.  $q = EC(1 - e^{-t/RC})$   
 43.  $26\,610 \text{ m}^2$     45.  $0.632$

**Exercises 28.4, page 862**

1. Change  $x dx$  to  $3x^2 dx$ .    3.  $\frac{1}{2} \sin 2x + C$   
 5.  $0.1 \tan 3\theta + C$     7.  $2 \sec \frac{1}{2}x + C$     9.  $0.6365$   
 11.  $\frac{3}{2} \ln|\sec \phi^2 + \tan \phi^2| + C$     13.  $\cos\left(\frac{1}{x}\right) + C$   
 15.  $\frac{1}{2}\sqrt{3}$     17.  $-2 \cos \frac{x}{2} + C$     19.  $\frac{1}{2} \ln|\sec 2x + \tan 2x| + C$   
 21.  $-\frac{1}{2} \ln \cos 2T + C$   
 23.  $\csc x - \cot x - \ln|\csc x - \cot x| + \ln|\sin x| + C$   
 25.  $\frac{1}{9} \pi + \frac{1}{3} \ln 2 = 0.580$     27.  $0.693$   
 29. Integral changes to  $\int (\sec^2 x - \sec x \tan x) dx$   
 31. (a)  $\frac{1}{2} \tan^2 x + C_1$ , (b)  $\frac{1}{2} \sec^2 x + C_2$ ;  $C_1 = C_2 + \frac{1}{2}$   
 33.  $\pi\sqrt{3} = 5.44$     35.  $\theta = 0.10 \cos 2.5t$   
 37.  $0.7726 \text{ m}$     39.  $0.707$

**Exercises 28.5, page 866**

1. Change  $dx$  to  $\cos 2x dx$ .    3.  $\frac{1}{3} \sin^3 x + C$   
 5.  $-\frac{1}{2} \cos 2x + \frac{1}{6} \cos^3 2x + C$     7.  $2 \sin 2\theta + C$   
 9.  $\frac{1}{24}(64 - 43\sqrt{2}) = 0.133$     11.  $\frac{1}{2}x - \frac{1}{4} \sin 2x + C$   
 13.  $\frac{1}{3}(9\phi + 4 \sin 3\phi + \sin 3\phi \cos 3\phi) + C$   
 15.  $\frac{1}{2} \tan^2 x + \ln|\cos x| + C$     17.  $\frac{3}{4}$   
 19.  $\frac{1}{6} \tan^3 2x - \frac{1}{2} \tan 2x + x + C$   
 21.  $\frac{1}{3} \sin^3 s + C$     23.  $x - \frac{1}{2} \cos 2x + C$   
 25.  $\frac{1}{4} \cot^4 x - \frac{1}{3} \cot^3 x + \frac{1}{2} \cot^2 x - \cot x + C$   
 27.  $1 + \frac{1}{2} \ln 2 = 1.35$     29.  $\frac{1}{5} \tan^5 x + \frac{2}{3} \tan^3 x + \tan x + C$   
 31.  $2 \sec x + C$     33.  $-\ln|\sec e^{-x} + \tan e^{-x}| + C$   
 35.  $\frac{1}{2} \cos x - \frac{1}{18} \cos 9x + C$     37.  $\frac{1}{2} \pi^2 = 4.93$   
 39.  $\sqrt{2} - 1 = 0.414$   
 41.  $\int \sin x \cos x dx = \frac{1}{2} \sin^2 x + C_1 = -\frac{1}{2} \cos^2 x + C_2$ ;  
 $C_2 = C_1 + \frac{1}{2}$   
 43.  $\int_0^\pi \sin^2 nx dx = \frac{1}{2} \int_0^\pi (1 - \cos 2nx) dx$   
 $= \frac{1}{2} x - \frac{\sin 2nx}{2} \Big|_0^\pi = \frac{\pi}{2}$   
 45.  $s = -\frac{1}{3} \cos t + \frac{1}{9} \cos^3 t + 6t + \frac{2}{9}$   
 47.  $\frac{4}{3}$     49.  $V = \sqrt{\frac{1}{1/60.0} \int_0^{1/60.0} (340\,120\pi t)^2 dt} = 240 \text{ V}$   
 51.  $\frac{aA}{2} + \frac{A}{2b\pi} \sin ab\pi \cos 2bc\pi$

**Exercises 28.6, page 870**

1. Change  $dx$  to  $-x dx$ .    3.  $\sin^{-1} \frac{1}{2}x + C$     5.  $\frac{1}{8} \tan^{-1} \frac{1}{8}x + C$   
 7.  $\sin^{-1} 4x^2 + C$     9.  $0.863$     11.  $\frac{2}{5}\sqrt{5} \sin^{-1} \frac{1}{5}\sqrt{5} = 0.415$   
 13.  $\frac{4}{9} \ln|9x^2 + 16| + C$     15.  $2.36$     17.  $\sin^{-1} e^x + C$

19.  $\tan^{-1}(T + 1) + C$     21.  $4 \sin^{-1} \frac{1}{2}(x + 2) + C$   
 23.  $-0.714$     25.  $2 \sin^{-1}(\frac{1}{2}x) + \sqrt{4 - x^2} + C$   
 27.  $\frac{1}{2}(\sin^{-1}x)^2 + C$     29.  $\frac{1}{3} \tan^{-1}x^3 + \frac{1}{2} \ln(x^6 + 1) + C$

31. (a) Inverse tangent,  $\int \frac{du}{a^2 + u^2}$  where  $u = 3x$ ,  
 $du = 3 dx$ ,  $a = 2$ ; numerator cannot fit  $du$  of denominator.  
 Positive  $9x^2$  leads to inverse tangent form.  
 (b) logarithmic,  $\int \frac{du}{u}$  where  $u = 4 + 9x$ ,  $du = 9 dx$   
 (c) general power,  $\int u^{-1/2} du$  where  $u = 4 + 9x^2$ ,  $du = 18x dx$

33. (a) General power,  $\int u^{-1/2} du$  where  $u = 4 - 9x^2$ ,  
 $du = -18x dx$ ; numerator can fit  $du$  of denominator. Square root  
 becomes  $-1/2$  power. Does not fit inverse sine form.  
 (b) Inverse sine,  $\int \frac{du}{\sqrt{a^2 - u^2}}$  where  
 $u = 3x$ ,  $du = 3 dx$ ,  $a = 2$  (c) Logarithmic,  $\int \frac{du}{u}$   
 where  $u = 4 - 9x$ ,  $du = -9 dx$

35. Form fits inverse tangent integral with  $u = \sqrt{x}$ ,  $du = \frac{dx}{2\sqrt{x}}$ .  
 Result is  $2 \tan^{-1} \sqrt{x} + C$ .  
 37.  $\tan^{-1} 2 = 1.11$     39.  $k \tan^{-1} \frac{x}{d} + C$   
 41.  $\sin^{-1} \frac{x}{A} = \sqrt{\frac{k}{m}} t + \sin^{-1} \frac{x_0}{A}$     43.  $0.220 \rho$

**Exercises 28.7, page 875**

1. No. Integral  $\int v du$  is more complex than the given integral.  
 3.  $\cos \theta + \theta \sin \theta + C$     5.  $2xe^{2x} - e^{2x} + C$   
 7.  $3 \ln|\sin x| - 3x \cot x + C$     9.  $2x \tan^{-1} x - \ln(1 + x^2) + C$   
 11.  $-\frac{32}{3}$     13.  $\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C$   
 15.  $\frac{1}{2} \phi \sin 2\phi - \frac{1}{4}(2\phi^2 - 1) \cos 2\phi + C$   
 17.  $\frac{1}{2}(e^{\pi/2} - 1) = 1.91$   
 19.  $\frac{1}{9}(x + 5)(x + 4)^9 - \frac{1}{90}(x + 4)^{10} + C$   
 21.  $\frac{1}{2}x[\cos(\ln x) + \sin(\ln x)] + C$   
 23.  $-2e^{-\sqrt{x}}(\sqrt{x} + 1) + C$     25.  $1 - \frac{3}{e^2} = 0.594$   
 27.  $0.110$     29.  $\frac{1}{2}\pi - 1 = 0.571$     31.  $0.756$   
 33.  $s = \frac{1}{3}[(t^2 - 2)\sqrt{t^2 + 1} + 2]$   
 35.  $q = \frac{1}{5}[e^{-2t}(\sin t - 2 \cos t) + 2]$

**Exercises 28.8, page 878**

1. Delete the  $x^2$  before the radical in the denominator.  
 3.  $x = 3 \sin \theta$ ,  $\int \cot^2 \theta d\theta$     5.  $x = \tan \theta$ ,  $\int \csc \theta \cot \theta d\theta$   
 7.  $x = \sec \theta$ ,  $\int d\theta$     9.  $-\frac{\sqrt{1 - x^2}}{x} - \sin^{-1} x + C$   
 11.  $2 \ln|x + \sqrt{x^2 - 4}| + C$     13.  $-\frac{2\sqrt{z^2 + 9}}{3z} + C$   
 15.  $\frac{x}{\sqrt{4 - x^2}} + C$     17.  $\frac{16 - 9\sqrt{3}}{24} = 0.017$

19.  $5 \ln|\sqrt{x^2 + 2x + 2} + x + 1| + C$  21. 0.0400

23.  $2 \sec^{-1} e^x + C$

25. (a)  $-\frac{1}{3}(1-x^2)^{3/2} + C$ , (b)  $-\frac{1}{3}(1-x^2)^{3/2} + C$

27.  $\pi$  29.  $\frac{1}{4}ma^2$  31. 2.68 33. 1.36

35.  $kQ \ln \frac{\sqrt{a^2 + b^2} + a}{\sqrt{a^2 + b^2} - a}$  37.  $\frac{2(x+1)^{5/2}}{5} - \frac{2(x+1)^{3/2}}{3} + C$

39.  $\frac{3}{8}(x-4)^{8/3} + \frac{12}{5}(x-4)^{5/3} + C$

**Exercises 28.9, page 882**

1.  $\frac{3}{x-1} - \frac{4}{x+2}$  3.  $\frac{A}{x} + \frac{B}{x+1}$

5.  $\frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-2}$  7.  $\ln \left| \frac{(x+1)^2}{x+2} \right| + C$

9.  $\frac{1}{4} \ln \left| \frac{x-2}{x+2} \right| + C$  11.  $x + \ln \left| \frac{x}{(x+3)^4} \right| + C$

13. 1.06 15.  $\ln \left| \frac{x^2(x-5)^3}{x+1} \right| + C$

17.  $\frac{1}{4} \ln \left| \frac{x^4(2x+1)^3}{2x-1} \right| + C$  19.  $1 + \ln \frac{16}{3} = 2.674$

21.  $\frac{1}{60} \ln \left| \frac{(V+2)^3(V-3)^2}{(V-2)^3(V+3)^2} \right| + C$  23.  $\ln \left| \frac{x(x-2)}{(x-1)^2} \right| + C$

25.  $\frac{1}{u(a+bu)} = \frac{A}{u} + \frac{B}{a+bu}$ ;  $1 = A(a+bu) + Bu$ ;  
 $A = \frac{1}{a}$ ,  $B = -\frac{b}{a}$

27.  $-\frac{1}{4} \ln \left| \frac{\sin \theta + 3}{\sin \theta - 1} \right| + C$  29.  $2\pi \ln \frac{6}{5} = 1.15$

31.  $y = \ln \left| \frac{x(x+5)^2}{36} \right|$  33. 0.163 N·cm

**Exercises 28.10, page 888**

1.  $\frac{2}{x(x+3)^2} = \frac{A}{x} + \frac{B}{x+3} + \frac{C}{(x+3)^2}$

3.  $\frac{4x+4}{x^3-9x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-9}$

5.  $\frac{3}{x-2} + 2 \ln \left| \frac{x-2}{x} \right| + C$  7.  $\frac{2}{x} + \ln \left| \frac{x-1}{x+1} \right| + C$

9.  $-\frac{5}{4}$  11.  $-\frac{2}{x+1} - \frac{1}{x-3} + \ln|x+1| + C$

13.  $\frac{1}{8}\pi + \ln 3 = 1.49$  15.  $-\frac{2}{x} + \frac{3}{2} \tan^{-1} \frac{x+2}{2} + C$

17.  $\frac{1}{4} \ln(4x^2+1) + \ln|x^2+6x+10| + \tan^{-1}(x+3) + C$

19.  $\tan^{-1}x + \frac{x}{x^2+1} + \ln|x+1| - \frac{1}{2} \ln|x^2+1| + C$

21.  $\frac{1-x}{(x-2)^2} + C$  23.  $2 + 4 \ln \frac{2}{3} = 0.378$

25.  $\pi \ln \frac{25}{9} = 3.21$  27. 0.919 m 29. 1.37

**Exercises 28.11, page 890**

1. Formula 3 3.  $u = y$ ,  $du = dy$ , Formula 7

5. Formula 25;  $u = x^2$ ,  $du = 2x dx$

7.  $u = x^2$ ,  $du = 2x dx$ , Formula 32

9.  $\frac{3}{25} [2 + 5x - 2 \ln|2 + 5x|] + C$  11.  $\frac{3544}{15} = 236$

13.  $\frac{y}{4\sqrt{y^2+4}} + C$  15.  $\frac{1}{2} \sin x - \frac{1}{10} \sin 5x + C$

17.  $\sqrt{4x^2-9} - 3 \sec^{-1} \left( \frac{2x}{3} \right) + C$

19.  $\frac{1}{20} \cos^4 4x \sin 4x + \frac{1}{5} \sin 4x - \frac{1}{15} \sin^3 4x + C$

21.  $3r^2 \tan^{-1} r^2 - \frac{3}{2} \ln(1+r^4) + C$

23.  $\frac{1}{4}(8\pi - 9\sqrt{3}) = 2.39$

25.  $-\ln \left( \frac{1 + \sqrt{4x^2+1}}{2x} \right) + C$

27.  $-8 \ln \left( \frac{1 + \sqrt{1-4x^2}}{2x} \right) + C$  29. 0.0208

31.  $\frac{1}{3}(\cos x^3 + x^3 \sin x^3) + C$  33.  $\frac{x^2}{\sqrt{1-x^4}} + C$

35. 4.89 37.  $\frac{1}{4}x^4(\ln x^2 - \frac{1}{2}) + C$  39.  $-\frac{3x^3}{\sqrt{x^6-1}} + C$

41.  $\frac{t^3}{12}(t^6+1)^{3/2} + \frac{t^3}{8}\sqrt{t^6+1} + \frac{1}{8} \ln(t^3 + \sqrt{t^6+1}) + C$

43.  $\frac{1}{24} \sin^4 4x \cos^2 4x + \frac{1}{48} \sin^4 4x + C$

45.  $\frac{1}{4}([2\sqrt{5} + \ln(2 + \sqrt{5})]) = 1.479$

47.  $\pi ab$  49. 32.7 kN 51. 187 000 m<sup>3</sup>

**Review Exercises for Chapter 28, page 892**

1.  $-\frac{1}{8}e^{-8x} + C$  3.  $-\frac{1}{\ln 2x} + C$  5.  $4 \ln 2 = 2.77$

7.  $\frac{25}{35} \tan^{-1} \frac{7}{5}x + C$  9. 0 11.  $\frac{1}{2} \ln 2 = 0.347$

13.  $\frac{2}{3} \sin^3 t - \cos t + C$  15.  $\tan^{-1} e^x + C$

17.  $\frac{1}{9} \tan^3 3x + \frac{1}{3} \tan 3x + C$  19.  $\ln \left| \frac{(x-1)^3}{x(2x+1)} \right| + C$

21.  $\frac{3}{4} \tan^{-1} \frac{x^2}{2} + C$  23.  $2 \ln|2x + \sqrt{4x^2-9}| + C$

25.  $\sqrt{e^{2x}+1} + C$  27.  $2 \ln|x| + \tan^{-1} \frac{x}{3} + C$

29.  $\frac{\pi}{4}$  31.  $-\frac{1}{2}x \cot 2x + \frac{1}{4} \ln|\sin 2x| + C$

33.  $\frac{2}{u} + \ln|3u+1| + C$  35.  $\frac{1}{2} \sin e^{2x} + C$

37.  $3 \sin 1 = 2.52$  39.  $\frac{1}{2}u^2 - 3u + \ln|u+3| + C$

41.  $\sqrt{5} \tan^{-1} \left( \frac{\sqrt{5} \cos x}{5} \right) - \cos x + C$

43.  $x = \sqrt{2} \sin \theta$ ;  $\frac{1}{\sqrt{2}} \int \csc \theta d\theta$  45.  $2x^2 + C, 2x^2 + C$

47. Formula 16,  $u = x^4$ ,  $du = 4x^3 dx$

49.  $\frac{1}{3}(e^x+1)^3 + C_1 = \frac{1}{3}e^{3x} + e^{2x} + e^x + C_2$ ;  $C_2 = C_1 + \frac{1}{3}$

51.  $x - \tan x + \sec x + C$

53. Power rule with  $u = x^2 + 4$ ,  $du = 2x dx$ , and  $n = -1/2$  is easier to use than a trigonometric substitution with  $x = 2 \tan \theta$ .



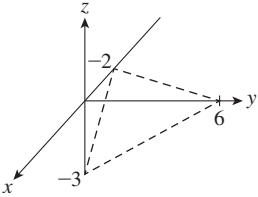
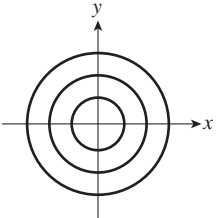
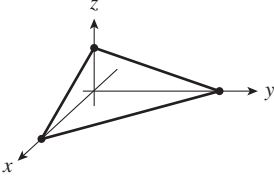
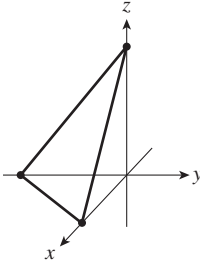
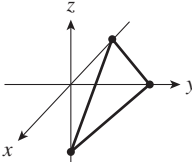
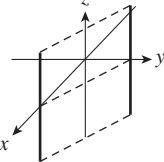
**B.76 ANSWERS TO ODD-NUMBERED EXERCISES**

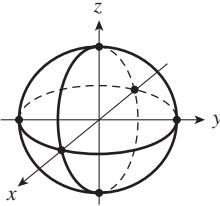
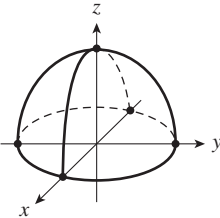
55.  $y = \frac{1}{3} \tan^3 x + \tan x$     57.  $2(e^3 - 1) = 38.2$     59. 11.2  
 61.  $\frac{1}{4}(8 \tan^{-1} 4 - \ln 17) = 1.94$     63.  $4\pi(e^2 - 1) = 80.3$   
 65.  $\frac{1}{8}\pi(e^{2\pi} - 1) = 210$     67.  $\ln 3 = 1.10$     69.  $1.01 \text{ N} \cdot \text{s}$   
 71.  $\Delta S = a \ln T + bT + \frac{1}{2}cT^2 + C$     73. 55.2 m  
 75.  $v = 98(1 - e^{-0.1t})$     77.  $\sqrt{2}$     79.  $\frac{2}{3}k$     81.  $3.47 \text{ cm}^3$   
 83.  $73.0 \text{ m}^2$

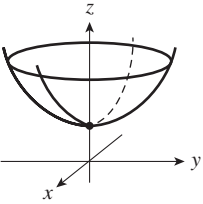
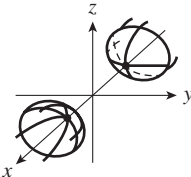
**Exercises 29.1, page 897**

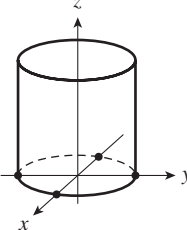
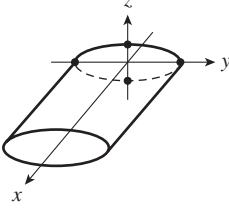
1. 7    3.  $V = \pi r^2 h$     5.  $A = \frac{2V}{r} + 2\pi r^2$   
 7.  $V = \frac{1}{4}\pi h(4r^2 - h^2)$     9. 24    11. -2  
 13.  $2 - 3y + 4y^2$     15.  $6xt + xt^2 + t^3$   
 17.  $\frac{p^2 + pq + kp - p + 2q^2 + 4kq + 2k^2 + 5q + 5k}{p + q + k}$   
 19.  $2hx - 2kx - 2hy + h^2 - 2hk - 4h$     21. 0  
 23.  $81z^6 - 9z^5 - 2z^3$     25.  $x = 0, y < 0$   
 27.  $y > 1$     29. 18 V    31. 150 Pa  
 33. For  $a, b,$  and  $T$  with same sign: circle if  $a = b,$  ellipse if  $a \neq b;$  for  $a$  and  $b$  of different signs, hyperbola  
 35. 0.0278 A    37.  $A = \frac{pw - 2w^2}{2}, 3850 \text{ cm}^2$   
 39.  $L = \frac{1.2 \times 10^4}{l^2} r^4$

**Exercises 29.2, page 904**

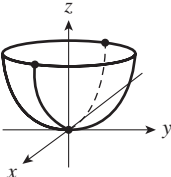
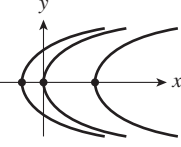
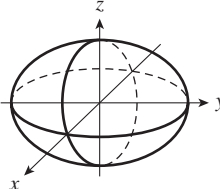
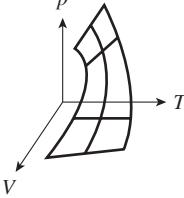
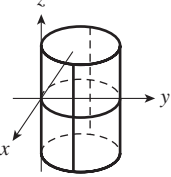
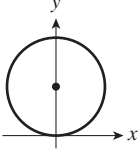
1. 
3. 
5. 
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25. (a)  $(\frac{3}{2}\sqrt{2}, \frac{3}{2}\sqrt{2}, 5)$  (b) (0, 2, 3) (c)  $(2, 2\sqrt{3}, 2)$   
 27. (a) cylinder, axis is z-axis,  $r = 2$   
 (b) plane  $\theta = 2$  for all  $r$  and  $z$   
 (c) plane  $z = 2$  for all  $r$  and  $\theta$   
 29.  $x^2 + y^2 = 4z$     31.

- 
- 
33. 
35. 
37. 
- 

**Exercises 29.3, page 908**

1.  $\frac{\partial z}{\partial x} = \frac{\ln y}{y^2 + 1}, \frac{\partial z}{\partial y} = \frac{x(y^2 + 1) - 2xy^2 \ln y}{y(y^2 + 1)^2}$   
 3.  $\frac{\partial z}{\partial x} = 5 + 8xy, \frac{\partial z}{\partial y} = 4x^2$

5.  $\frac{\partial f}{\partial x} = e^{2y}, \frac{\partial f}{\partial y} = 2xe^{2y}$   
 7.  $\frac{\partial f}{\partial x} = \frac{\sin x}{1 - \sec 3y}, \frac{\partial f}{\partial y} = \frac{3(2 + \cos x) \sec 3y \tan 3y}{(1 - \sec 3y)^2}$   
 9.  $\frac{\partial \phi}{\partial r} = \frac{1 + 3rs}{\sqrt{1 + 2rs}}, \frac{\partial \phi}{\partial s} = \frac{r^2}{\sqrt{1 + 2rs}}$   
 11.  $\frac{\partial z}{\partial x} = 4(2x + y^3)(x^2 + xy^3)^3, \frac{\partial z}{\partial y} = 12xy^2(x^2 + xy^3)^3$

13.  $\frac{\partial z}{\partial x} = y \cos xy, \frac{\partial z}{\partial y} = x \cos xy$   
 15.  $\frac{\partial y}{\partial r} = \frac{2r}{r^2 + s}, \frac{\partial y}{\partial s} = \frac{1}{r^2 + s}$   
 17.  $\frac{\partial f}{\partial x} = \frac{12 \sin^2 2x \cos 2x}{1 - 3y}, \frac{\partial f}{\partial y} = \frac{6 \sin^3 2x}{(1 - 3y)^2}$

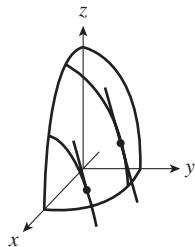
19.  $\frac{\partial z}{\partial x} = \frac{3y + x^2y - 2x\sqrt{1 - x^2y^2} \sin^{-1}xy}{(3 + x^2)^2\sqrt{1 - x^2y^2}},$   
 $\frac{\partial z}{\partial y} = \frac{x}{(3 + x^2)\sqrt{1 - x^2y^2}}$   
 21.  $\frac{\partial z}{\partial x} = \cos x - y \sin xy, \frac{\partial z}{\partial y} = -x \sin xy + \sin y$

23.  $\frac{\partial f}{\partial x} = e^x(\cos xy - y \sin xy) - 2e^{-2x} \tan y,$   
 $\frac{\partial f}{\partial y} = -xe^x \sin xy + e^{-2x} \sec^2 y$

25. -8      27.  $\frac{41}{4}$   
 29.  $\frac{\partial^2 z}{\partial x^2} = -6y, \frac{\partial^2 z}{\partial y^2} = 12xy, \frac{\partial^2 z}{\partial x \partial y} = 6y^2 - 6x$

31.  $\frac{\partial^2 z}{\partial x^2} = e^x \sin y, \frac{\partial^2 z}{\partial y^2} = \frac{2x}{y^3} - e^x \sin y,$   
 $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} = -\frac{1}{y^2} + e^x \cos y$

33. -4, -4      35.  $\left(\frac{R_2}{R_1 + R_2}\right)^2$

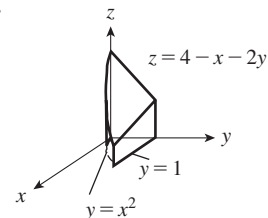


37. 114 cm<sup>2</sup>      39. 0.817      41. 3.75 × 10<sup>-3</sup> 1/Ω  
 43. -5e<sup>-t</sup> sin 4x =  $\frac{1}{16}(-80e^{-t} \sin 4x)$

**Exercises 29.4, page 912**

1.  $\frac{3}{20}$       3.  $\frac{28}{3}$       5.  $\frac{127}{14}$       7.  $\frac{1}{3}$       9.  $\frac{\pi - 6}{12}$   
 11. 1      13. 495      15.  $\frac{74}{5}$       17.  $\frac{32}{3}$       19. 8π

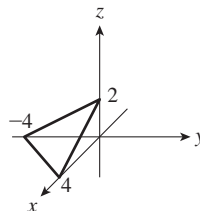
21.  $\frac{28}{3}$       23. 18      25. 300 cm<sup>3</sup>      27.



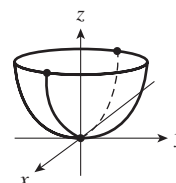
**Review Exercises for Chapter 29, page 913**

1. -52      3.  $\frac{8s^4 - s^2}{4}$

5.



7.



9.  $\frac{\partial z}{\partial x} = 15x^2y^2 - 2y^4, \frac{\partial z}{\partial y} = 10x^3y - 8xy^3$

11.  $\frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2 - 3y^2}}, \frac{\partial z}{\partial y} = \frac{-3y}{\sqrt{x^2 - 3y^2}}$

13.  $\frac{\partial z}{\partial x} = \frac{2 - 2x^2y + 6xy^2}{(x^2y + 1)^2}, \frac{\partial z}{\partial y} = -\frac{3 + 2x^3}{(x^2y + 1)^2}$

15.  $\frac{\partial u}{\partial x} = 2xy \cot(x^2 + 2y),$

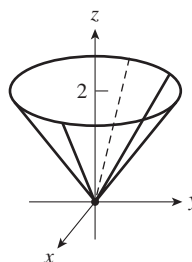
$\frac{\partial u}{\partial y} = 2y \cot(x^2 + 2y) + \ln \sin(x^2 + 2y)$

17.  $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = \frac{1}{2\sqrt{(x+y)(1-x-y)}}$

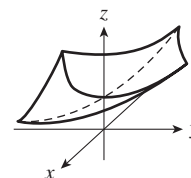
19.  $\frac{\partial^2 z}{\partial x^2} = 6y, \frac{\partial^2 z}{\partial y^2} = -6y, \frac{\partial^2 z}{\partial x \partial y} = 6x + 2$

21. 12      23.  $\frac{21}{2}$       25.  $\frac{e^2 - 3}{4} = 1.10$       27.  $\frac{1}{6}$

29.



31.



33. (a)  $\theta = 3$  represents a vertical plane.  
 (b)  $z = r^2$  represents a circular paraboloid.

35.  $\frac{\partial v}{\partial r} = \frac{ER}{(r + R)^2}, \frac{\partial v}{\partial R} = \frac{-rE}{(r + R)^2}$       37. 0.982

39.  $\frac{\pi}{\sqrt{gl}} = \frac{2\pi\sqrt{l/g}}{2l}$       41.  $\frac{k_2 + 2k_3FT}{L_0 + k_1F + k_2T + k_3FT^2}$

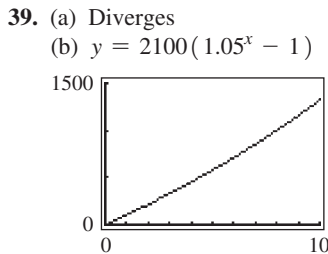
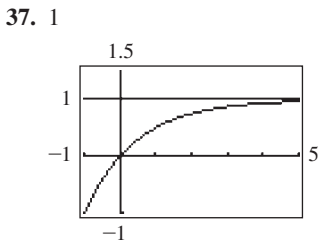
**B.78 ANSWERS TO ODD-NUMBERED EXERCISES**

43.  $\left(\frac{\partial V}{\partial T}\right)\left(\frac{\partial T}{\partial p}\right)\left(\frac{\partial p}{\partial V}\right) = -\left(\frac{nR}{p}\right)\left(\frac{V}{nR}\right)\left(\frac{pV}{V^2}\right) = -1$

45.  $32\left(\pi - \frac{2}{3}\right) = 79.20$ . The bounding surfaces are a cylinder with axis along the  $z$ -axis, and a plane parallel to the  $y$ -axis.

**Exercises 30.1, page 918**

1. Converges;  $S_1 = 0.5, S_2 = 0.75, S_3 = 0.875, S_4 = 0.9375$   
 3. 1, 4, 9, 16    5.  $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$   
 7. (a)  $-\frac{2}{5}, \frac{4}{25}, -\frac{8}{125}, \frac{16}{625}$  (b)  $-\frac{2}{5} + \frac{4}{25} - \frac{8}{125} + \frac{16}{625} - \dots$   
 9. (a) 1, 0, -1, 0 (b)  $1 + 0 - 1 + 0 + \dots$   
 11.  $a_n = \frac{1}{n+1}$     13.  $a_n = \frac{(-1)^{n+1}}{(n+1)(n+2)}$   
 15. 1, 1.125, 1.162 037 0, 1.177 662 0, 1.185 662 0; convergent; 1.2  
 17. 1, 1.5, 2.166 666 7, 2.916 666 7, 3.716 666 7; divergent  
 19. 0, 1, 2.4142, 4.1463, 6.1463; divergent  
 21. 0.75, 0.888 888 9, 0.937 500 0, 0.960 000 0, 0.972 222 2; convergent; 1  
 23. 0.210 367 7, 0.267 198 8, 0.269 403 8, 0.266 447 6, 0.265 511 1; convergent; 0.26    25. Divergent  
 27. Convergent,  $S = \frac{3}{4}$     29. Convergent,  $S = 100$   
 31. Convergent,  $S = \frac{4096}{9}$     33.  $3 < x < 5$   
 35. (a) 1 (b) Diverges



41.  $r = x; S = \frac{1}{1-x}$     43. 2, 4, 12, 48, 240

**Exercises 30.2, page 922**

1.  $\frac{2}{2+x} = 1 - \frac{1}{2}x + \frac{1}{4}x^2 - \frac{1}{8}x^3 + \dots$   
 3.  $1 + x + \frac{1}{2}x^2 + \dots$     5.  $1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \dots$   
 7.  $1 + \frac{1}{2}x - \frac{1}{8}x^2 + \dots$     9.  $1 - 2x + 2x^2 - \dots$   
 11.  $1 - 8\pi^2x^2 + \frac{32}{3}\pi^4x^4 - \dots$     13.  $1 + x + x^2 + \dots$   
 15.  $-2x - 2x^2 - \frac{8}{3}x^3 - \dots$     17.  $1 - x^2 + \frac{1}{3}x^4 - \dots$   
 19.  $\frac{\sqrt{2}}{2}\left(1 + x - \frac{1}{2}x^2 - \dots\right)$     21.  $x - \frac{1}{3}x^3 + \dots$   
 23.  $x + \frac{1}{3}x^3 + \dots$     25.  $-\frac{1}{2}x^2 - \frac{1}{12}x^4 - \dots$   
 27.  $1 + \frac{1}{2}x - \dots$

29. No. Functions are not defined at  $x = 0$ .

31.  $e^x = 1 + x + \frac{x^2}{2} + \dots, e^{x^2} = 1 + x^2 + \frac{x^4}{2} + \dots$

33.  $f(x) = 1 + 3x + \frac{9}{2}x^2 + \dots; L(x) = 1 + 3x$

35.  $1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$     37.  $f(x) = x^2$

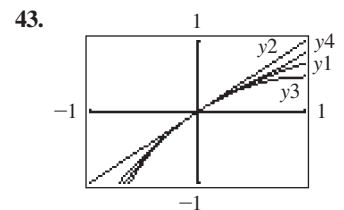
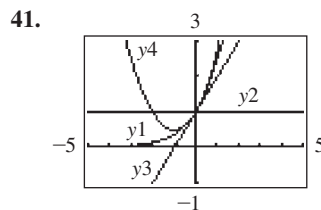
39.  $4 - 0.8t - 1.92t^2 + \dots$

41.  $R = e^{-0.001t} = 1 - 0.001t + (5 \times 10^{-7})t^2 - \dots$

**Exercises 30.3, page 927**

1.  $e^{2x^2} = 1 + 2x^2 + 2x^4 + \frac{4}{3}x^6 + \dots$   
 3.  $1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 + \dots$   
 5.  $\frac{x}{2} - \frac{x^3}{2^3 3!} + \frac{x^5}{2^5 5!} - \frac{x^7}{2^7 7!} + \dots$   
 7.  $x - 8x^3 + \frac{32}{3}x^5 - \frac{256}{45}x^7 + \dots$   
 9.  $x^2 - \frac{1}{2}x^4 + \frac{1}{3}x^6 - \frac{1}{4}x^8 + \dots$     11. 0.310  
 13. 0.190    15.  $2(1 + x^2 + x^4 + x^6 + \dots)$   
 17.  $x + x^2 + \frac{1}{3}x^3 + \dots$   
 19.  $-2x^3 - x^4 - \frac{2x^5}{3} - \frac{x^6}{2} - \dots$   
 21.  $x - \frac{1}{2}x^2 + \frac{1}{6}x^4 - \dots$   
 23.  $\frac{d}{dx}\left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots\right) = 1 + x + \frac{1}{2}x^2 + \dots$   
 25.  $\int \cos x \, dx = x - \frac{x^3}{3!} + \dots$   
 27.  $4 - 0.8t - 1.92t^2 + 0.4t^3 + \dots$   
 29.  $\int_0^1 e^x \, dx = 1.718 281 8,$   
 $\int_0^1 \left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3\right) dx = 1.708 333 3$   
 31.  $-\frac{1}{6}$     33. 0.003 10    35. 0.200  
 37.  $\left[\left(1 - \frac{v^2}{c^2}\right)^{-1/2} - 1\right] mc^2$   
 $= \left[1 + \left(-\frac{1}{2}\right)\left(-\frac{v^2}{c^2}\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}\left(-\frac{v^2}{c^2}\right)^2 + \dots - 1\right] mc^2$   
 $= 1 + \frac{1}{2}mv^2 + \frac{3}{8}m\frac{v^4}{c^2} + \dots - 1 = \frac{1}{2}mv^2$ , if  $v$  is much smaller than  $c$ .  
 39.  $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ ; differentiate on the left by the power rule and

on the right term by term.



**Exercises 30.4, page 930**

1. 0.905    3. 1.22, 1.221 402 8    5. 0.099 833 3, 0.099 833 4  
 7. 2.718 055 6, 2.718 281 8    9. 0.998 496 77, 0.998 497 15  
 11. 0.334 933 3, 0.336 472 2    13. 0.354 613 0, 0.354 612 9  
 15. -0.167 115 17, -0.167 117 72    17. 1.207 36, 1.208 03  
 19. 0.981 16, 0.981 118 50    21. 1.052 352 8    23. 0.987 446 2  
 25.  $8.3 \times 10^{-8}$     27.  $3.77 \times 10^{-7}$     29. 1.9799    31. 3.146  
 33. The terms of the expansion for  $e^x$  after those on the right side of the inequality have a positive value.

35. 1.59 years    37.  $i = \frac{E}{L} \left( t - \frac{Rt^2}{2L} \right)$ ; small values of  $t$

39. 18 m

**Exercises 30.5, page 933**

1.  $\sqrt{x} = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3 - \dots$

3. 3.32    5. 2.049    7. 0.5299    9. 0.492 88

11.  $e^{-2} \left[ 1 - (x-2) + \frac{(x-2)^2}{2!} - \dots \right]$

13.  $\frac{1}{2} \left[ \sqrt{3} + \left( x - \frac{1}{3}\pi \right) - \frac{\sqrt{3}}{2!} \left( x - \frac{1}{3}\pi \right)^2 - \dots \right]$

15.  $2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2 + \dots$

17.  $1 + 2 \left( x - \frac{1}{4}\pi \right) + 2 \left( x - \frac{1}{4}\pi \right)^2 + \dots$

19.  $e^{\pi/2} \left[ 1 + \left( x - \frac{\pi}{2} \right) - \frac{1}{3} \left( x - \frac{\pi}{2} \right)^3 + \dots \right]$

21.  $\frac{1}{5} - \frac{x-3}{25} + \frac{(x-3)^2}{125} - \dots$

23. 23.1308    25. 3.0496    27. 2.0247    29. 0.874 62

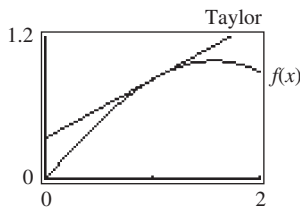
31. Use the indicated method.

33.  $2x^3 + x^2 - 3x + 5 = 5 + 5(x-1) + 14 \frac{(x-1)^2}{2} + 12 \frac{(x-1)^3}{6}$

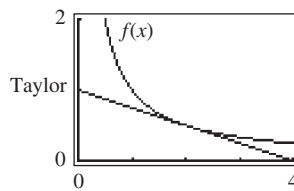
35. 0.515 040 8, 0.515 038 8, 0.515 038 1

37.  $6 \sin \frac{\pi^2}{2} + 6\pi \cos \frac{\pi^2}{2} \left( t - \frac{\pi}{2} \right) - 3\pi^2 \sin \frac{\pi^2}{3} \left( t - \frac{\pi}{2} \right)^2 + \dots$

39. Graph of part (b) fits well near  $x = \pi/3$ .



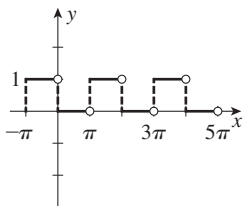
41. Graph of part (b) fits well near  $x = 2$ .



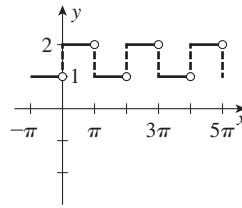
**Exercises 30.6, page 939**

1.  $f(x) = \frac{8}{\pi} \left( \sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \dots \right)$

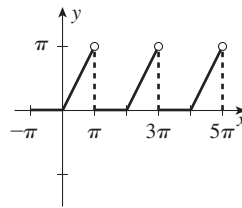
3.  $f(x) = \frac{1}{2} - \frac{2}{\pi} \sin x - \frac{2}{3\pi} \sin 3x - \dots$



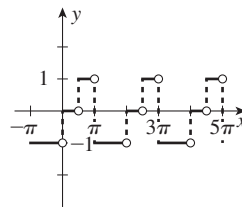
5.  $f(x) = \frac{3}{2} + \frac{2}{\pi} \sin x + \frac{2}{3\pi} \sin 3x + \dots$



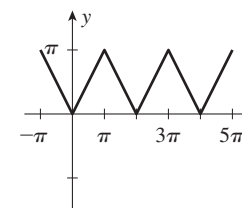
7.  $f(x) = \frac{\pi}{4} - \frac{2}{\pi} \left( \cos x + \frac{1}{9} \cos 3x + \dots \right) + \left( \sin x - \frac{1}{2} \sin 2x + \dots \right)$



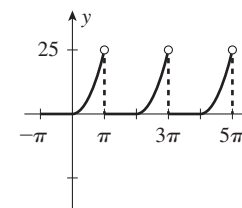
9.  $f(x) = -\frac{1}{4} - \frac{1}{\pi} \cos x + \frac{1}{3\pi} \cos 3x - \dots + \frac{3}{\pi} \sin x - \frac{1}{\pi} \sin 2x + \frac{1}{\pi} \sin 3x - \dots$

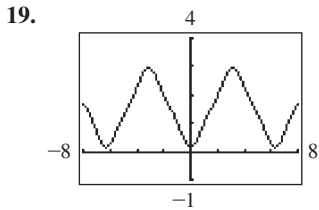
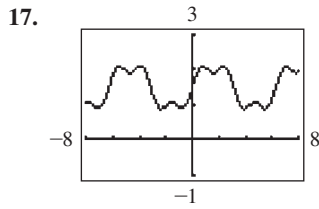
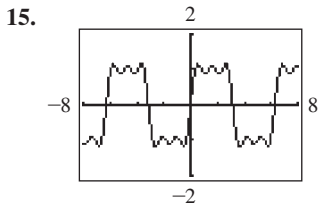


11.  $f(x) = \frac{\pi}{2} - \frac{4}{\pi} \cos x - \frac{4}{9\pi} \cos 3x - \dots$



13.  $\frac{e^{\pi} - e^{-\pi}}{2\pi} \left( 1 - \cos x + \frac{2}{5} \cos 2x - \dots + \sin x - \frac{4}{5} \sin 2x \dots \right)$





21. 
$$\frac{\pi(2\pi + 3)}{12} - \frac{2\pi + 2}{\pi} \cos t + \frac{1}{2} \cos 2t - \dots$$

$$+ \frac{\pi^2 + \pi - 4}{\pi} \sin t - \frac{\pi + 1}{2} \sin 2t + \dots$$

23. 
$$f(t) = \frac{2}{\pi} - \frac{4}{3\pi} \cos 2t - \frac{4}{15\pi} \cos 4t - \dots$$

**Exercises 30.7, page 945**

1.  $\frac{5}{2} + \frac{2}{\pi} (\cos x - \frac{1}{3} \cos 3x + \frac{1}{5} \cos 5x - \dots)$

3.  $-1 + \frac{4}{\pi} \sin \frac{\pi x}{4} + \frac{4}{3\pi} \sin \frac{3\pi x}{4} + \dots$

5. Neither    7. Even    9. Even    11. Odd

13. Sine terms    15. Sine terms and cosine terms

17. 
$$f(x) = \frac{5}{2} - \frac{10}{\pi} \left( \sin \frac{\pi x}{3} + \frac{1}{3} \sin \pi x - \dots \right)$$

19. 
$$f(x) = 1 + \frac{4}{\pi} \cos \frac{\pi x}{2} - \frac{4}{3\pi} \cos \frac{3\pi x}{2} + \dots$$

21. 
$$f(x) = 2 - \frac{16}{\pi^2} \left( \cos \frac{\pi x}{4} + \frac{1}{9} \cos \frac{3\pi x}{4} + \dots \right)$$

23. 
$$f(x) = \frac{4}{\pi} \left( \sin \frac{\pi x}{4} + \frac{1}{3} \sin \frac{3\pi x}{4} + \frac{1}{5} \sin \frac{5\pi x}{4} + \dots \right)$$

25. 
$$f(x) = \frac{4}{3} - \frac{16}{\pi^2} \left( \cos \frac{\pi x}{2} - \frac{1}{4} \cos \pi x + \frac{1}{9} \cos \frac{3\pi x}{2} - \dots \right)$$

27. 
$$f(t) = 2 + \frac{8}{\pi} \left( \cos \frac{\pi}{2} t - \frac{1}{3} \cos \frac{3\pi}{2} t + \dots + \sin \frac{\pi}{2} t \right.$$

$$\left. + \sin \pi t + \frac{1}{3} \sin \frac{3\pi}{2} t + \dots \right)$$

**Review Exercises for Chapter 30, page 946**

1.  $\frac{1}{2} - \frac{1}{4}x + \frac{1}{48}x^3 - \dots$     3.  $2x^2 - \frac{4}{3}x^6 + \frac{4}{15}x^{10} - \dots$

5.  $1 + \frac{1}{3}x - \frac{1}{9}x^2 + \dots$     7.  $x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \dots$

9.  $\cos a - (\sin a)x - (\cos a) \frac{x^2}{2} + \dots$     11. 0.82

13. 1.09    15. 0.921    17. -0.202    19. 0.953

21. 12.1655    23. 0.259

25.  $\frac{1}{2} - \frac{1}{2}\sqrt{3} \left( x - \frac{1}{3}\pi \right) - \frac{1}{4} \left( x - \frac{1}{3}\pi \right)^2 + \dots$

27. 
$$f(x) = \frac{\pi - 2}{4} - \frac{2}{\pi} \left( \cos x + \frac{1}{9} \cos 3x + \dots \right)$$

$$+ \left( \frac{\pi - 2}{\pi} \right) \sin x - \frac{1}{2} \sin 2x + \dots$$

29. 
$$f(x) = \pi + \frac{4}{\pi} \left( \sin \frac{\pi x}{4} + \frac{1}{3} \sin \frac{3\pi x}{4} + \dots \right)$$

31. 
$$f(x) = \frac{1}{2} + \frac{2}{\pi} \left( \cos x - \frac{1}{3} \cos 3x + \dots \right)$$

33. 
$$f(x) = \frac{4}{\pi} \left( \sin \frac{\pi x}{2} - \frac{1}{2} \sin \pi x + \frac{1}{3} \sin \frac{3\pi x}{2} - \dots \right)$$

35. Convergent,  $S = 5000$     37.  $S = 256$

39.  $\frac{1}{3}x^3 - \frac{1}{42}x^7 + \dots$

41.  $1 + 2 \left( x - \frac{\pi}{4} \right) + 2 \left( x - \frac{\pi}{2} \right)^2 + \dots$

43.  $2 + 4 \left( x - \frac{\pi}{4} \right) + \dots$

45. 
$$(x + h) - \frac{(x + h)^3}{3!} + \dots - (x - h) +$$

$$\frac{(x - h)^3}{3!} - \frac{1}{315}x^8 + \dots$$

$$= 2h - \frac{2hx^2}{2!} + \dots = 2h \left( 1 - \frac{x^2}{2!} + \dots \right)$$

47.  $4x - 2x^2 + \frac{4}{3}x^3 - x^4 + \dots$     49.  $x^2 - \frac{1}{3}x^4 + \frac{2}{45}x^6 - \dots$

51.  $1 + \frac{x^2}{2} + \frac{5x^4}{24} + \dots$     53.  $1 - x + x^2 - \dots$

55.  $\frac{1}{3} + \frac{4}{\pi^2} \left( -\cos \pi x + \frac{1}{4} \cos 2\pi x - \frac{1}{9} \cos 3\pi x + \dots \right)$

57. 2.4265, 2.4600, 2.459 603 1    59. 0.002 496 88

61.  $x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$

63.  $3.2 \left[ 1 - \frac{(880\pi t)^2}{2} + \frac{(880\pi t)^4}{24} - \dots \right]$

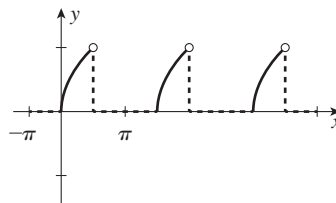
65.  $N_0 \left( 1 - \lambda t + \frac{\lambda^2 t^2}{2} - \frac{\lambda^3 t^3}{6} + \dots \right)$

67. 7.8 m    69.  $2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + \frac{2}{7}x^7 + \dots$

71.  $N_0 [ 1 + e^{-k/T} + (e^{-k/T})^2 + \dots ]$   
 $= N_0 ( 1 + e^{-k/T} + e^{-2k/T} + \dots )$

73. 
$$f(t) = \frac{1}{2\pi} + \frac{1}{\pi} \left( \frac{1}{2} \cos t - \frac{1}{3} \cos 2t + \dots \right)$$

$$+ \frac{1}{4} \sin t + \frac{2}{3\pi} \sin 2t + \dots$$



**Exercises 31.1, page 951**

1.  $(c_1e^{-x} + 4c_2e^{2x}) - (-c_1e^{-x} + 2c_2e^{2x}) = 2(c_1e^{-x} + c_2e^{2x});$   
 $(4e^{-x}) - (-4e^{-x}) = 2(4e^{-x})$     **3.** Particular solution
5. General solution  
 (The following "answers" are the unsimplified expressions obtained by substituting functions and derivatives.)
7.  $e^x - (e^x - 1) = 1; 5e^x - (5e^x - 1) = 1$
9.  $-12 \cos 2x + 4(3 \cos 2x) = 0;$   
 $(-4c_1 \sin 2x - 4c_2 \cos 2x) + 4(c_1 \sin 2x + c_2 \cos 2x) = 0$
11.  $2x = 2x$     **13.**  $1 - 3x^2 = 1 - 3x^2$
15.  $(-2ce^{-2x} + 1) + 2(ce^{-2x} + x - \frac{1}{2}) = 2x$
17.  $-\frac{1}{2} \cos x + \frac{9}{2} \cos x = 4 \cos x$
19.  $x^2 \left[ -\frac{c^2}{(x-c)^2} \right] + \left[ \frac{cx}{(x-c)} \right]^2 = 0$
21.  $x \left( -\frac{c_1}{x^2} \right) + \frac{c_1}{x} = 0$
23.  $(\cos x - \sin x + e^{-x}) + (\sin x + \cos x - e^{-x}) = 2 \cos x$
25.  $(c_1e^x + 4c_2e^{2x}) - 3(c_1e^x + 2c_2e^{2x}) +$   
 $2(c_1e^x + c_2e^{2x} + \frac{3}{2}) = 3$
27.  $\cos x \left[ \frac{(\sec x + \tan x) - (x+c)(\sec x \tan x + \sec^2 x)}{(\sec x + \tan x)^2} \right]$   
 $+ \sin x = 1 - \frac{x+c}{\sec x + \tan x}$
29.  $c^2 + cx = cx + c^2$     **31.**  $c_3e^x = c_3e^x$
33.  $y = x^3 + c_1x^2 - 4$

**Exercises 31.2, page 954**

1.  $y(x^2 + 1) = c$     **3.**  $y = c - x^2$     **5.**  $x - \frac{1}{y} = c$
7.  $\ln V = \frac{1}{P} + c$     **9.**  $\ln(x^3 + 5) + 3y = c$
11.  $y = 2x^2 + x - x \ln x + c$     **13.**  $4\sqrt{1-y} = e^{-x^2} + c$
15.  $e^x - e^{-y} = c; e^{x+y}$  is the same as  $e^xe^y$ . Divide each term by  $e^y$  and integrate.
17.  $\ln(y + 4) = x + c$     **19.**  $y(1 + \ln x)^2 + cy + 2 = 0$
21.  $\tan^2 x + 2 \ln y = c$     **23.**  $x^2 + 1 + x \ln y + cx = 0$
25.  $y - e^{\cos \theta} = c$     **27.**  $e^{x^2} = 2e^y + c$
29.  $i = c - (\ln t)^2$     **31.**  $(y^2 - 1)(x^3 + 1) = c$
33.  $3 \ln y + x^3 = 0$     **35.**  $2 \ln(1 - y) = 1 - 2 \sin x$
37.  $e^{2x} - \frac{2}{y} = 2(e^x - 1)$
39.  $\frac{1}{2} \ln(x^2 + 1) + \ln y = \ln c; \frac{1}{2} \ln 1 + \ln e = \ln c; c = e$
41.  $T = 10 + 30e^{-0.15t}$
43.  $y^2 + x^2 = c$ ; circles of radius  $\sqrt{c}$

**Exercises 31.3, page 957**

1.  $\ln xy + y^2 = c$     **3.**  $2xy + x^2 = c$     **5.**  $x^3 - 2y = cx - 4$
7.  $A^2r - r = cA$     **9.**  $y \sin x = x + c$
11.  $2\sqrt{x^2 + y^2} = x + c$     **13.**  $y = c - \frac{1}{2} \ln \sin(x^2 + y^2)$
15.  $\ln(y^2 - x^2) + 2x = c$ ; subtract  $x dx$  from each side and divide through by  $y^2 - x^2$ .
17.  $5xy^2 + y^3 = c$     **19.**  $2xy + x^3 = 5$
21.  $2x = 2xy^2 - 15y$     **23.**  $2x + 1 = \cos x$
25.  $(e^{-x} dy - ye^{-x} dx) - 2y dy = 0$

**Exercises 31.4, page 959**

1.  $y = x + cx^{-2}$     **3.**  $y = e^{-x}(x + c)$
5.  $y = -\frac{1}{2}e^{-4x} + ce^{-2x}$     **7.**  $y = -2 + ce^{2x}$
9.  $y = ce^{-x^3} + 2$     **11.**  $y = \frac{8}{7}x^3 + \frac{c}{\sqrt{x}}$
13.  $r = -\cot \theta + c \csc \theta$     **15.**  $y = (x + c) \csc x$
17.  $y = x + \frac{1}{2}e^x - 1 + ce^{-x}$     **19.**  $2s = e^{4t}(t^2 + c)$
21.  $y = \frac{1}{4} + ce^{-x^4}$     **23.**  $3y = x^4 - 6x^2 - 3 + cx$
25.  $y = ce^{-\sqrt{x^2+1}} - 1$
27.  $r = \sin \theta [\ln(\tan \theta + \sec \theta)] + c \sin \theta$
29. Can solve by separation of variables:  $\frac{dy}{1-y} = 2 dx$ .  
 Can also solve as linear differential equation of first order:  
 $dy + 2y dx = 2 dx; y = 1 + ce^{-2x}$
31.  $y = e^{-x}$     **33.**  $y = \frac{4}{3} \sin x - \csc^2 x$
35.  $y(\csc x - \cot x) = \ln \frac{(\sqrt{2} - 1)(\csc 2x - \cot 2x)}{\csc x - \cot x}$
37.  $u' - P(x)u = -Q(x)$     **39.**  $i = \frac{V}{R}(1 - e^{-Rt/L})$

**Exercises 31.5, page 962**

1. 

x	0.0	0.2	0.4	0.6	0.8	1.0
y	1.00	1.20	1.44	1.72	2.04	2.40
y (correct)	1.00	1.22	1.48	1.78	2.12	2.50

  
 $y = \frac{1}{2}x^2 + x + 1$
3. 

x	-0.2	-0.1	0.0	0.1	0.2	0.3
y	2.0000	2.1840	2.3937	2.6330	2.9069	3.2208
y (correct)	2.00	2.20	2.42	2.68	2.98	3.33

  
 $y = 2.4233e^{0.2x^2+x}$
5. 

x	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7
y	1.00	1.10	1.21	1.33	1.46	1.60	1.75	1.91
x	0.8	0.9	1.0					
y	2.08	2.26	2.45					

**B.82 ANSWERS TO ODD-NUMBERED EXERCISES**

7. (Not all values shown)

$x$	-0.2	-0.1	0.0	0.1	0.2	0.3
$y$	2.000	2.1903	2.4079	2.6573	2.9436	3.2732

9.

$x$	0.0	0.1	0.2	0.3	0.4
$y$	0.0000	0.1003	0.2027	0.3092	0.4220

11.

$x$	0.0	0.2	0.4	0.6	0.8	1.0
$y$	0.0000	0.2027	0.4232	0.6884	1.0588	1.7722

13.

$x$	0.0	0.1	0.2	0.3	0.4	0.5	0.6
$y$	1.5708	1.5660	1.5521	1.5302	1.5011	1.4656	1.4244

15.  $y_3 = 12$ ;  $y_{\text{actual}} = 36.2$ . Euler method is too inaccurate for larger values of  $x$  or  $\Delta x$ .

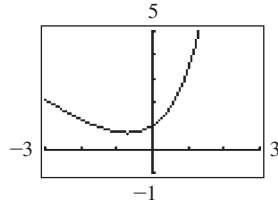
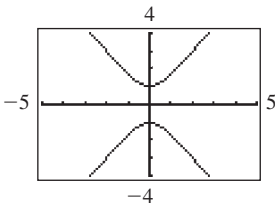
17.  $i_{\text{approx}} = 0.0804A$ ,  $i_{\text{exact}} = 0.0898A$

**Exercises 31.6, page 966**

1.  $y^2 + 2x^2 = c$     3. 2.46 kg

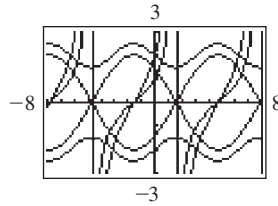
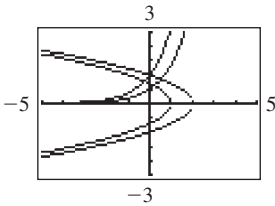
5.  $y^2 = 2x^2 + 1$

7.  $y = 2e^x - x - 1$



9.  $y^2 = c - 2x$

11.  $y^2 = c - 2 \sin x$



13. 76.9%    15. 3.82 days    17.  $N = \frac{r}{k}(1 - e^{-kt})$

19. 39.6 million    21.  $S = a + \frac{c}{r^2}$     23.  $5250e^{0.0196t}$

25. 12 min    27. \$1040.81    29.  $c = c_0(1 - e^{-kt})$

31.  $\lim_{t \rightarrow \infty} \frac{E}{R}(1 - e^{-Rt/L}) = \frac{E}{R}$

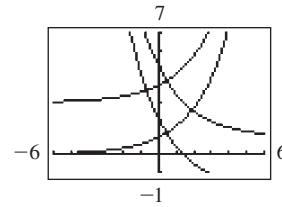
33.  $i = \frac{E}{R^2 + \omega^2 L^2}(R \sin \omega t - \omega L \cos \omega t + \omega L e^{-Rt/L})$

35.  $q = q_0 e^{-t/RC}$     37.  $v = 9.8(1 - e^{-t})$ , 9.8    39. 37 m/s

41.  $x = 3t^2 - t^3$ ,  $y = 6t^2 - 2t^3 - 9t^4 + 6t^5 - t^6$

43.  $p = 100(0.8)^{t/2000}$     45. \$4980    47.  $x = 0.20 + 0.15e^{-5.0t}$

49.  $y = 4e^{-x/2} + c_1$



**Exercises 31.7, page 972**

1.  $y = c_1 + c_2 e^{5x}$     3.  $y = c_1 e^{3x} + c_2 e^{-2x}$

5.  $y = c_1 e^{-x} + c_2 e^{-x/3}$     7.  $y = c_1 + c_2 e^{3x}$

9.  $y = c_1 e^{3x/2} + c_2 e^{-x}$     11.  $y = c_1 e^{6x} + c_2 e^{2x/3}$

13.  $y = c_1 e^{x/3} + c_2 e^{-3x}$     15.  $y = c_1 e^{x/3} + c_2 e^{-x}$

17.  $y = e^x(c_1 e^{x\sqrt{2}/2} + c_2 e^{-x\sqrt{2}/2})$

19.  $y = e^{3x/8}(c_1 e^{x\sqrt{41}/8} + c_2 e^{-x\sqrt{41}/8})$

21.  $y = e^{3x/2}(c_1 e^{x\sqrt{13}/2} + c_2 e^{-x\sqrt{13}/2})$

23.  $y = e^{-x/2}(c_1 e^{x\sqrt{33}/2} + c_2 e^{-x\sqrt{33}/2})$

25.  $y = c_1 e^{3ax/2} + c_2 e^{-4ax}$     27.  $y = \frac{1}{5}(3e^{7x} + 7e^{-3x})$

29.  $y = \frac{e^3}{e^7 - 1}(e^{4x} - e^{-3x})$     31.  $y = c_1 + c_2 e^{-x} + c_3 e^{3x}$

33.  $y = c_1 e^x + c_2 e^{-x} + c_3 e^{2x} + c_4 e^{-2x}$

35.  $v = c_1 e^{as} + c_2 e^{-as}$

**Exercises 31.8, page 976**

1.  $y = e^{-5x}(c_1 + c_2 x)$     3.  $y = c_1 + c_2 e^{10x}$

5.  $y = (c_1 + c_2 x)e^x$     7.  $y = (c_1 + c_2 x)e^{-6x}$

9.  $y = c_1 \sin 3x + c_2 \cos 3x$

11.  $y = e^{-x/2}(c_1 \sin \frac{1}{2}\sqrt{7}x + c_2 \cos \frac{1}{2}\sqrt{7}x)$

13.  $y = c_1 e^x + c_2 e^{-x} + c_3 \sin x + c_4 \cos x$

15.  $y = c_1 \sin \frac{1}{2}x + c_2 \cos \frac{1}{2}x$     17.  $y = (c_1 + c_2 x)e^{3x/4}$

19.  $y = c_1 \sin \frac{2}{5}x + c_2 \cos \frac{2}{5}x$

21.  $y = e^x(c_1 \cos \frac{1}{2}\sqrt{6}x + c_2 \sin \frac{1}{2}\sqrt{6}x)$

23.  $y = (c_1 + c_2 x)e^{4x/5}$

25.  $y = e^{3x/4}(c_1 e^{x\sqrt{17}/4} + c_2 e^{-x\sqrt{17}/4})$

27.  $y = c_1 e^{x(-6+\sqrt{42})/3} + c_2 e^{x(-6-\sqrt{42})/3}$

29.  $y = e^{2x}(c_1 + c_2 x + c_3 x^2)$

31.  $y = (c_1 + c_2 x) \sin x + (c_3 + c_4 x) \cos x$

33.  $y = e^{-x} \sin 3x$     35.  $y = e^{4x}(4 - 14x)$

37.  $D^2 y - 9y = 0$

39.  $D^2 y + 9y = 0$ . The sum of  $\cos 3x$  and  $\sin 3x$  with no exponential factor indicates imaginary roots with  $\alpha = 0$  and  $\beta = 3$ .

41. (a)  $y = 0$     (b)  $y = 0$

**Exercises 31.9, page 982**

1.  $y_p = A + Bx + Cx^2 + Ee^{-x}$

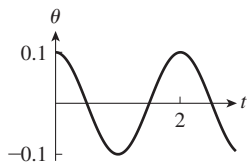
3.  $y = c_1 \sin 2x + c_2 \cos 2x + \frac{1}{3} \sin x - \frac{4}{5} e^{-x}$



5.  $y = c_1 e^{2x} + c_2 e^{-x} - 2$     7.  $y = c_1 e^{-x} + c_2 e^x - 4 - x^2$   
 9.  $y = c_1 + c_2 e^{3x} - \frac{3}{4} e^x - \frac{1}{2} x e^x$   
 11.  $y = c_1 e^{x/3} + c_2 e^{-x/3} - \frac{1}{10} \sin x$   
 13.  $y = (c_1 + c_2 x) e^x + 10 + 6x + x^2 - \frac{2}{25} \sin 3x + \frac{3}{50} \cos 3x$   
 15.  $y = c_1 \sin 2x + c_2 \cos 2x + 3x \cos 2x$   
 17.  $y = c_1 e^{-5x} + c_2 e^{6x} - \frac{1}{3}$     19.  $y = c_1 e^{4x/3} + c_2 e^{-x} + \frac{1}{4} e^{3x}$   
 21.  $y = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{5} \sin x - \frac{2}{5} \cos x$   
 23.  $y = c_1 \sin x + c_2 \cos x - \frac{1}{3} \sin 2x + 4$   
 25.  $y = c_1 e^{-x} + c_2 e^{-4x} - \frac{7}{100} e^x + \frac{1}{10} x e^x + 1$   
 27.  $y = c_1 + c_2 e^x + c_3 e^{-x} + \frac{1}{10} \cos 2x$   
 29.  $y = c_1 \sin x + c_2 \cos x + \frac{1}{2} x \sin x$   
 31.  $y = c_1 + c_2 e^{-2x} + 2x^2 - 2x - \frac{1}{2} x e^{-2x}$   
 33.  $y = \frac{1}{6} (11e^{3x} + 5e^{-2x} + e^x - 5)$   
 35.  $y = -\frac{2}{3} \sin x + \pi \cos x + x - \frac{1}{3} \sin 2x$   
 37.  $y = ce^x - x^2 - 2x - 2$ ; If solved as a first-order linear equation, integration is more complex.  
 39.  $\frac{16}{\pi} \left( \frac{1}{16 - \pi^2} \sin \frac{\pi x}{2} + \frac{1}{32 - 8\pi^2} \sin \frac{3\pi x}{2} + \dots \right)$

**Exercises 31.10, page 988**

1.  $x = \frac{1}{2} \sin 4t$     3.  $x = e^{-0.1t} (0.04 \sin 10t + 4 \cos 10t)$   
 5.  $\theta = 0.1 \cos 3.1t$     7. 10



9.  $s = (s_0 - L) \cos \sqrt{\frac{g}{e}} t + L$     11.  $y = 0.100 \cos 14.0t$   
 13.  $y = -\frac{7}{960} \sin 14t + 0.100 \cos 14t + \frac{49}{960} \sin 2t$   
 15.  $q = 2.24 \times 10^{-4} e^{-20t} \sin 2240t$   
 17.  $q = 0.01 (1 - \cos 316t)$   
 19.  $q = e^{-10t} (c_1 \sin 99.5t + c_2 \cos 99.5t) - 1.81 \times 10^{-3} \sin 120\pi t - 1.03 \times 10^{-4} \cos 120\pi t$   
 21.  $i = 10^{-6} (2.00 \cos (1.58 \times 10^4 t) + 158 \sin (1.58 \times 10^4 t) - 2.00 e^{-200t})$   
 23.  $i_p = 0.528 \sin 100t - 3.52 \cos 100t$   
 25.  $y = \frac{w}{24EI} (6L^2 x^2 - 4Lx^3 + x^4)$   
 27.  $y = c_1 \sin 8t + c_2 \cos 8t + \sum_{n=1}^{\infty} \frac{32(-1)^{n+1}}{n(64 - \pi^2 n^2)} \sin n\pi t$

**Exercises 31.11, page 993**

1.  $F(s) = \int_0^{\infty} e^{-st} dt = -\frac{1}{s} e^{-st} \Big|_0^{\infty} = \frac{1}{s}$   
 3.  $(s^2 - 2s)\mathcal{L}(f) - s + 2$     5.  $\frac{1}{s - 3}$     7.  $\frac{30}{(s + 2)^4}$   
 9.  $\frac{s - 2}{s^2 + 4}$     11.  $\frac{3}{s} + \frac{2(s^2 - 9)}{(s^2 + 9)^2}$     13.  $s^2 \mathcal{L}(f) + s \mathcal{L}(f)$   
 15.  $(2s^2 - s + 1)\mathcal{L}(f) - 2s + 1$     17.  $t^2$     19.  $\frac{15}{2} e^{-3t}$   
 21.  $\frac{1}{2} t^2 e^{-t}$     23.  $\frac{1}{54} (9t \sin 3t + 2 \sin 3t - 6t \cos 3t)$   
 25.  $-\frac{1}{3} e^{-t} - \frac{8}{3} e^{2t} + 7e^{3t}$     27.  $\frac{1}{2} e^t (4 \cos 2t + 5 \sin 2t)$   
 29.  $-\frac{d}{ds} \left( \frac{1}{s + a} \right) = \frac{1}{(s + a)^2}$

**Exercises 31.12, page 997**

1.  $y = 2e^{t/2}$     3.  $y = \frac{t}{2} \sin t + \sin t + 2 \cos t$     5.  $y = e^{-t}$   
 7.  $y = -e^{3t/2}$     9.  $y = (1 + t)e^{-3t}$     11.  $y = \frac{1}{2} \sin 2t$   
 13.  $y = e^{-t/2} \cos t$     15.  $y = e^{2t} \cos t$     17.  $y = 1 + \sin t$   
 19.  $y = e^{-t} \left( \frac{1}{2} t^2 + 3t + 1 \right)$     21.  $y = 2e^{3t} + 3e^{-2t}$   
 23.  $y = \frac{3}{2} e^t - \frac{3}{2} e^{-t} - \sin 2t$     25.  $v = 6(1 - e^{-t/2})$   
 27.  $y = 4 \cos 80t$     29.  $q = 1.6 \times 10^{-4} (1 - e^{-5000t})$   
 31.  $i = 5t \sin 50t$     33.  $y = \sin 3t - 3t \cos 3t$   
 35.  $i = 5.0e^{-50t} - 5.0e^{-100t}$     37.  $y = \frac{w}{24EI} (L^3 x - 2Lx^3 + x^4)$

**Review Exercises for Chapter 31, page 999**

1.  $2 \ln(x^2 + 1) - \frac{1}{2y^2} = c$     3.  $y^2 = 2x - 4 \sin x + c$   
 5.  $y = c_1 + c_2 e^{-x/2}$     7.  $y = (c_1 + c_2 x) e^{x/4}$   
 9.  $2x^2 + 4xy + y^4 = c$     11.  $P = cV^5 - \frac{1}{3} V^2$   
 13.  $y = \frac{2c}{1 - ce^{2x}}$     15.  $y = e^{-x} (c_1 \sin \sqrt{5}x + c_2 \cos \sqrt{5}x)$   
 17.  $y = e^{-2x} + ce^{-4x}$     19.  $y = \frac{1}{2} (c - x^2) \csc x$   
 21.  $s = c_1 e^t + c_2 e^{-3t/2} - 2$   
 23.  $y = e^{-x/2} (c_1 e^{x\sqrt{5}/2} + c_2 e^{-x\sqrt{5}/2}) + 2e^x$   
 25.  $y = c_1 e^{2x/3} + c_2 e^{4x/3} + \frac{1}{2} x + \frac{25}{8}$   
 27.  $y = c_1 e^x + c_2 \sin 3x + c_3 \cos 3x - \frac{1}{16} (\sin x + \cos x)$   
 29.  $y = c_1 e^{-x} + c_2 e^{8x} - \frac{2}{9} x e^{-x}$   
 31.  $y = c_1 \sin 5x + c_2 \cos 5x + 5x \sin 5x$   
 33.  $y^3 = 8 \sin^2 x$     35.  $y = 2x - 1 - e^{-2x}$   
 37.  $v = 2e^{-t/2} \sin \left( \frac{1}{2} \sqrt{15} t \right)$   
 39.  $y = \frac{1}{25} [16 \sin x + 12 \cos x - 3e^{-2x} (4 + 5x)]$   
 41.  $y = e^{t/4}$     43.  $y = \frac{1}{2} (e^{3t} - e^t)$     45.  $y = -4 \sin t$   
 47.  $y = \frac{1}{25} (3e^x - 3 \cos \frac{3x}{4} - 4 \sin \frac{3x}{4})$

**B.84** ANSWERS TO ODD-NUMBERED EXERCISES

**49.**

$x$	0	0.1	0.2	0.3	0.4
$y$	0	0.100	0.201	0.305	0.414

**51.** (a) and (b)  $y = e^{3x}$      **53.**  $x = t^2 + 1; y = \frac{1}{t^2 + 1}$

**55.**  $r = r_0 + kt$      **57.**  $m = m_0 e^{kt}$      **59.** 3.93 m/s

**61.**  $y^2 - 2xy - 8 = 0$      **63.** 40.0 s     **65.**  $5.31 \times 10^8$  years

**67.** 7.8 billion     **69.**  $5y^2 + x^2 = c$      **71.** 36.1 °C

**73.**  $i = 0.5(1 - e^{-20t})$

**75.**  $y = 0.25e^{-2t}(2 \cos 4t + \sin 4t)$ , underdamped

**77.**  $q = e^{-6t}(0.4 \cos 8t + 0.3 \sin 8t) - 0.4 \cos 10t$      **79.**  $i = 0$

**81.**  $i = 12(1 - e^{-t/2}); i(0.3) = 1.67$  A

**83.**  $q = 10^{-4}e^{-8t}(4.0 \cos 200t + 0.16 \sin 200t)$

**85.**  $y = 0.25t \sin 8t$      **87.** 2.47 L

**89.**  $y = \frac{10}{3EI} [100x^3 - x^4 + xL^2(L - 100)]$

**91.**  $i = 4.42e^{-66.7t} \sin(226t)$

# Solutions to Practice Test Problems

## Chapter 1

1.  $\sqrt{9 + 16} = \sqrt{25} = 5$

2.  $\frac{(7)(-3)(-2)}{(-6)(0)}$  is undefined (division by zero).

3.  $\frac{3.372 \times 10^{-3}}{7.526 \times 10^{12}} = 4.480 \times 10^{-16}$

4.  $\frac{(+6)(-2) - 3(-1)}{5 - 2} = \frac{-12 - (-3)}{3} = \frac{-12 + 3}{3}$   
 $= \frac{-9}{3} = -3$

5.  $\frac{346.4 - 23.5}{287.7} - \frac{0.944^3}{(3.46)(0.109)} = -1.108$

6.  $(2a^0b^{-2}c^3)^{-3} = 2^{-3}a^{0(-3)}b^{(-2)(-3)}c^{3(-3)} = 2^{-3}a^0b^6c^{-9}$   
 $= \frac{b^6}{8c^9}$

7.  $(2x + 3)^2 = (2x + 3)(2x + 3)$   
 $= 2x(2x) + 2x(3) + 3(2x) + 3(3)$   
 $= 4x^2 + 6x + 6x + 9$   
 $= 4x^2 + 12x + 9$

8.  $3m^2(am - 2m^3) = 3m^2(am) + 3m^2(-2m^3)$   
 $= 3am^3 - 6m^5$

9.  $\frac{8a^3x^2 - 4a^2x^4}{-2ax^2} = \frac{8a^3x^2}{-2ax^2} - \frac{4a^2x^4}{-2ax^2} = -4a^2 - (-2ax^2)$   
 $= -4a^2 + 2ax^2$

10. 
$$\begin{array}{r} 3x - 5 \quad \text{(quotient)} \\ 2x - 1 \overline{)6x^2 - 13x + 7} \\ \underline{6x^2 - 3x} \phantom{+ 7} \\ -10x + 7 \\ \underline{-10x + 5} \\ 2 \quad \text{(remainder)} \end{array}$$

11.  $(2x - 3)(x + 7)$   
 $= 2x(x) + 2x(7) + (-3)(x) + (-3)(7)$   
 $= 2x^2 + 14x - 3x - 21 = 2x^2 + 11x - 21$

12.  $3x - [4x - (3 - 2x)] = 3x - [4x - 3 + 2x]$   
 $= 3x - [6x - 3]$   
 $= 3x - 6x + 3 = -3x + 3$

13.  $5y - 2(y - 4) = 7$   
 $5y - 2y + 8 = 7$   
 $3y + 8 - 8 = 7 - 8$   
 $3y = -1$   
 $y = -1/3$

14.  $3(x - 3) = x - (2 - 3d)$   
 $3x - 9 = x - 2 + 3d$   
 $3x - 9 - x + 9 = x - 2 + 3d - x + 9$   
 $2x = 3d + 7$   
 $x = \frac{3d + 7}{2}$

15.  $0.0000036 = 3.6 \times 10^{-6}$  (six places to right)

16. 

$-\pi$	$-3$	$0.3$	$\sqrt{2}$	$ -4 $	(order)
$-3.14$	$-3$	$0.3$	$1.41$	$4$	(value)

17.  $3(5 + 8) = 3(5) + 3(8)$  illustrates distributive law.

18. (a) 5 (b) 3.0 (zero is significant)

19. Evaluation:  
 $1000(1 + 0.05/2)^{2(3)} = 1000(1.025)^6 = \$1159.69$

20.  $8(100 - x)^2 + x^2 = 8(100 - x)(100 - x) + x^2$   
 $= 8(10\,000 - 200x + x^2) + x^2$   
 $= 80\,000 - 1600x + 8x^2 + x^2$   
 $= 80\,000 - 1600x + 9x^2$

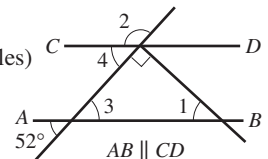
21.  $L = L_0[1 + \alpha(t_2 - t_1)]$   
 $= L_0[1 + \alpha t_2 - \alpha t_1]$   
 $= L_0 + \alpha L_0 t_2 - \alpha L_0 t_1$   
 $L - L_0 + \alpha L_0 t_1 = \alpha L_0 t_2$   
 $t_2 = \frac{L - L_0 + \alpha L_0 t_1}{\alpha L_0}$

22. Let  $n$  = number of newtons of second alloy  
 $0.3(20) + 0.8n = 0.6(n + 20)$   
 $6 + 0.8n = 0.6n + 12$   
 $0.2n = 6$   
 $n = 30$  N

## Chapter 2

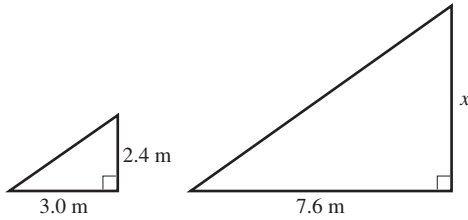
1.  $\angle 1 + \angle 3 + 90^\circ = 180^\circ$  (sum of angles of a triangle)  
 $\angle 3 = 52^\circ$  (vertical angles)  
 $\angle 1 = 180^\circ - 90^\circ - 52^\circ = 38^\circ$

2.  $\angle 2 + \angle 4 = 180^\circ$  (straight angle)  
 $\angle 4 = 52^\circ$  (corresponding angles)  
 $\angle 2 + 52^\circ = 180^\circ$   
 $\angle 2 = 128^\circ$



**C.2 SOLUTIONS TO PRACTICE TEST PROBLEMS**

3.



$$\frac{x}{2.4} = \frac{7.6}{3.0} \quad x = \frac{2.4(7.6)}{3.0} = 6.1 \text{ m}$$

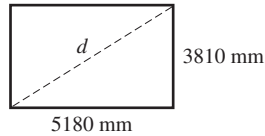
4. Use Hero's formula:

$$s = \frac{1}{2}(2.46 + 3.65 + 4.07) = 5.09 \text{ cm}$$

$$A = \sqrt{5.09(5.09 - 2.46)(5.09 - 3.65)(5.09 - 4.07)}$$

$$= 4.43 \text{ cm}^2$$

5.  $d^2 = 3810^2 + 5180^2$   
 $d = \sqrt{3810^2 + 5180^2}$   
 $= 6430 \text{ mm}$



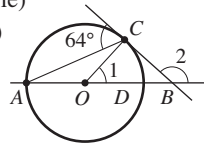
6.  $A = \frac{1}{2}h(b_1 + b_2)$   
 $= \frac{1}{2}(13.8)(49.8 + 23.5) = 506 \text{ mm}^2$

7. (a)  $A = 6s^2 = 6(4.50)^2 = 122 \text{ cm}^2$   
 (b)  $V = s^3 = 4.50^3 = 91.1 \text{ cm}^3$

8.  $c = 2\pi r$        $A = 4\pi r^2$   
 $21.0 = 2\pi r$        $= 4\pi\left(\frac{10.5}{\pi}\right)^2 = \frac{4(10.5^2)}{\pi}$   
 $r = \frac{10.5}{\pi}$        $= 140 \text{ cm}^2$

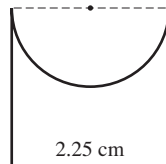
9.  $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(2.08^2)(1.78)$   
 $= 8.06 \text{ m}^3$

10.  $\angle ACO + 64^\circ = 90^\circ$  (tangent perpendicular to radius)  
 $\angle ACO = 26^\circ$   
 $\angle A = \angle ACO = 26^\circ$  (isosceles triangle)  
 $\frac{1}{2}\widehat{CD} = 26^\circ$  (intercepted arc)  
 $\widehat{CD} = 52^\circ$   
 $\angle 1 = 52^\circ$  (central angle)



11.  $\angle CBO + \angle 1 + 90^\circ = 180^\circ$  (sum of angles of triangle)  
 $\angle CBO + 52^\circ + 90^\circ = 180^\circ$   
 $\angle CBO = 180^\circ - 52^\circ - 90^\circ = 38^\circ$   
 $\angle 2 + \angle CBO = 180^\circ$  (straight angle)  
 $\angle 2 + 38^\circ = 180^\circ$   
 $\angle 2 = 142^\circ$

12.  $r = \frac{1}{2}(2.25) \text{ cm}$   
 $p = 3(2.25) + \frac{1}{2}(2\pi)\left[\frac{1}{2}(2.25)\right]$   
 $= 10.3 \text{ cm}$



13.  $A = 2.25^2 - \frac{1}{2}\pi\left[\frac{1}{2}(2.25)\right]^2 = 3.07 \text{ cm}^2$

14.  $A = \frac{1}{2}(50)[0 + 2(90) + 2(145) + 2(260)$   
 $+ 2(205) + 2(110) + 20]$   
 $= 41\,000 \text{ m}^2$

**Chapter 3**

1.  $f(x) = 2x - x^2 + \frac{8}{x}$

$$f(-4) = 2(-4) - (-4)^2 + \frac{8}{-4}$$

$$= -8 - 16 - 2$$

$$= -26$$

$$f(2.385) = 2(2.385) - (2.385)^2 + \frac{8}{2.385} = 2.436$$

2.  $w = 2000 - 10t$

3.  $f(x) = 4 - 2x$

$$y = 4 - 2x$$

$$y = 4 - 2(-1) = 6$$

$$y = 4 - 2(0) = 4$$

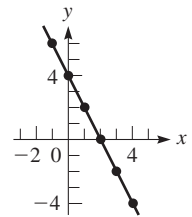
$$y = 4 - 2(1) = 2$$

$$y = 4 - 2(2) = 0$$

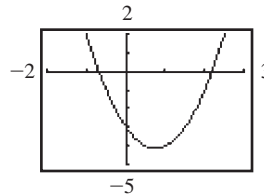
$$y = 4 - 2(3) = -2$$

$$y = 4 - 2(4) = -4$$

x	y
-1	6
0	4
1	2
2	0
3	-2
4	-4



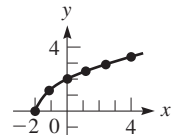
4.  $y = 2x^2 - 3x - 3$



$x = -0.7$  and  $x = 2.2$

5.  $y = \sqrt{4 + 2x}$   
 $y = \sqrt{4 + 2(-2)} = 0$   
 $y = \sqrt{4 + 2(-1)} = 1.4$   
 $y = \sqrt{4 + 2(0)} = 2$   
 $y = \sqrt{4 + 2(1)} = 2.4$   
 $y = \sqrt{4 + 2(2)} = 2.8$   
 $y = \sqrt{4 + 2(4)} = 3.5$

x	y
-2	0
-1	1.4
0	2
1	2.4
2	2.8
4	3.5



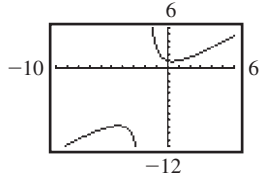
6. On negative x-axis

7.  $f(x) = \sqrt{6 - x}$

Domain:  $x \leq 6$ , or  $(-\infty, 6]$ ;  $x$  cannot be greater than 6 to have real values of  $f(x)$ .

Range:  $f(x) \geq 0$ , or  $[0, \infty)$ ;  $\sqrt{6 - x}$  is the principal square root of  $6 - x$  and cannot be negative.

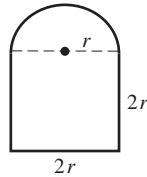
8. Shifting  $y = 2x^2 - 3$  to the right 1 and up 3 gives  
 $y = 2(x - 1)^2 - 3 + 3 = 2(x - 1)^2 = 2x^2 - 4x + 2$ .
9. Range:  $y < -8.9$  or  $y > 0.9$ , or  $(-\infty, -8.9)$  or  $(0.9, \infty)$



10. Let  $r =$  radius of circular part  
**square**      **semicircle**

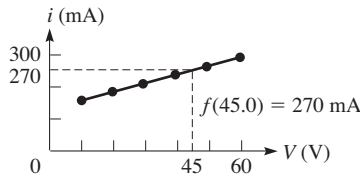
$$A = (2r)(2r) + \frac{1}{2}(\pi r^2)$$

$$= 4r^2 + \frac{1}{2}\pi r^2$$



11. 

Voltage $V$	10.0	20.0	30.0	40.0	50.0	60.0
Current $i$	145	188	220	255	285	315



12.  $V = 20.0$  V for  $i = 188$  mA  
 $V = 30.0$  V for  $i = 220$  mA

$$\frac{x}{10} = \frac{12}{32}, \quad x = 3.8 \quad (\text{rounded off})$$

$$V = 20.0 + 3.8 = 23.8 \text{ V}$$

for  $i = 200$  mA

$$10 \begin{bmatrix} x[ & & ]12 \\ & 20.0 & 188 \\ & 30.0 & 220 \end{bmatrix} 32$$

**Chapter 4**

- $39' = \left(\frac{39}{60}\right)^\circ = 0.65^\circ$   
 $37^\circ 39' = 37.65^\circ$
- $\tan 73.8^\circ = 3.44$
- $\cos \theta = 0.3726$ ;  $\theta = 68.12^\circ$

4. Let  $x =$  distance from course to east

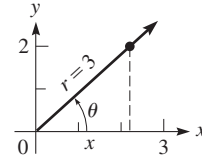
$$\frac{x}{22.62} = \sin 4.05^\circ$$

$$x = 22.62 \sin 4.05^\circ$$

$$= 1.598 \text{ km}$$



5.  $\sin \theta = \frac{2}{3}$   
 $x = \sqrt{3^2 - 2^2} = \sqrt{5}$   
 $\tan \theta = \frac{2}{\sqrt{5}}$



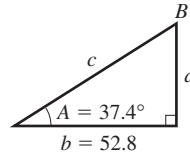
6.  $\tan \theta = 1.294$ ;  $\csc \theta = 1.264$

7.  $B = 90^\circ - 37.4^\circ = 52.6^\circ$        $\frac{52.8}{c} = \cos 37.4^\circ$

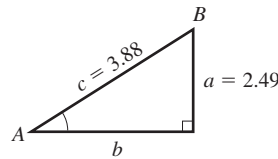
$$\frac{a}{52.8} = \tan 37.4^\circ$$

$$a = 52.8 \tan 37.4^\circ = 40.4$$

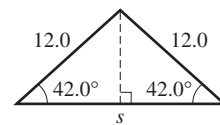
$$c = \frac{52.8}{\cos 37.4^\circ} = 66.5$$



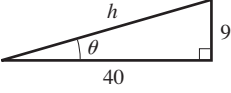
8.  $2.49^2 + b^2 = 3.88^2$   
 $b = \sqrt{3.88^2 - 2.49^2} = 2.98$   
 $\sin A = \frac{2.49}{3.88}$   
 $A = 39.9^\circ$        $B = 50.1^\circ$



9.  $\frac{s/2}{12.0} = \cos 42.0^\circ$   
 $s = 24.0 \cos 42.0^\circ = 17.8$

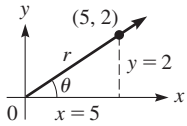


**C.4 SOLUTIONS TO PRACTICE TEST PROBLEMS**

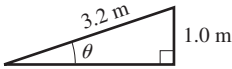
10.   $h = \sqrt{9^2 + 40^2} = 41$   
 $\sin \theta = \frac{9}{41}, \cos \theta = \frac{40}{41}$   
 $\frac{\sin \theta}{\cos \theta} = \frac{9/41}{40/41} = \frac{9}{40}$

11.  $\lambda = d \sin \theta$   
 $= 30.05 \sin 1.167^\circ$   
 $= 0.6120 \mu\text{m}$

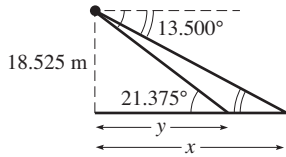
12.  $r = \sqrt{5^2 + 2^2} = \sqrt{29}$   
 $\sin \theta = \frac{2}{\sqrt{29}} = 0.3714$   $\csc \theta = \frac{\sqrt{29}}{2} = 2.693$   
 $\cos \theta = \frac{5}{\sqrt{29}} = 0.9285$   $\sec \theta = \frac{\sqrt{29}}{5} = 1.077$   
 $\tan \theta = \frac{2}{5} = 0.4000$   $\cot \theta = \frac{5}{2} = 2.500$



13.  $\sin \theta = 1.0/3.2$   
 $\theta = 18^\circ$



14. Distance between points is  $x - y$ .  
 $\frac{18.525}{x} = \tan 13.500^\circ$   $\frac{18.525}{y} = \tan 21.375^\circ$   
 $x - y = \frac{18.525}{\tan 13.500^\circ} - \frac{18.525}{\tan 21.375^\circ} = 29.831 \text{ m}$



**Chapter 5**

1. Points (2, -5) and (-1, 4)  
 $m = \frac{4 - (-5)}{-1 - 2} = \frac{9}{-3} = -3$

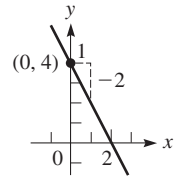
2.  $2x + 5y = 11$   $2(-2) + 5(3) = 11$  OK  
 $y - 5x = 12$   $3 - 5(-2) = 13$  No  
 $x = -2, y = 3$  Not a solution. Values do not satisfy second equation.

3.  $\begin{vmatrix} -1 & 3 & -2 \\ 4 & -3 & 0 \\ 5 & -4 & 2 \end{vmatrix} \begin{vmatrix} -1 & 3 \\ 4 & -3 \\ 5 & -4 \end{vmatrix} \begin{vmatrix} -1 & 3 \\ 4 & -3 \\ 5 & -4 \end{vmatrix}$   
 $4(-3) = 6 + 0 + 32 - 30 - 0 - 24 = -16$

4.  $x + 2y = 5$   $4y = 3 - 2(5 - 2y)$   
 $4y = 3 - 2x$   $4y = 3 - 10 + 4y$   
 $x = 5 - 2y$   $0 = -7$   
 Inconsistent

5.  $3x - 2y = 4$   
 $2x + 5y = -1$   
 $x = \frac{\begin{vmatrix} 4 & -2 \\ -1 & 5 \end{vmatrix}}{\begin{vmatrix} 3 & -2 \\ 2 & 5 \end{vmatrix}} = \frac{20 - 2}{15 - (-4)} = \frac{18}{19}$   
 $y = \frac{\begin{vmatrix} 3 & 4 \\ 2 & -1 \end{vmatrix}}{\begin{vmatrix} 3 & -2 \\ 2 & 5 \end{vmatrix}} = \frac{-3 - 8}{15 - (-4)} = -\frac{11}{19}$

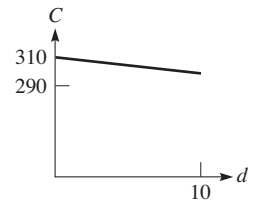
6.  $2x + y = 4$   
 $y = -2x + 4$   
 $m = -2$   $b = 4$



7.  $2l + 2w = 24$  (perimeter)  
 $l + w = 12$   
 $l - w = 6.0$   
 $2l = 18$   
 $l = 9 \text{ km}$   
 $9 = w + 6$   
 $w = 3 \text{ km}$

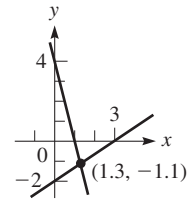
8.  $6N - 2P = 13$   
 $4N + 3P = -13$   
 $18N - 6P = 39$   
 $8N + 6P = -26$   
 $26N = 13$   
 $N = \frac{1}{2}$

9.  $C = 310 - 2d$



$6\left(\frac{1}{2}\right) - 2P = 13$   
 $-2P = 10$   
 $P = -5$

10.  $2x - 3y = 6$   $4x + y = 4$   
 $x = 0: y = -2$   $x = 0: y = 4$   
 $y = 0: x = 3$   $y = 0: x = 1$   
 Int: (0, -2), (3, 0) Int: (0, 4), (1, 0)  
 $x = 1.3$   $y = -1.1$



11. Let  $x$  = vol. of first alloy  
 $y$  = vol. of second alloy  
 $z$  = vol. of third alloy

$x + y + z = 100$	total vol.
$0.6x + 0.5y + 0.3z = 40$	copper
$0.3x + 0.3y = 15$	zinc

---

$x + y + z = 100$
$6x + 5y + 3z = 400$
$3x + 3y = 150$
$3x + 2y = 100$
$3x + 3y = 150$

$y = 50 \text{ cm}^3$   
 $x = 0 \text{ cm}^3$   
 $z = 50 \text{ cm}^3$

12.  $3x + 2y - z = 4$   
 $2x - y + 3z = -2$   
 $x + 4z = 5$

3	4	-1
2	-2	3
1	5	4

$y =$

3	2	-1
2	-1	3
1	0	4

$= \frac{-24 + 12 - 10 - 2 - 45 - 32}{-12 + 6 + 0 - 1 - 0 - 16}$   
 $= \frac{-101}{-23} = \frac{101}{23}$

**Chapter 6**

1.  $2x(2x - 3)^2 = 2x[(2x)^2 - 2(2x)(3) + 3^2]$   
 $= 2x(4x^2 - 12x + 9)$   
 $= 8x^3 - 24x^2 + 18x$

2.  $\frac{1}{R} = \frac{1}{R_1 + r} + \frac{1}{R_2}$   
 $\frac{RR_2(R_1 + r)}{R} = \frac{RR_2(R_1 + r)}{R_1 + r} + \frac{RR_2(R_1 + r)}{R_2}$   
 $R_2(R_1 + r) = RR_2 + R(R_1 + r)$   
 $R_1R_2 + rR_2 = RR_2 + RR_1 + rR$   
 $R_1R_2 - RR_1 = RR_2 + rR - rR_2$   
 $R_1(R_2 - R) = RR_2 + rR - rR_2$   
 $R_1 = \frac{RR_2 + rR - rR_2}{R_2 - R}$

3.  $\frac{2x^2 + 5x - 3}{2x^2 + 12x + 18} = \frac{(2x - 1)(x + 3)}{2(x + 3)^2} = \frac{2x - 1}{2(x + 3)}, x \neq -3$

4.  $4x^2 - 16y^2 = 4(x^2 - 4y^2) = 4(x + 2y)(x - 2y)$

5.  $pb^3 + 8a^3p = p(b^3 + 8a^3)$   
 $= p(b + 2a)(b^2 - 2ab + 4a^2)$

6.  $2a - 4T - ba + 2bT = 2(a - 2T) - b(a - 2T)$   
 $= (2 - b)(a - 2T)$

7.  $36x^2 + 14x - 16 = 2(18x^2 + 7x - 8)$   
 $= 2(9x + 8)(2x - 1)$

8.  $\frac{3}{4x^2} - \frac{2}{x^2 - x} - \frac{x}{2x - 2} = \frac{3}{4x^2} - \frac{2}{x(x - 1)} - \frac{x}{2(x - 1)}$   
 $= \frac{3(x - 1) - 2(4x) - x(2x^2)}{4x^2(x - 1)}$   
 $= \frac{3x - 3 - 8x - 2x^3}{4x^2(x - 1)}$   
 $= \frac{-2x^3 - 5x - 3}{4x^2(x - 1)}$

9.  $\frac{x^2 + x}{2 - x} \div \frac{x^2}{x^2 - 4x + 4} = \frac{x^2 + x}{2 - x} \times \frac{x^2 - 4x + 4}{x^2}$   
 $= \frac{x(x + 1)}{2 - x} \times \frac{(x - 2)^2}{x^2}$   
 $= \frac{x(x + 1)(x - 2)^2}{(x - 2)(x^2)}$   
 $= -\frac{(x + 1)(x - 2)}{x}, x \neq 2$

10.  $1 - \frac{3}{2x + 2} = \frac{2(x + 1) - 3}{2(x + 1)}$   
 $\frac{x}{5} - \frac{1}{2} = \frac{2x - 5}{10}$   
 $= \frac{2x + 2 - 3}{2(x + 1)} \times \frac{10}{2x - 5}$   
 $= \frac{5(2x - 1)}{(x + 1)(2x - 5)}$

11. Let  $t$  = time working together  $\frac{48t}{12} + \frac{48t}{16} = 48$   
 $\frac{t}{12} + \frac{t}{16} = 1$   $4t + 3t = 48$   
LCD of 12 and 16 is 48.  $t = \frac{48}{7} = 6.9$  days

12.  $\frac{3}{2x^2 - 3x} + \frac{1}{x} = \frac{3}{2x - 3}$  LCD =  $x(2x - 3), x \neq 0, 3/2$   
 $\frac{3x(2x - 3)}{x(2x - 3)} + \frac{x(2x - 3)}{x} = \frac{3x(2x - 3)}{2x - 3}$   
 $3 + (2x - 3) = 3x$   
 $x = 0$   
 No solution.

**Chapter 7**

1.  $2x^2 + 5x = 12$  2.  $x^2 = 3x + 5$   
 $2x^2 + 5x - 12 = 0$   $x^2 - 3x - 5 = 0$   
 $(2x - 3)(x + 4) = 0$   $a = 1, b = -3, c = -5$   
 $2x - 3 = 0 \quad x + 4 = 0$   $x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(-5)}}{2(1)}$   
 $x = \frac{3}{2} \quad \text{or} \quad x = -4$   $= \frac{3 \pm \sqrt{29}}{2}$



C.6 SOLUTIONS TO PRACTICE TEST PROBLEMS

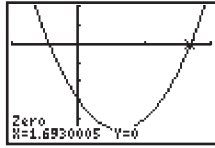
3.  $4x^2 - 5x - 3 = 0$

On calculator

$Y_1 = 4X^2 - 5X - 3$

From zero feature

$x = -0.44$  or  $x = 1.69$



4.  $2x^2 - x = 6 - 2x(3 - x)$

$2x^2 - x = 6 - 6x + 2x^2$

$5x = 6$  (not quadratic)

$x = \frac{6}{5}$

5.  $\frac{3}{x} - \frac{2}{x+2} = 1, x \neq 0, -2$

$\frac{3x(x+2)}{x} - \frac{2x(x+2)}{x+2} = x(x+2)$

$3(x+2) - 2x = x^2 + 2x$

$3x + 6 - 2x = x^2 + 2x$

$0 = x^2 + x - 6$

$(x+3)(x-2) = 0$

$x = -3, 2$

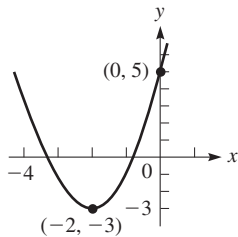
6.  $y = 2x^2 + 8x + 5$

$\frac{-b}{2a} = \frac{-8}{2(2)} = -2$

$y = 2(-2)^2 + 8(-2) + 5 = -3$

Min. pt. ( $a > 0$ ) is  $(-2, -3)$ ,

$c = 5$ , y-intercept is  $(0, 5)$ .



7.  $P = EI - RI^2$

$RI^2 - EI + P = 0$

$I = \frac{-(-E) \pm \sqrt{(-E)^2 - 4RP}}{2R}$

$= \frac{E \pm \sqrt{E^2 - 4RP}}{2R}$

8.  $x^2 - 6x - 9 = 0$

$x^2 - 6x = 9$

$x^2 - 6x + 9 = 9 + 9$

$(x-3)^2 = 18$

$x-3 = \pm\sqrt{18}$

$x = 3 \pm 3\sqrt{2}$

9. Let  $w$  = width of window

$h$  = height of window

$2w + 2h = 8.4$  (perimeter)

$w + h = 4.2$   $h = 4.2 - w$

$w(4.2 - w) = 3.8$  (area)

$-w^2 + 4.2w = 3.8$

$w^2 - 4.2w + 3.8 = 0$

$w = \frac{-(-4.2) \pm \sqrt{(-4.2)^2 - 4(3.8)}}{2}$

$= 2.88, 1.32$

$w = 1.3$  m,  $h = 2.9$  m or

$w = 2.9$  m,  $h = 1.3$  m

10.  $y = x^2 - 8x + 8$

$a = 1, b = -8$

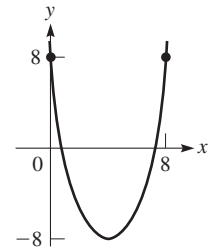
$\frac{-b}{2a} = \frac{-(-8)}{2(1)} = 4$

$y = 4^2 - 8(4) + 8 = -8$

$V(4, -8), c = 8, y$ -int  $(0, 8)$

For  $x = 8, y = 8$

$y = 0$  for  $x = 1.2, 6.8$



Chapter 8

1.  $150^\circ = \left(\frac{\pi}{180}\right)(150) = \frac{5\pi}{6}$

2. (a) + (b) - (c) +

3.  $\sin 205^\circ = -\sin(205^\circ - 180^\circ) = -\sin 25^\circ$

4.  $x = -9, y = 12$   
 $r = \sqrt{(-9)^2 + 12^2} = 15$

$\sin \theta = \frac{12}{15} = \frac{4}{5}$

$\sec \theta = \frac{15}{-9} = -\frac{5}{3}$

5.  $r = 1.40$  m

$\omega = 2200$  r/min  $= (2200$  r/min) $(2\pi$  rad/r)  
 $= 4400\pi$  rad/min

$v = \omega r = (4400\pi)(1.40) = 19\,000$  m/min

6.  $3.572 = 3.572\left(\frac{180^\circ}{\pi}\right) = 204.7^\circ$

7.  $\tan \theta = 0.2396$

$\theta_{\text{ref}} = 13.47^\circ$

$\theta = 13.47^\circ$  or

$\theta = 180^\circ + 13.47^\circ = 193.47^\circ$

8.  $\cos \theta = -0.8244, \csc \theta < 0;$

$\cos \theta$  negative,  $\csc \theta$  negative,

$\theta$  in third quadrant

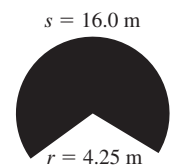
$\theta_{\text{ref}} = 0.6017, \theta = 3.7432$

9.  $s = r\theta, 16.0 = 4.25\theta$

$\theta = \frac{16.0}{4.25} = 3.76$  rad

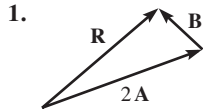
$A = \frac{1}{2}\theta r^2 = \frac{1}{2}(3.76)(4.25)^2$

$= 34.0$  m<sup>2</sup>

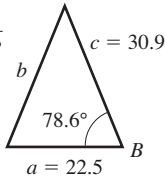


10.  $A = \frac{1}{2} \theta r^2$   
 $A = 38.5 \text{ cm}^2, d = 12.2 \text{ cm}, r = 6.10 \text{ cm}$   
 $38.5 = \frac{1}{2} \theta (6.10)^2, \theta = \frac{2(38.5)}{6.10^2} = 2.07$   
 $s = r\theta, s = 6.10(2.07) = 12.6 \text{ cm}$

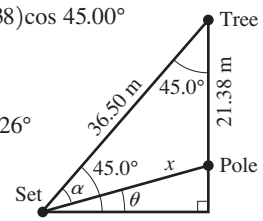
Chapter 9



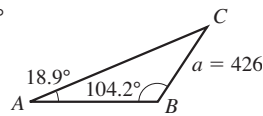
2.  $b^2 = a^2 + c^2 - 2ac \cos B$   
 $b = \sqrt{22.5^2 + 30.9^2 - 2(22.5)(30.9)\cos 78.6^\circ}$   
 $= 34.4$



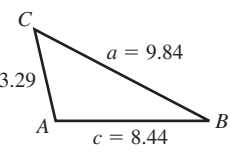
3.  $x^2 = 36.50^2 + 21.38^2 - 2(36.50)(21.38)\cos 45.00^\circ$   
 $x = 26.19 \text{ m}$   
 $\frac{21.38}{\sin \alpha} = \frac{26.19}{\sin 45.00^\circ}$   
 $\sin \alpha = \frac{21.38 \sin 45.00^\circ}{26.19}, \alpha = 35.26^\circ$   
 $\theta = 45.00^\circ - 35.26^\circ = 9.74^\circ$   
 Displacement is 26.19 m,  
 9.74° N of E.



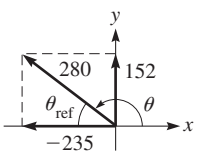
4.  $C = 180^\circ - (18.9^\circ + 104.2^\circ) = 56.9^\circ$   
 $\frac{c}{\sin C} = \frac{a}{\sin A}$   
 $c = \frac{426 \sin 56.9^\circ}{\sin 18.9^\circ} = 1100$



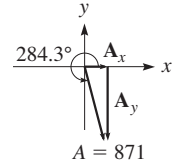
5. Since  $a$  is longest side, find  $A$  first.  
 $a^2 = b^2 + c^2 - 2bc \cos A$   
 $\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{3.29^2 + 8.44^2 - 9.84^2}{2(3.29)(8.44)}$   
 $A = 105.4^\circ$   
 $\frac{b}{\sin B} = \frac{a}{\sin A}$   
 $\sin B = \frac{b \sin A}{a} = \frac{3.29 \sin 105.4^\circ}{9.84}, b = 3.29$   
 $B = 18.8^\circ$   
 $C = 180^\circ - (105.4^\circ + 18.8^\circ) = 55.8^\circ$



6.  $R_x = -235, R_y = 152$   
 $R = \sqrt{(-235)^2 + 152^2} = 280$   
 $\tan \theta_{\text{ref}} = \left| \frac{152}{-235} \right| = \frac{152}{235}, \theta_{\text{ref}} = 32.9^\circ$   
 Quad. II:  $\theta = 180^\circ - 32.9^\circ = 147.1^\circ$

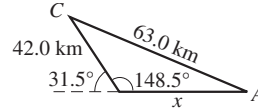


7.  $A_x = 871 \cos 284.3^\circ = 215$   
 $A_y = 871 \sin 284.3^\circ = -844$

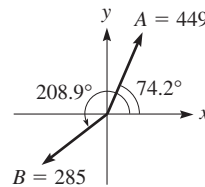


8.  $\frac{63.0}{\sin 148.5^\circ} = \frac{42.0}{\sin A}$   
 $\sin A = \frac{42.0 \sin 148.5^\circ}{63.0}$   
 $A = 20.4^\circ$   
 $C = 180^\circ - (148.5^\circ + 20.4^\circ) = 11.1^\circ$

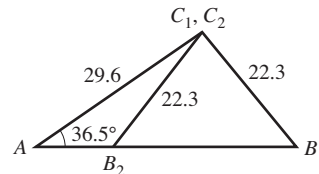
$\frac{x}{\sin 11.1^\circ} = \frac{63.0}{\sin 148.5^\circ}$   
 $x = \frac{63.0 \sin 11.1^\circ}{\sin 148.5^\circ} = 23.2 \text{ km}$



9.  $A_x = 449 \cos 74.2^\circ, B_x = 285 \cos 208.9^\circ$   
 $A_y = 449 \sin 74.2^\circ, B_y = 285 \sin 208.9^\circ$   
 $R_x = A_x + B_x = 449 \cos 74.2^\circ + 285 \cos 208.9^\circ = -127.3$   
 $R_y = A_y + B_y = 449 \sin 74.2^\circ + 285 \sin 208.9^\circ = 294.3$   
 $R = \sqrt{(-127.3)^2 + 294.3^2} = 321$   
 $\tan \theta_{\text{ref}} = \frac{294.3}{127.3}, \theta_{\text{ref}} = 66.6^\circ, \theta = 113.4^\circ$   
 $\theta$  is in second quadrant, since  $R_x$  is negative and  $R_y$  is positive.



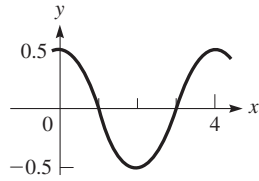
10.  $29.6 \sin 36.5^\circ = 17.6$   
 $17.6 < 22.3 < 29.6$  means two solutions.  
 $\frac{29.6}{\sin B} = \frac{22.3}{\sin 36.5^\circ}, \sin B = \frac{29.6 \sin 36.5^\circ}{22.3}$   
 $B_1 = 52.1^\circ, C_1 = 180^\circ - 36.5^\circ - 52.1^\circ = 91.4^\circ$   
 $B_2 = 180^\circ - 52.1^\circ = 127.9^\circ, C_2 = 180^\circ - 36.5^\circ - 127.9^\circ = 15.6^\circ$   
 $\frac{c_1}{\sin 91.4^\circ} = \frac{22.3}{\sin 36.5^\circ}, c_1 = \frac{22.3 \sin 91.4^\circ}{\sin 36.5^\circ} = 37.5$   
 $\frac{c_2}{\sin 15.6^\circ} = \frac{22.3}{\sin 36.5^\circ}, c_2 = \frac{22.3 \sin 15.6^\circ}{\sin 36.5^\circ} = 10.1$



Chapter 10

1.  $y = -3 \sin(4\pi x - \pi/3)$   
 $a = -3, b = 4\pi, c = -\pi/3$  amplitude =  $|-3| = 3$   
 period =  $\frac{2\pi}{4\pi} = \frac{1}{2}$  displacement =  $-\frac{-\pi/3}{4\pi} = \frac{1}{12}$

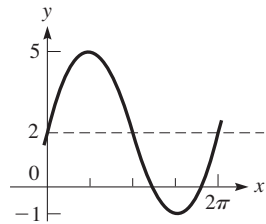
2.  $y = 0.5 \cos \frac{\pi}{2} x$   
 Amp. = 0.5, disp. = 0,  
 per. =  $\frac{2\pi}{\pi/2} = 4$



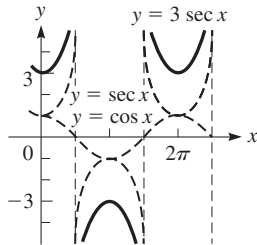
x	0	1	2	3	4
y	0.5	0	-0.5	0	0.5

3.  $y = 2 + 3 \sin x$   
 For  $y_1 = 3 \sin x$ ,  
 amp. = 3, per. =  $2\pi$ ,  
 disp. = 0

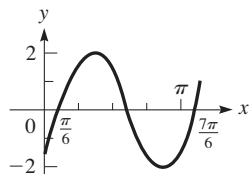
x	0	$\frac{\pi}{2}$	$\pi$	$\frac{3\pi}{2}$	$2\pi$
$y_1 = 3 \sin x$	0	3	0	-3	0
$y = 2 + 3 \sin x$	2	5	2	-1	2



4.  $y = 3 \sec x$   
 $\sec x = \frac{1}{\cos x}$



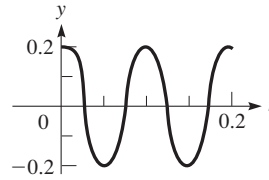
5.  $y = 2 \sin(2x - \frac{\pi}{3})$   
 Amp. = 2, per. =  $\frac{2\pi}{2} = \pi$ ,  
 disp. =  $-\frac{-\pi/3}{2} = \frac{\pi}{6}$



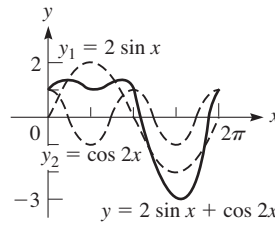
x	$\frac{\pi}{6}$	$\frac{5\pi}{12}$	$\frac{2\pi}{3}$	$\frac{11\pi}{12}$	$\frac{7\pi}{6}$
y	0	2	0	-2	0

$\frac{\pi}{6} + \frac{\pi}{4} = \frac{5\pi}{12}, \frac{\pi}{6} + \frac{\pi}{2} = \frac{2\pi}{3}, \frac{\pi}{6} + \frac{3\pi}{4} = \frac{11\pi}{12}$

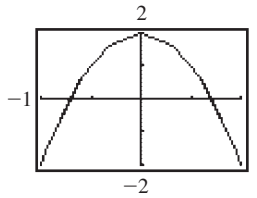
6.  $y = A \cos \frac{2\pi}{T} t$   
 $A = 0.200 \text{ cm}, T = 0.100 \text{ s},$   
 $y = 0.200 \cos 20\pi t,$   
 amp. = 0.200 cm, per. = 0.100 s



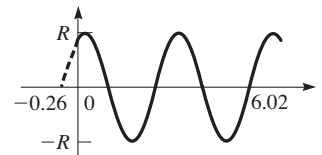
7.  $y = 2 \sin x + \cos 2x$   
 For  $y_1 = 2 \sin x$ ,  
 amp. = 2, per. =  $2\pi$ , disp. = 0  
 For  $y_2 = \cos 2x$ ,  
 amp. = 1, per. =  $\frac{2\pi}{2} = \pi$ , disp. = 0



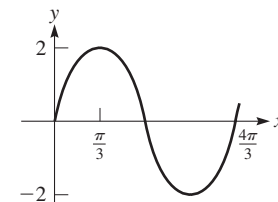
8.  $x = \sin \pi t, y = 2 \cos 2\pi t$



9.  $d = R \sin(\omega t + \frac{\pi}{6})$   
 $\omega = 2.00 \text{ rad/s}$   
 $d = R \sin(2.00t + \frac{\pi}{6})$   
 Amp. =  $R$ , per. =  $\frac{2\pi}{2.00} = \pi \text{ s} = 3.14 \text{ s},$   
 disp. =  $\frac{-\pi/6}{2.00} = -\frac{\pi}{12} \text{ s} = -0.26 \text{ s}$



10.  $y = 2 \sin bx$   
 $(\frac{\pi}{3}, 2)$  is first max. with  $x > 0$   
 period =  $\frac{2\pi}{b}, \frac{1}{4}(\frac{2\pi}{b}) = \frac{\pi}{3}, \frac{\pi}{2b} = \frac{\pi}{3}, b = \frac{3}{2}$   
 $y = 2 \sin \frac{3x}{2}$



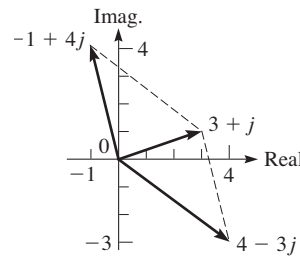
**Chapter 11**

1.  $-5y^0 = -5(1) = -5$
2.  $(3\pi x^{-4})^{-2} = (3\pi)^{-2}(x^{-4})^{-2} = \frac{x^8}{(3\pi)^2} = \frac{x^8}{9\pi^2}$
3.  $\frac{100^{3/2}}{8^{-2/3}} = (100^{3/2})(8^{2/3}) = [(100^{1/2})^3][(8^{1/3})^2]$   
 $= (10^3)(2^2) = 4000$
4.  $(as^{-1/3}t^{3/4})^{12} = a^{12}s^{(-1/3)(12)}t^{(3/4)(12)} = a^{12}s^{-4}t^9 = \frac{a^{12}t^9}{s^4}$
5.  $2\sqrt{20} - \sqrt{125} = 2\sqrt{4 \times 5} - \sqrt{25 \times 5}$   
 $= 2(2\sqrt{5}) - 5\sqrt{5}$   
 $= 4\sqrt{5} - 5\sqrt{5} = -\sqrt{5}$
6.  $(2x^{-1} + y^{-2})^{-1} = \frac{1}{2x^{-1} + y^{-2}} = \frac{1}{\frac{2}{x} + \frac{1}{y^2}} = \frac{1}{\frac{2y^2 + x}{xy^2}}$   
 $= \frac{xy^2}{2y^2 + x}$
7.  $(\sqrt{2x} - 3\sqrt{y})^2 = (\sqrt{2x})^2 - 2\sqrt{2x}(3\sqrt{y}) + (3\sqrt{y})^2$   
 $= 2x - 6\sqrt{2xy} + 9y$
8.  $\sqrt[3]{\sqrt[4]{4}} = \sqrt[3]{4} = \sqrt[3]{2^2} = 2^{2/12} = 2^{1/6} = \sqrt[6]{2}$
9.  $\frac{3 - 2\sqrt{2}}{2\sqrt{x}} = \frac{3 - 2\sqrt{2}}{2\sqrt{x}} \times \frac{\sqrt{x}}{\sqrt{x}} = \frac{3\sqrt{x} - 2\sqrt{2x}}{2x}$
10.  $\sqrt{27a^4b^3} = \sqrt{9 \times 3 \times (a^2)^2(b^2)(b)} = 3a^2b\sqrt{3b}$
11.  $2\sqrt{2}(3\sqrt{10} - \sqrt{6}) = 2\sqrt{2}(3\sqrt{10}) - 2\sqrt{2}(\sqrt{6})$   
 $= 6\sqrt{20} - 2\sqrt{12}$   
 $= 6(2\sqrt{5}) - 2(2\sqrt{3})$   
 $= 12\sqrt{5} - 4\sqrt{3}$
12.  $(2x + 3)^{1/2} + (x + 1)(2x + 3)^{-1/2}$   
 $= (2x + 3)^{1/2} + \frac{x + 1}{(2x + 3)^{1/2}}$   
 $= \frac{(2x + 3)^{1/2}(2x + 3)^{1/2} + x + 1}{(2x + 3)^{1/2}} = \frac{2x + 3 + x + 1}{(2x + 3)^{1/2}}$   
 $= \frac{3x + 4}{(2x + 3)^{1/2}}$
13.  $\left(\frac{4a^{-1/2}b^{3/4}}{b^{-2}}\right)\left(\frac{b^{-1}}{2a}\right) = \frac{4a^{-1/2}b^{3/4-1}}{2ab^{-2}} = \frac{2b^{-1/4}}{a^{1+1/2}} = \frac{2b^{7/4}}{a^{3/2}}$
14.  $\frac{2\sqrt{15} + \sqrt{3}}{\sqrt{15} - 2\sqrt{3}} = \frac{2\sqrt{15} + \sqrt{3}}{\sqrt{15} - 2\sqrt{3}} \times \frac{\sqrt{15} + 2\sqrt{3}}{\sqrt{15} + 2\sqrt{3}}$   
 $= \frac{2(15) + 4\sqrt{45} + \sqrt{45} + 2(3)}{15 - 4(3)}$   
 $= \frac{36 + 15\sqrt{5}}{3}$   
 $= 12 + 5\sqrt{5}$
15.  $\frac{3^{-1/2}}{2} = \frac{1}{2 \times 3^{1/2}} = \frac{1}{2\sqrt{3}}$   
 $= \frac{\sqrt{3}}{2\sqrt{3}\sqrt{3}} = \frac{\sqrt{3}}{6}$

16.  $0.220N^{-1/6} \quad N = 64 \times 10^6$   
 $0.220(64 \times 10^6)^{-1/6} = \frac{0.220}{(64 \times 10^6)^{1/6}} = \frac{0.220}{2 \times 10} = 0.011$

**Chapter 12**

1.  $(3 - \sqrt{-4}) + (5\sqrt{-9} - 1) = (3 - 2j) + [5(3j) - 1]$   
 $= 3 - 2j + 15j - 1$   
 $= 2 + 13j$
2.  $(2\angle 130^\circ)(3\angle 45^\circ) = (2)(3)\angle 130^\circ + 45^\circ = 6\angle 175^\circ$
3.  $2 - 7j: \quad r = \sqrt{2^2 + (-7)^2} = \sqrt{53} = 7.28$   
 $\tan \theta = \frac{-7}{2} = -3.500, \quad \theta_{\text{ref}} = 74.1^\circ, \quad \theta = 285.9^\circ$   
 $2 - 7j = 7.28(\cos 285.9^\circ + j \sin 285.9^\circ)$   
 $= 7.28\angle 285.9^\circ$
4. (a)  $-\sqrt{-64} = -(8j) = -8j$   
 (b)  $-j^{15} = -j^{12}j^3 = (-1)(-j) = j$
5.  $(4 - 3j) + (-1 + 4j) = 3 + j$



6.  $\frac{2 - 4j}{5 + 3j} = \frac{(2 - 4j)(5 - 3j)}{(5 + 3j)(5 - 3j)} = \frac{10 - 26j + 12j^2}{25 - 9j^2}$   
 $= \frac{10 - 26j - 12}{25 - 9(-1)} = \frac{-2 - 26j}{34} = -\frac{1 + 13j}{17}$
7.  $2.56(\cos 125.2^\circ + j \sin 125.2^\circ) = 2.56e^{2.185j}$   
 $125.2^\circ = \frac{125.2\pi}{180} = 2.185 \text{ rad}$
8.  $R = 3.50 \Omega, X_L = 6.20 \Omega, X_C = 7.35 \Omega$   
 $|Z| = \sqrt{R^2 + (X_L - X_C)^2}$   
 $= \sqrt{3.50^2 + (6.20 - 7.35)^2} = 3.68 \Omega$   
 $\tan \theta = \frac{X_L - X_C}{R} = \frac{6.20 - 7.35}{3.50} = \frac{-1.15}{3.50}$   
 $\theta = -18.2^\circ$
9.  $3.47 - 2.81j = 4.47e^{5.60j}$   
 $R = \sqrt{3.47^2 + (-2.81)^2} = 4.47$   
 $\tan \theta = \frac{-2.81}{3.47}, \quad \theta_{\text{ref}} = 39.0^\circ$   
 $\theta = 321.0^\circ = 5.60 \text{ rad}$
10.  $x + 2j - y = yj - 3xj$   
 $x - y + 3xj - yj = -2j$   
 $(x - y) + (3x - y)j = 0 - 2j$   
 $x - y = 0$   
 $3x - y = -2$   
 $2x = -2$   
 $x = -1, \quad y = -1$

**C.10 SOLUTIONS TO PRACTICE TEST PROBLEMS**

11.  $L = 8.75 \text{ mH} = 8.75 \times 10^{-3} \text{ H}$   
 $f = 600 \text{ kHz} = 6.00 \times 10^5 \text{ Hz}$

$$2\pi fL = \frac{1}{2\pi fC}$$

$$C = \frac{1}{(2\pi f)^2 L} = \frac{1}{(2\pi)^2 (6.00 \times 10^5)^2 (8.75 \times 10^{-3})}$$

$$= 8.04 \times 10^{-12} = 8.04 \text{ pF}$$

12.  $j = 1(\cos 90^\circ + j \sin 90^\circ)$

$$j^{1/3} = 1^{1/3} \left( \cos \frac{90^\circ}{3} + j \sin \frac{90^\circ}{3} \right)$$

$$= \cos 30^\circ + j \sin 30^\circ = 0.8660 + 0.5000j$$

$$= 1^{1/3} \left( \cos \frac{90^\circ + 360^\circ}{3} + j \sin \frac{90^\circ + 360^\circ}{3} \right)$$

$$= \cos 150^\circ + j \sin 150^\circ = -0.8660 + 0.5000j$$

$$= 1^{1/3} \left( \cos \frac{90^\circ + 720^\circ}{3} + j \sin \frac{90^\circ + 720^\circ}{3} \right)$$

$$= \cos 270^\circ + j \sin 270^\circ = -j$$

Cube roots of  $j$ :  $0.8660 + 0.5000j$   
 $-0.8660 + 0.5000j$   
 $-j$

**Chapter 13**

1.  $\log_9 x = -\frac{1}{2}$   
 $x = 9^{-1/2}$

$$= \frac{1}{9^{1/2}} = \frac{1}{3}$$

2.  $\log_3 x - \log_3 2 = 2$

$$\log_3 \frac{x}{2} = 2$$

$$\frac{x}{2} = 3^2$$

$$x = 2(3^2) = 18$$

3.  $\log_x 64 = 3$

$$64 = x^3$$

$$4^3 = x^3$$

$$x = 4$$

4.  $3^{3x+1} = 8$

$$(3x + 1)\log 3 = \log 8$$

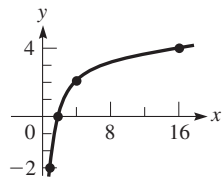
$$3x + 1 = \frac{\log 8}{\log 3}$$

$$x = \frac{1}{3} \left( \frac{\log 8}{\log 3} - 1 \right) = 0.298$$

5.  $y = 2 \log_4 x$

$x$	$\frac{1}{4}$	1	4	16
$y$	-2	0	2	4

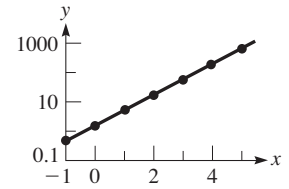
$$\log_4 \frac{1}{4} = -1, \log_4 16 = 2$$



6.  $y = 2(3^x)$

$x$	-1	0	1	2
$y$	0.7	2	6	18

$x$	3	4	5
$y$	54	162	486



7.  $\log_5 \left( \frac{4a^3}{7} \right) = \log_5 4a^3 - \log_5 7$

$$= \log_5 4 + \log_5 a^3 - \log_5 7$$

$$= \log_5 4 + 3 \log_5 a - \log_5 7$$

$$= 2 \log_5 2 + 3 \log_5 a - \log_5 7$$

8.  $3 \log_7 x - \log_7 y = 2$

$$\log_7 x^3 - \log_7 y = 2$$

$$\log_7 \frac{x^3}{y} = 2 \quad \frac{x^3}{y} = 7^2$$

$$x^3 = 49y \quad y = \frac{1}{49} x^3$$

9.  $\ln i - \ln I = -t/RC$

$$\ln \frac{i}{I} = -t/RC$$

$$\frac{i}{I} = e^{-t/RC} \quad i = Ie^{-t/RC}$$

10.  $\frac{2 \ln 0.9523}{\log 6066} = -0.02584$

11.  $\log_b x = \frac{\log_a x}{\log_a b}$

$$\log_5 7.32 = \frac{\log 7.32}{\log 5} = 1.237$$

12.  $A = A_0 e^{0.08t} \quad A = 2A_0$   
 $2A_0 = A_0 e^{0.08t} \quad 2 = e^{0.08t}$

$$\ln 2 = \ln e^{0.08t} = 0.08t \quad t = \frac{\ln 2}{0.08} = 8.66 \text{ years}$$

**Chapter 14**

1.  $x^{1/2} - 2x^{1/4} = 3$

$$\text{Let } y = x^{1/4}$$

$$y^2 - 2y - 3 = 0$$

$$(y - 3)(y + 1) = 0$$

$$y = 3, -1$$

$$x^{1/4} \neq -1$$

$$x^{1/4} = 3, \quad x = 81$$

$$\text{Check: } 81^{1/2} - 2(81^{1/4}) = 3$$

$$9 - 6 = 3$$

$$\text{Solution: } x = 81$$

$$\begin{aligned}
 2. \quad & 3\sqrt{x-2} - \sqrt{x+1} = 1 \\
 & 3\sqrt{x-2} = 1 + \sqrt{x+1} \\
 & 9(x-2) = 1 + 2\sqrt{x+1} + (x+1) \\
 & 8x - 20 = 2\sqrt{x+1} \\
 & 4x - 10 = \sqrt{x+1} \\
 & 16x^2 - 80x + 100 = x + 1 \\
 & 16x^2 - 81x + 99 = 0 \\
 & x = \frac{81 \pm \sqrt{81^2 - 4(16)(99)}}{32} \\
 & = \frac{81 \pm 15}{32} = 3, \frac{33}{16}
 \end{aligned}$$

$$\begin{aligned}
 \text{Check: } x = 3: & 3\sqrt{3-2} - \sqrt{3+1} \stackrel{?}{=} 1 \\
 & 3 - 2 = 1 \\
 x = \frac{33}{16}: & 3\sqrt{\frac{33}{16}-2} - \sqrt{\frac{33}{16}+1} \stackrel{?}{=} 1 \\
 & \frac{3}{4} - \frac{7}{4} \neq 1
 \end{aligned}$$

Solution:  $x = 3$

$$3. \quad x^4 - 17x^2 + 16 = 0$$

$$\begin{aligned}
 \text{Let } y &= x^2 \\
 y^2 - 17y + 16 &= 0 \\
 (y-1)(y-16) &= 0 \\
 y &= 1, 16 \\
 x^2 &= 1, 16 \\
 x &= -1, 1, -4, 4
 \end{aligned}$$

All values check.

$$4. \quad x^2 - 2y = 5$$

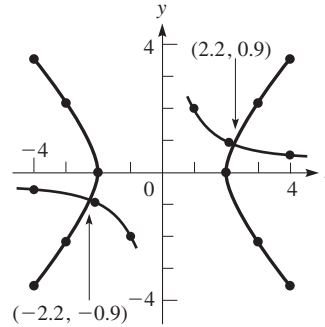
$$\begin{aligned}
 2x + 6y &= 1 \\
 y &= \frac{1}{2}(x^2 - 5) \\
 2x + 6\left(\frac{1}{2}\right)(x^2 - 5) &= 1 \\
 3x^2 + 2x - 16 &= 0 \\
 (3x+8)(x-2) &= 0 \\
 x &= -\frac{8}{3}, 2 \\
 x = -\frac{8}{3}: y &= \frac{1}{2}\left(\frac{64}{9} - 5\right) = \frac{19}{18} \\
 x = 2: y &= \frac{1}{2}(4 - 5) = -\frac{1}{2} \\
 x = -\frac{8}{3}, y &= \frac{19}{18} \\
 \text{or } x = 2, y &= -\frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 5. \quad & \sqrt[3]{2x+5} = 5 \\
 & 2x+5 = 125 \\
 & 2x = 120 \\
 & x = 60
 \end{aligned}$$

$$\begin{aligned}
 6. \quad & v = \sqrt{v_0^2 + 2gh} \\
 & v^2 = v_0^2 + 2gh \\
 & 2gh = v^2 - v_0^2 \\
 & h = \frac{v^2 - v_0^2}{2g}
 \end{aligned}$$

$$\begin{aligned}
 7. \quad & x^2 - y^2 = 4 \quad xy = 2 \\
 & y = \pm\sqrt{x^2 - 4} \quad y = \frac{2}{x}
 \end{aligned}$$

x	y	x	y
±2	0	-4	$-\frac{1}{2}$
±3	±2.2	-2	-1
±4	±3.5	-1	-2
		0	-
		1	2
		2	1
		4	$\frac{1}{2}$



$$x = 2.2, \quad y = 0.9; \quad x = -2.2, \quad y = -0.9$$

$$8. \quad \text{Let } l = \text{length, } w = \text{width}$$

$$\begin{aligned}
 2l + 2w &= 14 \\
 lw &= 10 \\
 \hline
 l + w &= 7 \\
 l &= 7 - w \\
 (7 - w)w &= 10 \\
 7w - w^2 &= 10 \\
 w^2 - 7w + 10 &= 0 \\
 (w - 5)(w - 2) &= 0 \\
 w &= 5, 2 \\
 l = 5.00 \text{ m, } w &= 2.00 \text{ m} \\
 (\text{or } l = 2.00 \text{ m, } w &= 5.00 \text{ m})
 \end{aligned}$$

### Chapter 15

$$\begin{aligned}
 1. \quad & f(x) = 2x^3 + 3x^2 + 7x - 6 \\
 & f(-3) = 2(-3)^3 + 3(-3)^2 + 7(-3) - 6 \\
 & = -54 + 27 - 21 - 6 = -54 \\
 & f(-3) \neq 0, \quad -3 \text{ is not a zero}
 \end{aligned}$$

$$\begin{aligned}
 2. \quad & x^4 - 2x^3 - 7x^2 + 20x - 12 = 0 \\
 & \begin{array}{r|rrrrr}
 1 & -2 & -7 & 20 & -12 & \\
 & & 2 & 0 & -14 & 12 \\
 \hline
 1 & 0 & -7 & 6 & & \\
 & & 2 & 4 & -6 & \\
 \hline
 1 & 2 & -3 & & & 
 \end{array}
 \end{aligned}$$

$$\begin{aligned}
 & x^2 + 2x - 3 = (x+3)(x-1) \\
 & \text{Other roots: } x = -3, 1
 \end{aligned}$$

$$\begin{aligned}
 3. \quad & (x^3 - 5x^2 + 4x - 9) \div (x - 3) \\
 & \begin{array}{r|rrrr}
 1 & -5 & 4 & -9 & \\
 & 3 & -6 & -6 & \\
 \hline
 1 & -2 & -2 & -15 & \\
 \hline
 \text{Quotient: } & x^2 - 2x - 2 & & & \\
 \text{Remainder} & = -15 & & & 
 \end{array}
 \end{aligned}$$

**C.12 SOLUTIONS TO PRACTICE TEST PROBLEMS**

4.  $f(x) = 2x^4 + 15x^3 + 23x^2 - 16$   
 $2x + 1 = 2(x + \frac{1}{2})$   

$$\begin{array}{r|rrrrr} 2 & 15 & 23 & 0 & -16 & \\ & -1 & -7 & -8 & 4 & \\ \hline & 2 & 14 & 16 & -8 & -12 \end{array}$$

Remainder is not zero;  
 $2x + 1$  is not a factor.

5.  $(x^3 + 4x^2 + 7x - 9) \div (x + 4)$   
 $f(x) = x^3 + 4x^2 + 7x - 9$   
 $f(-4) = (-4)^3 + 4(-4)^2 + 7(-4) - 9$   
 $= -64 + 64 - 28 - 9 = -37$   
 Remainder = -37

6.  $2x^4 - x^3 + 5x^2 - 4x - 12 = 0$   
 $f(x) = 2x^4 - x^3 + 5x^2 - 4x - 12$   
 $f(-x) = 2x^4 + x^3 + 5x^2 + 4x - 12$   
 $n = 4$ ; 4 roots  
 No more than 3 positive roots  
 One negative root  
 Rational roots: factors of 12 divided by factors of 2  
 Possible rational roots:  $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12, \pm \frac{1}{2}, \pm \frac{3}{2}$

$$\begin{array}{r|rrrrr} 2 & -1 & 5 & -4 & -12 & \\ & 4 & 6 & 22 & 36 & \\ \hline & 2 & 3 & 11 & 18 & 24 \end{array}$$

2 is too large.

$$\begin{array}{r|rrrrr} 2 & -1 & 5 & -4 & -12 & \\ & 3 & 3 & 12 & 12 & \\ \hline & 2 & 2 & 8 & 8 & 0 \end{array}$$

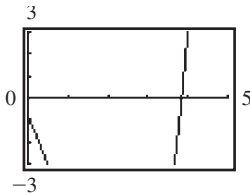
$$\begin{array}{r|rrrrr} 2 & -2 & 0 & -8 & & \\ & 2 & 0 & 8 & 0 & \\ \hline & 2 & 0 & 8 & 0 & \end{array}$$

$2x^2 + 8 = 0, \quad x^2 + 4 = 0, \quad x = \pm 2j$   
 roots:  $\frac{3}{2}, -1, 2j, -2j$

7.  $y = kx^2(x^3 + 436x - 4000)$   
 $y = 0, kx^2(x^3 + 436x - 4000) = 0$   
 $x = 0, x^3 + 436x - 4000 = 0$   

$$\begin{array}{r|rrrr} 1 & 0 & 436 & -4000 & \\ & 8 & 64 & 4000 & \\ \hline & 1 & 8 & 500 & 0 \end{array}$$
  
 $y = 0$  for  $x = 0$  m,  $x = 8$  m

8. Let  $x =$  length of edge.  
 $V = x^3, \quad 2V = (x + 1)^3$   
 $2x^3 = x^3 + 3x^2 + 3x + 1$   
 $x^3 - 3x^2 - 3x - 1 = 0$   
 $y = x^3 - 3x^2 - 3x - 1$   
 $x = 3.8$  mm



**Chapter 16**

1.  $A = \begin{bmatrix} 3 & -1 & 4 \\ 2 & 0 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 4 & 5 \\ -1 & -2 & 3 \end{bmatrix}$   
 $2B = \begin{bmatrix} 2 & 8 & 10 \\ -2 & -4 & 6 \end{bmatrix}$

$$A - 2B = \begin{bmatrix} 3 - 2 & -1 - 8 & 4 - 10 \\ 2 + 2 & 0 + 4 & -2 - 6 \end{bmatrix} = \begin{bmatrix} 1 & -9 & -6 \\ 4 & 4 & -8 \end{bmatrix}$$

2.  $\begin{bmatrix} 2x & x - y & z \\ x + z & 2y & y + z \end{bmatrix} = \begin{bmatrix} 6 & -2 & 4 \\ a & b & c \end{bmatrix}$   
 $2x = 6 \quad x - y = -2 \quad z = 4$   
 $x = 3 \quad 3 - y = -2 \quad y = 5$

$x + z = a \quad 2y = b \quad y + z = c$   
 $3 + 4 = a \quad 2(5) = b \quad 5 + 4 = c$   
 $a = 7 \quad b = 10 \quad c = 9$

3.  $CD = \begin{bmatrix} 1 & 0 & 4 \\ 2 & -2 & 1 \\ -1 & 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ 4 & -5 \\ 6 & 1 \end{bmatrix}$   
 $= \begin{bmatrix} 2 + 0 + 24 & -2 + 0 + 4 \\ 4 - 8 + 6 & -4 + 10 + 1 \\ -2 + 12 + 12 & 2 - 15 + 2 \end{bmatrix}$   
 $= \begin{bmatrix} 26 & 2 \\ 2 & 7 \\ 22 & -11 \end{bmatrix}$

$DC = \begin{bmatrix} 2 & -2 \\ 4 & -5 \\ 6 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 4 \\ 2 & -2 & 1 \\ -1 & 3 & 2 \end{bmatrix}$  not defined, since  $D$  has 2 columns and  $C$  has 3 rows

4.  $A = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \quad B = \begin{bmatrix} -2 & 5 \\ -1 & 2 \end{bmatrix}$

$AB = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} -2 & 5 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -4 + 5 & 10 - 10 \\ -2 + 2 & 5 - 4 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

$BA = \begin{bmatrix} -2 & 5 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} -4 + 5 & 10 - 10 \\ -2 + 2 & 5 - 4 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

$AB = BA = I, B = A^{-1}$

5.  $\left[ \begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 2 & -2 & 1 & 0 & 1 & 0 \\ -1 & 3 & 2 & 0 & 0 & 1 \end{array} \right] \rightarrow \left[ \begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & -2 & -7 & -2 & 1 & 0 \\ 0 & 3 & 6 & 1 & 0 & 1 \end{array} \right] \rightarrow$   
 $\left[ \begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & \frac{7}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 3 & 6 & 1 & 0 & 1 \end{array} \right] \rightarrow \left[ \begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & \frac{7}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{9}{2} & -2 & \frac{3}{2} & 1 \end{array} \right] \rightarrow$   
 $\left[ \begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & \frac{7}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{4}{9} & -\frac{1}{3} & -\frac{2}{9} \end{array} \right] \rightarrow \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{-7}{9} & \frac{4}{9} & \frac{8}{9} \\ 0 & 1 & 0 & \frac{-5}{9} & \frac{2}{9} & \frac{7}{9} \\ 0 & 0 & 1 & \frac{4}{9} & -\frac{1}{3} & -\frac{2}{9} \end{array} \right]$

$C^{-1} = \begin{bmatrix} -\frac{7}{9} & \frac{4}{9} & \frac{8}{9} \\ -\frac{5}{9} & \frac{2}{9} & \frac{7}{9} \\ \frac{4}{9} & -\frac{1}{3} & -\frac{2}{9} \end{bmatrix}$



6.  $2x - 3y = 11$   
 $x + 2y = 2$

$$A = \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix} \quad A^{-1} = \begin{bmatrix} \frac{2}{7} & \frac{3}{7} \\ -\frac{1}{7} & \frac{2}{7} \end{bmatrix} \quad C = \begin{bmatrix} 11 \\ 2 \end{bmatrix}$$

$$A^{-1}C = \begin{bmatrix} \frac{2}{7} & \frac{3}{7} \\ -\frac{1}{7} & \frac{2}{7} \end{bmatrix} \begin{bmatrix} 11 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{22}{7} + \frac{6}{7} \\ -\frac{11}{7} + \frac{4}{7} \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

$x = 4$        $y = -1$

7.  $2x - 3y = 11$

$x + 2y = 2$

$x + 2y = 2$       interchange equations

$2x - 3y = 11$

$x + 2y = 2$       subtract 2 times first equation

$-7y = 7$       from second equation

$x + 2y = 2$       divide second equation by -7

$y = -1$

$x + 2(-1) = 2$       substitute  $y = -1$  in first equation

$x = 4$

8. 
$$\begin{bmatrix} 2 & -4 & -3 \\ -3 & 6 & 2 \\ 5 & -1 & 5 \end{bmatrix} = 2 \begin{bmatrix} 6 & 2 \\ -1 & 5 \end{bmatrix} - (-4) \begin{bmatrix} -3 & 2 \\ 5 & 5 \end{bmatrix} + (-3) \begin{bmatrix} -3 & 6 \\ 5 & -1 \end{bmatrix}$$
  

$$= 2[30 - (-2)] + 4[-15 - 10] - 3[3 - 30]$$
  

$$= 2(32) + 4(-25) - 3(-27) = 64 - 100 + 81 = 45$$

9. 
$$\begin{bmatrix} 2 & -4 & -3 \\ -3 & 6 & 2 \\ 5 & -1 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -3 \\ -3 & 0 & 2 \\ 5 & 9 & 5 \end{bmatrix} = - \begin{bmatrix} 5 & 9 & 5 \\ -3 & 0 & 2 \\ 2 & 0 & -3 \end{bmatrix} = \begin{bmatrix} 9 & 5 & 5 \\ 0 & -3 & 2 \\ 0 & 2 & -3 \end{bmatrix}$$
  

$$= 9 \begin{bmatrix} -3 & 2 \\ 2 & -3 \end{bmatrix} = 9(9 - 4) = 45$$

(1)  $2 \times$  col. 1 added to col. 2; (2) rows 1 and 3 interchanged;

(3) cols. 1 and 2 interchanged; (4) expanded by col. 1

10. 
$$\begin{bmatrix} 7 & -2 & 1 & 1 & 0 & 0 \\ 2 & 3 & -4 & 0 & 1 & 0 \\ 4 & -5 & 2 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -\frac{2}{7} & \frac{1}{7} & \frac{1}{7} & 0 & 0 \\ 2 & 3 & -4 & 0 & 1 & 0 \\ 4 & -5 & 2 & 0 & 0 & 1 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} 1 & -\frac{2}{7} & \frac{1}{7} & \frac{1}{7} & 0 & 0 \\ 0 & \frac{25}{7} & -\frac{30}{7} & -\frac{2}{7} & 1 & 0 \\ 0 & -\frac{27}{7} & \frac{10}{7} & -\frac{4}{7} & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -\frac{2}{7} & \frac{1}{7} & \frac{1}{7} & 0 & 0 \\ 0 & 1 & -\frac{6}{5} & -\frac{2}{25} & \frac{7}{25} & 0 \\ 0 & -\frac{27}{7} & \frac{10}{7} & -\frac{4}{7} & 0 & 1 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} 1 & 0 & -\frac{1}{5} & \frac{3}{25} & \frac{2}{25} & 0 \\ 0 & 1 & -\frac{6}{5} & -\frac{2}{25} & \frac{7}{25} & 0 \\ 0 & 0 & -\frac{16}{5} & -\frac{22}{25} & \frac{27}{25} & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & \frac{7}{40} & \frac{1}{80} & -\frac{1}{16} \\ 0 & 1 & 0 & \frac{1}{4} & -\frac{1}{8} & -\frac{3}{8} \\ 0 & 0 & 1 & \frac{11}{40} & -\frac{27}{80} & -\frac{5}{16} \end{bmatrix}$$

$$A^{-1}C = \begin{bmatrix} \frac{7}{40} & \frac{1}{80} & -\frac{1}{16} \\ \frac{1}{4} & -\frac{1}{8} & -\frac{3}{8} \\ \frac{11}{40} & -\frac{27}{80} & -\frac{5}{16} \end{bmatrix} \begin{bmatrix} 6 \\ 6 \\ 10 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} \\ -3 \\ -\frac{7}{2} \end{bmatrix}$$

The solution is  $x = \frac{1}{2}$ ,  $y = -3$ ,  $z = -\frac{7}{2}$ .

11. Let  $A$  = number of shares of stock  $A$

$B$  = number of shares of stock  $B$

$50A + 30B = 2600$

$30A + 40B = 2000$

$5A + 3B = 260$

$3A + 4B = 200$

Let  $C$  = coefficient matrix

$$C = \begin{bmatrix} 5 & 3 \\ 3 & 4 \end{bmatrix}, \quad \begin{vmatrix} 5 & 3 \\ 3 & 4 \end{vmatrix} = 20 - 9 = 11$$

$$C^{-1} = \frac{1}{11} \begin{bmatrix} 4 & -3 \\ -3 & 5 \end{bmatrix} = \begin{bmatrix} \frac{4}{11} & -\frac{3}{11} \\ -\frac{3}{11} & \frac{5}{11} \end{bmatrix}$$

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} \frac{4}{11} & -\frac{3}{11} \\ -\frac{3}{11} & \frac{5}{11} \end{bmatrix} \begin{bmatrix} 260 \\ 200 \end{bmatrix} = \begin{bmatrix} \frac{4}{11}(260) - \frac{3}{11}(200) \\ -\frac{3}{11}(260) + \frac{5}{11}(200) \end{bmatrix}$$

$$= \begin{bmatrix} 40 \\ 20 \end{bmatrix}$$

$A = 40$  shares       $B = 20$  shares

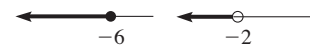
**Chapter 17**

1.  $x < 0, y > 0$       2.  $\frac{-x}{2} \geq 3$       3.  $3x + 1 < -5$

$$-x \geq 6 \qquad 3x < -6$$

$$x \leq -6 \qquad x < -2$$

$$(-\infty, -6] \qquad (-\infty, -2)$$



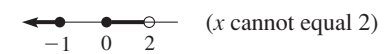
4.  $-1 < 1 - 2x < 5$   
 $-2 < -2x < 4$   
 $1 > x > -2$   
 $-2 < x < 1$   
 $(-2, 1)$



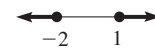
5.  $\frac{x^2 + x}{x - 2} \leq 0, \frac{x(x + 1)}{x - 2} \leq 0$

Interval	$\frac{x(x + 1)}{x - 2}$	Sign
$x < -1$	$\frac{-}{-}$	-
$-1 < x < 0$	$\frac{-}{+}$	+
$0 < x < 2$	$\frac{+}{-}$	-
$x > 2$	$\frac{+}{+}$	+

Solution:  $x \leq -1$  or  $0 \leq x < 2$   
 $(-\infty, -1] \cup [0, 2)$

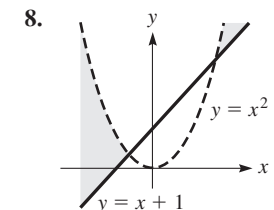


6.  $|2x + 1| \geq 3$   
 $2x + 1 \geq 3$   
 $2x \geq 2$   
 $x \geq 1$   
 $[1, \infty)$



$2x + 1 \leq -3$   
 $2x \leq -4$   
 or  $x \leq -2$   
 $(-\infty, -2]$

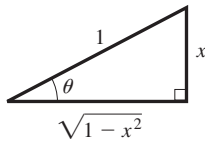
7.  $|2 - 3x| < 8$   
 $-8 < 2 - 3x < 8$   
 $-10 < -3x < 6$   
 $\frac{10}{3} > x > -2$   
 $-2 < x < \frac{10}{3}$   
 $(-2, 10/3)$



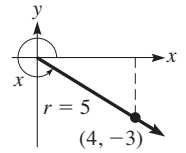


5.  $\sqrt{1-4x} = (1-4x)^{1/2}$   
 $= 1 + \left(\frac{1}{2}\right)(-4x) + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2}(-4x)^2 + \dots$   
 $= 1 - 2x - 2x^2 + \dots$
6.  $(2x-y)^5 = (2x)^5 + 5(2x)^4(-y) + \frac{5(4)}{2}(2x)^3(-y)^2$   
 $+ \frac{5(4)(3)}{2(3)}(2x)^2(-y)^3 + \frac{5(4)(3)(2)}{2(3)(4)}(2x)(-y)^4 + (-y)^5$   
 $= 32x^5 - 80x^4y + 80x^3y^2 - 40x^2y^3 + 10xy^4 - y^5$
7.  $5\% = 0.05$   
 Value after 1 year is  
 $2500 + 2500(0.05) = 2500(1.05)$   
 $V_{20} = 2500(1.05)^{20}$   
 $= \$6633.24$
8.  $2 + 4 + \dots + 200$   
 $a_1 = 2, a_{100} = 200, n = 100$   
 $S_{100} = \frac{100}{2}(2 + 200) = 10100$
9. Ball falls 8.00 m, rises 4.00 m, falls 4.00 m, etc.  
 Distance =  $8.00 + (4.00 + 4.00) + (2.00 + 2.00) + \dots$   
 $= 8.00 + 8.00 + 4.00 + 2.00 + \dots$   
 $= 8.00 + \frac{8.00}{1-0.5} = 8.00 + 16.0 = 24.0$  m

**Chapter 20**

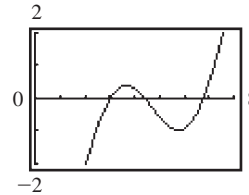
1.  $\sec \theta - \frac{\tan \theta}{\csc \theta} = \frac{1}{\cos \theta} - \frac{\frac{\sin \theta}{\cos \theta}}{\frac{1}{\sin \theta}} = \frac{1}{\cos \theta} - \frac{\sin^2 \theta}{\cos \theta}$   
 $= \frac{1 - \sin^2 \theta}{\cos \theta} = \frac{\cos^2 \theta}{\cos \theta} = \cos \theta$   
 $\cos \theta = \cos \theta$
2.  $\sin 2x + \sin x = 0$   
 $2 \sin x \cos x + \sin x = 0$   
 $\sin x(2 \cos x + 1) = 0$   
 $\sin x = 0 \quad \cos x = -\frac{1}{2}$   
 $x = 0, \pi, \frac{2\pi}{3}, \frac{4\pi}{3}$
3.  $\theta = \sin^{-1} x$   
 $\cos \theta = \cos(\sin^{-1} x)$   
 $= \sqrt{1-x^2}$
- 
4.  $i = 8.00e^{-20t}(1.73 \cos 10.0t - \sin 10.0t)$   
 Since  $8.00e^{-20t}$  will not equal zero,  $i = 0$  if  
 $1.73 \cos 10.0t - \sin 10.0t = 0$   
 $1.73 = \tan 10.0t$   
 $10.0t = \tan^{-1} 1.73$   
 $t = 0.105$  s
5.  $\frac{\tan \alpha + \tan \beta}{\tan \alpha - \tan \beta} = \frac{\frac{\sin \alpha}{\cos \alpha} + \frac{\sin \beta}{\cos \beta}}{\frac{\sin \alpha}{\cos \alpha} - \frac{\sin \beta}{\cos \beta}} = \frac{\sin \alpha \cos \beta + \sin \beta \cos \alpha}{\sin \alpha \cos \beta - \sin \beta \cos \alpha}$   
 $= \frac{\sin(\alpha + \beta)}{\sin(\alpha - \beta)}$

6.  $\cot^2 x - \cos^2 x = \frac{\cos^2 x}{\sin^2 x} - \cos^2 x =$   
 $\cos^2 x(\csc^2 x - 1) = \cos^2 x \cot^2 x;$   
 $\cos^2 x \cot^2 x = \cos^2 x \cot^2 x$
7.  $\sin x = -\frac{3}{5}$   
 $\cos x = \frac{4}{5}$   
 $\cos \frac{x}{2} = -\sqrt{\frac{1 + \left(\frac{4}{5}\right)}{2}} = -\sqrt{\frac{9}{10}} = -0.9487$   
 since  $270^\circ < x < 360^\circ$ ,  $135^\circ < \frac{x}{2} < 180^\circ$ ;  
 $\frac{x}{2}$  is in second quadrant, where  $\cos \frac{x}{2}$  is negative.



8.  $I = I_0 \sin 2\theta \cos 2\theta$   
 $\frac{2I}{I_0} = 2 \sin 2\theta \cos 2\theta = \sin 4\theta$   
 $4\theta = \sin^{-1} \frac{2I}{I_0}$   
 $\theta = \frac{1}{4} \sin^{-1} \frac{2I}{I_0}$

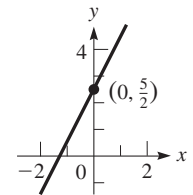
9.  $y = x - 2 \cos x - 5$   
 $x = 3.02, 4.42, 6.77$



10.  $\cos^{-1} x = -\tan^{-1}(-1)$   
 $\tan^{-1}(-1) = -\frac{\pi}{4}$   
 $\cos^{-1} x = -\left(-\frac{\pi}{4}\right) = \frac{\pi}{4}$   
 $x = \cos \frac{\pi}{4} = \frac{1}{2}\sqrt{2}$

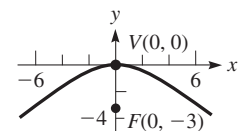
**Chapter 21**

1. Points:  $(4, -1), (6, 3)$   
 (a)  $d = \sqrt{(6-4)^2 + [3-(-1)]^2} = \sqrt{4+16} = 2\sqrt{5}$   
 (b) Between points  $m = \frac{3-(-1)}{6-4} = 2$ ; perp. line,  $m = -\frac{1}{2}$
2.  $2(x^2 + x) = 1 - y^2$   
 $2x^2 + y^2 + 2x - 1 = 0$   
 $A \neq C$  (same sign)  
 $B = 0$ : ellipse
3.  $4x - 2y + 5 = 0$   
 $y = 2x + \frac{5}{2}$   
 $m = 2 \quad b = \frac{5}{2}$



4.  $x^2 = 2x - y^2$   
 $x^2 + y^2 = 2x$   
 $r^2 = 2r \cos \theta$   
 $r = 2 \cos \theta$

5.  $x^2 = -12y$   
 $4p = -12, p = -3$   
 $V(0,0) \quad F(0,-3)$

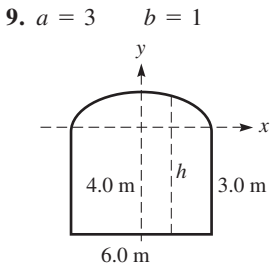
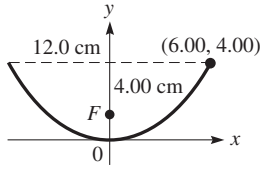


**C.16 SOLUTIONS TO PRACTICE TEST PROBLEMS**

6. Centre  $(-1, 2)$ ;  $h = -1, k = 2$  7.  $m = \frac{-2 - 1}{2 + 4} = -\frac{1}{2}$   
 $r = \sqrt{(2 + 1)^2 + (3 - 2)^2} = \sqrt{10}$   
 $(x + 1)^2 + (y - 2)^2 = 10$   $y - 1 = -\frac{1}{2}(x + 4)$   
 $x^2 + y^2 + 2x - 4y - 5 = 0$   $2y - 2 = -x - 4$   
 $x + 2y + 2 = 0$

or  
 $x^2 + y^2 + 2x - 4y - 5 = 0$

8.  $x^2 = 4py$   
 $(6.00)^2 = 4p(4.00)$   
 $p = 2.25$   
 Focus is 2.25 cm from vertex.



$$\frac{x^2}{9} + \frac{y^2}{1} = 1$$

Find  $y$  for  $x = 2$  m.

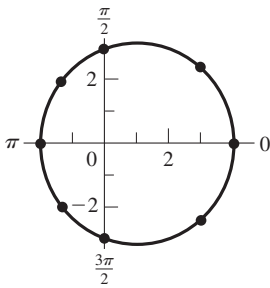
$$\frac{4}{9} + \frac{y^2}{1} = 1$$

$$y^2 = 5/9 \quad y = 0.7 \text{ m}$$

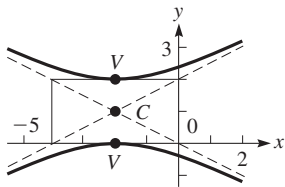
$$h = 3.0 + 0.7 = 3.7 \text{ m}$$

10.  $r = 3 + \cos \theta$

$r$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	$\pi$	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	$2\pi$
$\theta$	4.0	3.7	3.0	2.3	2.0	2.3	3.0	3.7	4.0



11.  $4y^2 - x^2 - 4x - 8y - 4 = 0$   
 $4(y^2 - 2y) - (x^2 + 4x) = 4$   
 $4(y^2 - 2y + 1) - (x^2 + 4x + 4) = 4 + 4 - 4$   
 $\frac{(y - 1)^2}{1^2} - \frac{(x + 2)^2}{2^2} = 1$   
 $C(-2, 1) \quad V(-2, 0) \quad V(-2, 2)$



12. Equation of curve:  $8x^2 - 4xy + 5y^2 = 36$ ;  $A = 8, B = -4, C = 5$   
 (a)  $B^2 - 4AC = (-4)^2 - 4(8)(5) = 16 - 160 < 0$ ; ellipse  
 (b)  $\tan 2\theta = \frac{B}{A - C} = \frac{-4}{8 - 5} = -\frac{4}{3}$ ;  $\theta = \frac{1}{2} \tan^{-1}\left(-\frac{4}{3}\right) = 63.4^\circ$

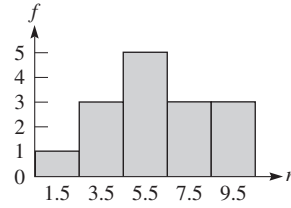
**Chapter 22**

1. Number	1	2	3	4	5	6	7	8	9	10
Frequency	1	0	1	2	3	2	1	2	2	1

$\sum f = 15$   
 Median is eighth number; median = 6.

2. 5 appears three times, and no other number appears more than twice; mode = 5.

3. Number	1-2	3-4	5-6	7-8	9-10
Frequency	1	3	5	3	3



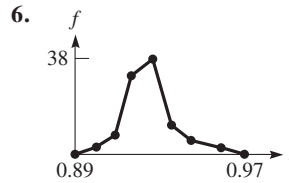
4. Let  $t =$  thickness

$$\bar{t} = \frac{3(0.90) + 9(0.91) + 31(0.92) + 38(0.93) + 12(0.94) + 5(0.95) + 2(0.96)}{3 + 9 + 31 + 38 + 12 + 5 + 2}$$

$$= \frac{92.7}{100} = 0.927 \text{ cm}$$

5.  $\sum x^2 = 3(0.90)^2 + 9(0.91)^2 + 31(0.92)^2 + 38(0.93)^2 + 12(0.94)^2 + 5(0.95)^2 + 2(0.96)^2 = 85.9464$   
 $(\sum x)^2 = 92.7^2$

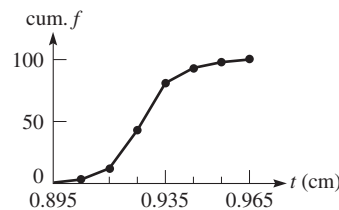
$$s = \sqrt{\frac{100(85.9464) - 92.7^2}{100(99)}} = 0.117 \text{ cm}$$



7. $t$ (cm)	0.90	0.91	0.92	0.93	0.94	0.95	0.96
%	3	9	31	38	12	5	2

8. $t$ (cm)	$< 0.905$	$< 0.915$	$< 0.925$	$< 0.935$
cum $f$	3	12	43	81

$t$ (cm)	$< 0.945$	$< 0.955$	$< 0.965$
cum $f$	93	98	100



9. A 95% confidence interval is given by

$$\bar{x} \pm z_{\alpha/2} \cdot \frac{s}{\sqrt{n}} = 143 \pm 1.96 \cdot \frac{5.2}{\sqrt{175}}$$

$$= 143 \pm 0.77$$

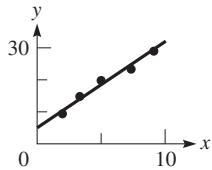
$$= (142.2, 143.8)$$

10. We have  $\hat{p} = \frac{33}{120}$ ,  $n\hat{p} = 33 \geq 5$ , and  $n(1 - \hat{p}) = 87 \geq 5$ .  
Therefore, we can apply Eq. (22.14) with  $z_{\alpha/2} = 2.575$ .  
The desired 99% confidence interval is

$$\begin{aligned} \hat{p} \pm z_{\alpha/2} \sqrt{\frac{\hat{p}(1 - \hat{p})}{n}} &= \frac{33}{120} \pm 2.575 \sqrt{\frac{\frac{33}{120} \cdot \frac{87}{120}}{120}} \\ &= 0.275 \pm 0.105 \\ &= (0.170, 0.380) \end{aligned}$$

11.  $0.5000 + 0.2257 = 0.7257$  (total area to the left)  
 $= 72.57\%$
12. Find the range of each subgroup (subtract the lowest value from the largest value). Find the mean  $R$  of these ranges (divide their sum by 20). This is the value of the central line. Multiply  $R$  by the appropriate control chart factors to obtain the upper control limit (UCL) and the lower control limit (LCL). Draw the central line, the UCL line, and the LCL line on a graph. Plot the points for each  $R$  corresponding to its subgroup and join successive points by straight-line segments.

$x$	$y$	$xy$	$x^2$
1	5	5	1
3	11	33	9
5	17	85	25
7	20	140	49
9	27	243	81
25	80	506	165

$$\begin{aligned} n &= 5 \\ m &= \frac{5(506) - (25)(80)}{5(165) - 25^2} \\ &= 2.65 \\ b &= \frac{165(80) - (506)(25)}{5(165) - 25^2} \\ &= 2.75 \\ y &= 2.65x + 2.75 \end{aligned}$$


14.

$x$	$\sqrt{x}$	$y$	$\sqrt{xy}$	$(\sqrt{x})^2 = x$
1.00	1.000	1.10	1.1000	1.000
3.00	1.732	1.90	3.2908	3.000
5.00	2.236	2.50	5.5900	5.000
7.00	2.646	2.90	7.6734	7.000
9.00	3.000	3.30	9.9000	9.000
	10.614	11.70	27.5542	25.000

$$\begin{aligned} n &= 5 \\ m &= \frac{5(27.5542) - (10.614)(11.70)}{5(25.000) - (10.614)^2} = 1.10 \\ b &= \frac{(25.000)(11.70) - (27.5542)(10.614)}{5(25.000) - (10.614)^2} = 0.00 \text{ (rounded)} \end{aligned}$$

off)

$$y = 1.10\sqrt{x}$$

Chapter 23

1.  $\lim_{x \rightarrow 1} \frac{x^2 - x}{x^2 - 1} = \lim_{x \rightarrow 1} \frac{x(x - 1)}{(x + 1)(x - 1)} = \lim_{x \rightarrow 1} \frac{x}{x + 1} = \frac{1}{2}$

2.  $\lim_{x \rightarrow \infty} \frac{1 - 4x^2}{x + 2x^2} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x^2} - 4}{\frac{1}{x} + 2} = -2$

3.  $y = 3x^2 - \frac{4}{x^2}$   
 $\frac{dy}{dx} = 6x + \frac{8}{x^3}$   
 $\left. \frac{dy}{dx} \right|_{x=2} = 6(2) + \frac{8}{2^3}$   
 $= 13$   
 $m_{\tan} = 13$

4.  $s = t\sqrt{10 - 2t}$   
 $v = \frac{ds}{dt} = t\left(\frac{1}{2}\right)(10 - 2t)^{-1/2}(-2) + (10 - 2t)^{1/2}(1)$   
 $= \frac{-t}{(10 - 2t)^{1/2}} + (10 - 2t)^{1/2} = \frac{10 - 3t}{(10 - 2t)^{1/2}}$   
 $v|_{t=4.00} = \frac{10 - 3(4.00)}{[10 - 2(4.00)]^{1/2}} = \frac{10 - 12.00}{2.00^{1/2}} = -1.41 \text{ cm/s}$

5.  $y = 4x^6 - 2x^4 + \pi^3$   
 $\frac{dy}{dx} = 4(6x^5) - 2(4x^3)$   $\pi^3$  is constant  
 $= 24x^5 - 8x^3$

6.  $y = 2x(5 - 3x)^4$   
 $\frac{dy}{dx} = 2x[4(5 - 3x)^3(-3)] + (5 - 3x)^4(2)$   
 $= -24x(5 - 3x)^3 + 2(5 - 3x)^4$   
 $= 2(5 - 3x)^3(-12x + 5 - 3x)$   
 $= 2(5 - 15x)(5 - 3x)^3$   
 $= 10(1 - 3x)(5 - 3x)^3$

7.  $(1 + y^2)^3 - x^2y = 7x$   
 $3(1 + y^2)^2(2yy') - x^2y' - y(2x) = 7$   
 $6y(1 + y^2)^2y' - x^2y' = 7 + 2xy$   
 $y' = \frac{7 + 2xy}{6y(1 + y^2)^2 - x^2}$

8.  $V = \frac{kq}{\sqrt{x^2 + b^2}} = kq(x^2 + b^2)^{-1/2}$   
 $\frac{dV}{dx} = kq\left(-\frac{1}{2}\right)(x^2 + b^2)^{-3/2}(2x) = \frac{-kqx}{(x^2 + b^2)^{3/2}}$

9.  $y = \frac{2x}{3x + 2}$   
 $\frac{dy}{dx} = \frac{(3x + 2)(2) - 2x(3)}{(3x + 2)^2} = \frac{4}{(3x + 2)^2} = 4(3x + 2)^{-2}$   
 $\frac{d^2y}{dx^2} = -2(4)(3x + 2)^{-3}(3) = \frac{-24}{(3x + 2)^3}$

10.  $y = 5x - 2x^2$   
 $f(x + h) = 5(x + h) - 2(x + h)^2$   
 $f(x + h) - f(x) = 5(x + h) - 2(x + h)^2 - (5x - 2x^2)$   
 $= 5h - 4xh - 2h^2$   
 $\frac{f(x + h) - f(x)}{h} = \frac{5h - 4xh - 2h^2}{h} = 5 - 4x - 2h$   
 $\lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h} = 5 - 4x$

Chapter 24

1.  $y = x^4 - 3x^2$

$$\frac{dy}{dx} = 4x^3 - 6x$$

$$\left. \frac{dy}{dx} \right|_{x=1} = 4(1^3) - 6(1)$$

$$= -2$$

$$y - (-2) = -2(x - 1)$$

$$y = -2x$$

2.  $y = 3x^2 - x$

$$\Delta y = f(3.1) - f(3)$$

$$[3(3.1)^2 - 3.1] - [3(3^2) - 3] = 1.73$$

$$dy = (6x - 1) dx$$

$$= [6(3) - 1](0.1) = 1.7$$

$$\Delta y - dy = 0.03$$

3.  $x = 3t^2$

$$y = 2t^3 - t^2$$

$$v_x = \frac{dx}{dt} = 6t$$

$$v_y = \frac{dy}{dt} = 6t^2 - 2t$$

$$a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2} = 6$$

$$a_y = \frac{dv_y}{dt} = \frac{d^2y}{dt^2} = 12t - 2$$

$$a_x|_{t=2} = 6$$

$$a_y|_{t=2} = 12(2) - 2 = 22$$

$$a|_{t=2} = \sqrt{6^2 + 22^2} = 22.8$$

$$\tan \theta = \frac{22}{6}, \theta = 74.7^\circ$$

4.  $P = \frac{144r}{(r + 0.6)^2}$

$$\frac{dP}{dr} = \frac{144[(r + 0.6)^2(1) - r(2)(r + 0.6)(1)]}{(r + 0.6)^4}$$

$$= \frac{144[(r + 0.6) - 2r]}{(r + 0.6)^3} = \frac{144(0.6 - r)}{(r + 0.6)^3}$$

$$\frac{dP}{dr} = 0; \quad 0.6 - r = 0, \quad r = 0.6 \Omega$$

$$\left( r < 0.6, \frac{dP}{dr} > 0; r > 0.6, \frac{dP}{dr} < 0 \right)$$

5.  $x^2 - \sqrt{4x + 1} = 0; \quad f(x) = x^2 - \sqrt{4x + 1}$

$$f'(x) = 2x - \frac{1}{2}(4x + 1)^{-1/2}(4) = 2x - \frac{2}{(4x + 1)^{1/2}}$$

$n$	$x_n$	$f(x_n)$	$f'(x_n)$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.5	-0.395 751 3	2.244 071 1	1.676 354 2
2	1.676 354 2	0.034 300 1	2.632 211 8	1.663 323 3

$$x_3 = 1.6633$$

6.  $y = \sqrt{2x + 4}, a = 6$

$$\frac{dy}{dx} = \frac{1}{2}(2x + 4)^{-1/2}(2) = \frac{1}{\sqrt{2x + 4}}$$

$$\left. \frac{dy}{dx} \right|_{x=6} = \frac{1}{\sqrt{2(6) + 4}} = \frac{1}{4}$$

$$f(6) = \sqrt{2(6) + 4} = 4$$

$$L(x) = 4 + \frac{1}{4}(x - 6) = \frac{1}{4}x + \frac{5}{2}$$

7.  $y = x^3 + 6x^2$

$$y' = 3x^2 + 12x = 3x(x + 4)$$

$$y'' = 6x + 12 = 6(x + 2)$$

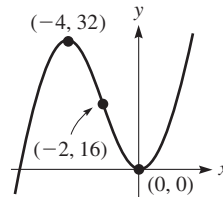
$$x < -4 \quad y \text{ inc.} \quad \text{Max. } (-4, 32)$$

$$-4 < x < 0 \quad y \text{ dec.} \quad \text{Min. } (0, 0)$$

$$x > 0 \quad y \text{ inc.} \quad \text{Infl. } (-2, 16)$$

$$x < -2 \quad y \text{ conc. down}$$

$$x > -2 \quad y \text{ conc. up}$$



8.  $y = \frac{4}{x^2} - x$

$$y' = -\frac{8}{x^3} - 1 = -\frac{8 + x^3}{x^3}$$

$$y'' = \frac{24}{x^4}$$

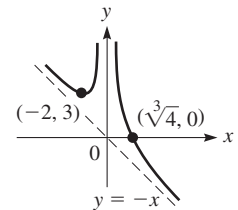
$$x < -2 \quad y \text{ dec.}$$

$$-2 < x < 0 \quad y \text{ inc.}$$

$$x > 0 \quad y \text{ dec., conc. up}$$

$$x < 0 \quad y \text{ conc. up}$$

Min.  $(-2, 3)$ , no infl., int.  $(\sqrt[3]{4}, 0)$ , sym. none; as  $x \rightarrow \pm\infty, y \rightarrow -x$ , asym.  $y = -x, x = 0$ . Domain: all real  $x$  except 0; range: all real  $y$ .



9. Let  $V$  = volume of cube

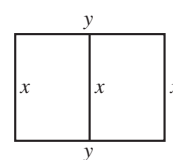
$s$  = edge of cube

$$V = s^3 \quad \frac{dV}{dt} = 3s^2 \frac{de}{dt}$$

$$\left. \frac{dV}{dt} \right|_{s=1.25} = 3(1.25)^2(-0.10)$$

$$= -0.47 \text{ m}^3/\text{s}$$

10.



$$3x + 2y = 6000$$

$$y = \frac{6000 - 3x}{2}$$

$$A = xy = x \left( \frac{6000 - 3x}{2} \right) = 3000x - \frac{3}{2}x^2$$

$$\frac{dA}{dx} = 3000 - 3x$$

$$3000 - 3x = 0, x = 1000 \text{ m}$$

$$y = 1500 \text{ m}$$

$$A_{\text{max}} = (1000)(1500) = 1.5 \times 10^6 \text{ m}^2$$

$$\left( x < 1000, \frac{dA}{dx} > 0; x > 1000, \frac{dA}{dx} < 0 \right)$$

Chapter 25

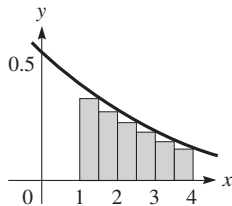
1. Power of  $x$  required for  $2x$  is 2. Therefore, multiply by  $1/2$ . Antiderivative of  $2x = \frac{1}{2}(2x^2) = x^2$ . Power of  $(1-x)^4$  required is 5. Derivative of  $(1-x)^5$  is  $5(1-x)^4(-1)$ . Writing  $-(1-x)^4$  as  $\frac{1}{5}[5(1-x)^4(-1)]$ , the antiderivative of  $-(1-x)^4$  is  $\frac{1}{5}(1-x)^5$ . Therefore, the antiderivative of  $2x - (1-x)^4$  is  $x^2 + \frac{1}{5}(1-x)^5$ .

2.  $\int x\sqrt{1-2x^2} dx = \int x(1-2x^2)^{1/2} dx$   
 $u = 1 - 2x^2 \quad du = -4x dx \quad n = \frac{1}{2} \quad n + 1 = \frac{3}{2}$   
 $\int x(1-2x^2)^{1/2} dx = -\frac{1}{4} \int (1-2x^2)^{1/2} (-4x dx)$   
 $= -\frac{1}{4} \left(\frac{2}{3}\right) (1-2x^2)^{3/2} + C$   
 $= -\frac{1}{6} (1-2x^2)^{3/2} + C$

3.  $\frac{dy}{dx} = (6-x)^4, dy = (6-x)^4 dx$   
 $\int dy = \int (6-x)^4 dx$   
 $y = -\int (6-x)^4 (-dx) = -\frac{1}{5} (6-x)^5 + C$   
 $2 = -\frac{1}{5} (6-5) + C, C = \frac{11}{5}$   
 $y = -\frac{1}{5} (6-x)^5 + \frac{11}{5}$

4.  $y = \frac{1}{x+2}, n = 6, \Delta x = \frac{4-1}{6} = \frac{1}{2}$   
 $A = \frac{1}{2} \left(\frac{2}{7} + \frac{1}{4} + \frac{2}{9} + \frac{1}{5} + \frac{2}{11} + \frac{1}{6}\right) = 0.6532$

$x$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	4
$y$	$\frac{1}{3}$	$\frac{2}{7}$	$\frac{1}{4}$	$\frac{2}{9}$	$\frac{1}{5}$	$\frac{2}{11}$	$\frac{1}{6}$



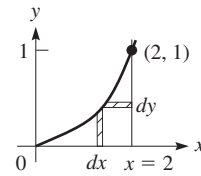
5. (See values for Problem 4.)  
 $\int_1^4 \frac{dx}{x+2} = \frac{1}{4} \left[ \frac{1}{3} + 2\left(\frac{2}{7}\right) + 2\left(\frac{1}{4}\right) + 2\left(\frac{2}{9}\right) + 2\left(\frac{1}{5}\right) + 2\left(\frac{2}{11}\right) + \frac{1}{6} \right]$   
 $= 0.6949$

6. (See values for Problem 4.)  
 $\int_1^4 \frac{dx}{x+2} = \frac{1}{6} \left[ \frac{1}{3} + 4\left(\frac{2}{7}\right) + 2\left(\frac{1}{4}\right) + 4\left(\frac{2}{9}\right) + 2\left(\frac{1}{5}\right) + 4\left(\frac{2}{11}\right) + \frac{1}{6} \right] = 0.6932$

7.  $i = \int_1^3 \left(t^2 + \frac{1}{t^2}\right) dt = \frac{1}{3}t^3 - \frac{1}{t} \Big|_1^3$   
 $= \frac{1}{3}(27) - \frac{1}{3} - \left(\frac{1}{3} - 1\right) = 9.3 \text{ A}$

Chapter 26

1.  $A = \int_0^2 \frac{1}{4} x^2 dx$   
 $= \frac{1}{12} x^3 \Big|_0^2 = \frac{2}{3}$

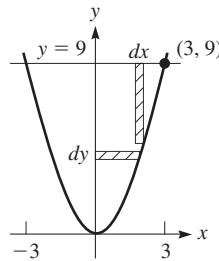


2.  $\bar{x} = \frac{\int_0^2 x \left(\frac{1}{4} x^2\right) dx}{\frac{2}{3}} = \frac{\frac{1}{4} \int_0^2 x^3 dx}{\frac{2}{3}} = \frac{\frac{1}{16} x^4 \Big|_0^2}{\frac{2}{3}} = \frac{1}{\frac{2}{3}} = \frac{3}{2}$

$\bar{y} = \frac{\int_0^1 y(2-2\sqrt{y}) dy}{\frac{2}{3}} = \frac{2 \int_0^1 (y - y^{3/2}) dy}{\frac{2}{3}}$   
 $= \frac{2 \left(\frac{1}{2} y^2 - \frac{2}{5} y^{5/2}\right) \Big|_0^1}{\frac{2}{3}}$   
 $= \frac{1 - \frac{4}{5}}{\frac{2}{3}} = \frac{1}{5} \times \frac{3}{2} = \frac{3}{10}$

3.  $V = \pi \int_0^2 \left(\frac{1}{4} x^2\right)^2 dx = \frac{\pi}{16} \int_0^2 x^4 dx = \frac{\pi}{80} x^5 \Big|_0^2$   
 $= \frac{32\pi}{80} = \frac{2\pi}{5}$

4.  $V = \pi \int_0^3 9^2 dx - \pi \int_0^3 (x^2)^2 dx = 81\pi x \Big|_0^3 - \frac{\pi}{5} x^5 \Big|_0^3$   
 $= 243\pi - \frac{243\pi}{5} = \frac{972\pi}{5}$   
 or  $V = 2\pi \int_0^9 xy dy = 2\pi \int_0^9 y^{1/2} dy = \frac{4\pi}{5} y^{5/2} \Big|_0^9$   
 $= \frac{4\pi}{5} (3^5 - 0) = \frac{972\pi}{5}$



5.  $I_y = \rho \int_0^3 x^2(9-x^2) dx = \rho \int_0^3 (9x^2 - x^4) dx$   
 $= \rho \left(3x^3 - \frac{1}{5}x^5\right) \Big|_0^3 = \rho \left(81 - \frac{243}{5}\right) = \frac{162\rho}{5}$

6.  $s = \int (60 - 4t) dt = 60t - 2t^2 + C$   
 $s = 10$  for  $t = 0, \quad 10 = 60(0) - 2(0^2) + C, \quad C = 10$   
 $s = 60t - 2t^2 + 10$

7.  $F = kx, 12 = k(2.0), k = 6.0 \text{ N/cm}$

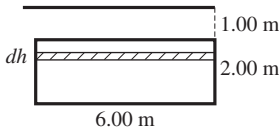
$W = \int_{2.0}^{6.0} 6.0x dx = 3.0x^2 \Big|_{2.0}^{6.0} = 3.0(36.0 - 4.0)$   
 $= 96 \text{ N} \cdot \text{cm}$



C.20 SOLUTIONS TO PRACTICE TEST PROBLEMS

$$8. F = 9.80 \int_{1.00}^{3.00} 6.00h \, dh = (9.80)(6.00) \frac{1}{2} h^2 \Big|_{1.00}^{3.00}$$

$$= (9.80)(3.00)(9.00 - 1.00) = 235 \text{ kN}$$



Chapter 27

$$1. y = \tan^3 2x + \tan^{-1} 2x$$

$$\frac{dy}{dx} = 3(\tan^2 2x)(\sec^2 2x)(2) + \frac{2}{1 + (2x)^2}$$

$$= 6 \tan^2 2x \sec^2 2x + \frac{2}{1 + 4x^2}$$

$$2. y = 2(3 + \cot 4x)^3$$

$$\frac{dy}{dx} = 2(3)(3 + \cot 4x)^2(-\csc^2 4x)(4)$$

$$= -24(3 + \cot 4x)^2 \csc^2 4x$$

$$3. y \sec 2x = \sin^{-1} 3y$$

$$y(\sec 2x \tan 2x)(2) + (\sec 2x)(y') = \frac{3y'}{\sqrt{1 - (3y)^2}}$$

$$\sqrt{1 - 9y^2} \sec 2x(2y \tan 2x + y') = 3y'$$

$$y' = \frac{2y\sqrt{1 - 9y^2} \sec 2x \tan 2x}{3 - \sqrt{1 - 9y^2} \sec 2x}$$

$$4. y = \frac{\cos^2(3x + 1)}{x}$$

$$dy = \frac{x\{2 \cos(3x + 1)[- \sin(3x + 1)(3)]\} - \cos^2(3x + 1)(1)}{x^2} dx$$

$$= \frac{-6x \cos(3x + 1) \sin(3x + 1) - \cos^2(3x + 1)}{x^2} dx$$

$$5. y = \ln \frac{2x - 1}{1 + x^2}$$

$$= \ln(2x - 1) - \ln(1 + x^2)$$

$$\frac{dy}{dx} = \frac{2}{2x - 1} - \frac{2x}{1 + x^2}$$

$$m_{\tan} = \left. \frac{dy}{dx} \right|_{x=2} = \frac{2}{4 - 1} - \frac{4}{1 + 4} = \frac{2}{3} - \frac{4}{5} = -\frac{2}{15}$$

$$6. i = 8e^{-t} \sin 10t$$

$$\frac{di}{dt} = 8[(e^{-t} \cos 10t)(10) + (\sin 10t)(-e^{-t})]$$

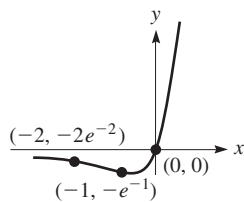
$$= 8e^{-t}(10 \cos 10t - \sin 10t)$$

$$7. y = xe^x$$

$$y' = xe^x + e^x = e^x(x + 1)$$

$$y'' = xe^x + e^x + e^x = e^x(x + 2)$$

$x < -1$   $y$  dec.  
 $x > -1$   $y$  inc.  
 $x < -2$   $y$  conc. down  
 $x > -2$   $y$  conc. up  
 Int.  $(0, 0)$ , min.  $(-1, -e^{-1})$ ,  
 infl.  $(-2, -2e^{-2})$ , asym.  $y = 0$

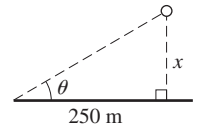


8. For  $t = 8.0$  s,  $x = 40$  m

$$\theta = \tan^{-1} \frac{x}{250}$$

$$\frac{d\theta}{dt} = \frac{1}{1 + \frac{x^2}{250^2}} \frac{dx/dt}{250} = \frac{250^2}{250^2 + x^2} \frac{dx/dt}{250}$$

$$\left. \frac{d\theta}{dt} \right|_{t=8.0} = \frac{250}{250^2 + 40^2} (5.0) = 0.020 \text{ rad/s}$$



9. Function is 0/0 indeterminate form.

$$\lim_{x \rightarrow 0} \frac{\tan^{-1} x}{2 \sin x} = \lim_{x \rightarrow 0} \frac{\frac{d}{dx} \tan^{-1} x}{\frac{d}{dx} (2 \sin x)} = \lim_{x \rightarrow 0} \frac{\frac{1}{1 + x^2}}{2 \cos x} = \frac{1}{2}$$

Chapter 28

$$1. \int (\sec x - \sec^3 x \tan x) dx$$

$$= \int \sec x dx - \int \sec^2 x (\sec x \tan x) dx$$

$$= \ln|\sec x + \tan x| - \frac{1}{3} \sec^3 x + C$$

$$2. \int \sin^3 x dx = \int \sin^2 x \sin x dx$$

$$= \int (1 - \cos^2 x) \sin x dx$$

$$= \int \sin x dx - \int \cos^2 x \sin x dx$$

$$= -\cos x + \frac{1}{3} \cos^3 x + C$$

$$3. \int \tan^3 2x dx = \int \tan 2x (\tan^2 2x) dx$$

$$= \int \tan 2x (\sec^2 2x - 1) dx$$

$$= \frac{1}{2} \int \tan 2x \sec^2 2x (2 dx) - \frac{1}{2} \int \tan 2x (2 dx)$$

$$= \frac{1}{4} \tan^2 2x + \frac{1}{2} \ln|\cos 2x| + C$$

$$4. \int \cos^2 4\theta d\theta = \frac{1}{2} \int (1 + \cos 8\theta) d\theta$$

$$= \frac{1}{2} \int d\theta + \frac{1}{16} \int \cos 8\theta (8 d\theta)$$

$$= \frac{1}{2} \theta + \frac{1}{16} \sin 8\theta + C$$

$$5. \text{ Let } x = 2 \sin \theta,$$

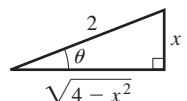
$$dx = 2 \cos \theta d\theta.$$

$$\int \frac{dx}{x^2 \sqrt{4 - x^2}} = \int \frac{2 \cos \theta d\theta}{4 \sin^2 \theta \sqrt{4 - 4 \sin^2 \theta}}$$

$$= \frac{1}{4} \int \frac{\cos \theta d\theta}{\sin^2 \theta \sqrt{\cos^2 \theta}} = \frac{1}{4} \int \csc^2 \theta d\theta$$

$$= -\frac{1}{4} \cot \theta + C$$

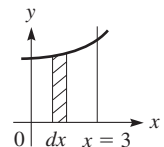
$$= -\frac{\sqrt{4 - x^2}}{4x} + C$$



6.  $\int x e^{-2x} dx; u = x, du = dx, dv = e^{-2x} dx, v = -\frac{1}{2}e^{-2x}$   
 $\int x e^{-2x} dx = x(-\frac{1}{2}e^{-2x}) - \int(-\frac{1}{2}e^{-2x}) dx$   
 $= -\frac{1}{2}x e^{-2x} - \frac{1}{4}e^{-2x} + C$

7.  $\int \frac{x^3 + 5x^2 + x + 2}{x^4 + x^2} dx$   
 $\frac{x^3 + 5x^2 + x + 2}{x^4 + x^2} = \frac{x^3 + 5x^2 + x + 2}{x^2(x^2 + 1)}$   
 $= \frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{x^2 + 1}$   
 $x^3 + 5x^2 + x + 2 = Ax(x^2 + 1) + B(x^2 + 1) + Cx^3 + Dx^2$   
 $x^3\text{-terms: } 1 = A + C \quad x^2\text{-terms: } 5 = B + D$   
 $x\text{-terms: } 1 = A \quad x = 0: 2 = B$   
 $A = 1, B = 2, C = 0, D = 3$   
 $\int \frac{x^3 + 5x^2 + x + 2}{x^4 + x^2} dx = \int \frac{dx}{x} + \int \frac{2 dx}{x^2} + \int \frac{3 dx}{x^2 + 1}$   
 $= \ln|x| - \frac{2}{x} + 3 \tan^{-1} x + C$

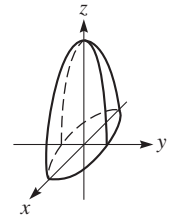
8.  $i = \int \frac{6t + 1}{4t^2 + 9} dt = \int \frac{6t dt}{4t^2 + 9} + \int \frac{dt}{4t^2 + 9}$   
 $= \frac{6}{8} \int \frac{8t dt}{4t^2 + 9} + \frac{1}{2} \int \frac{2 dt}{9 + (2t)^2}$   
 $= \frac{3}{4} \ln(4t^2 + 9) + \frac{1}{2} \left(\frac{1}{3}\right) \tan^{-1} \frac{2t}{3} + C$   
 $i = 0 \text{ for } t = 0: 0 = \frac{3}{4} \ln 9 + \frac{1}{6} \tan^{-1} 0 + C, C = -\frac{3}{4} \ln 9$   
 $i = \frac{3}{4} \ln(4t^2 + 9) + \frac{1}{6} \tan^{-1} \frac{2t}{3} - \frac{3}{4} \ln 9$   
 $= \frac{3}{4} \ln \frac{4t^2 + 9}{9} + \frac{1}{6} \tan^{-1} \frac{2t}{3}$

9.  $y = \frac{1}{\sqrt{16 - x^2}}; A = \int_0^3 y dx$   
  
 $= \int_0^3 \frac{dx}{\sqrt{16 - x^2}}$   
 $= \sin^{-1} \frac{x}{4} \Big|_0^3$   
 $= \sin^{-1} \frac{3}{4} = 0.8481$

**Chapter 29**

1.  $f(x, y) = \frac{2y}{x^2 - y^2}; f(-1, 3) = \frac{2(3)}{(-1)^2 + 3^2} = \frac{6}{10} = \frac{3}{5}$

2.  $z = 4 - x^2 - 4y^2$   
 Intercepts: (2, 0, 0), (-2, 0, 0), (0, 1, 0), (0, -1, 0), (0, 0, 4)  
 Traces:  
 in  $yz$ -plane:  $z = 4 - 4y^2$  (parabola)  
 in  $xz$ -plane:  $z = 4 - x^2$  (parabola)  
 in  $xy$ -plane:  $x^2 + 4y^2 = 4$  (ellipse)



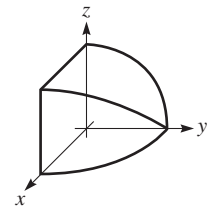
3.  $z = x e^{2xy}; \frac{\partial z}{\partial x} = x(e^{2xy})(2y) + e^{2xy}(1) = e^{2xy}(2xy + 1)$   
 $\frac{\partial z}{\partial y} = x e^{2xy}(2x) = 2x^2 e^{2xy}$

4.  $z = 3x^3 y + 2x^2 y^4; \frac{\partial z}{\partial y} = 3x^3(1) + 2x^2(4y^3) = 3x^3 + 8x^2 y^3$   
 $\frac{\partial^2 z}{\partial x \partial y} = 9x^2 + (16x)y^3 = 9x^2 + 16xy^3$

5.  $\int_0^2 \int_{x^2}^{2x} (x^3 + 4y) dy dx = \int_0^2 (x^3 y + 2y^2) \Big|_{x^2}^{2x} dx$   
 $= \int_0^2 (2x^4 + 8x^2 - x^5 - 2x^4) dx = \int_0^2 (8x^2 - x^5) dx$   
 $= \left(\frac{8}{3}x^3 - \frac{1}{6}x^6\right) \Big|_0^2 = \frac{8}{3}(8) - \frac{1}{6}(64) = \frac{64}{3} - \frac{32}{3} = \frac{32}{3}$

6.  $\int_1^{\ln 8} \int_0^{\ln y} e^{x+y} dx dy = \int_1^{\ln 8} e^{x+y} \Big|_0^{\ln y} dy$   
 $= \int_1^{\ln 8} (e^{y+\ln y} - e^y) dy$   
 $= \int_1^{\ln 8} (e^y e^{\ln y} - e^y) dy = \int_1^{\ln 8} (y e^y - e^y) dy$   
 $= e^y(y - 1) - e^y \Big|_1^{\ln 8} = e^{\ln 8}(\ln 8 - 1) - e^{\ln 8} + e$   
 $= 8(\ln 8 - 1) - 8 + e = 8 \ln 2^3 - 8 - 8 + e$   
 $= e - 16 + 24 \ln 2 = 3.354$

7.  $V = \int_0^3 \int_0^{\sqrt{9-y^2}} \sqrt{9-y^2} dx dy$   
 $= \int_0^3 x \sqrt{9-y^2} \Big|_0^{\sqrt{9-y^2}} dy$   
 $= \int_0^3 (9 - y^2) dy$   
 $= 9y - \frac{1}{3}y^3 \Big|_0^3 = 18$



$$8. f = \frac{k\sqrt{T}}{L}; \quad 30\text{Hz} = \frac{k\sqrt{65\text{N}}}{60\text{cm}}; \quad k = \frac{1800\text{ Hz}\cdot\text{cm}}{\sqrt{65}\text{ N}^{1/2}}$$

$$\frac{\partial f}{\partial T} = \frac{k}{L} \frac{1}{2\sqrt{T}} = \frac{1800}{\sqrt{65}L} \frac{1}{2\sqrt{T}} = \frac{900}{L\sqrt{65}T}$$

$$\left. \frac{\partial f}{\partial T} \right|_{L=60, T=65} = \frac{900}{60\sqrt{65} \times 65} = \frac{15}{65} = 0.23\text{ Hz/N}$$

**Chapter 30**

$$1. f(x) = (1 + e^x)^2 \quad f(0) = (1 + 1)^2 = 4$$

$$f'(x) = 2(1 + e^x)(e^x) \quad f'(0) = 2(1 + 1)(1) = 4$$

$$= 2e^x + 2e^{2x} \quad f''(0) = 2(1) + 4(1) = 6$$

$$f''(x) = 2e^x + 4e^{2x} \quad f'''(0) = 2(1) + 8(1) = 10$$

$$f'''(x) = 2e^x + 8e^{2x}$$

$$(1 + e^x)^2 = 4 + 4x + \frac{6}{2}x^2 + \frac{10}{6}x^3 + \dots$$

$$= 4 + 4x + 3x^2 + \frac{5}{3}x^3 + \dots$$

$$2. f(x) = \cos x \quad f\left(\frac{\pi}{3}\right) = \frac{1}{2}$$

$$f'(x) = -\sin x \quad f'\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

$$f''(x) = -\cos x \quad f''\left(\frac{\pi}{3}\right) = -\frac{1}{2}$$

$$\cos x = \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) + \frac{1}{2}\left(x - \frac{\pi}{3}\right)^2 - \dots$$

$$= \frac{1}{2}\left[1 - \sqrt{3}\left(x - \frac{\pi}{3}\right) + \left(x - \frac{\pi}{3}\right)^2 - \dots\right]$$

$$3. \ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

$$\ln 0.96 = \ln(1 - 0.04)$$

$$= -0.04 - \frac{(-0.04)^2}{2} + \frac{(-0.04)^3}{3} - \frac{(-0.04)^4}{4}$$

$$= -0.0408220$$

$$4. f(x) = \frac{1}{\sqrt{1 - 2x}} = (1 - 2x)^{-1/2}$$

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \dots$$

$$(1 - 2x)^{-1/2} = 1 + \left(-\frac{1}{2}\right)(-2x) + \frac{-\frac{1}{2}\left(-\frac{3}{2}\right)}{2}(-2x)^2 + \dots$$

$$= 1 + x + \frac{3}{2}x^2 + \dots$$

$$5. \int_0^1 x \cos x \, dx = \int_0^1 x\left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right) dx$$

$$= \int_0^1 \left(x - \frac{x^3}{2} + \frac{x^5}{24}\right) dx$$

$$= \frac{1}{2}x^2 - \frac{x^4}{8} + \frac{x^6}{144} \Big|_0^1$$

$$= \frac{1}{2} - \frac{1}{8} + \frac{1}{144} - 0 = 0.3819$$

$$6. f(t) = 0 \quad -\pi \leq t < 0$$

$$f(t) = 2 \quad 0 \leq t < \pi$$

$$a_0 = \frac{1}{2\pi} \int_0^\pi 2 \, dt = \frac{1}{\pi} t \Big|_0^\pi = 1$$

$$a_n = \frac{1}{\pi} \int_0^\pi 2 \cos nt \, dt = \frac{2}{n\pi} \sin nt \Big|_0^\pi$$

$$= 0 \quad \text{for all } n$$

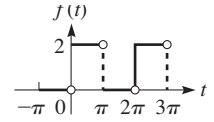
$$b_n = \frac{1}{\pi} \int_0^\pi 2 \sin nt \, dt = \frac{2}{n\pi} (-\cos nt) \Big|_0^\pi$$

$$= \frac{2}{n\pi} (1 - \cos n\pi)$$

$$b_1 = \frac{2}{\pi}(1 + 1) = \frac{4}{\pi} \quad b_2 = \frac{2}{2\pi}(1 - 1) = 0$$

$$b_3 = \frac{2}{3\pi}(1 + 1) = \frac{4}{3\pi}$$

$$f(t) = 1 + \frac{4}{\pi} \sin t + \frac{4}{3\pi} \sin 3t + \dots$$



$$7. f(x) = x^2 + 2, f(-x) = (-x)^2 + 2 = x^2 + 2, -f(-x) \neq f(x)$$

Since  $f(x) = f(-x)$ ,  $f(x)$  is an even function.

Since  $f(x) \neq -f(-x)$ ,  $f(x)$  is not an odd function.

$$g(x) = x^2 - 1, g(x) = f(x) - 3$$

Fourier series for  $g(x)$  is  $F(x) - 3$ .

**Chapter 31**

$$1. x \frac{dy}{dx} + 2y = 4$$

$$dy + \frac{2}{x}y \, dx = \frac{4}{x} \, dx$$

$$e^{\int \frac{2}{x} dx} = e^{2 \ln x} = x^2$$

$$yx^2 = \int \frac{4}{x} x^2 \, dx = \int 4x \, dx = 2x^2 + c$$

$$y = 2 + \frac{c}{x^2}$$

$$2. y'' + 2y' + 5y = 0$$

$$m^2 + 2m + 5 = 0$$

$$m = \frac{-2 \pm \sqrt{4 - 20}}{2} = -1 \pm 2j$$

$$y = e^{-x}(c_1 \sin 2x + c_2 \cos 2x)$$

$$3. x \, dx + y \, dy = x^2 \, dx + y^2 \, dx = (x^2 + y^2) \, dx$$

$$\frac{x \, dx + y \, dy}{x^2 + y^2} = dx$$

$$\frac{1}{2} \ln(x^2 + y^2) = x + \frac{1}{2}c \quad \ln(x^2 + y^2) = 2x + c$$

$$4. 2D^2y - Dy = 2 \cos x$$

$$2m^2 - m = 0 \quad m = 0, \frac{1}{2}$$

$$y_c = c_1 + c_2 e^{x/2}$$

$$y_p = A \sin x + B \cos x$$

$$Dy_p = A \cos x - B \sin x$$

$$D^2y_p = -A \sin x - B \cos x$$

$$2(-A \sin x - B \cos x) - (A \cos x - B \sin x) = 2 \cos x$$

$$-2A + B = 0 \quad -2B - A = 2 \quad A = -\frac{2}{5} \quad B = -\frac{4}{5}$$

$$y = c_1 + c_2 e^{x/2} - \frac{2}{5} \sin x - \frac{4}{5} \cos x$$

5.  $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 3x$

$m^2 - 4m + 4 = 0 \quad m = 2, 2$

$y_c = (c_1 + c_2x)e^{2x}$

$y_p = A + Bx \quad Dy_p = B \quad D^2y_p = 0$

$0 - 4B + 4(A + Bx) = 3x$

$4A - 4B = 0 \quad 4B = 3$

$B = \frac{3}{4} \quad A = \frac{3}{4}$

$y = (c_1 + c_2x)e^{2x} + \frac{3}{4} + \frac{3}{4}x$

6.  $D^2y - 2Dy - 8y = 4e^{-2x}$

$m^2 - 2m - 8 = 0 \quad (m + 2)(m - 4) = 0 \quad m = -2, 4$

$y_c = c_1e^{-2x} + c_2e^{4x}$

$y_p = Axe^{-2x}$  (factor of  $x$  necessary due to first term of  $y_c$ )

$Dy_p = Ae^{-2x} - 2Axe^{-2x}$

$D^2y_p = -2Ae^{-2x} - 2Ae^{-2x} + 4Axe^{-2x} = -4Ae^{-2x} + 4Axe^{-2x}$

$(-4Ae^{-2x} + 4Axe^{-2x}) - 2(Ae^{-2x} - 2Axe^{-2x})$

$-8Axe^{-2x} = 4e^{-2x}$

$-6Ae^{-2x} = 4e^{-2x} \quad A = -\frac{2}{3}$

$y = c_1e^{-2x} + c_2e^{4x} - \frac{2}{3}xe^{-2x}$

7.  $(xy + y)\frac{dy}{dx} = 2$

$y(x + 1)dy = 2 dx$

$y dy = \frac{2 dx}{x + 1}$

$\frac{1}{2}y^2 = 2 \ln(x + 1) + c$

$y = 2$  when  $x = 0$

$\frac{1}{2}(4) = 2 \ln(1) + c, \quad c = 2$

$\frac{1}{2}y^2 = 2 \ln(x + 1) + 2$

$y^2 = 4 \ln(x + 1) + 4$

8.  $\frac{dA}{dt} = rA$

$\frac{dA}{A} = r dt$

$\ln A = rt + \ln c$

$\ln \frac{A}{c} = rt$

$A = ce^{rt}$

$A_0 = ce^0$

$c = A_0$

$A = A_0e^{rt}$

9.  $y'' + 9y = 9$

$y(0) = 0 \quad y'(0) = 1$

$\mathcal{L}(y'') + 9\mathcal{L}(y) = \mathcal{L}(9)$

$s^2\mathcal{L}(y) - s(0) - 1 + 9\mathcal{L}(y) = \frac{9}{s}$

$(s^2 + 9)\mathcal{L}(y) = \frac{9}{s} + 1$

$\mathcal{L}(y) = \frac{9}{s(s^2 + 9)} + \frac{1}{s^2 + 9}$

$y = 1 - \cos 3t + \frac{1}{3} \sin 3t$

10.  $D^2y - Dy - 2y = 12 \quad y(0) = 0, y'(0) = 0$

$\mathcal{L}(y'') - \mathcal{L}(y') - 2\mathcal{L}(y) = \mathcal{L}(12)$

$s^2\mathcal{L}(y) - s(0) - 0 - [s\mathcal{L}(y) - 0] - 2\mathcal{L}(y) = \frac{12}{s}$

$\mathcal{L}(y)(s^2 - s - 2) = \frac{12}{s}$

$\mathcal{L}(y) = \frac{12}{s(s^2 - s - 2)} = \frac{12}{s(s + 1)(s - 2)}$

$= \frac{A}{s} + \frac{B}{s + 1} + \frac{C}{s - 2}$

$12 = A(s + 1)(s - 2) + Bs(s - 2) + Cs(s + 1)$

$s = 0: \quad 12 = -2A, A = -6$

$s = -1: \quad 12 = 3B, B = 4$

$s = 2: \quad 12 = 6C, C = 2$

$\mathcal{L}(y) = -\frac{6}{s} + \frac{4}{s + 1} + \frac{2}{s - 2}$

$y = -6 + 4e^{-t} + 2e^{2t}$

11.  $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} = E \quad L\frac{di}{dt} + Ri = E$

$L = 2 \text{ H} \quad R = 8 \Omega \quad E = 6 \text{ V}$

$2\frac{di}{dt} + 8i = 6 \quad di + 4i dt = 3 dt$

$e^{\int 4 dt} = e^{4t} \quad ie^{4t} = \int 3e^{4t} dt = \frac{3}{4}e^{4t} + c$

$i = 0$  for  $t = 0 \quad 0 = \frac{3}{4} + c \quad c = -\frac{3}{4}$

$ie^{4t} = \frac{3}{4}e^{4t} - \frac{3}{4}$

$i = \frac{3}{4}(1 - e^{-4t}) = 0.75(1 - e^{-4t})$

12.  $m = 0.5 \text{ kg}, k = 32 \text{ N/m}$

$mD^2x = -kx, \quad 0.5D^2x + 32x = 0$

$D^2x + 64x = 0$

$m^2 + 64 = 0; \quad m = \pm 8j$

$x = c_1 \sin 8t + c_2 \cos 8t$

$Dx = 8c_1 \cos 8t - 8c_2 \sin 8t$

$x = 0.3 \text{ ft} \quad Dx = 0$  for  $t = 0$

$0.3 = c_1 \sin 0 + c_2 \cos 0, \quad c_2 = 0.3$

$0 = 8c_1 \cos 0 - 8c_2 \sin 0, \quad c_1 = 0$

$x = 0.3 \cos 8t$

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# Index of Applications

## Acoustics

Acoustical intensity, 53(79), 312(18), 505(56, 59)  
Acoustics, 189(72)  
Doppler effect, 333(70)  
Loudness of siren, 319(76), 939(24)  
Loudness sensed by human ear, 329(64), 391(50), 395(35), 640(17), 833(51)  
Sound in seawater, 32(47)  
Sound of jet engine, 126(55), 614(106)  
Sound reflection in buildings, 587(51)  
Ultrasonics, 613(95)  
Velocity of sound, 152(59), 166(52), 178(84), 571(51), 726(34), 909(40)  
Wavelength of sound, 657(56)

## Aeronautics

Air traffic control, 292(37), 592(48), 606(59), 846(87)  
Aircraft emergency locator transmitter, 218(107)  
Aircraft on aircraft carrier, 781(11)  
Aircraft stabilizer, 812(43)  
Aircraft wing design, 77(7, 8), 498(17), 587(52)  
Airfoil design, 211(66), 218(104)  
Concorde gain in altitude, 293(47)  
Distance between jets, 726(39)  
Flying speed, 698(47)  
Fuel supply, 165(41), 218(92), 499(38)  
Glide-slope indicator, 83(50)  
Helicopter blade rotation, 259(48)  
Helicopter forces, 280(14)  
Helicopter path, 552(52)  
Helicopter position, 403(33), 545(55)  
Helicopter velocity, 822(48)  
Jet distance, 592(49)  
Jet speed, 11(64), 21(96), 216(55), 238(74, 76), 259(50), 504(40)  
Mach number, 485(36), 498(14), 657(53)  
Plane landing, 287(32), 612(82)  
Plane location, 95(42), 272(26), 294(65), 392(56), 482(63), 686(53), 741(10)  
Plane on radar, 138(71)  
Plane route, 258(35), 292(31), 294(66), 599(48), 614(105)  
Plane speed, 145(27), 236(46), 272(25), 280(2, 19), 340(96), 349(40), 406(43), 698(47, 54), 709(85), 722(19, 27), 725(15), 741(8), 742(40), 743(48), 749(65)  
Plane velocity, 722(27)  
Propeller rotation, 263(5)  
Radar, 38(59), 218(95)  
Rocket pursuit of aircraft, 329(68), 609(50), 709(87), 714(26), 819(55)  
Shock wave of supersonic jet, 599(47)  
Speed of SST Concorde, 51(22)

Wind lift and drag, 269(41), 505(42), 561(113), 894(80)  
Wind shear, 272(33)

## Architecture

Arch design, 232(50), 581(46), 709(83), 776(49, 50, 51), 846(85)  
Architecture, 185(67), 189(66), 216(42), 218(117)  
Attic room, 138(77)  
Bastion floor design, 287(27)  
BC Place Stadium (Vancouver), 84(60)  
Building design, 742(24), 743(51)  
Ceiling design, 138(63), 178(86)  
CN Tower (Toronto), 53(99), 134(10), 508(53)  
Courtyard design, 69(31)  
Deck design, 403(36)  
Dome design, 717(26), 891(51), 927(36)  
Door design, 432(44), 434(61), 803(26)  
Floor design, 77(12), 236(44), 287(27)  
Gateway Arch (St. Louis), 373(38), 847(97), 859(43)  
Gothic arch, 262(89)  
Great Pyramid (Egypt), 81(31)  
Hallway design, 258(27)  
Kitchen tiling, 54(127)  
Norman window, 238(80), 577(62), 750(74)  
Patio design, 83(52), 138(75), 140(95), 259(54), 609(47), 709(81), 914(47)  
Ramp for disabled, 83(49), 134(11)  
Roof design, 65(39), 413(47), 499(39), 592(47), 606(56)  
Room design, 232(55)  
Rotating restaurant, 262(78)  
Security fence, 69(38), 135(25), 236(45), 258(31), 402(32), 406(40), 742(22)  
Shelf support, 702(43)  
Spaceship Earth (Florida) design, 81(33)  
Sunroom design, 263(9)  
Support design, 69(34)  
Support panel, 673(33)  
Swimming pool design, 77(5, 6), 80(28), 742(34), 749(62), 787(48)  
Wall panel, 293(41), 856(47)  
Window design, 68(2), 69(32), 73(43), 114(10), 139(81), 140(93), 422(58), 581(48), 587(54), 742(28), 787(50), 829(36), 894(84)  
Winnipeg Art Gallery, 288(41)  
World Trade Center, 135(32)

## Astronomy

Astronomical unit, 29(61)  
Black holes, 53(84)

Centre of mass (Earth-Moon), 812(52)  
Cosmic radiation, 656(27, 28, 29, 30)  
Crater on Moon, 294(56)  
Diameter of Venus, 262(92)  
Earth's shadow, 84(62)  
Earth's velocity, 259(42)  
Electrons in Universe, 29(53)  
Energy from sun, 20(41)  
Galaxy movement, 503(5)  
Haley's comet, 587(48)  
Hubble telescope, 84(65), 581(49)  
Jupiter's moon, 135(24)  
Meteor velocity, 403(37), 722(28), 967(22)  
Meteorite, 808(17)  
Moon, 20(42), 262(79), 485(38)  
Parsec, 29(42)  
Planetary motion, 215(3), 259(41), 262(79), 340(88), 395(34)  
Space object, 587(47)  
Star brightness, 377(63), 398(93), 505(54)  
Star movement, 260(59)  
Star temperature, 519(45)

## Automotive Technology

Accident analysis, 280(16)  
Airbag in car, 781(9)  
Auto jack, 68(29)  
Auto repair, 219(128)  
Battery life, 219(125)  
Car speed, 232(57), 272(24), 875(33)  
Carburetor assembly, 165(46)  
Compression ratio, 340(94), 498(18)  
Cost of operating car, 113(72), 169(24)  
Drive shaft rotation, 259(52), 782(25)  
Engine coolant, 21(99)  
Engine cylinders, 725(22)  
Engine efficiency, 47(41), 332(68), 398(96), 493(77), 709(78), 809(33), 931(40)  
Engine emissions, 499(37)  
Engine flywheel, 259(44), 398(95)  
Engine power, 236(40), 498(24)  
Engine torque, 111(8, 16)  
Fuel consumption, 102(38), 195(60), 225(44), 627(16, 17, 18)  
Headlight angle, 134(7)  
Piston velocity, 303(59), 312(20), 318(63)  
Radiator antifreeze, 51(29)  
Rear axle ratio, 506(8)  
Seat belt design, 280(17)  
Speed in turn, 102(43), 258(25), 505(58)  
Speedometer calibration, 261(72)  
Stopping distance, 90(43), 493(72), 596(48), 619(21, 22, 23, 24), 627(23, 24), 706(48), 781(15)  
Tire angular velocity, 262(77)  
Tire lifetimes, 634(17, 18, 19, 20, 21, 22)  
Tire performance on ice, 280(11)  
Torque rotation, 645(5, 6, 7, 8)

## D.2 Index of Applications

Tractor-trailer, 377(70), 833(53)  
Universal joint, 561(108)  
Weight of car, 159(47)  
Wheel angular acceleration, 432(33)  
Windshield design, 138(73), 258(28)

### Biology and Medical Science

Ancestors, 519(52)  
Bacteria, 507(43), 999(58)  
Biology, 195(61)  
Blood cell, 29(40)  
Blood flow, 218(90), 725(23)  
Body burns, 749(63)  
Body heat, 391(47)  
Calcium density in bones, 316(36), 732(48)  
Cholesterol, 506(14)  
DNA sample, 507(30), 967(29)  
Drug dosage, 158(44), 180(9), 736(22)  
Drug testing, 238(69)  
Eye exposure to light, 668(54), 709(76)  
Flu epidemic, 967(30)  
Insecticide effectiveness, 530(85)  
Intravenous solution, 175(44)  
Lithotripter, 587(50)  
Lung flow rate, 829(33)  
Medical science, 46(15), 50(10)  
Medication, 20(40), 50(10), 382(66),  
391(53), 506(23), 650(5), 690(50)  
Muscle action, 650(8)  
Nutrition, 492(21)  
Population density, 382(65), 843(24)  
Population growth, 47(28), 238(67),  
387(52), 397(85), 519(40), 706(45),  
837(54), 856(43), 967(19, 20), 1000(67)  
Protein molecule, 397(92)  
Pulse rate, 316(37)  
Shearing effects in spinal column, 368(93)  
Tumour, 572(60)  
Ultracentrifuge, 262(87)  
Ultrasound, 307(40), 472(58)  
Vaccine, 113(73)  
Virus under microscope, 498(12)  
Windpipe, 741(12)  
X-ray dosage, 340(100), 613(96), 620(25,  
26), 627(25, 26)

### Business and Finance

Airline on-time flights, 655(1, 2, 3, 4, 5, 6,  
7, 8, 9, 10)  
Annuity value, 53(104)  
Appreciated value, 340(92)  
Benford's law, 398(98)  
Brand-switching model, 465(85)  
Business, 189(65), 216(48)  
Cell phones, 395(33)  
Commissions on sales, 165(47), 178(77),  
506(13)  
Compound interest, 38(57), 111(7, 15),  
112(26), 387(51), 846(72), 960(40)  
Contractor costs, 476(35)

Cost of calculators, 493(69)  
Cost of CDs/DVDs, 51(19)  
Cost of construction, 515(56)  
Cost of drilling well, 529(76)  
Cost of living index, 340(87)  
Cost of machine parts, 482(60)  
Cost of mouse pads, 36(52)  
Cost of operating copier, 113(70)  
Cost of production, 237(57), 787(46)  
Cost of TV commercial, 53(116)  
Cost of wire, 494(11)  
Demand and price, 503(6)  
Depreciation of car, 32(50), 515(53),  
968(45)  
Depreciation of equipment, 47(39), 108(46),  
527(51), 529(67)  
Depreciation of yard sale purchases,  
529(77)  
Distribution of gasoline, 51(27)  
Exchange rate for Canadian dollars, 111(2)  
Filling bottles, 634(24), 640(22), 645(15, 16)  
Finance, 325(69)  
Furniture leasing, 897(8)  
Helmet test damage, 640(20)  
Income and education, 658(71)  
Inflation, 846(78)  
Installation of cable, 94(29), 95(44)  
Insurance claim, 165(45)  
Interest on loan, 515(52)  
Investments, 38(57), 54(129), 112(3),  
178(78), 232(52), 373(33), 397(79),  
492(17), 506(24), 519(43), 529(75),  
530(93), 708(57), 919(39)  
Land developer, 103(49)  
Logging company plans, 454(30)  
Lottery, 51(20), 397(84)  
Mail order charges, 175(40)  
Manufacturing defects, 640(10), 656(43, 44)  
Manufacturing electronic equipment, 237(63)  
Marginal profit, 967(24)  
Money saved, 519(48, 51)  
Office rentals, 159(51)  
Petroleum rights, 50(9)  
Price increases, 530(80)  
Price of coffee, 653(9)  
Production of calculators, 492(19)  
Production of CD/DVD players, 179(90),  
494(82)  
Production of computer parts, 94(26), 492(22)  
Production of medical supplies, 169(19)  
Production of product, 492(20)  
Profit from table sales, 406(44)  
Profit on sales, 94(33), 113(65), 165(43),  
743(47), 760(58)  
Rent charges, 507(41)  
Resale value of truck, 102(39)  
RRSP account, 145(30)  
Salaries, 627(32, 33), 628(40, 41, 42)  
Sales of computer chips, 434(54)  
Sales of software, 53(115)  
Sales over time, 482(55), 690(48)

Sales report, 159(55)  
Sales volume, 507(38), 749(64)  
Satellite telephone leasing, 94(34)  
Size of DVDs, 20(35)  
Taxes, 90(46), 94(32)  
Telephone accounts, 646(17, 18)  
Toy assembly time required, 640(19)  
Trucking delivery routes, 51(16)  
Value of building lot, 493(68)  
Value of car, 50(5), 681(37, 38)  
Value of house, 530(86)  
Value of machinery, 709(84)  
Waiting time for luggage, 640(18)  
Washing machine replacement times, 640(16)  
Yard sales, 529(77)

### Chemistry

Acid solution, 571(52)  
Alcohol solution, 94(28)  
Brine solution, 54(126), 398(101), 782(31),  
966(6), 968(36)  
Chemical reactions, 54(117), 397(88),  
507(29), 709(72), 709(80), 725(25, 26),  
883(34), 893(72), 999(57)  
Chemistry, 46(6), 195(63)  
Dissolving, 891(50), 968(46)  
Evaporation, 968(48), 999(55)  
Fertilizer solubility, 114(81)  
Filtration, 262(82)  
Industrial cleaners, 114(80)  
Molecular interaction, 197(32), 894(79)  
Molecule size, 875(36)  
Oxygen-air mixture, 1001(87)  
Sodium in salt, 499(43)  
Weed-killing solution, 159(54)

### Civil Engineering

Aqueduct, 136(37)  
Bridge cable, 809(35)  
Bridge design, 51(13), 139(85), 218(108),  
581(45), 612(87), 657(59)  
Bridge support, 84(66), 113(64), 561(115)  
Chlorine in water supply, 506(26)  
City development, 494(83)  
City park, 68(3), 292(39)  
Civil engineering, 195(58)  
Conduit design, 73(44)  
Culvert design, 133(3), 488(52), 742(31),  
879(34)  
Dam design, 612(89)  
Drainage trough design, 66(54)  
Drainpipe, 593(50)  
Drawbridge construction, 135(22)  
Driveway design, 135(21)  
Gold mine, 288(39)  
Highway construction, 84(64), 287(34),  
967(23)  
Highway design, 258(33, 36), 750(80)  
Kejimikujik National Park area, 77(13)  
Levee, 138(76)



Parking lot design, 76(3), 493(74), 613(92)  
 Power line, 113(77), 893(73)  
 Railroad track, 84(67), 258(29)  
 River channel, 77(15)  
 River silt, 504(8)  
 Road banking, 138(67)  
 Road grade, 134(17), 498(20)  
 Road tunnel, 261(70), 587(53)  
 Roadbed, 54(121), 507(28)  
 Runaway truck ramp, 530(84)  
 Sidewalk design, 258(20)  
 Storm drain flow, 340(98)  
 Street plan, 135(33), 259(45), 287(31)  
 Suspension cable, 530(82), 714(25)  
 Tower support, 65(40), 84(53)  
 Traffic flow, 170(28), 257(17), 476(43),  
 572(59), 843(36)  
 Utility pole wires (catenary), 392(59),  
 582(54), 771(21), 837(39)  
 Utility poles, 238(79)  
 Water channel, 138(70)  
 Well locations, 406(36)

## Computer

Access time, 47(32)  
 Chip power, 572(55)  
 Chip price, 434(54)  
 Chip quality, 51(26), 499(45)  
 Chip size, 68(4), 406(35), 415(63), 482(61),  
 488(54), 742(19, 21), 746(28), 847(99)  
 Chip temperature, 485(34)  
 Computer design, 38(61), 185(63), 189(74)  
 Computer efficiency, 47(46)  
 Computer image, 197(27), 335(52), 609(52)  
 Computer memory, 6(41), 20(34), 21(97),  
 29(38), 53(97, 100), 169(26), 179(91),  
 237(66), 391(45), 498(21)  
 Computer memory test, 505(50), 572(61),  
 681(40)  
 Computer network, 113(76)  
 Computer program, 262(88)  
 Computer reliability, 708(63), 736(25),  
 836(51), 843(38), 923(41)  
 Computer speed, 29(54), 103(53), 377(69),  
 472(57), 619(13, 14, 15, 16), 627(19),  
 833(52)  
 Computer time, 494(79)  
 Cryptography, 449(42)  
 Disk design, 506(18), 833(54)  
 Disk speed, 722(25)  
 Download time, 504(31)  
 Email, 149(42), 159(45)  
 Hard disk files, 846(73)  
 Internet, 373(36), 619(1, 2, 3, 4), 846(83)  
 Printer speed, 258(26), 506(21)  
 Website, 11(63), 50(8)

## Construction

Backhoe use, 529(64)  
 Board cutting, 113(60), 499(41)

Board measurements, 134(14)  
 Cement bag weights, 634(9, 10, 11, 12)  
 Cement-sand mixture, 51(30)  
 Concrete drying, 238(73)  
 Concrete mix, 499(36)  
 Cutting speed of saw, 108(55), 281(29),  
 318(62), 819(53)  
 Demolition ball, 90(41), 725(12)  
 Gutter design, 847(95)  
 Leaning Tower of Pisa, 134(4)  
 Lumber from log, 103(47)  
 Nail resistance, 329(66), 893(69)  
 Paint supply, 746(26)  
 Painting crew, 216(52)  
 Pile driver, 529(61)  
 Pipe section, 813(54)  
 Plywood size, 134(15), 335(51), 415(64)  
 Riveter, 219(11)  
 Roof support, 529(65)  
 Steam-pipe construction, 59(33)  
 Structure support, 126(56), 537(61), 673(33)  
 Truss design, 59(23, 24, 25, 26, 27, 28),  
 169(23), 179(89), 294(62), 406(37),  
 561(111)  
 Wallboard size, 84(61)  
 Winding up rope, 767(45), 808(12), 811(37)

## Design

Airplane emergency chute, 138(72)  
 Aquarium, 90(37)  
 Banner, 415(61)  
 Barrel plug, 774(17)  
 Baseball field, 259(53), 291(30)  
 Basketball, 81(37)  
 Basketball rim, 73(39)  
 Boat deck, 776(56), 812(46)  
 Boat rudder, 879(32, 33)  
 Book, 750(76)  
 Box, 80(27), 108(53), 414(48), 482(64),  
 732(49), 742(33), 743(43), 750(69)  
 Calculator, 415(65)  
 Carton liner, 406(41)  
 CD/DVD, 238(82), 258(39), 747(33)  
 Computer monitor, 232(51), 415(60)  
 Container, 111(3), 114(82), 189(63),  
 197(30), 237(65), 681(34), 686(56),  
 702(44), 750(81), 793(38), 897(5)  
 Cooler, 108(47)  
 Dip stick, 81(40)  
 Drafting, 135(29), 319(71), 587(49),  
 613(104)  
 Drawer, 750(73)  
 Engineering, 47(25)  
 Filter, 847(98)  
 Funnel, 742(30)  
 Girder structure, 136(34)  
 Glass panel, 262(84)  
 Glass prism, 749(48)  
 Golden ratio, 231(47), 414(40)  
 Grain storage bin, 434(60)  
 Gutter, 847(95)  
 Highway pylon, 799(33)  
 Hot water tank, 85(69)  
 Igloo, 847(97)  
 Instrumentation, 218(99)  
 Label, 81(38)  
 Lacrosse field, 581(44)  
 Lawn roller, 81(36)  
 Machinery hood, 112(2), 262(85)  
 Manhole cover, 746(27)  
 Metal bracket, 847(94)  
 Metal frame, 825(54)  
 Metal sheet, 493(67)  
 Oil tank, 434(59), 736(29), 741(2)  
 Olympic Stadium (Montreal), 613(94)  
 Olympic technical ring, 613(94)  
 Paper cup, 81(32), 743(44, 52)  
 Peace Tower, 561(112)  
 Planter, 732(50)  
 Plastic band, 415(68)  
 Plastic pipe, 476(36)  
 Porthole, 577(59)  
 Poster, 742(29)  
 Propane tank, 81(34), 434(52)  
 Putting green, 812(40)  
 Race track, 743(45)  
 Radar, 53(94)  
 Railroad track, 548(40)  
 Ramp, 141(13), 281(32), 287(25)  
 Ranch, 69(37)  
 Safe, 432(42)  
 Sailboat sail, 65(43), 406(34)  
 Silo, 748(35), 846(89)  
 Silver and stainless steel container, 750(77)  
 Ski slope, 677(43), 748(36)  
 Solar energy panel, 749(60)  
 Spotlight reflector, 582(53)  
 Stairway handrail, 135(31)  
 Storage building, 108(52, 56), 228(36),  
 750(79)  
 Storage tank, 587(55)  
 Structural support, 139(83), 415(71), 599(43)  
 Structure, 548(44)  
 Surveillance camera, 134(19)  
 Tablet and foil, 434(56)  
 Tabletop, 69(36), 134(18), 338(66), 742(16),  
 787(49), 812(39)  
 Tarpaulin, 236(37)  
 Tennis ball, 85(8)  
 Tennis court, 232(53)  
 Tent, 85(70)  
 Terry Fox monument, 558(71)  
 Tower guardrail, 134(16)  
 Tray, 432(39)  
 Trellis, 139(86)  
 Trough, 415(66), 742(32), 812(42)  
 Tubing, 485(37), 714(29), 894(81)  
 TV screen, 32(49), 135(30), 238(77),  
 499(47), 830(38)  
 Valve, 41(51), 813(57)  
 Vent, 742(17)

## D.4 Index of Applications

Washer, 406(33), 410(38)  
Water purifying tablet, 434(56)  
Water slide, 138(74), 673(32), 819(52)  
Water tank, 709(82), 749(44)  
Wedge, 81(35), 912(25)  
Wire shape, 530(88)  
Workbench, 69(33)  
Zoo display, 742(20)

### Electricity and Electronics

Admittance, 355(39)  
AM radio signal, 126(57), 681(33), 748(34), 825(49)  
Ammeter, 90(40), 189(71), 259(46)  
Amplification factor, 909(42)  
Amplifier, 338(69), 498(11), 782(19), 819(54)  
Antenna, 113(59), 135(32), 609(48)  
Apparent power, 138(68), 828(20)  
Battery charge, 395(30)  
Battery life, 620(27, 28), 627(28), 859(45)  
Battery power, 613(93)  
Battery voltages, 645(9, 10, 11, 12)  
Bode diagram, 395(39, 40)  
Capacitors in series, 6(39), 216(45), 232(58), 339(71), 377(66), 434(58), 482(57), 498(15), 717(27), 736(23), 747(41), 749(67), 781(18)  
Cell phone, 494(81), 928(40)  
Charge on capacitor, 253(70), 319(81), 843(30), 846(82), 927(38), 988(8)  
Circuit board, 59(36)  
Circuit design, 68(30), 829(34)  
Circuit frequency, 340(99)  
Circuit tuning, 325(62)  
Current change with time, 955(42)  
Current density, 843(34)  
Discharge through hydrogen gas, 919(40)  
Electric charge, 743(42), 776(52, 55), 787(44), 891(52)  
Electric circuit, 837(53), 846(88), 856(45), 862(36), 893(76), 898(32), 962(17)  
Electric current, 11(60), 36(48), 41(52), 108(45), 114(11, 12), 149(41), 152(34), 159(56), 216(57), 236(39), 248(53), 307(38), 316(39), 335(50), 347(54, 57), 352(44), 361(58), 366(12), 373(35), 387(53), 398(102, 104), 392(60), 397(80), 422(55), 472(60), 498(22), 499(42, 46), 519(41), 529(66), 541(47), 552(46), 558(59), 561(109, 4), 572(64), 596(47), 657(57), 681(29), 690(47), 748(37), 749(58), 759(53), 774(18), 781(17), 809(31), 848(6), 852(39), 875(35), 883(32), 888(28), 894(8), 923(40), 948(6), 960(39)  
Electric field, 113(74), 269(42), 280(6), 352(43), 708(65), 714(27), 767(50), 782(29), 871(39), 968(50)  
Electric intensity, 472(59)  
Electric motors, 218(106)  
Electric potential, 706(46), 710(8), 742(14), 833(56), 879(35), 905(33), 948(69)  
Electric power, 856(48)  
Electric usage, 627(35, 36)  
Electricity, 46(5), 47(18), 53(83, 88), 185(72), 195(59), 211(65, 67), 218(114, 116, 123), 219(133, 2), 333(72)  
Electromagnetic wave, 313(22), 368(90)  
Electronics, 47(23, 34, 37)  
Electroplating, 476(41), 504(34)  
Energy, 843(35)  
Filter circuit, 333(71)  
FM transmission, 216(41)  
Force between parallel wires, 505(47)  
Force on electric charge, 29(44), 681(42), 736(28), 771(20)  
Full-wave rectifier, 939(23)  
Fuses, 646(19, 20)  
Generator, 541(44, 46), 741(7)  
Half-wave rectifier, 939(22)  
Heat developed, 102(41), 503(4)  
Impedance, 138(78), 319(79), 347(56, 58), 355(37), 365(2), 366(5, 6, 13, 14, 22, 23), 384(44), 596(50), 741(11), 749(50)  
Inductance, 366(15), 759(56), 782(27), 833(50)  
Kirchhoff's laws, 165(42), 169(22), 175(36), 178(74)  
Magnetic field, 135(23), 211(62), 294(53), 527(53), 725(19)  
Magnetic intensity, 281(35, 36), 395(38)  
Microwave, 47(48), 387(49), 505(60)  
Mutual conductance, 909(41)  
Mutual inductance, 111(5, 13)  
Ohm's law, 593(51)  
Oscillator frequency, 709(75), 759(54)  
Oscilloscope signal, 300(31), 316(41, 43, 44), 319(78)  
Permittivity, 325(64)  
Photoelectric effect, 609(51), 650(13)  
Potential from magnet, 846(74)  
Power, 29(37), 38(63), 51(18), 90(44), 749(53), 751(4), 829(30)  
Power gain, 382(68), 398(94), 505(55), 545(59), 698(52)  
Power generation, 482(53)  
Radar signal, 355(38)  
Radiation, 189(76), 948(71)  
Radio antennas, 253(72), 545(56)  
Radio circuit, 968(34), 988(19, 20)  
Radio communication, 413(43)  
Radio signal, 312(17), 548(42)  
Radio wave, 698(57), 714(28), 767(47)  
Resistance, 29(60), 32(46), 41(48), 47(45), 178(85), 215(39), 236(42), 406(39), 488(51), 493(76), 504(7), 505(50), 506(10), 515(50), 527(55), 612(79), 657(51), 686(49), 732(45), 742(18, 38), 812(51), 843(28)  
Resistance and temperature, 102(40), 145(28), 725(11)  
Resistors in parallel, 53(101), 114(86), 219(131), 225(52), 237(64), 262(73), 410(35), 432(43), 702(45), 725(16), 747(42), 908(35)  
Resonance, 95(41), 366(17, 18, 19), 413(42), 613(90), 654(10), 701(39), 898(40)  
RLC circuits, 232(49), 329(67), 366(7, 8, 9, 10, 16, 20), 415(54), 698(58), 701(39), 825(51), 859(41), 931(37), 966(2), 968(31), 997(29, 30, 31, 32, 33, 35, 36, 37), 1000(72, 73, 77, 78, 79, 80, 81, 82, 83, 84), 1001(91, 11)  
Root-mean-square current, 548(41), 867(46, 50), 894(77)  
Root-mean-square voltage, 867(49)  
Superconductivity, 613(91)  
Telephone channel, 397(91)  
Tracking device, 863(39)  
Thermocouple, 725(1)  
Transformer, 499(28)  
Transistors, 211(59)  
Tuner, 726(30)  
TV signal, 813(53)  
Voltage, 6(40), 11(59), 21(98), 44(48), 47(45), 54(119), 103(54), 138(65), 158(41), 166(51), 175(39), 225(41), 228(33), 238(70), 303(57), 312(9, 10), 316(34), 352(41), 361(55, 58), 366(3, 4, 11), 368(85), 432(37), 476(44), 485(40), 541(45), 571(50), 650(7), 706(49), 709(67), 725(14), 782(21, 22, 23, 24), 811(9), 891(48), 898(29), 914(35), 972(35)  
Voltage across inductor, 843(23)  
Wheatstone bridge, 499(27)  
Wire in magnetic field, 262(83)

### Energy Technology

Delivery and customer service, 494(80)  
Electric heater, 709(71)  
Energy conservation, 189(75)  
Heating element, 506(19)  
Heating unit efficiency, 493(73)  
Insulation, 529(79), 650(12)  
Jet engine power, 47(35)  
Joule, 325(63)  
Natural gas consumption, 759(51)  
Nuclear energy, 225(43), 237(56), 613(99)  
Oil burner heat, 111(4)  
Solar cell, 236(43), 415(67), 572(56), 599(44)  
Solar collector, 102(42)  
Solar energy, 11(55), 21(95), 218(94), 316(40), 325(65), 329(63), 398(103), 504(37), 537(59), 582(55), 681(35, 36), 698(55), 706(50), 732(44), 741(9), 760(57), 825(50), 852(38), 898(37)  
Solar heating, 53(92)  
Solar panel design, 108(49), 225(54), 403(34), 476(39)

Water power, 196(67), 259(49)  
 Wind power, 102(45), 159(53), 259(51),  
 505(45), 655(17, 18, 19, 20, 21, 22, 23,  
 24, 25, 26), 657(58), 725(10)

## Environmental Science

Air circulation, 968(47)  
 Air pollution, 50(11), 54(120), 215(40),  
 238(81), 384(30), 504(32), 529(62),  
 657(52), 742(41), 771(22), 846(80)  
 Beach erosion, 515(47), 530(83)  
 Cans recycled, 507(42), 523(32)  
 Chemical dump site, 216(50)  
 Forest fire, 54(124), 77(9, 10)  
 Insecticide DDT, 381(64), 391(46),  
 967(18)  
 Lake pollution, 108(50), 529(68)  
 Nuclear power plant discharge, 178(81),  
 967(17)  
 Oil spill, 84(63), 185(76), 232(54), 681(32),  
 749(59), 759(52)  
 Organic pollutants, 657(47, 48, 49, 50)  
 Ozone layer, 397(81)  
 Pond draining, 709(70)  
 River flow, 50(6), 721(13), 775(40), 811(4),  
 968(42)  
 Seawater corrosion, 725(26)  
 Stream pollution, 44(46), 54(118), 519(44)  
 Trash compacting, 751(9)  
 Water consumption, 829(24)

## Geodesy, Geology, Seismology

Arctic Circle, 135(28)  
 Athabasca glacier, 90(39)  
 Bermuda Triangle, 65(42)  
 Comparing earthquakes, 391(52)  
 Distance from Manila to Shanghai,  
 258(18)  
 Distance from Whitehorse to Russia,  
 262(81)  
 Distance to point on Lake Ontario, 931(39)  
 Drumlin surface, 793(35)  
 Earth's crust, 485(33)  
 Earth's radius, 73(37)  
 Earthquake centre, 613(100)  
 Earthquake intensity, 373(34), 577(60)  
 Easter Island area, 76(2)  
 Gravitational force on spacecraft, 53(102),  
 395(31), 811(38)  
 Haiti earthquake (2010), 384(42)  
 Hill surface, 677(44), 905(34)  
 Hours of daylight, 307(45), 319(73, 74),  
 828(18), 845(66), 863(40)  
 Jurassic geological period, 472(56)  
 Lake Erie length, 948(66)  
 Lake Ontario width, 75(2)  
 Moment magnitude scale, 384(42), 391(51)  
 Ocean tides, 303(60), 316(38), 557(57)  
 Richter scale for earthquakes, 384(41)  
 Sleeping Giant Provincial Park, 134(13)

## Hydrodynamics and Hydrostatics

Air pressure, 312(13, 14), 395(28, 36),  
 968(43)  
 Atmospheric pressure, 20(30)  
 Capillary tube, 294(50), 812(48)  
 Dam water release, 515(48)  
 Floating buoy, 432(36)  
 Floodgate, 799(34)  
 Fluid flow, 46(16), 189(68), 218(97),  
 237(58), 339(72), 340(16), 530(81),  
 686(57), 767(46), 867(52)  
 Fluid pressure, 506(9)  
 Force due to water pressure, 38(58),  
 690(57), 808(3, 21, 22, 23, 24, 25, 26,  
 27), 809(29, 30), 812(50), 813(8)  
 Hydraulic press, 84(58)  
 Hydrodynamics, 46(11), 218(111), 767(43)  
 Peterborough Lift Lock, 485(39)  
 Roman aqueduct, 139(79)  
 Streamlines of fluid flow, 955(43, 44)  
 Water discharge from tank, 111(17, 18, 19),  
 686(52), 968(44)  
 Water flow, 73(40), 236(36), 272(22),  
 377(68), 582(56), 596(49), 698(50),  
 718(30), 721(11), 725(21), 726(37)  
 Water pressure, 113(68), 114(78), 236(38),  
 387(50), 493(71)  
 Water pumping, 781(13), 808(18, 19, 20),  
 812(44)  
 Water waves, 112(1), 238(68), 698(53),  
 742(15), 828(17), 948(64)

## Machine Technology

Air pump, 519(39)  
 Ball bearings, 114(84), 218(102), 432(41),  
 434(62), 656(45, 46), 686(55), 747(34)  
 Bell crank, 77(16), 175(35)  
 Braking system, 709(69)  
 Cam design, 76(1), 111(11, 12), 258(22),  
 312(5, 6), 586(46), 606(57), 613(97),  
 686(54), 732(43), 787(47)  
 Cam follower, 776(57)  
 Chain saw, 262(86)  
 Chain saw engine fuel, 179(88)  
 Clutch, 690(53), 846(77)  
 Conveyor belt, 781(5)  
 Defective parts, 741(6)  
 Diesel engine design, 725(20)  
 Drill bit, 259(47), 803(25)  
 Emery wheel, 714(24)  
 Etching tool, 829(22, 23)  
 Flywheel, 218(105), 709(74), 803(27)  
 Friction drive, 596(51)  
 Gears, 51(12), 53(91), 73(48), 139(88),  
 262(75), 294(52), 413(45), 499(26),  
 542(49)  
 Grinding machine, 809(38)  
 Hoist mechanism, 781(16)  
 Hydraulic component failure, 776(58)  
 Jet engine design, 46(13)

Leather-cutting blade, 894(85)  
 Machine design, 46(9), 47(31), 189(64),  
 197(29), 215(38), 218(103), 231(45)  
 Machine parts, 72(3), 83(51), 292(40),  
 577(56), 606(60), 658(4, 5, 6, 12),  
 708(58), 749(66), 750(72), 798(7, 8, 9,  
 10), 811(28, 36), 879(36)  
 Machine tool, 149(39)  
 Mechanism design, 218(98, 113)  
 Metal plate, 189(69), 894(82)  
 Oil reservoir, 726(36)  
 Outboard engine fuel, 51(28), 54(122)  
 Packaging machine, 216(51)  
 Pipe, 189(70), 238(72), 294(60), 577(53),  
 912(26)  
 Piston, 552(50), 558(60), 710(4), 781(6)  
 Pressing machine, 945(27)  
 Pressure gauges, 47(30)  
 Pulley belt, 73(46), 113(67), 178(80),  
 257(4), 262(76), 558(74)  
 Pulleys, 47(21, 33), 507(40), 577(55), 714(30)  
 Pump design, 179(95), 219(126)  
 Rivet, 81(39)  
 Robotic link, 20(84), 53(105), 103(52),  
 113(61), 134(6), 165(39), 169(25),  
 236(35), 268(7), 272(27), 291(28),  
 300(32), 432(40), 552(47), 561(114),  
 609(49), 645(15), 698(49), 732(40),  
 822(49)  
 Robotic movement, 722(26)  
 Robotics, 196(68), 686(48), 690(49),  
 706(53), 709(68), 846(67), 859(44),  
 871(42)  
 Roller belt, 414(52)  
 Roller mechanism, 319(77), 406(38),  
 721(12), 726(44), 843(32, 33), 968(41)  
 Sanding machine, 799(32)  
 Saw tooth, 83(47)  
 Shearing pin, 294(52)  
 Slider mechanism, 397(89)  
 Spring mechanism, 310(29)  
 Springs, 158(42), 218(109), 225(51),  
 361(56), 434(55), 504(35), 657(54),  
 808(1, 5, 6, 8), 813(7), 847(96), 893(74)  
 Stamping machine, 883(33)  
 Taper, 135(27)  
 Torture testing, 523(39), 928(40)  
 V-gauge, 136(35)  
 Wrench, 292(33)

## Materials

Brass, 653(4)  
 Copper alloy, 180(11)  
 Copper ore, 178(79)  
 Fertilizer, 170(27)  
 Gold alloy, 54(128), 179(93)  
 Lead alloy, 498(3)  
 Nickel alloy, 175(41)  
 Plastic building material, 36(49)  
 Plastic sheet, 340(91)  
 Sterling silver, 166(50)

## D.6 Index of Applications

### Measurement

Agricultural test station, 596(52)  
Airplane altitude, 139(82)  
Antennas, 53(103)  
Artillery, 253(65)  
Balloon, 108(57), 725(2), 848(8), 1000(68)  
Banner, 136(36)  
Baseball batting average, 11(56)  
Beach shade, 65(46)  
Bicycle wheel, 253(67)  
Blueprint measurements, 506(25)  
Bolts, 507(31)  
Bottle volume, 634(24), 640(22), 645(15, 16)  
Box capacity, 506(20)  
Building height, 825(53)  
Building lots, 84(54), 238(71), 494(10)  
Buoy circumference, 95(46)  
Cable length, 159(46), 750(68)  
Calculator use, 54(129), 620(32)  
Canada Post package regulation, 742(26, 27)  
Carpet area, 138(66)  
Chair, 291(29)  
Chlorine in pool, 519(46)  
Coin tosses, 620(31)  
Communications cable, 844(41)  
Container weight, 6(38)  
Copy machine reductions, 519(42)  
Crate angle, 140(90)  
Crop-dusting area, 140(94)  
Cylinder, 506(5)  
Cylindrical cup area, 114(87)  
Desktop, 416(8)  
Distance between cities, 280(8), 287(30), 498(1)  
Distance between persons, 84(68), 403(38)  
Distance between points on blueprint, 506(25)  
Distance between streets, 59(35), 135(33)  
Distance from horizon to horizon, 504(38)  
Distance to bridge, 295(67)  
Distance to fire, 90(38), 294(63)  
Distance to hang glider, 140(91)  
Distance to helicopter, 95(37)  
Distance to mountain, 288(38)  
Distance to object, 948(67)  
Distance to rocket, 65(44), 133(2), 231(46)  
Distance to shoreline, 750(71)  
Distance travelled, 36(50), 90(45), 114(79), 175(43), 216(56), 219(130), 269(45), 493(70), 775(42)  
Fence enclosure, 415(62)  
Field dimensions, 95(40), 238(75), 415(62), 482(58), 581(44), 736(30), 741(1), 751(10)  
Flagpole height, 84(55)  
Flare shot, 11(58)  
Fossils, time dating and, 472(56)  
Glass cube, 506(6)  
Golf course holes, 175(38)  
Grey Cup blimp height, 133(1)

Googol, 29(52)  
Greyhound and fox, 216(58)  
Guy wire length, 65(49), 406(42)  
Heating object, 962(18)  
Height of cliff, 294(54)  
Iceberg height, 310(27)  
Ladder against wall, 65(45)  
Lawn sprinkler area, 258(23)  
Lead sphere, 83(48)  
Length of Natapoka Arc (Hudson Bay), 73(49)  
Lighted area, 599(45, 46)  
Log volume, 47(44)  
Map distance, 506(17), 509(77)  
Marshy area, 85(14)  
Mine shaft, 281(26)  
Paper thickness, 529(69)  
Paperweight, 136(39)  
Pile of sand, 750(75)  
Piled logs, 515(49)  
Pole on wall, 66(51), 577(54), 830(37)  
Pond, draining, 709(70)  
Ranch perimeter, 180(7)  
Rectangular plate, 73(42)  
Rope length, 613(102)  
Shadow, 846(68)  
Shot-put stopboard, 262(80)  
Soccer ball area, 140(89)  
Soccer player, 268(5)  
Spotlight beam, 258(24), 259(56)  
Sprinkler system flow, 159(48)  
Storage depot fencing, 112(4)  
Street light height, 134(20)  
Strobe light times, 619(17, 18, 19, 20), 627(21, 22)  
Tank volume, 681(26)  
Telephone pole, 85(3)  
Theater rows, 515(51)  
Time, 253(66), 472(54)  
Time on clock, 258(21)  
Time using ship bells, 515(54)  
Tree measurement, 73(41), 134(5)  
Wall pennant area, 65(41)

### Meteorology

Air temperature, 47(27)  
Annual rainfall in Vancouver, 627(40)  
Atmospheric pressure at given altitude, 395(36)  
Atmospheric temperature, 898(33)  
Cloud ceiling, 139(80)  
Daily temperature for Montreal, 316(35), 654(12)  
Daily temperature for Sydney, Australia, 846(69)  
El Niño, 113(63)  
Hurricane, 85(72)  
Isobars, 968(49)  
Isotherms, 1000(62)  
Jet stream, 843(37)

Rainfall, 80(26), 698(48), 736(27)  
Small craft warning, 268(6)  
Storm cloud, 293(46)  
Storm front, 280(13)  
Tornado, 504(9)  
Weather balloon, 80(25), 94(27), 95(43)  
Weather instruments, 844(40)  
Weather satellite, 606(58)  
Wind indicator, 863(37)  
Wind-chill factor, 111(6, 14)

### Motion

Falling object, 281(33), 395(27), 414(50), 415(57, 59), 416(6), 515(55), 529(71, 72), 650(6), 708(51), 721(16), 781(1, 3, 12), 829(25), 833(55), 840(39, 40), 893(75), 966(4), 968(37), 1000(60)  
Flare, 228(34)  
General  
Acceleration, 175(37), 612(80), 748(11, 12), 856(41), 867(45)  
Ambulance to and from accident, 219(132)  
Average speed of truck, 95(39)  
Baseball player, 749(61)  
Beach ball, 781(4)  
Bicycle, 523(35)  
Bouncing ball, 530(9), 545(53)  
Canadian Grand Prix race, 51(25)  
Car and plane, 750(70)  
Car passing semi, 51(31)  
Crate, 829(29)  
Distance moved, 94(23), 165(44), 529(70)  
Hydrofoil, 225(53)  
Motorboat, 280(7)  
Moving boat, 368(92), 726(40, 41), 968(39)  
Moving ladder, 726(38)  
Moving particle, 548(39), 1000(61)  
Moving shadow, 726(28), 726(43)  
Path of least time (brachistochrone), 537(62)  
Person on barge, 281(34)  
Rising balloon, 523(34)  
Searchlight beam, 829(27)  
Ship speed, 54(123)  
Ski slope, 51(21)  
Sphere in fluid, 422(56)  
Subway stopping distance, 708(64), 811(2)  
Terminal speed of skydiver, 32(48)  
Trains, 216(54), 415(70), 476(45)  
Trains in Eurotunnel, 51(23)  
Travel time, 51(24)  
Velocity, 9(7), 44(45), 215(37), 258(30), 272(24), 277(32), 293(44), 325(66, 67), 658(14), 668(55), 681(21, 22, 23), 685(44), 748(7, 8, 9, 10), 811(2), 867(44)



Walking speed and distance, 749(47)  
 Weather balloon, 726(42)  
**Oscillatory, Rotational, and Vibrational**  
 Buoy, 318(69)  
 Energy, 195(64)  
 Ferris wheel, 253(68), 259(40),  
 612(86)  
 Floating wood, 988(6)  
 Gyroscope, 185(70)  
 Mechanical vibrations, 218(121)  
 Motorboat propellor, 258(34), 318(67)  
 Musical notes, 219(127)  
 Object moving in circle, 505(57),  
 508(48), 750(78), 846(79), 898(30)  
 Object moving in conic section path,  
 828(2), 999(53, 54)  
 Oscillating plate, 253(69)  
 Oscillating spring, 316(33), 319(75),  
 338(65), 529(73), 654(11), 819(50),  
 844(39), 923(39), 939(21), 948(68,  
 70, 74), 988(10, 11, 12, 13, 14),  
 997(33, 34), 1000(75, 85, 86),  
 1001(12)  
 Pendulum, 138(64), 262(74), 312(15),  
 508(47), 725(13), 862(35), 914(39),  
 931(36), 988(5), 997(26)  
 Piano wire, 303(58)  
 River boat paddles, 258(37), 313(21)  
 Rotating beacon, 312(19)  
 Rotating drum, 258(32)  
 Rotating wheel, 537(60), 829(21),  
 847(91)  
 Rubber raft, 319(70)  
 Simple harmonic motion, 871(41),  
 988(1, 3, 4, 9)  
 Sprocket assembly, 259(43)  
 Stone moving in circle, 73(47)  
 Torsion pendulum, 318(66), 1001(88)  
 TV camera, 846(84)  
 Vibrating rod, 997(27, 28)  
 Vibrating wire, 503(3), 948(63)  
 Vibration analysis of three-storey build-  
 ing, 718(31)  
 Watch second-hand, 258(38)  
 Wave, 355(40), 541(46), 561(110)  
 Wave in string, 307(37), 312(11, 12),  
 319(83, 6), 505(48), 725(9), 909(44),  
 914(8)  
**Rectilinear and Curvilinear**  
 Acceleration, 552(45)  
 Airport walkway, 165(48)  
 Arrow, 781(14)  
 Balloon, 294(51)  
 Baseball, 732(39), 811(1, 6)  
 Bullet, 499(40), 706(47), 846(76)  
 Car on test track, 749(46)  
 Elevator, 216(53), 808(13), 829(28)  
 Exercise machine, 722(20)  
 Falling object, 53(98), 90(3)  
 Golf ball, 721(15), 811(5)  
 Hang glider, 257(3)

Missile, 47(47), 231(46), 236(41),  
 295(68), 657(55), 844(44)  
 Object on inclined plane, 280(22), 319(80),  
 612(81), 653(5), 747(30), 999(59)  
 Object through resisting medium,  
 856(46)  
 Parachutist, 749(45), 893(70), 968(40)  
 Projectile, 195(57), 237(61), 545(57),  
 681(27, 28), 776(54)  
 Radio-controlled car, 721(14)  
 Rock, 812(41)  
 Scuba diver, 281(28)  
 Ski tow, 269(43)  
 Skier, 722(22), 781(8)  
 Sky diver, 280(15)  
 Tennis ball machine, 781(7)  
 Velocity, 708(59, 66), 722(18, 23, 27),  
 813(6), 888(27)  
 Water skier, 272(30)  
 Wind speed, 281(25)

### Navigation

Boat's course, 281(23), 281(30), 288(37),  
 292(35, 36, 38), 295(69), 368(92),  
 414(51), 722(24)  
 Coast Guard boat, 139(84), 165(49), 281(31)  
 Ship's course, 140(4), 269(46), 277(30),  
 280(1, 20), 295(8), 415(69), 582(57)  
 Ship's location, 295(70), 414(49), 742(23)  
 Ship's location using LORAN, 593(53)  
 Ship's power, 505(43)  
 Submarine, 126(58), 272(21), 291(27)  
 Tugboats, 277(29)

### Nuclear and Atomic Physics

Alpha particle, 741(13)  
 Atomic mass, 29(58), 498(16)  
 Atomic particle, 867(47)  
 Atomic spectra, 847(92)  
 Atomic structure, 415(53), 894(78)  
 Atomic theory, 47(40)  
 Electron motion, 722(18), 808(9, 10),  
 813(53)  
 Force between electrons, 29(44)  
 Molecular orientation, 294(55)  
 Nuclear physics, 46(12), 218(100, 119)  
 Radioactive elements:  
 carbon, 384(29)  
 cesium-137, 397(82)  
 cobalt, 967(13), 999(56)  
 helium, 967(16)  
 iodine, 1000(64)  
 neon, 1000(63)  
 plutonium, 523(33)  
 potassium, 1000(65)  
 radium, 377(65), 397(87), 843(26), 967(14)  
 radon gas, 967(15)  
 strontium, 395(29)  
 substance, 948(65)  
 uranium, 29(55)

### Optics

Apparent depth in water, 292(32)  
 Electron microscope, 472(53)  
 Fibre-optic cable, 488(49), 506(27),  
 620(29, 30)  
 Fibre-optic system, 384(43), 519(47)  
 Fresnel integral, 927(35)  
 Holography, 681(41), 725(17), 747(31),  
 782(32), 846(75)  
 Illuminance, 397(86), 552(48), 847(93)  
 Illumination, 108(51)  
 Interference of light waves, 561(104)  
 Interference of waves, 318(68)  
 Interferometer, 410(37)  
 Laser beam, 29(43), 59(34), 139(87),  
 248(56), 262(91), 287(33), 310(28),  
 340(97), 482(62), 587(56), 698(56)  
 Laser energy, 29(43), 113(71), 690(56),  
 859(42)  
 Lasers, 185(66), 196(66), 612(88)  
 Lens combinations, 708(52)  
 Light dispersion, 410(36), 527(54)  
 Light emission, 237(62)  
 Light intensity, 398(97), 529(63), 552(48),  
 593(54), 867(51)  
 Light passing through narrow slit, 847(90)  
 Light reflection, 369(94), 542(50), 545(58),  
 582(53), 743(50)  
 Light refraction, 825(52)  
 Light waves, 318(64), 923(42)  
 Mirrors, 434(57), 593(52), 948(72)  
 Optic fibre, 103(51), 572(57), 819(56)  
 Optical lens, 325(70), 413(46), 507(39),  
 701(40), 799(31), 898(36), 931(38)  
 Optical prism, 80(30)  
 Polarized light, 316(42)  
 Refraction, 507(44)  
 Spectroscopy, 46(14), 415(56), 561(105)  
 Transmittance of a substance, 384(40)  
 Wavelength of light, 141(11), 548(43),  
 747(29)

### Petroleum

Athabasca oil pipeline, 80(29)  
 Blending gasoline, 102(37)  
 Diesel fuel production, 111(1)  
 Fuel consumption, 178(75, 76)  
 Fuel oil storage, 51(14)  
 Gas and diesel fuel production, 488(50),  
 507(32)  
 Gasoline blending, 44(49)  
 Gasoline in car, 20(31)  
 Gasoline storage tank, 898(38)  
 Land leases, 50(9)  
 Mixing, 149(40), 158(43)  
 Natural gas pipeline, 51(17)  
 Natural gas reserves, 258(19)  
 Oil additives, 54(125)  
 Oil pipeline, 294(59), 701(42), 743(49)  
 Oil pipeline pressure, 432(38), 650(11)

## D.8 Index of Applications

Oil pressure, 808(24), 809(28)  
Oil production, 485(35), 530(87),  
741(4), 767(44)  
Oil pump, 169(20)  
Oil spill, 681(32)  
Oil storage tank capacity, 94(31), 113(66),  
259(55), 476(46), 571(54), 717(28),  
749(57), 812(47), 871(40)  
Oil tanker, 216(49)  
Oil viscosity, 655(11, 12, 13, 14, 15, 16)  
Oil well drilling, 11(61), 47(26)  
Pipeline flow, 114(83), 189(67), 237(60),  
701(42)  
Specific gravity of oil, 498(19)  
Trans-Alaska pipeline, 262(90)

### Photography

Aerial photography, 84(57)  
Camera flash, 373(37), 391(48)  
Camera focal length, 232(48), 482(56)  
Camera lens, 136(40), 505(44)  
Camera magnification, 103(50)  
Camera on Mars, 134(9)  
Cartridge exposures, 36(51)  
Photographic film image, 45(52), 552(49)  
Photography, 47(20, 24), 185(65), 216(44),  
218(120)

### Physics

Bending moment, 434(53)  
Centre of mass, 811(27)  
Coefficient of friction, 498(13), 845(63, 65)  
Displacement, 277(34), 759(50)  
Elastic collision, 415(58)  
Elasticity, 189(73)  
Energy and momentum, 406(32)  
Force and acceleration, 504(41), 706(52),  
843(31), 909(38)  
Force components, 277(31)  
Gravitational force, 41(49), 747(32), 808(11)  
Kinetic energy, 47(19), 927(37)  
Mechanical advantage, 94(30)  
Mechanics, 46(10), 53(85)  
Moment of inertia, 803(3, 4, 5, 6), 811(35)  
Moments of forces, 53(87)  
Momentum, 53(90)  
Pressure, 898(34)  
Relative density, 219(129)  
Relativity, 340(93), 669(66), 927(37)  
Solid state physics, 218(89)  
Specific gravity, 717(25)  
Torque, 218(118)  
Work, 218(115)  
Work and power, 787(43)

### Police Science

Ballistics test, 47(22), 268(8), 293(45), 968(38)  
Bullet impact, 613(101)  
Bullet line of fire, 134(8), 294(58), 572(58)  
Drivers exceeding speed limit, 640(21)

Legal highway speed, 476(44)  
Police helicopter, 612(85), 829(26)  
Police radar, 656(31, 32, 33, 34, 35, 36)

### Refrigeration and Air Conditioning

Air conditioning, 46(8)  
Air conditioning damper, 294(57)  
Air conditioning duct, 94(36)  
Energy used by refrigerator-freezer, 836(50)  
Freezer temperature, 41(50), 114(85),  
690(51), 709(77)  
Performance coefficient, 504(36)  
Refrigerant freon, 653(7)  
Refrigeration, 47(29, 36)

### Space Technology

Apollo 11 (moon spacecraft), 613(98)  
Astronaut's weight, 102(46)  
Communications satellite, 53(106), 136(38),  
287(35), 577(58)  
Earth satellite, 84(59), 294(61), 307(39),  
312(7, 8), 338(68), 340(90), 472(55),  
613(103), 846(70)  
Escape velocity, 505(46), 1001(90)  
Lunar lander, 781(10)  
Lunar rover, 288(36)  
Object in space, 333(69), 726(31)  
*Pioneer* space probe to Jupiter, 272(34)  
Rocket flight path, 159(49), 293(48),  
318(61), 406(31), 485(41), 545(54),  
582(50), 809(36), 819(55)  
Rocket fuel, 218(110), 225(42), 482(54),  
808(15)  
Rocket height, 102(44), 108(48), 253(71),  
319(72), 493(78), 558(73), 650(9),  
717(24), 732(47), 749(54), 787(45),  
822(51)  
Rocket motion, 960(38)  
Rocket nose cone, 812(45), 875(34)  
Rocket thrust, 732(42)  
Rocket velocity, 280(10), 377(64), 387(54)  
Rocket weight, 94(25), 114(2)  
Satellite power, 519(50), 843(25)  
Satellite velocity, 281(24), 725(18)  
Space probe, 498(23)  
Space shuttle, 134(12), 178(83), 280(21),  
726(35)  
Space vehicle, 852(40), 867(48), 894(83)  
Spacecraft acceleration and velocity,  
721(15), 722(17), 726(32)  
Spacecraft altitude, 572(63), 746(25),  
829(31)  
Spacecraft circling Moon, 262(93)  
Spacecraft launch to moon, 582(51)  
Weight in space, 482(59)

### Statics

Chains support crate, 152(33)  
Force on weld, 211(63)

Forces on:  
balloon, 269(47)  
beam, 476(40)  
boat, 272(28), 349(39)  
bolt, 272(32), 293(49), 352(42), 507(45),  
560(101)  
box, 248(54), 846(86)  
car, 268(4)  
chain holding engine, 280(18)  
crane, 178(73)  
girder, 169(21)  
hockey puck, 280(3)  
lever, 45(50), 179(87), 272(29), 281(27),  
506(7), 507(37), 572(62)  
pin joint, 175(45)  
structure, 145(29), 338(70)  
tower, 340(101)  
tree, 280(4)  
wheelbarrow, 272(31)  
Object on inclined plane, 558(58)  
Removing equipment from mud, 280(5)  
Ropes support crate, 269(48), 280(12),  
287(28, 34), 292(34)  
Tension in cable, 21(100), 287(29), 368(91),  
811(11)  
Torque, 159(50)  
Unloading car from ship, 272(23)

### Strength of Materials

Beam deflection, 38(64), 47(38), 53(95),  
185(69), 432(35), 434(7, 53), 582(52),  
706(51), 709(73, 79), 732(46), 742(37),  
748(43), 776(54), 782(30), 927(35),  
988(26), 997(37), 1001(89)  
Beam design, 196(65), 216(46)  
Beam stiffness, 504(11), 743(46)  
Beam strength, 237(55), 628(50), 829(35)  
Cantilever beam, 803(28), 988(25)  
Cantilever column, 310(30)  
Concrete beam, 413(44)  
Concrete column, 218(124), 397(83)  
Concrete slab, 702(46)  
Diving board deflection, 686(50)  
Pillar crushing load, 898(39)  
Pipe deflection, 749(49)  
Slenderness ratio of column, 506(12)  
Steel column, 702(47)  
Strain on beam (Mohr's circle), 577(61)  
Strength of materials, 211(64), 686(51)  
Stress on bar, 545(60)  
Stress on pipe, 967(21)  
Stretched wire, 914(38)  
Support column, 432(34)  
Suspended cable, 415(55)  
Tensile strength, 845(64)  
Vinyl sheet, 909(39)

### Surveying

Azimuth, 561(107)  
Barometric levelling, 507(46)

Bridge height, 295(71)  
 Building location, 280(9)  
 Cliff height, 135(26)  
 Head-Smashed-In Buffalo Jump height, 294(54)  
 Irregular shoreline, 113(75)  
 Land elevations, 141(14)  
 Land measurements, 138(69), 178(82), 260(60), 261(71), 287(26), 293(42), 829(32)  
 Marker distances, 822(52)  
 River width, 66(52)  
 Security restricted area, 294(64)  
 Surveying, 46(7)  
 Surveyor's formula, 165(40)  
 Utility pole location, 295(3)

### Thermal

Boiling point of water, 237(59)  
 Cooling object, 668(53), 681(25), 681(31), 685(45, 46), 843(27), 909(37)  
 Cooling steam, 111(9, 10)  
 Heat flow, 185(64)  
 Heat intensity, 726(33)  
 Heat of vapourization, 506(11)  
 Heated liquid, 657(60)

Hot metal bar, 395(37), 909(43)  
 Ice forming on weather balloon, 748(33)  
 Isotherms, 714(23), 905(31)  
 Melting ice, 504(33), 685(46), 726(27)  
 Newton's law of cooling, 846(81), 967(25, 26)  
 Temperature and pressure, 398(99)  
 Temperature change, 571(53), 962(18)  
 Temperature in blast furnace, 759(55)  
 Temperature readings, 11(57), 38(60), 47(43), 108(58), 398(100), 476(37, 42), 519(49), 529(79), 571(49), 612(83, 84), 657(60), 782(28), 809(32), 837(52), 955(41), 1000(71)  
 Thermal expansion, 53(96), 185(71), 218(101), 434(8), 653(6), 718(29), 909(37), 914(41)  
 Thermal resistance, 53(93)

### Thermodynamics

Absolute zero, 651(14), 856(44)  
 Adiabatic expansion of gas, 395(32), 504(10), 698(51), 1000(66)  
 Boyle's law, 498(25), 726(29)  
 Energy in metal at lower temperature, 767(49)  
 Entropy, 893(71)  
 Expansion of gas, 698(51)

Gas molecules, 844(43), 852(37)  
 Gas volume and pressure, 856(42)  
 Gas-liquid equilibrium, 422(57)  
 General gas law, 505(52), 914(43)  
 Isothermal change, 377(67)  
 Otto cycle, 325(68)  
 Pressure of gas, 898(31)  
 Pressure, volume and temperature of a gas, 701(41)  
 Radiant energy emission, 38(62), 185(68), 189(76)  
 Speed of oxygen molecules, 505(49)  
 Thermodynamic surface, 905(35)  
 Thermodynamic temperature, 398(98), 527(52), 690(46, 54), 701(41), 898(35)  
 Thermodynamics, 195(62), 197(28, 31), 211(60), 216(43)  
 Vapour pressure, 843(29)  
 Volume expansivity, 914(36)

### Wastewater Technology

Chemical waste holding tank, 812(49)  
 Draining holding tank, 232(56), 504(39), 522(31)  
 Wastewater pump, 488(53)



# Index of Writing Exercises

The final exercise in each chapter is a writing exercise. Each of these 31 exercises will require at least a paragraph to provide a good explanation of the problem presented. There are also about 400 other exercises throughout the text marked with a green circle before the exercise number or instructions that require at least one complete sentence (up to a paragraph) to provide the explanation needed.

Following is a listing of these writing exercises. The first number shown is the page number, and the numbers in parentheses are the exercise numbers. The \* denotes the final review exercise of the chapter.

## Chapter 1: Basic Algebraic Operations

5(21, 22, 33)  
6(34, 35, 36, 43)  
10(50)  
11(51, 52, 53, 54, 56)  
20(83, 84)  
21(88, 94)  
26(58, 60)  
32(51)  
36(55, 56)  
38(53, 54)  
53(109, 110)  
54(129\*)

## Chapter 2: Geometry

59(39, 40)  
65(30, 32, 33, 34)  
68(25, 26, 29)  
69(39)  
73(33, 36, 47)  
76(1, 2, 3, 4)  
78(17, 18, 19, 20)  
83(43, 44)  
85(73\*)

## Chapter 3: Functions and Graphs

90(29, 30, 31, 32, 33, 34, 35, 36, 47)  
94(11, 12)  
95(41, 50)  
97(32, 34)  
103(55, 57, 58, 61, 62)  
108(54, 56, 59, 60)  
112(45)  
113(55, 56)  
114(87\*)

## Chapter 4: The Trigonometric Functions

122(36)  
126(45, 46, 47, 48)  
131(44)  
137(54)

138(56)  
140(89, 95\*)

## Chapter 5: Systems of Linear Equations; Determinants

159(55, 56)  
170(28)  
177(41, 42, 43, 44)  
178(82)  
179(95\*)

## Chapter 6: Factoring and Fractions

185(80)  
189(61, 62)  
196(74)  
197(36, 37, 38)  
202(65, 66, 67, 68)  
211(58)  
219(133\*)

## Chapter 7: Quadratic Equations

231(44)  
236(25, 26, 27, 28, 29, 30)  
238(78, 83\*)

## Chapter 8: Trigonometric Functions of Any Angle

248(45, 46, 47, 48)  
253(70)  
258(18)  
259(57)  
262(81, 93\*)

## Chapter 9: Vectors and Oblique Triangles

268(5, 6, 7, 8)  
281(24)  
291(22, 24, 25)  
292(40)  
293(38)  
295(72\*)

## Chapter 10: Graphs of the Trigonometric Functions

306(31, 32, 33, 34)  
307(41, 42, 43, 44)  
312(19)  
317(39, 40)  
318(68)  
319(74, 83\*)

## Chapter 11: Exponents and Radicals

325(58, 60)  
328(62)  
329(63)  
332(68)

335(46, 47)  
338(60)  
340(101\*)

## Chapter 12: Complex Numbers

345(61, 62, 64)  
347(60, 61, 62)  
349(35, 36, 37, 38)  
360(52)  
366(24)  
369(97\*)

## Chapter 13: Exponential and Logarithmic Functions

377(60, 62, 71, 72)  
381(57, 58, 59, 63)  
387(38)  
398(105\*)

## Chapter 14: Additional Types of Equations and Systems of Equations

403(36)  
413(19, 20, 41)  
414(39, 40)  
415(52, 71\*)

## Chapter 15: Equations of Higher Degree

422(47, 48, 53, 54)  
426(27, 28)  
432(46)  
433(41, 42, 45)  
434(46, 63\*)

## Chapter 16: Matrices; Systems of Linear Equations

439(35, 44)  
444(38)  
449(36, 38)  
453(29, 30)  
465(68)  
466(90\*)

## Chapter 17: Inequalities

471(1)  
472(25, 26, 27, 28, 48, 51)  
481(45, 46)  
485(29, 35)  
492(20)  
493(57, 58)  
494(83\*)

## Chapter 18: Variation

504(13, 14, 15, 16, 40, 41)  
505(42)  
509(78\*)

**Chapter 19: Sequences and the Binomial Theorem**

515(35, 36, 58)  
 519(29, 54, 55, 56)  
 522(30)  
 523(36)  
 527(39, 45)  
 530(91, 93\*)

**Chapter 20: Additional Topics in Trigonometry**

537(63, 64)  
 541(28, 42)  
 552(41)  
 557(1)  
 558(72)  
 561(115\*)

**Chapter 21: Plane Analytic Geometry**

571(39, 40, 53)  
 572(56, 59)  
 577(43, 44, 47)  
 581(43)  
 586(37, 38)  
 587(48)  
 592(46)  
 599(39, 40, 41, 42)  
 606(60)  
 609(53, 54)  
 614(107\*)

**Chapter 22: Introduction to Statistics**

620(31, 32)  
 628(43, 44, 45, 46)  
 633(1)  
 634(8, 21, 23, 24, 32)  
 640(22)  
 645(3, 4)  
 658(71\*)

**Chapter 23: The Derivative**

668(21, 22, 23, 24)  
 669(64, 65, 66)

677(39)  
 681(28)  
 685(40, 42)  
 689(36)  
 698(44)  
 708(49, 50)  
 709(84, 87\*)

**Chapter 24: Applications of the Derivative**

714(19)  
 717(19, 20, 22)  
 722(28)  
 731(33, 34, 35)  
 732(54)  
 736(19, 20)  
 747(36, 40)  
 749(52)  
 750(81\*)

**Chapter 25: Integration**

754(39)  
 759(41, 42, 43, 44, 45, 46)  
 760(60)  
 764(24, 25, 26, 27, 28)  
 767(35, 39, 46)  
 771(17)  
 774(2)  
 775(27, 28, 33, 37)  
 776(59\*)

**Chapter 26: Applications of Integration**

782(20)  
 787(31, 36)  
 793(27, 28, 32)  
 799(29)  
 803(28)  
 809(29, 34)  
 813(57\*)

**Chapter 27: Differentiation of Transcendental Functions**

818(37, 38)  
 822(39, 40)

825(35, 36, 40)  
 833(35, 36, 47, 48, 49)  
 836(33, 34)  
 843(15, 16)  
 847(99\*)

**Chapter 28: Methods of Integration**

855(32)  
 859(30)  
 862(31)  
 867(41)  
 871(31, 32, 33, 34, 35)  
 875(1, 2)  
 878(1, 2)  
 893(49, 50, 53)  
 875(85\*)

**Chapter 29: Partial Derivatives and Double Integrals**

898(27)  
 904(27)  
 909(36, 40)  
 912(26)  
 914(33, 45, 47\*)

**Chapter 30: Expansion of Functions in Series**

919(36)  
 923(29, 30)  
 930(33)  
 931(37)  
 934(39, 40, 41, 42)  
 948(75\*)

**Chapter 31: Differential Equations**

954(15)  
 960(29, 30)  
 962(15)  
 976(38, 39)  
 982(37)  
 993(28)  
 1001(75, 92\*)

# Index

- Abscissa, 96  
Absolute error, 745  
Absolute inequality, 468  
Absolute value, 3; of complex numbers, 350; in inequalities, 482; order of operations, 8  
Acceleration, 279, 704, 719, 778, 842; angular, 256  
Accuracy of number, 15, 123; approximate number, 18  
Addition: of algebraic expressions, 32; algebraic method of solving three linear equations, 166; of approximate numbers, 19; of complex number, 345, 357; of cubes, 196; of fractions, 206; of matrices, 437; of ordinates, 313; of radicals, 333; of signed numbers, 7; solution of system of equations, 404; of vectors, 266, 273  
Adjacent angles, 57  
Agnesi, Gaetana, 733  
Agnesi, Maria, 849  
Algebraic expressions, 24, 32  
Algebraic operations, 32  
Alternate-exterior angles, 58  
Alternate-interior angles, 58  
Alternating current, 311, 361, 531  
Altitude, 61  
Ambiguous case, 286  
Ampere, 364  
Amplitude of sine curve, 298  
Analytic geometry, 562  
Angle, 56, 116; acute, 56; bisectors, 61; central, 70; of depression, 132; double-angle formulas, 542; of elevation, 132; half-angle formulas, 545; inscribed, 71; negative, 116, 247; obtuse, 56; phase, 303; quadrantal, 117, 247; radian measure, 71; reference, 243; right, 56; of rotation, 600; standard position, 117; straight, 56; sum and difference of two, 537  
Angular acceleration, 256  
Angular velocity, 255  
Antiderivative, 753  
Antilogarithm, 382  
Apollo 11, 503  
Applied maximum and minimum problems, 737  
Approximate numbers, 15, 17, 121; operations, 18  
Arc, 70  
Arc length, 253  
Area: of circle, 70; of circular sector, 254; under a curve, 760; element of, 782; of geometric figures, 61; by integration, 782; lateral, 78; of quadrilateral, 67; of triangle, 62; between two curves, 783  
Argument of complex number, 350  
Aristotle, 86  
Arithmetic mean, 620  
Arithmetic sequence, 511  
Array, 616  
Associative law, 6, 438  
Asymptote, 100, 588, 733  
Attribute, 616; control chart for, 641  
Auxiliary equation of differential equation, 970; complex roots, 974; repeated roots, 973  
Average value, 807  
Average velocity, 659  
Axes: rotation of, 599; translation of, 593  
Axes, coordinate, 96  
Axis: of ellipse, 583; of hyperbola, 588; of parabola, 578; polar, 603  
Babbage, Charles, 417  
Bar, 34  
Barrow, Isaac, 752  
Base: of exponents, 21, 371; of logarithms, 382, 385; of solid, 78; of trapezoid, 66; units, 13  
Basic identities, 532  
Becquerel, Henri, 964  
bel, 383  
Bell, Alexander Graham, 383  
Benford's law, 398  
Benz, Karl, 737  
Bernoulli, Johann, 837  
Binomial, 33  
Binomial formula, 523, 525  
Binomial series, 526  
Binomial theorem, 523, 682  
Braces, 34  
Brackets, 8, 25, 34  
Briggs, Henry, 370  
Calculator, 11, 104, 150, 245, 370, 448; scientific notation, 28; solving systems of equations, 401  
Calculus, 659, 667, 711  
Cancellation, 199  
Capacitance, 361, 781  
Capacitor, voltage across, 779  
Carbon dating, 964  
Cardioid, 607  
Catenary, 392  
Cauchy, Augustin-Louis, 399, 667  
Cayley, Arthur, 435  
Center of mass, 793  
Central angle, 70  
Central line, 642  
Central tendency, 620  
Centre of circle, 69  
Centre of mass, 793  
Centroid, 61, 793; of solid of revolution, 797; of thin, flat plate, 795  
Chain rule, 691  
Change, time rate of, 827  
Character of roots, 230  
Charles' law, 500  
Chebyshev's theorem, 626  
Chord, 69  
Circle, 69, 400, 573  
Circular paraboloid, 901  
Circumference, 70  
Class, 616  
Class mark, 617  
Coefficient, 33, 155  
Cofunction, 128  
*Colossus* computer, 214  
Column of determinant, 160  
Common difference, 511  
Common logarithms, 382  
Common ratio, 516  
Commutative law, 6, 440  
Complementary angles, 57, 127  
Complementary solution of differential equation, 977  
Completing the square, 225, 575, 595  
Complex fraction, 209  
Complex number, 2, 341, 343, 345; division, 356; exponential form, 352; multiplication, 355; polar form, 350  
Complex plane, 348  
Complex roots, 424, 430  
Components of vector, 269, 273  
Composite function, 690  
Composite trigonometric curves, 313  
Computer, 903  
Concavity, 728  
Conditional equation, 41  
Conditional inequality, 468  
Cone, 78  
Confidence intervals, 634  
Congruent triangles, 64  
Conic sections, 400, 598  
Conjugate: axis of hyperbola, 588; of complex number, 344  
Constant, 4, 963; combining, 877; derivative of, 682; of integration, 755, 758; of proportionality, 500, 501  
Constraint, 488  
Continuity, 660  
Contradiction, 42  
Control charts, 641  
Convergent, 916  
Conversion of angle measurements, 249  
Conversion of angles, 116  
Conversion of units, 116  
Coordinates: cylindrical, 903; polar, 603; rectangular, 95  
Copernicus, Nicolaus, 586  
Correlation, 651  
Corresponding angles, 58, 119  
Corresponding segments, 58  
Corresponding sides, 63, 119  
Cosecant: of angle, 120; graph of, 307; integration of, 859

- Cosine: of angle, 120; derivative of, 823; of double angle, 542; graph of, 297; half angle, 545; half-range Fourier series, 944; integration of, 863; of sum of two angles, 537  
 Cosines, law of, 282, 288  
 Cotangent: of angle, 120; graph of, 307; integration of, 859  
 Coterminal angles, 116, 243  
 coulomb, 364  
 Cramer, Gabriel, 142, 162  
 Cramer's rule, 142, 162, 172, 460  
 Critical value, 477  
 Cube, 78; difference, 196; special products, 183–184; sum, 196  
 Cube roots, 358  
 Cumulative frequency, 618  
 Current, 361, 780  
 Curve: area between two, 783; area under, 760; finding, given slope, 963; graphing smooth, 109; identifying, 601; in polar coordinates, 606; sketching, 727, 732, 825, 841; slope of tangent to, 669; in three dimensions, 899  
 Curve in space, 900  
 Curvilinear motion, 718, 828  
 Cycle, logarithmic scale, 393  
 Cyclotron, 604  
 Cylinder, 78  
 Cylindrical coordinates, 903  
 Cylindrical shell, 789  
 Cylindrical surface, 903  
  
 d'Alembert, Jean, 264  
 Damped simple harmonic motion, 983  
 Dantzig, George, 467  
 Decibel, 383  
 Decimal, repeating, 2, 521  
 Decision variables, 488  
 Definite integral, 765  
 Deflection of beams, 986  
 Degree: of differential equation, 950; as measure of angle, 56, 116; of polynomial, 33; of term, 33  
 Delta method, 670  
 DeMoivre, Abraham, 357  
 DeMoivre's theorem, 357  
 Denominate number, 4, 323  
 Denominator: rationalizing of, 331  
 Dependent system of equations, 152, 163  
 Dependent variable, 87  
 Derivative, 659, 673; of a constant, 682; of a constant times a function, 683; of cosine function, 815; curve sketching, 727; of exponential function, 834; higher, 702; of implicit function, 700; of inverse trigonometric functions, 822; of logarithmic function, 830; partial, 895; of a polynomial, 682; of a power of a function, 690; of a product, 686; of a quotient, 686; as a rate of change, 677; second, 702; of sine function, 815; of a sum, 684; of trigonometric functions, 819  
 Derived units, 13  
 Descartes, René, 320, 343, 400, 427, 562  
 Descartes' rule of signs, 428  
 Determinant, 142; higher-order, 457; properties of, 459; second-order, 160; third-order, 170, 459  
 Deviation, 647; standard, 623  
 Diagonal, 66  
 Diagonal of determinant, 161  
 Diameter, 69  
 Difference: of matrices, 438; of squares, 665  
 Difference, common, 511  
 Difference engine, 417  
 Differential, 743, 827  
 Differential calculus, 659  
 Differential equation, 949; Laplace transforms, 994; linear, 969; numerical solutions, 960  
 Differentiation, 674; of implicit functions, 699; series formed by, 925; of transcendental functions, 814  
 Diocles, 220  
 Direct current, 531  
 Direction, 720  
 Directrix of parabola, 578  
 Direct variation, 500  
 Disc, 788  
 Displacement, 268, 778; of sine curve, 304  
 Distance, related rates and, 723  
 Distance formula, 563, 739  
 Distances, related rates and, 724  
 Distributive law, 6  
 Divergent, 916  
 Dividend, 39  
 Division: of algebraic expressions, 38; of complex numbers, 345, 356; of fractions, 202; of radicals, 336; with remainder, 418; of signed numbers, 7; synthetic, 419; by zero, 9, 99  
 Divisor, 39  
 Domain, 91, 675, 733  
 Double-angle formulas, 542  
 Double integral, 895, 909  
  
*e* (irrational number), 352, 385, 834  
 Edison, Thomas, 296, 362, 531  
 Einstein, Albert, 45  
 Electric circuits, 965, 985, 995  
 Element: of area, 782; of determinant, 160; of matrix, 436; of volume, 788, 910  
*Elements* (Euclid), 510  
 Elimination by addition or subtraction, 154, 404  
 Ellipse, 400, 582, 594, 601  
 Elliptic hyperboloid, 901  
 Engineering notation, 28  
 ENIAC, 214  
 Equal sign (=), 19  
 Equations, 41; conditional, 41; curves sketched from, 606; differential, 950; exponential, 388; graphical solution, 104, 146; higher-degree, 417; involving fractions, 212; linear, 142; literal, 45; logarithmic, 389; polynomial, 424; quadratic form, 407; with radicals, 410; roots of, 419, 423; second-degree, 596; solving graphically, 104–106; systems of linear, 144, 160, 166, 401, 460; systems of quadratic, 400; trigonometric, 548  
 Equilibrium, 279  
 Eratosthenes, 115  
 Estimating, 19  
 Euclid, 55, 510  
 Euler, Leonhard, 240, 352, 711, 814, 895, 949, 960  
 Euler's formula, 925  
 Euler's method, 960  
 Even function, 940  
 Exact number, 15; and approximate numbers, 19  
 Explicit functions, 699  
 Exponential equations, 388  
 Exponential form, 856; of complex number, 352  
 Exponential function, 370; derivative of, 834; graph of, 392; integration of, 854  
 Exponential value, 928  
 Exponents, 21, 320, 322  
 Extraneous roots, 407  
 Extraneous solution, 214  
 Extrapolation, 110  
 Extreme point, 233  
  
 Faces, 78  
 Factor, 32, 197, 419; canceling, 199  
 Factorial notation, 524  
 Factoring, 181, 185; common factor and difference of squares, 185; complete, 188; difference of two squares, 187; by grouping, 188, 194; quadratic equations, 222; trinomials, 190  
 Factor theorem, 419  
 Family of curves, 963  
 Farad, 364  
 Feasible point, 488  
 Fermat, Pierre de, 562, 752  
 First-derivative test, 727  
 First-quadrant angle, 116  
 Focus: of ellipse, 582; of hyperbola, 587; of parabola, 578  
 Folium, 701  
 Force: liquid pressure, 806; related rates, 724; unit of, 277  
 Forced vibrations, 985  
 Formula, 45; binomial, 523; distance, 563; quadratic, 228  
 Fourier, Jean, 915, 934  
 Fourier series, 934, 940, 980  
 Fraction, 2–3, 181; addition, 30, 206; complex, 209; division, 202; equations, 212; equivalent, 197; as exponent, 325; multiplication, 202; partial, 879, 883; simplest form, 198

## D.14 Index

- Frequency, 312; arithmetic mean, 621;  
distribution, 616; polygon, 617
- Frustum, 79
- Function, 87, 554; average value, 807;  
composite, 690; exponential, 371; finding  
zeros of, 104; functional notation, 88; im-  
plicit, 699; interval notation, 91; inverse,  
375, 553; logarithmic, 371; quadratic,  
220; series formed by functional notation,  
924; trigonometric, 120, 240; of two vari-  
ables, 896; from verbal statements, 92–94
- Fundamental laws of algebra, 6
- Fundamental principle of fractions, 197
- Fundamental theorem of algebra, 417, 423
- Gabor, Dennis, 413
- Galileo, 86, 220
- Galton, Francis, 615
- Gauss, Karl, 343, 417, 423, 454, 513, 628
- Gaussian elimination, 435, 454
- Gauss-Jordan method, 446
- General equation: of circle, 575; quadratic,  
220; of straight line, 568
- General power formula, 850
- General solution of differential equation, 950
- Geometric sequence, 510, 516
- Geometry, 55
- Googol, 29
- Graph: derivatives to find features of, 729;  
of exponential function, 371, 392; of  
function, 97; functions, 733; on graphing  
calculator, 104–107; inequalities, 468; of  
linear function, 146; of logarithmic func-  
tion, 375; on logarithmic paper, 393; in  
polar coordinates, 606; of quadratic func-  
tion, 232; sketch a graph, 841; of specific  
values, 619; of trigonometric equations,  
549; of trigonometric functions, 296; us-  
ing intercepts and traces to sketch, 900
- Graphical representation of complex  
numbers, 348
- Graphical solution of equations, 104, 149,  
232, 400
- Graphical solution of inequalities, 481, 485
- Graunt, John, 615
- Gravitation, universal law of, 495, 501
- Gravity Recovery and Interior Laboratory  
(GRAIL), 495
- Great circle, 254
- Greater than, 4, 468
- Grouping: factoring trinomial by, 194–195
- Half-angle formulas, 545
- Half-line, 56
- Half-range expansion, 943
- Half-wave rectifier, 938
- Halley, Edmond, 615
- Harmonic sequence, 515
- Heaviside, Oliver, 949
- Height: of solid geometric figures, 78; of  
triangle, 61
- henry, 364
- Hero's formula, 62
- Hertz, 312
- Hertz, Heinrich, 296, 311, 341
- Hexagon, 60
- Higher-order determinants, 457
- Hipparchus, 115
- Histogram, 617
- Holography, 413
- Homogeneous differential equation, 969
- Hooke, Robert, 804
- Hooke's law, 804
- Hyperbola, 101, 401, 587, 594, 601
- Hyperbolic cosine, 837
- Hypotenuse, 60, 120
- Identity, 7, 34, 39, 42; matrix, 441;  
trigonometric, 532
- Imaginary axis, 348
- Imaginary number, 2, 31, 342;  
graphing, 100
- Imaginary roots, 31
- Impedance, 362
- Implicit function, 699; derivative of, 700;  
differentiation of, 699
- Improper integral, 989
- Inclination, 564
- Inconsistent system of equations, 151, 163
- Increment, 744
- Indefinite integral, 755, 778
- Independent variable, 87
- Indeterminate, 9
- Indeterminate form, 837
- Index: of radical, 329
- Inductance, 361
- Inequalities, 4, 467; algebraic solution of,  
468; graphical solution of, 485; involving  
absolute values, 485; properties of, 469;  
with two variables, 485
- Infinite series, 520, 916
- Infinity, 521, 733; limit as  $x$  approaches, 664
- Initial point of vector, 266
- Initial side of angle, 116
- Instantaneous acceleration, 704
- Instantaneous rate of change, 659,  
672, 677
- Instantaneous velocity, 659, 678
- Integer, 2
- Integral: approximating with Simpson's  
rule, 772; definite, 765; double, 895, 909;  
indefinite, 755
- Integrating combinations, 955
- Integration, 752; areas by, 782; of exponen-  
tial forms, 854; of inverse trigonometric  
forms, 867; limits of, 765; of logarithmic  
forms, 850; methods of, 849; by partial  
fractions, 879, 883; by parts, 871; of  
powers, 850; as summation, 763; by ta-  
bles, 888; of trigonometric forms, 854,  
859; by trigonometric substitution, 876;  
by use of series, 926; volumes, 788
- Intercepts, 104, 148, 733
- Intersect, 56
- Interval of convergence, 920
- Inverse functions, 375, 553; trigonometric  
functions, 123, 554
- Inverse Laplace transform, 992
- Inverse logarithm, 383
- Inverse matrix, 441, 445
- Inverse trigonometric functions, 123, 246, 553;  
derivatives of, 822; integral forms, 867
- Inverse variation, 500
- Irrational numbers, 2
- Irrational roots, 430
- Iterative method, 715
- Jacobi, Carl, 905
- Joint variation, 501
- $j$ -operator, 364
- Jordan, Wilhelm, 446
- joule, 322
- Joule, James Prescott, 322
- kelvin, 326
- Kirchhoff, Gustav, 142
- Kirchhoff's current law, 142
- Kirchhoff's voltage law, 142
- Kutta, Martin, 961
- Lagrange, Joseph Louis, 264, 915
- Laplace, Pierre, 370, 628, 949, 989
- Laplace transforms, 989, 994
- Latitude, 254
- Law: of cosines, 282, 288; of sines, 282
- Least squares, method of, 647
- Least-squares line, 648
- Leibniz, Gottfried Wilhelm, 160, 765, 814,  
849, 895
- Lemniscate, 608
- Length, 66
- Less than, 4, 468
- L'Hospital, Marquis de, 837
- L'Hospital's rule, 837
- Libby, Willard, 964
- Like terms, 33
- Limaçon, 607
- Limit, 520, 660;  $e$  as a, 834; of a function,  
662; numerical verification of, 839; of  
 $\sin \theta / \theta$ , 815
- Limits of integration, 765
- Line, 56; slope of, 563
- Linear approximation, 745
- Linear differential equation, 957, 969
- Linear equation, 143; graph of, 146; matri-  
ces, 449; solving by determinants, 460
- Linear extrapolation, 110
- Linear factors, repeated, 883
- Linear inequalities, 472
- Linear interpolation, 110
- Linearity property, 991
- Linearization, 745
- Linear programming, 467, 488



- Linear regression, 646  
 Linear simultaneous systems, 144  
 Lissajous, Jules, 315  
 Lissajous figures, 314  
 Literal number, 4  
 Locus, 573  
 Logarithmic equations, 389  
 Logarithmic function, 373; derivative of, 830; integral form, 850  
 Logarithmic paper, 393  
 Logarithmic scale, 392  
 Logarithmic value, 929  
 Logarithms, 370, 385; to base 10, 382; basic form, 852; computations by, 383; natural, 385; of a product, 378; properties of, 377  
 Longitude, 254  
 Lord Kelvin, 326  
 Lower control limit (LCL), 642  
 Lowest common denominator (LCD), 206  
  
 Maclaurin, Colin, 142, 921  
 Maclaurin series, 919  
 Magnitude, 720  
 Major axis of ellipse, 583  
 Matrix, 171, 435; addition, 437; elements of, 436; identity, 441; inverse, 441, 445; linear equations, 449; multiplication, 439; square, 436; subtraction, 438; zero, 436  
 Maximum and minimum problems, 727  
 Maximum points, 233, 727  
 Maxwell, James, 341  
 Mean: arithmetic, 620; large sample confidence intervals, 635  
 Measurement, 12; control chart for, 641; estimating errors in, 745  
 Median, 61, 620  
 Members of inequality, 468  
 Method of least squares, 647  
 Method of partial fractions, 879  
 Method of undetermined coefficients, 977  
 Middle term, 190  
 Minimum points, 233, 727  
 Minor, 458  
 Minor axis of ellipse, 583  
 Minute (measure of angle), 116  
 Mode, 116, 622  
 Modulus of complex number, 350  
 Mohr's circle, 577  
 Moivre, Abraham de, 628  
 Moment of inertia, 799; of a mass, 793; of a solid, 801  
 Monomial, 33; common factors, 186; dividing, 38; multiplying, 36  
 Motion, in resisting medium, 966  
 Multinomial, 33  
 Multiplication: of algebraic expressions, 36; of complex numbers, 345, 355; of fractions, 202; of matrices, 439; order of operation, 322; of radicals, 335; scalar, 438; series formed by, 924; of signed numbers, 7  
 Multiplicity of a root, 222  
  
 Napier, John, 370  
 Natural logarithm, 385  
 Negative angle, 116, 247  
 Negative direction, 3  
 Negative exponents, 23, 124; solving an equation containing, 408  
 Negative numbers, 343; plotting, 98  
 Nested parentheses, 35  
 Newton (N), 277  
 Newton, Sir Isaac, 277, 279, 320, 435, 495, 501, 659, 715, 814, 849, 983  
 Newton's method, 714, 826, 842  
 Niépce, Joseph, 409  
 Nightingale, Florence, 615  
 Nonhomogeneous differential equations, 969, 977  
 Nonlinear inequalities, 476  
 Nonlinear regression, 651  
 Nonrepeated linear factors, 879  
 Nonrepeated quadratic factors, 885  
 Normal distribution, 628  
 Normal line, 712  
 Number: approximate, 15, 18; changing to scientific notation, 27; complex, 2, 341; denominate, 4; exact, 15; imaginary, 2, 31, 343; irrational, 2; natural, 2; negative, 2; rational, 2; real, 2; roots, 30  
 Numerical coefficient, 33  
 Numerical integration, 768  
  
 Objective function, 488  
 Oblique triangle, 264, 282, 290  
 Octant, 899  
 Odd function, 940  
 Ohm, 618  
 Ohm's law, 361  
 Operations with zero, 9  
 Operator, 969  
 Optimal solution, 489  
 Order: of differential equation, 950; of operations, 8, 24; of radical, 329; of trigonometric functions, 125  
 Ordinate, 96  
 Ordinates, addition of, 313  
 Origin, 3, 95  
 Orthogonal trajectories, 963  
  
 Parabola, 220, 232, 400, 578, 594, 601  
 Parallel, 56, 565  
 Parallelogram, 66; method of adding vectors, 267  
 Parameter, 616, 718  
 Parametric equations, 314  
 Parametric form, 718  
 Parentheses, 8, 34; nested, 35  
 Partial derivative, 895, 905  
 Partial fractions, 879  
 Partial sum, 916  
 Particular solution of differential equation, 950, 977  
 pascal, 322  
  
 Pascal, Blaise, 525, 777  
 Pascal's triangle, 525  
 Pentagon, 60  
 Perfect square, 30, 330  
 Perimeter, 61; of quadrilateral, 67; of triangle, 61  
 Period of sine curve, 298  
 Perpendicular, 56, 565  
 Phase angle, 303–306, 363  
 Phase shift, 304  
 Phasor, 364  
 Pi, 70  
 Pixel, 104  
 Plane, 56, 168, 900  
 Point, 56; continuous at a, 660; of inflection, 729; locating, 96, 348; polar coordinates of, 603  
 Point estimate, 634  
 Point-slope form of straight line, 567  
 Polar axis, 603  
 Polar coordinates, 603; curves, 606  
 Polar form of complex number, 350; addition, 357; division, 356; multiplication, 355  
 Pole, 603  
 Polygon, 60; method of adding vectors, 266  
 Polynomial, 33; antiderivative, 753; derivative of, 682; dividing, 39–40; indefinite integral, 755; multiplying, 36; rational roots, 427  
 Polynomial function, 418  
 Polyphase generator, 531  
 Population, 616  
 Power: antiderivative of, 753; of complex number, 357; derivative of, 682, 686; general power formula, 850; integration of, 850; of number, 25; series, 920  
 Power rule, 756  
 Precision, 15, 18  
 Pressure, 777; liquid, 806  
 Prime factor, 185  
 Principal root, 30  
 Prism, 78  
 Product: of complex numbers, 345, 355; derivative of, 687; logarithm of, 378; of matrices, 438; special, 182  
 Progression: arithmetic, 511; geometric, 516  
 Projection, 310  
 Proportion, 43, 496; large sample confidence intervals, 638  
 Pyramid, 78  
 Pythagoras, 62  
 Pythagorean theorem, 120  
  
 Quadrant, 95  
 Quadrantal angle, 117, 247  
 Quadratic equation, 220, 407; solving graphically, 235  
 Quadratic equation in form, 407  
 Quadratic factors, nonrepeated, 885  
 Quadratic formula, 228  
 Quadrilateral, 60, 66

## D.16 Index

- Quantitative variable, 616  
Quotient, 40; of complex numbers, 345, 356;  
derivative of, 688; of polynomials, 881
- Radian, 71, 117, 249, 297  
Radicals, 320, 329; addition of, 333;  
division of, 336; equations with, 410;  
multiplication of, 335; simplest form,  
329, 331; subtraction of, 333  
Radical sign, 30  
Radicand, 329  
Radioactivity, 964  
Radius, 69, 573; of gyration, 799; related  
rates, 723  
Range, 91, 106, 554, 623, 733  
Rate of change, 672  
Ratio, 43, 496; equality, 119  
Rationalizing: denominator, 331;  
numerator, 331, 665  
Rational number, 2  
Rational roots, 427  
Raw data, 616  
Ray, 56  
Reactance, 361  
Real axis, 348  
Real numbers, 2, 91  
Reciprocal, 4, 124  
Rectangle, 66, 409; area under a curve, 760  
Rectangular coordinate system, 95, 604, 899  
Rectangular form of complex number, 343  
Rectangular solid, 78  
Rectifier: half-wave, 938  
Reference angle, 244  
Regression, 646  
Related rates, 722  
Relation, 93  
Relative error, 745  
Relative frequency, 617  
Relative maximum and minimum  
points, 727  
Remainder, 418  
Remainder theorem, 418  
Repeated quadratic factors, 887  
Repeating decimal, 521  
Resistance, 361  
Resolving vector, 269  
Resonance, 365  
Resultant of vectors, 266  
Rhombus, 66  
Rice, Kellogg, 489  
Right triangle, 62, 126  
Roemer, Olaf, 591  
Root-mean-square value, 866  
Roots, 30, 427, 430; complex, 974; of com-  
plex numbers, 358; double, 222; of equa-  
tion, 419, 423; extraneous, 407; of linear  
equations, 143; of polynomial equation,  
429; of quadratic equation, 222; rational,  
427; repeated, 973; repeated complex, 975  
Rose, 608  
Rotation of angles, 116  
Rotation of axes, 599  
Rounding off, 17  
Row of determinant, 160  
Row operations, 446  
Runge, Carl, 961  
Runge-Kutta method, 961
- Salt solution, 965  
Sample, 616  
Sampling distributions, 632  
Scalar, 265; multiple of vector, 267;  
multiplication, 437  
Scale drawing, 63  
Scientific notation, 26  
Scott, David, 86  
Secant, 69; of angle, 120; graph of, 307;  
integration of, 863; line, 69, 669  
Second (measure of angle), 116  
Second-degree equation, 596  
Second derivative, 702  
Second-derivative test, 729  
Section, 901  
Sector, 70, 254  
Segment, 56, 71  
Semilogarithmic (semilog) paper, 392  
Semimajor axis of ellipse, 583  
Sense of inequality, 468  
Separation of variables, 952  
Sequence, 510, 511; arithmetic, 511; finite,  
511; geometric, 516; infinite, 916  
Series, 520, 923; binomial, 523; computa-  
tions with, 928; Fourier, 915, 934;  
Maclaurin, 921  
Shell (element of volume), 789  
Shewhart, Walter, 615  
Shifting a graph, 106  
Signed numbers, 7  
Significant digits, 15, 27, 123  
Signs: factors differing only in, 200; laws of,  
7; of trigonometric functions, 241  
Signs of inequality, 4  
Sikorsky, Igor, 284  
Similar terms, 33  
Simple harmonic motion, 310, 982, 995  
Simpson, Thomas, 75, 716  
Simpson's rule, 75, 771  
Sine: of angle, 120; derivative of, 823; graph  
of, 297; of half-angle, 545; half-range  
Fourier series, 944; integration of, 863;  
inverse, 553; of sum of two angles, 537;  
Taylor series, 932  
Sines, law of, 282  
SI prefixes, 14; engineering notation, 28;  
scientific notation, 28  
Sixth roots: by DeMoivre's theorem, 359  
Slant height, 78  
Slope, 146, 563; curve, finding, 963; of linear  
function, 146; of tangent line, 669, 818  
Slope-intercept form of straight line,  
147, 567  
Slug, 966
- Solution: of differential equation, 950; of  
equation, 144; of inequality, 468, 485; of  
linear equation, 143; of quadratic equa-  
tion, 222; of system of linear equations,  
144, 149, 160, 166, 450, 460; of triangle,  
126, 283; of trigonometric equations,  
548; of two sides and angle opposite one  
of them, 285  
Special products, 182  
Sphere, 78  
Spread, measures of, 623  
Square, 66  
Square matrix, 436  
Square root, 30, 181; antiderivative, 753;  
by DeMoivre's theorem, 359; evaluating  
using Taylor series, 932; solving an  
equation, 408  
Squares, difference of, 665  
Square wave, 936  
Standard deviation, 623  
Standard equation: of circle, 573; of ellipse,  
583; of hyperbola, 588; of parabola, 578  
Standard error of the mean, 632  
Standard errors, 632  
Standard normal distribution, 630; table of  
areas, 631  
Standard position of angle, 117  
Statistical process control, 640  
Statistics, 615, 616  
Steady-state solution, 986  
Straight line, 109, 146, 567  
Subscripts, 45, 46  
Substitution, 41; elimination by, 150, 404;  
solution of a system of equations by,  
153, 403; trigonometric, 876  
Subtraction: of algebraic expressions, 32;  
algebraic method of solving three linear  
equations, 166; of complex numbers,  
345; of cubes, 196; of fractions, 208; of  
matrices, 438; of radicals, 333; of signed  
numbers, 7; solution of system of equa-  
tions, 404; of vector, 267  
Summation symbol, 622  
Sum of  $n$  terms, 513, 517  
Supplementary angles, 57  
Supplementary units, 13  
Surface, 899  
Symbols of grouping, 8, 34  
Symmetry, 574, 733  
Synthetic division, 419  
System: of linear equations, 144, 149,  
160, 166, 403, 450, 460; of quadratic  
equations, 400
- Tables, integration by use of, 888  
Tangent, 69; of angle, 120; to curve, 669;  
derivative of, 824; of double angle, 542;  
to earth's surface, 930; graph of, 307;  
integration of, 863; line, 69, 669, 712  
Taylor, Brook, 931  
Taylor series, 931



- Temperature, 722  
 Terminal point of vector, 266  
 Terminal side of angle, 116  
 Terms, 32, 190; common factor same as, 186; of fractions, 199; of sequences, 511, 916; similar, 32  
 Tesla, Nikola, 531  
 3D, compared to 2D, 902  
 Time rate of change, 827, 842  
 Trace, 105, 900  
 Transcendental functions, 814; derivatives of, 815; differentiation of, 814  
 Transient term, 986  
 Translation of axes, 593  
 Transversal, 58  
 Transverse axis of hyperbola, 588  
 Trapezoidal rule, 74, 768  
 Triangle, 60; congruent, 64; equilateral, 60; isosceles, 60; oblique, 264, 288; Pascal's, 525; right, 60, 126; scalene, 60; similar, 63, 119; solution of, 126, 283  
 Trigonometric equations, 548  
 Trigonometric form of complex numbers, 350  
 Trigonometric functions, 115, 120; of angles measured in degrees, 122; of angles of right triangle, 120; of any angle, 240; derivatives of, 819; graphs of, 296; integration of, 859, 876; inverse, 553; of negative angles, 246; signs, 241  
 Trigonometric identities, 532  
 Trigonometric value, 928  
 Trinomial, 33, 190; factoring, 190  
 2D, compared to 3D, 902  
 Unbiased rounding, 17  
 Uncertainty, of measured value, 12  
 Units: conversion, 14; writing, 13  
 Units of measurement, 12, 322  
 Universal law of gravitation, 495, 501  
 Unknown, 41  
 Upper control limit (UCL), 642  
 Variable, 4–5, 616; dependent, 87; independent, 87  
 Variables, functions of two, 896  
 Variables, separation of, 952  
 Variation, 495, 500, 739  
 Vectors, 264  
 Velocity, 720, 778; angular, 255; approximation, 929; average value, 807; linear, 255; parametric form, 718  
 Vertex: of angle, 56, 116; of ellipse, 583; of hyperbola, 588; of parabola, 233, 578  
 Vertical angles, 57  
 Vertical asymptotes, 308  
 Vertical line test, 102  
 Viète, François, 181  
 volt, 364  
 Voltage, 361, 363, 722; across capacitor, 779; variations caused by, 641  
 Volumes: element of, 910; of geometric figures, 78; by integration, 788; under a plane, 911; related rates, 723; under a surface, 910, 911  
 Wallis, John, 320, 520  
 watt, 101  
 Watt, James, 615  
 Weighted mean, 621  
 Westinghouse, George, 362, 531  
 Width, 66  
 Witch of Agnesi, 733  
 Word problems, 48, 106, 151  
 Work, 804  
*x*-axis, 95; areas below, 785  
*x*-intercept, 104, 146  
*y*-axis, 95  
*y*-intercept, 147, 568  
 Zero, 2; as exponent, 23; matrix, 436; operations with, 9; trailing, 16  
 Zoom, 105

# Notes



# Notes



# Notes

# MATHNOTES

## A Study Chart for Technical Mathematics

PEARSON

Education  
Canada

JOHN JENNESS

BRITISH COLUMBIA INSTITUTE OF TECHNOLOGY

### ALGEBRA

#### ORDER OF OPERATIONS (BEDMAS)

1. Simplify contents of Brackets [ ], parentheses ( ), and braces { } working from the innermost outward, and working separately above and below the fraction lines.
2. Simplify Exponents and Roots working from left to right.
3. Do Multiplication and Division in the order that they appear from left to right.
4. Do Addition and Subtraction in the order that they appear from left to right.

#### PROPERTIES OF NUMBERS

Commutative Law:  $a+b = b+a$  and  $ab = ba$

Associative Law:  $a+(b+c) = (a+b)+c$  and  $a(bc) = (ab)c$

**Note:** Commutative Law does not apply to Subtraction or Division.

Distributive Law:  $a(b+c) = ab+ac$

Signs:  $a+(-b) = a-b$  and  $a-(-b) = a+b$  and  $-b = -(b-a)$

#### NUMBER SET DEFINITIONS

Natural (or Counting) Numbers: { 1, 2, 3, 4, ... }

Whole Numbers: { 0, 1, 2, 3, 4, ... }

Integers: { ... , -3, -2, -1, 0, 1, 2, 3, ... }

Rational Numbers: {  $x/y$  such that  $x, y$  are integers but  $y \neq 0$  }

Irrational Numbers:  $x$  is a real number but not a Rational number {e.g.  $\pi, e$  }

Real Numbers include both Rational and Irrational Numbers.

Imaginary Numbers: of the form  $xj$  (or  $xi$ )

where  $x$  is a real number and  $j$  is a number such that  $j^2 = -1$

Complex Numbers: of the form:  $x + yj$

where  $x$  is a real number and  $yj$  is an imaginary number.

#### EXPONENTS, ROOTS AND RADICALS

$$a^m \times a^n = a^{m+n}$$

$$\frac{a^m}{a^n} = a^{m-n}, a \neq 0 \text{ or } \frac{a^m}{a^{-n}} = \frac{1}{a^{n-m}}, a \neq 0$$

$$(a^m)^n = a^{mn}$$

$$(ab)^n = a^n b^n$$

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n} (b \neq 0)$$

$$a^0 = 1 (a \neq 0)$$

$$a^{-n} = \frac{1}{a^n} (a \neq 0)$$

$$a^{1/n} = \sqrt[n]{a}$$

$$a^{m/n} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$$

$$\sqrt[n]{a^m} = a^{m/n}$$

$$\sqrt[n]{a} \sqrt[n]{b} = \sqrt[n]{ab}$$

$$\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$$

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}} (b \neq 0)$$

#### FACTORING AND SPECIAL PRODUCTS

F.O.I.L. rule:  $(a+b)(c+d) = ac+ad+bc+bd$

$$a(x+y) = ax+ay$$

$$(x+y)(x-y) = x^2-y^2$$

$$(x+y)^2 = x^2+2xy+y^2$$

$$(x-y)^2 = x^2-2xy+y^2$$

$$(x+a)(x+b) = x^2+(a+b)x+ab$$

$$(ax+b)(cx+d) = acx^2+(ad+bc)x+bd$$

$$(x+y)^3 = x^3+3x^2y+3xy^2+y^3$$

$$(x-y)^3 = x^3-3x^2y+3xy^2-y^3$$

$$x^3+y^3 = (x+y)(x^2-xy+y^2)$$

$$x^3-y^3 = (x-y)(x^2+xy+y^2)$$

#### QUADRATIC EQUATION

$$ax^2+bx+c = 0 \text{ has solution form } x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

Four possible solutions based on discriminant

$$D = b^2-4ac$$

- If  $D > 0$  and a perfect square, then roots are real, rational, and unequal.

- If  $D > 0$  and not a perfect square, then roots are real, irrational, and unequal.

- If  $D = 0$ , then roots are real, rational, and equal.

- If  $D < 0$ , then roots contain imaginary numbers, and are unequal.

#### COMPLEX NUMBERS

- The symbol  $j$  represents the imaginary number  $\sqrt{-1}$  such that  $j^2 = -1$

- **Note:**  $\sqrt{-a} = j\sqrt{a}$  where  $a > 0$

Operations with complex numbers:

$$\text{Addition: } (a+bj)+(c+dj) = (a+c)+(b+d)j$$

$$\text{Subtraction: } (a+bj)-(c+dj) = (a-c)+(b-d)j$$

Multiplication:

$$(a+bj)(c+dj) = (ac-bd)+(ad+bc)j$$

Division:

$$\frac{a+bj}{c+dj} = \frac{(a+bj)(c-dj)}{(c+dj)(c-dj)} = \frac{(ac+bd)+(bc-ad)j}{c^2+d^2}$$

Rectangular form:  $x+yj$

$$\text{Polar form: } r(\cos\theta+j\sin\theta) = r\angle\theta$$

Product in polar form:

$$r_1(\cos\theta_1+j\sin\theta_1) \times r_2(\cos\theta_2+j\sin\theta_2) =$$

$$(r_1\angle\theta_1)(r_2\angle\theta_2) = r_1r_2\angle(\theta_1+\theta_2)$$

Quotient in polar form:

$$\frac{r_1(\cos\theta_1+j\sin\theta_1)}{r_2(\cos\theta_2+j\sin\theta_2)} = \frac{r_1\angle\theta_1}{r_2\angle\theta_2} = \frac{r_1}{r_2}\angle(\theta_1-\theta_2)$$

#### EXPONENTS AND LOGARITHMS

Exponential function:  $y = b^x$

Logarithmic function:  $y = \log_b x$

Properties of logarithms:

$$\log_b xy = \log_b x + \log_b y$$

$$\log_b \frac{x}{y} = \log_b x - \log_b y$$

$$\log_b (x^n) = n\log_b x$$

$$\log_b 1 = 0$$

$$\log_b b = 1$$

$$\log_b (b^n) = n$$

Changing bases of logarithms:

$$\log_b x = \frac{\log_a x}{\log_a b}$$

$$\ln x = \frac{\log x}{\log e}$$

$$\log x = \frac{\ln x}{\ln 10}$$

#### VARIATION, RATIO AND PROPORTIONS

ratio or proportion:  $\frac{a}{b} = \frac{c}{d}$

direct variation:  $y = kx$

inverse variation:  $y = k/x$

joint variation:  $y = kxz$

where:  $x, y, z$  are variables,

$k$  is constant of proportionality, and  $k \neq 0$

### GEOMETRY

#### PLANE SHAPES

**Triangles:** 3 sides, all angles add to 180°

Scalene: no two sides are equal in length

Isosceles: two sides are equal in length

Equilateral: all sides are equal in length

Right: one angle is 90°

$$\text{Area: } A = \frac{1}{2}bh$$

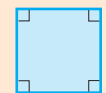
Hero's formula:  $A = \sqrt{s(s-a)(s-b)(s-c)}$

$$\text{where } s = \frac{1}{2}(a+b+c)$$

**Quadrilaterals:** 4 sides

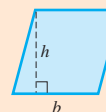
Square: all sides are equal in length, all angles 90°

$$\text{Area: } A = s^2$$



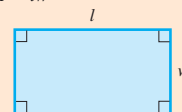
Rhombus: all sides are equal in length, angles not 90°

$$\text{Area: } A = bh$$



Rectangle: opposite sides are equal in length, all angles 90°

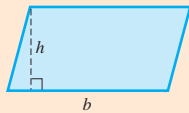
$$\text{Area: } A = lw$$





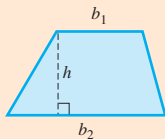
Parallelogram: opposite sides are equal in length, angles not 90°

Area:  $A = bh$



Trapezoid: sides of unequal length, angles not 90°

Area:  $A = \frac{1}{2}h(b_1 + b_2)$



**Circle:**  $360^\circ = 2\pi \text{ radians} = 1 \text{ revolution}$

and  $1 \text{ radian} = \frac{180^\circ}{\pi} \approx 57.30^\circ$

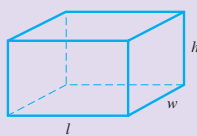
Perimeter (or circumference):  $c = 2\pi r$

Area:  $A = \pi r^2$

Arc length:  $s = \theta r$  where  $r$  is radius,  $\theta$  is angle in radians

**SOLIDS**

Rectangular solid:



Surface area:  $A = 2lw + 2lh + 2wh$

Volume:  $V = lwh$

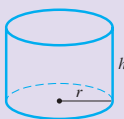
Cube:



Surface area:  $A = 6e^2$

Volume:  $V = e^3$

Right circular cylinder:

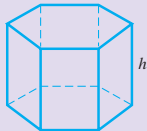


Surface area:  $A = 2\pi r^2 + 2\pi rh$

Volume:  $V = \pi r^2 h$

Lateral surface area (excluding base):  $S = 2\pi rh$

Right prism:

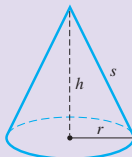


Lateral surface area (excluding ends):

$S = (\text{base perimeter}) \times h$

Volume:  $V = (\text{base area}) \times h$

Right circular cone:

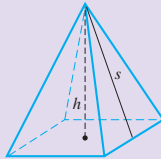


Surface area:  $A = \pi r^2 + \pi rs$

Volume:  $V = \frac{1}{3}\pi r^2 h$

Lateral surface area (excluding base):  $S = \pi rs$

Regular pyramid:

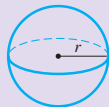


Lateral surface area (excluding base):

$S = \frac{1}{2}(\text{base perimeter}) \times s$

Volume:  $\frac{1}{3}(\text{base area}) \times h$

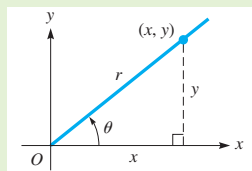
Sphere:



Surface area:  $4\pi r^2$

Volume:  $\frac{4}{3}\pi r^3$

**TRIGONOMETRIC FUNCTIONS**



Basic relationships:

$\sin \theta = \frac{y}{r} = \frac{\text{side opposite } \theta}{\text{hypotenuse}}$

$\cos \theta = \frac{x}{r} = \frac{\text{side adjacent } \theta}{\text{hypotenuse}}$

$\tan \theta = \frac{y}{x} = \frac{\text{side opposite } \theta}{\text{side adjacent } \theta}$

$\sec \theta = \frac{r}{x} = \frac{\text{hypotenuse}}{\text{side adjacent } \theta}$

$\csc \theta = \frac{r}{y} = \frac{\text{hypotenuse}}{\text{side opposite } \theta}$

$\cot \theta = \frac{x}{y} = \frac{\text{side adjacent } \theta}{\text{side opposite } \theta}$

$\csc \theta = \frac{1}{\sin \theta}$

$\sec \theta = \frac{1}{\cos \theta}$

$\cot \theta = \frac{1}{\tan \theta}$

$\tan \theta = \frac{\sin \theta}{\cos \theta}$

$\cot \theta = \frac{\cos \theta}{\sin \theta}$

$\sin^2 \theta + \cos^2 \theta = 1$

$1 + \tan^2 \theta = \sec^2 \theta$

$1 + \cot^2 \theta = \csc^2 \theta$

Sum and difference identities:

$\sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta$

$\cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta$

$\tan(\alpha \pm \beta) = \frac{\tan \alpha \pm \tan \beta}{1 \mp \tan \alpha \tan \beta}$

Double-angle formulae:

$\sin 2\alpha = 2\sin \alpha \cos \alpha$

$\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha$

$= 2\cos^2 \alpha - 1$

$= 1 - 2\sin^2 \alpha$

Half-angle formulae:

$\sin \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{2}}$

$\cos \frac{\alpha}{2} = \pm \sqrt{\frac{1 + \cos \alpha}{2}}$

Inverse trigonometric functions:

$y = \sin^{-1} x \quad (-\frac{\pi}{2} \leq y \leq \frac{\pi}{2})$

$-\frac{\pi}{2} \leq \sin^{-1} x \leq \frac{\pi}{2}$

$0 \leq \cos^{-1} x \leq \pi$

$-\frac{\pi}{2} < \tan^{-1} x < \frac{\pi}{2}$

$0 < \cot^{-1} x < \pi$

$0 \leq \sec^{-1} x \leq \pi \quad (\sec^{-1} x \neq \frac{\pi}{2})$

$-\frac{\pi}{2} \leq \csc^{-1} x \leq \frac{\pi}{2} \quad (\csc^{-1} x \neq 0)$

**TRIG FUNCTIONS OF ANY ANGLE**

– Unless specified otherwise all angles are measured counter-clockwise from the positive x axis.

– **Note:** When manipulating angles using a calculator, verify the quadrant of the answer.

Positive functions:

First quadrant: all

Second quadrant:  $\sin \theta$  and  $\csc \theta$

Third quadrant:  $\tan \theta$  and  $\cot \theta$

Fourth quadrant:  $\cos \theta$  and  $\sec \theta$

Law of Sines:  $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

Law of Cosines:  $a^2 = b^2 + c^2 - 2bc \cos A$

$b^2 = a^2 + c^2 - 2ac \cos B$

$c^2 = a^2 + b^2 - 2ab \cos C$

**VECTORS ADDITION**

Step 1: break original vectors into

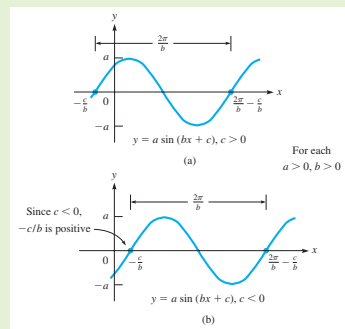
x and y components:  $\vec{A} = A_x + A_y$

Step 2: add x pieces to x pieces to get  $R_x$ , add y pieces to y pieces to get  $R_y$

Step 3: apply Pythagorean formula to get magnitude resultant  $R$ ,

Step 4: use  $\theta = \tan^{-1} \frac{R_y}{R_x}$  to get angle.

**GRAPHS OF TRIG FUNCTIONS**



amplitude =  $|a|$

period =  $\frac{2\pi}{b}$

displacement =  $-\frac{c}{b}$

**PLANE ANALYTIC GEOMETRY**

distance formula:  $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

slope:  $m = \frac{y_2 - y_1}{x_2 - x_1}$  and  $m = \tan \theta \quad (0^\circ \leq \theta < 180^\circ)$

parallel lines:  $m_1 = m_2$   
 perpendicular lines:  $m_2 = -\frac{1}{m_1}$  or  $m_1 m_2 = -1$   
 straight line: point slope  
 form:  $y - y_1 = m(x - x_1)$  or  
 slope-intercept form:  $y = mx + b$  or  
 general form:  $Ax + By + C = 0$   
 circles: centred on origin:  $x^2 + y^2 = r^2$   
 centred on:  $(h, k)$ :  $(x - h)^2 + (y - k)^2 = r^2$   
 parabolas: centred on origin parallel to  $x$  axis:  
 $y^2 = 4px$   
 centred on origin parallel to  $y$  axis:  
 $x^2 = 4py$   
 ellipses: centred on origin major axes parallel to  
 $x$  axis:  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > b$   
 centred on origin major axes parallel to  
 $y$  axis:  $\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1, a > b$   
 hyperbolae: centred on origin foci on  $x$  axis:  
 $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, a > b$   
 centred on origin foci on  $y$  axis:  
 $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1, a > b$   
 translation of axes:  $x = x' + h$  and  $y = y' + k$

## FUNCTIONS

### BASIC DEFINITIONS

A **function** is defined as a relationship between two variables such that for every value of the first (independent) variable, there is one corresponding value of the second (dependent) variable.

The complete set of possible values of the independent variable is called the **domain** of the function. The corresponding complete set of dependent variable values is called the **range** of the function.

Linear equation in one unknown:  $ax + b = 0$

Linear equation in two unknowns:  $ax + by = c$

### LINEAR EQUATIONS AND DETERMINANTS

Given two linear equations of the form:

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}$$

The determinant of the second order is defined as:

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

Cramer's Rule gives solution forms:

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \text{ and } y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

### EQUATIONS OF HIGHER DEGREE

Polynomial function:  $f(x) = a_0x^n + a_1x^{n-1} + \dots + a_n$

Remainder theorem:  $f(x) = (x - r)q(x) + R$

where  $f(r) = R$

Rational roots:  $r = \frac{\text{factor of } a_n}{\text{factor of } a_0}$

### MATRICES

A matrix is any rectangular array of numbers. If the number of rows and columns is equal, then it is a square matrix. A determinant is a specific value associated with a square matrix.

Basic laws for matrices:

Commutative law:  $A + B = B + A$

Associative law:  $A + (B + C) = (A + B) + C$

$k(A + B) = kA + kB$

$A + 0 = A$

$AA^{-1} = A^{-1}A = I$

A system of linear equations:  $a_1x + b_1y = c_1$

$$a_2x + b_2y = c_2$$

can be represented in matrix form as:  $AX = C$

where  $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ ,  $X = \begin{bmatrix} x \\ y \end{bmatrix}$  and  $C = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

and using the inverse:  $X = A^{-1}C$

## SEQUENCE AND SERIES

### SEQUENCES AND SERIES

Factorial notation:  $n! = n(n-1)(n-2)\dots(2)(1)$

Arithmetic sequences:  $a_n = a_{n-1} + d$

$n^{\text{th}}$  term:  $a_n = a_1 + (n-1)d$

Sum of  $n$  terms:  $S_n = \frac{n}{2}(a_1 + a_n)$

Geometric sequences:  $a_n = ra_{n-1}$

$n^{\text{th}}$  term:  $a_n = a_1r^{n-1}$

Sum of  $n$  terms:  $S_n = \frac{a_1(1-r^n)}{1-r}$  where  $(r \neq 1)$

Sum of geometric series:  $S = \lim_{n \rightarrow \infty} S_n = \frac{a_1}{1-r}$   
 where  $(|r| < 1)$

Binomial formula:

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \dots + b^n$$

Binomial series:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 +$$

$$\frac{n(n-1)(n-2)}{3!}x^3 + \dots \text{ where } (|x| < 1)$$

### EXPANSION OF FUNCTIONS IN SERIES

Infinite series:  $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots + a_n + \dots$

Sum of series:  $S = \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \sum_{i=1}^n a_i$

Power series:  $f(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots$

Maclaurin series:  $f(x) = f(0) + f'(0)x +$

$$\frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \dots + \frac{f^{(n)}(0)x^n}{n!} + \dots$$

Taylor series:  $f(x) = f(a) + f'(a)(x-a) +$

$$\frac{f''(a)(x-a)^2}{2!} + \dots$$

Special series:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$1n(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \text{ where } (|x| < 1)$$

Fourier series:

$$f(x) = a_0 + a_1 \cos x + a_2 \cos 2x + \dots + a_n \cos nx + \dots + b_1 \sin x + b_2 \sin 2x + \dots + b_n \sin nx + \dots$$

Fourier coefficients for period  $2\pi$ :

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx$$

Fourier coefficients for period  $2L$ :

$$a_0 = \frac{1}{2L} \int_{-L}^L f(x) dx$$

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx$$

## STATISTICS

### BASICS

Arithmetic mean  $\bar{x} = \frac{x_1f_1 + x_2f_2 + \dots + x_n f_n}{f_1 + f_2 + \dots + f_n} = \frac{\sum xf}{\sum f}$

Standard deviation:

$$s = \sqrt{\frac{\sum(x-\bar{x})^2}{n-1}} = \sqrt{\frac{n(\sum x^2) - (\sum x)^2}{n(n-1)}}$$

Normal distribution:  $y = \frac{e^{-(x-\mu)^2/2\sigma^2}}{\sigma\sqrt{2\pi}}$

Standard normal distribution:  $y = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

Standard (z) score:  $z = \frac{x-\mu}{\sigma}$

Standard error of  $\bar{x}$ :  $\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$

Standard error of  $s$ :  $\sigma_s = \frac{\sigma}{\sqrt{2n}}$

Least-squares line:  $y = mx + b$

$$m = \frac{n\sum xy - (\sum x)(\sum y)}{n\sum x^2 - (\sum x)^2}$$

$$b = \frac{(\sum x^2)(\sum y) - (\sum xy)(\sum x)}{n\sum x^2 - (\sum x)^2}$$

## DERIVATIVES

### BASICS RULES

The limiting value of the ratio  $\Delta x/\Delta y$  is known as the derivative of the function. The derivative can be interpreted as the instantaneous rate of change of the dependent variable with respect to the independent variable.

Derivative of a constant:  $\frac{dc}{dx} = 0$

Derivative of a polynomial:  $\frac{dx^n}{dx} = nx^{n-1}$

Derivative of a constant times a function:

$$\frac{d(cu)}{dx} = c \left( \frac{du}{dx} \right)$$

Derivative of a sum:  $\frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx}$

Product rule:  $\frac{d(uv)}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$

Quotient rule:  $\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{d(u/v)}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$

Chain rule:  $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$

General power rule:  $\frac{du^n}{dx} = nu^{n-1} \left( \frac{du}{dx} \right)$  and

$$\frac{du^{p/q}}{dx} = \frac{p}{q} u^{(p/q)-1} \left( \frac{du}{dx} \right)$$

### APPLICATIONS

Newton's Method:  $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

Curvilinear motion:

Velocity components:  $v_x = \frac{dx}{dt}$  and  $v_y = \frac{dy}{dt}$

Acceleration components:  $a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2}$  and

$$a_y = \frac{dv_y}{dt} = \frac{d^2y}{dt^2}$$

Magnitude:  $v = \sqrt{v_x^2 + v_y^2}$  and  $a = \sqrt{a_x^2 + a_y^2}$

Direction:  $\tan \theta_v = \frac{v_y}{v_x}$  and  $\tan \theta_a = \frac{a_y}{a_x}$

Differential form of a function  $y = f(x)$  is defined as  $dy = f'(x)dx$   
 Linearization:  $L(x) = f(a) + f'(a)(x - a)$

### DERIVATIVE OF TRANSCENDENTAL FUNCTIONS

$$\frac{d(\sin u)}{dx} = \cos u \frac{du}{dx}$$

$$\frac{d(\cos u)}{dx} = -\sin u \frac{du}{dx}$$

$$\frac{d(\tan u)}{dx} = \sec^2 u \frac{du}{dx}$$

$$\frac{d(\cot u)}{dx} = -\csc^2 u \frac{du}{dx}$$

$$\frac{d(\sec u)}{dx} = \sec u \tan u \frac{du}{dx}$$

$$\frac{d(\csc u)}{dx} = -\csc u \cot u \frac{du}{dx}$$

$$\frac{d(\sin^{-1} u)}{dx} = \frac{1}{\sqrt{1-u^2}} \frac{du}{dx}$$

$$\frac{d(\cos^{-1} u)}{dx} = -\frac{1}{\sqrt{1-u^2}} \frac{du}{dx}$$

$$\frac{d(\tan^{-1} u)}{dx} = \frac{1}{1+u^2} \frac{du}{dx}$$

$$\frac{d(\log_b u)}{dx} = \frac{1}{u} \log_b e \frac{du}{dx}$$

$$\frac{d(\ln u)}{dx} = \frac{1}{u} \frac{du}{dx}$$

$$\frac{d(b^u)}{dx} = b^u \ln b \frac{du}{dx}$$

$$\frac{d(e^u)}{dx} = e^u \frac{du}{dx}$$

### INTEGRATION

#### BASIC RULES

Indefinite integral:  $\int f(x)dx = F(x) + C$   
 Integral of a constant:  $\int c \, du = c \int du = cu + C$   
 Integral of sum:  $\int (du + dv) = u + v + C$   
 Power formula:  $\int u^n du = \frac{u^{n+1}}{n+1} + C$   
 where  $(n \neq -1)$   
 Area under a curve:  $A_{ab} = \left[ \int_a^b f(x)dx \right]_a^b = F(b) - F(a)$   
 Definite integral:  $\int_a^b f(x)dx = F(b) - F(a)$   
 Trapezoid rule:  $\int_a^b f(x)dx \approx \frac{\Delta x}{2} (y_0 + 2y_1 + 2y_2 + \dots + 2y_{n-1} + y_n)$   
 Simpson's rule:  $\int_a^b f(x)dx \approx \frac{\Delta x}{3} (y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \dots + 4y_{n-1} + y_n)$   
 where  $n$  is even

#### APPLICATIONS OF INTEGRATION

velocity:  $v = \int a \, dt = at + C_1$  and  
 displacement:  $s = \int v \, dt$   
 electric current:  $i = \frac{dq}{dt}$  and  
 electric charge:  $q = \int i \, dt$   
 voltage across a capacitor:  $V_c = \frac{1}{C} \int i \, dt$

Areas:

Between a curve and the x axis:

$$A = \int_a^b y \, dx = \int_a^b f(x)dx$$

Between a curve and the y axis:

$$A = \int_c^d x \, dy = \int_c^d g(y)dy$$

Between two curves on x axis:

$$A = \int_a^b (y_2 - y_1)dx$$

Between two curves on y axis:

$$A = \int_c^d (x_2 - x_1)dy$$

Volumes of rotation:

About x axis:  $V = \pi \int_a^b y^2 dx = \pi \int_a^b [f(x)]^2 dx$

About y axis:  $V = \pi \int_c^d x^2 dy = \pi \int_c^d [g(y)]^2 dy$

Shell method:  $dV = 2\pi (\text{radius}) \times (\text{height}) \times (\text{thickness})$

Disk method:  $dV = \pi (\text{radius})^2 \times (\text{thickness})$

Centre of mass:  $m_1 d_1 + m_2 d_2 + \dots + m_n d_n = (m_1 + m_2 + \dots + m_n) \bar{d}$

Centroid of area:  $\bar{x} = \frac{\int_a^b x(y_2 - y_1)dx}{\int_a^b (y_2 - y_1)dx}$  and

$$\bar{y} = \frac{\int_c^d y(x_2 - x_1)dy}{\int_c^d (x_2 - x_1)dy}$$

Radius of gyration:  $m_1 d_1^2 + m_2 d_2^2 + \dots + m_n d_n^2 = (m_1 + m_2 + \dots + m_n) R^2$

Moment of Inertia of area:  $I_y = k \int_a^b x^2 (y_2 - y_1) dx$   
 and  $I_x = k \int_c^d y^2 (x_2 - x_1) dy$

Work:  $W = \int_a^b f(x)dx$

Force due to liquid pressure:  $F = w \int_a^b lh \, dh$

Average value:  $y_{avg} = \frac{\int_a^b ydx}{b-a}$

Root-mean-square:  $y_{rms} = \sqrt{\frac{1}{T} \int_0^T y^2 dx}$

### INTEGRATION OF TRANSCENDENTAL FUNCTIONS

$$\int \frac{du}{u} = \ln|u| + C$$

$$\int e^u du = e^u + C$$

$$\int \sin u \, du = -\cos u + C$$

$$\int \cos u \, du = \sin u + C$$

$$\int \sec^2 u \, du = \tan u + C$$

$$\int \csc^2 u \, du = -\cot u + C$$

$$\int \sec u \tan u \, du = \sec u + C$$

$$\int \csc u \cot u \, du = -\csc u + C$$

$$\int \tan u \, du = -\ln|\cos u| + C$$

$$\int \cot u \, du = \ln|\sin u| + C$$

$$\int \sec u \, du = \ln|\sec u + \tan u| + C$$

$$\int \csc u \, du = \ln|\csc u - \cot u| + C$$

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1} \frac{u}{a} + C$$

$$\int \frac{du}{a^2 + u^2} = \frac{1}{a} \tan^{-1} \frac{u}{a} + C$$

#### INTEGRATION METHODS

Integration by parts:  $\int u \, dv = uv - \int v \, du$

Trig substitutions:

For  $\sqrt{a^2 - x^2}$  use  $x = a \sin \theta$

For  $\sqrt{a^2 + x^2}$  use  $x = a \tan \theta$

For  $\sqrt{x^2 - a^2}$  use  $x = a \sec \theta$

Square relation substitutions:

$$\cos^2 x + \sin^2 x = 1$$

$$1 + \tan^2 x = \sec^2 x$$

$$1 + \cot^2 x = \csc^2 x$$

Cosine double angle substitutions:

$$2 \cos^2 x = 1 + \cos 2x$$

$$2 \sin^2 x = 1 - \cos 2x$$

### DIFFERENTIAL EQUATIONS

#### BASICS

General form for  $n^{\text{th}}$  order differential equation:

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_{n-1} \frac{dy}{dx} + a_n y = b$$

alternately expressed using the differential operator notation  $D$ :

$$a_0 D^n y + a_1 D^{n-1} y + \dots + a_{n-1} D y + a_n y = b$$

Solving first-order differential equations:

From given form:  $M(x, y)dx + N(x, y)dy = 0$

Algebraically manipulate into general form:

$$A(x)dx + B(y)dy = 0$$

Using one of three methods:

1. Separation of variables
2. Integrable substitution combinations:

$$d(xy) = xdy + ydx$$

$$d(x^2 + y^2) = 2(xdx + ydy)$$

$$d\left(\frac{y}{x}\right) = \frac{x dy - y dx}{x^2}$$

$$d\left(\frac{x}{y}\right) = \frac{y dx - x dy}{y^2}$$

3. PQ Method for inseparable forms:

$dy + Py \, dx = Q \, dx$  where  $P, Q$  are

functions of  $x$  with solution of the form:

$$y e^{\int P dx} = \int Q e^{\int P dx} dx + C$$

Solving second-order differential equations:

General form:  $a_0 D^2 y + a_1 D y + a_2 y = b$

Homogeneous linear form where  $(b = 0)$

non-homogeneous form where  $(b \neq 0)$

Three possible homogeneous solution forms (see quadratic discriminant) using auxiliary equation:

$$a_0 m^2 + a_1 m + a_2 = 0$$

1. If discriminant  $> 0$ , then 2 real roots,

solution form:  $y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

2. If discriminant  $= 0$ , then 2 identical real roots,

solution form:  $y = e^{mx} (c_1 + c_2 x)$

3. If discriminant  $< 0$ , then 2 complex roots,

solution form:  $y = e^{\alpha x} (c_1 \sin \beta x + c_2 \cos \beta x)$

Non-homogeneous forms:  $y = y_c + y_p$

where:  $y_c$  is homogeneous solution,

$y_p$  is particular solution based on initial conditions.

#### APPLICATIONS

Electric RLC circuits:  $L \frac{d^2 q}{dt^2} + R \frac{dq}{dt} + \frac{q}{C} = E$

Motion in a resisting medium:  $m \frac{dv}{dt} = F - kv$

#### LAPLACE TRANSFORMS

$$F(s) = L(f) = \int_0^\infty e^{-st} f(t) dt$$

$$\mathcal{L}[af(t) + bg(t)] = a\mathcal{L}(f) + b\mathcal{L}(g)$$

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$$

$$\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0)$$

$$\mathcal{L}^{-1}(F) = f(t)$$

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