# Elementary Linear Algebra

**Ron Larson**

**Seventh Edition**

# INDEX OF APPLICATIONS

# Elementary Linear Algebra

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# Elementary Linear Algebra

## Seventh Edition

**Ron Larson**
The Pennsylvania State University
The Behrend College

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**Elementary Linear Algebra**
**Seventh Edition**

**Ron Larson**

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Library of Congress Control Number: 2011935270

Student Edition:
ISBN-13: 978-1-133-11087-3
ISBN-10: 1-133-11087-8

**Brooks/Cole**
20 Channel Center Street
Boston, MA 02210
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Printed in the United States of America
1 2 3 4 5 6 7 15 14 13 12 11

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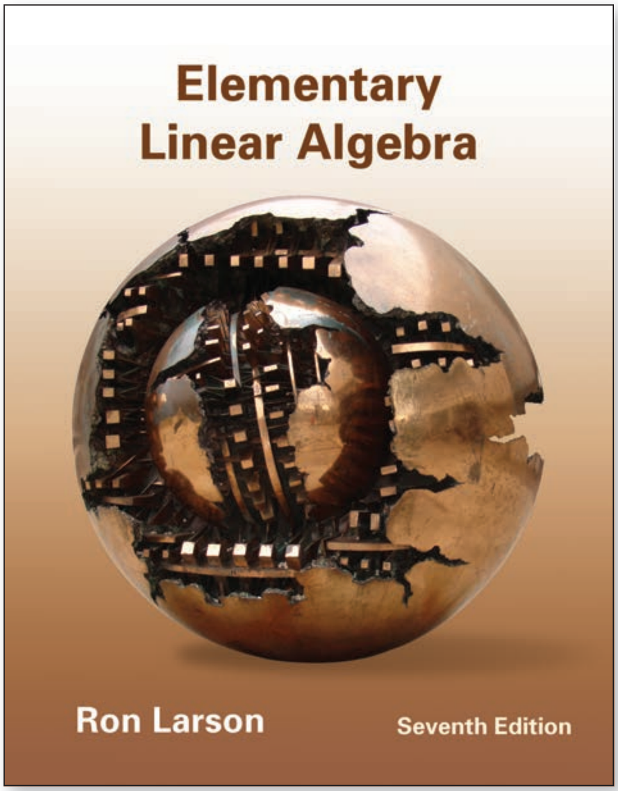
# Preface

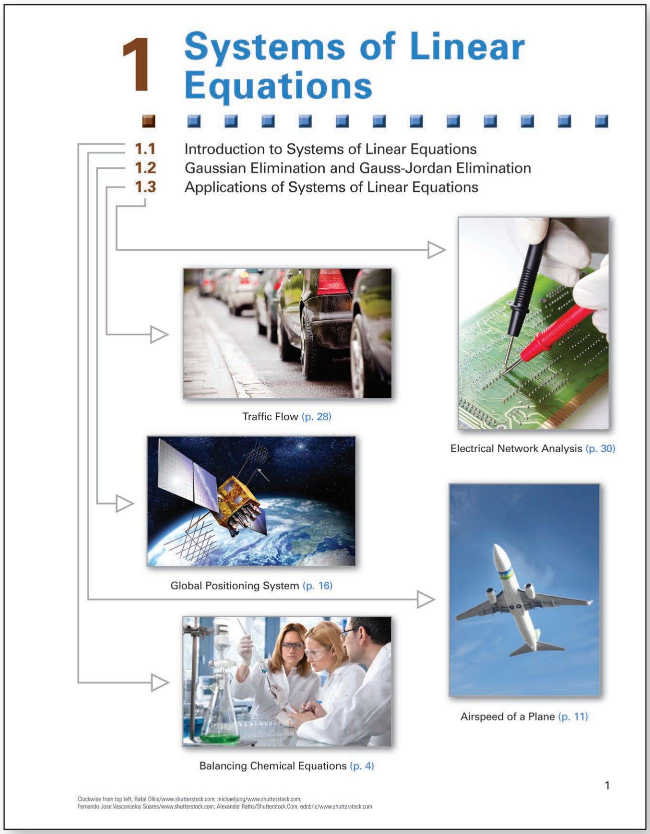
Welcome to the Seventh Edition of *Elementary Linear Algebra*. My primary goal is to present the major concepts of linear algebra clearly and concisely. To this end, I have carefully selected the examples and exercises to balance theory with applications and geometrical intuition. The order and coverage of topics were chosen for maximum efficiency, effectiveness, and balance. The new design complements the multitude of features and applications found throughout the book.




## *New To This Edition*

### NEW *Chapter Openers*

Each *Chapter Opener* highlights five real-life applications of linear algebra found throughout the chapter. Many of the applications reference the new *Linear Algebra Applied* featured (discussed below). You can find a full listing of the applications in the *Index of Applications* on the inside front cover.


1 Systems of Linear Equations

1.1 Introduction to Systems of Linear Equations
1.2 Gaussian Elimination and Gauss-Jordan Elimination
1.3 Applications of Systems of Linear Equations

Traffic Flow (p. 28)
Electrical Network Analysis (p. 30)
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Balancing Chemical Equations (p. 4)

1

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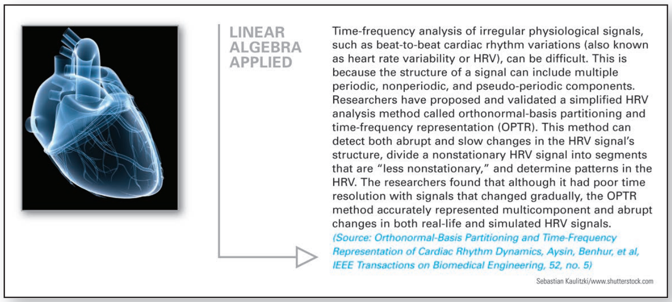
Fernando Jose Vasconcelos Soares/www.shutterstock.com; Alexander Raths/Shutterstock.Com; edobric/www.shutterstock.com

### NEW *Linear Algebra Applied*

*Linear Algebra Applied* describes a real-life application of concepts discussed in a section. These applications include biology and life sciences, business and economics, engineering and technology, physical sciences, and statistics and probability.


LINEAR ALGEBRA APPLIED

Time-frequency analysis of irregular physiological signals, such as beat-to-beat cardiac rhythm variations (also known as heart rate variability or HRV), can be difficult. This is because the structure of a signal can include multiple periodic, nonperiodic, and pseudo-periodic components. Researchers have proposed and validated a simplified HRV analysis method called orthonormal-basis partitioning and time-frequency representation (OPTR). This method can detect both abrupt and slow changes in the HRV signal's structure, divide a nonstationary HRV signal into segments that are "less nonstationary," and determine patterns in the HRV. The researchers found that although it had poor time resolution with signals that changed gradually, the OPTR method accurately represented multicomponent and abrupt changes in both real-life and simulated HRV signals. *(Source: Orthonormal-Basis Partitioning and Time-Frequency Representation of Cardiac Rhythm Dynamics, Aysin, Benhur, et al, IEEE Transactions on Biomedical Engineering, 52, no. 5)*

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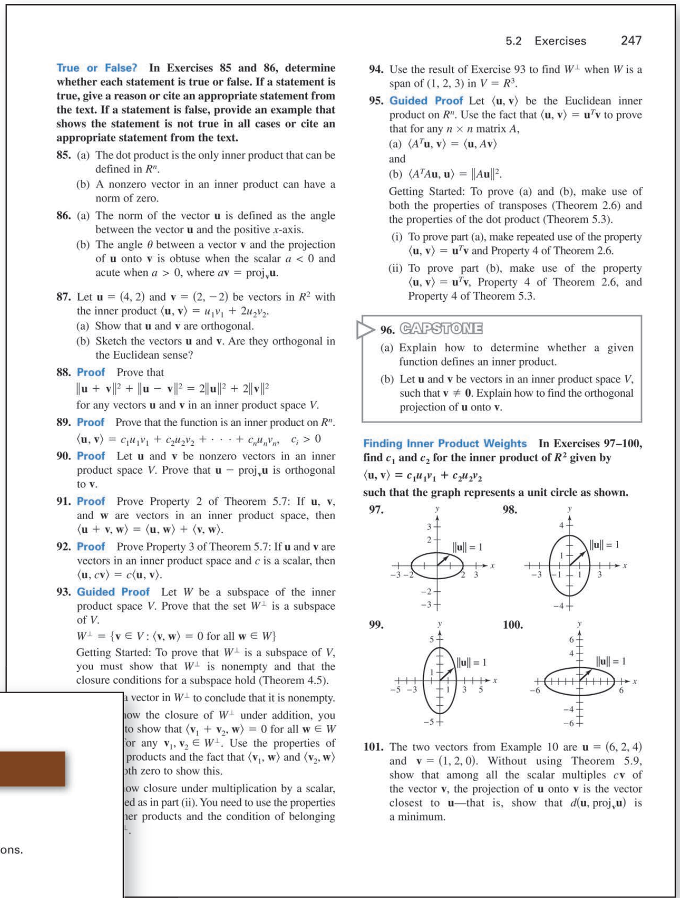
### NEW *Capstone Exercises*

The *Capstone* is a conceptual problem that synthesizes key topics to check students' understanding of the section concepts. I recommend it.

## REVISED *Exercise Sets*

The exercise sets have been carefully and extensively examined to ensure they are rigorous, relevant, and cover all topics suggested by our users. The exercises have been reorganized and titled so you can better see the connections between examples and exercises. Many new skill building, challenging, and application exercises have been added. As in earlier editions, the following pedagogically-proven types of exercises are included:

- **True or False Exercises** ask students to give examples or justifications to support their conclusions.
- **Proofs**
- **Guided Proofs** lead student through the initial steps of constructing proofs and then utilizing the results.
- **Writing Exercises**
- **Technology Exercises** are indicated throughout the text with .
- Exercises utilizing **electronic data sets** are indicated by and found at *www.cengagebrain.com.*



5.2 Exercises 247

**True or False?** **In Exercises 85 and 86, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.**

**85.** (a) The dot product is the only inner product that can be defined in $R^n$.
(b) A nonzero vector in an inner product can have a norm of zero.

**86.** (a) The norm of the vector $\mathbf{u}$ is defined as the angle between the vector $\mathbf{u}$ and the positive $x$-axis.
(b) The angle $\theta$ between a vector $\mathbf{v}$ and the projection of $\mathbf{u}$ onto $\mathbf{v}$ is obtuse when the scalar $a < 0$ and acute when $a > 0$, where $a\mathbf{v} = \text{proj}_{\mathbf{v}}\mathbf{u}$.

**87.** Let $\mathbf{u} = (4, 2)$ and $\mathbf{v} = (2, -2)$ be vectors in $R^2$ with the inner product $\langle \mathbf{u}, \mathbf{v} \rangle = u_1v_1 + 2u_2v_2$.
(a) Show that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.
(b) Sketch the vectors $\mathbf{u}$ and $\mathbf{v}$. Are they orthogonal in the Euclidean sense?

**88.** **Proof** Prove that
$\|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2$
for any vectors $\mathbf{u}$ and $\mathbf{v}$ in an inner product space $V$.

**89.** **Proof** Prove that the function is an inner product on $R^n$.
$\langle \mathbf{u}, \mathbf{v} \rangle = c_1u_1v_1 + c_2u_2v_2 + \cdot \cdot \cdot + c_nu_nv_n, \quad c_i > 0$

**90.** **Proof** Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors in an inner product space $V$. Prove that $\mathbf{u} - \text{proj}_{\mathbf{v}}\mathbf{u}$ is orthogonal to $\mathbf{v}$.

**91.** **Proof** Prove Property 2 of Theorem 5.7: If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in an inner product space, then $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$.

**92.** **Proof** Prove Property 3 of Theorem 5.7: If $\mathbf{u}$ and $\mathbf{v}$ are vectors in an inner product space and $c$ is a scalar, then $\langle \mathbf{u}, c\mathbf{v} \rangle = c\langle \mathbf{u}, \mathbf{v} \rangle$.

**93.** **Guided Proof** Let $W$ be a subspace of the inner product space $V$. Prove that the set $W^{\perp}$ is a subspace of $V$.
$W^{\perp} = \{\mathbf{v} \in V : \langle \mathbf{v}, \mathbf{w} \rangle = 0 \text{ for all } \mathbf{w} \in W\}$
Getting Started: To prove that $W^{\perp}$ is a subspace of $V$, you must show that $W^{\perp}$ is nonempty and that the closure conditions for a subspace hold (Theorem 4.5).
a vector in $W^{\perp}$ to conclude that it is nonempty.
ow the closure of $W^{\perp}$ under addition, you to show that $\langle \mathbf{v}_1 + \mathbf{v}_2, \mathbf{w} \rangle = 0$ for all $\mathbf{w} \in W$ or any $\mathbf{v}_1, \mathbf{v}_2 \in W^{\perp}$. Use the properties of products and the fact that $\langle \mathbf{v}_1, \mathbf{w} \rangle$ and $\langle \mathbf{v}_2, \mathbf{w} \rangle$ oth zero to show this.
ow closure under multiplication by a scalar, ed as in part (ii). You need to use the properties er products and the condition of belonging

**94.** Use the result of Exercise 93 to find $W^{\perp}$ when $W$ is a span of $(1, 2, 3)$ in $V = R^3$.

**95.** **Guided Proof** Let $\langle \mathbf{u}, \mathbf{v} \rangle$ be the Euclidean inner product on $R^n$. Use the fact that $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^T\mathbf{v}$ to prove that for any $n \times n$ matrix $A$,
(a) $\langle A^T\mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, A\mathbf{v} \rangle$
and
(b) $\langle A^TA\mathbf{u}, \mathbf{u} \rangle = \|A\mathbf{u}\|^2$.
Getting Started: To prove (a) and (b), make use of both the properties of transposes (Theorem 2.6) and the properties of the dot product (Theorem 5.3).
(i) To prove part (a), make repeated use of the property $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^T\mathbf{v}$ and Property 4 of Theorem 2.6.
(ii) To prove part (b), make use of the property $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^T\mathbf{v}$, Property 4 of Theorem 2.6, and Property 4 of Theorem 5.3.

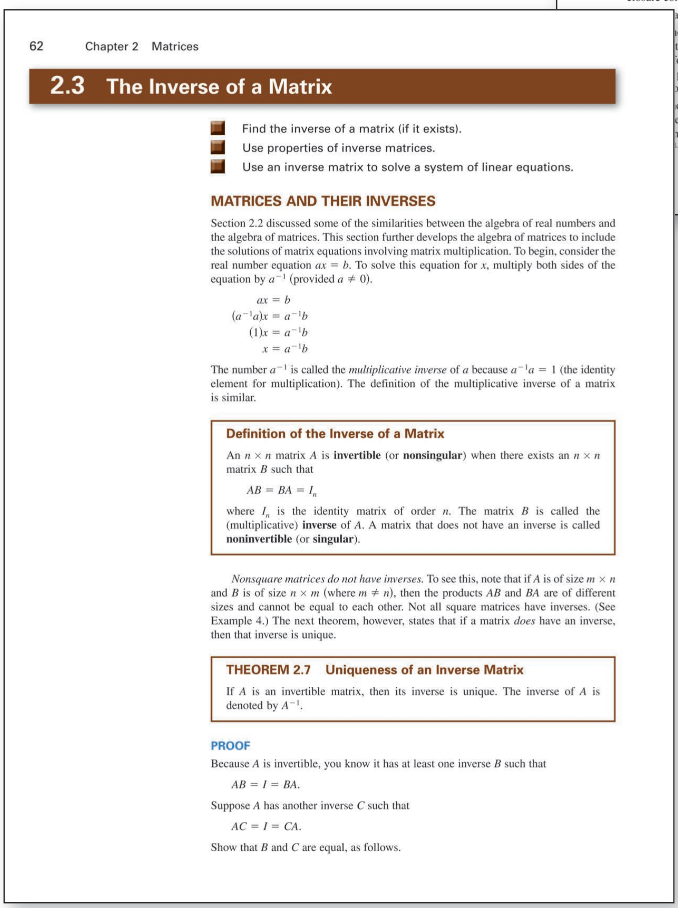
**96.** CAPSTONE
(a) Explain how to determine whether a given function defines an inner product.
(b) Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in an inner product space $V$, such that $\mathbf{v} \neq \mathbf{0}$. Explain how to find the orthogonal projection of $\mathbf{u}$ onto $\mathbf{v}$.

**Finding Inner Product Weights** **In Exercises 97–100, find $c_1$ and $c_2$ for the inner product of $R^2$ given by**
$\langle \mathbf{u}, \mathbf{v} \rangle = c_1u_1v_1 + c_2u_2v_2$
**such that the graph represents a unit circle as shown.**

**97.** **98.** **99.** **100.**



**101.** The two vectors from Example 10 are $\mathbf{u} = (6, 2, 4)$ and $\mathbf{v} = (1, 2, 0)$. Without using Theorem 5.9, show that among all the scalar multiples $c\mathbf{v}$ of the vector $\mathbf{v}$, the projection of $\mathbf{u}$ onto $\mathbf{v}$ is the vector closest to $\mathbf{u}$—that is, show that $d(\mathbf{u}, \text{proj}_{\mathbf{v}}\mathbf{u})$ is a minimum.



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**2.3 The Inverse of a Matrix**

- Find the inverse of a matrix (if it exists).
- Use properties of inverse matrices.
- Use an inverse matrix to solve a system of linear equations.

**MATRICES AND THEIR INVERSES**

Section 2.2 discussed some of the similarities between the algebra of real numbers and the algebra of matrices. This section further develops the algebra of matrices to include the solutions of matrix equations involving matrix multiplication. To begin, consider the real number equation $ax = b$. To solve this equation for $x$, multiply both sides of the equation by $a^{-1}$ (provided $a \neq 0$).

$$\begin{aligned} ax &= b \\ (a^{-1}a)x &= a^{-1}b \\ (1)x &= a^{-1}b \\ x &= a^{-1}b \end{aligned}$$

The number $a^{-1}$ is called the *multiplicative inverse* of $a$ because $a^{-1}a = 1$ (the identity element for multiplication). The definition of the multiplicative inverse of a matrix is similar.

**Definition of the Inverse of a Matrix**

An $n \times n$ matrix $A$ is **invertible** (or **nonsingular**) when there exists an $n \times n$ matrix $B$ such that

$$AB = BA = I_n$$

where $I_n$ is the identity matrix of order $n$. The matrix $B$ is called the (multiplicative) **inverse** of $A$. A matrix that does not have an inverse is called **noninvertible** (or **singular**).

*Nonsquare matrices do not have inverses.* To see this, note that if $A$ is of size $m \times n$ and $B$ is of size $n \times m$ (where $m \neq n$), then the products $AB$ and $BA$ are of different sizes and cannot be equal to each other. Not all square matrices have inverses. (See Example 4.) The next theorem, however, states that if a matrix *does* have an inverse, then that inverse is unique.

**THEOREM 2.7 Uniqueness of an Inverse Matrix**

If $A$ is an invertible matrix, then its inverse is unique. The inverse of $A$ is denoted by $A^{-1}$.

**PROOF**

Because $A$ is invertible, you know it has at least one inverse $B$ such that

$$AB = I = BA.$$

Suppose $A$ has another inverse $C$ such that

$$AC = I = CA.$$

Show that $B$ and $C$ are equal, as follows.

## *Table of Contents Changes*

Based on feedback from users, Section 3.4 (Introduction to Eigenvalues) in the previous edition has been removed and its content has been absorbed in Chapter 7 (Eigenvalues and Eigenvectors).

# *Trusted Features*

## *Section Objectives*

A bulleted list of learning objectives provides you the opportunity to preview what will be presented in the upcoming section. For the Seventh Edition, the section objectives are located by relevance at the beginning of each section.

## *Theorems, Definitions, and Properties*

Presented in clear and mathematically precise language, all theorems, definitions, and properties are highlighted for emphasis and easy reference.

## *Proofs in Outline Form*

In addition to proofs in the exercises, some proofs are presented in outline form, omitting the need for burdensome calculations.

## *Discovery*

Discovery helps you develop an intuitive understanding of mathematical concepts and relationships.

**EXAMPLE 4 Finding a Transition Matrix**

Find the transition matrix from $B$ to $B'$ for the following bases for $R^3$.

$$B = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\} \quad \text{and} \quad B' = \{(1, 0, 1), (0, -1, 2), (2, 3, -5)\}$$

**SOLUTION**

First use the vectors in the two bases to form the matrices $B$ and $B'$.

$$B = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} \quad \text{and} \quad B' = \begin{bmatrix}1 & 0 & 2\\0 & -1 & 3\\1 & 2 & -5\end{bmatrix}$$

Then form the matrix $[B' \; B]$ and use Gauss-Jordan elimination to rewrite $[B' \; B]$ as $[I_3 \;\; P^{-1}]$.

$$\left[\begin{array}{rrrrrr}1 & 0 & 2 & 1 & 0 & 0\\0 & -1 & 3 & 0 & 1 & 0\\1 & 2 & -5 & 0 & 0 & 1\end{array}\right] \rightarrow \left[\begin{array}{rrrrrr}1 & 0 & 0 & -1 & 4 & 2\\0 & 1 & 0 & 3 & -7 & -3\\0 & 0 & 1 & 1 & -2 & -1\end{array}\right]$$

From this, you can conclude that the transition matrix from $B$ to $B'$ is

$$P^{-1} = \begin{bmatrix}-1 & 4 & 2\\3 & -7 & -3\\1 & -2 & -1\end{bmatrix}.$$

Try multiplying $P^{-1}$ by the coordinate matrix of $\mathbf{x} = [1 \;\; 2 \;\; -1]^T$ to see that the result is the same as the one obtained in Example 3.

**DISCOVERY**

1. Let $B = \{(1, 0), (1, 2)\}$ and $B' = \{(1, 0), (0, 1)\}$. Form the matrix $[B' \; B]$.
2. Make a conjecture about the necessity of using Gauss-Jordan elimination to obtain the transition matrix $P^{-1}$ when the change of basis is from a nonstandard basis to a standard basis.

**SOLUTION**

After inspecting
zeros. So, to elim

$$|A| = 3(C_{13}) + 0(C_{23}) + 0(C_{33}) + 0(C_{43})$$

Because $C_{23}$, $C_{33}$, and $C_{43}$ have zero coefficients, you need only find the cofactor $C_{13}$. To do this, delete the first row and third column of $A$ and evaluate the determinant of the resulting matrix.

$$C_{13} = (-1)^{1+3}\begin{vmatrix}-1 & 1 & 2\\0 & 2 & 3\\3 & 4 & -2\end{vmatrix} \qquad \text{Delete 1st row and 3rd column.}$$

$$= \begin{vmatrix}-1 & 1 & 2\\0 & 2 & 3\\3 & 4 & -2\end{vmatrix} \qquad \text{Simplify.}$$

Expanding by cofactors in the second row yields

$$C_{13} = (0)(-1)^{2+1}\begin{vmatrix}1 & 2\\4 & -2\end{vmatrix} + (2)(-1)^{2+2}\begin{vmatrix}-1 & 2\\3 & -2\end{vmatrix} + (3)(-1)^{2+3}\begin{vmatrix}-1 & 1\\3 & 4\end{vmatrix}$$

$$= 0 + 2(1)(-4) + 3(-1)(-7)$$

$$= 13.$$

You obtain

$$|A| = 3(13)$$

$$= 39.$$

**TECHNOLOGY**

Many graphing utilities and software programs can calculate the determinant of a square matrix. If you use a graphing utility, then you may see something similar to the following for Example 4. **The Online Technology Guide,** available at *www.cengagebrain.com,* provides syntax for programs applicable to Example 4.

```
A
        [[1  -2 3 0 ]
         [-1  1 0 2 ]
         [0   2 0 3 ]
         [3   4 0 -2]]
det A
                   39
```

## *Technology Notes*

Technology notes show how you can use graphing utilities and software programs appropriately in the problem-solving process. Many of the technology notes reference the **Online Technology Guide,** located at *www.cengagebrain.com*.

## *Chapter Projects*

Two per chapter, these offer the opportunity for group activities or more extensive homework assignments, and are focused on theoretical concepts or applications. Many encourage the use of technology.

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**2 Projects**

| | *Test 1* | *Test 2* |
|---|---|---|
| Anna | 84 | 96 |
| Bruce | 56 | 72 |
| Chris | 78 | 83 |
| David | 82 | 91 |

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**1 Exploring Matrix Multiplication**

The table shows the first two test scores for Anna, Bruce, Chris, and David. Use the table to create a matrix $M$ to represent the data. Input $M$ into a software program or a graphing utility and use it to answer the following questions.

1. Which test was more difficult? Which was easier? Explain.
2. How would you rank the performances of the four students?
3. Describe the meanings of the matrix products $M\begin{bmatrix}1\\0\end{bmatrix}$ and $M\begin{bmatrix}0\\1\end{bmatrix}$.
4. Describe the meanings of the matrix products $[1 \;\; 0 \;\; 0 \;\; 0]M$ and $[0 \;\; 0 \;\; 1 \;\; 0]M$.
5. Describe the meanings of the matrix products $M\begin{bmatrix}1\\1\end{bmatrix}$ and $\frac{1}{2}M\begin{bmatrix}1\\1\end{bmatrix}$.
6. Describe the meanings of the matrix products $[1 \;\; 1 \;\; 1 \;\; 1]M$ and $\frac{1}{4}[1 \;\; 1 \;\; 1 \;\; 1]M$.
7. Describe the meaning of the matrix product $[1 \;\; 1 \;\; 1 \;\; 1]M\begin{bmatrix}1\\1\end{bmatrix}$.
8. Use matrix multiplication to find the combined overall average score on both tests.
9. How could you use matrix multiplication to scale the scores on test 1 by a factor of 1.1?

**2 Nilpotent Matrices**

Let $A$ be a nonzero square matrix. Is it possible that a positive integer $k$ exists such that $A^k = O$? For example, find $A^3$ for the matrix

$$A = \begin{bmatrix}0 & 1 & 2\\0 & 0 & 1\\0 & 0 & 0\end{bmatrix}.$$

A square matrix $A$ is **nilpotent of index** $k$ when $A \neq O$, $A^2 \neq O, \ldots, A^{k-1} \neq O$, but $A^k = O$. In this project you will explore nilpotent matrices.

1. The matrix in the example given above is nilpotent. What is its index?
2. Use a software program or a graphing utility to determine which of the following matrices are nilpotent and find their indices.

(a) $\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}$ (b) $\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$ (c) $\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}$

(d) $\begin{bmatrix}1 & 0\\1 & 0\end{bmatrix}$ (e) $\begin{bmatrix}0 & 0 & 1\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$ (f) $\begin{bmatrix}0 & 0 & 0\\1 & 0 & 0\\1 & 1 & 0\end{bmatrix}$

3. Find $3 \times 3$ nilpotent matrices of indices 2 and 3.
4. Find $4 \times 4$ nilpotent matrices of indices 2, 3, and 4.
5. Find a nilpotent matrix of index 5.
6. Are nilpotent matrices invertible? Prove your answer.
7. When $A$ is nilpotent, what can you say about $A^T$? Prove your answer.
8. Show that if $A$ is nilpotent, then $I - A$ is invertible.

# Instructor Resources

## Print

**Instructor's Solutions Manual**
ISBN-13: 978-1-133-36538-9
The *Instructor's Solutions Manual* provides worked-out solutions for all even-numbered exercises in the text.

## Media

**Solution Builder**
*www.cengage.com/solutionbuilder*
This online instructor database offers complete worked-out solutions of all exercises in the text. Solution Builder allows you to create customized, secure solutions printouts (in PDF format) matched exactly to the problems you assign in class.

**Diploma Computerized Testing**
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# Student Resources

## Print

**Student Solutions Manual**
ISBN-13: 978-1-133-11132-0
The *Student Solutions Manual* provides complete worked-out solutions to all odd-numbered exercises in the text. Also included are the solutions to all Cumulative Test problems.

## Media

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Enhanced WebAssign is an online homework system that lets instructors deliver, collect, grade, and record assignments via the web. Enhanced WebAssign includes Cengage YouBook—a media-rich interactive ebook, Personal Study Plans, a Show My Work feature, quizzes, videos, and more! Be sure to check with your instructor to find out if Enhanced WebAssign is required for your course.

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To access additional course materials and companion resources, please visit *www.cengagebrain.com*. At the CengageBrain.com home page, search for the ISBN of your title (from the back cover of your book) using the search box at the top of the page. This will take you to the product page where free companion resources can be found.

# Acknowledgements

I would like to thank the many people who have helped me during various stages of writing this new edition. In particular, I appreciate the efforts of the following colleagues who made many helpful suggestions along the way:

Michael Brown, *San Diego Mesa College*

Nasser Dastrange, *Buena Vista University*

Mike Daven, *Mount Saint Mary College*

David Hemmer, *University of Buffalo, SUNY*

Wai Lau, *Seattle Pacific University*

Jorge Sarmiento, *County College of Morris*

I would like to thank Bruce H. Edwards, University of Florida, and David C. Falvo, The Pennsylvania State University, The Behrend College, for their contributions to previous editions of *Elementary Linear Algebra*.

On a personal level, I am grateful to my spouse, Deanna Gilbert Larson, for her love, patience, and support. Also, a special thanks goes to R. Scott O'Neil.

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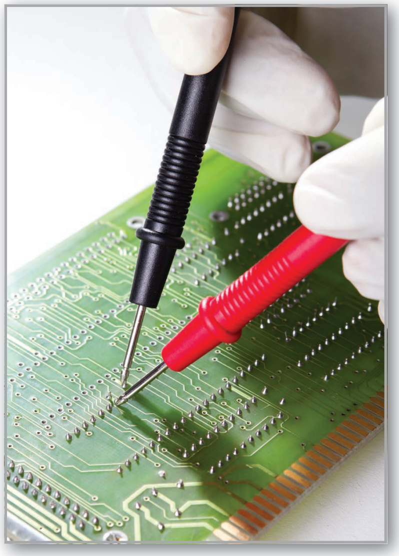
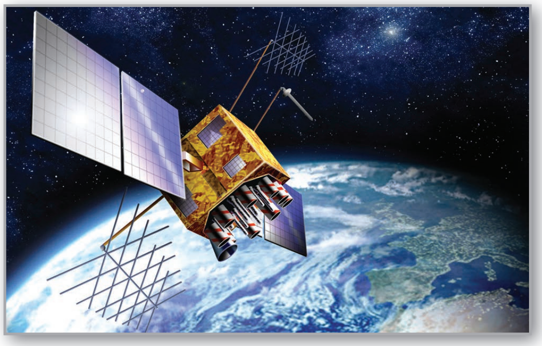
# 1 Systems of Linear Equations





Electrical Network Analysis (p. 30)

Traffic Flow (p. 28)



Global Positioning System (p. 16)

Airspeed of a Plane (p. 11)

Balancing Chemical Equations (p. 4)

# 1.1 Introduction to Systems of Linear Equations

- Recognize a linear equation in *n* variables.
- Find a parametric representation of a solution set.
- Determine whether a system of linear equations is consistent or inconsistent.
- Use back-substitution and Gaussian elimination to solve a system of linear equations.

## LINEAR EQUATIONS IN *n* VARIABLES

The study of linear algebra demands familiarity with algebra, analytic geometry, and trigonometry. Occasionally, you will find examples and exercises requiring a knowledge of calculus; these are clearly marked in the text.

Early in your study of linear algebra, you will discover that many of the solution methods involve multiple arithmetic steps, so it is essential to check your work. Use a computer or calculator to check your work and perform routine computations.

Although you will be familiar with some material in this chapter, you should carefully study the methods presented in this chapter. This will cultivate and clarify your intuition for the more abstract material that follows.

Recall from analytic geometry that the equation of a line in two-dimensional space has the form

$$a_1x + a_2y = b, \quad a_1, a_2, \text{ and } b \text{ are constants.}$$

This is a **linear equation in two variables** $x$ and $y$. Similarly, the equation of a plane in three-dimensional space has the form

$$a_1x + a_2y + a_3z = b, \quad a_1, a_2, a_3, \text{ and } b \text{ are constants.}$$

This is a **linear equation in three variables** $x$, $y$, and $z$. In general, a linear equation in $n$ variables is defined as follows.

**REMARK**

Letters that occur early in the alphabet are used to represent constants, and letters that occur late in the alphabet are used to represent variables.

### Definition of a Linear Equation in *n* Variables

A **linear equation in *n* variables** $x_1, x_2, x_3, \ldots, x_n$ has the form

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = b.$$

The **coefficients** $a_1, a_2, a_3, \ldots, a_n$ are real numbers, and the **constant term** $b$ is a real number. The number $a_1$ is the **leading coefficient,** and $x_1$ is the **leading variable.**

Linear equations have no products or roots of variables and no variables involved in trigonometric, exponential, or logarithmic functions. Variables appear only to the first power.

### EXAMPLE 1 Linear and Nonlinear Equations

Each equation is linear.

**a.** $3x + 2y = 7$ **b.** $\frac{1}{2}x + y - \pi z = \sqrt{2}$ **c.** $(\sin \pi)x_1 - 4x_2 = e^2$

Each equation is not linear.

**a.** $xy + z = 2$ **b.** $e^x - 2y = 4$ **c.** $\sin x_1 + 2x_2 - 3x_3 = 0$

## SOLUTIONS AND SOLUTION SETS

A **solution** of a linear equation in $n$ variables is a sequence of $n$ real numbers $s_1, s_2, s_3, \ldots, s_n$ arranged to satisfy the equation when you substitute the values

$$x_1 = s_1, \quad x_2 = s_2, \quad x_3 = s_3, \quad \ldots, \quad x_n = s_n$$

into the equation. For example, $x_1 = 2$ and $x_2 = 1$ satisfy the equation $x_1 + 2x_2 = 4$. Some other solutions are $x_1 = -4$ and $x_2 = 4$, $x_1 = 0$ and $x_2 = 2$, and $x_1 = -2$ and $x_2 = 3$.

The set of *all* solutions of a linear equation is called its **solution set,** and when you have found this set, you have **solved** the equation. To describe the entire solution set of a linear equation, use a **parametric representation,** as illustrated in Examples 2 and 3.

### EXAMPLE 2 Parametric Representation of a Solution Set

Solve the linear equation $x_1 + 2x_2 = 4$.

**SOLUTION**

To find the solution set of an equation involving two variables, solve for one of the variables in terms of the other variable. Solving for $x_1$ in terms of $x_2$, you obtain

$$x_1 = 4 - 2x_2.$$

In this form, the variable $x_2$ is **free,** which means that it can take on any real value. The variable $x_1$ is not free because its value depends on the value assigned to $x_2$. To represent the infinitely many solutions of this equation, it is convenient to introduce a third variable $t$ called a **parameter.** By letting $x_2 = t$, you can represent the solution set as

$$x_1 = 4 - 2t, \quad x_2 = t, \quad t \text{ is any real number.}$$

To obtain particular solutions, assign values to the parameter $t$. For instance, $t = 1$ yields the solution $x_1 = 2$ and $x_2 = 1$, and $t = 4$ yields the solution $x_1 = -4$ and $x_2 = 4$.

To parametrically represent the solution set of the linear equation in Example 2 another way, you could have chosen $x_1$ to be the free variable. The parametric representation of the solution set would then have taken the form

$$x_1 = s, \quad x_2 = 2 - \tfrac{1}{2}s, \quad s \text{ is any real number.}$$

For convenience, choose the variables that occur last in a given equation to be free variables.

### EXAMPLE 3 Parametric Representation of a Solution Set

Solve the linear equation $3x + 2y - z = 3$.

**SOLUTION**

Choosing $y$ and $z$ to be the free variables, solve for $x$ to obtain

$$\begin{aligned} 3x &= 3 - 2y + z \\ x &= 1 - \tfrac{2}{3}y + \tfrac{1}{3}z. \end{aligned}$$

Letting $y = s$ and $z = t$, you obtain the parametric representation

$$x = 1 - \tfrac{2}{3}s + \tfrac{1}{3}t, \quad y = s, \quad z = t$$

where $s$ and $t$ are any real numbers. Two particular solutions are

$$x = 1, y = 0, z = 0 \quad \text{and} \quad x = 1, y = 1, z = 2.$$

## SYSTEMS OF LINEAR EQUATIONS

A **system of *m* linear equations in *n* variables** is a set of $m$ equations, each of which is linear in the same $n$ variables:

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1\\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2\\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3\\
&\vdots\\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m.
\end{aligned}$$

**REMARK**

The double-subscript notation indicates $a_{ij}$ is the coefficient of $x_j$ in the $i$th equation.

A **solution** of a system of linear equations is a sequence of numbers $s_1, s_2, s_3, \ldots, s_n$ that is a solution of each of the linear equations in the system. For example, the system

$$\begin{aligned}
3x_1 + 2x_2 &= 3\\
-x_1 + x_2 &= 4
\end{aligned}$$

has $x_1 = -1$ and $x_2 = 3$ as a solution because $x_1 = -1$ and $x_2 = 3$ satisfy *both* equations. On the other hand, $x_1 = 1$ and $x_2 = 0$ is not a solution of the system because these values satisfy only the first equation in the system.

### DISCOVERY

1. Graph the two lines

$$\begin{aligned}
3x - y &= 1\\
2x - y &= 0
\end{aligned}$$

in the $xy$-plane. Where do they intersect? How many solutions does this system of linear equations have?

2. Repeat this analysis for the pairs of lines

$$\begin{aligned}
3x - y &= 1\\
3x - y &= 0
\end{aligned}$$

and

$$\begin{aligned}
3x - y &= 1\\
6x - 2y &= 2.
\end{aligned}$$

3. What basic types of solution sets are possible for a system of two equations in two unknowns?

**LINEAR ALGEBRA APPLIED**

In a chemical reaction, atoms reorganize in one or more substances. For instance, when methane gas ($CH_4$) combines with oxygen ($O_2$) and burns, carbon dioxide ($CO_2$) and water ($H_2O$) form. Chemists represent this process by a chemical equation of the form

$$(x_1)CH_4 + (x_2)O_2 \rightarrow (x_3)CO_2 + (x_4)H_2O.$$

Because a chemical reaction can neither create nor destroy atoms, all of the atoms represented on the left side of the arrow must be accounted for on the right side of the arrow. This is called *balancing* the chemical equation. In the given example, chemists can use a system of linear equations to find values of $x_1$, $x_2$, $x_3$, and $x_4$ that will balance the chemical equation.

It is possible for a system of linear equations to have exactly one solution, infinitely many solutions, or no solution. A system of linear equations is **consistent** when it has at least one solution and **inconsistent** when it has no solution.

### EXAMPLE 4 Systems of Two Equations in Two Variables

Solve and graph each system of linear equations.

**a.** $\begin{aligned} x + y &= 3 \\ x - y &= -1 \end{aligned}$ **b.** $\begin{aligned} x + y &= 3 \\ 2x + 2y &= 6 \end{aligned}$ **c.** $\begin{aligned} x + y &= 3 \\ x + y &= 1 \end{aligned}$

**SOLUTION**

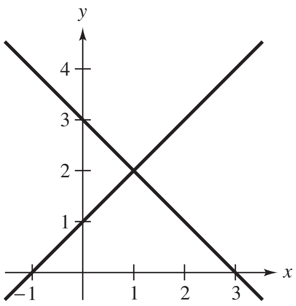
**a.** This system has exactly one solution, $x = 1$ and $y = 2$. One way to obtain the solution is to add the two equations to give $2x = 2$, which implies $x = 1$ and so $y = 2$. The graph of this system is two *intersecting* lines, as shown in Figure 1.1(a).

**b.** This system has infinitely many solutions because the second equation is the result of multiplying both sides of the first equation by 2. A parametric representation of the solution set is

$$x = 3 - t, \quad y = t, \quad t \text{ is any real number.}$$

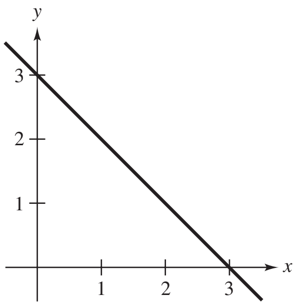
The graph of this system is two *coincident* lines, as shown in Figure 1.1(b).

**c.** This system has no solution because the sum of two numbers cannot be 3 and 1 simultaneously. The graph of this system is two *parallel* lines, as shown in Figure 1.1(c).



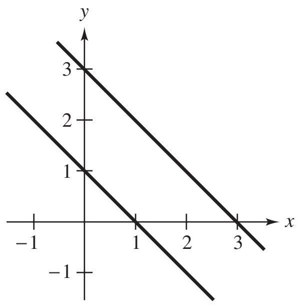
**a.** Two intersecting lines:

$$\begin{aligned} x + y &= 3 \\ x - y &= -1 \end{aligned}$$



**b.** Two coincident lines:

$$\begin{aligned} x + y &= 3 \\ 2x + 2y &= 6 \end{aligned}$$



**c.** Two parallel lines:

$$\begin{aligned} x + y &= 3 \\ x + y &= 1 \end{aligned}$$

**Figure 1.1**

Example 4 illustrates the three basic types of solution sets that are possible for a system of linear equations. This result is stated here without proof. (The proof is provided later in Theorem 2.5.)

**Number of Solutions of a System of Linear Equations**

For a system of linear equations, precisely one of the following is true.

1. The system has exactly one solution (consistent system).
2. The system has infinitely many solutions (consistent system).
3. The system has no solution (inconsistent system).

## SOLVING A SYSTEM OF LINEAR EQUATIONS

Which system is easier to solve algebraically?

$$\begin{aligned} x - 2y + 3z &= 9 \\ -x + 3y \phantom{+ 3z} &= -4 \\ 2x - 5y + 5z &= 17 \end{aligned} \qquad \begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ z &= 2 \end{aligned}$$

The system on the right is clearly easier to solve. This system is in **row-echelon form,** which means that it has a "stair-step" pattern with leading coefficients of 1. To solve such a system, use a procedure called **back-substitution.**

### EXAMPLE 5 Using Back-Substitution in Row-Echelon Form

Use back-substitution to solve the system.

$$\begin{aligned} x - 2y &= 5 && \text{Equation 1} \\ y &= -2 && \text{Equation 2} \end{aligned}$$

#### SOLUTION

From Equation 2, you know that $y = -2$. By substituting this value of $y$ into Equation 1, you obtain

$$\begin{aligned} x - 2(-2) &= 5 && \text{Substitute } -2 \text{ for } y. \\ x &= 1. && \text{Solve for } x. \end{aligned}$$

The system has exactly one solution: $x = 1$ and $y = -2$.

The term *back-substitution* implies that you work *backwards*. For instance, in Example 5, the second equation gives you the value of $y$. Then you substitute that value into the first equation to solve for $x$. Example 6 further demonstrates this procedure.

### EXAMPLE 6 Using Back-Substitution in Row-Echelon Form

Solve the system.

$$\begin{aligned} x - 2y + 3z &= 9 && \text{Equation 1} \\ y + 3z &= 5 && \text{Equation 2} \\ z &= 2 && \text{Equation 3} \end{aligned}$$

#### SOLUTION

From Equation 3, you know the value of $z$. To solve for $y$, substitute $z = 2$ into Equation 2 to obtain

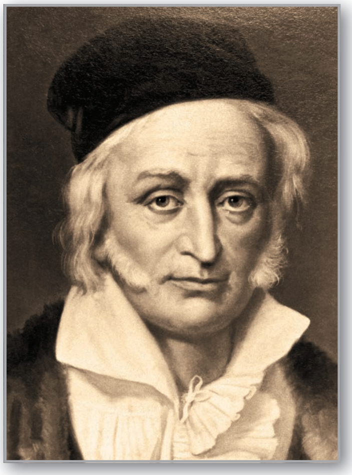
$$\begin{aligned} y + 3(2) &= 5 && \text{Substitute 2 for } z. \\ y &= -1. && \text{Solve for } y. \end{aligned}$$

Then, substitute $y = -1$ and $z = 2$ in Equation 1 to obtain

$$\begin{aligned} x - 2(-1) + 3(2) &= 9 && \text{Substitute } -1 \text{ for } y \text{ and 2 for } z. \\ x &= 1. && \text{Solve for } x. \end{aligned}$$

The solution is $x = 1$, $y = -1$, and $z = 2$.

Two systems of linear equations are **equivalent** when they have the same solution set. To solve a system that is not in row-echelon form, first convert it to an *equivalent* system that is in row-echelon form by using the operations listed on the next page.



**Carl Friedrich Gauss (1777–1855)**

German mathematician Carl Friedrich Gauss is recognized, with Newton and Archimedes, as one of the three greatest mathematicians in history. Gauss used a form of what is now known as Gaussian elimination in his research. Although this method was named in his honor, the Chinese used an almost identical method some 2000 years prior to Gauss.

### Operations That Produce Equivalent Systems

Each of the following operations on a system of linear equations produces an *equivalent* system.

1. Interchange two equations.
2. Multiply an equation by a nonzero constant.
3. Add a multiple of an equation to another equation.

Rewriting a system of linear equations in row-echelon form usually involves a *chain* of equivalent systems, each of which is obtained by using one of the three basic operations. This process is called **Gaussian elimination,** after the German mathematician Carl Friedrich Gauss (1777–1855).

## EXAMPLE 7 Using Elimination to Rewrite a System in Row-Echelon Form

Solve the system.

$$\begin{aligned} x - 2y + 3z &= 9 \\ -x + 3y \phantom{{}+3z} &= -4 \\ 2x - 5y + 5z &= 17 \end{aligned}$$

### SOLUTION

Although there are several ways to begin, you want to use a systematic procedure that is easily applicable to large systems. Work from the upper left corner of the system, saving the $x$ at the upper left and eliminating the other $x$-terms from the first column.

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ 2x - 5y + 5z &= 17 \end{aligned}$$

Adding the first equation to the second equation produces a new second equation.

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ -y - z &= -1 \end{aligned}$$

Adding $-2$ times the first equation to the third equation produces a new third equation.

Now that you have eliminated all but the first $x$ from the first column, work on the second column.

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ 2z &= 4 \end{aligned}$$

Adding the second equation to the third equation produces a new third equation.

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ z &= 2 \end{aligned}$$

Multiplying the third equation by $\frac{1}{2}$ produces a new third equation.

This is the same system you solved in Example 6, and, as in that example, the solution is

$$x = 1, \quad y = -1, \quad z = 2.$$

Each of the three equations in Example 7 represents a plane in a three-dimensional coordinate system. Because the unique solution of the system is the point

$$(x, y, z) = (1, -1, 2)$$

the three planes intersect at this point, as shown in Figure 1.2.

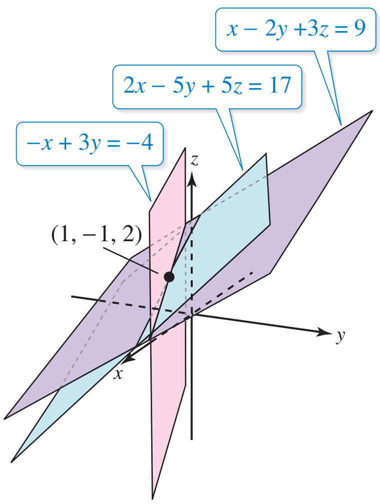



**Figure 1.2**

Because many steps are required to solve a system of linear equations, it is very easy to make arithmetic errors. So, you should develop the habit of *checking your solution by substituting it into each equation in the original system.* For instance, in Example 7, you can check the solution $x = 1$, $y = -1$, and $z = 2$ as follows.

$$\begin{aligned}
&\text{Equation 1:} && (1) - 2(-1) + 3(2) = 9\\
&\text{Equation 2:} && -(1) + 3(-1) = -4\\
&\text{Equation 3:} && 2(1) - 5(-1) + 5(2) = 17
\end{aligned}$$

Substitute solution in each equation of the original system.

The next example involves an inconsistent system—one that has no solution. The key to recognizing an inconsistent system is that at some stage of the elimination process, you obtain a false statement such as $0 = -2$.

## EXAMPLE 8 An Inconsistent System

Solve the system.

$$\begin{aligned}
x_1 - 3x_2 + x_3 &= 1\\
2x_1 - x_2 - 2x_3 &= 2\\
x_1 + 2x_2 - 3x_3 &= -1
\end{aligned}$$

**SOLUTION**

$$\begin{aligned}
x_1 - 3x_2 + x_3 &= 1\\
5x_2 - 4x_3 &= 0\\
x_1 + 2x_2 - 3x_3 &= -1
\end{aligned}$$

Adding $-2$ times the first equation to the second equation produces a new second equation.

$$\begin{aligned}
x_1 - 3x_2 + x_3 &= 1\\
5x_2 - 4x_3 &= 0\\
5x_2 - 4x_3 &= -2
\end{aligned}$$

Adding $-1$ times the first equation to the third equation produces a new third equation.

(Another way of describing this operation is to say that you *subtracted* the first equation from the third equation to produce a new third equation.)

$$\begin{aligned}
x_1 - 3x_2 + x_3 &= 1\\
5x_2 - 4x_3 &= 0\\
0 &= -2
\end{aligned}$$

Adding $-1$ times the second equation to the third equation produces a new third equation.

Because $0 = -2$ is a false statement, this system has no solution. Moreover, because this system is equivalent to the original system, the original system also has no solution.

As in Example 7, the three equations in Example 8 represent planes in a three-dimensional coordinate system. In this example, however, the system is inconsistent. So, the planes do not have a point in common, as shown in Figure 1.3.

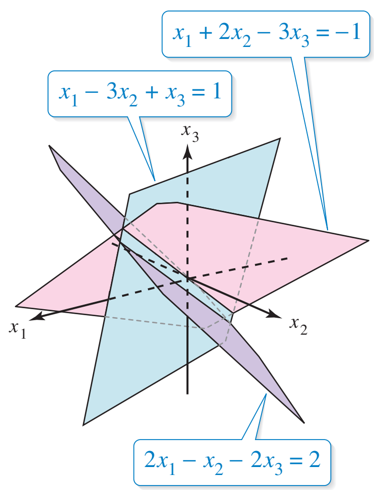



**Figure 1.3**

This section ends with an example of a system of linear equations that has infinitely many solutions. You can represent the solution set for such a system in parametric form, as you did in Examples 2 and 3.

## EXAMPLE 9 A System with Infinitely Many Solutions

Solve the system.

$$\begin{aligned} x_2 - x_3 &= 0 \\ x_1 \phantom{+3x_2} - 3x_3 &= -1 \\ -x_1 + 3x_2 \phantom{-3x_3} &= 1 \end{aligned}$$

### SOLUTION

Begin by rewriting the system in row-echelon form, as follows.

$$\begin{aligned} x_1 \phantom{+3x_2} - 3x_3 &= -1 \\ x_2 - x_3 &= 0 \\ -x_1 + 3x_2 \phantom{-3x_3} &= 1 \end{aligned}$$

Interchange the first two equations.

$$\begin{aligned} x_1 \phantom{+3x_2} - 3x_3 &= -1 \\ x_2 - x_3 &= 0 \\ 3x_2 - 3x_3 &= 0 \end{aligned}$$

Adding the first equation to the third equation produces a new third equation.

$$\begin{aligned} x_1 \phantom{+3x_2} - 3x_3 &= -1 \\ x_2 - x_3 &= 0 \\ 0 &= 0 \end{aligned}$$

Adding $-3$ times the second equation to the third equation eliminates the third equation.

Because the third equation is unnecessary, omit it to obtain the system shown below.

$$\begin{aligned} x_1 \phantom{+x_2} - 3x_3 &= -1 \\ x_2 - x_3 &= 0 \end{aligned}$$

To represent the solutions, choose $x_3$ to be the free variable and represent it by the parameter $t$. Because $x_2 = x_3$ and $x_1 = 3x_3 - 1$, you can describe the solution set as

$$x_1 = 3t - 1, \quad x_2 = t, \quad x_3 = t, \quad t \text{ is any real number.}$$

## DISCOVERY

1. Graph the two lines represented by the system of equations.

$$\begin{aligned} x - 2y &= 1 \\ -2x + 3y &= -3 \end{aligned}$$

2. Use Gaussian elimination to solve this system as follows.

$$\begin{aligned} x - 2y &= 1 \\ -1y &= -1 \end{aligned}$$

$$\begin{aligned} x - 2y &= 1 \\ y &= 1 \end{aligned}$$

$$\begin{aligned} x &= 3 \\ y &= 1 \end{aligned}$$

Graph the system of equations you obtain at each step of this process. What do you observe about the lines?

You are asked to repeat this graphical analysis for other systems in Exercises 89 and 90.

# 1.1 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

**Linear Equations** In Exercises 1–6, determine whether the equation is linear in the variables $x$ and $y$.

1. $2x - 3y = 4$
2. $3x - 4xy = 0$
3. $\dfrac{3}{y} + \dfrac{2}{x} - 1 = 0$
4. $x^2 + y^2 = 4$
5. $2 \sin x - y = 14$
6. $(\sin 2)x - y = 14$

**Parametric Representation** In Exercises 7–10, find a parametric representation of the solution set of the linear equation.

7. $2x - 4y = 0$
8. $3x - \frac{1}{2}y = 9$
9. $x + y + z = 1$
10. $13x_1 - 26x_2 + 39x_3 = 13$

**Graphical Analysis** In Exercises 11–24, graph the system of linear equations. Solve the system and interpret your answer.

11. $\begin{aligned} 2x + y &= 4 \\ x - y &= 2 \end{aligned}$
12. $\begin{aligned} x + 3y &= 2 \\ -x + 2y &= 3 \end{aligned}$
13. $\begin{aligned} x - y &= 1 \\ -2x + 2y &= 5 \end{aligned}$
14. $\begin{aligned} \tfrac{1}{2}x - \tfrac{1}{3}y &= 1 \\ -2x + \tfrac{4}{3}y &= -4 \end{aligned}$
15. $\begin{aligned} 3x - 5y &= 7 \\ 2x + y &= 9 \end{aligned}$
16. $\begin{aligned} -x + 3y &= 17 \\ 4x + 3y &= 7 \end{aligned}$
17. $\begin{aligned} 2x - y &= 5 \\ 5x - y &= 11 \end{aligned}$
18. $\begin{aligned} x - 5y &= 21 \\ 6x + 5y &= 21 \end{aligned}$
19. $\begin{aligned} \frac{x+3}{4} + \frac{y-1}{3} &= 1 \\ 2x - y &= 12 \end{aligned}$
20. $\begin{aligned} \frac{x-1}{2} + \frac{y+2}{3} &= 4 \\ x - 2y &= 5 \end{aligned}$
21. $\begin{aligned} 0.05x - 0.03y &= 0.07 \\ 0.07x + 0.02y &= 0.16 \end{aligned}$
22. $\begin{aligned} 0.2x - 0.5y &= -27.8 \\ 0.3x + 0.4y &= 68.7 \end{aligned}$
23. $\begin{aligned} \frac{x}{4} + \frac{y}{6} &= 1 \\ x - y &= 3 \end{aligned}$
24. $\begin{aligned} \frac{2x}{3} + \frac{y}{6} &= \frac{2}{3} \\ 4x + y &= 4 \end{aligned}$

**Back-Substitution** In Exercises 25–30, use back-substitution to solve the system.

25. $\begin{aligned} x_1 - x_2 &= 2 \\ x_2 &= 3 \end{aligned}$
26. $\begin{aligned} 2x_1 - 4x_2 &= 6 \\ 3x_2 &= 9 \end{aligned}$
27. $\begin{aligned} -x + y - z &= 0 \\ 2y + z &= 3 \\ \tfrac{1}{2}z &= 0 \end{aligned}$
28. $\begin{aligned} x - y \phantom{{}+z} &= 4 \\ 2y + z &= 6 \\ 3z &= 6 \end{aligned}$
29. $\begin{aligned} 5x_1 + 2x_2 + x_3 &= 0 \\ 2x_1 + x_2 &= 0 \end{aligned}$
30. $\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ x_2 &= 0 \end{aligned}$

**Graphical Analysis** In Exercises 31–36, complete the following for the system of equations.

(a) Use a graphing utility to graph the system.

(b) Use the graph to determine whether the system is consistent or inconsistent.

(c) If the system is consistent, approximate the solution.

(d) Solve the system algebraically.

(e) Compare the solution in part (d) with the approximation in part (c). What can you conclude?

31. $\begin{aligned} -3x - y &= 3 \\ 6x + 2y &= 1 \end{aligned}$
32. $\begin{aligned} 4x - 5y &= 3 \\ -8x + 10y &= 14 \end{aligned}$
33. $\begin{aligned} 2x - 8y &= 3 \\ \tfrac{1}{2}x + y &= 0 \end{aligned}$
34. $\begin{aligned} 9x - 4y &= 5 \\ \tfrac{1}{2}x + \tfrac{1}{3}y &= 0 \end{aligned}$
35. $\begin{aligned} 4x - 8y &= 9 \\ 0.8x - 1.6y &= 1.8 \end{aligned}$
36. $\begin{aligned} -5.3x + 2.1y &= 1.25 \\ 15.9x - 6.3y &= -3.75 \end{aligned}$

**System of Linear Equations** In Exercises 37–56, solve the system of linear equations.

37. $\begin{aligned} x_1 - x_2 &= 0 \\ 3x_1 - 2x_2 &= -1 \end{aligned}$
38. $\begin{aligned} 3x + 2y &= 2 \\ 6x + 4y &= 14 \end{aligned}$
39. $\begin{aligned} 2u + v &= 120 \\ u + 2v &= 120 \end{aligned}$
40. $\begin{aligned} x_1 - 2x_2 &= 0 \\ 6x_1 + 2x_2 &= 0 \end{aligned}$
41. $\begin{aligned} 9x - 3y &= -1 \\ \tfrac{1}{5}x + \tfrac{2}{5}y &= -\tfrac{1}{3} \end{aligned}$
42. $\begin{aligned} \tfrac{2}{3}x_1 + \tfrac{1}{6}x_2 &= 0 \\ 4x_1 + x_2 &= 0 \end{aligned}$
43. $\begin{aligned} \frac{x-2}{4} + \frac{y-1}{3} &= 2 \\ x - 3y &= 20 \end{aligned}$
44. $\begin{aligned} \frac{x_1+4}{3} + \frac{x_2+1}{2} &= 1 \\ 3x_1 - x_2 &= -2 \end{aligned}$
45. $\begin{aligned} 0.02x_1 - 0.05x_2 &= -0.19 \\ 0.03x_1 + 0.04x_2 &= 0.52 \end{aligned}$
46. $\begin{aligned} 0.05x_1 - 0.03x_2 &= 0.21 \\ 0.07x_1 + 0.02x_2 &= 0.17 \end{aligned}$
47. $\begin{aligned} x + y + z &= 6 \\ 2x - y + z &= 3 \\ 3x \phantom{{}-y} - z &= 0 \end{aligned}$
48. $\begin{aligned} x + y + z &= 2 \\ -x + 3y + 2z &= 8 \\ 4x + y \phantom{{}+2z} &= 4 \end{aligned}$
49. $\begin{aligned} 3x_1 - 2x_2 + 4x_3 &= 1 \\ x_1 + x_2 - 2x_3 &= 3 \\ 2x_1 - 3x_2 + 6x_3 &= 8 \end{aligned}$
50. $\begin{aligned} 5x_1 - 3x_2 + 2x_3 &= 3 \\ 2x_1 + 4x_2 - x_3 &= 7 \\ x_1 - 11x_2 + 4x_3 &= 3 \end{aligned}$

The symbol indicates an exercise in which you are instructed to use a graphing utility or a symbolic computer software program.

**51.**
$$\begin{aligned} 2x_1 + x_2 - 3x_3 &= 4 \\ 4x_1 + 2x_3 &= 10 \\ -2x_1 + 3x_2 - 13x_3 &= -8 \end{aligned}$$

**52.**
$$\begin{aligned} x_1 + 4x_3 &= 13 \\ 4x_1 - 2x_2 + x_3 &= 7 \\ 2x_1 - 2x_2 - 7x_3 &= -19 \end{aligned}$$

**53.**
$$\begin{aligned} x - 3y + 2z &= 18 \\ 5x - 15y + 10z &= 18 \end{aligned}$$

**54.**
$$\begin{aligned} x_1 - 2x_2 + 5x_3 &= 2 \\ 3x_1 + 2x_2 - x_3 &= -2 \end{aligned}$$

**55.**
$$\begin{aligned} x + y + z + w &= 6 \\ 2x + 3y - w &= 0 \\ -3x + 4y + z + 2w &= 4 \\ x + 2y - z + w &= 0 \end{aligned}$$

**56.**
$$\begin{aligned} x_1 + 3x_4 &= 4 \\ 2x_2 - x_3 - x_4 &= 0 \\ 3x_2 - 2x_4 &= 1 \\ 2x_1 - x_2 + 4x_3 &= 5 \end{aligned}$$

**System of Linear Equations** **In Exercises 57–60, use a software program or a graphing utility to solve the system of linear equations.**

**57.**
$$\begin{aligned} x_1 + 0.5x_2 + 0.33x_3 + 0.25x_4 &= 1.1 \\ 0.5x_1 + 0.33x_2 + 0.25x_3 + 0.21x_4 &= 1.2 \\ 0.33x_1 + 0.25x_2 + 0.2x_3 + 0.17x_4 &= 1.3 \\ 0.25x_1 + 0.2x_2 + 0.17x_3 + 0.14x_4 &= 1.4 \end{aligned}$$

**58.**
$$\begin{aligned} 120.2x + 62.4y - 36.5z &= 258.64 \\ 56.8x - 42.8y + 27.3z &= -71.44 \\ 88.1x + 72.5y - 28.5z &= 225.88 \end{aligned}$$

**59.**
$$\begin{aligned} \tfrac{1}{2}x_1 - \tfrac{3}{7}x_2 + \tfrac{2}{9}x_3 &= \tfrac{349}{630} \\ \tfrac{2}{3}x_1 + \tfrac{4}{9}x_2 - \tfrac{2}{5}x_3 &= -\tfrac{19}{45} \\ \tfrac{4}{5}x_1 - \tfrac{1}{8}x_2 + \tfrac{4}{3}x_3 &= \tfrac{139}{150} \end{aligned}$$

**60.**
$$\begin{aligned} \tfrac{1}{8}x - \tfrac{1}{7}y + \tfrac{1}{6}z - \tfrac{1}{5}w &= 1 \\ \tfrac{1}{7}x + \tfrac{1}{6}y - \tfrac{1}{5}z + \tfrac{1}{4}w &= 1 \\ \tfrac{1}{6}x - \tfrac{1}{5}y + \tfrac{1}{4}z - \tfrac{1}{3}w &= 1 \\ \tfrac{1}{5}x + \tfrac{1}{4}y - \tfrac{1}{3}z + \tfrac{1}{2}w &= 1 \end{aligned}$$

**Number of Solutions** **In Exercises 61–64, state why the system of equations must have at least one solution. Then solve the system and determine whether it has exactly one solution or infinitely many solutions.**

**61.**
$$\begin{aligned} 4x + 3y + 17z &= 0 \\ 5x + 4y + 22z &= 0 \\ 4x + 2y + 19z &= 0 \end{aligned}$$

**62.**
$$\begin{aligned} 2x + 3y &= 0 \\ 4x + 3y - z &= 0 \\ 8x + 3y + 3z &= 0 \end{aligned}$$

**63.**
$$\begin{aligned} 5x + 5y - z &= 0 \\ 10x + 5y + 2z &= 0 \\ 5x + 15y - 9z &= 0 \end{aligned}$$

**64.**
$$\begin{aligned} 12x + 5y + z &= 0 \\ 12x + 4y - z &= 0 \end{aligned}$$

**65.** **Nutrition** One eight-ounce glass of apple juice and one eight-ounce glass of orange juice contain a total of 177.4 milligrams of vitamin C. Two eight-ounce glasses of apple juice and three eight-ounce glasses of orange juice contain a total of 436.7 milligrams of vitamin C. How much vitamin C is in an eight-ounce glass of each type of juice?

**66.** **Airplane Speed** Two planes start from Los Angeles International Airport and fly in opposite directions. The second plane starts $\frac{1}{2}$ hour after the first plane, but its speed is 80 kilometers per hour faster. Find the airspeed of each plane if 2 hours after the first plane departs, the planes are 3200 kilometers apart.

**True or False?** **In Exercises 67 and 68, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.**

**67.** (a) A system of one linear equation in two variables is always consistent.

(b) A system of two linear equations in three variables is always consistent.

(c) If a linear system is consistent, then it has infinitely many solutions.

**68.** (a) A linear system can have exactly two solutions.

(b) Two systems of linear equations are equivalent when they have the same solution set.

(c) A system of three linear equations in two variables is always inconsistent.

**69.** Find a system of two equations in two variables, $x_1$ and $x_2$, that has the solution set given by the parametric representation $x_1 = t$ and $x_2 = 3t - 4$, where $t$ is any real number. Then show that the solutions to the system can also be written as

$$x_1 = \frac{4}{3} + \frac{t}{3} \quad \text{and} \quad x_2 = t.$$

**70.** Find a system of two equations in three variables, $x_1$, $x_2$, and $x_3$, that has the solution set given by the parametric representation

$$x_1 = t, \quad x_2 = s, \quad \text{and} \quad x_3 = 3 + s - t$$

where $s$ and $t$ are any real numbers. Then show that the solutions to the system can also be written as

$$x_1 = 3 + s - t, \quad x_2 = s, \quad \text{and} \quad x_3 = t.$$

---

The symbol indicates that electronic data sets for these exercises are available at *www.cengagebrain.com*. These data sets are compatible with each of the following technologies: MATLAB, *Mathematica*, *Maple*, TI-83 Plus, TI-84 Plus, TI-89, Voyage 200.

**Substitution** **In Exercises 71–74, solve the system of equations by letting $A = 1/x$, $B = 1/y$, and $C = 1/z$.**

**71.** $\frac{12}{x} - \frac{12}{y} = 7$
$\frac{3}{x} + \frac{4}{y} = 0$

**72.** $\frac{2}{x} + \frac{3}{y} = 0$
$\frac{3}{x} - \frac{4}{y} = -\frac{25}{6}$

**73.** $\frac{2}{x} + \frac{1}{y} - \frac{3}{z} = 4$
$\frac{4}{x} + \frac{2}{z} = 10$
$-\frac{2}{x} + \frac{3}{y} - \frac{13}{z} = -8$

**74.** $\frac{2}{x} + \frac{1}{y} - \frac{2}{z} = 5$
$\frac{3}{x} - \frac{4}{y} = -1$
$\frac{2}{x} + \frac{1}{y} + \frac{3}{z} = 0$

**Trigonometric Coefficients** **In Exercises 75 and 76, solve the system of linear equations for $x$ and $y$.**

**75.** $(\cos\theta)x + (\sin\theta)y = 1$
$(-\sin\theta)x + (\cos\theta)y = 0$

**76.** $(\cos\theta)x + (\sin\theta)y = 1$
$(-\sin\theta)x + (\cos\theta)y = 1$

**Coefficient Design** **In Exercises 77–82, determine the value(s) of $k$ such that the system of linear equations has the indicated number of solutions.**

**77.** Infinitely many solutions
$4x + ky = 6$
$kx + y = -3$

**78.** Infinitely many solutions
$kx + y = 4$
$2x - 3y = -12$

**79.** Exactly one solution
$x + ky = 0$
$kx + y = 0$

**80.** No solution
$x + ky = 2$
$kx + y = 4$

**81.** No solution
$x + 2y + kz = 6$
$3x + 6y + 8z = 4$

**82.** Exactly one solution
$kx + 2ky + 3kz = 4k$
$x + y + z = 0$
$2x - y + z = 1$

**83.** Determine the values of $k$ such that the system of linear equations does not have a unique solution.
$x + y + kz = 3$
$x + ky + z = 2$
$kx + y + z = 1$

**84.** **CAPSTONE** Find values of $a$, $b$, and $c$ such that the system of linear equations has (a) exactly one solution, (b) infinitely many solutions, and (c) no solution. Explain your reasoning.
$x + 5y + z = 0$
$x + 6y - z = 0$
$2x + ay + bz = c$

**85.** **Writing** Consider the system of linear equations in $x$ and $y$.
$a_1x + b_1y = c_1$
$a_2x + b_2y = c_2$
$a_3x + b_3y = c_3$
Describe the graphs of these three equations in the $xy$-plane when the system has (a) exactly one solution, (b) infinitely many solutions, and (c) no solution.

**86.** **Writing** Explain why the system of linear equations in Exercise 85 must be consistent when the constant terms $c_1$, $c_2$, and $c_3$ are all zero.

**87.** Show that if $ax^2 + bx + c = 0$ for all $x$, then $a = b = c = 0$.

**88.** Consider the system of linear equations in $x$ and $y$.
$ax + by = e$
$cx + dy = f$
Under what conditions will the system have exactly one solution?

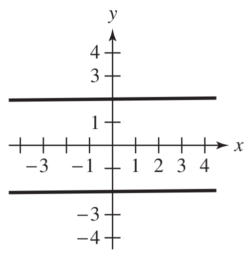
**Discovery** **In Exercises 89 and 90, sketch the lines represented by the system of equations. Then use Gaussian elimination to solve the system. At each step of the elimination process, sketch the corresponding lines. What do you observe about the lines?**

**89.** $x - 4y = -3$
$5x - 6y = 13$

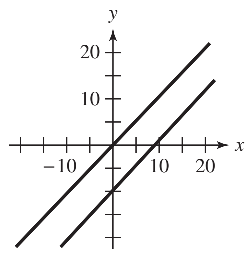
**90.** $2x - 3y = 7$
$-4x + 6y = -14$

**Writing** **In Exercises 91 and 92, the graphs of the two equations appear to be parallel. Solve the system of equations algebraically. Explain why the graphs are misleading.**

**91.** $100y - x = 200$
$99y - x = -198$




**92.** $21x - 20y = 0$
$13x - 12y = 120$



# 1.2 Gaussian Elimination and Gauss-Jordan Elimination

- Determine the size of a matrix and write an augmented or coefficient matrix from a system of linear equations.
- Use matrices and Gaussian elimination with back-substitution to solve a system of linear equations.
- Use matrices and Gauss-Jordan elimination to solve a system of linear equations.
- Solve a homogeneous system of linear equations.

## MATRICES

Section 1.1 introduced Gaussian elimination as a procedure for solving a system of linear equations. In this section, you will study this procedure more thoroughly, beginning with some definitions. The first is the definition of a **matrix.**

**REMARK**

The plural of matrix is *matrices*. If each entry of a matrix is a *real* number, then the matrix is called a **real matrix.** Unless stated otherwise, assume all matrices in this text are real matrices.

### Definition of a Matrix

If $m$ and $n$ are positive integers, an $m \times n$ (read "$m$ by $n$") matrix is a rectangular array

$$\begin{array}{c} \\ \text{Row 1} \\ \text{Row 2} \\ \text{Row 3} \\ \vdots \\ \text{Row } m \end{array} \begin{array}{c} \begin{array}{ccccc} \text{Column 1} & \text{Column 2} & \text{Column 3} & \cdots & \text{Column } n \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \end{array}$$

in which each **entry,** $a_{ij}$, of the matrix is a number. An $m \times n$ matrix has $m$ rows and $n$ columns. Matrices are usually denoted by capital letters.

The entry $a_{ij}$ is located in the $i$th row and the $j$th column. The index $i$ is called the **row subscript** because it identifies the row in which the entry lies, and the index $j$ is called the **column subscript** because it identifies the column in which the entry lies.

A matrix with $m$ rows and $n$ columns is said to be of **size** $m \times n$. When $m = n$, the matrix is called **square** of **order** $n$ and the entries $a_{11}, a_{22}, a_{33}, \ldots$ are called the **main diagonal** entries.

**EXAMPLE 1** **Sizes of Matrices**

Each matrix has the indicated size.

**a.** Size: $1 \times 1$ $\quad [2]$ **b.** Size: $2 \times 2$ $\quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ **c.** Size: $2 \times 3$ $\quad \begin{bmatrix} e & 2 & -7 \\ \pi & \sqrt{2} & 4 \end{bmatrix}$

**REMARK**

Begin by aligning the variables in the equations vertically. Use 0 to indicate coefficients of zero in the matrix. Note the fourth column of constant terms in the augmented matrix.

One common use of matrices is to represent systems of linear equations. The matrix derived from the coefficients and constant terms of a system of linear equations is called the **augmented matrix** of the system. The matrix containing only the coefficients of the system is called the **coefficient matrix** of the system. Here is an example.

$$\begin{array}{ccc} \text{System} & \text{Augmented Matrix} & \text{Coefficient Matrix} \\ \begin{aligned} x - 4y + 3z &= 5 \\ -x + 3y - z &= -3 \\ 2x \phantom{+3y} - 4z &= 6 \end{aligned} & \begin{bmatrix} 1 & -4 & 3 & 5 \\ -1 & 3 & -1 & -3 \\ 2 & 0 & -4 & 6 \end{bmatrix} & \begin{bmatrix} 1 & -4 & 3 \\ -1 & 3 & -1 \\ 2 & 0 & -4 \end{bmatrix} \end{array}$$

## ELEMENTARY ROW OPERATIONS

In the previous section, you studied three operations that produce equivalent systems of linear equations.

**1.** Interchange two equations.

**2.** Multiply an equation by a nonzero constant.

**3.** Add a multiple of an equation to another equation.

In matrix terminology, these three operations correspond to **elementary row operations.** An elementary row operation on an augmented matrix produces a new augmented matrix corresponding to a new (but equivalent) system of linear equations. Two matrices are **row-equivalent** when one can be obtained from the other by a finite sequence of elementary row operations.

### Elementary Row Operations

**1.** Interchange two rows.
**2.** Multiply a row by a nonzero constant.
**3.** Add a multiple of a row to another row.

Although elementary row operations are simple to perform, they involve a lot of arithmetic. Because it is easy to make a mistake, you should get in the habit of noting the elementary row operations performed in each step so that checking your work is easier.

Because solving some systems involves several steps, it is helpful to use a shorthand method of notation to keep track of each elementary row operation you perform. The next example introduces this notation.

### TECHNOLOGY

Many graphing utilities and software programs can perform elementary row operations on matrices. If you use a graphing utility, you may see something similar to the following for Example 2(c). The **Online Technology Guide,** available at *www.cengagebrain.com*, provides syntax for programs applicable to Example 2(c).

```
A
          [[1  2  -4  3 ]
           [0  3  -2  -1]
           [2  1  5   -2]]

mRAdd(-2,A,1,3)
          [[1  2   -4  3 ]
           [0  3   -2  -1]
           [0  -3  13  -8]]
```

### EXAMPLE 2 Elementary Row Operations

**a.** Interchange the first and second rows.

| Original Matrix | New Row-Equivalent Matrix | Notation |
|---|---|---|
| $\begin{bmatrix} 0 & 1 & 3 & 4 \\ -1 & 2 & 0 & 3 \\ 2 & -3 & 4 & 1 \end{bmatrix}$ | $\begin{bmatrix} -1 & 2 & 0 & 3 \\ 0 & 1 & 3 & 4 \\ 2 & -3 & 4 & 1 \end{bmatrix}$ | $R_1 \leftrightarrow R_2$ |

**b.** Multiply the first row by $\frac{1}{2}$ to produce a new first row.

| Original Matrix | New Row-Equivalent Matrix | Notation |
|---|---|---|
| $\begin{bmatrix} 2 & -4 & 6 & -2 \\ 1 & 3 & -3 & 0 \\ 5 & -2 & 1 & 2 \end{bmatrix}$ | $\begin{bmatrix} 1 & -2 & 3 & -1 \\ 1 & 3 & -3 & 0 \\ 5 & -2 & 1 & 2 \end{bmatrix}$ | $\left(\frac{1}{2}\right)R_1 \rightarrow R_1$ |

**c.** Add $-2$ times the first row to the third row to produce a new third row.

| Original Matrix | New Row-Equivalent Matrix | Notation |
|---|---|---|
| $\begin{bmatrix} 1 & 2 & -4 & 3 \\ 0 & 3 & -2 & -1 \\ 2 & 1 & 5 & -2 \end{bmatrix}$ | $\begin{bmatrix} 1 & 2 & -4 & 3 \\ 0 & 3 & -2 & -1 \\ 0 & -3 & 13 & -8 \end{bmatrix}$ | $R_3 + (-2)R_1 \rightarrow R_3$ |

Notice that adding $-2$ times row 1 to row 3 does not change row 1.

In Example 7 in Section 1.1, you used Gaussian elimination with back-substitution to solve a system of linear equations. The next example demonstrates the matrix version of Gaussian elimination. The two methods are essentially the same. The basic difference is that with matrices you do not need to keep writing the variables.

### EXAMPLE 3 Using Elementary Row Operations to Solve a System

| Linear System | Associated Augmented Matrix |
|---|---|
| $\begin{aligned} x - 2y + 3z &= 9 \\ -x + 3y \phantom{+ 3z} &= -4 \\ 2x - 5y + 5z &= 17 \end{aligned}$ | $\begin{bmatrix} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{bmatrix}$ |
| Add the first equation to the second equation. | Add the first row to the second row to produce a new second row. |
| $\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ 2x - 5y + 5z &= 17 \end{aligned}$ | $\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 2 & -5 & 5 & 17 \end{bmatrix}$ $R_2 + R_1 \rightarrow R_2$ |
| Add $-2$ times the first equation to the third equation. | Add $-2$ times the first row to the third row to produce a new third row. |
| $\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ -y - z &= -1 \end{aligned}$ | $\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{bmatrix}$ $R_3 + (-2)R_1 \rightarrow R_3$ |
| Add the second equation to the third equation. | Add the second row to the third row to produce a new third row. |
| $\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ 2z &= 4 \end{aligned}$ | $\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{bmatrix}$ $R_3 + R_2 \rightarrow R_3$ |
| Multiply the third equation by $\frac{1}{2}$. | Multiply the third row by $\frac{1}{2}$ to produce a new third row. |
| $\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ z &= 2 \end{aligned}$ | $\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ $\left(\frac{1}{2}\right)R_3 \rightarrow R_3$ |

Use back-substitution to find the solution, as in Example 6 in Section 1.1. The solution is $x = 1$, $y = -1$, and $z = 2$.

The last matrix in Example 3 is said to be in **row-echelon** form. The term *echelon* refers to the stair-step pattern formed by the nonzero elements of the matrix. To be in this form, a matrix must have the following properties.

#### Row-Echelon Form and Reduced Row-Echelon Form

A matrix in **row-echelon form** has the following properties.

1. Any rows consisting entirely of zeros occur at the bottom of the matrix.
2. For each row that does not consist entirely of zeros, the first nonzero entry is 1 (called a **leading 1**).
3. For two successive (nonzero) rows, the leading 1 in the higher row is farther to the left than the leading 1 in the lower row.

A matrix in row-echelon form is in **reduced row-echelon form** when every column that has a leading 1 has zeros in every position above and below its leading 1.

## EXAMPLE 4 Row-Echelon Form

Determine whether each matrix is in row-echelon form. If it is, determine whether the matrix is in reduced row-echelon form.

**a.** $\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$ **b.** $\begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & -4 \end{bmatrix}$

**c.** $\begin{bmatrix} 1 & -5 & 2 & -1 & 3 \\ 0 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ **d.** $\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

**e.** $\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & -3 \end{bmatrix}$ **f.** $\begin{bmatrix} 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

### SOLUTION

The matrices in (a), (c), (d), and (f) are in row-echelon form. The matrices in (d) and (f) are in *reduced* row-echelon form because every column that has a leading 1 has zeros in every position above and below its leading 1. The matrix in (b) is not in row-echelon form because a row of all zeros does not occur at the bottom of the matrix. The matrix in (e) is not in row-echelon form because the first nonzero entry in Row 2 is not a leading 1.

### TECHNOLOGY

Use a graphing utility or a software program to find the reduced row-echelon form of the matrix in Example 4(b). The **Online Technology Guide,** available at *www.cengagebrain.com*, provides syntax for programs applicable to Example 4(b). Similar exercises and projects are also available on the website.

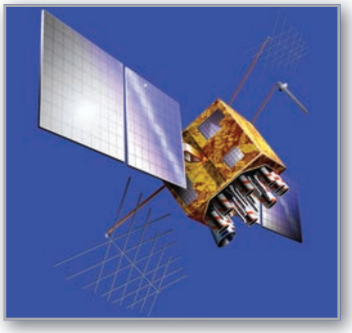
Every matrix is row-equivalent to a matrix in row-echelon form. For instance, in Example 4(e), multiplying the second row in the matrix by $\frac{1}{2}$ changes the matrix to row-echelon form.

The following summarizes the procedure for using Gaussian elimination with back-substitution.

### Gaussian Elimination with Back-Substitution

1. Write the augmented matrix of the system of linear equations.
2. Use elementary row operations to rewrite the matrix in row-echelon form.
3. Write the system of linear equations corresponding to the matrix in row-echelon form, and use back-substitution to find the solution.

Gaussian elimination with back-substitution works well for solving systems of linear equations by hand or with a computer. For this algorithm, the order in which you perform the elementary row operations is important. Operate from *left to right by columns*, using elementary row operations to obtain zeros in all entries directly below the leading 1's.



### LINEAR ALGEBRA APPLIED

The Global Positioning System (GPS) is a network of 24 satellites originally developed by the U.S. military as a navigational tool. Today, GPS receivers are used in a wide variety of civilian applications, such as determining directions, locating vessels lost at sea, and monitoring earthquakes. A GPS receiver works by using satellite readings to calculate its location. In three dimensions, the receiver uses signals from at least four satellites to "trilaterate" its position. In a simplified mathematical model, a system of three linear equations in four unknowns (three dimensions and time) is used to determine the coordinates of the receiver as functions of time.

**EXAMPLE 5** **Gaussian Elimination with Back-Substitution**

Solve the system.

$$\begin{aligned} x_2 + x_3 - 2x_4 &= -3 \\ x_1 + 2x_2 - x_3 \qquad &= 2 \\ 2x_1 + 4x_2 + x_3 - 3x_4 &= -2 \\ x_1 - 4x_2 - 7x_3 - x_4 &= -19 \end{aligned}$$

**SOLUTION**

The augmented matrix for this system is

$$\begin{bmatrix} 0 & 1 & 1 & -2 & -3 \\ 1 & 2 & -1 & 0 & 2 \\ 2 & 4 & 1 & -3 & -2 \\ 1 & -4 & -7 & -1 & -19 \end{bmatrix}.$$

Obtain a leading 1 in the upper left corner and zeros elsewhere in the first column.

$$\begin{bmatrix} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 2 & 4 & 1 & -3 & -2 \\ 1 & -4 & -7 & -1 & -19 \end{bmatrix}$$

Interchange the first two rows. $R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 3 & -3 & -6 \\ 1 & -4 & -7 & -1 & -19 \end{bmatrix}$$

Adding $-2$ times the first row to the third row produces a new third row. $R_3 + (-2)R_1 \rightarrow R_3$

$$\begin{bmatrix} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 3 & -3 & -6 \\ 0 & -6 & -6 & -1 & -21 \end{bmatrix}$$

Adding $-1$ times the first row to the fourth row produces a new fourth row. $R_4 + (-1)R_1 \rightarrow R_4$

Now that the first column is in the desired form, change the second column as follows.

$$\begin{bmatrix} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 3 & -3 & -6 \\ 0 & 0 & 0 & -13 & -39 \end{bmatrix}$$

Adding 6 times the second row to the fourth row produces a new fourth row. $R_4 + (6)R_2 \rightarrow R_4$

To write the third and fourth columns in proper form, multiply the third row by $\frac{1}{3}$ and the fourth row by $-\frac{1}{13}$.

$$\begin{bmatrix} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}$$

Multiplying the third row by $\frac{1}{3}$ and the fourth row by $-\frac{1}{13}$ produces new third and fourth rows. $\left(\frac{1}{3}\right)R_3 \rightarrow R_3$ $\left(-\frac{1}{13}\right)R_4 \rightarrow R_4$

The matrix is now in row-echelon form, and the corresponding system is as follows.

$$\begin{aligned} x_1 + 2x_2 - x_3 \qquad &= 2 \\ x_2 + x_3 - 2x_4 &= -3 \\ x_3 - x_4 &= -2 \\ x_4 &= 3 \end{aligned}$$

Use back-substitution to find that the solution is $x_1 = -1$, $x_2 = 2$, $x_3 = 1$, and $x_4 = 3$.

When solving a system of linear equations, remember that it is possible for the system to have no solution. If, in the elimination process, you obtain a row of all zeros except for the last entry, then it is unnecessary to continue the elimination process. You can simply conclude that the system has no solution, or is *inconsistent*.

## EXAMPLE 6 A System with No Solution

Solve the system.

$$\begin{aligned} x_1 - \phantom{2}x_2 + 2x_3 &= 4 \\ x_1 \phantom{- 3x_2} + \phantom{5}x_3 &= 6 \\ 2x_1 - 3x_2 + 5x_3 &= 4 \\ 3x_1 + 2x_2 - \phantom{5}x_3 &= 1 \end{aligned}$$

**SOLUTION**

The augmented matrix for this system is

$$\begin{bmatrix} 1 & -1 & 2 & 4 \\ 1 & 0 & 1 & 6 \\ 2 & -3 & 5 & 4 \\ 3 & 2 & -1 & 1 \end{bmatrix}.$$

Apply Gaussian elimination to the augmented matrix.

$$\begin{bmatrix} 1 & -1 & 2 & 4 \\ 0 & 1 & -1 & 2 \\ 2 & -3 & 5 & 4 \\ 3 & 2 & -1 & 1 \end{bmatrix} \quad R_2 + (-1)R_1 \rightarrow R_2$$

$$\begin{bmatrix} 1 & -1 & 2 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & -1 & 1 & -4 \\ 3 & 2 & -1 & 1 \end{bmatrix} \quad R_3 + (-2)R_1 \rightarrow R_3$$

$$\begin{bmatrix} 1 & -1 & 2 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & -1 & 1 & -4 \\ 0 & 5 & -7 & -11 \end{bmatrix} \quad R_4 + (-3)R_1 \rightarrow R_4$$

$$\begin{bmatrix} 1 & -1 & 2 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & -2 \\ 0 & 5 & -7 & -11 \end{bmatrix} \quad R_3 + R_2 \rightarrow R_3$$

Note that the third row of this matrix consists entirely of zeros except for the last entry. This means that the original system of linear equations is *inconsistent*. You can see why this is true by converting back to a system of linear equations.

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 4 \\ x_2 - x_3 &= 2 \\ 0 &= -2 \\ 5x_2 - 7x_3 &= -11 \end{aligned}$$

Because the third equation is not possible, the system has no solution.

## GAUSS-JORDAN ELIMINATION

With Gaussian elimination, you apply elementary row operations to a matrix to obtain a (row-equivalent) row-echelon form. A second method of elimination, called **Gauss-Jordan elimination,** after Carl Friedrich Gauss and Wilhelm Jordan (1842–1899), continues the reduction process until a *reduced* row-echelon form is obtained. Example 7 demonstrates this procedure.

### EXAMPLE 7 Gauss-Jordan Elimination

Use Gauss-Jordan elimination to solve the system.

$$\begin{aligned} x - 2y + 3z &= 9 \\ -x + 3y \phantom{+ 3z} &= -4 \\ 2x - 5y + 5z &= 17 \end{aligned}$$

**SOLUTION**

In Example 3, you used Gaussian elimination to obtain the row-echelon form

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

Now, apply elementary row operations until you obtain zeros above each of the leading 1's, as follows.

$$\begin{bmatrix} 1 & 0 & 9 & 19 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix} \qquad R_1 + (2)R_2 \to R_1$$

$$\begin{bmatrix} 1 & 0 & 9 & 19 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \qquad R_2 + (-3)R_3 \to R_2$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \qquad R_1 + (-9)R_3 \to R_1$$

The matrix is now in reduced row-echelon form. Converting back to a system of linear equations, you have

$$\begin{aligned} x &= 1 \\ y &= -1 \\ z &= 2. \end{aligned}$$

The elimination procedures described in this section sometimes result in fractional coefficients. For instance, in the elimination procedure for the system

$$\begin{aligned} 2x - 5y + 5z &= 17 \\ 3x - 2y + 3z &= 11 \\ -3x + 3y \phantom{+ 3z} &= -16 \end{aligned}$$

you may be inclined to multiply the first row by $\frac{1}{2}$ to produce a leading 1, which will result in working with fractional coefficients. Sometimes, judiciously choosing the order in which you apply elementary row operations enables you to avoid fractions.

**REMARK**

No matter which order you use, the reduced row-echelon form will be the same.

## DISCOVERY

1. Without doing any row operations, explain why the following system of linear equations is consistent.

$$\begin{aligned} 2x_1 + 3x_2 + 5x_3 &= 0 \\ -5x_1 + 6x_2 - 17x_3 &= 0 \\ 7x_1 - 4x_2 + 3x_3 &= 0 \end{aligned}$$

2. The following system has more variables than equations. Why does it have an infinite number of solutions?

$$\begin{aligned} 2x_1 + 3x_2 + 5x_3 + 2x_4 &= 0 \\ -5x_1 + 6x_2 - 17x_3 - 3x_4 &= 0 \\ 7x_1 - 4x_2 + 3x_3 + 13x_4 &= 0 \end{aligned}$$

The next example demonstrates how Gauss-Jordan elimination can be used to solve a system with infinitely many solutions.

### EXAMPLE 8 A System with Infinitely Many Solutions

Solve the system of linear equations.

$$\begin{aligned} 2x_1 + 4x_2 - 2x_3 &= 0 \\ 3x_1 + 5x_2 \qquad &= 1 \end{aligned}$$

**SOLUTION**

The augmented matrix for this system is

$$\begin{bmatrix} 2 & 4 & -2 & 0 \\ 3 & 5 & 0 & 1 \end{bmatrix}.$$

Using a graphing utility, a software program, or Gauss-Jordan elimination, verify that the reduced row-echelon form of the matrix is

$$\begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & -3 & -1 \end{bmatrix}.$$

The corresponding system of equations is

$$\begin{aligned} x_1 \qquad + 5x_3 &= 2 \\ x_2 - 3x_3 &= -1. \end{aligned}$$

Now, using the parameter $t$ to represent $x_3$, you have

$$x_1 = 2 - 5t, \quad x_2 = -1 + 3t, \quad x_3 = t, \quad \text{where } t \text{ is any real number.}$$

Note that in Example 8 an arbitrary parameter was assigned to the *nonleading* variable $x_3$. You subsequently solved for the leading variables $x_1$ and $x_2$ as functions of $t$.

You have looked at two elimination methods for solving a system of linear equations. Which is better? To some degree the answer depends on personal preference. In real-life applications of linear algebra, systems of linear equations are usually solved by computer. Most computer programs use a form of Gaussian elimination, with special emphasis on ways to reduce rounding errors and minimize storage of data. Because the examples and exercises in this text focus on the underlying concepts, you should know both elimination methods.

## HOMOGENEOUS SYSTEMS OF LINEAR EQUATIONS

Systems of linear equations in which each of the constant terms is zero are called **homogeneous.** A homogeneous system of $m$ equations in $n$ variables has the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= 0. \end{aligned}$$

A homogeneous system must have at least one solution. Specifically, if all variables in a homogeneous system have the value zero, then each of the equations must be satisfied. Such a solution is called **trivial** (or **obvious**).

**REMARK**

A homogeneous system of three equations in the three variables $x_1$, $x_2$, and $x_3$ must have $x_1 = 0$, $x_2 = 0$, and $x_3 = 0$ as a trivial solution.

### EXAMPLE 9 Solving a Homogeneous System of Linear Equations

Solve the system of linear equations.

$$\begin{aligned} x_1 - x_2 + 3x_3 &= 0 \\ 2x_1 + x_2 + 3x_3 &= 0 \end{aligned}$$

**SOLUTION**

Applying Gauss-Jordan elimination to the augmented matrix

$$\begin{bmatrix} 1 & -1 & 3 & 0 \\ 2 & 1 & 3 & 0 \end{bmatrix}$$

yields the following.

$$\begin{bmatrix} 1 & -1 & 3 & 0 \\ 0 & 3 & -3 & 0 \end{bmatrix} \qquad R_2 + (-2)R_1 \to R_2$$

$$\begin{bmatrix} 1 & -1 & 3 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix} \qquad \left(\tfrac{1}{3}\right)R_2 \to R_2$$

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix} \qquad R_1 + R_2 \to R_1$$

The system of equations corresponding to this matrix is

$$\begin{aligned} x_1 \qquad + 2x_3 &= 0 \\ x_2 - \phantom{2}x_3 &= 0. \end{aligned}$$

Using the parameter $t = x_3$, the solution set is $x_1 = -2t$, $x_2 = t$, and $x_3 = t$, where $t$ is any real number.

This system has infinitely many solutions, one of which is the trivial solution $(t = 0)$.

As illustrated in Example 9, a homogeneous system with fewer equations than variables has infinitely many solutions.

**THEOREM 1.1 The Number of Solutions of a Homogeneous System**

Every homogeneous system of linear equations is consistent. Moreover, if the system has fewer equations than variables, then it must have infinitely many solutions.

To prove Theorem 1.1, use the procedure in Example 9, but for a general matrix.

## 1.2 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

**Matrix Size** In Exercises 1–6, determine the size of the matrix.

**1.** $\begin{bmatrix} 1 & 2 & -4 \\ 3 & -4 & 6 \\ 0 & 1 & 2 \end{bmatrix}$ **2.** $\begin{bmatrix} 1 \\ 2 \\ -1 \\ -2 \end{bmatrix}$

**3.** $\begin{bmatrix} 2 & -1 & -1 & 1 \\ -6 & 2 & 0 & 1 \end{bmatrix}$ **4.** $\begin{bmatrix} -1 \end{bmatrix}$

**5.** $\begin{bmatrix} 8 & 6 & 4 & 1 & 3 \\ 2 & 1 & -7 & 4 & 1 \\ 1 & 1 & -1 & 2 & 1 \\ 1 & -1 & 2 & 0 & 0 \end{bmatrix}$

**6.** $\begin{bmatrix} 1 & 2 & 3 & 4 & -10 \end{bmatrix}$

**Elementary Row Operations** In Exercises 7–10, identify the elementary row operation(s) being performed to obtain the new row-equivalent matrix.

*Original Matrix* *New Row-Equivalent Matrix*

**7.** $\begin{bmatrix} -2 & 5 & 1 \\ 3 & -1 & -8 \end{bmatrix}$ $\begin{bmatrix} 13 & 0 & -39 \\ 3 & -1 & -8 \end{bmatrix}$

*Original Matrix* *New Row-Equivalent Matrix*

**8.** $\begin{bmatrix} 3 & -1 & -4 \\ -4 & 3 & 7 \end{bmatrix}$ $\begin{bmatrix} 3 & -1 & -4 \\ 5 & 0 & -5 \end{bmatrix}$

*Original Matrix* *New Row-Equivalent Matrix*

**9.** $\begin{bmatrix} 0 & -1 & -5 & 5 \\ -1 & 3 & -7 & 6 \\ 4 & -5 & 1 & 3 \end{bmatrix}$ $\begin{bmatrix} -1 & 3 & -7 & 6 \\ 0 & -1 & -5 & 5 \\ 0 & 7 & -27 & 27 \end{bmatrix}$

*Original Matrix* *New Row-Equivalent Matrix*

**10.** $\begin{bmatrix} -1 & -2 & 3 & -2 \\ 2 & -5 & 1 & -7 \\ 5 & 4 & -7 & 6 \end{bmatrix}$ $\begin{bmatrix} -1 & -2 & 3 & -2 \\ 0 & -9 & 7 & -11 \\ 0 & -6 & 8 & -4 \end{bmatrix}$

**Augmented Matrix** In Exercises 11–18, find the solution set of the system of linear equations represented by the augmented matrix.

**11.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}$ **12.** $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}$

**13.** $\begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$ **14.** $\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

**15.** $\begin{bmatrix} 2 & 1 & -1 & 3 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix}$ **16.** $\begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & -2 & 1 & -2 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

**17.** $\begin{bmatrix} 1 & 2 & 0 & 1 & 4 \\ 0 & 1 & 2 & 1 & 3 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$

**18.** $\begin{bmatrix} 1 & 2 & 0 & 1 & 3 \\ 0 & 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$

**Row-Echelon Form** In Exercises 19–24, determine whether the matrix is in row-echelon form. If it is, determine whether it is also in reduced row-echelon form.

**19.** $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

**20.** $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 1 \end{bmatrix}$

**21.** $\begin{bmatrix} 2 & 0 & 1 & 3 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

**22.** $\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

**23.** $\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{bmatrix}$

**24.** $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

**System of Linear Equations** In Exercises 25–38, solve the system using either Gaussian elimination with back-substitution or Gauss-Jordan elimination.

**25.** $\begin{aligned} x + 2y &= 7 \\ 2x + y &= 8 \end{aligned}$ **26.** $\begin{aligned} 2x + 6y &= 16 \\ -2x - 6y &= -16 \end{aligned}$

**27.** $\begin{aligned} -x + 2y &= 1.5 \\ 2x - 4y &= 3 \end{aligned}$

**28.** $\begin{aligned} 2x - y &= -0.1 \\ 3x + 2y &= 1.6 \end{aligned}$

**29.** $\begin{aligned} -3x + 5y &= -22 \\ 3x + 4y &= 4 \\ 4x - 8y &= 32 \end{aligned}$

**30.** $\begin{aligned} x + 2y &= 0 \\ x + y &= 6 \\ 3x - 2y &= 8 \end{aligned}$

**31.** $\begin{aligned} x_1 \phantom{{}+x_2} - 3x_3 &= -2 \\ 3x_1 + x_2 - 2x_3 &= 5 \\ 2x_1 + 2x_2 + x_3 &= 4 \end{aligned}$

**32.** $\begin{aligned} 2x_1 - x_2 + 3x_3 &= 24 \\ 2x_2 - x_3 &= 14 \\ 7x_1 - 5x_2 &= 6 \end{aligned}$

**33.** $\begin{aligned} 2x_1 + 3x_3 &= 3 \\ 4x_1 - 3x_2 + 7x_3 &= 5 \\ 8x_1 - 9x_2 + 15x_3 &= 10 \end{aligned}$

**34.** $\begin{aligned} x_1 + x_2 - 5x_3 &= 3 \\ x_1 - 2x_3 &= 1 \\ 2x_1 - x_2 - x_3 &= 0 \end{aligned}$

**35.** $\begin{aligned} 4x + 12y - 7z - 20w &= 22 \\ 3x + 9y - 5z - 28w &= 30 \end{aligned}$

**36.** $\begin{aligned} x + 2y + z &= 8 \\ -3x - 6y - 3z &= -21 \end{aligned}$

**37.** $\begin{aligned} 3x + 3y + 12z &= 6 \\ x + y + 4z &= 2 \\ 2x + 5y + 20z &= 10 \\ -x + 2y + 8z &= 4 \end{aligned}$

**38.** $\begin{aligned} 2x + y - z + 2w &= -6 \\ 3x + 4y + w &= 1 \\ x + 5y + 2z + 6w &= -3 \\ 5x + 2y - z - w &= 3 \end{aligned}$

**System of Linear Equations** **In Exercises 39 and 40, use a software program or a graphing utility to solve the system of linear equations.**

**39.** $\begin{aligned} x_1 - x_2 + 2x_3 + 2x_4 + 6x_5 &= 6 \\ 3x_1 - 2x_2 + 4x_3 + 4x_4 + 12x_5 &= 14 \\ x_2 - x_3 - x_4 - 3x_5 &= -3 \\ 2x_1 - 2x_2 + 4x_3 + 5x_4 + 15x_5 &= 10 \\ 2x_1 - 2x_2 + 4x_3 + 4x_4 + 13x_5 &= 13 \end{aligned}$

**40.** $\begin{aligned} x_1 + 2x_2 - 2x_3 + 2x_4 - x_5 + 3x_6 &= 0 \\ 2x_1 - x_2 + 3x_3 + x_4 - 3x_5 + 2x_6 &= 17 \\ x_1 + 3x_2 - 2x_3 + x_4 - 2x_5 - 3x_6 &= -5 \\ 3x_1 - 2x_2 + x_3 - x_4 + 3x_5 - 2x_6 &= -1 \\ -x_1 - 2x_2 + x_3 + 2x_4 - 2x_5 + 3x_6 &= 10 \\ x_1 - 3x_2 + x_3 + 3x_4 - 2x_5 + x_6 &= 11 \end{aligned}$

**Homogeneous System** **In Exercises 41–44, solve the homogeneous linear system corresponding to the given coefficient matrix.**

**41.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

**42.** $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$

**43.** $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

**44.** $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

**45.** **Finance** A small software corporation borrowed \$500,000 to expand its software line. The corporation borrowed some of the money at 9%, some at 10%, and some at 12%. Use a system of equations to determine how much was borrowed at each rate if the annual interest was \$52,000 and the amount borrowed at 10% was $2\frac{1}{2}$ times the amount borrowed at 9%. Solve the system using matrices.

**46.** **Tips** A food server examines the amount of money earned in tips after working an 8-hour shift. The server has a total of \$95 in denominations of \$1, \$5, \$10, and \$20 bills. The total number of paper bills is 26. The number of \$5 bills is 4 times the number of \$10 bills, and the number of \$1 bills is 1 less than twice the number of \$5 bills. Write a system of linear equations to represent the situation. Then use matrices to find the number of each denomination.

**Matrix Representation** **In Exercises 47 and 48, assume that the matrix is the *augmented* matrix of a system of linear equations, and (a) determine the number of equations and the number of variables, and (b) find the value(s) of $k$ such that the system is consistent. Then assume that the matrix is the *coefficient* matrix of a *homogeneous* system of linear equations, and repeat parts (a) and (b).**

**47.** $A = \begin{bmatrix} 1 & k & 2 \\ -3 & 4 & 1 \end{bmatrix}$

**48.** $A = \begin{bmatrix} 2 & -1 & 3 \\ -4 & 2 & k \\ 4 & -2 & 6 \end{bmatrix}$

**Coefficient Design** **In Exercises 49 and 50, find values of $a$, $b$, and $c$ (if possible) such that the system of linear equations has (a) a unique solution, (b) no solution, and (c) infinitely many solutions.**

**49.** $\begin{aligned} x + y &= 2 \\ y + z &= 2 \\ x + z &= 2 \\ ax + by + cz &= 0 \end{aligned}$

**50.** $\begin{aligned} x + y &= 0 \\ y + z &= 0 \\ x + z &= 0 \\ ax + by + cz &= 0 \end{aligned}$

**51.** The following system has one solution: $x = 1$, $y = -1$, and $z = 2$.

$$\begin{aligned} 4x - 2y + 5z &= 16 \quad &\text{Equation 1}\\ x + y \phantom{+ 5z} &= 0 &\text{Equation 2}\\ -x - 3y + 2z &= 6 &\text{Equation 3} \end{aligned}$$

Solve the systems provided by (a) Equations 1 and 2, (b) Equations 1 and 3, and (c) Equations 2 and 3. (d) How many solutions does each of these systems have?

**52.** Assume the system below has a unique solution.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \quad &\text{Equation 1}\\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 &\text{Equation 2}\\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 &\text{Equation 3} \end{aligned}$$

Does the system composed of Equations 1 and 2 have a unique solution, no solution, or infinitely many solutions?

**Row Equivalence** **In Exercises 53 and 54, find the reduced row-echelon matrix that is row-equivalent to the given matrix.**

**53.** $\begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix}$ **54.** $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

**55.** **Writing** Describe all possible $2 \times 2$ reduced row-echelon matrices. Support your answer with examples.

**56.** **Writing** Describe all possible $3 \times 3$ reduced row-echelon matrices. Support your answer with examples.

**True or False?** **In Exercises 57 and 58, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.**

**57.** (a) A $6 \times 3$ matrix has six rows.

(b) Every matrix is row-equivalent to a matrix in row-echelon form.

(c) If the row-echelon form of the augmented matrix of a system of linear equations contains the row $[1 \ \ 0 \ \ 0 \ \ 0 \ \ 0]$, then the original system is inconsistent.

(d) A homogeneous system of four linear equations in six variables has infinitely many solutions.

**58.** (a) A $4 \times 7$ matrix has four columns.

(b) Every matrix has a unique reduced row-echelon form.

(c) A homogeneous system of four linear equations in four variables is always consistent.

(d) Multiplying a row of a matrix by a constant is one of the elementary row operations.

**59.** **Writing** Is it possible for a system of linear equations with fewer equations than variables to have no solution? If so, give an example.

**60.** **Writing** Does a matrix have a unique row-echelon form? Illustrate your answer with examples. Is the reduced row-echelon form unique?

**Row Equivalence** **In Exercises 61 and 62, determine conditions on $a$, $b$, $c$, and $d$ such that the matrix**

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

**will be row-equivalent to the given matrix.**

**61.** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **62.** $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

**Homogeneous System** **In Exercises 63 and 64, find all values of $\lambda$ (the Greek letter lambda) for which the homogeneous linear system has nontrivial solutions.**

**63.** $\begin{aligned} (\lambda - 2)x + \phantom{(\lambda - 2)}y &= 0 \\ x + (\lambda - 2)y &= 0 \end{aligned}$

**64.** $\begin{aligned} (\lambda - 1)x + 2y &= 0 \\ x + \lambda y &= 0 \end{aligned}$

**65.** **Writing** Consider the $2 \times 2$ matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Perform the sequence of row operations.

(a) Add $(-1)$ times the second row to the first row.

(b) Add 1 times the first row to the second row.

(c) Add $(-1)$ times the second row to the first row.

(d) Multiply the first row by $(-1)$.

What happened to the original matrix? Describe, in general, how to interchange two rows of a matrix using only the second and third elementary row operations.

**66.** The augmented matrix represents a system of linear equations that has been reduced using Gauss-Jordan elimination. Write a system of equations with nonzero coefficients that the reduced matrix could represent.

$$\begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

There are many correct answers.

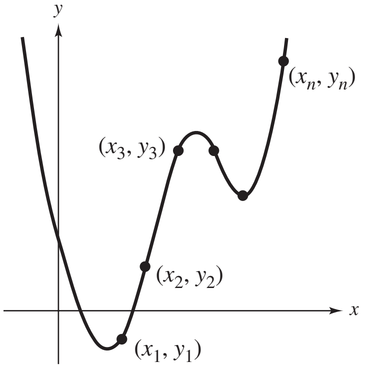
**67.** **Writing** Describe the row-echelon form of an augmented matrix that corresponds to a linear system that (a) is inconsistent, and (b) has infinitely many solutions.

**68.** **CAPSTONE** In your own words, describe the difference between a matrix in row-echelon form and a matrix in reduced row-echelon form. Include an example of each to support your explanation.

# 1.3 Applications of Systems of Linear Equations

- Set up and solve a system of equations to fit a polynomial function to a set of data points.
- Set up and solve a system of equations to represent a network.

Systems of linear equations arise in a wide variety of applications. In this section you will look at two applications, and you will see more in subsequent chapters. The first application shows how to fit a polynomial function to a set of data points in the plane. The second application focuses on networks and Kirchhoff's Laws for electricity.



Polynomial Curve Fitting

**Figure 1.4**

## POLYNOMIAL CURVE FITTING

Suppose $n$ points in the $xy$-plane

$$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$$

represent a collection of data and you are asked to find a polynomial function of degree $n - 1$

$$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1}$$

whose graph passes through the specified points. This procedure is called **polynomial curve fitting.** If all $x$-coordinates of the points are distinct, then there is precisely one polynomial function of degree $n - 1$ (or less) that fits the $n$ points, as shown in Figure 1.4.

To solve for the $n$ coefficients of $p(x)$, substitute each of the $n$ points into the polynomial function and obtain $n$ linear equations in $n$ variables $a_0, a_1, a_2, \ldots, a_{n-1}$.

$$\begin{aligned}
a_0 + a_1x_1 + a_2x_1^2 + \cdots + a_{n-1}x_1^{n-1} &= y_1 \\
a_0 + a_1x_2 + a_2x_2^2 + \cdots + a_{n-1}x_2^{n-1} &= y_2 \\
&\vdots \\
a_0 + a_1x_n + a_2x_n^2 + \cdots + a_{n-1}x_n^{n-1} &= y_n
\end{aligned}$$

Example 1 demonstrates this procedure with a second-degree polynomial.

### EXAMPLE 1 Polynomial Curve Fitting

Determine the polynomial $p(x) = a_0 + a_1x + a_2x^2$ whose graph passes through the points (1, 4), (2, 0), and (3, 12).

**SOLUTION**

Substituting $x = 1$, 2, and 3 into $p(x)$ and equating the results to the respective $y$-values produces the system of linear equations in the variables $a_0$, $a_1$, and $a_2$ shown below.

$$\begin{aligned}
p(1) &= a_0 + a_1(1) + a_2(1)^2 = a_0 + a_1 + a_2 = 4 \\
p(2) &= a_0 + a_1(2) + a_2(2)^2 = a_0 + 2a_1 + 4a_2 = 0 \\
p(3) &= a_0 + a_1(3) + a_2(3)^2 = a_0 + 3a_1 + 9a_2 = 12
\end{aligned}$$

The solution of this system is

$$a_0 = 24,\ a_1 = -28, \text{ and } a_2 = 8$$

so the polynomial function is

$$p(x) = 24 - 28x + 8x^2.$$

Figure 1.5 shows the graph of $p$.

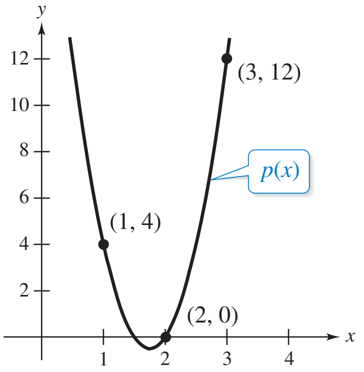



**Figure 1.5**

***Simulation***
Explore this concept further with an electronic simulation available at *www.cengagebrain.com*.

## EXAMPLE 2 Polynomial Curve Fitting

Find a polynomial that fits the points

$$(-2, 3), (-1, 5), (0, 1), (1, 4), \text{ and } (2, 10).$$

### SOLUTION

Because you are given five points, choose a fourth-degree polynomial function

$$p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4.$$

Substituting the given points into $p(x)$ produces the following system of linear equations.

$$\begin{aligned}
a_0 - 2a_1 + 4a_2 - 8a_3 + 16a_4 &= 3\\
a_0 - a_1 + a_2 - a_3 + a_4 &= 5\\
a_0 &= 1\\
a_0 + a_1 + a_2 + a_3 + a_4 &= 4\\
a_0 + 2a_1 + 4a_2 + 8a_3 + 16a_4 &= 10
\end{aligned}$$

The solution of these equations is

$$a_0 = 1, \quad a_1 = -\tfrac{30}{24}, \quad a_2 = \tfrac{101}{24}, \quad a_3 = \tfrac{18}{24}, \quad a_4 = -\tfrac{17}{24}$$

which means the polynomial function is

$$\begin{aligned}
p(x) &= 1 - \tfrac{30}{24}x + \tfrac{101}{24}x^2 + \tfrac{18}{24}x^3 - \tfrac{17}{24}x^4\\
&= \tfrac{1}{24}(24 - 30x + 101x^2 + 18x^3 - 17x^4).
\end{aligned}$$

Figure 1.6 shows the graph of $p$.

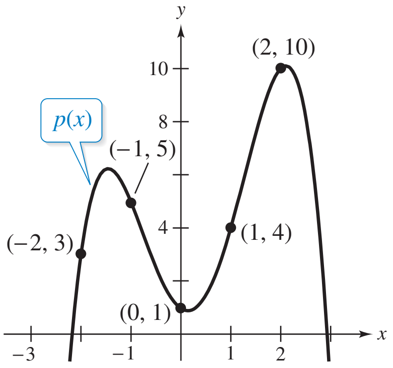



**Figure 1.6**

The system of linear equations in Example 2 is relatively easy to solve because the $x$-values are small. For a set of points with large $x$-values, it is usually best to *translate* the values before attempting the curve-fitting procedure. The next example demonstrates this approach.

## EXAMPLE 3 Translating Large *x*-Values Before Curve Fitting

Find a polynomial that fits the points

$$\overbrace{(2006, 3)}^{(x_1, y_1)}, \quad \overbrace{(2007, 5)}^{(x_2, y_2)}, \quad \overbrace{(2008, 1)}^{(x_3, y_3)}, \quad \overbrace{(2009, 4)}^{(x_4, y_4)}, \quad \overbrace{(2010, 10)}^{(x_5, y_5)}.$$

### SOLUTION

Because the given $x$-values are large, use the translation $z = x - 2008$ to obtain

$$\overbrace{(-2, 3)}^{(z_1, y_1)}, \quad \overbrace{(-1, 5)}^{(z_2, y_2)}, \quad \overbrace{(0, 1)}^{(z_3, y_3)}, \quad \overbrace{(1, 4)}^{(z_4, y_4)}, \quad \overbrace{(2, 10)}^{(z_5, y_5)}.$$

This is the same set of points as in Example 2. So, the polynomial that fits these points is

$$\begin{aligned}
p(z) &= \tfrac{1}{24}(24 - 30z + 101z^2 + 18z^3 - 17z^4)\\
&= 1 - \tfrac{5}{4}z + \tfrac{101}{24}z^2 + \tfrac{3}{4}z^3 - \tfrac{17}{24}z^4.
\end{aligned}$$

Letting $z = x - 2008$, you have

$$p(x) = 1 - \tfrac{5}{4}(x - 2008) + \tfrac{101}{24}(x - 2008)^2 + \tfrac{3}{4}(x - 2008)^3 - \tfrac{17}{24}(x - 2008)^4.$$

## EXAMPLE 4 An Application of Curve Fitting

Find a polynomial that relates the periods of the three planets that are closest to the Sun to their mean distances from the Sun, as shown in the table. Then test the accuracy of the fit by using the polynomial to calculate the period of Mars. (In the table, the mean distance is given in astronomical units, and the period is given in years.)

| *Planet* | *Mercury* | *Venus* | *Earth* | *Mars* |
|---|---|---|---|---|
| *Mean Distance* | 0.387 | 0.723 | 1.000 | 1.524 |
| *Period* | 0.241 | 0.615 | 1.000 | 1.881 |

### SOLUTION

Begin by fitting a quadratic polynomial function

$$p(x) = a_0 + a_1x + a_2x^2$$

to the points

$$(0.387, 0.241),\ (0.723, 0.615),\ \text{and } (1, 1).$$

The system of linear equations obtained by substituting these points into $p(x)$ is

$$\begin{aligned} a_0 + 0.387a_1 + (0.387)^2a_2 &= 0.241 \\ a_0 + 0.723a_1 + (0.723)^2a_2 &= 0.615 \\ a_0 + a_1 + a_2 &= 1. \end{aligned}$$

The approximate solution of the system is

$$a_0 \approx -0.0634, \quad a_1 \approx 0.6119, \quad a_2 \approx 0.4515$$

which means that an approximation of the polynomial function is

$$p(x) = -0.0634 + 0.6119x + 0.4515x^2.$$

Using $p(x)$ to evaluate the period of Mars produces

$$p(1.524) \approx 1.918 \text{ years.}$$

Note that the actual period of Mars is 1.881 years. Figure 1.7 compares the estimate with the actual period graphically.

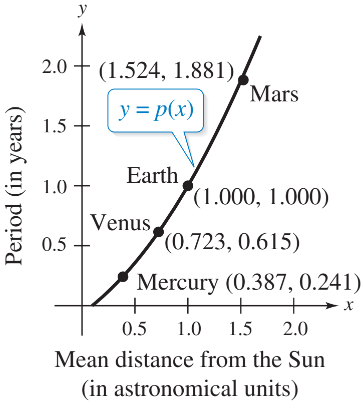



**Figure 1.7**

As illustrated in Example 4, a polynomial that fits some of the points in a data set is not necessarily an accurate model for other points in the data set. Generally, the farther the other points are from those used to fit the polynomial, the worse the fit. For instance, the mean distance of Jupiter from the Sun is 5.203 astronomical units. Using $p(x)$ in Example 4 to approximate the period gives 15.343 years—a poor estimate of Jupiter's actual period of 11.860 years.

The problem of curve fitting can be difficult. Types of functions other than polynomial functions may provide better fits. For instance, look again at the curve-fitting problem in Example 4. Taking the natural logarithms of the given distances and periods produces the following results.

| *Planet* | *Mercury* | *Venus* | *Earth* | *Mars* |
|---|---|---|---|---|
| *Mean Distance* $(x)$ | 0.387 | 0.723 | 1.000 | 1.524 |
| $\ln x$ | $-0.949$ | $-0.324$ | 0.0 | 0.421 |
| *Period* $(y)$ | 0.241 | 0.615 | 1.000 | 1.881 |
| $\ln y$ | $-1.423$ | $-0.486$ | 0.0 | 0.632 |

Now, fitting a polynomial to the logarithms of the distances and periods produces the *linear relationship*

$$\ln y = \tfrac{3}{2} \ln x$$

shown in Figure 1.8.

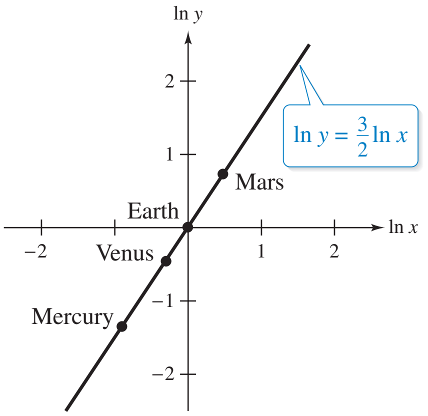



**Figure 1.8**

From $\ln y = \frac{3}{2} \ln x$, it follows that $y = x^{3/2}$, or $y^2 = x^3$. In other words, the square of the period (in years) of each planet is equal to the cube of its mean distance (in astronomical units) from the Sun. Johannes Kepler first discovered this relationship in 1619.

**LINEAR ALGEBRA APPLIED**

Researchers in Italy studying the acoustical noise levels from vehicular traffic at a busy three-way intersection on a college campus used a system of linear equations to model the traffic flow at the intersection. To help formulate the system of equations, "operators" stationed themselves at various locations along the intersection and counted the numbers of vehicles going by. *(Source: Acoustical Noise Analysis in Road Intersections: A Case Study, Guarnaccia, Claudio, Recent Advances in Acoustics & Music, Proceedings of the 11th WSEAS International Conference on Acoustics & Music: Theory & Applications, June, 2010)*

## NETWORK ANALYSIS

Networks composed of branches and junctions are used as models in such fields as economics, traffic analysis, and electrical engineering. In a network model, you assume that the total flow into a junction is equal to the total flow out of the junction. For instance, the junction shown in Figure 1.9 has 25 units flowing into it, so there must be 25 units flowing out of it. You can represent this with the linear equation

$$x_1 + x_2 = 25.$$

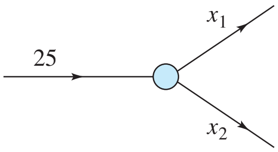



**Figure 1.9**

Because each junction in a network gives rise to a linear equation, you can analyze the flow through a network composed of several junctions by solving a system of linear equations. Example 5 illustrates this procedure.

### EXAMPLE 5 Analysis of a Network

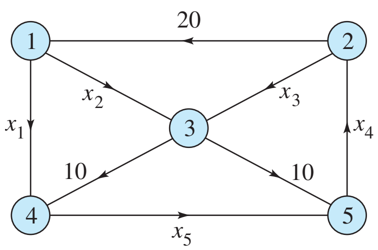



**Figure 1.10**

Set up a system of linear equations to represent the network shown in Figure 1.10. Then solve the system.

**SOLUTION**

Each of the network's five junctions gives rise to a linear equation, as follows.

$$\begin{aligned}
x_1 + x_2 \qquad\qquad\qquad &= \phantom{-}20 \qquad \text{Junction 1}\\
x_3 - x_4 \qquad &= -20 \qquad \text{Junction 2}\\
x_2 + x_3 \qquad\qquad &= \phantom{-}20 \qquad \text{Junction 3}\\
x_1 \qquad\qquad\qquad - x_5 &= -10 \qquad \text{Junction 4}\\
-x_4 + x_5 &= -10 \qquad \text{Junction 5}
\end{aligned}$$

The augmented matrix for this system is

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 20 \\ 0 & 0 & 1 & -1 & 0 & -20 \\ 0 & 1 & 1 & 0 & 0 & 20 \\ 1 & 0 & 0 & 0 & -1 & -10 \\ 0 & 0 & 0 & -1 & 1 & -10 \end{bmatrix}.$$

Gauss-Jordan elimination produces the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & -10 \\ 0 & 1 & 0 & 0 & 1 & 30 \\ 0 & 0 & 1 & 0 & -1 & -10 \\ 0 & 0 & 0 & 1 & -1 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

From the matrix above, you can see that

$$x_1 - x_5 = -10, \quad x_2 + x_5 = 30, \quad x_3 - x_5 = -10, \quad \text{and} \quad x_4 - x_5 = 10.$$

Letting $t = x_5$, you have

$$x_1 = t - 10, \quad x_2 = -t + 30, \quad x_3 = t - 10, \quad x_4 = t + 10, \quad x_5 = t$$

where $t$ is any real number, so this system has infinitely many solutions.

In Example 5, suppose you could control the amount of flow along the branch labeled $x_5$. Using the solution of Example 5, you could then control the flow represented by each of the other variables. For instance, letting $t = 10$ would reduce the flow of $x_1$ and $x_3$ to zero, as shown in Figure 1.11.

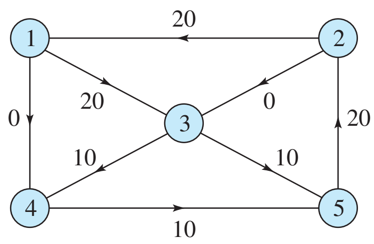



**Figure 1.11**

You may be able to see how the type of network analysis demonstrated in Example 5 could be used in problems dealing with the flow of traffic through the streets of a city or the flow of water through an irrigation system.

**REMARK**

A closed path is a sequence of branches such that the beginning point of the first branch coincides with the end point of the last branch.

An electrical network is another type of network where analysis is commonly applied. An analysis of such a system uses two properties of electrical networks known as **Kirchhoff's Laws.**

1. All the current flowing into a junction must flow out of it.
2. The sum of the products $IR$ ($I$ is current and $R$ is resistance) around a closed path is equal to the total voltage in the path.

In an electrical network, current is measured in amperes, or amps ($A$), resistance is measured in ohms ($\Omega$), and the product of current and resistance is measured in volts ($V$). The symbol  represents a battery. The larger vertical bar denotes where the current flows out of the terminal. The symbol  denotes resistance. An arrow in the branch indicates the direction of the current.

## EXAMPLE 6 Analysis of an Electrical Network

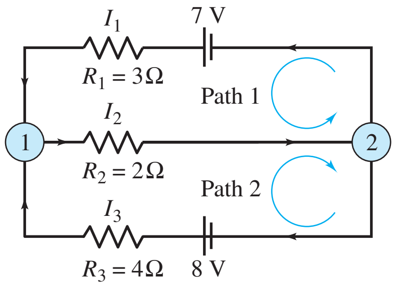



**Figure 1.12**

Determine the currents $I_1$, $I_2$, and $I_3$ for the electrical network shown in Figure 1.12.

### SOLUTION

Applying Kirchhoff's first law to either junction produces

$$I_1 + I_3 = I_2 \qquad \text{Junction 1 or Junction 2}$$

and applying Kirchhoff's second law to the two paths produces

$$\begin{aligned} R_1I_1 + R_2I_2 &= 3I_1 + 2I_2 = 7 && \text{Path 1} \\ R_2I_2 + R_3I_3 &= 2I_2 + 4I_3 = 8. && \text{Path 2} \end{aligned}$$

So, you have the following system of three linear equations in the variables $I_1$, $I_2$, and $I_3$.

$$\begin{aligned} I_1 - I_2 + I_3 &= 0 \\ 3I_1 + 2I_2 \phantom{+ 4I_3} &= 7 \\ 2I_2 + 4I_3 &= 8 \end{aligned}$$

Applying Gauss-Jordan elimination to the augmented matrix

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 3 & 2 & 0 & 7 \\ 0 & 2 & 4 & 8 \end{bmatrix}$$

produces the reduced row-echelon form

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

which means $I_1 = 1$ amp, $I_2 = 2$ amps, and $I_3 = 1$ amp.

## EXAMPLE 7 Analysis of an Electrical Network

Determine the currents $I_1$, $I_2$, $I_3$, $I_4$, $I_5$, and $I_6$ for the electrical network shown in Figure 1.13.

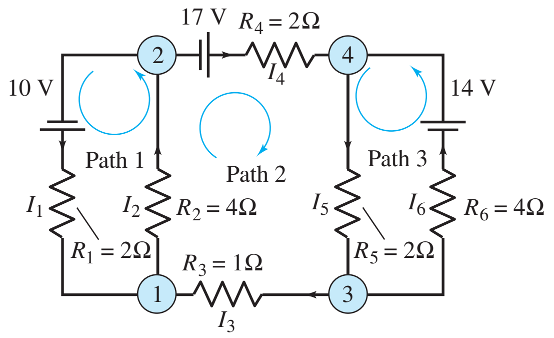



**Figure 1.13**

### SOLUTION

Applying Kirchhoff's first law to the four junctions produces

$$
\begin{aligned}
I_1 + I_3 &= I_2 && \text{Junction 1}\\
I_1 + I_4 &= I_2 && \text{Junction 2}\\
I_3 + I_6 &= I_5 && \text{Junction 3}\\
I_4 + I_6 &= I_5 && \text{Junction 4}
\end{aligned}
$$

and applying Kirchhoff's second law to the three paths produces

$$
\begin{aligned}
2I_1 + 4I_2 \qquad\qquad\qquad\qquad &= 10 && \text{Path 1}\\
4I_2 + I_3 + 2I_4 + 2I_5 \qquad &= 17 && \text{Path 2}\\
2I_5 + 4I_6 &= 14. && \text{Path 3}
\end{aligned}
$$

You now have the following system of seven linear equations in the variables $I_1$, $I_2$, $I_3$, $I_4$, $I_5$, and $I_6$.

$$
\begin{aligned}
I_1 - I_2 + I_3 \qquad\qquad\qquad\qquad &= 0\\
I_1 - I_2 \qquad + I_4 \qquad\qquad\quad &= 0\\
I_3 \qquad - I_5 + I_6 &= 0\\
I_4 - I_5 + I_6 &= 0\\
2I_1 + 4I_2 \qquad\qquad\qquad\qquad &= 10\\
4I_2 + I_3 + 2I_4 + 2I_5 \qquad &= 17\\
2I_5 + 4I_6 &= 14
\end{aligned}
$$

The augmented matrix for this system is

$$
\begin{bmatrix}
1 & -1 & 1 & 0 & 0 & 0 & 0\\
1 & -1 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & -1 & 1 & 0\\
0 & 0 & 0 & 1 & -1 & 1 & 0\\
2 & 4 & 0 & 0 & 0 & 0 & 10\\
0 & 4 & 1 & 2 & 2 & 0 & 17\\
0 & 0 & 0 & 0 & 2 & 4 & 14
\end{bmatrix}.
$$

Using Gauss-Jordan elimination, a graphing utility, or a software program, solve this system to obtain

$$I_1 = 1, \quad I_2 = 2, \quad I_3 = 1, \quad I_4 = 1, \quad I_5 = 3, \quad \text{and} \quad I_6 = 2$$

meaning $I_1 = 1$ amp, $I_2 = 2$ amps, $I_3 = 1$ amp, $I_4 = 1$ amp, $I_5 = 3$ amps, and $I_6 = 2$ amps.

## 1.3 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

**Polynomial Curve Fitting** **In Exercises 1–12, (a) determine the polynomial function whose graph passes through the given points, and (b) sketch the graph of the polynomial function, showing the given points.**

**1.** $(2, 5), (3, 2), (4, 5)$

**2.** $(0, 0), (2, -2), (4, 0)$

**3.** $(2, 4), (3, 6), (5, 10)$

**4.** $(2, 4), (3, 4), (4, 4)$

**5.** $(-1, 3), (0, 0), (1, 1), (4, 58)$

**6.** $(0, 42), (1, 0), (2, -40), (3, -72)$

**7.** $(-2, 28), (-1, 0), (0, -6), (1, -8), (2, 0)$

**8.** $(-4, 18), (0, 1), (4, 0), (6, 28), (8, 135)$

**9.** $(2006, 5), (2007, 7), (2008, 12)$

**10.** $(2005, 150), (2006, 180), (2007, 240), (2008, 360)$

**11.** $(0.072, 0.203), (0.120, 0.238), (0.148, 0.284)$

**12.** $(1, 1), (1.189, 1.587), (1.316, 2.080), (1.414, 2.520)$

**13.** Use $\sin 0 = 0$, $\sin\left(\frac{\pi}{2}\right) = 1$, and $\sin \pi = 0$ to estimate $\sin\left(\frac{\pi}{3}\right)$.

**14.** Use $\log_2 1 = 0$, $\log_2 2 = 1$, and $\log_2 4 = 2$ to estimate $\log_2 3$.

**15.** Find an equation of the circle passing through the points $(1, 3)$, $(-2, 6)$, and $(4, 2)$.

**16.** Find an equation of the ellipse passing through the points $(-5, 1)$, $(-3, 2)$, $(-1, 1)$, and $(-3, 0)$.

**17. Population** The U.S. census lists the population of the United States as 249 million in 1990, 281 million in 2000, and 309 million in 2010. Fit a second-degree polynomial passing through these three points and use it to predict the population in 2020 and in 2030. (Source: U.S. Census Bureau)

**18. Population** The table shows the U.S. population figures for the years 1940, 1950, 1960, and 1970. (Source: U.S. Census Bureau)

| *Year* | 1940 | 1950 | 1960 | 1970 |
|---|---|---|---|---|
| *Population (in millions)* | 132 | 151 | 179 | 203 |

(a) Find a cubic polynomial that fits these data and use it to estimate the population in 1980.

(b) The actual population in 1980 was 227 million. How does your estimate compare?

**19. Net Profit** The table shows the net profits (in millions of dollars) for Microsoft from 2003 through 2010. (Source: Microsoft Corp.)

| *Year* | 2003 | 2004 | 2005 | 2006 |
|---|---|---|---|---|
| *Net Profit* | 10,526 | 11,330 | 12,715 | 12,599 |

| *Year* | 2007 | 2008 | 2009 | 2010 |
|---|---|---|---|---|
| *Net Profit* | 14,065 | 17,681 | 14,569 | 18,760 |

(a) Set up a system of equations to fit the data for the years 2003, 2004, 2005, and 2006 to a cubic model.

(b) Solve the system. Does the solution produce a reasonable model for determining net profits after 2006? Explain.

**20. Sales** The table shows the sales (in billions of dollars) for Wal-Mart stores from 2002 through 2009. (Source: Wal-Mart Stores, Inc.)

| *Year* | 2002 | 2003 | 2004 | 2005 |
|---|---|---|---|---|
| *Sales* | 244.5 | 256.3 | 285.2 | 312.4 |

| *Year* | 2006 | 2007 | 2008 | 2009 |
|---|---|---|---|---|
| *Sales* | 345.0 | 374.5 | 401.2 | 405.0 |

(a) Set up a system of equations to fit the data for the years 2002, 2003, 2004, 2005, and 2006 to a quartic model.

(b) Solve the system. Does the solution produce a reasonable model for determining sales after 2006? Explain.

**21. Guided Proof** Prove that if a polynomial function $p(x) = a_0 + a_1x + a_2x^2$ is zero for $x = -1$, $x = 0$, and $x = 1$, then $a_0 = a_1 = a_2 = 0$.

Getting Started: Write a system of linear equations and solve the system for $a_0$, $a_1$, and $a_2$.

(i) Substitute $x = -1$, 0, and 1 into $p(x)$.

(ii) Set the result equal to 0.

(iii) Solve the resulting system of linear equations in the variables $a_0$, $a_1$, and $a_2$.

**22.** Generalizing the statement in Exercise 21, if a polynomial function

$$p(x) = a_0 + a_1 x + \cdots + a_{n-1}x^{n-1}$$

is zero for more than $n - 1$ $x$-values, then

$$a_0 = a_1 = \cdots = a_{n-1} = 0.$$

Use this result to prove that there is at most one polynomial function of degree $n - 1$ (or less) whose graph passes through $n$ points in the plane with distinct $x$-coordinates.

**23.** **Calculus** The graph of a cubic polynomial function has horizontal tangents at $(1, -2)$ and $(-1, 2)$. Find an equation for the cubic and sketch its graph.

**24.** **Calculus** The graph of a parabola passes through the points $(0, 1)$ and $\left(\frac{1}{2}, \frac{1}{2}\right)$ and has a horizontal tangent at $\left(\frac{1}{2}, \frac{1}{2}\right)$. Find an equation for the parabola and sketch its graph.

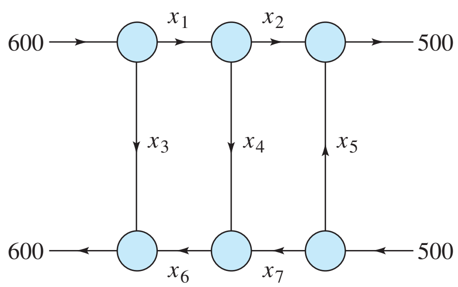
**25.** (a) The graph of a function $f$ passes through the points $(0, 1)$, $\left(2, \frac{1}{3}\right)$, and $\left(4, \frac{1}{5}\right)$. Find a quadratic function whose graph passes through these points.

(b) Find a polynomial function $p$ of degree 2 or less that passes through the points $(0, 1)$, $(2, 3)$, and $(4, 5)$. Then sketch the graph of $y = 1/p(x)$ and compare this graph with the graph of the polynomial function found in part (a).

**26.** **Writing** Try to fit the graph of a polynomial function to the values shown in the table. What happens, and why?

| | | | | | |
|---|---|---|---|---|---|
| $x$ | 1 | 2 | 3 | 3 | 4 |
| $y$ | 1 | 1 | 2 | 3 | 4 |

**27.** **Network Analysis** Water is flowing through a network of pipes (in thousands of cubic meters per hour), as shown in the figure.

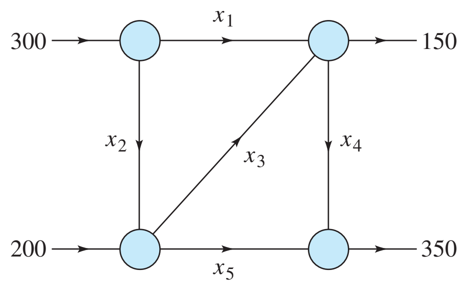



(a) Solve this system for the water flow represented by $x_i$, $i = 1, 2, \ldots, 7$.

(b) Find the water flow when $x_6 = x_7 = 0$.

(c) Find the water flow when $x_5 = 1000$ and $x_6 = 0$.

**28.** **Network Analysis** The figure shows the flow of traffic (in vehicles per hour) through a network of streets.

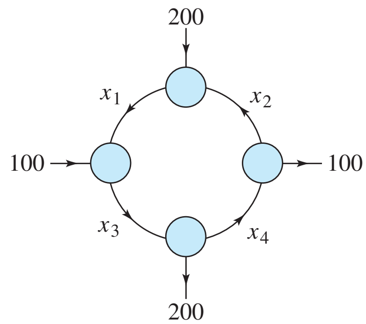



(a) Solve this system for $x_i$, $i = 1, 2, \ldots, 5$.

(b) Find the traffic flow when $x_2 = 200$ and $x_3 = 50$.

(c) Find the traffic flow when $x_2 = 150$ and $x_3 = 0$.

**29.** **Network Analysis** The figure shows the flow of traffic (in vehicles per hour) through a network of streets.

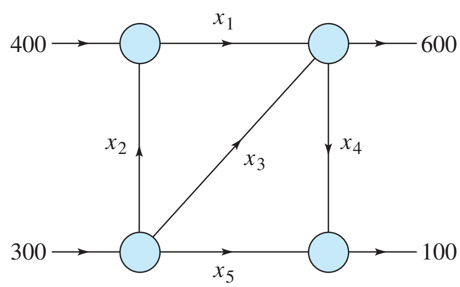



(a) Solve this system for $x_i$, $i = 1, 2, 3, 4$.

(b) Find the traffic flow when $x_4 = 0$.

(c) Find the traffic flow when $x_4 = 100$.

**30.** **Network Analysis** The figure shows the flow of traffic through a network of streets.

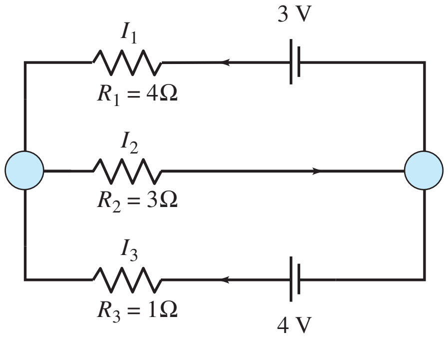



(a) Solve this system for $x_i$, $i = 1, 2, \ldots, 5$.

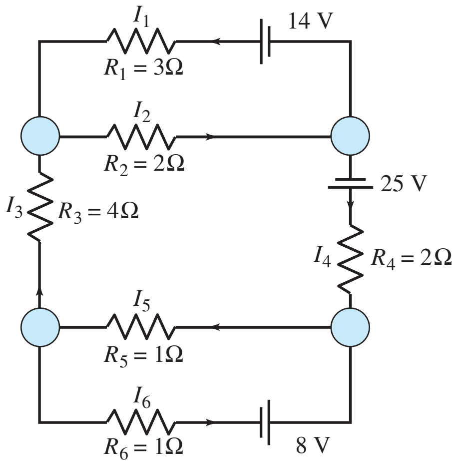
(b) Find the traffic flow when $x_3 = 0$ and $x_5 = 100$.

(c) Find the traffic flow when $x_3 = x_5 = 100$.

**31.** **Network Analysis** Determine the currents $I_1$, $I_2$, and $I_3$ for the electrical network shown in the figure.



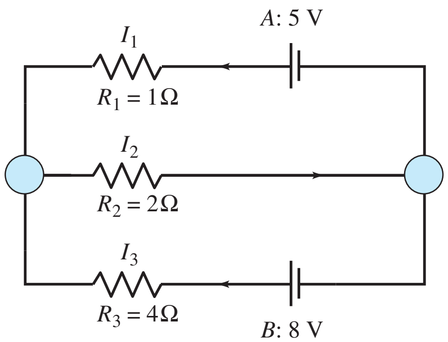

**32.** **Network Analysis** Determine the currents $I_1$, $I_2$, $I_3$, $I_4$, $I_5$, and $I_6$ for the electrical network shown in the figure.




**33.** **Network Analysis**

(a) Determine the currents $I_1$, $I_2$, and $I_3$ for the electrical network shown in the figure.

(b) How is the result affected when $A$ is changed to 2 volts and $B$ is changed to 6 volts?



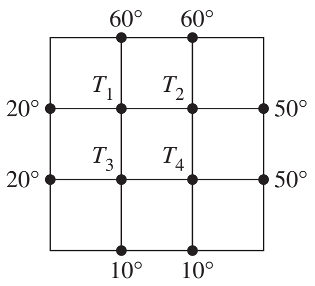

**34.** **CAPSTONE**

(a) Explain how to use systems of linear equations for polynomial curve fitting.

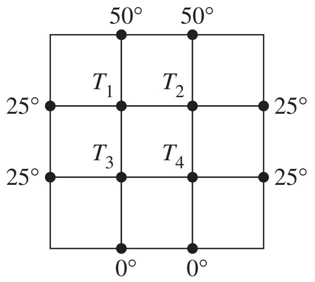
(b) Explain how to use systems of linear equations to perform network analysis.

**Temperature** **In Exercises 35 and 36, the figure shows the boundary temperatures (in degrees Celsius) of an insulated thin metal plate. The steady-state temperature at an interior junction is approximately equal to the mean of the temperatures at the four surrounding junctions. Use a system of linear equations to approximate the interior temperatures $T_1$, $T_2$, $T_3$, and $T_4$.**

**35.**




**36.**




**Partial Fraction Decomposition** **In Exercises 37 and 38, use a system of equations to write the partial fraction decomposition of the rational expression. Then solve the system using matrices.**

**37.** $\dfrac{4x^2}{(x+1)^2(x-1)} = \dfrac{A}{x-1} + \dfrac{B}{x+1} + \dfrac{C}{(x+1)^2}$

**38.** $\dfrac{3x^2 - 7x - 12}{(x+4)(x-4)^2} = \dfrac{A}{x+4} + \dfrac{B}{x-4} + \dfrac{C}{(x-4)^2}$

**Calculus** **In Exercises 39 and 40, find the values of $x$, $y$, and $\lambda$ that satisfy the system of equations. Such systems arise in certain problems of calculus, and $\lambda$ is called the Lagrange multiplier.**

**39.**
$$\begin{aligned} 2x \phantom{{}+2y} + \lambda \phantom{{}-4} &= 0 \\ 2y + \lambda \phantom{{}-4} &= 0 \\ x + y \phantom{{}+\lambda} - 4 &= 0 \end{aligned}$$

**40.**
$$\begin{aligned} 2y + 2\lambda + 2 &= 0 \\ 2x \phantom{{}+2y} + \lambda + 1 &= 0 \\ 2x + y \phantom{{}+\lambda} - 100 &= 0 \end{aligned}$$

# 1 Review Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

**Linear Equations** In Exercises 1–6, determine whether the equation is linear in the variables $x$ and $y$.

**1.** $2x - y^2 = 4$

**2.** $2xy - 6y = 0$

**3.** $(\sin \pi)x + y = 2$

**4.** $e^{-2}x + 5y = 8$

**5.** $\dfrac{2}{x} + 4y = 3$

**6.** $\dfrac{x}{2} - \dfrac{y}{4} = 0$

**Parametric Representation** In Exercises 7 and 8, find a parametric representation of the solution set of the linear equation.

**7.** $-4x + 2y - 6z = 1$

**8.** $3x_1 + 2x_2 - 4x_3 = 0$

**System of Linear Equations** In Exercises 9–20, solve the system of linear equations.

**9.** $\begin{aligned} x + y &= 2 \\ 3x - y &= 0 \end{aligned}$

**10.** $\begin{aligned} x + y &= -1 \\ 3x + 2y &= 0 \end{aligned}$

**11.** $\begin{aligned} 3y &= 2x \\ y &= x + 4 \end{aligned}$

**12.** $\begin{aligned} x &= y + 3 \\ 4x &= y + 10 \end{aligned}$

**13.** $\begin{aligned} y + x &= 0 \\ 2x + y &= 0 \end{aligned}$

**14.** $\begin{aligned} y &= -4x \\ y &= x \end{aligned}$

**15.** $\begin{aligned} x - y &= 9 \\ -x + y &= 1 \end{aligned}$

**16.** $\begin{aligned} 40x_1 + 30x_2 &= 24 \\ 20x_1 + 15x_2 &= -14 \end{aligned}$

**17.** $\begin{aligned} \tfrac{1}{2}x - \tfrac{1}{3}y &= 0 \\ 3x + 2(y + 5) &= 10 \end{aligned}$

**18.** $\begin{aligned} \tfrac{1}{3}x + \tfrac{4}{7}y &= 3 \\ 2x + 3y &= 15 \end{aligned}$

**19.** $\begin{aligned} 0.2x_1 + 0.3x_2 &= 0.14 \\ 0.4x_1 + 0.5x_2 &= 0.20 \end{aligned}$

**20.** $\begin{aligned} 0.2x - 0.1y &= 0.07 \\ 0.4x - 0.5y &= -0.01 \end{aligned}$

**Matrix Size** In Exercises 21 and 22, determine the size of the matrix.

**21.** $\begin{bmatrix} 2 & 3 & -1 \\ 0 & 5 & 1 \end{bmatrix}$

**22.** $\begin{bmatrix} 2 & 1 \\ -4 & -1 \\ 0 & 5 \end{bmatrix}$

**Augmented Matrix** In Exercises 23 and 24, find the solution set of the system of linear equations represented by the augmented matrix.

**23.** $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

**24.** $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

**Row-Echelon Form** In Exercises 25–28, determine whether the matrix is in row-echelon form. If it is, determine whether it is also in reduced row-echelon form.

**25.** $\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

**26.** $\begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

**27.** $\begin{bmatrix} -1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

**28.** $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

**System of Linear Equations** In Exercises 29–38, solve the system using either Gaussian elimination with back-substitution or Gauss-Jordan elimination.

**29.** $\begin{aligned} -x + y + 2z &= 1 \\ 2x + 3y + z &= -2 \\ 5x + 4y + 2z &= 4 \end{aligned}$

**30.** $\begin{aligned} 2x + 3y + z &= 10 \\ 2x - 3y - 3z &= 22 \\ 4x - 2y + 3z &= -2 \end{aligned}$

**31.** $\begin{aligned} 2x + 3y + 3z &= 3 \\ 6x + 6y + 12z &= 13 \\ 12x + 9y - z &= 2 \end{aligned}$

**32.** $\begin{aligned} 2x + y + 2z &= 4 \\ 2x + 2y &= 5 \\ 2x - y + 6z &= 2 \end{aligned}$

**33.** $\begin{aligned} x - 2y + z &= -6 \\ 2x - 3y &= -7 \\ -x + 3y - 3z &= 11 \end{aligned}$

**34.** $\begin{aligned} 2x + 6z &= -9 \\ 3x - 2y + 11z &= -16 \\ 3x - y + 7z &= -11 \end{aligned}$

**35.** $\begin{aligned} x + 2y + 6z &= 1 \\ 2x + 5y + 15z &= 4 \\ 3x + y + 3z &= -6 \end{aligned}$

**36.** $\begin{aligned} 2x_1 + 5x_2 - 19x_3 &= 34 \\ 3x_1 + 8x_2 - 31x_3 &= 54 \end{aligned}$

**37.** $\begin{aligned} 2x_1 + x_2 + x_3 + 2x_4 &= -1 \\ 5x_1 - 2x_2 + x_3 - 3x_4 &= 0 \\ -x_1 + 3x_2 + 2x_3 + 2x_4 &= 1 \\ 3x_1 + 2x_2 + 3x_3 - 5x_4 &= 12 \end{aligned}$

**38.** $\begin{aligned} x_1 + 5x_2 + 3x_3 &= 14 \\ 4x_2 + 2x_3 + 5x_4 &= 3 \\ 3x_3 + 8x_4 + 6x_5 &= 16 \\ 2x_1 + 4x_2 - 2x_5 &= 0 \\ 2x_1 - x_3 &= 0 \end{aligned}$

**System of Linear Equations** **In Exercises 39–42, use a software program or a graphing utility to solve the system of linear equations.**

**39.** $$\begin{aligned} 3x + 3y + 12z &= 6 \\ x + y + 4z &= 2 \\ 2x + 5y + 20z &= 10 \\ -x + 2y + 8z &= 4 \end{aligned}$$

**40.** $$\begin{aligned} 2x + 10y + 2z &= 6 \\ x + 5y + 2z &= 6 \\ x + 5y + z &= 3 \\ -3x - 15y + 3z &= -9 \end{aligned}$$

**41.** $$\begin{aligned} 2x + y - z + 2w &= -6 \\ 3x + 4y + w &= 1 \\ x + 5y + 2z + 6w &= -3 \\ 5x + 2y - z - w &= 3 \end{aligned}$$

**42.** $$\begin{aligned} x + 2y + z + 3w &= 0 \\ x - y + w &= 0 \\ 5y - z + 2w &= 0 \end{aligned}$$

**Homogeneous System** **In Exercises 43–46, solve the homogeneous system of linear equations.**

**43.** $$\begin{aligned} x_1 - 2x_2 - 8x_3 &= 0 \\ 3x_1 + 2x_2 &= 0 \end{aligned}$$

**44.** $$\begin{aligned} 2x_1 + 4x_2 - 7x_3 &= 0 \\ x_1 - 3x_2 + 9x_3 &= 0 \end{aligned}$$

**45.** $$\begin{aligned} 2x_1 - 8x_2 + 4x_3 &= 0 \\ 3x_1 - 10x_2 + 7x_3 &= 0 \\ 10x_2 + 5x_3 &= 0 \end{aligned}$$

**46.** $$\begin{aligned} x_1 + 3x_2 + 5x_3 &= 0 \\ x_1 + 4x_2 + \tfrac{1}{2}x_3 &= 0 \end{aligned}$$

**47.** Determine the values of $k$ such that the system of linear equations is inconsistent.
$$\begin{aligned} kx + y &= 0 \\ x + ky &= 1 \end{aligned}$$

**48.** Determine the values of $k$ such that the system of linear equations has exactly one solution.
$$\begin{aligned} x - y + 2z &= 0 \\ -x + y - z &= 0 \\ x + ky + z &= 0 \end{aligned}$$

**49.** Find values of $a$ and $b$ such that the system of linear equations has (a) no solution, (b) exactly one solution, and (c) infinitely many solutions.
$$\begin{aligned} x + 2y &= 3 \\ ax + by &= -9 \end{aligned}$$

**50.** Find (if possible) values of $a$, $b$, and $c$ such that the system of linear equations has (a) no solution, (b) exactly one solution, and (c) infinitely many solutions.
$$\begin{aligned} 2x - y + z &= a \\ x + y + 2z &= b \\ 3y + 3z &= c \end{aligned}$$

**51. Writing** Describe a method for showing that two matrices are row-equivalent. Are the two matrices below row-equivalent?
$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & 2 \\ 3 & 1 & 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 3 & 6 \\ 5 & 5 & 10 \end{bmatrix}$$

**52. Writing** Describe all possible $2 \times 3$ reduced row-echelon matrices. Support your answer with examples.

**53.** Let $n \geq 3$. Find the reduced row-echelon form of the $n \times n$ matrix.
$$\begin{bmatrix} 1 & 2 & 3 & \cdots & n \\ n+1 & n+2 & n+3 & \cdots & 2n \\ 2n+1 & 2n+2 & 2n+3 & \cdots & 3n \\ \vdots & \vdots & \vdots & & \vdots \\ n^2-n+1 & n^2-n+2 & n^2-n+3 & \cdots & n^2 \end{bmatrix}$$

**54.** Find all values of $\lambda$ for which the homogeneous system of linear equations has nontrivial solutions.
$$\begin{aligned} (\lambda + 2)x_1 - 2x_2 + 3x_3 &= 0 \\ -2x_1 + (\lambda - 1)x_2 + 6x_3 &= 0 \\ x_1 + 2x_2 + \lambda x_3 &= 0 \end{aligned}$$

**True or False?** **In Exercises 55 and 56, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.**

**55.** (a) There is only one way to parametrically represent the solution set of a linear equation.
(b) A consistent system of linear equations can have infinitely many solutions.

**56.** (a) A homogeneous system of linear equations must have at least one solution.
(b) A system of linear equations with fewer equations than variables always has at least one solution.

**57. Sports** In Super Bowl I, on January 15, 1967, the Green Bay Packers defeated the Kansas City Chiefs by a score of 35 to 10. The total points scored came from a combination of touchdowns, extra-point kicks, and field goals, worth 6, 1, and 3 points, respectively. The numbers of touchdowns and extra-point kicks were equal. There were six times as many touchdowns as field goals. Find the numbers of touchdowns, extra-point kicks, and field goals scored. (Source: National Football League)

**58. Agriculture** A mixture of 6 gallons of chemical A, 8 gallons of chemical B, and 13 gallons of chemical C is required to kill a destructive crop insect. Commercial spray X contains 1, 2, and 2 parts, respectively, of these chemicals. Commercial spray Y contains only chemical C. Commercial spray Z contains chemicals A, B, and C in equal amounts. How much of each type of commercial spray is needed to get the desired mixture?

**Partial Fraction Decomposition In Exercises 59 and 60, use a system of equations to write the partial fraction decomposition of the rational expression. Then solve the system using matrices.**

**59.** $\dfrac{8x^2}{(x-1)^2(x+1)} = \dfrac{A}{x+1} + \dfrac{B}{x-1} + \dfrac{C}{(x-1)^2}$

**60.** $\dfrac{3x^2+3x-2}{(x+1)^2(x-1)} = \dfrac{A}{x+1} + \dfrac{B}{x-1} + \dfrac{C}{(x+1)^2}$

**Polynomial Curve Fitting In Exercises 61 and 62, (a) determine the polynomial function whose graph passes through the given points, and (b) sketch the graph of the polynomial function, showing the given points.**

**61.** $(2, 5), (3, 0), (4, 20)$

**62.** $(-1, -1), (0, 0), (1, 1), (2, 4)$

**63. Sales** A company has sales (measured in millions) of \$50, \$60, and \$75 during three consecutive years. Find a quadratic function that fits these data, and use it to predict the sales during the fourth year.

**64.** The polynomial function

$$p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$$

is zero when $x = 1, 2, 3$, and 4. What are the values of $a_0, a_1, a_2$, and $a_3$?

**65. Deer Population** A wildlife management team studied the population of deer in one small tract of a wildlife preserve. The table shows the population and the number of years since the study began.

| *Year* | 0 | 4 | 80 |
|---|---|---|---|
| *Population* | 80 | 68 | 30 |

(a) Set up a system of equations to fit the data to a quadratic polynomial function.

(b) Solve the system.

(c) Use a graphing utility to fit the data to a quadratic model.

(d) Compare the quadratic polynomial function in part (b) with the model in part (c).

(e) Cite the statement from the text that verifies your results.

**66. Vertical Motion** An object moving vertically is at the given heights at the specified times. Find the position equation

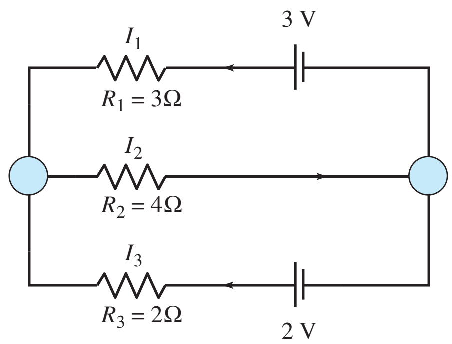
$$s = \tfrac{1}{2}at^2 + v_0t + s_0$$

for the object.

(a) At $t = 1$ second, $s = 134$ feet
At $t = 2$ seconds, $s = 86$ feet
At $t = 3$ seconds, $s = 6$ feet

(b) At $t = 1$ second, $s = 184$ feet
At $t = 2$ seconds, $s = 116$ feet
At $t = 3$ seconds, $s = 16$ feet

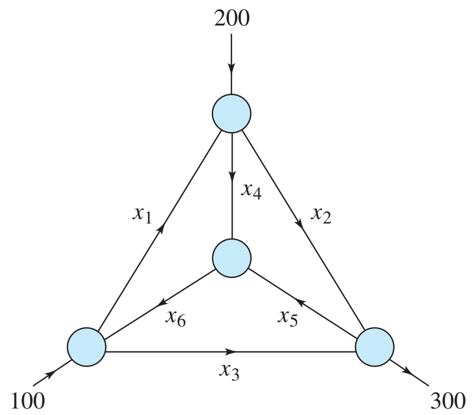
**67. Network Analysis** Determine the currents $I_1$, $I_2$, and $I_3$ for the electrical network shown in the figure.




**68. Network Analysis** The figure shows the flow through a network.

(a) Solve the system for $x_i$, $i = 1, 2, \ldots, 6$.

(b) Find the flow when $x_3 = 100$, $x_5 = 50$, and $x_6 = 50$.



# 1 Projects

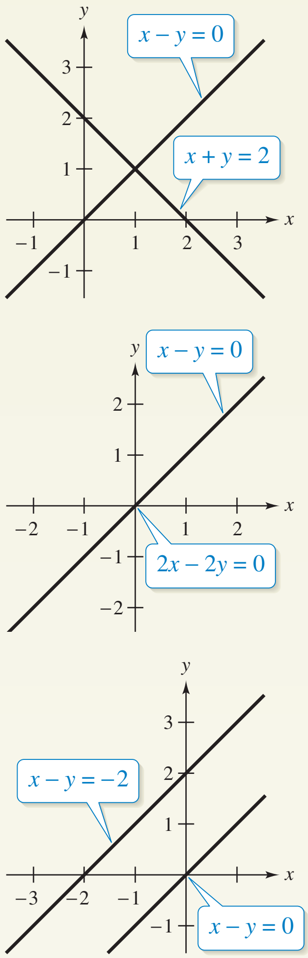



**Figure 1.14**

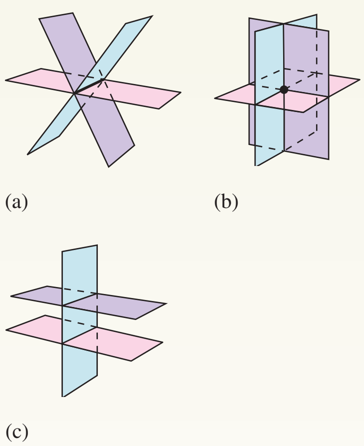



**Figure 1.15**

## 1 Graphing Linear Equations

You saw in Section 1.1 that you can represent a system of two linear equations in two variables $x$ and $y$ geometrically as two lines in the plane. These lines can intersect at a point, coincide, or be parallel, as indicated in Figure 1.14.

**1.** Consider the following system, where $a$ and $b$ are constants.

$$\begin{aligned} 2x - \phantom{b}y &= 3 \\ ax + by &= 6 \end{aligned}$$

(a) Find values of $a$ and $b$ for which the resulting system has a unique solution.

(b) Find values of $a$ and $b$ for which the resulting system has infinitely many solutions.

(c) Find values of $a$ and $b$ for which the resulting system has no solution.

(d) Graph the lines for each of the systems in parts (a), (b), and (c).

**2.** Now consider a system of three linear equations in $x$, $y$, and $z$. Each equation represents a plane in the three-dimensional coordinate system.

(a) Find an example of a system represented by three planes intersecting in a line, as shown in Figure 1.15(a).

(b) Find an example of a system represented by three planes intersecting at a point, as shown in Figure 1.15(b).

(c) Find an example of a system represented by three planes with no common intersection, as shown in Figure 1.15(c).

(d) Are there other configurations of three planes in addition to those given in Figure 1.15? Explain.

## 2 Underdetermined and Overdetermined Systems

The following system of linear equations is said to be **underdetermined** because there are more variables than equations.

$$\begin{aligned} x_1 + 2x_2 - 3x_3 &= 4 \\ 2x_1 - x_2 + 4x_3 &= -3 \end{aligned}$$

Similarly, the following system is **overdetermined** because there are more equations than variables.

$$\begin{aligned} x_1 + 3x_2 &= 5 \\ 2x_1 - 2x_2 &= -3 \\ -x_1 + 7x_2 &= 0 \end{aligned}$$

You can explore whether the number of variables and the number of equations have any bearing on the consistency of a system of linear equations. For Exercises 1–4, if an answer is yes, give an example. Otherwise, explain why the answer is no.

**1.** Can you find a consistent underdetermined linear system?

**2.** Can you find a consistent overdetermined linear system?

**3.** Can you find an inconsistent underdetermined linear system?

**4.** Can you find an inconsistent overdetermined linear system?

**5.** Explain why you would expect an overdetermined linear system to be inconsistent. Must this always be the case?

**6.** Explain why you would expect an underdetermined linear system to have infinitely many solutions. Must this always be the case?

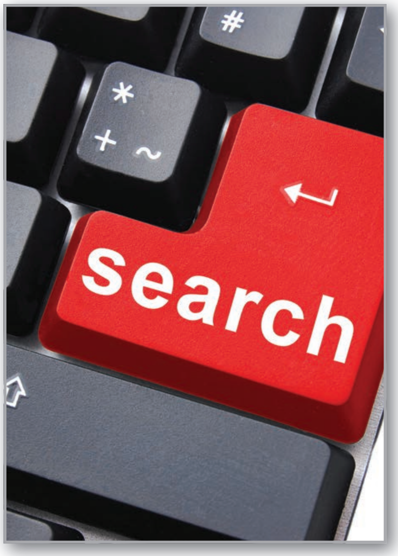
# 2 Matrices



Data Encryption (p. 87)

Computational Fluid Dynamics (p. 79)

Beam Deflection (p. 64)




Information Retrieval (p. 58)

Flight Crew Scheduling (p. 47)



Clockwise from top left, Cousin_Avi/www.shutterstock.com; Goncharuk/www.shutterstock.com;
Gunnar Pippel/www.shutterstock.com; © Sean Locke/iStockphoto.com; nostal6ie/www.shutterstock.com

# 2.1 Operations with Matrices

- Determine whether two matrices are equal.
- Add and subtract matrices and multiply a matrix by a scalar.
- Multiply two matrices.
- Use matrices to solve a system of linear equations.
- Partition a matrix and write a linear combination of column vectors.

## EQUALITY OF MATRICES

In Section 1.2, you used matrices to solve systems of linear equations. This chapter introduces some fundamentals of matrix theory and further applications of matrices.

It is standard mathematical convention to represent matrices in any one of the following three ways.

**1.** An uppercase letter such as $A$, $B$, or $C$

**2.** A representative element enclosed in brackets, such as $[a_{ij}]$, $[b_{ij}]$, or $[c_{ij}]$

**3.** A rectangular array of numbers

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

As mentioned in Chapter 1, the matrices in this text are primarily *real matrices*. That is, their entries contain real numbers.

Two matrices are *equal* when their corresponding entries are equal.

### Definition of Equality of Matrices

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are **equal** when they have the same size $(m \times n)$ and $a_{ij} = b_{ij}$ for $1 \le i \le m$ and $1 \le j \le n$.

**REMARK**

The phrase "if and only if" means the statement is true in both directions. For example, "*p* if and only if *q*" means that *p* implies *q* and *q* implies *p*.

### EXAMPLE 1 Equality of Matrices

Consider the four matrices

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 3 \end{bmatrix}, \quad \text{and} \quad D = \begin{bmatrix} 1 & 2 \\ x & 4 \end{bmatrix}.$$

Matrices $A$ and $B$ are **not** equal because they are of different sizes. Similarly, $B$ and $C$ are not equal. Matrices $A$ and $D$ are equal if and only if $x = 3$.

A matrix that has only one column, such as matrix $B$ in Example 1, is called a **column matrix** or **column vector.** Similarly, a matrix that has only one row, such as matrix $C$ in Example 1, is called a **row matrix** or **row vector.** Boldface lowercase letters often designate column matrices and row matrices. For instance, matrix $A$ in Example 1 can be partitioned into the two column matrices $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\mathbf{a}_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, as follows.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \left[\begin{array}{c|c} 1 & 2 \\ 3 & 4 \end{array}\right] = [\mathbf{a}_1 \mid \mathbf{a}_2]$$

## MATRIX ADDITION, SUBTRACTION, AND SCALAR MULTIPLICATION

To **add** two matrices (of the same size), add their corresponding entries.

### Definition of Matrix Addition

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are matrices of size $m \times n$, then their **sum** is the $m \times n$ matrix given by $A + B = [a_{ij} + b_{ij}]$.

The sum of two matrices of different sizes is undefined.

### EXAMPLE 2 Addition of Matrices

**a.** $\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -1+1 & 2+3 \\ 0+(-1) & 1+2 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ -1 & 3 \end{bmatrix}$

**b.** $\begin{bmatrix} 0 & 1 & -2 \\ 1 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ 1 & 2 & 3 \end{bmatrix}$

**c.** $\begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ **d.** $\begin{bmatrix} 2 & 1 & 0 \\ 4 & 0 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix}$ is undefined.

**REMARK**

It is often convenient to rewrite the scalar multiple $cA$ by factoring $c$ out of every entry in the matrix. For instance, factoring the scalar $\frac{1}{2}$ out of the matrix below gives

$$\begin{bmatrix} \frac{1}{2} & -\frac{3}{2} \\ \frac{5}{2} & \frac{1}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -3 \\ 5 & 1 \end{bmatrix}.$$

When working with matrices, real numbers are referred to as **scalars.** To multiply a matrix $A$ by a scalar $c$, multiply each entry in $A$ by $c$.

### Definition of Scalar Multiplication

If $A = [a_{ij}]$ is an $m \times n$ matrix and $c$ is a scalar, then the **scalar multiple** of $A$ by $c$ is the $m \times n$ matrix given by

$$cA = [ca_{ij}].$$

You can use $-A$ to represent the scalar product $(-1)A$. If $A$ and $B$ are of the same size, then $A - B$ represents the sum of $A$ and $(-1)B$. That is, $A - B = A + (-1)B$.

### EXAMPLE 3 Scalar Multiplication and Matrix Subtraction

For the matrices $A$ and $B$, find (a) $3A$, (b) $-B$, and (c) $3A - B$.

$$A = \begin{bmatrix} 1 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix}$$

**SOLUTION**

**a.** $3A = 3\begin{bmatrix} 1 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 3(1) & 3(2) & 3(4) \\ 3(-3) & 3(0) & 3(-1) \\ 3(2) & 3(1) & 3(2) \end{bmatrix} = \begin{bmatrix} 3 & 6 & 12 \\ -9 & 0 & -3 \\ 6 & 3 & 6 \end{bmatrix}$

**b.** $-B = (-1)\begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ -1 & 4 & -3 \\ 1 & -3 & -2 \end{bmatrix}$

**c.** $3A - B = \begin{bmatrix} 3 & 6 & 12 \\ -9 & 0 & -3 \\ 6 & 3 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 12 \\ -10 & 4 & -6 \\ 7 & 0 & 4 \end{bmatrix}$

## MATRIX MULTIPLICATION

Another basic matrix operation is **matrix multiplication.** To see the usefulness of this operation, consider the following application in which matrices are helpful for organizing information.

A football stadium has three concession areas, located in the south, north, and west stands. The top-selling items are peanuts, hot dogs, and soda. Sales for one day are recorded in the first matrix below, and the prices (in dollars) of the three items are given in the second matrix.

***Numbers of Items Sold***

$$\begin{array}{r} \\ \text{South Stand} \\ \text{North Stand} \\ \text{West Stand} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Peanuts} & \text{Hot Dogs} & \text{Sodas} \end{array} \\ \begin{bmatrix} 120 & 250 & 305 \\ 207 & 140 & 419 \\ 29 & 120 & 190 \end{bmatrix} \end{array} \quad \begin{array}{c} \text{Selling Price} \\ \begin{bmatrix} 2.00 \\ 3.00 \\ 2.75 \end{bmatrix} \begin{array}{l} \text{Peanuts} \\ \text{Hot Dogs} \\ \text{Sodas} \end{array} \end{array}$$

To calculate the total sales of the three top-selling items at the south stand, multiply each entry in the first row of the matrix on the left by the corresponding entry in the price column matrix on the right and add the results. The south stand sales are

$$(120)(2.00) + (250)(3.00) + (305)(2.75) = \$1828.75. \qquad \text{South stand sales}$$

Similarly, the sales for the other two stands are as follows.

$$(207)(2.00) + (140)(3.00) + (419)(2.75) = \$1986.25 \qquad \text{North stand sales}$$
$$(29)(2.00) + (120)(3.00) + (190)(2.75) = \$940.50 \qquad \text{West stand sales}$$

The preceding computations are examples of matrix multiplication. You can write the product of the $3 \times 3$ matrix indicating the number of items sold and the $3 \times 1$ matrix indicating the selling prices as follows.

$$\begin{bmatrix} 120 & 250 & 305 \\ 207 & 140 & 419 \\ 29 & 120 & 190 \end{bmatrix} \begin{bmatrix} 2.00 \\ 3.00 \\ 2.75 \end{bmatrix} = \begin{bmatrix} 1828.75 \\ 1986.25 \\ 940.50 \end{bmatrix}$$

The product of these matrices is the $3 \times 1$ matrix giving the total sales for each of the three stands.

The general definition of the product of two matrices shown below is based on the ideas just developed. Although at first glance this definition may seem unusual, you will see that it has many practical applications.

### Definition of Matrix Multiplication

If $A = [a_{ij}]$ is an $m \times n$ matrix and $B = [b_{ij}]$ is an $n \times p$ matrix, then the **product** $AB$ is an $m \times p$ matrix

$$AB = [c_{ij}]$$

where

$$c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdot \cdot \cdot + a_{in}b_{nj}.$$

This definition means that to find the entry in the $i$th row and the $j$th column of the product $AB$, multiply the entries in the $i$th row of $A$ by the corresponding entries in the $j$th column of $B$ and then add the results. The next example illustrates this process.

## EXAMPLE 4 Finding the Product of Two Matrices

Find the product $AB$, where

$$A = \begin{bmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix}.$$

### SOLUTION

First, note that the product $AB$ is defined because $A$ has size $3 \times 2$ and $B$ has size $2 \times 2$. Moreover, the product $AB$ has size $3 \times 2$, and will take the form

$$\begin{bmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix}.$$

To find $c_{11}$ (the entry in the first row and first column of the product), multiply corresponding entries in the first row of $A$ and the first column of $B$. That is,

$$c_{11} = (-1)(-3) + (3)(-4) = -9$$

$$\begin{bmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} -9 & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix}.$$

Similarly, to find $c_{12}$, multiply corresponding entries in the first row of $A$ and the second column of $B$ to obtain

$$c_{12} = (-1)(2) + (3)(1) = 1$$

$$\begin{bmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} -9 & 1 \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix}.$$

Continuing this pattern produces the following results.

$$\begin{aligned} c_{21} &= (4)(-3) + (-2)(-4) = -4 \\ c_{22} &= (4)(2) \quad + (-2)(1) \quad = 6 \\ c_{31} &= (5)(-3) + (0)(-4) \quad = -15 \\ c_{32} &= (5)(2) \quad + (0)(1) \quad\;\; = 10 \end{aligned}$$

The product is

$$AB = \begin{bmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} -9 & 1 \\ -4 & 6 \\ -15 & 10 \end{bmatrix}.$$

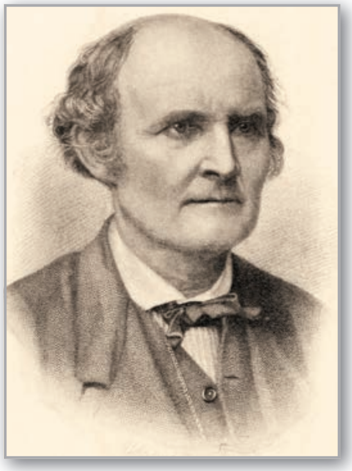
Be sure you understand that for the product of two matrices to be defined, the number of columns of the first matrix must equal the number of rows of the second matrix. That is,

$$\underset{m \times n}{A} \quad \underset{n \times p}{B} \quad = \quad \underset{m \times p}{AB}.$$

Equal

Size of $AB$

So, the product $BA$ is not defined for matrices such as $A$ and $B$ in Example 4.



**Arthur Cayley (1821–1895)**

British mathematician Arthur Cayley is credited with giving an abstract definition of a matrix. Cayley was a Cambridge University graduate and a lawyer by profession. He began his groundbreaking work on matrices as he studied the theory of transformations. Cayley also was instrumental in the development of determinants (discussed in Chapter 3). Cayley and two American mathematicians, Benjamin Peirce (1809–1880) and his son, Charles S. Peirce (1839–1914), are credited with developing "matrix algebra."

Bettmann/CORBIS

The general pattern for matrix multiplication is as follows. To obtain the element in the $i$th row and the $j$th column of the product $AB$, use the $i$th row of $A$ and the $j$th column of $B$.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2p} \\ b_{31} & b_{32} & \cdots & b_{3j} & \cdots & b_{3p} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nj} & \cdots & b_{np} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1j} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2j} & \cdots & c_{2p} \\ \vdots & \vdots & & \vdots & & \vdots \\ c_{i1} & c_{i2} & \cdots & c_{ij} & \cdots & c_{ip} \\ \vdots & \vdots & & \vdots & & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mj} & \cdots & c_{mp} \end{bmatrix}$$

$$a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{in}b_{nj} = c_{ij}$$

## DISCOVERY

Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}.$$

1. Calculate $A + B$ and $B + A$. Is matrix addition commutative?
2. Calculate $AB$ and $BA$. Is matrix multiplication commutative?

## EXAMPLE 5 Matrix Multiplication

**a.** $$\underset{2 \times 3}{\begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & -2 \end{bmatrix}} \underset{3 \times 3}{\begin{bmatrix} -2 & 4 & 2 \\ 1 & 0 & 0 \\ -1 & 1 & -1 \end{bmatrix}} = \underset{2 \times 3}{\begin{bmatrix} -5 & 7 & -1 \\ -3 & 6 & 6 \end{bmatrix}}$$

**b.** $$\underset{2 \times 2}{\begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}} \underset{2 \times 2}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} = \underset{2 \times 2}{\begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}}$$

**c.** $$\underset{2 \times 2}{\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}} \underset{2 \times 2}{\begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}} = \underset{2 \times 2}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$$

**d.** $$\underset{1 \times 3}{\begin{bmatrix} 1 & -2 & -3 \end{bmatrix}} \underset{3 \times 1}{\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}} = \underset{1 \times 1}{\begin{bmatrix} 1 \end{bmatrix}}$$

**e.** $$\underset{3 \times 1}{\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}} \underset{1 \times 3}{\begin{bmatrix} 1 & -2 & -3 \end{bmatrix}} = \underset{3 \times 3}{\begin{bmatrix} 2 & -4 & -6 \\ -1 & 2 & 3 \\ 1 & -2 & -3 \end{bmatrix}}$$

Note the difference between the two products in parts (d) and (e) of Example 5. In general, matrix multiplication is not commutative. It is usually not true that the product $AB$ is equal to the product $BA$. (See Section 2.2 for further discussion of the noncommutativity of matrix multiplication.)

## SYSTEMS OF LINEAR EQUATIONS

One practical application of matrix multiplication is representing a system of linear equations. Note how the system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}$$

can be written as the matrix equation $A\mathbf{x} = \mathbf{b}$, where $A$ is the coefficient matrix of the system, and $\mathbf{x}$ and $\mathbf{b}$ are column matrices.

$$\underset{A}{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}} \underset{\mathbf{x}}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} \underset{=}{=} \underset{\mathbf{b}}{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}$$

### EXAMPLE 6 Solving a System of Linear Equations

Solve the matrix equation $A\mathbf{x} = \mathbf{0}$, where

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 3 & -2 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \text{and} \quad \mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

#### SOLUTION

As a system of linear equations, $A\mathbf{x} = \mathbf{0}$ is

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_1 + 3x_2 - 2x_3 &= 0. \end{aligned}$$

Using Gauss-Jordan elimination on the augmented matrix of this system, you obtain

$$\begin{bmatrix} 1 & 0 & -\frac{1}{7} & 0 \\ 0 & 1 & -\frac{4}{7} & 0 \end{bmatrix}.$$

So, the system has infinitely many solutions. Here a convenient choice of a parameter is $x_3 = 7t$, and you can write the solution set as

$$x_1 = t, \quad x_2 = 4t, \quad x_3 = 7t, \quad t \text{ is any real number.}$$

In matrix terminology, you have found that the matrix equation

$$\begin{bmatrix} 1 & -2 & 1 \\ 2 & 3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

has infinitely many solutions represented by

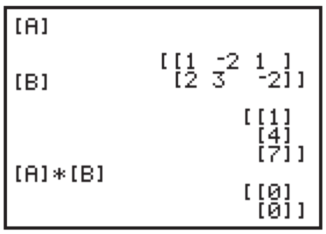
$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ 4t \\ 7t \end{bmatrix} = t \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \quad t \text{ is any scalar.}$$

That is, any scalar multiple of the column matrix on the right is a solution. Here are some sample solutions:

$$\begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 8 \\ 14 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{and} \begin{bmatrix} -1 \\ -4 \\ -7 \end{bmatrix}.$$

**TECHNOLOGY**

Many graphing utilities and software programs can perform matrix addition, scalar multiplication, and matrix multiplication. If you use a graphing utility to check one of the solutions in Example 6, you may see something similar to the following.




The **Online Technology Guide,** available at *www.cengagebrain.com*, provides syntax for programs applicable to Example 6.

## PARTITIONED MATRICES

The system $A\mathbf{x} = \mathbf{b}$ can be represented in a more convenient way by partitioning the matrices $A$ and $\mathbf{x}$ in the following manner. If

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

are the coefficient matrix, the column matrix of unknowns, and the right-hand side, respectively, of the $m \times n$ linear system $A\mathbf{x} = \mathbf{b}$, then

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \mathbf{b}$$

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix} = \mathbf{b}$$

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \mathbf{b}.$$

In other words,

$$A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

where $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ are the columns of the matrix $A$. The expression

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

is called a **linear combination** of the column matrices $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ with **coefficients** $x_1, x_2, \ldots, x_n$.

### Linear Combinations of Column Vectors

The matrix product $A\mathbf{x}$ is a linear combination of the column vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ that form the coefficient matrix $A$.

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

Furthermore, the system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ can be expressed as such a linear combination, where the coefficients of the linear combination are a solution of the system.

## EXAMPLE 7 Solving a System of Linear Equations

The linear system

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 0 \\ 4x_1 + 5x_2 + 6x_3 &= 3 \\ 7x_1 + 8x_2 + 9x_3 &= 6 \end{aligned}$$

can be rewritten as a matrix equation $A\mathbf{x} = \mathbf{b}$, as follows.

$$x_1\begin{bmatrix}1\\4\\7\end{bmatrix} + x_2\begin{bmatrix}2\\5\\8\end{bmatrix} + x_3\begin{bmatrix}3\\6\\9\end{bmatrix} = \begin{bmatrix}0\\3\\6\end{bmatrix}$$

Using Gaussian elimination, you can show that this system has infinitely many solutions, one of which is $x_1 = 1, x_2 = 1, x_3 = -1$.

$$1\begin{bmatrix}1\\4\\7\end{bmatrix} + 1\begin{bmatrix}2\\5\\8\end{bmatrix} + (-1)\begin{bmatrix}3\\6\\9\end{bmatrix} = \begin{bmatrix}0\\3\\6\end{bmatrix}$$

That is, $\mathbf{b}$ can be expressed as a linear combination of the columns of $A$. This representation of one column vector in terms of others is a fundamental theme of linear algebra.

Just as you partition $A$ into columns and $\mathbf{x}$ into rows, it is often useful to consider an $m \times n$ matrix partitioned into smaller matrices. For example, the following matrix can be partitioned as shown.

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ -1 & -2 & 2 & 1 \end{bmatrix} \qquad \left[\begin{array}{cc|cc} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ \hline -1 & -2 & 2 & 1 \end{array}\right]$$

You can also partition the matrix into column matrices

$$\left[\begin{array}{c|c|c|c} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ -1 & -2 & 2 & 1 \end{array}\right] = [\mathbf{c}_1 \quad \mathbf{c}_2 \quad \mathbf{c}_3 \quad \mathbf{c}_4]$$

or row matrices

$$\left[\begin{array}{cccc} 1 & 2 & 0 & 0 \\ \hline 3 & 4 & 0 & 0 \\ \hline -1 & -2 & 2 & 1 \end{array}\right] = \begin{bmatrix}\mathbf{r}_1\\\mathbf{r}_2\\\mathbf{r}_3\end{bmatrix}.$$

**LINEAR ALGEBRA APPLIED**

Many real-life applications of linear systems involve enormous numbers of equations and variables. For example, a flight crew scheduling problem for American Airlines required the manipulation of matrices with 837 rows and more than 12,750,000 columns. This application of *linear programming* required that the problem be partitioned into smaller pieces and then solved on a Cray supercomputer. *(Source: Very Large-Scale Linear Programming. A Case Study in Combining Interior Point and Simplex Methods, Bixby, Robert E., et al., Operations Research, 40, no. 5)*

# 2.1 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

**Equality of Matrices** In Exercises 1–4, find $x$ and $y$.

**1.** $\begin{bmatrix} x & -2 \\ 7 & y \end{bmatrix} = \begin{bmatrix} -4 & -2 \\ 7 & 22 \end{bmatrix}$

**2.** $\begin{bmatrix} -5 & x \\ y & 8 \end{bmatrix} = \begin{bmatrix} -5 & 13 \\ 12 & 8 \end{bmatrix}$

**3.** $\begin{bmatrix} 16 & 4 & 5 & 4 \\ -3 & 13 & 15 & 6 \\ 0 & 2 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 16 & 4 & 2x+1 & 4 \\ -3 & 13 & 15 & 3x \\ 0 & 2 & 3y-5 & 0 \end{bmatrix}$

**4.** $\begin{bmatrix} x+2 & 8 & -3 \\ 1 & 2y & 2x \\ 7 & -2 & y+2 \end{bmatrix} = \begin{bmatrix} 2x+6 & 8 & -3 \\ 1 & 18 & -8 \\ 7 & -2 & 11 \end{bmatrix}$

**Operations with Matrices** In Exercises 5–12, find, if possible, (a) $A + B$, (b) $A - B$, (c) $2A$, (d) $2A - B$, and (e) $B + \frac{1}{2}A$.

**5.** $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -1 \\ -1 & 8 \end{bmatrix}$

**6.** $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -3 & -2 \\ 4 & 2 \end{bmatrix}$

**7.** $A = \begin{bmatrix} 6 & -1 \\ 2 & 4 \\ -3 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 4 \\ -1 & 5 \\ 1 & 10 \end{bmatrix}$

**8.** $A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & -1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -3 & 4 \\ -3 & 1 & -2 \end{bmatrix}$

**9.** $A = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 4 & 5 \\ 0 & 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 2 & 1 \\ 5 & 4 & 2 \\ 2 & 1 & 0 \end{bmatrix}$

**10.** $A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 6 & 2 \\ 4 & 1 & 0 \\ -1 & 2 & 4 \end{bmatrix}$

**11.** $A = \begin{bmatrix} 6 & 0 & 3 \\ -1 & -4 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 8 & -1 \\ 4 & -3 \end{bmatrix}$

**12.** $A = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$, $B = \begin{bmatrix} -4 & 6 & 2 \end{bmatrix}$

**13.** Find (a) $c_{21}$ and (b) $c_{13}$, where $C = 2A - 3B$,

$$A = \begin{bmatrix} 5 & 4 & 4 \\ -3 & 1 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 2 & -7 \\ 0 & -5 & 1 \end{bmatrix}.$$

**14.** Find (a) $c_{23}$ and (b) $c_{32}$, where $C = 5A + 2B$,

$$A = \begin{bmatrix} 4 & 11 & -9 \\ 0 & 3 & 2 \\ -3 & 1 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 & 5 \\ -4 & 6 & 11 \\ -6 & 4 & 9 \end{bmatrix}.$$

**15.** Solve for $x$, $y$, and $z$ in the matrix equation

$$4\begin{bmatrix} x & y \\ z & -1 \end{bmatrix} = 2\begin{bmatrix} y & z \\ -x & 1 \end{bmatrix} + 2\begin{bmatrix} 4 & x \\ 5 & -x \end{bmatrix}.$$

**16.** Solve for $x$, $y$, $z$, and $w$ in the matrix equation

$$\begin{bmatrix} w & x \\ y & x \end{bmatrix} = \begin{bmatrix} -4 & 3 \\ 2 & -1 \end{bmatrix} + 2\begin{bmatrix} y & w \\ z & x \end{bmatrix}.$$

**Finding Products of Two Matrices** In Exercises 17–30, find, if possible, (a) $AB$ and (b) $BA$.

**17.** $A = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -1 \\ -1 & 8 \end{bmatrix}$

**18.** $A = \begin{bmatrix} 2 & -2 \\ -1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 1 \\ 2 & -2 \end{bmatrix}$

**19.** $A = \begin{bmatrix} 2 & -1 & 3 \\ 5 & 1 & -2 \\ 2 & 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 & 2 \\ -4 & 1 & 3 \\ -4 & -1 & -2 \end{bmatrix}$

**20.** $A = \begin{bmatrix} 1 & -1 & 7 \\ 2 & -1 & 8 \\ 3 & 1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 2 \end{bmatrix}$

**21.** $A = \begin{bmatrix} 2 & 1 \\ -3 & 4 \\ 1 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -1 & 0 \\ 4 & 0 & 2 \\ 8 & -1 & 7 \end{bmatrix}$

**22.** $A = \begin{bmatrix} 3 & 2 & 1 \\ -3 & 0 & 4 \\ 4 & -2 & -4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 2 & -1 \\ 1 & -2 \end{bmatrix}$

**23.** $A = \begin{bmatrix} 3 & 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$

**24.** $A = \begin{bmatrix} -1 \\ 2 \\ -2 \\ 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 & 3 & 2 \end{bmatrix}$

**25.** $A = \begin{bmatrix} -1 & 3 \\ 4 & -5 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 0 & 7 \end{bmatrix}$

**26.** $A = \begin{bmatrix} 2 & -3 \\ 5 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 1 & 3 \\ 2 & -1 \end{bmatrix}$

**27.** $A = \begin{bmatrix} 0 & -1 & 0 \\ 4 & 0 & 2 \\ 8 & -1 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$

**28.** $A = \begin{bmatrix} 2 & 1 & 2 \\ 3 & -1 & -2 \\ -2 & 1 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 0 & 1 & 3 \\ -1 & 2 & -3 & -1 \\ -2 & 1 & 4 & 3 \end{bmatrix}$

**29.** $A = \begin{bmatrix} 6 \\ -2 \\ 1 \\ 6 \end{bmatrix}, \quad B = \begin{bmatrix} 10 & 12 \end{bmatrix}$

**30.** $A = \begin{bmatrix} 1 & 0 & 3 & -2 & 4 \\ 6 & 13 & 8 & -17 & 20 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 6 \\ 4 & 2 \end{bmatrix}$

**Matrix Size** In Exercises 31–38, let $A, B, C, D,$ and $E$ be matrices with the following sizes.

$A$: $3 \times 4$ $B$: $3 \times 4$ $C$: $4 \times 2$ $D$: $4 \times 2$ $E$: $4 \times 3$

If defined, determine the size of the matrix. If not defined, explain why.

**31.** $A + B$

**32.** $C + E$

**33.** $\frac{1}{2}D$

**34.** $-4A$

**35.** $AC$

**36.** $BE$

**37.** $E - 2A$

**38.** $2D + C$

**Solving a Matrix Equation** In Exercises 39 and 40, solve the matrix equation $A\mathbf{x} = \mathbf{0}$.

**39.** $A = \begin{bmatrix} 2 & -1 & -1 \\ 1 & -2 & 2 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

**40.** $A = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 2 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

**Solving a System of Linear Equations** In Exercises 41–48, write the system of linear equations in the form $A\mathbf{x} = \mathbf{b}$ and solve this matrix equation for $\mathbf{x}$.

**41.** $\begin{aligned} -x_1 + x_2 &= 4 \\ -2x_1 + x_2 &= 0 \end{aligned}$

**42.** $\begin{aligned} 2x_1 + 3x_2 &= 5 \\ x_1 + 4x_2 &= 10 \end{aligned}$

**43.** $\begin{aligned} -2x_1 - 3x_2 &= -4 \\ 6x_1 + x_2 &= -36 \end{aligned}$

**44.** $\begin{aligned} -4x_1 + 9x_2 &= -13 \\ x_1 - 3x_2 &= 12 \end{aligned}$

**45.** $\begin{aligned} x_1 - 2x_2 + 3x_3 &= 9 \\ -x_1 + 3x_2 - x_3 &= -6 \\ 2x_1 - 5x_2 + 5x_3 &= 17 \end{aligned}$

**46.** $\begin{aligned} x_1 + x_2 - 3x_3 &= -1 \\ -x_1 + 2x_2 \phantom{- 3x_3} &= 1 \\ x_1 - x_2 + x_3 &= 2 \end{aligned}$

**47.** $\begin{aligned} x_1 - 5x_2 + 2x_3 &= -20 \\ -3x_1 + x_2 - x_3 &= 8 \\ -2x_2 + 5x_3 &= -16 \end{aligned}$

**48.** $\begin{aligned} x_1 - x_2 + 4x_3 &= 17 \\ x_1 + 3x_2 \phantom{+ 4x_3} &= -11 \\ -6x_2 + 5x_3 &= 40 \end{aligned}$

**Writing a Linear Combination** In Exercises 49–52, write the column matrix $\mathbf{b}$ as a linear combination of the columns of $A$.

**49.** $A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & -3 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -1 \\ 7 \end{bmatrix}$

**50.** $A = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$

**51.** $A = \begin{bmatrix} 1 & 1 & -5 \\ 1 & 0 & -1 \\ 2 & -1 & -1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$

**52.** $A = \begin{bmatrix} -3 & 5 \\ 3 & 4 \\ 4 & -8 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -22 \\ 4 \\ 32 \end{bmatrix}$

**Solving a Matrix Equation** In Exercises 53 and 54, solve for $A$.

**53.** $\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

**54.** $\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

**Solving a Matrix Equation** In Exercises 55 and 56, solve the matrix equation for $a, b, c,$ and $d$.

**55.** $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 19 & 2 \end{bmatrix}$

**56.** $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 17 \\ 4 & -1 \end{bmatrix}$

**Diagonal Matrix** In Exercises 57 and 58, find the product $AA$ for the diagonal matrix. A square matrix

$$A = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

is called a diagonal matrix when all entries that are not on the main diagonal are zero.

**57.** $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

**58.** $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

**Finding Products of Diagonal Matrices** In Exercises 59 and 60, find the products $AB$ and $BA$ for the diagonal matrices.

**59.** $A = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} -5 & 0 \\ 0 & 4 \end{bmatrix}$

**60.** $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -7 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 12 \end{bmatrix}$

**61. Guided Proof** Prove that if $A$ and $B$ are diagonal matrices (of the same size), then $AB = BA$.

Getting Started: To prove that the matrices $AB$ and $BA$ are equal, you need to show that their corresponding entries are equal.

(i) Begin your proof by letting $A = [a_{ij}]$ and $B = [b_{ij}]$ be two diagonal $n \times n$ matrices.

(ii) The $ij$th entry of the product $AB$ is $c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$.

(iii) Evaluate the entries $c_{ij}$ for the two cases $i \neq j$ and $i = j$.

(iv) Repeat this analysis for the product $BA$.

**62. Writing** Let $A$ and $B$ be $3 \times 3$ matrices, where $A$ is diagonal.

(a) Describe the product $AB$. Illustrate your answer with examples.

(b) Describe the product $BA$. Illustrate your answer with examples.

(c) How do the results in parts (a) and (b) change when the diagonal entries of $A$ are all equal?

**Trace of a Matrix In Exercises 63–66, find the trace of the matrix. The trace of an $n \times n$ matrix $A$ is the sum of the main diagonal entries. That is, $\text{Tr}(A) = a_{11} + a_{22} + \cdots + a_{nn}$.**

**63.** $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & 4 \\ 3 & 1 & 3 \end{bmatrix}$

**64.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

**65.** $\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & 2 \\ 4 & 2 & 1 & 0 \\ 0 & 0 & 5 & 1 \end{bmatrix}$

**66.** $\begin{bmatrix} 1 & 4 & 3 & 2 \\ 4 & 0 & 6 & 1 \\ 3 & 6 & 2 & 1 \\ 2 & 1 & 1 & -3 \end{bmatrix}$

**67. Proof** Prove that each statement is true when $A$ and $B$ are square matrices of order $n$ and $c$ is a scalar.

(a) $\text{Tr}(A + B) = \text{Tr}(A) + \text{Tr}(B)$

(b) $\text{Tr}(cA) = c\text{Tr}(A)$

**68. Proof** Prove that if $A$ and $B$ are square matrices of order $n$, then $\text{Tr}(AB) = \text{Tr}(BA)$.

**69.** Find conditions on $w$, $x$, $y$, and $z$ such that $AB = BA$ for the following matrices.

$$A = \begin{bmatrix} w & x \\ y & z \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

**70.** Verify $AB = BA$ for the following matrices.

$$A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}$$

**71.** Show that the matrix equation has no solution.

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

**72.** Show that no $2 \times 2$ matrices $A$ and $B$ exist that satisfy the matrix equation

$$AB - BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

**73. Exploration** Let $i = \sqrt{-1}$ and let

$$A = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}.$$

(a) Find $A^2$, $A^3$, and $A^4$. (*Note:* $A^2 = AA$, $A^3 = AAA = A^2A$, and so on.) Identify any similarities with $i^2$, $i^3$, and $i^4$.

(b) Find and identify $B^2$.

**74. Guided Proof** Prove that if the product $AB$ is a square matrix, then the product $BA$ is defined.

Getting Started: To prove that the product $BA$ is defined, you need to show that the number of columns of $B$ equals the number of rows of $A$.

(i) Begin your proof by noting that the number of columns of $A$ equals the number of rows of $B$.

(ii) Then assume that $A$ has size $m \times n$ and $B$ has size $n \times p$.

(iii) Use the hypothesis that the product $AB$ is a square matrix.

**75. Proof** Prove that if both products $AB$ and $BA$ are defined, then $AB$ and $BA$ are square matrices.

**76.** Let $A$ and $B$ be matrices such that the product $AB$ is defined. Show that if $A$ has two identical rows, then the corresponding two rows of $AB$ are also identical.

**77.** Let $A$ and $B$ be $n \times n$ matrices. Show that if the $i$th row of $A$ has all zero entries, then the $i$th row of $AB$ will have all zero entries. Give an example using $2 \times 2$ matrices to show that the converse is not true.

**78. CAPSTONE** Let matrices $A$ and $B$ be of sizes $3 \times 2$ and $2 \times 2$, respectively. Answer the following questions and explain your reasoning.

(a) Is it possible that $A = B$?

(b) Is $A + B$ defined?

(c) Is $AB$ defined? If so, is it possible that $AB = BA$?

**79. Agriculture** A fruit grower raises two crops, apples and peaches. The grower ships each of these crops to three different outlets. In the matrix

$$A = \begin{bmatrix} 125 & 100 & 75 \\ 100 & 175 & 125 \end{bmatrix}$$

$a_{ij}$ represents the number of units of crop $i$ that the grower ships to outlet $j$. The matrix

$$B = [\$3.50 \quad \$6.00]$$

represents the profit per unit. Find the product $BA$ and state what each entry of the matrix represents.

**80. Manufacturing** A corporation has three factories, each of which manufactures acoustic guitars and electric guitars. In the matrix

$$A = \begin{bmatrix} 70 & 50 & 25 \\ 35 & 100 & 70 \end{bmatrix}$$

$a_{ij}$ represents the number of guitars of type $i$ produced at factory $j$ in one day. Find the production levels when production increases by 20%.

**81. Politics** In the matrix

$$P = \begin{array}{ccc} & \overbrace{\begin{array}{ccc}\text{R} & \text{D} & \text{I}\end{array}}^{\text{From}} & \\ & \begin{bmatrix} 0.6 & 0.1 & 0.1 \\ 0.2 & 0.7 & 0.1 \\ 0.2 & 0.2 & 0.8 \end{bmatrix} & \left.\begin{array}{l}\text{R}\\ \text{D}\\ \text{I}\end{array}\right\}\text{To} \end{array}$$

each entry $p_{ij}$ $(i \neq j)$ represents the proportion of the voting population that changes from party $i$ to party $j$, and $p_{ii}$ represents the proportion that remains loyal to party $i$ from one election to the next. Find and interpret the product of $P$ with itself.

**82. Population** The matrices show the numbers of people (in thousands) who lived in each region of the United States in 2009 and the numbers of people (in thousands) projected to live in each region in 2015. The regional populations are separated into three age categories. (Source: U.S. Census Bureau)

**2009**

| | 0–17 | 18–64 | 65+ |
|---|---|---|---|
| Northeast | 12,399 | 35,137 | 7747 |
| Midwest | 16,047 | 41,902 | 8888 |
| South | 27,959 | 70,571 | 14,608 |
| Mountain | 5791 | 13,716 | 2616 |
| Pacific | 12,352 | 31,381 | 5712 |

**2015**

| | 0–17 | 18–64 | 65+ |
|---|---|---|---|
| Northeast | 12,441 | 35,289 | 8835 |
| Midwest | 16,363 | 42,250 | 9955 |
| South | 29,373 | 73,496 | 17,572 |
| Mountain | 6015 | 14,231 | 3337 |
| Pacific | 12,826 | 33,292 | 7086 |

(a) The total population in 2009 was approximately 307 million and the projected total population in 2015 is about 322 million. Rewrite the matrices to give the information as percents of the total population.

(b) Write a matrix that gives the projected changes in the percents of the population in each region and age group from 2009 to 2015.

(c) Based on the result of part (b), which age group(s) is (are) projected to show relative growth from 2009 to 2015?

**Block Multiplication** **In Exercises 83 and 84, perform the indicated block multiplication of matrices $A$ and $B$. If matrices $A$ and $B$ are each partitioned into four submatrices**

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

**then you can block multiply $A$ and $B$, provided the sizes of the submatrices are such that the matrix multiplications and additions are defined.**

$$AB = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$
$$= \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}$$

**83.** $A = \left[\begin{array}{cc|cc} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 2 & 1 \end{array}\right], \quad B = \left[\begin{array}{cc|c} 1 & 2 & 0 \\ -1 & 1 & 0 \\ \hline 0 & 0 & 1 \\ 0 & 0 & 3 \end{array}\right]$

**84.** $A = \left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right], \quad B = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ \hline 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{array}\right]$

**True or False?** **In Exercises 85 and 86, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.**

**85.** (a) For the product of two matrices to be defined, the number of columns of the first matrix must equal the number of rows of the second matrix.

(b) The system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ can be expressed as a linear combination of the columns of $A$, where the coefficients of the linear combination are a solution of the system.

**86.** (a) If $A$ is an $m \times n$ matrix and $B$ is an $n \times r$ matrix, then the product $AB$ is an $m \times r$ matrix.

(b) The matrix equation $A\mathbf{x} = \mathbf{b}$, where $A$ is the coefficient matrix and $\mathbf{x}$ and $\mathbf{b}$ are column matrices, can be used to represent a system of linear equations.

**87.** The columns of matrix $T$ show the coordinates of the vertices of a triangle. Matrix $A$ is a transformation matrix.

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad T = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \end{bmatrix}$$

(a) Find $AT$ and $AAT$. Then sketch the original triangle and the two transformed triangles. What transformation does $A$ represent?

(b) A triangle is determined by $AAT$. Describe the transformation process that produces the triangle determined by $AT$ and then the triangle determined by $T$.

# 2.2 Properties of Matrix Operations

- Use the properties of matrix addition, scalar multiplication, and zero matrices.
- Use the properties of matrix multiplication and the identity matrix.
- Find the transpose of a matrix.

## ALGEBRA OF MATRICES

In Section 2.1, you concentrated on the mechanics of the three basic matrix operations: matrix addition, scalar multiplication, and matrix multiplication. This section begins to develop the **algebra of matrices.** You will see that this algebra shares many (but not all) of the properties of the algebra of real numbers. Theorem 2.1 lists several properties of matrix addition and scalar multiplication.

**THEOREM 2.1 Properties of Matrix Addition and Scalar Multiplication**

If $A$, $B$, and $C$ are $m \times n$ matrices, and $c$ and $d$ are scalars, then the following properties are true.

**1.** $A + B = B + A$ — Commutative property of addition
**2.** $A + (B + C) = (A + B) + C$ — Associative property of addition
**3.** $(cd)A = c(dA)$ — Associative property of multiplication
**4.** $1A = A$ — Multiplicative identity
**5.** $c(A + B) = cA + cB$ — Distributive property
**6.** $(c + d)A = cA + dA$ — Distributive property

**PROOF**

The proofs of these six properties follow directly from the definitions of matrix addition, scalar multiplication, and the corresponding properties of real numbers. For example, to prove the commutative property of *matrix addition,* let $A = [a_{ij}]$ and $B = [b_{ij}]$. Then, using the commutative property of *addition of real numbers,* write

$$A + B = [a_{ij} + b_{ij}] = [b_{ij} + a_{ij}] = B + A.$$

Similarly, to prove Property 5, use the distributive property (for real numbers) of multiplication over addition to write

$$c(A + B) = [c(a_{ij} + b_{ij})] = [ca_{ij} + cb_{ij}] = cA + cB.$$

The proofs of the remaining four properties are left as exercises. (See Exercises 57–60.)

The preceding section defined matrix addition as the sum of *two* matrices, making it a binary operation. The associative property of matrix addition now allows you to write expressions such as $A + B + C$ as $(A + B) + C$ or as $A + (B + C)$. This same reasoning applies to sums of four or more matrices.

**EXAMPLE 1 Addition of More than Two Matrices**

To obtain the sum of four matrices, add corresponding entries as shown below.

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

One important property of the addition of real numbers is that the number 0 is the additive identity. That is, $c + 0 = c$ for any real number $c$. For matrices, a similar property holds. Specifically, if $A$ is an $m \times n$ matrix and $O_{mn}$ is the $m \times n$ matrix consisting entirely of zeros, then $A + O_{mn} = A$. The matrix $O_{mn}$ is called a **zero matrix,** and it is the **additive identity** for the set of all $m \times n$ matrices. For example, the following matrix is the additive identity for the set of all $2 \times 3$ matrices.

$$O_{23} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

When the size of the matrix is understood, you may denote a zero matrix simply by $O$ or $\mathbf{0}$.

The following properties of zero matrices are relatively easy to prove, and their proofs are left as an exercise. (See Exercise 61.)

**REMARK**

Property 2 can be described by saying that matrix $-A$ is the **additive inverse** of $A$.

### THEOREM 2.2 Properties of Zero Matrices

If $A$ is an $m \times n$ matrix and $c$ is a scalar, then the following properties are true.

**1.** $A + O_{mn} = A$
**2.** $A + (-A) = O_{mn}$
**3.** If $cA = O_{mn}$, then $c = 0$ or $A = O_{mn}$.

The algebra of real numbers and the algebra of matrices have many similarities. For example, compare the following solutions.

| Real Numbers (Solve for $x$.) | $m \times n$ Matrices (Solve for $X$.) |
|---|---|
| $x + a = b$ | $X + A = B$ |
| $x + a + (-a) = b + (-a)$ | $X + A + (-A) = B + (-A)$ |
| $x + 0 = b - a$ | $X + O = B - A$ |
| $x = b - a$ | $X = B - A$ |

Example 2 demonstrates the process of solving a matrix equation.

## EXAMPLE 2 Solving a Matrix Equation

Solve for $X$ in the equation $3X + A = B$, where

$$A = \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -3 & 4 \\ 2 & 1 \end{bmatrix}.$$

**SOLUTION**

Begin by solving the equation for $X$ to obtain

$$\begin{aligned} 3X &= B - A \\ X &= \tfrac{1}{3}(B - A). \end{aligned}$$

Now, using the matrices $A$ and $B$, you have

$$\begin{aligned} X &= \tfrac{1}{3}\left(\begin{bmatrix} -3 & 4 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}\right) \\ &= \tfrac{1}{3}\begin{bmatrix} -4 & 6 \\ 2 & -2 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{4}{3} & 2 \\ \frac{2}{3} & -\frac{2}{3} \end{bmatrix}. \end{aligned}$$

**REMARK**

Note that no commutative property of matrix multiplication is listed in Theorem 2.3. Although the product $AB$ is defined, the product $BA$ may not be defined. For instance, if $A$ is of size $2 \times 3$ and $B$ is of size $3 \times 3$, then the product $AB$ is defined but the product $BA$ is not.

## PROPERTIES OF MATRIX MULTIPLICATION

The next theorem extends the algebra of matrices to include some useful properties of matrix multiplication. The proof of Property 2 is below. The proofs of the remaining properties are left as an exercise. (See Exercise 62.)

### THEOREM 2.3 Properties of Matrix Multiplication

If $A$, $B$, and $C$ are matrices (with sizes such that the given matrix products are defined), and $c$ is a scalar, then the following properties are true.

1. $A(BC) = (AB)C$ Associative property of multiplication
2. $A(B + C) = AB + AC$ Distributive property
3. $(A + B)C = AC + BC$ Distributive property
4. $c(AB) = (cA)B = A(cB)$

### PROOF

To prove Property 2, show that the corresponding entries of matrices $A(B + C)$ and $AB + AC$ are equal. Assume $A$ has size $m \times n$, $B$ has size $n \times p$, and $C$ has size $n \times p$. Using the definition of matrix multiplication, the entry in the $i$th row and $j$th column of $A(B + C)$ is $a_{i1}(b_{1j} + c_{1j}) + a_{i2}(b_{2j} + c_{2j}) + \cdots + a_{in}(b_{nj} + c_{nj})$.

Moreover, the entry in the $i$th row and $j$th column of $AB + AC$ is

$$(a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}) + (a_{i1}c_{1j} + a_{i2}c_{2j} + \cdots + a_{in}c_{nj}).$$

By distributing and regrouping, you can see that these two $ij$th entries are equal. So,

$$A(B + C) = AB + AC.$$

The associative property of matrix multiplication permits you to write such matrix products as $ABC$ without ambiguity, as demonstrated in Example 3.

### EXAMPLE 3 Matrix Multiplication Is Associative

Find the matrix product $ABC$ by grouping the factors first as $(AB)C$ and then as $A(BC)$. Show that you obtain the same result from both processes.

$$A = \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 2 \\ 3 & -2 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 0 \\ 3 & 1 \\ 2 & 4 \end{bmatrix}$$

### SOLUTION

Grouping the factors as $(AB)C$, you have

$$\begin{aligned}(AB)C &= \left(\begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 3 & -2 & 1 \end{bmatrix}\right)\begin{bmatrix} -1 & 0 \\ 3 & 1 \\ 2 & 4 \end{bmatrix} \\ &= \begin{bmatrix} -5 & 4 & 0 \\ -1 & 2 & 3 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 3 & 1 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 17 & 4 \\ 13 & 14 \end{bmatrix}.\end{aligned}$$

Grouping the factors as $A(BC)$, you obtain the same result.

$$\begin{aligned}A(BC) &= \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}\left(\begin{bmatrix} 1 & 0 & 2 \\ 3 & -2 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 3 & 1 \\ 2 & 4 \end{bmatrix}\right) \\ &= \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 3 & 8 \\ -7 & 2 \end{bmatrix} = \begin{bmatrix} 17 & 4 \\ 13 & 14 \end{bmatrix}\end{aligned}$$

The next example shows that even when both products $AB$ and $BA$ are defined, they may not be equal.

## EXAMPLE 4 Noncommutativity of Matrix Multiplication

Show that $AB$ and $BA$ are not equal for the matrices

$$A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}.$$

**SOLUTION**

$$AB = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 4 & -4 \end{bmatrix}, \; BA = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 7 \\ 4 & -2 \end{bmatrix}$$

$AB \neq BA$

Do not conclude from Example 4 that the matrix products $AB$ and $BA$ are *never* equal. Sometimes they are equal. For example, find $AB$ and $BA$ for the following matrices.

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -2 & 4 \\ 2 & -2 \end{bmatrix}$$

You will see that the two products are equal. The point is that although $AB$ and $BA$ are sometimes equal, $AB$ and $BA$ are usually not equal.

Another important quality of matrix algebra is that it does not have a general cancellation property for matrix multiplication. That is, when $AC = BC$, it is not necessarily true that $A = B$. Example 5 demonstrates this. (In the next section you will see that, for some special types of matrices, cancellation is valid.)

## EXAMPLE 5 An Example in Which Cancellation Is Not Valid

Show that $AC = BC$.

$$A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix}$$

**SOLUTION**

$$AC = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix}, \; BC = \begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix}$$

$AC = BC$, even though $A \neq B$.

You will now look at a special type of *square* matrix that has 1's on the main diagonal and 0's elsewhere.

$$I_n = \underbrace{\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}}_{n \times n}$$

For instance, for $n = 1, 2,$ and $3$,

$$I_1 = [1], \quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

When the order of the matrix is understood to be $n$, you may denote $I_n$ simply as $I$.

As stated in Theorem 2.4 on the next page, the matrix $I_n$ serves as the **identity** for matrix multiplication; it is called the **identity matrix of order $n$.** The proof of this theorem is left as an exercise. (See Exercise 63.)

**REMARK**

Note that if $A$ is a *square* matrix of order $n$, then $AI_n = I_nA = A$.

**THEOREM 2.4 Properties of the Identity Matrix**

If $A$ is a matrix of size $m \times n$, then the following properties are true.

**1.** $AI_n = A$ **2.** $I_mA = A$

## EXAMPLE 6 Multiplication by an Identity Matrix

**a.** $\begin{bmatrix} 3 & -2 \\ 4 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 4 & 0 \\ -1 & 1 \end{bmatrix}$ **b.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$

For repeated multiplication of *square* matrices, use the same exponential notation used with real numbers. That is, $A^1 = A$, $A^2 = AA$, and for a positive integer $k$, $A^k$ is

$$A^k = \underbrace{AA \cdot \cdot \cdot A}_{k \text{ factors}}.$$

It is convenient also to define $A^0 = I_n$ (where $A$ is a square matrix of order $n$). These definitions allow you to establish the properties (1) $A^jA^k = A^{j+k}$ and (2) $(A^j)^k = A^{jk}$, where $j$ and $k$ are nonnegative integers.

## EXAMPLE 7 Repeated Multiplication of a Square Matrix

For the matrix $A = \begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix}$,

$$A^3 = \left(\begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix}\right)\begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 6 & -3 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} -4 & -1 \\ 3 & -6 \end{bmatrix}.$$

In Section 1.1, you saw that a system of linear equations must have exactly one solution, infinitely many solutions, or no solution. You can use matrix algebra to prove this.

**THEOREM 2.5 Number of Solutions of a Linear System**

For a system of linear equations, precisely one of the following is true.

1. The system has exactly one solution.
2. The system has infinitely many solutions.
3. The system has no solution.

**PROOF**

Represent the system by the matrix equation $A\mathbf{x} = \mathbf{b}$. If the system has exactly one solution or no solution, then there is nothing to prove. So, assume that the system has at least two distinct solutions $\mathbf{x}_1$ and $\mathbf{x}_2$. If you show that this assumption implies that the system has infinitely many solutions, then the proof will be complete. Because $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions, you have $A\mathbf{x}_1 = A\mathbf{x}_2 = \mathbf{b}$ and $A(\mathbf{x}_1 - \mathbf{x}_2) = O$. This implies that the (nonzero) column matrix $\mathbf{x}_h = \mathbf{x}_1 - \mathbf{x}_2$ is a solution of the homogeneous system of linear equations $A\mathbf{x} = O$. So, for any scalar $c$,

$$A(\mathbf{x}_1 + c\mathbf{x}_h) = A\mathbf{x}_1 + A(c\mathbf{x}_h) = \mathbf{b} + c(A\mathbf{x}_h) = \mathbf{b} + cO = \mathbf{b}.$$

So $\mathbf{x}_1 + c\mathbf{x}_h$ is a solution of $A\mathbf{x} = \mathbf{b}$ for any scalar $c$. Because there are infinitely many possible values of $c$ and each value produces a different solution, the system has infinitely many solutions.

## THE TRANSPOSE OF A MATRIX

The **transpose** of a matrix is formed by writing its columns as rows. For instance, if $A$ is the $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

Size: $m \times n$

then the transpose, denoted by $A^T$, is the $n \times m$ matrix

$$A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} & \cdots & a_{m1} \\ a_{12} & a_{22} & a_{32} & \cdots & a_{m2} \\ a_{13} & a_{23} & a_{33} & \cdots & a_{m3} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & a_{3n} & \cdots & a_{mn} \end{bmatrix}.$$

Size: $n \times m$

**DISCOVERY**

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}$.

1. Calculate $(AB)^T$, $A^TB^T$, and $B^TA^T$.
2. Make a conjecture about the transpose of a product of two square matrices.
3. Select two other square matrices to check your conjecture.

### EXAMPLE 8 The Transpose of a Matrix

Find the transpose of each matrix.

**a.** $A = \begin{bmatrix} 2 \\ 8 \end{bmatrix}$ **b.** $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ **c.** $C = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **d.** $D = \begin{bmatrix} 0 & 1 \\ 2 & 4 \\ 1 & -1 \end{bmatrix}$

**SOLUTION**

**a.** $A^T = \begin{bmatrix} 2 & 8 \end{bmatrix}$ **b.** $B^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$

**c.** $C^T = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **d.** $D^T = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 4 & -1 \end{bmatrix}$

**REMARK**

Note that the square matrix in part (c) is equal to its transpose. Such a matrix is called **symmetric.** A matrix $A$ is symmetric when $A = A^T$. From this definition it should be clear that a symmetric matrix must be square. Also, if $A = [a_{ij}]$ is a symmetric matrix, then $a_{ij} = a_{ji}$ for all $i \neq j$.

**THEOREM 2.6 Properties of Transposes**

If $A$ and $B$ are matrices (with sizes such that the given matrix operations are defined) and $c$ is a scalar, then the following properties are true.

1. $(A^T)^T = A$ Transpose of a transpose
2. $(A + B)^T = A^T + B^T$ Transpose of a sum
3. $(cA)^T = c(A^T)$ Transpose of a scalar multiple
4. $(AB)^T = B^TA^T$ Transpose of a product

**PROOF**

Because the transpose operation interchanges rows and columns, Property 1 seems to make sense. To prove Property 1, let $A$ be an $m \times n$ matrix. Observe that $A^T$ has size $n \times m$ and $(A^T)^T$ has size $m \times n$, the same as $A$. To show that $(A^T)^T = A$, you must show that the $ij$th entries are the same. Let $a_{ij}$ be the $ij$th entry of $A$. Then $a_{ij}$ is the $ji$th entry of $A^T$, and the $ij$th entry of $(A^T)^T$. This proves Property 1. The proofs of the remaining properties are left as an exercise. (See Exercise 64.)

**REMARK**

Remember that you *reverse the order* of multiplication when forming the transpose of a product. That is, the transpose of $AB$ is $(AB)^T = B^TA^T$ and is usually *not* equal to $A^TB^T$.

Properties 2 and 4 can be generalized to cover sums or products of any finite number of matrices. For instance, the transpose of the sum of three matrices is

$$(A + B + C)^T = A^T + B^T + C^T$$

and the transpose of the product of three matrices is

$$(ABC)^T = C^TB^TA^T.$$

## EXAMPLE 9 Finding the Transpose of a Product

Show that $(AB)^T$ and $B^TA^T$ are equal.

$$A = \begin{bmatrix} 2 & 1 & -2 \\ -1 & 0 & 3 \\ 0 & -2 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 1 \\ 2 & -1 \\ 3 & 0 \end{bmatrix}$$

**SOLUTION**

$$AB = \begin{bmatrix} 2 & 1 & -2 \\ -1 & 0 & 3 \\ 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 2 & -1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 6 & -1 \\ -1 & 2 \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} 2 & 6 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$B^TA^T = \begin{bmatrix} 3 & 2 & 3 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 2 & -1 & 0 \\ 1 & 0 & -2 \\ -2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 6 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$(AB)^T = B^TA^T$$

## EXAMPLE 10 The Product of a Matrix and Its Transpose

For the matrix $A = \begin{bmatrix} 1 & 3 \\ 0 & -2 \\ -2 & -1 \end{bmatrix}$, find the product $AA^T$ and show that it is symmetric.

**SOLUTION**

Because $AA^T = \begin{bmatrix} 1 & 3 \\ 0 & -2 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -2 \\ 3 & -2 & -1 \end{bmatrix} = \begin{bmatrix} 10 & -6 & -5 \\ -6 & 4 & 2 \\ -5 & 2 & 5 \end{bmatrix}$,

it follows that $AA^T = (AA^T)^T$, so $AA^T$ is symmetric.

**REMARK**

The property demonstrated in Example 10 is true in general. That is, for any matrix $A$, the matrix $AA^T$ is symmetric. The matrix $A^TA$ is also symmetric. You are asked to prove these results in Exercise 65.


**LINEAR ALGEBRA APPLIED**

Information retrieval systems such as Internet search engines make use of matrix theory and linear algebra to keep track of, for instance, keywords that occur in a database. To illustrate with a simplified example, suppose you wanted to perform a search on some of the $m$ available keywords in a database of $n$ documents. You could represent the search with the $m \times 1$ column matrix $\mathbf{x}$, in which a 1 entry represents a keyword you are searching and 0 represents a keyword you are not searching. You could represent the occurrences of the $m$ keywords in the $n$ documents with $A$, an $m \times n$ matrix in which an entry is 1 if the keyword occurs in the document and 0 if it does not occur in the document. Then, the $n \times 1$ matrix product $A^T\mathbf{x}$ would represent the number of keywords in your search that occur in each of the $n$ documents.

# 2.2 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

**Evaluating an Expression** In Exercises 1–6, evaluate the expression.

1. $\begin{bmatrix} -5 & 0 \\ 3 & -6 \end{bmatrix} + \begin{bmatrix} 7 & 1 \\ -2 & -1 \end{bmatrix} + \begin{bmatrix} -10 & -8 \\ 14 & 6 \end{bmatrix}$

2. $\begin{bmatrix} 6 & 8 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 5 \\ -3 & -1 \end{bmatrix} + \begin{bmatrix} -11 & -7 \\ 2 & -1 \end{bmatrix}$

3. $4\left(\begin{bmatrix} -4 & 0 & 1 \\ 0 & 2 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 1 & -2 \\ 3 & -6 & 0 \end{bmatrix}\right)$

4. $\frac{1}{2}([5 \quad -2 \quad 4 \quad 0] + [14 \quad 6 \quad -18 \quad 9])$

5. $-3\left(\begin{bmatrix} 0 & -3 \\ 7 & 2 \end{bmatrix} + \begin{bmatrix} -6 & 3 \\ 8 & 1 \end{bmatrix}\right) - 2\begin{bmatrix} 4 & -4 \\ 7 & -9 \end{bmatrix}$

6. $-\begin{bmatrix} 4 & 11 \\ -2 & -1 \\ 9 & 3 \end{bmatrix} + \frac{1}{6}\left(\begin{bmatrix} -5 & -1 \\ 3 & 4 \\ 0 & 13 \end{bmatrix} + \begin{bmatrix} 7 & 5 \\ -9 & -1 \\ 6 & -1 \end{bmatrix}\right)$

**Operations with Matrices** In Exercises 7–12, perform the indicated operations, given $a = 3$, $b = -4$, and

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}, \quad O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

7. $aA + bB$
8. $A + B$
9. $ab(B)$
10. $(a + b)B$
11. $(a - b)(A - B)$
12. $(ab)O$

13. Solve for $X$ in the equation, given

$$A = \begin{bmatrix} -4 & 0 \\ 1 & -5 \\ -3 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 \\ -2 & 1 \\ 4 & 4 \end{bmatrix}.$$

(a) $3X + 2A = B$ (b) $2A - 5B = 3X$
(c) $X - 3A + 2B = O$ (d) $6X - 4A - 3B = O$

14. Solve for $X$ in the equation, given

$$A = \begin{bmatrix} -2 & -1 \\ 1 & 0 \\ 3 & -4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 3 \\ 2 & 0 \\ -4 & -1 \end{bmatrix}.$$

(a) $X = 3A - 2B$ (b) $2X = 2A - B$
(c) $2X + 3A = B$ (d) $2A + 4B = -2X$

**Operations with Matrices** In Exercises 15–20, perform the indicated operations, given $c = -2$ and

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

15. $B(CA)$
16. $C(BC)$
17. $(B + C)A$
18. $B(C + O)$
19. $(cB)(C + C)$
20. $B(cA)$

**Associativity of Matrix Multiplication** In Exercises 21 and 22, find the matrix product $ABC$ by (a) grouping the factors as $(AB)C$, and (b) grouping the factors as $A(BC)$. Show that you obtain the same result from both processes.

21. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}, C = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$

22. $A = \begin{bmatrix} -4 & 2 \\ 1 & -3 \end{bmatrix}, B = \begin{bmatrix} 1 & -5 & 0 \\ -2 & 3 & 3 \end{bmatrix},$

$$C = \begin{bmatrix} -3 & 4 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}$$

**Noncommutativity of Matrix Multiplication** In Exercises 23 and 24, show that $AB$ and $BA$ are not equal for the given matrices.

23. $A = \begin{bmatrix} -2 & 1 \\ 0 & 3 \end{bmatrix}, B = \begin{bmatrix} 4 & 0 \\ -1 & 2 \end{bmatrix}$

24. $A = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, B = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} \end{bmatrix}$

**Equal Matrix Products** In Exercises 25 and 26, show that $AC = BC$, even though $A \neq B$.

25. $A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, C = \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}$

26. $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 4 \\ 3 & -2 & 1 \end{bmatrix}, B = \begin{bmatrix} 4 & -6 & 3 \\ 5 & 4 & 4 \\ -1 & 0 & 1 \end{bmatrix},$

$$C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 4 & -2 & 3 \end{bmatrix}$$

**Zero Matrix Product** In Exercises 27 and 28, show that $AB = O$, even though $A \neq O$ and $B \neq O$.

27. $A = \begin{bmatrix} 3 & 3 \\ 4 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

28. $A = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 \\ -\frac{1}{2} & 1 \end{bmatrix}$

**Operations with Matrices** In Exercises 29–34, perform the indicated operations when

$$A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}.$$

29. $IA$
30. $AI$
31. $A(I + A)$
32. $A + IA$
33. $A^2$
34. $A^4$

**Writing** In Exercises 35 and 36, explain why the formula is *not* valid for matrices. Illustrate your argument with examples.

**35.** $(A + B)(A - B) = A^2 - B^2$

**36.** $(A + B)(A + B) = A^2 + 2AB + B^2$

**Finding the Transpose of a Matrix** In Exercises 37 and 38, find the transpose of the matrix.

**37.** $D = \begin{bmatrix} 1 & -2 \\ -3 & 4 \\ 5 & -1 \end{bmatrix}$ **38.** $D = \begin{bmatrix} 6 & -7 & 19 \\ -7 & 0 & 23 \\ 19 & 23 & -32 \end{bmatrix}$

**Finding the Transpose of a Product of Two Matrices** In Exercises 39–42, verify that $(AB)^T = B^TA^T$.

**39.** $A = \begin{bmatrix} -1 & 1 & -2 \\ 2 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 0 \\ 1 & 2 \\ 1 & -1 \end{bmatrix}$

**40.** $A = \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & -1 \\ 2 & 1 \end{bmatrix}$

**41.** $A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ -2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 4 & -1 \end{bmatrix}$

**42.** $A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 3 \\ 4 & 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -2 \\ 0 & 1 & 3 \end{bmatrix}$

**Multiplication with the Transpose of a Matrix** In Exercises 43–46, find (a) $A^TA$ and (b) $AA^T$. Show that each of these products is symmetric.

**43.** $A = \begin{bmatrix} 4 & 2 & 1 \\ 0 & 2 & -1 \end{bmatrix}$ **44.** $A = \begin{bmatrix} 1 & -1 \\ 3 & 4 \\ 0 & -2 \end{bmatrix}$

**45.** $A = \begin{bmatrix} 0 & -4 & 3 & 2 \\ 8 & 4 & 0 & 1 \\ -2 & 3 & 5 & 1 \\ 0 & 0 & -3 & 2 \end{bmatrix}$

**46.** $A = \begin{bmatrix} 4 & -3 & 2 & 0 \\ 2 & 0 & 11 & -1 \\ -1 & -2 & 0 & 3 \\ 14 & -2 & 12 & -9 \\ 6 & 8 & -5 & 4 \end{bmatrix}$

**True or False?** In Exercises 47 and 48, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.

**47.** (a) Matrix addition is commutative.

(b) Matrix multiplication is associative.

(c) The transpose of the product of two matrices equals the product of their transposes; that is, $(AB)^T = A^TB^T$.

(d) For any matrix $C$, the matrix $CC^T$ is symmetric.

**48.** (a) Matrix multiplication is commutative.

(b) Every matrix $A$ has an additive inverse.

(c) If the matrices $A$, $B$, and $C$ satisfy $AB = AC$, then $B = C$.

(d) The transpose of the sum of two matrices equals the sum of their transposes.

**49.** Consider the following matrices.

$$X = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad Y = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \quad W = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

(a) Find scalars $a$ and $b$ such that $Z = aX + bY$.

(b) Show that there do not exist scalars $a$ and $b$ such that $W = aX + bY$.

(c) Show that if $aX + bY + cW = O$, then $a = b = c = 0$.

(d) Find scalars $a$, $b$, and $c$, not all equal to zero, such that $aX + bY + cZ = O$.

**50.** **CAPSTONE** In the matrix equation

$$aX + A(bB) = b(AB + IB)$$

$X$, $A$, $B$, and $I$ are square matrices, and $a$ and $b$ are nonzero scalars. Justify each step in the solution given below.

$$\begin{aligned} aX + (Ab)B &= b(AB + B) \\ aX + bAB &= bAB + bB \\ aX + bAB + (-bAB) &= bAB + bB + (-bAB) \\ aX &= bAB + bB + (-bAB) \\ aX &= bAB + (-bAB) + bB \\ aX &= bB \\ X &= \frac{b}{a}B \end{aligned}$$

**Finding a Power of a Matrix** In Exercises 51 and 52, compute the power of $A$ for the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

**51.** $A^{19}$ **52.** $A^{20}$

**Finding an *n*th Root of a Matrix** In Exercises 53 and 54, find the $n$th root of the matrix $B$. An $n$th root of a matrix $B$ is a matrix $A$ such that $A^n = B$.

**53.** $B = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}, \quad n = 2$

**54.** $B = \begin{bmatrix} 8 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 27 \end{bmatrix}, \quad n = 3$

**Polynomial Function** **In Exercises 55 and 56, use the given definition to find $f(A)$: If $f$ is the polynomial function**

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

**then for a square matrix $A$, $f(A)$ is defined to be**

$$f(A) = a_0I + a_1A + a_2A^2 + \cdots + a_nA^n.$$

**55.** $f(x) = x^2 - 5x + 2, \quad A = \begin{bmatrix} 2 & 0 \\ 4 & 5 \end{bmatrix}$

**56.** $f(x) = x^3 - 2x^2 + 5x - 10, \quad A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 2 \\ -1 & 1 & 3 \end{bmatrix}$

**57. Guided Proof** Prove the associative property of matrix addition: $A + (B + C) = (A + B) + C$.

Getting Started: To prove that $A + (B + C)$ and $(A + B) + C$ are equal, show that their corresponding entries are the same.

(i) Begin your proof by letting $A$, $B$, and $C$ be $m \times n$ matrices.

(ii) Observe that the $ij$th entry of $B + C$ is $b_{ij} + c_{ij}$.

(iii) Furthermore, the $ij$th entry of $A + (B + C)$ is $a_{ij} + (b_{ij} + c_{ij})$.

(iv) Determine the $ij$th entry of $(A + B) + C$.

**58. Proof** Prove the associative property of scalar multiplication: $(cd)A = c(dA)$.

**59. Proof** Prove that the scalar 1 is the identity for scalar multiplication: $1A = A$.

**60. Proof** Prove the following distributive property: $(c + d)A = cA + dA$.

**61. Proof** Prove Theorem 2.2.

**62. Proof** Complete the proof of Theorem 2.3.

(a) Prove the associative property of multiplication: $A(BC) = (AB)C$.

(b) Prove the distributive property: $(A + B)C = AC + BC$.

(c) Prove the property: $c(AB) = (cA)B = A(cB)$.

**63. Proof** Prove Theorem 2.4.

**64. Proof** Prove Properties 2, 3, and 4 of Theorem 2.6.

**65. Guided Proof** Prove that if $A$ is an $m \times n$ matrix, then $AA^T$ and $A^TA$ are symmetric matrices.

Getting Started: To prove that $AA^T$ is symmetric, you need to show that it is equal to its transpose, $(AA^T)^T = AA^T$.

(i) Begin your proof with the left-hand matrix expression $(AA^T)^T$.

(ii) Use the properties of the transpose operation to show that $(AA^T)^T$ can be simplified to equal the right-hand expression, $AA^T$.

(iii) Repeat this analysis for the product $A^TA$.

**66. Proof** Let $A$ and $B$ be two $n \times n$ symmetric matrices.

(a) Give an example to show that the product $AB$ is not necessarily symmetric.

(b) Prove that $AB$ is symmetric if and only if $AB = BA$.

**Symmetric and Skew-Symmetric Matrices** **In Exercises 67–70, determine whether the matrix is symmetric, skew-symmetric, or neither. A square matrix is called skew-symmetric when $A^T = -A$.**

**67.** $A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$ **68.** $A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$

**69.** $A = \begin{bmatrix} 0 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & 3 & 0 \end{bmatrix}$ **70.** $A = \begin{bmatrix} 0 & 2 & -1 \\ -2 & 0 & -3 \\ 1 & 3 & 0 \end{bmatrix}$

**71. Proof** Prove that the main diagonal of a skew-symmetric matrix consists entirely of zeros.

**72. Proof** Prove that if $A$ and $B$ are $n \times n$ skew-symmetric matrices, then $A + B$ is skew-symmetric.

**73. Proof** Let $A$ be a square matrix of order $n$.

(a) Show that $\frac{1}{2}(A + A^T)$ is symmetric.

(b) Show that $\frac{1}{2}(A - A^T)$ is skew-symmetric.

(c) Prove that $A$ can be written as the sum of a symmetric matrix $B$ and a skew-symmetric matrix $C$, $A = B + C$.

(d) Write the matrix

$$A = \begin{bmatrix} 2 & 5 & 3 \\ -3 & 6 & 0 \\ 4 & 1 & 1 \end{bmatrix}$$

as the sum of a skew-symmetric matrix and a symmetric matrix.

**74. Proof** Prove that if $A$ is an $n \times n$ matrix, then $A - A^T$ is skew-symmetric.

**75.** Consider matrices of the form

$$A = \begin{bmatrix} 0 & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & 0 & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a_{(n-1)n} \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

(a) Write a $2 \times 2$ matrix and a $3 \times 3$ matrix in the form of $A$.

(b) Use a software program or a graphing utility to raise each of the matrices to higher powers. Describe the result.

(c) Use the result of part (b) to make a conjecture about powers of $A$ when $A$ is a $4 \times 4$ matrix. Use a graphing utility to test your conjecture.

(d) Use the results of parts (b) and (c) to make a conjecture about powers of $A$ when $A$ is an $n \times n$ matrix.

# 2.3 The Inverse of a Matrix

- Find the inverse of a matrix (if it exists).
- Use properties of inverse matrices.
- Use an inverse matrix to solve a system of linear equations.

## MATRICES AND THEIR INVERSES

Section 2.2 discussed some of the similarities between the algebra of real numbers and the algebra of matrices. This section further develops the algebra of matrices to include the solutions of matrix equations involving matrix multiplication. To begin, consider the real number equation $ax = b$. To solve this equation for $x$, multiply both sides of the equation by $a^{-1}$ (provided $a \neq 0$).

$$\begin{aligned} ax &= b \\ (a^{-1}a)x &= a^{-1}b \\ (1)x &= a^{-1}b \\ x &= a^{-1}b \end{aligned}$$

The number $a^{-1}$ is called the *multiplicative inverse* of $a$ because $a^{-1}a = 1$ (the identity element for multiplication). The definition of the multiplicative inverse of a matrix is similar.

### Definition of the Inverse of a Matrix

An $n \times n$ matrix $A$ is **invertible** (or **nonsingular**) when there exists an $n \times n$ matrix $B$ such that

$$AB = BA = I_n$$

where $I_n$ is the identity matrix of order $n$. The matrix $B$ is called the (multiplicative) **inverse** of $A$. A matrix that does not have an inverse is called **noninvertible** (or **singular**).

*Nonsquare matrices do not have inverses.* To see this, note that if $A$ is of size $m \times n$ and $B$ is of size $n \times m$ (where $m \neq n$), then the products $AB$ and $BA$ are of different sizes and cannot be equal to each other. Not all square matrices have inverses. (See Example 4.) The next theorem, however, states that if a matrix *does* have an inverse, then that inverse is unique.

### THEOREM 2.7 Uniqueness of an Inverse Matrix

If $A$ is an invertible matrix, then its inverse is unique. The inverse of $A$ is denoted by $A^{-1}$.

### PROOF

Because $A$ is invertible, you know it has at least one inverse $B$ such that

$$AB = I = BA.$$

Suppose $A$ has another inverse $C$ such that

$$AC = I = CA.$$

Show that $B$ and $C$ are equal, as follows.

$$\begin{aligned} AB &= I \\ C(AB) &= CI \\ (CA)B &= C \\ IB &= C \\ B &= C \end{aligned}$$

Consequently $B = C$, and it follows that the inverse of a matrix is unique.

Because the inverse $A^{-1}$ of an invertible matrix $A$ is unique, you can call it *the* inverse of $A$ and write

$$AA^{-1} = A^{-1}A = I.$$

## EXAMPLE 1 The Inverse of a Matrix

Show that $B$ is the inverse of $A$, where

$$A = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}.$$

**REMARK**

Recall that it is not always true that $AB = BA$, even when both products are defined. If $A$ and $B$ are both square matrices and $AB = I_n$, however, then it can be shown that $BA = I_n$. (The proof of this is omitted.) So, in Example 1, you need only check that $AB = I_2$.

**SOLUTION**

Using the definition of an inverse matrix, show that $B$ is the inverse of $A$ by showing that $AB = I = BA$, as follows.

$$AB = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -1+2 & 2-2 \\ -1+1 & 2-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1+2 & 2-2 \\ -1+1 & 2-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The next example shows how to use a system of equations to find the inverse of a matrix.

## EXAMPLE 2 Finding the Inverse of a Matrix

Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}.$$

**SOLUTION**

To find the inverse of $A$, solve the matrix equation $AX = I$ for $X$.

$$\begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x_{11} + 4x_{21} & x_{12} + 4x_{22} \\ -x_{11} - 3x_{21} & -x_{12} - 3x_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Equating corresponding entries, you obtain two systems of linear equations.

$$\begin{aligned} x_{11} + 4x_{21} &= 1 \\ -x_{11} - 3x_{21} &= 0 \end{aligned} \qquad \begin{aligned} x_{12} + 4x_{22} &= 0 \\ -x_{12} - 3x_{22} &= 1 \end{aligned}$$

Solving the first system, you find that $x_{11} = -3$ and $x_{21} = 1$. Similarly, solving the second system, you find that $x_{12} = -4$ and $x_{22} = 1$. So, the inverse of $A$ is

$$X = A^{-1} = \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix}.$$

Try using matrix multiplication to check this result.

Generalizing the method used to solve Example 2 provides a convenient method for finding an inverse. Note that the two systems of linear equations

$$\begin{aligned} x_{11} + 4x_{21} &= 1 \\ -x_{11} - 3x_{21} &= 0 \end{aligned} \qquad \begin{aligned} x_{12} + 4x_{22} &= 0 \\ -x_{12} - 3x_{22} &= 1 \end{aligned}$$

have the *same coefficient matrix*. Rather than solve the two systems represented by

$$\begin{bmatrix} 1 & 4 & 1 \\ -1 & -3 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 4 & 0 \\ -1 & -3 & 1 \end{bmatrix}$$

separately, you can solve them simultaneously by **adjoining** the identity matrix to the coefficient matrix to obtain

$$\begin{bmatrix} 1 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{bmatrix}.$$

By applying Gauss-Jordan elimination to this matrix, solve *both* systems with a single elimination process, as follows.

$$\begin{bmatrix} 1 & 4 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \qquad R_2 + R_1 \rightarrow R_2$$

$$\begin{bmatrix} 1 & 0 & -3 & -4 \\ 0 & 1 & 1 & 1 \end{bmatrix} \qquad R_1 + (-4)R_2 \rightarrow R_1$$

Applying Gauss-Jordan elimination to the "doubly augmented" matrix $[A \quad I]$, you obtain the matrix $[I \quad A^{-1}]$.

$$\left[\underbrace{\begin{matrix} 1 & 4 \\ -1 & -3 \end{matrix}}_{A} \quad \underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{I}\right] \longrightarrow \left[\underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{I} \quad \underbrace{\begin{matrix} -3 & -4 \\ 1 & 1 \end{matrix}}_{A^{-1}}\right]$$

This procedure (or algorithm) works for an arbitrary $n \times n$ matrix. If $A$ cannot be row reduced to $I_n$, then $A$ is noninvertible (or singular). This procedure will be formally justified in the next section, after introducing the concept of an elementary matrix. For now, a summary of the algorithm is as follows.

### Finding the Inverse of a Matrix by Gauss-Jordan Elimination

Let $A$ be a square matrix of order $n$.

1. Write the $n \times 2n$ matrix that consists of the given matrix $A$ on the left and the $n \times n$ identity matrix $I$ on the right to obtain $[A \quad I]$. This process is called **adjoining** matrix $I$ to matrix $A$.
2. If possible, row reduce $A$ to $I$ using elementary row operations on the entire matrix $[A \quad I]$. The result will be the matrix $[I \quad A^{-1}]$. If this is not possible, then $A$ is noninvertible (or singular).
3. Check your work by multiplying to see that $AA^{-1} = I = A^{-1}A$.

**LINEAR ALGEBRA APPLIED**

Recall Hooke's law, which states that for relatively small deformations of an elastic object, the amount of deflection is directly proportional to the force causing the deformation. In a simply supported elastic beam subjected to multiple forces, deflection $\mathbf{d}$ is related to force $\mathbf{w}$ by the matrix equation

$$\mathbf{d} = F\mathbf{w}$$

where $F$ is a *flexibility matrix* whose entries depend on the material of the beam. The inverse of the flexibility matrix, $F^{-1}$, is called the *stiffness matrix*. In Exercises 61 and 62, you are asked to find the stiffness matrix $F^{-1}$ and the force matrix $\mathbf{w}$ for a given set of flexibility and deflection matrices.

## EXAMPLE 3 Finding the Inverse of a Matrix

Find the inverse of the matrix.

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{bmatrix}$$

### SOLUTION

Begin by adjoining the identity matrix to $A$ to form the matrix

$$[A \quad I] = \left[\begin{array}{rrrrrr} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ -6 & 2 & 3 & 0 & 0 & 1 \end{array}\right].$$

Use elementary row operations to obtain the form

$$[I \quad A^{-1}]$$

as follows.

$$\left[\begin{array}{rrrrrr} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ -6 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \qquad R_2 + (-1)R_1 \rightarrow R_2$$

$$\left[\begin{array}{rrrrrr} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & -4 & 3 & 6 & 0 & 1 \end{array}\right] \qquad R_3 + (6)R_1 \rightarrow R_3$$

$$\left[\begin{array}{rrrrrr} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 2 & 4 & 1 \end{array}\right] \qquad R_3 + (4)R_2 \rightarrow R_3$$

$$\left[\begin{array}{rrrrrr} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right] \qquad (-1)R_3 \rightarrow R_3$$

$$\left[\begin{array}{rrrrrr} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & -3 & -1 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right] \qquad R_2 + R_3 \rightarrow R_2$$

$$\left[\begin{array}{rrrrrr} 1 & 0 & 0 & -2 & -3 & -1 \\ 0 & 1 & 0 & -3 & -3 & -1 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right] \qquad R_1 + R_2 \rightarrow R_1$$

The matrix $A$ is invertible, and its inverse is

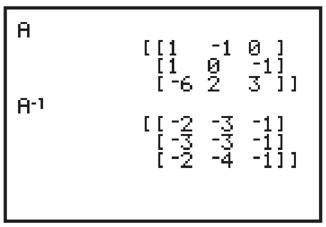
$$A^{-1} = \begin{bmatrix} -2 & -3 & -1 \\ -3 & -3 & -1 \\ -2 & -4 & -1 \end{bmatrix}.$$

Try confirming this by showing that

$$\begin{aligned} AA^{-1} &= I \\ &= A^{-1}A. \end{aligned}$$

### TECHNOLOGY

Many graphing utilities and software programs can calculate the inverse of a square matrix. If you use a graphing utility, you may see something similar to the following for Example 3. The **Online Technology Guide**, available at *www.cengagebrain.com*, provides syntax for programs applicable to Example 3.




The process shown in Example 3 applies to any $n \times n$ matrix $A$ and will find the inverse of $A$, if it exists. When $A$ has no inverse, the process will also tell you that. The next example applies the process to a singular matrix (one that has no inverse).

## EXAMPLE 4 A Singular Matrix

Show that the matrix has no inverse.

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ -2 & 3 & -2 \end{bmatrix}$$

### SOLUTION

Adjoin the identity matrix to $A$ to form

$$[A \quad I] = \begin{bmatrix} 1 & 2 & 0 & 1 & 0 & 0 \\ 3 & -1 & 2 & 0 & 1 & 0 \\ -2 & 3 & -2 & 0 & 0 & 1 \end{bmatrix}$$

and apply Gauss-Jordan elimination to obtain the following.

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -7 & 2 & -3 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{bmatrix}$$

Note that the "$A$ portion" of the matrix has a row of zeros. So it is not possible to rewrite the matrix $[A \quad I]$ in the form $[I \quad A^{-1}]$. This means that $A$ has no inverse, or is noninvertible (or singular).

Using Gauss-Jordan elimination to find the inverse of a matrix works well (even as a computer technique) for matrices of size $3 \times 3$ or greater. For $2 \times 2$ matrices, however, you can use a formula for the inverse rather than Gauss-Jordan elimination.

If $A$ is a $2 \times 2$ matrix given by

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then $A$ is invertible if and only if $ad - bc \neq 0$. Moreover, if $ad - bc \neq 0$, then the inverse is given by

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

**REMARK**

The denominator $ad - bc$ is called the **determinant** of $A$. You will study determinants in detail in Chapter 3.

## EXAMPLE 5 Finding the Inverse of a $2 \times 2$ Matrix

If possible, find the inverse of each matrix.

**a.** $A = \begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix}$ **b.** $B = \begin{bmatrix} 3 & -1 \\ -6 & 2 \end{bmatrix}$

### SOLUTION

**a.** For the matrix $A$, apply the formula for the inverse of a $2 \times 2$ matrix to obtain $ad - bc = (3)(2) - (-1)(-2) = 4$. Because this quantity is not zero, $A$ is invertible. Form the inverse by interchanging the entries on the main diagonal, changing the signs of the other two entries, and multiplying by the scalar $\frac{1}{4}$, as follows.

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{3}{4} \end{bmatrix}$$

**b.** For the matrix $B$, you have $ad - bc = (3)(2) - (-1)(-6) = 0$, which means that $B$ is noninvertible.

## PROPERTIES OF INVERSES

Theorem 2.8 below lists important properties of inverse matrices.

### THEOREM 2.8 Properties of Inverse Matrices

If $A$ is an invertible matrix, $k$ is a positive integer, and $c$ is a nonzero scalar, then $A^{-1}$, $A^k$, $cA$, and $A^T$ are invertible and the following are true.

**1.** $(A^{-1})^{-1} = A$

**2.** $(A^k)^{-1} = \underbrace{A^{-1}A^{-1} \cdot \cdot \cdot A^{-1}}_{k \text{ factors}} = (A^{-1})^k$

**3.** $(cA)^{-1} = \dfrac{1}{c}A^{-1}$

**4.** $(A^T)^{-1} = (A^{-1})^T$

### PROOF

The key to the proofs of Properties 1, 3, and 4 is the fact that the inverse of a matrix is unique (Theorem 2.7). That is, if $BC = CB = I$, then $C$ is the inverse of $B$.

Property 1 states that the inverse of $A^{-1}$ is $A$ itself. To prove this, observe that $A^{-1}A = AA^{-1} = I$, which means that $A$ is the inverse of $A^{-1}$. Thus, $A = (A^{-1})^{-1}$.

Similarly, Property 3 states that $\frac{1}{c}A^{-1}$ is the inverse of $(cA)$, $c \neq 0$. To prove this, use the properties of scalar multiplication given in Theorems 2.1 and 2.3, as follows.

$$(cA)\left(\frac{1}{c}A^{-1}\right) = \left(c\frac{1}{c}\right)AA^{-1} = (1)I = I$$

$$\left(\frac{1}{c}A^{-1}\right)(cA) = \left(\frac{1}{c}c\right)A^{-1}A = (1)I = I$$

So $\frac{1}{c}A^{-1}$ is the inverse of $(cA)$, which implies that $(cA)^{-1} = \frac{1}{c}A^{-1}$. Properties 2 and 4 are left for you to prove. (See Exercises 65 and 66.)

For nonsingular matrices, the exponential notation used for repeated multiplication of *square* matrices can be extended to include exponents that are negative integers. This may be done by defining $A^{-k}$ to be

$$A^{-k} = \underbrace{A^{-1}A^{-1} \cdot \cdot \cdot A^{-1}}_{k \text{ factors}} = (A^{-1})^k.$$

Using this convention you can show that the properties $A^jA^k = A^{j+k}$ and $(A^j)^k = A^{jk}$ are true for any integers $j$ and $k$.

### DISCOVERY

Let $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix}$.

1. Calculate $(AB)^{-1}$, $A^{-1}B^{-1}$, and $B^{-1}A^{-1}$.
2. Make a conjecture about the inverse of a product of two nonsingular matrices.
3. Select two other nonsingular matrices and see whether your conjecture holds.

### EXAMPLE 6 The Inverse of the Square of a Matrix

Compute $A^{-2}$ two different ways and show that the results are equal.

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}$$

**SOLUTION**

One way to find $A^{-2}$ is to find $(A^2)^{-1}$ by squaring the matrix $A$ to obtain

$$A^2 = \begin{bmatrix} 3 & 5 \\ 10 & 18 \end{bmatrix}$$

and using the formula for the inverse of a $2 \times 2$ matrix to obtain

$$(A^2)^{-1} = \frac{1}{4}\begin{bmatrix} 18 & -5 \\ -10 & 3 \end{bmatrix} = \begin{bmatrix} \frac{9}{2} & -\frac{5}{4} \\ -\frac{5}{2} & \frac{3}{4} \end{bmatrix}.$$

Another way to find $A^{-2}$ is to find $(A^{-1})^2$ by finding $A^{-1}$

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 4 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -\frac{1}{2} \\ -1 & \frac{1}{2} \end{bmatrix}$$

and then squaring this matrix to obtain

$$(A^{-1})^2 = \begin{bmatrix} \frac{9}{2} & -\frac{5}{4} \\ -\frac{5}{2} & \frac{3}{4} \end{bmatrix}.$$

Note that each method produces the same result.

The next theorem gives a formula for computing the inverse of a product of two matrices.

**THEOREM 2.9 The Inverse of a Product**

If $A$ and $B$ are invertible matrices of order $n$, then $AB$ is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}.$$

**PROOF**

To show that $B^{-1}A^{-1}$ is the inverse of $AB$, you need only show that it conforms to the definition of an inverse matrix. That is,

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = A(I)A^{-1} = (AI)A^{-1} = AA^{-1} = I.$$

In a similar way, $(B^{-1}A^{-1})(AB) = I$. So, $AB$ is invertible and has the indicated inverse.

Theorem 2.9 states that the inverse of a product of two invertible matrices is the product of their inverses taken in the *reverse* order. This can be generalized to include the product of several invertible matrices:

$$(A_1A_2A_3 \cdot \cdot \cdot A_n)^{-1} = A_n^{-1} \cdot \cdot \cdot A_3^{-1}A_2^{-1}A_1^{-1}.$$

(See Example 4 in Appendix A.)

## EXAMPLE 7 Finding the Inverse of a Matrix Product

Find $(AB)^{-1}$ for the matrices

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \\ 2 & 4 & 3 \end{bmatrix}$$

using the fact that $A^{-1}$ and $B^{-1}$ are

$$A^{-1} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad B^{-1} = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 1 & 0 \\ \frac{2}{3} & 0 & -\frac{1}{3} \end{bmatrix}.$$

**REMARK**

Note that you *reverse the order* of multiplication to find the inverse of $AB$. That is, $(AB)^{-1} = B^{-1}A^{-1}$, and the inverse of $AB$ is usually *not* equal to $A^{-1}B^{-1}$.

### SOLUTION

Using Theorem 2.9 produces

$$(AB)^{-1} = B^{-1}A^{-1} = \overbrace{\begin{bmatrix} 1 & -2 & 1 \\ -1 & 1 & 0 \\ \frac{2}{3} & 0 & -\frac{1}{3} \end{bmatrix}}^{B^{-1}} \overbrace{\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}}^{A^{-1}}$$

$$= \begin{bmatrix} 8 & -5 & -2 \\ -8 & 4 & 3 \\ 5 & -2 & -\frac{7}{3} \end{bmatrix}.$$

One important property in the algebra of real numbers is the cancellation property. That is, if $ac = bc$ $(c \neq 0)$, then $a = b$. *Invertible* matrices have similar cancellation properties.

### THEOREM 2.10 Cancellation Properties

If $C$ is an invertible matrix, then the following properties hold.

**1.** If $AC = BC$, then $A = B$. Right cancellation property

**2.** If $CA = CB$, then $A = B$. Left cancellation property

### PROOF

To prove Property 1, use the fact that $C$ is invertible and write

$$\begin{aligned} AC &= BC \\ (AC)C^{-1} &= (BC)C^{-1} \\ A(CC^{-1}) &= B(CC^{-1}) \\ AI &= BI \\ A &= B. \end{aligned}$$

The second property can be proved in a similar way. (See Exercise 68.)

Be sure to remember that Theorem 2.10 can be applied only when $C$ is an *invertible* matrix. If $C$ is not invertible, then cancellation is not usually valid. For instance, Example 5 in Section 2.2 gives an example of a matrix equation $AC = BC$ in which $A \neq B$, because $C$ is not invertible in the example.

## SYSTEMS OF EQUATIONS

For *square* systems (those having the same number of equations as variables), you can use the following theorem to determine whether the system has a unique solution.

**THEOREM 2.11 Systems of Equations with Unique Solutions**

If $A$ is an invertible matrix, then the system of linear equations $A\mathbf{x} = \mathbf{b}$ has a unique solution given by $\mathbf{x} = A^{-1}\mathbf{b}$.

**PROOF**

Because $A$ is nonsingular, the steps shown below are valid.

$$\begin{aligned} A\mathbf{x} &= \mathbf{b} \\ A^{-1}A\mathbf{x} &= A^{-1}\mathbf{b} \\ I\mathbf{x} &= A^{-1}\mathbf{b} \\ \mathbf{x} &= A^{-1}\mathbf{b} \end{aligned}$$

This solution is unique because if $\mathbf{x}_1$ and $\mathbf{x}_2$ were two solutions, then you could apply the cancellation property to the equation $A\mathbf{x}_1 = \mathbf{b} = A\mathbf{x}_2$ to conclude that $\mathbf{x}_1 = \mathbf{x}_2$.

One use of Theorem 2.11 is in solving *several* systems that all have the same coefficient matrix $A$. You could find the inverse matrix once and then solve each system by computing the product $A^{-1}\mathbf{b}$.

**EXAMPLE 8 Solving a System of Equations Using an Inverse Matrix**

Use an inverse matrix to solve each system.

**a.** $$\begin{aligned} 2x + 3y + z &= -1 \\ 3x + 3y + z &= 1 \\ 2x + 4y + z &= -2 \end{aligned}$$

**b.** $$\begin{aligned} 2x + 3y + z &= 4 \\ 3x + 3y + z &= 8 \\ 2x + 4y + z &= 5 \end{aligned}$$

**c.** $$\begin{aligned} 2x + 3y + z &= 0 \\ 3x + 3y + z &= 0 \\ 2x + 4y + z &= 0 \end{aligned}$$

**SOLUTION**

First note that the coefficient matrix for each system is $A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix}$.

Using Gauss-Jordan elimination, $A^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}$.

**a.** $\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$ The solution is $x = 2$, $y = -1$, and $z = -2$.

**b.** $\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}\begin{bmatrix} 4 \\ 8 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ -7 \end{bmatrix}$ The solution is $x = 4$, $y = 1$, and $z = -7$.

**c.** $\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ The solution is trivial: $x = 0$, $y = 0$, and $z = 0$.

# 2.3 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

**The Inverse of a Matrix** In Exercises 1–6, show that $B$ is the inverse of $A$.

1. $A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}, B = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$

2. $A = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

3. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$

4. $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix}$

5. $A = \begin{bmatrix} -2 & 2 & 3 \\ 1 & -1 & 0 \\ 0 & 1 & 4 \end{bmatrix}, \quad B = \frac{1}{3}\begin{bmatrix} -4 & -5 & 3 \\ -4 & -8 & 3 \\ 1 & 2 & 0 \end{bmatrix}$

6. $A = \begin{bmatrix} 2 & -17 & 11 \\ -1 & 11 & -7 \\ 0 & 3 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix}$

**Finding the Inverse of a Matrix** In Exercises 7–30, find the inverse of the matrix (if it exists).

7. $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

8. $\begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix}$

9. $\begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$

10. $\begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}$

11. $\begin{bmatrix} -7 & 33 \\ 4 & -19 \end{bmatrix}$

12. $\begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix}$

13. $\begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 4 \\ 3 & 6 & 5 \end{bmatrix}$

14. $\begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -1 & -4 & -7 \end{bmatrix}$

15. $\begin{bmatrix} 1 & 2 & -1 \\ 3 & 7 & -10 \\ 7 & 16 & -21 \end{bmatrix}$

16. $\begin{bmatrix} 10 & 5 & -7 \\ -5 & 1 & 4 \\ 3 & 2 & -2 \end{bmatrix}$

17. $\begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 0 \\ -2 & 0 & 3 \end{bmatrix}$

18. $\begin{bmatrix} 3 & 2 & 5 \\ 2 & 2 & 4 \\ -4 & 4 & 0 \end{bmatrix}$

19. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

20. $\begin{bmatrix} -\frac{5}{6} & \frac{1}{3} & \frac{11}{6} \\ 0 & \frac{2}{3} & 2 \\ 1 & -\frac{1}{2} & -\frac{5}{2} \end{bmatrix}$

21. $\begin{bmatrix} 0.6 & 0 & -0.3 \\ 0.7 & -1 & 0.2 \\ 1 & 0 & -0.9 \end{bmatrix}$

22. $\begin{bmatrix} 0.1 & 0.2 & 0.3 \\ -0.3 & 0.2 & 0.2 \\ 0.5 & 0.5 & 0.5 \end{bmatrix}$

23. $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 4 & 0 \\ 2 & 5 & 5 \end{bmatrix}$

24. $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 0 & 0 \\ 2 & 5 & 5 \end{bmatrix}$

25. $\begin{bmatrix} -8 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 \end{bmatrix}$

26. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

27. $\begin{bmatrix} 1 & -2 & -1 & -2 \\ 3 & -5 & -2 & -3 \\ 2 & -5 & -2 & -5 \\ -1 & 4 & 4 & 11 \end{bmatrix}$

28. $\begin{bmatrix} 4 & 8 & -7 & 14 \\ 2 & 5 & -4 & 6 \\ 0 & 2 & 1 & -7 \\ 3 & 6 & -5 & 10 \end{bmatrix}$

29. $\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 2 & 0 & 4 \\ 1 & 0 & 3 & 0 \\ 0 & 2 & 0 & 4 \end{bmatrix}$

30. $\begin{bmatrix} 1 & 3 & -2 & 0 \\ 0 & 2 & 4 & 6 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix}$

**Finding the Inverse of a 2 × 2 Matrix** In Exercises 31–36, use the formula on page 66 to find the inverse of the 2 × 2 matrix (if it exists).

31. $\begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix}$

32. $\begin{bmatrix} 1 & -2 \\ -3 & 2 \end{bmatrix}$

33. $\begin{bmatrix} -4 & -6 \\ 2 & 3 \end{bmatrix}$

34. $\begin{bmatrix} -12 & 3 \\ 5 & -2 \end{bmatrix}$

35. $\begin{bmatrix} \frac{7}{2} & -\frac{3}{4} \\ \frac{1}{5} & \frac{4}{5} \end{bmatrix}$

36. $\begin{bmatrix} -\frac{1}{4} & \frac{9}{4} \\ \frac{5}{3} & \frac{8}{9} \end{bmatrix}$

**Finding the Inverse of the Square of a Matrix** In Exercises 37–40, compute $A^{-2}$ two different ways and show that the results are equal.

37. $A = \begin{bmatrix} 0 & -2 \\ -1 & 3 \end{bmatrix}$

38. $A = \begin{bmatrix} 2 & 7 \\ -5 & 6 \end{bmatrix}$

39. $A = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

40. $A = \begin{bmatrix} 6 & 0 & 4 \\ -2 & 7 & -1 \\ 3 & 1 & 2 \end{bmatrix}$

**Finding the Inverses of Products and Transposes** In Exercises 41–44, use the inverse matrices to find (a) $(AB)^{-1}$, (b) $(A^T)^{-1}$, and (c) $(2A)^{-1}$.

41. $A^{-1} = \begin{bmatrix} 2 & 5 \\ -7 & 6 \end{bmatrix}, \quad B^{-1} = \begin{bmatrix} 7 & -3 \\ 2 & 0 \end{bmatrix}$

42. $A^{-1} = \begin{bmatrix} -\frac{2}{7} & \frac{1}{7} \\ \frac{3}{7} & \frac{2}{7} \end{bmatrix}, \quad B^{-1} = \begin{bmatrix} \frac{5}{11} & \frac{2}{11} \\ \frac{3}{11} & -\frac{1}{11} \end{bmatrix}$

43. $A^{-1} = \begin{bmatrix} 1 & -\frac{1}{2} & \frac{3}{4} \\ \frac{3}{2} & \frac{1}{2} & -2 \\ \frac{1}{4} & 1 & \frac{1}{2} \end{bmatrix}, \quad B^{-1} = \begin{bmatrix} 2 & 4 & \frac{5}{2} \\ -\frac{3}{4} & 2 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & 2 \end{bmatrix}$

44. $A^{-1} = \begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & 3 \\ 4 & 2 & 1 \end{bmatrix}, \quad B^{-1} = \begin{bmatrix} 6 & 5 & -3 \\ -2 & 4 & -1 \\ 1 & 3 & 4 \end{bmatrix}$

**Solving a System of Equations Using an Inverse** In Exercises 45–48, use an inverse matrix to solve each system of linear equations.

**45.** (a) $\begin{aligned} x + 2y &= -1 \\ x - 2y &= 3 \end{aligned}$ (b) $\begin{aligned} x + 2y &= 10 \\ x - 2y &= -6 \end{aligned}$

**46.** (a) $\begin{aligned} 2x - y &= -3 \\ 2x + y &= 7 \end{aligned}$ (b) $\begin{aligned} 2x - y &= -1 \\ 2x + y &= -3 \end{aligned}$

**47.** (a) $\begin{aligned} x_1 + 2x_2 + x_3 &= 2 \\ x_1 + 2x_2 - x_3 &= 4 \\ x_1 - 2x_2 + x_3 &= -2 \end{aligned}$

(b) $\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ x_1 + 2x_2 - x_3 &= 3 \\ x_1 - 2x_2 + x_3 &= -3 \end{aligned}$

**48.** (a) $\begin{aligned} x_1 + x_2 - 2x_3 &= 0 \\ x_1 - 2x_2 + x_3 &= 0 \\ x_1 - x_2 - x_3 &= -1 \end{aligned}$

(b) $\begin{aligned} x_1 + x_2 - 2x_3 &= -1 \\ x_1 - 2x_2 + x_3 &= 2 \\ x_1 - x_2 - x_3 &= 0 \end{aligned}$

**Solving a System of Equations Using an Inverse** In Exercises 49–52, use a software program or a graphing utility with matrix capabilities to solve the system of linear equations using an inverse matrix.

**49.** $\begin{aligned} x_1 + 2x_2 - x_3 + 3x_4 - x_5 &= -3 \\ x_1 - 3x_2 + x_3 + 2x_4 - x_5 &= -3 \\ 2x_1 + x_2 + x_3 - 3x_4 + x_5 &= 6 \\ x_1 - x_2 + 2x_3 + x_4 - x_5 &= 2 \\ 2x_1 + x_2 - x_3 + 2x_4 + x_5 &= -3 \end{aligned}$

**50.** $\begin{aligned} x_1 + x_2 - x_3 + 3x_4 - x_5 &= 3 \\ 2x_1 + x_2 + x_3 + x_4 + x_5 &= 4 \\ x_1 + x_2 - x_3 + 2x_4 - x_5 &= 3 \\ 2x_1 + x_2 + 4x_3 + x_4 - x_5 &= -1 \\ 3x_1 + x_2 + x_3 - 2x_4 + x_5 &= 5 \end{aligned}$

**51.** $\begin{aligned} 2x_1 - 3x_2 + x_3 - 2x_4 + x_5 - 4x_6 &= 20 \\ 3x_1 + x_2 - 4x_3 + x_4 - x_5 + 2x_6 &= -16 \\ 4x_1 + x_2 - 3x_3 + 4x_4 - x_5 + 2x_6 &= -12 \\ -5x_1 - x_2 + 4x_3 + 2x_4 - 5x_5 + 3x_6 &= -2 \\ x_1 + x_2 - 3x_3 + 4x_4 - 3x_5 + x_6 &= -15 \\ 3x_1 - x_2 + 2x_3 - 3x_4 + 2x_5 - 6x_6 &= 25 \end{aligned}$

**52.** $\begin{aligned} 4x_1 - 2x_2 + 4x_3 + 2x_4 - 5x_5 - x_6 &= 1 \\ 3x_1 + 6x_2 - 5x_3 - 6x_4 + 3x_5 + 3x_6 &= -11 \\ 2x_1 - 3x_2 + x_3 + 3x_4 - x_5 - 2x_6 &= 0 \\ -x_1 + 4x_2 - 4x_3 - 6x_4 + 2x_5 + 4x_6 &= -9 \\ 3x_1 - x_2 + 5x_3 + 2x_4 - 3x_5 - 5x_6 &= 1 \\ -2x_1 + 3x_2 - 4x_3 - 6x_4 + x_5 + 2x_6 &= -12 \end{aligned}$

**Matrix Equal to Its Own Inverse** In Exercises 53 and 54, find $x$ such that the matrix is equal to its own inverse.

**53.** $A = \begin{bmatrix} 3 & x \\ -2 & -3 \end{bmatrix}$ **54.** $A = \begin{bmatrix} 2 & x \\ -1 & -2 \end{bmatrix}$

**Singular Matrix** In Exercises 55 and 56, find $x$ such that the matrix is singular.

**55.** $A = \begin{bmatrix} 4 & x \\ -2 & -3 \end{bmatrix}$ **56.** $A = \begin{bmatrix} x & 2 \\ -3 & 4 \end{bmatrix}$

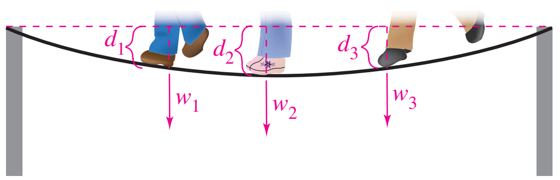
**Solving a Matrix Equation** In Exercises 57 and 58, find $A$.

**57.** $(2A)^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ **58.** $(4A)^{-1} = \begin{bmatrix} 2 & 4 \\ -3 & 2 \end{bmatrix}$

**Finding the Inverse of a Matrix** In Exercises 59 and 60, show that the matrix is invertible and find its inverse.

**59.** $A = \begin{bmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{bmatrix}$ **60.** $A = \begin{bmatrix} \sec\theta & \tan\theta \\ \tan\theta & \sec\theta \end{bmatrix}$

**Beam Deflection** In Exercises 61 and 62, forces $w_1$, $w_2$, and $w_3$ (in pounds) act on a simply supported elastic beam, resulting in deflections $d_1$, $d_2$, and $d_3$ (in inches) in the beam (see figure).




Use the matrix equation $\mathbf{d} = F\mathbf{w}$, where

$$\mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

and $F$ is the $3 \times 3$ *flexibility matrix* for the beam, to find the stiffness matrix $F^{-1}$ and the force matrix $\mathbf{w}$. The units of the entries of $F$ are given in inches per pound.

**61.** $F = \begin{bmatrix} 0.008 & 0.004 & 0.003 \\ 0.004 & 0.006 & 0.004 \\ 0.003 & 0.004 & 0.008 \end{bmatrix}, \mathbf{d} = \begin{bmatrix} 0.585 \\ 0.640 \\ 0.835 \end{bmatrix}$

**62.** $F = \begin{bmatrix} 0.017 & 0.010 & 0.008 \\ 0.010 & 0.012 & 0.010 \\ 0.008 & 0.010 & 0.017 \end{bmatrix}, \mathbf{d} = \begin{bmatrix} 0 \\ 0.15 \\ 0 \end{bmatrix}$

**True or False?** In Exercises 63 and 64, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.

**63.** (a) If the matrices $A$, $B$, and $C$ satisfy $BA = CA$ and $A$ is invertible, then $B = C$.

(b) The inverse of the product of two matrices is the product of their inverses; that is, $(AB)^{-1} = A^{-1}B^{-1}$.

(c) If $A$ can be row reduced to the identity matrix, then $A$ is nonsingular.

**64.** (a) The inverse of the inverse of a nonsingular matrix $A$, $(A^{-1})^{-1}$, is equal to $A$ itself.

(b) The matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible when $ab - dc \neq 0$.

(c) If $A$ is a square matrix, then the system of linear equations $A\mathbf{x} = \mathbf{b}$ has a unique solution.

**65.** **Proof** Prove Property 2 of Theorem 2.8: If $A$ is an invertible matrix and $k$ is a positive integer, then

$$(A^k)^{-1} = \underbrace{A^{-1}A^{-1} \cdot \cdot \cdot A^{-1}}_{k \text{ factors}} = (A^{-1})^k$$

**66.** **Proof** Prove Property 4 of Theorem 2.8: If $A$ is an invertible matrix, then $(A^T)^{-1} = (A^{-1})^T$.

**67.** **Guided Proof** Prove that the inverse of a symmetric nonsingular matrix is symmetric.

Getting Started: To prove that the inverse of $A$ is symmetric, you need to show that $(A^{-1})^T = A^{-1}$.

(i) Let $A$ be a symmetric, nonsingular matrix.

(ii) This means that $A^T = A$ and $A^{-1}$ exists.

(iii) Use the properties of the transpose to show that $(A^{-1})^T$ is equal to $A^{-1}$.

**68.** **Proof** Prove Property 2 of Theorem 2.10: If $C$ is an invertible matrix such that $CA = CB$, then $A = B$.

**69.** **Proof** Prove that if $A^2 = A$, then

$I - 2A = (I - 2A)^{-1}$.

**70.** **Proof** Prove that if $A$, $B$, and $C$ are square matrices and $ABC = I$, then $B$ is invertible and $B^{-1} = CA$.

**71.** **Proof** Prove that if $A$ is invertible and $AB = O$, then $B = O$.

**72.** **Guided Proof** Prove that if $A^2 = A$, then either $A = I$ or $A$ is singular.

Getting Started: You must show that either $A$ is singular or $A$ equals the identity matrix.

(i) Begin your proof by observing that $A$ is either singular or nonsingular.

(ii) If $A$ is singular, then you are done.

(iii) If $A$ is nonsingular, then use the inverse matrix $A^{-1}$ and the hypothesis $A^2 = A$ to show that $A = I$.

**73.** **Writing** Is the sum of two invertible matrices invertible? Explain why or why not. Illustrate your conclusion with appropriate examples.

**74.** **Writing** Under what conditions will the diagonal matrix

$$A = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

be invertible? Assume that $A$ is invertible and find its inverse.

**75.** Use the result of Exercise 74 to find $A^{-1}$ for each matrix.

(a) $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

(b) $A = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$

**76.** Let $A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$.

(a) Show that $A^2 - 2A + 5I = O$, where $I$ is the identity matrix of order 2.

(b) Show that $A^{-1} = \frac{1}{5}(2I - A)$.

(c) Show that for any square matrix satisfying $A^2 - 2A + 5I = 0$, the inverse of $A$ is given by $A^{-1} = \frac{1}{5}(2I - A)$.

**77.** **Proof** Let $\mathbf{u}$ be an $n \times 1$ column matrix satisfying $\mathbf{u}^T\mathbf{u} = I_1$. The $n \times n$ matrix $H = I_n - 2\mathbf{u}\mathbf{u}^T$ is called a **Householder matrix.**

(a) Prove that $H$ is symmetric and nonsingular.

(b) Let $\mathbf{u} = \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \\ 0 \end{bmatrix}$. Show that $\mathbf{u}^T\mathbf{u} = I_1$ and calculate the Householder matrix $H$.

**78.** **Proof** Let $A$ and $B$ be $n \times n$ matrices. Prove that if the matrix $I - AB$ is nonsingular, then so is $I - BA$.

**79.** Let $A$, $D$, and $P$ be $n \times n$ matrices satisfying $AP = PD$. Assume that $P$ is nonsingular and solve this equation for $A$. Must it be true that $A = D$?

**80.** Find an example of a singular $2 \times 2$ matrix satisfying $A^2 = A$.

**81.** **Writing** Explain how to determine whether the inverse of a matrix exists. If so, explain how to find the inverse.

**82.** **CAPSTONE** As mentioned on page 66, if $A$ is a $2 \times 2$ matrix given by

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then $A$ is invertible if and only if $ad - bc \neq 0$. Verify that the inverse of $A$ is given by

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

**83.** **Writing** Explain in your own words how to write a system of three linear equations in three variables as a matrix equation, $AX = B$, as well as how to solve the system using an inverse matrix.

# 2.4 Elementary Matrices

- Factor a matrix into a product of elementary matrices.
- Find and use an *LU*-factorization of a matrix to solve a system of linear equations.

## ELEMENTARY MATRICES AND ELEMENTARY ROW OPERATIONS

Section 1.2 introduced the three elementary row operations for matrices listed below.

**1.** Interchange two rows.

**2.** Multiply a row by a nonzero constant.

**3.** Add a multiple of a row to another row.

In this section, you will see how to use matrix multiplication to perform these operations.

**REMARK**

The identity matrix $I_n$ is elementary by this definition because it can be obtained from itself by multiplying any one of its rows by 1.

### Definition of an Elementary Matrix

An $n \times n$ matrix is called an **elementary matrix** when it can be obtained from the identity matrix $I_n$ by a single elementary row operation.

### EXAMPLE 1 Elementary Matrices and Nonelementary Matrices

Which of the following matrices are elementary? For those that are, describe the corresponding elementary row operation.

**a.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **b.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

**c.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ **d.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

**e.** $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ **f.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

#### SOLUTION

**a.** This matrix *is* elementary. To obtain it from $I_3$, multiply the second row of $I_3$ by 3.

**b.** This matrix is *not* elementary because it is not square.

**c.** This matrix is *not* elementary because to obtain it from $I_3$, you must multiply the third row of $I_3$ by 0 (row multiplication must be by a nonzero constant).

**d.** This matrix *is* elementary. To obtain it from $I_3$, interchange the second and third rows of $I_3$.

**e.** This matrix *is* elementary. To obtain it from $I_2$, multiply the first row of $I_2$ by 2 and add the result to the second row.

**f.** This matrix is *not* elementary because it requires two elementary row operations to obtain from $I_3$.

Elementary matrices are useful because they enable you to use matrix multiplication to perform elementary row operations, as demonstrated in Example 2.

**EXAMPLE 2** **Elementary Matrices and Elementary Row Operations**

**a.** In the matrix product below, $E$ is the elementary matrix in which the first two rows of $I_3$ are interchanged.

$$\overset{E}{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \overset{A}{\begin{bmatrix} 0 & 2 & 1 \\ 1 & -3 & 6 \\ 3 & 2 & -1 \end{bmatrix}} = \begin{bmatrix} 1 & -3 & 6 \\ 0 & 2 & 1 \\ 3 & 2 & -1 \end{bmatrix}$$

Note that the first two rows of $A$ are interchanged by multiplying *on the left* by $E$.

**b.** In the following matrix product, $E$ is the elementary matrix in which the second row of $I_3$ is multiplied by $\frac{1}{2}$.

$$\overset{E}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}} \overset{A}{\begin{bmatrix} 1 & 0 & -4 & 1 \\ 0 & 2 & 6 & -4 \\ 0 & 1 & 3 & 1 \end{bmatrix}} = \begin{bmatrix} 1 & 0 & -4 & 1 \\ 0 & 1 & 3 & -2 \\ 0 & 1 & 3 & 1 \end{bmatrix}$$

Note that the size of $A$ is $3 \times 4$. $A$ could, however, be any $3 \times n$ matrix and multiplication on the left by $E$ would still result in multiplying the second row of $A$ by $\frac{1}{2}$.

**c.** In the product below, $E$ is the elementary matrix in which 2 times the first row of $I_3$ is added to the second row.

$$\overset{E}{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \overset{A}{\begin{bmatrix} 1 & 0 & -1 \\ -2 & -2 & 3 \\ 0 & 4 & 5 \end{bmatrix}} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -2 & 1 \\ 0 & 4 & 5 \end{bmatrix}$$

Note that in the product $EA$, 2 times the first row of $A$ is added to the second row.

**REMARK**

Be sure to remember in Theorem 2.12 to multiply $A$ *on the left* by the elementary matrix $E$. This text does not consider right multiplication by elementary matrices, which involves column operations.

In each of the three products in Example 2, you are able to perform elementary row operations by multiplying *on the left* by an elementary matrix. The next theorem, stated without proof, generalizes this property of elementary matrices.

**THEOREM 2.12 Representing Elementary Row Operations**

Let $E$ be the elementary matrix obtained by performing an elementary row operation on $I_m$. If that same elementary row operation is performed on an $m \times n$ matrix $A$, then the resulting matrix is given by the product $EA$.

Most applications of elementary row operations require a sequence of operations. For instance, Gaussian elimination usually requires several elementary row operations to row reduce a matrix. This translates into multiplication on the left by several elementary matrices. The order of multiplication is important; the elementary matrix immediately to the left of $A$ corresponds to the row operation performed first. Example 3 demonstrates this process.

## EXAMPLE 3 Using Elementary Matrices

Find a sequence of elementary matrices that can be used to write the matrix $A$ in row-echelon form.

$$A = \begin{bmatrix} 0 & 1 & 3 & 5 \\ 1 & -3 & 0 & 2 \\ 2 & -6 & 2 & 0 \end{bmatrix}$$

**SOLUTION**

| Matrix | Elementary Row Operation | Elementary Matrix |
|---|---|---|
| $\begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 2 & -6 & 2 & 0 \end{bmatrix}$ | $R_1 \leftrightarrow R_2$ | $E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ |
| $\begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & -4 \end{bmatrix}$ | $R_3 + (-2)R_1 \rightarrow R_3$ | $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$ |
| $\begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & -2 \end{bmatrix}$ | $\left(\frac{1}{2}\right)R_3 \rightarrow R_3$ | $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$ |

The three elementary matrices $E_1$, $E_2$, and $E_3$ can be used to perform the same elimination.

$$\begin{aligned} B = E_3E_2E_1A &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 3 & 5 \\ 1 & -3 & 0 & 2 \\ 2 & -6 & 2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 2 & -6 & 2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & -4 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & -2 \end{bmatrix} \end{aligned}$$

**REMARK**

The procedure demonstrated in Example 3 is primarily of theoretical interest. In other words, this procedure is not a practical method for performing Gaussian elimination.

The two matrices in Example 3

$$A = \begin{bmatrix} 0 & 1 & 3 & 5 \\ 1 & -3 & 0 & 2 \\ 2 & -6 & 2 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

are row-equivalent because you can obtain $B$ by performing a sequence of row operations on $A$. That is, $B = E_3E_2E_1A$.

The definition of row-equivalent matrices is restated using elementary matrices as follows.

### Definition of Row Equivalence

Let $A$ and $B$ be $m \times n$ matrices. Matrix $B$ is **row-equivalent** to $A$ when there exists a finite number of elementary matrices $E_1, E_2, \ldots, E_k$ such that

$$B = E_kE_{k-1} \cdots E_2E_1A.$$

You know from Section 2.3 that not all square matrices are invertible. Every elementary matrix, however, is invertible. Moreover, the inverse of an elementary matrix is itself an elementary matrix.

**THEOREM 2.13 Elementary Matrices Are Invertible**

If $E$ is an elementary matrix, then $E^{-1}$ exists and is an elementary matrix.

The inverse of an elementary matrix $E$ is the elementary matrix that converts $E$ back to $I_n$. For instance, the inverses of the three elementary matrices in Example 3 are as follows.

**REMARK**

$E_2^{-1}$ is as shown because to convert $E_2$ back to $I_3$, in $E_2$ you would add 2 times row 1 to row 3.

| Elementary Matrix | | Inverse Matrix | |
|---|---|---|---|
| $E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ | $R_1 \leftrightarrow R_2$ | $E_1^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ | $R_1 \leftrightarrow R_2$ |
| $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$ | $R_3 + (-2)R_1 \rightarrow R_3$ | $E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$ | $R_3 + (2)R_1 \rightarrow R_3$ |
| $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$ | $(\frac{1}{2})R_3 \rightarrow R_3$ | $E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ | $(2)R_3 \rightarrow R_3$ |

Try using matrix multiplication to check these results.

The following theorem states that every invertible matrix can be written as the product of elementary matrices.

**THEOREM 2.14 A Property of Invertible Matrices**

A square matrix $A$ is invertible if and only if it can be written as the product of elementary matrices.

**PROOF**

The phrase "if and only if" means that there are actually two parts to the theorem. On the one hand, you have to show that *if* $A$ is invertible, *then* it can be written as the product of elementary matrices. Then you have to show that *if* $A$ can be written as the product of elementary matrices, *then* $A$ is invertible.

To prove the theorem in one direction, assume $A$ is invertible. From Theorem 2.11 you know that the system of linear equations represented by $A\mathbf{x} = O$ has only the trivial solution. But this implies that the augmented matrix $[A \quad O]$ can be rewritten in the form $[I \quad O]$ (using elementary row operations corresponding to $E_1, E_2, \ldots,$ and $E_k$). So, $E_k \cdots E_3E_2E_1A = I$ and it follows that $A = E_1^{-1}E_2^{-1}E_3^{-1} \cdots E_k^{-1}$. $A$ can be written as the product of elementary matrices.

To prove the theorem in the other direction, assume $A$ is the product of elementary matrices. Then, because every elementary matrix is invertible and the product of invertible matrices is invertible, it follows that $A$ is invertible. This completes the proof. ■

Example 4 illustrates the first part of this proof.

## EXAMPLE 4 Writing a Matrix as the Product of Elementary Matrices

Find a sequence of elementary matrices whose product is the nonsingular matrix

$$A = \begin{bmatrix} -1 & -2 \\ 3 & 8 \end{bmatrix}.$$

### SOLUTION

Begin by finding a sequence of elementary row operations that can be used to rewrite $A$ in reduced row-echelon form.

| Matrix | Elementary Row Operation | Elementary Matrix |
|---|---|---|
| $\begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix}$ | $(-1)R_1 \rightarrow R_1$ | $E_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ |
| $\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$ | $R_2 + (-3)R_1 \rightarrow R_2$ | $E_2 = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$ |
| $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ | $\left(\frac{1}{2}\right)R_2 \rightarrow R_2$ | $E_3 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$ |
| $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ | $R_1 + (-2)R_2 \rightarrow R_1$ | $E_4 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$ |

Now, from the matrix product $E_4E_3E_2E_1A = I$, solve for $A$ to obtain $A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}$. This implies that $A$ is a product of elementary matrices.

$$A = \overset{E_1^{-1}}{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}}\overset{E_2^{-1}}{\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}}\overset{E_3^{-1}}{\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}}\overset{E_4^{-1}}{\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}} = \begin{bmatrix} -1 & -2 \\ 3 & 8 \end{bmatrix}$$

In Section 2.3, you learned a process for finding the inverse of a nonsingular matrix $A$. There, you used Gauss-Jordan elimination to reduce the augmented matrix $[A \quad I]$ to $[I \quad A^{-1}]$. You can now use Theorem 2.14 to justify this procedure. Specifically, the proof of Theorem 2.14 allows you to write the product

$$I = E_k \cdot \cdot \cdot E_3E_2E_1A.$$

Multiplying both sides of this equation (on the right) by $A^{-1}$, $A^{-1} = E_k \cdot \cdot \cdot E_3E_2E_1I$. In other words, a sequence of elementary matrices that reduces $A$ to the identity $I$ also reduces the identity $I$ to $A^{-1}$. Applying the corresponding sequence of elementary row operations to the matrices $A$ and $I$ simultaneously, you have

$$E_k \cdot \cdot \cdot E_3E_2E_1[A \quad I] = [I \quad A^{-1}].$$

Of course, if $A$ is singular, then no such sequence is possible.

The next theorem ties together some important relationships between $n \times n$ matrices and systems of linear equations. The essential parts of this theorem have already been proved (see Theorems 2.11 and 2.14); it is left to you to fill in the other parts of the proof.

### THEOREM 2.15 Equivalent Conditions

If $A$ is an $n \times n$ matrix, then the following statements are equivalent.

1. $A$ is invertible.
2. $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $n \times 1$ column matrix $\mathbf{b}$.
3. $A\mathbf{x} = O$ has only the trivial solution.
4. $A$ is row-equivalent to $I_n$.
5. $A$ can be written as the product of elementary matrices.

## THE *LU*-FACTORIZATION

At the heart of the most efficient and modern algorithms for solving linear systems $A\mathbf{x} = \mathbf{b}$ is the $LU$-factorization, in which the square matrix $A$ is expressed as a product, $A = LU$. In this product, the square matrix $L$ is **lower triangular,** which means all the entries above the main diagonal are zero. The square matrix $U$ is **upper triangular,** which means all the entries below the main diagonal are zero.

$$\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$

$3 \times 3$ lower triangular matrix $\qquad$ $3 \times 3$ upper triangular matrix

### Definition of *LU*-Factorization

If the $n \times n$ matrix $A$ can be written as the product of a lower triangular matrix $L$ and an upper triangular matrix $U$, then $A = LU$ is an ***LU*-factorization** of $A$.

### EXAMPLE 5 *LU*-Factorizations

**a.** $\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} = LU$

is an $LU$-factorization of the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$$

as the product of the lower triangular matrix

$$L = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

and the upper triangular matrix

$$U = \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}.$$

**b.** $A = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 2 & -10 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -4 & 1 \end{bmatrix}\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 14 \end{bmatrix} = LU$

is an $LU$-factorization of the matrix $A$.

**LINEAR ALGEBRA APPLIED**

Computational fluid dynamics (CFD) is the computer-based analysis of such real-life phenomena as fluid flow, heat transfer, and chemical reactions. Solving the conservation of energy, mass, and momentum equations involved in a CFD analysis can involve large systems of linear equations. So, for efficiency in computing, CFD analyses often use matrix partitioning and *LU*-factorization in their algorithms. Aerospace companies such as Boeing and Airbus have used CFD analysis in aircraft design. For instance, engineers at Boeing used CFD analysis to simulate airflow around a virtual model of their 787 aircraft to help produce a faster and more efficient design than those of earlier Boeing aircraft.

If a square matrix $A$ row reduces to an upper triangular matrix $U$ using only the row operation of adding a multiple of one row to another row below it, then it is relatively easy to find an $LU$-factorization of the matrix $A$. All you need to do is keep track of the individual row operations, as shown in the following example.

## EXAMPLE 6 Finding an *LU*-Factorization of a Matrix

Find an $LU$-factorization of the matrix $A = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 2 & -10 & 2 \end{bmatrix}$.

### SOLUTION

Begin by row reducing $A$ to upper triangular form while keeping track of the elementary matrices used for each row operation.

| Matrix | Elementary Row Operation | Elementary Matrix |
|---|---|---|
| $\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & -4 & 2 \end{bmatrix}$ | $R_3 + (-2)R_1 \rightarrow R_3$ | $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$ |
| $\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 14 \end{bmatrix}$ | $R_3 + (4)R_2 \rightarrow R_3$ | $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}$ |

The reduced matrix $U$ above is upper triangular, and it follows that $E_2E_1A = U$, or $A = E_1^{-1}E_2^{-1}U$. Because the product of the lower triangular matrices

$$E_1^{-1}E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -4 & 1 \end{bmatrix}$$

is a lower triangular matrix $L$, the factorization $A = LU$ is complete. Notice that this is the same $LU$-factorization as in Example 5(b).

If $A$ row reduces to an upper triangular matrix $U$ using only the row operation of adding a multiple of one row to another row below it, then $A$ has an $LU$-factorization.

$$E_k \cdot \cdot \cdot E_2E_1A = U$$
$$A = E_1^{-1}E_2^{-1} \cdot \cdot \cdot E_k^{-1}U = LU$$

Here $L$ is the product of the inverses of the elementary matrices used in the row reduction.

Note that the multipliers in Example 6 are $-2$ and 4, which are the negatives of the corresponding entries in $L$. This is true in general. If $U$ can be obtained from $A$ using only the row operation of adding a multiple of one row to another row below, then the matrix $L$ is lower triangular with 1's along the diagonal. Furthermore, the negative of each multiplier is in the same position as that of the corresponding zero in $U$.

Once you have obtained an $LU$-factorization of a matrix $A$, you can then solve the system of $n$ linear equations in $n$ variables $A\mathbf{x} = \mathbf{b}$ very efficiently in two steps.

1. Write $\mathbf{y} = U\mathbf{x}$ and solve $L\mathbf{y} = \mathbf{b}$ for $\mathbf{y}$.
2. Solve $U\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$.

The column matrix $\mathbf{x}$ is the solution of the original system because

$$A\mathbf{x} = LU\mathbf{x} = L\mathbf{y} = \mathbf{b}.$$

The second step is just back-substitution, because the matrix $U$ is upper triangular. The first step is similar, except that it starts at the top of the matrix, because $L$ is lower triangular. For this reason, the first step is often called **forward substitution.**

**EXAMPLE 7** **Solving a Linear System Using *LU*-Factorization**

Solve the linear system.

$$\begin{aligned} x_1 - 3x_2 \qquad\quad &= -5 \\ x_2 + 3x_3 &= -1 \\ 2x_1 - 10x_2 + 2x_3 &= -20 \end{aligned}$$

**SOLUTION**

You obtained an $LU$-factorization of the coefficient matrix $A$ in Example 6.

$$\begin{aligned} A &= \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 2 & -10 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -4 & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 14 \end{bmatrix} \end{aligned}$$

First, let $\mathbf{y} = U\mathbf{x}$ and solve the system $L\mathbf{y} = \mathbf{b}$ for $\mathbf{y}$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -4 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \\ -20 \end{bmatrix}$$

Solve this system using forward substitution. Starting with the first equation, you have

$$y_1 = -5.$$

The second equation gives $y_2 = -1$. Finally, from the third equation,

$$\begin{aligned} 2y_1 - 4y_2 + y_3 &= -20 \\ y_3 &= -20 - 2y_1 + 4y_2 \\ y_3 &= -20 - 2(-5) + 4(-1) \\ y_3 &= -14. \end{aligned}$$

The solution of $L\mathbf{y} = \mathbf{b}$ is

$$\mathbf{y} = \begin{bmatrix} -5 \\ -1 \\ -14 \end{bmatrix}.$$

Now solve the system $U\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$ using back-substitution.

$$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \\ -14 \end{bmatrix}$$

From the bottom equation, $x_3 = -1$. Then, the second equation gives

$$x_2 + 3(-1) = -1$$

or $x_2 = 2$. Finally, the first equation gives

$$x_1 - 3(2) = -5$$

or $x_1 = 1$. So, the solution of the original system of equations is

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}.$$

## 2.4 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

**Elementary Matrices** **In Exercises 1–8, determine whether the matrix is elementary. If it is, state the elementary row operation used to produce it.**

**1.** $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ **2.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

**3.** $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ **4.** $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

**5.** $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ **6.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$

**7.** $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -5 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ **8.** $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -3 & 1 \end{bmatrix}$

**Finding an Elementary Matrix** **In Exercises 9–12, let $A$, $B$, and $C$ be**

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 2 \\ -1 & 2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 2 & -3 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 4 & -3 \\ 0 & 1 & 2 \\ -1 & 2 & 0 \end{bmatrix}.$$

**9.** Find an elementary matrix $E$ such that $EA = B$.

**10.** Find an elementary matrix $E$ such that $EA = C$.

**11.** Find an elementary matrix $E$ such that $EB = A$.

**12.** Find an elementary matrix $E$ such that $EC = A$.

**Finding a Sequence of Elementary Matrices** **In Exercises 13–16, find a sequence of elementary matrices that can be used to write the matrix in row-echelon form.**

**13.** $\begin{bmatrix} 0 & 1 & 7 \\ 5 & 10 & -5 \end{bmatrix}$ **14.** $\begin{bmatrix} 0 & 3 & -3 & 6 \\ 1 & -1 & 2 & -2 \\ 0 & 0 & 2 & 2 \end{bmatrix}$

**15.** $\begin{bmatrix} 1 & -2 & -1 & 0 \\ 0 & 4 & 8 & -4 \\ -6 & 12 & 8 & 1 \end{bmatrix}$ **16.** $\begin{bmatrix} 1 & 3 & 0 \\ 2 & 5 & -1 \\ 3 & -2 & -4 \end{bmatrix}$

**Finding the Inverse of an Elementary Matrix** **In Exercises 17–22, find the inverse of the elementary matrix.**

**17.** $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ **18.** $\begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$

**19.** $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ **20.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$

**21.** $\begin{bmatrix} k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **22.** $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & k & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$k \neq 0$

**Finding the Inverse of a Matrix** **In Exercises 23–26, find the inverse of the matrix using elementary matrices.**

**23.** $\begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$ **24.** $\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$

**25.** $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 6 & -1 \\ 0 & 0 & 4 \end{bmatrix}$ **26.** $\begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

**Finding a Sequence of Elementary Matrices** **In Exercises 27–34, find a sequence of elementary matrices whose product is the given nonsingular matrix.**

**27.** $\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$ **28.** $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

**29.** $\begin{bmatrix} 4 & -1 \\ 3 & -1 \end{bmatrix}$ **30.** $\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$

**31.** $\begin{bmatrix} 1 & -2 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **32.** $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 1 & 3 & 4 \end{bmatrix}$

**33.** $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 3 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$ **34.** $\begin{bmatrix} 4 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 2 \\ 1 & 0 & 0 & -2 \end{bmatrix}$

**True or False?** **In Exercises 35 and 36, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.**

**35.** (a) The identity matrix is an elementary matrix.

(b) If $E$ is an elementary matrix, then $2E$ is an elementary matrix.

(c) The inverse of an elementary matrix is an elementary matrix.

**36.** (a) The zero matrix is an elementary matrix.

(b) A square matrix is nonsingular when it can be written as the product of elementary matrices.

(c) $A\mathbf{x} = O$ has only the trivial solution if and only if $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $n \times 1$ column matrix $\mathbf{b}$.

**37.** **Writing** Is the product of two elementary matrices always elementary? Explain.

**38.** **Writing** $E$ is the elementary matrix obtained by interchanging two rows in $I_n$. $A$ is an $n \times n$ matrix.

(a) How will $EA$ compare with $A$? (b) Find $E^2$.

**39.** Use elementary matrices to find the inverse of

$$A = \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ b & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & c \end{bmatrix},$$

$c \neq 0$.

**40.** Use elementary matrices to find the inverse of

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & c \end{bmatrix}, \quad c \neq 0.$$

**Finding an *LU*-Factorization of a Matrix** **In Exercises 41–44, find an *LU*-factorization of the matrix.**

**41.** $\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$ **42.** $\begin{bmatrix} -2 & 1 \\ -6 & 4 \end{bmatrix}$

**43.** $\begin{bmatrix} 3 & 0 & 1 \\ 6 & 1 & 1 \\ -3 & 1 & 0 \end{bmatrix}$ **44.** $\begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 1 \\ 10 & 12 & 3 \end{bmatrix}$

**Solving a Linear System Using *LU*-Factorization** **In Exercises 45 and 46, solve the linear system $A\mathbf{x} = \mathbf{b}$ by**

**(a) finding an *LU*-factorization of the coefficient matrix $A$,**

**(b) solving the lower triangular system $L\mathbf{y} = \mathbf{b}$, and**

**(c) solving the upper triangular system $U\mathbf{x} = \mathbf{y}$.**

**45.**
$$\begin{aligned} 2x + y \phantom{{}+z} &= 1 \\ y - z &= 2 \\ -2x + y + z &= -2 \end{aligned}$$

**46.**
$$\begin{aligned} 2x_1 \phantom{{}+x_2-x_3-x_4} &= 4 \\ -2x_1 + x_2 - x_3 &= -4 \\ 6x_1 + 2x_2 + x_3 &= 15 \\ -x_4 &= -1 \end{aligned}$$

**47.** **Writing** Suppose you needed to solve many systems of linear equations $A\mathbf{x} = \mathbf{b}_i$, each having the same coefficient matrix $A$. Explain how you could use the $LU$-factorization technique to make the task easier, rather than solving each system individually using Gaussian elimination.

**48.** **CAPSTONE** Explain each of the following.

(a) How to find an elementary matrix

(b) How to use elementary matrices to find an $LU$-factorization of a matrix

(c) How to use $LU$-factorization to solve a linear system

**Idempotent Matrices** **In Exercises 49–52, determine whether the matrix is idempotent. A square matrix $A$ is idempotent when $A^2 = A$.**

**49.** $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ **50.** $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

**51.** $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ **52.** $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

**53.** Determine $a$ and $b$ such that $A$ is idempotent.

$$A = \begin{bmatrix} 1 & 0 \\ a & b \end{bmatrix}$$

**54.** **Guided Proof** Prove that $A$ is idempotent if and only if $A^T$ is idempotent.

Getting Started: The phrase "if and only if" means that you have to prove two statements:

**1.** If $A$ is idempotent, then $A^T$ is idempotent.

**2.** If $A^T$ is idempotent, then $A$ is idempotent.

(i) Begin your proof of the first statement by assuming that $A$ is idempotent.

(ii) This means that $A^2 = A$.

(iii) Use the properties of the transpose to show that $A^T$ is idempotent.

(iv) Begin your proof of the second statement by assuming that $A^T$ is idempotent.

**55.** **Proof** Prove that if $A$ is an $n \times n$ matrix that is idempotent and invertible, then $A = I_n$.

**56.** **Proof** Prove that if $A$ and $B$ are idempotent and $AB = BA$, then $AB$ is idempotent.

**57.** **Guided Proof** Prove that if $A$ is row-equivalent to $B$ and $B$ is row-equivalent to $C$, then $A$ is row-equivalent to $C$.

Getting Started: To prove that $A$ is row-equivalent to $C$, you have to find elementary matrices $E_1, \ldots, E_k$ such that $A = E_k \cdots E_1 C$.

(i) Begin your proof by observing that $A$ is row-equivalent to $B$.

(ii) Meaning, there exist elementary matrices $F_1, \ldots, F_n$ such that $A = F_n \cdots F_1 B$.

(iii) There exist elementary matrices $G_1, \ldots, G_m$ such that $B = G_1 \cdots G_m C$.

(iv) Combine the matrix equations from steps (ii) and (iii).

**58.** **Proof** Prove that if $A$ is row-equivalent to $B$, then $B$ is row-equivalent to $A$.

**59.** **Proof** Let $A$ be a nonsingular matrix. Prove that if $B$ is row-equivalent to $A$, then $B$ is also nonsingular.

**60.** Show that the following matrix does not have an $LU$-factorization.

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

# 2.5 Applications of Matrix Operations

- Write and use a stochastic matrix.
- Use matrix multiplication to encode and decode messages.
- Use matrix algebra to analyze an economic system (Leontief input-output model).
- Find the least squares regression line for a set of data.

## STOCHASTIC MATRICES

Many types of applications involve a finite set of *states* $\{S_1, S_2, \ldots, S_n\}$ of a given population. For instance, residents of a city may live downtown or in the suburbs. Voters may vote Democrat, Republican, or Independent. Soft drink consumers may buy Coca-Cola, Pepsi Cola, or another brand.

The probability that a member of a population will change from the $j$th state to the $i$th state is represented by a number $p_{ij}$, where $0 \le p_{ij} \le 1$. A probability of $p_{ij} = 0$ means that the member is certain *not* to change from the $j$th state to the $i$th state, whereas a probability of $p_{ij} = 1$ means that the member is certain to change from the $j$th state to the $i$th state.

$$\begin{array}{ccc} & \overbrace{\begin{array}{cccc} S_1 & S_2 & \cdots & S_n \end{array}}^{\text{From}} & \\ P = & \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix} & \left.\begin{array}{c} S_1 \\ S_2 \\ \vdots \\ S_n \end{array}\right\} \text{To} \end{array}$$

$P$ is called the **matrix of transition probabilities** because it gives the probabilities of each possible type of transition (or change) within the population.

At each transition, each member in a given state must either stay in that state or change to another state. For probabilities, this means that the sum of the entries in any column of $P$ is 1. For instance, in the first column,

$$p_{11} + p_{21} + \cdots + p_{n1} = 1.$$

Such a matrix is called **stochastic** (the term "stochastic" means "regarding conjecture"). That is, an $n \times n$ matrix $P$ is a **stochastic matrix** when each entry is a number between 0 and 1 inclusive, and the sum of the entries in each column of $P$ is 1.

**EXAMPLE 1** **Examples of Stochastic Matrices and Nonstochastic Matrices**

The matrices in parts (a) and (b) are stochastic, but the matrices in parts (c) and (d) are not.

**a.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **b.** $\begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{3}{4} \\ \frac{1}{4} & \frac{2}{3} & 0 \end{bmatrix}$

**c.** $\begin{bmatrix} 0.1 & 0.2 & 0.3 \\ 0.2 & 0.3 & 0.4 \\ 0.3 & 0.4 & 0.5 \end{bmatrix}$ **d.** $\begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{4} & \frac{3}{4} & 0 \end{bmatrix}$

Example 2 describes the use of a stochastic matrix to measure consumer preferences.

## EXAMPLE 2 A Consumer Preference Model

Two competing companies offer satellite television service to a city with 100,000 households. Figure 2.1 shows the changes in satellite subscriptions each year. Company A now has 15,000 subscribers and Company B has 20,000 subscribers. How many subscribers will each company have in one year?

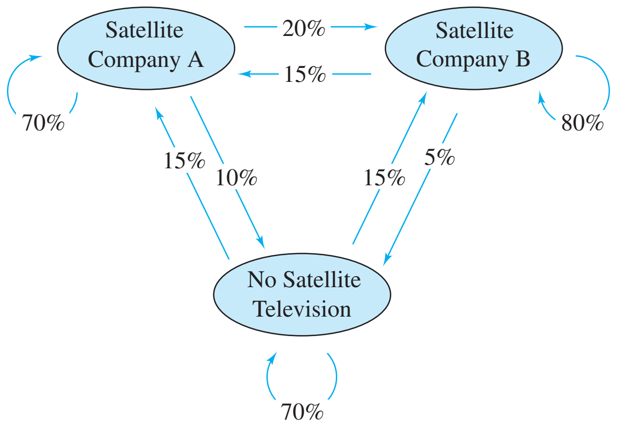



**Figure 2.1**

### SOLUTION

The matrix representing the given transition probabilities is

$$\begin{array}{c} \overbrace{\begin{array}{ccc} \text{A} & \text{B} & \text{None} \end{array}}^{\text{From}} \\ P = \begin{bmatrix} 0.70 & 0.15 & 0.15 \\ 0.20 & 0.80 & 0.15 \\ 0.10 & 0.05 & 0.70 \end{bmatrix} \end{array} \left.\begin{array}{l} \text{A} \\ \text{B} \\ \text{None} \end{array}\right\} \text{To}$$

and the **state matrix** representing the current populations in the three states is

$$X = \begin{bmatrix} 15{,}000 \\ 20{,}000 \\ 65{,}000 \end{bmatrix}. \qquad \begin{array}{l} \text{A} \\ \text{B} \\ \text{None} \end{array}$$

To find the state matrix representing the populations in the three states in one year, multiply $P$ by $X$ to obtain

$$PX = \begin{bmatrix} 0.70 & 0.15 & 0.15 \\ 0.20 & 0.80 & 0.15 \\ 0.10 & 0.05 & 0.70 \end{bmatrix} \begin{bmatrix} 15{,}000 \\ 20{,}000 \\ 65{,}000 \end{bmatrix}$$

$$= \begin{bmatrix} 23{,}250 \\ 28{,}750 \\ 48{,}000 \end{bmatrix}.$$

In one year, Company A will have 23,250 subscribers and Company B will have 28,750 subscribers.

One appeal of the matrix solution in Example 2 is that once you have created the model, it becomes relatively easy to find the state matrices representing future years by repeatedly multiplying by the matrix $P$. Example 3 demonstrates this process.

## EXAMPLE 3 A Consumer Preference Model

Assuming the matrix of transition probabilities from Example 2 remains the same year after year, find the number of subscribers each satellite television company will have after (a) 3 years, (b) 5 years, and (c) 10 years. Round each answer to the nearest whole number.

**SOLUTION**

**a.** From Example 2, you know that the numbers of subscribers after 1 year are

$$PX = \begin{bmatrix} 23{,}250 \\ 28{,}750 \\ 48{,}000 \end{bmatrix}. \quad \begin{matrix} \text{A} \\ \text{B} \\ \text{None} \end{matrix} \quad \text{After 1 year}$$

Because the matrix of transition probabilities is the same from the first year to the third year, the numbers of subscribers after 3 years are

$$P^3X \approx \begin{bmatrix} 30{,}283 \\ 39{,}042 \\ 30{,}675 \end{bmatrix}. \quad \begin{matrix} \text{A} \\ \text{B} \\ \text{None} \end{matrix} \quad \text{After 3 years}$$

After 3 years, Company A will have 30,283 subscribers and Company B will have 39,042 subscribers.

**b.** The numbers of subscribers after 5 years are

$$P^5X \approx \begin{bmatrix} 32{,}411 \\ 43{,}812 \\ 23{,}777 \end{bmatrix}. \quad \begin{matrix} \text{A} \\ \text{B} \\ \text{None} \end{matrix} \quad \text{After 5 years}$$

After 5 years, Company A will have 32,411 subscribers and Company B will have 43,812 subscribers.

**c.** The numbers of subscribers after 10 years are

$$P^{10}X \approx \begin{bmatrix} 33{,}287 \\ 47{,}147 \\ 19{,}566 \end{bmatrix}. \quad \begin{matrix} \text{A} \\ \text{B} \\ \text{None} \end{matrix} \quad \text{After 10 years}$$

After 10 years, Company A will have 33,287 subscribers and Company B will have 47,147 subscribers.

In Example 3, notice that there is little difference between the numbers of subscribers after 5 years and after 10 years. If you continue the process shown in this example, then the numbers of subscribers eventually reach a **steady state.** That is, as long as the matrix $P$ does not change, the matrix product $P^nX$ approaches a limit $\overline{X}$. In Example 3, the limit is the steady state matrix

$$\overline{X} = \begin{bmatrix} 33{,}333 \\ 47{,}619 \\ 19{,}048 \end{bmatrix}. \quad \begin{matrix} \text{A} \\ \text{B} \\ \text{None} \end{matrix} \quad \text{Steady state}$$

Check to see that $P\overline{X} = \overline{X}$, as follows.

$$P\overline{X} = \begin{bmatrix} 0.70 & 0.15 & 0.15 \\ 0.20 & 0.80 & 0.15 \\ 0.10 & 0.05 & 0.70 \end{bmatrix} \begin{bmatrix} 33{,}333 \\ 47{,}619 \\ 19{,}048 \end{bmatrix}$$

$$\approx \begin{bmatrix} 33{,}333 \\ 47{,}619 \\ 19{,}048 \end{bmatrix} = \overline{X}$$

## CRYPTOGRAPHY

A **cryptogram** is a message written according to a secret code (the Greek word *kryptos* means “hidden”). The following describes a method of using matrix multiplication to **encode** and **decode** messages.

To begin, assign a number to each letter in the alphabet (with 0 assigned to a blank space), as follows.

| | |
|---|---|
| 0 = __ | 14 = N |
| 1 = A | 15 = O |
| 2 = B | 16 = P |
| 3 = C | 17 = Q |
| 4 = D | 18 = R |
| 5 = E | 19 = S |
| 6 = F | 20 = T |
| 7 = G | 21 = U |
| 8 = H | 22 = V |
| 9 = I | 23 = W |
| 10 = J | 24 = X |
| 11 = K | 25 = Y |
| 12 = L | 26 = Z |
| 13 = M | |

Then convert the message to numbers and partition it into **uncoded row matrices,** each having $n$ entries, as demonstrated in Example 4.

### EXAMPLE 4 Forming Uncoded Row Matrices

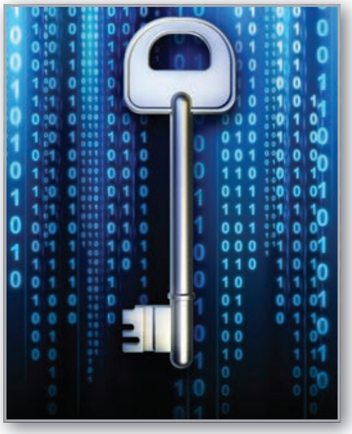
Write the uncoded row matrices of size $1 \times 3$ for the message MEET ME MONDAY.

**SOLUTION**

Partitioning the message (including blank spaces, but ignoring punctuation) into groups of three produces the following uncoded row matrices.

$$\begin{array}{ccccc} [13 \quad 5 \quad 5] & [20 \quad 0 \quad 13] & [5 \quad 0 \quad 13] & [15 \quad 14 \quad 4] & [1 \quad 25 \quad 0] \\ M \quad E \quad E & T \quad \_\_ \quad M & E \quad \_\_ \quad M & O \quad N \quad D & A \quad Y \quad \_\_ \end{array}$$

Note the use of a blank space to fill out the last uncoded row matrix.



**LINEAR ALGEBRA APPLIED**

Because of the heavy use of the Internet to conduct business, Internet security is of the utmost importance. If a malicious party should receive confidential information such as passwords, personal identification numbers, credit card numbers, social security numbers, bank account details, or corporate secrets, the effects can be damaging. To protect the confidentiality and integrity of such information, the most popular forms of Internet security use data *encryption,* the process of encoding information so that the only way to decode it, apart from a brute force “exhaustion attack,” is to use a *key*. Data encryption technology uses algorithms based on the material presented here, but on a much more sophisticated level, to prevent malicious parties from discovering the key.

To **encode** a message, choose an $n \times n$ invertible matrix $A$ and multiply the uncoded row matrices (on the right) by $A$ to obtain **coded row matrices.** Example 5 demonstrates this process.

Andrea Danti/Shutterstock.com

### EXAMPLE 5 Encoding a Message

Use the following invertible matrix

$$A = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix}$$

to encode the message MEET ME MONDAY.

**SOLUTION**

Obtain the coded row matrices by multiplying each of the uncoded row matrices found in Example 4 by the matrix $A$, as follows.

Uncoded Row Matrix — Encoding Matrix $A$ — Coded Row Matrix

$$[13 \quad 5 \quad 5]\begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix} = [13 \quad -26 \quad 21]$$

$$[20 \quad 0 \quad 13]\begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix} = [33 \quad -53 \quad -12]$$

$$[5 \quad 0 \quad 13]\begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix} = [18 \quad -23 \quad -42]$$

$$[15 \quad 14 \quad 4]\begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix} = [5 \quad -20 \quad 56]$$

$$[1 \quad 25 \quad 0]\begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix} = [-24 \quad 23 \quad 77]$$

The sequence of coded row matrices is

$$[13 \quad -26 \quad 21][33 \quad -53 \quad -12][18 \quad -23 \quad -42][5 \quad -20 \quad 56][-24 \quad 23 \quad 77].$$

Finally, removing the matrix notation produces the following cryptogram.

$$13 \;\; -26 \;\; 21 \;\; 33 \;\; -53 \;\; -12 \;\; 18 \;\; -23 \;\; -42 \;\; 5 \;\; -20 \;\; 56 \;\; -24 \;\; 23 \;\; 77$$

For those who do not know the encoding matrix $A$, decoding the cryptogram found in Example 5 is difficult. But for an authorized receiver who knows the encoding matrix $A$, decoding is relatively simple. The receiver just needs to multiply the coded row matrices by $A^{-1}$ to retrieve the uncoded row matrices. In other words, if

$$X = [x_1 \;\; x_2 \cdot \cdot \cdot \; x_n]$$

is an uncoded $1 \times n$ matrix, then $Y = XA$ is the corresponding encoded matrix. The receiver of the encoded matrix can decode $Y$ by multiplying on the right by $A^{-1}$ to obtain

$$YA^{-1} = (XA)A^{-1} = X.$$

Example 6 demonstrates this procedure.

## EXAMPLE 6 Decoding a Message

***Simulation***
Explore this concept further with an electronic simulation available at *www.cengagebrain.com*.

Use the inverse of the matrix

$$A = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix}$$

to decode the cryptogram

$$13 \quad -26 \quad 21 \quad 33 \quad -53 \quad -12 \quad 18 \quad -23 \quad -42 \quad 5 \quad -20 \quad 56 \quad -24 \quad 23 \quad 77.$$

### SOLUTION

Begin by using Gauss-Jordan elimination to find $A^{-1}$.

$$\overset{[A \qquad I]}{\begin{bmatrix} 1 & -2 & 2 & 1 & 0 & 0 \\ -1 & 1 & 3 & 0 & 1 & 0 \\ 1 & -1 & -4 & 0 & 0 & 1 \end{bmatrix}} \rightarrow \overset{[I \qquad A^{-1}]}{\begin{bmatrix} 1 & 0 & 0 & -1 & -10 & -8 \\ 0 & 1 & 0 & -1 & -6 & -5 \\ 0 & 0 & 1 & 0 & -1 & -1 \end{bmatrix}}$$

Now, to decode the message, partition the message into groups of three to form the coded row matrices

$$[13 \quad -26 \quad 21][33 \quad -53 \quad -12][18 \quad -23 \quad -42][5 \quad -20 \quad 56][-24 \quad 23 \quad 77].$$

To obtain the decoded row matrices, multiply each coded row matrix by $A^{-1}$ (on the right).

| Coded Row Matrix | Decoding Matrix $A^{-1}$ | Decoded Row Matrix |
|---|---|---|
| $[13 \quad -26 \quad 21]$ | $\begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix}$ | $= [13 \quad 5 \quad 5]$ |
| $[33 \quad -53 \quad -12]$ | $\begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix}$ | $= [20 \quad 0 \quad 13]$ |
| $[18 \quad -23 \quad -42]$ | $\begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix}$ | $= [5 \quad 0 \quad 13]$ |
| $[5 \quad -20 \quad 56]$ | $\begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix}$ | $= [15 \quad 14 \quad 4]$ |
| $[-24 \quad 23 \quad 77]$ | $\begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix}$ | $= [1 \quad 25 \quad 0]$ |

The sequence of decoded row matrices is

$$[13 \quad 5 \quad 5][20 \quad 0 \quad 13][5 \quad 0 \quad 13][15 \quad 14 \quad 4][1 \quad 25 \quad 0]$$

and the message is

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 13 | 5 | 5 | 20 | 0 | 13 | 5 | 0 | 13 | 15 | 14 | 4 | 1 | 25 | 0. |
| *M* | *E* | *E* | *T* | __ | *M* | *E* | __ | *M* | *O* | *N* | *D* | *A* | *Y* | __ |

## LEONTIEF INPUT-OUTPUT MODELS

In 1936, American economist Wassily W. Leontief (1906–1999) published a model concerning the input and output of an economic system. In 1973, Leontief received a Nobel prize for his work in economics. A brief discussion of Leontief's model follows.

Suppose that an economic system has $n$ different industries $I_1, I_2, \ldots, I_n$, each of which has **input** needs (raw materials, utilities, etc.) and an **output** (finished product). In producing each unit of output, an industry may use the outputs of other industries, including itself. For example, an electric utility uses outputs from other industries, such as coal and water, and also uses its own electricity.

Let $d_{ij}$ be the amount of output the $j$th industry needs from the $i$th industry to produce one unit of output per year. The matrix of these coefficients is called the **input-output matrix.**

$$D = \begin{array}{cl} \overbrace{\begin{array}{cccc} I_1 & I_2 & \cdots & I_n \end{array}}^{\text{User (Output)}} & \\ \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1n} \\ d_{21} & d_{22} & \cdots & d_{2n} \\ \vdots & \vdots & & \vdots \\ d_{n1} & d_{n2} & \cdots & d_{nn} \end{bmatrix} & \left.\begin{array}{c} I_1 \\ I_2 \\ \vdots \\ I_n \end{array}\right\} \text{Supplier (Input)} \end{array}$$

To understand how to use this matrix, consider $d_{12} = 0.4$. This means that for Industry 2 to produce one unit of its product, it must use 0.4 unit of Industry 1's product. If $d_{33} = 0.2$, then Industry 3 needs 0.2 unit of its own product to produce one unit. For this model to work, the values of $d_{ij}$ must satisfy $0 \leq d_{ij} \leq 1$ and the sum of the entries in any column must be less than or equal to 1.

### EXAMPLE 7 Forming an Input-Output Matrix

Consider a simple economic system consisting of three industries: electricity, water, and coal. Production, or output, of one unit of electricity requires 0.5 unit of itself, 0.25 unit of water, and 0.25 unit of coal. Production of one unit of water requires 0.1 unit of electricity, 0.6 unit of itself, and 0 units of coal. Production of one unit of coal requires 0.2 unit of electricity, 0.15 unit of water, and 0.5 unit of itself. Find the input-output matrix for this system.

**SOLUTION**

The column entries show the amounts each industry requires from the others, and from itself, to produce one unit of output.

$$\begin{array}{cl} \overbrace{\begin{array}{ccc} \text{E} & \text{W} & \text{C} \end{array}}^{\text{User (Output)}} & \\ \begin{bmatrix} 0.5 & 0.1 & 0.2 \\ 0.25 & 0.6 & 0.15 \\ 0.25 & 0 & 0.5 \end{bmatrix} & \left.\begin{array}{c} \text{E} \\ \text{W} \\ \text{C} \end{array}\right\} \text{Supplier (Input)} \end{array}$$

The row entries show the amounts each industry supplies to the others, and to itself, for that industry to produce one unit of output. For instance, the electricity industry supplies 0.5 unit to itself, 0.1 unit to water, and 0.2 unit to coal.

To develop the Leontief input-output model further, let the total output of the $i$th industry be denoted by $x_i$. If the economic system is **closed** (meaning that it sells its products only to industries within the system, as in the example above), then the total output of the $i$th industry is given by the linear equation

$$x_i = d_{i1}x_1 + d_{i2}x_2 + \cdots + d_{in}x_n. \qquad \text{Closed system}$$

On the other hand, if the industries within the system sell products to nonproducing groups (such as governments or charitable organizations) outside the system, then the system is **open** and the total output of the $i$th industry is given by

$$x_i = d_{i1}x_1 + d_{i2}x_2 + \cdot \cdot \cdot + d_{in}x_n + e_i \qquad \text{Open system}$$

where $e_i$ represents the external demand for the $i$th industry's product. The following system of $n$ linear equations represents the collection of total outputs for an open system.

$$\begin{aligned} x_1 &= d_{11}x_1 + d_{12}x_2 + \cdot \cdot \cdot + d_{1n}x_n + e_1 \\ x_2 &= d_{21}x_1 + d_{22}x_2 + \cdot \cdot \cdot + d_{2n}x_n + e_2 \\ &\vdots \\ x_n &= d_{n1}x_1 + d_{n2}x_2 + \cdot \cdot \cdot + d_{nn}x_n + e_n \end{aligned}$$

The matrix form of this system is $X = DX + E$, where $X$ is the **output matrix** and $E$ is the **external demand matrix.**

## EXAMPLE 8 Solving for the Output Matrix of an Open System

An economic system composed of three industries has the following input-output matrix.

$$D = \begin{array}{c} \overbrace{\begin{array}{ccc} \text{A} & \text{B} & \text{C} \end{array}}^{\text{User (Output)}} \\ \begin{bmatrix} 0.1 & 0.43 & 0 \\ 0.15 & 0 & 0.37 \\ 0.23 & 0.03 & 0.02 \end{bmatrix} \end{array} \begin{array}{c} \\ \left.\begin{array}{c} \text{A} \\ \text{B} \\ \text{C} \end{array}\right\} \text{Supplier (Input)} \end{array}$$

Find the output matrix $X$ when the external demands are

$$E = \begin{bmatrix} 20{,}000 \\ 30{,}000 \\ 25{,}000 \end{bmatrix}. \qquad \begin{array}{c} \text{A} \\ \text{B} \\ \text{C} \end{array}$$

Round each matrix entry to the nearest whole number.

### SOLUTION

Letting $I$ be the identity matrix, write the equation $X = DX + E$ as $IX - DX = E$, which means that $(I - D)X = E$. Using the matrix $D$ above produces

$$I - D = \begin{bmatrix} 0.9 & -0.43 & 0 \\ -0.15 & 1 & -0.37 \\ -0.23 & -0.03 & 0.98 \end{bmatrix}.$$

Finally, applying Gauss-Jordan elimination to the system of linear equations represented by $(I - D)X = E$ produces

$$\begin{bmatrix} 0.9 & -0.43 & 0 & 20{,}000 \\ -0.15 & 1 & -0.37 & 30{,}000 \\ -0.23 & -0.03 & 0.98 & 25{,}000 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 46{,}616 \\ 0 & 1 & 0 & 51{,}058 \\ 0 & 0 & 1 & 38{,}014 \end{bmatrix}.$$

So, the output matrix is

$$X = \begin{bmatrix} 46{,}616 \\ 51{,}058 \\ 38{,}014 \end{bmatrix}. \qquad \begin{array}{c} \text{A} \\ \text{B} \\ \text{C} \end{array}$$

To produce the given external demands, the outputs of the three industries must be 46,616 units for industry A, 51,058 units for industry B, and 38,014 units for industry C.

**REMARK**

The economic systems described in Examples 7 and 8 are, of course, simple ones. In the real world, an economic system would include many industries or industrial groups. A detailed analysis using the Leontief input-output model could easily require an input-output matrix greater than $100 \times 100$ in size. Clearly, this type of analysis would require the aid of a computer.

## LEAST SQUARES REGRESSION ANALYSIS

You will now look at a procedure used in statistics to develop linear models. The next example demonstrates a visual method for approximating a line of best fit for a given set of data points.

### EXAMPLE 9 A Visual Straight-Line Approximation

Determine a line that appears to best fit the points $(1, 1)$, $(2, 2)$, $(3, 4)$, $(4, 4)$, and $(5, 6)$.

**SOLUTION**

Plot the points, as shown in Figure 2.2. It appears that a good choice would be the line whose slope is 1 and whose $y$-intercept is 0.5. The equation of this line is

$$y = 0.5 + x.$$

An examination of the line in Figure 2.2 reveals that you can improve the fit by rotating the line counterclockwise slightly, as shown in Figure 2.3. It seems clear that this line, whose equation is $y = 1.2x$, fits the given points better than the original line.

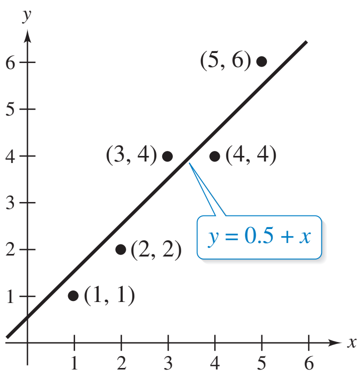


**Figure 2.2**

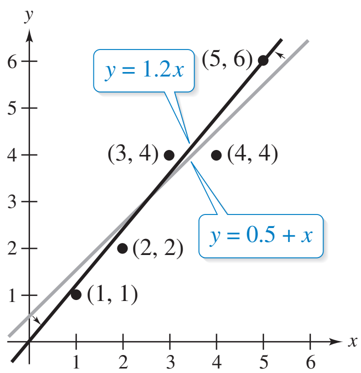


**Figure 2.3**

One way of measuring how well a function $y = f(x)$ fits a set of points

$$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$$

is to compute the differences between the values from the function $f(x_i)$ and the actual values $y_i$. These values are shown in Figure 2.4. By squaring the differences and summing the results, you obtain a measure of error called the **sum of squared error.** The table shows the sums of squared errors for the two linear models.

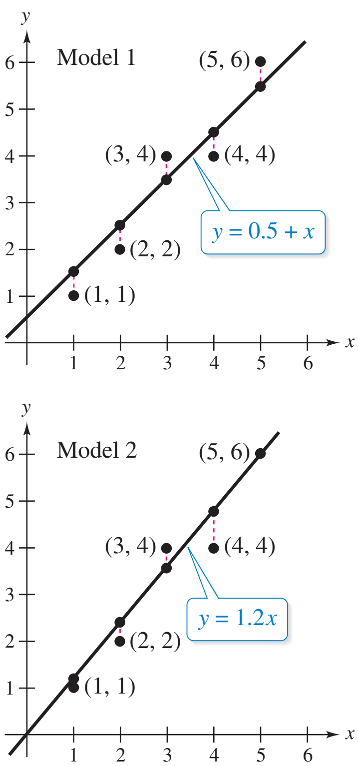


**Figure 2.4**

| ***Model 1:*** $f(x) = 0.5 + x$ | | | | ***Model 2:*** $f(x) = 1.2x$ | | | |
|---|---|---|---|---|---|---|---|
| $x_i$ | $y_i$ | $f(x_i)$ | $[y_i - f(x_i)]^2$ | $x_i$ | $y_i$ | $f(x_i)$ | $[y_i - f(x_i)]^2$ |
| 1 | 1 | 1.5 | $(-0.5)^2$ | 1 | 1 | 1.2 | $(-0.2)^2$ |
| 2 | 2 | 2.5 | $(-0.5)^2$ | 2 | 2 | 2.4 | $(-0.4)^2$ |
| 3 | 4 | 3.5 | $(+0.5)^2$ | 3 | 4 | 3.6 | $(+0.4)^2$ |
| 4 | 4 | 4.5 | $(-0.5)^2$ | 4 | 4 | 4.8 | $(-0.8)^2$ |
| 5 | 6 | 5.5 | $(+0.5)^2$ | 5 | 6 | 6.0 | $(0.0)^2$ |
| Sum | | | 1.25 | Sum | | | 1.00 |

The sums of squared errors confirm that the second model fits the given points better than the first model.

Of all possible linear models for a given set of points, the model that has the best fit is defined to be the one that minimizes the sum of squared error. This model is called the **least squares regression line,** and the procedure for finding it is called the **method of least squares.**

## Definition of Least Squares Regression Line

For a set of points

$$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$$

the **least squares regression line** is given by the linear function

$$f(x) = a_0 + a_1x$$

that minimizes the sum of squared error

$$[y_1 - f(x_1)]^2 + [y_2 - f(x_2)]^2 + \cdots + [y_n - f(x_n)]^2.$$

To find the least squares regression line for a set of points, begin by forming the system of linear equations

$$\begin{aligned} y_1 &= f(x_1) + [y_1 - f(x_1)] \\ y_2 &= f(x_2) + [y_2 - f(x_2)] \\ &\vdots \\ y_n &= f(x_n) + [y_n - f(x_n)] \end{aligned}$$

where the right-hand term,

$$[y_i - f(x_i)]$$

of each equation is the error in the approximation of $y_i$ by $f(x_i)$. Then write this error as

$$e_i = y_i - f(x_i)$$

and write the system of equations in the form

$$\begin{aligned} y_1 &= (a_0 + a_1x_1) + e_1 \\ y_2 &= (a_0 + a_1x_2) + e_2 \\ &\vdots \\ y_n &= (a_0 + a_1x_n) + e_n. \end{aligned}$$

Now, if you define $Y$, $X$, $A$, and $E$ as

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \quad X = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}, \quad A = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}, \quad E = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix}$$

then the $n$ linear equations may be replaced by the matrix equation

$$Y = XA + E.$$

Note that the matrix $X$ has a column of 1's (corresponding to $a_0$) and a column containing the $x_i$'s. This matrix equation can be used to determine the coefficients of the least squares regression line, as follows.

## Matrix Form for Linear Regression

For the regression model $Y = XA + E$, the coefficients of the least squares regression line are given by the matrix equation

$$A = (X^TX)^{-1}X^TY$$

and the sum of squared error is

$$E^TE.$$

**REMARK**
You will learn more about this procedure in Section 5.4.

Example 10 demonstrates the use of this procedure to find the least squares regression line for the set of points from Example 9.

### EXAMPLE 10 Finding the Least Squares Regression Line

Find the least squares regression line for the points (1, 1), (2, 2), (3, 4), (4, 4), and (5, 6).

**SOLUTION**

The matrices $X$ and $Y$ are

$$X = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 4 \\ 6 \end{bmatrix}.$$

This means that

$$X^TX = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 5 & 15 \\ 15 & 55 \end{bmatrix}$$

and

$$X^TY = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 4 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 17 \\ 63 \end{bmatrix}.$$

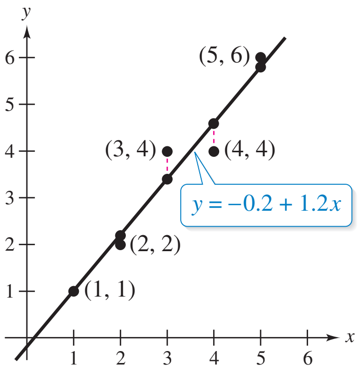
Now, using $(X^TX)^{-1}$ to find the coefficient matrix $A$, you have

$$\begin{aligned} A &= (X^TX)^{-1}X^TY \\ &= \frac{1}{50}\begin{bmatrix} 55 & -15 \\ -15 & 5 \end{bmatrix}\begin{bmatrix} 17 \\ 63 \end{bmatrix} \\ &= \begin{bmatrix} -0.2 \\ 1.2 \end{bmatrix}. \end{aligned}$$

So, the least squares regression line is

$$y = -0.2 + 1.2x$$

as shown in Figure 2.5. The sum of squared error for this line is 0.8 (verify this), which means that this line fits the data better than either of the two experimental linear models determined earlier.



Least Squares Regression Line

**Figure 2.5**

## 2.5 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

**Stochastic Matrices** **In Exercises 1–4, determine whether the matrix is stochastic.**

**1.** $\begin{bmatrix} \frac{2}{5} & -\frac{2}{5} \\ \frac{3}{5} & \frac{7}{5} \end{bmatrix}$

**2.** $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

**3.** $\begin{bmatrix} 0.3 & 0.1 & 0.8 \\ 0.5 & 0.2 & 0.1 \\ 0.2 & 0.7 & 0.1 \end{bmatrix}$

**4.** $\begin{bmatrix} 0.\overline{3} & 0.1\overline{6} & 0.25 \\ 0.\overline{3} & 0.\overline{6} & 0.25 \\ 0.\overline{3} & 0.1\overline{6} & 0.5 \end{bmatrix}$

**5. Purchase of a Product** The market research department at a manufacturing plant determines that 20% of the people who purchase the plant's product during any month will not purchase it the next month. On the other hand, 30% of the people who do not purchase the product during any month will purchase it the next month. In a population of 1000 people, 100 people purchased the product this month. How many will purchase the product (a) next month and (b) in 2 months?

**6. Spread of a Virus** A medical researcher is studying the spread of a virus in a population of 1000 laboratory mice. During any week, there is an 80% probability that an infected mouse will overcome the virus, and during the same week there is a 10% probability that a noninfected mouse will become infected. One hundred mice are currently infected with the virus. How many will be infected (a) next week and (b) in 2 weeks?

**7. Smokers and Nonsmokers** A population of 10,000 is grouped as follows: 5000 nonsmokers, 2500 smokers of one pack or less per day, and 2500 smokers of more than one pack per day. During any month, there is a 5% probability that a nonsmoker will begin smoking a pack or less per day, and a 2% probability that a nonsmoker will begin smoking more than a pack per day. For smokers who smoke a pack or less per day, there is a 10% probability of quitting and a 10% probability of increasing to more than a pack per day. For smokers who smoke more than a pack per day, there is a 5% probability of quitting and a 10% probability of dropping to a pack or less per day. How many people will be in each group (a) in 1 month and (b) in 2 months?

**8. Television Watching** A college dormitory houses 200 students. Those who watch an hour or more of television on any day always watch for less than an hour the next day. One-fourth of those who watch television for less than an hour one day will watch an hour or more the next day. Half of the students watched television for an hour or more today. How many will watch television for an hour or more (a) tomorrow, (b) in 2 days, and (c) in 30 days?

**9. Consumer Preference** A population of 100,000 consumers is grouped as follows: 20,000 users of Brand A, 30,000 users of Brand B, and 50,000 who use neither brand. During any month, a Brand A user has a 20% probability of switching to Brand B and a 5% probability of not using either brand. A Brand B user has a 15% probability of switching to Brand A and a 10% probability of not using either brand. A nonuser has a 10% probability of purchasing Brand A and a 15% probability of purchasing Brand B. How many people will be in each group (a) in 1 month, (b) in 2 months, and (c) in 3 months?

**10.** For the matrix of transition probabilities

$$P = \begin{bmatrix} 0.6 & 0.1 & 0.1 \\ 0.2 & 0.7 & 0.1 \\ 0.2 & 0.2 & 0.8 \end{bmatrix}$$

find $P^2X$ and $P^3X$ for the state matrix

$$X = \begin{bmatrix} 100 \\ 100 \\ 800 \end{bmatrix}.$$

Then find the steady state matrix for $P$.

**Encoding a Message** **In Exercises 11–14, write the uncoded row matrices of the indicated size for the given message. Then encode the message using the matrix $A$.**

**11.** *Message:* SELL CONSOLIDATED

*Row Matrix Size:* $1 \times 3$

*Encoding Matrix:* $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{bmatrix}$

**12.** *Message:* PLEASE SEND MONEY

*Row Matrix Size:* $1 \times 3$

*Encoding Matrix:* $A = \begin{bmatrix} 4 & 2 & 1 \\ -3 & -3 & -1 \\ 3 & 2 & 1 \end{bmatrix}$

**13.** *Message:* COME HOME SOON

*Row Matrix Size:* $1 \times 2$

*Encoding Matrix:* $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$

**14.** *Message:* HELP IS COMING

*Row Matrix Size:* $1 \times 4$

*Encoding Matrix:* $A = \begin{bmatrix} -2 & 3 & -1 & -1 \\ -1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 2 \\ 3 & 1 & -2 & -4 \end{bmatrix}$

**Decoding a Message** **In Exercises 15–18, use $A^{-1}$ to decode the cryptogram.**

**15.** $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$,

11 21 64 112 25 50 29 53 23 46 40 75 55 92

**16.** $A = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$,

85 120 6 8 10 15 84 117 42 56 90 125 60 80 30 45 19 26

**17.** $A = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -1 & -4 & -7 \end{bmatrix}$,

13 19 10 −1 −33 −77 3 −2 −14 4 1 −9 −5 −25 −47 4 1 −9

**18.** $A = \begin{bmatrix} 3 & -4 & 2 \\ 0 & 2 & 1 \\ 4 & -5 & 3 \end{bmatrix}$,

112 −140 83 19 −25 13 72 −76 61 95 −118 71 20 21 38 35 −23 36 42 −48 32

**19. Decoding a Message** The following cryptogram was encoded with a $2 \times 2$ matrix.

8 21 −15 −10 −13 −13 5 10 5 25 5 19 −1 6 20 40 −18 −18 1 16

The last word of the message is __RON. What is the message?

**20. Decoding a Message** The following cryptogram was encoded with a $2 \times 2$ matrix.

5 2 25 11 −2 −7 −15 −15 32 14 −8 −13 38 19 −19 −19 37 16

The last word of the message is __SUE. What is the message?

**21. Decoding a Message** Use a software program or a graphing utility with matrix capabilities to decode the cryptogram.

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

38 −14 29 56 −15 62 17 3 38 18 20 76 18 −5 21 29 −7 32 32 9 77 36 −8 48 33 −5 51 41 3 79 12 1 26 58 −22 49 63 −19 69 28 8 67 31 −11 27 41 −18 28

**22. Decoding a Message** A code breaker intercepted the following encoded message.

45 −35 38 −30 18 −18 35 −30 81 −60 42 −28 75 −55 2 −2 22 −21 15 −10

Let the inverse of the encoding matrix be

$$A^{-1} = \begin{bmatrix} w & x \\ y & z \end{bmatrix}.$$

(a) You know that $[45 \quad -35]A^{-1} = [10 \quad 15]$ and $[38 \quad -30]A^{-1} = [8 \quad 14]$. Write and solve two systems of equations to find $w$, $x$, $y$, and $z$.

(b) Decode the message.

**23. Industrial System** A system composed of two industries, coal and steel, has the following input requirements.

(a) To produce \$1.00 worth of output, the coal industry requires \$0.10 of its own product and \$0.80 of steel.

(b) To produce \$1.00 worth of output, the steel industry requires \$0.10 of its own product and \$0.20 of coal.

Find $D$, the input-output matrix for this system. Then solve for the output matrix $X$ in the equation $X = DX + E$, where $E$ is the external demand matrix

$$E = \begin{bmatrix} 10{,}000 \\ 20{,}000 \end{bmatrix}.$$

**24. Industrial System** An industrial system has two industries with the following input requirements.

(a) To produce \$1.00 worth of output, Industry A requires \$0.30 of its own product and \$0.40 of Industry B's product.

(b) To produce \$1.00 worth of output, Industry B requires \$0.20 of its own product and \$0.40 of Industry A's product.

Find $D$, the input-output matrix for this system. Then solve for the output matrix $X$ in the equation $X = DX + E$, where $E$ is the external demand matrix

$$E = \begin{bmatrix} 10{,}000 \\ 20{,}000 \end{bmatrix}.$$

**25. Solving for the Output Matrix** A small community includes a farmer, a baker, and a grocer and has the following input-output matrix $D$ and external demand matrix $E$.

$$D = \begin{array}{c} \begin{array}{ccc} \text{Farmer} & \text{Baker} & \text{Grocer} \end{array} \\ \begin{bmatrix} 0.4 & 0.5 & 0.5 \\ 0.3 & 0.0 & 0.3 \\ 0.2 & 0.2 & 0.0 \end{bmatrix} \end{array} \begin{array}{l} \text{Farmer} \\ \text{Baker} \\ \text{Grocer} \end{array} \quad \text{and} \quad E = \begin{bmatrix} 1000 \\ 1000 \\ 1000 \end{bmatrix}$$

Solve for the output matrix $X$ in the equation $X = DX + E$.

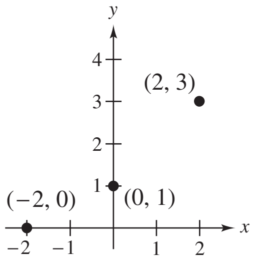
**26. Solving for the Output Matrix** An industrial system with three industries has the following input-output matrix $D$ and external demand matrix $E$.

$$D = \begin{bmatrix} 0.2 & 0.4 & 0.4 \\ 0.4 & 0.2 & 0.2 \\ 0.0 & 0.2 & 0.2 \end{bmatrix} \quad \text{and} \quad E = \begin{bmatrix} 5000 \\ 2000 \\ 8000 \end{bmatrix}$$

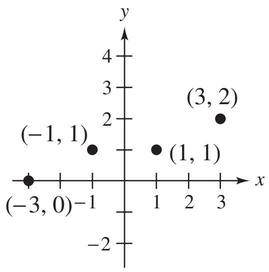
Solve for the output matrix $X$ in the equation $X = DX + E$.

**Least Squares Regression Analysis** In Exercises 27–30, (a) sketch the line that appears to be the best fit for the given points, (b) find the least squares regression line, and (c) calculate the sum of squared error.

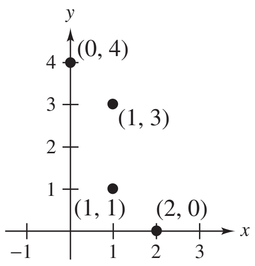
**27.**



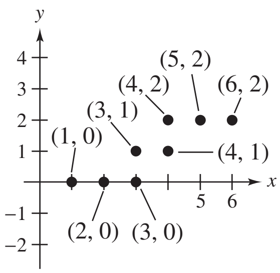

**28.**




**29.**




**30.**




**Finding the Least Squares Regression Line** In Exercises 31–38, find the least squares regression line.

**31.** (0, 0), (1, 1), (2, 4)

**32.** (1, 0), (3, 3), (5, 6)

**33.** (−2, 0), (−1, 1), (0, 1), (1, 2)

**34.** (−4, −1), (−2, 0), (2, 4), (4, 5)

**35.** (−5, 1), (1, 3), (2, 3), (2, 5)

**36.** (−3, 4), (−1, 2), (1, 1), (3, 0)

**37.** (−5, 10), (−1, 8), (3, 6), (7, 4), (5, 5)

**38.** (0, 6), (4, 3), (5, 0), (8, −4), (10, −5)

**39. Demand** A fuel refiner wants to know the demand for a grade of gasoline as a function of price. The table shows the daily sales $y$ (in gallons) for three different prices.

| Price, $x$ | \$3.00 | \$3.25 | \$3.50 |
|---|---|---|---|
| Demand, $y$ | 4500 | 3750 | 3300 |

(a) Find the least squares regression line for the data.

(b) Estimate the demand when the price is \$3.40.

**40. Demand** A hardware retailer wants to know the demand for a rechargeable power drill as a function of price. The table shows the monthly sales $y$ for four different prices.

| Price, $x$ | \$25 | \$30 | \$35 | \$40 |
|---|---|---|---|---|
| Demand, $y$ | 82 | 75 | 67 | 55 |

(a) Find the least squares regression line for the data.

(b) Estimate the demand when the price is \$32.95.

**41. Motor Vehicle Registration** The table shows the numbers $y$ of motor vehicle registrations (in millions) in the United States from 2004 through 2008. (Source: U.S. Federal Highway Administration)

| Year | 2004 | 2005 | 2006 | 2007 | 2008 |
|---|---|---|---|---|---|
| Number, $y$ | 237.2 | 241.2 | 244.2 | 247.3 | 248.2 |

(a) Find the least squares regression line for the data. Let $t$ represent the year, with $t = 4$ corresponding to 2004.

(b) Use the linear regression capabilities of a graphing utility to find a linear model for the data. Let $t$ represent the year, with $t = 4$ corresponding to 2004.

**42. Wildlife** A wildlife management team studied the reproduction rates of deer in three tracts of a wildlife preserve. The team recorded the number of females $x$ in each tract and the percent of females $y$ in each tract that had offspring the following year. The table shows the results.

| Number, $x$ | 100 | 120 | 140 |
|---|---|---|---|
| Percent, $y$ | 75 | 68 | 55 |

(a) Find the least squares regression line for the data.

(b) Use a graphing utility to graph the model and the data in the same viewing window.

(c) Use the model to create a table of estimated values for $y$. Compare the estimated values with the actual data.

(d) Use the model to estimate the percent of females that had offspring when there were 170 females.

(e) Use the model to estimate the number of females when 40% of the females had offspring.

**43.** Use your school's library, the Internet, or some other reference source to derive the matrix form for linear regression given at the top of page 94.

**44. CAPSTONE** Explain each of the following.

(a) How to write and use a stochastic matrix

(b) How to use matrix multiplication to encode and decode messages

(c) How to use a Leontief input-output model to analyze an economic system

(d) How to use matrices to find the least squares regression line for a set of data

**45. Proof** Prove that the product of two $2 \times 2$ stochastic matrices is stochastic.

**46. Proof** Let $P$ be a $2 \times 2$ stochastic matrix. Prove that there exists a $2 \times 1$ state matrix $X$ with nonnegative entries such that $PX = X$.

# 2 Review Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

**Operations with Matrices** In Exercises 1–6, perform the matrix operations.

1. $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 5 & -4 \end{bmatrix} - 3\begin{bmatrix} 5 & 3 & -6 \\ 0 & -2 & 5 \end{bmatrix}$

2. $-2\begin{bmatrix} 1 & 2 \\ 5 & -4 \\ 6 & 0 \end{bmatrix} + 8\begin{bmatrix} 7 & 1 \\ 1 & 2 \\ 1 & 4 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 2 \\ 5 & -4 \\ 6 & 0 \end{bmatrix}\begin{bmatrix} 6 & -2 & 8 \\ 4 & 0 & 0 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 5 \\ 2 & -4 \end{bmatrix}\begin{bmatrix} 6 & -2 & 8 \\ 4 & 0 & 0 \end{bmatrix}$

5. $\begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & -4 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 4 & -3 & 2 \\ 0 & 3 & -1 \\ 0 & 0 & 2 \end{bmatrix}$

6. $\begin{bmatrix} 2 & 1 \\ 6 & 0 \end{bmatrix}\begin{bmatrix} 4 & 2 \\ -3 & 1 \end{bmatrix} + \begin{bmatrix} -2 & 4 \\ 0 & 4 \end{bmatrix}$

**Solving a System of Linear Equations** In Exercises 7–10, write the system of linear equations in the form $A\mathbf{x} = \mathbf{b}$. Then use Gaussian elimination to solve this matrix equation for $\mathbf{x}$.

7. $\begin{aligned} 2x_1 + x_2 &= -8 \\ x_1 + 4x_2 &= -4 \end{aligned}$

8. $\begin{aligned} 2x_1 - x_2 &= 5 \\ 3x_1 + 2x_2 &= -4 \end{aligned}$

9. $\begin{aligned} -3x_1 - x_2 + x_3 &= 0 \\ 2x_1 + 4x_2 - 5x_3 &= -3 \\ x_1 - 2x_2 + 3x_3 &= 1 \end{aligned}$

10. $\begin{aligned} 2x_1 + 3x_2 + x_3 &= 10 \\ 2x_1 - 3x_2 - 3x_3 &= 22 \\ 4x_1 - 2x_2 + 3x_3 &= -2 \end{aligned}$

**Finding and Multiplying with a Transpose** In Exercises 11–14, find $A^T$, $A^TA$, and $AA^T$.

11. $\begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 2 \end{bmatrix}$

12. $\begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}$

13. $\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$

14. $\begin{bmatrix} 1 & -2 & -3 \end{bmatrix}$

**Finding the Inverse of a Matrix** In Exercises 15–18, find the inverse of the matrix (if it exists).

15. $\begin{bmatrix} 3 & -1 \\ 2 & -1 \end{bmatrix}$

16. $\begin{bmatrix} 4 & -1 \\ -8 & 2 \end{bmatrix}$

17. $\begin{bmatrix} 2 & 3 & 1 \\ 2 & -3 & -3 \\ 4 & 0 & 3 \end{bmatrix}$

18. $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

**Using the Inverse of a Matrix** In Exercises 19–26, use an inverse matrix to solve each matrix equation or system of linear equations.

19. $\begin{bmatrix} 5 & 4 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -15 \\ -6 \end{bmatrix}$

20. $\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$

21. $\begin{bmatrix} 0 & 1 & -2 \\ -1 & 3 & 1 \\ 2 & -2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

22. $\begin{bmatrix} 0 & 1 & 2 \\ 3 & 2 & 1 \\ 4 & -3 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ -7 \end{bmatrix}$

23. $\begin{aligned} 5x_1 + 4x_2 &= 2 \\ -x_1 + x_2 &= -22 \end{aligned}$

24. $\begin{aligned} 3x_1 + 2x_2 &= 1 \\ x_1 + 4x_2 &= -3 \end{aligned}$

25. $\begin{aligned} -x_1 + x_2 + 2x_3 &= 1 \\ 2x_1 + 3x_2 + x_3 &= -2 \\ 5x_1 + 4x_2 + 2x_3 &= 4 \end{aligned}$

26. $\begin{aligned} x_1 + x_2 + 2x_3 &= 0 \\ x_1 - x_2 + x_3 &= -1 \\ 2x_1 + x_2 + x_3 &= 2 \end{aligned}$

**Solving a Matrix Equation** In Exercises 27 and 28, find $A$.

27. $(3A)^{-1} = \begin{bmatrix} 4 & -1 \\ 2 & 3 \end{bmatrix}$

28. $(2A)^{-1} = \begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix}$

**Nonsingular Matrix** In Exercises 29 and 30, find $x$ such that the matrix $A$ is nonsingular.

29. $A = \begin{bmatrix} 3 & 1 \\ x & -1 \end{bmatrix}$

30. $A = \begin{bmatrix} 2 & x \\ 1 & 4 \end{bmatrix}$

**Finding the Inverse of an Elementary Matrix** In Exercises 31 and 32, find the inverse of the elementary matrix.

31. $\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

32. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

**Finding a Sequence of Elementary Matrices** In Exercises 33–36, find a sequence of elementary matrices whose product is the given nonsingular matrix.

33. $\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$

34. $\begin{bmatrix} -3 & 13 \\ 1 & -4 \end{bmatrix}$

35. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 4 \end{bmatrix}$

36. $\begin{bmatrix} 3 & 0 & 6 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix}$

**37.** Find two $2 \times 2$ matrices $A$ such that $A^2 = I$.

**38.** Find two $2 \times 2$ matrices $A$ such that $A^2 = O$.

**39.** Find three $2 \times 2$ idempotent matrices. (Recall that a square matrix $A$ is *idempotent* when $A^2 = A$.)

**40.** Find $2 \times 2$ matrices $A$ and $B$ such that $AB = O$ but $BA \neq O$.

**41.** Consider the following matrices.

$$X = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \quad Y = \begin{bmatrix} -1 \\ 0 \\ 3 \\ 2 \end{bmatrix}, \quad Z = \begin{bmatrix} 3 \\ 4 \\ -1 \\ 2 \end{bmatrix}, \quad W = \begin{bmatrix} 3 \\ 2 \\ -4 \\ -1 \end{bmatrix}$$

(a) Find scalars $a$, $b$, and $c$ such that $W = aX + bY + cZ$.

(b) Show that there do not exist scalars $a$ and $b$ such that $Z = aX + bY$.

(c) Show that if $aX + bY + cZ = O$, then $a = b = c = 0$.

**42. Proof** Let $A$, $B$, and $A + B$ be nonsingular matrices. Prove that $A^{-1} + B^{-1}$ is nonsingular by showing that

$(A^{-1} + B^{-1})^{-1} = A(A + B)^{-1}B$.

**Finding an *LU*-Factorization of a Matrix** **In Exercises 43 and 44, find an *LU*-factorization of the matrix.**

**43.** $\begin{bmatrix} 2 & 5 \\ 6 & 14 \end{bmatrix}$

**44.** $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

**Solving a Linear System Using *LU*-Factorization** **In Exercises 45 and 46, use an *LU*-factorization of the coefficient matrix to solve the linear system.**

**45.**
$$\begin{aligned} x \phantom{{}+2y} + z &= 3 \\ 2x + y + 2z &= 7 \\ 3x + 2y + 6z &= 8 \end{aligned}$$

**46.**
$$\begin{aligned} 2x_1 + x_2 + x_3 - x_4 &= 7 \\ 3x_2 + x_3 - x_4 &= -3 \\ -2x_3 &= 2 \\ 2x_1 + x_2 + x_3 - 2x_4 &= 8 \end{aligned}$$

**True or False?** **In Exercises 47–50, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.**

**47.** (a) Addition of matrices is not commutative.

(b) The transpose of the sum of matrices is equal to the sum of the transposes of the matrices.

**48.** (a) The product of a $2 \times 3$ matrix and a $3 \times 5$ matrix is a $5 \times 2$ matrix.

(b) The transpose of a product is equal to the product of transposes in reverse order.

**49.** (a) All $n \times n$ matrices are invertible.

(b) If an $n \times n$ matrix $A$ is not symmetric, then $A^TA$ is not symmetric.

**50.** (a) If $A$ and $B$ are $n \times n$ matrices and $A$ is invertible, then $(ABA^{-1})^2 = AB^2A^{-1}$.

(b) If $A$ and $B$ are nonsingular $n \times n$ matrices, then $A + B$ is a nonsingular matrix.

**51. Manufacturing** A corporation has four factories, each of which manufactures sport utility vehicles and pickup trucks. In the matrix

$$A = \begin{bmatrix} 100 & 90 & 70 & 30 \\ 40 & 20 & 60 & 60 \end{bmatrix}$$

$a_{ij}$ represents the number of vehicles of type $i$ produced at factory $j$ in one day. Find the production levels when production increases by 10%.

**52. Manufacturing** A company manufactures tables and chairs at two locations. Matrix $C$ gives the costs of manufacturing at each location.

$$C = \begin{array}{c} \text{Location 1} \quad \text{Location 2} \\ \begin{bmatrix} 627 & 681 \\ 135 & 150 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Tables} \\ \text{Chairs} \end{array}$$

(a) If labor accounts for $\frac{2}{3}$ of the cost, determine the matrix $L$ that gives the labor costs at each location.

(b) Find the matrix $M$ that gives material costs at each location. (Assume there are only labor and material costs.)

**53. Gasoline Sales** Matrix $A$ shows the numbers of gallons of 87-octane, 89-octane, and 93-octane gasoline sold at a convenience store over a weekend.

$$A = \begin{array}{c} \overbrace{\begin{array}{ccc} 87 & 89 & 93 \end{array}}^{\text{Octane}} \\ \begin{bmatrix} 580 & 840 & 320 \\ 560 & 420 & 160 \\ 860 & 1020 & 540 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Friday} \\ \text{Saturday} \\ \text{Sunday} \end{array}$$

Matrix $B$ gives the selling prices (in dollars per gallon) and the profits (in dollars per gallon) for the three grades of gasoline.

$$B = \begin{array}{c} \text{Selling Price} \quad \text{Profit} \\ \begin{bmatrix} 3.05 & 0.05 \\ 3.15 & 0.08 \\ 3.25 & 0.10 \end{bmatrix} \end{array} \begin{array}{l} \\ \left.\begin{array}{l} 87 \\ 89 \\ 93 \end{array}\right\} \text{Octane} \end{array}$$

(a) Find $AB$ and interpret the result.

(b) Find the convenience store's profit from gasoline sales for the weekend.

**54. Final Grades** The final grades in a course at a liberal arts college are determined by two midterms and a final exam. The following matrices show the grades for six students and two possible grading systems.

$$A = \begin{array}{c} \begin{array}{ccc} \text{Midterm 1} & \text{Midterm 2} & \text{Final Exam} \end{array} \\ \begin{bmatrix} 78 & 82 & 80 \\ 84 & 88 & 85 \\ 92 & 93 & 90 \\ 88 & 86 & 90 \\ 74 & 78 & 80 \\ 96 & 95 & 98 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Student 1} \\ \text{Student 2} \\ \text{Student 3} \\ \text{Student 4} \\ \text{Student 5} \\ \text{Student 6} \end{array}$$

$$B = \begin{array}{c} \begin{array}{cc} \text{Grading System 1} & \text{Grading System 2} \end{array} \\ \begin{bmatrix} 0.25 & 0.20 \\ 0.25 & 0.20 \\ 0.50 & 0.60 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Midterm 1} \\ \text{Midterm 2} \\ \text{Final Exam} \end{array}$$

(a) Describe the grading systems in matrix $B$.

(b) Compute the numerical grades for the six students (to the nearest whole number) using the two grading systems.

(c) Assign each student a letter grade for each grading system using the following table.

| *Numerical Grade Range* | *Letter Grade* |
|---|---|
| 90–100 | A |
| 80–89 | B |
| 70–79 | C |
| 60–69 | D |
| 0–59 | F |

**Polynomial Function** **In Exercises 55 and 56, use the given definition to find $f(A)$: If $f$ is the polynomial function**

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

**then for a square matrix $A$, $f(A)$ is defined to be**

$$f(A) = a_0I + a_1A + a_2A^2 + \cdots + a_nA^n.$$

**55.** $f(x) = x^2 - 7x + 6, \quad A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

**56.** $f(x) = x^3 - 3x + 2, \quad A = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$

**Stochastic Matrices** **In Exercises 57 and 58, determine whether the matrix is stochastic.**

**57.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0.1 \\ 0 & 0.1 & 0.5 \end{bmatrix}$

**58.** $\begin{bmatrix} 0.3 & 0.4 & 0.1 \\ 0.2 & 0.4 & 0.5 \\ 0.5 & 0.2 & 0.4 \end{bmatrix}$

**Finding State Matrices** **In Exercises 59 and 60, use the matrix of transition probabilities $P$ and state matrix $X$ to find the state matrices $PX$, $P^2X$, and $P^3X$.**

**59.** $P = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{3}{4} \end{bmatrix}, \quad X = \begin{bmatrix} 128 \\ 64 \end{bmatrix}$

**60.** $P = \begin{bmatrix} 0.6 & 0.2 & 0.0 \\ 0.2 & 0.7 & 0.1 \\ 0.2 & 0.1 & 0.9 \end{bmatrix}, \quad X = \begin{bmatrix} 1000 \\ 1000 \\ 1000 \end{bmatrix}$

**61. Population Migration** A country is divided into three regions. Each year, 10% of the residents of Region 1 move to Region 2 and 5% move to Region 3, 15% of the residents of Region 2 move to Region 1 and 5% move to Region 3, and 10% of the residents of Region 3 move to Region 1 and 10% move to Region 2. This year, each region has a population of 100,000. Find the populations of each region (a) in 1 year and (b) in 3 years.

**62. Population Migration** Find the steady state matrix for the populations described in Exercise 61.

**Encoding a Message** **In Exercises 63 and 64, write the uncoded row matrices of the indicated size for the given message. Then encode the message using the matrix $A$.**

**63.** *Message:* ONE IF BY LAND

*Row Matrix Size:* $1 \times 2$

*Encoding Matrix:* $A = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$

**64.** *Message:* BEAM ME UP SCOTTY

*Row Matrix Size:* $1 \times 3$

*Encoding Matrix:* $A = \begin{bmatrix} 2 & 1 & 4 \\ 3 & 1 & 3 \\ -2 & -1 & -3 \end{bmatrix}$

**Decoding a Message** **In Exercises 65–68, use $A^{-1}$ to decode the cryptogram.**

**65.** $A = \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix}$,

$-45$ 34 36 $-24$ $-43$ 37 $-23$ 22 $-37$ 29 57 $-38$ $-39$ 31

**66.** $A = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}$,

11 52 $-8$ $-9$ $-13$ $-39$ 5 20 12 56 5 20 $-2$ 7 9 41 25 100

**67.** $A = \begin{bmatrix} 2 & -1 & -1 \\ -5 & 2 & 2 \\ 5 & -1 & -2 \end{bmatrix}$,

58 $-3$ $-25$ $-48$ 28 19 $-40$ 13 13 $-98$ 39 39 118 $-25$ $-48$ 28 $-14$ $-14$

**68.** $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$,

23 20 132 54 128 102 32 21 203 6 10 23 21 15 129 36 46 173 29 72 45

**Decoding a Message** **In Exercises 69 and 70, use a software program or a graphing utility with matrix capabilities to decode the cryptogram.**

**69.** $\begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix}$

$-2$ 2 5 39 $-53$ $-72$ $-6$ $-9$ 93 4 $-12$ 27 31 $-49$ $-16$ 19 $-24$ $-46$ $-8$ $-7$ 99

**70.** $\begin{bmatrix} 2 & 0 & 1 \\ 2 & -1 & 0 \\ 1 & 2 & -4 \end{bmatrix}$

66 27 $-31$ 37 5 $-9$ 61 46 $-73$ 46 $-14$ 9 94 21 $-49$ 32 $-4$ 12 66 31 $-53$ 47 33 $-67$

**71.** **Industrial System** An industrial system has two industries with the following input requirements.

(a) To produce \$1.00 worth of output, Industry A requires \$0.20 of its own product and \$0.30 of Industry B's product.

(b) To produce \$1.00 worth of output, Industry B requires \$0.10 of its own product and \$0.50 of Industry A's product.

Find $D$, the input-output matrix for this system. Then solve for the output matrix $X$ in the equation $X = DX + E$, where $E$ is the external demand matrix

$$E = \begin{bmatrix} 40{,}000 \\ 80{,}000 \end{bmatrix}.$$

**72.** **Industrial System** An industrial system with three industries has the following input-output matrix $D$ and external demand matrix $E$.

$$D = \begin{bmatrix} 0.1 & 0.3 & 0.2 \\ 0.0 & 0.2 & 0.3 \\ 0.4 & 0.1 & 0.1 \end{bmatrix} \quad \text{and} \quad E = \begin{bmatrix} 3000 \\ 3500 \\ 8500 \end{bmatrix}$$

Solve for the output matrix $X$ in the equation $X = DX + E$.

**Finding the Least Squares Regression Line** **In Exercises 73–76, find the least squares regression line.**

**73.** (1, 5), (2, 4), (3, 2)

**74.** (2, 1), (3, 3), (4, 2), (5, 4), (6, 4)

**75.** (1, 1), (1, 3), (1, 2), (1, 4), (2, 5)

**76.** $(-2, 4), (-1, 2), (0, 1), (1, -2), (2, -3)$

**77.** **Agriculture** A farmer used four test plots to determine the relationship between wheat yield (in kilograms per square kilometer) and the amount of fertilizer (in hundreds of kilograms per square kilometer). The table shows the results.

| *Fertilizer, x* | 1.0 | 1.5 | 2.0 | 2.5 |
|---|---|---|---|---|
| *Yield, y* | 32 | 41 | 48 | 53 |

(a) Find the least squares regression line for the data.

(b) Estimate the yield when the amount of fertilizer is 160 kilograms per square kilometer.

**78.** **Cellular Phone Subscribers** The table shows the numbers of cellular phone subscribers $y$ (in millions) in the United States from 2004 through 2009. (Source: CTIA–The Wireless Association)

| *Year* | 2004 | 2005 | 2006 |
|---|---|---|---|
| *Subscribers, y* | 182.1 | 207.9 | 233.0 |

| *Year* | 2007 | 2008 | 2009 |
|---|---|---|---|
| *Subscribers, y* | 255.4 | 270.3 | 285.6 |

(a) Find the least squares regression line for the data. Let $x$ represent the year, with $x = 4$ corresponding to 2004.

(b) Use the linear regression capabilities of a graphing utility to find a linear model for the data. How does this model compare with the model obtained in part (a)?

(c) Use the linear model to create a table of estimated values for $y$. Compare the estimated values with the actual data.

**79.** **Major League Baseball Salaries** The table shows the average salaries $y$ (in millions of dollars) of Major League Baseball players on opening day of baseball season from 2005 through 2010. (Source: Major League Baseball)

| *Year* | 2005 | 2006 | 2007 | 2008 | 2009 | 2010 |
|---|---|---|---|---|---|---|
| *Salary, y* | 2.6 | 2.9 | 2.9 | 3.2 | 3.2 | 3.3 |

(a) Find the least squares regression line for the data. Let $x$ represent the year, with $x = 5$ corresponding to 2005.

(b) Use the linear regression capabilities of a graphing utility to find a linear model for the data. How does this model compare with the model obtained in part (a)?

(c) Use the linear model to create a table of estimated values for $y$. Compare the estimated values with the actual data.

# 2 Projects

## 1 Exploring Matrix Multiplication

| | Test 1 | Test 2 |
|---|---|---|
| Anna | 84 | 96 |
| Bruce | 56 | 72 |
| Chris | 78 | 83 |
| David | 82 | 91 |

The table shows the first two test scores for Anna, Bruce, Chris, and David. Use the table to create a matrix $M$ to represent the data. Input $M$ into a software program or a graphing utility and use it to answer the following questions.

**1.** Which test was more difficult? Which was easier? Explain.

**2.** How would you rank the performances of the four students?

**3.** Describe the meanings of the matrix products $M\begin{bmatrix}1\\0\end{bmatrix}$ and $M\begin{bmatrix}0\\1\end{bmatrix}$.

**4.** Describe the meanings of the matrix products $[1 \quad 0 \quad 0 \quad 0]M$ and $[0 \quad 0 \quad 1 \quad 0]M$.

**5.** Describe the meanings of the matrix products $M\begin{bmatrix}1\\1\end{bmatrix}$ and $\frac{1}{2}M\begin{bmatrix}1\\1\end{bmatrix}$.

**6.** Describe the meanings of the matrix products $[1 \quad 1 \quad 1 \quad 1]M$ and $\frac{1}{4}[1 \quad 1 \quad 1 \quad 1]M$.

**7.** Describe the meaning of the matrix product $[1 \quad 1 \quad 1 \quad 1]M\begin{bmatrix}1\\1\end{bmatrix}$.

**8.** Use matrix multiplication to find the combined overall average score on both tests.

**9.** How could you use matrix multiplication to scale the scores on test 1 by a factor of 1.1?

## 2 Nilpotent Matrices

Let $A$ be a nonzero square matrix. Is it possible that a positive integer $k$ exists such that $A^k = O$? For example, find $A^3$ for the matrix

$$A = \begin{bmatrix}0 & 1 & 2\\0 & 0 & 1\\0 & 0 & 0\end{bmatrix}.$$

A square matrix $A$ is **nilpotent of index $k$** when $A \neq O, A^2 \neq O, \ldots, A^{k-1} \neq O$, but $A^k = O$. In this project you will explore nilpotent matrices.

**1.** The matrix in the example given above is nilpotent. What is its index?

**2.** Use a software program or a graphing utility to determine which of the following matrices are nilpotent and find their indices.

(a) $\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}$ (b) $\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$ (c) $\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}$

(d) $\begin{bmatrix}1 & 0\\1 & 0\end{bmatrix}$ (e) $\begin{bmatrix}0 & 0 & 1\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$ (f) $\begin{bmatrix}0 & 0 & 0\\1 & 0 & 0\\1 & 1 & 0\end{bmatrix}$

**3.** Find $3 \times 3$ nilpotent matrices of indices 2 and 3.

**4.** Find $4 \times 4$ nilpotent matrices of indices 2, 3, and 4.

**5.** Find a nilpotent matrix of index 5.

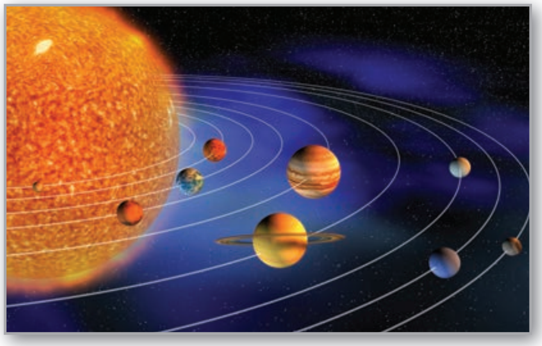
**6.** Are nilpotent matrices invertible? Prove your answer.

**7.** When $A$ is nilpotent, what can you say about $A^T$? Prove your answer.

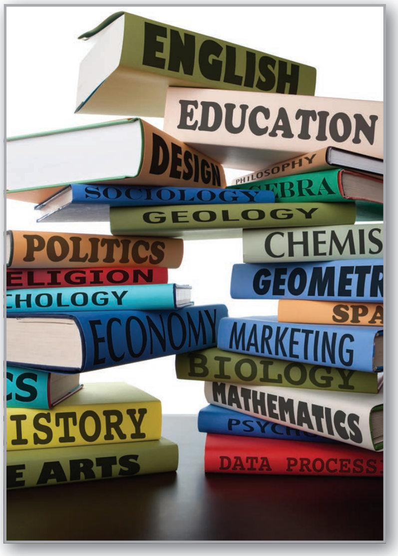
**8.** Show that if $A$ is nilpotent, then $I - A$ is invertible.

# 3 Determinants





Planetary Orbits (p. 135)




Textbook Publishing (p. 137)

Engineering and Control (p. 124)

Sudoku (p. 114)

Volume of a Solid (p. 108)

# 3.1 The Determinant of a Matrix

- Find the determinant of a $2 \times 2$ matrix.
- Find the minors and cofactors of a matrix.
- Use expansion by cofactors to find the determinant of a matrix.
- Find the determinant of a triangular matrix.

## THE DETERMINANT OF A 2 × 2 MATRIX

**REMARK**

In this text, $\det(A)$ and $|A|$ are used interchangeably to represent the determinant of $A$. Although vertical bars are also used to denote the absolute value of a real number, the context will show which use is intended. Furthermore, it is common practice to delete the matrix brackets and write

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

instead of

$$\left|\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right|.$$

Every *square* matrix can be associated with a real number called its *determinant*. Historically, the use of determinants arose from the recognition of special patterns that occur in the solutions of systems of linear equations. For instance, the system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}$$

has the solution

$$x_1 = \frac{b_1a_{22} - b_2a_{12}}{a_{11}a_{22} - a_{21}a_{12}} \quad \text{and} \quad x_2 = \frac{b_2a_{11} - b_1a_{21}}{a_{11}a_{22} - a_{21}a_{12}}$$

when $a_{11}a_{22} - a_{21}a_{12} \neq 0$. (See Exercise 69.) Note that both fractions have the same denominator, $a_{11}a_{22} - a_{21}a_{12}$. This quantity is called the *determinant* of the coefficient matrix of the system.

### Definition of the Determinant of a 2 × 2 Matrix

The **determinant** of the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is given by $\det(A) = |A| = a_{11}a_{22} - a_{21}a_{12}$.

The following diagram shows a convenient method for remembering the formula for the determinant of a $2 \times 2$ matrix.

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

The determinant is the difference of the products of the two diagonals of the matrix. Note that the order of the products is important.

**REMARK**

Notice that the determinant of a matrix can be positive, zero, or negative.

**EXAMPLE 1** **The Determinant of a Matrix of Order 2**

**a.** For $A = \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix}$, $|A| = \begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix} = 2(2) - 1(-3) = 4 + 3 = 7.$

**b.** For $B = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$, $|B| = \begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} = 2(2) - 4(1) = 4 - 4 = 0.$

**c.** For $C = \begin{bmatrix} 0 & \frac{3}{2} \\ 2 & 4 \end{bmatrix}$, $|C| = \begin{vmatrix} 0 & \frac{3}{2} \\ 2 & 4 \end{vmatrix} = 0(4) - 2(\frac{3}{2}) = 0 - 3 = -3.$

## MINORS AND COFACTORS

To define the determinant of a square matrix of order higher than 2, it is convenient to use *minors* and *cofactors*.

### Minors and Cofactors of a Square Matrix

If $A$ is a square matrix, then the **minor** $M_{ij}$ of the entry $a_{ij}$ is the determinant of the matrix obtained by deleting the $i$th row and $j$th column of $A$. The **cofactor** $C_{ij}$ of the entry $a_{ij}$ is $C_{ij} = (-1)^{i+j}M_{ij}$.

For example, if $A$ is a $3 \times 3$ matrix, then the minors and cofactors of $a_{21}$ and $a_{22}$ are as follows.

Minor of $a_{21}$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

Delete row 2 and column 1.

Minor of $a_{22}$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

Delete row 2 and column 2.

Cofactor of $a_{21}$

$$C_{21} = (-1)^{2+1}M_{21} = -M_{21}$$

Cofactor of $a_{22}$

$$C_{22} = (-1)^{2+2}M_{22} = M_{22}$$

As you can see, the minors and cofactors of a matrix can differ only in sign. To obtain the cofactors of a matrix, first find the minors and then apply the checkerboard pattern of $+$'s and $-$'s shown at the left. Note that *odd* positions (where $i + j$ is odd) have negative signs, and even positions (where $i + j$ is even) have positive signs.

Sign Pattern for Cofactors

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$3 \times 3$ matrix

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

$4 \times 4$ matrix

$$\begin{bmatrix} + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

$n \times n$ matrix

### EXAMPLE 2 Minors and Cofactors of a Matrix

Find all the minors and cofactors of

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}.$$

**SOLUTION**

To find the minor $M_{11}$, delete the first row and first column of $A$ and evaluate the determinant of the resulting matrix.

$$\begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}, \quad M_{11} = \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -1(1) - 0(2) = -1$$

Verify that the minors are

$$\begin{array}{lll} M_{11} = -1 & M_{12} = -5 & M_{13} = 4 \\ M_{21} = 2 & M_{22} = -4 & M_{23} = -8 \\ M_{31} = 5 & M_{32} = -3 & M_{33} = -6. \end{array}$$

Now, to find the cofactors, combine these minors with the checkerboard pattern of signs for a $3 \times 3$ matrix shown above.

$$\begin{array}{lll} C_{11} = -1 & C_{12} = 5 & C_{13} = 4 \\ C_{21} = -2 & C_{22} = -4 & C_{23} = 8 \\ C_{31} = 5 & C_{32} = 3 & C_{33} = -6 \end{array}$$

## THE DETERMINANT OF A SQUARE MATRIX

**REMARK**

The determinant of a matrix of order 1 is defined simply as the entry of the matrix. For instance, if $A = [-2]$, then

$\det(A) = -2.$

The definition below is called **inductive** because it uses the determinant of a square matrix of order $n - 1$ to define the determinant of a square matrix of order $n$.

### Definition of the Determinant of a Square Matrix

If $A$ is a square matrix of order $n \geq 2$, then the determinant of $A$ is the sum of the entries in the first row of $A$ multiplied by their respective cofactors. That is,

$$\det(A) = |A| = \sum_{j=1}^{n} a_{1j}C_{1j} = a_{11}C_{11} + a_{12}C_{12} + \cdot \cdot \cdot + a_{1n}C_{1n}.$$

Try confirming that, for $2 \times 2$ matrices, this definition yields

$$|A| = a_{11}a_{22} - a_{21}a_{12}$$

as previously defined.

When you use this definition to evaluate a determinant, you are **expanding by cofactors in the first row.** Example 3 demonstrates this procedure.

### EXAMPLE 3 The Determinant of a Matrix of Order 3

Find the determinant of

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}.$$

**SOLUTION**

This is the same matrix as in Example 2. There you found the cofactors of the entries in the first row to be

$$C_{11} = -1, \quad C_{12} = 5, \quad C_{13} = 4.$$

So, by the definition of a determinant, you have

$$\begin{aligned} |A| &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} && \text{First row expansion} \\ &= 0(-1) + 2(5) + 1(4) \\ &= 14. \end{aligned}$$

Although the determinant is defined as an expansion by the cofactors in the first row, it can be shown that the determinant can be evaluated by expanding in *any* row or column. For instance, you could expand the matrix in Example 3 in the second row to obtain

$$\begin{aligned} |A| &= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} && \text{Second row expansion} \\ &= 3(-2) + (-1)(-4) + 2(8) \\ &= 14 \end{aligned}$$

or in the first column to obtain

$$\begin{aligned} |A| &= a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} && \text{First column expansion} \\ &= 0(-1) + 3(-2) + 4(5) \\ &= 14. \end{aligned}$$

Try other possibilities to confirm that the determinant of $A$ can be evaluated by expanding in *any* row or column. The following theorem states this, and is known as Laplace's Expansion of a Determinant, after the French mathematician Pierre Simon de Laplace (1749–1827).

**THEOREM 3.1 Expansion by Cofactors**

Let $A$ be a square matrix of order $n$. Then the determinant of $A$ is given by

$$\det(A) = |A| = \sum_{j=1}^{n} a_{ij}C_{ij} = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdot \cdot \cdot + a_{in}C_{in}$$ *i*th row expansion

or

$$\det(A) = |A| = \sum_{i=1}^{n} a_{ij}C_{ij} = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdot \cdot \cdot + a_{nj}C_{nj}.$$ *j*th column expansion

When expanding by cofactors, you do not need to find cofactors of zero entries, because zero times its cofactor is zero.

$$\begin{aligned} a_{ij}C_{ij} &= (0)C_{ij} \\ &= 0 \end{aligned}$$

The row (or column) containing the most zeros is usually the best choice for expansion by cofactors. The next example demonstrates this.

## EXAMPLE 4 The Determinant of a Matrix of Order 4

Find the determinant of

$$A = \begin{bmatrix} 1 & -2 & 3 & 0 \\ -1 & 1 & 0 & 2 \\ 0 & 2 & 0 & 3 \\ 3 & 4 & 0 & -2 \end{bmatrix}.$$

**SOLUTION**

After inspecting this matrix, you can see that three of the entries in the third column are zeros. So, to eliminate some of the work in the expansion, use the third column.

$$|A| = 3(C_{13}) + 0(C_{23}) + 0(C_{33}) + 0(C_{43})$$

Because $C_{23}$, $C_{33}$, and $C_{43}$ have zero coefficients, you need only find the cofactor $C_{13}$. To do this, delete the first row and third column of $A$ and evaluate the determinant of the resulting matrix.

$$C_{13} = (-1)^{1+3}\begin{vmatrix} -1 & 1 & 2 \\ 0 & 2 & 3 \\ 3 & 4 & -2 \end{vmatrix}$$ Delete 1st row and 3rd column.

$$= \begin{vmatrix} -1 & 1 & 2 \\ 0 & 2 & 3 \\ 3 & 4 & -2 \end{vmatrix}$$ Simplify.

Expanding by cofactors in the second row yields

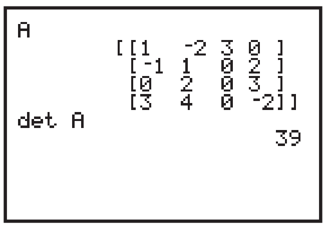
$$\begin{aligned} C_{13} &= (0)(-1)^{2+1}\begin{vmatrix} 1 & 2 \\ 4 & -2 \end{vmatrix} + (2)(-1)^{2+2}\begin{vmatrix} -1 & 2 \\ 3 & -2 \end{vmatrix} + (3)(-1)^{2+3}\begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix} \\ &= 0 + 2(1)(-4) + 3(-1)(-7) \\ &= 13. \end{aligned}$$

You obtain

$$\begin{aligned} |A| &= 3(13) \\ &= 39. \end{aligned}$$

### TECHNOLOGY

Many graphing utilities and software programs can calculate the determinant of a square matrix. If you use a graphing utility, then you may see something similar to the following for Example 4. **The Online Technology Guide,** available at *www.cengagebrain.com,* provides syntax for programs applicable to Example 4.



An alternative method is commonly used to evaluate the determinant of a $3 \times 3$ matrix $A$. To apply this method, copy the first and second columns of $A$ to form fourth and fifth columns. Then obtain the determinant of $A$ by adding (or subtracting) the products of the six diagonals, as shown in the following diagram.

Subtract these three products.

$$\begin{array}{ccccc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}$$

Add these three products.

Try confirming that the determinant of $A$ is

$$|A| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}.$$

## EXAMPLE 5 The Determinant of a Matrix of Order 3

Find the determinant of $A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & -4 & 1 \end{bmatrix}$.

***Simulation***

To explore this concept further with an electronic simulation, and for syntax regarding specific programs involving Example 5, please visit *www.cengagebrain.com.* Similar exercises and projects are also available on the website.

### SOLUTION

Begin by copying the first two columns and then computing the six diagonal products as follows.

$$\begin{array}{ccccc} & & -4 & 0 & 6 \\ 0 & 2 & 1 & 0 & 2 \\ 3 & -1 & 2 & 3 & -1 \\ 4 & -4 & 1 & 4 & -4 \\ & & 0 & 16 & -12 \end{array}$$

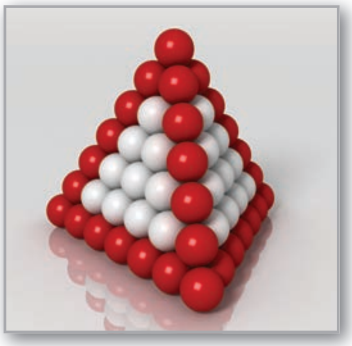
$\leftarrow$ Subtract these products.

$\leftarrow$ Add these products.

Now, by adding the lower three products and subtracting the upper three products, you can find the determinant of $A$ to be

$$|A| = 0 + 16 + (-12) - (-4) - 0 - 6 = 2.$$

The diagonal process illustrated in Example 5 is valid *only* for matrices of order 3. For matrices of higher order, you must use another method.



**LINEAR ALGEBRA APPLIED**

If $x$, $y$, and $z$ are continuous functions of $u$, $v$, and $w$ with continuous first partial derivatives, then the **Jacobians** $J(u, v)$ and $J(u, v, w)$ are defined as the determinants

$$J(u, v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} \quad \text{and} \quad J(u, v, w) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}.$$

One practical use of Jacobians is in finding the volume of a solid region. In Section 3.4, you will study a formula, which also uses determinants, for finding the volume of a tetrahedron. In the Chapter 3 Review, you are asked to find the Jacobian of a given set of functions. (See Review Exercises 49–52.)

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## TRIANGULAR MATRICES

Upper Triangular Matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

Lower Triangular Matrix

$$\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

Recall from Section 2.4 that a square matrix is *upper triangular* when it has all zero entries below its main diagonal, and *lower triangular* when it has all zero entries above its main diagonal, as shown in the diagram at the left. A matrix that is both upper and lower triangular is called **diagonal.** That is, a diagonal matrix is one in which all entries above and below the main diagonal are zero.

To find the determinant of a triangular matrix, simply form the product of the entries on the main diagonal. It is easy to see that this procedure is valid for triangular matrices of order 2 or 3. For instance, to find the determinant of

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

expand in the third row to obtain

$$\begin{aligned} |A| &= 0(-1)^{3+1}\begin{vmatrix} 3 & -1 \\ -1 & 2 \end{vmatrix} + 0(-1)^{3+2}\begin{vmatrix} 2 & -1 \\ 0 & 2 \end{vmatrix} + 3(-1)^{3+3}\begin{vmatrix} 2 & 3 \\ 0 & -1 \end{vmatrix} \\ &= 3(1)(-2) = -6 \end{aligned}$$

which is the product of the entries on the main diagonal.

**THEOREM 3.2 Determinant of a Triangular Matrix**

If $A$ is a triangular matrix of order $n$, then its determinant is the product of the entries on the main diagonal. That is,

$$\det(A) = |A| = a_{11}a_{22}a_{33} \cdot \cdot \cdot a_{nn}.$$

### PROOF

Use *mathematical induction** to prove this theorem for the case in which $A$ is an upper triangular matrix. The proof of the case in which $A$ is lower triangular is similar. If $A$ has order 1, then $A = [a_{11}]$ and the determinant is $|A| = a_{11}$. Assuming the theorem is true for any upper triangular matrix of order $k - 1$, consider an upper triangular matrix $A$ of order $k$. Expanding in the $k$th row, you obtain

$$|A| = 0C_{k1} + 0C_{k2} + \cdot \cdot \cdot + 0C_{k(k-1)} + a_{kk}C_{kk} = a_{kk}C_{kk}.$$

Now, note that $C_{kk} = (-1)^{2k}M_{kk} = M_{kk}$, where $M_{kk}$ is the determinant of the upper triangular matrix formed by deleting the $k$th row and $k$th column of $A$. Because this matrix is of order $k - 1$, apply the induction assumption to write

$$|A| = a_{kk}M_{kk} = a_{kk}(a_{11}a_{22}a_{33} \cdot \cdot \cdot a_{k-1\,k-1}) = a_{11}a_{22}a_{33} \cdot \cdot \cdot a_{kk}.$$

### EXAMPLE 6 The Determinant of a Triangular Matrix

The determinant of the lower triangular matrix

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 4 & -2 & 0 & 0 \\ -5 & 6 & 1 & 0 \\ 1 & 5 & 3 & 3 \end{bmatrix}$$

is

$$|A| = (2)(-2)(1)(3) = -12.$$

*See Appendix for a discussion of mathematical induction.

## 3.1 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

**The Determinant of a Matrix** In Exercises 1–12, find the determinant of the matrix.

1. $[1]$

2. $[-3]$

3. $\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$

4. $\begin{bmatrix} -3 & 1 \\ 5 & 2 \end{bmatrix}$

5. $\begin{bmatrix} 5 & 2 \\ -6 & 3 \end{bmatrix}$

6. $\begin{bmatrix} 2 & -2 \\ 4 & 3 \end{bmatrix}$

7. $\begin{bmatrix} -7 & 6 \\ \frac{1}{2} & 3 \end{bmatrix}$

8. $\begin{bmatrix} \frac{1}{3} & 5 \\ 4 & -9 \end{bmatrix}$

9. $\begin{bmatrix} 0 & 6 \\ 0 & 3 \end{bmatrix}$

10. $\begin{bmatrix} 2 & -3 \\ -6 & 9 \end{bmatrix}$

11. $\begin{bmatrix} \lambda - 3 & 2 \\ 4 & \lambda - 1 \end{bmatrix}$

12. $\begin{bmatrix} \lambda - 2 & 0 \\ 4 & \lambda - 4 \end{bmatrix}$

**Finding the Minors and Cofactors of a Matrix** In Exercises 13–16, find all (a) minors and (b) cofactors of the matrix.

13. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

14. $\begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix}$

15. $\begin{bmatrix} -3 & 2 & 1 \\ 4 & 5 & 6 \\ 2 & -3 & 1 \end{bmatrix}$

16. $\begin{bmatrix} -3 & 4 & 2 \\ 6 & 3 & 1 \\ 4 & -7 & -8 \end{bmatrix}$

17. Find the determinant of the matrix in Exercise 15 using the method of expansion by cofactors. Use (a) the second row and (b) the second column.

18. Find the determinant of the matrix in Exercise 16 using the method of expansion by cofactors. Use (a) the third row and (b) the first column.

**Finding a Determinant** In Exercises 19–32, use expansion by cofactors to find the determinant of the matrix.

19. $\begin{bmatrix} 1 & 4 & -2 \\ 3 & 2 & 0 \\ -1 & 4 & 3 \end{bmatrix}$

20. $\begin{bmatrix} 2 & -1 & 3 \\ 1 & 4 & 4 \\ 1 & 0 & 2 \end{bmatrix}$

21. $\begin{bmatrix} 2 & 4 & 6 \\ 0 & 3 & 1 \\ 0 & 0 & -5 \end{bmatrix}$

22. $\begin{bmatrix} -3 & 0 & 0 \\ 7 & 11 & 0 \\ 1 & 2 & 2 \end{bmatrix}$

23. $\begin{bmatrix} -0.4 & 0.4 & 0.3 \\ 0.2 & 0.2 & 0.2 \\ 0.3 & 0.2 & 0.2 \end{bmatrix}$

24. $\begin{bmatrix} 0.1 & 0.2 & 0.3 \\ -0.3 & 0.2 & 0.2 \\ 0.5 & 0.4 & 0.4 \end{bmatrix}$

25. $\begin{bmatrix} x & y & 1 \\ 2 & 3 & 1 \\ 0 & -1 & 1 \end{bmatrix}$

26. $\begin{bmatrix} x & y & 1 \\ -2 & -2 & 1 \\ 1 & 5 & 1 \end{bmatrix}$

27. $\begin{bmatrix} 5 & 3 & 0 & 6 \\ 4 & 6 & 4 & 12 \\ 0 & 2 & -3 & 4 \\ 0 & 1 & -2 & 2 \end{bmatrix}$

28. $\begin{bmatrix} 3 & 0 & 7 & 0 \\ 2 & 6 & 11 & 12 \\ 4 & 1 & -1 & 2 \\ 1 & 5 & 2 & 10 \end{bmatrix}$

29. $\begin{bmatrix} w & x & y & z \\ 21 & -15 & 24 & 30 \\ -10 & 24 & -32 & 18 \\ -40 & 22 & 32 & -35 \end{bmatrix}$

30. $\begin{bmatrix} w & x & y & z \\ 10 & 15 & -25 & 30 \\ -30 & 20 & -15 & -10 \\ 30 & 35 & -25 & -40 \end{bmatrix}$

31. $\begin{bmatrix} 5 & 2 & 0 & 0 & -2 \\ 0 & 1 & 4 & 3 & 2 \\ 0 & 0 & 2 & 6 & 3 \\ 0 & 0 & 3 & 4 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$

32. $\begin{bmatrix} 4 & 3 & -2 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & -7 & 13 & 12 \\ 6 & -2 & 5 & 6 & 7 \\ 1 & 4 & 2 & 0 & 9 \end{bmatrix}$

**Finding a Determinant** In Exercises 33 and 34, use the method demonstrated in Example 5 to find the determinant of the matrix.

33. $\begin{bmatrix} 4 & 0 & 2 \\ -3 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix}$

34. $\begin{bmatrix} 3 & 8 & -7 \\ 0 & -5 & 4 \\ 8 & 1 & 6 \end{bmatrix}$

**Finding a Determinant** In Exercises 35–38, use a software program or a graphing utility with matrix capabilities to find the determinant of the matrix.

35. $\begin{bmatrix} 0.25 & -1 & 0.6 \\ 0.50 & 0.8 & -0.2 \\ 0.75 & 0.9 & -0.4 \end{bmatrix}$

36. $\begin{bmatrix} 4 & 3 & 2 & 5 \\ 1 & 6 & -1 & 2 \\ -3 & 2 & 4 & 5 \\ 6 & 1 & 3 & -2 \end{bmatrix}$

37. $\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 2 & -2 \\ 0 & 3 & 2 & -1 \\ 1 & 2 & 0 & -2 \end{bmatrix}$

38. $\begin{bmatrix} 8 & 5 & 1 & -2 & 0 \\ -1 & 0 & 7 & 1 & 6 \\ 0 & 8 & 6 & 5 & -3 \\ 1 & 2 & 5 & -8 & 4 \\ 2 & 6 & -2 & 0 & 6 \end{bmatrix}$

**Finding the Determinant of a Triangular Matrix** **In Exercises 39–42, find the determinant of the triangular matrix.**

**39.** $\begin{bmatrix} -2 & 0 & 0 \\ 4 & 6 & 0 \\ -3 & 7 & 2 \end{bmatrix}$ **40.** $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -3 \end{bmatrix}$

**41.** $\begin{bmatrix} 5 & 8 & -4 & 2 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & -1 \end{bmatrix}$ **42.** $\begin{bmatrix} 4 & 0 & 0 & 0 \\ -1 & \frac{1}{2} & 0 & 0 \\ 3 & 5 & 3 & 0 \\ -8 & 7 & 0 & -2 \end{bmatrix}$

**True or False?** **In Exercises 43 and 44, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.**

**43.** (a) The determinant of a $2 \times 2$ matrix $A$ is $a_{21}a_{12} - a_{11}a_{22}$.

(b) The determinant of a matrix of order 1 is the entry of the matrix.

(c) The $ij$-cofactor of a square matrix $A$ is the matrix defined by deleting the $i$th row and $j$th column of $A$.

**44.** (a) To find the determinant of a triangular matrix, add the entries on the main diagonal.

(b) To find the determinant of a matrix, expand by cofactors in any row or column.

(c) When expanding by cofactors, you need not evaluate the cofactors of zero entries.

**Solving an Equation** **In Exercises 45–48, solve for $x$.**

**45.** $\begin{vmatrix} x+3 & 2 \\ 1 & x+2 \end{vmatrix} = 0$ **46.** $\begin{vmatrix} x+1 & -2 \\ 1 & x-2 \end{vmatrix} = 0$

**47.** $\begin{vmatrix} x-1 & 2 \\ 3 & x-2 \end{vmatrix} = 0$ **48.** $\begin{vmatrix} x+3 & 1 \\ -4 & x-1 \end{vmatrix} = 0$

**Solving an Equation** **In Exercises 49–52, find the values of $\lambda$ for which the determinant is zero.**

**49.** $\begin{vmatrix} \lambda+2 & 2 \\ 1 & \lambda \end{vmatrix}$ **50.** $\begin{vmatrix} \lambda-1 & 1 \\ 4 & \lambda-3 \end{vmatrix}$

**51.** $\begin{vmatrix} \lambda & 2 & 0 \\ 0 & \lambda+1 & 2 \\ 0 & 1 & \lambda \end{vmatrix}$ **52.** $\begin{vmatrix} \lambda & 0 & 1 \\ 0 & \lambda & 3 \\ 2 & 2 & \lambda-2 \end{vmatrix}$

**Entries Involving Expressions** **In Exercises 53–58, evaluate the determinant, in which the entries are functions. Determinants of this type occur when changes of variables are made in calculus.**

**53.** $\begin{vmatrix} 4u & -1 \\ -1 & 2v \end{vmatrix}$ **54.** $\begin{vmatrix} 3x^2 & -3y^2 \\ 1 & 1 \end{vmatrix}$

**55.** $\begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{vmatrix}$ **56.** $\begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & (1-x)e^{-x} \end{vmatrix}$

**57.** $\begin{vmatrix} x & \ln x \\ 1 & 1/x \end{vmatrix}$ **58.** $\begin{vmatrix} x & x \ln x \\ 1 & 1 + \ln x \end{vmatrix}$

**59.** The determinant of a $2 \times 2$ matrix involves two products. The determinant of a $3 \times 3$ matrix involves six triple products. Show that the determinant of a $4 \times 4$ matrix involves 24 quadruple products.

**60.** Show that the system of linear equations

$$ax + by = e$$
$$cx + dy = f$$

has a unique solution if and only if the determinant of the coefficient matrix is nonzero.

**Verifying an Equation** **In Exercises 61–66, evaluate the determinants to verify the equation.**

**61.** $\begin{vmatrix} w & x \\ y & z \end{vmatrix} = -\begin{vmatrix} y & z \\ w & x \end{vmatrix}$

**62.** $\begin{vmatrix} w & cx \\ y & cz \end{vmatrix} = c\begin{vmatrix} w & x \\ y & z \end{vmatrix}$

**63.** $\begin{vmatrix} w & x \\ y & z \end{vmatrix} = \begin{vmatrix} w & x+cw \\ y & z+cy \end{vmatrix}$ **64.** $\begin{vmatrix} w & x \\ cw & cx \end{vmatrix} = 0$

**65.** $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = (y-x)(z-x)(z-y)$

**66.** $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$

**67.** You are given the equation

$$\begin{vmatrix} x & 0 & c \\ -1 & x & b \\ 0 & -1 & a \end{vmatrix} = ax^2 + bx + c.$$

(a) Verify the equation.

(b) Use the equation as a model to find a determinant that is equal to $ax^3 + bx^2 + cx + d$.

**68.** **CAPSTONE** If $A$ is an $n \times n$ matrix, explain how to find the following.

(a) The minor $M_{ij}$ of the entry $a_{ij}$.

(b) The cofactor $C_{ij}$ of the entry $a_{ij}$.

(c) The determinant of $A$.

**69.** Show that the system of linear equations

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

has the solution

$$x_1 = \frac{b_1a_{22} - b_2a_{12}}{a_{11}a_{22} - a_{21}a_{12}} \quad \text{and} \quad x_2 = \frac{b_2a_{11} - b_1a_{21}}{a_{11}a_{22} - a_{21}a_{12}}$$

when $a_{11}a_{22} - a_{21}a_{12} \neq 0$.

# 3.2 Determinants and Elementary Operations

- Use elementary row operations to evaluate a determinant.
- Use elementary column operations to evaluate a determinant.
- Recognize conditions that yield zero determinants.

## DETERMINANTS AND ELEMENTARY ROW OPERATIONS

Which of the following two determinants is easier to evaluate?

$$|A| = \begin{vmatrix} 1 & -2 & 3 & 1 \\ 4 & -6 & 3 & 2 \\ -2 & 4 & -9 & -3 \\ 3 & -6 & 9 & 2 \end{vmatrix} \quad \text{or} \quad |B| = \begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 2 & -9 & -2 \\ 0 & 0 & -3 & -1 \\ 0 & 0 & 0 & -1 \end{vmatrix}$$

Given what you know about the determinant of a triangular matrix, it is clear that the second determinant is *much* easier to evaluate. Its determinant is simply the product of the entries on the main diagonal. That is, $|B| = (1)(2)(-3)(-1) = 6$. Using expansion by cofactors (the only technique discussed so far) to evaluate the first determinant is messy. For instance, when you expand by cofactors in the first row, you have

$$|A| = 1\begin{vmatrix} -6 & 3 & 2 \\ 4 & -9 & -3 \\ -6 & 9 & 2 \end{vmatrix} + 2\begin{vmatrix} 4 & 3 & 2 \\ -2 & -9 & -3 \\ 3 & 9 & 2 \end{vmatrix} + 3\begin{vmatrix} 4 & -6 & 2 \\ -2 & 4 & -3 \\ 3 & -6 & 2 \end{vmatrix} - 1\begin{vmatrix} 4 & -6 & 3 \\ -2 & 4 & -9 \\ 3 & -6 & 9 \end{vmatrix}.$$

Evaluating the determinants of these four $3 \times 3$ matrices produces

$$|A| = (1)(-60) + (2)(39) + (3)(-10) - (1)(-18) = 6.$$

Note that $|A|$ and $|B|$ have the same value. Also note that you can obtain matrix $B$ from matrix $A$ by adding multiples of the first row to the second, third, and fourth rows. (Try verifying this.) In this section, you will see the effects of elementary row (and column) operations on the value of a determinant.

### EXAMPLE 1 The Effects of Elementary Row Operations on a Determinant

**a.** The matrix $B$ was obtained from $A$ by interchanging the rows of $A$.

$$|A| = \begin{vmatrix} 2 & -3 \\ 1 & 4 \end{vmatrix} = 11 \quad \text{and} \quad |B| = \begin{vmatrix} 1 & 4 \\ 2 & -3 \end{vmatrix} = -11$$

**b.** The matrix $B$ was obtained from $A$ by adding $-2$ times the first row of $A$ to the second row of $A$.

$$|A| = \begin{vmatrix} 1 & -3 \\ 2 & -4 \end{vmatrix} = 2 \quad \text{and} \quad |B| = \begin{vmatrix} 1 & -3 \\ 0 & 2 \end{vmatrix} = 2$$

**c.** The matrix $B$ was obtained from $A$ by multiplying the first row of $A$ by $\frac{1}{2}$.

$$|A| = \begin{vmatrix} 2 & -8 \\ -2 & 9 \end{vmatrix} = 2 \quad \text{and} \quad |B| = \begin{vmatrix} 1 & -4 \\ -2 & 9 \end{vmatrix} = 1$$

In Example 1, you can see that interchanging two rows of a matrix changes the sign of its determinant. Adding a multiple of one row to another does not change the determinant. Finally, multiplying a row by a nonzero constant multiplies the determinant by that same constant. The next theorem generalizes these observations.

**REMARK**

Note that the third property enables you to divide a row by the common factor. For instance,

$$\begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix} = 2\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}.$$ Factor 2 out of first row.

### THEOREM 3.3 Elementary Row Operations and Determinants

Let $A$ and $B$ be square matrices.

1. When $B$ is obtained from $A$ by interchanging two rows of $A$, $\det(B) = -\det(A)$.
2. When $B$ is obtained from $A$ by adding a multiple of a row of $A$ to another row of $A$, $\det(B) = \det(A)$.
3. When $B$ is obtained from $A$ by multiplying a row of $A$ by a nonzero constant $c$, $\det(B) = c\det(A)$.

**PROOF**

The proof of the first property follows. The proofs of the other two properties are left as exercises. (See Exercises 47 and 48.) Assume that $A$ and $B$ are $2 \times 2$ matrices

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{bmatrix}.$$

Then, you have $|A| = a_{11}a_{22} - a_{21}a_{12}$ and $|B| = a_{21}a_{12} - a_{11}a_{22}$. So $|B| = -|A|$. Using mathematical induction, assume the property is true for matrices of order $(n - 1)$. Let $A$ be an $n \times n$ matrix such that $B$ is obtained from $A$ by interchanging two rows of $A$. Then, to find $|A|$ and $|B|$, expand in a row other than the two interchanged rows. By the induction assumption, the cofactors of $B$ will be the negatives of the cofactors of $A$ because the corresponding $(n - 1) \times (n - 1)$ matrices have two rows interchanged. Finally, $|B| = -|A|$ and the proof is complete. ■

**Augustin-Louis Cauchy (1789–1857)**

Cauchy's contributions to the study of mathematics were revolutionary, and he is often credited with bringing rigor to modern mathematics. For example, he was the first to rigorously define limits, continuity, and the convergence of an infinite series. In addition to being known for his work in complex analysis, he contributed to the theories of determinants and differential equations. It is interesting to note that Cauchy's work on determinants preceded Cayley's development of matrices.

Theorem 3.3 provides a practical way to evaluate determinants. To find the determinant of a matrix $A$, use elementary row operations to obtain a triangular matrix $B$ that is row-equivalent to $A$. For each step in the elimination process, use Theorem 3.3 to determine the effect of the elementary row operation on the determinant. Finally, find the determinant of $B$ by multiplying the entries on its main diagonal.

### EXAMPLE 2 Finding a Determinant Using Elementary Row Operations

Find the determinant of

$$A = \begin{bmatrix} 2 & -3 & 10 \\ 1 & 2 & -2 \\ 0 & 0 & -3 \end{bmatrix}.$$

**SOLUTION**

Using elementary row operations, rewrite $A$ in triangular form as follows.

$$\begin{vmatrix} 2 & -3 & 10 \\ 1 & 2 & -2 \\ 0 & 0 & -3 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & -2 \\ 2 & -3 & 10 \\ 0 & 0 & -3 \end{vmatrix}$$ Interchange the first two rows.

$$= -\begin{vmatrix} 1 & 2 & -2 \\ 0 & -7 & 14 \\ 0 & 0 & -3 \end{vmatrix}$$ Add $-2$ times the first row to the second row to produce a new second row.

$$= -21\begin{vmatrix} 1 & 2 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{vmatrix}$$ Factor $-7$ out of the second row. Factor $-3$ out of the third row.

Now, because the above matrix is triangular, you can conclude that the determinant is

$$|A| = -21(1)(1)(1) = -21.$$ ■

## DETERMINANTS AND ELEMENTARY COLUMN OPERATIONS

Although Theorem 3.3 is stated in terms of elementary *row* operations, the theorem remains valid when the word “column” replaces the word “row.” Operations performed on the columns (rather than the rows) of a matrix are called **elementary column operations,** and two matrices are called **column-equivalent** when one can be obtained from the other by elementary column operations. The following illustrate the column version of Theorem 3.3.

$$\begin{vmatrix} 2 & 1 & -3 \\ 4 & 0 & 1 \\ 0 & 0 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & -3 \\ 0 & 4 & 1 \\ 0 & 0 & 2 \end{vmatrix}$$

Interchange the first two columns.

$$\begin{vmatrix} 2 & 3 & -5 \\ 4 & 1 & 0 \\ -2 & 4 & -3 \end{vmatrix} = 2\begin{vmatrix} 1 & 3 & -5 \\ 2 & 1 & 0 \\ -1 & 4 & -3 \end{vmatrix}$$

Factor 2 out of the first column.

In evaluating a determinant, it is occasionally convenient to use elementary column operations, as shown in Example 3.

### EXAMPLE 3 Finding a Determinant Using Elementary Column Operations

Find the determinant of

$$A = \begin{bmatrix} -1 & 2 & 2 \\ 3 & -6 & 4 \\ 5 & -10 & -3 \end{bmatrix}.$$

#### SOLUTION

Because the first two columns of $A$ are multiples of each other, you can obtain a column of zeros by adding 2 times the first column to the second column, as follows.

$$\begin{vmatrix} -1 & 2 & 2 \\ 3 & -6 & 4 \\ 5 & -10 & -3 \end{vmatrix} = \begin{vmatrix} -1 & 0 & 2 \\ 3 & 0 & 4 \\ 5 & 0 & -3 \end{vmatrix}$$

At this point, you do not need to continue to rewrite the matrix in triangular form. Because there is an entire column of zeros, simply conclude that the determinant is zero. The validity of this conclusion follows from Theorem 3.1. Specifically, by expanding by cofactors in the second column, you have

$$\begin{aligned} |A| &= (0)C_{12} + (0)C_{22} + (0)C_{32} \\ &= 0. \end{aligned}$$

**LINEAR ALGEBRA APPLIED**

In the number-placement puzzle Sudoku, the object is to fill out a partially completed $9 \times 9$ grid of boxes with numbers from 1 to 9 so that each column, row, and $3 \times 3$ sub-grid contains each of these numbers without repetition. For a completed Sudoku grid to be valid, no two rows (or columns) will have the numbers in the same order. If this should happen in a row or column, then the determinant of the matrix formed by the numbers in the grid will be zero. This is a direct result of condition 2 of Theorem 3.4 on the next page.

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## MATRICES AND ZERO DETERMINANTS

Example 3 shows that when two columns of a matrix are scalar multiples of each other, the determinant of the matrix is zero. This is one of three conditions that yield a determinant of zero.

**THEOREM 3.4 Conditions That Yield a Zero Determinant**

If $A$ is a square matrix and any one of the following conditions is true, then $\det(A) = 0$.

1. An entire row (or an entire column) consists of zeros.
2. Two rows (or columns) are equal.
3. One row (or column) is a multiple of another row (or column).

### PROOF

Verify each part of this theorem by using elementary row operations and expansion by cofactors. For example, if an entire row or column is zero, then each cofactor in the expansion is multiplied by zero. When condition 2 or 3 is true, use elementary row or column operations to create an entire row or column of zeros.

Recognizing the conditions listed in Theorem 3.4 can make evaluating a determinant much easier. For instance,

$$\underbrace{\begin{vmatrix} 0 & 0 & 0 \\ 2 & 4 & -5 \\ 3 & -5 & 2 \end{vmatrix}}_{\text{The first row has all zeros.}} = 0, \quad \underbrace{\begin{vmatrix} 1 & -2 & 4 \\ 0 & 1 & 2 \\ 1 & -2 & 4 \end{vmatrix}}_{\text{The first and third rows are the same.}} = 0, \quad \underbrace{\begin{vmatrix} 1 & 2 & -3 \\ 2 & -1 & -6 \\ -2 & 0 & 6 \end{vmatrix}}_{\text{The third column is a multiple of the first column.}} = 0.$$

Do not conclude, however, that Theorem 3.4 gives the *only* conditions that produce a zero determinant. This theorem is often used indirectly. That is, you may begin with a matrix that does not satisfy any of the conditions of Theorem 3.4 and, through elementary row or column operations, obtain a matrix that does satisfy one of the conditions. Example 4 demonstrates this.

### EXAMPLE 4 A Matrix with a Zero Determinant

Find the determinant of

$$A = \begin{bmatrix} 1 & 4 & 1 \\ 2 & -1 & 0 \\ 0 & 18 & 4 \end{bmatrix}.$$

### SOLUTION

Adding $-2$ times the first row to the second row produces

$$|A| = \begin{vmatrix} 1 & 4 & 1 \\ 2 & -1 & 0 \\ 0 & 18 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 4 & 1 \\ 0 & -9 & -2 \\ 0 & 18 & 4 \end{vmatrix}.$$

Because the second and third rows are multiples of each other, the determinant is zero.

In Example 4, you could have obtained a matrix with a row of all zeros by performing an additional elementary row operation (adding 2 times the second row to the third row). This is true in general. That is, a square matrix has a determinant of zero if and only if it is row- (or column-) equivalent to a matrix that has at least one row (or column) consisting entirely of zeros. The proof of this is given in the next section.

You have now surveyed two general methods for evaluating determinants. Of these, the method of using elementary row operations to reduce the matrix to triangular form is usually faster than cofactor expansion along a row or column. If the matrix is large, then the number of arithmetic operations needed for cofactor expansion can become extremely large. For this reason, most computer and calculator algorithms use the method involving elementary row operations. The following table shows the numbers of additions (plus subtractions) and multiplications (plus divisions) needed for each of these two methods for matrices of orders 3, 5, and 10.

| | *Cofactor Expansion* | | *Row Reduction* | |
|---|---|---|---|---|
| *Order n* | *Additions* | *Multiplications* | *Additions* | *Multiplications* |
| 3 | 5 | 9 | 5 | 10 |
| 5 | 119 | 205 | 30 | 45 |
| 10 | 3,628,799 | 6,235,300 | 285 | 339 |

In fact, the number of additions alone for the cofactor expansion of an $n \times n$ matrix is $n! - 1$. Because $30! \approx 2.65 \times 10^{32}$, even a relatively small $30 \times 30$ matrix would require more than $10^{32}$ operations. If a computer could do one trillion operations per second, it would still take more than one trillion years to compute the determinant of this matrix using cofactor expansion. Yet, row reduction would take only a few seconds.

When evaluating a determinant *by hand*, you sometimes save steps by using elementary row (or column) operations to create a row (or column) having zeros in all but one position and then using cofactor expansion to reduce the order of the matrix by 1. The next two examples illustrate this approach.

## EXAMPLE 5 Finding a Determinant

Find the determinant of

$$A = \begin{bmatrix} -3 & 5 & 2 \\ 2 & -4 & -1 \\ -3 & 0 & 6 \end{bmatrix}.$$

### SOLUTION

Notice that the matrix $A$ already has one zero in the third row. Create another zero in the third row by adding 2 times the first column to the third column, as follows.

$$|A| = \begin{vmatrix} -3 & 5 & 2 \\ 2 & -4 & -1 \\ -3 & 0 & 6 \end{vmatrix} = \begin{vmatrix} -3 & 5 & -4 \\ 2 & -4 & 3 \\ -3 & 0 & 0 \end{vmatrix}$$

Expanding by cofactors in the third row produces

$$|A| = \begin{vmatrix} -3 & 5 & -4 \\ 2 & -4 & 3 \\ -3 & 0 & 0 \end{vmatrix} = -3(-1)^4 \begin{vmatrix} 5 & -4 \\ -4 & 3 \end{vmatrix} = -3(1)(-1) = 3.$$

## EXAMPLE 6 Finding a Determinant

Find the determinant of

$$A = \begin{bmatrix} 2 & 0 & 1 & 3 & -2 \\ -2 & 1 & 3 & 2 & -1 \\ 1 & 0 & -1 & 2 & 3 \\ 3 & -1 & 2 & 4 & -3 \\ 1 & 1 & 3 & 2 & 0 \end{bmatrix}.$$

### SOLUTION

Because the second column of this matrix already has two zeros, choose it for cofactor expansion. Create two additional zeros in the second column by adding the second row to the fourth row, and then adding $-1$ times the second row to the fifth row.

$$\begin{aligned} |A| &= \begin{vmatrix} 2 & 0 & 1 & 3 & -2 \\ -2 & 1 & 3 & 2 & -1 \\ 1 & 0 & -1 & 2 & 3 \\ 3 & -1 & 2 & 4 & -3 \\ 1 & 1 & 3 & 2 & 0 \end{vmatrix} \\ &= \begin{vmatrix} 2 & 0 & 1 & 3 & -2 \\ -2 & 1 & 3 & 2 & -1 \\ 1 & 0 & -1 & 2 & 3 \\ 1 & 0 & 5 & 6 & -4 \\ 3 & 0 & 0 & 0 & 1 \end{vmatrix} \\ &= (1)(-1)^4 \begin{vmatrix} 2 & 1 & 3 & -2 \\ 1 & -1 & 2 & 3 \\ 1 & 5 & 6 & -4 \\ 3 & 0 & 0 & 1 \end{vmatrix} \end{aligned}$$

You have now reduced the problem of finding the determinant of a $5 \times 5$ matrix to the problem of finding the determinant of a $4 \times 4$ matrix. Because the fourth row already has two zeros, choose it for the next cofactor expansion. Add $-3$ times the fourth column to the first column to produce the following.

$$\begin{aligned} |A| = \begin{vmatrix} 2 & 1 & 3 & -2 \\ 1 & -1 & 2 & 3 \\ 1 & 5 & 6 & -4 \\ 3 & 0 & 0 & 1 \end{vmatrix} &= \begin{vmatrix} 8 & 1 & 3 & -2 \\ -8 & -1 & 2 & 3 \\ 13 & 5 & 6 & -4 \\ 0 & 0 & 0 & 1 \end{vmatrix} \\ &= (1)(-1)^8 \begin{vmatrix} 8 & 1 & 3 \\ -8 & -1 & 2 \\ 13 & 5 & 6 \end{vmatrix} \end{aligned}$$

Add the second row to the first row and then expand by cofactors in the first row.

$$\begin{aligned} |A| = \begin{vmatrix} 8 & 1 & 3 \\ -8 & -1 & 2 \\ 13 & 5 & 6 \end{vmatrix} &= \begin{vmatrix} 0 & 0 & 5 \\ -8 & -1 & 2 \\ 13 & 5 & 6 \end{vmatrix} \\ &= 5(-1)^4 \begin{vmatrix} -8 & -1 \\ 13 & 5 \end{vmatrix} \\ &= 5(1)(-27) \\ &= -135 \end{aligned}$$

## 3.2 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

**Properties of Determinants** In Exercises 1–20, determine which property of determinants the equation illustrates.

**1.** $\begin{vmatrix} 2 & -6 \\ 1 & -3 \end{vmatrix} = 0$

**2.** $\begin{vmatrix} -4 & 5 \\ 12 & -15 \end{vmatrix} = 0$

**3.** $\begin{vmatrix} 1 & 4 & 2 \\ 0 & 0 & 0 \\ 5 & 6 & -7 \end{vmatrix} = 0$

**4.** $\begin{vmatrix} -4 & 3 & 2 \\ 8 & 0 & 0 \\ -4 & 3 & 2 \end{vmatrix} = 0$

**5.** $\begin{vmatrix} 1 & 3 & 4 \\ -7 & 2 & -5 \\ 6 & 1 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & 4 & 3 \\ -7 & -5 & 2 \\ 6 & 2 & 1 \end{vmatrix}$

**6.** $\begin{vmatrix} 1 & 3 & 4 \\ -2 & 2 & 0 \\ 1 & 6 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & 6 & 2 \\ -2 & 2 & 0 \\ 1 & 3 & 4 \end{vmatrix}$

**7.** $\begin{vmatrix} 5 & 10 \\ 2 & -7 \end{vmatrix} = 5\begin{vmatrix} 1 & 2 \\ 2 & -7 \end{vmatrix}$

**8.** $\begin{vmatrix} 4 & 1 \\ 2 & 8 \end{vmatrix} = 2\begin{vmatrix} 2 & 1 \\ 1 & 8 \end{vmatrix}$

**9.** $\begin{vmatrix} 1 & 8 & -3 \\ 3 & -12 & 6 \\ 7 & 4 & 9 \end{vmatrix} = 12\begin{vmatrix} 1 & 2 & -1 \\ 3 & -3 & 2 \\ 7 & 1 & 3 \end{vmatrix}$

**10.** $\begin{vmatrix} 1 & 2 & 3 \\ 4 & -8 & 6 \\ 5 & 4 & 12 \end{vmatrix} = 6\begin{vmatrix} 1 & 1 & 1 \\ 4 & -4 & 2 \\ 5 & 2 & 4 \end{vmatrix}$

**11.** $\begin{vmatrix} 5 & 0 & 10 \\ 25 & -30 & 40 \\ -15 & 5 & 20 \end{vmatrix} = 5^3\begin{vmatrix} 1 & 0 & 2 \\ 5 & -6 & 8 \\ -3 & 1 & 4 \end{vmatrix}$

**12.** $\begin{vmatrix} 6 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 6 \end{vmatrix} = 6^4\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$

**13.** $\begin{vmatrix} 2 & -3 \\ 8 & 7 \end{vmatrix} = \begin{vmatrix} 2 & -3 \\ 0 & 19 \end{vmatrix}$

**14.** $\begin{vmatrix} 2 & 1 \\ 0 & -1 \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 4 & 1 \end{vmatrix}$

**15.** $\begin{vmatrix} 1 & -3 & 2 \\ 5 & 2 & -1 \\ -1 & 0 & 6 \end{vmatrix} = \begin{vmatrix} 1 & -3 & 2 \\ 0 & 17 & -11 \\ -1 & 0 & 6 \end{vmatrix}$

**16.** $\begin{vmatrix} 3 & 2 & 4 \\ -2 & 1 & 5 \\ 5 & -7 & -20 \end{vmatrix} = \begin{vmatrix} 3 & 2 & -6 \\ -2 & 1 & 0 \\ 5 & -7 & 15 \end{vmatrix}$

**17.** $\begin{vmatrix} 5 & 4 & 2 \\ 4 & -3 & 4 \\ 7 & 6 & 3 \end{vmatrix} = -\begin{vmatrix} 5 & 4 & 2 \\ -4 & 3 & -4 \\ 7 & 6 & 3 \end{vmatrix}$

**18.** $\begin{vmatrix} 2 & 1 & -1 \\ 0 & 1 & 4 \\ 5 & 3 & 1 \end{vmatrix} = -\begin{vmatrix} 2 & 1 & -1 \\ 5 & 3 & 1 \\ 0 & 1 & 4 \end{vmatrix}$

**19.** $\begin{vmatrix} 2 & 1 & -1 & 0 & 4 \\ 1 & 0 & 1 & 3 & 2 \\ 3 & 6 & 1 & -3 & 6 \\ 0 & 4 & 0 & 2 & 0 \\ -1 & 8 & 5 & 3 & -2 \end{vmatrix} = 0$

**20.** $\begin{vmatrix} 4 & 3 & 1 & 9 & 9 \\ 9 & -1 & 2 & 3 & -3 \\ 3 & 4 & 6 & 9 & 12 \\ 5 & 2 & 0 & 6 & 6 \\ 6 & 0 & 3 & 0 & 0 \end{vmatrix} = 0$

**Finding a Determinant** In Exercises 21–24, use either elementary row or column operations, or cofactor expansion, to find the determinant by hand. Then use a software program or a graphing utility to verify your answer.

**21.** $\begin{vmatrix} 1 & 0 & 2 \\ -1 & 1 & 4 \\ 2 & 0 & 3 \end{vmatrix}$

**22.** $\begin{vmatrix} -1 & 3 & 2 \\ 0 & 2 & 0 \\ 1 & 1 & -1 \end{vmatrix}$

**23.** $\begin{vmatrix} 1 & 2 & 1 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 3 & -1 & 1 \\ 0 & 0 & 4 & 1 \end{vmatrix}$

**24.** $\begin{vmatrix} 3 & 2 & 1 & 1 \\ -1 & 0 & 2 & 0 \\ 4 & 1 & -1 & 0 \\ 3 & 1 & 1 & 0 \end{vmatrix}$

**Finding a Determinant** In Exercises 25–36, use elementary row or column operations to find the determinant.

**25.** $\begin{vmatrix} 1 & 7 & -3 \\ 1 & 3 & 1 \\ 4 & 8 & 1 \end{vmatrix}$

**26.** $\begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & -2 \\ 1 & -2 & -1 \end{vmatrix}$

**27.** $\begin{vmatrix} 2 & -1 & -1 \\ 1 & 3 & 2 \\ -6 & 3 & 3 \end{vmatrix}$

**28.** $\begin{vmatrix} 3 & 0 & 6 \\ 2 & -3 & 4 \\ 1 & -2 & 2 \end{vmatrix}$

**29.** $\begin{vmatrix} 4 & 3 & -2 \\ 5 & 4 & 1 \\ -2 & 3 & 4 \end{vmatrix}$

**30.** $\begin{vmatrix} 3 & 8 & -7 \\ 0 & -5 & 4 \\ 6 & 1 & 6 \end{vmatrix}$

**31.** $\begin{vmatrix} 4 & -7 & 9 & 1 \\ 6 & 2 & 7 & 0 \\ 3 & 6 & -3 & 3 \\ 0 & 7 & 4 & -1 \end{vmatrix}$

**32.** $\begin{vmatrix} 9 & -4 & 2 & 5 \\ 2 & 7 & 6 & -5 \\ 4 & 1 & -2 & 0 \\ 7 & 3 & 4 & 10 \end{vmatrix}$

**33.** $\begin{vmatrix} 1 & -2 & 7 & 9 \\ 3 & -4 & 5 & 5 \\ 3 & 6 & 1 & -1 \\ 4 & 5 & 3 & 2 \end{vmatrix}$

**34.** $\begin{vmatrix} 0 & -3 & 8 & 2 \\ 8 & 1 & -1 & 6 \\ -4 & 6 & 0 & 9 \\ -7 & 0 & 0 & 14 \end{vmatrix}$

**35.** $\begin{vmatrix} 1 & -1 & 8 & 4 & 2 \\ 2 & 6 & 0 & -4 & 3 \\ 2 & 0 & 2 & 6 & 2 \\ 0 & 2 & 8 & 0 & 0 \\ 0 & 1 & 1 & 2 & 2 \end{vmatrix}$

**36.** $\begin{vmatrix} 3 & -2 & 4 & 3 & 1 \\ -1 & 0 & 2 & 1 & 0 \\ 5 & -1 & 0 & 3 & 2 \\ 4 & 7 & -8 & 0 & 0 \\ 1 & 2 & 3 & 0 & 2 \end{vmatrix}$

**True or False? In Exercises 37 and 38, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.**

**37.** (a) Interchanging two rows of a given matrix changes the sign of its determinant.

(b) Multiplying a column of a matrix by a nonzero constant results in the determinant being multiplied by the same nonzero constant.

(c) If two rows of a square matrix are equal, then its determinant is 0.

**38.** (a) Adding a multiple of one column of a matrix to another column changes only the sign of the determinant.

(b) Two matrices are column-equivalent when one matrix can be obtained by performing elementary column operations on the other.

(c) If one row of a square matrix is a multiple of another row, then the determinant is 0.

**Finding the Determinant of an Elementary Matrix In Exercises 39–42, find the determinant of the elementary matrix. (Assume $k \neq 0$.)**

**39.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{bmatrix}$

**40.** $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

**41.** $\begin{bmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

**42.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & k & 1 \end{bmatrix}$

**43.** **Proof** Prove the property.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} b_{11} & a_{12} & a_{13} \\ b_{21} & a_{22} & a_{23} \\ b_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} (a_{11}+b_{11}) & a_{12} & a_{13} \\ (a_{21}+b_{21}) & a_{22} & a_{23} \\ (a_{31}+b_{31}) & a_{32} & a_{33} \end{vmatrix}$$

**44.** **Proof** Prove the property.

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right),$$

$a \neq 0, \quad b \neq 0, \quad c \neq 0$

**45.** Find each determinant.

(a) $\begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix}$ (b) $\begin{vmatrix} \sin\theta & 1 \\ 1 & \sin\theta \end{vmatrix}$

**46.** **CAPSTONE** Evaluate each determinant when $a = 1$, $b = 4$, and $c = -3$.

(a) $\begin{vmatrix} 0 & b & 0 \\ a & 0 & 0 \\ 0 & 0 & c \end{vmatrix}$ (b) $\begin{vmatrix} a & 0 & 1 \\ 0 & c & 0 \\ b & 0 & -16 \end{vmatrix}$

**47.** **Guided Proof** Prove Property 2 of Theorem 3.3: If $B$ is obtained from $A$ by adding a multiple of a row of $A$ to another row of $A$, then $\det(B) = \det(A)$.

Getting Started: To prove that the determinant of $B$ is equal to the determinant of $A$, you need to show that their respective cofactor expansions are equal.

(i) Begin by letting $B$ be the matrix obtained by adding $c$ times the $j$th row of $A$ to the $i$th row of $A$.

(ii) Find the determinant of $B$ by expanding in this $i$th row.

(iii) Distribute and then group the terms containing a coefficient of $c$ and those not containing a coefficient of $c$.

(iv) Show that the sum of the terms not containing a coefficient of $c$ is the determinant of $A$, and the sum of the terms containing a coefficient of $c$ is equal to 0.

**48.** **Guided Proof** Prove Property 3 of Theorem 3.3: If $B$ is obtained from $A$ by multiplying a row of $A$ by a nonzero constant $c$, then $\det(B) = c\det(A)$.

Getting Started: To prove that the determinant of $B$ is equal to $c$ times the determinant of $A$, you need to show that the determinant of $B$ is equal to $c$ times the cofactor expansion of the determinant of $A$.

(i) Begin by letting $B$ be the matrix obtained by multiplying $c$ times the $i$th row of $A$.

(ii) Find the determinant of $B$ by expanding in this $i$th row.

(iii) Factor out the common factor $c$.

(iv) Show that the result is $c$ times the determinant of $A$.

# 3.3 Properties of Determinants

- Find the determinant of a matrix product and a scalar multiple of a matrix.
- Find the determinant of an inverse matrix and recognize equivalent conditions for a nonsingular matrix.
- Find the determinant of the transpose of a matrix.

## MATRIX PRODUCTS AND SCALAR MULTIPLES

In this section, you will learn several important properties of determinants. You will begin by considering the determinant of the product of two matrices.

### EXAMPLE 1 The Determinant of a Matrix Product

Find $|A|$, $|B|$, and $|AB|$ for the matrices

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & -2 \\ 3 & 1 & -2 \end{bmatrix}.$$

**SOLUTION**

$|A|$ and $|B|$ have the values

$$|A| = \begin{vmatrix} 1 & -2 & 2 \\ 0 & 3 & 2 \\ 1 & 0 & 1 \end{vmatrix} = -7 \quad \text{and} \quad |B| = \begin{vmatrix} 2 & 0 & 1 \\ 0 & -1 & -2 \\ 3 & 1 & -2 \end{vmatrix} = 11.$$

The matrix product $AB$ is

$$AB = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & -2 \\ 3 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 8 & 4 & 1 \\ 6 & -1 & -10 \\ 5 & 1 & -1 \end{bmatrix}.$$

Finally, $|AB|$ has the value $|AB| = \begin{vmatrix} 8 & 4 & 1 \\ 6 & -1 & -10 \\ 5 & 1 & -1 \end{vmatrix} = -77.$

**REMARK**

Theorem 3.5 can be extended to include the product of any finite number of matrices. That is,

$$|A_1A_2A_3 \cdot \cdot \cdot A_k| = |A_1||A_2||A_3| \cdot \cdot \cdot |A_k|.$$

In Example 1, note that $|AB| = |A||B|$, or $-77 = (-7)(11)$. This is true in general, as indicated in the next theorem.

**THEOREM 3.5 Determinant of a Matrix Product**

If $A$ and $B$ are square matrices of order $n$, then $\det(AB) = \det(A)\det(B)$.

**PROOF**

To begin, observe that if $E$ is an elementary matrix, then, by Theorem 3.3, the next three statements are true. If you obtain $E$ from $I$ by interchanging two rows, then $|E| = -1$. If you obtain $E$ by multiplying a row of $I$ by a nonzero constant $c$, then $|E| = c$. If you obtain $E$ by adding a multiple of one row of $I$ to another row of $I$, then $|E| = 1$. Additionally, by Theorem 2.12, if $E$ results from performing an elementary row operation on $I$ and the same elementary row operation is performed on $B$, then the matrix $EB$ results. It follows that $|EB| = |E|\,|B|$.

This can be generalized to conclude that $|E_k \cdots E_2E_1B| = |E_k| \cdots |E_2|\,|E_1|\,|B|$, where $E_i$ is an elementary matrix. Now consider the matrix $AB$. If $A$ is *nonsingular*, then, by Theorem 2.14, it can be written as the product $A = E_k \cdots E_2E_1$, and

$$\begin{aligned}|AB| &= |E_k \cdots E_2E_1B| \\ &= |E_k| \cdots |E_2|\,|E_1|\,|B| = |E_k \cdots E_2E_1|\,|B| = |A|\,|B|.\end{aligned}$$

If $A$ is *singular,* then $A$ is row-equivalent to a matrix with an entire row of zeros. From Theorem 3.4, you can conclude that $|A| = 0$. Moreover, because $A$ is singular, it follows that $AB$ is also singular. (If $AB$ were nonsingular, then $A[B(AB)^{-1}] = I$ would imply that $A$ is nonsingular.) So, $|AB| = 0$, and you can conclude that $|AB| = |A||B|$.

The next theorem shows the relationship between $|A|$ and $|cA|$.

**THEOREM 3.6 Determinant of a Scalar Multiple of a Matrix**

If $A$ is a square matrix of order $n$ and $c$ is a scalar, then the determinant of $cA$ is

$$\det(cA) = c^n \det(A).$$

**PROOF**

This formula can be proven by repeated applications of Property 3 of Theorem 3.3. Factor the scalar $c$ out of each of the $n$ rows of $|cA|$ to obtain

$$|cA| = c^n|A|.$$

**EXAMPLE 2 The Determinant of a Scalar Multiple of a Matrix**

Find the determinant of the matrix.

$$A = \begin{bmatrix} 10 & -20 & 40 \\ 30 & 0 & 50 \\ -20 & -30 & 10 \end{bmatrix}$$

**SOLUTION**

Because

$$A = 10\begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & 5 \\ -2 & -3 & 1 \end{bmatrix} \quad \text{and} \quad \begin{vmatrix} 1 & -2 & 4 \\ 3 & 0 & 5 \\ -2 & -3 & 1 \end{vmatrix} = 5,$$

apply Theorem 3.6 to conclude that

$$|A| = 10^3\begin{vmatrix} 1 & -2 & 4 \\ 3 & 0 & 5 \\ -2 & -3 & 1 \end{vmatrix} = 1000(5) = 5000.$$

Theorems 3.5 and 3.6 give formulas for the determinants of the product of two matrices and a scalar multiple of a matrix. These theorems do not, however, give a formula for the determinant of the *sum* of two matrices. The sum of the determinants of two matrices usually does not equal the determinant of their sum. That is, in general, $|A| + |B| \neq |A + B|$. For instance, if

$$A = \begin{bmatrix} 6 & 2 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 7 \\ 0 & -1 \end{bmatrix}$$

then $|A| = 2$ and $|B| = -3$, but $A + B = \begin{bmatrix} 9 & 9 \\ 2 & 0 \end{bmatrix}$ and $|A + B| = -18$.

## DETERMINANTS AND THE INVERSE OF A MATRIX

It can be difficult to tell simply by inspection whether or not a matrix has an inverse. Can you tell which of the following two matrices is invertible?

$$A = \begin{bmatrix} 0 & 2 & -1 \\ 3 & -2 & 1 \\ 3 & 2 & -1 \end{bmatrix} \quad \text{or} \quad B = \begin{bmatrix} 0 & 2 & -1 \\ 3 & -2 & 1 \\ 3 & 2 & 1 \end{bmatrix}$$

The next theorem shows that determinants are useful for classifying square matrices as invertible or noninvertible.

### THEOREM 3.7 Determinant of an Invertible Matrix

A square matrix $A$ is invertible (nonsingular) if and only if $\det(A) \neq 0$.

### PROOF

To prove the theorem in one direction, assume $A$ is invertible. Then $AA^{-1} = I$, and by Theorem 3.5 you can write $|A||A^{-1}| = |I|$. Now, because $|I| = 1$, you know that neither determinant on the left is zero. Specifically, $|A| \neq 0$.

To prove the theorem in the other direction, assume the determinant of $A$ is nonzero. Then, using Gauss-Jordan elimination, find a matrix $B$, in reduced row-echelon form, that is row-equivalent to $A$. Because $B$ is in reduced row-echelon form, it must be the identity matrix $I$ or it must have at least one row that consists entirely of zeros. But if $B$ has a row of all zeros, then by Theorem 3.4 you know that $|B| = 0$, which would imply that $|A| = 0$. Because you assumed that $|A|$ is nonzero, you can conclude that $B = I$. $A$ is, therefore, row-equivalent to the identity matrix, and by Theorem 2.15 you know that $A$ is invertible.

**DISCOVERY**

Let

$$A = \begin{bmatrix} 6 & 4 & 1 \\ 0 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}.$$

1. Use a software program or a graphing utility to find $A^{-1}$.
2. Compare $\det(A^{-1})$ with $\det(A)$.
3. Make a conjecture about the determinant of the inverse of a matrix.

### EXAMPLE 3 Classifying Square Matrices as Singular or Nonsingular

Which of the matrices has an inverse?

**a.** $\begin{bmatrix} 0 & 2 & -1 \\ 3 & -2 & 1 \\ 3 & 2 & -1 \end{bmatrix}$ **b.** $\begin{bmatrix} 0 & 2 & -1 \\ 3 & -2 & 1 \\ 3 & 2 & 1 \end{bmatrix}$

### SOLUTION

**a.** $\begin{vmatrix} 0 & 2 & -1 \\ 3 & -2 & 1 \\ 3 & 2 & -1 \end{vmatrix} = 0$

so you can conclude that this matrix has no inverse (it is singular).

**b.** $\begin{vmatrix} 0 & 2 & -1 \\ 3 & -2 & 1 \\ 3 & 2 & 1 \end{vmatrix} = -12 \neq 0$

so you can conclude that this matrix has an inverse (it is nonsingular).

The next theorem provides a way to find the determinant of the inverse of a matrix.

### THEOREM 3.8 Determinant of an Inverse Matrix

If $A$ is an $n \times n$ invertible matrix, then $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

**PROOF**

Because $A$ is invertible, $AA^{-1} = I$, and you can apply Theorem 3.5 to conclude that $|A||A^{-1}| = |I| = 1$. Because $A$ is invertible, you also know that $|A| \neq 0$, and you can divide each side by $|A|$ to obtain

$$|A^{-1}| = \frac{1}{|A|}.$$

## EXAMPLE 4 The Determinant of the Inverse of a Matrix

Find $|A^{-1}|$ for the matrix

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 2 \\ 2 & 1 & 0 \end{bmatrix}.$$

**REMARK**

The inverse of $A$ is

$$A^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{3}{4} & \frac{3}{4} \\ 1 & -\frac{3}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}.$$

Try evaluating the determinant of this matrix directly. Then compare your answer with that obtained in Example 4.

**SOLUTION**

One way to solve this problem is to find $A^{-1}$ and then evaluate its determinant. It is easier, however, to apply Theorem 3.8, as follows. Find the determinant of $A$,

$$|A| = \begin{vmatrix} 1 & 0 & 3 \\ 0 & -1 & 2 \\ 2 & 1 & 0 \end{vmatrix} = 4$$

and then use the formula $|A^{-1}| = 1/|A|$ to conclude that $|A^{-1}| = \frac{1}{4}$.

Note that Theorem 3.7 provides another equivalent condition that can be added to the list in Theorem 2.15. The following summarizes all six conditions.

**REMARK**

In Section 3.2, you saw that a square matrix $A$ has a determinant of zero when $A$ is row-equivalent to a matrix that has at least one row consisting entirely of zeros. The validity of this statement follows from the equivalence of Statements 4 and 6.

### Equivalent Conditions for a Nonsingular Matrix

If $A$ is an $n \times n$ matrix, then the following statements are equivalent.

1. $A$ is invertible.
2. $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $n \times 1$ column matrix $\mathbf{b}$.
3. $A\mathbf{x} = O$ has only the trivial solution.
4. $A$ is row-equivalent to $I_n$.
5. $A$ can be written as the product of elementary matrices.
6. $\det(A) \neq 0$

## EXAMPLE 5 Systems of Linear Equations

Which of the systems has a unique solution?

**a.**
$$\begin{aligned} 2x_2 - x_3 &= -1 \\ 3x_1 - 2x_2 + x_3 &= 4 \\ 3x_1 + 2x_2 - x_3 &= -4 \end{aligned}$$

**b.**
$$\begin{aligned} 2x_2 - x_3 &= -1 \\ 3x_1 - 2x_2 + x_3 &= 4 \\ 3x_1 + 2x_2 + x_3 &= -4 \end{aligned}$$

**SOLUTION**

From Example 3, you know that the coefficient matrices for these two systems have the following determinants.

**a.** $\begin{vmatrix} 0 & 2 & -1 \\ 3 & -2 & 1 \\ 3 & 2 & -1 \end{vmatrix} = 0$ **b.** $\begin{vmatrix} 0 & 2 & -1 \\ 3 & -2 & 1 \\ 3 & 2 & 1 \end{vmatrix} = -12$

Using the preceding list of equivalent conditions, you can conclude that only the second system has a unique solution.

## DETERMINANTS AND THE TRANSPOSE OF A MATRIX

The next theorem tells you that the determinant of the transpose of a square matrix is equal to the determinant of the original matrix. This theorem can be proven using mathematical induction and Theorem 3.1, which states that a determinant can be evaluated using cofactor expansion in a row or a column. The details of the proof are left to you. (See Exercise 64.)

**THEOREM 3.9 Determinant of a Transpose**

If $A$ is a square matrix, then

$$\det(A) = \det(A^T).$$

### EXAMPLE 6 The Determinant of a Transpose

Show that $|A| = |A^T|$ for the matrix below.

$$A = \begin{bmatrix} 3 & 1 & -2 \\ 2 & 0 & 0 \\ -4 & -1 & 5 \end{bmatrix}$$

**SOLUTION**

To find the determinant of $A$, expand by cofactors in the second *row* to obtain

$$\begin{aligned} |A| &= 2(-1)^3 \begin{vmatrix} 1 & -2 \\ -1 & 5 \end{vmatrix} \\ &= (2)(-1)(3) \\ &= -6. \end{aligned}$$

To find the determinant of

$$A^T = \begin{bmatrix} 3 & 2 & -4 \\ 1 & 0 & -1 \\ -2 & 0 & 5 \end{bmatrix}$$

expand by cofactors in the second *column* to obtain

$$\begin{aligned} |A^T| &= 2(-1)^3 \begin{vmatrix} 1 & -1 \\ -2 & 5 \end{vmatrix} \\ &= (2)(-1)(3) \\ &= -6. \end{aligned}$$

So, $|A| = |A^T|$.

**LINEAR ALGEBRA APPLIED**

Systems of linear differential equations often arise in engineering and control theory. For a function $f(t)$ that is defined for all positive values of $t$, the **Laplace transform** of $f(t)$ is given by

$$F(s) = \int_0^{\infty} e^{-st} f(t)\,dt.$$

Laplace transforms and Cramer's Rule, which uses determinants to solve a system of linear equations, can often be used to solve a system of differential equations. You will study Cramer's Rule in the next section.

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# 3.3 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

**The Determinant of a Matrix Product** In Exercises 1–6, find (a) $|A|$, (b) $|B|$, (c) $AB$, and (d) $|AB|$. Then verify that $|A||B| = |AB|$.

**1.** $A = \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$

**2.** $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 \\ 3 & 0 \end{bmatrix}$

**3.** $A = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

**4.** $A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 2 \\ 3 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -1 & 4 \\ 0 & 1 & 3 \\ 3 & -2 & 1 \end{bmatrix}$

**5.** $A = \begin{bmatrix} 2 & 0 & 1 & 1 \\ 1 & -1 & 0 & 1 \\ 2 & 3 & 1 & 0 \\ 1 & 2 & 3 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & 2 \\ 1 & 1 & -1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix}$

**6.** $A = \begin{bmatrix} 3 & 2 & 4 & 0 \\ 1 & -1 & 2 & 1 \\ 0 & 0 & 3 & 1 \\ -1 & 1 & 1 & 0 \end{bmatrix}$,

$B = \begin{bmatrix} 4 & 2 & -1 & 0 \\ 1 & 1 & 2 & -1 \\ 0 & 0 & 2 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix}$

**The Determinant of a Scalar Multiple of a Matrix** In Exercises 7–12, use the fact that $|cA| = c^n|A|$ to evaluate the determinant of the $n \times n$ matrix.

**7.** $A = \begin{bmatrix} 4 & 2 \\ 6 & -8 \end{bmatrix}$

**8.** $A = \begin{bmatrix} 5 & 15 \\ 10 & -20 \end{bmatrix}$

**9.** $A = \begin{bmatrix} -3 & 6 & 9 \\ 6 & 9 & 12 \\ 9 & 12 & 15 \end{bmatrix}$

**10.** $A = \begin{bmatrix} 4 & 16 & 0 \\ 12 & -8 & 8 \\ 16 & 20 & -4 \end{bmatrix}$

**11.** $A = \begin{bmatrix} 2 & -4 & 6 \\ -4 & 6 & -8 \\ 6 & -8 & 10 \end{bmatrix}$

**12.** $A = \begin{bmatrix} 40 & 25 & 10 \\ 30 & 5 & 20 \\ 15 & 35 & 45 \end{bmatrix}$

**The Determinant of a Matrix Sum** In Exercises 13–16, find (a) $|A|$, (b) $|B|$, and (c) $|A + B|$. Then verify that $|A| + |B| \neq |A + B|$.

**13.** $A = \begin{bmatrix} -1 & 1 \\ 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 \\ -2 & 0 \end{bmatrix}$

**14.** $A = \begin{bmatrix} 1 & -2 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -2 \\ 0 & 0 \end{bmatrix}$

**15.** $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$

**16.** $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & -1 & 0 \\ 2 & 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 & -1 \\ 2 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

**Classifying Matrices as Singular or Nonsingular** In Exercises 17–24, use a determinant to decide whether the matrix is singular or nonsingular.

**17.** $\begin{bmatrix} 5 & 4 \\ 10 & 8 \end{bmatrix}$

**18.** $\begin{bmatrix} 3 & -6 \\ 4 & 2 \end{bmatrix}$

**19.** $\begin{bmatrix} 14 & 5 & 7 \\ -2 & 0 & 3 \\ 1 & -5 & -10 \end{bmatrix}$

**20.** $\begin{bmatrix} 1 & 0 & 4 \\ 0 & 6 & 3 \\ 2 & -1 & 4 \end{bmatrix}$

**21.** $\begin{bmatrix} \frac{1}{2} & \frac{3}{2} & 2 \\ \frac{2}{3} & -\frac{1}{3} & 0 \\ 1 & 1 & 1 \end{bmatrix}$

**22.** $\begin{bmatrix} 2 & -\frac{1}{2} & 8 \\ 1 & -\frac{1}{4} & 4 \\ -\frac{5}{2} & \frac{3}{2} & 8 \end{bmatrix}$

**23.** $\begin{bmatrix} 1 & 0 & -8 & 2 \\ 0 & 8 & -1 & 10 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

**24.** $\begin{bmatrix} 0.8 & 0.2 & -0.6 & 0.1 \\ -1.2 & 0.6 & 0.6 & 0 \\ 0.7 & -0.3 & 0.1 & 0 \\ 0.2 & -0.3 & 0.6 & 0 \end{bmatrix}$

**The Determinant of the Inverse of a Matrix** In Exercises 25–30, find $|A^{-1}|$. Begin by finding $A^{-1}$, and then evaluate its determinant. Verify your result by finding $|A|$ and then applying the formula from Theorem 3.8, $|A^{-1}| = \dfrac{1}{|A|}$.

**25.** $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$

**26.** $A = \begin{bmatrix} 1 & -2 \\ 2 & 2 \end{bmatrix}$

**27.** $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 1 & -2 & 2 \end{bmatrix}$

**28.** $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 2 \\ 1 & -2 & 3 \end{bmatrix}$

**29.** $A = \begin{bmatrix} 1 & 0 & -1 & 3 \\ 1 & 0 & 3 & -2 \\ 2 & 0 & 2 & -1 \\ 1 & -3 & 1 & 2 \end{bmatrix}$

**30.** $A = \begin{bmatrix} 0 & 1 & 0 & 3 \\ 1 & -2 & -3 & 1 \\ 0 & 0 & 2 & -2 \\ 1 & -2 & -4 & 1 \end{bmatrix}$

**System of Linear Equations** **In Exercises 31–36, use the determinant of the coefficient matrix to determine whether the system of linear equations has a unique solution.**

**31.** $\begin{aligned} x_1 - 3x_2 &= 2 \\ 2x_1 + x_2 &= 1 \end{aligned}$

**32.** $\begin{aligned} 3x_1 - 4x_2 &= 2 \\ \tfrac{2}{3}x_1 - \tfrac{8}{9}x_2 &= 1 \end{aligned}$

**33.** $\begin{aligned} x_1 - x_2 + x_3 &= 4 \\ 2x_1 - x_2 + x_3 &= 6 \\ 3x_1 - 2x_2 + 2x_3 &= 0 \end{aligned}$

**34.** $\begin{aligned} x_1 + x_2 - x_3 &= 4 \\ 2x_1 - x_2 + x_3 &= 6 \\ 3x_1 - 2x_2 + 2x_3 &= 0 \end{aligned}$

**35.** $\begin{aligned} 2x_1 + x_2 + 5x_3 + x_4 &= 5 \\ x_1 + x_2 - 3x_3 - 4x_4 &= -1 \\ 2x_1 + 2x_2 + 2x_3 - 3x_4 &= 2 \\ x_1 + 5x_2 - 6x_3 \phantom{- 3x_4} &= 3 \end{aligned}$

**36.** $\begin{aligned} x_1 - x_2 - x_3 - x_4 &= 0 \\ x_1 + x_2 - x_3 - x_4 &= 0 \\ x_1 + x_2 + x_3 - x_4 &= 0 \\ x_1 + x_2 + x_3 + x_4 &= 6 \end{aligned}$

**Finding Determinants** **In Exercises 37–44, find (a) $|A^T|$, (b) $|A^2|$, (c) $|AA^T|$, (d) $|2A|$, and (e) $|A^{-1}|$.**

**37.** $A = \begin{bmatrix} 6 & -11 \\ 4 & -5 \end{bmatrix}$

**38.** $A = \begin{bmatrix} -4 & 10 \\ 5 & 6 \end{bmatrix}$

**39.** $A = \begin{bmatrix} 5 & 0 & 0 \\ 1 & -3 & 0 \\ 0 & -1 & 2 \end{bmatrix}$

**40.** $A = \begin{bmatrix} 1 & 5 & 4 \\ 0 & -6 & 2 \\ 0 & 0 & -3 \end{bmatrix}$

**41.** $A = \begin{bmatrix} 2 & 0 & 5 \\ 4 & -1 & 6 \\ 3 & 2 & 1 \end{bmatrix}$

**42.** $A = \begin{bmatrix} 4 & 1 & 9 \\ -1 & 0 & -2 \\ -3 & 3 & 0 \end{bmatrix}$

**43.** $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

**44.** $A = \begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

**Finding Determinants** **In Exercises 45–50, use a software program or a graphing utility with matrix capabilities to find (a) $|A|$, (b) $|A^T|$, (c) $|A^2|$, (d) $|2A|$, and (e) $|A^{-1}|$.**

**45.** $A = \begin{bmatrix} 4 & 2 \\ -1 & 5 \end{bmatrix}$

**46.** $A = \begin{bmatrix} -2 & 4 \\ 6 & 8 \end{bmatrix}$

**47.** $A = \begin{bmatrix} 3 & 1 & -2 \\ 2 & -1 & 3 \\ -3 & 1 & 2 \end{bmatrix}$

**48.** $A = \begin{bmatrix} \frac{3}{4} & \frac{2}{3} & -\frac{1}{4} \\ \frac{2}{3} & 1 & \frac{1}{3} \\ -\frac{1}{4} & \frac{1}{3} & \frac{3}{4} \end{bmatrix}$

**49.** $A = \begin{bmatrix} 4 & -2 & 1 & 5 \\ 3 & 8 & 2 & -1 \\ 6 & 8 & 9 & 2 \\ 2 & 3 & -1 & 0 \end{bmatrix}$

**50.** $A = \begin{bmatrix} 6 & 5 & 1 & -1 \\ -2 & 4 & 3 & 5 \\ 6 & 1 & -4 & -2 \\ 2 & 2 & 1 & 3 \end{bmatrix}$

**51.** Let $A$ and $B$ be square matrices of order 4 such that $|A| = -5$ and $|B| = 3$. Find (a) $|A^2|$, (b) $|B^2|$, (c) $|A^3|$, and (d) $|B^4|$.

**52.** **CAPSTONE** Let $A$ and $B$ be square matrices of order 3 such that $|A| = 4$ and $|B| = 5$.

(a) Find $|AB|$.

(b) Find $|2A|$.

(c) Are $A$ and $B$ singular or nonsingular? Explain.

(d) If $A$ and $B$ are nonsingular, find $|A^{-1}|$ and $|B^{-1}|$.

(e) Find $|(AB)^T|$.

**Singular Matrices** **In Exercises 53–56, find the value(s) of $k$ such that $A$ is singular.**

**53.** $A = \begin{bmatrix} k-1 & 3 \\ 2 & k-2 \end{bmatrix}$

**54.** $A = \begin{bmatrix} k-1 & 2 \\ 2 & k+2 \end{bmatrix}$

**55.** $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & 0 \\ 4 & 2 & k \end{bmatrix}$

**56.** $A = \begin{bmatrix} 1 & k & 2 \\ -2 & 0 & -k \\ 3 & 1 & -4 \end{bmatrix}$

**57. Proof** Let $A$ and $B$ be $n \times n$ matrices such that $AB = I$. Prove that $|A| \neq 0$ and $|B| \neq 0$.

**58. Proof** Let $A$ and $B$ be $n \times n$ matrices such that $AB$ is singular. Prove that either $A$ or $B$ is singular.

**59.** Find two $2 \times 2$ matrices such that

$$|A| + |B| = |A + B|.$$

**60.** Verify the equation.

$$\begin{vmatrix} a+b & a & a \\ a & a+b & a \\ a & a & a+b \end{vmatrix} = b^2(3a + b)$$

**61.** Let $A$ be an $n \times n$ matrix in which the entries of each row add up to zero. Find $|A|$.

**62.** Illustrate the result of Exercise 61 with the matrix

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -3 & 1 & 2 \\ 0 & -2 & 2 \end{bmatrix}.$$

**63. Guided Proof** Prove that the determinant of an invertible matrix $A$ is equal to $\pm 1$ when all of the entries of $A$ and $A^{-1}$ are integers.

Getting Started: Denote $\det(A)$ as $x$ and $\det(A^{-1})$ as $y$. Note that $x$ and $y$ are real numbers. To prove that $\det(A)$ is equal to $\pm 1$, you must show that both $x$ and $y$ are integers such that their product $xy$ is equal to 1.

(i) Use the property for the determinant of a matrix product to show that $xy = 1$.

(ii) Use the definition of a determinant and the fact that the entries of $A$ and $A^{-1}$ are integers to show that both $x = \det(A)$ and $y = \det(A^{-1})$ are integers.

(iii) Conclude that $x = \det(A)$ must be either 1 or $-1$ because these are the only integer solutions to the equation $xy = 1$.

**64. Guided Proof** Prove Theorem 3.9: If $A$ is a square matrix, then $\det(A) = \det(A^T)$.

Getting Started: To prove that the determinants of $A$ and $A^T$ are equal, you need to show that their cofactor expansions are equal. Because the cofactors are $\pm$ determinants of smaller matrices, you need to use mathematical induction.

(i) Initial step for induction: If $A$ is of order 1, then

$A = [a_{11}] = A^T$

so

$\det(A) = \det(A^T) = a_{11}$.

(ii) Assume the inductive hypothesis holds for all matrices of order $n - 1$. Let $A$ be a square matrix of order $n$. Write an expression for the determinant of $A$ by expanding in the first row.

(iii) Write an expression for the determinant of $A^T$ by expanding in the first column.

(iv) Compare the expansions in (ii) and (iii). The entries of the first row of $A$ are the same as the entries of the first column of $A^T$. Compare cofactors (these are the $\pm$ determinants of smaller matrices that are transposes of one another) and use the inductive hypothesis to conclude that they are equal as well.

**True or False?** **In Exercises 65 and 66, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows that the statement is not true in all cases or cite an appropriate statement from the text.**

**65.** (a) If $A$ is an $n \times n$ matrix and $c$ is a nonzero scalar, then the determinant of the matrix $cA$ is given by $nc \cdot \det(A)$.

(b) If $A$ is an invertible matrix, then the determinant of $A^{-1}$ is equal to the reciprocal of the determinant of $A$.

(c) If $A$ is an invertible $n \times n$ matrix, then $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$.

**66.** (a) In general, the determinant of the sum of two matrices equals the sum of the determinants of the matrices.

(b) If $A$ and $B$ are square matrices of order $n$, and $\det(A) = \det(B)$, then $\det(AB) = \det(A^2)$.

(c) If the determinant of an $n \times n$ matrix $A$ is nonzero, then $A\mathbf{x} = O$ has only the trivial solution.

**67. Writing** Let $A$ and $P$ be $n \times n$ matrices, where $P$ is invertible. Does $P^{-1}AP = A$? Illustrate your conclusion with appropriate examples. What can you say about the two determinants $|P^{-1}AP|$ and $|A|$?

**68. Writing** Let $A$ be an $n \times n$ nonzero matrix satisfying $A^{10} = O$. Explain why $A$ must be singular. What properties of determinants are you using in your argument?

**69. Proof** A square matrix is called **skew-symmetric** when $A^T = -A$. Prove that if $A$ is an $n \times n$ skew-symmetric matrix, then $|A| = (-1)^n|A|$.

**70. Proof** Let $A$ be a skew-symmetric matrix of odd order. Use the result of Exercise 69 to prove that $|A| = 0$.

**Orthogonal Matrices** **In Exercises 71–76, determine whether the matrix is orthogonal. An invertible square matrix $A$ is orthogonal when $A^{-1} = A^T$.**

**71.** $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ **72.** $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

**73.** $\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}$ **74.** $\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ -1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

**75.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ **76.** $\begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$

**77. Proof** Prove that if $A$ is an orthogonal matrix, then $|A| = \pm 1$.

**Orthogonal Matrices** **In Exercises 78 and 79, use a graphing utility with matrix capabilities to determine whether $A$ is orthogonal. To test for orthogonality, find (a) $A^{-1}$, (b) $A^T$, and (c) $|A|$, and verify that $A^{-1} = A^T$ and $|A| = \pm 1$.**

**78.** $A = \begin{bmatrix} \frac{3}{5} & 0 & -\frac{4}{5} \\ 0 & 1 & 0 \\ \frac{4}{5} & 0 & \frac{3}{5} \end{bmatrix}$ **79.** $A = \begin{bmatrix} \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix}$

**80. Proof** If $A$ is an idempotent matrix ($A^2 = A$), then prove that the determinant of $A$ is either 0 or 1.

**81. Proof** Let $S$ be an $n \times n$ singular matrix. Prove that for any $n \times n$ matrix $B$, the matrix $SB$ is also singular.

# 3.4 Applications of Determinants

- Find the adjoint of a matrix and use it to find the inverse of the matrix.
- Use Cramer's Rule to solve a system of *n* linear equations in *n* variables.
- Use determinants to find area, volume, and the equations of lines and planes.

## THE ADJOINT OF A MATRIX

So far in this chapter, you have studied procedures for evaluating, and properties of, determinants. In this section, you will study an explicit formula for the inverse of a nonsingular matrix and use this formula to derive a theorem known as Cramer's Rule. You will then solve several applications of determinants.

Recall from Section 3.1 that the cofactor $C_{ij}$ of a square matrix $A$ is defined as $(-1)^{i+j}$ times the determinant of the matrix obtained by deleting the $i$th row and the $j$th column of $A$. The **matrix of cofactors** of $A$ has the form

$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}.$$

The transpose of this matrix is called the **adjoint** of $A$ and is denoted by adj$(A)$. That is,

$$\operatorname{adj}(A) = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}.$$

### EXAMPLE 1 Finding the Adjoint of a Square Matrix

Find the adjoint of $A = \begin{bmatrix} -1 & 3 & 2 \\ 0 & -2 & 1 \\ 1 & 0 & -2 \end{bmatrix}$.

**SOLUTION**

The cofactor $C_{11}$ is given by

$$\begin{bmatrix} -1 & 3 & 2 \\ 0 & -2 & 1 \\ 1 & 0 & -2 \end{bmatrix} \longrightarrow C_{11} = (-1)^2 \begin{vmatrix} -2 & 1 \\ 0 & -2 \end{vmatrix} = 4.$$

Continuing this process produces the following matrix of cofactors of $A$.

$$\begin{bmatrix} 4 & 1 & 2 \\ 6 & 0 & 3 \\ 7 & 1 & 2 \end{bmatrix}$$

The transpose of this matrix is the adjoint of $A$. That is, $\operatorname{adj}(A) = \begin{bmatrix} 4 & 6 & 7 \\ 1 & 0 & 1 \\ 2 & 3 & 2 \end{bmatrix}$.

The adjoint of a matrix $A$ is useful for finding the inverse of $A$, as indicated in the next theorem.

**REMARK**

Theorem 3.10 is not particularly efficient for calculating inverses. The Gauss-Jordan elimination method discussed in Section 2.3 is much better. Theorem 3.10 is theoretically useful, however, because it provides a concise formula for the inverse of a matrix.

**THEOREM 3.10 The Inverse of a Matrix Given by Its Adjoint**

If $A$ is an $n \times n$ invertible matrix, then $A^{-1} = \dfrac{1}{\det(A)}\operatorname{adj}(A)$.

**PROOF**

Begin by proving that the product of $A$ and its adjoint is equal to the product of the determinant of $A$ and $I_n$. Consider the product

$$A[\operatorname{adj}(A)] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{j1} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{j2} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{jn} & \cdots & C_{nn} \end{bmatrix}.$$

The entry in the $i$th row and $j$th column of this product is

$$a_{i1}C_{j1} + a_{i2}C_{j2} + \cdots + a_{in}C_{jn}.$$

If $i = j$, then this sum is simply the cofactor expansion of $A$ in its $i$th row, which means that the sum is the determinant of $A$. On the other hand, if $i \neq j$, then the sum is zero. (Try verifying this.)

$$A[\operatorname{adj}(A)] = \begin{bmatrix} \det(A) & 0 & \cdots & 0 \\ 0 & \det(A) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \det(A) \end{bmatrix} = \det(A)I$$

Because $A$ is invertible, $\det(A) \neq 0$ and you can write

$$\frac{1}{\det(A)}A[\operatorname{adj}(A)] = I \quad \text{or} \quad A\left[\frac{1}{\det(A)}\operatorname{adj}(A)\right] = I.$$

By Theorem 2.7 and the definition of the inverse of a matrix, it follows that

$$\frac{1}{\det(A)}\operatorname{adj}(A) = A^{-1}.$$

**REMARK**

If $A$ is a $2 \times 2$ matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then the adjoint of $A$ is simply

$$\operatorname{adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Moreover, if $A$ is invertible, then from Theorem 3.10 you have

$$A^{-1} = \frac{1}{|A|}\operatorname{adj}(A) = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

which agrees with the result in Section 2.3.

## EXAMPLE 2 Using the Adjoint of a Matrix to Find Its Inverse

Use the adjoint of $A$ to find $A^{-1}$, where $A = \begin{bmatrix} -1 & 3 & 2 \\ 0 & -2 & 1 \\ 1 & 0 & -2 \end{bmatrix}$.

**SOLUTION**

The determinant of this matrix is 3. Using the adjoint of $A$ (found in Example 1), the inverse of $A$ is

$$A^{-1} = \frac{1}{|A|}\operatorname{adj}(A) = \frac{1}{3}\begin{bmatrix} 4 & 6 & 7 \\ 1 & 0 & 1 \\ 2 & 3 & 2 \end{bmatrix} = \begin{bmatrix} \frac{4}{3} & 2 & \frac{7}{3} \\ \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{2}{3} & 1 & \frac{2}{3} \end{bmatrix}.$$

Check that this matrix is the inverse of $A$ by showing that $AA^{-1} = I = A^{-1}A$.

## CRAMER'S RULE

Cramer's Rule, named after Gabriel Cramer (1704–1752), uses determinants to solve a system of $n$ linear equations in $n$ variables. This rule applies only to systems with unique solutions. To see how Cramer's Rule works, take another look at the solution described at the beginning of Section 3.1. There, it was pointed out that the system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}$$

has the solution

$$x_1 = \frac{b_1a_{22} - b_2a_{12}}{a_{11}a_{22} - a_{21}a_{12}} \quad \text{and} \quad x_2 = \frac{b_2a_{11} - b_1a_{21}}{a_{11}a_{22} - a_{21}a_{12}}$$

when $a_{11}a_{22} - a_{21}a_{12} \neq 0$. Each numerator and denominator in this solution can be represented as a determinant, as follows.

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}, \quad x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}, \quad a_{11}a_{22} - a_{21}a_{12} \neq 0$$

The denominator for $x_1$ and $x_2$ is simply the determinant of the coefficient matrix $A$ of the original system. The numerators for $x_1$ and $x_2$ are formed by using the column of constants as replacements for the coefficients of $x_1$ and $x_2$ in $|A|$. These two determinants are denoted by $|A_1|$ and $|A_2|$, as follows.

$$|A_1| = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix} \quad \text{and} \quad |A_2| = \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}$$

You have $x_1 = \dfrac{|A_1|}{|A|}$ and $x_2 = \dfrac{|A_2|}{|A|}$. This determinant form of the solution is called **Cramer's Rule.**

### EXAMPLE 3 Using Cramer's Rule

Use Cramer's Rule to solve the system of linear equations.

$$\begin{aligned} 4x_1 - 2x_2 &= 10 \\ 3x_1 - 5x_2 &= 11 \end{aligned}$$

#### SOLUTION

First find the determinant of the coefficient matrix.

$$|A| = \begin{vmatrix} 4 & -2 \\ 3 & -5 \end{vmatrix} = -14$$

Because $|A| \neq 0$, you know the system has a unique solution, and applying Cramer's Rule produces

$$x_1 = \frac{|A_1|}{|A|} = \frac{\begin{vmatrix} 10 & -2 \\ 11 & -5 \end{vmatrix}}{-14} = \frac{-28}{-14} = 2$$

and

$$x_2 = \frac{|A_2|}{|A|} = \frac{\begin{vmatrix} 4 & 10 \\ 3 & 11 \end{vmatrix}}{-14} = \frac{14}{-14} = -1.$$

The solution is $x_1 = 2$ and $x_2 = -1$.

Cramer's Rule generalizes easily to systems of $n$ linear equations in $n$ variables. The value of each variable is given as the quotient of two determinants. The denominator is the determinant of the coefficient matrix, and the numerator is the determinant of the matrix formed by replacing the column corresponding to the variable being solved for with the column representing the constants. For instance, the solution for $x_3$ in the system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned} \quad \text{is} \quad x_3 = \frac{|A_3|}{|A|} = \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}.$$

**THEOREM 3.11 Cramer's Rule**

If a system of $n$ linear equations in $n$ variables has a coefficient matrix $A$ with a nonzero determinant $|A|$, then the solution of the system is

$$x_1 = \frac{\det(A_1)}{\det(A)}, \quad x_2 = \frac{\det(A_2)}{\det(A)}, \quad . . . , \quad x_n = \frac{\det(A_n)}{\det(A)}$$

where the $i$th column of $A_i$ is the column of constants in the system of equations.

**PROOF**

Let the system be represented by $AX = B$. Because $|A|$ is nonzero, you can write

$$X = A^{-1}B = \frac{1}{|A|}\operatorname{adj}(A)B = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

If the entries of $B$ are $b_1, b_2, . . . , b_n$, then $x_i = \dfrac{1}{|A|}(b_1C_{1i} + b_2C_{2i} + \cdot \cdot \cdot + b_nC_{ni})$, but the sum (in parentheses) is precisely the cofactor expansion of $A_i$, which means that $x_i = |A_i|/|A|$, and the proof is complete.

## EXAMPLE 4 Using Cramer's Rule

Use Cramer's Rule to solve the system of linear equations for $x$.

$$\begin{aligned} -x + 2y - 3z &= 1 \\ 2x \qquad + z &= 0 \\ 3x - 4y + 4z &= 2 \end{aligned}$$

**SOLUTION**

The determinant of the coefficient matrix is $|A| = \begin{vmatrix} -1 & 2 & -3 \\ 2 & 0 & 1 \\ 3 & -4 & 4 \end{vmatrix} = 10.$

Because $|A| \neq 0$, you know that the solution is unique, so apply Cramer's Rule to solve for $x$, as follows.

$$x = \frac{\begin{vmatrix} 1 & 2 & -3 \\ 0 & 0 & 1 \\ 2 & -4 & 4 \end{vmatrix}}{10} = \frac{(1)(-1)^5\begin{vmatrix} 1 & 2 \\ 2 & -4 \end{vmatrix}}{10} = \frac{(1)(-1)(-8)}{10} = \frac{4}{5}$$

**REMARK**

Try applying Cramer's Rule to solve for $y$ and $z$. You will see that the solution is $y = -\frac{3}{2}$ and $z = -\frac{8}{5}$.

## AREA, VOLUME, AND EQUATIONS OF LINES AND PLANES

Determinants have many applications in analytic geometry. One application is in finding the area of a triangle in the *xy*-plane.

### Area of a Triangle in the *xy*-Plane

The area of a triangle with vertices

$$(x_1, y_1),\ (x_2, y_2),\ \text{and}\ (x_3, y_3)$$

is

$$\text{Area} = \pm\tfrac{1}{2}\det\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}$$

where the sign $(\pm)$ is chosen to give a positive area.

**PROOF**

Prove the case for $y_i > 0$. Assume that $x_1 \le x_3 \le x_2$ and that $(x_3, y_3)$ lies above the line segment connecting $(x_1, y_1)$ and $(x_2, y_2)$, as shown in Figure 3.1. Consider the three trapezoids whose vertices are

Trapezoid 1: $(x_1, 0), (x_1, y_1), (x_3, y_3), (x_3, 0)$
Trapezoid 2: $(x_3, 0), (x_3, y_3), (x_2, y_2), (x_2, 0)$
Trapezoid 3: $(x_1, 0), (x_1, y_1), (x_2, y_2), (x_2, 0)$.

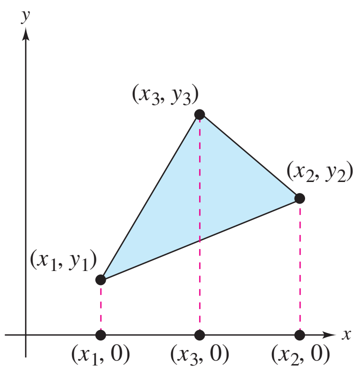



**Figure 3.1**

The area of the triangle is equal to the sum of the areas of the first two trapezoids minus the area of the third trapezoid. So,

$$\begin{aligned}\text{Area} &= \tfrac{1}{2}(y_1 + y_3)(x_3 - x_1) + \tfrac{1}{2}(y_3 + y_2)(x_2 - x_3) - \tfrac{1}{2}(y_1 + y_2)(x_2 - x_1) \\ &= \tfrac{1}{2}(x_1y_2 + x_2y_3 + x_3y_1 - x_1y_3 - x_2y_1 - x_3y_2) \\ &= \tfrac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.\end{aligned}$$

If the vertices do not occur in the order $x_1 \le x_3 \le x_2$ or if the vertex $(x_3, y_3)$ is not above the line segment connecting the other two vertices, then the formula above may yield the negative of the area. So, use $\pm$ and choose the correct sign to give a positive area.

**EXAMPLE 5** **Finding the Area of a Triangle**

Find the area of the triangle whose vertices are

$$(1, 0),\quad (2, 2),\quad \text{and}\quad (4, 3).$$

**SOLUTION**

It is not necessary to know the relative positions of the three vertices. Simply evaluate the determinant

$$\tfrac{1}{2}\begin{vmatrix} 1 & 0 & 1 \\ 2 & 2 & 1 \\ 4 & 3 & 1 \end{vmatrix} = -\tfrac{3}{2}$$

and conclude that the area of the triangle is $\frac{3}{2}$ square units.

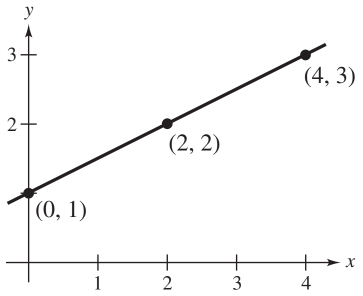



**Figure 3.2**

Suppose the three points in Example 5 had been on the same line. What would have happened had you applied the area formula to three such points? The answer is that the determinant would have been zero. Consider, for instance, the three collinear points (0, 1), (2, 2), and (4, 3), as shown in Figure 3.2. The determinant that yields the area of the "triangle" that has these three points as vertices is

$$\frac{1}{2}\begin{vmatrix} 0 & 1 & 1 \\ 2 & 2 & 1 \\ 4 & 3 & 1 \end{vmatrix} = 0.$$

If three points in the $xy$-plane lie on the same line, then the determinant in the formula for the area of a triangle is zero. The following generalizes this result.

### Test for Collinear Points in the *xy*-Plane

Three points $(x_1, y_1)$, $(x_2, y_2)$, and $(x_3, y_3)$ are collinear if and only if

$$\det\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} = 0.$$

The test for collinear points can be adapted to another use. That is, when you are given two points in the $xy$-plane, you can find an equation of the line passing through the two points, as follows.

### Two-Point Form of the Equation of a Line

An equation of the line passing through the distinct points $(x_1, y_1)$ and $(x_2, y_2)$ is given by

$$\det\begin{bmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{bmatrix} = 0.$$

## EXAMPLE 6 Finding an Equation of the Line Passing Through Two Points

Find an equation of the line passing through the two points

(2, 4) and (−1, 3).

**SOLUTION**

Let $(x_1, y_1) = (2, 4)$ and $(x_2, y_2) = (-1, 3)$. Applying the determinant formula for the equation of a line produces

$$\begin{vmatrix} x & y & 1 \\ 2 & 4 & 1 \\ -1 & 3 & 1 \end{vmatrix} = 0.$$

To evaluate this determinant, expand by cofactors in the first row to obtain the following.

$$\begin{aligned} x\begin{vmatrix} 4 & 1 \\ 3 & 1 \end{vmatrix} - y\begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} + 1\begin{vmatrix} 2 & 4 \\ -1 & 3 \end{vmatrix} &= 0 \\ x(1) - y(3) + 1(10) &= 0 \\ x - 3y + 10 &= 0 \end{aligned}$$

So, an equation of the line is $x - 3y = -10$.

The formula for the area of a triangle in the plane has a straightforward generalization to three-dimensional space, which is presented without proof as follows.

### Volume of a Tetrahedron

The volume of a tetrahedron with vertices $(x_1, y_1, z_1)$, $(x_2, y_2, z_2)$, $(x_3, y_3, z_3)$, and $(x_4, y_4, z_4)$ is

$$\text{Volume} = \pm\tfrac{1}{6}\det\begin{bmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{bmatrix}$$

where the sign $(\pm)$ is chosen to give a positive volume.

## EXAMPLE 7 Finding the Volume of a Tetrahedron

Find the volume of the tetrahedron whose vertices are (0, 4, 1), (4, 0, 0), (3, 5, 2), and (2, 2, 5), as shown in Figure 3.3.

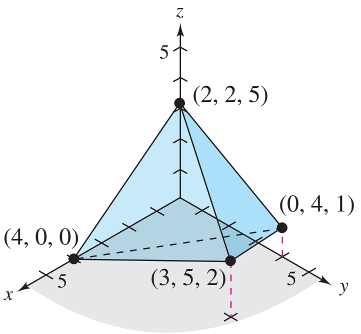



**Figure 3.3**

### SOLUTION

Using the determinant formula for the volume of a tetrahedron produces

$$\frac{1}{6}\begin{vmatrix} 0 & 4 & 1 & 1 \\ 4 & 0 & 0 & 1 \\ 3 & 5 & 2 & 1 \\ 2 & 2 & 5 & 1 \end{vmatrix} = \tfrac{1}{6}(-72) = -12.$$

So, the volume of the tetrahedron is 12 cubic units.

If four points in three-dimensional space lie in the same plane, then the determinant in the formula for the volume of a tetrahedron is zero. So, you have the following test.

### Test for Coplanar Points in Space

Four points $(x_1, y_1, z_1)$, $(x_2, y_2, z_2)$, $(x_3, y_3, z_3)$, and $(x_4, y_4, z_4)$ are coplanar if and only if

$$\det\begin{bmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{bmatrix} = 0.$$

An adaptation of this test is the determinant form of an equation of a plane passing through three points in space, as follows.

**Three-Point Form of the Equation of a Plane**

An equation of the plane passing through the distinct points $(x_1, y_1, z_1)$, $(x_2, y_2, z_2)$, and $(x_3, y_3, z_3)$ is given by

$$\det\begin{bmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{bmatrix} = 0.$$

**EXAMPLE 8** **Finding an Equation of the Plane Passing Through Three Points**

Find an equation of the plane passing through the three points $(0, 1, 0)$, $(-1, 3, 2)$, and $(-2, 0, 1)$.

**SOLUTION**

Using the determinant form of the equation of a plane produces

$$\begin{vmatrix} x & y & z & 1 \\ 0 & 1 & 0 & 1 \\ -1 & 3 & 2 & 1 \\ -2 & 0 & 1 & 1 \end{vmatrix} = 0.$$

To evaluate this determinant, subtract the fourth column from the second column to obtain

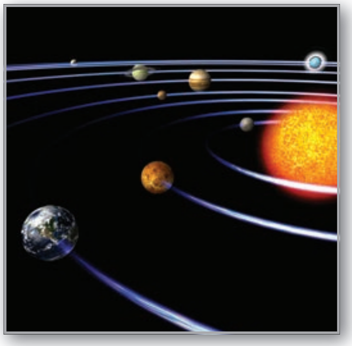
$$\begin{vmatrix} x & y-1 & z & 1 \\ 0 & 0 & 0 & 1 \\ -1 & 2 & 2 & 1 \\ -2 & -1 & 1 & 1 \end{vmatrix} = 0.$$

Now, expanding by cofactors in the second row yields

$$x\begin{vmatrix} 2 & 2 \\ -1 & 1 \end{vmatrix} - (y-1)\begin{vmatrix} -1 & 2 \\ -2 & 1 \end{vmatrix} + z\begin{vmatrix} -1 & 2 \\ -2 & -1 \end{vmatrix} = 0$$

$$x(4) - (y-1)(3) + z(5) = 0.$$

This produces the equation $4x - 3y + 5z = -3$.



**LINEAR ALGEBRA APPLIED**

According to Kepler's First Law of Planetary Motion, the orbits of the planets are ellipses, with the sun at one focus of the ellipse. The general equation of a conic section (such as an ellipse) is

$$ax^2 + bxy + cy^2 + dx + ey + f = 0.$$

To determine the equation of the orbit of a planet, an astronomer can find the coordinates of the planet along its orbit at five different points $(x_i, y_i)$, where $i = 1, 2, 3, 4$, and 5, and then use the determinant

$$\begin{vmatrix} x^2 & xy & y^2 & x & y & 1 \\ x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & x_4y_4 & y_4^2 & x_4 & y_4 & 1 \\ x_5^2 & x_5x_5 & y_5^2 & x_5 & y_5 & 1 \end{vmatrix}.$$

Ralf Juergen Kraft/Shutterstock.com

# 3.4 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

**Finding the Adjoint and Inverse of a Matrix** **In Exercises 1–8, find the adjoint of the matrix $A$. Then use the adjoint to find the inverse of $A$ (if possible).**

**1.** $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ **2.** $A = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}$

**3.** $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 6 \\ 0 & -4 & -12 \end{bmatrix}$ **4.** $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 2 & 2 & 2 \end{bmatrix}$

**5.** $A = \begin{bmatrix} -3 & -5 & -7 \\ 2 & 4 & 3 \\ 0 & 1 & -1 \end{bmatrix}$ **6.** $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 3 \\ -1 & -1 & -2 \end{bmatrix}$

**7.** $A = \begin{bmatrix} -1 & 2 & 0 & 1 \\ 3 & -1 & 4 & 1 \\ 0 & 0 & 1 & 2 \\ -1 & 1 & 1 & 2 \end{bmatrix}$

**8.** $A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$

**9. Proof** Prove that if $|A| = 1$ and all entries of $A$ are integers, then all entries of $|A^{-1}|$ must also be integers.

**10. Proof** Prove that if an $n \times n$ matrix $A$ is not invertible, then $A[\operatorname{adj}(A)]$ is the zero matrix.

**Proof** **In Exercises 11 and 12, prove the formula for a nonsingular $n \times n$ matrix $A$. Assume $n \geq 2$.**

**11.** $|\operatorname{adj}(A)| = |A|^{n-1}$ **12.** $\operatorname{adj}[\operatorname{adj}(A)] = |A|^{n-2}A$

**13.** Illustrate the formula in Exercise 11 using the matrix

$$A = \begin{bmatrix} 1 & 0 \\ 1 & -2 \end{bmatrix}.$$

**14.** Illustrate the formula in Exercise 12 using the matrix

$$A = \begin{bmatrix} -1 & 3 \\ 1 & 2 \end{bmatrix}.$$

**15. Proof** Prove that if $A$ is an $n \times n$ invertible matrix, then $\operatorname{adj}(A^{-1}) = [\operatorname{adj}(A)]^{-1}$.

**16.** Illustrate the formula in Exercise 15 using the matrix

$$A = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}.$$

**Using Cramer's Rule** **In Exercises 17–30, use Cramer's Rule to solve (if possible) the system of linear equations.**

**17.** $\begin{aligned} x_1 + 2x_2 &= 5 \\ -x_1 + x_2 &= 1 \end{aligned}$ **18.** $\begin{aligned} 2x_1 - x_2 &= -10 \\ 3x_1 + 2x_2 &= -1 \end{aligned}$

**19.** $\begin{aligned} 3x_1 + 4x_2 &= -2 \\ 5x_1 + 3x_2 &= 4 \end{aligned}$ **20.** $\begin{aligned} 18x_1 + 12x_2 &= 13 \\ 30x_1 + 24x_2 &= 23 \end{aligned}$

**21.** $\begin{aligned} 20x_1 + 8x_2 &= 11 \\ 12x_1 - 24x_2 &= 21 \end{aligned}$ **22.** $\begin{aligned} 13x_1 - 6x_2 &= 17 \\ 26x_1 - 12x_2 &= 8 \end{aligned}$

**23.** $\begin{aligned} -0.4x_1 + 0.8x_2 &= 1.6 \\ 2x_1 - 4x_2 &= 5.0 \end{aligned}$ **24.** $\begin{aligned} -0.4x_1 + 0.8x_2 &= 1.6 \\ 0.2x_1 + 0.3x_2 &= 0.6 \end{aligned}$

**25.** $\begin{aligned} 4x_1 - x_2 - x_3 &= 1 \\ 2x_1 + 2x_2 + 3x_3 &= 10 \\ 5x_1 - 2x_2 - 2x_3 &= -1 \end{aligned}$

**26.** $\begin{aligned} 4x_1 - 2x_2 + 3x_3 &= -2 \\ 2x_1 + 2x_2 + 5x_3 &= 16 \\ 8x_1 - 5x_2 - 2x_3 &= 4 \end{aligned}$

**27.** $\begin{aligned} 3x_1 + 4x_2 + 4x_3 &= 11 \\ 4x_1 - 4x_2 + 6x_3 &= 11 \\ 6x_1 - 6x_2 \qquad &= 3 \end{aligned}$

**28.** $\begin{aligned} 14x_1 - 21x_2 - 7x_3 &= -21 \\ -4x_1 + 2x_2 - 2x_3 &= 2 \\ 56x_1 - 21x_2 + 7x_3 &= 7 \end{aligned}$

**29.** $\begin{aligned} 4x_1 - x_2 + x_3 &= -5 \\ 2x_1 + 2x_2 + 3x_3 &= 10 \\ 5x_1 - 2x_2 + 6x_3 &= 1 \end{aligned}$

**30.** $\begin{aligned} 2x_1 + 3x_2 + 5x_3 &= 4 \\ 3x_1 + 5x_2 + 9x_3 &= 7 \\ 5x_1 + 9x_2 + 17x_3 &= 13 \end{aligned}$

**Using Cramer's Rule** **In Exercises 31–34, use a software program or a graphing utility with matrix capabilities and Cramer's Rule to solve (if possible) the system of linear equations.**

**31.** $\begin{aligned} \tfrac{5}{6}x_1 - x_2 &= -20 \\ \tfrac{4}{3}x_1 - \tfrac{7}{2}x_2 &= -51 \end{aligned}$

**32.** $\begin{aligned} -8x_1 + 7x_2 - 10x_3 &= -151 \\ 12x_1 + 3x_2 - 5x_3 &= 86 \\ 15x_1 - 9x_2 + 2x_3 &= 187 \end{aligned}$

**33.** $\begin{aligned} 3x_1 - 2x_2 + 9x_3 + 4x_4 &= 35 \\ -x_1 \qquad - 9x_3 - 6x_4 &= -17 \\ 3x_3 + x_4 &= 5 \\ 2x_1 + 2x_2 \qquad + 8x_4 &= -4 \end{aligned}$

**34.** $\begin{aligned} -x_1 - x_2 \qquad + x_4 &= -8 \\ 3x_1 + 5x_2 + 5x_3 \qquad &= 24 \\ 2x_3 + x_4 &= -6 \\ -2x_1 - 3x_2 - 3x_3 \qquad &= -15 \end{aligned}$

**35.** Use Cramer's Rule to solve the system of linear equations for $x$ and $y$.

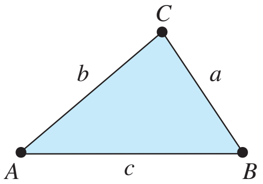
$$\begin{aligned} kx + (1-k)y &= 1 \\ (1-k)x + ky &= 3 \end{aligned}$$

For what value(s) of $k$ will the system be inconsistent?

**36.** Verify the following system of linear equations in $\cos A$, $\cos B$, and $\cos C$ for the triangle shown in the figure.

$$\begin{aligned} c\cos B + b\cos C &= a \\ c\cos A \phantom{+ b\cos C} + a\cos C &= b \\ b\cos A + a\cos B \phantom{+ a\cos C} &= c \end{aligned}$$

Then use Cramer's Rule to solve for $\cos C$, and use the result to verify the Law of Cosines, $c^2 = a^2 + b^2 - 2ab\cos C$.




**Finding the Area of a Triangle** **In Exercises 37–40, find the area of the triangle with the given vertices.**

**37.** $(0, 0), (2, 0), (0, 3)$ **38.** $(1, 1), (2, 4), (4, 2)$

**39.** $(-1, 2), (2, 2), (-2, 4)$ **40.** $(1, 1), (-1, 1), (0, -2)$

**Testing for Collinear Points** **In Exercises 41–44, determine whether the points are collinear.**

**41.** $(1, 2), (3, 4), (5, 6)$ **42.** $(-1, 0), (1, 1), (3, 3)$

**43.** $(-2, 5), (0, -1), (3, -9)$

**44.** $(-1, -3), (-4, 7), (2, -13)$

**Finding an Equation of a Line** **In Exercises 45–48, find an equation of the line passing through the given points.**

**45.** $(0, 0), (3, 4)$ **46.** $(-4, 7), (2, 4)$

**47.** $(-2, 3), (-2, -4)$ **48.** $(1, 4), (3, 4)$

**Finding the Volume of a Tetrahedron** **In Exercises 49–52, find the volume of the tetrahedron with the given vertices.**

**49.** $(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1)$

**50.** $(1, 1, 1), (0, 0, 0), (2, 1, -1), (-1, 1, 2)$

**51.** $(3, -1, 1), (4, -4, 4), (1, 1, 1), (0, 0, 1)$

**52.** $(0, 0, 0), (0, 2, 0), (3, 0, 0), (1, 1, 4)$

**Testing for Coplanar Points** **In Exercises 53–56, determine whether the points are coplanar.**

**53.** $(-4, 1, 0), (0, 1, 2), (4, 3, -1), (0, 0, 1)$

**54.** $(1, 2, 3), (-1, 0, 1), (0, -2, -5), (2, 6, 11)$

**55.** $(0, 0, -1), (0, -1, 0), (1, 1, 0), (2, 1, 2)$

**56.** $(1, 2, 7), (-3, 6, 6), (4, 4, 2), (3, 3, 4)$

**Finding an Equation of a Plane** **In Exercises 57–60, find an equation of the plane passing through the given points.**

**57.** $(1, -2, 1), (-1, -1, 7), (2, -1, 3)$

**58.** $(0, -1, 0), (1, 1, 0), (2, 1, 2)$

**59.** $(0, 0, 0), (1, -1, 0), (0, 1, -1)$

**60.** $(1, 2, 7), (4, 4, 2), (3, 3, 4)$

**Using Cramer's Rule** **In Exercises 61 and 62, determine whether Cramer's Rule is used correctly to solve for the variable. If not, identify the mistake.**

**61.** $\begin{aligned} x + 2y + z &= 2 \\ -x + 3y - 2z &= 4 \\ 4x + y - z &= 6 \end{aligned}$ $\qquad y = \dfrac{\begin{vmatrix} 1 & 2 & 1 \\ -1 & 3 & -2 \\ 4 & 1 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 1 \\ -1 & 4 & -2 \\ 4 & 6 & -1 \end{vmatrix}}$

**62.** $\begin{aligned} 5x - 2y + z &= 15 \\ 3x - 3y - z &= -7 \\ 2x - y - 7z &= -3 \end{aligned}$ $\qquad x = \dfrac{\begin{vmatrix} 15 & -2 & 1 \\ -7 & -3 & -1 \\ -3 & -1 & -7 \end{vmatrix}}{\begin{vmatrix} 5 & -2 & 1 \\ 3 & -3 & -1 \\ 2 & -1 & -7 \end{vmatrix}}$

**63. Textbook Publishing** The table shows the estimated revenues (in millions of dollars) of textbook publishers in the United States from 2007 through 2009. (Source: U.S. Census Bureau)

| *Year, t* | *Revenues, y* |
|---|---|
| 2007 | 10,697 |
| 2008 | 11,162 |
| 2009 | 9891 |

(a) Create a system of linear equations for the data to fit the curve

$y = at^2 + bt + c$

where $t = 7$ corresponds to 2007, and $y$ is the revenue.

(b) Use Cramer's Rule to solve the system.

(c) Use a graphing utility to plot the data and graph the polynomial function in the same viewing window.

(d) Briefly describe how well the polynomial function fits the data.

**64. CAPSTONE** Consider the system of linear equations

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$$

where $a_1$, $b_1$, $c_1$, $a_2$, $b_2$, and $c_2$ represent real numbers. What must be true about the lines represented by the equations when

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0?$$

# 3 Review Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

**The Determinant of a Matrix** In Exercises 1–18, find the determinant of the matrix.

**1.** $\begin{bmatrix} 4 & -1 \\ 2 & 2 \end{bmatrix}$

**2.** $\begin{bmatrix} 0 & -3 \\ 1 & 2 \end{bmatrix}$

**3.** $\begin{bmatrix} -3 & 1 \\ 6 & -2 \end{bmatrix}$

**4.** $\begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}$

**5.** $\begin{bmatrix} 1 & 4 & -2 \\ 0 & -3 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

**6.** $\begin{bmatrix} 5 & 0 & 2 \\ 0 & -1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$

**7.** $\begin{bmatrix} -2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

**8.** $\begin{bmatrix} -15 & 0 & 4 \\ 3 & 0 & -5 \\ 12 & 0 & 6 \end{bmatrix}$

**9.** $\begin{bmatrix} -3 & 6 & 9 \\ 9 & 12 & -3 \\ 0 & 15 & -6 \end{bmatrix}$

**10.** $\begin{bmatrix} -15 & 0 & 3 \\ 3 & 9 & -6 \\ 12 & -3 & 6 \end{bmatrix}$

**11.** $\begin{bmatrix} 2 & 0 & -1 & 4 \\ -1 & 2 & 0 & 3 \\ 3 & 0 & 1 & 2 \\ -2 & 0 & 3 & 1 \end{bmatrix}$

**12.** $\begin{bmatrix} 2 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 4 & -1 & 3 & 0 \\ 5 & 2 & 1 & -1 \end{bmatrix}$

**13.** $\begin{bmatrix} -4 & 1 & 2 & 3 \\ 1 & -2 & 1 & 2 \\ 2 & -1 & 3 & 4 \\ 1 & 2 & 2 & -1 \end{bmatrix}$

**14.** $\begin{bmatrix} 3 & -1 & 2 & 1 \\ -2 & 0 & 1 & -3 \\ -1 & 2 & -3 & 4 \\ -2 & 1 & -2 & 1 \end{bmatrix}$

**15.** $\begin{bmatrix} -1 & 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 \\ 1 & 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 1 & -1 \\ 0 & 1 & 1 & -1 & 1 \end{bmatrix}$

**16.** $\begin{bmatrix} 1 & 2 & -1 & 3 & 4 \\ 2 & 3 & -1 & 2 & -2 \\ 1 & 2 & 0 & 1 & -1 \\ 1 & 0 & 2 & -1 & 0 \\ 0 & -1 & 1 & 0 & 2 \end{bmatrix}$

**17.** $\begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$

**18.** $\begin{bmatrix} 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \end{bmatrix}$

**Properties of Determinants** In Exercises 19–22, determine which property of determinants the equation illustrates.

**19.** $\begin{vmatrix} 2 & -1 \\ 6 & -3 \end{vmatrix} = 0$

**20.** $\begin{vmatrix} 1 & 2 & -1 \\ 2 & 0 & 3 \\ 4 & -1 & 1 \end{vmatrix} = -\begin{vmatrix} 1 & -1 & 2 \\ 2 & 3 & 0 \\ 4 & 1 & -1 \end{vmatrix}$

**21.** $\begin{vmatrix} 2 & -4 & 3 & 2 \\ 0 & 4 & 6 & 1 \\ 1 & 8 & 9 & 0 \\ 6 & 12 & -6 & 1 \end{vmatrix} = -12\begin{vmatrix} 2 & 1 & 1 & 2 \\ 0 & -1 & 2 & 1 \\ 1 & -2 & 3 & 0 \\ 6 & -3 & -2 & 1 \end{vmatrix}$

**22.** $\begin{vmatrix} 1 & 3 & 1 \\ 0 & -1 & 2 \\ 1 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 1 \\ 2 & 5 & 4 \\ 1 & 2 & 1 \end{vmatrix}$

**The Determinant of a Matrix Product** In Exercises 23 and 24, find (a) $|A|$, (b) $|B|$, (c) $AB$, and (d) $|AB|$. Then verify that $|A||B| = |AB|$.

**23.** $A = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}$

**24.** $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 2 & 3 \end{bmatrix}$

**Finding Determinants** In Exercises 25 and 26, find (a) $|A^T|$, (b) $|A^3|$, (c) $|A^TA|$, and (d) $|5A|$.

**25.** $A = \begin{bmatrix} -2 & 6 \\ 1 & 3 \end{bmatrix}$

**26.** $A = \begin{bmatrix} 3 & 0 & 1 \\ -1 & 0 & 0 \\ 2 & 1 & 2 \end{bmatrix}$

**Finding Determinants** In Exercises 27 and 28, find (a) $|A|$ and (b) $|A^{-1}|$.

**27.** $A = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 3 & 2 \\ -2 & 7 & 6 \end{bmatrix}$

**28.** $A = \begin{bmatrix} 2 & -1 & 4 \\ 5 & 0 & 3 \\ 1 & -2 & 0 \end{bmatrix}$

**The Determinant of the Inverse of a Matrix** In Exercises 29–32, find $|A^{-1}|$. Begin by finding $A^{-1}$, and then evaluate its determinant. Verify your result by finding $|A|$ and then applying the formula from Theorem 3.8, $|A^{-1}| = \dfrac{1}{|A|}$.

**29.** $\begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$

**30.** $\begin{bmatrix} 10 & 2 \\ -2 & 7 \end{bmatrix}$

**31.** $\begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 4 \\ 2 & 6 & 0 \end{bmatrix}$

**32.** $\begin{bmatrix} -1 & 1 & 2 \\ 2 & 4 & 8 \\ 1 & -1 & 0 \end{bmatrix}$

**Solving a System of Linear Equations** **In Exercises 33–36, solve the system of linear equations by each of the following methods.**

**(a) Gaussian elimination with back-substitution**

**(b) Gauss-Jordan elimination**

**(c) Cramer's Rule**

**33.** $$\begin{aligned} 3x_1 + 3x_2 + 5x_3 &= 1 \\ 3x_1 + 5x_2 + 9x_3 &= 2 \\ 5x_1 + 9x_2 + 17x_3 &= 4 \end{aligned}$$

**34.** $$\begin{aligned} 2x_1 + x_2 + 2x_3 &= 6 \\ -x_1 + 2x_2 - 3x_3 &= 0 \\ 3x_1 + 2x_2 - x_3 &= 6 \end{aligned}$$

**35.** $$\begin{aligned} x_1 + 2x_2 - x_3 &= -7 \\ 2x_1 - 2x_2 - 2x_3 &= -8 \\ -x_1 + 3x_2 + 4x_3 &= 8 \end{aligned}$$

**36.** $$\begin{aligned} 2x_1 + 3x_2 + 5x_3 &= 4 \\ 3x_1 + 5x_2 + 9x_3 &= 7 \\ 5x_1 + 9x_2 + 13x_3 &= 17 \end{aligned}$$

**System of Linear Equations** **In Exercises 37–42, use the determinant of the coefficient matrix to determine whether the system of linear equations has a unique solution.**

**37.** $$\begin{aligned} 5x + 4y &= 2 \\ -x + y &= -22 \end{aligned}$$

**38.** $$\begin{aligned} 2x - 5y &= 2 \\ 3x - 7y &= 1 \end{aligned}$$

**39.** $$\begin{aligned} -x + y + 2z &= 1 \\ 2x + 3y + z &= -2 \\ 5x + 4y + 2z &= 4 \end{aligned}$$

**40.** $$\begin{aligned} 2x + 3y + z &= 10 \\ 2x - 3y - 3z &= 22 \\ 8x + 6y \phantom{+ z} &= -2 \end{aligned}$$

**41.** $$\begin{aligned} x_1 + 2x_2 + 6x_3 &= 1 \\ 2x_1 + 5x_2 + 15x_3 &= 4 \\ 3x_1 + x_2 + 3x_3 &= -6 \end{aligned}$$

**42.** $$\begin{aligned} x_1 + 5x_2 + 3x_3 \phantom{+ 8x_4 + 6x_5} &= 14 \\ 4x_1 + 2x_2 + 5x_3 \phantom{+ 8x_4 + 6x_5} &= 3 \\ 3x_3 + 8x_4 + 6x_5 &= 16 \\ 2x_1 + 4x_2 \phantom{+ 5x_3 + 8x_4} - 2x_5 &= 0 \\ 2x_1 \phantom{+ 4x_2} - x_3 \phantom{+ 8x_4 + 6x_5} &= 0 \end{aligned}$$

**43.** Let $A$ and $B$ be square matrices of order 4 such that $|A| = 4$ and $|B| = 2$. Find (a) $|BA|$, (b) $|B^2|$, (c) $|2A|$, (d) $|(AB)^T|$, and (e) $|B^{-1}|$.

**44.** Let $A$ and $B$ be square matrices of order 3 such that $|A| = -2$ and $|B| = 5$. Find (a) $|BA|$, (b) $|B^4|$, (c) $|2A|$, (d) $|(AB)^T|$, and (e) $|B^{-1}|$.

**45. Proof** Prove the following property.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} + c_{31} & a_{32} + c_{32} & a_{33} + c_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ c_{31} & c_{32} & c_{33} \end{vmatrix}$$

**46.** Illustrate the property in Exercise 45 with the following.

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & -1 & 2 \\ 2 & 1 & -1 \end{bmatrix}, \quad c_{31} = 3, \quad c_{32} = 0, \quad c_{33} = 1$$

**47.** Find the determinant of the $n \times n$ matrix.

$$\begin{bmatrix} 1-n & 1 & 1 & \cdots & 1 \\ 1 & 1-n & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \cdots & 1-n \end{bmatrix}$$

**48.** Show that

$$\begin{vmatrix} a & 1 & 1 & 1 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \\ 1 & 1 & 1 & a \end{vmatrix} = (a+3)(a-1)^3.$$

**Calculus** **In Exercises 49–52, find the Jacobians of the functions. If $x$, $y$, and $z$ are continuous functions of $u$, $v$, and $w$ with continuous first partial derivatives, then the Jacobians $J(u, v)$ and $J(u, v, w)$ are defined as**

$$J(u, v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} \text{ and } J(u, v, w) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}.$$

**49.** $x = \frac{1}{2}(v - u), \quad y = \frac{1}{2}(v + u)$

**50.** $x = au + bv, \quad y = cu + dv$

**51.** $x = \frac{1}{2}(u + v), \quad y = \frac{1}{2}(u - v), \quad z = 2uvw$

**52.** $x = u - v + w, \quad y = 2uv, \quad z = u + v + w$

**53. Writing** Compare the various methods for calculating the determinant of a matrix. Which method requires the least amount of computation? Which method do you prefer when the matrix has very few zeros?

**54. Writing** A computer operator charges \$0.001 (one tenth of a cent) for each addition and subtraction, and \$0.003 for each multiplication and division. Use the table on page 116 to compare and contrast the costs of calculating the determinant of a $10 \times 10$ matrix by cofactor expansion and then by row reduction. Which method would you prefer to use for calculating determinants?

**55. Writing** Solve the equation for $x$, if possible. Explain your result.

$$\begin{vmatrix} \cos x & 0 & \sin x \\ \sin x & 0 & \cos x \\ \sin x - \cos x & 1 & \sin x + \cos x \end{vmatrix} = 0$$

**56. Proof** Prove that if $|A| = |B| \neq 0$, and $A$ and $B$ are of the same size, then there exists a matrix $C$ such that $|C| = 1$ and $A = CB$.

**Finding the Adjoint of a Matrix** **In Exercises 57 and 58, find the adjoint of the matrix.**

**57.** $\begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix}$ **58.** $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix}$

**System of Linear Equations** **In Exercises 59–62, use the determinant of the coefficient matrix to determine whether the system of linear equations has a unique solution. If it does, use Cramer's Rule to find the solution.**

**59.** $\begin{aligned} 0.2x - 0.1y &= 0.07 \\ 0.4x - 0.5y &= -0.01 \end{aligned}$ **60.** $\begin{aligned} 2x + y &= 0.3 \\ 3x - y &= -1.3 \end{aligned}$

**61.** $\begin{aligned} 2x_1 + 3x_2 + 3x_3 &= 3 \\ 6x_1 + 6x_2 + 12x_3 &= 13 \\ 12x_1 + 9x_2 - x_3 &= 2 \end{aligned}$

**62.** $\begin{aligned} 4x_1 + 4x_2 + 4x_3 &= 5 \\ 4x_1 - 2x_2 - 8x_3 &= 1 \\ 8x_1 + 2x_2 - 4x_3 &= 6 \end{aligned}$

**Using Cramer's Rule** **In Exercises 63 and 64, use a software program or a graphing utility with matrix capabilities and Cramer's Rule to solve (if possible) the system of linear equations.**

**63.** $\begin{aligned} 0.2x_1 - 0.6x_2 &= 2.4 \\ -x_1 + 1.4x_2 &= -8.8 \end{aligned}$

**64.** $\begin{aligned} 4x_1 - x_2 + x_3 &= -5 \\ 2x_1 + 2x_2 + 3x_3 &= 10 \\ 5x_1 - 2x_2 + 6x_3 &= 1 \end{aligned}$

**Finding the Area of a Triangle** **In Exercises 65 and 66, use a determinant to find the area of the triangle with the given vertices.**

**65.** $(1, 0), (5, 0), (5, 8)$ **66.** $(-4, 0), (4, 0), (0, 6)$

**Finding an Equation of a Line** **In Exercises 67 and 68, use a determinant to find an equation of the line passing through the given points.**

**67.** $(-4, 0), (4, 4)$ **68.** $(2, 5), (6, -1)$

**Finding an Equation of a Plane** **In Exercises 69 and 70, use a determinant to find an equation of the plane passing through the given points.**

**69.** $(0, 0, 0), (1, 0, 3), (0, 3, 4)$

**70.** $(0, 0, 0), (2, -1, 1), (-3, 2, 5)$

**71. Using Cramer's Rule** Determine whether Cramer's Rule is used correctly to solve for the variable. If not, identify the mistake.

$$\begin{aligned} x - 4y - z &= -1 \\ 2x - 3y + z &= 6 \\ x + y - 4z &= 1 \end{aligned} \qquad z = \frac{\begin{vmatrix} -1 & -4 & -1 \\ 6 & -3 & 1 \\ 1 & 1 & -4 \end{vmatrix}}{\begin{vmatrix} 1 & -4 & -1 \\ 2 & -3 & 1 \\ 1 & 1 & -4 \end{vmatrix}}$$

**72. Health Care Expenditures** The table shows annual personal health care expenditures (in billions of dollars) in the United States from 2007 through 2009. (Source: Bureau of Economic Analysis)

| *Year, t* | 2007 | 2008 | 2009 |
|---|---|---|---|
| *Amount, y* | 1465 | 1547 | 1623 |

(a) Create a system of linear equations for the data to fit the curve

$y = at^2 + bt + c$

where $t = 7$ corresponds to 2007, and $y$ is the amount of the expenditure.

(b) Use Cramer's Rule to solve the system.

(c) Use a graphing utility to plot the data and graph the polynomial function in the same viewing window.

(d) Briefly describe how well the polynomial function fits the data.

**True or False?** **In Exercises 73–76, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.**

**73.** (a) The cofactor $C_{22}$ of a given matrix is always a positive number.

(b) If a square matrix $B$ is obtained from $A$ by interchanging two rows, then $\det(B) = \det(A)$.

(c) If one column of a square matrix is a multiple of another column, then the determinant is 0.

(d) If $A$ is a square matrix of order $n$, then $\det(A) = -\det(A^T)$.

**74.** (a) If $A$ and $B$ are square matrices of order $n$ such that $\det(AB) = -1$, then both $A$ and $B$ are nonsingular.

(b) If $A$ is a $3 \times 3$ matrix with $\det(A) = 5$, then $\det(2A) = 10$.

(c) If $A$ and $B$ are square matrices of order $n$, then $\det(A + B) = \det(A) + \det(B)$.

**75.** (a) In Cramer's Rule, the value of $x_i$ is the quotient of two determinants, where the numerator is the determinant of the coefficient matrix.

(b) Three points $(x_1, y_1)$, $(x_2, y_2)$, and $(x_3, y_3)$ are collinear when the determinant of the matrix that has the coordinates as entries in the first two columns and 1's as entries in the third column is nonzero.

**76.** (a) If $A$ is a square matrix, then the matrix of cofactors of $A$ is called the adjoint of $A$.

(b) In Cramer's Rule, the denominator is the determinant of the matrix formed by replacing the column corresponding to the variable being solved for with the column representing the constants.

# 3 Projects


## 1 Stochastic Matrices

In Section 2.5, you studied a consumer preference model for competing satellite television companies. The matrix representing the transition probabilities was

$$P = \begin{bmatrix} 0.70 & 0.15 & 0.15 \\ 0.20 & 0.80 & 0.15 \\ 0.10 & 0.05 & 0.70 \end{bmatrix}.$$

When you were given the initial state matrix $X$, you observed that the number of subscribers after 1 year was the product $PX$.

$$X = \begin{bmatrix} 15{,}000 \\ 20{,}000 \\ 65{,}000 \end{bmatrix}$$

$$PX = \begin{bmatrix} 0.70 & 0.15 & 0.15 \\ 0.20 & 0.80 & 0.15 \\ 0.10 & 0.05 & 0.70 \end{bmatrix}\begin{bmatrix} 15{,}000 \\ 20{,}000 \\ 65{,}000 \end{bmatrix} = \begin{bmatrix} 23{,}250 \\ 28{,}750 \\ 48{,}000 \end{bmatrix}$$

After 10 years, the number of subscribers had nearly reached a **steady state.**

$$P^{10}X \approx \begin{bmatrix} 33{,}287 \\ 47{,}147 \\ 19{,}566 \end{bmatrix}$$

That is, for large values of $n$, the product $P^nX$ approached a limit $\overline{X}$, $P\overline{X} = \overline{X}$.

Because $P\overline{X} = \overline{X} = 1\overline{X}$, 1 is an *eigenvalue* of $P$ with corresponding *eigenvector* $\overline{X}$. You will study eigenvalues and eigenvectors in more detail in Chapter 7.

**1.** Use a software program or a graphing utility to show that the following are eigenvalues and eigenvectors of $P$. That is, show that $P\mathbf{x}_i = \lambda_i \mathbf{x}_i$ for $i = 1, 2$, and 3.

*Eigenvalues:* $\lambda_1 = 1$, $\lambda_2 = 0.65$, $\lambda_3 = 0.55$

*Eigenvectors:* $\mathbf{x}_1 = \begin{bmatrix} 7 \\ 10 \\ 4 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$, $\mathbf{x}_3 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

**2.** Let $S$ be the matrix whose columns are the eigenvectors of $P$. Show that $S^{-1}PS$ is a diagonal matrix $D$. What are the entries along the diagonal of $D$?

**3.** Show that $P^n = (SDS^{-1})^n = SD^nS^{-1}$. Use this result to calculate $P^{10}X$ and verify the result from Section 2.5.

## 2 The Cayley-Hamilton Theorem

The **characteristic polynomial** of a square matrix $A$ is given by the determinant $|\lambda I - A|$. If the order of $A$ is $n$, then the characteristic polynomial $p(\lambda)$ is an $n$th-degree polynomial in the variable $\lambda$.

$$p(\lambda) = \det(\lambda I - A) = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_2\lambda^2 + c_1\lambda + c_0$$

The Cayley-Hamilton Theorem asserts that every square matrix satisfies its characteristic polynomial. That is, for the $n \times n$ matrix $A$, $p(A) = O$, or

$$A^n + c_{n-1}A^{n-1} + \cdots + c_2A^2 + c_1A + c_0I = O.$$

Note that this is a matrix equation. The zero on the right is the $n \times n$ zero matrix, and the coefficient $c_0$ is multiplied by the $n \times n$ identity matrix $I$.

**1.** Verify the Cayley-Hamilton Theorem for the matrix

$$\begin{bmatrix} 2 & -2 \\ -2 & -1 \end{bmatrix}.$$

**2.** Verify the Cayley-Hamilton Theorem for the matrix

$$\begin{bmatrix} 6 & 0 & 4 \\ -2 & 1 & 3 \\ 2 & 0 & 4 \end{bmatrix}.$$

**3.** Verify the Cayley-Hamilton Theorem for a general $2 \times 2$ matrix $A$,

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

**4.** If $A$ is a nonsingular and $n \times n$ matrix, show that

$$A^{-1} = \frac{1}{c_0}(-A^{n-1} - c_{n-1}A^{n-2} - \cdot \cdot \cdot - c_2A - c_1I).$$

Use this result to find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}.$$

**5.** The Cayley-Hamilton Theorem is useful for calculating powers $A^n$ of the square matrix $A$. For example, the characteristic polynomial of the matrix

$$A = \begin{bmatrix} 3 & -1 \\ 2 & -1 \end{bmatrix}$$

is $p(\lambda) = \lambda^2 - 2\lambda - 1$.

Using the Cayley-Hamilton Theorem gives

$$A^2 - 2A - I = O \quad \text{or} \quad A^2 = 2A + I.$$

So, $A^2$ is written in terms of $A$ and $I$.

$$A^2 = 2A + I = 2\begin{bmatrix} 3 & -1 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -2 \\ 4 & -1 \end{bmatrix}$$

Similarly, multiplying both sides of the equation $A^2 = 2A + I$ by $A$ gives $A^3$ in terms of $A^2$, $A$, and $I$. Moreover, you can write $A^3$ in terms of $A$ and $I$ by replacing $A^2$ with $2A + I$, as follows.

$$A^3 = 2A^2 + A = 2(2A + I) + A = 5A + 2I$$

(a) Write $A^4$ in terms of $A$ and $I$.

(b) Find $A^5$ for the matrix

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 2 & -1 \\ 1 & 0 & 2 \end{bmatrix}.$$

(*Hint:* Find the characteristic polynomial of $A$, then use the Cayley-Hamilton Theorem to write $A^3$ in terms of $A^2$, $A$, and $I$. Inductively write $A^5$ in terms of $A^2$, $A$, and $I$.)

# 1–3 Cumulative Test

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

**Take this test to review the material in Chapters 1–3. After you are finished, check your work against the answers in the back of the book.**

**In Exercises 1 and 2, determine whether the equation is linear in the variables $x$ and $y$.**

**1.** $\frac{4}{y} - x = 10$

**2.** $\frac{3}{5}x + \frac{7}{10}y = 2$

**In Exercises 3 and 4, use Gaussian elimination to solve the system of linear equations.**

**3.** $\begin{aligned} x - 2y &= 5 \\ 3x + y &= 1 \end{aligned}$

**4.** $\begin{aligned} 4x_1 + x_2 - 3x_3 &= 11 \\ 2x_1 - 3x_2 + 2x_3 &= 9 \\ x_1 + x_2 + x_3 &= -3 \end{aligned}$

**5.** Use a software program or a graphing utility to solve the system of linear equations.

$$\begin{aligned} 0.1x - 2.5y + 1.2z - 0.75w &= 108 \\ 2.4x + 1.5y - 1.8z + 0.25w &= -81 \\ 0.4x - 3.2y + 1.6z - 1.4w &= 148.8 \\ 1.6x + 1.2y - 3.2z + 0.6w &= -143.2 \end{aligned}$$

**6.** Find the solution set of the system of linear equations represented by the augmented matrix.

$$\begin{bmatrix} 0 & 1 & -1 & 0 & 2 \\ 1 & 0 & 2 & -1 & 0 \\ 1 & 2 & 0 & -1 & 4 \end{bmatrix}$$

**7.** Solve the homogeneous linear system corresponding to the coefficient matrix.

$$\begin{bmatrix} 1 & 2 & 1 & -2 \\ 0 & 0 & 2 & -4 \\ -2 & -4 & 1 & -2 \end{bmatrix}$$

**8.** Determine the value(s) of $k$ such that the system is consistent.

$$\begin{aligned} x + 2y - z &= 3 \\ -x - y + z &= 2 \\ -x + y + z &= k \end{aligned}$$

**9.** Solve for $x$ and $y$ in the matrix equation $2A - B = I$, given

$$A = \begin{bmatrix} -1 & 1 \\ 2 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} x & 2 \\ y & 5 \end{bmatrix}.$$

**10.** Find $A^TA$ for the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$. Show that this product is symmetric.

**In Exercises 11–14, find the inverse of the matrix (if it exists).**

**11.** $\begin{bmatrix} -2 & 3 \\ 4 & 6 \end{bmatrix}$

**12.** $\begin{bmatrix} -2 & 3 \\ 3 & 6 \end{bmatrix}$

**13.** $\begin{bmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 3 \end{bmatrix}$

**14.** $\begin{bmatrix} 1 & 1 & 0 \\ -3 & 6 & 5 \\ 0 & 1 & 0 \end{bmatrix}$

**In Exercises 15 and 16, use an inverse matrix to solve the system of linear equations.**

**15.** $\begin{aligned} x + 2y &= -3 \\ x - 2y &= 0 \end{aligned}$

**16.** $\begin{aligned} 2x - y &= 6 \\ 2x + y &= 10 \end{aligned}$

**17.** Find a sequence of elementary matrices whose product is the following nonsingular matrix.

$$\begin{bmatrix} 2 & -4 \\ 1 & 0 \end{bmatrix}$$

**18.** Find the determinant of the matrix

$$\begin{bmatrix} 5 & 1 & 2 & 4 \\ 1 & 0 & -2 & -3 \\ 1 & 1 & 6 & 1 \\ 1 & 0 & 0 & -4 \end{bmatrix}.$$

**19.** Find (a) $|A|$, (b) $|B|$, (c) $AB$, and (d) $|AB|$. Then verify that $|A||B| = |AB|$.

$$A = \begin{bmatrix} 1 & -3 \\ 4 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & 1 \\ 0 & 5 \end{bmatrix}$$

**20.** Find (a) $|A|$ and (b) $|A^{-1}|$.

$$A = \begin{bmatrix} 5 & -2 & -3 \\ -1 & 0 & 4 \\ 6 & -8 & 2 \end{bmatrix}$$

**21.** If $|A| = 7$ and $A$ is of order 4, then find each determinant.

(a) $|3A|$ (b) $|A^T|$ (c) $|A^{-1}|$ (d) $|A^3|$

**22.** Use the adjoint of

$$A = \begin{bmatrix} 1 & -5 & -1 \\ 0 & -2 & 1 \\ 1 & 0 & 2 \end{bmatrix}$$

to find $A^{-1}$.

**23.** Let $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$, and $\mathbf{b}$ be the following column matrices.

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{x}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Find constants $a$, $b$, and $c$ such that $a\mathbf{x}_1 + b\mathbf{x}_2 + c\mathbf{x}_3 = \mathbf{b}$.

**24.** Use a system of linear equations to find the parabola $y = ax^2 + bx + c$ that passes through the points $(-1, 2)$, $(0, 1)$, and $(2, 6)$.

**25.** Use a determinant to find an equation of the line passing through the points $(1, 4)$ and $(5, -2)$.

**26.** Use a determinant to find the area of the triangle with vertices $(3, 1)$, $(7, 1)$, and $(7, 9)$.

**27.** Determine the currents $I_1$, $I_2$, and $I_3$ for the electrical network shown in the figure at the left.

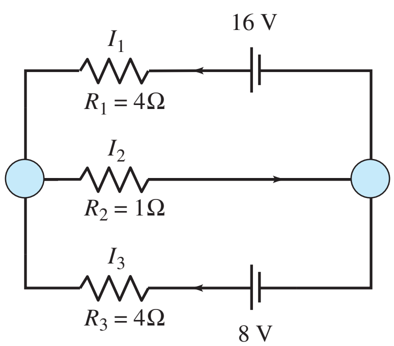



**Figure for 27**

**28.** A manufacturer produces three different models of a product and ships them to two warehouses. In the matrix

$$A = \begin{bmatrix} 200 & 300 \\ 600 & 350 \\ 250 & 400 \end{bmatrix}$$

$a_{ij}$ represents the number of units of model $i$ that the manufacturer ships to warehouse $j$. The matrix

$$B = \begin{bmatrix} 12.50 & 9.00 & 21.50 \end{bmatrix}$$

represents the prices of the three models in dollars per unit. Find the product $BA$ and state what each entry of the matrix represents.

**29.** Let $A$, $B$, and $C$ be three nonzero $n \times n$ matrices such that $AC = BC$. Does it follow that $A = B$? Prove your answer.

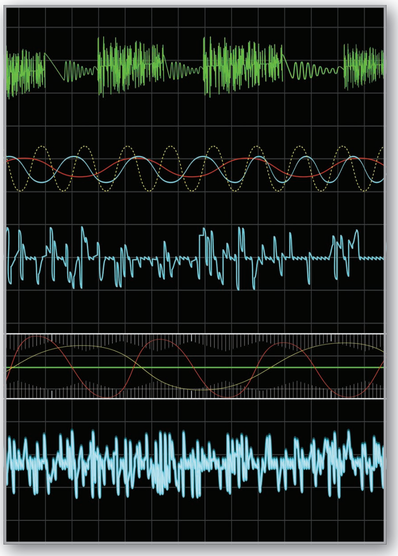
# 4 Vector Spaces



Satellite Dish (p. 217)

Crystallography (p. 207)

Image Morphing (p. 174)



Digital Sampling (p. 166)

Force (p. 151)



Clockwise from top left, Marty Ellis/Shutterstock.com; LVV/Shutterstock.com;
Richard Laschon/Shutterstock.com; Anne Kitzman/Shutterstock.com; Kent Mathews/Getty Images

# 4.1 Vectors in $R^n$

- Represent a vector as a directed line segment.
- Perform basic vector operations in $R^2$ and represent them graphically.
- Perform basic vector operations in $R^n$.
- Write a vector as a linear combination of other vectors.

## VECTORS IN THE PLANE

In physics and engineering, a *vector* is characterized by two quantities (length and direction) and is represented by a *directed line segment*. In this chapter you will see that these are only two special types of vectors. Their geometric representations can help you understand the more general definition of a vector.

Geometrically, a **vector in the plane** is represented by a **directed line segment** with its **initial point** at the origin and its **terminal point** at $(x_1, x_2)$, as shown in Figure 4.1.

**REMARK**

The term *vector* derives from the Latin word *vectus*, meaning "to carry." The idea is that if you were to carry something from the origin to the point $(x_1, x_2)$, then the trip could be represented by the directed line segment from $(0, 0)$ to $(x_1, x_2)$. Vectors are represented by lowercase letters set in boldface type (such as **u**, **v**, **w**, and **x**).

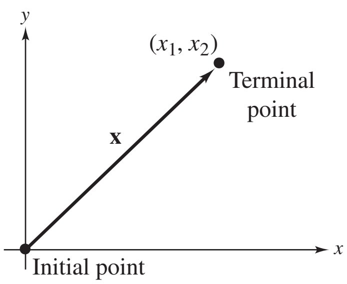



**Figure 4.1**

The same **ordered pair** used to represent its terminal point also represents the vector. That is, $\mathbf{x} = (x_1, x_2)$. The coordinates $x_1$ and $x_2$ are called the **components** of the vector $\mathbf{x}$. Two vectors in the plane $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ are **equal** if and only if $u_1 = v_1$ and $u_2 = v_2$.

**EXAMPLE 1** Vectors in the Plane

**a.** To represent $\mathbf{u} = (2, 3)$, draw a directed line segment from the origin to the point $(2, 3)$, as shown in Figure 4.2(a).

**b.** To represent $\mathbf{v} = (-1, 2)$, draw a directed line segment from the origin to the point $(-1, 2)$, as shown in Figure 4.2(b).

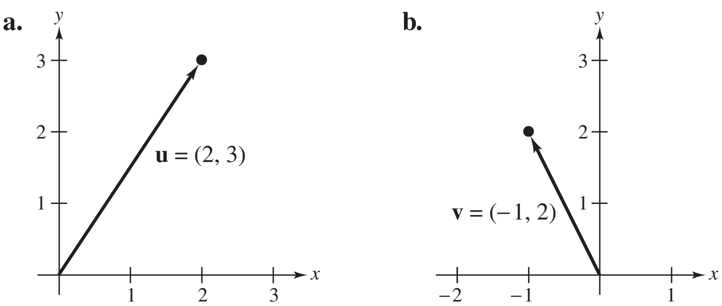



**Figure 4.2**

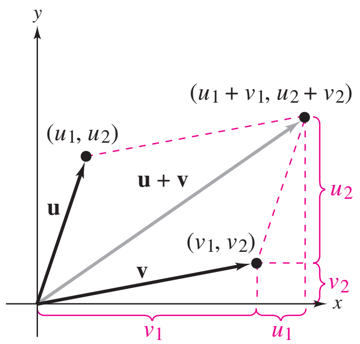
## VECTOR OPERATIONS

One basic vector operation is **vector addition.** To add two vectors in the plane, add their corresponding components. That is, the **sum** of $\mathbf{u}$ and $\mathbf{v}$ is the vector

$$\mathbf{u} + \mathbf{v} = (u_1, u_2) + (v_1, v_2) = (u_1 + v_1, u_2 + v_2).$$

Geometrically, the sum of two vectors in the plane can be represented by the diagonal of a parallelogram having $\mathbf{u}$ and $\mathbf{v}$ as its adjacent sides, as shown in Figure 4.3.

In the next example, one of the vectors you will add is the vector (0, 0), called the **zero vector.** The zero vector is denoted by $\mathbf{0}$.



Vector Addition

**Figure 4.3**

### EXAMPLE 2 Adding Two Vectors in the Plane

Find the sum of the vectors.

**a.** $\mathbf{u} = (1, 4)$, $\mathbf{v} = (2, -2)$

**b.** $\mathbf{u} = (3, -2)$, $\mathbf{v} = (-3, 2)$

**c.** $\mathbf{u} = (2, 1)$, $\mathbf{v} = (0, 0)$

**SOLUTION**

**a.** $\mathbf{u} + \mathbf{v} = (1, 4) + (2, -2) = (3, 2)$

**b.** $\mathbf{u} + \mathbf{v} = (3, -2) + (-3, 2) = (0, 0) = \mathbf{0}$

**c.** $\mathbf{u} + \mathbf{v} = (2, 1) + (0, 0) = (2, 1)$

Figure 4.4 gives a graphical representation of each sum.

**a.** **b.** **c.**

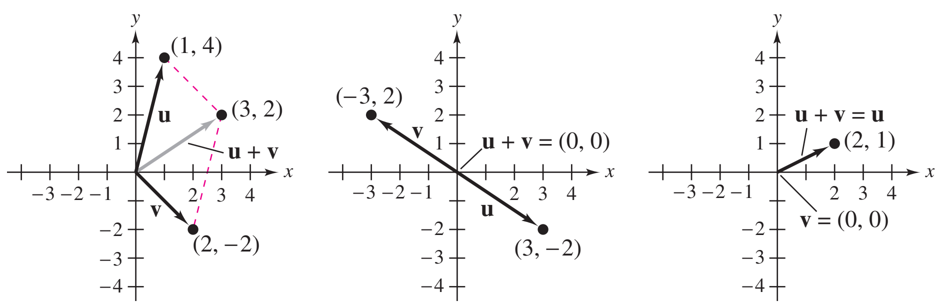



**Figure 4.4**

***Simulation***

To explore this concept further with an electronic simulation, and for syntax regarding specific programs involving Example 2, please visit *www.cengagebrain.com*. Similar exercises and projects are also available on the website.

Another basic vector operation is **scalar multiplication.** To multiply a vector $\mathbf{v}$ by a scalar $c$, multiply each of the components of $\mathbf{v}$ by $c$. That is,

$$c\mathbf{v} = c(v_1, v_2) = (cv_1, cv_2).$$

Recall from Chapter 2 that the word *scalar* is used to mean a real number. Historically, this usage arose from the fact that multiplying a vector by a real number changes the "scale" of the vector. For instance, when a vector $\mathbf{v}$ is multiplied by 2, the resulting vector $2\mathbf{v}$ is a vector having the same direction as $\mathbf{v}$ and twice the length. In general, for a scalar $c$, the vector $c\mathbf{v}$ will be $|c|$ times as long as $\mathbf{v}$. If $c$ is positive, then $c\mathbf{v}$ and $\mathbf{v}$ have the same direction, and if $c$ is negative, then $c\mathbf{v}$ and $\mathbf{v}$ have opposite directions. Figure 4.5 shows this.

The product of a vector $\mathbf{v}$ and the scalar $-1$ is denoted by

$$-\mathbf{v} = (-1)\mathbf{v}.$$

The vector $-\mathbf{v}$ is called the **negative** of $\mathbf{v}$. The **difference** of $\mathbf{u}$ and $\mathbf{v}$ is defined as

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$$

and you can say $\mathbf{v}$ is **subtracted** from $\mathbf{u}$.

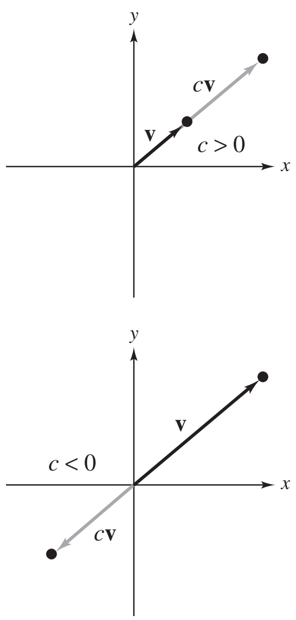



**Figure 4.5**

## EXAMPLE 3 Operations with Vectors in the Plane

Let $\mathbf{v} = (-2, 5)$ and $\mathbf{u} = (3, 4)$. Find each of the following vectors.

**a.** $\frac{1}{2}\mathbf{v}$ **b.** $\mathbf{u} - \mathbf{v}$ **c.** $\frac{1}{2}\mathbf{v} + \mathbf{u}$

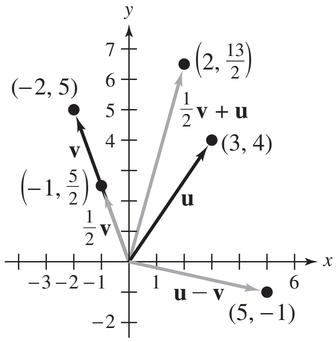



**Figure 4.6**

### SOLUTION

**a.** Because $\mathbf{v} = (-2, 5)$, $\frac{1}{2}\mathbf{v} = \left(\frac{1}{2}(-2), \frac{1}{2}(5)\right) = \left(-1, \frac{5}{2}\right)$.

**b.** By the definition of vector subtraction, $\mathbf{u} - \mathbf{v} = (3 - (-2), 4 - 5) = (5, -1)$.

**c.** Using the result of part (a), $\frac{1}{2}\mathbf{v} + \mathbf{u} = \left(-1, \frac{5}{2}\right) + (3, 4) = \left(2, \frac{13}{2}\right)$.

Figure 4.6 gives a graphical representation of these vector operations.

Vector addition and scalar multiplication share many properties with matrix addition and scalar multiplication. The ten properties listed in the next theorem play a fundamental role in linear algebra. In fact, in the next section you will see that it is precisely these ten properties that help define a vector space.

### THEOREM 4.1 Properties of Vector Addition and Scalar Multiplication in the Plane

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in the plane, and let $c$ and $d$ be scalars.

| | |
|---|---|
| **1.** $\mathbf{u} + \mathbf{v}$ is a vector in the plane. | Closure under addition |
| **2.** $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ | Commutative property of addition |
| **3.** $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ | Associative property of addition |
| **4.** $\mathbf{u} + \mathbf{0} = \mathbf{u}$ | Additive identity property |
| **5.** $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ | Additive inverse property |
| **6.** $c\mathbf{u}$ is a vector in the plane. | Closure under scalar multiplication |
| **7.** $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ | Distributive property |
| **8.** $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ | Distributive property |
| **9.** $c(d\mathbf{u}) = (cd)\mathbf{u}$ | Associative property of multiplication |
| **10.** $1(\mathbf{u}) = \mathbf{u}$ | Multiplicative identity property |

### REMARK

Note that the associative property of vector addition allows you to write such expressions as $\mathbf{u} + \mathbf{v} + \mathbf{w}$ without ambiguity, because you obtain the same vector sum regardless of which addition is performed first.

### PROOF

The proof of each property is straightforward. For instance, to prove the associative property of vector addition, write

$$\begin{aligned}
(\mathbf{u} + \mathbf{v}) + \mathbf{w} &= [(u_1, u_2) + (v_1, v_2)] + (w_1, w_2) \\
&= (u_1 + v_1, u_2 + v_2) + (w_1, w_2) \\
&= ((u_1 + v_1) + w_1, (u_2 + v_2) + w_2) \\
&= (u_1 + (v_1 + w_1), u_2 + (v_2 + w_2)) \\
&= (u_1, u_2) + (v_1 + w_1, v_2 + w_2) \\
&= \mathbf{u} + (\mathbf{v} + \mathbf{w}).
\end{aligned}$$

Similarly, to prove the right distributive property of scalar multiplication over addition, write

$$\begin{aligned}
(c + d)\mathbf{u} &= (c + d)(u_1, u_2) \\
&= ((c + d)u_1, (c + d)u_2) \\
&= (cu_1 + du_1, cu_2 + du_2) \\
&= (cu_1, cu_2) + (du_1, du_2) \\
&= c\mathbf{u} + d\mathbf{u}.
\end{aligned}$$

The proofs of the other eight properties are left as an exercise. (See Exercise 57.)

## VECTORS IN $R^n$

The discussion of vectors in the plane can be extended to a discussion of vectors in $n$-space. An **ordered $n$-tuple** represents a vector in $n$-space. For instance, an ordered triple has the form $(x_1, x_2, x_3)$, an ordered quadruple has the form $(x_1, x_2, x_3, x_4)$, and a general ordered $n$-tuple has the form $(x_1, x_2, x_3, \ldots, x_n)$. The set of all $n$-tuples is called **$n$-space** and is denoted by $R^n$.

$$\begin{aligned}
R^1 &= \text{1-space} = \text{set of all real numbers}\\
R^2 &= \text{2-space} = \text{set of all ordered pairs of real numbers}\\
R^3 &= \text{3-space} = \text{set of all ordered triples of real numbers}\\
&\vdots\\
R^n &= n\text{-space} = \text{set of all ordered } n\text{-tuples of real numbers}
\end{aligned}$$

An $n$-tuple $(x_1, x_2, x_3, \ldots, x_n)$ can be viewed as a **point** in $R^n$ with the $x_i$'s as its coordinates or as a **vector**

$$\mathbf{x} = (x_1, x_2, x_3, \ldots, x_n) \qquad \text{Vector in } R^n$$

with the $x_i$'s as its components. As with vectors in the plane (or $R^2$), two vectors in $R^n$ are **equal** if and only if corresponding components are equal. [In the case of $n = 2$ or $n = 3$, the familiar $(x, y)$ or $(x, y, z)$ notation is used occasionally.]

The sum of two vectors in $R^n$ and the scalar multiple of a vector in $R^n$ are called the **standard operations in $R^n$** and are defined as follows.

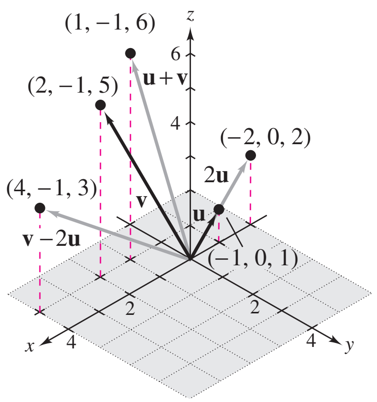



**Figure 4.7**

### Definitions of Vector Addition and Scalar Multiplication in $R^n$

Let $\mathbf{u} = (u_1, u_2, u_3, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, v_3, \ldots, v_n)$ be vectors in $R^n$ and let $c$ be a real number. Then the sum of $\mathbf{u}$ and $\mathbf{v}$ is defined as the vector

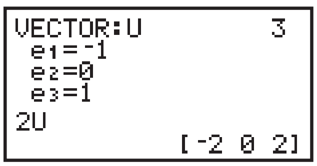
$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3, \ldots, u_n + v_n)$$

and the **scalar multiple** of $\mathbf{u}$ by $c$ is defined as the vector

$$c\mathbf{u} = (cu_1, cu_2, cu_3, \ldots, cu_n).$$

### TECHNOLOGY

Many graphing utilities and software programs can perform vector addition and scalar multiplication. If you use a graphing utility, then you may verify Example 4(b) as follows. The **Online Technology Guide,** available at *www.cengagebrain.com*, provides syntax for programs applicable to Example 4(b).




As with 2-space, the **negative** of a vector in $R^n$ is defined as

$$-\mathbf{u} = (-u_1, -u_2, -u_3, \ldots, -u_n)$$

and the **difference** of two vectors in $R^n$ is defined as

$$\mathbf{u} - \mathbf{v} = (u_1 - v_1, u_2 - v_2, u_3 - v_3, \ldots, u_n - v_n).$$

The **zero vector** in $R^n$ is denoted by $\mathbf{0} = (0, 0, \ldots, 0)$.

### EXAMPLE 4 Vector Operations in $R^3$

Let $\mathbf{u} = (-1, 0, 1)$ and $\mathbf{v} = (2, -1, 5)$ in $R^3$. Find each of the following vectors.

**a.** $\mathbf{u} + \mathbf{v}$ **b.** $2\mathbf{u}$ **c.** $\mathbf{v} - 2\mathbf{u}$

**SOLUTION**

**a.** To add two vectors, add their corresponding components, as follows.

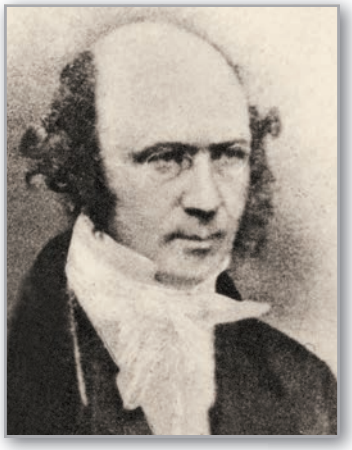
$$\mathbf{u} + \mathbf{v} = (-1, 0, 1) + (2, -1, 5) = (1, -1, 6)$$

**b.** To multiply a vector by a scalar, multiply each component by the scalar, as follows.

$$2\mathbf{u} = 2(-1, 0, 1) = (-2, 0, 2)$$

**c.** Using the result of part (b), $\mathbf{v} - 2\mathbf{u} = (2, -1, 5) - (-2, 0, 2) = (4, -1, 3)$.

Figure 4.7 gives a graphical representation of these vector operations in $R^3$.



**William Rowan Hamilton (1805–1865)**

Hamilton is considered to be Ireland's most famous mathematician. In 1828, he published an impressive work on optics entitled *A Theory of Systems of Rays*. In it, Hamilton included some of his own methods for working with systems of linear equations. He also introduced the notion of the characteristic equation of a matrix (see Section 7.1). Hamilton's work led to the development of modern vector notation. We still use his **i**, **j**, and **k** notation for the standard unit vectors in $R^3$ (see Section 5.1).

The following properties of vector addition and scalar multiplication for vectors in $R^n$ are similar to those listed in Theorem 4.1 for vectors in $R^2$. Their proofs, based on the definitions of vector addition and scalar multiplication in $R^n$, are left as an exercise. (See Exercise 58.)

### THEOREM 4.2 Properties of Vector Addition and Scalar Multiplication in $R^n$

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in $R^n$, and let $c$ and $d$ be scalars.

| | |
|---|---|
| **1.** $\mathbf{u} + \mathbf{v}$ is a vector in $R^n$. | Closure under addition |
| **2.** $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ | Commutative property of addition |
| **3.** $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ | Associative property of addition |
| **4.** $\mathbf{u} + \mathbf{0} = \mathbf{u}$ | Additive identity property |
| **5.** $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ | Additive inverse property |
| **6.** $c\mathbf{u}$ is a vector in $R^n$. | Closure under scalar multiplication |
| **7.** $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ | Distributive property |
| **8.** $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ | Distributive property |
| **9.** $c(d\mathbf{u}) = (cd)\mathbf{u}$ | Associative property of multiplication |
| **10.** $1(\mathbf{u}) = \mathbf{u}$ | Multiplicative identity property |

Using the ten properties from Theorem 4.2, you can perform algebraic manipulations with vectors in $R^n$ in much the same way as you do with real numbers, as demonstrated in the next example.

### EXAMPLE 5 Vector Operations in $R^4$

Let $\mathbf{u} = (2, -1, 5, 0)$, $\mathbf{v} = (4, 3, 1, -1)$, and $\mathbf{w} = (-6, 2, 0, 3)$ be vectors in $R^4$. Find $\mathbf{x}$ using each equation.

**a.** $\mathbf{x} = 2\mathbf{u} - (\mathbf{v} + 3\mathbf{w})$

**b.** $3(\mathbf{x} + \mathbf{w}) = 2\mathbf{u} - \mathbf{v} + \mathbf{x}$

**SOLUTION**

**a.** Using the properties listed in Theorem 4.2, you have

$$\begin{aligned}
\mathbf{x} &= 2\mathbf{u} - (\mathbf{v} + 3\mathbf{w}) \\
&= 2\mathbf{u} - \mathbf{v} - 3\mathbf{w} \\
&= 2(2, -1, 5, 0) - (4, 3, 1, -1) - 3(-6, 2, 0, 3) \\
&= (4, -2, 10, 0) - (4, 3, 1, -1) - (-18, 6, 0, 9) \\
&= (4 - 4 + 18, -2 - 3 - 6, 10 - 1 - 0, 0 + 1 - 9) \\
&= (18, -11, 9, -8).
\end{aligned}$$

**b.** Begin by solving for $\mathbf{x}$ as follows.

$$\begin{aligned}
3(\mathbf{x} + \mathbf{w}) &= 2\mathbf{u} - \mathbf{v} + \mathbf{x} \\
3\mathbf{x} + 3\mathbf{w} &= 2\mathbf{u} - \mathbf{v} + \mathbf{x} \\
3\mathbf{x} - \mathbf{x} &= 2\mathbf{u} - \mathbf{v} - 3\mathbf{w} \\
2\mathbf{x} &= 2\mathbf{u} - \mathbf{v} - 3\mathbf{w} \\
\mathbf{x} &= \tfrac{1}{2}(2\mathbf{u} - \mathbf{v} - 3\mathbf{w})
\end{aligned}$$

Using the result of part (a),

$$\begin{aligned}
\mathbf{x} &= \tfrac{1}{2}(18, -11, 9, -8) \\
&= \left(9, -\tfrac{11}{2}, \tfrac{9}{2}, -4\right).
\end{aligned}$$

The zero vector $\mathbf{0}$ in $R^n$ is called the **additive identity** in $R^n$. Similarly, the vector $-\mathbf{v}$ is called the **additive inverse** of $\mathbf{v}$. The following theorem summarizes several important properties of the additive identity and additive inverse in $R^n$.

**REMARK**

Note that in Properties 3 and 5, two different zeros are used, the scalar 0 and the vector **0**.

**THEOREM 4.3 Properties of Additive Identity and Additive Inverse**

Let $\mathbf{v}$ be a vector in $R^n$, and let $c$ be a scalar. Then the following properties are true.

1. The additive identity is unique. That is, if $\mathbf{v} + \mathbf{u} = \mathbf{v}$, then $\mathbf{u} = \mathbf{0}$.
2. The additive inverse of $\mathbf{v}$ is unique. That is, if $\mathbf{v} + \mathbf{u} = \mathbf{0}$, then $\mathbf{u} = -\mathbf{v}$.
3. $0\mathbf{v} = \mathbf{0}$
4. $c\mathbf{0} = \mathbf{0}$
5. If $c\mathbf{v} = \mathbf{0}$, then $c = 0$ or $\mathbf{v} = \mathbf{0}$.
6. $-(-\mathbf{v}) = \mathbf{v}$

**PROOF**

To prove the first property, assume $\mathbf{v} + \mathbf{u} = \mathbf{v}$. Then Theorem 4.2 justifies the following steps.

| | |
|---|---|
| $\mathbf{v} + \mathbf{u} = \mathbf{v}$ | Given |
| $(\mathbf{v} + \mathbf{u}) + (-\mathbf{v}) = \mathbf{v} + (-\mathbf{v})$ | Add $-\mathbf{v}$ to both sides. |
| $(\mathbf{v} + \mathbf{u}) + (-\mathbf{v}) = \mathbf{0}$ | Additive inverse |
| $(\mathbf{u} + \mathbf{v}) + (-\mathbf{v}) = \mathbf{0}$ | Commutative property |
| $\mathbf{u} + (\mathbf{v} + (-\mathbf{v})) = \mathbf{0}$ | Associative property |
| $\mathbf{u} + \mathbf{0} = \mathbf{0}$ | Additive inverse |
| $\mathbf{u} = \mathbf{0}$ | Additive identity |

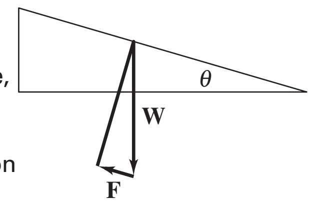
To prove the second property, assume $\mathbf{v} + \mathbf{u} = \mathbf{0}$, and again use Theorem 4.2 to justify the following steps.

| | |
|---|---|
| $\mathbf{v} + \mathbf{u} = \mathbf{0}$ | Given |
| $(-\mathbf{v}) + (\mathbf{v} + \mathbf{u}) = (-\mathbf{v}) + \mathbf{0}$ | Add $-\mathbf{v}$ to both sides. |
| $(-\mathbf{v}) + (\mathbf{v} + \mathbf{u}) = -\mathbf{v}$ | Additive identity |
| $[(-\mathbf{v}) + \mathbf{v}] + \mathbf{u} = -\mathbf{v}$ | Associative property |
| $\mathbf{0} + \mathbf{u} = -\mathbf{v}$ | Additive inverse |
| $\mathbf{u} + \mathbf{0} = -\mathbf{v}$ | Commutative property |
| $\mathbf{u} = -\mathbf{v}$ | Additive identity |

As you gain experience in reading and writing proofs involving vector algebra, you will not need to list as many steps as shown above. For now, however, it is a good idea to list as many steps as possible. The proofs of the other four properties are left as exercises. (See Exercises 61–64.)

**LINEAR ALGEBRA APPLIED**

Vectors have a wide variety of applications in engineering and the physical sciences. For instance, to determine the amount of force required to pull an object up a ramp that has an angle of elevation $\theta$, use the figure at the right.




In the figure, the vector labeled **W** represents the weight of the object, and the vector labeled **F** represents the required force. Using similar triangles and some trigonometry, the required force is $\mathbf{F} = \mathbf{W} \sin \theta$. Try verifying this.

## LINEAR COMBINATIONS OF VECTORS

An important type of problem in linear algebra involves writing one vector $\mathbf{x}$ as the sum of scalar multiples of other vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots,$ and $\mathbf{v}_n$. That is, for scalars $c_1, c_2, \ldots, c_n$,

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n.$$

The vector $\mathbf{x}$ is called a **linear combination** of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots,$ and $\mathbf{v}_n$.

**DISCOVERY**

1. Is the vector (1, 1) a linear combination of the vectors (1, 2) and (−2, −4)? Graph these vectors and explain your answer geometrically.
2. Similarly, determine whether the vector (1, 1) is a linear combination of the vectors (1, 2) and (2, 1).
3. What is the geometric significance of questions 1 and 2?
4. Is every vector in $R^2$ a linear combination of the vectors (1, 2) and (2, 1)? Give a geometric explanation for your answer.

### EXAMPLE 6 Writing a Vector as a Linear Combination of Other Vectors

Let $\mathbf{x} = (-1, -2, -2)$, $\mathbf{u} = (0, 1, 4)$, $\mathbf{v} = (-1, 1, 2)$, and $\mathbf{w} = (3, 1, 2)$ in $R^3$. Find scalars $a$, $b$, and $c$ such that

$$\mathbf{x} = a\mathbf{u} + b\mathbf{v} + c\mathbf{w}.$$

**SOLUTION**

Write

$$\begin{aligned}\overbrace{(-1, -2, -2)}^{\mathbf{x}} &= a\overbrace{(0, 1, 4)}^{\mathbf{u}} + b\overbrace{(-1, 1, 2)}^{\mathbf{v}} + c\overbrace{(3, 1, 2)}^{\mathbf{w}} \\ &= (-b + 3c, a + b + c, 4a + 2b + 2c)\end{aligned}$$

and equate corresponding components so that they form the following system of three linear equations in $a$, $b$, and $c$.

$$\begin{aligned} -b + 3c &= -1 &&\text{Equation from first component}\\ a + b + c &= -2 &&\text{Equation from second component}\\ 4a + 2b + 2c &= -2 &&\text{Equation from third component}\end{aligned}$$

Solve for $a$, $b$, and $c$ to get $a = 1$, $b = -2$, and $c = -1$. As a linear combination of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$,

$$\mathbf{x} = \mathbf{u} - 2\mathbf{v} - \mathbf{w}.$$

Try using vector addition and scalar multiplication to check this result.

You will often find it useful to represent a vector $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ in $R^n$ as either a $1 \times n$ row matrix (row vector) or an $n \times 1$ column matrix (column vector). This approach is valid because the matrix operations of addition and scalar multiplication give the same results as the corresponding vector operations. That is, the matrix sums

$$\begin{aligned}\mathbf{u} + \mathbf{v} &= [u_1 \quad u_2 \quad \cdots \quad u_n] + [v_1 \quad v_2 \quad \cdots \quad v_n]\\ &= [u_1 + v_1 \quad u_2 + v_2 \quad \cdots \quad u_n + v_n]\end{aligned}$$

and

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix}$$

yield the same results as the vector operation of addition,

$$\begin{aligned}\mathbf{u} + \mathbf{v} &= (u_1, u_2, \ldots, u_n) + (v_1, v_2, \ldots, v_n)\\ &= (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n).\end{aligned}$$

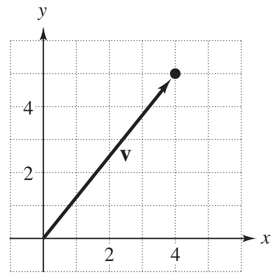
The same argument applies to scalar multiplication. The only difference in the three notations is how the components are displayed.

# 4.1 Exercises

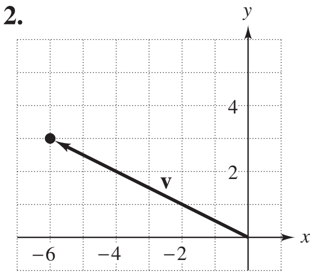
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

**Finding the Component Form of a Vector** In Exercises 1 and 2, find the component form of the given vector.

**1.**




**2.**




**Representing a Vector** In Exercises 3–6, use a directed line segment to represent the vector.

**3.** $\mathbf{u} = (2, -4)$

**4.** $\mathbf{v} = (-2, 3)$

**5.** $\mathbf{u} = (-3, -4)$

**6.** $\mathbf{v} = (-2, -5)$

**Finding the Sum of Two Vectors** In Exercises 7–10, find the sum of the vectors and illustrate the sum geometrically.

**7.** $\mathbf{u} = (1, 3), \mathbf{v} = (2, -2)$

**8.** $\mathbf{u} = (-1, 4), \mathbf{v} = (4, -3)$

**9.** $\mathbf{u} = (2, -3), \mathbf{v} = (-3, -1)$

**10.** $\mathbf{u} = (4, -2), \mathbf{v} = (-2, -3)$

**Vector Operations** In Exercises 11–16, find the vector v and illustrate the specified vector operations geometrically, where $\mathbf{u} = (-2, 3)$ and $\mathbf{w} = (-3, -2)$.

**11.** $\mathbf{v} = \frac{3}{2}\mathbf{u}$

**12.** $\mathbf{v} = \mathbf{u} + \mathbf{w}$

**13.** $\mathbf{v} = \mathbf{u} + 2\mathbf{w}$

**14.** $\mathbf{v} = -\mathbf{u} + \mathbf{w}$

**15.** $\mathbf{v} = \frac{1}{2}(3\mathbf{u} + \mathbf{w})$

**16.** $\mathbf{v} = \mathbf{u} - 2\mathbf{w}$

**17.** Given the vector $\mathbf{v} = (2, 1)$, sketch (a) $2\mathbf{v}$, (b) $-3\mathbf{v}$, and (c) $\frac{1}{2}\mathbf{v}$.

**18.** Given the vector $\mathbf{v} = (3, -2)$, sketch (a) $4\mathbf{v}$, (b) $-\frac{1}{2}\mathbf{v}$, and (c) $0\mathbf{v}$.

**Vector Operations** In Exercises 19–24, let $\mathbf{u} = (1, 2, 3)$, $\mathbf{v} = (2, 2, -1)$, and $\mathbf{w} = (4, 0, -4)$.

**19.** Find $\mathbf{u} - \mathbf{v}$ and $\mathbf{v} - \mathbf{u}$.

**20.** Find $\mathbf{u} - \mathbf{v} + 2\mathbf{w}$.

**21.** Find $2\mathbf{u} + 4\mathbf{v} - \mathbf{w}$.

**22.** Find $5\mathbf{u} - 3\mathbf{v} - \frac{1}{2}\mathbf{w}$.

**23.** Find $\mathbf{z}$, where $2\mathbf{z} - 3\mathbf{u} = \mathbf{w}$.

**24.** Find $\mathbf{z}$, where $2\mathbf{u} + \mathbf{v} - \mathbf{w} + 3\mathbf{z} = \mathbf{0}$.

**25.** Given the vector $\mathbf{v} = (1, 2, 2)$, sketch (a) $2\mathbf{v}$, (b) $-\mathbf{v}$, and (c) $\frac{1}{2}\mathbf{v}$.

**26.** Given the vector $\mathbf{v} = (2, 0, 1)$, sketch (a) $-\mathbf{v}$, (b) $2\mathbf{v}$, and (c) $\frac{1}{2}\mathbf{v}$.

**27.** Which of the following vectors are scalar multiples of $\mathbf{z} = (3, 2, -5)$?

(a) $\mathbf{v} = \left(2, \frac{4}{3}, -\frac{10}{3}\right)$ (b) $\mathbf{w} = (6, 4, 10)$

**28.** Which of the following vectors are scalar multiples of $\mathbf{z} = \left(\frac{1}{2}, -\frac{2}{3}, \frac{3}{4}\right)$?

(a) $\mathbf{u} = (6, -4, 9)$

(b) $\mathbf{v} = \left(-1, \frac{4}{3}, -\frac{3}{2}\right)$

**Vector Operations** In Exercises 29 and 30, find (a) $\mathbf{u} - \mathbf{v}$, (b) $2(\mathbf{u} + 3\mathbf{v})$, and (c) $2\mathbf{v} - \mathbf{u}$.

**29.** $\mathbf{u} = (4, 0, -3, 5)$, $\mathbf{v} = (0, 2, 5, 4)$

**30.** $\mathbf{u} = (0, 4, 3, 4, 4)$, $\mathbf{v} = (6, 8, -3, 3, -5)$

**Vector Operations** In Exercises 31 and 32, use a graphing utility with matrix capabilities to find the following, where $\mathbf{u} = (1, 2, -3, 1)$, $\mathbf{v} = (0, 2, -1, -2)$, and $\mathbf{w} = (2, -2, 1, 3)$.

**31.** (a) $\mathbf{u} + 2\mathbf{v}$
(b) $\mathbf{w} - 3\mathbf{u}$
(c) $4\mathbf{v} + \frac{1}{2}\mathbf{u} - \mathbf{w}$

**32.** (a) $\mathbf{v} + 3\mathbf{w}$
(b) $2\mathbf{w} - \frac{1}{2}\mathbf{u}$
(c) $\frac{1}{2}(4\mathbf{v} - 3\mathbf{u} + \mathbf{w})$

**Solving a Vector Equation** In Exercises 33–36, solve for w, where $\mathbf{u} = (1, -1, 0, 1)$ and $\mathbf{v} = (0, 2, 3, -1)$.

**33.** $2\mathbf{w} = \mathbf{u} - 3\mathbf{v}$

**34.** $\mathbf{w} + \mathbf{u} = -\mathbf{v}$

**35.** $\frac{1}{2}\mathbf{w} = 2\mathbf{u} + 3\mathbf{v}$

**36.** $\mathbf{w} + 3\mathbf{v} = -2\mathbf{u}$

**Solving a Vector Equation** In Exercises 37 and 38, find w such that $2\mathbf{u} + \mathbf{v} - 3\mathbf{w} = \mathbf{0}$.

**37.** $\mathbf{u} = (0, 2, 7, 5)$, $\mathbf{v} = (-3, 1, 4, -8)$

**38.** $\mathbf{u} = (0, 0, -8, 1)$, $\mathbf{v} = (1, -8, 0, 7)$

**Writing a Linear Combination** In Exercises 39–44, write v as a linear combination of u and w, if possible, where $\mathbf{u} = (1, 2)$ and $\mathbf{w} = (1, -1)$.

**39.** $\mathbf{v} = (2, 1)$

**40.** $\mathbf{v} = (0, 3)$

**41.** $\mathbf{v} = (3, 0)$

**42.** $\mathbf{v} = (1, -1)$

**43.** $\mathbf{v} = (-1, -2)$

**44.** $\mathbf{v} = (1, -4)$

**Writing a Linear Combination** In Exercises 45–48, write v as a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$, if possible.

**45.** $\mathbf{v} = (10, 1, 4)$, $\mathbf{u}_1 = (2, 3, 5)$, $\mathbf{u}_2 = (1, 2, 4)$, $\mathbf{u}_3 = (-2, 2, 3)$

**46.** $\mathbf{v} = (-1, 7, 2)$, $\mathbf{u}_1 = (1, 3, 5)$, $\mathbf{u}_2 = (2, -1, 3)$, $\mathbf{u}_3 = (-3, 2, -4)$

**47.** $\mathbf{v} = (0, 5, 3, 0)$, $\mathbf{u}_1 = (1, 1, 2, 2)$, $\mathbf{u}_2 = (2, 3, 5, 6)$, $\mathbf{u}_3 = (-3, 1, -4, 2)$

**48.** $\mathbf{v} = (2, 5, -4, 0)$, $\mathbf{u}_1 = (1, 3, 2, 1)$, $\mathbf{u}_2 = (2, -2, -5, 4)$, $\mathbf{u}_3 = (2, -1, 3, 6)$

**Writing a Linear Combination** **In Exercises 49 and 50, use a software program or a graphing utility with matrix capabilities to write v as a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$, $\mathbf{u}_4$, and $\mathbf{u}_5$. Then verify your solution.**

**49.** $\mathbf{v} = (5, 3, -11, 11, 9)$
$\mathbf{u}_1 = (1, 2, -3, 4, -1)$
$\mathbf{u}_2 = (1, 2, 0, 2, 1)$
$\mathbf{u}_3 = (0, 1, 1, 1, -4)$
$\mathbf{u}_4 = (2, 1, -1, 2, 1)$
$\mathbf{u}_5 = (0, 2, 2, -1, -1)$

**50.** $\mathbf{v} = (5, 8, 7, -2, 4)$
$\mathbf{u}_1 = (1, 1, -1, 2, 1)$
$\mathbf{u}_2 = (2, 1, 2, -1, 1)$
$\mathbf{u}_3 = (1, 2, 0, 1, 2)$
$\mathbf{u}_4 = (0, 2, 0, 1, -4)$
$\mathbf{u}_5 = (1, 1, 2, -1, 2)$

**True or False?** **In Exercises 51 and 52, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.**

**51.** (a) Two vectors in $R^n$ are equal if and only if their corresponding components are equal.
(b) The vector $-\mathbf{v}$ is called the additive identity of the vector $\mathbf{v}$.

**52.** (a) To subtract two vectors in $R^n$, subtract their corresponding components.
(b) The zero vector $\mathbf{0}$ in $R^n$ is defined as the additive inverse of a vector.

**Writing a Linear Combination** **In Exercises 53 and 54, the zero vector $\mathbf{0} = (0, 0, 0)$ can be written as a linear combination of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ because $\mathbf{0} = 0\mathbf{v}_1 + 0\mathbf{v}_2 + 0\mathbf{v}_3$. This is called the *trivial* solution. Can you find a *nontrivial* way of writing $\mathbf{0}$ as a linear combination of the three vectors?**

**53.** $\mathbf{v}_1 = (1, 0, 1)$, $\mathbf{v}_2 = (-1, 1, 2)$, $\mathbf{v}_3 = (0, 1, 4)$

**54.** $\mathbf{v}_1 = (1, 0, 1)$, $\mathbf{v}_2 = (-1, 1, 2)$, $\mathbf{v}_3 = (0, 1, 3)$

**55.** Illustrate properties 1–10 of Theorem 4.2 for $\mathbf{u} = (2, -1, 3, 6)$, $\mathbf{v} = (1, 4, 0, 1)$, $\mathbf{w} = (3, 0, 2, 0)$, $c = 5$, and $d = -2$.

**56.** Illustrate properties 1–10 of Theorem 4.2 for $\mathbf{u} = (2, -1, 3)$, $\mathbf{v} = (3, 4, 0)$, $\mathbf{w} = (7, 8, -4)$, $c = 2$, and $d = -1$.

**57.** **Proof** Complete the proof of Theorem 4.1.

**58.** **Proof** Prove each property of vector addition and scalar multiplication from Theorem 4.2.

**59.** **Writing** Let $A\mathbf{x} = \mathbf{b}$ be a system of $m$ linear equations in $n$ variables. Designate the columns of $A$ as $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$. When $\mathbf{b}$ is a linear combination of these $n$ column vectors, explain why this implies that the linear system is consistent. Illustrate your answer with appropriate examples. What can you conclude about the linear system when $\mathbf{b}$ is not a linear combination of the columns of $A$?

**60.** CAPSTONE Consider the vectors $\mathbf{u} = (3, -4)$ and $\mathbf{v} = (9, 1)$.
(a) Use directed line segments to represent each vector graphically.
(b) Find $\mathbf{u} + \mathbf{v}$.
(c) Find $2\mathbf{v} - \mathbf{u}$.
(d) Write $\mathbf{w} = (39, 0)$ as a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

**Proof** **In Exercises 61–64, complete the proofs of the remaining properties of Theorem 4.3 by supplying the justification for each step. Use the properties of vector addition and scalar multiplication from Theorem 4.2.**

**61.** Property 3: $0\mathbf{v} = \mathbf{0}$

| | |
|---|---|
| $0\mathbf{v} = (0 + 0)\mathbf{v}$ | a. ______ |
| $0\mathbf{v} = 0\mathbf{v} + 0\mathbf{v}$ | b. ______ |
| $0\mathbf{v} + (-0\mathbf{v}) = (0\mathbf{v} + 0\mathbf{v}) + (-0\mathbf{v})$ | c. ______ |
| $\mathbf{0} = 0\mathbf{v} + (0\mathbf{v} + (-0\mathbf{v}))$ | d. ______ |
| $\mathbf{0} = 0\mathbf{v} + \mathbf{0}$ | e. ______ |
| $\mathbf{0} = 0\mathbf{v}$ | f. ______ |

**62.** Property 4: $c\mathbf{0} = \mathbf{0}$

| | |
|---|---|
| $c\mathbf{0} = c(\mathbf{0} + \mathbf{0})$ | a. ______ |
| $c\mathbf{0} = c\mathbf{0} + c\mathbf{0}$ | b. ______ |
| $c\mathbf{0} + (-c\mathbf{0}) = (c\mathbf{0} + c\mathbf{0}) + (-c\mathbf{0})$ | c. ______ |
| $\mathbf{0} = c\mathbf{0} + (c\mathbf{0} + (-c\mathbf{0}))$ | d. ______ |
| $\mathbf{0} = c\mathbf{0} + \mathbf{0}$ | e. ______ |
| $\mathbf{0} = c\mathbf{0}$ | f. ______ |

**63.** Property 5: If $c\mathbf{v} = \mathbf{0}$, then $c = 0$ or $\mathbf{v} = \mathbf{0}$. If $c = 0$, then you are done. If $c \neq 0$, then $c^{-1}$ exists, and you have

| | |
|---|---|
| $c^{-1}(c\mathbf{v}) = c^{-1}\mathbf{0}$ | a. ______ |
| $(c^{-1}c)\mathbf{v} = \mathbf{0}$ | b. ______ |
| $1\mathbf{v} = \mathbf{0}$ | c. ______ |
| $\mathbf{v} = \mathbf{0}.$ | d. ______ |

**64.** Property 6: $-(-\mathbf{v}) = \mathbf{v}$

| | |
|---|---|
| $-(-\mathbf{v}) + (-\mathbf{v}) = \mathbf{0}$ and $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$ | a. ______ |
| $-(-\mathbf{v}) + (-\mathbf{v}) = \mathbf{v} + (-\mathbf{v})$ | b. ______ |
| $-(-\mathbf{v}) + (-\mathbf{v}) + \mathbf{v} = \mathbf{v} + (-\mathbf{v}) + \mathbf{v}$ | c. ______ |
| $-(-\mathbf{v}) + ((-\mathbf{v}) + \mathbf{v}) = \mathbf{v} + ((-\mathbf{v}) + \mathbf{v})$ | d. ______ |
| $-(-\mathbf{v}) + \mathbf{0} = \mathbf{v} + \mathbf{0}$ | e. ______ |
| $-(-\mathbf{v}) = \mathbf{v}$ | f. ______ |

**65.** **Writing** How could you describe vector subtraction geometrically? What is the relationship between vector subtraction and the basic vector operations of addition and scalar multiplication?

# 4.2 Vector Spaces

- Define a vector space and recognize some important vector spaces.
- Show that a given set is not a vector space.

## DEFINITION OF A VECTOR SPACE

Theorem 4.2 lists ten special properties of vector addition and scalar multiplication in $R^n$. Suitable definitions of addition and scalar multiplication reveal that many other mathematical quantities (such as matrices, polynomials, and functions) also share these ten properties. *Any* set that satisfies these properties (or **axioms**) is called a **vector space,** and the objects in the set are called **vectors.**

It is important to realize that the following definition of a vector space is precisely that—a *definition*. You do not need to prove anything because you are simply listing the axioms required of vector spaces. This type of definition is called an **abstraction** because you are abstracting a collection of properties from a particular setting, $R^n$, to form the axioms for a more general setting.

### Definition of a Vector Space

Let $V$ be a set on which two operations (**vector addition** and **scalar multiplication**) are defined. If the listed axioms are satisfied for every $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $V$ and every scalar (real number) $c$ and $d$, then $V$ is called a **vector space.**

*Addition:*

| | |
|---|---|
| **1.** $\mathbf{u} + \mathbf{v}$ is in $V$. | Closure under addition |
| **2.** $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ | Commutative property |
| **3.** $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ | Associative property |
| **4.** $V$ has a **zero vector 0** such that for every $\mathbf{u}$ in $V$, $\mathbf{u} + \mathbf{0} = \mathbf{u}$. | Additive identity |
| **5.** For every $\mathbf{u}$ in $V$, there is a vector in $V$ denoted by $-\mathbf{u}$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$. | Additive inverse |

*Scalar Multiplication:*

| | |
|---|---|
| **6.** $c\mathbf{u}$ is in $V$. | Closure under scalar multiplication |
| **7.** $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ | Distributive property |
| **8.** $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ | Distributive property |
| **9.** $c(d\mathbf{u}) = (cd)\mathbf{u}$ | Associative property |
| **10.** $1(\mathbf{u}) = \mathbf{u}$ | Scalar identity |

It is important to realize that a vector space consists of four entities: a set of vectors, a set of scalars, and two operations. When you refer to a vector space $V$, be sure that all four entities are clearly stated or understood. Unless stated otherwise, assume that the set of scalars is the set of real numbers.

The first two examples of vector spaces should not be surprising. They are, in fact, the models used to form the ten vector space axioms.

**EXAMPLE 1** **$R^2$ with the Standard Operations Is a Vector Space**

The set of all ordered pairs of real numbers $R^2$ with the standard operations is a vector space. To verify this, look back at Theorem 4.1. Vectors in this space have the form

$$\mathbf{v} = (v_1, v_2).$$

**REMARK**

From Example 2 you can conclude that $R^1$, the set of real numbers (with the usual operations of addition and multiplication), is a vector space.

### EXAMPLE 2 $R^n$ with the Standard Operations Is a Vector Space

The set of all ordered $n$-tuples of real numbers $R^n$ with the standard operations is a vector space. Theorem 4.2 verifies this. Vectors in this space are of the form

$$\mathbf{v} = (v_1, v_2, v_3, \ldots, v_n).$$

The next three examples describe vector spaces in which the basic set $V$ does not consist of ordered $n$-tuples. Each example describes the set $V$ and defines the two vector operations. To show that the set is a vector space, you must verify all ten axioms.

**REMARK**

In the same way you are able to show that the set of all $2 \times 3$ matrices is a vector space, you can show that the set of all $m \times n$ matrices, denoted by $M_{m,n}$, is a vector space.

### EXAMPLE 3 The Vector Space of All 2 × 3 Matrices

Show that the set of all $2 \times 3$ matrices with the operations of matrix addition and scalar multiplication is a vector space.

**SOLUTION**

If $A$ and $B$ are $2 \times 3$ matrices and $c$ is a scalar, then $A + B$ and $cA$ are also $2 \times 3$ matrices. The set is, therefore, closed under matrix addition and scalar multiplication. Moreover, the other eight vector space axioms follow directly from Theorems 2.1 and 2.2 (see Section 2.2). So, the set is a vector space. Vectors in this space have the form

$$\mathbf{a} = A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}.$$

### EXAMPLE 4 The Vector Space of All Polynomials of Degree 2 or Less

Let $P_2$ be the set of all polynomials of the form $p(x) = a_2x^2 + a_1x + a_0$, where $a_0$, $a_1$, and $a_2$ are real numbers. The *sum* of two polynomials $p(x) = a_2x^2 + a_1x + a_0$ and $q(x) = b_2x^2 + b_1x + b_0$ is defined in the usual way by

$$p(x) + q(x) = (a_2 + b_2)x^2 + (a_1 + b_1)x + (a_0 + b_0)$$

and the *scalar multiple* of $p(x)$ by the scalar $c$ is defined by

$$cp(x) = ca_2x^2 + ca_1x + ca_0.$$

Show that $P_2$ is a vector space.

**SOLUTION**

Verification of each of the ten vector space axioms is a straightforward application of the properties of real numbers. For instance, because the set of real numbers is closed under addition, it follows that $a_2 + b_2$, $a_1 + b_1$, and $a_0 + b_0$ are real numbers, and

$$p(x) + q(x) = (a_2 + b_2)x^2 + (a_1 + b_1)x + (a_0 + b_0)$$

is in the set $P_2$ because it is a polynomial of degree 2 or less. $P_2$ is closed under addition. To verify the commutative axiom of addition, write

$$\begin{aligned} p(x) + q(x) &= (a_2x^2 + a_1x + a_0) + (b_2x^2 + b_1x + b_0) \\ &= (a_2 + b_2)x^2 + (a_1 + b_1)x + (a_0 + b_0) \\ &= (b_2 + a_2)x^2 + (b_1 + a_1)x + (b_0 + a_0) \\ &= (b_2x^2 + b_1x + b_0) + (a_2x^2 + a_1x + a_0) \\ &= q(x) + p(x). \end{aligned}$$

**REMARK**

Even though the zero polynomial $\mathbf{0}(x) = 0$ has no degree, $P_2$ is often defined as the set of all polynomials of degree 2 *or less*.

Can you see where the commutative property of addition of real numbers was used? The zero vector in this space is the zero polynomial given by $\mathbf{0}(x) = 0x^2 + 0x + 0$, for all $x$. Try verifying the other vector space axioms to show that $P_2$ is a vector space.

$P_n$ is defined as the set of all polynomials of degree $n$ or less (together with the zero polynomial). The procedure used to verify that $P_2$ is a vector space can be extended to show that $P_n$, with the usual operations of polynomial addition and scalar multiplication, is a vector space.

## EXAMPLE 5 The Vector Space of Continuous Functions (Calculus)

Let $C(-\infty, \infty)$ be the set of all real-valued continuous functions defined on the entire real line. This set consists of all polynomial functions and all other continuous functions on the entire real line. For instance, $f(x) = \sin x$ and $g(x) = e^x$ are members of this set.

Addition is defined by

$$(f + g)(x) = f(x) + g(x)$$

as shown in Figure 4.8. Scalar multiplication is defined by

$$(cf)(x) = c[f(x)].$$

Show that $C(-\infty, \infty)$ is a vector space.

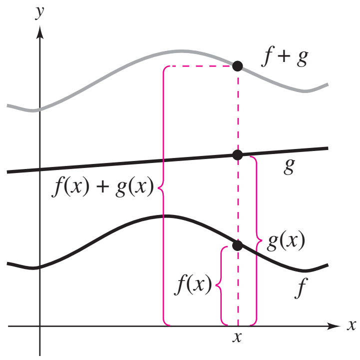



**Figure 4.8**

### SOLUTION

To verify that the set $C(-\infty, \infty)$ is closed under addition and scalar multiplication, use a result from calculus—the sum of two continuous functions is continuous and the product of a scalar and a continuous function is continuous. To verify that the set $C(-\infty, \infty)$ has an additive identity, consider the function $f_0$ that has a value of zero for all $x$, meaning that

$$f_0(x) = 0, \quad \text{where } x \text{ is any real number.}$$

This function is continuous on the entire real line (its graph is simply the line $y = 0$), which means that it is in the set $C(-\infty, \infty)$. Moreover, if $f$ is any other function that is continuous on the entire real line, then

$$(f + f_0)(x) = f(x) + f_0(x) = f(x) + 0 = f(x).$$

This shows that $f_0$ is the additive identity in $C(-\infty, \infty)$. The verification of the other vector space axioms is left to you.

For convenience, the following summary lists some important vector spaces frequently referenced in the remainder of this text. The operations are the standard operations in each case.

### Summary of Important Vector Spaces

$R$ = set of all real numbers
$R^2$ = set of all ordered pairs
$R^3$ = set of all ordered triples
$R^n$ = set of all $n$-tuples
$C(-\infty, \infty)$ = set of all continuous functions defined on the real number line
$C[a, b]$ = set of all continuous functions defined on a closed interval $[a, b]$
$P$ = set of all polynomials
$P_n$ = set of all polynomials of degree $\leq n$
$M_{m,n}$ = set of all $m \times n$ matrices
$M_{n,n}$ = set of all $n \times n$ square matrices

You have seen the versatility of the concept of a vector space. For instance, a vector can be a real number, an $n$-tuple, a matrix, a polynomial, a continuous function, and so on. But what is the purpose of this abstraction, and why bother to define it? There are several reasons, but the most important reason applies to efficiency. This abstraction turns out to be mathematically efficient because general results that apply to all vector spaces can now be derived. Once a theorem has been proved for an abstract vector space, you need not give separate proofs for $n$-tuples, matrices, and polynomials. Simply point out that the theorem is true for any vector space, regardless of the particular form the vectors happen to take. This process is illustrated in Theorem 4.4.

### THEOREM 4.4 Properties of Scalar Multiplication

Let $\mathbf{v}$ be any element of a vector space $V$, and let $c$ be any scalar. Then the following properties are true.

**1.** $0\mathbf{v} = \mathbf{0}$

**2.** $c\mathbf{0} = \mathbf{0}$

**3.** If $c\mathbf{v} = \mathbf{0}$, then $c = 0$ or $\mathbf{v} = \mathbf{0}$.

**4.** $(-1)\mathbf{v} = -\mathbf{v}$

### PROOF

To prove these properties, you are restricted to using the ten vector space axioms. For instance, to prove the second property, note from axiom 4 that $\mathbf{0} = \mathbf{0} + \mathbf{0}$. This allows you to write the following steps.

$$
\begin{aligned}
c\mathbf{0} &= c(\mathbf{0} + \mathbf{0}) && \text{Additive identity} \\
c\mathbf{0} &= c\mathbf{0} + c\mathbf{0} && \text{Left distributive property} \\
c\mathbf{0} + (-c\mathbf{0}) &= (c\mathbf{0} + c\mathbf{0}) + (-c\mathbf{0}) && \text{Add } -c\mathbf{0} \text{ to both sides.} \\
c\mathbf{0} + (-c\mathbf{0}) &= c\mathbf{0} + [c\mathbf{0} + (-c\mathbf{0})] && \text{Associative property} \\
\mathbf{0} &= c\mathbf{0} + \mathbf{0} && \text{Additive inverse} \\
\mathbf{0} &= c\mathbf{0} && \text{Additive identity}
\end{aligned}
$$

To prove the third property, suppose that $c\mathbf{v} = \mathbf{0}$. To show that this implies either $c = 0$ or $\mathbf{v} = \mathbf{0}$, assume that $c \neq 0$. (When $c = 0$, you have nothing more to prove.) Now, because $c \neq 0$, you can use the reciprocal $1/c$ to show that $\mathbf{v} = \mathbf{0}$, as follows.

$$\mathbf{v} = 1\mathbf{v} = \left(\frac{1}{c}\right)(c)\mathbf{v} = \frac{1}{c}(c\mathbf{v}) = \frac{1}{c}(\mathbf{0}) = \mathbf{0}$$

Note that the last step uses Property 2 (the one you just proved). The proofs of the first and fourth properties are left as exercises. (See Exercises 47 and 48.)

**LINEAR ALGEBRA APPLIED**

In a mass-spring system, motion is assumed to occur in only the vertical direction. That is, the system has one *degree of freedom*. When the mass is pulled downward and then released, the system will oscillate. If the system is undamped, meaning that there are no forces present to slow or stop the oscillation, then the system will oscillate indefinitely. Applying Newton's Second Law of Motion to the mass yields the second order differential equation

$$x'' + \omega^2 x = 0$$

where $x$ is the displacement at time $t$, and $\omega$ is a fixed constant called the *natural frequency* of the system. The general solution of this differential equation is

$$x(t) = a_1 \sin \omega t + a_2 \cos \omega t$$

where $a_1$ and $a_2$ are arbitrary constants. (Try verifying this.) In Exercise 41, you are asked to show that the set of all functions $x(t)$ is a vector space.

## SETS THAT ARE NOT VECTOR SPACES

The remaining examples in this section describe some sets (with operations) that *do not* form vector spaces. To show that a set is not a vector space, you need only find one axiom that is not satisfied.

**REMARK**

Notice that a single failure of one of the ten vector space axioms suffices to show that a set is not a vector space.

### EXAMPLE 6 The Set of Integers Is Not a Vector Space

The set of all integers (with the standard operations) does not form a vector space because it is not closed under scalar multiplication. For example,

$$\underbrace{\tfrac{1}{2}}_{\text{Scalar}}\underbrace{(1)}_{\text{Integer}} = \underbrace{\tfrac{1}{2}}_{\text{Noninteger}}.$$

In Example 4, it was shown that the set of all polynomials of degree 2 or less forms a vector space. You will now see that the set of all polynomials whose degree is exactly 2 does not form a vector space.

### EXAMPLE 7 The Set of Second-Degree Polynomials Is Not a Vector Space

The set of all second-degree polynomials is not a vector space because it is not closed under addition. To see this, consider the second-degree polynomials $p(x) = x^2$ and $q(x) = -x^2 + x + 1$, whose sum is the first-degree polynomial $p(x) + q(x) = x + 1$.

The sets in Examples 6 and 7 are not vector spaces because they fail one or both closure axioms. In the next example, you will look at a set that passes both tests for closure but still fails to be a vector space.

### EXAMPLE 8 A Set That Is Not a Vector Space

Let $V = R^2$, the set of all ordered pairs of real numbers, with the standard operation of addition and the *nonstandard* definition of scalar multiplication listed as follows.

$$c(x_1, x_2) = (cx_1, 0)$$

Show that $V$ is not a vector space.

**SOLUTION**

In this example, the operation of scalar multiplication is not the standard one. For instance, the product of the scalar 2 and the ordered pair (3, 4) does not equal (6, 8). Instead, the second component of the product is 0,

$$2(3, 4) = (2 \cdot 3, 0) = (6, 0).$$

This example is interesting because it actually satisfies the first nine axioms of the definition of a vector space (try showing this). In attempting to verify the tenth axiom, the nonstandard definition of scalar multiplication gives you

$$1(1, 1) = (1, 0) \neq (1, 1).$$

The tenth axiom is not verified and the set (together with the two operations) is not a vector space.

Do not be confused by the notation used for scalar multiplication in Example 8. In writing $c(x_1, x_2) = (cx_1, 0)$, the scalar multiple of $(x_1, x_2)$ by $c$ is *defined* to be $(cx_1, 0)$ in this example.

## 4.2 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

**Describing the Additive Identity** **In Exercises 1–6, describe the zero vector (the additive identity) of the vector space.**

**1.** $R^4$ **2.** $C(-\infty, \infty)$ **3.** $M_{2,3}$

**4.** $M_{1,4}$ **5.** $P_3$ **6.** $M_{2,2}$

**Describing the Additive Inverse** **In Exercises 7–12, describe the additive inverse of a vector in the vector space.**

**7.** $R^4$ **8.** $C(-\infty, \infty)$ **9.** $M_{2,3}$

**10.** $M_{1,4}$ **11.** $P_3$ **12.** $M_{2,2}$

**Testing for a Vector Space** **In Exercises 13–34, determine whether the set, together with the indicated operations, is a vector space. If it is not, then identify at least one of the ten vector space axioms that fails.**

**13.** $M_{4,6}$ with the standard operations

**14.** $M_{1,1}$ with the standard operations

**15.** The set of all third-degree polynomials with the standard operations

**16.** The set of all fifth-degree polynomials with the standard operations

**17.** The set of all first-degree polynomial functions $ax$, $a \neq 0$, whose graphs pass through the origin with the standard operations

**18.** The set of all first-degree polynomial functions $ax + b$, $a, b \neq 0$, whose graphs *do not* pass through the origin with the standard operations

**19.** The set of all polynomials of degree four or less with the standard operations

**20.** The set of all quadratic functions whose graphs pass through the origin with the standard operations

**21.** The set

$$\{(x, y)\colon\ x \geq 0, y \text{ is a real number}\}$$

with the standard operations in $R^2$

**22.** The set

$$\{(x, y)\colon\ x \geq 0, y \geq 0\}$$

with the standard operations in $R^2$

**23.** The set

$$\{(x, x)\colon\ x \text{ is a real number}\}$$

with the standard operations

**24.** The set

$$\left\{\left(x, \tfrac{1}{2}x\right)\colon\ x \text{ is a real number}\right\}$$

with the standard operations

**25.** The set of all $2 \times 2$ matrices of the form

$$\begin{bmatrix} a & b \\ c & 0 \end{bmatrix}$$

with the standard operations

**26.** The set of all $2 \times 2$ matrices of the form

$$\begin{bmatrix} a & b \\ c & 1 \end{bmatrix}$$

with the standard operations

**27.** The set of all $3 \times 3$ matrices of the form

$$\begin{bmatrix} 0 & a & b \\ c & 0 & d \\ e & f & 0 \end{bmatrix}$$

with the standard operations

**28.** The set of all $3 \times 3$ matrices of the form

$$\begin{bmatrix} 1 & a & b \\ c & 1 & d \\ e & f & 1 \end{bmatrix}$$

with the standard operations

**29.** The set of all $2 \times 2$ singular matrices with the standard operations

**30.** The set of all $2 \times 2$ nonsingular matrices with the standard operations

**31.** The set of all $2 \times 2$ diagonal matrices with the standard operations

**32.** The set of all $3 \times 3$ upper triangular matrices with the standard operations

**33.** $C[0, 1]$, the set of all continuous functions defined on the interval $[0, 1]$, with the standard operations

**34.** $C[-1, 1]$, the set of all continuous functions defined on the interval $[-1, 1]$, with the standard operations

**35.** Rather than use the standard definitions of addition and scalar multiplication in $R^2$, suppose these two operations are defined as follows.

(a) $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$
$c(x, y) = (cx, y)$

(b) $(x_1, y_1) + (x_2, y_2) = (x_1, 0)$
$c(x, y) = (cx, cy)$

(c) $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$
$c(x, y) = \left(\sqrt{c}\,x, \sqrt{c}\,y\right)$

With these new definitions, is $R^2$ a vector space? Justify your answers.

**36.** Rather than use the standard definitions of addition and scalar multiplication in $R^3$, suppose these two operations are defined as follows.

(a) $(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$
$c(x, y, z) = (cx, cy, 0)$

(b) $(x_1, y_1, z_1) + (x_2, y_2, z_2) = (0, 0, 0)$
$c(x, y, z) = (cx, cy, cz)$

(c) $(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2 + 1, y_1 + y_2 + 1, z_1 + z_2 + 1)$
$c(x, y, z) = (cx, cy, cz)$

(d) $(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2 + 1, y_1 + y_2 + 1, z_1 + z_2 + 1)$
$c(x, y, z) = (cx + c - 1, cy + c - 1, cz + c - 1)$

With these new definitions, is $R^3$ a vector space? Justify your answers.

**37.** **Proof** Prove in full detail that $M_{2,2}$, with the standard operations, is a vector space.

**38.** **Proof** Prove in full detail that the set $\{(x, 2x): x \text{ is a real number}\}$, with the standard operations in $R^2$, is a vector space.

**39.** Determine whether the set $R^2$ with the operations

$$(x_1, y_1) + (x_2, y_2) = (x_1x_2, y_1y_2)$$

and

$$c(x_1, y_1) = (cx_1, cy_1)$$

is a vector space. If it is, then verify each vector space axiom; if it is not, state all vector space axioms that fail.

**40. CAPSTONE**

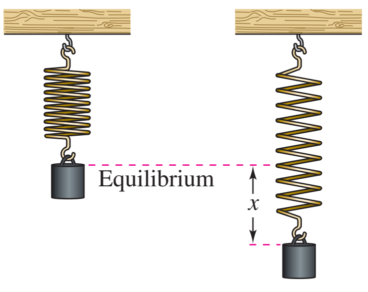
(a) Describe the conditions under which a set may be classified as a vector space.

(b) Give an example of a set that is a vector space and an example of a set that is not a vector space.

**41.** **Mass-Spring System** The mass in a mass-spring system (see figure) is pulled downward and then released, causing the system to oscillate according to

$$x(t) = a_1 \sin \omega t + a_2 \cos \omega t$$

where $x$ is the displacement at time $t$, $a_1$ and $a_2$ are arbitrary constants, and $\omega$ is a fixed constant. Show that the set of all functions $x(t)$ is a vector space.




**42.** Let $R^\infty$ be the set of all infinite sequences of real numbers, with the operations

$$\mathbf{u} + \mathbf{v} = (u_1, u_2, u_3, \ldots) + (v_1, v_2, v_3, \ldots) = (u_1 + v_1, u_2 + v_2, u_3 + v_3, \ldots)$$

and

$$c\mathbf{u} = c(u_1, u_2, u_3, \ldots) = (cu_1, cu_2, cu_3, \ldots).$$

Determine whether $R^\infty$ is a vector space. If it is, then verify each vector space axiom; if it is not, then state all vector space axioms that fail.

**43.** Let $V$ be the set of all positive real numbers. Determine whether $V$ is a vector space with the following operations.

$x + y = xy$ Addition

$cx = x^c$ Scalar multiplication

If it is, then verify each vector space axiom; if it is not, then state all vector space axioms that fail.

**44.** **Proof** Complete the proof of the cancellation property of vector addition by supplying the justification for each step.

Prove that if $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in a vector space $V$ such that $\mathbf{u} + \mathbf{w} = \mathbf{v} + \mathbf{w}$, then $\mathbf{u} = \mathbf{v}$.

| | |
|---|---|
| $\mathbf{u} + \mathbf{w} = \mathbf{v} + \mathbf{w}$ | Given |
| $(\mathbf{u} + \mathbf{w}) + (-\mathbf{w}) = (\mathbf{v} + \mathbf{w}) + (-\mathbf{w})$ | a. ______ |
| $\mathbf{u} + (\mathbf{w} + (-\mathbf{w})) = \mathbf{v} + (\mathbf{w} + (-\mathbf{w}))$ | b. ______ |
| $\mathbf{u} + \mathbf{0} = \mathbf{v} + \mathbf{0}$ | c. ______ |
| $\mathbf{u} = \mathbf{v}$ | d. ______ |

**True or False?** **In Exercises 45 and 46, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.**

**45.** (a) A vector space consists of four entities: a set of vectors, a set of scalars, and two operations.

(b) The set of all integers with the standard operations is a vector space.

(c) The set of all ordered triples $(x, y, z)$ of real numbers, where $y \geq 0$, with the standard operations on $R^3$ is a vector space.

**46.** (a) To show that a set is not a vector space, it is sufficient to show that just one axiom is not satisfied.

(b) The set of all first-degree polynomials with the standard operations is a vector space.

(c) The set of all pairs of real numbers of the form $(0, y)$, with the standard operations on $R^2$, is a vector space.

**47.** **Proof** Prove Property 1 of Theorem 4.4.

**48.** **Proof** Prove Property 4 of Theorem 4.4.

# 4.3 Subspaces of Vector Spaces

- Determine whether a subset $W$ of a vector space $V$ is a subspace of $V$.
- Determine subspaces of $R^n$.

## SUBSPACES

In many applications in linear algebra, vector spaces occur as **subspaces** of larger spaces. For instance, you will see that the solution set of a homogeneous system of linear equations in $n$ variables is a subspace of $R^n$. (See Theorem 4.16.)

A nonempty subset of a vector space is a subspace when it is a vector space (with the *same* operations defined in the original vector space), as stated in the next definition.

**REMARK**

Note that if $W$ is a subspace of $V$, then it must be closed under the operations inherited from $V$.

### Definition of Subspace of a Vector Space

A nonempty subset $W$ of a vector space $V$ is called a **subspace** of $V$ when $W$ is a vector space under the operations of addition and scalar multiplication defined in $V$.

### EXAMPLE 1 A Subspace of $R^3$

Show that the set $W = \{(x_1, 0, x_3): x_1 \text{ and } x_3 \text{ are real numbers}\}$ is a subspace of $R^3$ with the standard operations.

**SOLUTION**

The set $W$ is nonempty because it contains the zero vector $(0, 0, 0)$.

Graphically, the set $W$ can be interpreted as the $xz$-plane, as shown in Figure 4.9. The set $W$ is closed under addition because the sum of any two vectors in the $xz$-plane must also lie in the $xz$-plane. That is, if $(x_1, 0, x_3)$ and $(y_1, 0, y_3)$ are in $W$, then their sum $(x_1 + y_1, 0, x_3 + y_3)$ is also in $W$. Similarly, to see that $W$ is closed under scalar multiplication, let $(x_1, 0, x_3)$ be in $W$ and let $c$ be a scalar. Then $c(x_1, 0, x_3) = (cx_1, 0, cx_3)$ has zero as its second component and must be in $W$. The verifications of the other eight vector space axioms are left to you.

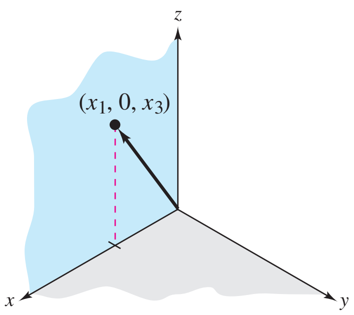



**Figure 4.9**

To establish that a set $W$ is a vector space, you must verify all ten vector space axioms. If $W$ is a nonempty subset of a larger vector space $V$ (and the operations defined on $W$ are the *same* as those defined on $V$), however, then most of the ten properties are *inherited* from the larger space and need no verification. The next theorem states that it is sufficient to test for closure in order to establish that a nonempty subset of a vector space is a subspace.

### THEOREM 4.5 Test for a Subspace

If $W$ is a nonempty subset of a vector space $V$, then $W$ is a subspace of $V$ if and only if the following closure conditions hold.

**1.** If $\mathbf{u}$ and $\mathbf{v}$ are in $W$, then $\mathbf{u} + \mathbf{v}$ is in $W$.
**2.** If $\mathbf{u}$ is in $W$ and $c$ is any scalar, then $c\mathbf{u}$ is in $W$.

**PROOF**

The proof of this theorem in one direction is straightforward. That is, if $W$ is a subspace of $V$, then $W$ is a vector space and must be closed under addition and scalar multiplication.

To prove the theorem in the other direction, assume that $W$ is closed under addition and scalar multiplication. Note that if $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are in $W$, then they are also in $V$. Consequently, vector space axioms 2, 3, 7, 8, 9, and 10 are satisfied automatically. Because $W$ is closed under addition and scalar multiplication, it follows that for any $\mathbf{v}$ in $W$ and scalar $c = 0$, $c\mathbf{v} = \mathbf{0}$ and $(-1)\mathbf{v} = -\mathbf{v}$ both lie in $W$, which satisfies axioms 4 and 5.

**REMARK**

Note that if $W$ is a subspace of a vector space $V$, then both $W$ and $V$ must have the same zero vector $\mathbf{0}$. (In Exercise 55, you are asked to prove this.)

Because a subspace of a vector space is a vector space, it must contain the zero vector. In fact, the simplest subspace of a vector space $V$ is the one consisting of only the zero vector, $W = \{\mathbf{0}\}$. This subspace is called the **zero subspace.** Another subspace of $V$ is $V$ itself. Every vector space contains these two trivial subspaces, and subspaces other than these are called **proper** (or nontrivial) subspaces.

## EXAMPLE 2 A Subspace of $M_{2,2}$

Let $W$ be the set of all $2 \times 2$ symmetric matrices. Show that $W$ is a subspace of the vector space $M_{2,2}$, with the standard operations of matrix addition and scalar multiplication.

### SOLUTION

Recall that a square matrix is *symmetric* when it is equal to its own transpose. Because $M_{2,2}$ is a vector space, you only need to show that $W$ (a subset of $M_{2,2}$) satisfies the conditions of Theorem 4.5. Begin by observing that $W$ is *nonempty*. $W$ is closed under addition because for matrices $A_1$ and $A_2$ in $W$, $A_1 = A_1^T$ and $A_2 = A_2^T$, which implies that

$$(A_1 + A_2)^T = A_1^T + A_2^T = A_1 + A_2.$$

So, if $A_1$ and $A_2$ are symmetric matrices of order 2, then so is $A_1 + A_2$. Similarly, $W$ is closed under scalar multiplication because $A = A^T$ implies that $(cA)^T = cA^T = cA$. If $A$ is a symmetric matrix of order 2, then so is $cA$.

The result of Example 2 can be generalized. That is, for any positive integer $n$, the set of symmetric matrices of order $n$ is a subspace of the vector space $M_{n,n}$ with the standard operations. The next example describes a subset of $M_{n,n}$ that is not a subspace.

## EXAMPLE 3 The Set of Singular Matrices Is Not a Subspace of $M_{n,n}$

Let $W$ be the set of singular matrices of order 2. Show that $W$ is not a subspace of $M_{2,2}$ with the standard operations.

### SOLUTION

By Theorem 4.5, show that a subset $W$ is not a subspace by showing that $W$ is empty, $W$ is not closed under addition, or $W$ is not closed under scalar multiplication. For this particular set, $W$ is nonempty and closed under scalar multiplication, but it is not closed under addition. To see this, let $A$ and $B$ be

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then $A$ and $B$ are both singular (noninvertible), but their sum

$$A + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is nonsingular (invertible). So $W$ is not closed under addition, and by Theorem 4.5, $W$ is not a subspace of $M_{2,2}$.

## EXAMPLE 4 A Subset of $R^2$ That Is Not a Subspace

Show that $W = \{(x_1, x_2): x_1 \geq 0 \text{ and } x_2 \geq 0\}$, with the standard operations, is not a subspace of $R^2$.

### SOLUTION

This set is nonempty and closed under addition. It is not, however, closed under scalar multiplication. To see this, note that $(1, 1)$ is in $W$, but the scalar multiple $(-1)(1, 1) = (-1, -1)$ is not in $W$. So $W$ is not a subspace of $R^2$.

You will often encounter sequences of nested subspaces. For instance, consider the vector spaces $P_0, P_1, P_2, P_3, \ldots, P_n$, where $P_k$ is the set of all polynomials of degree less than or equal to $k$, with the standard operations. You can write $P_0 \subset P_1 \subset P_2 \subset P_3 \subset \cdots \subset P_n$. If $j \leq k$, then $P_j$ is a subspace of $P_k$. (In Exercise 45, you are asked to show this.) Another nesting of subspaces is described in Example 5.

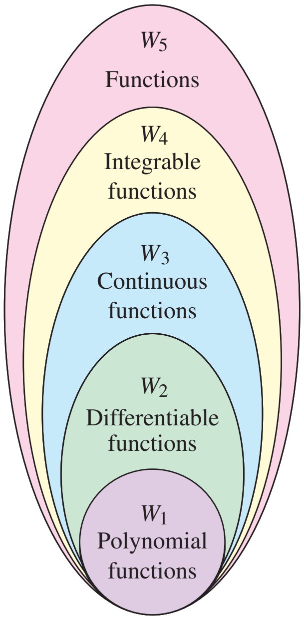



**Figure 4.10**

## EXAMPLE 5 Subspaces of Functions (Calculus)

Let $W_5$ be the *vector space* of all functions defined on $[0, 1]$, and let $W_1$, $W_2$, $W_3$, and $W_4$ be defined as follows.

$W_1$ = set of all polynomial functions that are defined on $[0, 1]$

$W_2$ = set of all functions that are differentiable on $[0, 1]$

$W_3$ = set of all functions that are continuous on $[0, 1]$

$W_4$ = set of all functions that are integrable on $[0, 1]$

Show that $W_1 \subset W_2 \subset W_3 \subset W_4 \subset W_5$ and that $W_i$ is a subspace of $W_j$ for $i \leq j$.

### SOLUTION

From calculus you know that every polynomial function is differentiable on $[0, 1]$. So, $W_1 \subset W_2$. Moreover, because every differentiable function is continuous, every continuous function is integrable, and every integrable function is a function, $W_2 \subset W_3 \subset W_4 \subset W_5$. So, you have $W_1 \subset W_2 \subset W_3 \subset W_4 \subset W_5$, as shown in Figure 4.10. It is left to you to show that $W_i$ is a subspace of $W_j$ for $i \leq j$. (See Exercise 46.)

Note in Example 5 that if $U$, $V$, and $W$ are vector spaces such that $W$ is a subspace of $V$ and $V$ is a subspace of $U$, then $W$ is also a subspace of $U$. This special case of the next theorem tells you that the intersection of two subspaces is a subspace, as shown in Figure 4.11.

**REMARK**

Theorem 4.6 states that the *intersection* of two subspaces is a subspace. In Exercise 56 you are asked to show that the *union* of two subspaces is not necessarily a subspace.

**THEOREM 4.6 The Intersection of Two Subspaces Is a Subspace**

If $V$ and $W$ are both subspaces of a vector space $U$, then the intersection of $V$ and $W$ (denoted by $V \cap W$) is also a subspace of $U$.

### PROOF

Because $V$ and $W$ are both subspaces of $U$, you know that both contain the zero vector, which means that $V \cap W$ is nonempty. To show that $V \cap W$ is closed under addition, let $\mathbf{v}_1$ and $\mathbf{v}_2$ be any two vectors in $V \cap W$. Then, because $V$ and $W$ are both subspaces of $U$, you know that both are closed under addition. Because $\mathbf{v}_1$ and $\mathbf{v}_2$ are both in $V$, their sum $\mathbf{v}_1 + \mathbf{v}_2$ must be in $V$. Similarly, $\mathbf{v}_1 + \mathbf{v}_2$ is in $W$ because $\mathbf{v}_1$ and $\mathbf{v}_2$ are also both in $W$. But this implies that $\mathbf{v}_1 + \mathbf{v}_2$ is in $V \cap W$, and it follows that $V \cap W$ is closed under addition. It is left to you to show (by a similar argument) that $V \cap W$ is closed under scalar multiplication. (See Exercise 59.)

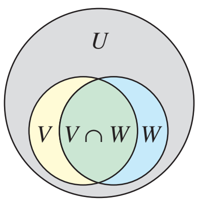



**Figure 4.11** The intersection of two subspaces is a subspace.

## SUBSPACES OF $R^n$

$R^n$ is a convenient source for examples of vector spaces, and the remainder of this section is devoted to looking at subspaces of $R^n$.

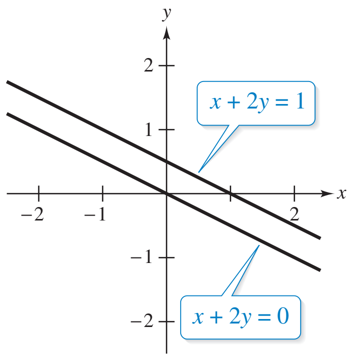



**Figure 4.12**

### EXAMPLE 6 Determining Subspaces of $R^2$

Which of these two subsets is a subspace of $R^2$?

**a.** The set of points on the line $x + 2y = 0$

**b.** The set of points on the line $x + 2y = 1$

**SOLUTION**

**a.** Solving for $x$, you can see that a point in $R^2$ is on the line $x + 2y = 0$ if and only if it has the form $(-2t, t)$, where $t$ is any real number. (See Figure 4.12.)

To show that this set is closed under addition, let $\mathbf{v}_1 = (-2t_1, t_1)$ and $\mathbf{v}_2 = (-2t_2, t_2)$ be any two points on the line. Then you have

$$\mathbf{v}_1 + \mathbf{v}_2 = (-2t_1, t_1) + (-2t_2, t_2) = (-2(t_1 + t_2), t_1 + t_2) = (-2t_3, t_3)$$

where $t_3 = t_1 + t_2$. $\mathbf{v}_1 + \mathbf{v}_2$ lies on the line, and the set is closed under addition. In a similar way, you can show that the set is closed under scalar multiplication. So, this set is a subspace of $R^2$.

**b.** This subset of $R^2$ is *not* a subspace of $R^2$ because every subspace must contain the zero vector $(0, 0)$, which is not on the line $x + 2y = 1$. (See Figure 4.12.)

Of the two lines in Example 6, the one that is a subspace of $R^2$ is the one that passes through the origin. This is characteristic of subspaces of $R^2$. That is, if $W$ is a subset of $R^2$, then it is a subspace if and only if one of the following three possibilities is true.

1. $W$ consists of the *single point* $(0, 0)$.
2. $W$ consists of all points on a *line* that passes through the origin.
3. $W$ consists of all of $R^2$.

Figure 4.13 shows these three possibilities graphically.

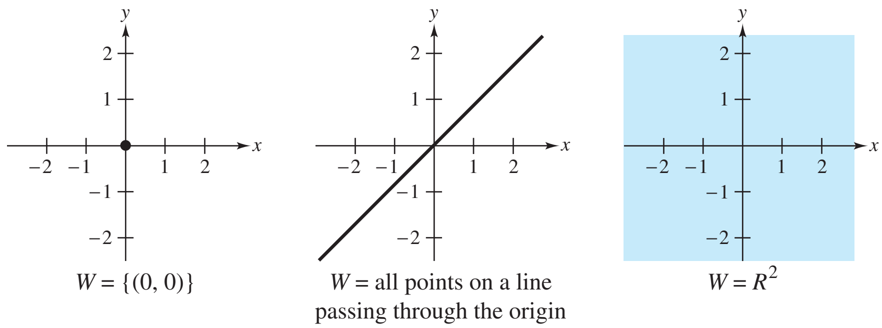



**Figure 4.13**

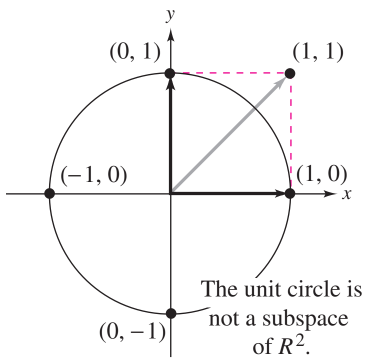



**Figure 4.14**

**REMARK**

Another way to tell that the subset shown in Figure 4.14 is not a subspace of $R^2$ is by noting that it does not contain the zero vector (the origin).

### EXAMPLE 7 A Subset of $R^2$ That Is Not a Subspace

Show that the subset of $R^2$ consisting of all points on $x^2 + y^2 = 1$ is not a subspace.

**SOLUTION**

This subset of $R^2$ is *not* a subspace because the points $(1, 0)$ and $(0, 1)$ are in the subset, but their sum $(1, 1)$ is not. (See Figure 4.14.) So, this subset is not closed under addition.

## EXAMPLE 8 Determining Subspaces of $R^3$

Which of the following subsets is a subspace of $R^3$?

**a.** $W = \{(x_1, x_2, 1): x_1 \text{ and } x_2 \text{ are real numbers}\}$

**b.** $W = \{(x_1, x_1 + x_3, x_3): x_1 \text{ and } x_3 \text{ are real numbers}\}$

### SOLUTION

**a.** Because $\mathbf{0} = (0, 0, 0)$ is not in $W$, you know that $W$ is *not* a subspace of $R^3$.

**b.** This set is nonempty because it contains the zero vector $(0, 0, 0)$. Let

$$\mathbf{v} = (v_1, v_1 + v_3, v_3) \quad \text{and} \quad \mathbf{u} = (u_1, u_1 + u_3, u_3)$$

be two vectors in $W$, and let $c$ be any real number. $W$ is closed under addition because

$$\begin{aligned}\mathbf{v} + \mathbf{u} &= (v_1 + u_1, v_1 + v_3 + u_1 + u_3, v_3 + u_3)\\ &= (v_1 + u_1, (v_1 + u_1) + (v_3 + u_3), v_3 + u_3)\\ &= (x_1, x_1 + x_3, x_3)\end{aligned}$$

where $x_1 = v_1 + u_1$ and $x_3 = v_3 + u_3$, which means that $\mathbf{v} + \mathbf{u}$ is in $W$. Similarly, $W$ is closed under scalar multiplication because

$$\begin{aligned}c\mathbf{v} &= (cv_1, c(v_1 + v_3), cv_3)\\ &= (cv_1, cv_1 + cv_3, cv_3)\\ &= (x_1, x_1 + x_3, x_3)\end{aligned}$$

where $x_1 = cv_1$ and $x_3 = cv_3$, which means that $c\mathbf{v}$ is in $W$. So, $W$ is a subspace of $R^3$. ■

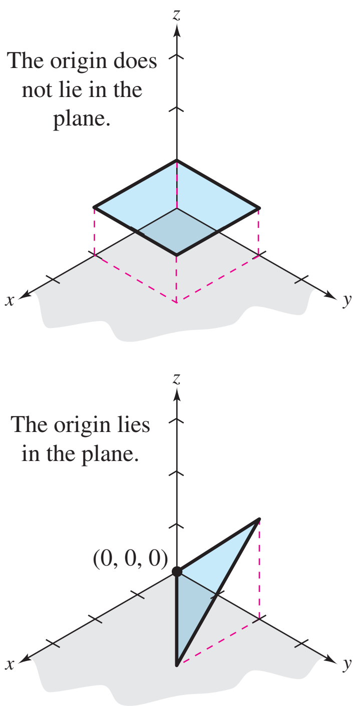



**Figure 4.15**

In Example 8, note that the graph of each subset is a plane in $R^3$, but the only subset that is a *subspace* is the one represented by a plane that passes through the origin. (See Figure 4.15.)

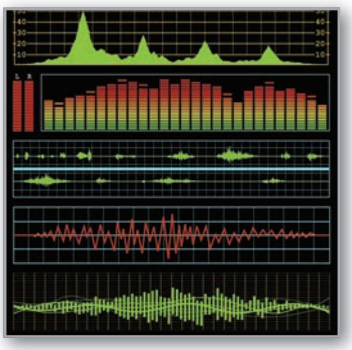
You can show that a subset $W$ of $R^3$ is a subspace of $R^3$ if and only if it has one of the following forms.

**1.** $W$ consists of the *single point* $(0, 0, 0)$.

**2.** $W$ consists of all points on a *line* that passes through the origin.

**3.** $W$ consists of all points in a *plane* that passes through the origin.

**4.** $W$ consists of all of $R^3$.



zphoto/Shutterstock.com

**LINEAR ALGEBRA APPLIED**

Digital signal processing depends on sampling, which converts continuous signals into discrete sequences that can be used by digital devices. Traditionally, sampling is uniform and pointwise, and is obtained from a single vector space. Then, the resulting sequence is reconstructed into a continuous-domain signal. Such a process, however, can involve a significant reduction in information, which could result in a low-quality reconstructed signal. In applications such as radar, geophysics, and wireless communications, researchers have determined situations in which sampling from a *union* of vector subspaces can be more appropriate. *(Source: Sampling Signals from a Union of Subspaces—A New Perspective for the Extension of This Theory, Lu, Y.M. and Do, M.N., IEEE Signal Processing Magazine, March, 2008.)*

## 4.3 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

**Verifying Subspaces** **In Exercises 1–6, verify that $W$ is a subspace of $V$. In each case, assume that $V$ has the standard operations.**

1. $W = \{(x_1, x_2, x_3, 0): x_1, x_2, \text{ and } x_3 \text{ are real numbers}\}$
   $V = R^4$
2. $W = \{(x, y, 2x - 3y): x \text{ and } y \text{ are real numbers}\}$
   $V = R^3$
3. $W$ is the set of all $2 \times 2$ matrices of the form
   $$\begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}.$$
   $V = M_{2,2}$
4. $W$ is the set of all $3 \times 2$ matrices of the form
   $$\begin{bmatrix} a & b \\ a+b & 0 \\ 0 & c \end{bmatrix}.$$
   $V = M_{3,2}$
5. **Calculus** $W$ is the set of all functions that are continuous on $[-1, 1]$. $V$ is the set of all functions that are integrable on $[-1, 1]$.
6. **Calculus** $W$ is the set of all functions that are differentiable on $[-1, 1]$. $V$ is the set of all functions that are continuous on $[-1, 1]$.

**Subsets That Are Not Subspaces** **In Exercises 7–20, $W$ is not a subspace of the vector space. Verify this by giving a specific example that violates the test for a vector subspace (Theorem 4.5).**

7. $W$ is the set of all vectors in $R^3$ whose third component is $-1$.
8. $W$ is the set of all vectors in $R^2$ whose second component is 1.
9. $W$ is the set of all vectors in $R^2$ whose components are rational numbers.
10. $W$ is the set of all vectors in $R^2$ whose components are integers.
11. $W$ is the set of all nonnegative functions in $C(-\infty, \infty)$.
12. $W$ is the set of all linear functions $ax + b$, $a \neq 0$, in $C(-\infty, \infty)$.
13. $W$ is the set of all vectors in $R^3$ whose components are nonnegative.
14. $W$ is the set of all vectors in $R^3$ whose components are Pythagorean triples.
15. $W$ is the set of all matrices in $M_{3,3}$ of the form
   $$\begin{bmatrix} 0 & a & b \\ c & 0 & d \\ e & f & 1 \end{bmatrix}.$$
16. $W$ is the set of all matrices in $M_{3,1}$ of the form
   $\begin{bmatrix} a & 0 & \sqrt{a} \end{bmatrix}^T$.
17. $W$ is the set of all matrices in $M_{n,n}$ with determinants equal to 1.
18. $W$ is the set of all matrices in $M_{n,n}$ such that $A^2 = A$.
19. $W$ is the set of all vectors in $R^2$ whose second component is the cube of the first.
20. $W$ is the set of all vectors in $R^2$ whose second component is the square of the first.

**Determining Subspaces** **In Exercises 21–28, determine whether the subset of $C(-\infty, \infty)$ is a subspace of $C(-\infty, \infty)$.**

21. The set of all positive functions: $f(x) > 0$
22. The set of all negative functions: $f(x) < 0$
23. The set of all even functions: $f(-x) = f(x)$
24. The set of all odd functions: $f(-x) = -f(x)$
25. The set of all constant functions: $f(x) = c$
26. The set of all exponential functions $f(x) = a^x$, where $a > 0$
27. The set of all functions such that $f(0) = 0$
28. The set of all functions such that $f(0) = 1$

**Determining Subspaces** **In Exercises 29–36, determine whether the subset of $M_{n,n}$ is a subspace of $M_{n,n}$ with the standard operations of matrix addition and scalar multiplication.**

29. The set of all $n \times n$ upper triangular matrices
30. The set of all $n \times n$ diagonal matrices
31. The set of all $n \times n$ matrices with integer entries
32. The set of all $n \times n$ matrices $A$ that commute with a given matrix $B$; that is, $AB = BA$
33. The set of all $n \times n$ singular matrices
34. The set of all $n \times n$ invertible matrices
35. The set of all $n \times n$ matrices whose entries add up to zero
36. The set of all $n \times n$ matrices whose trace is nonzero

**Determining Subspaces** **In Exercises 37–42, determine whether the set $W$ is a subspace of $R^3$ with the standard operations. Justify your answer.**

37. $W = \{(x_1, x_2, 0): x_1 \text{ and } x_2 \text{ are real numbers}\}$
38. $W = \{(x_1, x_2, 4): x_1 \text{ and } x_2 \text{ are real numbers}\}$
39. $W = \{(a, b, a + 2b): a \text{ and } b \text{ are real numbers}\}$
40. $W = \{(s, s - t, t): s \text{ and } t \text{ are real numbers}\}$
41. $W = \{(x_1, x_2, x_1x_2): x_1 \text{ and } x_2 \text{ are real numbers}\}$
42. $W = \{(x_1, 1/x_1, x_3): x_1 \text{ and } x_3 \text{ are real numbers}, x_1 \neq 0\}$

**True or False?** **In Exercises 43 and 44, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.**

**43.** (a) Every vector space $V$ contains at least one subspace that is the zero subspace.

(b) If $V$ and $W$ are both subspaces of a vector space $U$, then the intersection of $V$ and $W$ is also a subspace.

(c) If $U$, $V$, and $W$ are vector spaces such that $W$ is a subspace of $V$ and $U$ is a subspace of $V$, then $W = U$.

**44.** (a) Every vector space $V$ contains two proper subspaces that are the zero subspace and itself.

(b) If $W$ is a subspace of $R^2$, then $W$ must contain the vector $(0, 0)$.

(c) If $W$ is a subspace of a vector space $V$, then it has closure under addition as defined in $V$.

**45.** Consider the vector spaces

$P_0, P_1, P_2, \ldots, P_n$

where $P_k$ is the set of all polynomials of degree less than or equal to $k$, with the standard operations. Show that if $j \leq k$, then $P_j$ is a subspace of $P_k$.

**46. Calculus** Let $W_1$, $W_2$, $W_3$, $W_4$, and $W_5$ be defined as in Example 5. Show that $W_i$ is a subspace of $W_j$ for $i \leq j$.

**47. Calculus** Let $F(-\infty, \infty)$ be the vector space of real-valued functions defined on the entire real line. Show that each of the following is a subspace of $F(-\infty, \infty)$.

(a) $C(-\infty, \infty)$

(b) The set of all differentiable functions $f$ defined on the real number line

(c) The set of all differentiable functions $f$ defined on the real number line that satisfy the differential equation

$f' - 3f = 0$

**48. Calculus** Determine whether the set

$$S = \left\{ f \in C[0, 1]: \int_0^1 f(x)\, dx = 0 \right\}$$

is a subspace of $C[0, 1]$. Prove your answer.

**49.** Let $W$ be the subset of $R^3$ consisting of all points on a line that passes through the origin. Such a line can be represented by the parametric equations

$x = at$, $y = bt$, and $z = ct$.

Use these equations to show that $W$ is a subspace of $R^3$.

**50. CAPSTONE** Explain why it is sufficient to test for closure in order to establish that a nonempty subset of a vector space is a subspace.

**51. Guided Proof** Prove that a nonempty set $W$ is a subspace of a vector space $V$ if and only if $a\mathbf{x} + b\mathbf{y}$ is an element of $W$ for all scalars $a$ and $b$ and all vectors $\mathbf{x}$ and $\mathbf{y}$ in $W$.

Getting Started: In one direction, assume $W$ is a subspace, and show by using closure axioms that $a\mathbf{x} + b\mathbf{y}$ is an element of $W$. In the other direction, assume $a\mathbf{x} + b\mathbf{y}$ is an element of $W$ for all scalars $a$ and $b$ and all vectors $\mathbf{x}$ and $\mathbf{y}$ in $W$, and verify that $W$ is closed under addition and scalar multiplication.

(i) If $W$ is a subspace of $V$, then use scalar multiplication closure to show that $a\mathbf{x}$ and $b\mathbf{y}$ are in $W$. Now use additive closure to get the desired result.

(ii) Conversely, assume $a\mathbf{x} + b\mathbf{y}$ is in $W$. By cleverly assigning specific values to $a$ and $b$, show that $W$ is closed under addition and scalar multiplication.

**52.** Let $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ be vectors in a vector space $V$. Show that the set of all linear combinations of $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$,

$W = \{a\mathbf{x} + b\mathbf{y} + c\mathbf{z}$: $a$, $b$, and $c$ are scalars$\}$

is a subspace of $V$. This subspace is called the **span** of $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$.

**53. Proof** Let $A$ be a fixed $2 \times 3$ matrix. Prove that the set

$$W = \left\{ \mathbf{x} \in R^3 : A\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$$

is not a subspace of $R^3$.

**54. Proof** Let $A$ be a fixed $m \times n$ matrix. Prove that the set

$W = \{\mathbf{x} \in R^n : A\mathbf{x} = \mathbf{0}\}$

is a subspace of $R^n$.

**55. Proof** Let $W$ be a subspace of the vector space $V$. Prove that the zero vector in $V$ is also the zero vector in $W$.

**56.** Give an example showing that the union of two subspaces of a vector space $V$ is not necessarily a subspace of $V$.

**57. Proof** Let $A$ and $B$ be fixed $2 \times 2$ matrices. Prove that the set

$W = \{X : XAB = BAX\}$

is a subspace of $M_{2,2}$.

**58. Proof** Let $V$ and $W$ be two subspaces of a vector space $U$.

(a) Prove that the set

$V + W = \{\mathbf{u} : \mathbf{u} = \mathbf{v} + \mathbf{w}$, where $\mathbf{v} \in V$ and $\mathbf{w} \in W\}$

is a subspace of $U$.

(b) Describe $V + W$ when $V$ and $W$ are the subspaces of $U = R^2$:

$V = \{(x, 0) : x$ is a real number$\}$ and $W = \{(0, y) : y$ is a real number$\}$.

**59. Proof** Complete the proof of Theorem 4.6 by showing that the intersection of two subspaces of a vector space is closed under scalar multiplication.

# 4.4 Spanning Sets and Linear Independence

- Write a linear combination of a set of vectors in a vector space $V$.
- Determine whether a set $S$ of vectors in a vector space $V$ is a spanning set of $V$.
- Determine whether a set of vectors in a vector space $V$ is linearly independent.

## LINEAR COMBINATIONS OF VECTORS IN A VECTOR SPACE

This section begins to develop procedures for representing each vector in a vector space as a **linear combination** of a select number of vectors in the space.

### Definition of Linear Combination of Vectors

A vector $\mathbf{v}$ in a vector space $V$ is called a **linear combination** of the vectors $\mathbf{u}_1, \mathbf{u}_2, . . . , \mathbf{u}_k$ in $V$ when $\mathbf{v}$ can be written in the form

$$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdot\cdot\cdot + c_k\mathbf{u}_k$$

where $c_1, c_2, . . . , c_k$ are scalars.

Often, one or more of the vectors in a set can be written as linear combinations of other vectors in the set. Examples 1, 2, and 3 illustrate this possibility.

### EXAMPLE 1 Examples of Linear Combinations

**a.** For the set of vectors in $R^3$

$$S = \{\overset{\mathbf{v}_1}{(1, 3, 1)}, \overset{\mathbf{v}_2}{(0, 1, 2)}, \overset{\mathbf{v}_3}{(1, 0, -5)}\}$$

$\mathbf{v}_1$ is a linear combination of $\mathbf{v}_2$ and $\mathbf{v}_3$ because

$$\mathbf{v}_1 = 3\mathbf{v}_2 + \mathbf{v}_3 = 3(0, 1, 2) + (1, 0, -5) = (1, 3, 1).$$

**b.** For the set of vectors in $M_{2,2}$

$$S = \left\{\overset{\mathbf{v}_1}{\begin{bmatrix} 0 & 8 \\ 2 & 1 \end{bmatrix}}, \overset{\mathbf{v}_2}{\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}}, \overset{\mathbf{v}_3}{\begin{bmatrix} -1 & 3 \\ 1 & 2 \end{bmatrix}}, \overset{\mathbf{v}_4}{\begin{bmatrix} -2 & 0 \\ 1 & 3 \end{bmatrix}}\right\}$$

$\mathbf{v}_1$ is a linear combination of $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$ because

$$\begin{aligned}\mathbf{v}_1 &= \mathbf{v}_2 + 2\mathbf{v}_3 - \mathbf{v}_4 \\ &= \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} + 2\begin{bmatrix} -1 & 3 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} -2 & 0 \\ 1 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 8 \\ 2 & 1 \end{bmatrix}.\end{aligned}$$

In Example 1, it is easy to verify that one of the vectors in the set $S$ is a linear combination of the other vectors because you are given the appropriate coefficients to form the linear combination. In the next example, a procedure for finding the coefficients is demonstrated.

## EXAMPLE 2 Finding a Linear Combination

Write the vector $\mathbf{w} = (1, 1, 1)$ as a linear combination of vectors in the set $S$.

$$S = \{\overset{\mathbf{v}_1}{(1, 2, 3)}, \overset{\mathbf{v}_2}{(0, 1, 2)}, \overset{\mathbf{v}_3}{(-1, 0, 1)}\}$$

### SOLUTION

You need to find scalars $c_1$, $c_2$, and $c_3$ such that

$$\begin{aligned}(1, 1, 1) &= c_1(1, 2, 3) + c_2(0, 1, 2) + c_3(-1, 0, 1)\\ &= (c_1, 2c_1, 3c_1) + (0, c_2, 2c_2) + (-c_3, 0, c_3)\\ &= (c_1 - c_3, 2c_1 + c_2, 3c_1 + 2c_2 + c_3).\end{aligned}$$

Equating corresponding components yields the following system of linear equations.

$$\begin{aligned} c_1 \qquad\quad - c_3 &= 1\\ 2c_1 + \phantom{2}c_2 \qquad &= 1\\ 3c_1 + 2c_2 + c_3 &= 1\end{aligned}$$

Using Gauss-Jordan elimination, the augmented matrix of this system row reduces to

$$\begin{bmatrix} 1 & 0 & -1 & 1\\ 0 & 1 & 2 & -1\\ 0 & 0 & 0 & 0\end{bmatrix}.$$

So, this system has infinitely many solutions, each of the form

$$c_1 = 1 + t, \quad c_2 = -1 - 2t, \quad c_3 = t.$$

To obtain one solution, you could let $t = 1$. Then $c_3 = 1$, $c_2 = -3$, and $c_1 = 2$, and you have

$$\mathbf{w} = 2\mathbf{v}_1 - 3\mathbf{v}_2 + \mathbf{v}_3.$$

Other choices for $t$ would yield other ways to write $\mathbf{w}$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

## EXAMPLE 3 Finding a Linear Combination

If possible, write the vector

$$\mathbf{w} = (1, -2, 2)$$

as a linear combination of vectors in the set $S$ in Example 2.

### SOLUTION

Following the procedure from Example 2 results in the system

$$\begin{aligned} c_1 \qquad\quad - c_3 &= 1\\ 2c_1 + \phantom{2}c_2 \qquad &= -2\\ 3c_1 + 2c_2 + c_3 &= 2.\end{aligned}$$

The augmented matrix of this system row reduces to

$$\begin{bmatrix} 1 & 0 & -1 & 0\\ 0 & 1 & 2 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}.$$

From the third row you can conclude that the system of equations is inconsistent, which means that there is no solution. Consequently, $\mathbf{w}$ *cannot* be written as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

## SPANNING SETS

If every vector in a vector space can be written as a linear combination of vectors in a set $S$, then $S$ is called a **spanning set** of the vector space.

> **Definition of Spanning Set of a Vector Space**
>
> Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a subset of a vector space $V$. The set $S$ is called a **spanning set** of $V$ when *every* vector in $V$ can be written as a linear combination of vectors in $S$. In such cases it is said that $S$ **spans** $V$.

### EXAMPLE 4 Examples of Spanning Sets

**a.** The set $S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ spans $R^3$ because any vector $\mathbf{u} = (u_1, u_2, u_3)$ in $R^3$ can be written as

$$\mathbf{u} = u_1(1, 0, 0) + u_2(0, 1, 0) + u_3(0, 0, 1) = (u_1, u_2, u_3).$$

**b.** The set $S = \{1, x, x^2\}$ spans $P_2$ because any polynomial function $p(x) = a + bx + cx^2$ in $P_2$ can be written as

$$\begin{aligned} p(x) &= a(1) + b(x) + c(x^2) \\ &= a + bx + cx^2. \end{aligned}$$

The spanning sets in Example 4 are called the **standard spanning sets** of $R^3$ and $P_2$, respectively. (You will learn more about standard spanning sets in the next section.) In the next example, you will look at a nonstandard spanning set of $R^3$.

### EXAMPLE 5 A Spanning Set of $R^3$

Show that the set $S = \{(1, 2, 3), (0, 1, 2), (-2, 0, 1)\}$ spans $R^3$.

**SOLUTION**

Let $\mathbf{u} = (u_1, u_2, u_3)$ be *any* vector in $R^3$. You need to find scalars $c_1$, $c_2$, and $c_3$ such that

$$\begin{aligned} (u_1, u_2, u_3) &= c_1(1, 2, 3) + c_2(0, 1, 2) + c_3(-2, 0, 1) \\ &= (c_1 - 2c_3, 2c_1 + c_2, 3c_1 + 2c_2 + c_3). \end{aligned}$$

This vector equation produces the system

$$\begin{aligned} c_1 \phantom{{}+2c_2} - 2c_3 &= u_1 \\ 2c_1 + c_2 \phantom{{}+c_3} &= u_2 \\ 3c_1 + 2c_2 + c_3 &= u_3. \end{aligned}$$

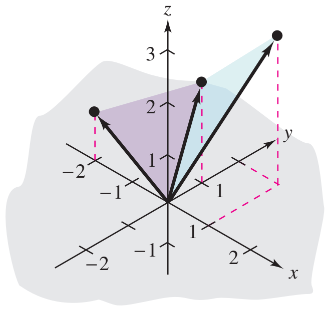
The coefficient matrix of this system has a nonzero determinant (try verifying that it is equal to $-1$), and it follows from the list of equivalent conditions given in Section 3.3 that the system has a unique solution. So, any vector in $R^3$ can be written as a linear combination of the vectors in $S$, and you can conclude that the set $S$ spans $R^3$.

### EXAMPLE 6 A Set That Does Not Span $R^3$

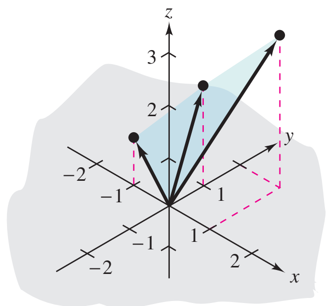
From Example 3 you know that the set

$$S = \{(1, 2, 3), (0, 1, 2), (-1, 0, 1)\}$$

*does not* span $R^3$ because $\mathbf{w} = (1, -2, 2)$ is in $R^3$ and cannot be expressed as a linear combination of the vectors in $S$.




$S_1 = \{(1, 2, 3), (0, 1, 2), (-2, 0, 1)\}$
The vectors in $S_1$ do not lie in a common plane.




$S_2 = \{(1, 2, 3), (0, 1, 2), (-1, 0, 1)\}$
The vectors in $S_2$ lie in a common plane.

**Figure 4.16**

Comparing the sets of vectors in Examples 5 and 6, note that the sets are the same except for a seemingly insignificant difference in the third vector.

$S_1 = \{(1, 2, 3), (0, 1, 2), (-2, 0, 1)\}$ Example 5

$S_2 = \{(1, 2, 3), (0, 1, 2), (-1, 0, 1)\}$ Example 6

The difference, however, is significant, because the set $S_1$ spans $R^3$ whereas the set $S_2$ does not. The reason for this difference can be seen in Figure 4.16. The vectors in $S_2$ lie in a common plane; the vectors in $S_1$ do not.

Although the set $S_2$ does not span all of $R^3$, it does span a subspace of $R^3$—namely, the plane in which the three vectors of $S_2$ lie. This subspace is called the **span of $S_2$,** as indicated in the next definition.

### Definition of the Span of a Set

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a set of vectors in a vector space $V$, then the **span of $S$** is the set of all linear combinations of the vectors in $S$,

$$\text{span}(S) = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k : c_1, c_2, \ldots, c_k \text{ are real numbers}\}.$$

The span of $S$ is denoted by

$$\text{span}(S) \quad \text{or} \quad \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}.$$

When $\text{span}(S) = V$, it is said that $V$ is **spanned** by $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$, or that $S$ **spans** $V$.

The following theorem tells you that the span of any finite nonempty subset of a vector space $V$ is a subspace of $V$.

### THEOREM 4.7 Span($S$) Is a Subspace of $V$

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a set of vectors in a vector space $V$, then span($S$) is a subspace of $V$. Moreover, span($S$) is the smallest subspace of $V$ that contains $S$, in the sense that every other subspace of $V$ that contains $S$ must contain span($S$).

### PROOF

To show that span($S$), the set of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$, is a subspace of $V$, show that it is closed under addition and scalar multiplication. Consider any two vectors $\mathbf{u}$ and $\mathbf{v}$ in span($S$),

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$$
$$\mathbf{v} = d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \cdots + d_k\mathbf{v}_k$$

where

$$c_1, c_2, \ldots, c_k \quad \text{and} \quad d_1, d_2, \ldots, d_k$$

are scalars. Then

$$\mathbf{u} + \mathbf{v} = (c_1 + d_1)\mathbf{v}_1 + (c_2 + d_2)\mathbf{v}_2 + \cdots + (c_k + d_k)\mathbf{v}_k$$

and

$$c\mathbf{u} = (cc_1)\mathbf{v}_1 + (cc_2)\mathbf{v}_2 + \cdots + (cc_k)\mathbf{v}_k$$

which means that $\mathbf{u} + \mathbf{v}$ and $c\mathbf{u}$ are also in span($S$) because they can be written as linear combinations of vectors in $S$. So, span($S$) is a subspace of $V$. It is left to you to prove that span($S$) is the smallest subspace of $V$ that contains $S$. (See Exercise 55.) ■

## LINEAR DEPENDENCE AND LINEAR INDEPENDENCE

For a given set of vectors

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$$

in a vector space $V$, the vector equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$$

always has the **trivial solution**

$$c_1 = 0, c_2 = 0, \ldots, c_k = 0.$$

Often, however, there are also **nontrivial** solutions. For instance, in Example 1(a) you saw that in the set

$$S = \{\overset{\mathbf{v}_1}{(1, 3, 1)}, \overset{\mathbf{v}_2}{(0, 1, 2)}, \overset{\mathbf{v}_3}{(1, 0, -5)}\}$$

the vector $\mathbf{v}_1$ can be written as a linear combination of the other two vectors, as follows.

$$\mathbf{v}_1 = 3\mathbf{v}_2 + \mathbf{v}_3$$

So, the vector equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$$

has a nontrivial solution in which the coefficients are *not all zero:*

$$c_1 = 1, \quad c_2 = -3, \quad c_3 = -1.$$

This characteristic is described by saying that the set $S$ is **linearly dependent.** Had the only solution been the trivial one $(c_1 = c_2 = c_3 = 0)$, then the set $S$ would have been **linearly independent.** This concept is essential to the study of linear algebra and is formally stated in the next definition.

### Definition of Linear Dependence and Linear Independence

A set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ in a vector space $V$ is called **linearly independent** when the vector equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$$

has only the trivial solution

$$c_1 = 0, c_2 = 0, \ldots, c_k = 0.$$

If there are also nontrivial solutions, then $S$ is called **linearly dependent.**

### EXAMPLE 7 Examples of Linearly Dependent Sets

**a.** The set $S = \{(1, 2), (2, 4)\}$ in $R^2$ is linearly dependent because

$$-2(1, 2) + (2, 4) = (0, 0).$$

**b.** The set $S = \{(1, 0), (0, 1), (-2, 5)\}$ in $R^2$ is linearly dependent because

$$2(1, 0) - 5(0, 1) + (-2, 5) = (0, 0).$$

**c.** The set $S = \{(0, 0), (1, 2)\}$ in $R^2$ is linearly dependent because

$$1(0, 0) + 0(1, 2) = (0, 0).$$

The next example demonstrates a testing procedure for determining whether a set of vectors is linearly independent or linearly dependent.

## EXAMPLE 8 Testing for Linear Independence

Determine whether the set of vectors in $R^3$ is linearly independent or linearly dependent.

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \{(1, 2, 3), (0, 1, 2), (-2, 0, 1)\}$$

### SOLUTION

To test for linear independence or linear dependence, form the vector equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}.$$

If the only solution of this equation is $c_1 = c_2 = c_3 = 0$, then the set $S$ is linearly independent. Otherwise, $S$ is linearly dependent. Expanding this equation, you have

$$\begin{aligned} c_1(1, 2, 3) + c_2(0, 1, 2) + c_3(-2, 0, 1) &= (0, 0, 0) \\ (c_1 - 2c_3, 2c_1 + c_2, 3c_1 + 2c_2 + c_3) &= (0, 0, 0) \end{aligned}$$

which yields the following homogeneous system of linear equations in $c_1$, $c_2$, and $c_3$.

$$\begin{aligned} c_1 \qquad\;\; - 2c_3 &= 0 \\ 2c_1 + \;\; c_2 \qquad\;\; &= 0 \\ 3c_1 + 2c_2 + \;\; c_3 &= 0 \end{aligned}$$

The augmented matrix of this system reduces by Gauss-Jordan elimination as follows.

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

This implies that the only solution is the trivial solution $c_1 = c_2 = c_3 = 0$. So, $S$ is linearly independent.

The steps in Example 8 are summarized as follows.

### Testing for Linear Independence and Linear Dependence

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, . . . , \mathbf{v}_k\}$ be a set of vectors in a vector space $V$. To determine whether $S$ is linearly independent or linearly dependent, use the following steps.

**1.** From the vector equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdot \cdot \cdot + c_k\mathbf{v}_k = \mathbf{0}$, write a system of linear equations in the variables $c_1$, $c_2$, . . . , and $c_k$.

**2.** Use Gaussian elimination to determine whether the system has a unique solution.

**3.** If the system has only the trivial solution, $c_1 = 0$, $c_2 = 0$, . . . , $c_k = 0$, then the set $S$ is linearly independent. If the system also has nontrivial solutions, then $S$ is linearly dependent.

### LINEAR ALGEBRA APPLIED

Image morphing is the process by which one image is transformed into another by generating a sequence of synthetic intermediate images. Morphing has a wide variety of applications, including movie special effects, wound healing and cosmetic surgery results simulation, and age progression software. Morphing an image makes use of a process called warping, in which a piece of an image is distorted. The mathematics behind warping and morphing can include forming a linear combination of the linearly independent vectors that bound a triangular piece of an image, and performing an *affine transformation* to form new vectors and an image piece that is distorted.

**EXAMPLE 9** **Testing for Linear Independence**

Determine whether the set of vectors in $P_2$ is linearly independent or linearly dependent.

$$S = \{\underbrace{1 + x - 2x^2}_{\mathbf{v}_1},\ \underbrace{2 + 5x - x^2}_{\mathbf{v}_2},\ \underbrace{x + x^2}_{\mathbf{v}_3}\}$$

**SOLUTION**

Expanding the equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$ produces

$$c_1(1 + x - 2x^2) + c_2(2 + 5x - x^2) + c_3(x + x^2) = 0 + 0x + 0x^2$$
$$(c_1 + 2c_2) + (c_1 + 5c_2 + c_3)x + (-2c_1 - c_2 + c_3)x^2 = 0 + 0x + 0x^2.$$

Equating corresponding coefficients of powers of $x$ produces the following homogeneous system of linear equations in $c_1$, $c_2$, and $c_3$.

$$\begin{aligned} c_1 + 2c_2 \phantom{{}+c_3} &= 0 \\ c_1 + 5c_2 + c_3 &= 0 \\ -2c_1 - c_2 + c_3 &= 0 \end{aligned}$$

The augmented matrix of this system reduces by Gaussian elimination as follows.

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 1 & 5 & 1 & 0 \\ -2 & -1 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This implies that the system has infinitely many solutions. So, the system must have nontrivial solutions, and you can conclude that the set $S$ is linearly dependent.

One nontrivial solution is

$$c_1 = 2, \quad c_2 = -1, \quad \text{and} \quad c_3 = 3$$

which yields the nontrivial linear combination

$$(2)(1 + x - 2x^2) + (-1)(2 + 5x - x^2) + (3)(x + x^2) = 0.$$

**EXAMPLE 10** **Testing for Linear Independence**

Determine whether the set of vectors in $M_{2,2}$ is linearly independent or linearly dependent.

$$S = \left\{\underbrace{\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}}_{\mathbf{v}_1}, \underbrace{\begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix}}_{\mathbf{v}_2}, \underbrace{\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}}_{\mathbf{v}_3}\right\}$$

**SOLUTION**

From the equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$, you have

$$c_1\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} + c_2\begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix} + c_3\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

which produces the following system of linear equations in $c_1$, $c_2$, and $c_3$.

$$\begin{aligned} 2c_1 + 3c_2 + c_3 &= 0 \\ c_1 \phantom{{}+3c_2+c_3} &= 0 \\ 2c_2 + 2c_3 &= 0 \\ c_1 + c_2 \phantom{{}+c_3} &= 0 \end{aligned}$$

Use Gaussian elimination to show that the system has only the trivial solution, which means that the set $S$ is linearly independent.

**EXAMPLE 11** **Testing for Linear Independence**

Determine whether the set of vectors in $M_{4,1}$ is linearly independent or linearly dependent.

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\} = \left\{\begin{bmatrix}1\\0\\-1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\\2\end{bmatrix}, \begin{bmatrix}0\\3\\1\\-2\end{bmatrix}, \begin{bmatrix}0\\1\\-1\\2\end{bmatrix}\right\}$$

**SOLUTION**

From the equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 = \mathbf{0}$, you obtain

$$c_1\begin{bmatrix}1\\0\\-1\\0\end{bmatrix} + c_2\begin{bmatrix}1\\1\\0\\2\end{bmatrix} + c_3\begin{bmatrix}0\\3\\1\\-2\end{bmatrix} + c_4\begin{bmatrix}0\\1\\-1\\2\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}.$$

This equation produces the following system of linear equations in $c_1$, $c_2$, $c_3$, and $c_4$.

$$\begin{aligned} c_1 + \phantom{2}c_2 \phantom{{}+3c_3+c_4} &= 0\\ c_2 + 3c_3 + \phantom{2}c_4 &= 0\\ -c_1 \phantom{{}+c_2} + \phantom{3}c_3 - \phantom{2}c_4 &= 0\\ 2c_2 - 2c_3 + 2c_4 &= 0 \end{aligned}$$

Use Gaussian elimination to show that the system has only the trivial solution, which means that the set $S$ is linearly independent.

If a set of vectors is linearly dependent, then by definition the equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdot\cdot\cdot + c_k\mathbf{v}_k = \mathbf{0}$ has a nontrivial solution (a solution for which not all the $c_i$'s are zero). For instance, if $c_1 \neq 0$, then you can solve this equation for $\mathbf{v}_1$ and write $\mathbf{v}_1$ as a linear combination of the other vectors $\mathbf{v}_2, \mathbf{v}_3, \ldots$, and $\mathbf{v}_k$. In other words, the vector $\mathbf{v}_1$ *depends* on the other vectors in the set. This property is characteristic of a linearly dependent set.

**THEOREM 4.8 A Property of Linearly Dependent Sets**

A set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$, $k \geq 2$, is linearly dependent if and only if at least one of the vectors $\mathbf{v}_j$ can be written as a linear combination of the other vectors in $S$.

**PROOF**

To prove the theorem in one direction, assume $S$ is a linearly dependent set. Then there exist scalars $c_1, c_2, c_3, \ldots, c_k$ (not all zero) such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + \cdot\cdot\cdot + c_k\mathbf{v}_k = \mathbf{0}.$$

Because one of the coefficients must be nonzero, no generality is lost by assuming $c_1 \neq 0$. Then solving for $\mathbf{v}_1$ as a linear combination of the other vectors produces

$$\begin{aligned} c_1\mathbf{v}_1 &= -c_2\mathbf{v}_2 - c_3\mathbf{v}_3 - \cdot\cdot\cdot - c_k\mathbf{v}_k\\ \mathbf{v}_1 &= -\frac{c_2}{c_1}\mathbf{v}_2 - \frac{c_3}{c_1}\mathbf{v}_3 - \cdot\cdot\cdot - \frac{c_k}{c_1}\mathbf{v}_k. \end{aligned}$$

Conversely, suppose the vector $\mathbf{v}_1$ in $S$ is a linear combination of the other vectors. That is,

$$\mathbf{v}_1 = c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + \cdot\cdot\cdot + c_k\mathbf{v}_k.$$

Then the equation $-\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + \cdot\cdot\cdot + c_k\mathbf{v}_k = \mathbf{0}$ has at least one coefficient, $-1$, that is nonzero, and you can conclude that $S$ is linearly dependent.

## EXAMPLE 12 Writing a Vector as a Linear Combination of Other Vectors

In Example 9, you determined that the set

$$S = \{\overset{\mathbf{v}_1}{1 + x - 2x^2}, \overset{\mathbf{v}_2}{2 + 5x - x^2}, \overset{\mathbf{v}_3}{x + x^2}\}$$

is linearly dependent. Show that one of the vectors in this set can be written as a linear combination of the other two.

### SOLUTION

In Example 9, the equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$ produced the system

$$\begin{aligned} c_1 + 2c_2 \phantom{{}+c_3} &= 0 \\ c_1 + 5c_2 + c_3 &= 0 \\ -2c_1 - c_2 + c_3 &= 0. \end{aligned}$$

This system has infinitely many solutions represented by $c_3 = 3t$, $c_2 = -t$, and $c_1 = 2t$. Letting $t = 1$ results in the equation $2\mathbf{v}_1 - \mathbf{v}_2 + 3\mathbf{v}_3 = \mathbf{0}$. So, $\mathbf{v}_2$ can be written as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_3$, as follows.

$$\mathbf{v}_2 = 2\mathbf{v}_1 + 3\mathbf{v}_3$$

A check yields

$$2 + 5x - x^2 = 2(1 + x - 2x^2) + 3(x + x^2) = 2 + 5x - x^2.$$

Theorem 4.8 has a practical corollary that provides a simple test for determining whether *two* vectors are linearly dependent. In Exercise 73 you are asked to prove this corollary.

**REMARK**

The zero vector is always a scalar multiple of another vector in a vector space.

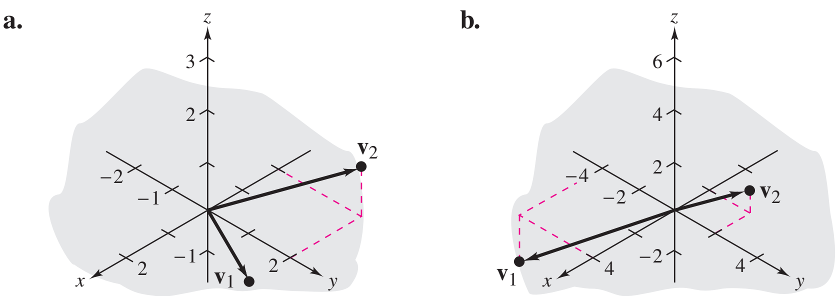
**THEOREM 4.8 Corollary**

Two vectors $\mathbf{u}$ and $\mathbf{v}$ in a vector space $V$ are linearly dependent if and only if one is a scalar multiple of the other.

## EXAMPLE 13 Testing for Linear Dependence of Two Vectors

**a.** The set $S = \{\mathbf{v}_1, \mathbf{v}_2\} = \{(1, 2, 0), (-2, 2, 1)\}$ is linearly independent because $\mathbf{v}_1$ and $\mathbf{v}_2$ are not scalar multiples of each other, as shown in Figure 4.17(a).

**b.** The set $S = \{\mathbf{v}_1, \mathbf{v}_2\} = \{(4, -4, -2), (-2, 2, 1)\}$ is linearly dependent because $\mathbf{v}_1 = -2\mathbf{v}_2$, as shown in Figure 4.17(b).




a. $S = \{(1, 2, 0), (-2, 2, 1)\}$
The set $S$ is linearly independent.

b. $S = \{(4, -4, -2), (-2, 2, 1)\}$
The set $S$ is linearly dependent.

**Figure 4.17**

# 4.4 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

**Linear Combinations** **In Exercises 1–4, write each vector as a linear combination of the vectors in $S$ (if possible).**

1. $S = \{(2, -1, 3), (5, 0, 4)\}$
   (a) $\mathbf{z} = (-1, -2, 2)$ (b) $\mathbf{v} = \left(8, -\frac{1}{4}, \frac{27}{4}\right)$
   (c) $\mathbf{w} = (1, -8, 12)$ (d) $\mathbf{u} = (1, 1, -1)$
2. $S = \{(1, 2, -2), (2, -1, 1)\}$
   (a) $\mathbf{z} = (-4, -3, 3)$ (b) $\mathbf{v} = (-2, -6, 6)$
   (c) $\mathbf{w} = (-1, -22, 22)$ (d) $\mathbf{u} = (1, -5, -5)$
3. $S = \{(2, 0, 7), (2, 4, 5), (2, -12, 13)\}$
   (a) $\mathbf{u} = (-1, 5, -6)$ (b) $\mathbf{v} = (-3, 15, 18)$
   (c) $\mathbf{w} = \left(\frac{1}{3}, \frac{4}{3}, \frac{1}{2}\right)$ (d) $\mathbf{z} = (2, 20, -3)$
4. $S = \{(6, -7, 8, 6), (4, 6, -4, 1)\}$
   (a) $\mathbf{u} = (-42, 113, -112, -60)$
   (b) $\mathbf{v} = \left(\frac{49}{2}, \frac{99}{4}, -14, \frac{19}{2}\right)$
   (c) $\mathbf{w} = \left(-4, -14, \frac{27}{2}, \frac{53}{8}\right)$
   (d) $\mathbf{z} = \left(8, 4, -1, \frac{17}{4}\right)$

**Linear Combinations** **In Exercises 5–8, for the matrices**

$$A = \begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 5 \\ 1 & -2 \end{bmatrix}$$

**in $M_{2,2}$, determine whether the given matrix is a linear combination of $A$ and $B$.**

5. $\begin{bmatrix} 6 & -19 \\ 10 & 7 \end{bmatrix}$
6. $\begin{bmatrix} 6 & 2 \\ 9 & 11 \end{bmatrix}$
7. $\begin{bmatrix} -2 & 28 \\ 1 & -11 \end{bmatrix}$
8. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

**Spanning Sets** **In Exercises 9–20, determine whether the set $S$ spans $R^2$. If the set does not span $R^2$, then give a geometric description of the subspace that it does span.**

9. $S = \{(2, 1), (-1, 2)\}$
10. $S = \{(1, -1), (2, 1)\}$
11. $S = \{(5, 0), (5, -4)\}$
12. $S = \{(2, 0), (0, 1)\}$
13. $S = \{(-3, 5)\}$
14. $S = \{(1, 1)\}$
15. $S = \{(1, 3), (-2, -6), (4, 12)\}$
16. $S = \left\{(1, 2), (-2, -4), \left(\frac{1}{2}, 1\right)\right\}$
17. $S = \{(-1, 2), (2, -4)\}$
18. $S = \{(0, 2), (1, 4)\}$
19. $S = \{(-1, 4), (4, -1), (1, 1)\}$
20. $S = \{(-1, 2), (2, -1), (1, 1)\}$

**Spanning Sets** **In Exercises 21–26, determine whether the set $S$ spans $R^3$. If the set does not span $R^3$, then give a geometric description of the subspace that it does span.**

21. $S = \{(4, 7, 3), (-1, 2, 6), (2, -3, 5)\}$
22. $S = \{(6, 7, 6), (3, 2, -4), (1, -3, 2)\}$
23. $S = \{(-2, 5, 0), (4, 6, 3)\}$
24. $S = \{(1, 0, 1), (1, 1, 0), (0, 1, 1)\}$
25. $S = \{(1, -2, 0), (0, 0, 1), (-1, 2, 0)\}$
26. $S = \{(1, 0, 3), (2, 0, -1), (4, 0, 5), (2, 0, 6)\}$

27. Determine whether the set $S = \{1, x^2, x^2 + 2\}$ spans $P_2$.
28. Determine whether the set
    $S = \{x^2 - 2x, x^3 + 8, x^3 - x^2, x^2 - 4\}$
    spans $P_3$.

**Testing for Linear Independence** **In Exercises 29–40, determine whether the set $S$ is linearly independent or linearly dependent.**

29. $S = \{(-2, 2), (3, 5)\}$
30. $S = \{(3, -6), (-1, 2)\}$
31. $S = \{(0, 0), (1, -1)\}$
32. $S = \{(1, 0), (1, 1), (2, -1)\}$
33. $S = \{(1, -4, 1), (6, 3, 2)\}$
34. $S = \{(6, 2, 1), (-1, 3, 2)\}$
35. $S = \{(1, 1, 1), (2, 2, 2), (3, 3, 3)\}$
36. $S = \left\{\left(\frac{3}{4}, \frac{5}{2}, \frac{3}{2}\right), \left(3, 4, \frac{7}{2}\right), \left(-\frac{3}{2}, 6, 2\right)\right\}$
37. $S = \{(-4, -3, 4), (1, -2, 3), (6, 0, 0)\}$
38. $S = \{(1, 0, 0), (0, 4, 0), (0, 0, -6), (1, 5, -3)\}$
39. $S = \{(4, -3, 6, 2), (1, 8, 3, 1), (3, -2, -1, 0)\}$
40. $S = \{(0, 0, 0, 1), (0, 0, 1, 1), (0, 1, 1, 1), (1, 1, 1, 1)\}$

**Testing for Linear Independence** **In Exercises 41–44, determine whether the set of vectors in $P_2$ is linearly independent or linearly dependent.**

41. $S = \{2 - x, 2x - x^2, 6 - 5x + x^2\}$
42. $S = \{x^2 - 1, 2x + 5\}$
43. $S = \{x^2 + 3x + 1, 2x^2 + x - 1, 4x\}$
44. $S = \{x^2, x^2 + 1\}$

**Testing for Linear Independence** **In Exercises 45–48, determine whether the set of vectors in $M_{2,2}$ is linearly independent or linearly dependent.**

45. $A = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, C = \begin{bmatrix} -2 & 1 \\ 1 & 4 \end{bmatrix}$
46. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$
47. $A = \begin{bmatrix} 1 & -1 \\ 4 & 5 \end{bmatrix}, B = \begin{bmatrix} 4 & 3 \\ -2 & 3 \end{bmatrix}, C = \begin{bmatrix} 1 & -8 \\ 22 & 23 \end{bmatrix}$
48. $A = \begin{bmatrix} 2 & 0 \\ -3 & 1 \end{bmatrix}, B = \begin{bmatrix} -4 & -1 \\ 0 & 5 \end{bmatrix}, C = \begin{bmatrix} -8 & -3 \\ -6 & 17 \end{bmatrix}$

**Showing Linear Dependence** **In Exercises 49–52, show that the set is linearly dependent by finding a nontrivial linear combination of vectors in the set whose sum is the zero vector. Then express one of the vectors in the set as a linear combination of the other vectors in the set.**

**49.** $S = \{(3, 4), (-1, 1), (2, 0)\}$

**50.** $S = \{(2, 4), (-1, -2), (0, 6)\}$

**51.** $S = \{(1, 1, 1), (1, 1, 0), (0, 1, 1), (0, 0, 1)\}$

**52.** $S = \{(1, 2, 3, 4), (1, 0, 1, 2), (1, 4, 5, 6)\}$

**53.** For which values of $t$ is each set linearly independent?

(a) $S = \{(t, 1, 1), (1, t, 1), (1, 1, t)\}$

(b) $S = \{(t, 1, 1), (1, 0, 1), (1, 1, 3t)\}$

**54.** For which values of $t$ is each set linearly independent?

(a) $S = \{(t, 0, 0), (0, 1, 0), (0, 0, 1)\}$

(b) $S = \{(t, t, t), (t, 1, 0), (t, 0, 1)\}$

**55.** **Proof** Complete the proof of Theorem 4.7.

**56.** **CAPSTONE** By inspection, determine why each of the sets is linearly dependent.

(a) $S = \{(1, -2), (2, 3), (-2, 4)\}$

(b) $S = \{(1, -6, 2), (2, -12, 4)\}$

(c) $S = \{(0, 0), (1, 0)\}$

**Spanning the Same Subspace** **In Exercises 57 and 58, show that the sets $S_1$ and $S_2$ span the same subspace of $R^3$.**

**57.** $S_1 = \{(1, 2, -1), (0, 1, 1), (2, 5, -1)\}$

$S_2 = \{(-2, -6, 0), (1, 1, -2)\}$

**58.** $S_1 = \{(0, 0, 1), (0, 1, 1), (2, 1, 1)\}$

$S_2 = \{(1, 1, 1), (1, 1, 2), (2, 1, 1)\}$

**True or False?** **In Exercises 59 and 60, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.**

**59.** (a) A set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ in a vector space is called linearly dependent when the vector equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$ has only the trivial solution.

(b) The set $S = \{(1, 0, 0, 0), (0, -1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}$ spans $R^4$.

**60.** (a) A set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$, $k \geq 2$, is linearly independent if and only if at least one of the vectors $\mathbf{v}_j$ can be written as a linear combination of the other vectors.

(b) If a subset $S$ spans a vector space $V$, then every vector in $V$ can be written as a linear combination of the vectors in $S$.

**Proof** **In Exercises 61 and 62, prove that the set of vectors is linearly independent and spans $R^3$.**

**61.** $B = \{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$

**62.** $B = \{(1, 2, 3), (3, 2, 1), (0, 0, 1)\}$

**63.** **Guided Proof** Prove that a nonempty subset of a finite set of linearly independent vectors is linearly independent.

Getting Started: You need to show that a subset of a linearly independent set of vectors cannot be linearly dependent.

(i) Suppose $S$ is a set of linearly independent vectors. Let $T$ be a subset of $S$.

(ii) If $T$ is linearly dependent, then there exist constants not all zero satisfying the vector equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$.

(iii) Use this fact to derive a contradiction and conclude that $T$ is linearly independent.

**64.** **Proof** Prove that if $S_1$ is a nonempty subset of the finite set $S_2$, and $S_1$ is linearly dependent, then so is $S_2$.

**65.** **Proof** Prove that any set of vectors containing the zero vector is linearly dependent.

**66.** **Proof** When $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is a linearly independent set of vectors and the set $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n, \mathbf{v}\}$ is linearly dependent, prove that $\mathbf{v}$ is a linear combination of the $\mathbf{u}_i$'s.

**67.** **Proof** Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a linearly independent set of vectors in a vector space $V$. Delete the vector $\mathbf{v}_k$ from this set and prove that the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{k-1}\}$ cannot span $V$.

**68.** **Proof** When $V$ is spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ and one of these vectors can be written as a linear combination of the other $k - 1$ vectors, prove that the span of these $k - 1$ vectors is also $V$.

**69.** **Proof** Let $S = \{\mathbf{u}, \mathbf{v}\}$ be a linearly independent set. Prove that the set $\{\mathbf{u} + \mathbf{v}, \mathbf{u} - \mathbf{v}\}$ is linearly independent.

**70.** Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be any three vectors from a vector space $V$. Determine whether the set of vectors $\{\mathbf{v} - \mathbf{u}, \mathbf{w} - \mathbf{v}, \mathbf{u} - \mathbf{w}\}$ is linearly independent or linearly dependent.

**71.** **Proof** Let $A$ be a nonsingular matrix of order 3. Prove that if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set in $M_{3,1}$, then the set $\{A\mathbf{v}_1, A\mathbf{v}_2, A\mathbf{v}_3\}$ is also linearly independent. Explain, by means of an example, why this is not true when $A$ is singular.

**72.** **Writing** Under what conditions will a set consisting of a single vector be linearly independent?

**73.** **Proof** Prove the corollary to Theorem 4.8: Two vectors $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent if and only if one is a scalar multiple of the other.

# 4.5 Basis and Dimension

- Recognize bases in the vector spaces $R^n$, $P_n$, and $M_{m,n}$.
- Find the dimension of a vector space.

## BASIS FOR A VECTOR SPACE

**REMARK**

This definition tells you that a basis has two features. A basis $S$ must have *enough vectors* to span $V$, but *not so many vectors* that one of them could be written as a linear combination of the other vectors in $S$.

In this section, you will continue your study of spanning sets. In particular, you will look at spanning sets in a vector space that are both linearly independent *and* span the entire space. Such a set forms a **basis** for the vector space. (The plural of *basis* is *bases*.)

**Definition of Basis**

A set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, . . . , \mathbf{v}_n\}$ in a vector space $V$ is called a **basis** for $V$ when the following conditions are true.

**1.** $S$ spans $V$. **2.** $S$ is linearly independent.

This definition does not imply that every vector space has a basis consisting of a finite number of vectors. This text, however, restricts the discussion of bases to those consisting of a finite number of vectors. Moreover, if a vector space $V$ has a basis consisting of a finite number of vectors, then $V$ is **finite dimensional.** Otherwise, $V$ is called **infinite dimensional.** [The vector space $P$ of *all* polynomials is infinite dimensional, as is the vector space $C(-\infty, \infty)$ of all continuous functions defined on the real line.] The vector space $V = \{\mathbf{0}\}$, consisting of the zero vector alone, is finite dimensional.

**EXAMPLE 1** **The Standard Basis for $R^3$**

Show that the following set is a basis for $R^3$.

$$S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$$

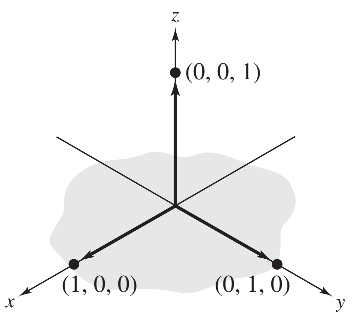



**Figure 4.18**

**SOLUTION**

Example 4(a) in Section 4.4 showed that $S$ spans $R^3$. Furthermore, $S$ is linearly independent because the vector equation

$$c_1(1, 0, 0) + c_2(0, 1, 0) + c_3(0, 0, 1) = (0, 0, 0)$$

has only the trivial solution

$$c_1 = c_2 = c_3 = 0.$$

(Try verifying this.) So, $S$ is a basis for $R^3$. (See Figure 4.18.)

The basis $S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ is called the **standard basis** for $R^3$. This can be generalized to $n$-space. That is, the vectors

$$\begin{aligned}\mathbf{e}_1 &= (1, 0, . . . , 0)\\ \mathbf{e}_2 &= (0, 1, . . . , 0)\\ &\vdots\\ \mathbf{e}_n &= (0, 0, . . . , 1)\end{aligned}$$

form a basis for $R^n$ called the **standard basis** for $R^n$.

The next two examples describe nonstandard bases for $R^2$ and $R^3$.

## EXAMPLE 2 A Nonstandard Basis for $R^2$

Show that the set

$$S = \{\overset{\mathbf{v}_1}{(1, 1)}, \overset{\mathbf{v}_2}{(1, -1)}\}$$

is a basis for $R^2$.

### SOLUTION

According to the definition of a basis for a vector space, you must show that $S$ spans $R^2$ and $S$ is linearly independent.

To verify that $S$ spans $R^2$, let

$$\mathbf{x} = (x_1, x_2)$$

represent an arbitrary vector in $R^2$. To show that $\mathbf{x}$ can be written as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, consider the equation

$$\begin{aligned} c_1\mathbf{v}_1 + c_2\mathbf{v}_2 &= \mathbf{x} \\ c_1(1, 1) + c_2(1, -1) &= (x_1, x_2) \\ (c_1 + c_2, c_1 - c_2) &= (x_1, x_2). \end{aligned}$$

Equating corresponding components yields the following system of linear equations.

$$\begin{aligned} c_1 + c_2 &= x_1 \\ c_1 - c_2 &= x_2 \end{aligned}$$

Because the coefficient matrix of this system has a nonzero determinant, you know that the system has a unique solution. So, $S$ spans $R^2$.

To show that $S$ is linearly independent, consider the linear combination

$$\begin{aligned} c_1\mathbf{v}_1 + c_2\mathbf{v}_2 &= \mathbf{0} \\ c_1(1, 1) + c_2(1, -1) &= (0, 0) \\ (c_1 + c_2, c_1 - c_2) &= (0, 0). \end{aligned}$$

Equating corresponding components yields the homogeneous system

$$\begin{aligned} c_1 + c_2 &= 0 \\ c_1 - c_2 &= 0. \end{aligned}$$

Because the coefficient matrix of this system has a nonzero determinant, you know that the system has only the trivial solution

$$c_1 = c_2 = 0.$$

So, $S$ is linearly independent.

You can conclude that $S$ is a basis for $R^2$ because it is a spanning set for $R^2$ and it is linearly independent.

## EXAMPLE 3 A Nonstandard Basis for $R^3$

From Examples 5 and 8 in the preceding section, you know that

$$S = \{(1, 2, 3), (0, 1, 2), (-2, 0, 1)\}$$

spans $R^3$ and is linearly independent. So, $S$ is a basis for $R^3$.

## EXAMPLE 4 A Basis for Polynomials

Show that the vector space $P_3$ has the basis

$$S = \{1, x, x^2, x^3\}.$$

### SOLUTION

It is clear that $S$ spans $P_3$ because the span of $S$ consists of all polynomials of the form

$$a_0 + a_1x + a_2x^2 + a_3x^3, \quad a_0, a_1, a_2, \text{ and } a_3 \text{ are real}$$

which is precisely the form of all polynomials in $P_3$.

To verify the linear independence of $S$, recall that the zero vector $\mathbf{0}$ in $P_3$ is the polynomial $\mathbf{0}(x) = 0$ for all $x$. The test for linear independence yields the equation

$$a_0 + a_1x + a_2x^2 + a_3x^3 = \mathbf{0}(x) = 0, \quad \text{for all } x.$$

This third-degree polynomial is said to be *identically equal to zero*. From algebra you know that for a polynomial to be identically equal to zero, all of its coefficients must be zero; that is,

$$a_0 = a_1 = a_2 = a_3 = 0.$$

So, $S$ is linearly independent and is a basis for $P_3$.

**REMARK**

The basis $S = \{1, x, x^2, x^3\}$ is called the **standard basis** for $P_3$. Similarly, the **standard basis for** $P_n$ is

$$S = \{1, x, x^2, . . . , x^n\}.$$

## EXAMPLE 5 A Basis for $M_{2,2}$

The set

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

is a basis for $M_{2,2}$. This set is called the **standard basis** for $M_{2,2}$. In a similar manner, the standard basis for the vector space $M_{m,n}$ consists of the $mn$ distinct $m \times n$ matrices having a single entry equal to 1 and all the other entries equal to 0.

### THEOREM 4.9 Uniqueness of Basis Representation

If $S = \{\mathbf{v}_1, \mathbf{v}_2, . . . , \mathbf{v}_n\}$ is a basis for a vector space $V$, then every vector in $V$ can be written in one and only one way as a linear combination of vectors in $S$.

### PROOF

The existence portion of the proof is straightforward. That is, because $S$ spans $V$, you know that an arbitrary vector $\mathbf{u}$ in $V$ can be expressed as $\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdot \cdot \cdot + c_n\mathbf{v}_n$.

To prove uniqueness (that a vector can be represented in only one way), suppose $\mathbf{u}$ has another representation

$$\mathbf{u} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdot \cdot \cdot + b_n\mathbf{v}_n.$$

Subtracting the second representation from the first produces

$$\mathbf{u} - \mathbf{u} = (c_1 - b_1)\mathbf{v}_1 + (c_2 - b_2)\mathbf{v}_2 + \cdot \cdot \cdot + (c_n - b_n)\mathbf{v}_n = \mathbf{0}.$$

Because $S$ is linearly independent, however, the only solution to this equation is the trivial solution

$$c_1 - b_1 = 0, \quad c_2 - b_2 = 0, \quad . . . , \quad c_n - b_n = 0$$

which means that $c_i = b_i$ for all $i = 1, 2, . . . , n$. So, $\mathbf{u}$ has only one representation for the basis $S$.

## EXAMPLE 6 Uniqueness of Basis Representation

Let $\mathbf{u} = (u_1, u_2, u_3)$ be any vector in $R^3$. Show that the equation $\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$ has a unique solution for the basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \{(1, 2, 3), (0, 1, 2), (-2, 0, 1)\}$.

### SOLUTION

From the equation

$$\begin{aligned}(u_1, u_2, u_3) &= c_1(1, 2, 3) + c_2(0, 1, 2) + c_3(-2, 0, 1)\\ &= (c_1 - 2c_3, 2c_1 + c_2, 3c_1 + 2c_2 + c_3)\end{aligned}$$

you obtain the following system of linear equations.

$$\begin{aligned} c_1 \qquad\quad - 2c_3 &= u_1\\ 2c_1 + \quad c_2 \qquad\quad &= u_2\\ 3c_1 + 2c_2 + \quad c_3 &= u_3 \end{aligned} \qquad \underset{A}{\begin{bmatrix}1 & 0 & -2\\ 2 & 1 & 0\\ 3 & 2 & 1\end{bmatrix}}\underset{\mathbf{c}}{\begin{bmatrix}c_1\\ c_2\\ c_3\end{bmatrix}} = \underset{\mathbf{u}}{\begin{bmatrix}u_1\\ u_2\\ u_3\end{bmatrix}}$$

Because the matrix $A$ is invertible, you know this system has a unique solution, $\mathbf{c} = A^{-1}\mathbf{u}$. Verify by finding $A^{-1}$ that

$$\begin{aligned} c_1 &= -u_1 + 4u_2 - 2u_3\\ c_2 &= \ \ 2u_1 - 7u_2 + 4u_3\\ c_3 &= -u_1 + 2u_2 - \ \ u_3. \end{aligned}$$

For instance, $\mathbf{u} = (1, 0, 0)$ can be represented uniquely as $-\mathbf{v}_1 + 2\mathbf{v}_2 - \mathbf{v}_3$.

You will now study two important theorems concerning bases.

> **THEOREM 4.10 Bases and Linear Dependence**
>
> If $S = \{\mathbf{v}_1, \mathbf{v}_2, . . . , \mathbf{v}_n\}$ is a basis for a vector space $V$, then every set containing more than $n$ vectors in $V$ is linearly dependent.

### PROOF

Let $S_1 = \{\mathbf{u}_1, \mathbf{u}_2, . . . , \mathbf{u}_m\}$ be any set of $m$ vectors in $V$, where $m > n$. To show that $S_1$ is linearly *dependent,* you need to find scalars $k_1, k_2, . . . , k_m$ (not all zero) such that

$$k_1\mathbf{u}_1 + k_2\mathbf{u}_2 + \cdots + k_m\mathbf{u}_m = \mathbf{0}. \qquad \text{Equation 1}$$

Because $S$ is a basis for $V$, each $\mathbf{u}_i$ is a linear combination of vectors in $S$:

$$\begin{aligned} \mathbf{u}_1 &= c_{11}\mathbf{v}_1 + c_{21}\mathbf{v}_2 + \cdots + c_{n1}\mathbf{v}_n\\ \mathbf{u}_2 &= c_{12}\mathbf{v}_1 + c_{22}\mathbf{v}_2 + \cdots + c_{n2}\mathbf{v}_n\\ &\ \vdots\\ \mathbf{u}_m &= c_{1m}\mathbf{v}_1 + c_{2m}\mathbf{v}_2 + \cdots + c_{nm}\mathbf{v}_n. \end{aligned}$$

Substituting into Equation 1 and regrouping terms produces

$$d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \cdots + d_n\mathbf{v}_n = \mathbf{0}$$

where $d_i = c_{i1}k_1 + c_{i2}k_2 + \cdots + c_{im}k_m$. Because the $\mathbf{v}_i$'s form a linearly independent set, each $d_i = 0$. So, you obtain the following system of equations.

$$\begin{aligned} c_{11}k_1 + c_{12}k_2 + \cdots + c_{1m}k_m &= 0\\ c_{21}k_1 + c_{22}k_2 + \cdots + c_{2m}k_m &= 0\\ &\ \vdots\\ c_{n1}k_1 + c_{n2}k_2 + \cdots + c_{nm}k_m &= 0 \end{aligned}$$

But this homogeneous system has fewer equations than variables $k_1, k_2, . . . , k_m$, and from Theorem 1.1, it has *nontrivial* solutions. Consequently, $S_1$ is linearly dependent.

## EXAMPLE 7 Linearly Dependent Sets in $R^3$ and $P_3$

**a.** Because $R^3$ has a basis consisting of three vectors, the set

$$S = \{(1, 2, -1), (1, 1, 0), (2, 3, 0), (5, 9, -1)\}$$

must be linearly dependent.

**b.** Because $P_3$ has a basis consisting of four vectors, the set

$$S = \{1, 1 + x, 1 - x, 1 + x + x^2, 1 - x + x^2\}$$

must be linearly dependent.

Because $R^n$ has the standard basis consisting of $n$ vectors, it follows from Theorem 4.10 that every set of vectors in $R^n$ containing more than $n$ vectors must be linearly dependent. Another significant consequence of Theorem 4.10 is given in the next theorem.

### THEOREM 4.11 Number of Vectors in a Basis

If a vector space $V$ has one basis with $n$ vectors, then every basis for $V$ has $n$ vectors.

**PROOF**

Let $S_1 = \{\mathbf{v}_1, \mathbf{v}_2, . . . , \mathbf{v}_n\}$ be the basis for $V$, and let $S_2 = \{\mathbf{u}_1, \mathbf{u}_2, . . . , \mathbf{u}_m\}$ be any other basis for $V$. Because $S_1$ is a basis and $S_2$ is linearly independent, Theorem 4.10 implies that $m \leq n$. Similarly, $n \leq m$ because $S_1$ is linearly independent and $S_2$ is a basis. Consequently, $n = m$.

## EXAMPLE 8 Spanning Sets and Bases

Use Theorem 4.11 to explain why each of the following statements is true.

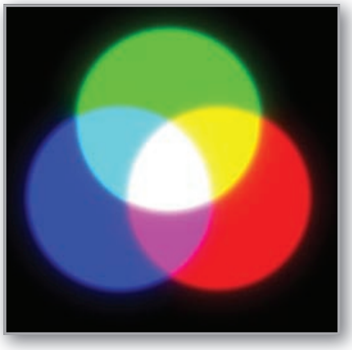
**a.** The set $S_1 = \{(3, 2, 1), (7, -1, 4)\}$ is not a basis for $R^3$.

**b.** The set $S_2 = \{x + 2, x^2, x^3 - 1, 3x + 1, x^2 - 2x + 3\}$ is not a basis for $P_3$.

**SOLUTION**

**a.** The standard basis for $R^3$ has three vectors, and $S_1$ has only two vectors. By Theorem 4.11, $S_1$ cannot be a basis for $R^3$.

**b.** The standard basis for $P_3$, $S = \{1, x, x^2, x^3\}$, has four vectors. By Theorem 4.11, the set $S_2$ has too many vectors to be a basis for $P_3$.



**LINEAR ALGEBRA APPLIED**

The RGB color model uses the theory that all visible colors are combinations of the colors red (**r**), green (**g**), and blue (**b**), known as the *primary additive colors*. Using the standard basis for $R^3$, where $\mathbf{r} = (1, 0, 0)$, $\mathbf{g} = (0, 1, 0)$, and $\mathbf{b} = (0, 0, 1)$, any visible color can be represented as a linear combination $c_1\mathbf{r} + c_2\mathbf{g} + c_3\mathbf{b}$ of the primary additive colors. The coefficients $c_i$ are values between 0 and a specified maximum $a$, inclusive. When $c_1 = c_2 = c_3$, the color is *grayscale*, with $c_i = 0$ representing black and $c_i = a$ representing white. The RGB color model is commonly used in computer monitors, smart phones, televisions, and other electronic equipment.

Chernetskiy/Shutterstock.com

## THE DIMENSION OF A VECTOR SPACE

By Theorem 4.11, if a vector space $V$ has a basis consisting of $n$ vectors, then every other basis for the space also has $n$ vectors. This number $n$ is called the **dimension** of $V$.

### Definition of Dimension of a Vector Space

If a vector space $V$ has a basis consisting of $n$ vectors, then the number $n$ is called the **dimension** of $V$, denoted by $\dim(V) = n$. When $V$ consists of the zero vector alone, the dimension of $V$ is defined as zero.

This definition allows you to observe the characteristics of the dimensions of the familiar vector spaces listed below. In each case, the dimension is determined by simply counting the number of vectors in the standard basis.

1. The dimension of $R^n$ with the standard operations is $n$.
2. The dimension of $P_n$ with the standard operations is $n + 1$.
3. The dimension of $M_{m,n}$ with the standard operations is $mn$.

If $W$ is a subspace of an $n$-dimensional vector space, then it can be shown that $W$ is finite dimensional and that the dimension of $W$ is less than or equal to $n$. (See Exercise 75.) The next three examples show a technique for determining the dimension of a subspace. Basically, you determine the dimension by finding a set of linearly independent vectors that spans the subspace. This set is a basis for the subspace, and the dimension of the subspace is the number of vectors in the basis.

### EXAMPLE 9 Finding the Dimension of a Subspace

Determine the dimension of each subspace of $R^3$.

**a.** $W = \{(d, c - d, c)\text{: } c \text{ and } d \text{ are real numbers}\}$

**b.** $W = \{(2b, b, 0)\text{: } b \text{ is a real number}\}$

**SOLUTION**

**a.** By writing the representative vector $(d, c - d, c)$ as

$$(d, c - d, c) = (0, c, c) + (d, -d, 0) = c(0, 1, 1) + d(1, -1, 0)$$

you can see that $W$ is spanned by the set $S = \{(0, 1, 1), (1, -1, 0)\}$. Using the techniques described in the preceding section, you can show that this set is linearly independent. So, it is a basis for $W$, and $W$ is a two-dimensional subspace of $R^3$.

**b.** By writing the representative vector $(2b, b, 0)$ as $b(2, 1, 0)$, you can see that $W$ is spanned by the set $S = \{(2, 1, 0)\}$. So, $W$ is a one-dimensional subspace of $R^3$. ■

**REMARK**

In Example 9(a), the subspace $W$ is a two-dimensional plane in $R^3$ determined by the vectors $(0, 1, 1)$ and $(1, -1, 0)$. In Example 9(b), the subspace is a one-dimensional line.

### EXAMPLE 10 Finding the Dimension of a Subspace

Find the dimension of the subspace $W$ of $R^4$ spanned by

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \{(-1, 2, 5, 0), (3, 0, 1, -2), (-5, 4, 9, 2)\}.$$

**SOLUTION**

Although $W$ is spanned by the set $S$, $S$ is not a basis for $W$ because $S$ is a linearly dependent set. Specifically, $\mathbf{v}_3$ can be written as $\mathbf{v}_3 = 2\mathbf{v}_1 - \mathbf{v}_2$. This means that $W$ is spanned by the set $S_1 = \{\mathbf{v}_1, \mathbf{v}_2\}$. Moreover, $S_1$ is linearly independent because neither vector is a scalar multiple of the other, and you can conclude that the dimension of $W$ is 2. ■

### EXAMPLE 11 Finding the Dimension of a Subspace

Let $W$ be the subspace of all symmetric matrices in $M_{2,2}$. What is the dimension of $W$?

**SOLUTION**

Every $2 \times 2$ symmetric matrix has the following form.

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

So, the set

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

spans $W$. Moreover, $S$ can be shown to be linearly independent, and you can conclude that the dimension of $W$ is 3.

Usually, to conclude that a set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space $V$, you must show that $S$ satisfies two conditions: $S$ spans $V$ and is linearly independent. If $V$ is known to have a dimension of $n$, however, then the next theorem tells you that you do not need to check both conditions: either one will suffice. The proof is left as an exercise. (See Exercise 74.)

**THEOREM 4.12 Basis Tests in an $n$-Dimensional Space**

Let $V$ be a vector space of dimension $n$.

1. If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a linearly independent set of vectors in $V$, then $S$ is a basis for $V$.
2. If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ spans $V$, then $S$ is a basis for $V$.

### EXAMPLE 12 Testing for a Basis in an $n$-Dimensional Space

Show that the set of vectors is a basis for $M_{5,1}$.

$$S = \left\{ \overset{\mathbf{v}_1}{\begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \\ 4 \end{bmatrix}}, \overset{\mathbf{v}_2}{\begin{bmatrix} 0 \\ 1 \\ 3 \\ -2 \\ 3 \end{bmatrix}}, \overset{\mathbf{v}_3}{\begin{bmatrix} 0 \\ 0 \\ 2 \\ -1 \\ 5 \end{bmatrix}}, \overset{\mathbf{v}_4}{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \\ -3 \end{bmatrix}}, \overset{\mathbf{v}_5}{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -2 \end{bmatrix}} \right\}$$

**SOLUTION**

Because $S$ has five vectors and the dimension of $M_{5,1}$ is 5, apply Theorem 4.12 to verify that $S$ is a basis by showing either that $S$ is linearly independent or that $S$ spans $M_{5,1}$. To show that $S$ is linearly independent, form the vector equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 + c_5\mathbf{v}_5 = \mathbf{0}$, which yields the following linear system.

$$\begin{aligned} c_1 \qquad\qquad\qquad\qquad\qquad &= 0 \\ 2c_1 + c_2 \qquad\qquad\qquad\qquad &= 0 \\ -c_1 + 3c_2 + 2c_3 \qquad\qquad &= 0 \\ 3c_1 - 2c_2 - c_3 + 2c_4 \qquad &= 0 \\ 4c_1 + 3c_2 + 5c_3 - 3c_4 - 2c_5 &= 0 \end{aligned}$$

Because this system has only the trivial solution, $S$ must be linearly independent. So, by Theorem 4.12, $S$ is a basis for $M_{5,1}$.

# 4.5 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

**Writing the Standard Basis** In Exercises 1–6, write the standard basis for the vector space.

1. $R^6$
2. $R^4$
3. $M_{2,4}$
4. $M_{4,1}$
5. $P_4$
6. $P_2$

**Writing** In Exercises 7–14, explain why $S$ is not a basis for $R^2$.

7. $S = \{(1, 2), (1, 0), (0, 1)\}$
8. $S = \{(-1, 2), (1, -2), (2, 4)\}$
9. $S = \{(-4, 5), (0, 0)\}$
10. $S = \{(2, 3), (6, 9)\}$
11. $S = \{(6, -5), (12, -10)\}$
12. $S = \{(4, -3), (8, -6)\}$
13. $S = \{(-3, 2)\}$
14. $S = \{(-1, 2)\}$

**Writing** In Exercises 15–20, explain why $S$ is not a basis for $R^3$.

15. $S = \{(1, 3, 0), (4, 1, 2), (-2, 5, -2)\}$
16. $S = \{(2, 1, -2), (-2, -1, 2), (4, 2, -4)\}$
17. $S = \{(7, 0, 3), (8, -4, 1)\}$
18. $S = \{(1, 1, 2), (0, 2, 1)\}$
19. $S = \{(0, 0, 0), (1, 0, 0), (0, 1, 0)\}$
20. $S = \{(6, 4, 1), (3, -5, 1), (8, 13, 6), (0, 6, 9)\}$

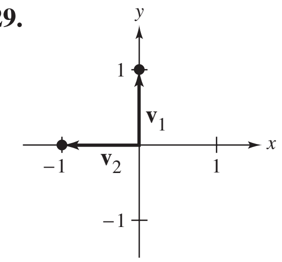
**Writing** In Exercises 21–24, explain why $S$ is not a basis for $P_2$.

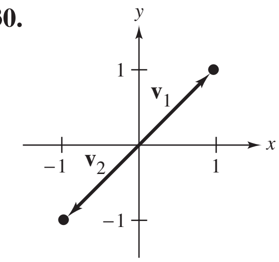
21. $S = \{1, 2x, x^2 - 4, 5x\}$
22. $S = \{2, x, x + 3, 3x^2\}$
23. $S = \{1 - x, 1 - x^2, 3x^2 - 2x - 1\}$
24. $S = \{6x - 3, 3x^2, 1 - 2x - x^2\}$

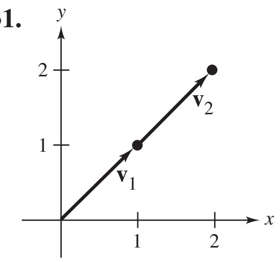
**Writing** In Exercises 25–28, explain why $S$ is not a basis for $M_{2,2}$.

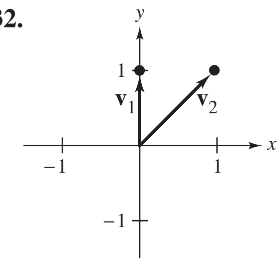
25. $S = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right\}$
26. $S = \left\{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right\}$
27. $S = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 8 & -4 \\ -4 & 3 \end{bmatrix}\right\}$
28. $S = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right\}$

**Determining Whether a Set Is a Basis** In Exercises 29–32, determine whether the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for $R^2$.

29.



30.



31.



32.



**Determining Whether a Set Is a Basis** In Exercises 33–40, determine whether $S$ is a basis for the indicated vector space.

33. $S = \{(3, -2), (4, 5)\}$ for $R^2$
34. $S = \{(1, 2), (1, -1)\}$ for $R^2$
35. $S = \{(1, 5, 3), (0, 1, 2), (0, 0, 6)\}$ for $R^3$
36. $S = \{(2, 1, 0), (0, -1, 1)\}$ for $R^3$
37. $S = \{(0, 3, -2), (4, 0, 3), (-8, 15, -16)\}$ for $R^3$
38. $S = \{(0, 0, 0), (1, 5, 6), (6, 2, 1)\}$ for $R^3$
39. $S = \{(-1, 2, 0, 0), (2, 0, -1, 0), (3, 0, 0, 4), (0, 0, 5, 0)\}$ for $R^4$
40. $S = \{(1, 0, 0, 1), (0, 2, 0, 2), (1, 0, 1, 0), (0, 2, 2, 0)\}$ for $R^4$

**Determining Whether a Set Is a Basis** In Exercises 41–44, determine whether $S$ is a basis for $P_3$.

41. $S = \{t^3 - 2t^2 + 1, t^2 - 4, t^3 + 2t, 5t\}$
42. $S = \{4t - t^2, 5 + t^3, 3t + 5, 2t^3 - 3t^2\}$
43. $S = \{4 - t, t^3, 6t^2, t^3 + 3t, 4t - 1\}$
44. $S = \{t^3 - 1, 2t^2, t + 3, 5 + 2t + 2t^2 + t^3\}$

**Determining Whether a Set Is a Basis** In Exercises 45 and 46, determine whether $S$ is a basis for $M_{2,2}$.

45. $S = \left\{\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}\right\}$
46. $S = \left\{\begin{bmatrix} 1 & 2 \\ -5 & 4 \end{bmatrix}, \begin{bmatrix} 2 & -7 \\ 6 & 2 \end{bmatrix}, \begin{bmatrix} 4 & -9 \\ 11 & 12 \end{bmatrix}, \begin{bmatrix} 12 & -16 \\ 17 & 42 \end{bmatrix}\right\}$

**Determining Whether a Set Is a Basis** **In Exercises 47–50, determine whether $S$ is a basis for $R^3$. If it is, then write $\mathbf{u} = (8, 3, 8)$ as a linear combination of the vectors in $S$.**

**47.** $S = \{(4, 3, 2), (0, 3, 2), (0, 0, 2)\}$

**48.** $S = \{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}$

**49.** $S = \{(0, 0, 0), (1, 3, 4), (6, 1, -2)\}$

**50.** $S = \left\{\left(\frac{2}{3}, \frac{5}{2}, 1\right), \left(1, \frac{3}{2}, 0\right), (2, 12, 6)\right\}$

**Determining the Dimension of a Vector Space** **In Exercises 51–56, determine the dimension of the vector space.**

**51.** $R^6$

**52.** $R$

**53.** $P_7$

**54.** $P_4$

**55.** $M_{2,3}$

**56.** $M_{3,2}$

**57.** Find a basis for the vector space of all $3 \times 3$ diagonal matrices. What is the dimension of this vector space?

**58.** Find a basis for the vector space of all $3 \times 3$ symmetric matrices. What is the dimension of this vector space?

**59.** Find all subsets of the set

$S = \{(1, 0), (0, 1), (1, 1)\}$

that form a basis for $R^2$.

**60.** Find all subsets of the set

$S = \{(1, 3, -2), (-4, 1, 1), (-2, 7, -3), (2, 1, 1)\}$

that form a basis for $R^3$.

**61.** Find a basis for $R^2$ that includes the vector $(2, 2)$.

**62.** Find a basis for $R^3$ that includes the vectors $(1, 0, 2)$ and $(0, 1, 1)$.

**Geometric Description, Basis, and Dimension** **In Exercises 63 and 64, (a) give a geometric description of, (b) find a basis for, and (c) determine the dimension of the subspace $W$ of $R^2$.**

**63.** $W = \{(2t, t)\text{: } t \text{ is a real number}\}$

**64.** $W = \{(0, t)\text{: } t \text{ is a real number}\}$

**Geometric Description, Basis, and Dimension** **In Exercises 65 and 66, (a) give a geometric description of, (b) find a basis for, and (c) determine the dimension of the subspace $W$ of $R^3$.**

**65.** $W = \{(2t, t, -t)\text{: } t \text{ is a real number}\}$

**66.** $W = \{(2s - t, s, t)\text{: } s \text{ and } t \text{ are real numbers}\}$

**Basis and Dimension** **In Exercises 67–70, find (a) a basis for and (b) the dimension of the subspace $W$ of $R^4$.**

**67.** $W = \{(2s - t, s, t, s)\text{: } s \text{ and } t \text{ are real numbers}\}$

**68.** $W = \{(5t, -3t, t, t)\text{: } t \text{ is a real number}\}$

**69.** $W = \{(0, 6t, t, -t)\text{: } t \text{ is a real number}\}$

**70.** $W = \{(s + 4t, t, s, 2s - t)\text{: } s \text{ and } t \text{ are real numbers}\}$

**True or False?** **In Exercises 71 and 72, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.**

**71.** (a) If $\dim(V) = n$, then there exists a set of $n - 1$ vectors in $V$ that will span $V$.

(b) If $\dim(V) = n$, then there exists a set of $n + 1$ vectors in $V$ that will span $V$.

**72.** (a) If $\dim(V) = n$, then any set of $n + 1$ vectors in $V$ must be linearly dependent.

(b) If $\dim(V) = n$, then any set of $n - 1$ vectors in $V$ must be linearly independent.

**73.** **Proof** Prove that if $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space $V$ and $c$ is a nonzero scalar, then the set $S_1 = \{c\mathbf{v}_1, c\mathbf{v}_2, \ldots, c\mathbf{v}_n\}$ is also a basis for $V$.

**74.** **Proof** Prove Theorem 4.12.

**75.** **Proof** Prove that if $W$ is a subspace of a finite dimensional vector space $V$, then $\dim(W) \leq \dim(V)$.

**76.** **CAPSTONE**

(a) A set $S_1$ consists of two vectors of the form $\mathbf{u} = (u_1, u_2, u_3)$. Explain why $S_1$ is not a basis for $R^3$.

(b) A set $S_2$ consists of four vectors of the form $\mathbf{u} = (u_1, u_2, u_3)$. Explain why $S_2$ is not a basis for $R^3$.

(c) A set $S_3$ consists of three vectors of the form $\mathbf{u} = (u_1, u_2, u_3)$. Determine the conditions under which $S_3$ is a basis for $R^3$.

**77.** **Proof** Let $S$ be a linearly independent set of vectors from the finite dimensional vector space $V$. Prove that there exists a basis for $V$ containing $S$.

**78.** **Guided Proof** Let $S$ be a spanning set for the finite dimensional vector space $V$. Prove that there exists a subset $S'$ of $S$ that forms a basis for $V$.

Getting Started: $S$ is a spanning set, but it may not be a basis because it may be linearly dependent. You need to remove extra vectors so that a subset $S'$ is a spanning set and is also linearly independent.

(i) If $S$ is a linearly independent set, then you are done. If not, remove some vector $\mathbf{v}$ from $S$ that is a linear combination of the other vectors in $S$. Call this set $S_1$.

(ii) If $S_1$ is a linearly independent set, then you are done. If not, then continue to remove dependent vectors until you produce a linearly independent subset $S'$.

(iii) Conclude that this subset is the minimal spanning set $S'$.

# 4.6 Rank of a Matrix and Systems of Linear Equations

- Find a basis for the row space, a basis for the column space, and the rank of a matrix.
- Find the nullspace of a matrix.
- Find the solution of a consistent system $Ax = b$ in the form $x_p + x_h$.

## ROW SPACE, COLUMN SPACE, AND RANK OF A MATRIX

In this section, you will investigate the vector space spanned by the **row vectors** (or **column vectors**) of a matrix. Then you will see how such vector spaces relate to solutions of systems of linear equations.

To begin, you need to know some terminology. For an $m \times n$ matrix $A$, the $n$-tuples corresponding to the rows of $A$ are called the **row vectors** of $A$.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Row Vectors of $A$

$$\begin{matrix} (a_{11}, a_{12}, \ldots, a_{1n}) \\ (a_{21}, a_{22}, \ldots, a_{2n}) \\ \vdots \\ (a_{m1}, a_{m2}, \ldots, a_{mn}) \end{matrix}$$

Similarly, the $m \times 1$ matrices corresponding to the columns of $A$ are called the **column vectors** of $A$. You will find it useful to preserve the column notation for column vectors.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Column Vectors of $A$

$$\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} \cdots \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

**EXAMPLE 1** **Row Vectors and Column Vectors**

For the matrix $A = \begin{bmatrix} 0 & 1 & -1 \\ -2 & 3 & 4 \end{bmatrix}$, the row vectors are $(0, 1, -1)$ and $(-2, 3, 4)$ and the column vectors are $\begin{bmatrix} 0 \\ -2 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$, and $\begin{bmatrix} -1 \\ 4 \end{bmatrix}$.

In Example 1, note that for an $m \times n$ matrix $A$, the row vectors are vectors in $R^n$ and the column vectors are vectors in $R^m$. This leads to the following definitions of the **row space** and **column space** of a matrix.

### Definitions of Row Space and Column Space of a Matrix

Let $A$ be an $m \times n$ matrix.

1. The **row space** of $A$ is the subspace of $R^n$ spanned by the row vectors of $A$.
2. The **column space** of $A$ is the subspace of $R^m$ spanned by the column vectors of $A$.

Recall that two matrices are row-equivalent when one can be obtained from the other by elementary row operations. The next theorem tells you that row-equivalent matrices have the same row space.

**REMARK**

Theorem 4.13 states that elementary row operations do not change the row space of a matrix. Elementary row operations can, however, change the *column* space of a matrix.

**THEOREM 4.13 Row-Equivalent Matrices Have the Same Row Space**

If an $m \times n$ matrix $A$ is row-equivalent to an $m \times n$ matrix $B$, then the row space of $A$ is equal to the row space of $B$.

**PROOF**

Because the rows of $B$ can be obtained from the rows of $A$ by elementary row operations (scalar multiplication and addition), it follows that the row vectors of $B$ can be written as linear combinations of the row vectors of $A$. The row vectors of $B$ lie in the row space of $A$, and the subspace spanned by the row vectors of $B$ is contained in the row space of $A$. But it is also true that the rows of $A$ can be obtained from the rows of $B$ by elementary row operations. So, the two row spaces are subspaces of each other, making them equal.

If a matrix $B$ is in row-echelon form, then its nonzero row vectors form a linearly independent set. (Try verifying this.) Consequently, they form a basis for the row space of $B$, and by Theorem 4.13 they also form a basis for the row space of $A$. The next theorem states this important result.

**THEOREM 4.14 Basis for the Row Space of a Matrix**

If a matrix $A$ is row-equivalent to a matrix $B$ in row-echelon form, then the nonzero row vectors of $B$ form a basis for the row space of $A$.

## EXAMPLE 2 Finding a Basis for a Row Space

Find a basis for the row space of

$$A = \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & -2 \end{bmatrix}.$$

**SOLUTION**

Using elementary row operations, rewrite $A$ in row-echelon form as follows.

$$B = \begin{array}{c} \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{l} \mathbf{w}_1 \\ \mathbf{w}_2 \\ \mathbf{w}_3 \\ \\ \\ \end{array}$$

By Theorem 4.14, the nonzero row vectors of $B$, $\mathbf{w}_1 = (1, 3, 1, 3)$, $\mathbf{w}_2 = (0, 1, 1, 0)$, and $\mathbf{w}_3 = (0, 0, 0, 1)$, form a basis for the row space of $A$.

The technique used in Example 2 to find a basis for the row space of a matrix can be used to find a basis for the subspace spanned by the set $S = \{\mathbf{v}_1, \mathbf{v}_2, . . . , \mathbf{v}_k\}$ in $R^n$. By using the vectors in $S$ to form the rows of a matrix $A$, you can use elementary row operations to rewrite $A$ in row-echelon form. The nonzero rows of this matrix will then form a basis for the subspace spanned by $S$. This is demonstrated in Example 3.

## EXAMPLE 3 Finding a Basis for a Subspace

Find a basis for the subspace of $R^3$ spanned by

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \{(-1, 2, 5), (3, 0, 3), (5, 1, 8)\}.$$

### SOLUTION

Use $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ to form the rows of a matrix $A$. Then write $A$ in row-echelon form.

$$A = \begin{array}{c} \begin{bmatrix} -1 & 2 & 5 \\ 3 & 0 & 3 \\ 5 & 1 & 8 \end{bmatrix} \end{array} \begin{array}{l} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{array} \longrightarrow B = \begin{bmatrix} 1 & -2 & -5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \mathbf{w}_1 \\ \mathbf{w}_2 \\ \phantom{0} \end{array}$$

So, the nonzero row vectors of $B$, $\mathbf{w}_1 = (1, -2, -5)$ and $\mathbf{w}_2 = (0, 1, 3)$, form a basis for the row space of $A$. That is, they form a basis for the subspace spanned by $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

To find a basis for the column space of a matrix $A$, you have two options. On the one hand, you could use the fact that the column space of $A$ is equal to the row space of $A^T$ and apply the technique of Example 2 to the matrix $A^T$. On the other hand, observe that although row operations can change the column space of a matrix, they do not change the dependency relationships among columns. (You are asked to prove this fact in Exercise 75.) For example, consider the row-equivalent matrices $A$ and $B$ from Example 2.

$$A = \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\qquad \mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{a}_4 \qquad\qquad \mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{b}_3 \quad \mathbf{b}_4$$

Notice that columns 1, 2, and 3 of matrix $B$ satisfy $\mathbf{b}_3 = -2\mathbf{b}_1 + \mathbf{b}_2$, and the corresponding columns of matrix $A$ satisfy $\mathbf{a}_3 = -2\mathbf{a}_1 + \mathbf{a}_2$. Similarly, the column vectors $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_4$ of matrix $B$ are linearly independent, as are the corresponding columns of matrix $A$.

The next two examples show how to find a basis for the column space of a matrix using these methods.

## EXAMPLE 4 Finding a Basis for the Column Space of a Matrix (Method 1)

Find a basis for the column space of matrix $A$ from Example 2 by finding a basis for the row space of $A^T$.

### SOLUTION

Take the transpose of $A$ and use elementary row operations to write $A^T$ in row-echelon form.

$$A^T = \begin{bmatrix} 1 & 0 & -3 & 3 & 2 \\ 3 & 1 & 0 & 4 & 0 \\ 1 & 1 & 6 & -2 & -4 \\ 3 & 0 & -1 & 1 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -3 & 3 & 2 \\ 0 & 1 & 9 & -5 & -6 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \mathbf{w}_1 \\ \mathbf{w}_2 \\ \mathbf{w}_3 \\ \phantom{0} \end{array}$$

So, $\mathbf{w}_1 = (1, 0, -3, 3, 2)$, $\mathbf{w}_2 = (0, 1, 9, -5, -6)$, and $\mathbf{w}_3 = (0, 0, 1, -1, -1)$ form a basis for the row space of $A^T$. This is equivalent to saying that the column vectors $[1 \quad 0 \quad -3 \quad 3 \quad 2]^T$, $[0 \quad 1 \quad 9 \quad -5 \quad -6]^T$, and $[0 \quad 0 \quad 1 \quad -1 \quad -1]^T$ form a basis for the column space of $A$.

### EXAMPLE 5 Finding a Basis for the Column Space of a Matrix (Method 2)

Find a basis for the column space of matrix $A$ from Example 2 by using the dependency relationships among columns.

**SOLUTION**

In Example 2, row operations were used on the original matrix $A$ to obtain its row-echelon form $B$. As mentioned earlier, in matrix $B$, the first, second, and fourth column vectors are linearly independent (these columns have the leading 1's), as are the corresponding columns of matrix $A$. So, a basis for the column space of $A$ consists of the vectors

$$\begin{bmatrix} 1 \\ 0 \\ -3 \\ 3 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 1 \\ 0 \\ 4 \\ 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 3 \\ 0 \\ -1 \\ 1 \\ -2 \end{bmatrix}.$$

**REMARK**

Notice that the row-echelon form $B$ indicates which columns of $A$ form the basis for the column space. You do not use the column vectors of $B$ to form the basis.

Notice that the basis for the column space obtained in Example 5 is different than that obtained in Example 4. Verify that these bases both span the column space of $A$.

Also notice in Examples 2, 4, and 5 that both the row space and the column space of $A$ have a dimension of 3 (because there are *three* vectors in both bases). The next theorem generalizes this.

**THEOREM 4.15 Row and Column Spaces Have Equal Dimensions**

If $A$ is an $m \times n$ matrix, then the row space and column space of $A$ have the same dimension.

**PROOF**

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ be the row vectors and $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ be the column vectors of

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

Suppose the row space of $A$ has dimension $r$ and basis $S = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_r\}$, where $\mathbf{b}_i = (b_{i1}, b_{i2}, \ldots, b_{in})$. Using this basis, you can write the row vectors of $A$ as

$$\begin{aligned} \mathbf{v}_1 &= c_{11}\mathbf{b}_1 + c_{12}\mathbf{b}_2 + \cdots + c_{1r}\mathbf{b}_r \\ \mathbf{v}_2 &= c_{21}\mathbf{b}_1 + c_{22}\mathbf{b}_2 + \cdots + c_{2r}\mathbf{b}_r \\ &\ \vdots \\ \mathbf{v}_m &= c_{m1}\mathbf{b}_1 + c_{m2}\mathbf{b}_2 + \cdots + c_{mr}\mathbf{b}_r. \end{aligned}$$

Rewrite this system of vector equations as follows.

$$\begin{aligned} [a_{11} a_{12} \cdots a_{1n}] &= c_{11}[b_{11} b_{12} \cdots b_{1n}] + c_{12}[b_{21} b_{22} \cdots b_{2n}] + \cdots + c_{1r}[b_{r1} b_{r2} \cdots b_{rn}] \\ [a_{21} a_{22} \cdots a_{2n}] &= c_{21}[b_{11} b_{12} \cdots b_{1n}] + c_{22}[b_{21} b_{22} \cdots b_{2n}] + \cdots + c_{2r}[b_{r1} b_{r2} \cdots b_{rn}] \\ &\ \vdots \\ [a_{m1} a_{m2} \cdots a_{mn}] &= c_{m1}[b_{11} b_{12} \cdots b_{1n}] + c_{m2}[b_{21} b_{22} \cdots b_{2n}] + \cdots + c_{mr}[b_{r1} b_{r2} \cdots b_{rn}] \end{aligned}$$

Now, take only entries corresponding to the first column of matrix $A$ to obtain the following system of scalar equations.

$$\begin{aligned}
a_{11} &= c_{11}b_{11} + c_{12}b_{21} + \cdots + c_{1r}b_{r1} \\
a_{21} &= c_{21}b_{11} + c_{22}b_{21} + \cdots + c_{2r}b_{r1} \\
&\vdots \\
a_{m1} &= c_{m1}b_{11} + c_{m2}b_{21} + \cdots + c_{mr}b_{r1}
\end{aligned}$$

Similarly, for the entries of the $j$th column, you can obtain the following system.

$$\begin{aligned}
a_{1j} &= c_{11}b_{1j} + c_{12}b_{2j} + \cdots + c_{1r}b_{rj} \\
a_{2j} &= c_{21}b_{1j} + c_{22}b_{2j} + \cdots + c_{2r}b_{rj} \\
&\vdots \\
a_{mj} &= c_{m1}b_{1j} + c_{m2}b_{2j} + \cdots + c_{mr}b_{rj}
\end{aligned}$$

Now, let the vectors

$$\mathbf{c}_i = [c_{1i} \ \ c_{2i} \ \ . \ . \ . \ \ c_{mi}]^T.$$

Then the system for the $j$th column can be rewritten in a vector form as

$$\mathbf{u}_j = b_{1j}\mathbf{c}_1 + b_{2j}\mathbf{c}_2 + \cdots + b_{rj}\mathbf{c}_r.$$

Put all column vectors together to obtain

$$\begin{aligned}
\mathbf{u}_1 &= [a_{11} \ \ a_{21} \cdots a_{m1}]^T = b_{11}\mathbf{c}_1 + b_{21}\mathbf{c}_2 + \cdots + b_{r1}\mathbf{c}_r \\
\mathbf{u}_2 &= [a_{12} \ \ a_{22} \cdots a_{m2}]^T = b_{12}\mathbf{c}_1 + b_{22}\mathbf{c}_2 + \cdots + b_{r2}\mathbf{c}_r \\
&\vdots \\
\mathbf{u}_n &= [a_{1n} \ \ a_{2n} \cdots a_{mn}]^T = b_{1n}\mathbf{c}_1 + b_{2n}\mathbf{c}_2 + \cdots + b_{rn}\mathbf{c}_r .
\end{aligned}$$

Because each column vector of $A$ is a linear combination of $r$ vectors, you know that the dimension of the column space of $A$ is less than or equal to $r$ (the dimension of the row space of $A$). That is,

$$\dim(\text{column space of } A) \le \dim(\text{row space of } A).$$

Repeating this procedure for $A^T$, you can conclude that the dimension of the column space of $A^T$ is less than or equal to the dimension of the row space of $A^T$. But this implies that the dimension of the row space of $A$ is less than or equal to the dimension of the column space of $A$. That is,

$$\dim(\text{row space of } A) \le \dim(\text{column space of } A).$$

So, the two dimensions must be equal.

**REMARK**

Some texts distinguish between the *row rank* and the *column rank* of a matrix, but because these ranks are equal (Theorem 4.15), this text will not distinguish between them.

The dimension of the row (or column) space of a matrix is called the **rank** of the matrix.

### Definition of the Rank of a Matrix

The dimension of the row (or column) space of a matrix $A$ is called the **rank** of $A$ and is denoted by $\operatorname{rank}(A)$.

## EXAMPLE 6 Finding the Rank of a Matrix

To find the rank of the matrix $A$ shown below, convert to row-echelon form as given in the matrix $B$.

$$A = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 2 & 1 & 5 & -3 \\ 0 & 1 & 3 & 5 \end{bmatrix} \longrightarrow B = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Because $B$ has three nonzero rows, the rank of $A$ is 3.

## THE NULLSPACE OF A MATRIX

Row and column spaces and rank have some important applications to systems of linear equations. Consider first the homogeneous linear system

$$A\mathbf{x} = \mathbf{0}$$

where $A$ is an $m \times n$ matrix, $\mathbf{x} = [x_1 \quad x_2 \quad . \ . \ . \quad x_n]^T$ is the column vector of unknowns, and $\mathbf{0} = [0 \quad 0 \quad . \ . \ . \quad 0]^T$ is the zero vector in $R^m$. The next theorem tells you that the set of all solutions of this homogeneous system is a subspace of $R^n$.

**REMARK**

The nullspace of $A$ is also called the **solution space** of the system $A\mathbf{x} = \mathbf{0}$.

**THEOREM 4.16 Solutions of a Homogeneous System**

If $A$ is an $m \times n$ matrix, then the set of all solutions of the homogeneous system of linear equations $A\mathbf{x} = \mathbf{0}$ is a subspace of $R^n$ called the **nullspace** of $A$ and is denoted by $N(A)$. So,

$$N(A) = \{\mathbf{x} \in R^n : A\mathbf{x} = \mathbf{0}\}.$$

The dimension of the nullspace of $A$ is called the **nullity** of $A$.

**PROOF**

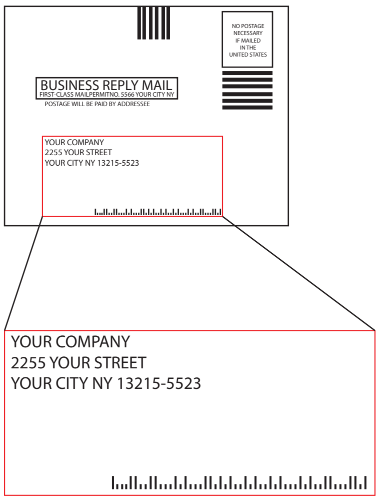
Because $A$ is an $m \times n$ matrix, you know that $\mathbf{x}$ has size $n \times 1$. So, the set of all solutions of the system is a *subset* of $R^n$. This set is clearly nonempty, because $A\mathbf{0} = \mathbf{0}$. You can verify that it is a subspace by showing that it is closed under the operations of addition and scalar multiplication. Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be two solution vectors of the system $A\mathbf{x} = \mathbf{0}$, and let $c$ be a scalar. Because $A\mathbf{x}_1 = \mathbf{0}$ and $A\mathbf{x}_2 = \mathbf{0}$, you know that

$$A(\mathbf{x}_1 + \mathbf{x}_2) = A\mathbf{x}_1 + A\mathbf{x}_2 = \mathbf{0} + \mathbf{0} = \mathbf{0} \qquad \text{Addition}$$

and

$$A(c\mathbf{x}_1) = c(A\mathbf{x}_1) = c(\mathbf{0}) = \mathbf{0}. \qquad \text{Scalar multiplication}$$

So, both $(\mathbf{x}_1 + \mathbf{x}_2)$ and $c\mathbf{x}_1$ are solutions of $A\mathbf{x} = \mathbf{0}$, and you can conclude that the set of all solutions forms a subspace of $R^n$. ■




**LINEAR ALGEBRA APPLIED**

The U.S. Postal Service uses barcodes to represent such information as ZIP codes and delivery addresses. The ZIP + 4 barcode shown at the left starts with a long bar, has a series of short and long bars to represent each digit in the ZIP + 4 code and an additional digit for error checking, and ends with a long bar. The following is the code for the digits.


The error checking digit is such that when it is summed with the digits in the ZIP + 4 code, the result is a multiple of 10. (Verify this, as well as whether the ZIP + 4 code shown is coded correctly.) More sophisticated barcodes will also include error correcting digit(s). In an analogous way, matrices can be used to check for errors in transmitted messages. Information in the form of column vectors can be multiplied by an error detection matrix. When the resulting product is in the nullspace of the error detection matrix, no error in transmission exists. Otherwise, an error exists somewhere in the message. If the error detection matrix also has error correction, then the resulting matrix product will also indicate where the error is occurring.

## EXAMPLE 7 Finding the Nullspace of a Matrix

Find the nullspace of the matrix.

$$A = \begin{bmatrix} 1 & 2 & -2 & 1 \\ 3 & 6 & -5 & 4 \\ 1 & 2 & 0 & 3 \end{bmatrix}$$

### SOLUTION

The nullspace of $A$ is the solution space of the homogeneous system $A\mathbf{x} = \mathbf{0}$.

To solve this system, write the augmented matrix $[A \quad \mathbf{0}]$ in reduced row-echelon form. Because the system of equations is homogeneous, the right-hand column of the augmented matrix consists entirely of zeros and will not change as you perform row operations. So, it is sufficient to find the reduced row-echelon form of $A$.

$$A = \begin{bmatrix} 1 & 2 & -2 & 1 \\ 3 & 6 & -5 & 4 \\ 1 & 2 & 0 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The system of equations corresponding to the reduced row-echelon form is

$$\begin{aligned} x_1 + 2x_2 \quad\quad + 3x_4 &= 0 \\ x_3 + \quad x_4 &= 0. \end{aligned}$$

Choose $x_2$ and $x_4$ as free variables to represent the solutions in parametric form.

$$x_1 = -2s - 3t, \quad x_2 = s, \quad x_3 = -t, \quad x_4 = t$$

This means that the solution space of $A\mathbf{x} = \mathbf{0}$ consists of all solution vectors $\mathbf{x}$ of the following form.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2s - 3t \\ s \\ -t \\ t \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

So, a basis for the nullspace of $A$ consists of the vectors

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -3 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

In other words, these two vectors are solutions of $A\mathbf{x} = \mathbf{0}$, and all solutions of this homogeneous system are linear combinations of these two vectors.

**REMARK**

Although the basis in Example 7 proved that the vectors spanned the solution set, it did not prove that they were linearly independent. When homogeneous systems are solved from the reduced row-echelon form, the spanning set is always linearly independent.

In Example 7, matrix $A$ has four columns. Furthermore, the rank of the matrix is 2, and the dimension of the nullspace is 2. So, you can see that

Number of columns $=$ rank $+$ nullity.

One way to see this is to look at the reduced row-echelon form of $A$.

$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The columns with the leading 1's (columns 1 and 3) determine the rank of the matrix. The other columns (2 and 4) determine the nullity of the matrix because they correspond to the free variables. The next theorem generalizes this relationship.

**THEOREM 4.17 Dimension of the Solution Space**

If $A$ is an $m \times n$ matrix of rank $r$, then the dimension of the solution space of $A\mathbf{x} = \mathbf{0}$ is $n - r$. That is, $n = \text{rank}(A) + \text{nullity}(A)$.

**PROOF**

Because $A$ has rank $r$, you know it is row-equivalent to a reduced row-echelon matrix $B$ with $r$ nonzero rows. No generality is lost by assuming that the upper left corner of $B$ has the form of the $r \times r$ identity matrix $I_r$. Moreover, because the zero rows of $B$ contribute nothing to the solution, you can discard them to form the $r \times n$ matrix $B'$, where $B'$ is the augmented matrix $[I_r \;\; C]$. The matrix $C$ has $n - r$ columns corresponding to the variables $x_{r+1}, x_{r+2}, \ldots, x_n$. So, the solution space of $A\mathbf{x} = \mathbf{0}$ can be represented by the system

$$\begin{aligned}
x_1 + \qquad\qquad & c_{11}x_{r+1} + c_{12}x_{r+2} + \cdots + c_{1,n-r}x_n = 0 \\
x_2 + \qquad & c_{21}x_{r+1} + c_{22}x_{r+2} + \cdots + c_{2,n-r}x_n = 0 \\
& \quad \vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \qquad\quad \vdots \\
x_r + {} & c_{r1}x_{r+1} + c_{r2}x_{r+2} + \cdots + c_{r,n-r}x_n = 0.
\end{aligned}$$

Solving for the first $r$ variables in terms of the last $n - r$ variables produces $n - r$ vectors in the basis for the solution space, so the solution space has dimension $n - r$.

Example 8 illustrates this theorem and further explores the column space of a matrix.

**EXAMPLE 8** **Rank, Nullity of a Matrix, and Basis for the Column Space**

Let the column vectors of the matrix $A$ be denoted by $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, $\mathbf{a}_4$, and $\mathbf{a}_5$.

$$A = \begin{array}{c} \begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & -1 & -3 & 1 & 3 \\ -2 & -1 & 1 & -1 & 3 \\ 0 & 3 & 9 & 0 & -12 \end{bmatrix} \\ \begin{array}{ccccc} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 \end{array} \end{array}$$

**a.** Find the rank and nullity of $A$.

**b.** Find a subset of the column vectors of $A$ that forms a basis for the column space of $A$.

**SOLUTION**

Let $B$ be the reduced row-echelon form of $A$.

$$A = \begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & -1 & -3 & 1 & 3 \\ -2 & -1 & 1 & -1 & 3 \\ 0 & 3 & 9 & 0 & -12 \end{bmatrix} \longrightarrow B = \begin{bmatrix} 1 & 0 & -2 & 0 & 1 \\ 0 & 1 & 3 & 0 & -4 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

**a.** Because $B$ has three nonzero rows, the rank of $A$ is 3. Also, the number of columns of $A$ is $n = 5$, which implies that the nullity of $A$ is $n - \text{rank} = 5 - 3 = 2$.

**b.** Because the first, second, and fourth column vectors of $B$ are linearly independent, the corresponding column vectors of $A$,

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 0 \\ -1 \\ -1 \\ 3 \end{bmatrix}, \quad \text{and} \quad \mathbf{a}_4 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

form a basis for the column space of $A$.

## SOLUTIONS OF SYSTEMS OF LINEAR EQUATIONS

You now know that the set of all solution vectors of the *homogeneous* linear system $A\mathbf{x} = \mathbf{0}$ is a subspace. The set of all solution vectors of the *nonhomogeneous* system $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} \neq \mathbf{0}$, is *not* a subspace because the zero vector is never a solution of a nonhomogeneous system. There is a relationship, however, between the sets of solutions of the two systems $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{x} = \mathbf{b}$. Specifically, if $\mathbf{x}_p$ is a *particular* solution of the nonhomogeneous system $A\mathbf{x} = \mathbf{b}$, then *every* solution of this system can be written in the form $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$, where $\mathbf{x}_h$ is a solution of the corresponding homogeneous system $A\mathbf{x} = \mathbf{0}$. The next theorem states this important concept.

### THEOREM 4.18 Solutions of a Nonhomogeneous Linear System

If $\mathbf{x}_p$ is a particular solution of the nonhomogeneous system $A\mathbf{x} = \mathbf{b}$, then every solution of this system can be written in the form $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$, where $\mathbf{x}_h$ is a solution of the corresponding homogeneous system $A\mathbf{x} = \mathbf{0}$.

### PROOF

Let $\mathbf{x}$ be any solution of $A\mathbf{x} = \mathbf{b}$. Then $(\mathbf{x} - \mathbf{x}_p)$ is a solution of the homogeneous system $A\mathbf{x} = \mathbf{0}$, because

$$A(\mathbf{x} - \mathbf{x}_p) = A\mathbf{x} - A\mathbf{x}_p = \mathbf{b} - \mathbf{b} = \mathbf{0}.$$

Letting $\mathbf{x}_h = \mathbf{x} - \mathbf{x}_p$, you have $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$.

### EXAMPLE 9 Finding the Solution Set of a Nonhomogeneous System

Find the set of all solution vectors of the system of linear equations.

$$\begin{aligned} x_1 \quad\;\; - 2x_3 + \;\; x_4 &= \;\;\, 5 \\ 3x_1 + \;\; x_2 - 5x_3 \quad\quad\;\; &= \;\;\, 8 \\ x_1 + 2x_2 \quad\quad\;\; - 5x_4 &= -9 \end{aligned}$$

### SOLUTION

The augmented matrix for the system $A\mathbf{x} = \mathbf{b}$ reduces as follows.

$$\begin{bmatrix} 1 & 0 & -2 & 1 & 5 \\ 3 & 1 & -5 & 0 & 8 \\ 1 & 2 & 0 & -5 & -9 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -2 & 1 & 5 \\ 0 & 1 & 1 & -3 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The system of linear equations corresponding to the reduced row-echelon matrix is

$$\begin{aligned} x_1 \quad - 2x_3 + \;\; x_4 &= \;\;\, 5 \\ x_2 + \;\; x_3 - 3x_4 &= -7. \end{aligned}$$

Letting $x_3 = s$ and $x_4 = t$, you can write a representative solution vector of $A\mathbf{x} = \mathbf{b}$ as follows.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2s - t + 5 \\ -s + 3t - 7 \\ s + 0t + 0 \\ 0s + t + 0 \end{bmatrix} = s\begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 3 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 5 \\ -7 \\ 0 \\ 0 \end{bmatrix} = s\mathbf{u}_1 + t\mathbf{u}_2 + \mathbf{x}_p$$

$\mathbf{x}_p$ is a *particular* solution vector of $A\mathbf{x} = \mathbf{b}$, and $\mathbf{x}_h = s\mathbf{u}_1 + t\mathbf{u}_2$ represents an arbitrary vector in the solution space of $A\mathbf{x} = \mathbf{0}$.

The next theorem describes how the column space of a matrix can be used to determine whether a system of linear equations is consistent.

**THEOREM 4.19 Solutions of a System of Linear Equations**

The system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is in the column space of $A$.

**PROOF**

For the system $A\mathbf{x} = \mathbf{b}$, let $A$, $\mathbf{x}$, and $\mathbf{b}$ be the $m \times n$ coefficient matrix, the $n \times 1$ column matrix of unknowns, and the $m \times 1$ right-hand side, respectively. Then

$$A\mathbf{x} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}.$$

So, $A\mathbf{x} = \mathbf{b}$ if and only if $\mathbf{b} = [b_1 \quad b_2 \quad \cdots \quad b_m]^T$ is a linear combination of the columns of $A$. That is, the system is consistent if and only if $\mathbf{b}$ is in the subspace of $R^m$ spanned by the columns of $A$.

**EXAMPLE 10 Consistency of a System of Linear Equations**

Consider the system of linear equations

$$\begin{aligned} x_1 + x_2 - x_3 &= -1 \\ x_1 \qquad + x_3 &= 3 \\ 3x_1 + 2x_2 - x_3 &= 1. \end{aligned}$$

The rank of the coefficient matrix is equal to the rank of the augmented matrix. (Try verifying this.)

$$[A \quad \mathbf{b}] = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 0 & 1 & 3 \\ 3 & 2 & -1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & -2 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

As shown above, $\mathbf{b}$ is in the column space of $A$, so the system of linear equations is consistent.

The following summary presents several major results involving systems of linear equations, matrices, determinants, and vector spaces.

**Summary of Equivalent Conditions for Square Matrices**

If $A$ is an $n \times n$ matrix, then the following conditions are equivalent.

1. $A$ is invertible.
2. $A\mathbf{x} = \mathbf{b}$ has a unique solution for any $n \times 1$ matrix $\mathbf{b}$.
3. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
4. $A$ is row-equivalent to $I_n$.
5. $|A| \neq 0$
6. $\text{Rank}(A) = n$
7. The $n$ row vectors of $A$ are linearly independent.
8. The $n$ column vectors of $A$ are linearly independent.

# 4.6 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

**Row Vectors and Column Vectors** In Exercises 1–4, write (a) the row vectors and (b) the column vectors of the matrix.

1. $\begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix}$

2. $\begin{bmatrix} 6 & 5 & -1 \end{bmatrix}$

3. $\begin{bmatrix} 4 & 3 & 1 \\ 1 & -4 & 0 \end{bmatrix}$

4. $\begin{bmatrix} 0 & 2 & -3 \\ 3 & 1 & 0 \\ -2 & -1 & 2 \end{bmatrix}$

**Finding a Basis for a Row Space and Rank** In Exercises 5–10, find (a) a basis for the row space and (b) the rank of the matrix.

5. $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

6. $\begin{bmatrix} 0 & 1 & -2 \end{bmatrix}$

7. $\begin{bmatrix} 1 & -3 & 2 \\ 4 & 2 & 1 \end{bmatrix}$

8. $\begin{bmatrix} 2 & -3 & 1 \\ 5 & 10 & 6 \\ 8 & -7 & 5 \end{bmatrix}$

9. $\begin{bmatrix} -2 & -4 & 4 & 5 \\ 3 & 6 & -6 & -4 \\ -2 & -4 & 4 & 9 \end{bmatrix}$

10. $\begin{bmatrix} 4 & 0 & 2 & 3 & 1 \\ 2 & -1 & 2 & 0 & 1 \\ 5 & 2 & 2 & 1 & -1 \\ 4 & 0 & 2 & 2 & 1 \\ 2 & -2 & 0 & 0 & 1 \end{bmatrix}$

**Finding a Basis for a Subspace** In Exercises 11–14, find a basis for the subspace of $R^3$ spanned by $S$.

11. $S = \{(1, 2, 4), (-1, 3, 4), (2, 3, 1)\}$
12. $S = \{(4, 2, -1), (1, 2, -8), (0, 1, 2)\}$
13. $S = \{(4, 4, 8), (1, 1, 2), (1, 1, 1)\}$
14. $S = \{(1, 2, 2), (-1, 0, 0), (1, 1, 1)\}$

**Finding a Basis for a Subspace** In Exercises 15–18, find a basis for the subspace of $R^4$ spanned by $S$.

15. $S = \{(2, 9, -2, 53), (-3, 2, 3, -2), (8, -3, -8, 17), (0, -3, 0, 15)\}$
16. $S = \{(6, -3, 6, 34), (3, -2, 3, 19), (8, 3, -9, 6), (-2, 0, 6, -5)\}$
17. $S = \{(-3, 2, 5, 28), (-6, 1, -8, -1), (14, -10, 12, -10), (0, 5, 12, 50)\}$
18. $S = \{(2, 5, -3, -2), (-2, -3, 2, -5), (1, 3, -2, 2), (-1, -5, 3, 5)\}$

**Finding a Basis for a Column Space and Rank** In Exercises 19–24, find (a) a basis for the column space and (b) the rank of the matrix.

19. $\begin{bmatrix} 2 & 4 \\ 1 & 6 \end{bmatrix}$

20. $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

21. $\begin{bmatrix} 1 & 2 & 4 \\ -1 & 2 & 1 \end{bmatrix}$

22. $\begin{bmatrix} 4 & 20 & 31 \\ 6 & -5 & -6 \\ 2 & -11 & -16 \end{bmatrix}$

23. $\begin{bmatrix} 2 & 4 & -3 & -6 \\ 7 & 14 & -6 & -3 \\ -2 & -4 & 1 & -2 \\ 2 & 4 & -2 & -2 \end{bmatrix}$

24. $\begin{bmatrix} 2 & 4 & -2 & 1 & 1 \\ 2 & 5 & 4 & -2 & 2 \\ 4 & 3 & 1 & 1 & 2 \\ 2 & -4 & 2 & -1 & 1 \\ 0 & 1 & 4 & 2 & -1 \end{bmatrix}$

**Finding the Nullspace of a Matrix** In Exercises 25–36, find the nullspace of the matrix.

25. $A = \begin{bmatrix} 2 & -1 \\ -6 & 3 \end{bmatrix}$

26. $A = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$

27. $A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

28. $A = \begin{bmatrix} 1 & 4 & 2 \end{bmatrix}$

29. $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \end{bmatrix}$

30. $A = \begin{bmatrix} 1 & 4 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

31. $A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & -1 & 4 \\ 4 & 3 & -2 \end{bmatrix}$

32. $A = \begin{bmatrix} 3 & -6 & 21 \\ -2 & 4 & -14 \\ 1 & -2 & 7 \end{bmatrix}$

33. $A = \begin{bmatrix} 1 & 3 & -2 & 4 \\ 0 & 1 & -1 & 2 \\ -2 & -6 & 4 & -8 \end{bmatrix}$

34. $A = \begin{bmatrix} 1 & 4 & 2 & 1 \\ 0 & 1 & 1 & -1 \\ -2 & -8 & -4 & -2 \end{bmatrix}$

35. $A = \begin{bmatrix} 2 & 6 & 3 & 1 \\ 2 & 1 & 0 & -2 \\ 3 & -2 & 1 & 1 \\ 0 & 6 & 2 & 0 \end{bmatrix}$

36. $A = \begin{bmatrix} 1 & 4 & 2 & 1 \\ 2 & -1 & 1 & 1 \\ 4 & 2 & 1 & 1 \\ 0 & 4 & 2 & 0 \end{bmatrix}$

**Finding a Basis and Dimension** In Exercises 37–46, find (a) a basis for and (b) the dimension of the solution space of the homogeneous system of linear equations.

**37.** $\begin{aligned} x - 4y &= 0 \\ 3x - 12y &= 0 \end{aligned}$

**38.** $\begin{aligned} x - y &= 0 \\ -x + y &= 0 \end{aligned}$

**39.** $\begin{aligned} -x + y + z &= 0 \\ 3x - y \phantom{+ z} &= 0 \\ 2x - 4y - 5z &= 0 \end{aligned}$

**40.** $\begin{aligned} 4x - y + 2z &= 0 \\ 2x + 3y - z &= 0 \\ 3x + y + z &= 0 \end{aligned}$

**41.** $\begin{aligned} x - 2y + 3z &= 0 \\ -3x + 6y - 9z &= 0 \end{aligned}$

**42.** $\begin{aligned} x + 2y - 4z &= 0 \\ -3x - 6y + 12z &= 0 \end{aligned}$

**43.** $\begin{aligned} 3x_1 + 3x_2 + 15x_3 + 11x_4 &= 0 \\ x_1 - 3x_2 + x_3 + x_4 &= 0 \\ 2x_1 + 3x_2 + 11x_3 + 8x_4 &= 0 \end{aligned}$

**44.** $\begin{aligned} 2x_1 + 2x_2 + 4x_3 - 2x_4 &= 0 \\ x_1 + 2x_2 + x_3 + 2x_4 &= 0 \\ -x_1 + x_2 + 4x_3 - 2x_4 &= 0 \end{aligned}$

**45.** $\begin{aligned} 9x_1 - 4x_2 - 2x_3 - 20x_4 &= 0 \\ 12x_1 - 6x_2 - 4x_3 - 29x_4 &= 0 \\ 3x_1 - 2x_2 \phantom{- x_3} - 7x_4 &= 0 \\ 3x_1 - 2x_2 - x_3 - 8x_4 &= 0 \end{aligned}$

**46.** $\begin{aligned} x_1 + 3x_2 + 2x_3 + 22x_4 + 13x_5 &= 0 \\ x_1 \phantom{+ 3x_2} + x_3 - 2x_4 + x_5 &= 0 \\ 3x_1 + 6x_2 + 5x_3 + 42x_4 + 27x_5 &= 0 \end{aligned}$

**Rank, Nullity, Bases, and Linear Independence** In Exercises 47 and 48, use the fact that matrices $A$ and $B$ are row-equivalent.

(a) Find the rank and nullity of $A$.

(b) Find a basis for the nullspace of $A$.

(c) Find a basis for the row space of $A$.

(d) Find a basis for the column space of $A$.

(e) Determine whether the rows of $A$ are linearly independent.

(f) Let the columns of $A$ be denoted by $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$, and $\mathbf{a}_5$. Which of the following sets is (are) linearly independent?

(i) $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4\}$ (ii) $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ (iii) $\{\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_5\}$

**47.** $A = \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 2 & 5 & 1 & 1 & 0 \\ 3 & 7 & 2 & 2 & -2 \\ 4 & 9 & 3 & -1 & 4 \end{bmatrix}$

$B = \begin{bmatrix} 1 & 0 & 3 & 0 & -4 \\ 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

**48.** $A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$

$B = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

**Nonhomogeneous System** In Exercises 49–54, (a) determine whether the nonhomogeneous system $A\mathbf{x} = \mathbf{b}$ is consistent, and (b) if the system is consistent, then write the solution in the form $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$, where $\mathbf{x}_p$ is a particular solution of $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x}_h$ is a solution of $A\mathbf{x} = \mathbf{0}$.

**49.** $\begin{aligned} x + 3y + 10z &= 18 \\ -2x + 7y + 32z &= 29 \\ -x + 3y + 14z &= 12 \\ x + y + 2z &= 8 \end{aligned}$

**50.** $\begin{aligned} 2x - 4y + 5z &= 8 \\ -7x + 14y + 4z &= -28 \\ 3x - 6y + z &= 12 \end{aligned}$

**51.** $\begin{aligned} 3x - 8y + 4z \phantom{+ 4w} &= 19 \\ -6y + 2z + 4w &= 5 \\ 5x \phantom{- 6y} + 22z + w &= 29 \\ x - 2y + 2z \phantom{+ 4w} &= 8 \end{aligned}$

**52.** $\begin{aligned} 3w - 2x + 16y - 2z &= -7 \\ -w + 5x - 14y + 18z &= 29 \\ 3w - x + 14y + 2z &= 1 \end{aligned}$

**53.** $\begin{aligned} x_1 + 2x_2 + x_3 + x_4 + 5x_5 &= 0 \\ -5x_1 - 10x_2 + 3x_3 + 3x_4 + 55x_5 &= -8 \\ x_1 + 2x_2 + 2x_3 - 3x_4 - 5x_5 &= 14 \\ -x_1 - 2x_2 + x_3 + x_4 + 15x_5 &= -2 \end{aligned}$

**54.** $\begin{aligned} 5x_1 - 4x_2 + 12x_3 - 33x_4 + 14x_5 &= -4 \\ -2x_1 + x_2 - 6x_3 + 12x_4 - 8x_5 &= 1 \\ 2x_1 - x_2 + 6x_3 - 12x_4 + 8x_5 &= -1 \end{aligned}$

**Consistency of $A\mathbf{x} = \mathbf{b}$** In Exercises 55–58, determine whether $\mathbf{b}$ is in the column space of $A$. If it is, then write $\mathbf{b}$ as a linear combination of the column vectors of $A$.

**55.** $A = \begin{bmatrix} -1 & 2 \\ 4 & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

**56.** $A = \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

**57.** $A = \begin{bmatrix} 1 & 3 & 2 \\ -1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

**58.** $A = \begin{bmatrix} 1 & 3 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$

**59. Proof** Prove that if $A$ is not square, then either the row vectors of $A$ or the column vectors of $A$ form a linearly dependent set.

**60.** Give an example showing that the rank of the product of two matrices can be less than the rank of either matrix.

**61.** Give examples of matrices $A$ and $B$ of the same size such that

(a) $\text{rank}(A + B) < \text{rank}(A)$ and $\text{rank}(A + B) < \text{rank}(B)$

(b) $\text{rank}(A + B) = \text{rank}(A)$ and $\text{rank}(A + B) = \text{rank}(B)$

(c) $\text{rank}(A + B) > \text{rank}(A)$ and $\text{rank}(A + B) > \text{rank}(B)$.

**62. Proof** Prove that the nonzero row vectors of a matrix in row-echelon form are linearly independent.

**63.** Let $A$ be an $m \times n$ matrix (where $m < n$) whose rank is $r$.

(a) What is the largest value $r$ can be?

(b) How many vectors are in a basis for the row space of $A$?

(c) How many vectors are in a basis for the column space of $A$?

(d) Which vector space $R^k$ has the row space as a subspace?

(e) Which vector space $R^k$ has the column space as a subspace?

**64.** Show that the three points $(x_1, y_1)$, $(x_2, y_2)$, and $(x_3, y_3)$ in a plane are collinear if and only if the matrix

$$\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}$$

has rank less than 3.

**65.** Given matrices $A$ and $B$, show that the row vectors of $AB$ are in the row space of $B$ and the column vectors of $AB$ are in the column space of $A$.

**66.** Find the ranks of the matrix

$$\begin{bmatrix} 1 & 2 & 3 & \cdots & n \\ n+1 & n+2 & n+3 & \cdots & 2n \\ 2n+1 & 2n+2 & 2n+3 & \cdots & 3n \\ \vdots & \vdots & \vdots & & \vdots \\ n^2-n+1 & n^2-n+2 & n^2-n+3 & \cdots & n^2 \end{bmatrix}$$

for $n = 2, 3,$ and $4$. Can you find a pattern in these ranks?

**67. Proof** Prove each property of the system of linear equations in $n$ variables $A\mathbf{x} = \mathbf{b}$.

(a) If $\text{rank}(A) = \text{rank}([A \ \ \mathbf{b}]) = n$, then the system has a unique solution.

(b) If $\text{rank}(A) = \text{rank}([A \ \ \mathbf{b}]) < n$, then the system has infinitely many solutions.

(c) If $\text{rank}(A) < \text{rank}([A \ \ \mathbf{b}])$, then the system is inconsistent.

**68. Proof** Let $A$ be an $m \times n$ matrix. Prove that $N(A) \subset N(A^TA)$.

**True or False?** **In Exercises 69–71, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.**

**69.** (a) The nullspace of a matrix $A$ is also called the solution space of $A$.

(b) The nullspace of a matrix $A$ is the solution space of the homogeneous system $A\mathbf{x} = \mathbf{0}$.

**70.** (a) If an $m \times n$ matrix $A$ is row-equivalent to an $m \times n$ matrix $B$, then the row space of $A$ is equivalent to the row space of $B$.

(b) If $A$ is an $m \times n$ matrix of rank $r$, then the dimension of the solution space of $A\mathbf{x} = \mathbf{0}$ is $m - r$.

**71.** (a) If an $m \times n$ matrix $B$ can be obtained from elementary row operations on an $m \times n$ matrix $A$, then the column space of $B$ is equal to the column space of $A$.

(b) The system of linear equations $A\mathbf{x} = \mathbf{b}$ is inconsistent if and only if $\mathbf{b}$ is in the column space of $A$.

(c) The column space of a matrix $A$ is equal to the row space of $A^T$.

**72. CAPSTONE** The dimension of the row space of a $3 \times 5$ matrix $A$ is 2.

(a) What is the dimension of the column space of $A$?

(b) What is the rank of $A$?

(c) What is the nullity of $A$?

(d) What is the dimension of the solution space of the homogeneous system $A\mathbf{x} = \mathbf{0}$?

**73.** Let $A$ and $B$ be square matrices of order $n$ satisfying $A\mathbf{x} = B\mathbf{x}$ for all $\mathbf{x}$ in $R^n$.

(a) Find the rank and nullity of $A - B$.

(b) Show that $A$ and $B$ must be identical.

**74. Proof** Let $A$ be an $m \times n$ matrix.

(a) Prove that the system of linear equations $A\mathbf{x} = \mathbf{b}$ is consistent for all column vectors $\mathbf{b}$ if and only if the rank of $A$ is $m$.

(b) Prove that the homogeneous system of linear equations $A\mathbf{x} = \mathbf{0}$ has only the trivial solution if and only if the columns of $A$ are linearly independent.

**75. Proof** Prove that row operations do not change the dependency relationships among the columns of an $m \times n$ matrix.

**76. Writing** Explain why the row vectors of a $4 \times 3$ matrix form a linearly dependent set. (Assume all matrix entries are distinct.)

# 4.7 Coordinates and Change of Basis

- Find a coordinate matrix relative to a basis in $R^n$.
- Find the transition matrix from the basis $B$ to the basis $B'$ in $R^n$.
- Represent coordinates in general $n$-dimensional spaces.

## COORDINATE REPRESENTATION IN $R^n$

In Theorem 4.9, you saw that if $B$ is a basis for a vector space $V$, then every vector $\mathbf{x}$ in $V$ can be expressed in one and only one way as a linear combination of vectors in $B$. The coefficients in the linear combination are the **coordinates of x relative to *B*.** In the context of coordinates, the order of the vectors in the basis is important, so this will sometimes be emphasized by referring to the basis $B$ as an *ordered* basis.

### Coordinate Representation Relative to a Basis

Let $B = \{\mathbf{v}_1, \mathbf{v}_2, . . . , \mathbf{v}_n\}$ be an ordered basis for a vector space $V$ and let $\mathbf{x}$ be a vector in $V$ such that

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdot \cdot \cdot + c_n\mathbf{v}_n.$$

The scalars $c_1, c_2, . . . , c_n$ are called the **coordinates of x relative to the basis *B*.** The **coordinate matrix** (or **coordinate vector**) **of x relative to *B*** is the column matrix in $R^n$ whose components are the coordinates of $\mathbf{x}$.

$$[\mathbf{x}]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

In $R^n$, column notation is used for the coordinate matrix. For the vector $\mathbf{x} = (x_1, x_2, . . . , x_n)$, the $x_i$'s are the coordinates of $\mathbf{x}$ *relative to the standard basis S for* $R^n$. So, you have

$$[\mathbf{x}]_S = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

### EXAMPLE 1 Coordinates and Components in $R^n$

Find the coordinate matrix of $\mathbf{x} = (-2, 1, 3)$ in $R^3$ relative to the standard basis

$$S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}.$$

#### SOLUTION

Because $\mathbf{x}$ can be written as $\mathbf{x} = (-2, 1, 3) = -2(1, 0, 0) + 1(0, 1, 0) + 3(0, 0, 1)$, the coordinate matrix of $\mathbf{x}$ relative to the standard basis is simply

$$[\mathbf{x}]_S = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}.$$

So, the components of $\mathbf{x}$ are the same as its coordinates relative to the standard basis. ■

## EXAMPLE 2 Finding a Coordinate Matrix Relative to a Standard Basis

The coordinate matrix of $\mathbf{x}$ in $R^2$ relative to the (nonstandard) ordered basis $B = \{\mathbf{v}_1, \mathbf{v}_2\} = \{(1, 0), (1, 2)\}$ is

$$[\mathbf{x}]_B = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

Find the coordinate matrix of $\mathbf{x}$ relative to the standard basis $B' = \{\mathbf{u}_1, \mathbf{u}_2\} = \{(1, 0), (0, 1)\}$.

### SOLUTION

Because $[\mathbf{x}]_B = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, you can write $\mathbf{x} = 3\mathbf{v}_1 + 2\mathbf{v}_2 = 3(1, 0) + 2(1, 2) = (5, 4)$. Moreover, because $(5, 4) = 5(1, 0) + 4(0, 1)$, it follows that the coordinate matrix of $\mathbf{x}$ relative to $B'$ is

$$[\mathbf{x}]_{B'} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}.$$

Figure 4.19 compares these two coordinate representations.

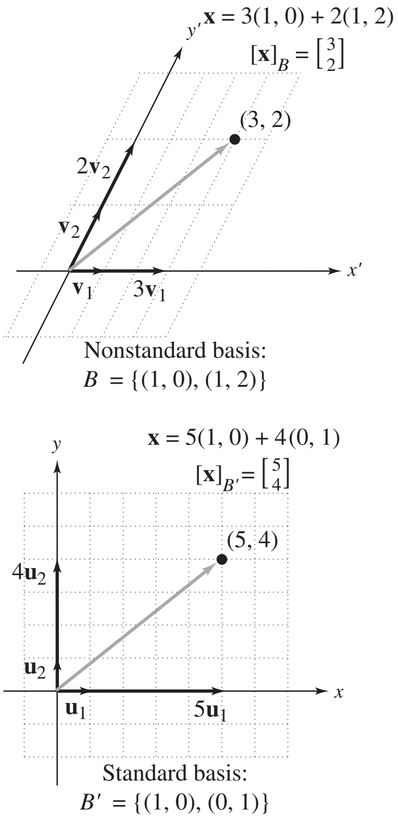



**Figure 4.19**

Example 2 shows that the procedure for finding the coordinate matrix relative to a *standard* basis is straightforward. It is more difficult, however, to find the coordinate matrix relative to a *nonstandard* basis. Here is an example.

## EXAMPLE 3 Finding a Coordinate Matrix Relative to a Nonstandard Basis

Find the coordinate matrix of $\mathbf{x} = (1, 2, -1)$ in $R^3$ relative to the (nonstandard) basis

$$B' = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} = \{(1, 0, 1), (0, -1, 2), (2, 3, -5)\}.$$

### SOLUTION

Begin by writing $\mathbf{x}$ as a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$.

$$\mathbf{x} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3$$
$$(1, 2, -1) = c_1(1, 0, 1) + c_2(0, -1, 2) + c_3(2, 3, -5)$$

Equating corresponding components produces the following system of linear equations and corresponding matrix equation.

$$\begin{aligned} c_1 \qquad\;\; + 2c_3 &= \;\;\,1 \\ -c_2 + 3c_3 &= \;\;\,2 \\ c_1 + \;2c_2 - 5c_3 &= -1 \end{aligned}$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 3 \\ 1 & 2 & -5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

The solution of this system is $c_1 = 5$, $c_2 = -8$, and $c_3 = -2$. So,

$$\mathbf{x} = 5(1, 0, 1) + (-8)(0, -1, 2) + (-2)(2, 3, -5)$$

and the coordinate matrix of $\mathbf{x}$ relative to $B'$ is

$$[\mathbf{x}]_{B'} = \begin{bmatrix} 5 \\ -8 \\ -2 \end{bmatrix}.$$

**REMARK**

It would be incorrect to write the coordinate matrix as

$$\mathbf{x} = \begin{bmatrix} 5 \\ -8 \\ -2 \end{bmatrix}.$$

Do you see why?

## CHANGE OF BASIS IN $R^n$

The procedure demonstrated in Examples 2 and 3 is called a **change of basis.** That is, you were given the coordinates of a vector relative to a basis $B$ and were asked to find the coordinates relative to another basis $B'$.

For instance, if in Example 3 you let $B$ be the standard basis, then the problem of finding the coordinate matrix of $\mathbf{x} = (1, 2, -1)$ relative to the basis $B'$ becomes one of solving for $c_1$, $c_2$, and $c_3$ in the matrix equation

$$\underbrace{\begin{bmatrix}1 & 0 & 2\\ 0 & -1 & 3\\ 1 & 2 & -5\end{bmatrix}}_{P}\underbrace{\begin{bmatrix}c_1\\ c_2\\ c_3\end{bmatrix}}_{[\mathbf{x}]_{B'}} = \underbrace{\begin{bmatrix}1\\ 2\\ -1\end{bmatrix}}_{[\mathbf{x}]_B}.$$

The matrix $P$ is called the **transition matrix from $B'$ to $B$,** where $[\mathbf{x}]_{B'}$ is the coordinate matrix of $\mathbf{x}$ relative to $B'$, and $[\mathbf{x}]_B$ is the coordinate matrix of $\mathbf{x}$ relative to $B$. Multiplication by the transition matrix $P$ changes a coordinate matrix relative to $B'$ into a coordinate matrix relative to $B$. That is,

$$P[\mathbf{x}]_{B'} = [\mathbf{x}]_B. \qquad \text{Change of basis from } B' \text{ to } B$$

To perform a change of basis from $B$ to $B'$, use the matrix $P^{-1}$ (the **transition matrix from $B$ to $B'$**) and write

$$[\mathbf{x}]_{B'} = P^{-1}[\mathbf{x}]_B. \qquad \text{Change of basis from } B \text{ to } B'$$

So, the change of basis problem in Example 3 can be represented by the matrix equation

$$\begin{bmatrix}c_1\\ c_2\\ c_3\end{bmatrix} = \underbrace{\begin{bmatrix}-1 & 4 & 2\\ 3 & -7 & -3\\ 1 & -2 & -1\end{bmatrix}}_{P^{-1}}\underbrace{\begin{bmatrix}1\\ 2\\ -1\end{bmatrix}}_{[\mathbf{x}]_B} = \underbrace{\begin{bmatrix}5\\ -8\\ -2\end{bmatrix}}_{[\mathbf{x}]_{B'}}.$$

This discussion generalizes as follows. Suppose that

$$B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \quad \text{and} \quad B' = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$$

are two ordered bases for $R^n$. If $\mathbf{x}$ is a vector in $R^n$ and

$$[\mathbf{x}]_B = \begin{bmatrix}c_1\\ c_2\\ \vdots\\ c_n\end{bmatrix} \quad \text{and} \quad [\mathbf{x}]_{B'} = \begin{bmatrix}d_1\\ d_2\\ \vdots\\ d_n\end{bmatrix}$$

are the coordinate matrices of $\mathbf{x}$ relative to $B$ and $B'$, then the **transition matrix $P$ from $B'$ to $B$** is the matrix $P$ such that

$$[\mathbf{x}]_B = P[\mathbf{x}]_{B'}.$$

The next theorem tells you that the transition matrix $P$ is invertible and its inverse is the **transition matrix from $B$ to $B'$.** That is,

$$\underbrace{[\mathbf{x}]_{B'}}_{\substack{\text{Coordinate}\\ \text{matrix of } \mathbf{x}\\ \text{relative to } B'}} = \underbrace{P^{-1}}_{\substack{\text{Transition}\\ \text{matrix}\\ \text{from } B \text{ to } B'}}\underbrace{[\mathbf{x}]_B}_{\substack{\text{Coordinate}\\ \text{matrix of } \mathbf{x}\\ \text{relative to } B}}.$$

### THEOREM 4.20 The Inverse of a Transition Matrix

If $P$ is the transition matrix from a basis $B'$ to a basis $B$ in $R^n$, then $P$ is invertible and the transition matrix from $B$ to $B'$ is given by $P^{-1}$.

Before proving Theorem 4.20, it is necessary to look at and prove a preliminary lemma.

## LEMMA

Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and $B' = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ be two bases for a vector space $V$. If

$$\begin{aligned}
\mathbf{v}_1 &= c_{11}\mathbf{u}_1 + c_{21}\mathbf{u}_2 + \cdots + c_{n1}\mathbf{u}_n\\
\mathbf{v}_2 &= c_{12}\mathbf{u}_1 + c_{22}\mathbf{u}_2 + \cdots + c_{n2}\mathbf{u}_n\\
&\vdots\\
\mathbf{v}_n &= c_{1n}\mathbf{u}_1 + c_{2n}\mathbf{u}_2 + \cdots + c_{nn}\mathbf{u}_n
\end{aligned}$$

then the transition matrix from $B$ to $B'$ is

$$Q = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n}\\ c_{21} & c_{22} & \cdots & c_{2n}\\ \vdots & \vdots & & \vdots\\ c_{n1} & c_{n2} & \cdots & c_{nn}\end{bmatrix}.$$

### PROOF (OF LEMMA)

Let $\mathbf{v} = d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \cdots + d_n\mathbf{v}_n$ be an arbitrary vector in $V$. The coordinate matrix of $\mathbf{v}$ with respect to the basis $B$ is

$$[\mathbf{v}]_B = \begin{bmatrix} d_1\\ d_2\\ \vdots\\ d_n\end{bmatrix}.$$

Then you have

$$Q[\mathbf{v}]_B = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n}\\ c_{21} & c_{22} & \cdots & c_{2n}\\ \vdots & \vdots & & \vdots\\ c_{n1} & c_{n2} & \cdots & c_{nn}\end{bmatrix}\begin{bmatrix} d_1\\ d_2\\ \vdots\\ d_n\end{bmatrix} = \begin{bmatrix} c_{11}d_1 + c_{12}d_2 + \cdots + c_{1n}d_n\\ c_{21}d_1 + c_{22}d_2 + \cdots + c_{2n}d_n\\ \vdots \\ c_{n1}d_1 + c_{n2}d_2 + \cdots + c_{nn}d_n\end{bmatrix}.$$

On the other hand, you can write

$$\begin{aligned}
\mathbf{v} &= d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \cdots + d_n\mathbf{v}_n\\
&= d_1(c_{11}\mathbf{u}_1 + c_{21}\mathbf{u}_2 + \cdots + c_{n1}\mathbf{u}_n) + d_2(c_{12}\mathbf{u}_1 + c_{22}\mathbf{u}_2 + \cdots + c_{n2}\mathbf{u}_n) + \cdots\\
&\qquad + d_n(c_{1n}\mathbf{u}_1 + c_{2n}\mathbf{u}_2 + \cdots + c_{nn}\mathbf{u}_n)\\
&= (d_1c_{11} + d_2c_{12} + \cdots + d_nc_{1n})\mathbf{u}_1 + (d_1c_{21} + d_2c_{22} + \cdots + d_nc_{2n})\mathbf{u}_2 + \cdots\\
&\qquad + (d_1c_{n1} + d_2c_{n2} + \cdots + d_nc_{nn})\mathbf{u}_n
\end{aligned}$$

which implies

$$[\mathbf{v}]_{B'} = \begin{bmatrix} c_{11}d_1 + c_{12}d_2 + \cdots + c_{1n}d_n\\ c_{21}d_1 + c_{22}d_2 + \cdots + c_{2n}d_n\\ \vdots \\ c_{n1}d_1 + c_{n2}d_2 + \cdots + c_{nn}d_n\end{bmatrix}.$$

So, $Q[\mathbf{v}]_B = [\mathbf{v}]_{B'}$ and you can conclude that $Q$ is the transition matrix from $B$ to $B'$. ■

### PROOF (OF THEOREM 4.20)

From the preceding lemma, let $Q$ be the transition matrix from $B$ to $B'$. Then $[\mathbf{v}]_B = P[\mathbf{v}]_{B'}$ and $[\mathbf{v}]_{B'} = Q[\mathbf{v}]_B$, which implies that $[\mathbf{v}]_B = PQ[\mathbf{v}]_B$ for every vector $\mathbf{v}$ in $R^n$. From this it follows that $PQ = I$. So, $P$ is invertible and $P^{-1}$ is equal to $Q$, the transition matrix from $B$ to $B'$. ■

Gauss-Jordan elimination can be used to find the transition matrix $P^{-1}$. First define two matrices $B$ and $B'$ whose columns correspond to the vectors in $B$ and $B'$. That is,

$$B = \begin{array}{c} \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ \vdots & \vdots & & \vdots \\ v_{n1} & v_{n2} & \cdots & v_{nn} \end{bmatrix} \\ \begin{array}{cccc} \mathbf{v}_1 & \mathbf{v}_2 & \quad & \mathbf{v}_n \end{array} \end{array} \quad \text{and} \quad B' = \begin{array}{c} \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ u_{21} & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & & \vdots \\ u_{n1} & u_{n2} & \cdots & u_{nn} \end{bmatrix} \\ \begin{array}{cccc} \mathbf{u}_1 & \mathbf{u}_2 & \quad & \mathbf{u}_n \end{array} \end{array}.$$

Then, by reducing the $n \times 2n$ matrix $[B' \;\; B]$ so that the identity matrix $I_n$ occurs in place of $B'$, you obtain the matrix $[I_n \;\; P^{-1}]$. This procedure is stated formally in the next theorem.

### THEOREM 4.21 Transition Matrix from $B$ to $B'$

Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and $B' = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ be two bases for $R^n$. Then the transition matrix $P^{-1}$ from $B$ to $B'$ can be found by using Gauss-Jordan elimination on the $n \times 2n$ matrix $[B' \;\; B]$, as follows.

$$[B' \;\; B] \longrightarrow [I_n \;\; P^{-1}]$$

### PROOF

To begin, let

$$\begin{aligned} \mathbf{v}_1 &= c_{11}\mathbf{u}_1 + c_{21}\mathbf{u}_2 + \cdots + c_{n1}\mathbf{u}_n \\ \mathbf{v}_2 &= c_{12}\mathbf{u}_1 + c_{22}\mathbf{u}_2 + \cdots + c_{n2}\mathbf{u}_n \\ &\vdots \\ \mathbf{v}_n &= c_{1n}\mathbf{u}_1 + c_{2n}\mathbf{u}_2 + \cdots + c_{nn}\mathbf{u}_n \end{aligned}$$

which implies that

$$c_{1i}\begin{bmatrix} u_{11} \\ u_{21} \\ \vdots \\ u_{n1} \end{bmatrix} + c_{2i}\begin{bmatrix} u_{12} \\ u_{22} \\ \vdots \\ u_{n2} \end{bmatrix} + \cdots + c_{ni}\begin{bmatrix} u_{1n} \\ u_{2n} \\ \vdots \\ u_{nn} \end{bmatrix} = \begin{bmatrix} v_{1i} \\ v_{2i} \\ \vdots \\ v_{ni} \end{bmatrix}$$

for $i = 1, 2, \ldots, n$. From these vector equations you can write the $n$ systems of linear equations

$$\begin{aligned} u_{11}c_{1i} + u_{12}c_{2i} + \cdots + u_{1n}c_{ni} &= v_{1i} \\ u_{21}c_{1i} + u_{22}c_{2i} + \cdots + u_{2n}c_{ni} &= v_{2i} \\ &\vdots \\ u_{n1}c_{1i} + u_{n2}c_{2i} + \cdots + u_{nn}c_{ni} &= v_{ni} \end{aligned}$$

for $i = 1, 2, \ldots, n$. Because each of the $n$ systems has the same coefficient matrix, you can reduce all $n$ systems simultaneously using the following augmented matrix.

$$\left[\begin{array}{cccc|cccc} u_{11} & u_{12} & \cdots & u_{1n} & v_{11} & v_{12} & \cdots & v_{1n} \\ u_{21} & u_{22} & \cdots & u_{2n} & v_{21} & v_{22} & \cdots & v_{2n} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \underbrace{u_{n1}}_{} & u_{n2} & \cdots & u_{nn} & v_{n1} & v_{n2} & \cdots & v_{nn} \end{array}\right]$$

$$\qquad\qquad B' \qquad\qquad\qquad\qquad B$$

Applying Gauss-Jordan elimination to this matrix produces

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & c_{11} & c_{12} & \cdots & c_{1n} \\ 0 & 1 & \cdots & 0 & c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix}.$$

By the lemma following Theorem 4.20, however, the right-hand side of this matrix is $Q = P^{-1}$, which implies that the matrix has the form $[I \quad P^{-1}]$, which proves the theorem.

In the next example, you will apply this procedure to the change of basis problem from Example 3.

## EXAMPLE 4 Finding a Transition Matrix

Find the transition matrix from $B$ to $B'$ for the following bases for $R^3$.

$$B = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\} \quad \text{and} \quad B' = \{(1, 0, 1), (0, -1, 2), (2, 3, -5)\}$$

### SOLUTION

First use the vectors in the two bases to form the matrices $B$ and $B'$.

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad B' = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 3 \\ 1 & 2 & -5 \end{bmatrix}$$

Then form the matrix $[B' \quad B]$ and use Gauss-Jordan elimination to rewrite $[B' \quad B]$ as $[I_3 \quad P^{-1}]$.

$$\begin{bmatrix} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & 3 & 0 & 1 & 0 \\ 1 & 2 & -5 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & -1 & 4 & 2 \\ 0 & 1 & 0 & 3 & -7 & -3 \\ 0 & 0 & 1 & 1 & -2 & -1 \end{bmatrix}$$

From this, you can conclude that the transition matrix from $B$ to $B'$ is

$$P^{-1} = \begin{bmatrix} -1 & 4 & 2 \\ 3 & -7 & -3 \\ 1 & -2 & -1 \end{bmatrix}.$$

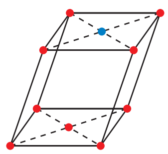
Try multiplying $P^{-1}$ by the coordinate matrix of $\mathbf{x} = [1 \quad 2 \quad -1]^T$ to see that the result is the same as the one obtained in Example 3.

### DISCOVERY

1. Let $B = \{(1, 0), (1, 2)\}$ and $B' = \{(1, 0), (0, 1)\}$. Form the matrix $[B' \quad B]$.
2. Make a conjecture about the necessity of using Gauss-Jordan elimination to obtain the transition matrix $P^{-1}$ when the change of basis is from a nonstandard basis to a standard basis.

### LINEAR ALGEBRA APPLIED

Crystallography is the science of the forms and structures of crystals. In a crystal, atoms are in a repeating pattern called a *lattice*. The simplest repeating unit in a lattice is called a *unit cell*. Crystallographers can use bases and coordinate matrices in $R^3$ to designate the locations of atoms in a unit cell. For instance, the following figure shows the unit cell known as *end-centered monoclinic*.



The coordinate matrix for the top end-centered (blue) atom could be given as $[\mathbf{x}]_{B'} = [\frac{1}{2} \;\; \frac{1}{2} \;\; 1]^T$.

Note that when $B$ is the standard basis, as in Example 4, the process of changing $[B' \;\; B]$ to $[I_n \;\; P^{-1}]$ becomes

$$[B' \;\; I_n] \longrightarrow [I_n \;\; P^{-1}].$$

But this is the same process that was used to find inverse matrices in Section 2.3. In other words, if $B$ is the standard basis for $R^n$, then the transition matrix from $B$ to $B'$ is

$$P^{-1} = (B')^{-1}. \qquad \text{Standard basis to nonstandard basis}$$

The process is even simpler when $B'$ is the standard basis, because the matrix $[B' \;\; B]$ is already in the form

$$[I_n \;\; B] = [I_n \;\; P^{-1}].$$

In this case, the transition matrix is simply

$$P^{-1} = B. \qquad \text{Nonstandard basis to standard basis}$$

For instance, the transition matrix in Example 2 from $B = \{(1, 0), (1, 2)\}$ to $B' = \{(1, 0), (0, 1)\}$ is

$$P^{-1} = B = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}.$$

## EXAMPLE 5 Finding a Transition Matrix

Find the transition matrix from $B$ to $B'$ for the following bases for $R^2$.

$$B = \{(-3, 2), (4, -2)\} \quad \text{and} \quad B' = \{(-1, 2), (2, -2)\}$$

### SOLUTION

Begin by forming the matrix

$$[B' \;\; B] = \begin{bmatrix} -1 & 2 & -3 & 4 \\ 2 & -2 & 2 & -2 \end{bmatrix}$$

and use Gauss-Jordan elimination to obtain the transition matrix $P^{-1}$ from $B$ to $B'$:

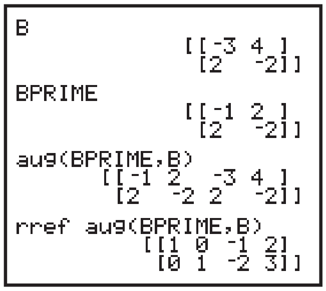
$$[I_2 \;\; P^{-1}] = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & -2 & 3 \end{bmatrix}.$$

So, you have

$$P^{-1} = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}.$$

**TECHNOLOGY**

Many graphing utilities and software programs can form an augmented matrix and find its reduced row-echelon form. If you use a graphing utility, then you may see something similar to the following for Example 5.




The **Online Technology Guide,** available at *www.cengagebrain.com*, provides syntax for programs applicable to Example 5.

In Example 5, if you had found the transition matrix from $B'$ to $B$ (rather than from $B$ to $B'$), then you would have obtained

$$[B' \;\; B] = \begin{bmatrix} -3 & 4 & -1 & 2 \\ 2 & -2 & 2 & -2 \end{bmatrix}$$

which reduces to

$$[I_2 \;\; P] = \begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 1 & 2 & -1 \end{bmatrix}.$$

The transition matrix from $B'$ to $B$ is

$$P = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}.$$

Verify that this is the inverse of the transition matrix found in Example 5 by multiplying $PP^{-1}$ to obtain $I_2$.

## COORDINATE REPRESENTATION IN GENERAL *n*-DIMENSIONAL SPACES

One benefit of coordinate representation is that it enables you to represent vectors in an arbitrary $n$-dimensional space using the same notation used in $R^n$. For instance, in Example 6, note that the coordinate matrix of a vector in $P_3$ is a vector in $R^4$.

### EXAMPLE 6 Coordinate Representation in $P_3$

Find the coordinate matrix of

$$p = 3x^3 - 2x^2 + 4$$

relative to the standard basis for $P_3$,

$$S = \{1, x, x^2, x^3\}.$$

**SOLUTION**

Write $p$ as a linear combination of the basis vectors (in the given order).

$$p = 4(1) + 0(x) + (-2)(x^2) + 3(x^3)$$

This tells you that the coordinate matrix of $p$ relative to $S$ is

$$[p]_S = \begin{bmatrix} 4 \\ 0 \\ -2 \\ 3 \end{bmatrix}.$$

In the next example, the coordinate matrix of a vector in $M_{3,1}$ is a vector in $R^3$.

### EXAMPLE 7 Coordinate Representation in $M_{3,1}$

Find the coordinate matrix of

$$X = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}$$

relative to the standard basis for $M_{3,1}$,

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

**SOLUTION**

Because $X$ can be written as

$$X = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix} = (-1)\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

the coordinate matrix of $X$ relative to $S$ is

$$[X]_S = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}.$$

**REMARK**

In Section 6.2 you will learn more about the use of $R^n$ to represent an arbitrary $n$-dimensional vector space.

Theorems 4.20 and 4.21 can be generalized to cover arbitrary $n$-dimensional spaces. This text, however, does not cover the generalizations of these theorems.

# 4.7 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

**Finding a Coordinate Matrix** In Exercises 1–4, find the coordinate matrix of $\mathbf{x}$ in $R^n$ relative to the standard basis.

1. $\mathbf{x} = (5, -2)$
2. $\mathbf{x} = (1, -3, 0)$
3. $\mathbf{x} = (7, -4, -1, 2)$
4. $\mathbf{x} = (-6, 12, -4, 9, -8)$

**Finding a Coordinate Matrix** In Exercises 5–10, given the coordinate matrix of $\mathbf{x}$ relative to a (nonstandard) basis $B$ for $R^n$, find the coordinate matrix of $\mathbf{x}$ relative to the standard basis.

5. $B = \{(2, -1), (0, 1)\}$, $[\mathbf{x}]_B = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$
6. $B = \{(-1, 4), (4, -1)\}$, $[\mathbf{x}]_B = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$
7. $B = \{(1, 0, 1), (1, 1, 0), (0, 1, 1)\}$, $[\mathbf{x}]_B = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$
8. $B = \left\{\left(\frac{3}{4}, \frac{5}{2}, \frac{3}{2}\right), \left(3, 4, \frac{7}{2}\right), \left(-\frac{3}{2}, 6, 2\right)\right\}$, $[\mathbf{x}]_B = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}$
9. $B = \{(0, 0, 0, 1), (0, 0, 1, 1), (0, 1, 1, 1), (1, 1, 1, 1)\}$, $[\mathbf{x}]_B = \begin{bmatrix} 1 \\ -2 \\ 3 \\ -1 \end{bmatrix}$
10. $B = \{(4, 0, 7, 3), (0, 5, -1, -1), (-3, 4, 2, 1), (0, 1, 5, 0)\}$, $[\mathbf{x}]_B = \begin{bmatrix} -2 \\ 3 \\ 4 \\ 1 \end{bmatrix}$

**Finding a Coordinate Matrix** In Exercises 11–16, find the coordinate matrix of $\mathbf{x}$ in $R^n$ relative to the basis $B'$.

11. $B' = \{(4, 0), (0, 3)\}$, $\mathbf{x} = (12, 6)$
12. $B' = \{(-6, 7), (4, -3)\}$, $\mathbf{x} = (-26, 32)$
13. $B' = \{(8, 11, 0), (7, 0, 10), (1, 4, 6)\}$, $\mathbf{x} = (3, 19, 2)$
14. $B' = \left\{\left(\frac{3}{2}, 4, 1\right), \left(\frac{3}{4}, \frac{5}{2}, 0\right), \left(1, \frac{1}{2}, 2\right)\right\}$, $\mathbf{x} = \left(3, -\frac{1}{2}, 8\right)$
15. $B' = \{(4, 3, 3), (-11, 0, 11), (0, 9, 2)\}$, $\mathbf{x} = (11, 18, -7)$
16. $B' = \{(9, -3, 15, 4), (3, 0, 0, 1), (0, -5, 6, 8), (3, -4, 2, -3)\}$, $\mathbf{x} = (0, -20, 7, 15)$

**Finding a Transition Matrix** In Exercises 17–24, find the transition matrix from $B$ to $B'$.

17. $B = \{(1, 0), (0, 1)\}$, $B' = \{(2, 4), (1, 3)\}$
18. $B = \{(1, 0), (0, 1)\}$, $B' = \{(1, 1), (5, 6)\}$
19. $B = \{(2, 4), (-1, 3)\}$, $B' = \{(1, 0), (0, 1)\}$
20. $B = \{(1, 1), (1, 0)\}$, $B' = \{(1, 0), (0, 1)\}$
21. $B = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$, $B' = \{(1, 0, 0), (0, 2, 8), (6, 0, 12)\}$
22. $B = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$, $B' = \{(1, 3, -1), (2, 7, -4), (2, 9, -7)\}$
23. $B = \{(3, 4, 0), (-2, -1, 1), (1, 0, -3)\}$, $B' = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$
24. $B = \{(1, 3, 2), (2, -1, 2), (5, 6, 1)\}$, $B' = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$

**Finding a Transition Matrix** In Exercises 25–34, use a software program or a graphing utility with matrix capabilities to find the transition matrix from $B$ to $B'$.

25. $B = \{(2, 5), (1, 2)\}$, $B' = \{(2, 1), (-1, 2)\}$
26. $B = \{(-2, 1), (3, 2)\}$, $B' = \{(1, 2), (-1, 0)\}$
27. $B = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$, $B' = \{(1, 3, 3), (1, 5, 6), (1, 4, 5)\}$
28. $B = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$, $B' = \{(2, -1, 4), (0, 2, 1), (-3, 2, 1)\}$
29. $B = \{(1, 2, 4), (-1, 2, 0), (2, 4, 0)\}$, $B' = \{(0, 2, 1), (-2, 1, 0), (1, 1, 1)\}$
30. $B = \{(3, 2, 1), (1, 1, 2), (1, 2, 0)\}$, $B' = \{(1, 1, -1), (0, 1, 2), (-1, 4, 0)\}$
31. $B = \{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}$, $B' = \{(1, 3, 2, -1), (-2, -5, -5, 4), (-1, -2, -2, 4), (-2, -3, -5, 11)\}$
32. $B = \{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}$, $B' = \{(1, 1, 1, 1), (0, 1, 1, 1), (0, 0, 1, 1), (0, 0, 0, 1)\}$
33. $B = \{(1, 0, 0, 0, 0), (0, 1, 0, 0, 0), (0, 0, 1, 0, 0), (0, 0, 0, 1, 0), (0, 0, 0, 0, 1)\}$, $B' = \{(1, 2, 4, -1, 2), (-2, -3, 4, 2, 1), (0, 1, 2, -2, 1), (0, 1, 2, 2, 1), (1, -1, 0, 1, 2)\}$
34. $B = \{(1, 0, 0, 0, 0), (0, 1, 0, 0, 0), (0, 0, 1, 0, 0), (0, 0, 0, 1, 0), (0, 0, 0, 0, 1)\}$, $B' = \{(2, 4, -2, 1, 0), (3, -1, 0, 1, 2), (0, 0, -2, 4, 5), (2, -1, 2, 1, 1), (0, 1, 2, -3, 1)\}$

**Finding Transition and Coordinate Matrices** **In Exercises 35–38, (a) find the transition matrix from $B$ to $B'$, (b) find the transition matrix from $B'$ to $B$, (c) verify that the two transition matrices are inverses of each other, and (d) find the coordinate matrix $[\mathbf{x}]_B$, given the coordinate matrix $[\mathbf{x}]_{B'}$.**

**35.** $B = \{(1, 3), (-2, -2)\}$, $B' = \{(-12, 0), (-4, 4)\}$,
$[\mathbf{x}]_{B'} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

**36.** $B = \{(2, -2), (6, 3)\}$, $B' = \{(1, 1), (32, 31)\}$,
$[\mathbf{x}]_{B'} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

**37.** $B = \{(1, 0, 2), (0, 1, 3), (1, 1, 1)\}$,
$B' = \{(2, 1, 1), (1, 0, 0), (0, 2, 1)\}$,
$[\mathbf{x}]_{B'} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

**38.** $B = \{(1, 1, 1), (1, -1, 1), (0, 0, 1)\}$,
$B' = \{(2, 2, 0), (0, 1, 1), (1, 0, 1)\}$,
$[\mathbf{x}]_{B'} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$

**Finding Transition and Coordinate Matrices** **In Exercises 39–41, use a software program or a graphing utility with matrix capabilities to (a) find the transition matrix from $B$ to $B'$, (b) find the transition matrix from $B'$ to $B$, (c) verify that the two transition matrices are inverses of each other, and (d) find the coordinate matrix $[\mathbf{x}]_B$, given the coordinate matrix $[\mathbf{x}]_{B'}$.**

**39.** $B = \{(4, 2, -4), (6, -5, -6), (2, -1, 8)\}$,
$B' = \{(1, 0, 4), (4, 2, 8), (2, 5, -2)\}$,
$[\mathbf{x}]_{B'} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

**40.** $B = \{(1, 3, 4), (2, -5, 2), (-4, 2, -6)\}$,
$B' = \{(1, 2, -2), (4, 1, -4), (-2, 5, 8)\}$,
$[\mathbf{x}]_{B'} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

**41.** $B = \{(2, 0, -1), (0, -1, 3), (1, -3, -2)\}$,
$B' = \{(0, -1, -3), (-1, 3, -2), (-3, -2, 0)\}$,
$[\mathbf{x}]_{B'} = \begin{bmatrix} 4 \\ -3 \\ -2 \end{bmatrix}$

**42.** **CAPSTONE** Let $B$ and $B'$ be two bases for $R^n$.

(a) When $B = I_n$, write the transition matrix from $B$ to $B'$ in terms of $B'$.

(b) When $B' = I_n$, write the transition matrix from $B$ to $B'$ in terms of $B$.

(c) When $B = I_n$, write the transition matrix from $B'$ to $B$ in terms of $B'$.

(d) When $B' = I_n$, write the transition matrix from $B'$ to $B$ in terms of $B$.

**Coordinate Representation in $P_3$** **In Exercises 43–46, find the coordinate matrix of $p$ relative to the standard basis for $P_3$.**

**43.** $p = 2x^3 + x^2 + 11x + 4$

**44.** $p = 3x^2 + 114x + 13$

**45.** $p = x^3 - 2x^2 + 5x + 1$ **46.** $p = 4x^3 - 3x - 2$

**Coordinate Representation in $M_{3,1}$** **In Exercises 47–50, find the coordinate matrix of $X$ relative to the standard basis for $M_{3,1}$.**

**47.** $X = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$ **48.** $X = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$

**49.** $X = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ **50.** $X = \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix}$

**True or False?** **In Exercises 51 and 52, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.**

**51.** (a) If $P$ is the transition matrix from a basis $B$ to $B'$, then the equation $P[\mathbf{x}]_{B'} = [\mathbf{x}]_B$ represents the change of basis from $B$ to $B'$.

(b) For any $4 \times 1$ matrix $X$, the coordinate matrix $[X]_S$ relative to the standard basis for $M_{4,1}$ is equal to $X$ itself.

**52.** (a) To perform the change of basis from a nonstandard basis $B'$ to the standard basis $B$, the transition matrix $P^{-1}$ is simply $B'$.

(b) The coordinate matrix of $p = 5x^2 + x - 3$ relative to the standard basis for $P_2$ is $[p]_S = [5 \quad 1 \quad -3]^T$.

**53.** Let $P$ be the transition matrix from $B''$ to $B'$, and let $Q$ be the transition matrix from $B'$ to $B$. What is the transition matrix from $B''$ to $B$?

**54.** Let $P$ be the transition matrix from $B''$ to $B'$, and let $Q$ be the transition matrix from $B'$ to $B$. What is the transition matrix from $B$ to $B''$?

# 4.8 Applications of Vector Spaces

- Use the Wronskian to test a set of solutions of a linear homogeneous differential equation for linear independence.
- Identify and sketch the graph of a conic section and perform a rotation of axes.

## LINEAR DIFFERENTIAL EQUATIONS (CALCULUS)

A **linear differential equation of order *n*** is of the form

$$y^{(n)} + g_{n-1}(x)y^{(n-1)} + \cdots + g_1(x)y' + g_0(x)y = f(x)$$

where $g_0, g_1, \ldots, g_{n-1}$ and $f$ are functions of $x$ with a common domain. If $f(x) = 0$, then the equation is **homogeneous.** Otherwise it is **nonhomogeneous.** A function $y$ is called a **solution** of the linear differential equation if the equation is satisfied when $y$ and its first $n$ derivatives are substituted into the equation.

### EXAMPLE 1 A Second-Order Linear Differential Equation

Show that both $y_1 = e^x$ and $y_2 = e^{-x}$ are solutions of the second-order linear differential equation $y'' - y = 0$.

#### SOLUTION

For the function $y_1 = e^x$, you have $y_1' = e^x$ and $y_1'' = e^x$. So,

$$y_1'' - y_1 = e^x - e^x = 0$$

which means that $y_1 = e^x$ is a solution of the differential equation. Similarly, for $y_2 = e^{-x}$, you have

$$y_2' = -e^{-x} \quad \text{and} \quad y_2'' = e^{-x}.$$

This implies that

$$y_2'' - y_2 = e^{-x} - e^{-x} = 0.$$

So, $y_2 = e^{-x}$ is also a solution of the linear differential equation.

There are two important observations you can make about Example 1. The first is that in the vector space $C''(-\infty, \infty)$ of all twice differentiable functions defined on the entire real line, the two solutions $y_1 = e^x$ and $y_2 = e^{-x}$ are *linearly independent*. This means that the only solution of

$$C_1y_1 + C_2y_2 = 0$$

that is valid for all $x$ is $C_1 = C_2 = 0$. The second observation is that every *linear combination* of $y_1$ and $y_2$ is also a solution of the linear differential equation. To see this, let $y = C_1y_1 + C_2y_2$. Then

$$\begin{aligned} y &= C_1e^x + C_2e^{-x} \\ y' &= C_1e^x - C_2e^{-x} \\ y'' &= C_1e^x + C_2e^{-x}. \end{aligned}$$

Substituting into the differential equation $y'' - y = 0$ produces

$$y'' - y = (C_1e^x + C_2e^{-x}) - (C_1e^x + C_2e^{-x}) = 0.$$

So, $y = C_1e^x + C_2e^{-x}$ is a solution.

The next theorem, which is stated without proof, generalizes these observations.

**REMARK**

The solution

$y = C_1y_1 + C_2y_2 + \cdot\cdot\cdot + C_ny_n$

is called the **general solution** of the given differential equation.

### Solutions of a Linear Homogeneous Differential Equation

Every $n$th-order linear homogeneous differential equation

$$y^{(n)} + g_{n-1}(x)y^{(n-1)} + \cdot\cdot\cdot + g_1(x)y' + g_0(x)y = 0$$

has $n$ linearly independent solutions. Moreover, if $\{y_1, y_2, . . . , y_n\}$ is a set of linearly independent solutions, then every solution is of the form

$$y = C_1y_1 + C_2y_2 + \cdot\cdot\cdot + C_ny_n$$

where $C_1, C_2, . . . , C_n$ are real numbers.

In light of the preceding theorem, you can see the importance of being able to determine whether a set of solutions is linearly independent. Before describing a way of testing for linear independence, you are given the following preliminary definition.

**REMARK**

The Wronskian of a set of functions is named after the Polish mathematician Josef Maria Wronski (1778–1853).

### Definition of the Wronskian of a Set of Functions

Let $\{y_1, y_2, . . . , y_n\}$ be a set of functions, each of which has $n - 1$ derivatives on an interval $I$. The determinant

$$W(y_1, y_2, . . . , y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

is called the **Wronskian** of the given set of functions.

**EXAMPLE 2** **Finding the Wronskian of a Set of Functions**

**a.** The Wronskian of the set $\{1 - x, 1 + x, 2 - x\}$ is

$$W = \begin{vmatrix} 1 - x & 1 + x & 2 - x \\ -1 & 1 & -1 \\ 0 & 0 & 0 \end{vmatrix} = 0.$$

**b.** The Wronskian of the set $\{x, x^2, x^3\}$ is

$$W = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} = 2x^3.$$

The Wronskian in part (a) of Example 2 is said to be **identically equal to zero,** because it is zero for any value of $x$. The Wronskian in part (b) is not identically equal to zero because values of $x$ exist for which this Wronskian is nonzero.

The next theorem shows how the Wronskian of a set of functions can be used to test for linear independence.

**REMARK**

This test does *not* apply to an arbitrary set of functions. Each of the functions $y_1, y_2, . . . ,$ and $y_n$ must be a solution of the same linear homogeneous differential equation of order $n$.

### Wronskian Test for Linear Independence

Let $\{y_1, y_2, . . . , y_n\}$ be a set of $n$ solutions of an $n$th-order linear homogeneous differential equation. This set is linearly independent if and only if the Wronskian is not identically equal to zero.

The proof of this theorem for the case where $n = 2$ is left as an exercise. (See Exercise 40.)

## EXAMPLE 3 Testing a Set of Solutions for Linear Independence

Determine whether $\{1, \cos x, \sin x\}$ is a set of linearly independent solutions of the linear homogeneous differential equation

$$y''' + y' = 0.$$

**SOLUTION**

Begin by observing that each of the functions is a solution of $y''' + y' = 0$. (Try checking this.) Next, testing for linear independence produces the Wronskian of the three functions, as follows.

$$\begin{aligned} W &= \begin{vmatrix} 1 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{vmatrix} \\ &= \sin^2 x + \cos^2 x = 1 \end{aligned}$$

Because $W$ is not identically equal to zero, the set

$$\{1, \cos x, \sin x\}$$

is linearly independent. Moreover, because this set consists of three linearly independent solutions of a third-order linear homogeneous differential equation, the general solution is

$$y = C_1 + C_2 \cos x + C_3 \sin x$$

where $C_1$, $C_2$, and $C_3$ are real numbers.

## EXAMPLE 4 Testing a Set of Solutions for Linear Independence

Determine whether $\{e^x, xe^x, (x + 1)e^x\}$ is a set of linearly independent solutions of the linear homogeneous differential equation

$$y''' - 3y'' + 3y' - y = 0.$$

**SOLUTION**

As in Example 3, begin by verifying that each of the functions is actually a solution of $y''' - 3y'' + 3y' - y = 0$. (This verification is left to you.) Testing for linear independence produces the Wronskian of the three functions, as follows.

$$W = \begin{vmatrix} e^x & xe^x & (x+1)e^x \\ e^x & (x+1)e^x & (x+2)e^x \\ e^x & (x+2)e^x & (x+3)e^x \end{vmatrix} = 0$$

So, the set $\{e^x, xe^x, (x + 1)e^x\}$ is linearly dependent.

In Example 4, the Wronskian is used to determine that the set

$$\{e^x, xe^x, (x + 1)e^x\}$$

is linearly dependent. Another way to determine the linear dependence of this set is to observe that the third function is a linear combination of the first two. That is,

$$(x + 1)e^x = e^x + xe^x.$$

Try showing that a different set, $\{e^x, xe^x, x^2e^x\}$, forms a linearly independent set of solutions of the differential equation

$$y''' - 3y'' + 3y' - y = 0.$$

## CONIC SECTIONS AND ROTATION

Every conic section in the $xy$-plane has an equation that can be written in the form

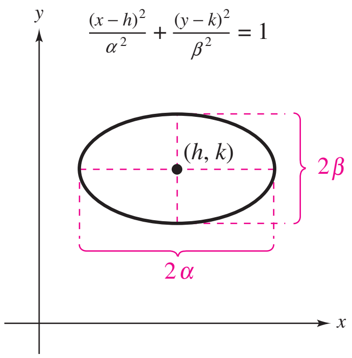
$$ax^2 + bxy + cy^2 + dx + ey + f = 0.$$

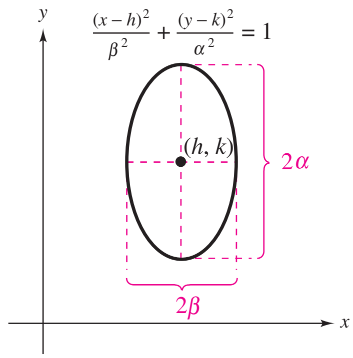
Identifying the graph of this equation is fairly simple as long as $b$, the coefficient of the $xy$-term, is zero. When $b$ is zero, the conic axes are parallel to the coordinate axes, and the identification is accomplished by writing the equation in standard (completed square) form. The standard forms of the equations of the four basic conics are given in the following summary. For circles, ellipses, and hyperbolas, the point $(h, k)$ is the center. For parabolas, the point $(h, k)$ is the vertex.

### Standard Forms of Equations of Conics

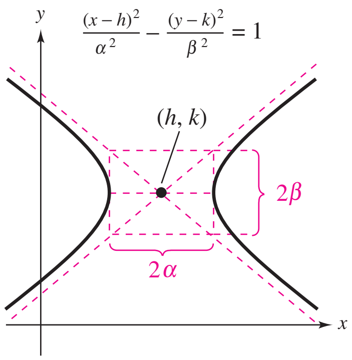
**Circle** ($r$ = radius): $(x - h)^2 + (y - k)^2 = r^2$

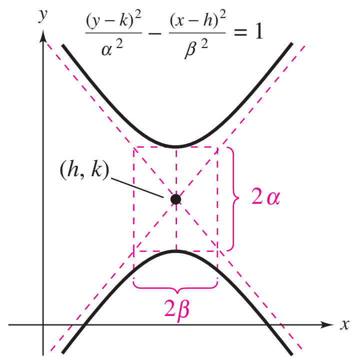
**Ellipse** ($2\alpha$ = major axis length, $2\beta$ = minor axis length):



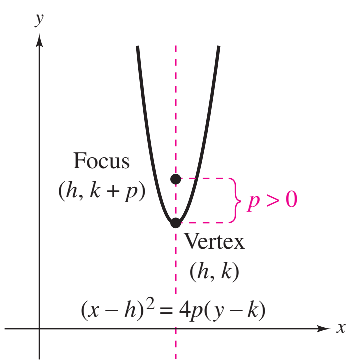



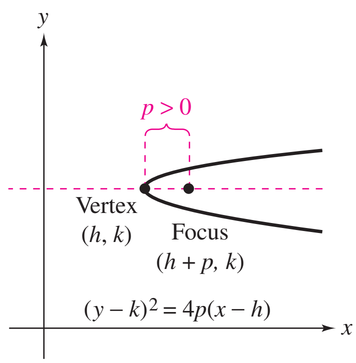

**Hyperbola** ($2\alpha$ = transverse axis length, $2\beta$ = conjugate axis length):




**Parabola** ($p$ = directed distance from vertex to focus):



**EXAMPLE 5** **Identifying Conic Sections**

**a.** The standard form of $x^2 - 2x + 4y - 3 = 0$ is

$$(x - 1)^2 = 4(-1)(y - 1).$$

The graph of this equation is a parabola with the vertex at $(h, k) = (1, 1)$. The axis of the parabola is vertical. Because $p = -1$, the focus is the point $(1, 0)$. Finally, because the focus lies below the vertex, the parabola opens downward, as shown in Figure 4.20(a).

**b.** The standard form of $x^2 + 4y^2 + 6x - 8y + 9 = 0$ is

$$\frac{(x + 3)^2}{4} + \frac{(y - 1)^2}{1} = 1.$$

The graph of this equation is an ellipse with its center at $(h, k) = (-3, 1)$. The major axis is horizontal, and its length is $2\alpha = 4$. The length of the minor axis is $2\beta = 2$. The vertices of this ellipse occur at $(-5, 1)$ and $(-1, 1)$, and the endpoints of the minor axis occur at $(-3, 2)$ and $(-3, 0)$, as shown in Figure 4.20(b).

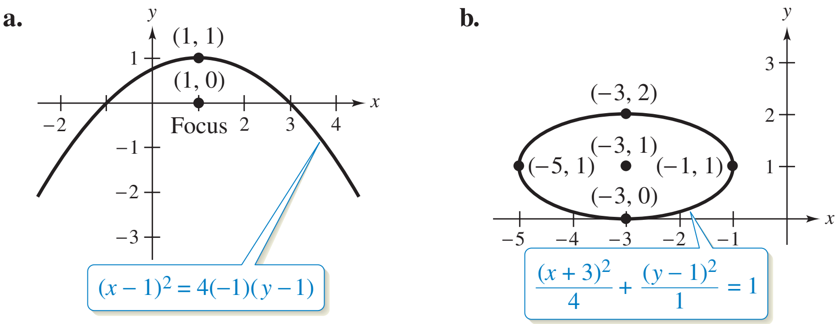



**Figure 4.20**

The equations of the conics in Example 5 have no $xy$-term. Consequently, the axes of the graphs of these conics are parallel to the coordinate axes. For second-degree equations that have an $xy$-term, the axes of the graphs of the corresponding conics are not parallel to the coordinate axes. In such cases it is helpful to *rotate* the standard axes to form a new $x'$-axis and $y'$-axis. The required rotation angle $\theta$ (measured counterclockwise) is $\cot 2\theta = (a - c)/b$. With this rotation, the standard basis for $R^2$,

$$B = \{(1, 0), (0, 1)\}$$

is rotated to form the new basis

$$B' = \{(\cos\theta, \sin\theta), (-\sin\theta, \cos\theta)\}$$

as shown in Figure 4.21.

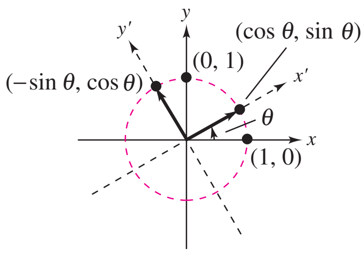



**Figure 4.21**

To find the coordinates of a point $(x, y)$ relative to this new basis, you can use a transition matrix, as demonstrated in Example 6.

## EXAMPLE 6 A Transition Matrix for Rotation in $R^2$

Find the coordinates of a point $(x, y)$ in $R^2$ relative to the basis

$$B' = \{(\cos\theta, \sin\theta), (-\sin\theta, \cos\theta)\}.$$

**SOLUTION**

By Theorem 4.21 you have

$$[B' \; B] = \left[\begin{array}{cc|cc} \cos\theta & -\sin\theta & 1 & 0 \\ \sin\theta & \cos\theta & 0 & 1 \end{array}\right].$$

Because $B$ is the standard basis for $R^2$, $P^{-1}$ is represented by $(B')^{-1}$. You can use the formula given in Section 2.3 (page 66) for the inverse of a $2 \times 2$ matrix to find $(B')^{-1}$. This results in

$$[I \; P^{-1}] = \left[\begin{array}{cc|cc} 1 & 0 & \cos\theta & \sin\theta \\ 0 & 1 & -\sin\theta & \cos\theta \end{array}\right].$$

By letting $(x', y')$ be the coordinates of $(x, y)$ relative to $B'$, you can use the transition matrix $P^{-1}$ as follows.

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

The $x'$- and $y'$-coordinates are $x' = x\cos\theta + y\sin\theta$ and $y' = -x\sin\theta + y\cos\theta$.

The last two equations in Example 6 give the $x'y'$-coordinates in terms of the $xy$-coordinates. To perform a rotation of axes for a general second-degree equation, it is helpful to express the $xy$-coordinates in terms of the $x'y'$-coordinates. To do this, solve the last two equations in Example 6 for $x$ and $y$ to obtain

$$x = x'\cos\theta - y'\sin\theta \quad \text{and} \quad y = x'\sin\theta + y'\cos\theta.$$

Substituting these expressions for $x$ and $y$ into the given second-degree equation produces a second-degree equation in $x'$ and $y'$ that has no $x'y'$-term.

### Rotation of Axes

The general second-degree equation $ax^2 + bxy + cy^2 + dx + ey + f = 0$ can be written in the form

$$a'(x')^2 + c'(y')^2 + d'x' + e'y' + f' = 0$$

by rotating the coordinate axes counterclockwise through the angle $\theta$, where $\theta$ is defined by $\cot 2\theta = \dfrac{a-c}{b}$. The coefficients of the new equation are obtained from the substitutions

$$x = x'\cos\theta - y'\sin\theta \quad \text{and} \quad y = x'\sin\theta + y'\cos\theta.$$

The proof of the above result is left to you. (See Exercise 82.)

**LINEAR ALGEBRA APPLIED**

A satellite dish is an antenna that is designed to transmit or receive signals of a specific type. A standard satellite dish consists of a bowl-shaped surface and a *feed horn* that is aimed toward the surface. The bowl-shaped surface is typically in the shape of an elliptic paraboloid. (See Section 7.4.) The cross section of the surface is typically in the shape of a rotated parabola.

Example 7 demonstrates how to identify the graph of a second-degree equation by rotating the coordinate axes.

## EXAMPLE 7 Rotation of a Conic Section

Perform a rotation of axes to eliminate the $xy$-term in

$$5x^2 - 6xy + 5y^2 + 14\sqrt{2}x - 2\sqrt{2}y + 18 = 0$$

and sketch the graph of the resulting equation in the $x'y'$-plane.

**SOLUTION**

The angle of rotation is given by

$$\cot 2\theta = \frac{a-c}{b} = \frac{5-5}{-6} = 0.$$

This implies that $\theta = \pi/4$. So,

$$\sin\theta = \frac{1}{\sqrt{2}} \quad \text{and} \quad \cos\theta = \frac{1}{\sqrt{2}}.$$

By substituting

$$x = x'\cos\theta - y'\sin\theta = \frac{1}{\sqrt{2}}(x' - y')$$

and

$$y = x'\sin\theta + y'\cos\theta = \frac{1}{\sqrt{2}}(x' + y')$$

into the original equation and simplifying, you obtain

$$(x')^2 + 4(y')^2 + 6x' - 8y' + 9 = 0.$$

Finally, by completing the square, you find the standard form of this equation to be

$$\frac{(x'+3)^2}{2^2} + \frac{(y'-1)^2}{1^2} = \frac{(x'+3)^2}{4} + \frac{(y'-1)^2}{1} = 1$$

which is the equation of an ellipse, as shown in Figure 4.22.

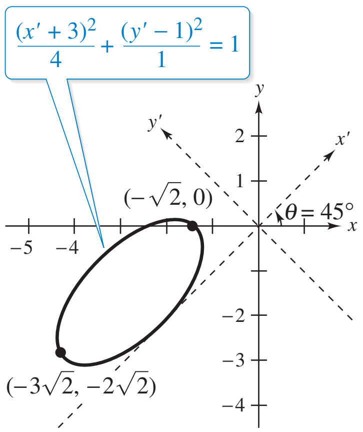


Figure 4.22

In Example 7 the new (rotated) basis for $R^2$ is

$$B' = \left\{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right\}$$

and the coordinates of the vertices of the ellipse relative to $B'$ are

$$\begin{bmatrix} -5 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

To find the coordinates of the vertices relative to the standard basis $B = \{(1, 0), (0, 1)\}$, use the equations

$$x = \frac{1}{\sqrt{2}}(x' - y')$$

and

$$y = \frac{1}{\sqrt{2}}(x' + y')$$

to obtain $\left(-3\sqrt{2}, -2\sqrt{2}\right)$ and $\left(-\sqrt{2}, 0\right)$, as shown in Figure 4.22.

# 4.8 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

**Determining Solutions of a Differential Equation** **In Exercises 1–12, determine which functions are solutions of the linear differential equation.**

1. $y'' + y = 0$
   (a) $e^x$ (b) $\sin x$
   (c) $\cos x$ (d) $\sin x - \cos x$

2. $y''' + y = 0$
   (a) $e^x$ (b) $e^{-x}$ (c) $e^{-2x}$ (d) $2e^{-x}$

3. $y''' + y'' + y' + y = 0$
   (a) $x$ (b) $e^x$ (c) $e^{-x}$ (d) $xe^{-x}$

4. $y'' + 4y' + 4y = 0$
   (a) $e^{-2x}$ (b) $xe^{-2x}$
   (c) $x^2e^{-2x}$ (d) $(x + 2)e^{-2x}$

5. $y^{(4)} + y''' - 2y'' = 0$
   (a) $1$ (b) $x$ (c) $x^2$ (d) $e^x$

6. $y^{(4)} - 16y = 0$
   (a) $3 \cos x$ (b) $3 \cos 2x$
   (c) $e^{-2x}$ (d) $3e^{2x} - 4 \sin 2x$

7. $x^2y'' - 2y = 0$
   (a) $\dfrac{1}{x^2}$ (b) $x^2$ (c) $e^{x^2}$ (d) $e^{-x^2}$

8. $y'(2x - 1)y = 0$
   (a) $e^{x-x^2}$ (b) $2e^{x-x^2}$
   (c) $3e^{x-x^2}$ (d) $4e^{x-x^2}$

9. $xy' - 2y = 0$
   (a) $\sqrt{x}$ (b) $x$ (c) $x^2$ (d) $x^3$

10. $xy'' + 2y' = 0$
    (a) $x$ (b) $\dfrac{1}{x}$ (c) $xe^x$ (d) $xe^{-x}$

11. $y'' - y' - 2y = 0$
    (a) $xe^{2x}$ (b) $2e^{2x}$ (c) $2e^{-2x}$ (d) $xe^{-x}$

12. $y' - 2xy = 0$
    (a) $3e^{x^2}$ (b) $xe^{x^2}$ (c) $x^2e^x$ (d) $xe^{-x}$

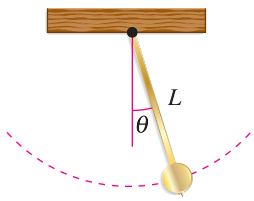
**Finding the Wronskian for a Set of Functions** **In Exercises 13–26, find the Wronskian for the set of functions.**

13. $\{x, \cos x\}$
14. $\{e^{2x}, \cos 3x\}$
15. $\{e^x, e^{-x}\}$
16. $\{e^{x^2}, e^{-x^2}\}$
17. $\{x, \sin x, \cos x\}$
18. $\{x, -\sin x, \cos x\}$
19. $\{e^{-x}, xe^{-x}, (x + 3)e^{-x}\}$
20. $\{x, e^{-x}, e^x\}$
21. $\{1, e^x, e^{2x}\}$
22. $\{x^2, e^{x^2}, x^2e^x\}$
23. $\{1, x, x^2, x^3\}$
24. $\{x, x^2, e^x, e^{-x}\}$
25. $\{1, x, \cos x, e^{-x}\}$
26. $\{x, e^x, \sin x, \cos x\}$

**Testing for Linear Independence** **In Exercises 27–34, (a) verify that each solution satisfies the differential equation, (b) test the set of solutions for linear independence, and (c) if the set is linearly independent, then write the general solution of the differential equation.**

| | *Differential Equation* | *Solutions* |
|---|---|---|
| **27.** | $y'' + 16y = 0$ | $\{\sin 4x, \cos 4x\}$ |
| **28.** | $y'' - 2y' + y = 0$ | $\{e^x, xe^x\}$ |
| **29.** | $y''' + 4y'' + 4y' = 0$ | $\{e^{-2x}, xe^{-2x}, (2x + 1)e^{-2x}\}$ |
| **30.** | $y''' + 4y' = 0$ | $\{2, 2 \sin 2x, 1 + \sin 2x\}$ |
| **31.** | $y''' + 4y' = 0$ | $\{1, \sin 2x, \cos 2x\}$ |
| **32.** | $y''' + 3y'' + 3y' + y = 0$ | $\{e^{-x}, xe^{-x}, x^2e^{-x}\}$ |
| **33.** | $y''' + 3y'' + 3y' + y = 0$ | $\{e^{-x}, xe^{-x}, e^{-x} + xe^{-x}\}$ |
| **34.** | $y^{(4)} - 2y''' + y'' = 0$ | $\{1, x, e^x, xe^x\}$ |

35. **Pendulum** Consider a pendulum of length $L$ that swings by the force of gravity only.




For small values of $\theta = \theta(t)$, the motion of the pendulum can be approximated by the differential equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0$$

where $g$ is the acceleration due to gravity.

(a) Verify that

$$\left\{\sin\sqrt{\frac{g}{L}}t, \cos\sqrt{\frac{g}{L}}t\right\}$$

is a set of linearly independent solutions of the differential equation.

(b) Find the general solution of the differential equation and show that it can be written in the form

$$\theta(t) = A \cos\left[\sin\sqrt{\frac{g}{L}}(t + \phi)\right].$$

36. **Proof** Prove that $y = C_1 \cos ax + C_2 \sin ax$ is the general solution of $y'' + a^2y = 0$, $a \neq 0$.

**Proof** **In Exercises 37–39, prove that the given set of solutions of a second-order linear homogeneous differential equation is linearly independent.**

37. $\{e^{ax}, e^{bx}\}$, $a \neq b$
38. $\{e^{ax}, xe^{ax}\}$
39. $\{e^{ax} \cos bx, e^{ax} \sin bx\}$, $b \neq 0$

**40.** **Proof** Let $\{y_1, y_2\}$ be a set of solutions of a second-order linear homogeneous differential equation. Prove that this set is linearly independent if and only if the Wronskian is not identically equal to zero.

**41.** **Writing** Is the sum of two solutions of a nonhomogeneous linear differential equation also a solution? Explain your answer.

**42.** **Writing** Is the scalar multiple of a solution of a nonhomogeneous linear differential equation also a solution? Explain your answer.

**Identifying and Graphing a Conic Section** **In Exercises 43–60, identify and sketch the graph of the conic section.**

**43.** $y^2 + x = 0$ **44.** $y^2 + 8x = 0$

**45.** $x^2 + 4y^2 - 16 = 0$ **46.** $5x^2 + 3y^2 - 15 = 0$

**47.** $\dfrac{x^2}{9} - \dfrac{y^2}{16} - 1 = 0$ **48.** $\dfrac{x^2}{16} - \dfrac{y^2}{25} = 1$

**49.** $x^2 - 2x + 8y + 17 = 0$

**50.** $y^2 - 6y - 4x + 21 = 0$

**51.** $9x^2 + 25y^2 - 36x - 50y + 61 = 0$

**52.** $4x^2 + y^2 - 8x + 3 = 0$

**53.** $9x^2 - y^2 + 54x + 10y + 55 = 0$

**54.** $4y^2 - 2x^2 - 4y - 8x - 15 = 0$

**55.** $x^2 + 4y^2 + 4x + 32y + 64 = 0$

**56.** $4y^2 + 4x^2 - 24x + 35 = 0$

**57.** $2x^2 - y^2 + 4x + 10y - 22 = 0$

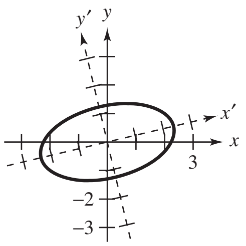
**58.** $4x^2 - y^2 + 4x + 2y - 1 = 0$

**59.** $x^2 + 4x + 6y - 2 = 0$

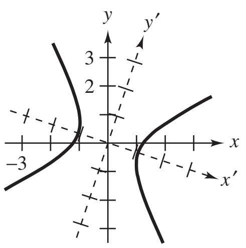
**60.** $y^2 + 8x + 6y + 25 = 0$

**Matching an Equation with a Graph** **In Exercises 61–64, match the graph with its equation. [The graphs are labeled (a), (b), (c), and (d).]**

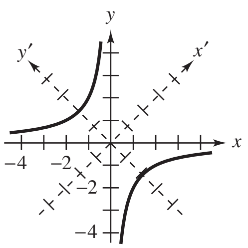
(a)



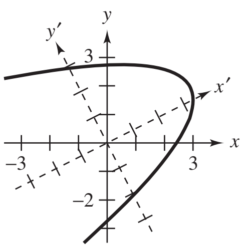
(b)



(c)



(d)



**61.** $xy + 2 = 0$

**62.** $-2x^2 + 3xy + 2y^2 + 3 = 0$

**63.** $x^2 - xy + 3y^2 - 5 = 0$

**64.** $x^2 - 4xy + 4y^2 + 10x - 30 = 0$

**Rotation of a Conic Section** **In Exercises 65–76, perform a rotation of axes to eliminate the $xy$-term, and sketch the graph of the conic.**

**65.** $xy + 1 = 0$ **66.** $xy - 2 = 0$

**67.** $4x^2 + 2xy + 4y^2 - 15 = 0$

**68.** $x^2 + 2xy + y^2 - 8x + 8y = 0$

**69.** $2x^2 - 3xy - 2y^2 + 10 = 0$

**70.** $5x^2 - 2xy + 5y^2 - 24 = 0$

**71.** $9x^2 + 24xy + 16y^2 + 80x - 60y = 0$

**72.** $5x^2 - 6xy + 5y^2 - 12 = 0$

**73.** $13x^2 + 6\sqrt{3}xy + 7y^2 - 16 = 0$

**74.** $7x^2 - 2\sqrt{3}xy + 5y^2 = 16$

**75.** $3x^2 - 2\sqrt{3}xy + y^2 + 2x + 2\sqrt{3}\,y = 0$

**76.** $x^2 + 2\sqrt{3}xy + 3y^2 - 2\sqrt{3}x + 2y + 16 = 0$

**Rotation of a Degenerate Conic Section** **In Exercises 77–80, perform a rotation of axes to eliminate the $xy$-term, and sketch the graph of the "degenerate" conic.**

**77.** $x^2 - 2xy + y^2 = 0$ **78.** $5x^2 - 2xy + 5y^2 = 0$

**79.** $x^2 + 2xy + y^2 - 1 = 0$ **80.** $x^2 - 10xy + y^2 = 0$

**81.** **Proof** Prove that a rotation of $\theta = \pi/4$ will eliminate the $xy$-term from the equation

$$ax^2 + bxy + ay^2 + dx + ey + f = 0.$$

**82.** **Proof** Prove that a rotation of $\theta$, where $\cot 2\theta = (a - c)/b$, will eliminate the $xy$-term from the equation

$$ax^2 + bxy + cy^2 + dx + ey + f = 0.$$

**83.** **Proof** For the equation $ax^2 + bxy + cy^2 = 0$, define the matrix $A$ as

$$A = \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}.$$

(a) Prove that if $|A| = 0$, then the graph of $ax^2 + bxy + cy^2 = 0$ is a line.

(b) Prove that if $|A| \neq 0$, then the graph of $ax^2 + bxy + cy^2 = 0$ is two intersecting lines.

**84.** **CAPSTONE** Explain each of the following.

(a) How to use the Wronskian to test a set of solutions of a linear homogeneous differential equation for linear independence

(b) How to eliminate the $xy$-term if it appears in the general equation of a conic section

# 4 Review Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

**Vector Operations** **In Exercises 1–4, find (a) $\mathbf{u} + \mathbf{v}$, (b) $2\mathbf{v}$, (c) $\mathbf{u} - \mathbf{v}$, and (d) $3\mathbf{u} - 2\mathbf{v}$.**

1. $\mathbf{u} = (-1, 2, 3), \quad \mathbf{v} = (1, 0, 2)$
2. $\mathbf{u} = (-1, 2, 1), \quad \mathbf{v} = (0, 1, 1)$
3. $\mathbf{u} = (3, -1, 2, 3), \quad \mathbf{v} = (0, 2, 2, 1)$
4. $\mathbf{u} = (0, 1, -1, 2), \quad \mathbf{v} = (1, 0, 0, 2)$

**Solving a Vector Equation** **In Exercises 5–8, solve for $\mathbf{x}$, where $\mathbf{u} = (1, -1, 2)$, $\mathbf{v} = (0, 2, 3)$, and $\mathbf{w} = (0, 1, 1)$.**

5. $2\mathbf{x} - \mathbf{u} + 3\mathbf{v} + \mathbf{w} = \mathbf{0}$
6. $3\mathbf{x} + 2\mathbf{u} - \mathbf{v} + 2\mathbf{w} = \mathbf{0}$
7. $5\mathbf{u} - 2\mathbf{x} = 3\mathbf{v} + \mathbf{w}$
8. $2\mathbf{u} + 3\mathbf{x} = 2\mathbf{v} - \mathbf{w}$

**Writing a Linear Combination** **In Exercises 9–12, write $\mathbf{v}$ as a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$, if possible.**

9. $\mathbf{v} = (3, 0, -6), \quad \mathbf{u}_1 = (1, -1, 2), \quad \mathbf{u}_2 = (2, 4, -2),$
   $\mathbf{u}_3 = (1, 2, -4)$
10. $\mathbf{v} = (4, 4, 5), \quad \mathbf{u}_1 = (1, 2, 3), \quad \mathbf{u}_2 = (-2, 0, 1),$
    $\mathbf{u}_3 = (1, 0, 0)$
11. $\mathbf{v} = (1, 2, 3, 5), \quad \mathbf{u}_1 = (1, 2, 3, 4),$
    $\mathbf{u}_2 = (-1, -2, -3, 4), \quad \mathbf{u}_3 = (0, 0, 1, 1)$
12. $\mathbf{v} = (4, -13, -5, -4), \quad \mathbf{u}_1 = (1, -2, 1, 1),$
    $\mathbf{u}_2 = (-1, 2, 3, 2), \quad \mathbf{u}_3 = (0, -1, -1, -1)$

**Describing the Zero Vector and the Additive Inverse** **In Exercises 13–16, describe the zero vector and the additive inverse of a vector in the vector space.**

13. $M_{3,4}$
14. $P_8$
15. $R^3$
16. $M_{2,3}$

**Determining Subspaces** **In Exercises 17–24, determine whether $W$ is a subspace of the vector space.**

17. $W = \{(x, y): x = 2y\}, \quad V = R^2$
18. $W = \{(x, y): x - y = 1\}, \quad V = R^2$
19. $W = \{(x, y): y = ax, a \text{ is an integer}\}, \quad V = R^2$
20. $W = \{(x, y): y = ax^2\}, \quad V = R^2$
21. $W = \{(x, 2x, 3x): x \text{ is a real number}\}, \quad V = R^3$
22. $W = \{(x, y, z): x \geq 0\}, \quad V = R^3$
23. $W = \{f: f(0) = -1\}, \quad V = C[-1, 1]$
24. $W = \{f: f(-1) = 0\}, \quad V = C[-1, 1]$

25. Which of the subsets of $R^3$ is a subspace of $R^3$?
    (a) $W = \{(x_1, x_2, x_3): x_1^2 + x_2^2 + x_3^2 = 0\}$
    (b) $W = \{(x_1, x_2, x_3): x_1^2 + x_2^2 + x_3^2 = 1\}$
26. Which of the subsets of $R^3$ is a subspace of $R^3$?
    (a) $W = \{(x_1, x_2, x_3): x_1 + x_2 + x_3 = 0\}$
    (b) $W = \{(x_1, x_2, x_3): x_1 + x_2 + x_3 = 1\}$

**Spanning Sets, Linear Independence, and Bases** **In Exercises 27–32, determine whether the set (a) spans $R^3$, (b) is linearly independent, and (c) is a basis for $R^3$.**

27. $S = \{(1, -5, 4), (11, 6, -1), (2, 3, 5)\}$
28. $S = \{(4, 0, 1), (0, -3, 2), (5, 10, 0)\}$
29. $S = \{\left(-\frac{1}{2}, \frac{3}{4}, -1\right), (5, 2, 3), (-4, 6, -8)\}$
30. $S = \{(2, 0, 1), (2, -1, 1), (4, 2, 0)\}$
31. $S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (-1, 2, -3)\}$
32. $S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (2, -1, 0)\}$

33. Determine whether

    $$S = \{1 - t, 2t + 3t^2, t^2 - 2t^3, 2 + t^3\}$$

    is a basis for $P_3$.
34. Determine whether $S = \{1, t, 1 + t^2\}$ is a basis for $P_2$.

**Determining Whether a Set Is a Basis** **In Exercises 35 and 36, determine whether the set is a basis for $M_{2,2}$.**

35. $S = \left\{\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} -3 & -3 \\ 1 & 3 \end{bmatrix}\right\}$
36. $S = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right\}$

**Finding the Nullspace, Nullity, and Rank of a Matrix** **In Exercises 37–42, find (a) the nullspace, (b) the nullity, and (c) the rank of the matrix $A$. Then verify that $\text{rank}(A) + \text{nullity}(A) = n$, where $n$ is the number of columns of $A$.**

37. $A = \begin{bmatrix} 5 & -8 \\ -10 & 16 \end{bmatrix}$
38. $A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$
39. $A = \begin{bmatrix} 2 & -3 & -6 & -4 \\ 1 & 5 & -3 & 11 \\ 2 & 7 & -6 & 16 \end{bmatrix}$
40. $A = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 4 & -2 & 4 & -2 \\ -2 & 0 & 1 & 3 \end{bmatrix}$
41. $A = \begin{bmatrix} 1 & 3 & 2 \\ 4 & -1 & -18 \\ -1 & 3 & 10 \\ 1 & 2 & 0 \end{bmatrix}$
42. $A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 4 & 0 & 3 \\ -2 & 3 & 0 & 2 \\ 1 & 2 & 6 & 1 \end{bmatrix}$

**Finding a Basis and Dimension** In Exercises 43–46, find (a) a basis for and (b) the dimension of the solution space of the homogeneous system of linear equations.

**43.** $$\begin{aligned} 2x_1 + 4x_2 + 3x_3 - 6x_4 &= 0 \\ x_1 + 2x_2 + 2x_3 - 5x_4 &= 0 \\ 3x_1 + 6x_2 + 5x_3 - 11x_4 &= 0 \end{aligned}$$

**44.** $$\begin{aligned} 3x_1 + 8x_2 + 2x_3 + 3x_4 &= 0 \\ 4x_1 + 6x_2 + 2x_3 - x_4 &= 0 \\ 3x_1 + 4x_2 + x_3 - 3x_4 &= 0 \end{aligned}$$

**45.** $$\begin{aligned} x_1 - 3x_2 + x_3 + x_4 &= 0 \\ 2x_1 + x_2 - x_3 + 2x_4 &= 0 \\ x_1 + 4x_2 - 2x_3 + x_4 &= 0 \\ 5x_1 - 8x_2 + 2x_3 + 5x_4 &= 0 \end{aligned}$$

**46.** $$\begin{aligned} -x_1 + 2x_2 - x_3 + 2x_4 &= 0 \\ -2x_1 + 2x_2 + x_3 + 4x_4 &= 0 \\ 3x_1 + 2x_2 + 2x_3 + 5x_4 &= 0 \\ -3x_1 + 8x_2 + 5x_3 + 17x_4 &= 0 \end{aligned}$$

**Finding a Basis for a Row Space and Rank** In Exercises 47–52, find (a) a basis for the row space and (b) the rank of the matrix.

**47.** $\begin{bmatrix} 1 & 2 \\ -4 & 3 \\ 6 & 1 \end{bmatrix}$

**48.** $\begin{bmatrix} 2 & -1 & 4 \\ 1 & 5 & 6 \\ 1 & 16 & 14 \end{bmatrix}$

**49.** $\begin{bmatrix} 1 & -4 & 0 & 4 \end{bmatrix}$

**50.** $\begin{bmatrix} 1 & 2 & -1 \end{bmatrix}$

**51.** $\begin{bmatrix} 7 & 0 & 2 \\ 4 & 1 & 6 \\ -1 & 16 & 14 \end{bmatrix}$

**52.** $\begin{bmatrix} 1 & 2 & 0 \\ -1 & 4 & 1 \\ 0 & 1 & 3 \end{bmatrix}$

**Finding a Coordinate Matrix** In Exercises 53–58, given the coordinate matrix of x relative to a (nonstandard) basis $B$ for $R^n$, find the coordinate matrix of x relative to the standard basis.

**53.** $B = \{(1, 1), (-1, 1)\}, \quad [\mathbf{x}]_B = [3 \quad 5]^T$

**54.** $B = \{(2, 0), (3, 3)\}, \quad [\mathbf{x}]_B = [1 \quad 1]^T$

**55.** $B = \{(\frac{1}{2}, \frac{1}{2}), (1, 0)\}, \quad [\mathbf{x}]_B = [\frac{1}{2} \quad \frac{1}{2}]^T$

**56.** $B = \{(4, 2), (1, -1)\}, \quad [\mathbf{x}]_B = [2 \quad 1]^T$

**57.** $B = \{(1, 0, 0), (1, 1, 0), (0, 1, 1)\},$
$[\mathbf{x}]_B = [2 \quad 0 \quad -1]^T$

**58.** $B = \{(1, 0, 1), (0, 1, 0), (0, 1, 1)\}, \quad [\mathbf{x}]_B = [4 \quad 0 \quad 2]^T$

**Finding a Coordinate Matrix** In Exercises 59–64, find the coordinate matrix of x in $R^n$ relative to the basis $B'$.

**59.** $B' = \{(5, 0), (0, -8)\}, \quad \mathbf{x} = (2, 2)$

**60.** $B' = \{(1, 1), (0, -2)\}, \quad \mathbf{x} = (2, -1)$

**61.** $B' = \{(1, 2, 3), (1, 2, 0), (0, -6, 2)\}, \quad \mathbf{x} = (3, -3, 0)$

**62.** $B' = \{(1, 0, 0), (0, 1, 0), (1, 1, 1)\}, \quad \mathbf{x} = (4, -2, 9)$

**63.** $B' = \{(9, -3, 15, 4), (-3, 0, 0, -1), (0, -5, 6, 8), (-3, 4, -2, 3)\}, \quad \mathbf{x} = (21, -5, 43, 14)$

**64.** $B' = \{(1, -1, 2, 1), (1, 1, -4, 3), (1, 2, 0, 3), (1, 2, -2, 0)\}, \quad \mathbf{x} = (5, 3, -6, 2)$

**Finding a Transition Matrix** In Exercises 65–68, find the transition matrix from $B$ to $B'$.

**65.** $B = \{(1, -1), (3, 1)\},$
$B' = \{(1, 0), (0, 1)\}$

**66.** $B = \{(1, -1), (3, 1)\},$
$B' = \{(1, 2), (-1, 0)\}$

**67.** $B = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\},$
$B' = \{(0, 0, 1), (0, 1, 0), (1, 0, 0)\}$

**68.** $B = \{(1, 1, 1), (1, 1, 0), (1, 0, 0)\},$
$B' = \{(1, 2, 3), (0, 1, 0), (1, 0, 1)\}$

**Finding Transition and Coordinate Matrices** In Exercises 69–72, (a) find the transition matrix from $B$ to $B'$, (b) find the transition matrix from $B'$ to $B$, (c) verify that the two transition matrices are inverses of each other, and (d) find the coordinate matrix $[x]_{B'}$, given the coordinate matrix $[x]_B$.

**69.** $B = \{(1, 1), (-1, 1)\}, \quad B' = \{(0, 1), (1, 2)\},$
$[\mathbf{x}]_B = [3 \quad -3]^T$

**70.** $B = \{(1, 0), (1, -1)\}, \quad B' = \{(1, 1), (1, -1)\},$
$[\mathbf{x}]_B = [2 \quad -2]^T$

**71.** $B = \{(1, 0, 0), (1, 1, 0), (1, 1, 1)\},$
$B' = \{(0, 0, 1), (0, 1, 1), (1, 1, 1)\},$
$[\mathbf{x}]_B = [-1 \quad 2 \quad -3]^T$

**72.** $B = \{(1, 1, -1), (1, 1, 0), (1, -1, 0)\},$
$B' = \{(1, -1, 2), (2, 2, -1), (2, 2, 2)\},$
$[\mathbf{x}]_B = [2 \quad 2 \quad -1]^T$

**73.** Let $W$ be the subspace of $P_3$ [the set of all polynomials $p(x)$ of degree 3 or less] such that $p(0) = 0$, and let $U$ be the subspace of $P_3$ such that $p(1) = 0$. Find a basis for $W$, a basis for $U$, and a basis for their intersection $W \cap U$.

**74. Calculus** Let $V = C'(-\infty, \infty)$, the vector space of all continuously differentiable functions on the real line.

(a) Prove that $W = \{f: f' = 4f\}$ is a subspace of $V$.

(b) Prove that $U = \{f: f' = f + 1\}$ is not a subspace of $V$.

**75. Writing** Let $B = \{p_1(x), p_2(x), \ldots, p_n(x), p_{n+1}(x)\}$ be a basis for $P_n$. Must $B$ contain a polynomial of each degree $0, 1, 2, \ldots, n$? Explain your reasoning.

**76. Proof** Let $A$ and $B$ be $n \times n$ matrices with $A \neq O$ and $B \neq O$. Prove that if $A$ is symmetric and $B$ is skew-symmetric ($B^T = -B$), then $\{A, B\}$ is a linearly independent set.

**77. Proof** Let $V = P_5$ and consider the set $W$ of all polynomials of the form $(x^3 + x)p(x)$, where $p(x)$ is in $P_2$. Is $W$ a subspace of $V$? Prove your answer.

**78.** Let $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ be three linearly independent vectors in a vector space $V$. Is the set $\{\mathbf{v}_1 - 2\mathbf{v}_2, 2\mathbf{v}_2 - 3\mathbf{v}_3, 3\mathbf{v}_3 - \mathbf{v}_1\}$ linearly dependent or linearly independent?

**79. Proof** Let $A$ be an $n \times n$ square matrix. Prove that the row vectors of $A$ are linearly dependent if and only if the column vectors of $A$ are linearly dependent.

**80. Proof** Let $A$ be an $n \times n$ square matrix, and let $\lambda$ be a scalar. Prove that the set

$$S = \{\mathbf{x}: A\mathbf{x} = \lambda\mathbf{x}\}$$

is a subspace of $R^n$. Determine the dimension of $S$ when $\lambda = 3$ and

$$A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

**81.** Let $f(x) = x$ and $g(x) = |x|$.

(a) Show that $f$ and $g$ are linearly independent in $C[-1, 1]$.

(b) Show that $f$ and $g$ are linearly dependent in $C[0, 1]$.

**82.** Given a set of functions, describe how its domain can influence whether the set is linearly independent or dependent.

**True or False?** **In Exercises 83–86, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.**

**83.** (a) The standard operations in $R^n$ are vector addition and scalar multiplication.

(b) The additive inverse of a vector is not unique.

(c) A vector space consists of four entities: a set of vectors, a set of scalars, and two operations.

**84.** (a) The set $W = \{(0, x^2, x^3): x^2 \text{ and } x^3 \text{ are real numbers}\}$ is a subspace of $R^3$.

(b) A linearly independent spanning set $S$ is called a basis for a vector space $V$.

(c) If $A$ is an invertible $n \times n$ matrix, then the $n$ row vectors of $A$ are linearly dependent.

**85.** (a) The set of all $n$-tuples is called $n$-space and is denoted by $R^n$.

(b) The additive identity of a vector space is not unique.

(c) Once a theorem has been proved for an abstract vector space, you need not give separate proofs for $n$-tuples, matrices, and polynomials.

**86.** (a) The set of points on the line $x + y = 0$ is a subspace of $R^2$.

(b) Elementary row operations preserve the column space of the matrix $A$.

## Determining Solutions of a Differential Equation

**In Exercises 87–90, determine which functions are solutions of the linear differential equation.**

**87.** $y'' - y' - 6y = 0$

(a) $e^{3x}$ (b) $e^{2x}$ (c) $e^{-3x}$ (d) $e^{-2x}$

**88.** $y^{(4)} - y = 0$

(a) $e^x$ (b) $e^{-x}$ (c) $\cos x$ (d) $\sin x$

**89.** $y' + 2y = 0$

(a) $e^{-2x}$ (b) $xe^{-2x}$ (c) $x^2e^{-x}$ (d) $2xe^{-2x}$

**90.** $y'' + 9y = 0$

(a) $\sin 3x + \cos 3x$ (b) $3 \sin x + 3 \cos x$

(c) $\sin 3x$ (d) $\cos 3x$

## Finding the Wronskian for a Set of Functions

**In Exercises 91–94, find the Wronskian for the set of functions.**

**91.** $\{1, x, e^x\}$

**92.** $\{1, x, 2 + x\}$

**93.** $\{1, \sin 2x, \cos 2x\}$

**94.** $\{x, \sin^2 x, \cos^2 x\}$

**Testing for Linear Independence** **In Exercises 95–98, (a) verify that each solution satisfies the differential equation, (b) test the set of solutions for linear independence, and (c) if the set is linearly independent, then write the general solution of the differential equation.**

| | *Differential Equation* | *Solutions* |
|---|---|---|
| **95.** | $y'' + 6y' + 9y = 0$ | $\{e^{-3x}, xe^{-3x}\}$ |
| **96.** | $y'' + 6y' + 9y = 0$ | $\{e^{-3x}, 3e^{-3x}\}$ |
| **97.** | $y''' - 6y'' + 11y' - 6y = 0$ | $\{e^x, e^{2x}, e^x - e^{2x}\}$ |
| **98.** | $y'' + 4y = 0$ | $\{\sin 2x, \cos 2x\}$ |

## Identifying and Graphing a Conic Section

**In Exercises 99–106, identify and sketch the graph of the conic section.**

**99.** $x^2 + y^2 - 4x + 2y - 4 = 0$

**100.** $9x^2 + 9y^2 + 18x - 18y + 14 = 0$

**101.** $x^2 - y^2 + 2x - 3 = 0$

**102.** $4x^2 - y^2 + 8x - 6y + 4 = 0$

**103.** $2x^2 - 20x - y + 46 = 0$

**104.** $y^2 - 4x - 4 = 0$

**105.** $4x^2 + y^2 + 32x + 4y + 63 = 0$

**106.** $16x^2 + 25y^2 - 32x - 50y + 16 = 0$

**Rotation of a Conic Section** **In Exercises 107–110, perform a rotation of axes to eliminate the $xy$-term, and sketch the graph of the conic.**

**107.** $xy = 3$

**108.** $9x^2 + 4xy + 9y^2 - 20 = 0$

**109.** $16x^2 - 24xy + 9y^2 - 60x - 80y + 100 = 0$

**110.** $7x^2 + 6\sqrt{3}xy + 13y^2 - 16 = 0$

# 4 Projects

## 1 Solutions of Linear Systems

Write a paragraph to answer the question. Do not perform any calculations, but instead base your explanations on appropriate properties from the text.

**1.** One solution of the homogeneous linear system

$$\begin{aligned} x + 2y + z + 3w &= 0 \\ x - \ y \qquad + \ w &= 0 \\ y - z + 2w &= 0 \end{aligned}$$

is $x = -2$, $y = -1$, $z = 1$, and $w = 1$. Explain why $x = 4$, $y = 2$, $z = -2$, and $w = -2$ is also a solution.

**2.** The vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions of the homogeneous linear system $A\mathbf{x} = \mathbf{0}$. Explain why the vector $2\mathbf{x}_1 - 3\mathbf{x}_2$ is also a solution.

**3.** Consider the two systems represented by the augmented matrices.

$$\begin{bmatrix} 1 & 1 & -5 & 3 \\ 1 & 0 & -2 & 1 \\ 2 & -1 & -1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & -5 & -9 \\ 1 & 0 & -2 & -3 \\ 2 & -1 & -1 & 0 \end{bmatrix}$$

If the first system is consistent, then why is the second system also consistent?

**4.** The vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions of the linear system $A\mathbf{x} = \mathbf{b}$. Is the vector $2\mathbf{x}_1 - 3\mathbf{x}_2$ also a solution? Why or why not?

**5.** The linear systems $A\mathbf{x} = \mathbf{b}_1$ and $A\mathbf{x} = \mathbf{b}_2$ are consistent. Is the system $A\mathbf{x} = \mathbf{b}_1 + \mathbf{b}_2$ necessarily consistent? Why or why not?

## 2 Direct Sum

In this project, you will explore the **sum** and **direct sum** of subspaces. In Exercise 58 in Section 4.3, you proved that for two subspaces $U$ and $W$ of a vector space $V$, the sum $U + W$ of the subspaces, defined as $U + W = \{\mathbf{u} + \mathbf{w} : \mathbf{u} \in U, \mathbf{w} \in W\}$, is also a subspace of $V$.

**1.** Consider the following subspaces of $V = R^3$.

$$\begin{aligned} U &= \{(x, y, x - y) : x, y \in R\} \\ W &= \{(x, 0, x) : x \in R\} \\ Z &= \{(x, x, x) : x \in R\} \end{aligned}$$

Find $U + W$, $U + Z$, and $W + Z$.

**2.** If $U$ and $W$ are subspaces of $V$ such that $V = U + W$ and $U \cap W = \{\mathbf{0}\}$, then prove that every vector in $V$ has a *unique* representation of the form $\mathbf{u} + \mathbf{w}$, where $\mathbf{u}$ is in $U$ and $\mathbf{w}$ is in $W$. $V$ is called the **direct sum** of $U$ and $W$, and is written as

$V = U \oplus W$. Direct sum

Which of the sums in part (1) are direct sums?

**3.** Let $V = U \oplus W$ and suppose that $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ is a basis for the subspace $U$ and $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m\}$ is a basis for the subspace $W$. Prove that the set $\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{w}_1, \ldots, \mathbf{w}_m\}$ is a basis for $V$.

**4.** Consider the subspaces $U = \{(x, 0, y) : x, y \in R\}$ and $W = \{(0, x, y) : x, y \in R\}$ of $V = R^3$. Show that $R^3 = U + W$. Is $R^3$ the *direct* sum of $U$ and $W$? What are the dimensions of $U$, $W$, $U \cap W$, and $U + W$? Formulate a conjecture that relates the dimensions of $U$, $W$, $U \cap W$, and $U + W$.

**5.** Do there exist two two-dimensional subspaces of $R^3$ whose intersection is the zero vector? Why or why not?

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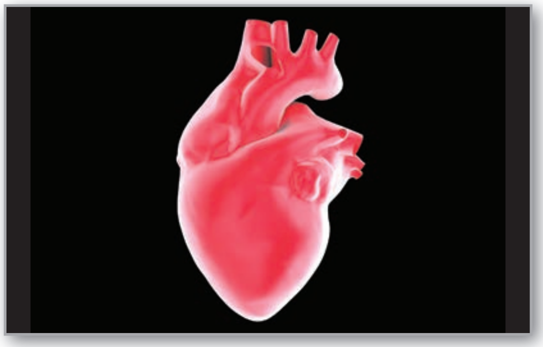
# 5 Inner Product Spaces



Torque (p. 271)

Revenue (p. 260)



Heart Rhythm Analysis (p. 249)

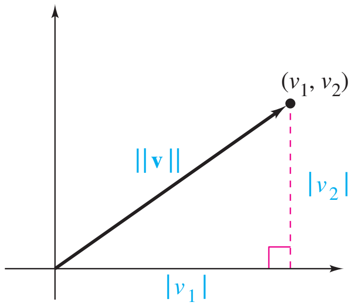
Work (p. 242)

Electric/Magnetic Flux (p. 234)

# 5.1 Length and Dot Product in $R^n$

- Find the length of a vector and find a unit vector.
- Find the distance between two vectors.
- Find a dot product and the angle between two vectors, determine orthogonality, and verify the Cauchy-Schwarz Inequality, the triangle inequality, and the Pythagorean Theorem.
- Use a matrix product to represent a dot product.

## VECTOR LENGTH AND UNIT VECTORS




$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2}$

**Figure 5.1**

Section 4.1 mentioned that vectors in the plane are directed line segments characterized by two quantities, *length* and *direction*. In this section, $R^2$ will be used as a model for defining these and other geometric properties (such as distance and angle) of vectors in $R^n$. In the next section, these ideas will be extended to general vector spaces.

You will begin by reviewing the definition of the length of a vector in $R^2$. If $\mathbf{v} = (v_1, v_2)$ is a vector in $R^2$, then the **length,** or **norm,** of $\mathbf{v}$, denoted by $\|\mathbf{v}\|$, is defined as

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2}.$$

This definition corresponds to the usual concept of length in Euclidean geometry. That is, $\|\mathbf{v}\|$ is thought of as the length of the hypotenuse of a right triangle whose legs have lengths of $|v_1|$ and $|v_2|$, as shown in Figure 5.1. Applying the Pythagorean Theorem produces

$$\|\mathbf{v}\|^2 = |v_1|^2 + |v_2|^2 = v_1^2 + v_2^2.$$

Using $R^2$ as a model, the length of a vector in $R^n$ is defined as follows.

### REMARK

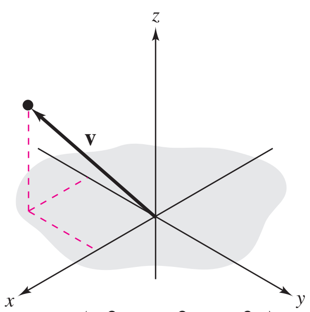
The length of a vector is also called its **magnitude.** If $\|\mathbf{v}\| = 1$, then the vector $\mathbf{v}$ is called a **unit vector.**

### Definition of Length of a Vector in $R^n$

The **length,** or **norm,** of a vector $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ in $R^n$ is given by

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}.$$

This definition shows that the length of a vector cannot be negative. That is, $\|\mathbf{v}\| \geq 0$. Moreover, $\|\mathbf{v}\| = 0$ if and only if $\mathbf{v}$ is the zero vector $\mathbf{0}$.




$\mathbf{v} = \left(\frac{2}{\sqrt{17}}, -\frac{2}{\sqrt{17}}, \frac{3}{\sqrt{17}}\right)$

**Figure 5.2**

### EXAMPLE 1 The Length of a Vector in $R^n$

**a.** In $R^5$, the length of $\mathbf{v} = (0, -2, 1, 4, -2)$ is

$$\|\mathbf{v}\| = \sqrt{0^2 + (-2)^2 + 1^2 + 4^2 + (-2)^2} = \sqrt{25} = 5.$$

**b.** In $R^3$, the length of $\mathbf{v} = \left(2/\sqrt{17}, -2/\sqrt{17}, 3/\sqrt{17}\right)$ is

$$\|\mathbf{v}\| = \sqrt{\left(2/\sqrt{17}\right)^2 + \left(-2/\sqrt{17}\right)^2 + \left(3\sqrt{17}\right)^2} = \sqrt{17/17} = 1.$$

Because its length is 1, $\mathbf{v}$ is a unit vector, as shown in Figure 5.2.

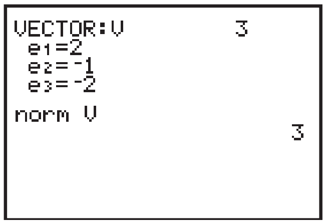
Each vector in the standard basis for $R^n$ has length 1 and is called a **standard unit vector** in $R^n$. It is common to denote the standard unit vectors in $R^2$ and $R^3$ as

$$\{\mathbf{i}, \mathbf{j}\} = \{(1, 0), (0, 1)\} \quad \text{and} \quad \{\mathbf{i}, \mathbf{j}, \mathbf{k}\} = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}.$$

Two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in $R^n$ are **parallel** when one is a scalar multiple of the other—that is, $\mathbf{u} = c\mathbf{v}$. Moreover, if $c > 0$, then $\mathbf{u}$ and $\mathbf{v}$ have the **same direction,** and if $c < 0$, then $\mathbf{u}$ and $\mathbf{v}$ have **opposite directions.** The next theorem gives a formula for finding the length of a scalar multiple of a vector.

### TECHNOLOGY

You can use a graphing utility or software program to find the length of a vector. For instance, if you use a graphing utility to find the length of the vector $\mathbf{v} = (2, -1, -2)$, then you may see something similar to the following.




Use a graphing utility or software program to verify the lengths of the vectors given in Example 1.

**THEOREM 5.1 Length of a Scalar Multiple**

Let $\mathbf{v}$ be a vector in $R^n$ and let $c$ be a scalar. Then

$$\|c\mathbf{v}\| = |c|\,\|\mathbf{v}\|$$

where $|c|$ is the absolute value of $c$.

**PROOF**

Because $c\mathbf{v} = (cv_1, cv_2, \ldots, cv_n)$, it follows that

$$\begin{aligned}\|c\mathbf{v}\| &= \|(cv_1, cv_2, \ldots, cv_n)\| \\ &= \sqrt{(cv_1)^2 + (cv_2)^2 + \cdots + (cv_n)^2} \\ &= |c|\sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} \\ &= |c|\,\|\mathbf{v}\|.\end{aligned}$$

One important use of Theorem 5.1 is in finding a unit vector having the same direction as a given vector. The following theorem provides a procedure for doing this.

**THEOREM 5.2 Unit Vector in the Direction of v**

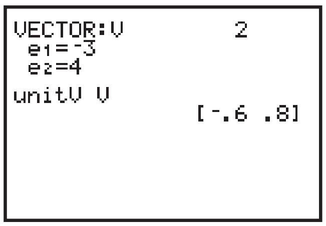
If $\mathbf{v}$ is a nonzero vector in $R^n$, then the vector

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$$

has length 1 and has the same direction as $\mathbf{v}$. This vector $\mathbf{u}$ is called the **unit vector in the direction of v.**

### TECHNOLOGY

You can use a graphing utility or software program to find a unit vector. For instance, if you use a graphing utility to find the unit vector in the direction of $\mathbf{v} = (-3, 4)$, then you may see something similar to the following.




**PROOF**

Because $\mathbf{v} \neq \mathbf{0}$, you know $\|\mathbf{v}\| \neq 0$. So, $1/\|\mathbf{v}\|$ is positive, and you can write $\mathbf{u}$ as a positive scalar multiple of $\mathbf{v}$.

$$\mathbf{u} = \left(\frac{1}{\|\mathbf{v}\|}\right)\mathbf{v}$$

It follows that $\mathbf{u}$ has the same direction as $\mathbf{v}$, and $\mathbf{u}$ has length 1 because

$$\|\mathbf{u}\| = \left\|\frac{\mathbf{v}}{\|\mathbf{v}\|}\right\| = \frac{1}{\|\mathbf{v}\|}\|\mathbf{v}\| = 1.$$

The process of finding the unit vector in the direction of $\mathbf{v}$ is called **normalizing** the vector $\mathbf{v}$. The next example demonstrates this procedure.

## EXAMPLE 2 Finding a Unit Vector

Find the unit vector in the direction of $\mathbf{v} = (3, -1, 2)$, and verify that this vector has length 1.

**SOLUTION**

The unit vector in the direction of $\mathbf{v}$ is

$$\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{(3, -1, 2)}{\sqrt{3^2 + (-1)^2 + 2^2}} = \frac{1}{\sqrt{14}}(3, -1, 2) = \left(\frac{3}{\sqrt{14}}, -\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}\right)$$

which is a unit vector because

$$\sqrt{\left(\frac{3}{\sqrt{14}}\right)^2 + \left(-\frac{1}{\sqrt{14}}\right)^2 + \left(\frac{2}{\sqrt{14}}\right)^2} = \sqrt{\frac{14}{14}} = 1. \text{ (See Figure 5.3.)}$$

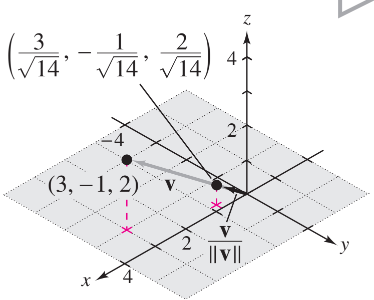



**Figure 5.3**

## DISTANCE BETWEEN TWO VECTORS IN $R^n$

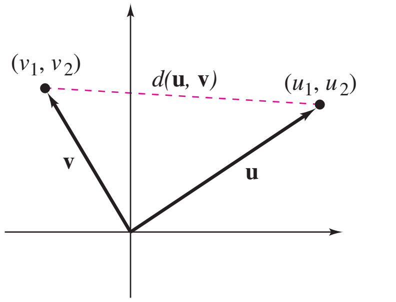
To define the **distance between two vectors** in $R^n$, $R^2$ will be used as a model. The Distance Formula from analytic geometry tells you that the distance $d$ between two points in $R^2$, $(u_1, u_2)$ and $(v_1, v_2)$, is

$$d = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2}.$$

In vector terminology, this distance can be viewed as the length of $\mathbf{u} - \mathbf{v}$, where $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$, as shown in Figure 5.4. That is,

$$\|\mathbf{u} - \mathbf{v}\| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2}$$

which leads to the next definition.




$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2}$

**Figure 5.4**

**Olga Taussky-Todd (1906–1995)**

Taussky-Todd was born in what is now the Czech Republic. She became interested in mathematics at an early age. During her life, Taussky-Todd was a distinguished and prolific mathematician. She wrote many research papers in such areas as matrix theory, group theory, algebraic number theory, and numerical analysis. Taussky-Todd received many honors and distinctions for her work. For instance, her paper on the sum of squares earned her the Ford Prize from the Mathematical Association of America.

### Definition of Distance Between Two Vectors

The distance between two vectors $\mathbf{u}$ and $\mathbf{v}$ in $R^n$ is

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|.$$

Try verifying the following three properties of distance.

1. $d(\mathbf{u}, \mathbf{v}) \geq 0$
2. $d(\mathbf{u}, \mathbf{v}) = 0$ if and only if $\mathbf{u} = \mathbf{v}$.
3. $d(\mathbf{u}, \mathbf{v}) = d(\mathbf{v}, \mathbf{u})$

### EXAMPLE 3 Finding the Distance Between Two Vectors

**a.** The distance between $\mathbf{u} = (-1, -4)$ and $\mathbf{v} = (2, 3)$ is

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \|(-1 - 2, -4 - 3)\| = \sqrt{(-3)^2 + (-7)^2} = \sqrt{58}.$$

**b.** The distance between $\mathbf{u} = (0, 2, 2)$ and $\mathbf{v} = (2, 0, 1)$ is

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \|(0 - 2, 2 - 0, 2 - 1)\| = \sqrt{(-2)^2 + 2^2 + 1^2} = 3.$$

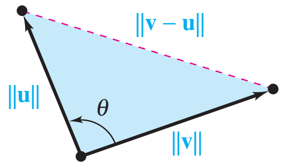
**c.** The distance between $\mathbf{u} = (3, -1, 0, -3)$ and $\mathbf{v} = (4, 0, 1, 2)$ is

$$\begin{aligned} d(\mathbf{u}, \mathbf{v}) &= \|\mathbf{u} - \mathbf{v}\| \\ &= \|(3 - 4, -1 - 0, 0 - 1, -3 - 2)\| \\ &= \sqrt{(-1)^2 + (-1)^2 + (-1)^2 + (-5)^2} \\ &= \sqrt{28} \\ &= 2\sqrt{7}. \end{aligned}$$

## DOT PRODUCT AND THE ANGLE BETWEEN TWO VECTORS

To find the angle $\theta$ $(0 \le \theta \le \pi)$ between two nonzero vectors $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ in $R^2$, apply the Law of Cosines to the triangle shown in Figure 5.5 to obtain

$$\|\mathbf{v} - \mathbf{u}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\,\|\mathbf{v}\| \cos \theta.$$




Angle Between Two Vectors

**Figure 5.5**

Expanding and solving for $\cos \theta$ yields

$$\cos \theta = \frac{u_1 v_1 + u_2 v_2}{\|\mathbf{u}\|\,\|\mathbf{v}\|}.$$

The numerator of the quotient above is defined as the **dot product** of **u** and **v** and is denoted by

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2.$$

The following generalizes this definition to $R^n$.

### REMARK

Notice that the dot product of two vectors is a scalar, not another vector.

### Definition of Dot Product in $R^n$

The **dot product** of $\mathbf{u} = (u_1, u_2, . . . , u_n)$ and $\mathbf{v} = (v_1, v_2, . . . , v_n)$ is the *scalar* quantity

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdot \cdot \cdot + u_n v_n.$$

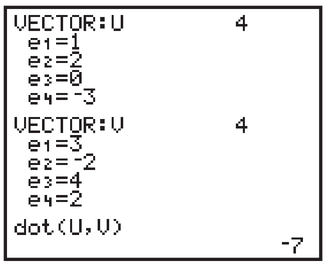
### EXAMPLE 4 Finding the Dot Product of Two Vectors

The dot product of $\mathbf{u} = (1, 2, 0, -3)$ and $\mathbf{v} = (3, -2, 4, 2)$ is

$$\mathbf{u} \cdot \mathbf{v} = (1)(3) + (2)(-2) + (0)(4) + (-3)(2) = -7.$$

### TECHNOLOGY

You can use a graphing utility or software program to find the dot product of two vectors. If you use a graphing utility, then you may verify Example 4 as follows.




The **Online Technology Guide,** available at *www.cengagebrain.com,* provides syntax for programs applicable to Example 4.

### THEOREM 5.3 Properties of the Dot Product

If **u**, **v**, and **w** are vectors in $R^n$ and $c$ is a scalar, then the following properties are true.

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
3. $c(\mathbf{u} \cdot \mathbf{v}) = (c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v})$
4. $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$
5. $\mathbf{v} \cdot \mathbf{v} \ge 0$, and $\mathbf{v} \cdot \mathbf{v} = 0$ if and only if $\mathbf{v} = \mathbf{0}$.

### PROOF

The proofs of these properties follow from the definition of dot product. For example, to prove the first property, you can write

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= u_1 v_1 + u_2 v_2 + \cdot \cdot \cdot + u_n v_n \\ &= v_1 u_1 + v_2 u_2 + \cdot \cdot \cdot + v_n u_n \\ &= \mathbf{v} \cdot \mathbf{u}. \end{aligned}$$

In Section 4.1, $R^n$ was defined as the set of all ordered $n$-tuples of real numbers. When $R^n$ is combined with the standard operations of vector addition, scalar multiplication, vector length, and the dot product, the resulting vector space is called **Euclidean $n$-space.** In the remainder of this text, unless stated otherwise, assume that $R^n$ has the standard Euclidean operations.

## EXAMPLE 5 Finding Dot Products

Let $\mathbf{u} = (2, -2)$, $\mathbf{v} = (5, 8)$, and $\mathbf{w} = (-4, 3)$. Find each quantity.

**a.** $\mathbf{u} \cdot \mathbf{v}$

**b.** $(\mathbf{u} \cdot \mathbf{v})\mathbf{w}$

**c.** $\mathbf{u} \cdot (2\mathbf{v})$

**d.** $\|\mathbf{w}\|^2$

**e.** $\mathbf{u} \cdot (\mathbf{v} - 2\mathbf{w})$

### SOLUTION

**a.** By definition, you have

$$\mathbf{u} \cdot \mathbf{v} = 2(5) + (-2)(8) = -6.$$

**b.** Using the result in part (a), you have

$$(\mathbf{u} \cdot \mathbf{v})\mathbf{w} = -6\mathbf{w} = -6(-4, 3) = (24, -18).$$

**c.** By Property 3 of Theorem 5.3, you have

$$\mathbf{u} \cdot (2\mathbf{v}) = 2(\mathbf{u} \cdot \mathbf{v}) = 2(-6) = -12.$$

**d.** By Property 4 of Theorem 5.3, you have

$$\|\mathbf{w}\|^2 = \mathbf{w} \cdot \mathbf{w} = (-4)(-4) + (3)(3) = 25.$$

**e.** Because $2\mathbf{w} = (-8, 6)$, you have

$$\mathbf{v} - 2\mathbf{w} = (5 - (-8), 8 - 6) = (13, 2).$$

Consequently,

$$\mathbf{u} \cdot (\mathbf{v} - 2\mathbf{w}) = 2(13) + (-2)(2) = 26 - 4 = 22.$$

## EXAMPLE 6 Using Properties of the Dot Product

Given two vectors $\mathbf{u}$ and $\mathbf{v}$ in $R^n$ such that $\mathbf{u} \cdot \mathbf{u} = 39$, $\mathbf{u} \cdot \mathbf{v} = -3$, and $\mathbf{v} \cdot \mathbf{v} = 79$, evaluate $(\mathbf{u} + 2\mathbf{v}) \cdot (3\mathbf{u} + \mathbf{v})$.

### SOLUTION

Using Theorem 5.3, rewrite the dot product as

$$\begin{aligned}(\mathbf{u} + 2\mathbf{v}) \cdot (3\mathbf{u} + \mathbf{v}) &= \mathbf{u} \cdot (3\mathbf{u} + \mathbf{v}) + (2\mathbf{v}) \cdot (3\mathbf{u} + \mathbf{v}) \\ &= \mathbf{u} \cdot (3\mathbf{u}) + \mathbf{u} \cdot \mathbf{v} + (2\mathbf{v}) \cdot (3\mathbf{u}) + (2\mathbf{v}) \cdot \mathbf{v} \\ &= 3(\mathbf{u} \cdot \mathbf{u}) + \mathbf{u} \cdot \mathbf{v} + 6(\mathbf{v} \cdot \mathbf{u}) + 2(\mathbf{v} \cdot \mathbf{v}) \\ &= 3(\mathbf{u} \cdot \mathbf{u}) + 7(\mathbf{u} \cdot \mathbf{v}) + 2(\mathbf{v} \cdot \mathbf{v}) \\ &= 3(39) + 7(-3) + 2(79) \\ &= 254.\end{aligned}$$

**DISCOVERY**

1. Let $\mathbf{u} = (1, 1)$ and $\mathbf{v} = (-4, -3)$. Calculate $\mathbf{u} \cdot \mathbf{v}$ and $\|\mathbf{u}\|\,\|\mathbf{v}\|$.
2. Repeat this experiment with other choices for $\mathbf{u}$ and $\mathbf{v}$.
3. Formulate a conjecture about the relationship between the dot product of two vectors and the product of their lengths.

To define the angle $\theta$ between two vectors $\mathbf{u}$ and $\mathbf{v}$ in $R^n$, use the formula in $R^2$

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}.$$

For such a definition to make sense, however, the absolute value of the right-hand side of this formula cannot exceed 1. This fact comes from a famous theorem named after the French mathematician Augustin-Louis Cauchy (1789–1857) and the German mathematician Hermann Schwarz (1843–1921).

### THEOREM 5.4 The Cauchy-Schwarz Inequality

If $\mathbf{u}$ and $\mathbf{v}$ are vectors in $R^n$, then

$$|\mathbf{u} \cdot \mathbf{v}| \le \|\mathbf{u}\|\,\|\mathbf{v}\|$$

where $|\mathbf{u} \cdot \mathbf{v}|$ denotes the *absolute value* of $\mathbf{u} \cdot \mathbf{v}$.

**PROOF**

*Case 1.* If $\mathbf{u} = \mathbf{0}$, then it follows that $|\mathbf{u} \cdot \mathbf{v}| = |\mathbf{0} \cdot \mathbf{v}| = 0$ and $\|\mathbf{u}\|\,\|\mathbf{v}\| = 0\|\mathbf{v}\| = 0$. So, the theorem is true when $\mathbf{u} = \mathbf{0}$.

*Case 2.* When $\mathbf{u} \neq \mathbf{0}$, let $t$ be any real number and consider the vector $t\mathbf{u} + \mathbf{v}$. Because $(t\mathbf{u} + \mathbf{v}) \cdot (t\mathbf{u} + \mathbf{v}) \ge 0$, it follows that

$$(t\mathbf{u} + \mathbf{v}) \cdot (t\mathbf{u} + \mathbf{v}) = t^2(\mathbf{u} \cdot \mathbf{u}) + 2t(\mathbf{u} \cdot \mathbf{v}) + \mathbf{v} \cdot \mathbf{v} \ge 0.$$

Now, let $a = \mathbf{u} \cdot \mathbf{u}$, $b = 2(\mathbf{u} \cdot \mathbf{v})$, and $c = \mathbf{v} \cdot \mathbf{v}$ to obtain the quadratic inequality $at^2 + bt + c \ge 0$. Because this quadratic is never negative, it has either no real roots or a single repeated real root. But by the Quadratic Formula, this implies that the discriminant, $b^2 - 4ac$, is less than or equal to zero.

$$\begin{aligned} b^2 - 4ac &\le 0 \\ b^2 &\le 4ac \\ 4(\mathbf{u} \cdot \mathbf{v})^2 &\le 4(\mathbf{u} \cdot \mathbf{u})(\mathbf{v} \cdot \mathbf{v}) \\ (\mathbf{u} \cdot \mathbf{v})^2 &\le (\mathbf{u} \cdot \mathbf{u})(\mathbf{v} \cdot \mathbf{v}) \end{aligned}$$

Taking the square roots of both sides produces

$$|\mathbf{u} \cdot \mathbf{v}| \le \sqrt{\mathbf{u} \cdot \mathbf{u}}\,\sqrt{\mathbf{v} \cdot \mathbf{v}} = \|\mathbf{u}\|\,\|\mathbf{v}\|.$$

### EXAMPLE 7 Verifying the Cauchy-Schwarz Inequality

Verify the Cauchy-Schwarz Inequality for $\mathbf{u} = (1, -1, 3)$ and $\mathbf{v} = (2, 0, -1)$.

**SOLUTION**

Because $\mathbf{u} \cdot \mathbf{v} = -1$, $\mathbf{u} \cdot \mathbf{u} = 11$, and $\mathbf{v} \cdot \mathbf{v} = 5$, you have

$$|\mathbf{u} \cdot \mathbf{v}| = |-1| = 1$$

and

$$\begin{aligned} \|\mathbf{u}\|\,\|\mathbf{v}\| &= \sqrt{\mathbf{u} \cdot \mathbf{u}}\,\sqrt{\mathbf{v} \cdot \mathbf{v}} \\ &= \sqrt{11}\sqrt{5} \\ &= \sqrt{55}. \end{aligned}$$

The inequality $|\mathbf{u} \cdot \mathbf{v}| \le \|\mathbf{u}\|\,\|\mathbf{v}\|$ holds, because $1 \le \sqrt{55}$.

The Cauchy-Schwarz Inequality leads to the definition of the angle between two nonzero vectors in $R^n$.

**REMARK**

The angle between the zero vector and another vector is not defined.

### Definition of the Angle Between Two Vectors in $R^n$

The **angle** $\theta$ between two nonzero vectors in $R^n$ is given by

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}, \quad 0 \le \theta \le \pi.$$

## EXAMPLE 8 Finding the Angle Between Two Vectors

The angle between $\mathbf{u} = (-4, 0, 2, -2)$ and $\mathbf{v} = (2, 0, -1, 1)$ is

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \frac{-12}{\sqrt{24}\,\sqrt{6}} = -\frac{12}{\sqrt{144}} = -1.$$

Consequently, $\theta = \pi$. It makes sense that $\mathbf{u}$ and $\mathbf{v}$ should have opposite directions, because $\mathbf{u} = -2\mathbf{v}$.

Note that because $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$ are always positive, $\mathbf{u}\cdot\mathbf{v}$ and $\cos\theta$ will always have the same sign. Moreover, because the cosine is positive in the first quadrant and negative in the second quadrant, the sign of the dot product of two vectors can be used to determine whether the angle between them is acute or obtuse, as shown in Figure 5.6.

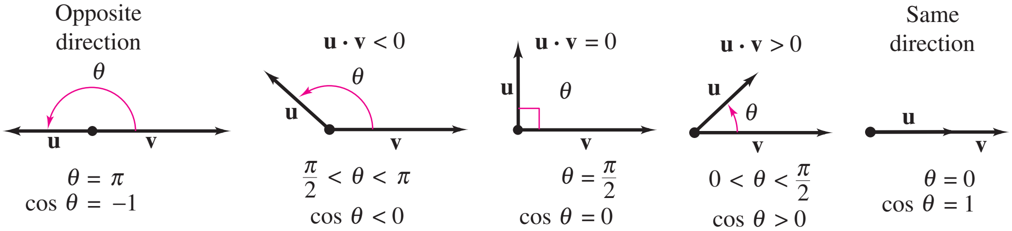



**Figure 5.6**

**REMARK**

Even though the angle between the zero vector and another vector is not defined, it is convenient to extend the definition of orthogonality to include the zero vector. In other words, the vector $\mathbf{0}$ is said to be orthogonal to every vector.

Figure 5.6 shows that two nonzero vectors meet at a right angle if and only if their dot product is zero. Two such vectors are said to be **orthogonal** (or perpendicular).

### Definition of Orthogonal Vectors

Two vectors $\mathbf{u}$ and $\mathbf{v}$ in $R^n$ are **orthogonal** when

$$\mathbf{u}\cdot\mathbf{v} = 0.$$

## EXAMPLE 9 Orthogonal Vectors in $R^n$

**a.** The vectors $\mathbf{u} = (1, 0, 0)$ and $\mathbf{v} = (0, 1, 0)$ are orthogonal because

$$\mathbf{u}\cdot\mathbf{v} = (1)(0) + (0)(1) + (0)(0) = 0.$$

**b.** The vectors $\mathbf{u} = (3, 2, -1, 4)$ and $\mathbf{v} = (1, -1, 1, 0)$ are orthogonal because

$$\mathbf{u}\cdot\mathbf{v} = (3)(1) + (2)(-1) + (-1)(1) + (4)(0) = 0.$$

## EXAMPLE 10 Finding Orthogonal Vectors

Determine all vectors in $R^2$ that are orthogonal to $\mathbf{u} = (4, 2)$.

### SOLUTION

Let $\mathbf{v} = (v_1, v_2)$ be orthogonal to $\mathbf{u}$. Then

$$\mathbf{u}\cdot\mathbf{v} = (4, 2)\cdot(v_1, v_2) = 4v_1 + 2v_2 = 0$$

which implies that $2v_2 = -4v_1$ and $v_2 = -2v_1$. So, every vector that is orthogonal to $(4, 2)$ is of the form

$$\mathbf{v} = (t, -2t) = t(1, -2)$$

where $t$ is a real number. (See Figure 5.7.)

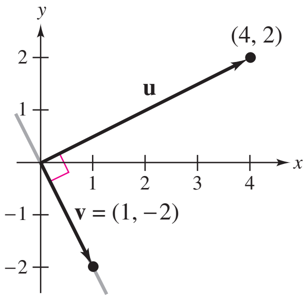



**Figure 5.7**

The Cauchy-Schwarz Inequality can be used to prove another well-known inequality called the **triangle inequality** (Theorem 5.5 below). The name "triangle inequality" is derived from the interpretation of the theorem in $R^2$, illustrated for the vectors $\mathbf{u}$ and $\mathbf{v}$ in Figure 5.8(a). When you consider

$$\|\mathbf{u}\| \text{ and } \|\mathbf{v}\|$$

to be the lengths of two sides of a triangle, you can see that the length of the third side is

$$\|\mathbf{u} + \mathbf{v}\|.$$

Moreover, because the length of any side of a triangle cannot be greater than the sum of the lengths of the other two sides, you have

$$\|\mathbf{u} + \mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|.$$

Figure 5.8(b) illustrates the triangle inequality for the vectors $\mathbf{u}$ and $\mathbf{v}$ in $R^3$. The following theorem generalizes these results to $R^n$.

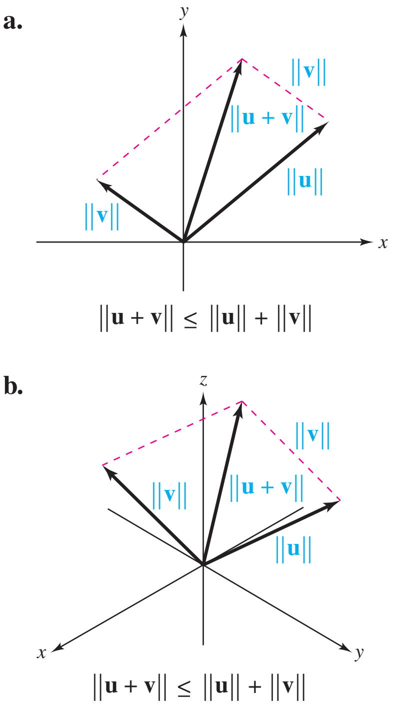



**Figure 5.8**

### THEOREM 5.5 The Triangle Inequality

If $\mathbf{u}$ and $\mathbf{v}$ are vectors in $R^n$, then

$$\|\mathbf{u} + \mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|.$$

### PROOF

Using the properties of the dot product, you have

$$\begin{aligned}\|\mathbf{u} + \mathbf{v}\|^2 &= (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) \\ &= \mathbf{u} \cdot (\mathbf{u} + \mathbf{v}) + \mathbf{v} \cdot (\mathbf{u} + \mathbf{v}) \\ &= \mathbf{u} \cdot \mathbf{u} + 2(\mathbf{u} \cdot \mathbf{v}) + \mathbf{v} \cdot \mathbf{v} \\ &= \|\mathbf{u}\|^2 + 2(\mathbf{u} \cdot \mathbf{v}) + \|\mathbf{v}\|^2 \\ &\le \|\mathbf{u}\|^2 + 2|\mathbf{u} \cdot \mathbf{v}| + \|\mathbf{v}\|^2.\end{aligned}$$

#### REMARK

Equality occurs in the triangle inequality if and only if the vectors **u** and **v** have the same direction. (See Exercise 84.)

Now, by the Cauchy-Schwarz Inequality, $|\mathbf{u} \cdot \mathbf{v}| \le \|\mathbf{u}\|\,\|\mathbf{v}\|$, and

$$\begin{aligned}\|\mathbf{u} + \mathbf{v}\|^2 &\le \|\mathbf{u}\|^2 + 2|\mathbf{u} \cdot \mathbf{v}| + \|\mathbf{v}\|^2 \\ &\le \|\mathbf{u}\|^2 + 2\|\mathbf{u}\|\,\|\mathbf{v}\| + \|\mathbf{v}\|^2 \\ &= (\|\mathbf{u}\| + \|\mathbf{v}\|)^2.\end{aligned}$$

Because both $\|\mathbf{u} + \mathbf{v}\|$ and $(\|\mathbf{u}\| + \|\mathbf{v}\|)$ are nonnegative, taking the square roots of both sides yields

$$\|\mathbf{u} + \mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|.$$

From the proof of the triangle inequality, you have

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + 2(\mathbf{u} \cdot \mathbf{v}) + \|\mathbf{v}\|^2.$$

If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, then $\mathbf{u} \cdot \mathbf{v} = 0$, and you have the extension of the **Pythagorean Theorem** to $R^n$, as follows.

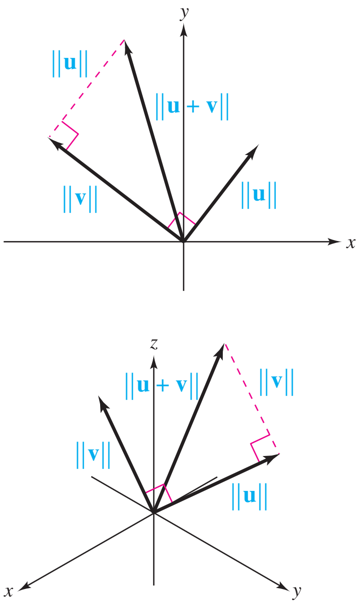



**Figure 5.9**

### THEOREM 5.6 The Pythagorean Theorem

If $\mathbf{u}$ and $\mathbf{v}$ are vectors in $R^n$, then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2.$$

Figure 5.9 illustrates this relationship graphically for $R^2$ and $R^3$.

## THE DOT PRODUCT AND MATRIX MULTIPLICATION

It is often useful to represent a vector $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ in $R^n$ as an $n \times 1$ column matrix. In this notation, the dot product of two vectors

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

can be represented as the matrix product of the transpose of $\mathbf{u}$ multiplied by $\mathbf{v}$.

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T\mathbf{v} = [u_1 \ u_2 \ \ldots \ u_n] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = [u_1v_1 + u_2v_2 + \cdots + u_nv_n]$$

### EXAMPLE 11 Using Matrix Multiplication to Find Dot Products

**a.** The dot product of the vectors

$$\mathbf{u} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

is $\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T\mathbf{v} = [2 \ \ 0]\begin{bmatrix} 3 \\ 1 \end{bmatrix} = [(2)(3) + (0)(1)] = 6.$

**b.** The dot product of the vectors

$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$$

is $\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T\mathbf{v} = [1 \quad 2 \quad -1]\begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix} = [(1)(3) + (2)(-2) + (-1)(4)] = -5.$

Many of the properties of the dot product are direct consequences of the corresponding properties of matrix multiplication. In Exercise 85, you are asked to use the properties of matrix multiplication to prove the first three properties of Theorem 5.3.

**LINEAR ALGEBRA APPLIED**

Electrical engineers can use the dot product to calculate electric or magnetic *flux*, which is a measure of the strength of the electric or magnetic field penetrating a surface. Consider an arbitrarily shaped surface with an element of area $dA$, normal (perpendicular) vector $d\mathbf{A}$, electric field vector $\mathbf{E}$, and magnetic field vector $\mathbf{B}$. The electric flux $\Phi_e$ can be found using the surface integral $\Phi_e = \int \mathbf{E} \cdot d\mathbf{A}$ and the magnetic flux $\Phi_m$ can be found using the surface integral $\Phi_m = \int \mathbf{B} \cdot d\mathbf{A}$. It is interesting to note that for a given closed surface that surrounds an electrical charge, the net electric flux is proportional to the charge, but the net magnetic flux is zero. This is because electric fields initiate at positive charges and terminate at negative charges, but magnetic fields form closed loops, so they do not initiate or terminate at any point. This means that the magnetic field entering a closed surface must equal the magnetic field leaving the closed surface.

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# 5.1 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

**Finding the Length of a Vector** In Exercises 1–4, find the length of the vector.

1. $\mathbf{v} = (4, 3)$
2. $\mathbf{v} = (0, 1)$
3. $\mathbf{v} = (1, 2, 2)$
4. $\mathbf{v} = (2, 0, -5, 5)$

**Finding the Length of a Vector** In Exercises 5–8, find (a) $\|\mathbf{u}\|$, (b) $\|\mathbf{v}\|$, and (c) $\|\mathbf{u} + \mathbf{v}\|$.

5. $\mathbf{u} = \left(-1, \frac{1}{4}\right)$, $\mathbf{v} = \left(4, -\frac{1}{8}\right)$
6. $\mathbf{u} = \left(1, \frac{1}{2}\right)$, $\mathbf{v} = \left(2, -\frac{1}{2}\right)$
7. $\mathbf{u} = (1, 2, 1)$, $\mathbf{v} = (0, 2, -2)$
8. $\mathbf{u} = (0, 1, -1, 2)$, $\mathbf{v} = (1, 1, 3, 0)$

**Finding a Unit Vector** In Exercises 9–12, find a unit vector (a) in the direction of u and (b) in the direction opposite that of u.

9. $\mathbf{u} = (-5, 12)$
10. $\mathbf{u} = (1, -1)$
11. $\mathbf{u} = (3, 2, -5)$
12. $\mathbf{u} = (-1, 3, 4)$

**Finding a Vector** In Exercises 13–16, find the vector v with the given length and the same direction as u.

13. $\|\mathbf{v}\| = 4$, $\mathbf{u} = (1, 1)$
14. $\|\mathbf{v}\| = 4$, $\mathbf{u} = (-1, 1)$
15. $\|\mathbf{v}\| = 2$, $\mathbf{u} = \left(\sqrt{3}, 3, 0\right)$
16. $\|\mathbf{v}\| = 3$, $\mathbf{u} = (0, 2, 1, -1)$

17. Given the vector $\mathbf{v} = (-1, 3, 0, 4)$, find $\mathbf{u}$ such that
    (a) $\mathbf{u}$ has the same direction as $\mathbf{v}$ and one-half its length.
    (b) $\mathbf{u}$ has the direction opposite that of $\mathbf{v}$ and twice its length.
18. For what values of $c$ is $\|c(1, 2, 3)\| = 1$?

**Finding the Distance Between Two Vectors** In Exercises 19–22, find the distance between u and v.

19. $\mathbf{u} = (1, -1)$, $\mathbf{v} = (-1, 1)$
20. $\mathbf{u} = (1, 1, 2)$, $\mathbf{v} = (-1, 3, 0)$
21. $\mathbf{u} = (1, 2, 0)$, $\mathbf{v} = (-1, 4, 1)$
22. $\mathbf{u} = (0, 1, -1, 2)$, $\mathbf{v} = (1, 1, 2, 2)$

**Finding Dot Products** In Exercises 23–26, find (a) $\mathbf{u} \cdot \mathbf{v}$, (b) $\mathbf{v} \cdot \mathbf{v}$, (c) $\|\mathbf{u}\|^2$, (d) $(\mathbf{u} \cdot \mathbf{v})\mathbf{v}$, and (e) $\mathbf{u} \cdot (5\mathbf{v})$.

23. $\mathbf{u} = (3, 4)$, $\mathbf{v} = (2, -3)$
24. $\mathbf{u} = (-1, 2)$, $\mathbf{v} = (2, -2)$
25. $\mathbf{u} = (-1, 1, -2)$, $\mathbf{v} = (1, -3, -2)$
26. $\mathbf{u} = (4, 0, -3, 5)$, $\mathbf{v} = (0, 2, 5, 4)$

27. Find $(\mathbf{u} + \mathbf{v}) \cdot (2\mathbf{u} - \mathbf{v})$, given that $\mathbf{u} \cdot \mathbf{u} = 4$, $\mathbf{u} \cdot \mathbf{v} = -5$, and $\mathbf{v} \cdot \mathbf{v} = 10$.
28. Find $(3\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - 3\mathbf{v})$, given that $\mathbf{u} \cdot \mathbf{u} = 8$, $\mathbf{u} \cdot \mathbf{v} = 7$, and $\mathbf{v} \cdot \mathbf{v} = 6$.

**Finding Norms, Unit Vectors, and Dot Products** In Exercises 29–32, use a software program or a graphing utility with vector capabilities to find (a) the norms of u and v, (b) a unit vector in the direction of v, (c) a unit vector in the direction opposite that of u, (d) $\mathbf{u} \cdot \mathbf{v}$, (e) $\mathbf{u} \cdot \mathbf{u}$, and (f) $\mathbf{v} \cdot \mathbf{v}$.

29. $\mathbf{u} = \left(1, \frac{1}{8}, \frac{2}{5}\right)$, $\mathbf{v} = \left(0, \frac{1}{4}, \frac{1}{5}\right)$
30. $\mathbf{u} = \left(-1, \frac{1}{2}, \frac{1}{4}\right)$, $\mathbf{v} = \left(0, \frac{1}{4}, -\frac{1}{2}\right)$
31. $\mathbf{u} = \left(0, 1, \sqrt{2}\right)$, $\mathbf{v} = \left(-1, \sqrt{2}, -1\right)$
32. $\mathbf{u} = \left(-1, \sqrt{3}, 2\right)$, $\mathbf{v} = \left(\sqrt{2}, -1, -\sqrt{2}\right)$

**Verifying the Cauchy-Schwarz Inequality** In Exercises 33–36, verify the Cauchy-Schwarz Inequality for the given vectors.

33. $\mathbf{u} = (3, 4)$, $\mathbf{v} = (2, -3)$
34. $\mathbf{u} = (-1, 0)$, $\mathbf{v} = (1, 1)$
35. $\mathbf{u} = (1, 1, -2)$, $\mathbf{v} = (1, -3, -2)$
36. $\mathbf{u} = (1, -1, 0)$, $\mathbf{v} = (0, 1, -1)$

**Finding the Angle Between Two Vectors** In Exercises 37–44, find the angle $\theta$ between the vectors.

37. $\mathbf{u} = (3, 1)$, $\mathbf{v} = (-2, 4)$
38. $\mathbf{u} = (2, -1)$, $\mathbf{v} = (2, 0)$
39. $\mathbf{u} = \left(\cos\dfrac{\pi}{6}, \sin\dfrac{\pi}{6}\right)$, $\mathbf{v} = \left(\cos\dfrac{3\pi}{4}, \sin\dfrac{3\pi}{4}\right)$
40. $\mathbf{u} = \left(\cos\dfrac{\pi}{3}, \sin\dfrac{\pi}{3}\right)$, $\mathbf{v} = \left(\cos\dfrac{\pi}{4}, \sin\dfrac{\pi}{4}\right)$
41. $\mathbf{u} = (1, 1, 1)$, $\mathbf{v} = (2, 1, -1)$
42. $\mathbf{u} = (2, 3, 1)$, $\mathbf{v} = (-3, 2, 0)$
43. $\mathbf{u} = (0, 1, 0, 1)$, $\mathbf{v} = (3, 3, 3, 3)$
44. $\mathbf{u} = (1, -1, 0, 1)$, $\mathbf{v} = (-1, 2, -1, 0)$

**Determining a Relationship Between Two Vectors** In Exercises 45–52, determine whether u and v are orthogonal, parallel, or neither.

45. $\mathbf{u} = (2, 18)$, $\mathbf{v} = \left(\frac{3}{2}, -\frac{1}{6}\right)$
46. $\mathbf{u} = (4, 3)$, $\mathbf{v} = \left(\frac{1}{2}, -\frac{2}{3}\right)$
47. $\mathbf{u} = \left(-\frac{1}{3}, \frac{2}{3}\right)$, $\mathbf{v} = (2, -4)$
48. $\mathbf{u} = (1, -1)$, $\mathbf{v} = (0, -1)$
49. $\mathbf{u} = (0, 1, 0)$, $\mathbf{v} = (1, -2, 0)$
50. $\mathbf{u} = (0, 1, 6)$, $\mathbf{v} = (1, -2, -1)$
51. $\mathbf{u} = (-2, 5, 1, 0)$, $\mathbf{v} = \left(\frac{1}{4}, -\frac{5}{4}, 0, 1\right)$
52. $\mathbf{u} = \left(4, \frac{3}{2}, -1, \frac{1}{2}\right)$, $\mathbf{v} = \left(-2, -\frac{3}{4}, \frac{1}{2}, -\frac{1}{4}\right)$

**Finding Orthogonal Vectors** **In Exercises 53–56, determine all vectors v that are orthogonal to u.**

**53.** $\mathbf{u} = (0, 5)$ **54.** $\mathbf{u} = (2, 7)$

**55.** $\mathbf{u} = (2, -1, 1)$ **56.** $\mathbf{u} = (4, -1, 0)$

**Verifying the Triangle Inequality** **In Exercises 57–60, verify the triangle inequality for the vectors u and v.**

**57.** $\mathbf{u} = (4, 0),\ \mathbf{v} = (1, 1)$ **58.** $\mathbf{u} = (-1, 1),\ \mathbf{v} = (2, 0)$

**59.** $\mathbf{u} = (1, 1, 1),\ \mathbf{v} = (0, 1, -2)$

**60.** $\mathbf{u} = (1, -1, 0),\ \mathbf{v} = (0, 1, 2)$

**Verifying the Pythagorean Theorem** **In Exercises 61–64, verify the Pythagorean Theorem for the vectors u and v.**

**61.** $\mathbf{u} = (1, -1),\quad \mathbf{v} = (1, 1)$

**62.** $\mathbf{u} = (3, -2),\quad \mathbf{v} = (4, 6)$

**63.** $\mathbf{u} = (3, 4, -2),\quad \mathbf{v} = (4, -3, 0)$

**64.** $\mathbf{u} = (4, 1, -5),\quad \mathbf{v} = (2, -3, 1)$

**65.** Rework Exercise 23 using matrix multiplication.

**66.** Rework Exercise 24 using matrix multiplication.

**67.** Rework Exercise 25 using matrix multiplication.

**68.** Rework Exercise 26 using matrix multiplication.

**Writing** **In Exercises 69 and 70, determine whether the vectors are orthogonal, parallel, or neither. Then explain your reasoning.**

**69.** $\mathbf{u} = (\cos\theta, \sin\theta, -1),\quad \mathbf{v} = (\sin\theta, -\cos\theta, 0)$

**70.** $\mathbf{u} = (-\sin\theta, \cos\theta, 1),\quad \mathbf{v} = (\sin\theta, -\cos\theta, 0)$

**True or False?** **In Exercises 71 and 72, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.**

**71.** (a) The length or norm of a vector is

$$\|\mathbf{v}\| = |v_1 + v_2 + v_3 + \cdot \cdot \cdot + v_n|.$$

(b) The dot product of two vectors **u** and **v** is another vector represented by

$$\mathbf{u} \cdot \mathbf{v} = (u_1v_1, u_2v_2, u_3v_3, \ldots, u_nv_n).$$

**72.** (a) If **v** is a nonzero vector in $R^n$, then the unit vector in the direction of **v** is $\mathbf{u} = \|\mathbf{v}\|/\mathbf{v}$.

(b) If $\mathbf{u} \cdot \mathbf{v} < 0$, then the angle $\theta$ between **u** and **v** is acute.

**Writing** **In Exercises 73 and 74, explain why each expression involving dot product(s) is meaningless. Assume that u and v are vectors in $R^n$, and that $c$ is a scalar.**

**73.** (a) $(\mathbf{u} \cdot \mathbf{v}) - \mathbf{v}$ (b) $\mathbf{u} + (\mathbf{u} \cdot \mathbf{v})$

**74.** (a) $(\mathbf{u} \cdot \mathbf{v}) \cdot \mathbf{u}$ (b) $c \cdot (\mathbf{u} \cdot \mathbf{v})$

**Orthogonal Vectors** **In Exercises 75 and 76, let $\mathbf{v} = (v_1, v_2)$ be a vector in $R^2$. Show that $(v_2, -v_1)$ is orthogonal to v, and use this fact to find two unit vectors orthogonal to the given vector.**

**75.** $\mathbf{v} = (12, 5)$ **76.** $\mathbf{v} = (8, 15)$

**77. Revenue** The vector $\mathbf{u} = (3140, 2750)$ gives the numbers of hamburgers and hot dogs, respectively, sold at a fast-food stand in one month. The vector $\mathbf{v} = (2.25, 1.75)$ gives the prices (in dollars) of the food items. Find the dot product $\mathbf{u} \cdot \mathbf{v}$ and interpret the result in the context of the problem.

**78. Revenue** The vector $\mathbf{u} = (4600, 4290, 5250)$ gives the numbers of units of three models of cellular phones produced by a telecommunications company. The vector $\mathbf{v} = (79.99, 89.99, 99.99)$ gives the prices in dollars of the three models of cellular phones, respectively. Find the dot product $\mathbf{u} \cdot \mathbf{v}$ and interpret the result in the context of the problem.

**79.** Find the angle between the diagonal of a cube and one of its edges.

**80.** Find the angle between the diagonal of a cube and the diagonal of one of its sides.

**81. Guided Proof** Prove that if **u** is orthogonal to **v** and **w**, then **u** is orthogonal to $c\mathbf{v} + d\mathbf{w}$ for any scalars $c$ and $d$.

Getting Started: To prove that **u** is orthogonal to $c\mathbf{v} + d\mathbf{w}$, you need to show that the dot product of **u** and $c\mathbf{v} + d\mathbf{w}$ is 0.

(i) Rewrite the dot product of **u** and $c\mathbf{v} + d\mathbf{w}$ as a linear combination of $(\mathbf{u} \cdot \mathbf{v})$ and $(\mathbf{u} \cdot \mathbf{w})$ using Properties 2 and 3 of Theorem 5.3.

(ii) Use the fact that **u** is orthogonal to **v** and **w**, and the result of part (i), to lead to the conclusion that **u** is orthogonal to $c\mathbf{v} + d\mathbf{w}$.

**82. Proof** Prove that if **u** and **v** are vectors in $R^n$, then

$$\mathbf{u} \cdot \mathbf{v} = \tfrac{1}{4}\|\mathbf{u} + \mathbf{v}\|^2 - \tfrac{1}{4}\|\mathbf{u} - \mathbf{v}\|^2.$$

**83. Proof** Prove that the vectors $\mathbf{u} = (\cos\theta, -\sin\theta)$ and $\mathbf{v} = (\sin\theta, \cos\theta)$ are orthogonal unit vectors for any value of $\theta$. Graph **u** and **v** for $\theta = \pi/3$.

**84. Proof** Prove that $\|\mathbf{u} + \mathbf{v}\| = \|\mathbf{u}\| + \|\mathbf{v}\|$ if and only if **u** and **v** have the same direction.

**85. Proof** Use the properties of matrix multiplication to prove the first three properties of Theorem 5.3.

**86. CAPSTONE** What is known about $\theta$, the angle between two nonzero vectors **u** and **v**, under each condition?

(a) $\mathbf{u} \cdot \mathbf{v} = 0$ (b) $\mathbf{u} \cdot \mathbf{v} > 0$ (c) $\mathbf{u} \cdot \mathbf{v} < 0$

**87. Writing** Let **x** be a solution to the $m \times n$ homogeneous linear system of equations $A\mathbf{x} = \mathbf{0}$. Explain why **x** is orthogonal to the row vectors of $A$.

# 5.2 Inner Product Spaces

- Determine whether a function defines an inner product, and find the inner product of two vectors in $R^n$, $M_{m,n}$, $P_n$, and $C[a, b]$.
- Find an orthogonal projection of a vector onto another vector in an inner product space.

## INNER PRODUCTS

In Section 5.1, the concepts of length, distance, and angle were extended from $R^2$ to $R^n$. This section extends these concepts one step further—to general vector spaces—by using the idea of an **inner product** of two vectors.

You already have one example of an inner product: the *dot product* in $R^n$. The dot product, called the **Euclidean inner product,** is only one of several inner products that can be defined on $R^n$. To distinguish between the standard inner product and other possible inner products, use the following notation.

$\mathbf{u} \cdot \mathbf{v} =$ dot product (Euclidean inner product for $R^n$)

$\langle \mathbf{u}, \mathbf{v} \rangle =$ general inner product for a vector space $V$

A general inner product is defined in much the same way that a general vector space is defined—that is, in order for a function to qualify as an inner product, it must satisfy a set of axioms. The following axioms parallel Properties 1, 2, 3, and 5 of the dot product given in Theorem 5.3.

### Definition of Inner Product

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in a vector space $V$, and let $c$ be any scalar. An **inner product** on $V$ is a function that associates a real number $\langle \mathbf{u}, \mathbf{v} \rangle$ with each pair of vectors $\mathbf{u}$ and $\mathbf{v}$ and satisfies the following axioms.

**1.** $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$
**2.** $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$
**3.** $c\langle \mathbf{u}, \mathbf{v} \rangle = \langle c\mathbf{u}, \mathbf{v} \rangle$
**4.** $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$, and $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ if and only if $\mathbf{v} = \mathbf{0}$.

A vector space $V$ with an inner product is called an **inner product space.** Whenever an inner product space is referred to, assume that the set of scalars is the set of real numbers.

### EXAMPLE 1 The Euclidean Inner Product for $R^n$

Show that the dot product in $R^n$ satisfies the four axioms of an inner product.

**SOLUTION**

In $R^n$, the dot product of two vectors $\mathbf{u} = (u_1, u_2, . . . , u_n)$ and $\mathbf{v} = (v_1, v_2, . . . , v_n)$ is

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + \cdot \cdot \cdot + u_nv_n.$$

By Theorem 5.3, you know that this dot product satisfies the required four axioms, which verifies that it is an inner product on $R^n$.

The Euclidean inner product is not the only inner product that can be defined on $R^n$. Example 2 illustrates a different inner product. To show that a function is an inner product, you must show that it satisfies the four inner product axioms.

## EXAMPLE 2 A Different Inner Product for $R^2$

Show that the following function defines an inner product on $R^2$, where $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$.

$$\langle \mathbf{u}, \mathbf{v} \rangle = u_1v_1 + 2u_2v_2$$

### SOLUTION

**1.** Because the product of real numbers is commutative,

$$\langle \mathbf{u}, \mathbf{v} \rangle = u_1v_1 + 2u_2v_2 = v_1u_1 + 2v_2u_2 = \langle \mathbf{v}, \mathbf{u} \rangle.$$

**2.** Let $\mathbf{w} = (w_1, w_2)$. Then

$$\begin{aligned}\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle &= u_1(v_1 + w_1) + 2u_2(v_2 + w_2)\\ &= u_1v_1 + u_1w_1 + 2u_2v_2 + 2u_2w_2\\ &= (u_1v_1 + 2u_2v_2) + (u_1w_1 + 2u_2w_2)\\ &= \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle.\end{aligned}$$

**3.** If $c$ is any scalar, then

$$c\langle \mathbf{u}, \mathbf{v} \rangle = c(u_1v_1 + 2u_2v_2) = (cu_1)v_1 + 2(cu_2)v_2 = \langle c\mathbf{u}, \mathbf{v} \rangle.$$

**4.** Because the square of a real number is nonnegative,

$$\langle \mathbf{v}, \mathbf{v} \rangle = v_1^2 + 2v_2^2 \geq 0.$$

Moreover, this expression is equal to zero if and only if $\mathbf{v} = \mathbf{0}$ (that is, if and only if $v_1 = v_2 = 0$).

Example 2 can be generalized. The function

$$\langle \mathbf{u}, \mathbf{v} \rangle = c_1u_1v_1 + c_2u_2v_2 + \cdot \cdot \cdot + c_nu_nv_n, \quad c_i > 0$$

is an inner product on $R^n$. (In Exercise 89, you are asked to prove this.) The positive constants $c_1, \ldots, c_n$ are called **weights.** If any $c_i$ is negative or 0, then this function does not define an inner product.

## EXAMPLE 3 A Function That Is Not an Inner Product

Show that the following function is not an inner product on $R^3$, where $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$.

$$\langle \mathbf{u}, \mathbf{v} \rangle = u_1v_1 - 2u_2v_2 + u_3v_3$$

### SOLUTION

Observe that Axiom 4 is not satisfied. For example, let $\mathbf{v} = (1, 2, 1)$. Then $\langle \mathbf{v}, \mathbf{v} \rangle = (1)(1) - 2(2)(2) + (1)(1) = -6$, which is less than zero.

## EXAMPLE 4 An Inner Product on $M_{2,2}$

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ be matrices in the vector space $M_{2,2}$. The function

$$\langle A, B \rangle = a_{11}b_{11} + a_{21}b_{21} + a_{12}b_{12} + a_{22}b_{22}$$

is an inner product on $M_{2,2}$. The verification of the four inner product axioms is left to you. (See Exercise 27.)

You obtain the inner product in the next example from calculus. The verification of the inner product properties depends on the properties of the definite integral.

**EXAMPLE 5** **An Inner Product Defined by a Definite Integral (Calculus)**

Let $f$ and $g$ be real-valued continuous functions in the vector space $C[a, b]$. Show that

$$\langle f, g \rangle = \int_a^b f(x)g(x)\, dx$$

defines an inner product on $C[a, b]$.

**SOLUTION**

Use familiar properties from calculus to verify the four parts of the definition.

**1.** $\langle f, g \rangle = \int_a^b f(x)g(x)\, dx = \int_a^b g(x)f(x)\, dx = \langle g, f \rangle$

**2.** $$\begin{aligned}\langle f, g + h \rangle &= \int_a^b f(x)[g(x) + h(x)]\, dx = \int_a^b [f(x)g(x) + f(x)h(x)]\, dx \\ &= \int_a^b f(x)g(x)\, dx + \int_a^b f(x)h(x)\, dx = \langle f, g \rangle + \langle f, h \rangle\end{aligned}$$

**3.** $c\langle f, g \rangle = c\int_a^b f(x)g(x)\, dx = \int_a^b cf(x)g(x)\, dx = \langle cf, g \rangle$

**4.** Because $[f(x)]^2 \geq 0$ for all $x$, you know from calculus that

$$\langle f, f \rangle = \int_a^b [f(x)]^2\, dx \geq 0$$

with

$$\langle f, f \rangle = \int_a^b [f(x)]^2\, dx = 0$$

if and only if $f$ is the zero function in $C[a, b]$, or when $a = b$.

The next theorem lists some properties of inner products.

**THEOREM 5.7** **Properties of Inner Products**

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in an inner product space $V$, and let $c$ be any real number.

**1.** $\langle \mathbf{0}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{0} \rangle = 0$
**2.** $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$
**3.** $\langle \mathbf{u}, c\mathbf{v} \rangle = c\langle \mathbf{u}, \mathbf{v} \rangle$

**PROOF**

The proof of the first property follows. The proofs of the other two properties are left as exercises. (See Exercises 91 and 92.) From the definition of an inner product, you know $\langle \mathbf{0}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{0} \rangle$, so you only need to show one of these to be zero. Using the fact that $0(\mathbf{v}) = \mathbf{0}$,

$$\begin{aligned}\langle \mathbf{0}, \mathbf{v} \rangle &= \langle 0(\mathbf{v}), \mathbf{v} \rangle \\ &= 0\langle \mathbf{v}, \mathbf{v} \rangle \\ &= 0.\end{aligned}$$

The definitions of length (or norm), distance, and angle for general inner product spaces closely parallel those for Euclidean $n$-space.

### Definitions of Length, Distance, and Angle

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in an inner product space $V$.

1. The **length** (or **norm**) of $\mathbf{u}$ is $\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle}$.
2. The **distance** between $\mathbf{u}$ and $\mathbf{v}$ is $d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$.
3. The **angle** between two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ is given by

$$\cos \theta = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\| \, \|\mathbf{v}\|}, \quad 0 \le \theta \le \pi.$$

4. $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal** when $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

If $\|\mathbf{u}\| = 1$, then $\mathbf{u}$ is called a **unit vector.** Moreover, if $\mathbf{v}$ is any nonzero vector in an inner product space $V$, then the vector $\mathbf{u} = \mathbf{v}/\|\mathbf{v}\|$ is a unit vector and is called the **unit vector in the direction of v.**

Note that the definition of the angle $\theta$ between $\mathbf{u}$ and $\mathbf{v}$ presumes that

$$-1 \le \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\| \, \|\mathbf{v}\|} \le 1$$

for a general inner product (as with Euclidean $n$-space), which follows from the Cauchy-Schwarz Inequality given later in Theorem 5.8.

### EXAMPLE 6 Finding Inner Products

For polynomials $p = a_0 + a_1x + \cdot \cdot \cdot + a_nx^n$ and $q = b_0 + b_1x + \cdot \cdot \cdot + b_nx^n$ in the vector space $P_n$, the function $\langle p, q \rangle = a_0b_0 + a_1b_1 + \cdot \cdot \cdot + a_nb_n$ is an inner product. (In Exercise 34, you are asked to show this.) Let $p(x) = 1 - 2x^2$, $q(x) = 4 - 2x + x^2$, and $r(x) = x + 2x^2$ be polynomials in $P_2$, and find each quantity.

**a.** $\langle p, q \rangle$ **b.** $\langle q, r \rangle$ **c.** $\|q\|$ **d.** $d(p, q)$

#### SOLUTION

**a.** The inner product of $p$ and $q$ is

$$\langle p, q \rangle = a_0b_0 + a_1b_1 + a_2b_2 = (1)(4) + (0)(-2) + (-2)(1) = 2.$$

**b.** The inner product of $q$ and $r$ is $\langle q, r \rangle = (4)(0) + (-2)(1) + (1)(2) = 0.$

Notice that the vectors $q$ and $r$ are orthogonal.

**c.** The length of $q$ is $\|q\| = \sqrt{\langle q, q \rangle} = \sqrt{4^2 + (-2)^2 + 1^2} = \sqrt{21}.$

**d.** The distance between $p$ and $q$ is

$$\begin{aligned} d(p, q) &= \|p - q\| \\ &= \|(1 - 2x^2) - (4 - 2x + x^2)\| \\ &= \|-3 + 2x - 3x^2\| \\ &= \sqrt{(-3)^2 + 2^2 + (-3)^2} \\ &= \sqrt{22}. \end{aligned}$$

Orthogonality depends on the inner product. That is, two vectors may be orthogonal with respect to one inner product but not to another. Try reworking Example 6 using the inner product $\langle p, q \rangle = a_0b_0 + a_1b_1 + 2a_2b_2$. With this inner product, $p$ and $q$ is an orthogonal pair, but $q$ and $r$ is not.

## EXAMPLE 7 Using the Inner Product on $C[0, 1]$ (Calculus)

Use the inner product defined in Example 5 and the functions $f(x) = x$ and $g(x) = x^2$ in $C[0, 1]$ to find each quantity.

**a.** $\|f\|$ **b.** $d(f, g)$

### SOLUTION

**a.** Because $f(x) = x$, you have

$$\|f\|^2 = \langle f, f \rangle = \int_0^1 (x)(x)\,dx = \int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}.$$

So, $\|f\| = \dfrac{1}{\sqrt{3}}$.

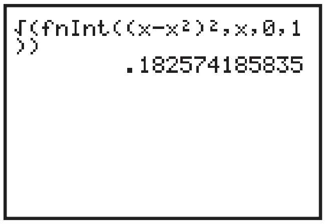
**b.** To find $d(f, g)$, write

$$\begin{aligned}[d(f, g)]^2 &= \langle f - g, f - g \rangle \\ &= \int_0^1 [f(x) - g(x)]^2\,dx = \int_0^1 [x - x^2]^2\,dx \\ &= \int_0^1 [x^2 - 2x^3 + x^4]\,dx = \left[\frac{x^3}{3} - \frac{x^4}{2} + \frac{x^5}{5}\right]_0^1 = \frac{1}{30}.\end{aligned}$$

So, $d(f, g) = \dfrac{1}{\sqrt{30}}$.

**TECHNOLOGY**

Many graphing utilities and software programs can approximate definite integrals. For instance, if you use a graphing utility, then you may verify Example 7(b), as follows.




The result should be approximately $0.183 \approx \dfrac{1}{\sqrt{30}}$.

The **Online Technology Guide,** available at *www.cengagebrain.com,* provides syntax for programs applicable to Example 7(b).

In Example 7, the distance between the functions $f(x) = x$ and $g(x) = x^2$ in $C[0, 1]$ is $1/\sqrt{30} \approx 0.183$. In practice, the distance between a pair of vectors is not as useful as the *relative* distance(s) between more than one pair. For instance, the distance between $g(x) = x^2$ and $h(x) = x^2 + 1$ in $C[0, 1]$ is 1. (Verify this.) From Figure 5.10, it seems reasonable to say that $f$ and $g$ are closer than $g$ and $h$.

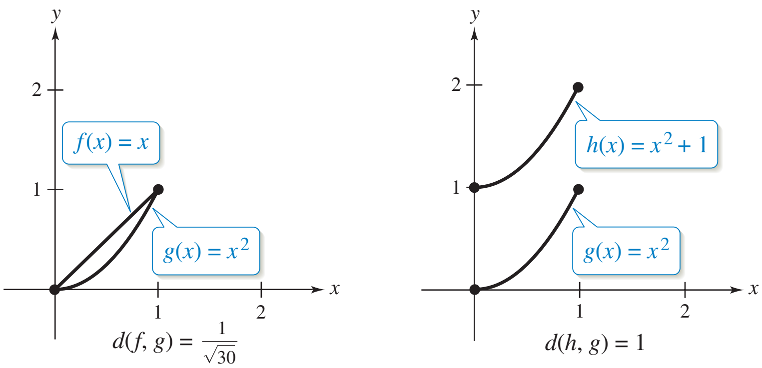



**Figure 5.10**

The properties of length and distance listed for $R^n$ in the preceding section also hold for general inner product spaces. For instance, if $\mathbf{u}$ and $\mathbf{v}$ are vectors in an inner product space, then the following properties are true.

| *Properties of Length* | *Properties of Distance* |
|---|---|
| **1.** $\|\mathbf{u}\| \geq 0$ | **1.** $d(\mathbf{u}, \mathbf{v}) \geq 0$ |
| **2.** $\|\mathbf{u}\| = 0$ if and only if $\mathbf{u} = \mathbf{0}$. | **2.** $d(\mathbf{u}, \mathbf{v}) = 0$ if and only if $\mathbf{u} = \mathbf{v}$. |
| **3.** $\|c\mathbf{u}\| = \lvert c \rvert\, \|\mathbf{u}\|$ | **3.** $d(\mathbf{u}, \mathbf{v}) = d(\mathbf{v}, \mathbf{u})$ |

Theorem 5.8 lists the general inner product space versions of the Cauchy-Schwarz Inequality, the triangle inequality, and the Pythagorean Theorem.

**THEOREM 5.8**

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in an inner product space $V$.

1. Cauchy-Schwarz Inequality: $|\langle \mathbf{u}, \mathbf{v}\rangle| \leq \|\mathbf{u}\|\,\|\mathbf{v}\|$
2. Triangle inequality: $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$
3. Pythagorean Theorem: $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2.$$

The proof of each part of Theorem 5.8 parallels the proofs of Theorems 5.4, 5.5, and 5.6, respectively. Simply substitute $\langle \mathbf{u}, \mathbf{v}\rangle$ for the Euclidean inner product $\mathbf{u} \cdot \mathbf{v}$ in each proof.

## EXAMPLE 8 An Example of the Cauchy-Schwarz Inequality (Calculus)

Let $f(x) = 1$ and $g(x) = x$ be functions in the vector space $C[0, 1]$, with the inner product defined in Example 5. Verify that $|\langle f, g\rangle| \leq \|f\|\,\|g\|$.

**SOLUTION**

For the left side of this inequality, you have

$$\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx = \int_0^1 x\,dx = \frac{x^2}{2}\bigg]_0^1 = \frac{1}{2}.$$

For the right side of the inequality, you have

$$\|f\|^2 = \int_0^1 f(x)f(x)\,dx = \int_0^1 dx = x\bigg]_0^1 = 1$$

and

$$\|g\|^2 = \int_0^1 g(x)g(x)\,dx = \int_0^1 x^2\,dx = \frac{x^3}{3}\bigg]_0^1 = \frac{1}{3}.$$

So,

$$\|f\|\,\|g\| = \sqrt{(1)\left(\frac{1}{3}\right)} = \frac{1}{\sqrt{3}} \approx 0.577, \text{ and } |\langle f, g\rangle| \leq \|f\|\,\|g\|.$$

**LINEAR ALGEBRA APPLIED**

The concept of work is important to scientists and engineers for determining the energy needed to perform various jobs. If a constant force $\mathbf{F}$ acts at an angle $\theta$ with the line of motion of an object to move the object from point $A$ to point $B$ (see figure below), then the work $W$ done by the force is given by

$$\begin{aligned} W &= (\cos\theta)\|\mathbf{F}\|\,\|\overrightarrow{AB}\| \\ &= \mathbf{F} \cdot \overrightarrow{AB} \end{aligned}$$

where $\overrightarrow{AB}$ represents the directed line segment from $A$ to $B$. The quantity $(\cos\theta)\|\mathbf{F}\|$ is the length of the *orthogonal projection* of $\mathbf{F}$ onto $\overrightarrow{AB}$. Orthogonal projections are discussed on the next page.

## ORTHOGONAL PROJECTIONS IN INNER PRODUCT SPACES

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $R^2$. If $\mathbf{v}$ is nonzero, then $\mathbf{u}$ can be orthogonally projected onto $\mathbf{v}$, as shown in Figure 5.11. This projection is denoted by $\text{proj}_{\mathbf{v}}\mathbf{u}$. Because $\text{proj}_{\mathbf{v}}\mathbf{u}$ is a scalar multiple of $\mathbf{v}$, you can write

$$\text{proj}_{\mathbf{v}}\mathbf{u} = a\mathbf{v}.$$

If $a > 0$, as shown in Figure 5.11(a), then $\cos\theta > 0$ and the length of $\text{proj}_{\mathbf{v}}\mathbf{u}$ is

$$\|a\mathbf{v}\| = |a|\,\|\mathbf{v}\| = a\|\mathbf{v}\| = \|\mathbf{u}\|\cos\theta = \frac{\|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta}{\|\mathbf{v}\|} = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|}$$

which implies that $a = (\mathbf{u}\cdot\mathbf{v})/\|\mathbf{v}\|^2 = (\mathbf{u}\cdot\mathbf{v})/(\mathbf{v}\cdot\mathbf{v})$. So,

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v}.$$

If $a < 0$, as shown in Figure 5.11(b), then the orthogonal projection of $\mathbf{u}$ onto $\mathbf{v}$ can be found using the same formula. (Verify this.)

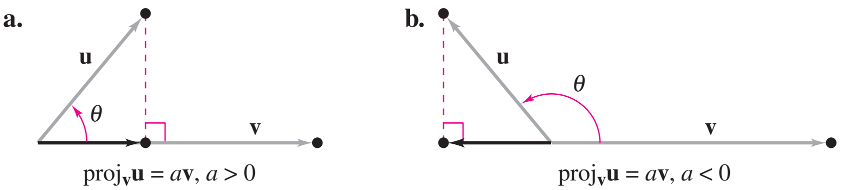



**Figure 5.11**

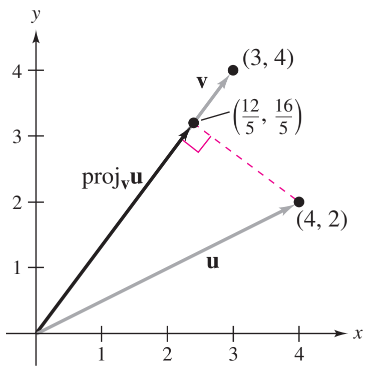



**Figure 5.12**

## EXAMPLE 9 Finding the Orthogonal Projection of u onto v

In $R^2$, the orthogonal projection of $\mathbf{u} = (4, 2)$ onto $\mathbf{v} = (3, 4)$ is

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = \frac{(4, 2)\cdot(3, 4)}{(3, 4)\cdot(3, 4)}(3, 4) = \frac{20}{25}(3, 4) = \left(\frac{12}{5}, \frac{16}{5}\right)$$

as shown in Figure 5.12.

**REMARK**

If $\mathbf{v}$ is a unit vector, then $\langle\mathbf{v}, \mathbf{v}\rangle = \|\mathbf{v}\|^2 = 1$, and the formula for the orthogonal projection of $\mathbf{u}$ onto $\mathbf{v}$ takes the simpler form

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \langle\mathbf{u}, \mathbf{v}\rangle\mathbf{v}.$$

An orthogonal projection in a general inner product space is defined as follows.

### Definition of Orthogonal Projection

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in an inner product space $V$, such that $\mathbf{v} \neq \mathbf{0}$. Then the **orthogonal projection** of $\mathbf{u}$ onto $\mathbf{v}$ is given by

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\langle\mathbf{u}, \mathbf{v}\rangle}{\langle\mathbf{v}, \mathbf{v}\rangle}\mathbf{v}.$$

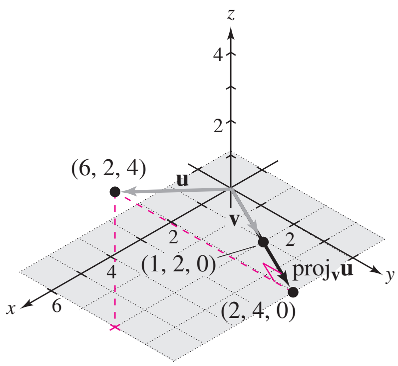



**Figure 5.13**

## EXAMPLE 10 Finding an Orthogonal Projection in $R^3$

Use the Euclidean inner product in $R^3$ to find the orthogonal projection of $\mathbf{u} = (6, 2, 4)$ onto $\mathbf{v} = (1, 2, 0)$.

**SOLUTION**

Because $\mathbf{u}\cdot\mathbf{v} = 10$ and $\mathbf{v}\cdot\mathbf{v} = 5$, the orthogonal projection of $\mathbf{u}$ onto $\mathbf{v}$ is

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = \frac{10}{5}(1, 2, 0) = 2(1, 2, 0) = (2, 4, 0)$$

as shown in Figure 5.13.

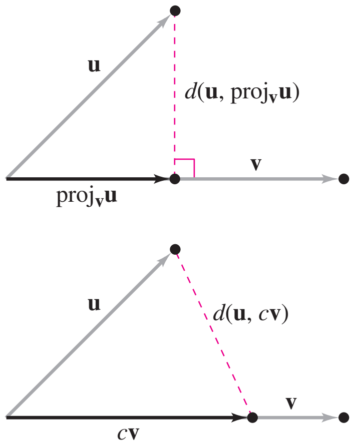



**Figure 5.14**

Note in Example 10 that $\mathbf{u} - \text{proj}_{\mathbf{v}}\mathbf{u} = (6, 2, 4) - (2, 4, 0) = (4, -2, 4)$ is orthogonal to $\mathbf{v} = (1, 2, 0)$. This is true in general: if $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors in an inner product space, then $\mathbf{u} - \text{proj}_{\mathbf{v}}\mathbf{u}$ is orthogonal to $\mathbf{v}$. (See Exercise 90.)

An important property of orthogonal projections used in mathematical modeling (see Section 5.4) is given in the next theorem. It states that, of all possible scalar multiples of a vector $\mathbf{v}$, the orthogonal projection of $\mathbf{u}$ onto $\mathbf{v}$ is the one closest to $\mathbf{u}$, as shown in Figure 5.14. For instance, in Example 10, this theorem implies that, of all the scalar multiples of the vector $\mathbf{v} = (1, 2, 0)$, the vector $\text{proj}_{\mathbf{v}}\mathbf{u} = (2, 4, 0)$ is closest to $\mathbf{u} = (6, 2, 4)$. You are asked to prove this explicitly in Exercise 101.

### THEOREM 5.9 Orthogonal Projection and Distance

Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors in an inner product space $V$, such that $\mathbf{v} \neq \mathbf{0}$. Then

$$d(\mathbf{u}, \text{proj}_{\mathbf{v}}\mathbf{u}) < d(\mathbf{u}, c\mathbf{v}), \quad c \neq \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle}.$$

**PROOF**

Let $b = \langle \mathbf{u}, \mathbf{v} \rangle / \langle \mathbf{v}, \mathbf{v} \rangle$. Then

$$\|\mathbf{u} - c\mathbf{v}\|^2 = \|(\mathbf{u} - b\mathbf{v}) + (b - c)\mathbf{v}\|^2$$

where $(\mathbf{u} - b\mathbf{v})$ and $(b - c)\mathbf{v}$ are orthogonal. You can verify this by using the inner product axioms to show that $\langle (\mathbf{u} - b\mathbf{v}), (b - c)\mathbf{v} \rangle = 0$. Now, by the Pythagorean Theorem,

$$\|(\mathbf{u} - b\mathbf{v}) + (b - c)\mathbf{v}\|^2 = \|\mathbf{u} - b\mathbf{v}\|^2 + \|(b - c)\mathbf{v}\|^2$$

which implies that

$$\|\mathbf{u} - c\mathbf{v}\|^2 = \|\mathbf{u} - b\mathbf{v}\|^2 + (b - c)^2\|\mathbf{v}\|^2.$$

Because $b \neq c$ and $\mathbf{v} \neq \mathbf{0}$, you know that $(b - c)^2\|\mathbf{v}\|^2 > 0$. So,

$$\|\mathbf{u} - b\mathbf{v}\|^2 < \|\mathbf{u} - c\mathbf{v}\|^2$$

and it follows that $d(\mathbf{u}, b\mathbf{v}) < d(\mathbf{u}, c\mathbf{v})$.

The next example discusses an orthogonal projection in the inner product space $C[a, b]$.

### EXAMPLE 11 Finding an Orthogonal Projection in $C[a, b]$ (Calculus)

Let $f(x) = 1$ and $g(x) = x$ be functions in $C[0, 1]$. Use the inner product on $C[a, b]$ defined in Example 5,

$$\langle f, g \rangle = \int_a^b f(x)g(x)\, dx$$

to find the orthogonal projection of $f$ onto $g$.

**SOLUTION**

From Example 8, you know that

$$\langle f, g \rangle = \frac{1}{2} \quad \text{and} \quad \langle g, g \rangle = \|g\|^2 = \frac{1}{3}.$$

So, the orthogonal projection of $f$ onto $g$ is

$$\text{proj}_g f = \frac{\langle f, g \rangle}{\langle g, g \rangle} g = \frac{1/2}{1/3} x = \frac{3}{2} x.$$

# 5.2 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

**Showing That a Function Is an Inner Product** In Exercises 1–4, show that the function defines an inner product on $R^2$, where $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$.

1. $\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1v_1 + u_2v_2$
2. $\langle \mathbf{u}, \mathbf{v} \rangle = u_1v_1 + 5u_2v_2$
3. $\langle \mathbf{u}, \mathbf{v} \rangle = \frac{1}{2}u_1v_1 + \frac{1}{4}u_2v_2$
4. $\langle \mathbf{u}, \mathbf{v} \rangle = 2u_1v_2 + u_2v_1 + u_1v_2 + 2u_2v_2$

**Showing That a Function Is an Inner Product** In Exercises 5–8, show that the function defines an inner product on $R^3$, where $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$.

5. $\langle \mathbf{u}, \mathbf{v} \rangle = 2u_1v_1 + 3u_2v_2 + u_3v_3$
6. $\langle \mathbf{u}, \mathbf{v} \rangle = u_1v_1 + 2u_2v_2 + u_3v_3$
7. $\langle \mathbf{u}, \mathbf{v} \rangle = 2u_1v_1 + u_2v_2 + 2u_3v_3$
8. $\langle \mathbf{u}, \mathbf{v} \rangle = \frac{1}{2}u_1v_1 + \frac{1}{4}u_2v_2 + \frac{1}{2}u_3v_3$

**Showing That a Function Is Not an Inner Product** In Exercises 9–12, show that the function *does not* define an inner product on $R^2$, where $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$.

9. $\langle \mathbf{u}, \mathbf{v} \rangle = u_1v_1$
10. $\langle \mathbf{u}, \mathbf{v} \rangle = u_1v_1 - 2u_2v_2$
11. $\langle \mathbf{u}, \mathbf{v} \rangle = u_1^2v_1^2 - u_2^2v_2^2$
12. $\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1v_2 - u_2v_1$

**Showing That a Function Is Not an Inner Product** In Exercises 13–16, show that the function *does not* define an inner product on $R^3$, where $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$.

13. $\langle \mathbf{u}, \mathbf{v} \rangle = -u_1u_2u_3$
14. $\langle \mathbf{u}, \mathbf{v} \rangle = u_1v_1 - u_2v_2 - u_3v_3$
15. $\langle \mathbf{u}, \mathbf{v} \rangle = u_1^2v_1^2 + u_2^2v_2^2 + u_3^2v_3^2$
16. $\langle \mathbf{u}, \mathbf{v} \rangle = u_1u_2 + v_1v_2 + u_3v_3$

**Finding Inner Product, Length, and Distance** In Exercises 17–26, find (a) $\langle \mathbf{u}, \mathbf{v} \rangle$, (b) $\|\mathbf{u}\|$, (c) $\|\mathbf{v}\|$, and (d) $d(\mathbf{u}, \mathbf{v})$ for the given inner product defined on $R^n$.

17. $\mathbf{u} = (3, 4)$, $\mathbf{v} = (5, -12)$, $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v}$
18. $\mathbf{u} = (1, 1)$, $\mathbf{v} = (7, 9)$, $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v}$
19. $\mathbf{u} = (-4, 3)$, $\mathbf{v} = (0, 5)$, $\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1v_1 + u_2v_2$
20. $\mathbf{u} = (0, -6)$, $\mathbf{v} = (-1, 1)$, $\langle \mathbf{u}, \mathbf{v} \rangle = u_1v_1 + 2u_2v_2$
21. $\mathbf{u} = (0, 9, 4)$, $\mathbf{v} = (9, -2, -4)$, $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v}$
22. $\mathbf{u} = (0, 1, 2)$, $\mathbf{v} = (1, 2, 0)$, $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v}$
23. $\mathbf{u} = (8, 0, -8)$, $\mathbf{v} = (8, 3, 16)$,
    $\langle \mathbf{u}, \mathbf{v} \rangle = 2u_1v_1 + 3u_2v_2 + u_3v_3$
24. $\mathbf{u} = (1, 1, 1)$, $\mathbf{v} = (2, 5, 2)$,
    $\langle \mathbf{u}, \mathbf{v} \rangle = u_1v_1 + 2u_2v_2 + u_3v_3$
25. $\mathbf{u} = (2, 0, 1, -1)$, $\mathbf{v} = (2, 2, 0, 1)$, $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v}$
26. $\mathbf{u} = (1, -1, 2, 0)$, $\mathbf{v} = (2, 1, 0, -1)$,
    $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v}$

**Showing That a Function Is an Inner Product** In Exercises 27 and 28, let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

be matrices in the vector space $M_{2,2}$. Show that the function defines an inner product on $M_{2,2}$.

27. $\langle A, B \rangle = a_{11}b_{11} + a_{21}b_{21} + a_{12}b_{12} + a_{22}b_{22}$
28. $\langle A, B \rangle = 2a_{11}b_{11} + a_{21}b_{21} + a_{12}b_{12} + 2a_{22}b_{22}$

**Finding Inner Product, Length, and Distance** In Exercises 29–32, use the inner product $\langle A, B \rangle = 2a_{11}b_{11} + a_{21}b_{21} + a_{12}b_{12} + 2a_{22}b_{22}$ to find (a) $\langle A, B \rangle$, (b) $\|A\|$, (c) $\|B\|$, and (d) $d(A, B)$ for the matrices in $M_{2,2}$.

29. $A = \begin{bmatrix} -1 & 3 \\ 4 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -2 \\ 1 & 1 \end{bmatrix}$
30. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
31. $A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$
32. $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$

**Showing That a Function Is an Inner Product** In Exercises 33 and 34, show that the function defines an inner product.

33. $\langle p, q \rangle = a_0b_0 + 2a_1b_1 + a_2b_2$, for $p(x) = a_0 + a_1x + a_2x^2$ and $q(x) = b_0 + b_1x + b_2x^2$ in $P_2$
34. $\langle p, q \rangle = a_0b_0 + a_1b_1 + \cdots + a_nb_n$, for $p(x) = a_0 + a_1x + \cdots + a_nx^n$ and $q(x) = b_0 + b_1x + \cdots + b_nx^n$ in $P_n$

**Finding Inner Product, Length, and Distance** In Exercises 35–38, use the inner product $\langle p, q \rangle = a_0b_0 + a_1b_1 + a_2b_2$ to find (a) $\langle p, q \rangle$, (b) $\|p\|$, (c) $\|q\|$, and (d) $d(p, q)$ for the polynomials in $P_2$.

35. $p(x) = 1 - x + 3x^2$, $q(x) = x - x^2$
36. $p(x) = 1 + x + \frac{1}{2}x^2$, $q(x) = 1 + 2x^2$
37. $p(x) = 1 + x^2$, $q(x) = 1 - x^2$
38. $p(x) = 1 - 2x - x^2$, $q(x) = x - x^2$

**Calculus** In Exercises 39–42, use the functions $f$ and $g$ in $C[-1, 1]$ to find (a) $\langle f, g \rangle$, (b) $\|f\|$, (c) $\|g\|$, and (d) $d(f, g)$ for the inner product

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\, dx.$$

39. $f(x) = 1$, $g(x) = 3x^2 - 1$
40. $f(x) = -x$, $g(x) = x^2 - x + 2$
41. $f(x) = x$, $g(x) = e^x$
42. $f(x) = x$, $g(x) = e^{-x}$

**Finding the Angle Between Two Vectors** In Exercises 43–52, find the angle $\theta$ between the vectors.

43. $\mathbf{u} = (3, 4)$, $\mathbf{v} = (5, -12)$, $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v}$
44. $\mathbf{u} = (2, -1)$, $\mathbf{v} = \left(\frac{1}{2}, 1\right)$, $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v}$
45. $\mathbf{u} = (-4, 3)$, $\mathbf{v} = (0, 5)$, $\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1v_1 + u_2v_2$
46. $\mathbf{u} = \left(\frac{1}{4}, -1\right)$, $\mathbf{v} = (2, 1)$,
$\langle \mathbf{u}, \mathbf{v} \rangle = 2u_1v_1 + u_2v_2$
47. $\mathbf{u} = (1, 1, 1)$, $\mathbf{v} = (2, -2, 2)$,
$\langle \mathbf{u}, \mathbf{v} \rangle = u_1v_1 + 2u_2v_2 + u_3v_3$
48. $\mathbf{u} = (0, 1, -1)$, $\mathbf{v} = (1, 2, 3)$, $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v}$
49. $p(x) = 1 - x + x^2$, $q(x) = 1 + x + x^2$,
$\langle p, q \rangle = a_0b_0 + a_1b_1 + a_2b_2$
50. $p(x) = 1 + x^2$, $q(x) = x - x^2$,
$\langle p, q \rangle = a_0b_0 + 2a_1b_1 + a_2b_2$
51. **Calculus** $f(x) = x$, $g(x) = x^2$,
$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\,dx$
52. **Calculus** $f(x) = 1$, $g(x) = x^2$,
$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\,dx$

**Verifying Inequalities** In Exercises 53–64, verify (a) the Cauchy-Schwarz Inequality and (b) the triangle inequality for the given vectors and inner products.

53. $\mathbf{u} = (5, 12)$, $\mathbf{v} = (3, 4)$, $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v}$
54. $\mathbf{u} = (-1, 1)$, $\mathbf{v} = (1, -1)$, $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v}$
55. $\mathbf{u} = (1, 0, 4)$, $\mathbf{v} = (-5, 4, 1)$, $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v}$
56. $\mathbf{u} = (1, 0, 2)$, $\mathbf{v} = (1, 2, 0)$, $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v}$
57. $p(x) = 2x$, $q(x) = 3x^2 + 1$,
$\langle p, q \rangle = a_0b_0 + a_1b_1 + a_2b_2$
58. $p(x) = x$, $q(x) = 1 - x^2$,
$\langle p, q \rangle = a_0b_0 + 2a_1b_1 + a_2b_2$
59. $A = \begin{bmatrix} 0 & 3 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 1 \\ 4 & 3 \end{bmatrix}$,
$\langle A, B \rangle = a_{11}b_{11} + a_{21}b_{21} + a_{12}b_{12} + a_{22}b_{22}$
60. $A = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}$,
$\langle A, B \rangle = a_{11}b_{11} + a_{21}b_{21} + a_{12}b_{12} + a_{22}b_{22}$
61. **Calculus** $f(x) = \sin x$, $g(x) = \cos x$,
$\langle f, g \rangle = \int_{0}^{\pi/4} f(x)g(x)\,dx$
62. **Calculus** $f(x) = x$, $g(x) = \cos \pi x$,
$\langle f, g \rangle = \int_{0}^{2} f(x)g(x)\,dx$
63. **Calculus** $f(x) = x$, $g(x) = e^x$,
$\langle f, g \rangle = \int_{0}^{1} f(x)g(x)\,dx$
64. **Calculus** $f(x) = x$, $g(x) = e^{-x}$,
$\langle f, g \rangle = \int_{0}^{1} f(x)g(x)\,dx$

**Calculus** In Exercises 65–68, show that $f$ and $g$ are orthogonal in the inner product space $C[a, b]$ with the inner product

$$\langle f, g \rangle = \int_a^b f(x)g(x)\,dx.$$

65. $C[-\pi/2, \pi/2]$, $f(x) = \cos x$, $g(x) = \sin x$
66. $C[-1, 1]$, $f(x) = x$, $g(x) = \frac{1}{2}(3x^2 - 1)$
67. $C[-1, 1]$, $f(x) = x$, $g(x) = \frac{1}{2}(5x^3 - 3x)$
68. $C[0, \pi]$, $f(x) = 1$, $g(x) = \cos(2nx)$,
$n = 1, 2, 3, \ldots$

**Finding and Graphing Orthogonal Projections in $R^2$** In Exercises 69–72, (a) find $\text{proj}_{\mathbf{v}}\mathbf{u}$, (b) find $\text{proj}_{\mathbf{u}}\mathbf{v}$, and (c) sketch a graph of both $\text{proj}_{\mathbf{v}}\mathbf{u}$ and $\text{proj}_{\mathbf{u}}\mathbf{v}$. Use the Euclidean inner product.

69. $\mathbf{u} = (1, 2)$, $\mathbf{v} = (2, 1)$
70. $\mathbf{u} = (-1, -2)$, $\mathbf{v} = (4, 2)$
71. $\mathbf{u} = (-1, 3)$, $\mathbf{v} = (4, 4)$
72. $\mathbf{u} = (2, -2)$, $\mathbf{v} = (3, 1)$

**Finding Orthogonal Projections** In Exercises 73–76, find (a) $\text{proj}_{\mathbf{v}}\mathbf{u}$ and (b) $\text{proj}_{\mathbf{u}}\mathbf{v}$. Use the Euclidean inner product.

73. $\mathbf{u} = (1, 3, -2)$, $\mathbf{v} = (0, -1, 1)$
74. $\mathbf{u} = (1, 2, -1)$, $\mathbf{v} = (-1, 2, -1)$
75. $\mathbf{u} = (0, 1, 3, -6)$, $\mathbf{v} = (-1, 1, 2, 2)$
76. $\mathbf{u} = (-1, 4, -2, 3)$, $\mathbf{v} = (2, -1, 2, -1)$

**Calculus** In Exercises 77–84, find the orthogonal projection of $f$ onto $g$. Use the inner product in $C[a, b]$

$$\langle f, g \rangle = \int_a^b f(x)g(x)\,dx.$$

77. $C[-1, 1]$, $f(x) = x$, $g(x) = 1$
78. $C[-1, 1]$, $f(x) = x^3 - x$, $g(x) = 2x - 1$
79. $C[0, 1]$, $f(x) = x$, $g(x) = e^x$
80. $C[0, 1]$, $f(x) = x$, $g(x) = e^{-x}$
81. $C[-\pi, \pi]$, $f(x) = \sin x$, $g(x) = \cos x$
82. $C[-\pi, \pi]$, $f(x) = \sin 2x$, $g(x) = \cos 2x$
83. $C[-\pi, \pi]$, $f(x) = x$, $g(x) = \sin 2x$
84. $C[-\pi, \pi]$, $f(x) = x$, $g(x) = \cos 2x$

**True or False?** **In Exercises 85 and 86, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.**

**85.** (a) The dot product is the only inner product that can be defined in $R^n$.

(b) A nonzero vector in an inner product can have a norm of zero.

**86.** (a) The norm of the vector $\mathbf{u}$ is defined as the angle between the vector $\mathbf{u}$ and the positive $x$-axis.

(b) The angle $\theta$ between a vector $\mathbf{v}$ and the projection of $\mathbf{u}$ onto $\mathbf{v}$ is obtuse when the scalar $a < 0$ and acute when $a > 0$, where $a\mathbf{v} = \text{proj}_{\mathbf{v}}\mathbf{u}$.

**87.** Let $\mathbf{u} = (4, 2)$ and $\mathbf{v} = (2, -2)$ be vectors in $R^2$ with the inner product $\langle \mathbf{u}, \mathbf{v} \rangle = u_1v_1 + 2u_2v_2$.

(a) Show that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

(b) Sketch the vectors $\mathbf{u}$ and $\mathbf{v}$. Are they orthogonal in the Euclidean sense?

**88. Proof** Prove that

$$\|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2$$

for any vectors $\mathbf{u}$ and $\mathbf{v}$ in an inner product space $V$.

**89. Proof** Prove that the function is an inner product on $R^n$.

$$\langle \mathbf{u}, \mathbf{v} \rangle = c_1u_1v_1 + c_2u_2v_2 + \cdots + c_nu_nv_n, \quad c_i > 0$$

**90. Proof** Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors in an inner product space $V$. Prove that $\mathbf{u} - \text{proj}_{\mathbf{v}}\mathbf{u}$ is orthogonal to $\mathbf{v}$.

**91. Proof** Prove Property 2 of Theorem 5.7: If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in an inner product space, then $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$.

**92. Proof** Prove Property 3 of Theorem 5.7: If $\mathbf{u}$ and $\mathbf{v}$ are vectors in an inner product space and $c$ is a scalar, then $\langle \mathbf{u}, c\mathbf{v} \rangle = c\langle \mathbf{u}, \mathbf{v} \rangle$.

**93. Guided Proof** Let $W$ be a subspace of the inner product space $V$. Prove that the set $W^\perp$ is a subspace of $V$.

$$W^\perp = \{\mathbf{v} \in V : \langle \mathbf{v}, \mathbf{w} \rangle = 0 \text{ for all } \mathbf{w} \in W\}$$

Getting Started: To prove that $W^\perp$ is a subspace of $V$, you must show that $W^\perp$ is nonempty and that the closure conditions for a subspace hold (Theorem 4.5).

(i) Find a vector in $W^\perp$ to conclude that it is nonempty.

(ii) To show the closure of $W^\perp$ under addition, you need to show that $\langle \mathbf{v}_1 + \mathbf{v}_2, \mathbf{w} \rangle = 0$ for all $\mathbf{w} \in W$ and for any $\mathbf{v}_1, \mathbf{v}_2 \in W^\perp$. Use the properties of inner products and the fact that $\langle \mathbf{v}_1, \mathbf{w} \rangle$ and $\langle \mathbf{v}_2, \mathbf{w} \rangle$ are both zero to show this.

(iii) To show closure under multiplication by a scalar, proceed as in part (ii). You need to use the properties of inner products and the condition of belonging to $W^\perp$.

**94.** Use the result of Exercise 93 to find $W^\perp$ when $W$ is a span of (1, 2, 3) in $V = R^3$.

**95. Guided Proof** Let $\langle \mathbf{u}, \mathbf{v} \rangle$ be the Euclidean inner product on $R^n$. Use the fact that $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^T\mathbf{v}$ to prove that for any $n \times n$ matrix $A$,

(a) $\langle A^T\mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, A\mathbf{v} \rangle$

and

(b) $\langle A^TA\mathbf{u}, \mathbf{u} \rangle = \|A\mathbf{u}\|^2$.

Getting Started: To prove (a) and (b), make use of both the properties of transposes (Theorem 2.6) and the properties of the dot product (Theorem 5.3).

(i) To prove part (a), make repeated use of the property $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^T\mathbf{v}$ and Property 4 of Theorem 2.6.

(ii) To prove part (b), make use of the property $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^T\mathbf{v}$, Property 4 of Theorem 2.6, and Property 4 of Theorem 5.3.

**96. CAPSTONE**

(a) Explain how to determine whether a given function defines an inner product.

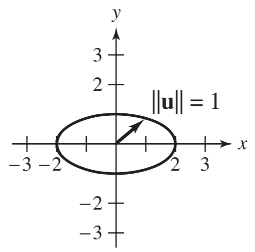
(b) Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in an inner product space $V$, such that $\mathbf{v} \neq \mathbf{0}$. Explain how to find the orthogonal projection of $\mathbf{u}$ onto $\mathbf{v}$.

**Finding Inner Product Weights** **In Exercises 97–100, find $c_1$ and $c_2$ for the inner product of $R^2$ given by**

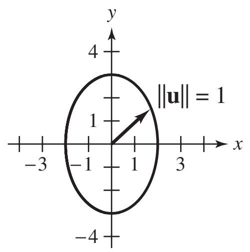
$$\langle \mathbf{u}, \mathbf{v} \rangle = c_1u_1v_1 + c_2u_2v_2$$

**such that the graph represents a unit circle as shown.**

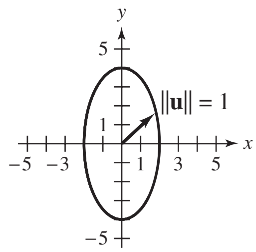
**97.**



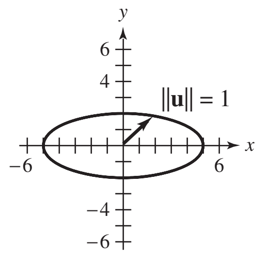

**98.**




**99.**




**100.**




**101.** The two vectors from Example 10 are $\mathbf{u} = (6, 2, 4)$ and $\mathbf{v} = (1, 2, 0)$. Without using Theorem 5.9, show that among all the scalar multiples $c\mathbf{v}$ of the vector $\mathbf{v}$, the projection of $\mathbf{u}$ onto $\mathbf{v}$ is the vector closest to $\mathbf{u}$—that is, show that $d(\mathbf{u}, \text{proj}_{\mathbf{v}}\mathbf{u})$ is a minimum.

# 5.3 Orthonormal Bases: Gram-Schmidt Process

- Show that a set of vectors is orthogonal and forms an orthonormal basis, and represent a vector relative to an orthonormal basis.
- Apply the Gram-Schmidt orthonormalization process.

## ORTHOGONAL AND ORTHONORMAL SETS

You saw in Section 4.7 that a vector space can have many different bases. While studying that section, you may have noticed that some bases are more convenient than others. For example, $R^3$ has the basis $B = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$. This set is the *standard* basis for $R^3$ because it has important characteristics that are particularly useful. One important characteristic is that the three vectors in the basis are *mutually orthogonal.* That is,

$$(1, 0, 0) \cdot (0, 1, 0) = 0$$
$$(1, 0, 0) \cdot (0, 0, 1) = 0$$
$$(0, 1, 0) \cdot (0, 0, 1) = 0.$$

A second important characteristic is that each vector in the basis is a *unit* vector.

This section identifies some advantages of using bases consisting of mutually orthogonal unit vectors and develops a procedure for constructing such bases, known as the *Gram-Schmidt orthonormalization process.*

### Definitions of Orthogonal and Orthonormal Sets

A set $S$ of vectors in an inner product space $V$ is called **orthogonal** when every pair of vectors in $S$ is orthogonal. If, in addition, each vector in the set is a unit vector, then $S$ is called **orthonormal.**

For $S = \{\mathbf{v}_1, \mathbf{v}_2, . . . , \mathbf{v}_n\}$, this definition has the following form.

| *Orthogonal* | *Orthonormal* |
|---|---|
| **1.** $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0, \ i \neq j$ | **1.** $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0, \ i \neq j$ |
| | **2.** $\|\mathbf{v}_i\| = 1, \ i = 1, 2, . . . , n$ |

If $S$ is a *basis,* then it is an **orthogonal basis** or an **orthonormal basis,** respectively.

The standard basis for $R^n$ is orthonormal, but it is not the only orthonormal basis for $R^n$. For instance, a nonstandard orthonormal basis for $R^3$ can be formed by rotating the standard basis about the $z$-axis, resulting in

$$B = \{(\cos \theta, \sin \theta, 0), (-\sin \theta, \cos \theta, 0), (0, 0, 1)\}$$

as shown in Figure 5.15. Try verifying that the dot product of any two distinct vectors in $B$ is zero, and that each vector in $B$ is a unit vector.

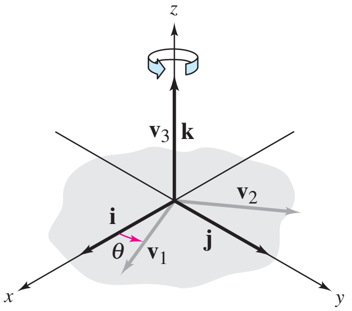



**Figure 5.15**

Example 1 describes another nonstandard orthonormal basis for $R^3$.

## EXAMPLE 1 A Nonstandard Orthonormal Basis for $R^3$

Show that the set is an orthonormal basis for $R^3$.

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \left\{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \left(-\frac{\sqrt{2}}{6}, \frac{\sqrt{2}}{6}, \frac{2\sqrt{2}}{3}\right), \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)\right\}$$

**SOLUTION**

First show that the three vectors are mutually orthogonal.

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = -\frac{1}{6} + \frac{1}{6} + 0 = 0$$

$$\mathbf{v}_1 \cdot \mathbf{v}_3 = \frac{2}{3\sqrt{2}} - \frac{2}{3\sqrt{2}} + 0 = 0$$

$$\mathbf{v}_2 \cdot \mathbf{v}_3 = -\frac{\sqrt{2}}{9} - \frac{\sqrt{2}}{9} + \frac{2\sqrt{2}}{9} = 0$$

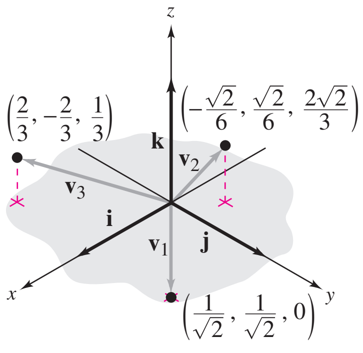



**Figure 5.16**

Now, each vector is of length 1 because

$$\|\mathbf{v}_1\| = \sqrt{\mathbf{v}_1 \cdot \mathbf{v}_1} = \sqrt{\tfrac{1}{2} + \tfrac{1}{2} + 0} = 1$$

$$\|\mathbf{v}_2\| = \sqrt{\mathbf{v}_2 \cdot \mathbf{v}_2} = \sqrt{\tfrac{2}{36} + \tfrac{2}{36} + \tfrac{8}{9}} = 1$$

$$\|\mathbf{v}_3\| = \sqrt{\mathbf{v}_3 \cdot \mathbf{v}_3} = \sqrt{\tfrac{4}{9} + \tfrac{4}{9} + \tfrac{1}{9}} = 1.$$

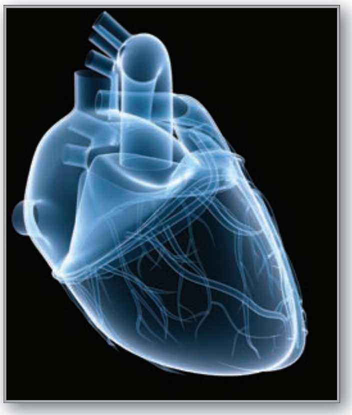
So, $S$ is an orthonormal set. Because the three vectors do not lie in the same plane (see Figure 5.16), you know that they span $R^3$. By Theorem 4.12, they form a (nonstandard) orthonormal basis for $R^3$.

## EXAMPLE 2 An Orthonormal Basis for $P_3$

In $P_3$, with the inner product

$$\langle p, q \rangle = a_0b_0 + a_1b_1 + a_2b_2 + a_3b_3$$

the standard basis $B = \{1, x, x^2, x^3\}$ is orthonormal. The verification of this is left as an exercise. (See Exercise 19.)



**LINEAR ALGEBRA APPLIED**

Time-frequency analysis of irregular physiological signals, such as beat-to-beat cardiac rhythm variations (also known as heart rate variability or HRV), can be difficult. This is because the structure of a signal can include multiple periodic, nonperiodic, and pseudo-periodic components. Researchers have proposed and validated a simplified HRV analysis method called orthonormal-basis partitioning and time-frequency representation (OPTR). This method can detect both abrupt and slow changes in the HRV signal's structure, divide a nonstationary HRV signal into segments that are "less nonstationary," and determine patterns in the HRV. The researchers found that although it had poor time resolution with signals that changed gradually, the OPTR method accurately represented multicomponent and abrupt changes in both real-life and simulated HRV signals. *(Source: Orthonormal-Basis Partitioning and Time-Frequency Representation of Cardiac Rhythm Dynamics, Aysin, Benhur, et al, IEEE Transactions on Biomedical Engineering, 52, no. 5)*

The orthogonal set in the next example is used to construct Fourier approximations of continuous functions. (See Section 5.5.)

## EXAMPLE 3 An Orthogonal Set in $C[0, 2\pi]$ (Calculus)

In $C[0, 2\pi]$, with the inner product

$$\langle f, g \rangle = \int_0^{2\pi} f(x)g(x)\,dx$$

show that the set $S = \{1, \sin x, \cos x, \sin 2x, \cos 2x, \ldots, \sin nx, \cos nx\}$ is orthogonal.

### SOLUTION

To show that this set is orthogonal, verify the following inner products, where $m$ and $n$ are positive integers.

$$\langle 1, \sin nx \rangle = \int_0^{2\pi} \sin nx\,dx = 0$$

$$\langle 1, \cos nx \rangle = \int_0^{2\pi} \cos nx\,dx = 0$$

$$\langle \sin mx, \cos nx \rangle = \int_0^{2\pi} \sin mx \cos nx\,dx = 0$$

$$\langle \sin mx, \sin nx \rangle = \int_0^{2\pi} \sin mx \sin nx\,dx = 0, \quad m \neq n$$

$$\langle \cos mx, \cos nx \rangle = \int_0^{2\pi} \cos mx \cos nx\,dx = 0, \quad m \neq n$$

One of these products is verified below, and the others are left to you. If $m \neq n$, then use the formula for rewriting a product of trigonometric functions as a sum to obtain

$$\int_0^{2\pi} \sin mx \cos nx\,dx = \frac{1}{2}\int_0^{2\pi} [\sin(m + n)x + \sin(m - n)x]\,dx = 0.$$

If $m = n$, then

$$\int_0^{2\pi} \sin mx \cos mx\,dx = \frac{1}{2m}\Big[\sin^2 mx\Big]_0^{2\pi} = 0.$$

**Jean-Baptiste Joseph Fourier (1768–1830)**

Fourier was born in Auxerre, France. He is credited as a significant contributor to the field of education for scientists, mathematicians, and engineers. His research led to important results pertaining to eigenvalues (Section 7.1), differential equations, and what would later become known as Fourier series (representations of functions using trigonometric series). His work forced mathematicians to reconsider the accepted, but narrow, definition of a function.

The set $S$ in Example 3 is orthogonal but not orthonormal. An orthonormal set can be formed, however, by normalizing each vector in $S$. That is, because

$$\|1\|^2 = \int_0^{2\pi} dx = 2\pi$$

$$\|\sin nx\|^2 = \int_0^{2\pi} \sin^2 nx\,dx = \pi$$

$$\|\cos nx\|^2 = \int_0^{2\pi} \cos^2 nx\,dx = \pi$$

it follows that the set

$$\left\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\sin x, \frac{1}{\sqrt{\pi}}\cos x, \ldots, \frac{1}{\sqrt{\pi}}\sin nx, \frac{1}{\sqrt{\pi}}\cos nx\right\}$$

is orthonormal.

Each set in Examples 1, 2, and 3 is linearly independent. Linear independence is a characteristic of any orthogonal set of nonzero vectors, as stated in the next theorem.

**THEOREM 5.10 Orthogonal Sets Are Linearly Independent**

If $S = \{\mathbf{v}_1, \mathbf{v}_2, . . . , \mathbf{v}_n\}$ is an orthogonal set of *nonzero* vectors in an inner product space $V$, then $S$ is linearly independent.

**PROOF**

You need to show that the vector equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdot \cdot \cdot + c_n\mathbf{v}_n = \mathbf{0}$$

implies $c_1 = c_2 = \cdot \cdot \cdot = c_n = 0$. To do this, form the inner product of both sides of the equation with each vector in $S$. That is, for each $i$,

$$\langle(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdot \cdot \cdot + c_i\mathbf{v}_i + \cdot \cdot \cdot + c_n\mathbf{v}_n), \mathbf{v}_i\rangle = \langle\mathbf{0}, \mathbf{v}_i\rangle$$
$$c_1\langle\mathbf{v}_1, \mathbf{v}_i\rangle + c_2\langle\mathbf{v}_2, \mathbf{v}_i\rangle + \cdot \cdot \cdot + c_i\langle\mathbf{v}_i, \mathbf{v}_i\rangle + \cdot \cdot \cdot + c_n\langle\mathbf{v}_n, \mathbf{v}_i\rangle = 0.$$

Now, because $S$ is orthogonal, $\langle\mathbf{v}_i, \mathbf{v}_j\rangle = 0$ for $j \neq i$, and the equation reduces to

$$c_i\langle\mathbf{v}_i, \mathbf{v}_i\rangle = 0.$$

But because each vector in $S$ is nonzero, you know that

$$\langle\mathbf{v}_i, \mathbf{v}_i\rangle = \|\mathbf{v}_i\|^2 \neq 0.$$

So, every $c_i$ must be zero and the set must be linearly independent.

As a consequence of Theorems 4.12 and 5.10, you have the following corollary.

**THEOREM 5.10 Corollary**

If $V$ is an inner product space of dimension $n$, then any orthogonal set of $n$ nonzero vectors is a basis for $V$.

## EXAMPLE 4 Using Orthogonality to Test for a Basis

Show that the following set is a basis for $R^4$.

$$S = \{\overset{\mathbf{v}_1}{(2, 3, 2, -2)}, \overset{\mathbf{v}_2}{(1, 0, 0, 1)}, \overset{\mathbf{v}_3}{(-1, 0, 2, 1)}, \overset{\mathbf{v}_4}{(-1, 2, -1, 1)}\}$$

**SOLUTION**

The set $S$ has four nonzero vectors. By the corollary to Theorem 5.10, you can show that $S$ is a basis for $R^4$ by showing that it is an orthogonal set, as follows.

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = 2 + 0 + 0 - 2 = 0$$
$$\mathbf{v}_1 \cdot \mathbf{v}_3 = -2 + 0 + 4 - 2 = 0$$
$$\mathbf{v}_1 \cdot \mathbf{v}_4 = -2 + 6 - 2 - 2 = 0$$
$$\mathbf{v}_2 \cdot \mathbf{v}_3 = -1 + 0 + 0 + 1 = 0$$
$$\mathbf{v}_2 \cdot \mathbf{v}_4 = -1 + 0 + 0 + 1 = 0$$
$$\mathbf{v}_3 \cdot \mathbf{v}_4 = 1 + 0 - 2 + 1 = 0$$

$S$ is orthogonal, and by the corollary to Theorem 5.10, it is a basis for $R^4$.

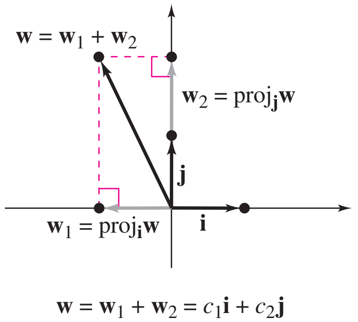



**Figure 5.17**

Section 4.7 discusses a technique for finding a coordinate representation relative to a nonstandard basis. When the basis is *orthonormal,* this procedure can be streamlined.

Before looking at this procedure, consider an example in $R^2$. Figure 5.17 shows that $\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$ form an orthonormal basis for $R^2$. Any vector $\mathbf{w}$ in $R^2$ can be represented as $\mathbf{w} = \mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{w}_1 = \text{proj}_{\mathbf{i}}\mathbf{w}$ and $\mathbf{w}_2 = \text{proj}_{\mathbf{j}}\mathbf{w}$. Because $\mathbf{i}$ and $\mathbf{j}$ are unit vectors, it follows that $\mathbf{w}_1 = (\mathbf{w} \cdot \mathbf{i})\mathbf{i}$ and $\mathbf{w}_2 = (\mathbf{w} \cdot \mathbf{j})\mathbf{j}$. Consequently,

$$\mathbf{w} = \mathbf{w}_1 + \mathbf{w}_2 = (\mathbf{w} \cdot \mathbf{i})\mathbf{i} + (\mathbf{w} \cdot \mathbf{j})\mathbf{j} = c_1\mathbf{i} + c_2\mathbf{j}$$

which shows that the coefficients $c_1$ and $c_2$ are simply the dot products of $\mathbf{w}$ with the respective basis vectors. The next theorem generalizes this.

**THEOREM 5.11 Coordinates Relative to an Orthonormal Basis**

If $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthonormal basis for an inner product space $V$, then the coordinate representation of a vector $\mathbf{w}$ relative to $B$ is

$$\mathbf{w} = \langle \mathbf{w}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{w}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \cdots + \langle \mathbf{w}, \mathbf{v}_n \rangle \mathbf{v}_n.$$

### PROOF

Because $B$ is a basis for $V$, there must exist unique scalars $c_1, c_2, \ldots, c_n$ such that

$$\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n.$$

Taking the inner product (with $\mathbf{v}_i$) of both sides of this equation, you have

$$\begin{aligned}\langle \mathbf{w}, \mathbf{v}_i \rangle &= \langle (c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n), \mathbf{v}_i \rangle \\ &= c_1\langle \mathbf{v}_1, \mathbf{v}_i \rangle + c_2\langle \mathbf{v}_2, \mathbf{v}_i \rangle + \cdots + c_n\langle \mathbf{v}_n, \mathbf{v}_i \rangle\end{aligned}$$

and by the orthogonality of $B$, this equation reduces to

$$\langle \mathbf{w}, \mathbf{v}_i \rangle = c_i\langle \mathbf{v}_i, \mathbf{v}_i \rangle.$$

Because $B$ is orthonormal, you have $\langle \mathbf{v}_i, \mathbf{v}_i \rangle = \|\mathbf{v}_i\|^2 = 1$, and it follows that $\langle \mathbf{w}, \mathbf{v}_i \rangle = c_i$.

In Theorem 5.11, the coordinates of $\mathbf{w}$ relative to the *orthonormal* basis $B$ are called the **Fourier coefficients** of $\mathbf{w}$ relative to $B$, after the French mathematician Jean-Baptiste Joseph Fourier (1768–1830). The corresponding coordinate matrix of $\mathbf{w}$ relative to $B$ is

$$[\mathbf{w}]_B = [c_1 \ c_2 \ \cdots \ c_n]^T = [\langle \mathbf{w}, \mathbf{v}_1 \rangle \ \langle \mathbf{w}, \mathbf{v}_2 \rangle \ \cdots \ \langle \mathbf{w}, \mathbf{v}_n \rangle]^T.$$

## EXAMPLE 5 Representing Vectors Relative to an Orthonormal Basis

Find the coordinate matrix of $\mathbf{w} = (5, -5, 2)$ relative to the following orthonormal basis for $R^3$.

$$B = \Big\{\underbrace{\left(\tfrac{3}{5}, \tfrac{4}{5}, 0\right)}^{\mathbf{v}_1}, \overbrace{\left(-\tfrac{4}{5}, \tfrac{3}{5}, 0\right)}^{\mathbf{v}_2}, \overbrace{(0, 0, 1)}^{\mathbf{v}_3}\Big\}$$

### SOLUTION

Because $B$ is orthonormal, use Theorem 5.11 to find the coordinates.

$$\begin{aligned}\mathbf{w} \cdot \mathbf{v}_1 &= (5, -5, 2) \cdot \left(\tfrac{3}{5}, \tfrac{4}{5}, 0\right) = -1 \\ \mathbf{w} \cdot \mathbf{v}_2 &= (5, -5, 2) \cdot \left(-\tfrac{4}{5}, \tfrac{3}{5}, 0\right) = -7 \\ \mathbf{w} \cdot \mathbf{v}_3 &= (5, -5, 2) \cdot (0, 0, 1) = 2\end{aligned}$$

So, the coordinate matrix relative to $B$ is $[\mathbf{w}]_B = [-1 \quad -7 \quad 2]^T$.

## GRAM-SCHMIDT ORTHONORMALIZATION PROCESS

**REMARK**

The Gram-Schmidt orthonormalization process leads to a matrix factorization similar to the *LU*-factorization you studied in Chapter 2. You are asked to investigate this *QR-factorization* in Project 1 on page 287.

Having seen one of the advantages of orthonormal bases (the straightforwardness of coordinate representation), you will now look at a procedure for finding such a basis. This procedure is called the **Gram-Schmidt orthonormalization process,** after the Danish mathematician Jorgen Pederson Gram (1850–1916) and the German mathematician Erhardt Schmidt (1876–1959). It has three steps.

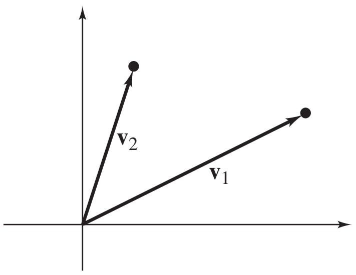
1. Begin with a basis for the inner product space. It need not be orthogonal nor consist of unit vectors.
2. Convert the given basis to an orthogonal basis.
3. Normalize each vector in the orthogonal basis to form an orthonormal basis.

**THEOREM 5.12 Gram-Schmidt Orthonormalization Process**

1. Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for an inner product space $V$.
2. Let $B' = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$, where $\mathbf{w}_i$ is given by

$$\begin{aligned}
\mathbf{w}_1 &= \mathbf{v}_1\\
\mathbf{w}_2 &= \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{w}_1\rangle}{\langle \mathbf{w}_1, \mathbf{w}_1\rangle}\mathbf{w}_1\\
\mathbf{w}_3 &= \mathbf{v}_3 - \frac{\langle \mathbf{v}_3, \mathbf{w}_1\rangle}{\langle \mathbf{w}_1, \mathbf{w}_1\rangle}\mathbf{w}_1 - \frac{\langle \mathbf{v}_3, \mathbf{w}_2\rangle}{\langle \mathbf{w}_2, \mathbf{w}_2\rangle}\mathbf{w}_2\\
&\ \vdots\\
\mathbf{w}_n &= \mathbf{v}_n - \frac{\langle \mathbf{v}_n, \mathbf{w}_1\rangle}{\langle \mathbf{w}_1, \mathbf{w}_1\rangle}\mathbf{w}_1 - \frac{\langle \mathbf{v}_n, \mathbf{w}_2\rangle}{\langle \mathbf{w}_2, \mathbf{w}_2\rangle}\mathbf{w}_2 - \cdots - \frac{\langle \mathbf{v}_n, \mathbf{w}_{n-1}\rangle}{\langle \mathbf{w}_{n-1}, \mathbf{w}_{n-1}\rangle}\mathbf{w}_{n-1}.
\end{aligned}$$

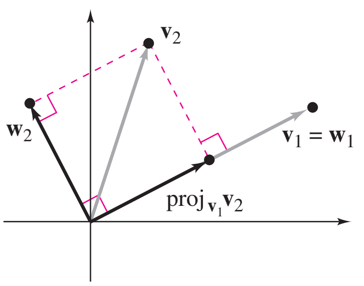
   Then $B'$ is an *orthogonal* basis for $V$.
3. Let $\mathbf{u}_i = \dfrac{\mathbf{w}_i}{\|\mathbf{w}_i\|}$. Then the set $B'' = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is an *orthonormal* basis for $V$. Moreover, span$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ = span$\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ for $k = 1, 2, \ldots, n$.




$\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for $R^2$.

**Figure 5.18**

Rather than give a proof of this theorem, it is more instructive to discuss a special case for which you can use a geometric model. Let $\{\mathbf{v}_1, \mathbf{v}_2\}$ be a basis for $R^2$, as shown in Figure 5.18. To determine an orthogonal basis for $R^2$, first choose one of the original vectors, say $\mathbf{v}_1$. Now you want to find a second vector orthogonal to $\mathbf{v}_1$. Figure 5.19 shows that $\mathbf{v}_2 - \text{proj}_{\mathbf{v}_1}\mathbf{v}_2$ has this property.




$\mathbf{w}_2 = \mathbf{v}_2 - \text{proj}_{\mathbf{v}_1}\mathbf{v}_2$ is orthogonal to $\mathbf{w}_1 = \mathbf{v}_1$.

**Figure 5.19**

By letting

$$\mathbf{w}_1 = \mathbf{v}_1$$

and

$$\mathbf{w}_2 = \mathbf{v}_2 - \text{proj}_{\mathbf{v}_1}\mathbf{v}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1}\mathbf{w}_1$$

you can conclude that the set $\{\mathbf{w}_1, \mathbf{w}_2\}$ is orthogonal. By the corollary to Theorem 5.10, it is a basis for $R^2$. Finally, by normalizing $\mathbf{w}_1$ and $\mathbf{w}_2$, you obtain the following orthonormal basis for $R^2$.

$$\{\mathbf{u}_1, \mathbf{u}_2\} = \left\{\frac{\mathbf{w}_1}{\|\mathbf{w}_1\|}, \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|}\right\}$$

**REMARK**

An orthonormal set derived by the Gram-Schmidt orthonormalization process depends on the order of the vectors in the basis. For instance, try reworking Example 6 with the original basis ordered as $\{\mathbf{v}_2, \mathbf{v}_1\}$ rather than $\{\mathbf{v}_1, \mathbf{v}_2\}$.

## EXAMPLE 6 Applying the Gram-Schmidt Orthonormalization Process

Apply the Gram-Schmidt orthonormalization process to the following basis for $R^2$.

$$B = \{\overset{\mathbf{v}_1}{(1, 1)}, \overset{\mathbf{v}_2}{(0, 1)}\}$$

**SOLUTION**

The Gram-Schmidt orthonormalization process produces

$$\mathbf{w}_1 = \mathbf{v}_1 = (1, 1)$$

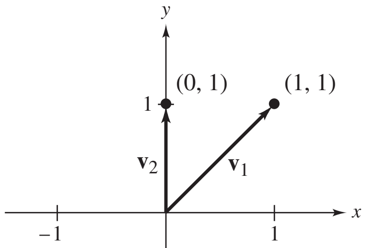
$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1}\mathbf{w}_1 = (0, 1) - \frac{1}{2}(1, 1) = \left(-\frac{1}{2}, \frac{1}{2}\right).$$

The set $B' = \{\mathbf{w}_1, \mathbf{w}_2\}$ is an orthogonal basis for $R^2$. By normalizing each vector in $B'$, you obtain

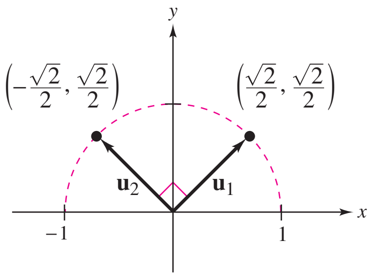
$$\mathbf{u}_1 = \frac{\mathbf{w}_1}{\|\mathbf{w}_1\|} = \frac{1}{\sqrt{2}}(1, 1) = \frac{\sqrt{2}}{2}(1, 1) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$

$$\mathbf{u}_2 = \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|} = \frac{1}{1/\sqrt{2}}\left(-\frac{1}{2}, \frac{1}{2}\right) = \sqrt{2}\left(-\frac{1}{2}, \frac{1}{2}\right) = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right).$$

So, $B'' = \{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthonormal basis for $R^2$. See Figure 5.20.




Given basis: $B = \{\mathbf{v}_1, \mathbf{v}_2\}$




Orthonormal basis: $B'' = \{\mathbf{u}_1, \mathbf{u}_2\}$

**Figure 5.20**

## EXAMPLE 7 Applying the Gram-Schmidt Orthonormalization Process

Apply the Gram-Schmidt orthonormalization process to the following basis for $R^3$.

$$B = \{\overset{\mathbf{v}_1}{(1, 1, 0)}, \overset{\mathbf{v}_2}{(1, 2, 0)}, \overset{\mathbf{v}_3}{(0, 1, 2)}\}$$

**SOLUTION**

Applying the Gram-Schmidt orthonormalization process produces

$$\mathbf{w}_1 = \mathbf{v}_1 = (1, 1, 0)$$

$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1}\mathbf{w}_1 = (1, 2, 0) - \frac{3}{2}(1, 1, 0) = \left(-\frac{1}{2}, \frac{1}{2}, 0\right)$$

$$\begin{aligned}\mathbf{w}_3 &= \mathbf{v}_3 - \frac{\mathbf{v}_3 \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1}\mathbf{w}_1 - \frac{\mathbf{v}_3 \cdot \mathbf{w}_2}{\mathbf{w}_2 \cdot \mathbf{w}_2}\mathbf{w}_2 \\ &= (0, 1, 2) - \frac{1}{2}(1, 1, 0) - \frac{1/2}{1/2}\left(-\frac{1}{2}, \frac{1}{2}, 0\right) \\ &= (0, 0, 2).\end{aligned}$$

The set $B' = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is an orthogonal basis for $R^3$. Normalizing each vector in $B'$ produces

$$\mathbf{u}_1 = \frac{\mathbf{w}_1}{\|\mathbf{w}_1\|} = \frac{1}{\sqrt{2}}(1, 1, 0) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right)$$

$$\mathbf{u}_2 = \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|} = \frac{1}{1/\sqrt{2}}\left(-\frac{1}{2}, \frac{1}{2}, 0\right) = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right)$$

$$\mathbf{u}_3 = \frac{\mathbf{w}_3}{\|\mathbf{w}_3\|} = \frac{1}{2}(0, 0, 2) = (0, 0, 1).$$

So, $B'' = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal basis for $R^3$.

Examples 6 and 7 apply the Gram-Schmidt orthonormalization process to bases for $R^2$ and $R^3$. The process works equally well for a subspace of an inner product space. The next example demonstrates this procedure.

## EXAMPLE 8 Applying the Gram-Schmidt Orthonormalization Process

The vectors $\mathbf{v}_1 = (0, 1, 0)$ and $\mathbf{v}_2 = (1, 1, 1)$ span a plane in $R^3$. Find an orthonormal basis for this subspace.

### SOLUTION

Applying the Gram-Schmidt orthonormalization process produces

$$\mathbf{w}_1 = \mathbf{v}_1 = (0, 1, 0)$$

$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1}\mathbf{w}_1 = (1, 1, 1) - \frac{1}{1}(0, 1, 0) = (1, 0, 1).$$

Normalizing $\mathbf{w}_1$ and $\mathbf{w}_2$ produces the orthonormal set

$$\mathbf{u}_1 = \frac{\mathbf{w}_1}{\|\mathbf{w}_1\|} = (0, 1, 0)$$

$$\mathbf{u}_2 = \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|} = \frac{1}{\sqrt{2}}(1, 0, 1) = \left(\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right).$$

See Figure 5.21.

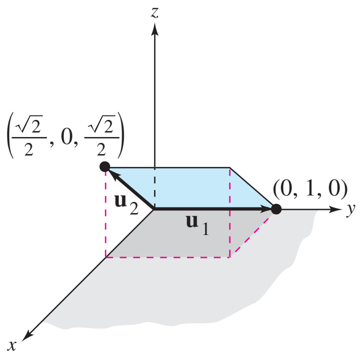



**Figure 5.21**

## EXAMPLE 9 Applying the Gram-Schmidt Orthonormalization Process (Calculus)

Apply the Gram-Schmidt orthonormalization process to the basis $B = \{1, x, x^2\}$ in $P_2$, using the inner product

$$\langle p, q \rangle = \int_{-1}^{1} p(x)q(x)\, dx.$$

### SOLUTION

Let $B = \{1, x, x^2\} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Then you have

$$\mathbf{w}_1 = \mathbf{v}_1 = 1$$

$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{w}_1 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle}\mathbf{w}_1 = x - \frac{0}{2}(1) = x$$

$$\begin{aligned}\mathbf{w}_3 &= \mathbf{v}_3 - \frac{\langle \mathbf{v}_3, \mathbf{w}_1 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle}\mathbf{w}_1 - \frac{\langle \mathbf{v}_3, \mathbf{w}_2 \rangle}{\langle \mathbf{w}_2, \mathbf{w}_2 \rangle}\mathbf{w}_2 \\ &= x^2 - \frac{2/3}{2}(1) - \frac{0}{2/3}(x) \\ &= x^2 - 1/3.\end{aligned}$$

Now, by normalizing $B' = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$, you have

$$\mathbf{u}_1 = \frac{\mathbf{w}_1}{\|\mathbf{w}_1\|} = \frac{1}{\sqrt{2}}(1) = \frac{1}{\sqrt{2}}$$

$$\mathbf{u}_2 = \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|} = \frac{1}{\sqrt{2/3}}(x) = \frac{\sqrt{3}}{\sqrt{2}}x$$

$$\mathbf{u}_3 = \frac{\mathbf{w}_3}{\|\mathbf{w}_3\|} = \frac{1}{\sqrt{8/45}}\left(x^2 - \frac{1}{3}\right) = \frac{\sqrt{5}}{2\sqrt{2}}(3x^2 - 1).$$

In Exercises 45–50, you are asked to verify these calculations.

**REMARK**

The polynomials $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ in Example 9 are called the first three **normalized Legendre polynomials,** after the French mathematician Adrien-Marie Legendre (1752–1833).

The computations in the Gram-Schmidt orthonormalization process are sometimes simpler when each vector $\mathbf{w}_i$ is normalized *before* it is used to determine the next vector. This **alternative form of the Gram-Schmidt orthonormalization process** has the following steps.

$$\mathbf{u}_1 = \frac{\mathbf{w}_1}{\|\mathbf{w}_1\|} = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|}$$

$$\mathbf{u}_2 = \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|}, \text{ where } \mathbf{w}_2 = \mathbf{v}_2 - \langle \mathbf{v}_2, \mathbf{u}_1 \rangle \mathbf{u}_1$$

$$\mathbf{u}_3 = \frac{\mathbf{w}_3}{\|\mathbf{w}_3\|}, \text{ where } \mathbf{w}_3 = \mathbf{v}_3 - \langle \mathbf{v}_3, \mathbf{u}_1 \rangle \mathbf{u}_1 - \langle \mathbf{v}_3, \mathbf{u}_2 \rangle \mathbf{u}_2$$

$$\vdots$$

$$\mathbf{u}_n = \frac{\mathbf{w}_n}{\|\mathbf{w}_n\|}, \text{ where } \mathbf{w}_n = \mathbf{v}_n - \langle \mathbf{v}_n, \mathbf{u}_1 \rangle \mathbf{u}_1 - \cdots - \langle \mathbf{v}_n, \mathbf{u}_{n-1} \rangle \mathbf{u}_{n-1}$$

## EXAMPLE 10 Alternative Form of the Gram-Schmidt Orthonormalization Process

Find an orthonormal basis for the solution space of the homogeneous linear system.

$$\begin{aligned} x_1 + x_2 \phantom{{}+2x_3} + 7x_4 &= 0 \\ 2x_1 + x_2 + 2x_3 + 6x_4 &= 0 \end{aligned}$$

### SOLUTION

The augmented matrix for this system reduces as follows.

$$\begin{bmatrix} 1 & 1 & 0 & 7 & 0 \\ 2 & 1 & 2 & 6 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 2 & -1 & 0 \\ 0 & 1 & -2 & 8 & 0 \end{bmatrix}$$

If you let $x_3 = s$ and $x_4 = t$, then each solution of the system has the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2s + t \\ 2s - 8t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} -2 \\ 2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ -8 \\ 0 \\ 1 \end{bmatrix}.$$

So, one basis for the solution space is

$$B = \{\mathbf{v}_1, \mathbf{v}_2\} = \{(-2, 2, 1, 0), (1, -8, 0, 1)\}.$$

To find an orthonormal basis $B' = \{\mathbf{u}_1, \mathbf{u}_2\}$, use the alternative form of the Gram-Schmidt orthonormalization process, as follows.

$$\begin{aligned} \mathbf{u}_1 &= \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} \\ &= \left(-\frac{2}{3}, \frac{2}{3}, \frac{1}{3}, 0\right) \end{aligned}$$

$$\begin{aligned} \mathbf{w}_2 &= \mathbf{v}_2 - \langle \mathbf{v}_2, \mathbf{u}_1 \rangle \mathbf{u}_1 \\ &= (1, -8, 0, 1) - \left[(1, -8, 0, 1) \cdot \left(-\frac{2}{3}, \frac{2}{3}, \frac{1}{3}, 0\right)\right]\left(-\frac{2}{3}, \frac{2}{3}, \frac{1}{3}, 0\right) \\ &= (-3, -4, 2, 1) \end{aligned}$$

$$\begin{aligned} \mathbf{u}_2 &= \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|} \\ &= \left(-\frac{3}{\sqrt{30}}, -\frac{4}{\sqrt{30}}, \frac{2}{\sqrt{30}}, \frac{1}{\sqrt{30}}\right) \end{aligned}$$

# 5.3 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

**Orthogonal and Orthonormal Sets** **In Exercises 1–14, (a) determine whether the set of vectors in $R^n$ is orthogonal, (b) if the set is orthogonal, then determine whether it is also orthonormal, and (c) determine whether the set is a basis for $R^n$.**

**1.** $\{(2, -4), (2, 1)\}$
**2.** $\{(3, -2), (-4, -6)\}$
**3.** $\{(-4, 6), (5, 0)\}$
**4.** $\{(11, 4), (8, -3)\}$
**5.** $\left\{\left(\frac{3}{5}, \frac{4}{5}\right), \left(-\frac{4}{5}, \frac{3}{5}\right)\right\}$
**6.** $\left\{(1, 2), \left(-\frac{2}{5}, \frac{1}{5}\right)\right\}$
**7.** $\{(4, -1, 1), (-1, 0, 4), (-4, -17, -1)\}$
**8.** $\{(2, -4, 2), (0, 2, 4), (-10, -4, 2)\}$
**9.** $\left\{\left(\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right), \left(-\frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{3}, \frac{\sqrt{6}}{6}\right), \left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}\right)\right\}$
**10.** $\left\{\left(\frac{\sqrt{2}}{3}, 0, -\frac{\sqrt{2}}{6}\right), \left(0, \frac{2\sqrt{5}}{5}, -\frac{\sqrt{5}}{5}\right), \left(\frac{\sqrt{5}}{5}, 0, \frac{1}{2}\right)\right\}$
**11.** $\{(2, 5, -3), (4, 2, 6)\}$
**12.** $\{(-6, 3, 2, 1), (2, 0, 6, 0)\}$
**13.** $\left\{\left(\frac{\sqrt{2}}{2}, 0, 0, \frac{\sqrt{2}}{2}\right), \left(0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right), \left(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right)\right\}$
**14.** $\left\{\left(\frac{\sqrt{10}}{10}, 0, 0, \frac{3\sqrt{10}}{10}\right), (0, 0, 1, 0), (0, 1, 0, 0), \left(-\frac{3\sqrt{10}}{10}, 0, 0, \frac{\sqrt{10}}{10}\right)\right\}$

**Normalizing an Orthogonal Set** **In Exercises 15–18, (a) show that the set of vectors in $R^n$ is orthogonal, and (b) normalize the set to produce an orthonormal set.**

**15.** $\{(-1, 4), (8, 2)\}$
**16.** $\{(2, -5), (10, 4)\}$
**17.** $\left\{\left(\sqrt{3}, \sqrt{3}, \sqrt{3}\right), \left(-\sqrt{2}, 0, \sqrt{2}\right)\right\}$
**18.** $\left\{\left(-\frac{2}{15}, \frac{1}{15}, \frac{2}{15}\right), \left(\frac{1}{15}, \frac{2}{15}, 0\right)\right\}$

**19.** Complete Example 2 by verifying that $\{1, x, x^2, x^3\}$ is an orthonormal basis for $P_3$ with the inner product $\langle p, q\rangle = a_0b_0 + a_1b_1 + a_2b_2 + a_3b_3$.

**20.** Verify that $\{(\sin\theta, \cos\theta), (\cos\theta, -\sin\theta)\}$ is an orthonormal basis for $R^2$.

**Finding a Coordinate Matrix** **In Exercises 21–26, find the coordinate matrix of x relative to the orthonormal basis $B$ in $R^n$.**

**21.** $\mathbf{x} = (1, 2)$, $B = \left\{\left(-\frac{2\sqrt{13}}{13}, \frac{3\sqrt{13}}{13}\right), \left(\frac{3\sqrt{13}}{13}, \frac{2\sqrt{13}}{13}\right)\right\}$
**22.** $\mathbf{x} = (-3, 4)$, $B = \left\{\left(\frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5}\right), \left(-\frac{2\sqrt{5}}{5}, \frac{\sqrt{5}}{5}\right)\right\}$
**23.** $\mathbf{x} = (2, -2, 1)$,
$B = \left\{\left(\frac{\sqrt{10}}{10}, 0, \frac{3\sqrt{10}}{10}\right), (0, 1, 0), \left(-\frac{3\sqrt{10}}{10}, 0, \frac{\sqrt{10}}{10}\right)\right\}$
**24.** $\mathbf{x} = (3, -5, 11)$, $B = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$
**25.** $\mathbf{x} = (5, 10, 15)$, $B = \left\{\left(\frac{3}{5}, \frac{4}{5}, 0\right), \left(-\frac{4}{5}, \frac{3}{5}, 0\right), (0, 0, 1)\right\}$
**26.** $\mathbf{x} = (2, -1, 4, 3)$,
$B = \left\{\left(\frac{5}{13}, 0, \frac{12}{13}, 0\right), (0, 1, 0, 0), \left(-\frac{12}{13}, 0, \frac{5}{13}, 0\right), (0, 0, 0, 1)\right\}$

**Applying the Gram-Schmidt Process** **In Exercises 27–36, apply the Gram-Schmidt orthonormalization process to transform the given basis for $R^n$ into an orthonormal basis. Use the Euclidean inner product on $R^n$ and use the vectors in the order in which they are given.**

**27.** $B = \{(3, 4), (1, 0)\}$
**28.** $B = \{(1, 2), (-1, 0)\}$
**29.** $B = \{(0, 1), (2, 5)\}$
**30.** $B = \{(4, -3), (3, 2)\}$
**31.** $B = \{(1, -2, 2), (2, 2, 1), (2, -1, -2)\}$
**32.** $B = \{(1, 0, 0), (1, 1, 1), (1, 1, -1)\}$
**33.** $B = \{(4, -3, 0), (1, 2, 0), (0, 0, 4)\}$
**34.** $B = \{(0, 1, 2), (2, 0, 0), (1, 1, 1)\}$
**35.** $B = \{(0, 1, 1), (1, 1, 0), (1, 0, 1)\}$
**36.** $B = \{(3, 4, 0, 0), (-1, 1, 0, 0), (2, 1, 0, -1), (0, 1, 1, 0)\}$

**Applying the Gram-Schmidt Process** **In Exercises 37–42, apply the Gram-Schmidt orthonormalization process to transform the given basis for a subspace of $R^n$ into an orthonormal basis for the subspace. Use the Euclidean inner product on $R^n$ and use the vectors in the order in which they are given.**

**37.** $B = \{(-8, 3, 5)\}$
**38.** $B = \{(4, -7, 6)\}$
**39.** $B = \{(3, 4, 0), (2, 0, 0)\}$
**40.** $B = \{(1, 2, 0), (2, 0, -2)\}$
**41.** $B = \{(1, 2, -1, 0), (2, 2, 0, 1), (1, 1, -1, 0)\}$
**42.** $B = \{(7, 24, 0, 0), (0, 0, 1, 1), (0, 0, 1, -2)\}$

**43.** Use the inner product $\langle \mathbf{u}, \mathbf{v}\rangle = 2u_1v_1 + u_2v_2$ in $R^2$ and the Gram-Schmidt orthonormalization process to transform $\{(2, -1), (-2, 10)\}$ into an orthonormal basis.

**44. Writing** Explain why the result of Exercise 43 is not an orthonormal basis when the Euclidean inner product on $R^2$ is used.

**Calculus** **In Exercises 45–50, let $B = \{1, x, x^2\}$ be a basis for $P_2$ with the inner product**

$$\langle p, q\rangle = \int_{-1}^{1} p(x)q(x)\,dx.$$

**Complete Example 9 by verifying the indicated inner products.**

**45.** $\langle x, 1\rangle = 0$
**46.** $\langle 1, 1\rangle = 2$
**47.** $\langle x^2, 1\rangle = \frac{2}{3}$
**48.** $\langle x^2, x\rangle = 0$
**49.** $\langle x, x\rangle = \frac{2}{3}$
**50.** $\left\langle x^2 - \frac{1}{3}, x^2 - \frac{1}{3}\right\rangle = \frac{8}{45}$

**Applying the Alternative Form of the Gram-Schmidt Process** **In Exercises 51–55, apply the alternative form of the Gram-Schmidt orthonormalization process to find an orthonormal basis for the solution space of the homogeneous linear system.**

**51.** $\begin{aligned} 2x_1 + x_2 - 6x_3 + 2x_4 &= 0 \\ x_1 + 2x_2 - 3x_3 + 4x_4 &= 0 \\ x_1 + x_2 - 3x_3 + 2x_4 &= 0 \end{aligned}$

**52.** $\begin{aligned} x_1 + x_2 - x_3 - x_4 &= 0 \\ 2x_1 + x_2 - 2x_3 - 2x_4 &= 0 \end{aligned}$

**53.** $\begin{aligned} x_1 - x_2 + x_3 + x_4 &= 0 \\ x_1 - 2x_2 + x_3 + x_4 &= 0 \end{aligned}$

**54.** $x_1 + 3x_2 - 3x_3 = 0$ **55.** $x_1 - 2x_2 + x_3 = 0$

**56.** **CAPSTONE** Let $B$ be a basis for an inner product space $V$. Explain how to apply the Gram-Schmidt orthonormalization process to form an orthonormal basis $B'$ for $V$.

**True or False?** **In Exercises 57 and 58, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.**

**57.** (a) A set $S$ of vectors in an inner product space $V$ is orthogonal when every pair of vectors in $S$ is orthogonal.

(b) An orthonormal basis derived by the Gram-Schmidt orthonormalization process does not depend on the order of the vectors in the basis.

**58.** (a) A set $S$ of vectors in an inner product space $V$ is orthonormal when every vector is a unit vector and each pair of vectors is orthogonal.

(b) If a set of nonzero vectors $S$ in an inner product space $V$ is orthogonal, then $S$ is linearly independent.

**Orthonormal Sets in $P_2$** **In Exercises 59–64, let $p(x) = a_0 + a_1x + a_2x^2$ and $q(x) = b_0 + b_1x + b_2x^2$ be vectors in $P_2$ with $\langle p, q\rangle = a_0b_0 + a_1b_1 + a_2b_2$. Determine whether the given second-degree polynomials form an orthonormal set, and if not, then apply the Gram-Schmidt orthonormalization process to form an orthonormal set.**

**59.** $\left\{\dfrac{x^2+1}{\sqrt{2}}, \dfrac{x^2+x-1}{\sqrt{3}}\right\}$

**60.** $\left\{\sqrt{2}(x^2-1), \sqrt{2}(x^2+x+2)\right\}$

**61.** $\{x^2, x^2+2x, x^2+2x+1\}$ **62.** $\{1, x, x^2\}$

**63.** $\left\{\dfrac{3x^2+4x}{5}, \dfrac{-4x^2+3x}{5}, 1\right\}$ **64.** $\{x^2-1, x-1\}$

**65.** **Proof** Let $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ be an orthonormal basis for $R^n$. Prove that

$$\|\mathbf{v}\|^2 = |\mathbf{v}\cdot\mathbf{u}_1|^2 + |\mathbf{v}\cdot\mathbf{u}_2|^2 + \cdots + |\mathbf{v}\cdot\mathbf{u}_n|^2$$

for any vector $\mathbf{v}$ in $R^n$. This equation is called **Parseval's equality.**

**66.** **Guided Proof** Prove that if $\mathbf{w}$ is orthogonal to each vector in $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, then $\mathbf{w}$ is orthogonal to every linear combination of vectors in $S$.

Getting Started: To prove that $\mathbf{w}$ is orthogonal to every linear combination of vectors in $S$, you need to show that their dot product is 0.

(i) Write $\mathbf{v}$ as a linear combination of vectors, with arbitrary scalars $c_1, \ldots, c_n$, in $S$.

(ii) Form the inner product of $\mathbf{w}$ and $\mathbf{v}$.

(iii) Use the properties of inner products to rewrite the inner product $\langle\mathbf{w}, \mathbf{v}\rangle$ as a linear combination of the inner products $\langle\mathbf{w}, \mathbf{v}_i\rangle$, $i = 1, \ldots, n$.

(iv) Use the fact that $\mathbf{w}$ is orthogonal to each vector in $S$ to lead to the conclusion that $\mathbf{w}$ is orthogonal to $\mathbf{v}$.

**67.** **Proof** Let $P$ be an $n \times n$ matrix. Prove that the following conditions are equivalent.

(a) $P^{-1} = P^T$. (Such a matrix is called *orthogonal*.)

(b) The row vectors of $P$ form an orthonormal basis for $R^n$.

(c) The column vectors of $P$ form an orthonormal basis for $R^n$.

**68.** **Proof** Let $W$ be a subspace of $R^n$. Prove that the intersection of $W$ and $W^\perp$ is $\{\mathbf{0}\}$, where $W^\perp$ is the subspace of $R^n$ given by

$W^\perp = \{\mathbf{v}: \mathbf{w}\cdot\mathbf{v} = 0 \text{ for every } \mathbf{w} \text{ in } W\}$.

**Fundamental Subspaces** **In Exercises 69 and 70, find bases for the four fundamental subspaces of the matrix $A$, as follows.**

**$N(A)$ = nullspace of $A$ $\quad N(A^T)$ = nullspace of $A^T$**

**$R(A)$ = column space of $A$ $\quad R(A^T)$ = column space of $A^T$**

**Then show that $N(A) = R(A^T)^\perp$ and $N(A^T) = R(A)^\perp$.**

**69.** $\begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 1 \\ 1 & 3 & 0 \end{bmatrix}$ **70.** $\begin{bmatrix} 0 & 1 & -1 \\ 0 & -2 & 2 \\ 0 & -1 & 1 \end{bmatrix}$

**71.** Let $A$ be a general $m \times n$ matrix.

(a) Explain why $R(A^T)$ is the same as the row space of $A$.

(b) Prove that $N(A) \subset R(A^T)^\perp$.

(c) Prove that $N(A) = R(A^T)^\perp$.

(d) Prove that $N(A^T) = R(A)^\perp$.

**72.** Find an orthonormal basis for $R^4$ that includes the vectors

$$\mathbf{v}_1 = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0\right) \text{ and } \mathbf{v}_2 = \left(0, -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right).$$

# 5.4 Mathematical Models and Least Squares Analysis

- Define the least squares problem.
- Find the orthogonal complement of a subspace and the projection of a vector onto a subspace.
- Find the four fundamental subspaces of a matrix.
- Solve a least squares problem.
- Use least squares for mathematical modeling.

## THE LEAST SQUARES PROBLEM

In this section, you will study *inconsistent* systems of linear equations and learn how to find the "best possible solution" of such a system. The necessity of "solving" inconsistent systems arises in the computation of least squares regression lines, as illustrated in Example 1.

### EXAMPLE 1 Least Squares Regression Line

Let $(1, 0)$, $(2, 1)$, and $(3, 3)$ be three points in $R^2$, as shown in Figure 5.22. How can you find the line $y = c_0 + c_1x$ that "best fits" these points? One way is to note that if the three points were collinear, then the following system of equations would be consistent.

$$\begin{aligned} c_0 + \phantom{2}c_1 &= 0 \\ c_0 + 2c_1 &= 1 \\ c_0 + 3c_1 &= 3 \end{aligned}$$

This system can be written in the matrix form $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}, \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} c_0 \\ c_1 \end{bmatrix}.$$

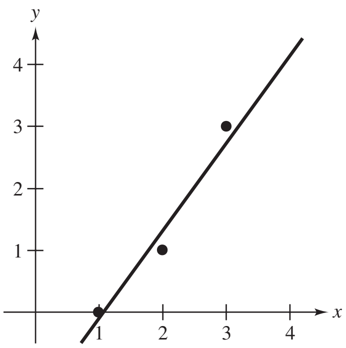


**Figure 5.22**

Because the points are not collinear, however, the system is inconsistent. Although it is impossible to find $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{b}$, you can look for an $\mathbf{x}$ that *minimizes* the norm of the error $\|A\mathbf{x} - \mathbf{b}\|$. The solution $\mathbf{x} = [c_0 \quad c_1]^T$ of this minimization problem is called the **least squares regression line** $y = c_0 + c_1x$.

In Section 2.5, you briefly studied the least squares regression line and how to calculate it using matrices. Now you will combine the ideas of orthogonality and projection to develop this concept in more generality. To begin, consider the linear system $A\mathbf{x} = \mathbf{b}$, where $A$ is an $m \times n$ matrix and $\mathbf{b}$ is a column vector in $R^m$. You know how to use Gaussian elimination with back-substitution to solve for $\mathbf{x}$ when the system is consistent. When the system is inconsistent, however, it is still useful to find the "best possible" solution; that is, the value of $\mathbf{x}$ for which the difference between $A\mathbf{x}$ and $\mathbf{b}$ is smallest. One way to define "best possible" is to require that the norm of $A\mathbf{x} - \mathbf{b}$ be minimized. This definition is the heart of the **least squares problem.**

**REMARK**

The term **least squares** comes from the fact that minimizing $\|A\mathbf{x} - \mathbf{b}\|$ is equivalent to minimizing $\|A\mathbf{x} - \mathbf{b}\|^2$, which is a sum of squares.

**Least Squares Problem**

Given an $m \times n$ matrix $A$ and a vector $\mathbf{b}$ in $R^m$, the **least squares problem** is to find $\mathbf{x}$ in $R^n$ such that $\|A\mathbf{x} - \mathbf{b}\|^2$ is minimized.

## ORTHOGONAL SUBSPACES

To solve the least squares problem, you first need to develop the concept of orthogonal subspaces. Two subspaces of $R^n$ are **orthogonal** when the vectors in each subspace are orthogonal to the vectors in the other subspace.

### Definition of Orthogonal Subspaces

The subspaces $S_1$ and $S_2$ of $R^n$ are **orthogonal** when $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$ for all $\mathbf{v}_1$ in $S_1$ and all $\mathbf{v}_2$ in $S_2$.

### EXAMPLE 2 Orthogonal Subspaces

The subspaces

$$S_1 = \operatorname{span}\left(\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}\right) \quad \text{and} \quad S_2 = \operatorname{span}\left(\begin{bmatrix}-1\\1\\1\end{bmatrix}\right)$$

are orthogonal because the dot product of any vector in $S_1$ and any vector in $S_2$ is zero.

Notice in Example 2 that the zero vector is the only vector common to both $S_1$ and $S_2$. This is true in general. If $S_1$ and $S_2$ are orthogonal subspaces of $R^n$, then their intersection consists of only the zero vector. You are asked to prove this in Exercise 43.

Given a subspace $S$ of $R^n$, the set of all vectors orthogonal to every vector in $S$ is called the **orthogonal complement** of $S$, as stated in the next definition.

### Definition of Orthogonal Complement

If $S$ is a subspace of $R^n$, then the **orthogonal complement** of $S$ is the set $S^{\perp} = \{\mathbf{u} \in R^n : \mathbf{v} \cdot \mathbf{u} = 0 \text{ for all vectors } \mathbf{v} \in S\}$.

The orthogonal complement of the trivial subspace $\{\mathbf{0}\}$ is all of $R^n$, and, conversely, the orthogonal complement of $R^n$ is the trivial subspace $\{\mathbf{0}\}$. In Example 2, the subspace $S_1$ is the orthogonal complement of $S_2$, and the subspace $S_2$ is the orthogonal complement of $S_1$. The orthogonal complement of a subspace of $R^n$ is itself a subspace of $R^n$ (see Exercise 44). You can find the orthogonal complement of a subspace of $R^n$ by finding the nullspace of a matrix, as illustrated in the next example.

**LINEAR ALGEBRA APPLIED**

The least squares problem has a wide variety of real-life applications. To illustrate, in Examples 9 and 10 and Exercises 37, 38, and 39, you will use least squares analysis to solve problems involving such diverse subject matter as world population, astronomy, doctoral degrees awarded, revenues for General Dynamics Corporation, and galloping speeds of animals. In each of these applications, you will be given a set of data and you will be asked to come up with mathematical model(s) for the data. For instance, in Exercise 38, you are given the annual revenues from 2005 through 2010 for General Dynamics Corporation, and you are asked to find the least squares regression quadratic and cubic polynomials for the data. With these models, you are asked to predict the revenue for the year 2015, and you are asked to decide which of the models appears to be more accurate for predicting future revenues.

## EXAMPLE 3 Finding the Orthogonal Complement

Find the orthogonal complement of the subspace $S$ of $R^4$ spanned by the two column vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ of the matrix $A$.

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$\qquad\ \ \mathbf{v}_1 \quad \mathbf{v}_2$$

### SOLUTION

A vector $\mathbf{u} \in R^4$ will be in the orthogonal complement of $S$ when its dot product with the two columns of $A$, $\mathbf{v}_1$ and $\mathbf{v}_2$, is zero. The orthogonal complement of $S$ consists of all the vectors $\mathbf{u}$ such that $A^T\mathbf{u} = \mathbf{0}$.

$$A^T\mathbf{u} = \mathbf{0}$$

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

That is, the orthogonal complement of $S$ is the nullspace of the matrix $A^T$:

$$S^\perp = N(A^T).$$

Using the techniques for solving homogeneous linear systems, you can find that a basis for the orthogonal complement consists of the vectors

$$\mathbf{u}_1 = [-2 \quad 1 \quad 0 \quad 0]^T \quad \text{and} \quad \mathbf{u}_2 = [-1 \quad 0 \quad 1 \quad 0]^T.$$

Notice that $R^4$ in Example 3 is split into two subspaces, $S = \text{span}(\mathbf{v}_1, \mathbf{v}_2)$ and $S^\perp = \text{span}(\mathbf{u}_1, \mathbf{u}_2)$. In fact, the four vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{u}_1$, and $\mathbf{u}_2$ form a basis for $R^4$. Each vector in $R^4$ can be *uniquely* written as a sum of a vector from $S$ and a vector from $S^\perp$. The next definition generalizes this concept.

### Definition of Direct Sum

Let $S_1$ and $S_2$ be two subspaces of $R^n$. If each vector $\mathbf{x} \in R^n$ can be uniquely written as a sum of a vector $\mathbf{s}_1$ from $S_1$ and a vector $\mathbf{s}_2$ from $S_2$, $\mathbf{x} = \mathbf{s}_1 + \mathbf{s}_2$, then $R^n$ is the **direct sum** of $S_1$ and $S_2$, and you can write $R^n = S_1 \oplus S_2$.

## EXAMPLE 4 Direct Sum

**a.** From Example 2, you can see that $R^3$ is the direct sum of the subspaces

$$S_1 = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right) \quad \text{and} \quad S_2 = \text{span}\left(\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}\right).$$

**b.** From Example 3, you can see that $R^4 = S \oplus S^\perp$, where

$$S = \text{span}\left(\begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right) \quad \text{and} \quad S^\perp = \text{span}\left(\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right).$$

The next theorem lists some important facts about orthogonal complements and direct sums.

**THEOREM 5.13 Properties of Orthogonal Subspaces**

Let $S$ be a subspace of $R^n$. Then the following properties are true.

**1.** $\dim(S) + \dim(S^\perp) = n$
**2.** $R^n = S \oplus S^\perp$
**3.** $(S^\perp)^\perp = S$

**PROOF**

**1.** If $S = R^n$ or $S = \{\mathbf{0}\}$, then Property 1 is trivial. So let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_t\}$ be a basis for $S$, $0 < t < n$. Let $A$ be the $n \times t$ matrix whose columns are the basis vectors $\mathbf{v}_i$. Then $S = R(A)$ (the column space of $A$), which implies that $S^\perp = N(A^T)$, where $A^T$ is a $t \times n$ matrix of rank $t$ (see Section 5.3, Exercise 71). Because the dimension of $N(A^T)$ is $n - t$, you have shown that

$$\dim(S) + \dim(S^\perp) = t + (n - t) = n.$$

**2.** If $S = R^n$ or $S = \{\mathbf{0}\}$, then Property 2 is trivial. So let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_t\}$ be a basis for $S$ and let $\{\mathbf{v}_{t+1}, \mathbf{v}_{t+2}, \ldots, \mathbf{v}_n\}$ be a basis for $S^\perp$. It can be shown that the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_t, \mathbf{v}_{t+1}, \ldots, \mathbf{v}_n\}$ is linearly independent and forms a basis for $R^n$. Let $\mathbf{x} \in R^n$, $\mathbf{x} = c_1\mathbf{v}_1 + \cdots + c_t\mathbf{v}_t + c_{t+1}\mathbf{v}_{t+1} + \cdots + c_n\mathbf{v}_n$. If you write $\mathbf{v} = c_1\mathbf{v}_1 + \cdots + c_t\mathbf{v}_t$ and $\mathbf{w} = c_{t+1}\mathbf{v}_{t+1} + \cdots + c_n\mathbf{v}_n$, then you have expressed an arbitrary vector $\mathbf{x}$ as the sum of a vector from $S$ and a vector from $S^\perp$, $\mathbf{x} = \mathbf{v} + \mathbf{w}$.

To show the uniqueness of this representation, assume $\mathbf{x} = \mathbf{v} + \mathbf{w} = \hat{\mathbf{v}} + \hat{\mathbf{w}}$ (where $\hat{\mathbf{v}}$ is in $S$ and $\hat{\mathbf{w}}$ is in $S^\perp$). This implies that $\hat{\mathbf{v}} - \mathbf{v} = \mathbf{w} - \hat{\mathbf{w}}$. So, the two vectors $\hat{\mathbf{v}} - \mathbf{v}$ and $\mathbf{w} - \hat{\mathbf{w}}$ are in both $S$ and $S^\perp$. Because $S \cap S^\perp = \{\mathbf{0}\}$, you must have $\hat{\mathbf{v}} = \mathbf{v}$ and $\mathbf{w} = \hat{\mathbf{w}}$.

**3.** Let $\mathbf{v} \in S$. Then $\mathbf{v} \cdot \mathbf{u} = 0$ for all $\mathbf{u} \in S^\perp$, which implies that $\mathbf{v} \in (S^\perp)^\perp$. On the other hand, if $\mathbf{v} \in (S^\perp)^\perp$, then, because $R^n = S \oplus S^\perp$, you can write $\mathbf{v}$ as the unique sum of the vector from $S$ and a vector from $S^\perp$, $\mathbf{v} = \mathbf{s} + \mathbf{w}$, $\mathbf{s} \in S$, $\mathbf{w} \in S^\perp$. Because $\mathbf{w}$ is in $S^\perp$, it is orthogonal to every vector in $S$, and in particular to $\mathbf{v}$. So,

$$0 = \mathbf{w} \cdot \mathbf{v} = \mathbf{w} \cdot (\mathbf{s} + \mathbf{w}) = \mathbf{w} \cdot \mathbf{s} + \mathbf{w} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{w}.$$

This implies that $\mathbf{w} = \mathbf{0}$ and $\mathbf{v} = \mathbf{s} + \mathbf{w} = \mathbf{s} \in S$.

You studied the projection of one vector onto another in Section 5.2. This is now generalized to projections of a vector $\mathbf{v}$ onto a subspace $S$. Because $R^n = S \oplus S^\perp$, every vector $\mathbf{v}$ in $R^n$ can be uniquely written as a sum of a vector from $S$ and a vector from $S^\perp$:

$$\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2, \quad \mathbf{v}_1 \in S, \quad \mathbf{v}_2 \in S^\perp.$$

The vector $\mathbf{v}_1$ is called the **projection** of $\mathbf{v}$ onto the subspace $S$, and is denoted by $\mathbf{v}_1 = \operatorname{proj}_S\mathbf{v}$. So, $\mathbf{v}_2 = \mathbf{v} - \mathbf{v}_1 = \mathbf{v} - \operatorname{proj}_S\mathbf{v}$, which implies that the vector $\mathbf{v} - \operatorname{proj}_S\mathbf{v}$ is orthogonal to the subspace $S$.

Given a subspace $S$ of $R^n$, you can apply the Gram-Schmidt orthonormalization process to calculate an orthonormal basis for $S$. It is then a relatively easy matter to compute the projection of a vector $\mathbf{v}$ onto $S$ using the next theorem. (You are asked to prove this theorem in Exercise 45.)

**THEOREM 5.14 Projection onto a Subspace**

If $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_t\}$ is an orthonormal basis for the subspace $S$ of $R^n$, and $\mathbf{v} \in R^n$, then

$$\operatorname{proj}_S\mathbf{v} = (\mathbf{v} \cdot \mathbf{u}_1)\mathbf{u}_1 + (\mathbf{v} \cdot \mathbf{u}_2)\mathbf{u}_2 + \cdots + (\mathbf{v} \cdot \mathbf{u}_t)\mathbf{u}_t.$$

## EXAMPLE 5 Projection Onto a Subspace

Find the projection of the vector $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$ onto the subspace $S$ of $R^3$ spanned by the vectors

$$\mathbf{w}_1 = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{w}_2 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}.$$

### SOLUTION

By normalizing $\mathbf{w}_1$ and $\mathbf{w}_2$, you obtain an orthonormal basis for $S$.

$$\{\mathbf{u}_1, \mathbf{u}_2\} = \left\{\frac{1}{\sqrt{10}}\mathbf{w}_1, \frac{1}{2}\mathbf{w}_2\right\} = \left\{\begin{bmatrix} 0 \\ \frac{3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}$$

Use Theorem 5.14 to find the projection of $\mathbf{v}$ onto $S$.

$$\begin{aligned} \operatorname{proj}_S \mathbf{v} &= (\mathbf{v} \cdot \mathbf{u}_1)\mathbf{u}_1 + (\mathbf{v} \cdot \mathbf{u}_2)\mathbf{u}_2 \\ &= \frac{6}{\sqrt{10}} \begin{bmatrix} 0 \\ \frac{3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{bmatrix} + 1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{9}{5} \\ \frac{3}{5} \end{bmatrix} \end{aligned}$$

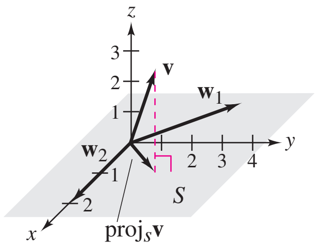



Figure 5.23

Figure 5.23 illustrates the projection of $\mathbf{v}$ onto the plane $S$.

Theorem 5.9 states that among all the scalar multiples of a vector $\mathbf{u}$, the orthogonal projection of $\mathbf{v}$ onto $\mathbf{u}$ is the one closest to $\mathbf{v}$. Example 5 suggests that this property is also true for projections onto subspaces. That is, among all the vectors in the subspace $S$, the vector $\operatorname{proj}_S \mathbf{v}$ is the closest vector to $\mathbf{v}$. Figure 5.24 illustrates these two results.

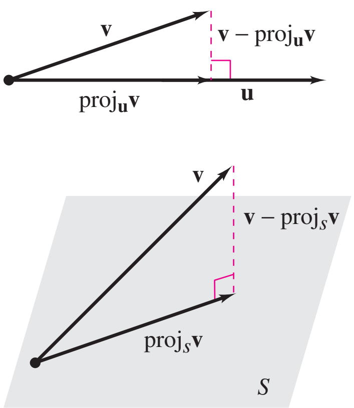



Figure 5.24

### THEOREM 5.15 Orthogonal Projection and Distance

Let $S$ be a subspace of $R^n$ and let $\mathbf{v} \in R^n$. Then, for all $\mathbf{u} \in S$, $\mathbf{u} \neq \operatorname{proj}_S \mathbf{v}$,

$$\|\mathbf{v} - \operatorname{proj}_S \mathbf{v}\| < \|\mathbf{v} - \mathbf{u}\|.$$

### PROOF

Let $\mathbf{u} \in S$, $\mathbf{u} \neq \operatorname{proj}_S \mathbf{v}$. By adding and subtracting the same quantity $\operatorname{proj}_S \mathbf{v}$ to and from the vector $\mathbf{v} - \mathbf{u}$, you obtain

$$\mathbf{v} - \mathbf{u} = (\mathbf{v} - \operatorname{proj}_S \mathbf{v}) + (\operatorname{proj}_S \mathbf{v} - \mathbf{u}).$$

Observe that $(\operatorname{proj}_S \mathbf{v} - \mathbf{u})$ is in $S$ and $(\mathbf{v} - \operatorname{proj}_S \mathbf{v})$ is orthogonal to $S$. So, $(\mathbf{v} - \operatorname{proj}_S \mathbf{v})$ and $(\operatorname{proj}_S \mathbf{v} - \mathbf{u})$ are orthogonal vectors, and you can use the Pythagorean Theorem (Theorem 5.6) to obtain

$$\|\mathbf{v} - \mathbf{u}\|^2 = \|\mathbf{v} - \operatorname{proj}_S \mathbf{v}\|^2 + \|\operatorname{proj}_S \mathbf{v} - \mathbf{u}\|^2.$$

Because $\mathbf{u} \neq \operatorname{proj}_S \mathbf{v}$, the second term on the right is positive, and you have

$$\|\mathbf{v} - \operatorname{proj}_S \mathbf{v}\| < \|\mathbf{v} - \mathbf{u}\|.$$

## FUNDAMENTAL SUBSPACES OF A MATRIX

Recall that if $A$ is an $m \times n$ matrix, then the column space of $A$ is a subspace of $R^m$ consisting of all vectors of the form $A\mathbf{x}$, $\mathbf{x} \in R^n$. The four **fundamental subspaces** of the matrix $A$ are defined as follows (see Exercises 69 and 70 in Section 5.3).

$$N(A) = \text{nullspace of } A \qquad N(A^T) = \text{nullspace of } A^T$$
$$R(A) = \text{column space of } A \qquad R(A^T) = \text{column space of } A^T$$

These subspaces play a crucial role in the solution of the least squares problem.

### EXAMPLE 6 Fundamental Subspaces

Find the four fundamental subspaces of the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

**SOLUTION**

The column space of $A$ is simply the span of the first and third columns, because the second column is a scalar multiple of the first column. The column space of $A^T$ is equivalent to the row space of $A$, which is spanned by the first two rows. The nullspace of $A$ is a solution space of the homogeneous system $A\mathbf{x} = \mathbf{0}$. Finally, the nullspace of $A^T$ is a solution space of the homogeneous system whose coefficient matrix is $A^T$. The following summarizes these results.

$$R(A) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right) \qquad R(A^T) = \text{span}\left(\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$$

$$N(A) = \text{span}\left(\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}\right) \qquad N(A^T) = \text{span}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right)$$

In Example 6, observe that $R(A)$ and $N(A^T)$ are orthogonal subspaces of $R^4$, and $R(A^T)$ and $N(A)$ are orthogonal subspaces of $R^3$. These and other properties of the four fundamental subspaces are stated in the next theorem.

### THEOREM 5.16 Fundamental Subspaces of a Matrix

If $A$ is an $m \times n$ matrix, then

**1.** $R(A)$ and $N(A^T)$ are orthogonal subspaces of $R^m$.
**2.** $R(A^T)$ and $N(A)$ are orthogonal subspaces of $R^n$.
**3.** $R(A) \oplus N(A^T) = R^m$.
**4.** $R(A^T) \oplus N(A) = R^n$.

**PROOF**

To prove Property 1, let $\mathbf{v} \in R(A)$ and $\mathbf{u} \in N(A^T)$. Because the column space of $A$ is equal to the row space of $A^T$, you can see that $A^T\mathbf{u} = \mathbf{0}$ implies $\mathbf{u} \cdot \mathbf{v} = 0$. Property 2 follows from applying Property 1 to $A^T$.

To prove Property 3, observe that $R(A)^\perp = N(A^T)$ and $R^m = R(A) \oplus R(A)^\perp$. So, $R^m = R(A) \oplus N(A^T)$. A similar argument applied to $R(A^T)$ proves Property 4.

## SOLVING THE LEAST SQUARES PROBLEM

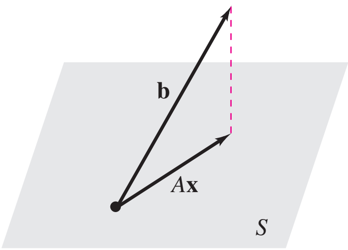


**Figure 5.25**

You have now developed all the tools needed to solve the least squares problem. Recall that you are attempting to find a vector $\mathbf{x}$ that minimizes $\|A\mathbf{x} - \mathbf{b}\|$, where $A$ is an $m \times n$ matrix and $\mathbf{b}$ is a vector in $R^m$. Let $S$ be the column space of $A$: $S = R(A)$. Assume that $\mathbf{b}$ is not in $S$, because otherwise the system $A\mathbf{x} = \mathbf{b}$ would be consistent. You are looking for a vector $A\mathbf{x}$ in $S$ that is as close as possible to $\mathbf{b}$, as indicated in Figure 5.25.

From Theorem 5.15, you know that the desired vector is the projection of $\mathbf{b}$ onto $S$. So, $A\mathbf{x} = \text{proj}_S\mathbf{b}$ and you can see that $A\mathbf{x} - \mathbf{b} = \text{proj}_S\mathbf{b} - \mathbf{b}$ is orthogonal to $S = R(A)$. However, this implies that $A\mathbf{x} - \mathbf{b}$ is in $R(A)^\perp$, which equals $N(A^T)$. This is the crucial observation: $A\mathbf{x} - \mathbf{b}$ is in the nullspace of $A^T$. So, you have

$$\begin{aligned} A^T(A\mathbf{x} - \mathbf{b}) &= \mathbf{0} \\ A^TA\mathbf{x} - A^T\mathbf{b} &= \mathbf{0} \\ A^TA\mathbf{x} &= A^T\mathbf{b}. \end{aligned}$$

The solution of the least squares problem comes down to solving the $n \times n$ linear system of equations $A^TA\mathbf{x} = A^T\mathbf{b}$. These equations are called the **normal equations** of the least squares problem $A\mathbf{x} = \mathbf{b}$.

### EXAMPLE 7 Solving the Normal Equations

Find the solution of the least squares problem

$$A\mathbf{x} = \mathbf{b}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$$

from Example 1.

**SOLUTION**

Begin by calculating the following matrix products.

$$A^TA = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}$$

$$A^T\mathbf{b} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 11 \end{bmatrix}$$

The normal equations are

$$A^TA\mathbf{x} = A^T\mathbf{b}$$

$$\begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}\begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 4 \\ 11 \end{bmatrix}.$$

The solution of this system of equations is

$$\mathbf{x} = \begin{bmatrix} -\frac{5}{3} \\ \frac{3}{2} \end{bmatrix}$$

which implies that the least squares regression line for the data is

$$y = \tfrac{3}{2}x - \tfrac{5}{3}$$

as shown in Figure 5.26.

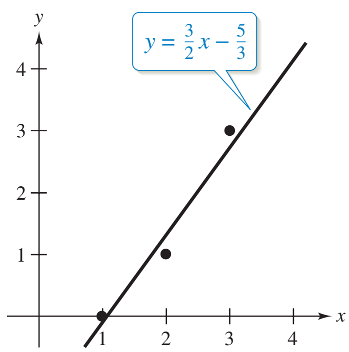


**Figure 5.26**

**TECHNOLOGY**

Many graphing utilities and software programs can find the least squares regression line for a set of data points. Use a graphing utility or software program to verify the result of Example 7. The **Online Technology Guide,** available at *www.cengagebrain.com,* provides syntax for programs applicable to Example 7.

For an $m \times n$ matrix $A$, the normal equations form an $n \times n$ system of linear equations. This system is always consistent, but it may have infinitely many solutions. It can be shown, however, that there is a unique solution when the rank of $A$ is $n$.

The next example illustrates how to solve the projection problem from Example 5 using normal equations.

### EXAMPLE 8 Orthogonal Projection Onto a Subspace

Find the orthogonal projection of the vector

$$\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$

onto the column space $S$ of the matrix

$$A = \begin{bmatrix} 0 & 2 \\ 3 & 0 \\ 1 & 0 \end{bmatrix}.$$

**SOLUTION**

To find the orthogonal projection of $\mathbf{b}$ onto $S$, first solve the least squares problem $A\mathbf{x} = \mathbf{b}$. As in Example 7, calculate the matrix products $A^TA$ and $A^T\mathbf{b}$.

$$\begin{aligned} A^TA &= \begin{bmatrix} 0 & 3 & 1 \\ 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 3 & 0 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 10 & 0 \\ 0 & 4 \end{bmatrix} \\ A^T\mathbf{b} &= \begin{bmatrix} 0 & 3 & 1 \\ 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 6 \\ 2 \end{bmatrix} \end{aligned}$$

The normal equations are

$$A^TA\mathbf{x} = A^T\mathbf{b}$$

$$\begin{bmatrix} 10 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}.$$

The solution of these equations is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} \\ \frac{1}{2} \end{bmatrix}.$$

Finally, the projection of $\mathbf{b}$ onto $S$ is

$$A\mathbf{x} = \begin{bmatrix} 0 & 2 \\ 3 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{3}{5} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{9}{5} \\ \frac{3}{5} \end{bmatrix}$$

which agrees with the solution obtained in Example 5.

## MATHEMATICAL MODELING

Least squares problems play a fundamental role in mathematical modeling of real-life phenomena. The next example shows how to model the world population using a least squares quadratic polynomial.

### EXAMPLE 9 World Population

The table shows the world population (in billions) for six different years. (Source: U.S. Census Bureau)

| *Year* | 1985 | 1990 | 1995 | 2000 | 2005 | 2010 |
|---|---|---|---|---|---|---|
| *Population (y)* | 4.9 | 5.3 | 5.7 | 6.1 | 6.5 | 6.9 |

Let $x = 5$ represent the year 1985. Find the least squares regression quadratic polynomial $y = c_0 + c_1x + c_2x^2$ for the data and use the model to estimate the population for the year 2015.

### SOLUTION

By substituting the data points $(5, 4.9)$, $(10, 5.3)$, $(15, 5.7)$, $(20, 6.1)$, $(25, 6.5)$, and $(30, 6.9)$ into the quadratic polynomial $y = c_0 + c_1x + c_2x^2$, you obtain the following system of linear equations.

$$\begin{aligned}
c_0 + 5c_1 + 25c_2 &= 4.9\\
c_0 + 10c_1 + 100c_2 &= 5.3\\
c_0 + 15c_1 + 225c_2 &= 5.7\\
c_0 + 20c_1 + 400c_2 &= 6.1\\
c_0 + 25c_1 + 625c_2 &= 6.5\\
c_0 + 30c_1 + 900c_2 &= 6.9
\end{aligned}$$

This produces the least squares problem

$$A\mathbf{x} = \mathbf{b}$$

$$\begin{bmatrix} 1 & 5 & 25\\ 1 & 10 & 100\\ 1 & 15 & 225\\ 1 & 20 & 400\\ 1 & 25 & 625\\ 1 & 30 & 900 \end{bmatrix}\begin{bmatrix} c_0\\ c_1\\ c_2 \end{bmatrix} = \begin{bmatrix} 4.9\\ 5.3\\ 5.7\\ 6.1\\ 6.5\\ 6.9 \end{bmatrix}.$$

The normal equations are

$$A^TA\mathbf{x} = A^T\mathbf{b}$$

$$\begin{bmatrix} 6 & 105 & 2275\\ 105 & 2275 & 55{,}125\\ 2275 & 55{,}125 & 1{,}421{,}875 \end{bmatrix}\begin{bmatrix} c_0\\ c_1\\ c_2 \end{bmatrix} = \begin{bmatrix} 35.4\\ 654.5\\ 14{,}647.5 \end{bmatrix}$$

and their solution is $\mathbf{x} = \begin{bmatrix} c_0\\ c_1\\ c_2 \end{bmatrix} = \begin{bmatrix} 4.5\\ 0.08\\ 0 \end{bmatrix}$.

Note that $c_2 = 0$. So, the least squares polynomial is the linear polynomial $y = 4.5 + 0.08x$. Evaluating this polynomial at $x = 35$ gives the estimate of the world population for the year 2015: $y = 4.5 + 0.08(35) = 7.3$ billion.

Least squares models can arise in many other contexts. Section 5.5 explores some applications of least squares models to approximations of functions. The next example uses data from Section 1.3 to find a nonlinear relationship between the period of a planet and its mean distance from the Sun.

## EXAMPLE 10 Application to Astronomy

The table shows the mean distances $x$ and the periods $y$ of the six planets that are closest to the Sun. The mean distances are given in astronomical units and the periods are given in years. Find a model for the data.

| *Planet* | *Mercury* | *Venus* | *Earth* | *Mars* | *Jupiter* | *Saturn* |
|---|---|---|---|---|---|---|
| *Distance (x)* | 0.387 | 0.723 | 1.000 | 1.524 | 5.203 | 9.555 |
| *Period (y)* | 0.241 | 0.615 | 1.000 | 1.881 | 11.860 | 29.420 |

### SOLUTION

When you plot the data as given, they do not lie in a straight line. By taking the natural logarithm of each coordinate, however, you obtain points of the form $(\ln x, \ln y)$, as follows.

| *Planet* | *Mercury* | *Venus* | *Earth* | *Mars* | *Jupiter* | *Saturn* |
|---|---|---|---|---|---|---|
| **ln x** | $-0.949$ | $-0.324$ | 0.0 | 0.421 | 1.649 | 2.257 |
| **ln y** | $-1.423$ | $-0.486$ | 0.0 | 0.632 | 2.473 | 3.382 |

### TECHNOLOGY

You can use a graphing utility or software program to verify the result of Example 10. For instance, using the data in the first table and a graphing utility, a power regression program would give a result of $y \approx 1.00008x^{1.49873}$. The **Online Technology Guide,** available at *www.cengagebrain.com,* provides syntax for programs applicable to Example 10.

Figure 5.27 shows a plot of the transformed points and suggests that the least squares regression line would be a good fit.

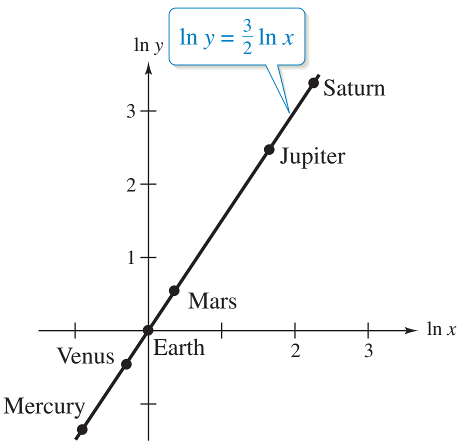



**Figure 5.27**

Use the techniques of this section on the system

$$
\begin{aligned}
c_0 - 0.949c_1 &= -1.423 \\
c_0 - 0.324c_1 &= -0.486 \\
c_0 \qquad\qquad &= \phantom{-}0.0 \\
c_0 + 0.421c_1 &= \phantom{-}0.632 \\
c_0 + 1.649c_1 &= \phantom{-}2.473 \\
c_0 + 2.257c_1 &= \phantom{-}3.382
\end{aligned}
$$

to verify that the equation of the line is $\ln y = \frac{3}{2} \ln x$ or $y = x^{3/2}$.

# 5.4 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

**Least Squares Regression Line** **In Exercises 1–4, determine whether the points are collinear. If so, find the line $y = c_0 + c_1 x$ that fits the points.**

**1.** $(0, 1), (1, 3), (2, 5)$ **2.** $(0, 0), (3, 1), (4, 2)$

**3.** $(-1, 0), (0, 1), (1, 1)$ **4.** $(-1, 5), (1, -1), (1, -4)$

**Orthogonal Subspaces** **In Exercises 5–8, determine whether the subspaces are orthogonal.**

**5.** $S_1 = \text{span}\left\{\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right\}$ $\quad S_2 = \text{span}\left\{\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}\right\}$

**6.** $S_1 = \text{span}\left\{\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}\right\}$ $\quad S_2 = \text{span}\left\{\begin{bmatrix} 2 \\ 1 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$

**7.** $S_1 = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right\}$ $\quad S_2 = \text{span}\left\{\begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -2 \\ 0 \end{bmatrix}\right\}$

**8.** $S_1 = \text{span}\left\{\begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \end{bmatrix}\right\}$ $\quad S_2 = \text{span}\left\{\begin{bmatrix} 3 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ 2 \end{bmatrix}\right\}$

**Finding the Orthogonal Complement and Direct Sum** **In Exercises 9–12, (a) find the orthogonal complement $S^{\perp}$, and (b) find the direct sum $S \oplus S^{\perp}$.**

**9.** $S$ is the subspace of $R^3$ consisting of the $xz$-plane.

**10.** $S$ is the subspace of $R^5$ consisting of all vectors whose third and fourth components are zero.

**11.** $S = \text{span}\left\{\begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}\right\}$ **12.** $S = \text{span}\left\{\begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right\}$

**13.** Find the orthogonal complement of the solution of Exercise 11(a).

**14.** Find the orthogonal complement of the solution of Exercise 12(a).

**Projection Onto a Subspace** **In Exercises 15–18, find the projection of the vector v onto the subspace $S$.**

**15.** $S = \text{span}\left\{\begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right\}, \quad \mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

**16.** $S = \text{span}\left\{\begin{bmatrix} -1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right\}, \quad \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

**17.** $S = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right\}, \quad \mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$

**18.** $S = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}\right\}, \quad \mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$

**Fundamental Subspaces** **In Exercises 19–22, find bases for the four fundamental subspaces of the matrix $A$.**

**19.** $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \end{bmatrix}$ **20.** $A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

**21.** $A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 2 & 2 & 3 \end{bmatrix}$

**22.** $A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

**Finding the Least Squares Solution** **In Exercises 23–26, find the least squares solution of the system $A\mathbf{x} = \mathbf{b}$.**

**23.** $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 1 & 1 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}$

**24.** $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 2 \end{bmatrix}$

**25.** $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 4 \\ -1 \\ 0 \\ 1 \end{bmatrix}$

**26.** $A = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 1 & -1 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & -1 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$

**Orthogonal Projection Onto a Subspace** **In Exercises 27 and 28, use the method of Example 8 to find the orthogonal projection of $\mathbf{b} = [2 \quad -2 \quad 1]^T$ onto the column space of the matrix $A$.**

**27.** $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$ **28.** $A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \\ 1 & 3 \end{bmatrix}$

**Finding the Least Squares Regression Line** **In Exercises 29–32, find the least squares regression line for the data points. Graph the points and the line on the same set of axes.**

**29.** $(-1, 1), (1, 0), (3, -3)$

**30.** $(1, 1), (2, 3), (4, 5)$

**31.** $(-2, 1), (-1, 2), (0, 1), (1, 2), (2, 1)$

**32.** $(-2, 0), (-1, 2), (0, 3), (1, 5), (2, 6)$

**Finding the Least Squares Quadratic Polynomial** **In Exercises 33–36, find the least squares regression quadratic polynomial for the data points.**

**33.** $(0, 0), (2, 2), (3, 6), (4, 12)$

**34.** $(0, 2), \left(1, \frac{3}{2}\right), \left(2, \frac{5}{2}\right), (3, 4)$

**35.** $(-2, 0), (-1, 0), (0, 1), (1, 2), (2, 5)$

**36.** $(-2, 6), (-1, 5), \left(0, \frac{7}{2}\right), (1, 2), (2, -1)$

**37. Doctoral Degrees** The table shows the numbers of doctoral degrees $y$ (in thousands) awarded in the United States from 2005 through 2008. Find the least squares regression line for the data. Then use the model to predict the number of degrees awarded in 2015. Let $t$ represent the year, with $t = 5$ corresponding to 2005. (Source: U.S. National Center for Education Statistics)

| *Year* | 2005 | 2006 | 2007 | 2008 |
|---|---|---|---|---|
| *Doctoral Degrees, y* | 52.6 | 56.1 | 60.6 | 63.7 |

**38. Revenue** The table shows the revenues $y$ (in billions of dollars) for General Dynamics Corporation from 2005 through 2010. Find the least squares regression quadratic and cubic polynomials for the data. Then use each model to predict the revenue in 2015. Let $t$ represent the year, with $t = 5$ corresponding to 2005. Which model appears to be more accurate for predicting future revenues? Explain. (Source: General Dynamics Corporation)

| *Year* | 2005 | 2006 | 2007 |
|---|---|---|---|
| *Revenue, y* | 21.2 | 24.1 | 27.2 |

| *Year* | 2008 | 2009 | 2010 |
|---|---|---|---|
| *Revenue, y* | 29.3 | 32.0 | 32.5 |

**39. Galloping Speeds of Animals** Four-legged animals run with two different types of motion: trotting and galloping. An animal that is trotting has at least one foot on the ground at all times, whereas an animal that is galloping has all four feet off the ground at some point in its stride. The number of strides per minute at which an animal breaks from a trot to a gallop depends on the weight of the animal. Use the table and the method of Example 10 to find an equation that relates an animal's weight $x$ (in pounds) and its lowest galloping speed $y$ (in strides per minute).

| *Weight, x* | 25 | 35 | 50 |
|---|---|---|---|
| *Galloping Speed, y* | 191.5 | 182.7 | 173.8 |

| *Weight, x* | 75 | 500 | 1000 |
|---|---|---|---|
| *Galloping Speed, y* | 164.2 | 125.9 | 114.2 |

**40.** CAPSTONE Explain how orthogonality, orthogonal complements, the projection of a vector, and fundamental subspaces are used to find the solution of a least squares problem.

**True or False?** **In Exercises 41 and 42, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.**

**41.** (a) The orthogonal complement of $R^n$ is the empty set.

(b) If each vector $\mathbf{v} \in R^n$ can be uniquely written as a sum of a vector $\mathbf{s}_1$ from $S_1$ and a vector $\mathbf{s}_2$ from $S_2$, then $R^n$ is called the direct sum of $S_1$ and $S_2$.

**42.** (a) If $A$ is an $m \times n$ matrix, then $R(A)$ and $N(A^T)$ are orthogonal subspaces of $R^n$.

(b) The set of all vectors orthogonal to every vector in a subspace $S$ is called the orthogonal complement of $S$.

(c) Given an $m \times n$ matrix $A$ and a vector $\mathbf{b}$ in $R^m$, the least squares problem is to find $\mathbf{x}$ in $R^n$ such that $\|A\mathbf{x} - \mathbf{b}\|^2$ is minimized.

**43. Proof** Prove that if $S_1$ and $S_2$ are orthogonal subspaces of $R^n$, then their intersection consists of only the zero vector.

**44. Proof** Prove that the orthogonal complement of a subspace of $R^n$ is itself a subspace of $R^n$.

**45. Proof** Prove Theorem 5.14.

**46. Proof** Prove that if $S_1$ and $S_2$ are subspaces of $R^n$ and if $R^n = S_1 \oplus S_2$, then $S_1 \cap S_2 = \{\mathbf{0}\}$.

# 5.5 Applications of Inner Product Spaces

- Find the cross product of two vectors in $R^3$.
- Find the linear or quadratic least squares approximation of a function.
- Find the $n$th-order Fourier approximation of a function.

## THE CROSS PRODUCT OF TWO VECTORS IN $R^3$

Here you will look at a vector product that yields a vector in $R^3$ orthogonal to two vectors. This vector product is called the **cross product,** and it is most conveniently defined and calculated with vectors written in standard unit vector form

$$\mathbf{v} = (v_1, v_2, v_3) = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}.$$

**REMARK**

The cross product is defined only for vectors in $R^3$. The cross product of two vectors in $R^n$, $n \neq 3$, is not defined here.

### Definition of the Cross Product of Two Vectors

Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ be vectors in $R^3$. The **cross product** of $\mathbf{u}$ and $\mathbf{v}$ is the vector

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}.$$

A convenient way to remember the formula for the cross product $\mathbf{u} \times \mathbf{v}$ is to use the following determinant form.

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \begin{array}{l} \\ \longleftarrow \text{Components of } \mathbf{u} \\ \longleftarrow \text{Components of } \mathbf{v} \end{array}$$

Technically this is not a determinant because it represents a vector and not a real number. Nevertheless, it is useful because it can help you remember the cross product formula. Using cofactor expansion in the first row produces

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\mathbf{k} \\ &= (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k} \end{aligned}$$

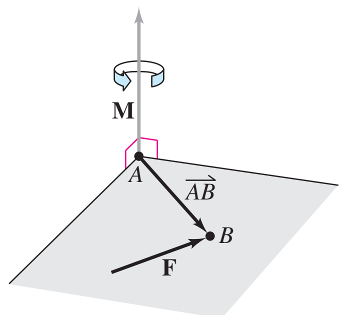
which yields the formula in the definition. Be sure to note that the $\mathbf{j}$-component is preceded by a minus sign.

**LINEAR ALGEBRA APPLIED**

In physics, the cross product can be used to measure *torque*—the moment $\mathbf{M}$ of a force $\mathbf{F}$ about a point $A$, as shown in the figure below. When the point of application of the force is $B$, the moment of $\mathbf{F}$ about $A$ is given by

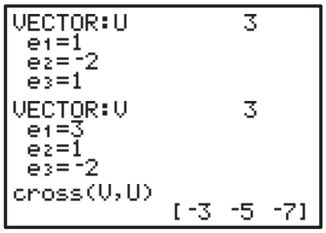
$$\mathbf{M} = \overrightarrow{AB} \times \mathbf{F}$$

where $\overrightarrow{AB}$ represents the vector whose initial point is $A$ and whose terminal point is $B$. The magnitude of the moment $\mathbf{M}$ measures the tendency of $\overrightarrow{AB}$ to rotate counterclockwise about an axis directed along the vector $\mathbf{M}$.



## TECHNOLOGY

Many graphing utilities and software programs can find a cross product. For instance, if you use a graphing utility to verify the result of Example 1(b), then you may see something similar to the following.




**Simulation**

Explore this concept further with an electronic simulation, and for syntax regarding specific programs involving Example 1, please visit *www.cengagebrain.com*. Similar exercises and projects are also available on the website.

## EXAMPLE 1 Finding the Cross Product of Two Vectors

Let $\mathbf{u} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$. Find each cross product.

**a.** $\mathbf{u} \times \mathbf{v}$ **b.** $\mathbf{v} \times \mathbf{u}$ **c.** $\mathbf{v} \times \mathbf{v}$

### SOLUTION

**a.**
$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 1 \\ 3 & 1 & -2 \end{vmatrix}$$
$$= \begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix}\mathbf{k}$$
$$= 3\mathbf{i} + 5\mathbf{j} + 7\mathbf{k}$$

**b.**
$$\mathbf{v} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & -2 \\ 1 & -2 & 1 \end{vmatrix}$$
$$= \begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & -2 \\ 1 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & 1 \\ 1 & -2 \end{vmatrix}\mathbf{k}$$
$$= -3\mathbf{i} - 5\mathbf{j} - 7\mathbf{k}$$

Note that this result is the negative of that in part (a).

**c.**
$$\mathbf{v} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & -2 \\ 3 & 1 & -2 \end{vmatrix}$$
$$= \begin{vmatrix} 1 & -2 \\ 1 & -2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & -2 \\ 3 & -2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & 1 \\ 3 & 1 \end{vmatrix}\mathbf{k}$$
$$= 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$$

The results obtained in Example 1 suggest some interesting *algebraic* properties of the cross product. For instance,

$$\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u}) \quad \text{and} \quad \mathbf{v} \times \mathbf{v} = \mathbf{0}.$$

These properties, along with several others, are stated in Theorem 5.17.

### THEOREM 5.17 Algebraic Properties of the Cross Product

If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in $R^3$ and $c$ is a scalar, then the following properties are true.

1. $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$
2. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$
3. $c(\mathbf{u} \times \mathbf{v}) = c\mathbf{u} \times \mathbf{v} = \mathbf{u} \times c\mathbf{v}$
4. $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$
5. $\mathbf{u} \times \mathbf{u} = \mathbf{0}$
6. $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$

### PROOF

The proof of the first property is given here. The proofs of the other properties are left to you. (See Exercises 53–57.) Let $\mathbf{u}$ and $\mathbf{v}$ be

$$\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$$

and

$$\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}.$$

Then $\mathbf{u} \times \mathbf{v}$ is

$$\begin{aligned}\mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \\ &= (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}\end{aligned}$$

and $\mathbf{v} \times \mathbf{u}$ is

$$\begin{aligned}\mathbf{v} \times \mathbf{u} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \end{vmatrix} \\ &= (v_2u_3 - v_3u_2)\mathbf{i} - (v_1u_3 - v_3u_1)\mathbf{j} + (v_1u_2 - v_2u_1)\mathbf{k} \\ &= -(u_2v_3 - u_3v_2)\mathbf{i} + (u_1v_3 - u_3v_1)\mathbf{j} - (u_1v_2 - u_2v_1)\mathbf{k} \\ &= -(\mathbf{v} \times \mathbf{u}).\end{aligned}$$

Property 1 of Theorem 5.17 tells you that the vectors $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{u}$ have equal lengths but opposite directions. The geometric implication of this will be discussed after establishing some geometric properties of the cross product of two vectors.

**THEOREM 5.18 Geometric Properties of the Cross Product**

If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors in $R^3$, then the following properties are true.

1. $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.
2. The angle $\theta$ between $\mathbf{u}$ and $\mathbf{v}$ is given by $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\,\|\mathbf{v}\| \sin\theta$.
3. $\mathbf{u}$ and $\mathbf{v}$ are parallel if and only if $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.
4. The parallelogram having $\mathbf{u}$ and $\mathbf{v}$ as adjacent sides has an area of $\|\mathbf{u} \times \mathbf{v}\|$.

**PROOF**

The proof of Property 4 follows. The proofs of the other properties are left to you. (See Exercises 58–60.) Let $\mathbf{u}$ and $\mathbf{v}$ represent adjacent sides of a parallelogram, as shown in Figure 5.28. By Property 2, the area of the parallelogram is

$$\text{Area} = \overbrace{\|\mathbf{u}\|}^{\text{Base}}\ \overbrace{\|\mathbf{v}\| \sin\theta}^{\text{Height}} = \|\mathbf{u} \times \mathbf{v}\|.$$

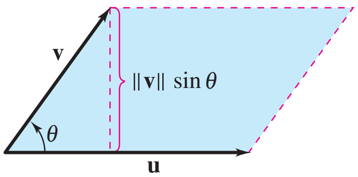


Figure 5.28

Property 1 states that the vector $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$. This implies that $\mathbf{u} \times \mathbf{v}$ (and $\mathbf{v} \times \mathbf{u}$) is orthogonal to the plane determined by $\mathbf{u}$ and $\mathbf{v}$. One way to remember the orientation of the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} \times \mathbf{v}$ is to compare them with the unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, as shown in Figure 5.29. The three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} \times \mathbf{v}$ form a *right-handed system,* whereas the three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{v} \times \mathbf{u}$ form a *left-handed system.*

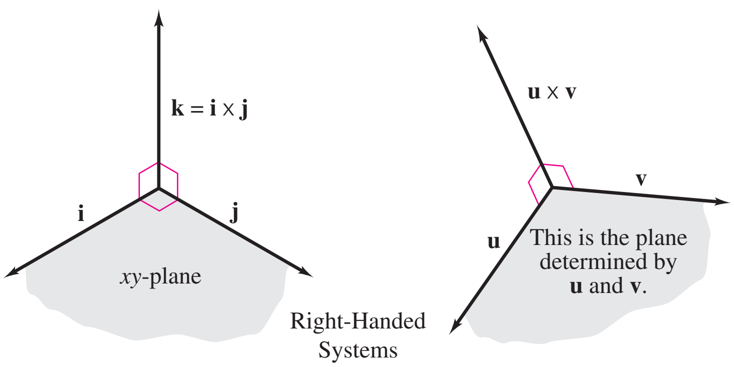


Figure 5.29

## EXAMPLE 2 Finding a Vector Orthogonal to Two Given Vectors

Find a unit vector orthogonal to both

$$\mathbf{u} = \mathbf{i} - 4\mathbf{j} + \mathbf{k}$$

and

$$\mathbf{v} = 2\mathbf{i} + 3\mathbf{j}.$$

### SOLUTION

From Property 1 of Theorem 5.18, you know that the cross product

$$\begin{aligned}\mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -4 & 1 \\ 2 & 3 & 0 \end{vmatrix} \\ &= -3\mathbf{i} + 2\mathbf{j} + 11\mathbf{k}\end{aligned}$$

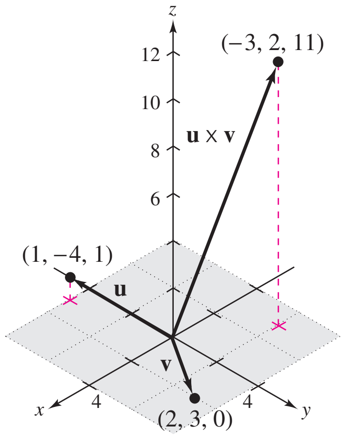


**Figure 5.30**

is orthogonal to both **u** and **v**, as shown in Figure 5.30. Then, by dividing by the length of **u** × **v**,

$$\begin{aligned}\|\mathbf{u} \times \mathbf{v}\| &= \sqrt{(-3)^2 + 2^2 + 11^2} \\ &= \sqrt{134}\end{aligned}$$

you obtain the unit vector

$$\frac{\mathbf{u} \times \mathbf{v}}{\|\mathbf{u} \times \mathbf{v}\|} = -\frac{3}{\sqrt{134}}\mathbf{i} + \frac{2}{\sqrt{134}}\mathbf{j} + \frac{11}{\sqrt{134}}\mathbf{k}$$

which is orthogonal to both **u** and **v**, because

$$\left(-\frac{3}{\sqrt{134}}, \frac{2}{\sqrt{134}}, \frac{11}{\sqrt{134}}\right) \cdot (1, -4, 1) = 0$$

and

$$\left(-\frac{3}{\sqrt{134}}, \frac{2}{\sqrt{134}}, \frac{11}{\sqrt{134}}\right) \cdot (2, 3, 0) = 0.$$

## EXAMPLE 3 Finding the Area of a Parallelogram

Find the area of the parallelogram that has

$$\mathbf{u} = -3\mathbf{i} + 4\mathbf{j} + \mathbf{k}$$

and

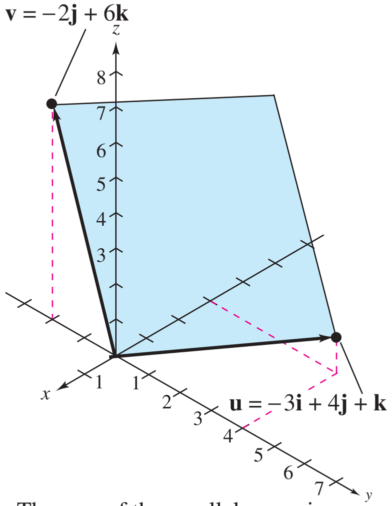
$$\mathbf{v} = -2\mathbf{j} + 6\mathbf{k}$$

as adjacent sides, as shown in Figure 5.31.

### SOLUTION

From Property 4 of Theorem 5.18, you know that the area of this parallelogram is $\|\mathbf{u} \times \mathbf{v}\|$. Because

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 4 & 1 \\ 0 & -2 & 6 \end{vmatrix} = 26\mathbf{i} + 18\mathbf{j} + 6\mathbf{k}$$



The area of the parallelogram is $\|\mathbf{u} \times \mathbf{v}\| = \sqrt{1036}$.

**Figure 5.31**

the area of the parallelogram is

$$\|\mathbf{u} \times \mathbf{v}\| = \sqrt{26^2 + 18^2 + 6^2} = \sqrt{1036} \approx 32.19 \text{ square units.}$$

## LEAST SQUARES APPROXIMATIONS (CALCULUS)

Many problems in the physical sciences and engineering involve an approximation of a function $f$ by another function $g$. If $f$ is in $C[a, b]$ (the inner product space of all continuous functions on $[a, b]$), then $g$ is usually chosen from a subspace $W$ of $C[a, b]$. For instance, to approximate the function

$$f(x) = e^x, \quad 0 \le x \le 1$$

you could choose one of the following forms of $g$.

1. $g(x) = a_0 + a_1 x, \quad 0 \le x \le 1$ — Linear
2. $g(x) = a_0 + a_1 x + a_2 x^2, \quad 0 \le x \le 1$ — Quadratic
3. $g(x) = a_0 + a_1 \cos x + a_2 \sin x, \quad 0 \le x \le 1$ — Trigonometric

Before discussing ways of finding the function $g$, you must define how one function can "best" approximate another function. One natural way would require the area bounded by the graphs of $f$ and $g$ on the interval $[a, b]$,

$$\text{Area} = \int_a^b |f(x) - g(x)|\, dx$$

to be a minimum with respect to other functions in the subspace $W$, as shown in Figure 5.32.

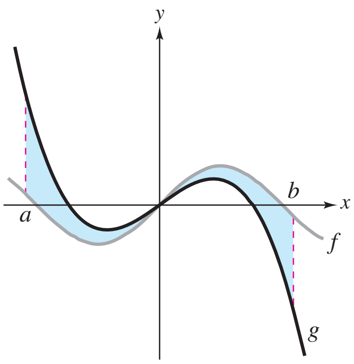



**Figure 5.32**

Because integrands involving absolute value are often difficult to evaluate, however, it is more common to square the integrand to obtain

$$\int_a^b [f(x) - g(x)]^2\, dx.$$

With this criterion, the function $g$ is called the **least squares approximation** of $f$ with respect to the inner product space $W$.

### Definition of Least Squares Approximation

Let $f$ be continuous on $[a, b]$, and let $W$ be a subspace of $C[a, b]$. A function $g$ in $W$ is called a **least squares approximation** of $f$ with respect to $W$ when the value of

$$I = \int_a^b [f(x) - g(x)]^2\, dx$$

is a minimum with respect to all other functions in $W$.

Note that if the subspace $W$ in this definition is the entire space $C[a, b]$, then $g(x) = f(x)$, which implies that $I = 0$.

## EXAMPLE 4 Finding a Least Squares Approximation

Find the least squares approximation $g(x) = a_0 + a_1x$ of

$$f(x) = e^x, \quad 0 \le x \le 1.$$

### SOLUTION

For this approximation you need to find the constants $a_0$ and $a_1$ that minimize the value of

$$\begin{aligned} I &= \int_0^1 [f(x) - g(x)]^2\,dx \\ &= \int_0^1 (e^x - a_0 - a_1x)^2\,dx. \end{aligned}$$

Evaluating this integral, you have

$$\begin{aligned} I &= \int_0^1 (e^x - a_0 - a_1x)^2\,dx \\ &= \int_0^1 (e^{2x} - 2a_0e^x - 2a_1xe^x + a_0^2 + 2a_0a_1x + a_1^2x^2)\,dx \\ &= \left[\frac{1}{2}e^{2x} - 2a_0e^x - 2a_1e^x(x-1) + a_0^2x + a_0a_1x^2 + a_1^2\frac{x^3}{3}\right]_0^1 \\ &= \frac{1}{2}(e^2 - 1) - 2a_0(e-1) - 2a_1 + a_0^2 + a_0a_1 + \frac{1}{3}a_1^2. \end{aligned}$$

Now, considering $I$ to be a function of the variables $a_0$ and $a_1$, use calculus to determine the values of $a_0$ and $a_1$ that minimize $I$. Specifically, by setting the partial derivatives

$$\begin{aligned} \frac{\partial I}{\partial a_0} &= 2a_0 - 2e + 2 + a_1 \\ \frac{\partial I}{\partial a_1} &= a_0 + \frac{2}{3}a_1 - 2 \end{aligned}$$

equal to zero, you obtain the following two linear equations in $a_0$ and $a_1$.

$$\begin{aligned} 2a_0 + \phantom{2}a_1 &= 2(e-1) \\ 3a_0 + 2a_1 &= 6 \end{aligned}$$

The solution of this system is

$$a_0 = 4e - 10 \approx 0.873 \quad \text{and} \quad a_1 = 18 - 6e \approx 1.690.$$

(Verify this.) So, the best *linear approximation* of $f(x) = e^x$ on the interval $[0, 1]$ is

$$g(x) = 4e - 10 + (18 - 6e)x \approx 0.873 + 1.690x.$$

Figure 5.33 shows the graphs of $f$ and $g$ on $[0, 1]$.

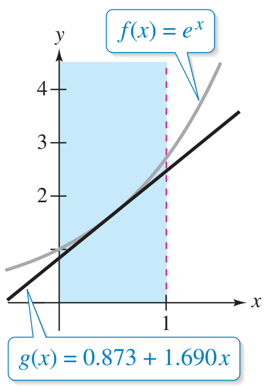



**Figure 5.33**

Of course, whether the approximation obtained in Example 4 is the best approximation depends on the definition of the best approximation. For instance, if the definition of the best approximation had been the *Taylor polynomial of degree 1* centered at 0.5, then the approximating function $g$ would have been

$$\begin{aligned} g(x) &= f(0.5) + f'(0.5)(x - 0.5) \\ &= e^{0.5} + e^{0.5}(x - 0.5) \\ &\approx 0.824 + 1.649x. \end{aligned}$$

Moreover, the function $g$ obtained in Example 4 is only the best *linear* approximation of $f$ (according to the least squares criterion). In Example 5 you will find the best *quadratic* approximation.

## EXAMPLE 5 Finding a Least Squares Approximation

Find the least squares approximation $g(x) = a_0 + a_1x + a_2x^2$ of $f(x) = e^x$, $0 \le x \le 1$.

### SOLUTION

For this approximation you need to find the values of $a_0$, $a_1$, and $a_2$ that minimize the value of

$$\begin{aligned} I &= \int_0^1 [f(x) - g(x)]^2\, dx \\ &= \int_0^1 (e^x - a_0 - a_1x - a_2x^2)^2\, dx \\ &= \frac{1}{2}(e^2 - 1) + 2a_0(1 - e) + 2a_2(2 - e) \\ &\qquad + a_0^2 + a_0a_1 + \frac{2}{3}a_0a_2 + \frac{1}{2}a_1a_2 + \frac{1}{3}a_1^2 + \frac{1}{5}a_2^2 - 2a_1. \end{aligned}$$

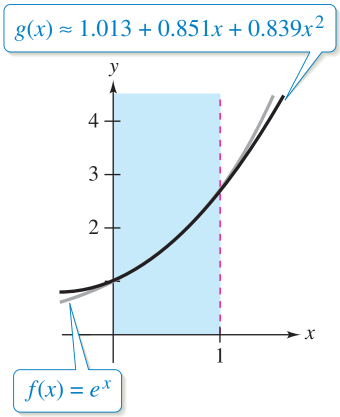



**Figure 5.34**

Setting the partial derivatives of $I$ (with respect to $a_0$, $a_1$, and $a_2$) equal to zero produces the following system of linear equations.

$$\begin{aligned} 6a_0 + 3a_1 + 2a_2 &= 6(e - 1) \\ 6a_0 + 4a_1 + 3a_2 &= 12 \\ 20a_0 + 15a_1 + 12a_2 &= 60(e - 2) \end{aligned}$$

The solution of this system is

$$\begin{aligned} a_0 &= -105 + 39e \approx 1.013 \\ a_1 &= 588 - 216e \approx 0.851 \\ a_2 &= -570 + 210e \approx 0.839. \end{aligned}$$

(Verify this.) So, the approximating function $g$ is $g(x) \approx 1.013 + 0.851x + 0.839x^2$. Figure 5.34 shows the graphs of $f$ and $g$.

The integral $I$ (given in the definition of the least squares approximation) can be expressed in vector form. To do this, use the inner product defined in Example 5 in Section 5.2:

$$\langle f, g \rangle = \int_a^b f(x)g(x)\, dx.$$

With this inner product you have

$$I = \int_a^b [f(x) - g(x)]^2\, dx = \langle f - g, f - g \rangle = \|f - g\|^2.$$

This means that the least squares approximating function $g$ is the function that minimizes $\|f - g\|^2$ or, equivalently, minimizes $\|f - g\|$. In other words, the least squares approximation of a function $f$ is the function $g$ (in the subspace $W$) closest to $f$ in terms of the inner product $\langle f, g \rangle$. The next theorem gives you a way of determining the function $g$.

### THEOREM 5.19 Least Squares Approximation

Let $f$ be continuous on $[a, b]$, and let $W$ be a finite-dimensional subspace of $C[a, b]$. The least squares approximating function of $f$ with respect to $W$ is given by

$$g = \langle f, \mathbf{w}_1 \rangle \mathbf{w}_1 + \langle f, \mathbf{w}_2 \rangle \mathbf{w}_2 + \cdot \cdot \cdot + \langle f, \mathbf{w}_n \rangle \mathbf{w}_n$$

where $B = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$ is an orthonormal basis for $W$.

**PROOF**

To show that $g$ is the least squares approximating function of $f$, prove that the inequality $\|f - g\| \le \|f - \mathbf{w}\|$ is true for any vector $\mathbf{w}$ in $W$. By writing $f - g$ as

$$f - g = f - \langle f, \mathbf{w}_1\rangle\mathbf{w}_1 - \langle f, \mathbf{w}_2\rangle\mathbf{w}_2 - \cdot\cdot\cdot - \langle f, \mathbf{w}_n\rangle\mathbf{w}_n$$

you can see that $f - g$ is orthogonal to each $\mathbf{w}_i$, which in turn implies that it is orthogonal to each vector in $W$. In particular, $f - g$ is orthogonal to $g - \mathbf{w}$. This allows you to apply the Pythagorean Theorem to the vector sum $f - \mathbf{w} = (f - g) + (g - \mathbf{w})$ to conclude that $\|f - \mathbf{w}\|^2 = \|f - g\|^2 + \|g - \mathbf{w}\|^2$. So, it follows that $\|f - g\|^2 \le \|f - \mathbf{w}\|^2$, which then implies that $\|f - g\| \le \|f - \mathbf{w}\|$.

Now observe how Theorem 5.19 can be used to produce the least squares approximation obtained in Example 4. First apply the Gram-Schmidt orthonormalization process to the standard basis $\{1, x\}$ to obtain the orthonormal basis $B = \left\{1, \sqrt{3}(2x - 1)\right\}$. (Verify this.) Then, by Theorem 5.19, the least squares approximation of $e^x$ in the subspace of all linear functions is

$$\begin{aligned} g(x) &= \langle e^x, 1\rangle(1) + \left\langle e^x, \sqrt{3}(2x - 1)\right\rangle\sqrt{3}(2x - 1) \\ &= \int_0^1 e^x\,dx + \sqrt{3}(2x - 1)\int_0^1 \sqrt{3}e^x(2x - 1)\,dx \\ &= \int_0^1 e^x\,dx + 3(2x - 1)\int_0^1 e^x(2x - 1)\,dx \\ &= 4e - 10 + (18 - 6e)x \end{aligned}$$

which agrees with the result obtained in Example 4.

## EXAMPLE 6 Finding a Least Squares Approximation

Find the least squares approximation of $f(x) = \sin x$, $0 \le x \le \pi$, with respect to the subspace $W$ of polynomial functions of degree 2 or less.

**SOLUTION**

To use Theorem 5.19, apply the Gram-Schmidt orthonormalization process to the standard basis for $W$, $\{1, x, x^2\}$, to obtain the orthonormal basis

$$B = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\} = \left\{\frac{1}{\sqrt{\pi}}, \frac{\sqrt{3}}{\pi\sqrt{\pi}}(2x - \pi), \frac{\sqrt{5}}{\pi^2\sqrt{\pi}}(6x^2 - 6\pi x + \pi^2)\right\}.$$

(Verify this.) The least squares approximating function $g$ is

$$g(x) = \langle f, \mathbf{w}_1\rangle\mathbf{w}_1 + \langle f, \mathbf{w}_2\rangle\mathbf{w}_2 + \langle f, \mathbf{w}_3\rangle\mathbf{w}_3$$

and you have

$$\begin{aligned} \langle f, \mathbf{w}_1\rangle &= \frac{1}{\sqrt{\pi}}\int_0^\pi \sin x\,dx = \frac{2}{\sqrt{\pi}} \\ \langle f, \mathbf{w}_2\rangle &= \frac{\sqrt{3}}{\pi\sqrt{\pi}}\int_0^\pi \sin x(2x - \pi)\,dx = 0 \\ \langle f, \mathbf{w}_3\rangle &= \frac{\sqrt{5}}{\pi^2\sqrt{\pi}}\int_0^\pi \sin x(6x^2 - 6\pi x + \pi^2)\,dx = \frac{2\sqrt{5}}{\pi^2\sqrt{\pi}}(\pi^2 - 12). \end{aligned}$$

So, $g$ is

$$g(x) = \frac{2}{\pi} + \frac{10(\pi^2 - 12)}{\pi^5}(6x^2 - 6\pi x + \pi^2) \approx -0.4177x^2 + 1.3122x - 0.0505.$$

Figure 5.35 shows the graphs of $f$ and $g$.

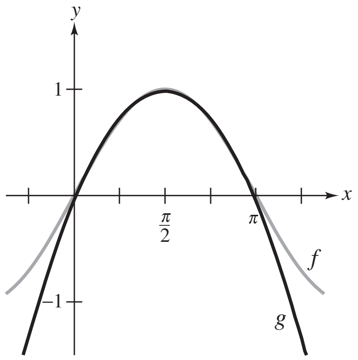



Figure 5.35

## FOURIER APPROXIMATIONS (CALCULUS)

You will now look at a special type of least squares approximation called a **Fourier approximation.** For this approximation, consider functions of the form

$$g(x) = \frac{a_0}{2} + a_1 \cos x + \cdots + a_n \cos nx + b_1 \sin x + \cdots + b_n \sin nx$$

in the subspace $W$ of

$$C[0, 2\pi]$$

spanned by the basis

$$S = \{1, \cos x, \cos 2x, \ldots, \cos nx, \sin x, \sin 2x, \ldots, \sin nx\}.$$

These $2n + 1$ vectors are orthogonal in the inner product space $C[0, 2\pi]$ because

$$\langle f, g\rangle = \int_0^{2\pi} f(x)g(x)\,dx = 0, \quad f \neq g$$

as demonstrated in Example 3 in Section 5.3. Moreover, by normalizing each function in this basis, you obtain the orthonormal basis

$$\begin{aligned} B &= \{\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_n, \mathbf{w}_{n+1}, \ldots, \mathbf{w}_{2n}\} \\ &= \left\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos x, \ldots, \frac{1}{\sqrt{\pi}}\cos nx, \frac{1}{\sqrt{\pi}}\sin x, \ldots, \frac{1}{\sqrt{\pi}}\sin nx\right\}. \end{aligned}$$

With this orthonormal basis, you can apply Theorem 5.19 to write

$$g(x) = \langle f, \mathbf{w}_0\rangle\mathbf{w}_0 + \langle f, \mathbf{w}_1\rangle\mathbf{w}_1 + \cdots + \langle f, \mathbf{w}_{2n}\rangle\mathbf{w}_{2n}.$$

The coefficients

$$a_0, a_1, \ldots, a_n, b_1, \ldots, b_n$$

for $g(x)$ in the equation

$$g(x) = \frac{a_0}{2} + a_1 \cos x + \cdots + a_n \cos nx + b_1 \sin x + \cdots + b_n \sin nx$$

are given by the following integrals.

$$\begin{aligned} a_0 &= \langle f, \mathbf{w}_0\rangle \frac{2}{\sqrt{2\pi}} = \frac{2}{\sqrt{2\pi}}\int_0^{2\pi} f(x)\frac{1}{\sqrt{2\pi}}\,dx = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx \\ a_1 &= \langle f, \mathbf{w}_1\rangle \frac{1}{\sqrt{\pi}} = \frac{1}{\sqrt{\pi}}\int_0^{2\pi} f(x)\frac{1}{\sqrt{\pi}}\cos x\,dx = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos x\,dx \\ &\vdots \\ a_n &= \langle f, \mathbf{w}_n\rangle \frac{1}{\sqrt{\pi}} = \frac{1}{\sqrt{\pi}}\int_0^{2\pi} f(x)\frac{1}{\sqrt{\pi}}\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx \\ b_1 &= \langle f, \mathbf{w}_{n+1}\rangle \frac{1}{\sqrt{\pi}} = \frac{1}{\sqrt{\pi}}\int_0^{2\pi} f(x)\frac{1}{\sqrt{\pi}}\sin x\,dx = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin x\,dx \\ &\vdots \\ b_n &= \langle f, \mathbf{w}_{2n}\rangle \frac{1}{\sqrt{\pi}} = \frac{1}{\sqrt{\pi}}\int_0^{2\pi} f(x)\frac{1}{\sqrt{\pi}}\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx \end{aligned}$$

The function $g(x)$ is called the ***n*th-order Fourier approximation** of $f$ on the interval $[0, 2\pi]$. Like Fourier coefficients, this function is named after the French mathematician Jean-Baptiste Joseph Fourier (1768–1830). This brings you to Theorem 5.20.

### THEOREM 5.20 Fourier Approximation

On the interval $[0, 2\pi]$, the least squares approximation of a continuous function $f$ with respect to the vector space spanned by

$$\{1, \cos x, \ldots, \cos nx, \sin x, \ldots, \sin nx\}$$

is

$$g(x) = \frac{a_0}{2} + a_1 \cos x + \cdots + a_n \cos nx + b_1 \sin x + \cdots + b_n \sin nx$$

where the **Fourier coefficients** $a_0, a_1, \ldots, a_n, b_1, \ldots, b_n$ are

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$a_j = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos jx\,dx, \quad j = 1, 2, \ldots, n$$

$$b_j = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin jx\,dx, \quad j = 1, 2, \ldots, n.$$

### EXAMPLE 7 Finding a Fourier Approximation

Find the third-order Fourier approximation of $f(x) = x$, $0 \le x \le 2\pi$.

**SOLUTION**

Using Theorem 5.20, you have

$$g(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x$$

where

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} x\,dx = \frac{1}{\pi}2\pi^2 = 2\pi$$

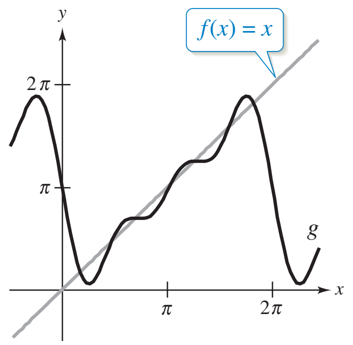
$$a_j = \frac{1}{\pi}\int_0^{2\pi} x\cos jx\,dx = \left[\frac{1}{\pi j^2}\cos jx + \frac{x}{\pi j}\sin jx\right]_0^{2\pi} = 0$$

$$b_j = \frac{1}{\pi}\int_0^{2\pi} x\sin jx\,dx = \left[\frac{1}{\pi j^2}\sin jx - \frac{x}{\pi j}\cos jx\right]_0^{2\pi} = -\frac{2}{j}.$$

This implies that $a_0 = 2\pi$, $a_1 = 0$, $a_2 = 0$, $a_3 = 0$, $b_1 = -2$, $b_2 = -\frac{2}{2} = -1$, and $b_3 = -\frac{2}{3}$. So, you have

$$\begin{aligned} g(x) &= \frac{2\pi}{2} - 2\sin x - \sin 2x - \frac{2}{3}\sin 3x \\ &= \pi - 2\sin x - \sin 2x - \frac{2}{3}\sin 3x. \end{aligned}$$

Figure 5.36 compares the graphs of $f$ and $g$.




Third-Order Fourier Approximation
**Figure 5.36**

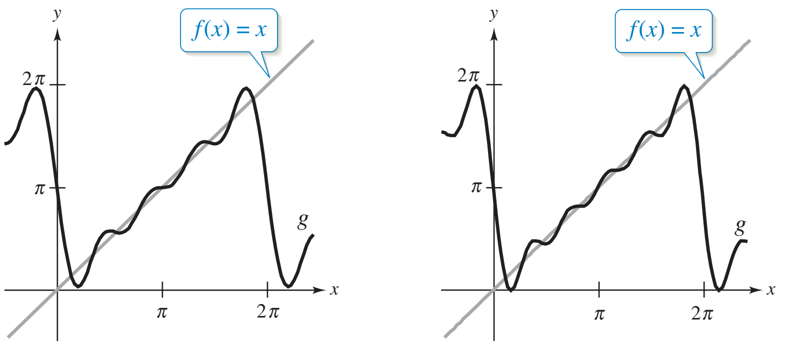
In Example 7, the pattern for the Fourier coefficients appears to be $a_0 = 2\pi$, $a_1 = a_2 = \cdot \cdot \cdot = a_n = 0$, and

$$b_1 = -\frac{2}{1},\ b_2 = -\frac{2}{2},\ . \ . \ .,\ b_n = -\frac{2}{n}.$$

The $n$th-order Fourier approximation of $f(x) = x$ is

$$g(x) = \pi - 2\left(\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \cdot \cdot \cdot + \frac{1}{n}\sin nx\right).$$

As $n$ increases, the Fourier approximation improves. For instance, Figure 5.37 shows the fourth- and fifth-order Fourier approximations of $f(x) = x,\ 0 \le x \le 2\pi$.




Fourth-Order Fourier Approximation Fifth-Order Fourier Approximation

**Figure 5.37**

In advanced courses it is shown that as $n \to \infty$, the approximation error $\|f - g\|$ approaches zero for all $x$ in the interval $(0, 2\pi)$. The infinite *series* for $g(x)$ is called a *Fourier series*.

## EXAMPLE 8 Finding a Fourier Approximation

Find the fourth-order Fourier approximation of $f(x) = |x - \pi|,\ 0 \le x \le 2\pi$.

### SOLUTION

Using Theorem 5.20, find the Fourier coefficients as follows.

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} |x - \pi|\, dx = \pi$$

$$\begin{aligned} a_j &= \frac{1}{\pi}\int_0^{2\pi} |x - \pi| \cos jx\, dx \\ &= \frac{2}{\pi}\int_0^{\pi} (\pi - x)\cos jx\, dx \\ &= \frac{2}{\pi j^2}(1 - \cos j\pi) \end{aligned}$$

$$b_j = \frac{1}{\pi}\int_0^{2\pi} |x - \pi| \sin jx\, dx = 0$$

So, $a_0 = \pi,\ a_1 = 4/\pi,\ a_2 = 0,\ a_3 = 4/9\pi,\ a_4 = 0,\ b_1 = 0,\ b_2 = 0,\ b_3 = 0$, and $b_4 = 0$, which means that the fourth-order Fourier approximation of $f$ is

$$g(x) = \frac{\pi}{2} + \frac{4}{\pi}\cos x + \frac{4}{9\pi}\cos 3x.$$

Figure 5.38 compares the graphs of $f$ and $g$.

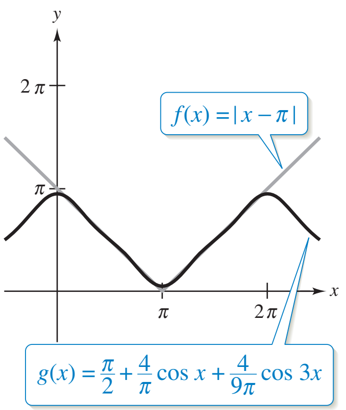



**Figure 5.38**

# 5.5 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

**Finding the Cross Product** In Exercises 1–6, find the cross product of the unit vectors [where $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$, and $\mathbf{k} = (0, 0, 1)$]. Sketch your result.

1. $\mathbf{j} \times \mathbf{i}$
2. $\mathbf{i} \times \mathbf{j}$
3. $\mathbf{j} \times \mathbf{k}$
4. $\mathbf{k} \times \mathbf{j}$
5. $\mathbf{i} \times \mathbf{k}$
6. $\mathbf{k} \times \mathbf{i}$

**Finding the Cross Product** In Exercises 7–12, find (a) $\mathbf{u} \times \mathbf{v}$, (b) $\mathbf{v} \times \mathbf{u}$, and (c) $\mathbf{v} \times \mathbf{v}$.

7. $\mathbf{u} = \mathbf{i} - \mathbf{j}$, $\mathbf{v} = \mathbf{j} + \mathbf{k}$
8. $\mathbf{u} = 2\mathbf{i} + \mathbf{k}$, $\mathbf{v} = \mathbf{i} + 3\mathbf{k}$
9. $\mathbf{u} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $\mathbf{v} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$
10. $\mathbf{u} = \mathbf{i} - \mathbf{j} - \mathbf{k}$, $\mathbf{v} = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$
11. $\mathbf{u} = (3, -2, 4)$, $\mathbf{v} = (1, 5, -3)$
12. $\mathbf{u} = (-2, 9, -3)$, $\mathbf{v} = (4, 6, -5)$

**Finding the Cross Product** In Exercises 13–22, find $\mathbf{u} \times \mathbf{v}$ and show that it is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

13. $\mathbf{u} = (0, 1, -2)$, $\mathbf{v} = (1, -1, 0)$
14. $\mathbf{u} = (-1, 1, 2)$, $\mathbf{v} = (0, 1, -1)$
15. $\mathbf{u} = (12, -3, 1)$, $\mathbf{v} = (-2, 5, 1)$
16. $\mathbf{u} = (-2, 1, 1)$, $\mathbf{v} = (4, 2, 0)$
17. $\mathbf{u} = (2, -3, 1)$, $\mathbf{v} = (1, -2, 1)$
18. $\mathbf{u} = (4, 1, 0)$, $\mathbf{v} = (3, 2, -2)$
19. $\mathbf{u} = \mathbf{j} + 6\mathbf{k}$, $\mathbf{v} = 2\mathbf{i} - \mathbf{k}$
20. $\mathbf{u} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{v} = 3\mathbf{i} - \mathbf{j}$
21. $\mathbf{u} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{v} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$
22. $\mathbf{u} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$, $\mathbf{v} = -\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$

**Finding the Cross Product** In Exercises 23–30, use a graphing utility with vector capabilities to find $\mathbf{u} \times \mathbf{v}$, and then show that it is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

23. $\mathbf{u} = (1, 2, -1)$, $\mathbf{v} = (2, 1, 2)$
24. $\mathbf{u} = (1, 2, -3)$, $\mathbf{v} = (-1, 1, 2)$
25. $\mathbf{u} = (0, 1, -1)$, $\mathbf{v} = (1, 2, 0)$
26. $\mathbf{u} = (0, 1, -2)$, $\mathbf{v} = (0, 1, 4)$
27. $\mathbf{u} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{v} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$
28. $\mathbf{u} = 3\mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{v} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$
29. $\mathbf{u} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$, $\mathbf{v} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$
30. $\mathbf{u} = 4\mathbf{i} + 2\mathbf{j}$, $\mathbf{v} = \mathbf{i} - 4\mathbf{k}$

**Using the Cross Product** In Exercises 31–38, find a unit vector orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

31. $\mathbf{u} = (2, -3, 4)$, $\mathbf{v} = (0, -1, 1)$
32. $\mathbf{u} = (2, -1, 3)$, $\mathbf{v} = (1, 0, -2)$
33. $\mathbf{u} = 3\mathbf{i} + \mathbf{j}$, $\mathbf{v} = \mathbf{j} + \mathbf{k}$
34. $\mathbf{u} = \mathbf{i} + 2\mathbf{j}$, $\mathbf{v} = \mathbf{i} - 3\mathbf{k}$
35. $\mathbf{u} = -3\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}$, $\mathbf{v} = \frac{1}{2}\mathbf{i} - \frac{3}{4}\mathbf{j} + \frac{1}{10}\mathbf{k}$
36. $\mathbf{u} = 7\mathbf{i} - 14\mathbf{j} + 5\mathbf{k}$, $\mathbf{v} = 14\mathbf{i} + 28\mathbf{j} - 15\mathbf{k}$
37. $\mathbf{u} = \mathbf{i} + \mathbf{j} - \mathbf{k}$, $\mathbf{v} = \mathbf{i} + \mathbf{j} + \mathbf{k}$
38. $\mathbf{u} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$, $\mathbf{v} = 2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$

**Finding the Area of a Parallelogram** In Exercises 39–42, find the area of the parallelogram that has the vectors as adjacent sides.

39. $\mathbf{u} = \mathbf{j}$, $\mathbf{v} = \mathbf{j} + \mathbf{k}$
40. $\mathbf{u} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{v} = \mathbf{j} + \mathbf{k}$
41. $\mathbf{u} = (3, 2, -1)$, $\mathbf{v} = (1, 2, 3)$
42. $\mathbf{u} = (2, -1, 0)$, $\mathbf{v} = (-1, 2, 0)$

**Geometric Application of the Cross Product** In Exercises 43 and 44, verify that the points are the vertices of a parallelogram, and then find its area.

43. $(1, 1, 1)$, $(2, 3, 4)$, $(6, 5, 2)$, $(7, 7, 5)$
44. $(2, -1, 1)$, $(5, 1, 4)$, $(0, 1, 1)$, $(3, 3, 4)$

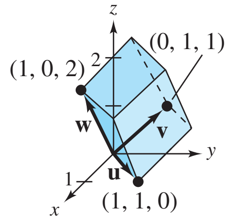
**Triple Scalar Product** In Exercises 45–48, find $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$. This quantity is called the triple scalar product of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

45. $\mathbf{u} = \mathbf{i}$, $\mathbf{v} = \mathbf{j}$, $\mathbf{w} = \mathbf{k}$
46. $\mathbf{u} = -\mathbf{i}$, $\mathbf{v} = -\mathbf{j}$, $\mathbf{w} = \mathbf{k}$
47. $\mathbf{u} = (1, 1, 1)$, $\mathbf{v} = (2, 1, 0)$, $\mathbf{w} = (0, 0, 1)$
48. $\mathbf{u} = (2, 0, 1)$, $\mathbf{v} = (0, 3, 0)$, $\mathbf{w} = (0, 0, 1)$

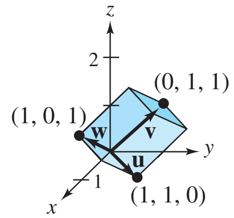
49. **Volume of a Parallelepiped** Show that the volume of a parallelepiped having $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ as adjacent edges is given by $|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$.

50. **Finding the Volume of a Parallelepiped** Use the result of Exercise 49 to find the volume of each parallelepiped.

(a) $\mathbf{u} = \mathbf{i} + \mathbf{j}$, $\mathbf{v} = \mathbf{j} + \mathbf{k}$, $\mathbf{w} = \mathbf{i} + 2\mathbf{k}$

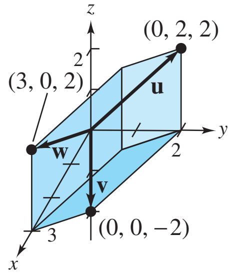


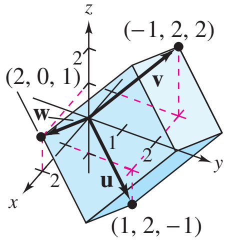

(b) $\mathbf{u} = \mathbf{i} + \mathbf{j}$, $\mathbf{v} = \mathbf{j} + \mathbf{k}$, $\mathbf{w} = \mathbf{i} + \mathbf{k}$



(c) $\mathbf{u} = (0, 2, 2)$
$\mathbf{v} = (0, 0, -2)$
$\mathbf{w} = (3, 0, 2)$

(d) $\mathbf{u} = (1, 2, -1)$
$\mathbf{v} = (-1, 2, 2)$
$\mathbf{w} = (2, 0, 1)$




**Finding the Area of a Triangle** **In Exercises 51 and 52, find the area of the triangle with the given vertices. Use the fact that the area of the triangle having u and v as adjacent sides is given by $A = \frac{1}{2}\|\mathbf{u} \times \mathbf{v}\|$.**

**51.** $(1, 3, 5), (3, 3, 0), (-2, 0, 5)$

**52.** $(2, -3, 4), (0, 1, 2), (-1, 2, 0)$

**53.** **Proof** Prove that $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$.

**54.** **Proof** Prove that $c\mathbf{u} \times \mathbf{v} = c(\mathbf{u} \times \mathbf{v}) = \mathbf{u} \times c\mathbf{v}$.

**55.** **Proof** Prove that $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$.

**56.** **Proof** Prove that $\mathbf{u} \times \mathbf{u} = \mathbf{0}$.

**57.** **Proof** Prove that $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$.

**58.** **Proof** Prove that $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

**59.** **Proof** Prove that the angle $\theta$ between $\mathbf{u}$ and $\mathbf{v}$ is given by $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\,\|\mathbf{v}\| \sin \theta$.

**60.** **Proof** Prove that $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are parallel.

**61.** **Proof** Prove **Lagrange's Identity:**

$\|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2$.

**62.** **Proof**

(a) Prove that

$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$.

(b) Find an example for which

$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \neq (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$.

**Finding a Least Squares Approximation** **In Exercises 63–68, (a) find the least squares approximation $g(x) = a_0 + a_1x$ of the function $f$, and (b) use a graphing utility to graph $f$ and $g$ in the same viewing window.**

**63.** $f(x) = x^2, \quad 0 \le x \le 1$

**64.** $f(x) = \sqrt{x}, \quad 1 \le x \le 4$

**65.** $f(x) = e^{2x}, \quad 0 \le x \le 1$

**66.** $f(x) = e^{-2x}, \quad 0 \le x \le 1$

**67.** $f(x) = \cos x, \quad 0 \le x \le \pi$

**68.** $f(x) = \sin x, \quad 0 \le x \le \pi/2$

**Finding a Least Squares Approximation** **In Exercises 69–72, (a) find the least squares approximation $g(x) = a_0 + a_1x + a_2x^2$ of the function $f$, and (b) use a graphing utility to graph $f$ and $g$ in the same viewing window.**

**69.** $f(x) = x^3, \quad 0 \le x \le 1$

**70.** $f(x) = \sqrt{x}, \quad 1 \le x \le 4$

**71.** $f(x) = \sin x, \quad -\pi/2 \le x \le \pi/2$

**72.** $f(x) = \cos x, \quad -\pi/2 \le x \le \pi/2$

**Finding a Fourier Approximation** **In Exercises 73–84, find the Fourier approximation with the specified order of the function on the interval $[0, 2\pi]$.**

**73.** $f(x) = \pi - x$, third order

**74.** $f(x) = \pi - x$, fourth order

**75.** $f(x) = (x - \pi)^2$, third order

**76.** $f(x) = (x - \pi)^2$, fourth order

**77.** $f(x) = e^{-x}$, first order

**78.** $f(x) = e^{-x}$, second order

**79.** $f(x) = e^{-2x}$, first order

**80.** $f(x) = e^{-2x}$, second order

**81.** $f(x) = 1 + x$, third order

**82.** $f(x) = 1 + x$, fourth order

**83.** $f(x) = 2 \sin x \cos x$, fourth order

**84.** $f(x) = \sin^2 x$, fourth order

**85.** Use the results of Exercises 73 and 74 to find the $n$th-order Fourier approximation of

$f(x) = \pi - x$

on the interval $[0, 2\pi]$.

**86.** Use the results of Exercises 75 and 76 to find the $n$th-order Fourier approximation of

$f(x) = (x - \pi)^2$

on the interval $[0, 2\pi]$.

**87.** Use the results of Exercises 77 and 78 to find the $n$th-order Fourier approximation of

$f(x) = e^{-x}$

on the interval $[0, 2\pi]$.

**88.** **CAPSTONE** Explain how to find each of the following.

(a) The cross product of two vectors in $R^3$

(b) The least squares approximation of a function $f \in C[a, b]$ with respect to a subspace $W$ of $C[a, b]$

(c) The $n$th-order Fourier approximation on the interval $[0, 2\pi]$ of a continuous function $f$ with respect to the vector space spanned by

$\{1, \cos x, \ldots, \cos nx, \sin x, \ldots, \sin nx\}$

# 5 Review Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

**Finding Lengths, Dot Product, and Distance** **In Exercises 1–8, find (a) $\|\mathbf{u}\|$, (b) $\|\mathbf{v}\|$, (c) $\mathbf{u} \cdot \mathbf{v}$, and (d) $d(\mathbf{u}, \mathbf{v})$.**

1. $\mathbf{u} = (1, 2), \quad \mathbf{v} = (4, 1)$
2. $\mathbf{u} = (-1, 2), \quad \mathbf{v} = (2, 3)$
3. $\mathbf{u} = (2, 1, 1), \quad \mathbf{v} = (3, 2, -1)$
4. $\mathbf{u} = (1, -1, 2), \quad \mathbf{v} = (2, 3, 1)$
5. $\mathbf{u} = (1, -2, 0, 1), \quad \mathbf{v} = (1, 1, -1, 0)$
6. $\mathbf{u} = (1, -2, 2, 0), \quad \mathbf{v} = (2, -1, 0, 2)$
7. $\mathbf{u} = (0, 1, -1, 1, 2), \quad \mathbf{v} = (0, 1, -2, 1, 1)$
8. $\mathbf{u} = (1, -1, 0, 1, 1), \quad \mathbf{v} = (0, 1, -2, 2, 1)$

**Finding Length and a Unit Vector** **In Exercises 9–12, find $\|\mathbf{v}\|$ and find a unit vector in the direction of v.**

9. $\mathbf{v} = (5, 3, -2)$
10. $\mathbf{v} = (1, -2, 1)$
11. $\mathbf{v} = (-1, 1, 2)$
12. $\mathbf{v} = (0, 2, -1)$

13. Given the vector $\mathbf{v} = (8, 8, 6)$, find $\mathbf{u}$ such that
    (a) $\mathbf{u}$ has the same direction as $\mathbf{v}$ and one-half its length.
    (b) $\mathbf{u}$ has the direction opposite that of $\mathbf{v}$ and one-fourth its length.
    (c) $\mathbf{u}$ has the direction opposite that of $\mathbf{v}$ and twice its length.
14. For what values of $c$ is $\|c(2, 2, -1)\| = 3$?

**Finding the Angle Between Two Vectors** **In Exercises 15–20, find the angle $\theta$ between the two vectors.**

15. $\mathbf{u} = (2, 2), \quad \mathbf{v} = (-3, 3)$
16. $\mathbf{u} = (1, -1), \quad \mathbf{v} = (0, 1)$
17. $\mathbf{u} = \left(\cos \frac{3\pi}{4}, \sin \frac{3\pi}{4}\right), \quad \mathbf{v} = \left(\cos \frac{2\pi}{3}, \sin \frac{2\pi}{3}\right)$
18. $\mathbf{u} = \left(\cos \frac{\pi}{6}, \sin \frac{\pi}{6}\right), \quad \mathbf{v} = \left(\cos \frac{5\pi}{6}, \sin \frac{5\pi}{6}\right)$
19. $\mathbf{u} = (10, -5, 15), \quad \mathbf{v} = (-2, 1, -3)$
20. $\mathbf{u} = (1, 0, -3, 0), \quad \mathbf{v} = (2, -2, 1, 1)$

**Finding Orthogonal Vectors** **In Exercises 21–24, determine all vectors that are orthogonal to u.**

21. $\mathbf{u} = (0, -4, 3)$
22. $\mathbf{u} = (1, -1, 2)$
23. $\mathbf{u} = (1, -2, 2, 1)$
24. $\mathbf{u} = (0, 1, 2, -1)$

25. For $\mathbf{u} = \left(2, -\frac{1}{2}, 1\right)$ and $\mathbf{v} = \left(\frac{3}{2}, 2, -1\right)$, (a) find the inner product represented by $\langle \mathbf{u}, \mathbf{v} \rangle = u_1v_1 + 2u_2v_2 + 3u_3v_3$, and (b) use this inner product to find the distance between $\mathbf{u}$ and $\mathbf{v}$.
26. For $\mathbf{u} = \left(0, 3, \frac{1}{3}\right)$ and $\mathbf{v} = \left(\frac{4}{3}, 1, -3\right)$, (a) find the inner product represented by $\langle \mathbf{u}, \mathbf{v} \rangle = 2u_1v_1 + u_2v_2 + 2u_3v_3$, and (b) use this inner product to find the distance between $\mathbf{u}$ and $\mathbf{v}$.
27. Verify the triangle inequality and the Cauchy-Schwarz Inequality for $\mathbf{u}$ and $\mathbf{v}$ from Exercise 25. (Use the inner product given in Exercise 25.)
28. Verify the triangle inequality and the Cauchy-Schwarz Inequality for $\mathbf{u}$ and $\mathbf{v}$ from Exercise 26. (Use the inner product given in Exercise 26.)

**Calculus** **In Exercises 29 and 30, (a) find the inner product, (b) determine whether the vectors are orthogonal, and (c) verify the Cauchy-Schwarz Inequality for the vectors.**

29. $f(x) = x,\ g(x) = \dfrac{1}{x^2 + 1},\ \langle f, g \rangle = \displaystyle\int_{-1}^{1} f(x)g(x)\, dx$
30. $f(x) = x,\ g(x) = 4x^2,\ \langle f, g \rangle = \displaystyle\int_{0}^{1} f(x)g(x)\, dx$

**Finding an Orthogonal Projection** **In Exercises 31–36, find $\text{proj}_{\mathbf{v}}\mathbf{u}$.**

31. $\mathbf{u} = (2, 4), \quad \mathbf{v} = (1, -5)$
32. $\mathbf{u} = (2, 3), \quad \mathbf{v} = (0, 4)$
33. $\mathbf{u} = (1, 2), \quad \mathbf{v} = (2, 5)$
34. $\mathbf{u} = (2, 5), \quad \mathbf{v} = (0, 5)$
35. $\mathbf{u} = (0, -1, 2), \quad \mathbf{v} = (3, 2, 4)$
36. $\mathbf{u} = (1, 2, -1), \quad \mathbf{v} = (0, 2, 3)$

**Applying the Gram-Schmidt Process** **In Exercises 37–40, apply the Gram-Schmidt orthonormalization process to transform the given basis for $R^n$ into an orthonormal basis. Use the Euclidean inner product for $R^n$ and use the vectors in the order in which they are given.**

37. $B = \{(1, 1), (0, 2)\}$
38. $B = \{(3, 4), (1, 2)\}$
39. $B = \{(0, 3, 4), (1, 0, 0), (1, 1, 0)\}$
40. $B = \{(0, 0, 2), (0, 1, 1), (1, 1, 1)\}$

41. Let $B = \{(0, 2, -2), (1, 0, -2)\}$ be a basis for a subspace of $R^3$, and let $\mathbf{x} = (-1, 4, -2)$ be a vector in the subspace.
    (a) Write $\mathbf{x}$ as a linear combination of the vectors in $B$. That is, find the coordinates of $\mathbf{x}$ relative to $B$.
    (b) Apply the Gram-Schmidt orthonormalization process to transform $B$ into an orthonormal set $B'$.
    (c) Write $\mathbf{x}$ as a linear combination of the vectors in $B'$. That is, find the coordinates of $\mathbf{x}$ relative to $B'$.

**42.** Repeat Exercise 41 for $B = \{(-1, 2, 2), (1, 0, 0)\}$ and $\mathbf{x} = (-3, 4, 4)$.

**Calculus** **In Exercises 43–46, let $f$ and $g$ be functions in the vector space $C[a, b]$ with inner product**

$$\langle f, g\rangle = \int_a^b f(x)g(x)\,dx.$$

**43.** Show that $f(x) = \sin x$ and $g(x) = \cos x$ are orthogonal in $C[0, \pi]$.

**44.** Show that $f(x) = \sqrt{1 - x^2}$ and $g(x) = 2x\sqrt{1 - x^2}$ are orthogonal in $C[-1, 1]$.

**45.** Let $f(x) = x$ and $g(x) = x^3$ be vectors in $C[0, 1]$.

(a) Find $\langle f, g\rangle$.

(b) Find $\|g\|$.

(c) Find $d(f, g)$.

(d) Orthonormalize the set $B = \{f, g\}$.

**46.** Let $f(x) = x + 2$ and $g(x) = 15x - 8$ be vectors in $C[0, 1]$.

(a) Find $\langle f, g\rangle$.

(b) Find $\langle -4f, g\rangle$.

(c) Find $\|f\|$.

(d) Orthonormalize the set $B = \{f, g\}$.

**47.** Find an orthonormal basis for the following subspace of Euclidean 3-space.

$W = \{(x_1, x_2, x_3): x_1 + x_2 + x_3 = 0\}$

**48.** Find an orthonormal basis for the solution space of the homogeneous system of linear equations.

$$\begin{aligned} x + y - z + w &= 0 \\ 2x - y + z + 2w &= 0 \end{aligned}$$

**49.** **Proof** Prove that if $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in $R^n$, then

$(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$.

**50.** **Proof** Prove that if $\mathbf{u}$ and $\mathbf{v}$ are vectors in $R^n$, then

$\|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2$.

**51.** **Proof** Prove that if $\mathbf{u}$ and $\mathbf{v}$ are vectors in an inner product space such that $\|\mathbf{u}\| \le 1$ and $\|\mathbf{v}\| \le 1$, then

$|\langle \mathbf{u}, \mathbf{v}\rangle| \le 1$.

**52.** **Proof** Prove that if $\mathbf{u}$ and $\mathbf{v}$ are vectors in an inner product space $V$, then

$\big|\|\mathbf{u}\| - \|\mathbf{v}\|\big| \le \|\mathbf{u} \pm \mathbf{v}\|$.

**53.** **Proof** Let $V$ be an $m$-dimensional subspace of $R^n$ such that $m < n$. Prove that any vector $\mathbf{u}$ in $R^n$ can be uniquely written in the form $\mathbf{u} = \mathbf{v} + \mathbf{w}$, where $\mathbf{v}$ is in $V$ and $\mathbf{w}$ is orthogonal to every vector in $V$.

**54.** Let $V$ be the two-dimensional subspace of $R^4$ spanned by $(0, 1, 0, 1)$ and $(0, 2, 0, 0)$. Write the vector $\mathbf{u} = (1, 1, 1, 1)$ in the form $\mathbf{u} = \mathbf{v} + \mathbf{w}$, where $\mathbf{v}$ is in $V$ and $\mathbf{w}$ is orthogonal to every vector in $V$.

**55.** **Proof** Let $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$ be an orthonormal subset of $R^n$, and let $\mathbf{v}$ be any vector in $R^n$. Prove that

$$\|\mathbf{v}\|^2 \ge \sum_{i=1}^{m} (\mathbf{v} \cdot \mathbf{u}_i)^2.$$

(This inequality is called **Bessel's Inequality**.)

**56.** **Proof** Let $\{x_1, x_2, \ldots, x_n\}$ be a set of real numbers. Use the Cauchy-Schwarz Inequality to prove that

$(x_1 + x_2 + \cdots + x_n)^2 \le n(x_1^2 + x_2^2 + \cdots + x_n^2)$.

**57.** **Proof** Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in an inner product space $V$. Prove that $\|\mathbf{u} + \mathbf{v}\| = \|\mathbf{u} - \mathbf{v}\|$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

**58.** **Writing** Let $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ be a dependent set of vectors in an inner product space $V$. Describe the result of applying the Gram-Schmidt orthonormalization process to this set.

**59.** Find the orthogonal complement $S^\perp$ of the subspace $S$ of $R^3$ spanned by the two column vectors of the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 0 & -1 \end{bmatrix}.$$

**60.** Find the projection of the vector $\mathbf{v} = [1 \quad 0 \quad -2]^T$ onto the subspace

$$S = \operatorname{span}\left\{\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right\}.$$

**61.** Find bases for the four fundamental subspaces of the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -3 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

**62.** Find the least squares regression line for the set of data points

$\{(-2, 2), (-1, 1), (0, 1), (1, 3)\}$.

Graph the points and the line on the same set of axes.

**63.** **Energy Consumption** The table shows the world energy consumptions $y$ (in quadrillions of Btu) from 2001 through 2008. Find the least squares regression line for the data. Then use the model to predict the energy consumption in 2015. Let $t$ represent the year, with $t = 1$ corresponding to 2001. (Source: U.S. Energy Information Administration)

| *Year* | 2001 | 2002 | 2003 | 2004 |
|---|---|---|---|---|
| *Energy Consumption, y* | 400.5 | 410.7 | 425.7 | 448.9 |

| *Year* | 2005 | 2006 | 2007 | 2008 |
|---|---|---|---|---|
| *Energy Consumption, y* | 461.6 | 470.9 | 482.3 | 493.0 |

**64. Revenue** The table shows the revenues $y$ (in billions of dollars) for Google, Incorporated from 2003 through 2010. Find the least squares regression quadratic and cubic polynomials for the data. Then use each model to predict the revenue in 2015. Let $t$ represent the year, with $t = 3$ corresponding to 2003. Which model appears to be more accurate for predicting future revenues? Explain. (Source: Google, Incorporated)

| *Year* | 2003 | 2004 | 2005 | 2006 |
|---|---|---|---|---|
| *Revenue, y* | 1.5 | 3.2 | 6.1 | 10.6 |

| *Year* | 2007 | 2008 | 2009 | 2010 |
|---|---|---|---|---|
| *Revenue, y* | 16.6 | 21.8 | 23.7 | 29.3 |

**Finding the Cross Product** **In Exercises 65–68, find $\mathbf{u} \times \mathbf{v}$ and show that it is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.**

**65.** $\mathbf{u} = (1, 1, 1)$, $\mathbf{v} = (1, 0, 0)$

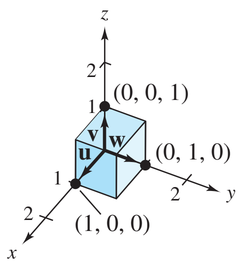
**66.** $\mathbf{u} = (1, -1, 1)$, $\mathbf{v} = (0, 1, 1)$

**67.** $\mathbf{u} = \mathbf{j} + 6\mathbf{k}$, $\mathbf{v} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$

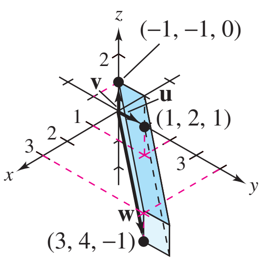
**68.** $\mathbf{u} = 2\mathbf{i} - \mathbf{k}$, $\mathbf{v} = \mathbf{i} + \mathbf{j} - \mathbf{k}$

**Finding the Volume of a Parallelepiped** **In Exercises 69–72, find the volume of the parallelepiped. Recall from Exercises 49 and 50 in Section 5.5 that the volume of a parallelepiped having $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ as adjacent edges is given by $|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$.**

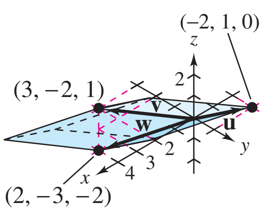
**69.** $\mathbf{u} = (1, 0, 0)$
$\mathbf{v} = (0, 0, 1)$
$\mathbf{w} = (0, 1, 0)$



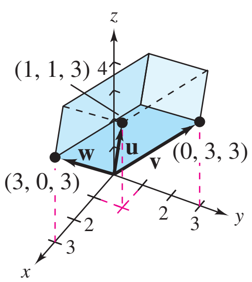

**70.** $\mathbf{u} = (1, 2, 1)$
$\mathbf{v} = (-1, -1, 0)$
$\mathbf{w} = (3, 4, -1)$




**71.** $\mathbf{u} = -2\mathbf{i} + \mathbf{j}$
$\mathbf{v} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$
$\mathbf{w} = 2\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}$




**72.** $\mathbf{u} = \mathbf{i} + \mathbf{j} + 3\mathbf{k}$
$\mathbf{v} = 3\mathbf{j} + 3\mathbf{k}$
$\mathbf{w} = 3\mathbf{i} + 3\mathbf{k}$




**73. Finding the Area of a Parallelogram** Find the area of the parallelogram that has

$\mathbf{u} = (1, 3, 0)$ and $\mathbf{v} = (-1, 0, 2)$

as adjacent sides.

**74. Proof** Prove that

$\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\,\|\mathbf{v}\|$

if and only if $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

**Finding a Least Squares Approximation** **In Exercises 75–78, (a) find the least squares approximation $g(x) = a_0 + a_1x$ of the function $f$, and (b) use a graphing utility to graph $f$ and $g$ in the same viewing window.**

**75.** $f(x) = x^3$, $-1 \le x \le 1$

**76.** $f(x) = x^3$, $0 \le x \le 2$

**77.** $f(x) = \sin 2x$, $0 \le x \le \pi/2$

**78.** $f(x) = \sin x \cos x$, $0 \le x \le \pi$

**Finding a Least Squares Approximation** **In Exercises 79 and 80, (a) find the least squares approximation $g(x) = a_0 + a_1x + a_2x^2$ of the function $f$, and (b) use a graphing utility to graph $f$ and $g$ in the same viewing window.**

**79.** $f(x) = \sqrt{x}$, $0 \le x \le 1$ **80.** $f(x) = \dfrac{1}{x}$, $1 \le x \le 2$

**Finding a Fourier Approximation** **In Exercises 81 and 82, find the Fourier approximation with the specified order of the function on the interval $[-\pi, \pi]$.**

**81.** $f(x) = x^2$, first order **82.** $f(x) = x$, second order

**True or False?** **In Exercises 83 and 84, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.**

**83.** (a) The cross product of two nonzero vectors in $R^3$ yields a vector orthogonal to the two vectors that produced it.

(b) The cross product of two nonzero vectors in $R^3$ is commutative.

(c) The least squares approximation of a function $f$ is the function $g$ (in the subspace $W$) closest to $f$ in terms of the inner product $\langle f, g\rangle$.

**84.** (a) The vectors $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{u}$ in $R^3$ have equal lengths but opposite directions.

(b) If $\mathbf{u}$ and $\mathbf{v}$ are two nonzero vectors in $R^3$, then $\mathbf{u}$ and $\mathbf{v}$ are parallel if and only if $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.

(c) A special type of least squares approximation, the Fourier approximation, is spanned by the basis $S = \{1, \cos x, \cos 2x, \ldots, \cos nx, \sin x, \sin 2x, \ldots, \sin nx\}$.

# 5 Projects

## 1 The *QR*-Factorization

The Gram-Schmidt orthonormalization process leads to an important factorization of matrices called the ***QR*-factorization.** If $A$ is an $m \times n$ matrix of rank $n$, then $A$ can be expressed as the product $A = QR$ of an $m \times n$ matrix $Q$ and an $n \times n$ matrix $R$, where $Q$ has orthonormal columns and $R$ is upper triangular.

The columns of $A$ can be considered a basis for a subspace of $R^m$, and the columns of $Q$ are the result of applying the Gram-Schmidt orthonormalization process to this set of column vectors.

Recall that Example 7, Section 5.3, used the Gram-Schmidt orthonormalization process on the column vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ of the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

to produce an orthonormal basis for $R^3$, which is labeled here as $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$.

$$\mathbf{q}_1 = \left(\sqrt{2}/2, \sqrt{2}/2, 0\right),\ \mathbf{q}_2 = \left(-\sqrt{2}/2, \sqrt{2}/2, 0\right),\ \mathbf{q}_3 = (0, 0, 1)$$

These vectors form the columns of the matrix $Q$.

$$Q = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The upper triangular matrix $R$ is

$$R = \begin{bmatrix} \mathbf{v}_1 \cdot \mathbf{q}_1 & \mathbf{v}_2 \cdot \mathbf{q}_1 & \mathbf{v}_3 \cdot \mathbf{q}_1 \\ 0 & \mathbf{v}_2 \cdot \mathbf{q}_2 & \mathbf{v}_3 \cdot \mathbf{q}_2 \\ 0 & 0 & \mathbf{v}_3 \cdot \mathbf{q}_3 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & 3\sqrt{2}/2 & \sqrt{2}/2 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & 0 & 2 \end{bmatrix}$$

Verify that $A = QR$.

In general, if $A$ is an $m \times n$ matrix of rank $n$ with columns $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, then the $QR$-factorization of $A$ is given by

$$A = QR$$

$$[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n] = [\mathbf{q}_1 \quad \mathbf{q}_2 \quad \cdots \quad \mathbf{q}_n] \begin{bmatrix} \mathbf{v}_1 \cdot \mathbf{q}_1 & \mathbf{v}_2 \cdot \mathbf{q}_1 & \cdots & \mathbf{v}_n \cdot \mathbf{q}_1 \\ 0 & \mathbf{v}_2 \cdot \mathbf{q}_2 & \cdots & \mathbf{v}_n \cdot \mathbf{q}_2 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \mathbf{v}_n \cdot \mathbf{q}_n \end{bmatrix}$$

where the columns $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n$ of the $m \times n$ matrix $Q$ are the orthonormal vectors that result from the Gram-Schmidt orthonormalization process.

**1.** Find the $QR$-factorization of each matrix.

(a) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$ (c) $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

**2.** Let $A = QR$ be the $QR$-factorization of the $m \times n$ matrix $A$ of rank $n$. Show how the least squares problem can be solved using the $QR$-factorization.

**3.** Use the result of part 2 to solve the least squares problem $A\mathbf{x} = \mathbf{b}$ when $A$ is the matrix from part 1(a) and $\mathbf{b} = [-1 \quad 1 \quad -1]^T$.

**REMARK**

The *QR*-factorization of a matrix forms the basis for many algorithms of linear algebra. Algorithms for the computation of eigenvalues (see Chapter 7) are based on this factorization, as are algorithms for computing the least squares regression line for a set of data points. It should also be mentioned that, in practice, techniques other than the Gram-Schmidt orthonormalization process are used to compute the *QR*-factorization of a matrix.

## 2 Orthogonal Matrices and Change of Basis

Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be an ordered basis for the vector space $V$. Recall that the coordinate matrix of a vector $\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$ in $V$ is the column vector

$$[\mathbf{x}]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}.$$

If $B'$ is another basis for $V$, then the transition matrix $P$ from $B'$ to $B$ changes a coordinate matrix relative to $B'$ into a coordinate matrix relative to $B$,

$$P[\mathbf{x}]_{B'} = [\mathbf{x}]_B.$$

The question you will explore now is whether there are transition matrices $P$ that preserve the length of the coordinate matrix—that is, given $P[\mathbf{x}]_{B'} = [\mathbf{x}]_B$, does $\|[\mathbf{x}]_{B'}\| = \|[\mathbf{x}]_B\|$?

For example, consider the transition matrix from Example 5 in Section 4.7,

$$P = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}$$

relative to the bases for $R^2$,

$$B = \{(-3, 2), (4, -2)\} \quad \text{and} \quad B' = \{(-1, 2), (2, -2)\}.$$

If $\mathbf{x} = (-1, 2)$, then $[\mathbf{x}]_{B'} = [1 \quad 0]^T$ and $[\mathbf{x}]_B = P[\mathbf{x}]_{B'} = [3 \quad 2]^T$. (Verify this.) So, using the Euclidean norm for $R^2$,

$$\|[\mathbf{x}]_{B'}\| = 1 \neq \sqrt{13} = \|[\mathbf{x}]_B\|.$$

You will see in this project that if the transition matrix $P$ is **orthogonal,** then the norm of the coordinate vector will remain unchanged. You may recall working with orthogonal matrices in Section 3.3 (Exercises 71–79) and Section 5.3 (Exercise 67).

### Definition of Orthogonal Matrix

The square matrix $P$ is **orthogonal** when it is invertible and $P^{-1} = P^T$.

**1.** Show that the matrix $P$ defined previously is *not* orthogonal.

**2.** Show that for any real number $\theta$, the matrix

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

is orthogonal.

**3.** Show that a matrix is orthogonal if and only if its columns are pairwise orthogonal.

**4.** Prove that the inverse of an orthogonal matrix is orthogonal.

**5.** Is the sum of orthogonal matrices orthogonal? Is the product of orthogonal matrices orthogonal? Illustrate your answers with appropriate examples.

**6.** Prove that if $P$ is an $m \times n$ orthogonal matrix, then $\|P\mathbf{x}\| = \|\mathbf{x}\|$ for all vectors $\mathbf{x}$ in $R^n$.

**7.** Verify the result of part 6 using the bases $B = \{(1, 0), (0, 1)\}$ and

$$B' = \left\{\left(-\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right), \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)\right\}.$$

# 4 and 5 Cumulative Test

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

**Take this test to review the material in Chapters 4 and 5. After you are finished, check your work against the answers in the back of the book.**

**1.** Given the vectors $\mathbf{v} = (1, -2)$ and $\mathbf{w} = (2, -5)$, find and sketch each vector.

(a) $\mathbf{v} + \mathbf{w}$ (b) $3\mathbf{v}$ (c) $2\mathbf{v} - 4\mathbf{w}$

**2.** Write $\mathbf{w} = (2, 4, 1)$ as a linear combination of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ (if possible).

$\mathbf{v}_1 = (1, 2, 0)$, $\mathbf{v}_2 = (-1, 0, 1)$, $\mathbf{v}_3 = (0, 3, 0)$

**3.** Write the third column of the matrix as a linear combination of the first two columns (if possible).

$$\begin{bmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \\ 4 & 5 & 7 \end{bmatrix}$$

**4.** Use a software program or a graphing utility with matrix capabilities to write $\mathbf{v}$ as a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$, $\mathbf{u}_4$, $\mathbf{u}_5$, and $\mathbf{u}_6$. Then verify your solution.

$\mathbf{v} = (10, 30, -13, 14, -7, 27)$
$\mathbf{u}_1 = (1, 2, -3, 4, -1, 2)$
$\mathbf{u}_2 = (1, -2, 1, -1, 2, 1)$
$\mathbf{u}_3 = (0, 2, -1, 2, -1, -1)$
$\mathbf{u}_4 = (1, 0, 3, -4, 1, 2)$
$\mathbf{u}_5 = (1, -2, 1, -1, 2, -3)$
$\mathbf{u}_6 = (3, 2, 1, -2, 3, 0)$

**5.** Prove that the set of all singular $3 \times 3$ matrices is not a vector space.

**6.** Determine whether the set is a subspace of $R^4$.

$\{(x, x + y, y, y): x, y \in R\}$

**7.** Determine whether the set is a subspace of $R^3$.

$\{(x, xy, y): x, y \in R\}$

**8.** Determine whether the columns of matrix $A$ span $R^4$.

$$A = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

**9.** (a) Define what it means to say that a set of vectors is *linearly independent.*

(b) Determine whether the set $S$ is linearly dependent or independent.

$S = \{(1, 0, 1, 0), (0, 3, 0, 1), (1, 1, 2, 2), (3, -4, 2, -3)\}$

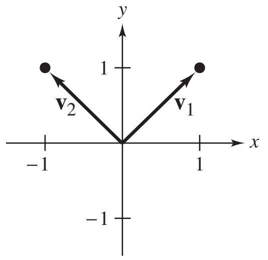



**Figure for 10(b)**

**10.** (a) Define a *basis* for a vector space.

(b) Determine whether the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ shown in the figure at the left is a basis for $R^2$.

(c) Determine whether the following set is a basis for $R^3$.

$\{(1, 2, 1), (0, 1, 2), (2, 1, -3)\}$

**11.** Find a basis for the solution space of $A\mathbf{x} = \mathbf{0}$ when

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -2 & -2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}.$$

**12.** Find the coordinates $[\mathbf{v}]_B$ of the vector $\mathbf{v} = (1, 2, -3)$ relative to the basis

$B = \{(0, 1, 1), (1, 1, 1), (1, 0, 1)\}$.

**13.** Find the transition matrix from the basis $B = \{(2, 1, 0), (1, 0, 0), (0, 1, 1)\}$ to the basis

$B' = \{(1, 1, 2), (1, 1, 1), (0, 1, 2)\}$.

**14.** Let $\mathbf{u} = (1, 0, 2)$ and $\mathbf{v} = (-2, 1, 3)$.

(a) Find $\|\mathbf{u}\|$.

(b) Find the distance between $\mathbf{u}$ and $\mathbf{v}$.

(c) Find $\mathbf{u} \cdot \mathbf{v}$.

(d) Find the angle $\theta$ between $\mathbf{u}$ and $\mathbf{v}$.

**15.** Find the inner product of $f(x) = x^2$ and $g(x) = x + 2$ from $C[0, 1]$ using the integral

$$\langle f, g \rangle = \int_0^1 f(x)g(x)\, dx.$$

**16.** Apply the Gram-Schmidt orthonormalization process to transform the following set of vectors into an orthonormal basis for $R^3$.

$\{(2, 0, 0), (1, 1, 1), (0, 1, 2)\}$

**17.** Let $\mathbf{u} = (1, 2)$ and $\mathbf{v} = (-3, 2)$. Find $\text{proj}_{\mathbf{v}}\mathbf{u}$, and graph $\mathbf{u}$, $\mathbf{v}$, and $\text{proj}_{\mathbf{v}}\mathbf{u}$ on the same set of coordinate axes.

**18.** Find the four fundamental subspaces of the matrix

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

**19.** Find the orthogonal complement $S^\perp$ of the set

$$S = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right).$$

**20.** Suppose that $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are linearly independent vectors and that $\mathbf{y}$ is a vector not in their span. Prove that the vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$ and $\mathbf{y}$ are linearly independent.

**21.** Find the least squares regression line for the points $\{(1, 1), (2, 0), (5, -5)\}$. Graph the points and the line.

**22.** The two matrices $A$ and $B$ are row-equivalent.

$$A = \begin{bmatrix} 2 & -4 & 0 & 1 & 7 & 11 \\ 1 & -2 & -1 & 1 & 9 & 12 \\ -1 & 2 & 1 & 3 & -5 & 16 \\ 4 & -8 & 1 & -1 & 6 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -2 & 0 & 0 & 3 & 2 \\ 0 & 0 & 1 & 0 & -5 & -3 \\ 0 & 0 & 0 & 1 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Find the rank of $A$.

(b) Find a basis for the row space of $A$.

(c) Find a basis for the column space of $A$.

(d) Find a basis for the nullspace of $A$.

(e) Is the last column of $A$ in the span of the first three columns?

(f) Are the first three columns of $A$ linearly independent?

(g) Is the last column of $A$ in the span of columns 1, 3, and 4?

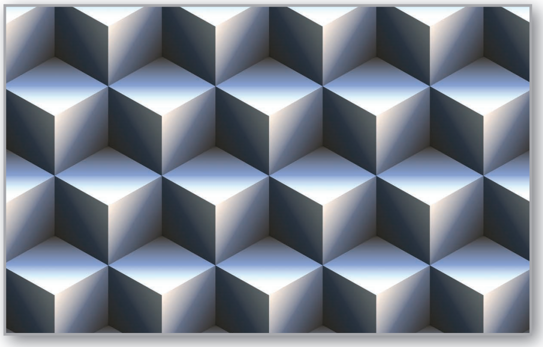
(h) Are columns 1, 3, and 4 linearly dependent?

**23.** Let $S_1$ and $S_2$ be two-dimensional subspaces of $R^3$. Is it possible that $S_1 \cap S_2 = \{(0, 0, 0)\}$? Explain.

**24.** Let $V$ be a vector space of dimension $n$. Prove that any set of less than $n$ vectors cannot span $V$.

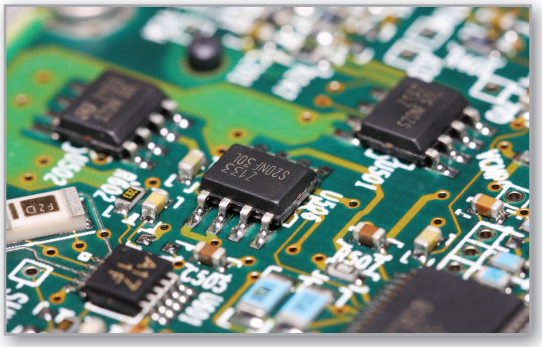
# 6 Linear Transformations





Computer Graphics (p. 332)

Weather (p. 325)



Circuit Design (p. 316)

Control Systems (p. 308)

Multivariate Statistics (p. 298)



Clockwise from top left, Matt Antonino/Shutterstock.Com; Etienne du Preez/shutterstock.com;
Mark Yuill/Shutterstock.com; A1Stock /Shutterstock.com; Mau Horng/Shutterstock

# 6.1 Introduction to Linear Transformations

- Find the image and preimage of a function.
- Show that a function is a linear transformation, and find a linear transformation.

## IMAGES AND PREIMAGES OF FUNCTIONS

In this chapter, you will learn about functions that **map** a vector space $V$ into a vector space $W$. This type of function is denoted by

$$T: V \to W.$$

The standard function terminology is used for such functions. For instance, $V$ is called the **domain** of $T$, and $W$ is called the **codomain** of $T$. If $\mathbf{v}$ is in $V$ and $\mathbf{w}$ is in $W$ such that $T(\mathbf{v}) = \mathbf{w}$, then $\mathbf{w}$ is called the **image** of $\mathbf{v}$ under $T$. The set of all images of vectors in $V$ is called the **range** of $T$, and the set of all $\mathbf{v}$ in $V$ such that $T(\mathbf{v}) = \mathbf{w}$ is called the **preimage** of $\mathbf{w}$. (See Figure 6.1.)

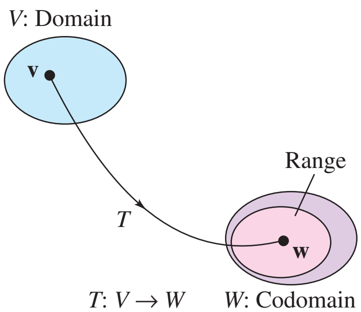



**Figure 6.1**

**REMARK**

For a vector $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ in $R^n$, it would be technically correct to use double parentheses to denote $T(\mathbf{v})$ as $T(\mathbf{v}) = T((v_1, v_2, \ldots, v_n))$. For convenience, however, drop one set of parentheses to produce

$$T(\mathbf{v}) = T(v_1, v_2, \ldots, v_n).$$

### EXAMPLE 1 A Function from $R^2$ into $R^2$

For any vector $\mathbf{v} = (v_1, v_2)$ in $R^2$, define $T: R^2 \to R^2$ by

$$T(v_1, v_2) = (v_1 - v_2, v_1 + 2v_2).$$

**a.** Find the image of $\mathbf{v} = (-1, 2)$.

**b.** Find the image of $\mathbf{v} = (0, 0)$.

**c.** Find the preimage of $\mathbf{w} = (-1, 11)$.

**SOLUTION**

**a.** For $\mathbf{v} = (-1, 2)$, you have

$$T(-1, 2) = (-1 - 2, -1 + 2(2)) = (-3, 3).$$

**b.** If $\mathbf{v} = (0, 0)$, then

$$T(0, 0) = (0 - 0, 0 + 2(0)) = (0, 0).$$

**c.** If $T(\mathbf{v}) = (v_1 - v_2, v_1 + 2v_2) = (-1, 11)$, then

$$\begin{aligned} v_1 - \phantom{2}v_2 &= -1 \\ v_1 + 2v_2 &= 11. \end{aligned}$$

This system of equations has the unique solution $v_1 = 3$ and $v_2 = 4$. So, the preimage of $(-1, 11)$ is the set in $R^2$ consisting of the single vector $(3, 4)$.

## LINEAR TRANSFORMATIONS

This chapter centers on functions (that map one vector space into another) that preserve the operations of vector addition and scalar multiplication. Such functions are called **linear transformations.**

### Definition of a Linear Transformation

Let $V$ and $W$ be vector spaces. The function

$$T: V \rightarrow W$$

is called a **linear transformation** of $V$ into $W$ when the following two properties are true for all $\mathbf{u}$ and $\mathbf{v}$ in $V$ and for any scalar $c$.

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
2. $T(c\mathbf{u}) = cT(\mathbf{u})$

A linear transformation is *operation preserving* because the same result occurs whether you perform the operations of addition and scalar multiplication before or after applying the linear transformation. Although the same symbols denote the vector operations in both $V$ and $W$, you should note that the operations may be different, as indicated in the following diagram.

$$T(\mathbf{u} \overset{\text{Addition in } V}{\downarrow +} \mathbf{v}) = T(\mathbf{u}) \overset{\text{Addition in } W}{\downarrow +} T(\mathbf{v}) \qquad \overset{\text{Scalar multiplication in } V}{T(c\mathbf{u})} = \overset{\text{Scalar multiplication in } W}{cT(\mathbf{u})}$$

## EXAMPLE 2 Verifying a Linear Transformation from $R^2$ into $R^2$

Show that the function given in Example 1 is a linear transformation from $R^2$ into $R^2$.

$$T(v_1, v_2) = (v_1 - v_2, v_1 + 2v_2)$$

### SOLUTION

To show that the function $T$ is a linear transformation, you must show that it preserves vector addition and scalar multiplication. To do this, let $\mathbf{v} = (v_1, v_2)$ and $\mathbf{u} = (u_1, u_2)$ be vectors in $R^2$ and let $c$ be any real number. Then, using the properties of vector addition and scalar multiplication, you have the following two statements.

1. Because $\mathbf{u} + \mathbf{v} = (u_1, u_2) + (v_1, v_2) = (u_1 + v_1, u_2 + v_2)$, you have

$$\begin{aligned} T(\mathbf{u} + \mathbf{v}) &= T(u_1 + v_1, u_2 + v_2) \\ &= ((u_1 + v_1) - (u_2 + v_2), (u_1 + v_1) + 2(u_2 + v_2)) \\ &= ((u_1 - u_2) + (v_1 - v_2), (u_1 + 2u_2) + (v_1 + 2v_2)) \\ &= (u_1 - u_2, u_1 + 2u_2) + (v_1 - v_2, v_1 + 2v_2) \\ &= T(\mathbf{u}) + T(\mathbf{v}). \end{aligned}$$

**REMARK**

A linear transformation $T: V \rightarrow V$ from a vector space into itself (as in Example 2) is called a **linear operator.**

2. Because $c\mathbf{u} = c(u_1, u_2) = (cu_1, cu_2)$, you have

$$\begin{aligned} T(c\mathbf{u}) &= T(cu_1, cu_2) \\ &= (cu_1 - cu_2, cu_1 + 2cu_2) \\ &= c(u_1 - u_2, u_1 + 2u_2) \\ &= cT(\mathbf{u}). \end{aligned}$$

So, $T$ is a linear transformation.

Most of the common functions studied in calculus are not linear transformations.

### EXAMPLE 3 Some Functions That Are Not Linear Transformations

**a.** $f(x) = \sin x$ is not a linear transformation from $R$ into $R$ because, in general, $\sin(x_1 + x_2) \neq \sin x_1 + \sin x_2$. For instance,

$$\sin[(\pi/2) + (\pi/3)] \neq \sin(\pi/2) + \sin(\pi/3).$$

**b.** $f(x) = x^2$ is not a linear transformation from $R$ into $R$ because, in general, $(x_1 + x_2)^2 \neq x_1^2 + x_2^2$. For instance, $(1 + 2)^2 \neq 1^2 + 2^2$.

**c.** $f(x) = x + 1$ is not a linear transformation from $R$ into $R$ because

$$f(x_1 + x_2) = x_1 + x_2 + 1$$

whereas

$$f(x_1) + f(x_2) = (x_1 + 1) + (x_2 + 1) = x_1 + x_2 + 2.$$

So $f(x_1 + x_2) \neq f(x_1) + f(x_2)$.

**REMARK**

The function in Example 3(c) suggests two uses of the term *linear*. The function $f(x) = x + 1$ is called a linear function because its graph is a line. It is not a linear transformation from the vector space $R$ into $R$, however, because it preserves neither vector addition nor scalar multiplication.

Two simple linear transformations are the **zero transformation** and the **identity transformation,** which are defined as follows.

1. $T(\mathbf{v}) = \mathbf{0}$, for all $\mathbf{v}$ — Zero transformation ($T: V \to W$)
2. $T(\mathbf{v}) = \mathbf{v}$, for all $\mathbf{v}$ — Identity transformation ($T: V \to V$)

The verifications of the linearity of these two transformations are left as an exercise. (See Exercise 77.)

Note that the linear transformation in Example 1 has the property that the zero vector maps to itself. That is, $T(\mathbf{0}) = \mathbf{0}$, as shown in Example 1(b). This property is true for all linear transformations, as stated in the first property of the following theorem.

**REMARK**

One advantage of Theorem 6.1 is that it provides a quick way to spot functions that are not linear transformations. That is, because all four conditions of the theorem must be true of a linear transformation, it follows that if any one of the properties is not satisfied for a function $T$, then the function is not a linear transformation. For example, the function

$$T(x_1, x_2) = (x_1 + 1, x_2)$$

is not a linear transformation from $R^2$ into $R^2$ because $T(0, 0) \neq (0, 0)$.

### THEOREM 6.1 Properties of Linear Transformations

Let $T$ be a linear transformation from $V$ into $W$, where $\mathbf{u}$ and $\mathbf{v}$ are in $V$. Then the following properties are true.

1. $T(\mathbf{0}) = \mathbf{0}$
2. $T(-\mathbf{v}) = -T(\mathbf{v})$
3. $T(\mathbf{u} - \mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v})$
4. If $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$, then
$T(\mathbf{v}) = T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + \cdots + c_nT(\mathbf{v}_n)$.

**PROOF**

To prove the first property, note that $0\mathbf{v} = \mathbf{0}$. Then it follows that

$$T(\mathbf{0}) = T(0\mathbf{v}) = 0T(\mathbf{v}) = \mathbf{0}.$$

The second property follows from $-\mathbf{v} = (-1)\mathbf{v}$, which implies that

$$T(-\mathbf{v}) = T[(-1)\mathbf{v}] = (-1)T(\mathbf{v}) = -T(\mathbf{v}).$$

The third property follows from $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$, which implies that

$$T(\mathbf{u} - \mathbf{v}) = T[\mathbf{u} + (-1)\mathbf{v}] = T(\mathbf{u}) + (-1)T(\mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v}).$$

The proof of the fourth property is left to you.

Property 4 of Theorem 6.1 tells you that a linear transformation $T: V \to W$ is determined completely by its action on a basis for $V$. In other words, if $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for the vector space $V$ and if $T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_n)$ are given, then $T(\mathbf{v})$ is determined for *any* $\mathbf{v}$ in $V$. Example 4 demonstrates the use of this property.

## EXAMPLE 4 Linear Transformations and Bases

Let $T: R^3 \rightarrow R^3$ be a linear transformation such that

$$\begin{aligned} T(1, 0, 0) &= (2, -1, 4) \\ T(0, 1, 0) &= (1, 5, -2) \\ T(0, 0, 1) &= (0, 3, 1). \end{aligned}$$

Find $T(2, 3, -2)$.

### SOLUTION

Because $(2, 3, -2) = 2(1, 0, 0) + 3(0, 1, 0) - 2(0, 0, 1)$, you can use Property 4 of Theorem 6.1 to write

$$\begin{aligned} T(2, 3, -2) &= 2T(1, 0, 0) + 3T(0, 1, 0) - 2T(0, 0, 1) \\ &= 2(2, -1, 4) + 3(1, 5, -2) - 2(0, 3, 1) \\ &= (7, 7, 0). \end{aligned}$$

In the next example, a matrix defines a linear transformation from $R^2$ into $R^3$. The vector $\mathbf{v} = (v_1, v_2)$ is in the matrix form

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

so it can be multiplied *on the left* by a matrix of size $3 \times 2$.

## EXAMPLE 5 A Linear Transformation Defined by a Matrix

Define the function $T: R^2 \rightarrow R^3$ as follows.

$$T(\mathbf{v}) = A\mathbf{v} = \begin{bmatrix} 3 & 0 \\ 2 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

**a.** Find $T(\mathbf{v})$, where $\mathbf{v} = (2, -1)$.

**b.** Show that $T$ is a linear transformation from $R^2$ into $R^3$.

### SOLUTION

**a.** Because $\mathbf{v} = (2, -1)$, you have

$$T(\mathbf{v}) = A\mathbf{v} = \begin{bmatrix} 3 & 0 \\ 2 & 1 \\ -1 & -2 \end{bmatrix} \underbrace{\begin{bmatrix} 2 \\ -1 \end{bmatrix}}_{\text{Vector in } R^2} = \underbrace{\begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix}}_{\text{Vector in } R^3}.$$

So, you have $T(2, -1) = (6, 3, 0)$.

**b.** Begin by observing that $T$ maps a vector in $R^2$ to a vector in $R^3$. To show that $T$ is a linear transformation, use the properties of matrix multiplication given in Theorem 2.3. For any vectors $\mathbf{u}$ and $\mathbf{v}$ in $R^2$, the distributive property of matrix multiplication over addition produces

$$T(\mathbf{u} + \mathbf{v}) = A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = T(\mathbf{u}) + T(\mathbf{v}).$$

Similarly, for any vector $\mathbf{u}$ in $R^2$ and any scalar $c$, the commutative property of scalar multiplication with matrix multiplication produces

$$T(c\mathbf{u}) = A(c\mathbf{u}) = c(A\mathbf{u}) = cT(\mathbf{u}).$$

Example 5 illustrates an important result regarding the representation of linear transformations from $R^n$ into $R^m$. This result is presented in two stages. Theorem 6.2 below states that every $m \times n$ matrix represents a linear transformation from $R^n$ into $R^m$. Then, in Section 6.3, you will see the converse—that every linear transformation from $R^n$ into $R^m$ can be represented by an $m \times n$ matrix.

Note that part (b) of Example 5 makes no reference to the matrix $A$. This verification serves as a general proof that the function defined by any $m \times n$ matrix is a linear transformation from $R^n$ into $R^m$.

**REMARK**

The $m \times n$ zero matrix corresponds to the zero transformation from $R^n$ into $R^m$, and the $n \times n$ identity matrix $I_n$ corresponds to the identity transformation from $R^n$ into $R^n$.

### THEOREM 6.2 The Linear Transformation Given by a Matrix

Let $A$ be an $m \times n$ matrix. The function $T$ defined by

$$T(\mathbf{v}) = A\mathbf{v}$$

is a linear transformation from $R^n$ into $R^m$. In order to conform to matrix multiplication with an $m \times n$ matrix, $n \times 1$ matrices represent the vectors in $R^n$ and $m \times 1$ matrices represent the vectors in $R^m$.

Be sure you see that an $m \times n$ matrix $A$ defines a linear transformation from $R^n$ into $R^m$:

$$A\mathbf{v} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}}_{\substack{\text{Vector} \\ \text{in } R^n}} = \underbrace{\begin{bmatrix} a_{11}v_1 + a_{12}v_2 + \cdots + a_{1n}v_n \\ a_{21}v_1 + a_{22}v_2 + \cdots + a_{2n}v_n \\ \vdots \\ a_{m1}v_1 + a_{m2}v_2 + \cdots + a_{mn}v_n \end{bmatrix}}_{\substack{\text{Vector} \\ \text{in } R^m}}.$$

## EXAMPLE 6 Linear Transformation Given by Matrices

Consider the linear transformation $T: R^n \rightarrow R^m$ defined by $T(\mathbf{v}) = A\mathbf{v}$. Find the dimensions of $R^n$ and $R^m$ for the linear transformation represented by each matrix.

**a.** $A = \begin{bmatrix} 0 & 1 & -1 \\ 2 & 3 & 0 \\ 4 & 2 & 1 \end{bmatrix}$ **b.** $A = \begin{bmatrix} 2 & -3 \\ -5 & 0 \\ 0 & -2 \end{bmatrix}$ **c.** $A = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 3 & 1 & 0 & 0 \end{bmatrix}$

**SOLUTION**

**a.** Because the size of this matrix is $3 \times 3$, it defines a linear transformation from $R^3$ into $R^3$.

$$A\mathbf{v} = \begin{bmatrix} 0 & 1 & -1 \\ 2 & 3 & 0 \\ 4 & 2 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}}_{\substack{\text{Vector} \\ \text{in } R^3}} = \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}}_{\substack{\text{Vector} \\ \text{in } R^3}}$$

**b.** Because the size of this matrix is $3 \times 2$, it defines a linear transformation from $R^2$ into $R^3$.

**c.** Because the size of this matrix is $2 \times 4$, it defines a linear transformation from $R^4$ into $R^2$.

The next example discusses a common type of linear transformation from $R^2$ into $R^2$.

## EXAMPLE 7 Rotation in $R^2$

Show that the linear transformation $T: R^2 \to R^2$ represented by the matrix

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

has the property that it rotates every vector in $R^2$ counterclockwise about the origin through the angle $\theta$.

### SOLUTION

From Theorem 6.2, you know that $T$ is a linear transformation. To show that it rotates every vector in $R^2$ counterclockwise through the angle $\theta$, let $\mathbf{v} = (x, y)$ be a vector in $R^2$. Using polar coordinates, you can write $\mathbf{v}$ as

$$\begin{aligned} \mathbf{v} &= (x, y) \\ &= (r\cos\alpha, r\sin\alpha) \end{aligned}$$

where $r$ is the length of $\mathbf{v}$ and $\alpha$ is the angle from the positive $x$-axis counterclockwise to the vector $\mathbf{v}$. Now, applying the linear transformation $T$ to $\mathbf{v}$ produces

$$\begin{aligned} T(\mathbf{v}) &= A\mathbf{v} \\ &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} r\cos\alpha \\ r\sin\alpha \end{bmatrix} \\ &= \begin{bmatrix} r\cos\theta\cos\alpha - r\sin\theta\sin\alpha \\ r\sin\theta\cos\alpha + r\cos\theta\sin\alpha \end{bmatrix} \\ &= \begin{bmatrix} r\cos(\theta + \alpha) \\ r\sin(\theta + \alpha) \end{bmatrix}. \end{aligned}$$

So, the vector $T(\mathbf{v})$ has the same length as $\mathbf{v}$. Furthermore, because the angle from the positive $x$-axis to $T(\mathbf{v})$ is

$$\theta + \alpha$$

$T(\mathbf{v})$ is the vector that results from rotating the vector $\mathbf{v}$ counterclockwise through the angle $\theta$, as shown in Figure 6.2.

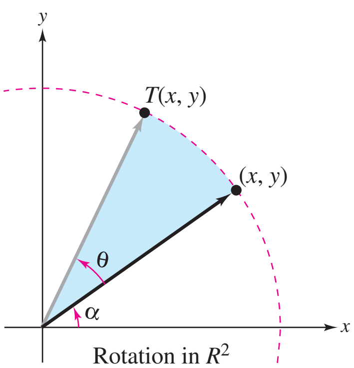



**Figure 6.2**

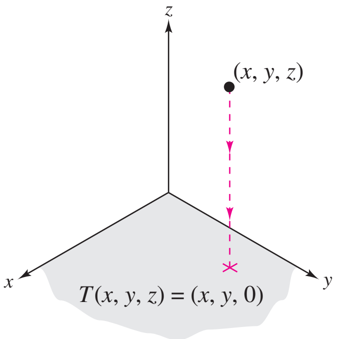
The linear transformation in Example 7 is called a **rotation** in $R^2$. Rotations in $R^2$ preserve both vector length and the angle between two vectors. That is, the angle between $\mathbf{u}$ and $\mathbf{v}$ is equal to the angle between $T(\mathbf{u})$ and $T(\mathbf{v})$.

## EXAMPLE 8 A Projection in $R^3$

The linear transformation $T: R^3 \rightarrow R^3$ represented by

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is called a **projection** in $R^3$. If $\mathbf{v} = (x, y, z)$ is a vector in $R^3$, then $T(\mathbf{v}) = (x, y, 0)$. In other words, $T$ maps every vector in $R^3$ to its orthogonal projection in the $xy$-plane, as shown in Figure 6.3.



Projection onto $xy$-plane

**Figure 6.3**

So far, only linear transformations from $R^n$ into $R^m$ or from $R^n$ into $R^n$ have been discussed. The remainder of this section considers some linear transformations involving vector spaces other than $R^n$.

## EXAMPLE 9 A Linear Transformation from $M_{m,n}$ into $M_{n,m}$

Let $T: M_{m,n} \rightarrow M_{n,m}$ be the function that maps an $m \times n$ matrix $A$ to its transpose. That is,

$$T(A) = A^T.$$

Show that $T$ is a linear transformation.

### SOLUTION

Let $A$ and $B$ be $m \times n$ matrices. From Theorem 2.6 you have

$$T(A + B) = (A + B)^T = A^T + B^T = T(A) + T(B)$$

and

$$T(cA) = (cA)^T = c(A^T) = cT(A).$$

So, $T$ is a linear transformation from $M_{m,n}$ into $M_{n,m}$.

**LINEAR ALGEBRA APPLIED**

Many multivariate statistical methods can use linear transformations. For instance, in a *multiple regression analysis,* there are two or more independent variables and a single dependent variable. A linear transformation is useful for finding weights to be assigned to the independent variables to predict the value of the dependent variable. Also, in a *canonical correlation analysis,* there are two or more independent variables and two or more dependent variables. Linear transformations can help find a linear combination of the independent variables to predict the value of a linear combination of the dependent variables.

## EXAMPLE 10 The Differential Operator (Calculus)

Let $C'[a, b]$ be the set of all functions whose derivatives are continuous on $[a, b]$. Show that the differential operator $D_x$ defines a linear transformation from $C'[a, b]$ into $C[a, b]$.

### SOLUTION

Using operator notation, you can write

$$D_x(f) = \frac{d}{dx}[f]$$

where $f$ is in $C'[a, b]$. To show that $D_x$ is a linear transformation, you must use calculus. Specifically, because the derivative of the sum of two differentiable functions is equal to the sum of their derivatives, you have

$$D_x(f + g) = \frac{d}{dx}[f + g] = \frac{d}{dx}[f] + \frac{d}{dx}[g] = D_x(f) + D_x(g)$$

where $g$ is also in $C'[a, b]$. Similarly, because the derivative of a scalar multiple of a differentiable function is equal to the scalar multiple of the derivative, you have

$$D_x(cf) = \frac{d}{dx}[cf] = c\left(\frac{d}{dx}[f]\right) = cD_x(f).$$

Because the sum of two continuous functions is continuous, and because the scalar multiple of a continuous function is continuous, $D_x$ is a linear transformation from $C'[a, b]$ into $C[a, b]$.

The linear transformation $D_x$ in Example 10 is called the **differential operator.** For polynomials, the differential operator is a linear transformation from $P_n$ into $P_{n-1}$ because the derivative of a polynomial function of degree $n$ is a polynomial function of degree $n - 1$. That is,

$$D_x(a_nx^n + \cdot \cdot \cdot + a_1x + a_0) = na_nx^{n-1} + \cdot \cdot \cdot + a_1.$$

The next example describes a linear transformation from the vector space of polynomial functions $P$ into the vector space of real numbers $R$.

## EXAMPLE 11 The Definite Integral as a Linear Transformation (Calculus)

Consider $T: P \rightarrow R$ defined by

$$T(p) = \int_a^b p(x)\, dx$$

where $p$ is a polynomial function. Show that $T$ is a linear transformation from $P$, the vector space of polynomial functions, into $R$, the vector space of real numbers.

### SOLUTION

Using properties of definite integrals, you can write

$$T(p + q) = \int_a^b [p(x) + q(x)]\, dx = \int_a^b p(x)\, dx + \int_a^b q(x)\, dx = T(p) + T(q)$$

where $q$ is a polynomial function, and

$$T(cp) = \int_a^b [cp(x)]\, dx = c\int_a^b p(x)\, dx = cT(p).$$

So, $T$ is a linear transformation.

# 6.1 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

**Finding an Image and a Preimage** In Exercises 1–8, use the function to find (a) the image of $\mathbf{v}$ and (b) the preimage of $\mathbf{w}$.

1. $T(v_1, v_2) = (v_1 + v_2, v_1 - v_2)$, $\mathbf{v} = (3, -4)$, $\mathbf{w} = (3, 19)$
2. $T(v_1, v_2) = (2v_2 - v_1, v_1, v_2)$, $\mathbf{v} = (0, 6)$, $\mathbf{w} = (3, 1, 2)$
3. $T(v_1, v_2, v_3) = (v_2 - v_1, v_1 + v_2, 2v_1)$, $\mathbf{v} = (2, 3, 0)$, $\mathbf{w} = (-11, -1, 10)$
4. $T(v_1, v_2, v_3) = (2v_1 + v_2, 2v_2 - 3v_1, v_1 - v_3)$, $\mathbf{v} = (-4, 5, 1)$, $\mathbf{w} = (4, 1, -1)$
5. $T(v_1, v_2, v_3) = (4v_2 - v_1, 4v_1 + 5v_2)$, $\mathbf{v} = (2, -3, -1)$, $\mathbf{w} = (3, 9)$
6. $T(v_1, v_2, v_3) = (2v_1 + v_2, v_1 - v_2)$, $\mathbf{v} = (2, 1, 4)$, $\mathbf{w} = (-1, 2)$
7. $T(v_1, v_2) = \left(\frac{\sqrt{2}}{2}v_1 - \frac{\sqrt{2}}{2}v_2, v_1 + v_2, 2v_1 - v_2\right)$, $\mathbf{v} = (1, 1)$, $\mathbf{w} = \left(-5\sqrt{2}, -2, -16\right)$
8. $T(v_1, v_2) = \left(\frac{\sqrt{3}}{2}v_1 - \frac{1}{2}v_2, v_1 - v_2, v_2\right)$, $\mathbf{v} = (2, 4)$, $\mathbf{w} = \left(\sqrt{3}, 2, 0\right)$

**Linear Transformations** In Exercises 9–22, determine whether the function is a linear transformation.

9. $T: R^2 \to R^2$, $T(x, y) = (x, 1)$
10. $T: R^2 \to R^2$, $T(x, y) = (x^2, y)$
11. $T: R^3 \to R^3$, $T(x, y, z) = (x + y, x - y, z)$
12. $T: R^3 \to R^3$, $T(x, y, z) = (x + 1, y + 1, z + 1)$
13. $T: R^2 \to R^3$, $T(x, y) = \left(\sqrt{x}, xy, \sqrt{y}\right)$
14. $T: R^2 \to R^3$, $T(x, y) = \left(x^2, xy, y^2\right)$
15. $T: M_{2,2} \to R$, $T(A) = |A|$
16. $T: M_{2,2} \to R$, $T(A) = a + b + c + d$, where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.
17. $T: M_{2,2} \to R$, $T(A) = a + b - c + d$, where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.
18. $T: M_{2,2} \to R$, $T(A) = b^2$, where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.
19. $T: M_{3,3} \to M_{3,3}$, $T(A) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A$
20. $T: M_{3,3} \to M_{3,3}$, $T(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} A$
21. $T: P_2 \to P_2$, $T(a_0 + a_1x + a_2x^2) = (a_0 + a_1 + a_2) + (a_1 + a_2)x + a_2x^2$
22. $T: P_2 \to P_2$, $T(a_0 + a_1x + a_2x^2) = a_1 + 2a_2x$
23. Let $T$ be a linear transformation from $R^2$ into $R^2$ such that $T(1, 0) = (1, 1)$ and $T(0, 1) = (-1, 1)$. Find $T(1, 4)$ and $T(-2, 1)$.
24. Let $T$ be a linear transformation from $R^2$ into $R^2$ such that $T(1, 1) = (1, 0)$ and $T(1, -1) = (0, 1)$. Find $T(1, 0)$ and $T(0, 2)$.

**Linear Transformation and Bases** In Exercises 25–28, let $T: R^3 \to R^3$ be a linear transformation such that $T(1, 0, 0) = (2, 4, -1)$, $T(0, 1, 0) = (1, 3, -2)$, and $T(0, 0, 1) = (0, -2, 2)$. Find the indicated image.

25. $T(0, 3, -1)$
26. $T(2, -1, 0)$
27. $T(2, -4, 1)$
28. $T(-2, 4, -1)$

**Linear Transformation and Bases** In Exercises 29–32, let $T: R^3 \to R^3$ be a linear transformation such that $T(1, 1, 1) = (2, 0, -1)$, $T(0, -1, 2) = (-3, 2, -1)$, and $T(1, 0, 1) = (1, 1, 0)$. Find the indicated image.

29. $T(2, 1, 0)$
30. $T(0, 2, -1)$
31. $T(2, -1, 1)$
32. $T(-2, 1, 0)$

**Linear Transformation Given by a Matrix** In Exercises 33–38, define the linear transformation $T: R^n \to R^m$ by $T(\mathbf{v}) = A\mathbf{v}$. Find the dimensions of $R^n$ and $R^m$.

33. $A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$
34. $A = \begin{bmatrix} 1 & 2 \\ -2 & 4 \\ -2 & 2 \end{bmatrix}$
35. $A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
36. $A = \begin{bmatrix} -1 & 2 & 1 & 3 & 4 \\ 0 & 0 & 2 & -1 & 0 \end{bmatrix}$
37. $A = \begin{bmatrix} 0 & 1 & -2 & 1 \\ -1 & 4 & 5 & 0 \\ 0 & 1 & 3 & 1 \end{bmatrix}$
38. $A = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$

**39.** For the linear transformation from Exercise 33, find (a) $T(1, 1)$, (b) the preimage of $(1, 1)$, and (c) the preimage of $(0, 0)$.

**40.** **Writing** For the linear transformation from Exercise 34, find (a) $T(2, 4)$ and (b) the preimage of $(-1, 2, 2)$. (c) Then explain why the vector $(1, 1, 1)$ has no preimage under this transformation.

**41.** For the linear transformation from Exercise 35, find (a) $T(1, 1, 1, 1)$ and (b) the preimage of $(1, 1, 1, 1)$.

**42.** For the linear transformation from Exercise 36, find (a) $T(1, 0, -1, 3, 0)$ and (b) the preimage of $(-1, 8)$.

**43.** For the linear transformation from Exercise 37, find (a) $T(1, 0, 2, 3)$ and (b) the preimage of $(0, 0, 0)$.

**44.** For the linear transformation from Exercise 38, find (a) $T(1, 0, 1, 0, 1)$, (b) the preimage of $(0, 0, 0)$, and (c) the preimage of $(1, 1, 1)$.

**45.** Let $T$ be a linear transformation from $R^2$ into $R^2$ such that $T(x, y) = (x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta)$.

Find (a) $T(4, 4)$ for $\theta = 45°$, (b) $T(4, 4)$ for $\theta = 30°$, and (c) $T(5, 0)$ for $\theta = 120°$.

**46.** For the linear transformation from Exercise 45, let $\theta = 45°$ and find the preimage of $\mathbf{v} = (1, 1)$.

**47.** Find the inverse of the matrix $A$ given in Example 7. What linear transformation from $R^2$ into $R^2$ does $A^{-1}$ represent?

**48.** For the linear transformation $T\colon R^2 \to R^2$ given by

$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

find $a$ and $b$ such that $T(12, 5) = (13, 0)$.

**Projection in $R^3$** **In Exercises 49 and 50, let the matrix $A$ represent the linear transformation $T\colon R^3 \to R^3$. Describe the orthogonal projection to which $T$ maps every vector in $R^3$.**

**49.** $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **50.** $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

**Linear Transformation Given by a Matrix** **In Exercises 51–54, determine whether the function involving the $n \times n$ matrix $A$ is a linear transformation.**

**51.** $T\colon M_{n,n} \to M_{n,n}$, $T(A) = A^{-1}$

**52.** $T\colon M_{n,n} \to M_{n,n}$, $T(A) = AX - XA$, where $X$ is a fixed $n \times n$ matrix

**53.** $T\colon M_{n,n} \to M_{n,m}$, $T(A) = AB$, where $B$ is a fixed $n \times m$ matrix

**54.** $T\colon M_{n,n} \to R$, $T(A) = a_{11} \cdot a_{22} \cdot \cdot \cdot \cdot \cdot a_{nn}$, where $A = [a_{ij}]$

**55.** Let $T$ be a linear transformation from $P_2$ into $P_2$ such that $T(1) = x$, $T(x) = 1 + x$, and $T(x^2) = 1 + x + x^2$. Find $T(2 - 6x + x^2)$.

**56.** Let $T$ be a linear transformation from $M_{2,2}$ into $M_{2,2}$ such that

$$T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}, \quad T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix},$$

$$T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix}.$$

Find $T\left(\begin{bmatrix} 1 & 3 \\ -1 & 4 \end{bmatrix}\right)$.

**Calculus** **In Exercises 57–60, let $D_x$ be the linear transformation from $C'[a, b]$ into $C[a, b]$ from Example 10. Determine whether each statement is true or false. Explain your reasoning.**

**57.** $D_x(e^{x^2} + 2x) = D_x(e^{x^2}) + 2D_x(x)$

**58.** $D_x(x^2 - \ln x) = D_x(x^2) - D_x(\ln x)$

**59.** $D_x(\sin 2x) = 2D_x(\sin x)$

**60.** $D_x\left(\cos \dfrac{x}{2}\right) = \dfrac{1}{2}D_x(\cos x)$

**Calculus** **In Exercises 61–64, for the linear transformation from Example 10, find the preimage of each function.**

**61.** $D_x(f) = 2x + 1$ **62.** $D_x(f) = e^x$

**63.** $D_x(f) = \sin x$ **64.** $D_x(f) = \dfrac{1}{x}$

**65.** **Calculus** Let $T$ be a linear transformation from $P$ into $R$ such that

$$T(p) = \int_0^1 p(x)\, dx.$$

Find (a) $T(3x^2 - 2)$, (b) $T(x^3 - x^5)$, and (c) $T(4x - 6)$.

**66.** **Calculus** Let $T$ be the linear transformation from $P_2$ into $R$ given by the integral in Exercise 65. Find the preimage of 1. That is, find the polynomial function(s) of degree 2 or less such that $T(p) = 1$.

**True or False?** **In Exercises 67 and 68, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.**

**67.** (a) The function $f(x) = \cos x$ is a linear transformation from $R$ into $R$.

(b) For polynomials, the differential operator $D_x$ is a linear transformation from $P_n$ into $P_{n-1}$.

**68.** (a) The function $g(x) = x^3$ is a linear transformation from $R$ into $R$.

(b) Any linear function of the form $f(x) = ax + b$ is a linear transformation from $R$ into $R$.

**69.** **Writing** Suppose $T: R^2 \to R^2$ such that $T(1, 0) = (1, 0)$ and $T(0, 1) = (0, 0)$.

(a) Determine $T(x, y)$ for $(x, y)$ in $R^2$.

(b) Give a geometric description of $T$.

**70.** **Writing** Suppose $T: R^2 \to R^2$ such that $T(1, 0) = (0, 1)$ and $T(0, 1) = (1, 0)$.

(a) Determine $T(x, y)$ for $(x, y)$ in $R^2$.

(b) Give a geometric description of $T$.

**71.** **Proof** Let $T$ be the function that maps $R^2$ into $R^2$ such that $T(\mathbf{u}) = \text{proj}_{\mathbf{v}}\mathbf{u}$, where $\mathbf{v} = (1, 1)$.

(a) Find $T(x, y)$. (b) Find $T(5, 0)$.

(c) Prove that $T$ is a linear transformation from $R^2$ into $R^2$.

**72.** **Writing** Find $T(3, 4)$ and $T(T(3, 4))$ from Exercise 71 and give geometric descriptions of the results.

**73.** Show that $T$ from Exercise 71 is represented by the matrix

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

**74.** **CAPSTONE** Explain how to determine whether a function $T: V \to W$ is a linear transformation.

**75.** **Proof** Use the concept of a fixed point of a linear transformation $T: V \to V$. A vector $\mathbf{u}$ is a **fixed point** when $T(\mathbf{u}) = \mathbf{u}$.

(a) Prove that $\mathbf{0}$ is a fixed point of any linear transformation $T: V \to V$.

(b) Prove that the set of fixed points of a linear transformation $T: V \to V$ is a subspace of $V$.

(c) Determine all fixed points of the linear transformation $T: R^2 \to R^2$ represented by $T(x, y) = (x, 2y)$.

(d) Determine all fixed points of the linear transformation $T: R^2 \to R^2$ represented by $T(x, y) = (y, x)$.

**76.** A **translation** in $R^2$ is a function of the form $T(x, y) = (x - h, y - k)$, where at least one of the constants $h$ and $k$ is nonzero.

(a) Show that a translation in $R^2$ is not a linear transformation.

(b) For the translation $T(x, y) = (x - 2, y + 1)$, determine the images of $(0, 0)$, $(2, -1)$, and $(5, 4)$.

(c) Show that a translation in $R^2$ has no fixed points.

**77.** **Proof** Prove that (a) the zero transformation and (b) the identity transformation are linear transformations.

**78.** Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a set of linearly independent vectors in $R^3$. Find a linear transformation $T$ from $R^3$ into $R^3$ such that the set $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ is linearly dependent.

**79.** **Proof** Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a set of linearly dependent vectors in $V$, and let $T$ be a linear transformation from $V$ into $V$. Prove that the set

$$\{T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_n)\}$$

is linearly dependent.

**80.** **Proof** Let $V$ be an inner product space. For a fixed vector $\mathbf{v}_0$ in $V$, define $T: V \to R$ by $T(\mathbf{v}) = \langle \mathbf{v}, \mathbf{v}_0 \rangle$. Prove that $T$ is a linear transformation.

**81.** **Proof** Define $T: M_{n,n} \to R$ by

$$T(A) = a_{11} + a_{22} + \cdots + a_{nn}$$

(the trace of $A$). Prove that $T$ is a linear transformation.

**82.** Let $V$ be an inner product space with a subspace $W$ having $B = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$ as an orthonormal basis. Show that the function $T: V \to W$ represented by

$$T(\mathbf{v}) = \langle \mathbf{v}, \mathbf{w}_1 \rangle \mathbf{w}_1 + \langle \mathbf{v}, \mathbf{w}_2 \rangle \mathbf{w}_2 + \cdots + \langle \mathbf{v}, \mathbf{w}_n \rangle \mathbf{w}_n$$

is a linear transformation. $T$ is called the **orthogonal projection of $V$ onto $W$.**

**83.** **Guided Proof** Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for a vector space $V$. Prove that if a linear transformation $T: V \to V$ satisfies $T(\mathbf{v}_i) = \mathbf{0}$ for $i = 1, 2, \ldots, n$, then $T$ is the zero transformation.

Getting Started: To prove that $T$ is the zero transformation, you need to show that $T(\mathbf{v}) = \mathbf{0}$ for every vector $\mathbf{v}$ in $V$.

(i) Let $\mathbf{v}$ be an arbitrary vector in $V$ such that

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n.$$

(ii) Use the definition and properties of linear transformations to rewrite $T(\mathbf{v})$ as a linear combination of $T(\mathbf{v}_i)$.

(iii) Use the fact that $T(\mathbf{v}_i) = \mathbf{0}$ to conclude that $T(\mathbf{v}) = \mathbf{0}$, making $T$ the zero transformation.

**84.** **Guided Proof** Prove that $T: V \to W$ is a linear transformation if and only if

$$T(a\mathbf{u} + b\mathbf{v}) = aT(\mathbf{u}) + bT(\mathbf{v})$$

for all vectors $\mathbf{u}$ and $\mathbf{v}$ and all scalars $a$ and $b$.

Getting Started: Because this is an "if and only if" statement, you need to prove the statement in both directions. To prove that $T$ is a linear transformation, you need to show that the function satisfies the definition of a linear transformation. In the other direction, suppose $T$ is a linear transformation. You can use the definition and properties of a linear transformation to prove that $T(a\mathbf{u} + b\mathbf{v}) = aT(\mathbf{u}) + bT(\mathbf{v})$.

(i) Suppose $T(a\mathbf{u} + b\mathbf{v}) = aT(\mathbf{u}) + bT(\mathbf{v})$. Show that $T$ preserves the properties of vector addition and scalar multiplication by choosing appropriate values of $a$ and $b$.

(ii) To prove the statement in the other direction, assume that $T$ is a linear transformation. Use the properties and definition of a linear transformation to show that $T(a\mathbf{u} + b\mathbf{v}) = aT(\mathbf{u}) + bT(\mathbf{v})$.

# 6.2 The Kernel and Range of a Linear Transformation

- Find the kernel of a linear transformation.
- Find a basis for the range, the rank, and the nullity of a linear transformation.
- Determine whether a linear transformation is one-to-one or onto.
- Determine whether vector spaces are isomorphic.

## THE KERNEL OF A LINEAR TRANSFORMATION

You know from Theorem 6.1 that for any linear transformation $T: V \rightarrow W$, the zero vector in $V$ maps to the zero vector in $W$. That is, $T(\mathbf{0}) = \mathbf{0}$. The first question you will consider in this section is whether there are *other* vectors $\mathbf{v}$ such that $T(\mathbf{v}) = \mathbf{0}$. The collection of all such elements is called the **kernel** of $T$. Note that the symbol $\mathbf{0}$ represents the zero vector in both $V$ and $W$, although these two zero vectors are often different.

### Definition of Kernel of a Linear Transformation

Let $T: V \rightarrow W$ be a linear transformation. Then the set of all vectors $\mathbf{v}$ in $V$ that satisfy $T(\mathbf{v}) = \mathbf{0}$ is called the **kernel** of $T$ and is denoted by $\ker(T)$.

Sometimes the kernel of a transformation can be found by inspection, as demonstrated in Examples 1, 2, and 3.

### EXAMPLE 1 Finding the Kernel of a Linear Transformation

Let $T: M_{3,2} \rightarrow M_{2,3}$ be the linear transformation that maps a $3 \times 2$ matrix $A$ to its transpose. That is, $T(A) = A^T$. Find the kernel of $T$.

**SOLUTION**

For this linear transformation, the $3 \times 2$ zero matrix is clearly the only matrix in $M_{3,2}$ whose transpose is the zero matrix in $M_{2,3}$. So, the kernel of $T$ consists of a single element: the zero matrix in $M_{3,2}$.

### EXAMPLE 2 The Kernels of the Zero and Identity Transformations

**a.** The kernel of the zero transformation $T: V \rightarrow W$ consists of all of $V$ because $T(\mathbf{v}) = \mathbf{0}$ for every $\mathbf{v}$ in $V$. That is, $\ker(T) = V$.

**b.** The kernel of the identity transformation $T: V \rightarrow V$ consists of the single element $\mathbf{0}$. That is, $\ker(T) = \{\mathbf{0}\}$.

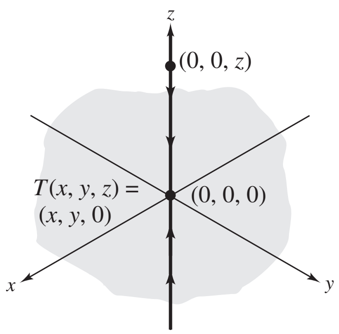
### EXAMPLE 3 Finding the Kernel of a Linear Transformation

Find the kernel of the projection $T: R^3 \rightarrow R^3$ represented by $T(x, y, z) = (x, y, 0)$.

**SOLUTION**

This linear transformation projects the vector $(x, y, z)$ in $R^3$ to the vector $(x, y, 0)$ in the $xy$-plane. The kernel consists of all vectors lying on the $z$-axis. That is,

$$\ker(T) = \{(0, 0, z): z \text{ is a real number}\}. \text{ (See Figure 6.4.)}$$




The kernel of $T$ is the set of all vectors on the $z$-axis.

**Figure 6.4**

Finding the kernels of the linear transformations in Examples 1, 2, and 3 is relatively easy. Generally, the kernel of a linear transformation is not so obvious, as illustrated in the next two examples.

## EXAMPLE 4 Finding the Kernel of a Linear Transformation

Find the kernel of the linear transformation $T: R^2 \to R^3$ represented by

$$T(x_1, x_2) = (x_1 - 2x_2, 0, -x_1).$$

**SOLUTION**

To find ker($T$), you need to find all $\mathbf{x} = (x_1, x_2)$ in $R^2$ such that

$$T(x_1, x_2) = (x_1 - 2x_2, 0, -x_1) = (0, 0, 0).$$

This leads to the homogeneous system

$$\begin{aligned} x_1 - 2x_2 &= 0 \\ 0 &= 0 \\ -x_1 \qquad &= 0 \end{aligned}$$

which has only the trivial solution $(x_1, x_2) = (0, 0)$. So, you have

$$\ker(T) = \{(0, 0)\} = \{\mathbf{0}\}.$$

## EXAMPLE 5 Finding the Kernel of a Linear Transformation

Find the kernel of the linear transformation $T: R^3 \to R^2$ defined by $T(\mathbf{x}) = A\mathbf{x}$, where

$$A = \begin{bmatrix} 1 & -1 & -2 \\ -1 & 2 & 3 \end{bmatrix}.$$

**SOLUTION**

The kernel of $T$ is the set of all $\mathbf{x} = (x_1, x_2, x_3)$ in $R^3$ such that $T(x_1, x_2, x_3) = (0, 0)$. From this equation, you can write the homogeneous system

$$\begin{bmatrix} 1 & -1 & -2 \\ -1 & 2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \rightarrow \quad \begin{aligned} x_1 - x_2 - 2x_3 &= 0 \\ -x_1 + 2x_2 + 3x_3 &= 0. \end{aligned}$$

Writing the augmented matrix of this system in reduced row-echelon form produces

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \quad \rightarrow \quad \begin{aligned} x_1 &= x_3 \\ x_2 &= -x_3. \end{aligned}$$

Using the parameter $t = x_3$ produces the family of solutions

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ -t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

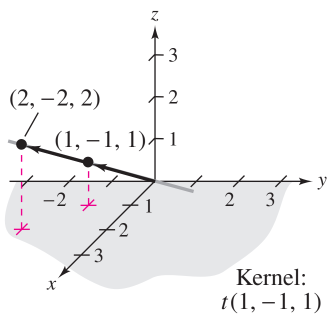


**Figure 6.5**

So, the kernel of $T$ is

$$\ker(T) = \{t(1, -1, 1)\text{: } t \text{ is a real number}\} = \text{span}\{(1, -1, 1)\}. \text{ (See Figure 6.5.)}$$

Note in Example 5 that the kernel of $T$ contains infinitely many vectors. Of course, the zero vector is in ker($T$), but the kernel also contains such nonzero vectors as $(1, -1, 1)$ and $(2, -2, 2)$, as shown in Figure 6.5. The figure also shows that the kernel is a line passing through the origin, which implies that it is a subspace of $R^3$. Theorem 6.3 on the next page states that the kernel of every linear transformation $T: V \to W$ is a subspace of $V$.

**THEOREM 6.3 The Kernel Is a Subspace of $V$**

The kernel of a linear transformation $T: V \to W$ is a subspace of the domain $V$.

**PROOF**

From Theorem 6.1, you know that $\ker(T)$ is a nonempty subset of $V$. So, by Theorem 4.5, you can show that $\ker(T)$ is a subspace of $V$ by showing that it is closed under vector addition and scalar multiplication. To do so, let $\mathbf{u}$ and $\mathbf{v}$ be vectors in the kernel of $T$. Then $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) = \mathbf{0} + \mathbf{0} = \mathbf{0}$, which implies that $\mathbf{u} + \mathbf{v}$ is in the kernel. Moreover, if $c$ is any scalar, then $T(c\mathbf{u}) = cT(\mathbf{u}) = c\mathbf{0} = \mathbf{0}$, which implies that $c\mathbf{u}$ is in the kernel.

**REMARK**

The kernel of $T$ is sometimes called the **nullspace** of $T$.

The next example shows how to find a basis for the kernel of a transformation defined by a matrix.

**EXAMPLE 6 Finding a Basis for the Kernel**

Define $T: R^5 \to R^4$ by $T(\mathbf{x}) = A\mathbf{x}$, where $\mathbf{x}$ is in $R^5$ and

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & -1 \\ 2 & 1 & 3 & 1 & 0 \\ -1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 2 & 8 \end{bmatrix}.$$

Find a basis for $\ker(T)$ as a subspace of $R^5$.

**SOLUTION**

Using the procedure shown in Example 5, write the augmented matrix $[A \quad \mathbf{0}]$ in reduced row-echelon form as follows.

$$\begin{bmatrix} 1 & 0 & 2 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \longrightarrow \quad \begin{aligned} x_1 &= -2x_3 + x_5 \\ x_2 &= x_3 + 2x_5 \\ x_4 &= -4x_5 \end{aligned}$$

Letting $x_3 = s$ and $x_5 = t$, you have

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2s + t \\ s + 2t \\ s + 0t \\ 0s - 4t \\ 0s + t \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 2 \\ 0 \\ -4 \\ 1 \end{bmatrix}.$$

So a basis for the kernel of $T$ is $B = \{(-2, 1, 1, 0, 0), (1, 2, 0, -4, 1)\}$.

**DISCOVERY**

1. What is the rank of the matrix $A$ in Example 6?
2. Formulate a conjecture relating the dimension of the kernel, the rank, and the number of columns of $A$.
3. Verify your conjecture for the matrix in Example 5.

In the solution of Example 6, a basis for the kernel of $T$ was found by solving the homogeneous system represented by $A\mathbf{x} = \mathbf{0}$. This procedure is a familiar one—it is the same procedure used to find the *solution space* of $A\mathbf{x} = \mathbf{0}$. In other words, the kernel of $T$ is the nullspace of the matrix $A$, as stated in the following corollary to Theorem 6.3.

**THEOREM 6.3 Corollary**

Let $T: R^n \to R^m$ be the linear transformation given by $T(\mathbf{x}) = A\mathbf{x}$. Then the kernel of $T$ is equal to the solution space of $A\mathbf{x} = \mathbf{0}$.

## THE RANGE OF A LINEAR TRANSFORMATION

The kernel is one of two critical subspaces associated with a linear transformation. The other is the **range** of $T$, denoted by range($T$). Recall from Section 6.1 that the range of $T: V \rightarrow W$ is the set of all vectors $\mathbf{w}$ in $W$ that are images of vectors in $V$. That is,

$$\text{range}(T) = \{T(\mathbf{v}): \mathbf{v} \text{ is in } V\}.$$

### THEOREM 6.4 The Range of $T$ Is a Subspace of $W$

The range of a linear transformation $T: V \rightarrow W$ is a subspace of $W$.

### PROOF

The range of $T$ is nonempty because $T(\mathbf{0}) = \mathbf{0}$ implies that the range contains the zero vector. To show that it is closed under vector addition, let $T(\mathbf{u})$ and $T(\mathbf{v})$ be vectors in the range of $T$. Because $\mathbf{u}$ and $\mathbf{v}$ are in $V$, it follows that $\mathbf{u} + \mathbf{v}$ is also in $V$. So, the sum

$$T(\mathbf{u}) + T(\mathbf{v}) = T(\mathbf{u} + \mathbf{v})$$

is in the range of $T$.

To show closure under scalar multiplication, let $T(\mathbf{u})$ be a vector in the range of $T$ and let $c$ be a scalar. Because $\mathbf{u}$ is in $V$, it follows that $c\mathbf{u}$ is also in $V$. So, the scalar multiple $cT(\mathbf{u}) = T(c\mathbf{u})$ is in the range of $T$.

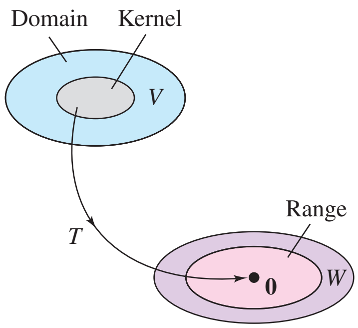



**Figure 6.6**

Note that the kernel and range of a linear transformation $T: V \rightarrow W$ are subspaces of $V$ and $W$, respectively, as illustrated in Figure 6.6.

To find a basis for the range of a linear transformation defined by $T(\mathbf{x}) = A\mathbf{x}$, observe that the range consists of all vectors $\mathbf{b}$ such that the system $A\mathbf{x} = \mathbf{b}$ is consistent. By writing the system

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

in the form

$$A\mathbf{x} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} = \mathbf{b}$$

you can see that $\mathbf{b}$ is in the range of $T$ if and only if $\mathbf{b}$ is a linear combination of the column vectors of $A$. So *the column space of the matrix $A$ is the same as the range of $T$.*

### THEOREM 6.4 Corollary

Let $T: R^n \rightarrow R^m$ be the linear transformation given by $T(\mathbf{x}) = A\mathbf{x}$. Then the column space of $A$ is equal to the range of $T$.

In Examples 4 and 5 in Section 4.6, you saw two procedures for finding a basis for the column space of a matrix. The next example uses the procedure from Example 5 in Section 4.6 to find a basis for the range of a linear transformation defined by a matrix.

## EXAMPLE 7 Finding a Basis for the Range of a Linear Transformation

For the linear transformation $R^5 \to R^4$ from Example 6, find a basis for the range of $T$.

### SOLUTION

Use the reduced row-echelon form of $A$ from Example 6.

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & -1 \\ 2 & 1 & 3 & 1 & 0 \\ -1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 2 & 8 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & -1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Because the leading 1's appear in columns 1, 2, and 4 of the reduced matrix on the right, the corresponding column vectors of $A$ form a basis for the column space of $A$. So, a basis for the range of $T$ is

$$B = \{(1, 2, -1, 0), (2, 1, 0, 0), (1, 1, 0, 2)\}.$$

The following definition gives the dimensions of the kernel and range of a linear transformation.

**REMARK**

If $T$ is given by a matrix $A$, then the rank of $T$ is equal to the rank of $A$, and the nullity of $T$ is equal to the nullity of $A$, as defined in Section 4.6.

### Definition of Rank and Nullity of a Linear Transformation

Let $T: V \to W$ be a linear transformation. The dimension of the kernel of $T$ is called the **nullity** of $T$ and is denoted by **nullity**$(T)$. The dimension of the range of $T$ is called the **rank** of $T$ and is denoted by **rank**$(T)$.

In Examples 6 and 7, the rank and nullity of $T$ are related to the dimension of the domain as follows.

$$\text{rank}(T) + \text{nullity}(T) = 3 + 2 = 5 = \text{dimension of domain}$$

This relationship is true for any linear transformation from a finite-dimensional vector space, as stated in the next theorem.

### THEOREM 6.5 Sum of Rank and Nullity

Let $T: V \to W$ be a linear transformation from an $n$-dimensional vector space $V$ into a vector space $W$. Then the sum of the dimensions of the range and kernel is equal to the dimension of the domain. That is,

$$\text{rank}(T) + \text{nullity}(T) = n \quad \text{or} \quad \dim(\text{range}) + \dim(\text{kernel}) = \dim(\text{domain}).$$

### PROOF

The proof provided here covers the case in which $T$ is represented by an $m \times n$ matrix $A$. The general case will follow in the next section, where you will see that any linear transformation from an $n$-dimensional space into an $m$-dimensional space can be represented by a matrix. To prove this theorem, assume that the matrix $A$ has a rank of $r$. Then you have

$$\text{rank}(T) = \dim(\text{range of } T) = \dim(\text{column space}) = \text{rank}(A) = r.$$

From Theorem 4.17, however, you know that

$$\text{nullity}(T) = \dim(\text{kernel of } T) = \dim(\text{solution space of } A\mathbf{x} = \mathbf{0}) = n - r.$$

So, it follows that $\text{rank}(T) + \text{nullity}(T) = r + (n - r) = n.$

## EXAMPLE 8 Finding the Rank and Nullity of a Linear Transformation

Find the rank and nullity of the linear transformation $T: R^3 \to R^3$ defined by the matrix

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

### SOLUTION

Because $A$ is in row-echelon form and has two nonzero rows, it has a rank of 2. So, the rank of $T$ is 2, and the nullity is $\dim(\text{domain}) - \text{rank} = 3 - 2 = 1$.

One way to visualize the relationship between the rank and the nullity of a linear transformation provided by a matrix is to observe that the number of leading 1's determines the rank, and the number of free variables (columns without leading 1's) determines the nullity. Their sum must be the total number of columns in the matrix, which is the dimension of the domain. In Example 8, the first two columns have leading 1's, indicating that the rank is 2. The third column corresponds to a free variable, indicating that the nullity is 1.

## EXAMPLE 9 Finding the Rank and Nullity of a Linear Transformation

Let $T: R^5 \to R^7$ be a linear transformation.

**a.** Find the dimension of the kernel of $T$ when the dimension of the range is 2.

**b.** Find the rank of $T$ when the nullity of $T$ is 4.

**c.** Find the rank of $T$ when $\ker(T) = \{\mathbf{0}\}$.

### SOLUTION

**a.** By Theorem 6.5, with $n = 5$, you have

$$\dim(\text{kernel}) = n - \dim(\text{range}) = 5 - 2 = 3.$$

**b.** Again by Theorem 6.5, you have

$$\text{rank}(T) = n - \text{nullity}(T) = 5 - 4 = 1.$$

**c.** In this case, the nullity of $T$ is 0. So

$$\text{rank}(T) = n - \text{nullity}(T) = 5 - 0 = 5.$$

### LINEAR ALGEBRA APPLIED

A control system, such as the one shown for a dairy factory, processes an input signal $\mathbf{x}_k$ and produces an output signal $\mathbf{x}_{k+1}$. Without external feedback, the **difference equation**

$$\mathbf{x}_{k+1} = A\mathbf{x}_k$$

a linear transformation where $\mathbf{x}_i$ is an $n \times 1$ vector and $A$ is an $n \times n$ matrix, can model the relationship between the input and output signals. Typically, however, a control system has external feedback, so the relationship becomes

$$\mathbf{x}_{k+1} = A\mathbf{x}_k + B\mathbf{u}_k$$

where $B$ is an $n \times m$ matrix and $\mathbf{u}_k$ is an $m \times 1$ input, or control, vector. A system is called *controllable* when it can reach any desired final state from its initial state in $n$ or fewer steps. If $A$ and $B$ make up a controllable system, then the rank of the *controllability matrix*

$$[B \quad AB \quad A^2B \quad . \; . \; . \quad A^{n-1}B]$$

is equal to $n$.

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## ONE-TO-ONE AND ONTO LINEAR TRANSFORMATIONS

This section began with a question: How many vectors in the domain of a linear transformation are mapped to the zero vector? Theorem 6.6 (below) states that if the zero vector is the only vector $\mathbf{v}$ such that $T(\mathbf{v}) = \mathbf{0}$, then $T$ is *one-to-one*. A function $T: V \rightarrow W$ is called **one-to-one** when the preimage of every $\mathbf{w}$ in the range consists of a single vector, as shown in Figure 6.7. This is equivalent to saying that $T$ is one-to-one if and only if, for all $\mathbf{u}$ and $\mathbf{v}$ in $V$, $T(\mathbf{u}) = T(\mathbf{v})$ implies $\mathbf{u} = \mathbf{v}$.

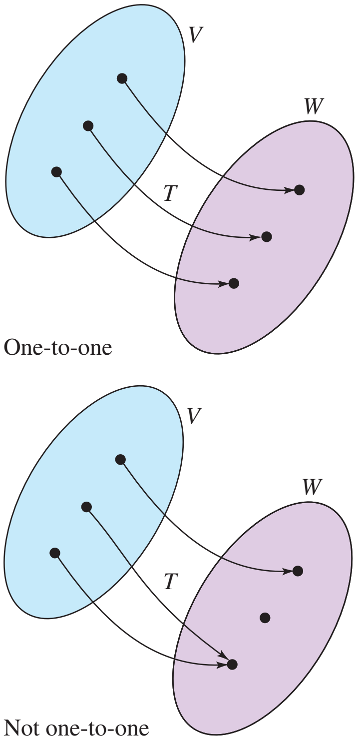



**Figure 6.7**

> **THEOREM 6.6 One-to-One Linear Transformations**
>
> Let $T: V \rightarrow W$ be a linear transformation. Then $T$ is one-to-one if and only if $\ker(T) = \{\mathbf{0}\}$.

**PROOF**

Suppose $T$ is one-to-one. Then $T(\mathbf{v}) = \mathbf{0}$ can have only one solution: $\mathbf{v} = \mathbf{0}$. In that case, $\ker(T) = \{\mathbf{0}\}$. Conversely, suppose $\ker(T) = \{\mathbf{0}\}$ and $T(\mathbf{u}) = T(\mathbf{v})$. Because $T$ is a linear transformation, it follows that

$$T(\mathbf{u} - \mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v}) = \mathbf{0}.$$

This implies that the vector $\mathbf{u} - \mathbf{v}$ lies in the kernel of $T$ and must equal $\mathbf{0}$. So, $\mathbf{u} = \mathbf{v}$, which means that $T$ is one-to-one.

**EXAMPLE 10 One-to-One and Not One-to-One Linear Transformations**

**a.** The linear transformation $T: M_{m,n} \rightarrow M_{n,m}$ represented by $T(A) = A^T$ is one-to-one because its kernel consists of only the $m \times n$ zero matrix.

**b.** The zero transformation $T: R^3 \rightarrow R^3$ is not one-to-one because its kernel is all of $R^3$.

A function $T: V \to W$ is said to be **onto** when every element in $W$ has a preimage in $V$. In other words, $T$ is onto $W$ when $W$ is equal to the range of $T$. The proof of the following related theorem is left as an exercise. (See Exercise 65.)

### THEOREM 6.7 Onto Linear Transformations

Let $T: V \to W$ be a linear transformation, where $W$ is finite dimensional. Then $T$ is onto if and only if the rank of $T$ is equal to the dimension of $W$.

For vector spaces of equal dimensions, you can combine the results of Theorems 6.5, 6.6, and 6.7 to obtain the next theorem relating the concepts of one-to-one and onto.

### THEOREM 6.8 One-to-One and Onto Linear Transformations

Let $T: V \to W$ be a linear transformation with vector spaces $V$ and $W$, *both* of dimension $n$. Then $T$ is one-to-one if and only if it is onto.

**PROOF**

If $T$ is one-to-one, then by Theorem 6.6 $\ker(T) = \{\mathbf{0}\}$, and $\dim(\ker(T)) = 0$. In that case, Theorem 6.5 produces

$$\dim(\text{range of } T) = n - \dim(\ker(T)) = n = \dim(W).$$

Consequently, by Theorem 6.7, $T$ is onto. Similarly, if $T$ is onto, then

$$\dim(\text{range of } T) = \dim(W) = n$$

which by Theorem 6.5 implies that $\dim(\ker(T)) = 0$. By Theorem 6.6, $T$ is one-to-one.

The next example brings together several concepts related to the kernel and range of a linear transformation.

### EXAMPLE 11 Summarizing Several Results

Consider the linear transformation $T: R^n \to R^m$ represented by $T(\mathbf{x}) = A\mathbf{x}$. Find the nullity and rank of $T$, and determine whether $T$ is one-to-one, onto, or neither.

**a.** $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ **b.** $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

**c.** $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \end{bmatrix}$ **d.** $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

**SOLUTION**

Note that each matrix is already in row-echelon form, so its rank can be determined by inspection.

| $T: R^n \to R^m$ | Dim(domain) | Dim(range) Rank($T$) | Dim(kernel) Nullity($T$) | One-to-One | Onto |
|---|---|---|---|---|---|
| **a.** $T: R^3 \to R^3$ | 3 | 3 | 0 | Yes | Yes |
| **b.** $T: R^2 \to R^3$ | 2 | 2 | 0 | Yes | No |
| **c.** $T: R^3 \to R^2$ | 3 | 2 | 1 | No | Yes |
| **d.** $T: R^3 \to R^3$ | 3 | 2 | 1 | No | No |

## ISOMORPHISMS OF VECTOR SPACES

Distinct vector spaces such as $R^3$ and $M_{3,1}$ can be thought of as being "essentially the same"—at least with respect to the operations of vector and scalar multiplication. Such spaces are said to be **isomorphic** to each other. (The Greek word *isos* means "equal.")

### Definition of Isomorphism

A linear transformation $T: V \rightarrow W$ that is one-to-one and onto is called an **isomorphism.** Moreover, if $V$ and $W$ are vector spaces such that there exists an isomorphism from $V$ to $W$, then $V$ and $W$ are said to be **isomorphic** to each other.

Isomorphic vector spaces are of the same finite dimension, and vector spaces of the same finite dimension are isomorphic, as stated in the next theorem.

### THEOREM 6.9 Isomorphic Spaces and Dimension

Two finite-dimensional vector spaces $V$ and $W$ are isomorphic if and only if they are of the same dimension.

### PROOF

Assume $V$ is isomorphic to $W$, where $V$ has dimension $n$. By the definition of isomorphic spaces, you know there exists a linear transformation $T: V \rightarrow W$ that is one-to-one and onto. Because $T$ is one-to-one, it follows that dim(kernel) $= 0$, which also implies that

$$\dim(\text{range}) = \dim(\text{domain}) = n.$$

In addition, because $T$ is onto, you can conclude that dim(range) $=$ dim($W$) $= n$.

To prove the theorem in the other direction, assume $V$ and $W$ both have dimension $n$. Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for $V$, and let $B' = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$ be a basis for $W$. Then an arbitrary vector in $V$ can be represented as

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$$

and you can define a linear transformation $T: V \rightarrow W$ as follows.

$$T(\mathbf{v}) = c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + \cdots + c_n\mathbf{w}_n$$

It can be shown that this linear transformation is both one-to-one and onto. So, $V$ and $W$ are isomorphic. ■

**REMARK**

Your study of vector spaces has included much greater coverage to $R^n$ than to other vector spaces. This preference for $R^n$ stems from its notational convenience and from the geometric models available for $R^2$ and $R^3$.

Example 12 lists some vector spaces that are isomorphic to $R^4$.

### EXAMPLE 12 Isomorphic Vector Spaces

The following vector spaces are isomorphic to each other.

**a.** $R^4 =$ 4-space
**b.** $M_{4,1} =$ space of all $4 \times 1$ matrices
**c.** $M_{2,2} =$ space of all $2 \times 2$ matrices
**d.** $P_3 =$ space of all polynomials of degree 3 or less
**e.** $V = \{(x_1, x_2, x_3, x_4, 0): x_i \text{ is a real number}\}$ (subspace of $R^5$) ■

Example 12 tells you that the elements in these spaces behave in the same way as vectors.

# 6.2 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

**Finding the Kernel of a Linear Transformation** **In Exercises 1–10, find the kernel of the linear transformation.**

1. $T: R^3 \to R^3$, $T(x, y, z) = (0, 0, 0)$
2. $T: R^3 \to R^3$, $T(x, y, z) = (x, 0, z)$
3. $T: R^4 \to R^4$, $T(x, y, z, w) = (y, x, w, z)$
4. $T: R^3 \to R^3$, $T(x, y, z) = (z, y, x)$
5. $T: P_3 \to R$, $T(a_0 + a_1x + a_2x^2 + a_3x^3) = a_0$
6. $T: P_2 \to R$, $T(a_0 + a_1x + a_2x^2) = a_0$
7. $T: P_2 \to P_1$, $T(a_0 + a_1x + a_2x^2) = a_1 + 2a_2x$
8. $T: P_3 \to P_2$,
$T(a_0 + a_1x + a_2x^2 + a_3x^3) = a_1 + 2a_2x + 3a_3x^2$
9. $T: R^2 \to R^2$, $T(x, y) = (x + 2y, y - x)$
10. $T: R^2 \to R^2$, $T(x, y) = (x - y, y - x)$

**Finding Bases for the Kernel and Range** **In Exercises 11–18, $T(\mathbf{v}) = A\mathbf{v}$ represents the linear transformation $T$. Find a basis for (a) the kernel of $T$ and (b) the range of $T$.**

11. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$
12. $A = \begin{bmatrix} 1 & 2 \\ -2 & -4 \end{bmatrix}$
13. $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$
14. $A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & 1 \end{bmatrix}$
15. $A = \begin{bmatrix} 1 & 2 \\ -1 & -2 \\ 1 & 1 \end{bmatrix}$
16. $A = \begin{bmatrix} 1 & 1 \\ -1 & 2 \\ 0 & 1 \end{bmatrix}$
17. $A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 3 & 1 & 2 & -1 \\ -4 & -3 & -1 & -3 \\ -1 & -2 & 1 & 1 \end{bmatrix}$
18. $A = \begin{bmatrix} -1 & 3 & 2 & 1 & 4 \\ 2 & 3 & 5 & 0 & 0 \\ 2 & 1 & 2 & 1 & 0 \end{bmatrix}$

**Finding the Kernel, Nullity, Range, and Rank** **In Exercises 19–30, define the linear transformation $T$ by $T(\mathbf{x}) = A\mathbf{x}$. Find (a) ker$(T)$, (b) nullity$(T)$, (c) range$(T)$, and (d) rank$(T)$.**

19. $A = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$
20. $A = \begin{bmatrix} 3 & 2 \\ -9 & -6 \end{bmatrix}$
21. $A = \begin{bmatrix} 5 & -3 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}$
22. $A = \begin{bmatrix} 4 & 1 \\ 0 & 0 \\ 2 & -3 \end{bmatrix}$
23. $A = \begin{bmatrix} \frac{9}{10} & \frac{3}{10} \\ \frac{3}{10} & \frac{1}{10} \end{bmatrix}$
24. $A = \begin{bmatrix} \frac{1}{26} & -\frac{5}{26} \\ -\frac{5}{26} & \frac{25}{26} \end{bmatrix}$
25. $A = \begin{bmatrix} \frac{4}{9} & -\frac{4}{9} & \frac{2}{9} \\ -\frac{4}{9} & \frac{4}{9} & -\frac{2}{9} \\ \frac{2}{9} & -\frac{2}{9} & \frac{1}{9} \end{bmatrix}$
26. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
27. $A = \begin{bmatrix} 0 & -2 & 3 \\ 4 & 0 & 11 \end{bmatrix}$
28. $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$
29. $A = \begin{bmatrix} 2 & 2 & -3 & 1 & 13 \\ 1 & 1 & 1 & 1 & -1 \\ 3 & 3 & -5 & 0 & 14 \\ 6 & 6 & -2 & 4 & 16 \end{bmatrix}$
30. $A = \begin{bmatrix} 3 & -2 & 6 & -1 & 15 \\ 4 & 3 & 8 & 10 & -14 \\ 2 & -3 & 4 & -4 & 20 \end{bmatrix}$

**Finding the Nullity and Describing the Kernel and Range** **In Exercises 31–38, let $T: R^3 \to R^3$ be a linear transformation. Use the given information to find the nullity of $T$, and give a geometric description of the kernel and range of $T$.**

31. rank$(T) = 2$
32. rank$(T) = 1$
33. rank$(T) = 0$
34. rank$(T) = 3$
35. $T$ is the counterclockwise rotation of 45° about the $z$-axis:
$$T(x, y, z) = \left(\frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y, \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y, z\right)$$
36. $T$ is the reflection through the $yz$-coordinate plane:
$T(x, y, z) = (-x, y, z)$
37. $T$ is the projection onto the vector $\mathbf{v} = (1, 2, 2)$:
$$T(x, y, z) = \frac{x + 2y + 2z}{9}(1, 2, 2)$$
38. $T$ is the projection onto the $xy$-coordinate plane:
$T(x, y, z) = (x, y, 0)$

**Finding the Nullity of a Linear Transformation** **In Exercises 39–42, find the nullity of $T$.**

39. $T: R^4 \to R^2$, rank$(T) = 2$
40. $T: R^5 \to R^2$, rank$(T) = 2$
41. $T: R^4 \to R^4$, rank$(T) = 0$
42. $T: P_3 \to P_1$, rank$(T) = 2$

**Verifying That $T$ Is One-to-One and Onto** **In Exercises 43–46, verify that the matrix defines a linear function $T$ that is one-to-one and onto.**

43. $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
44. $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

**45.** $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$  **46.** $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 4 \\ 0 & 4 & 1 \end{bmatrix}$

**Determining Whether $T$ Is One-to-One, Onto, or Neither** **In Exercises 47–50, determine whether the linear transformation is one-to-one, onto, or neither.**

**47.** $T$ as given in Exercise 3

**48.** $T$ as given in Exercise 10

**49.** $T: R^2 \to R^3$, $T(\mathbf{x}) = A\mathbf{x}$, where $A$ is given in Exercise 21

**50.** $T: R^5 \to R^3$, $T(\mathbf{x}) = A\mathbf{x}$, where $A$ is given in Exercise 18

**51.** Identify the zero element and standard basis for each of the isomorphic vector spaces in Example 12.

**52.** Which vector spaces are isomorphic to $R^6$?

(a) $M_{2,3}$ (b) $P_6$ (c) $C[0, 6]$

(d) $M_{6,1}$ (e) $P_5$

(f) $\{(x_1, x_2, x_3, 0, x_5, x_6, x_7): x_i \text{ is a real number}\}$

**53.** **Calculus** Define $T: P_4 \to P_3$ by $T(p) = p'$. What is the kernel of $T$?

**54.** **Calculus** Define $T: P_2 \to R$ by

$$T(p) = \int_0^1 p(x)\, dx.$$

What is the kernel of $T$?

**55.** Let $T: R^3 \to R^3$ be the linear transformation that projects $\mathbf{u}$ onto $\mathbf{v} = (2, -1, 1)$.

(a) Find the rank and nullity of $T$.

(b) Find a basis for the kernel of $T$.

**56.** Repeat Exercise 55 for $\mathbf{v} = (3, 0, 4)$.

**57.** For the transformation $T: R^n \to R^n$ represented by $T(\mathbf{v}) = A\mathbf{v}$, what can be said about the rank of $T$ when (a) $\det(A) \neq 0$ and (b) $\det(A) = 0$?

**58.** **CAPSTONE** Consider the linear transformation $T: R^4 \to R^3$ represented by $T(\mathbf{x}) = A\mathbf{x}$, where

$$A = \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(a) Find the dimension of the domain.

(b) Find the dimension of the range.

(c) Find the dimension of the kernel.

(d) Is $T$ one-to-one? Explain.

(e) Is $T$ onto? Explain.

(f) Is $T$ an isomorphism? Explain.

**59.** Define $T: M_{n,n} \to M_{n,n}$ by $T(A) = A - A^T$. Show that the kernel of $T$ is the set of $n \times n$ symmetric matrices.

**60.** Determine a relationship among $m$, $n$, $j$, and $k$ such that $M_{m,n}$ is isomorphic to $M_{j,k}$.

**True or False?** **In Exercises 61 and 62, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.**

**61.** (a) The set of all vectors mapped from a vector space $V$ into another vector space $W$ by a linear transformation $T$ is the kernel of $T$.

(b) The range of a linear transformation from a vector space $V$ into a vector space $W$ is a subspace of $V$.

(c) The vector spaces $R^3$ and $M_{3,1}$ are isomorphic to each other.

**62.** (a) The dimension of a linear transformation $T$ from a vector space $V$ into a vector space $W$ is called the rank of $T$.

(b) A linear transformation $T$ from $V$ into $W$ is one-to-one when the preimage of every $\mathbf{w}$ in the range consists of a single vector $\mathbf{v}$.

(c) The vector spaces $R^2$ and $P_1$ are isomorphic to each other.

**63.** **Guided Proof** Let $B$ be an invertible $n \times n$ matrix. Prove that the linear transformation $T: M_{n,n} \to M_{n,n}$ represented by $T(A) = AB$ is an isomorphism.

Getting Started: To show that the linear transformation is an isomorphism, you need to show that $T$ is both onto and one-to-one.

(i) Because $T$ is a linear transformation with vector spaces of equal dimension, then by Theorem 6.8, you only need to show that $T$ is one-to-one.

(ii) To show that $T$ is one-to-one, you need to determine the kernel of $T$ and show that it is $\{\mathbf{0}\}$ (Theorem 6.6). Use the fact that $B$ is an invertible $n \times n$ matrix and that $T(A) = AB$.

(iii) Conclude that $T$ is an isomorphism.

**64.** **Proof** Let $T: V \to W$ be a linear transformation. Prove that $T$ is one-to-one if and only if the rank of $T$ equals the dimension of $V$.

**65.** **Proof** Prove Theorem 6.7.

**66.** **Proof** Let $T: V \to W$ be a linear transformation, and let $U$ be a subspace of $W$. Prove that the set $T^{-1}(U) = \{\mathbf{v} \in V: T(\mathbf{v}) \in U\}$ is a subspace of $V$. What is $T^{-1}(U)$ when $U = \{\mathbf{0}\}$?

**67.** **Writing** Let $T: R^m \to R^n$ be a linear transformation. Explain the differences between the concepts of one-to-one and onto. What can you say about $m$ and $n$ when $T$ is onto? What can you say about $m$ and $n$ when $T$ is one-to-one?

# 6.3 Matrices for Linear Transformations

- Find the standard matrix for a linear transformation.
- Find the standard matrix for the composition of linear transformations and find the inverse of an invertible linear transformation.
- Find the matrix for a linear transformation relative to a nonstandard basis.

## THE STANDARD MATRIX FOR A LINEAR TRANSFORMATION

Which representation of $T: R^3 \rightarrow R^3$ is better:

$$T(x_1, x_2, x_3) = (2x_1 + x_2 - x_3, -x_1 + 3x_2 - 2x_3, 3x_2 + 4x_3)$$

or

$$T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 2 & 1 & -1 \\ -1 & 3 & -2 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}?$$

The second representation is better than the first for at least three reasons: it is simpler to write, simpler to read, and more easily adapted for computer use. Later, you will see that matrix representation of linear transformations also has some theoretical advantages. In this section, you will see that for linear transformations involving finite-dimensional vector spaces, matrix representation is always possible.

The key to representing a linear transformation $T: V \rightarrow W$ by a matrix is to determine how it acts on a basis for $V$. Once you know the image of every vector in the basis, you can use the properties of linear transformations to determine $T(\mathbf{v})$ for any $\mathbf{v}$ in $V$.

Recall that the standard basis for $R^n$, written in column vector notation, is

$$B = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right\}.$$

**THEOREM 6.10 Standard Matrix for a Linear Transformation**

Let $T: R^n \rightarrow R^m$ be a linear transformation such that, for the standard basis vectors $\mathbf{e}_i$ of $R^n$,

$$T(\mathbf{e}_1) = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, T(\mathbf{e}_2) = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \ldots, T(\mathbf{e}_n) = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}.$$

Then the $m \times n$ matrix whose $n$ columns correspond to $T(\mathbf{e}_i)$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

is such that $T(\mathbf{v}) = A\mathbf{v}$ for every $\mathbf{v}$ in $R^n$. $A$ is called the **standard matrix** for $T$.

**PROOF**

To show that $T(\mathbf{v}) = A\mathbf{v}$ for any $\mathbf{v}$ in $R^n$, you can write

$$\mathbf{v} = [v_1 \quad v_2 \quad \cdots \quad v_n]^T = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \cdots + v_n\mathbf{e}_n.$$

Because $T$ is a linear transformation, you have

$$\begin{aligned} T(\mathbf{v}) &= T(v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \cdots + v_n\mathbf{e}_n) \\ &= T(v_1\mathbf{e}_1) + T(v_2\mathbf{e}_2) + \cdots + T(v_n\mathbf{e}_n) \\ &= v_1T(\mathbf{e}_1) + v_2T(\mathbf{e}_2) + \cdots + v_nT(\mathbf{e}_n). \end{aligned}$$

On the other hand, the matrix product $A\mathbf{v}$ is

$$\begin{aligned} A\mathbf{v} &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \\ &= \begin{bmatrix} a_{11}v_1 + a_{12}v_2 + \cdots + a_{1n}v_n \\ a_{21}v_1 + a_{22}v_2 + \cdots + a_{2n}v_n \\ \vdots \\ a_{m1}v_1 + a_{m2}v_2 + \cdots + a_{mn}v_n \end{bmatrix} \\ &= v_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + v_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + v_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} \\ &= v_1T(\mathbf{e}_1) + v_2T(\mathbf{e}_2) + \cdots + v_nT(\mathbf{e}_n). \end{aligned}$$

So, $T(\mathbf{v}) = A\mathbf{v}$ for each $\mathbf{v}$ in $R^n$.

## EXAMPLE 1 Finding the Standard Matrix for a Linear Transformation

Find the standard matrix for the linear transformation $T: R^3 \rightarrow R^2$ defined by

$$T(x, y, z) = (x - 2y, 2x + y).$$

**SOLUTION**

Begin by finding the images of $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$.

| Vector Notation | Matrix Notation |
|---|---|
| $T(\mathbf{e}_1) = T(1, 0, 0) = (1, 2)$ | $T(\mathbf{e}_1) = T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ |
| $T(\mathbf{e}_2) = T(0, 1, 0) = (-2, 1)$ | $T(\mathbf{e}_2) = T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ |
| $T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0)$ | $T(\mathbf{e}_3) = T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ |

By Theorem 6.10, the columns of $A$ consist of $T(\mathbf{e}_1)$, $T(\mathbf{e}_2)$, and $T(\mathbf{e}_3)$, and you have

$$A = [T(\mathbf{e}_1) \quad T(\mathbf{e}_2) \quad T(\mathbf{e}_3)] = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 0 \end{bmatrix}.$$

**REMARK**

As a check, note that

$$A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x - 2y \\ 2x + y \end{bmatrix}$$

which is equivalent to

$$T(x, y, z) = (x - 2y, 2x + y).$$

A little practice will enable you to determine the standard matrix for a linear transformation, such as the one in Example 1, by inspection. For instance, to find the standard matrix for the linear transformation defined by

$$T(x_1, x_2, x_3) = (x_1 - 2x_2 + 5x_3,\ 2x_1 + 3x_3,\ 4x_1 + x_2 - 2x_3)$$

use the coefficients of $x_1, x_2$, and $x_3$ to form the rows of $A$, as follows.

$$A = \begin{bmatrix} 1 & -2 & 5 \\ 2 & 0 & 3 \\ 4 & 1 & -2 \end{bmatrix} \begin{matrix} \leftarrow & 1x_1 - 2x_2 + 5x_3 \\ \leftarrow & 2x_1 + 0x_2 + 3x_3 \\ \leftarrow & 4x_1 + 1x_2 - 2x_3 \end{matrix}$$

## EXAMPLE 2 Finding the Standard Matrix for a Linear Transformation

The linear transformation $T: R^2 \to R^2$ is given by projecting each point in $R^2$ onto the $x$-axis, as shown in Figure 6.8. Find the standard matrix for $T$.

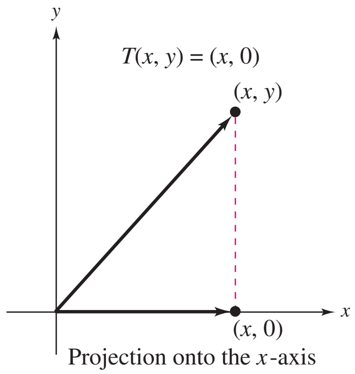



**Figure 6.8**

### SOLUTION

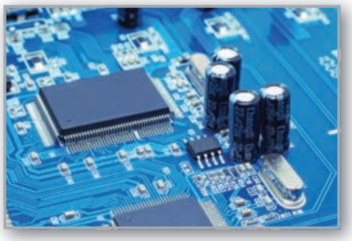
This linear transformation is represented by

$$T(x, y) = (x, 0).$$

So, the standard matrix for $T$ is

$$\begin{aligned} A &= [T(1, 0) \quad T(0, 1)] \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

The standard matrix for the zero transformation from $R^n$ into $R^m$ is the $m \times n$ zero matrix, and the standard matrix for the identity transformation from $R^n$ into $R^n$ is $I_n$.



### LINEAR ALGEBRA APPLIED

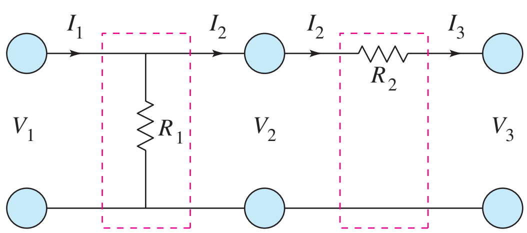
Ladder networks are useful tools for electrical engineers involved in circuit design. In a ladder network, the output voltage $V$ and current $I$ of one circuit are the input voltage and current of the circuit next to it. In the ladder network shown below, linear transformations can relate the input and output of an individual circuit (enclosed in a dashed box). Using Kirchhoff's Voltage and Current Laws and Ohm's Law,

$$\begin{bmatrix} V_2 \\ I_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1/R_1 & 1 \end{bmatrix} \begin{bmatrix} V_1 \\ I_1 \end{bmatrix}$$

and

$$\begin{bmatrix} V_3 \\ I_3 \end{bmatrix} = \begin{bmatrix} 1 & -R_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}.$$

A *composition* can relate the input and output of the entire ladder network, that is, $V_1$ and $I_1$ to $V_3$ and $I_3$. Discussion on the composition of linear transformations begins on the following page.



## COMPOSITION OF LINEAR TRANSFORMATIONS

The **composition,** $T$, of $T_1: R^n \to R^m$ with $T_2: R^m \to R^p$ is defined by

$$T(\mathbf{v}) = T_2(T_1(\mathbf{v}))$$

where $\mathbf{v}$ is a vector in $R^n$. This composition is denoted by

$$T = T_2 \circ T_1.$$

The domain of $T$ is defined as the domain of $T_1$. Moreover, the composition is not defined unless the range of $T_1$ lies within the domain of $T_2$, as shown in Figure 6.9.

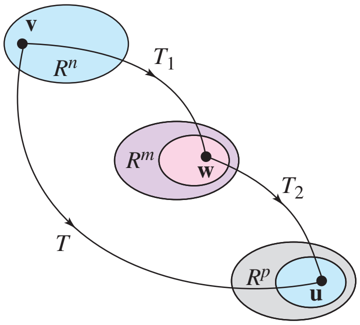



**Figure 6.9**

The next theorem emphasizes the usefulness of matrices for representing linear transformations. This theorem not only states that the composition of two linear transformations is a linear transformation, but also says that the standard matrix for the composition is the product of the standard matrices for the two original linear transformations.

### THEOREM 6.11 Composition of Linear Transformations

Let $T_1: R^n \to R^m$ and $T_2: R^m \to R^p$ be linear transformations with standard matrices $A_1$ and $A_2$, respectively. The **composition** $T: R^n \to R^p$, defined by $T(\mathbf{v}) = T_2(T_1(\mathbf{v}))$, is a linear transformation. Moreover, the standard matrix $A$ for $T$ is given by the matrix product

$$A = A_2 A_1.$$

### PROOF

To show that $T$ is a linear transformation, let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $R^n$ and let $c$ be any scalar. Then, because $T_1$ and $T_2$ are linear transformations, you can write

$$\begin{aligned}
T(\mathbf{u} + \mathbf{v}) &= T_2(T_1(\mathbf{u} + \mathbf{v})) \\
&= T_2(T_1(\mathbf{u}) + T_1(\mathbf{v})) \\
&= T_2(T_1(\mathbf{u})) + T_2(T_1(\mathbf{v})) \\
&= T(\mathbf{u}) + T(\mathbf{v}) \\
T(c\mathbf{v}) &= T_2(T_1(c\mathbf{v})) \\
&= T_2(cT_1(\mathbf{v})) \\
&= cT_2(T_1(\mathbf{v})) \\
&= cT(\mathbf{v}).
\end{aligned}$$

Now, to show that $A_2 A_1$ is the standard matrix for $T$, use the associative property of matrix multiplication to write

$$T(\mathbf{v}) = T_2(T_1(\mathbf{v})) = T_2(A_1\mathbf{v}) = A_2(A_1\mathbf{v}) = (A_2 A_1)\mathbf{v}.$$

Theorem 6.11 can be generalized to cover the composition of $n$ linear transformations. That is, if the standard matrices of $T_1, T_2, \ldots, T_n$ are $A_1, A_2, \ldots, A_n$, respectively, then the standard matrix for the composition $T(\mathbf{v}) = T_n(T_{n-1} \cdots (T_2(T_1(\mathbf{v}))) \cdots)$ is represented by $A = A_n A_{n-1} \cdots A_2 A_1$.

Because matrix multiplication is not commutative, order is important when forming the compositions of linear transformations. In general, the composition $T_2 \circ T_1$ is not the same as $T_1 \circ T_2$, as demonstrated in the next example.

## EXAMPLE 3 The Standard Matrix for a Composition

Let $T_1$ and $T_2$ be linear transformations from $R^3$ into $R^3$ such that

$$T_1(x, y, z) = (2x + y, 0, x + z) \quad \text{and} \quad T_2(x, y, z) = (x - y, z, y).$$

Find the standard matrices for the compositions $T = T_2 \circ T_1$ and $T' = T_1 \circ T_2$.

**SOLUTION**

The standard matrices for $T_1$ and $T_2$ are

$$A_1 = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

By Theorem 6.11, the standard matrix for $T$ is

$$\begin{aligned} A &= A_2 A_1 \\ &= \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

and the standard matrix for $T'$ is

$$\begin{aligned} A' &= A_2 A_1 \\ &= \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2 & -2 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \end{aligned}$$

Another benefit of matrix representation is that it can represent the **inverse** of a linear transformation. Before seeing how this works, consider the next definition.

### Definition of Inverse Linear Transformation

If $T_1: R^n \rightarrow R^n$ and $T_2: R^n \rightarrow R^n$ are linear transformations such that for every $\mathbf{v}$ in $R^n$,

$$T_2(T_1(\mathbf{v})) = \mathbf{v} \quad \text{and} \quad T_1(T_2(\mathbf{v})) = \mathbf{v}$$

then $T_2$ is called the **inverse** of $T_1$, and $T_1$ is said to be **invertible.**

Not every linear transformation has an inverse. If the transformation $T_1$ is invertible, however, then the inverse is unique and is denoted by $T_1^{-1}$.

Just as the inverse of a function of a real variable can be thought of as undoing what the function did, the inverse of a linear transformation $T$ can be thought of as undoing the mapping done by $T$. For instance, if $T$ is a linear transformation from $R^3$ into $R^3$ such that

$$T(1, 4, -5) = (2, 3, 1)$$

and if $T^{-1}$ exists, then $T^{-1}$ maps $(2, 3, 1)$ back to its preimage under $T$. That is,

$$T^{-1}(2, 3, 1) = (1, 4, -5).$$

The next theorem states that a linear transformation is invertible if and only if it is an isomorphism (one-to-one and onto). You are asked to prove this theorem in Exercise 56.

**REMARK**

Several other conditions are equivalent to the three given in Theorem 6.12; see the summary of equivalent conditions for square matrices in Section 4.6.

**THEOREM 6.12 Existence of an Inverse Transformation**

Let $T: R^n \to R^n$ be a linear transformation with standard matrix $A$. Then the following conditions are equivalent.

**1.** $T$ is invertible.
**2.** $T$ is an isomorphism.
**3.** $A$ is invertible.

If $T$ is invertible with standard matrix $A$, then the standard matrix for $T^{-1}$ is $A^{-1}$.

## EXAMPLE 4 Finding the Inverse of a Linear Transformation

Consider the linear transformation $T: R^3 \to R^3$ defined by

$$T(x_1, x_2, x_3) = (2x_1 + 3x_2 + x_3, 3x_1 + 3x_2 + x_3, 2x_1 + 4x_2 + x_3).$$

Show that $T$ is invertible, and find its inverse.

**SOLUTION**

The standard matrix for $T$ is

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix}.$$

Using the techniques for matrix inversion (see Section 2.3), you can find that $A$ is invertible, and its inverse is

$$A^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}.$$

So, $T$ is invertible and its standard matrix is $A^{-1}$.

Using the standard matrix for the inverse, you can find the rule for $T^{-1}$ by computing the image of an arbitrary vector $\mathbf{v} = (x_1, x_2, x_3)$.

$$A^{-1}\mathbf{v} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} -x_1 + x_2 \\ -x_1 + x_3 \\ 6x_1 - 2x_2 - 3x_3 \end{bmatrix}$$

Or,

$$T^{-1}(x_1, x_2, x_3) = (-x_1 + x_2, -x_1 + x_3, 6x_1 - 2x_2 - 3x_3).$$

## NONSTANDARD BASES AND GENERAL VECTOR SPACES

You will now consider the more general problem of finding a matrix for a linear transformation $T: V \rightarrow W$, where $B$ and $B'$ are ordered bases for $V$ and $W$, respectively. Recall that the coordinate matrix of $\mathbf{v}$ relative to $B$ is denoted by $[\mathbf{v}]_B$. In order to represent the linear transformation $T$, multiply $A$ by a *coordinate matrix relative to* $B$ to obtain a *coordinate matrix relative to* $B'$. That is, $[T(\mathbf{v})]_{B'} = A[\mathbf{v}]_B$. $A$ is called the **matrix of $T$ relative to the bases $B$ and $B'$.**

To find the matrix $A$, you will use a procedure similar to the one used to find the standard matrix for $T$. That is, the images of the vectors in $B$ are written as coordinate matrices relative to the basis $B'$. These coordinate matrices form the columns of $A$.

### Transformation Matrix for Nonstandard Bases

Let $V$ and $W$ be finite-dimensional vector spaces with bases $B$ and $B'$, respectively, where

$$B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}.$$

If $T: V \rightarrow W$ is a linear transformation such that

$$[T(\mathbf{v}_1)]_{B'} = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, [T(\mathbf{v}_2)]_{B'} = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \ldots, [T(\mathbf{v}_n)]_{B'} = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

then the $m \times n$ matrix whose $n$ columns correspond to $[T(\mathbf{v}_i)]_{B'}$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

is such that $[T(\mathbf{v})]_{B'} = A[\mathbf{v}]_B$ for every $\mathbf{v}$ in $V$.

### EXAMPLE 5 Finding a Matrix Relative to Nonstandard Bases

Let $T: R^2 \rightarrow R^2$ be a linear transformation defined by $T(x_1, x_2) = (x_1 + x_2, 2x_1 - x_2)$. Find the matrix for $T$ relative to the bases

$$B = \{\overset{\mathbf{v}_1}{(1, 2)}, \overset{\mathbf{v}_2}{(-1, 1)}\} \quad \text{and} \quad B' = \{\overset{\mathbf{w}_1}{(1, 0)}, \overset{\mathbf{w}_2}{(0, 1)}\}.$$

**SOLUTION**

By the definition of $T$, you have

$$T(\mathbf{v}_1) = T(1, 2) = (3, 0) = 3\mathbf{w}_1 + 0\mathbf{w}_2$$
$$T(\mathbf{v}_2) = T(-1, 1) = (0, -3) = 0\mathbf{w}_1 - 3\mathbf{w}_2.$$

The coordinate matrices for $T(\mathbf{v}_1)$ and $T(\mathbf{v}_2)$ relative to $B'$ are

$$[T(\mathbf{v}_1)]_{B'} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \quad \text{and} \quad [T(\mathbf{v}_2)]_{B'} = \begin{bmatrix} 0 \\ -3 \end{bmatrix}.$$

Form the matrix for $T$ relative to $B$ and $B'$ by using these coordinate matrices as columns to produce

$$A = \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}.$$

## EXAMPLE 6 Using a Matrix to Represent a Linear Transformation

For the linear transformation $T: R^2 \to R^2$ in Example 5, use the matrix $A$ to find $T(\mathbf{v})$, where $\mathbf{v} = (2, 1)$.

### SOLUTION

Using the basis $B = \{(1, 2), (-1, 1)\}$, you find that $\mathbf{v} = (2, 1) = 1(1, 2) - 1(-1, 1)$, which implies

$$[\mathbf{v}]_B = [1 \quad -1]^T.$$

So, $[T(\mathbf{v})]_{B'}$ is

$$A[\mathbf{v}]_B = \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}.$$

Finally, because $B' = \{(1, 0), (0, 1)\}$, it follows that

$$T(\mathbf{v}) = 3(1, 0) + 3(0, 1) = (3, 3).$$

Check this result by directly calculating $T(\mathbf{v})$ using the definition of $T$ in Example 5: $T(2, 1) = (2 + 1, 2(2) - 1) = (3, 3)$.

For the special case where $V = W$ and $B = B'$, the matrix $A$ is called the **matrix of $T$ relative to the basis $B$.** In such cases, the matrix of the identity transformation is simply $I_n$. To see this, let $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$. Because the identity transformation maps each $\mathbf{v}_i$ to itself, you have $[T(\mathbf{v}_1)]_B = [1 \ 0 \ \cdots \ 0]^T$, $[T(\mathbf{v}_2)]_B = [0 \ 1 \ \cdots \ 0]^T, \ldots, [T(\mathbf{v}_n)]_B = [1 \ 0 \ \cdots \ 1]^T$, and it follows that $A = I_n$.

In the next example, you will construct a matrix representing the differential operator discussed in Example 10 in Section 6.1.

## EXAMPLE 7 A Matrix for the Differential Operator (Calculus)

Let $D_x: P_2 \to P_1$ be the differential operator that maps a polynomial $p$ of degree 2 or less onto its derivative $p'$. Find the matrix for $D_x$ using the bases

$$B = \{1, x, x^2\} \quad \text{and} \quad B' = \{1, x\}.$$

### SOLUTION

The derivatives of the basis vectors are

$$\begin{aligned} D_x(1) &= \ 0 \ = 0(1) + 0(x) \\ D_x(x) &= \ 1 \ = 1(1) + 0(x) \\ D_x(x^2) &= 2x = 0(1) + 2(x). \end{aligned}$$

So, the coordinate matrices relative to $B'$ are

$$[D_x(1)]_{B'} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad [D_x(x)]_{B'} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad [D_x(x^2)]_{B'} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

and the matrix for $D_x$ is

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Note that this matrix *does* produce the derivative of a quadratic polynomial $p(x) = a + bx + cx^2$.

$$Ap = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} b \\ 2c \end{bmatrix} \implies b + 2cx = D_x[a + bx + cx^2]$$

# 6.3 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

**The Standard Matrix for a Linear Transformation** **In Exercises 1–6, find the standard matrix for the linear transformation $T$.**

1. $T(x, y) = (x + 2y, x - 2y)$
2. $T(x, y) = (2x - 3y, x - y, y - 4x)$
3. $T(x, y, z) = (x + y, x - y, z - x)$
4. $T(x, y) = (4x + y, 0, 2x - 3y)$
5. $T(x, y, z) = (3x - 2z, 2y - z)$
6. $T(x_1, x_2, x_3, x_4) = (0, 0, 0, 0)$

**Finding the Image of a Vector** **In Exercises 7–10, use the standard matrix for the linear transformation $T$ to find the image of the vector $\mathbf{v}$.**

7. $T(x, y, z) = (2x + y, 3y - z)$, $\mathbf{v} = (0, 1, -1)$
8. $T(x, y) = (x + y, x - y, 2x, 2y)$, $\mathbf{v} = (3, -3)$
9. $T(x, y) = (x - y, x + 2y, y)$, $\mathbf{v} = (2, -2)$
10. $T(x_1, x_2, x_3, x_4) = (x_1 - x_3, x_2 - x_4, x_3 - x_1, x_2 + x_4)$, $\mathbf{v} = (1, 2, 3, -2)$

**Finding the Standard Matrix and the Image** **In Exercises 11–22, (a) find the standard matrix $A$ for the linear transformation $T$, (b) use $A$ to find the image of the vector $\mathbf{v}$, and (c) sketch the graph of $\mathbf{v}$ and its image.**

11. $T$ is the reflection through the origin in $R^2$: $T(x, y) = (-x, -y)$, $\mathbf{v} = (3, 4)$.
12. $T$ is the reflection in the line $y = x$ in $R^2$: $T(x, y) = (y, x)$, $\mathbf{v} = (3, 4)$.
13. $T$ is the reflection in the $y$-axis in $R^2$: $T(x, y) = (-x, y)$, $\mathbf{v} = (2, -3)$.
14. $T$ is the reflection in the $x$-axis in $R^2$: $T(x, y) = (x, -y)$, $\mathbf{v} = (4, -1)$.
15. $T$ is the counterclockwise rotation of 45° in $R^2$, $\mathbf{v} = (2, 2)$.
16. $T$ is the counterclockwise rotation of 120° in $R^2$, $\mathbf{v} = (2, 2)$.
17. $T$ is the clockwise rotation ($\theta$ is negative) of 60° in $R^2$, $\mathbf{v} = (1, 2)$.
18. $T$ is the clockwise rotation ($\theta$ is negative) of 30° in $R^2$, $\mathbf{v} = (2, 1)$.
19. $T$ is the reflection through the $xy$-coordinate plane in $R^3$: $T(x, y, z) = (x, y, -z)$, $\mathbf{v} = (3, 2, 2)$.
20. $T$ is the reflection through the $yz$-coordinate plane in $R^3$: $T(x, y, z) = (-x, y, z)$, $\mathbf{v} = (2, 3, 4)$.
21. $T$ is the projection onto the vector $\mathbf{w} = (3, 1)$ in $R^2$: $T(\mathbf{v}) = \operatorname{proj}_{\mathbf{w}}\mathbf{v}$, $\mathbf{v} = (1, 4)$.
22. $T$ is the reflection through the vector $\mathbf{w} = (3, 1)$ in $R^2$: $T(\mathbf{v}) = 2\operatorname{proj}_{\mathbf{w}}\mathbf{v} - \mathbf{v}$, $\mathbf{v} = (1, 4)$.

**Finding the Standard Matrix and the Image** **In Exercises 23–26, (a) find the standard matrix $A$ for the linear transformation $T$ and (b) use $A$ to find the image of the vector $\mathbf{v}$. Use a software program or a graphing utility to verify your result.**

23. $T(x, y, z) = (2x + 3y - z, 3x - 2z, 2x - y + z)$, $\mathbf{v} = (1, 2, -1)$
24. $T(x, y, z) = (3x - 2y + z, 2x - 3y, y - 4z)$, $\mathbf{v} = (2, -1, -1)$
25. $T(x_1, x_2, x_3, x_4) = (x_1 - x_2, x_3, x_1 + 2x_2 - x_4, x_4)$, $\mathbf{v} = (1, 0, 1, -1)$
26. $T(x_1, x_2, x_3, x_4) = (x_1 + 2x_2, x_2 - x_1, 2x_3 - x_4, x_1)$, $\mathbf{v} = (0, 1, -1, 1)$

**Finding Standard Matrices for Compositions** **In Exercises 27–30, find the standard matrices $A$ and $A'$ for $T = T_2 \circ T_1$ and $T' = T_1 \circ T_2$.**

27. $T_1: R^2 \rightarrow R^2$, $T_1(x, y) = (x - 2y, 2x + 3y)$
    $T_2: R^2 \rightarrow R^2$, $T_2(x, y) = (y, 0)$
28. $T_1: R^3 \rightarrow R^3$, $T_1(x, y, z) = (x, y, z)$
    $T_2: R^3 \rightarrow R^3$, $T_2(x, y, z) = (0, x, 0)$
29. $T_1: R^2 \rightarrow R^3$, $T_1(x, y) = (-x + 2y, x + y, x - y)$
    $T_2: R^3 \rightarrow R^2$, $T_2(x, y, z) = (x - 3y, z + 3x)$
30. $T_1: R^2 \rightarrow R^3$, $T_1(x, y) = (x, y, y)$
    $T_2: R^3 \rightarrow R^2$, $T_2(x, y, z) = (y, z)$

**Finding the Inverse of a Linear Transformation** **In Exercises 31–36, determine whether the linear transformation is invertible. If it is, find its inverse.**

31. $T(x, y) = (-2x, 2y)$
32. $T(x, y) = (2x, 0)$
33. $T(x, y) = (x + y, 3x + 3y)$
34. $T(x, y) = (x + y, x - y)$
35. $T(x_1, x_2, x_3) = (x_1, x_1 + x_2, x_1 + x_2 + x_3)$
36. $T(x_1, x_2, x_3, x_4) = (x_1 - 2x_2, x_2, x_3 + x_4, x_3)$

**Finding the Image Two Ways** **In Exercises 37–42, find $T(\mathbf{v})$ by using (a) the standard matrix and (b) the matrix relative to $B$ and $B'$.**

37. $T: R^2 \rightarrow R^3$, $T(x, y) = (x + y, x, y)$, $\mathbf{v} = (5, 4)$,
    $B = \{(1, -1), (0, 1)\}$,
    $B' = \{(1, 1, 0), (0, 1, 1), (1, 0, 1)\}$
38. $T: R^3 \rightarrow R^2$, $T(x, y, z) = (x - y, y - z)$, $\mathbf{v} = (1, 2, 3)$,
    $B = \{(1, 1, 1), (1, 1, 0), (0, 1, 1)\}$, $B' = \{(1, 2), (1, 1)\}$
39. $T: R^3 \rightarrow R^4$, $T(x, y, z) = (2x, x + y, y + z, x + z)$,
    $\mathbf{v} = (1, -5, 2)$, $B = \{(2, 0, 1), (0, 2, 1), (1, 2, 1)\}$,
    $B' = \{(1, 0, 0, 1), (0, 1, 0, 1), (1, 0, 1, 0), (1, 1, 0, 0)\}$

**40.** $T: R^4 \to R^2$,
$T(x_1, x_2, x_3, x_4) = (x_1 + x_2 + x_3 + x_4, x_4 - x_1)$,
$\mathbf{v} = (4, -3, 1, 1)$,
$B = \{(1, 0, 0, 1), (0, 1, 0, 1), (1, 0, 1, 0), (1, 1, 0, 0)\}$,
$B' = \{(1, 1), (2, 0)\}$

**41.** $T: R^3 \to R^3$, $T(x, y, z) = (x + y + z, 2z - x, 2y - z)$,
$\mathbf{v} = (4, -5, 10)$,
$B = \{(2, 0, 1), (0, 2, 1), (1, 2, 1)\}$,
$B' = \{(1, 1, 1), (1, 1, 0), (0, 1, 1)\}$

**42.** $T: R^2 \to R^2$, $T(x, y) = (2x - 12y, x - 5y)$, $\mathbf{v} = (10, 5)$,
$B = B' = \{(4, 1), (3, 1)\}$

**43.** Let $T: P_2 \to P_3$ be given by $T(p) = xp$. Find the matrix for $T$ relative to the bases $B = \{1, x, x^2\}$ and $B' = \{1, x, x^2, x^3\}$.

**44.** Let $T: P_2 \to P_4$ be given by $T(p) = x^2p$. Find the matrix for $T$ relative to the bases $B = \{1, x, x^2\}$ and $B' = \{1, x, x^2, x^3, x^4\}$.

**45.** **Calculus** Let $B = \{1, x, e^x, xe^x\}$ be a basis for a subspace $W$ of the space of continuous functions, and let $D_x$ be the differential operator on $W$. Find the matrix for $D_x$ relative to the basis $B$.

**46.** **Calculus** Repeat Exercise 45 for $B = \{e^{2x}, xe^{2x}, x^2e^{2x}\}$.

**47.** **Calculus** Use the matrix from Exercise 45 to evaluate $D_x[3x - 2xe^x]$.

**48.** **Calculus** Use the matrix from Exercise 46 to evaluate $D_x[5e^{2x} - 3xe^{2x} + x^2e^{2x}]$.

**49.** **Calculus** Let $B = \{1, x, x^2, x^3\}$ be a basis for $P_3$, and let $T: P_3 \to P_4$ be the linear transformation represented by

$$T(x^k) = \int_0^x t^k\, dt.$$

(a) Find the matrix $A$ for $T$ with respect to $B$ and the standard basis for $P_4$.

(b) Use $A$ to integrate $p(x) = 6 - 2x + 3x^3$.

**50.** **CAPSTONE** Explain how to find each of the following.

(a) The standard matrix for a linear transformation

(b) A composition of linear transformations

(c) The inverse of a linear transformation

(d) The transformation matrix relative to nonstandard bases

**51.** Define $T: M_{2,3} \to M_{3,2}$ by $T(A) = A^T$.

(a) Find the matrix for $T$ relative to the standard bases for $M_{2,3}$ and $M_{3,2}$.

(b) Show that $T$ is an isomorphism.

(c) Find the matrix for the inverse of $T$.

**52.** Let $T$ be a linear transformation such that $T(\mathbf{v}) = k\mathbf{v}$ for $\mathbf{v}$ in $R^n$. Find the standard matrix for $T$.

**True or False?** **In Exercises 53 and 54, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.**

**53.** (a) If $T: R^n \to R^m$ is a linear transformation such that

$$\begin{aligned} T[\mathbf{e}_1] &= [a_{11} \;\; a_{21} \;\; \cdots \;\; a_{m1}]^T \\ T[\mathbf{e}_2] &= [a_{12} \;\; a_{22} \;\; \cdots \;\; a_{m2}]^T \\ &\vdots \\ T[\mathbf{e}_n] &= [a_{1n} \;\; a_{2n} \;\; \cdots \;\; a_{mn}]^T \end{aligned}$$

then the $m \times n$ matrix $A = [a_{ij}]$ whose columns correspond to $T(\mathbf{e}_i)$ and is such that $T(\mathbf{v}) = A\mathbf{v}$ for every $\mathbf{v}$ in $R^n$ is called the standard matrix for $T$.

(b) All linear transformations $T$ have a unique inverse $T^{-1}$.

**54.** (a) The composition $T$ of linear transformations $T_1$ and $T_2$, represented by $T(\mathbf{v}) = T_2(T_1(\mathbf{v}))$, is defined when the range of $T_1$ lies within the domain of $T_2$.

(b) In general, the compositions $T_2 \circ T_1$ and $T_1 \circ T_2$ have the same standard matrix $A$.

**55.** **Guided Proof** Let $T_1: V \to V$ and $T_2: V \to V$ be one-to-one linear transformations. Prove that the composition $T = T_2 \circ T_1$ is one-to-one and that $T^{-1}$ exists and is equal to $T_1^{-1} \circ T_2^{-1}$.

Getting Started: To show that $T$ is one-to-one, use the definition of a one-to-one transformation and show that $T(\mathbf{u}) = T(\mathbf{v})$ implies $\mathbf{u} = \mathbf{v}$. For the second statement, you first need to use Theorems 6.8 and 6.12 to show that $T$ is invertible, and then show that $T \circ (T_1^{-1} \circ T_2^{-1})$ and $(T_1^{-1} \circ T_2^{-1}) \circ T$ are identity transformations.

(i) Let $T(\mathbf{u}) = T(\mathbf{v})$. Recall that $(T_2 \circ T_1)(\mathbf{v}) = T_2(T_1(\mathbf{v}))$ for all vectors $\mathbf{v}$. Now use the fact that $T_2$ and $T_1$ are one-to-one to conclude that $\mathbf{u} = \mathbf{v}$.

(ii) Use Theorems 6.8 and 6.12 to show that $T_1$, $T_2$, and $T$ are all invertible transformations. So, $T_1^{-1}$ and $T_2^{-1}$ exist.

(iii) Form the composition $T' = T_1^{-1} \circ T_2^{-1}$. It is a linear transformation from $V$ into $V$. To show that it is the inverse of $T$, you need to determine whether the composition of $T$ with $T'$ on both sides gives an identity transformation.

**56.** **Proof** Prove Theorem 6.12.

**57.** **Writing** Is it always preferable to use the standard basis for $R^n$? Discuss the advantages and disadvantages of using different bases.

**58.** **Writing** Look back at Theorem 4.19 and rephrase it in terms of what you have learned in this chapter.

# 6.4 Transition Matrices and Similarity

- Find and use a matrix for a linear transformation.
- Show that two matrices are similar and use the properties of similar matrices.

## THE MATRIX FOR A LINEAR TRANSFORMATION

In Section 6.3, you saw that the matrix for a linear transformation $T: V \to V$ depends on the basis for $V$. In other words, the matrix for $T$ relative to a basis $B$ is different from the matrix for $T$ relative to another basis $B'$.

A classical problem in linear algebra is determining whether it is possible to find a basis $B$ such that the matrix for $T$ relative to $B$ is diagonal. The solution of this problem is discussed in Chapter 7. This section lays a foundation for solving the problem. You will see how the matrices for a linear transformation relative to two different bases are related. In this section, $A$, $A'$, $P$, and $P^{-1}$ represent the following four square matrices.

**1.** Matrix for $T$ relative to $B$: $A$

**2.** Matrix for $T$ relative to $B'$: $A'$

**3.** Transition matrix from $B'$ to $B$: $P$

**4.** Transition matrix from $B$ to $B'$: $P^{-1}$

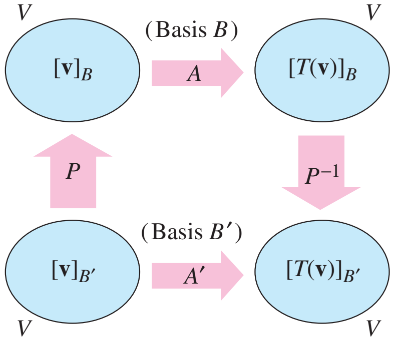



**Figure 6.10**

Note that in Figure 6.10, there are two ways to get from the coordinate matrix $[\mathbf{v}]_{B'}$ to the coordinate matrix $[T(\mathbf{v})]_{B'}$. One way is direct, using the matrix $A'$ to obtain

$$A'[\mathbf{v}]_{B'} = [T(\mathbf{v})]_{B'}.$$

The other way is indirect, using the matrices $P$, $A$, and $P^{-1}$ to obtain

$$P^{-1}AP[\mathbf{v}]_{B'} = [T(\mathbf{v})]_{B'}.$$

This implies that $A' = P^{-1}AP$. Example 1 demonstrates this relationship.

### EXAMPLE 1 Finding a Matrix for a Linear Transformation

Find the matrix $A'$ for $T: R^2 \to R^2$, where $T(x_1, x_2) = (2x_1 - 2x_2, -x_1 + 3x_2)$, relative to the basis $B' = \{(1, 0), (1, 1)\}$.

**SOLUTION**

The standard matrix for $T$ is $A = \begin{bmatrix} 2 & -2 \\ -1 & 3 \end{bmatrix}$.

Furthermore, using the techniques of Section 4.7, you can find that the transition matrix from $B'$ to the standard basis $B = \{(1, 0), (0, 1)\}$ is

$$P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

The inverse of this matrix is the transition matrix from $B$ to $B'$,

$$P^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

So, the matrix for $T$ relative to $B'$ is

$$A' = P^{-1}AP = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & -2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}.$$

In Example 1, the basis $B$ is the standard basis for $R^2$. In the next example, both $B$ and $B'$ are nonstandard bases.

## EXAMPLE 2 Finding a Matrix for a Linear Transformation

Let $B = \{(-3, 2), (4, -2)\}$ and $B' = \{(-1, 2), (2, -2)\}$ be bases for $R^2$, and let

$$A = \begin{bmatrix} -2 & 7 \\ -3 & 7 \end{bmatrix}$$

be the matrix for $T: R^2 \rightarrow R^2$ relative to $B$. Find $A'$, the matrix of $T$ relative to $B'$.

**SOLUTION**

In Example 5 in Section 4.7, you found that $P = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}$ and $P^{-1} = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}$.

So, the matrix of $T$ relative to $B'$ is

$$A' = P^{-1}AP = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} -2 & 7 \\ -3 & 7 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}.$$

The diagram in Figure 6.10 should help you to remember the roles of the matrices $A$, $A'$, $P$, and $P^{-1}$.

## EXAMPLE 3 Using a Matrix for a Linear Transformation

For the linear transformation $T: R^2 \rightarrow R^2$ from Example 2, find $[\mathbf{v}]_B$, $[T(\mathbf{v})]_B$, and $[T(\mathbf{v})]_{B'}$ for the vector $\mathbf{v}$ whose coordinate matrix is $[\mathbf{v}]_{B'} = [-3 \quad -1]^T$.

**SOLUTION**

To find $[\mathbf{v}]_B$, use the transition matrix $P$ from $B'$ to $B$.

$$[\mathbf{v}]_B = P[\mathbf{v}]_{B'} = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} -3 \\ -1 \end{bmatrix} = \begin{bmatrix} -7 \\ -5 \end{bmatrix}$$

To find $[T(\mathbf{v})]_B$, multiply $[\mathbf{v}]_B$ on the left by the matrix $A$ to obtain

$$[T(\mathbf{v})]_B = A[\mathbf{v}]_B = \begin{bmatrix} -2 & 7 \\ -3 & 7 \end{bmatrix}\begin{bmatrix} -7 \\ -5 \end{bmatrix} = \begin{bmatrix} -21 \\ -14 \end{bmatrix}.$$

To find $[T(\mathbf{v})]_{B'}$, multiply $[T(\mathbf{v})]_B$ on the left by $P^{-1}$ to obtain

$$[T(\mathbf{v})]_{B'} = P^{-1}[T(\mathbf{v})]_B = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} -21 \\ -14 \end{bmatrix} = \begin{bmatrix} -7 \\ 0 \end{bmatrix}$$

or multiply $[\mathbf{v}]_{B'}$ on the left by $A'$ to obtain

$$[T(\mathbf{v})]_{B'} = A'[\mathbf{v}]_{B'} = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} -3 \\ -1 \end{bmatrix} = \begin{bmatrix} -7 \\ 0 \end{bmatrix}.$$

**REMARK**

It is instructive to note that the rule $T(x, y) = \left(x - \frac{3}{2}y, 2x + 4y\right)$ represents the transformation $T$ in Examples 2 and 3. Verify the results of Example 3 by showing that $\mathbf{v} = (1, -4)$ and $T(\mathbf{v}) = (7, -14)$.

**LINEAR ALGEBRA APPLIED**

In Section 2.5, you studied stochastic matrices and state matrices. A **Markov chain** is a sequence $\{\mathbf{x}_n\}$ of state matrices that are probability vectors related by the linear transformation $\mathbf{x}_{k+1} = P\mathbf{x}_k$, where $P$, the transition matrix from one state to the next, is a stochastic matrix $[p_{ij}]$. For instance, suppose that it has been established, through studying extensive weather records, that the probability $p_{21}$ of a stormy day following a sunny day is 0.1 and the probability $p_{22}$ of a stormy day following a stormy day is 0.2. The transition matrix can be written as $P = \begin{bmatrix} 0.9 & 0.8 \\ 0.1 & 0.2 \end{bmatrix}$.

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## SIMILAR MATRICES

Two square matrices $A$ and $A'$ that are related by an equation $A' = P^{-1}AP$ are called **similar** matrices, as indicated in the next definition.

### Definition of Similar Matrices

For square matrices $A$ and $A'$ of order $n$, $A'$ is said to be **similar** to $A$ when there exists an invertible matrix $P$ such that $A' = P^{-1}AP$.

If $A'$ is similar to $A$, then it is also true that $A$ is similar to $A'$, as stated in the next theorem. So, it makes sense to say simply that **$A$ and $A'$ are similar.**

### THEOREM 6.13 Properties of Similar Matrices

Let $A$, $B$, and $C$ be square matrices of order $n$. Then the following properties are true.

**1.** $A$ is similar to $A$.
**2.** If $A$ is similar to $B$, then $B$ is similar to $A$.
**3.** If $A$ is similar to $B$ and $B$ is similar to $C$, then $A$ is similar to $C$.

### PROOF

The first property follows from the fact that $A = I_n A I_n$. To prove the second property, write

$$\begin{aligned} A &= P^{-1}BP \\ PAP^{-1} &= P(P^{-1}BP)P^{-1} \\ PAP^{-1} &= B \\ Q^{-1}AQ &= B, \text{ where } Q = P^{-1}. \end{aligned}$$

The proof of the third property is left to you. (See Exercise 29.)

From the definition of similarity, it follows that any two matrices that represent the same linear transformation $T: V \rightarrow V$ with respect to different bases must be similar.

### EXAMPLE 4 Similar Matrices

**a.** From Example 1, the matrices

$$A = \begin{bmatrix} 2 & -2 \\ -1 & 3 \end{bmatrix} \quad \text{and} \quad A' = \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}$$

are similar because $A' = P^{-1}AP$, where $P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

**b.** From Example 2, the matrices

$$A = \begin{bmatrix} -2 & 7 \\ -3 & 7 \end{bmatrix} \quad \text{and} \quad A' = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$$

are similar because $A' = P^{-1}AP$, where $P = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}$.

You have seen that the matrix for a linear transformation $T: V \rightarrow V$ depends on the basis used for $V$. This observation leads naturally to the question: What choice of basis will make the matrix for $T$ as simple as possible? Is it always the *standard* basis? Not necessarily, as the next example demonstrates.

**EXAMPLE 5** **A Comparison of Two Matrices for a Linear Transformation**

Suppose

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

is the matrix for $T: R^3 \rightarrow R^3$ relative to the standard basis. Find the matrix for $T$ relative to the basis $B' = \{(1, 1, 0), (1, -1, 0), (0, 0, 1)\}$.

**SOLUTION**

The transition matrix from $B'$ to the standard matrix has columns consisting of the vectors in $B'$,

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and it follows that

$$P^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

So, the matrix for $T$ relative to $B'$ is

$$\begin{aligned} A' &= P^{-1}AP \\ &= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}. \end{aligned}$$

Note that matrix $A'$ is diagonal.

Diagonal matrices have many computational advantages over nondiagonal matrices. For instance, for the diagonal matrix

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

the $k$th power of $D$ is

$$D^k = \begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n^k \end{bmatrix}.$$

Also, a diagonal matrix is its own transpose. Moreover, if all the diagonal elements are nonzero, then the inverse of a diagonal matrix is the matrix whose main diagonal elements are the reciprocals of corresponding elements in the original matrix. With such computational advantages, it is important to find ways (if possible) to choose a basis for $V$ such that the transformation matrix is diagonal, as it is in Example 5. You will pursue this problem in the next chapter.

## 6.4 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

**Finding a Matrix for a Linear Transformation** **In Exercises 1–8, find the matrix $A'$ for $T$ relative to the basis $B'$.**

**1.** $T: R^2 \to R^2$, $T(x, y) = (2x - y, y - x)$,
$B' = \{(1, -2), (0, 3)\}$

**2.** $T: R^2 \to R^2$, $T(x, y) = (2x + y, x - 2y)$,
$B' = \{(1, 2), (0, 4)\}$

**3.** $T: R^2 \to R^2$, $T(x, y) = (x + y, 4y)$,
$B' = \{(-4, 1), (1, -1)\}$

**4.** $T: R^2 \to R^2$, $T(x, y) = (x - 2y, 4x)$,
$B' = \{(-2, 1), (-1, 1)\}$

**5.** $T: R^3 \to R^3$, $T(x, y, z) = (x, y, z)$,
$B' = \{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$

**6.** $T: R^3 \to R^3$, $T(x, y, z) = (0, 0, 0)$,
$B' = \{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$

**7.** $T: R^3 \to R^3$,
$T(x, y, z) = (x - y + 2z, 2x + y - z, x + 2y + z)$,
$B' = \{(1, 0, 1), (0, 2, 2), (1, 2, 0)\}$

**8.** $T: R^3 \to R^3$, $T(x, y, z) = (x, x + 2y, x + y + 3z)$,
$B' = \{(1, -1, 0), (0, 0, 1), (0, 1, -1)\}$

**9.** Let $B = \{(1, 3), (-2, -2)\}$ and $B' = \{(-12, 0), (-4, 4)\}$ be bases for $R^2$, and let

$$A = \begin{bmatrix} 3 & 2 \\ 0 & 4 \end{bmatrix}$$

be the matrix for $T: R^2 \to R^2$ relative to $B$.

(a) Find the transition matrix $P$ from $B'$ to $B$.

(b) Use the matrices $P$ and $A$ to find $[\mathbf{v}]_B$ and $[T(\mathbf{v})]_B$, where

$[\mathbf{v}]_{B'} = [-1 \quad 2]^T$.

(c) Find $P^{-1}$ and $A'$ (the matrix for $T$ relative to $B'$).

(d) Find $[T(\mathbf{v})]_{B'}$ two ways.

**10.** Repeat Exercise 9 for $B = \{(1, 1), (-2, 3)\}$, $B' = \{(1, -1), (0, 1)\}$, and

$[\mathbf{v}]_{B'} = [1 \quad -3]^T$.

(Use matrix $A$ given in Exercise 9.)

**11.** Let $B = \{(1, 2), (-1, -1)\}$ and $B' = \{(-4, 1), (0, 2)\}$ be bases for $R^2$, and let

$$A = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$$

be the matrix for $T: R^2 \to R^2$ relative to $B$.

(a) Find the transition matrix $P$ from $B'$ to $B$.

(b) Use the matrices $P$ and $A$ to find $[\mathbf{v}]_B$ and $[T(\mathbf{v})]_B$, where

$[\mathbf{v}]_{B'} = [-1 \quad 4]^T$.

(c) Find $P^{-1}$ and $A'$ (the matrix for $T$ relative to $B'$).

(d) Find $[T(\mathbf{v})]_{B'}$ two ways.

**12.** Repeat Exercise 11 for $B = \{(1, -1), (-2, 1)\}$, $B' = \{(-1, 1), (1, 2)\}$, and

$[\mathbf{v}]_{B'} = [1 \quad -4]^T$.

(Use matrix $A$ given in Exercise 11.)

**13.** Let $B = \{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$ and $B' = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ be bases for $R^3$, and let

$$A = \begin{bmatrix} \frac{3}{2} & -1 & -\frac{1}{2} \\ -\frac{1}{2} & 2 & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{5}{2} \end{bmatrix}$$

be the matrix for $T: R^3 \to R^3$ relative to $B$.

(a) Find the transition matrix $P$ from $B'$ to $B$.

(b) Use the matrices $P$ and $A$ to find $[\mathbf{v}]_B$ and $[T(\mathbf{v})]_B$, where

$[\mathbf{v}]_{B'} = [1 \quad 0 \quad -1]^T$.

(c) Find $P^{-1}$ and $A'$ (the matrix for $T$ relative to $B'$).

(d) Find $[T(\mathbf{v})]_{B'}$ two ways.

**14.** Repeat Exercise 13 for

$B = \{(1, 1, -1), (1, -1, 1), (-1, 1, 1)\}$,

$B' = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$,

and

$[\mathbf{v}]_{B'} = [2 \quad 1 \quad 1]^T$.

(Use matrix $A$ given in Exercise 13.)

**Similar Matrices** **In Exercises 15–18, use the matrix $P$ to show that the matrices $A$ and $A'$ are similar.**

**15.** $P = \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix}, A = \begin{bmatrix} 12 & 7 \\ -20 & -11 \end{bmatrix}, A' = \begin{bmatrix} 1 & -2 \\ 4 & 0 \end{bmatrix}$

**16.** $P = A = A' = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$

**17.** $P = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}, A = \begin{bmatrix} 5 & 10 & 0 \\ 8 & 4 & 0 \\ 0 & 9 & 6 \end{bmatrix}, A' = \begin{bmatrix} 5 & 8 & 0 \\ 10 & 4 & 0 \\ 0 & 12 & 6 \end{bmatrix}$

**18.** $P = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}, A' = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 2 & 3 \end{bmatrix}$

**Diagonal Matrix for a Linear Transformation** **In Exercises 19 and 20, suppose $A$ is the matrix for $T: R^3 \to R^3$ relative to the standard basis. Find the diagonal matrix $A'$ for $T$ relative to the basis $B'$.**

**19.** $A = \begin{bmatrix} 0 & 2 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$

$B' = \{(-1, 1, 0), (2, 1, 0), (0, 0, 1)\}$

**20.** $A = \begin{bmatrix} \frac{3}{2} & -1 & -\frac{1}{2} \\ -\frac{1}{2} & 2 & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{5}{2} \end{bmatrix},$

$B' = \{(1, 1, -1), (1, -1, 1), (-1, 1, 1)\}$

**21.** **Proof** Prove that if $A$ and $B$ are similar, then

$|A| = |B|.$

Is the converse true?

**22.** Illustrate the result of Exercise 21 using the matrices

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, B = \begin{bmatrix} 11 & 7 & 10 \\ 10 & 8 & 10 \\ -18 & -12 & -17 \end{bmatrix},$$

$$P = \begin{bmatrix} -1 & 1 & 0 \\ 2 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}, P^{-1} = \begin{bmatrix} -1 & -1 & 2 \\ 0 & -1 & 2 \\ 1 & 2 & -3 \end{bmatrix},$$

where $B = P^{-1}AP$.

**23.** **Proof** Prove that if $A$ and $B$ are similar, then there exists a matrix $P$ such that $B^k = P^{-1}A^kP$.

**24.** Use the result of Exercise 23 to find $B^4$, where $B = P^{-1}AP$, for the matrices

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} -4 & -15 \\ 2 & 7 \end{bmatrix},$$

$$P = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}.$$

**25.** Determine all $n \times n$ matrices that are similar to $I_n$.

**26.** **Proof** Prove that if $A$ is idempotent and $B$ is similar to $A$, then $B$ is idempotent. (Recall that an $n \times n$ matrix $A$ is idempotent when $A = A^2$.)

**27.** **Proof** Let $A$ be an $n \times n$ matrix such that $A^2 = O$. Prove that if $B$ is similar to $A$, then $B^2 = O$.

**28.** **Proof** Let $B = P^{-1}AP$. Prove that if $A\mathbf{x} = \mathbf{x}$, then $PBP^{-1}\mathbf{x} = \mathbf{x}$.

**29.** **Proof** Complete the proof of Theorem 6.13 by proving that if $A$ is similar to $B$ and $B$ is similar to $C$, then $A$ is similar to $C$.

**30.** **Writing** Suppose $A$ and $B$ are similar. Explain why they have the same rank.

**31.** **Proof** Prove that if $A$ and $B$ are similar, then $A^T$ and $B^T$ are similar.

**32.** **Proof** Prove that if $A$ and $B$ are similar and $A$ is nonsingular, then $B$ is also nonsingular and $A^{-1}$ and $B^{-1}$ are similar.

**33.** **Proof** Let $A = CD$, where $C$ is an invertible $n \times n$ matrix. Prove that the matrix $DC$ is similar to $A$.

**34.** **Proof** Let $B = P^{-1}AP$, where $A = [a_{ij}]$, $P = [p_{ij}]$, and $B$ is a diagonal matrix with main diagonal entries $b_{11}, b_{22}, \ldots, b_{nn}$. Prove that

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} p_{1i} \\ p_{2i} \\ \vdots \\ p_{ni} \end{bmatrix} = b_{ii} \begin{bmatrix} p_{1i} \\ p_{2i} \\ \vdots \\ p_{ni} \end{bmatrix}$$

for $i = 1, 2, \ldots, n$.

**35.** **Writing** Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for the vector space $V$, let $B'$ be the standard basis, and consider the identity transformation $I: V \to V$. What can you say about the matrix for $I$ relative to $B$? relative to $B'$? when the domain has the basis $B$ and the range has the basis $B'$?

**36.** **CAPSTONE**

(a) Given two bases $B$ and $B'$ for a vector space $V$ and the matrix $A$ for the linear transformation $T: V \to V$ relative to $B$, explain how to obtain the coordinate matrix $[T(\mathbf{v})]_{B'}$ from the coordinate matrix $[\mathbf{v}]_{B'}$, where $\mathbf{v}$ is a vector in $V$.

(b) Explain how to determine whether two square matrices $A$ and $A'$ of order $n$ are similar.

**True or False?** **In Exercises 37 and 38, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.**

**37.** (a) The matrix for a linear transformation $A'$ relative to the basis $B'$ is equal to the product $P^{-1}AP$, where $P^{-1}$ is the transition matrix from $B$ to $B'$, $A$ is the matrix for the linear transformation relative to basis $B$, and $P$ is the transition matrix from $B'$ to $B$.

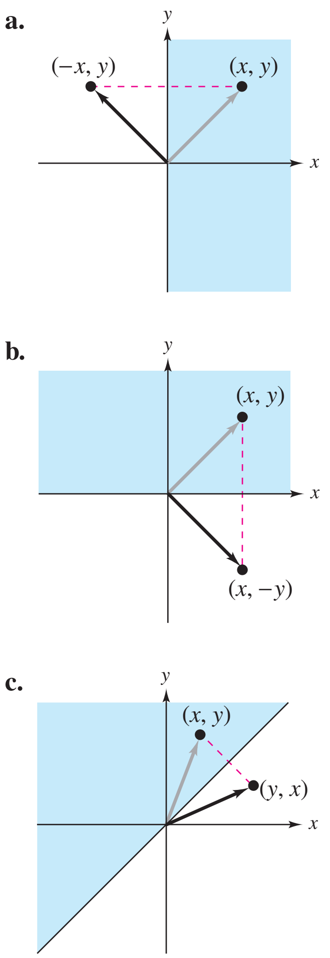
(b) Two matrices that represent the same linear transformation $T: V \to V$ with respect to different bases are not necessarily similar.

**38.** (a) The matrix for a linear transformation $A$ relative to the basis $B$ is equal to the product $PA'P^{-1}$, where $P$ is the transition matrix from $B'$ to $B$, $A'$ is the matrix for the linear transformation relative to basis $B'$, and $P^{-1}$ is the transition matrix from $B$ to $B'$.

(b) The standard basis for $R^n$ will always make the coordinate matrix for the linear transformation $T$ the simplest matrix possible.

# 6.5 Applications of Linear Transformations

- Identify linear transformations defined by reflections, expansions, contractions, or shears in $R^2$.
- Use a linear transformation to rotate a figure in $R^3$.




Reflections in $R^2$
**Figure 6.11**

## THE GEOMETRY OF LINEAR TRANSFORMATIONS IN $R^2$

This section gives geometric interpretations of linear transformations represented by $2 \times 2$ elementary matrices. Following a summary of the various types of $2 \times 2$ elementary matrices are examples that examine each type of matrix in more detail.

### Elementary Matrices for Linear Transformations in $R^2$

Reflection in $y$-Axis

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Reflection in $x$-Axis

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Reflection in Line $y = x$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Horizontal Expansion $(k > 1)$ or Contraction $(0 < k < 1)$

$$A = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$$

Vertical Expansion $(k > 1)$ or Contraction $(0 < k < 1)$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$$

Horizontal Shear

$$A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

Vertical Shear

$$A = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$$

**EXAMPLE 1** **Reflections in $R^2$**

The transformations defined by the following matrices are called **reflections.** These have the effect of mapping a point in the $xy$-plane to its "mirror image" with respect to one of the coordinate axes or the line $y = x$, as shown in Figure 6.11.

**a.** Reflection in the $y$-axis:

$$T(x, y) = (-x, y)$$

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}$$

**b.** Reflection in the $x$-axis:

$$T(x, y) = (x, -y)$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$$

**c.** Reflection in the line $y = x$:

$$T(x, y) = (y, x)$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$$

## EXAMPLE 2 Expansions and Contractions in $R^2$

The transformations defined by the following matrices are called **expansions** or **contractions,** depending on the value of the positive scalar $k$.

**a.** Horizontal contractions and expansions: $T(x, y) = (kx, y)$

$$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} kx \\ y \end{bmatrix}$$

**b.** Vertical contractions and expansions: $T(x, y) = (x, ky)$

$$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ ky \end{bmatrix}$$

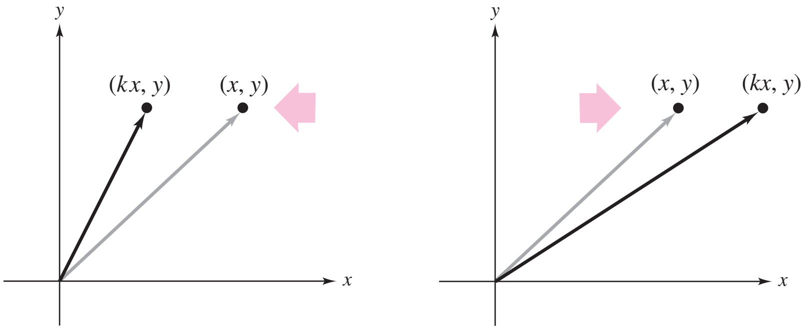
Note in Figures 6.12 and 6.13 that the distance the point $(x, y)$ moves by a contraction or an expansion is proportional to its $x$- or $y$-coordinate. For instance, under the transformation represented by

$$T(x, y) = (2x, y)$$

the point $(1, 3)$ would move one unit to the right, but the point $(4, 3)$ would move four units to the right. Under the transformation represented by

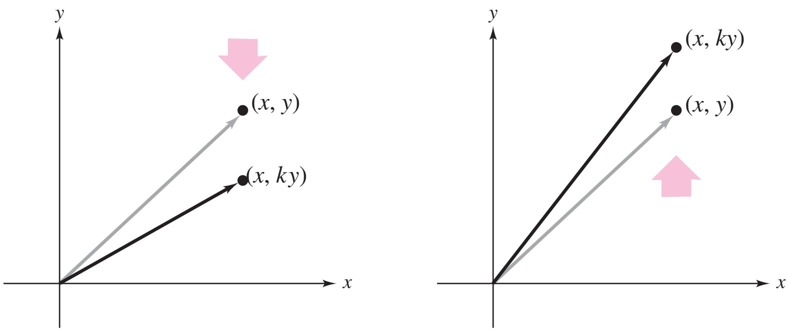
$$T(x, y) = \left(x, \tfrac{1}{2}y\right)$$

the point $(1, 4)$ would move two units down, but the point $(1, 2)$ would move one unit down.




Contraction $(0 < k < 1)$ Expansion $(k > 1)$

**Figure 6.12**




Contraction $(0 < k < 1)$ Expansion $(k > 1)$

**Figure 6.13**

Another type of linear transformation in $R^2$ corresponding to an elementary matrix is called a **shear,** as described in Example 3.

## EXAMPLE 3 Shears in $R^2$

The transformations defined by the following matrices are shears.

$$T(x, y) = (x + ky, y) \qquad\qquad T(x, y) = (x, y + kx)$$

$$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + ky \\ y \end{bmatrix} \qquad\qquad \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ kx + y \end{bmatrix}$$

**a.** A horizontal shear represented by

$$T(x, y) = (x + 2y, y)$$

is shown in Figure 6.14. Under this transformation, points in the upper half-plane "shear" to the right by amounts proportional to their $y$-coordinates. Points in the lower half-plane "shear" to the left by amounts proportional to the absolute values of their $y$-coordinates. Points on the $x$-axis do not move by this transformation.

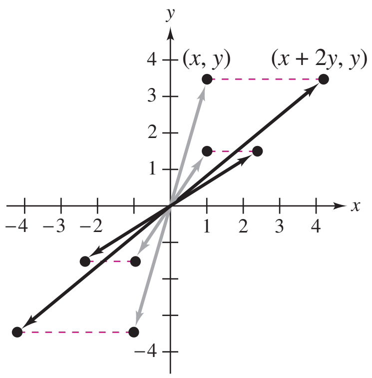



**Figure 6.14**

**b.** A vertical shear represented by

$$T(x, y) = (x, y + 2x)$$

is shown in Figure 6.15. Here, points in the right half-plane "shear" upward by amounts proportional to their $x$-coordinates. Points in the left half-plane "shear" downward by amounts proportional to the absolute values of their $x$-coordinates. Points on the $y$-axis do not move.

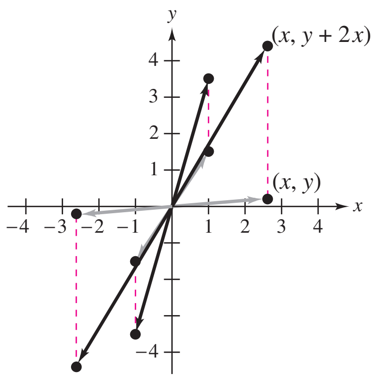
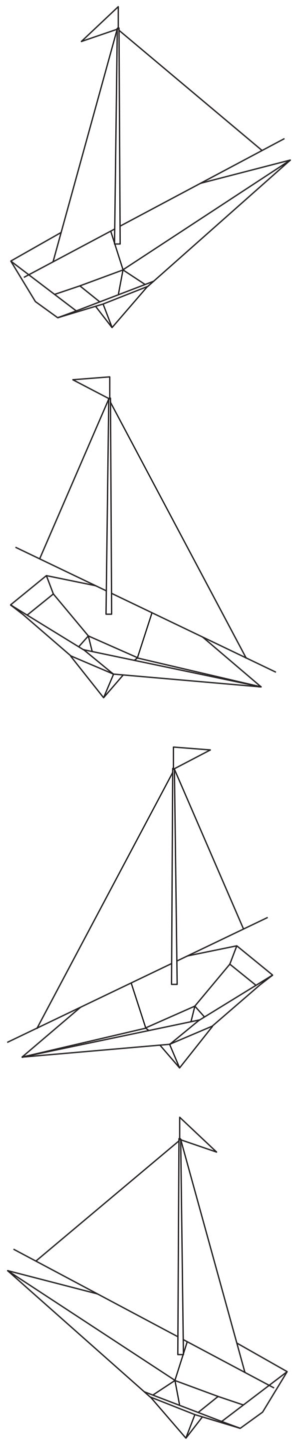



**Figure 6.15**



**LINEAR ALGEBRA APPLIED**

The use of computer graphics is common in many fields. By using graphics software, a designer can "see" an object before it is physically created. Linear transformations can be useful in computer graphics. To illustrate with a simplified example, only 23 points in $R^3$ make up the images of the toy boat shown in the figure at the left. Most graphics software can use such minimal information to generate views of an image from any perspective, as well as color, shade, and render as appropriate. Linear transformations, specifically those that produce rotations in $R^3$, can represent the different views. The remainder of this section discusses rotation in $R^3$.

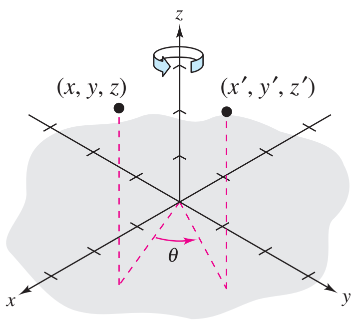



**Figure 6.16**

## ROTATION IN $R^3$

In Example 7 in Section 6.1, you saw how a linear transformation can be used to rotate figures in $R^2$. Here you will see how linear transformations can be used to rotate figures in $R^3$.

Suppose you want to rotate the point $(x, y, z)$ counterclockwise about the $z$-axis through an angle $\theta$, as shown in Figure 6.16. Letting the coordinates of the rotated point be $(x', y', z')$, you have

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \\ z \end{bmatrix}.$$

Example 4 shows how to use this matrix to rotate a figure in three-dimensional space.

### EXAMPLE 4 Rotation About the $z$-Axis

The eight vertices of the rectangular prism shown in Figure 6.17 are as follows.

$V_1(0, 0, 0)$ $V_2(1, 0, 0)$

$V_3(1, 2, 0)$ $V_4(0, 2, 0)$

$V_5(0, 0, 3)$ $V_6(1, 0, 3)$

$V_7(1, 2, 3)$ $V_8(0, 2, 3)$

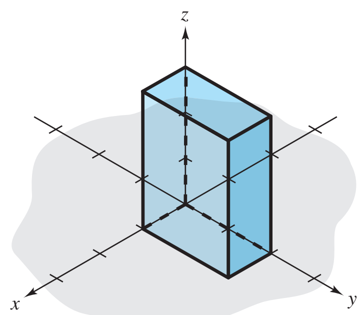



**Figure 6.17**

Find the coordinates of the vertices after the prism is rotated counterclockwise about the $z$-axis through (a) $\theta = 60°$, (b) $\theta = 90°$, and (c) $\theta = 120°$.

**SOLUTION**

**a.** The matrix that yields a rotation of 60° is

$$A = \begin{bmatrix} \cos 60° & -\sin 60° & 0 \\ \sin 60° & \cos 60° & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Multiplying this matrix by the column vectors corresponding to each vertex produces the following rotated vertices.

$V_1'(0, 0, 0)$ $V_2'(0.5, 0.87, 0)$ $V_3'(-1.23, 1.87, 0)$ $V_4'(-1.73, 1, 0)$

$V_5'(0, 0, 3)$ $V_6'(0.5, 0.87, 3)$ $V_7'(-1.23, 1.87, 3)$ $V_8'(-1.73, 1, 3)$

Figure 6.18(a) shows a graph of the rotated prism.

**b.** The matrix that yields a rotation of 90° is

$$A = \begin{bmatrix} \cos 90° & -\sin 90° & 0 \\ \sin 90° & \cos 90° & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and Figure 6.18(b) shows a graph of the rotated prism.

**c.** The matrix that yields a rotation of 120° is

$$A = \begin{bmatrix} \cos 120° & -\sin 120° & 0 \\ \sin 120° & \cos 120° & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and Figure 6.18(c) shows a graph of the rotated prism.

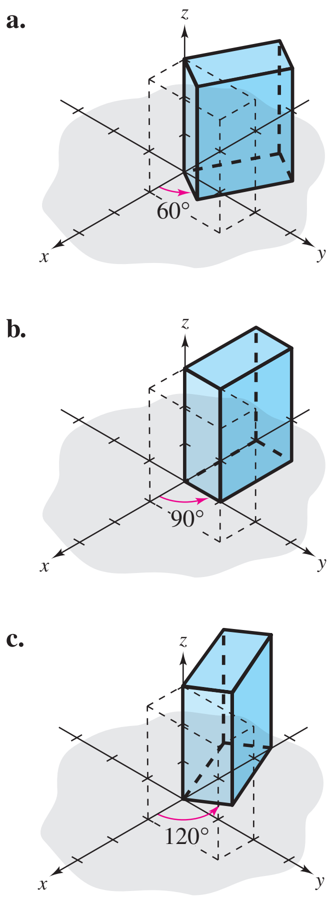
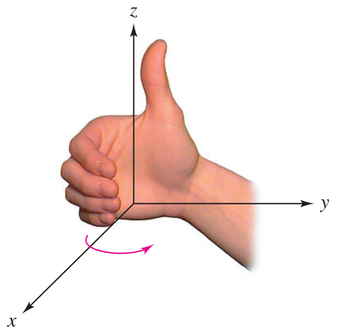



**Figure 6.18**

**REMARK**

To illustrate the right-hand rule, imagine the thumb of your right hand pointing in the positive direction of an axis. The cupped fingers will point in the direction of counterclockwise rotation. The figure below shows counterclockwise rotation about the $z$-axis.




Example 4 uses matrices to perform rotations about the $z$-axis. Similarly, you can use matrices to rotate figures about the $x$- or $y$-axis. The following summarizes all three types of rotations.

**Rotation About the $x$-Axis**

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

**Rotation About the $y$-Axis**

$$\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

**Rotation About the $z$-Axis**

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

In each case, the rotation is oriented counterclockwise (using the "right-hand rule") relative to the indicated axis, as shown in Figure 6.19.

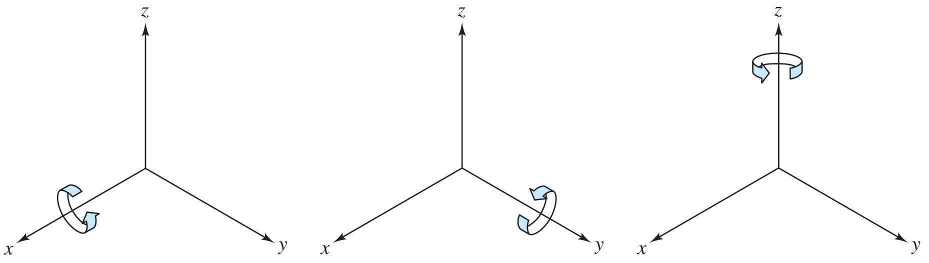



**Figure 6.19**

## EXAMPLE 5 Rotation About the $x$-Axis and $y$-Axis

**a.** The matrix that yields a rotation of 90° about the $x$-axis is

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 90^\circ & -\sin 90^\circ \\ 0 & \sin 90^\circ & \cos 90^\circ \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Figure 6.20(a) shows the prism from Example 4 rotated 90° about the $x$-axis.

**b.** The matrix that yields a rotation of 90° about the $y$-axis is

$$A = \begin{bmatrix} \cos 90^\circ & 0 & \sin 90^\circ \\ 0 & 1 & 0 \\ -\sin 90^\circ & 0 & \cos 90^\circ \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

Figure 6.20(b) shows the prism from Example 4 rotated 90° about the $y$-axis.

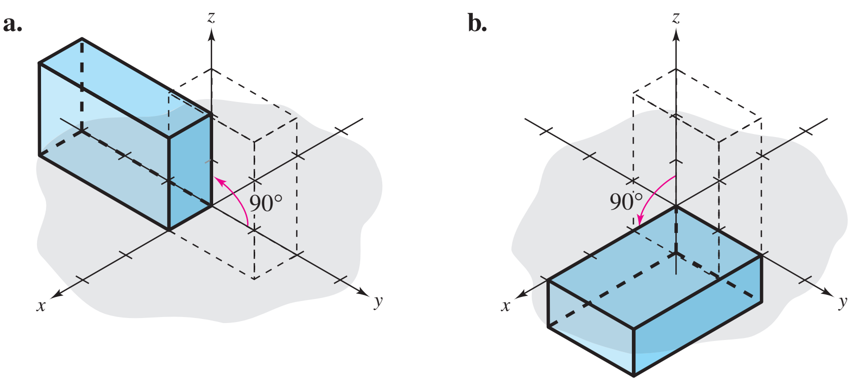



**Figure 6.20**

***Simulation***

Explore this concept further with an electronic simulation available at *www.cengagebrain.com.*

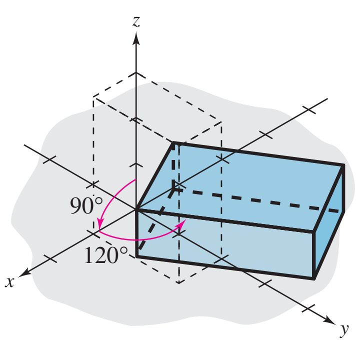



**Figure 6.21**

Rotations about the coordinate axes can be combined to produce any desired view of a figure. For instance, Figure 6.21 shows the prism from Example 4 rotated 90° about the $y$-axis and then 120° about the $z$-axis.

## 6.5 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

**1.** Let $T: R^2 \to R^2$ be a reflection in the $x$-axis. Find the image of each vector.
(a) $(3, 5)$ (b) $(2, -1)$ (c) $(a, 0)$
(d) $(0, b)$ (e) $(-c, d)$ (f) $(f, -g)$

**2.** Let $T: R^2 \to R^2$ be a reflection in the $y$-axis. Find the image of each vector.
(a) $(2, 5)$ (b) $(-4, -1)$ (c) $(a, 0)$
(d) $(0, b)$ (e) $(c, -d)$ (f) $(f, g)$

**3.** Let $T: R^2 \to R^2$ be a reflection in the line $y = x$. Find the image of each vector.
(a) $(0, 1)$ (b) $(-1, 3)$ (c) $(a, 0)$
(d) $(0, b)$ (e) $(-c, d)$ (f) $(f, -g)$

**4.** Let $T: R^2 \to R^2$ be a reflection in the line $y = -x$. Find the image of each vector.
(a) $(-1, 2)$ (b) $(2, 3)$ (c) $(a, 0)$
(d) $(0, b)$ (e) $(e, -d)$ (f) $(-f, g)$

**5.** Let $T(1, 0) = (2, 0)$ and $T(0, 1) = (0, 1)$.
(a) Determine $T(x, y)$ for any $(x, y)$.
(b) Give a geometric description of $T$.

**6.** Let $T(1, 0) = (1, 1)$ and $T(0, 1) = (0, 1)$.
(a) Determine $T(x, y)$ for any $(x, y)$.
(b) Give a geometric description of $T$.

**Identifying and Representing a Transformation** In Exercises 7–14, (a) identify the transformation, and (b) graphically represent the transformation for an arbitrary vector in $R^2$.

**7.** $T(x, y) = (x, y/2)$
**8.** $T(x, y) = (x/4, y)$
**9.** $T(x, y) = (4x, y)$
**10.** $T(x, y) = (x, 3y)$
**11.** $T(x, y) = (x + 3y, y)$
**12.** $T(x, y) = (x + 4y, y)$
**13.** $T(x, y) = (x, 5x + y)$
**14.** $T(x, y) = (x, 4x + y)$

**Finding Fixed Points of a Linear Transformation** In Exercises 15–22, find all fixed points of the linear transformation. Recall that the vector $\mathbf{v}$ is a fixed point of $T$ when $T(\mathbf{v}) = \mathbf{v}$.

**15.** A reflection in the $y$-axis
**16.** A reflection in the $x$-axis
**17.** A reflection in the line $y = x$
**18.** A reflection in the line $y = -x$
**19.** A vertical contraction
**20.** A horizontal expansion
**21.** A horizontal shear
**22.** A vertical shear

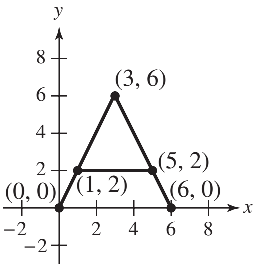
**Sketching an Image of the Unit Square** In Exercises 23–28, sketch the image of the unit square [a square with vertices at (0, 0), (1, 0), (1, 1), and (0, 1)] under the specified transformation.

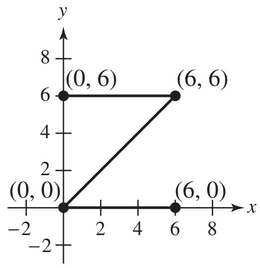
**23.** $T$ is a reflection in the $x$-axis.
**24.** $T$ is a reflection in the line $y = x$.
**25.** $T$ is the contraction represented by $T(x, y) = (x/2, y)$.
**26.** $T$ is the expansion represented by $T(x, y) = (x, 3y)$.
**27.** $T$ is the shear represented by $T(x, y) = (x + 2y, y)$.
**28.** $T$ is the shear represented by $T(x, y) = (x, y + 3x)$.

**Sketching an Image of a Rectangle** In Exercises 29–34, sketch the image of the rectangle with vertices at (0, 0), (1, 0), (1, 2), and (0, 2) under the specified transformation.

**29.** $T$ is a reflection in the $y$-axis.
**30.** $T$ is a reflection in the line $y = x$.
**31.** $T$ is the contraction represented by $T(x, y) = (x, y/2)$.
**32.** $T$ is the expansion represented by $T(x, y) = (2x, y)$.
**33.** $T$ is the shear represented by $T(x, y) = (x + y, y)$.
**34.** $T$ is the shear represented by $T(x, y) = (x, y + 2x)$.

**Sketching an Image of a Figure** In Exercises 35–38, sketch each of the images under the specified transformation.

**(a)**



**(b)**



**35.** $T$ is the shear represented by $T(x, y) = (x + y, y)$.
**36.** $T$ is the shear represented by $T(x, y) = (x, x + y)$.
**37.** $T$ is the expansion and contraction represented by
$T(x, y) = \left(2x, \frac{1}{2}y\right)$.
**38.** $T$ is the expansion and contraction represented by
$T(x, y) = \left(\frac{1}{2}x, 2y\right)$.

**39.** The linear transformation defined by a diagonal matrix with positive main diagonal elements is called a **magnification.** Find the images of (1, 0), (0, 1), and (2, 2) under the linear transformation $A$ and graphically interpret your result.

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

**40.** Repeat Exercise 39 for the linear transformation defined by

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}.$$

**Giving a Geometric Description** **In Exercises 41–46, give a geometric description of the linear transformation defined by the elementary matrix.**

**41.** $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ **42.** $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

**43.** $A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$ **44.** $A = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$

**45.** $A = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$ **46.** $A = \begin{bmatrix} -\frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}$

**Giving a Geometric Description** **In Exercises 47 and 48, give a geometric description of the linear transformation defined by the matrix product.**

**47.** $A = \begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

**48.** $A = \begin{bmatrix} 0 & 3 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$

**Finding a Matrix to Produce a Rotation** **In Exercises 49–52, find the matrix that produces the given rotation.**

**49.** 30° about the $z$-axis **50.** 60° about the $x$-axis

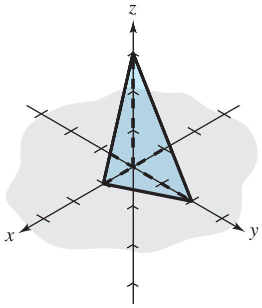
**51.** 60° about the $y$-axis **52.** 120° about the $x$-axis

**Finding the Image of a Vector** **In Exercises 53–56, find the image of the vector (1, 1, 1) for the given rotation.**

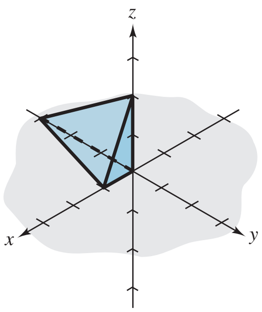
**53.** 30° about the $z$-axis **54.** 60° about the $x$-axis

**55.** 60° about the $y$-axis **56.** 120° about the $x$-axis

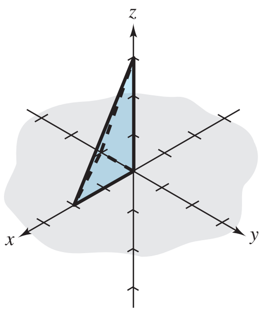
**Determining a Rotation** **In Exercises 57–62, determine which single counterclockwise rotation about the $x$-, $y$-, or $z$-axis produces the rotated tetrahedron. The figure at the right shows the tetrahedron before rotation.**



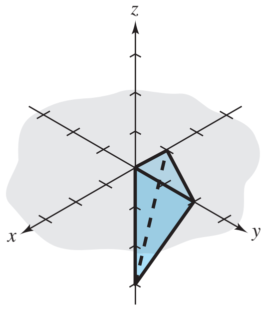

**57.**



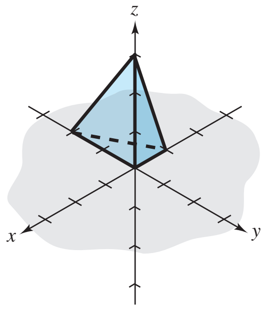

**58.**



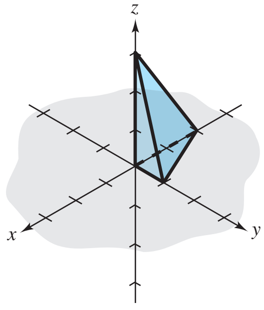

**59.**



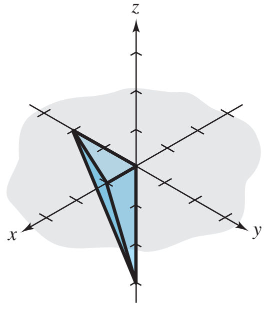

**60.**




**61.**




**62.**




**Determining a Matrix to Produce a Pair of Rotations** **In Exercises 63–67, determine the matrix that produces the given pair of rotations. Then find the image of the vector (1, 1, 1) under these rotations.**

**63.** 90° about the $x$-axis followed by 90° about the $y$-axis

**64.** 45° about the $y$-axis followed by 90° about the $z$-axis

**65.** 30° about the $z$-axis followed by 60° about the $y$-axis

**66.** 45° about the $z$-axis followed by 135° about the $x$-axis

**67.** 120° about the $x$-axis followed by 135° about the $z$-axis

**68.** **CAPSTONE** Describe the transformation defined by each matrix. Assume $k$ and $\theta$ are positive scalars.

(a) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$, $k > 1$

(e) $\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$, $0 < k < 1$ (f) $\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$, $k > 1$

(g) $\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$, $0 < k < 1$ (h) $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$

(i) $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$ (j) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$

(k) $\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$

(l) $\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

# 6 Review Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

**Finding an Image and a Preimage** In Exercises 1–4, find (a) the image of v and (b) the preimage of w for the linear transformation.

1. $T: R^2 \to R^2$, $T(v_1, v_2) = (v_1, v_1 + 2v_2)$, $\mathbf{v} = (2, -3)$, $\mathbf{w} = (4, 12)$
2. $T: R^2 \to R^2$, $T(v_1, v_2) = (v_1 + v_2, 2v_2)$, $\mathbf{v} = (4, -1)$, $\mathbf{w} = (8, 4)$
3. $T: R^3 \to R^3$, $T(v_1, v_2, v_3) = (0, v_1 + v_2, v_2 + v_3)$, $\mathbf{v} = (-3, 2, 5)$, $\mathbf{w} = (0, 2, 5)$
4. $T: R^3 \to R^3$, $T(v_1, v_2, v_3) = (v_1 + v_2, v_2 + v_3, v_3)$, $\mathbf{v} = (-2, 1, 2)$, $\mathbf{w} = (0, 1, 2)$

**Linear Transformations and Standard Matrices** In Exercises 5–12, determine whether the function is a linear transformation. If it is, then find its standard matrix $A$.

5. $T: R^2 \to R^2$, $T(x_1, x_2) = (x_1 + 2x_2, -x_1 - x_2)$
6. $T: R^2 \to R^2$, $T(x_1, x_2) = (x_1 + 3, x_2)$
7. $T: R^2 \to R^2$, $T(x, y) = (x - 2y, 2y - x)$
8. $T: R^2 \to R^2$, $T(x, y) = (x + y, y)$
9. $T: R^2 \to R^2$, $T(x, y) = (x + h, y + k)$, $h \neq 0$ or $k \neq 0$ (translation in $R^2$)
10. $T: R^2 \to R^2$, $T(x, y) = (|x|, |y|)$
11. $T: R^3 \to R^3$, $T(x_1, x_2, x_3) = (x_1 - x_2, x_2 - x_3, x_3 - x_1)$
12. $T: R^3 \to R^3$, $T(x, y, z) = (z, y, x)$

13. Let $T$ be a linear transformation from $R^2$ into $R^2$ such that $T(2, 0) = (1, 1)$ and $T(0, 3) = (3, 3)$. Find $T(1, 1)$ and $T(0, 1)$.
14. Let $T$ be a linear transformation from $R^3$ into $R$ such that $T(1, 1, 1) = 1$, $T(1, 1, 0) = 2$, and $T(1, 0, 0) = 3$. Find $T(0, 1, 1)$.
15. Let $T$ be a linear transformation from $R^2$ into $R^2$ such that $T(1, 1) = (2, 3)$ and $T(2, -1) = (1, 0)$. Find $T(0, -1)$.
16. Let $T$ be a linear transformation from $R^2$ into $R^2$ such that $T(1, -1) = (2, -3)$ and $T(0, 2) = (0, 8)$. Find $T(2, 4)$.

**Linear Transformation Given by a Matrix** In Exercises 17–24, define the linear transformation $T: R^n \to R^m$ by $T(\mathbf{v}) = A\mathbf{v}$. Use the matrix $A$ to (a) determine the dimensions of $R^n$ and $R^m$, (b) find the image of v, and (c) find the preimage of w.

17. $A = \begin{bmatrix} 0 & 1 & 2 \\ -2 & 0 & 0 \end{bmatrix}$, $\mathbf{v} = (6, 1, 1)$, $\mathbf{w} = (3, 5)$
18. $A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix}$, $\mathbf{v} = (5, 2, 2)$, $\mathbf{w} = (4, 2)$
19. $A = \begin{bmatrix} 1 & 1 \end{bmatrix}$, $\mathbf{v} = (2, 3)$, $\mathbf{w} = 4$
20. $A = \begin{bmatrix} 2 & -1 \end{bmatrix}$, $\mathbf{v} = (1, 2)$, $\mathbf{w} = -1$
21. $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, $\mathbf{v} = (2, 1, -5)$, $\mathbf{w} = (6, 4, 2)$
22. $A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$, $\mathbf{v} = (8, 4)$, $\mathbf{w} = (5, 2)$
23. $A = \begin{bmatrix} 4 & 0 \\ 0 & 5 \\ 1 & 1 \end{bmatrix}$, $\mathbf{v} = (2, 2)$, $\mathbf{w} = (4, -5, 0)$
24. $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 1 & 2 \end{bmatrix}$, $\mathbf{v} = (1, 2)$, $\mathbf{w} = (2, -5, 12)$

25. Use the standard matrix for counterclockwise rotation in $R^2$ to rotate the triangle with vertices $(3, 5)$, $(5, 3)$, and $(3, 0)$ counterclockwise 90° about the origin. Graph the triangles.
26. Rotate the triangle in Exercise 25 counterclockwise 90° about the point $(5, 3)$. Graph the triangles.

**Finding Bases for the Kernel and Range** In Exercises 27–30, find a basis for (a) ker($T$) and (b) range($T$).

27. $T: R^4 \to R^3$,
$T(w, x, y, z) = (2w + 4x + 6y + 5z, -w - 2x + 2y, 8y + 4z)$
28. $T: R^3 \to R^3$, $T(x, y, z) = (x + 2y, y + 2z, z + 2x)$
29. $T: R^3 \to R^3$, $T(x, y, z) = (x, y + 2z, z)$
30. $T: R^3 \to R^3$, $T(x, y, z) = (x + y, y + z, x - z)$

**Finding the Kernel, Nullity, Range, and Rank** In Exercises 31–34, define the linear transformation $T$ by $T(\mathbf{v}) = A\mathbf{v}$. Find (a) ker($T$), (b) nullity($T$), (c) range($T$), and (d) rank($T$).

31. $A = \begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 1 & 1 \end{bmatrix}$
32. $A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ 2 & 1 \end{bmatrix}$
33. $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & 0 \\ 0 & 1 & -3 \end{bmatrix}$
34. $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

35. Given $T: R^5 \to R^3$ and nullity($T$) $= 2$, find rank($T$).
36. Given $T: P_5 \to P_3$ and nullity($T$) $= 4$, find rank($T$).
37. Given $T: P_4 \to R^5$ and rank($T$) $= 3$, find nullity($T$).
38. Given $T: M_{2,2} \to M_{2,2}$ and rank($T$) $= 3$, find nullity($T$).

**Finding a Power of a Standard Matrix** **In Exercises 39–42, find the indicated power of $A$, the standard matrix for $T$.**

**39.** $T: R^3 \to R^3$, reflection in the $xy$-plane. Find $A^2$.

**40.** $T: R^3 \to R^3$, projection onto the $xy$-plane. Find $A^2$.

**41.** $T: R^2 \to R^2$, counterclockwise rotation through the angle $\theta$. Find $A^3$.

**42.** **Calculus** $T: P_3 \to P_3$, differential operator $D_x$. Find $A^2$.

**Finding Standard Matrices for Compositions** **In Exercises 43 and 44, find the standard matrices for $T = T_2 \circ T_1$ and $T' = T_1 \circ T_2$.**

**43.** $T_1: R^2 \to R^3$, $T_1(x, y) = (x, x + y, y)$

$T_2: R^3 \to R^2$, $T_2(x, y, z) = (0, y)$

**44.** $T_1: R \to R^2$, $T_1(x) = (x, 3x)$

$T_2: R^2 \to R$, $T_2(x, y) = (y + 2x)$

**Finding the Inverse of a Linear Transformation** **In Exercises 45–48, determine whether the linear transformation is invertible. If it is, find its inverse.**

**45.** $T: R^2 \to R^2$, $T(x, y) = (0, y)$

**46.** $T: R^2 \to R^2$,

$T(x, y) = (x \cos \theta - y \sin \theta, x \sin \theta + y \cos \theta)$

**47.** $T: R^2 \to R^2$, $T(x, y) = (x, -y)$

**48.** $T: R^3 \to R^2$, $T(x, y, z) = (x + y, y - z)$

**One-to-One, Onto, and Invertible Transformations** **In Exercises 49–52, determine whether the linear transformation represented by the matrix $A$ is (a) one-to-one, (b) onto, and (c) invertible.**

**49.** $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

**50.** $A = \begin{bmatrix} 1 & \frac{1}{4} \\ 0 & 1 \end{bmatrix}$

**51.** $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

**52.** $A = \begin{bmatrix} 4 & 0 & 7 \\ 5 & 5 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

**Finding the Image Two Ways** **In Exercises 53 and 54, find $T(\mathbf{v})$ by using (a) the standard matrix and (b) the matrix relative to $B$ and $B'$.**

**53.** $T: R^2 \to R^3$, $T(x, y) = (-x, y, x + y)$, $\mathbf{v} = (0, 1)$,

$B = \{(1, 1), (1, -1)\}$, $B' = \{(0, 1, 0), (0, 0, 1), (1, 0, 0)\}$

**54.** $T: R^2 \to R^2$, $T(x, y) = (2y, 0)$, $\mathbf{v} = (-1, 3)$,

$B = \{(2, 1), (-1, 0)\}$, $B' = \{(-1, 0), (2, 2)\}$

**Finding a Matrix for a Linear Transformation** **In Exercises 55 and 56, find the matrix $A'$ for $T$ relative to the basis $B'$.**

**55.** $T: R^2 \to R^2$,

$T(x, y) = (x - 3y, y - x)$, $B' = \{(1, -1), (1, 1)\}$

**56.** $T: R^3 \to R^3$, $T(x, y, z) = (x + 3y, 3x + y, -2z)$,

$B' = \{(1, 1, 0), (1, -1, 0), (0, 0, 1)\}$

**Similar Matrices** **In Exercises 57 and 58, use the matrix $P$ to show that the matrices $A$ and $A'$ are similar.**

**57.** $P = \begin{bmatrix} 3 & -5 \\ 1 & -4 \end{bmatrix}$, $A = \begin{bmatrix} 18 & -19 \\ 11 & -12 \end{bmatrix}$, $A' = \begin{bmatrix} 5 & -3 \\ -4 & 1 \end{bmatrix}$

**58.** $P = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix}$, $A' = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

**59.** Define $T: R^3 \to R^3$ by $T(\mathbf{v}) = \text{proj}_{\mathbf{u}}\mathbf{v}$, where $\mathbf{u} = (0, 1, 2)$.

(a) Find $A$, the standard matrix for $T$.

(b) Let $S$ be the linear transformation represented by $I - A$. Show that $S$ is of the form

$S(\mathbf{v}) = \text{proj}_{\mathbf{w}_1}\mathbf{v} + \text{proj}_{\mathbf{w}_2}\mathbf{v}$

where $\mathbf{w}_1$ and $\mathbf{w}_2$ are fixed vectors in $R^3$.

(c) Show that the kernel of $T$ is equal to the range of $S$.

**60.** Define $T: R^2 \to R^2$ by $T(\mathbf{v}) = \text{proj}_{\mathbf{u}}\mathbf{v}$, where $\mathbf{u} = (4, 3)$.

(a) Find $A$, the standard matrix for $T$, and show that $A^2 = A$.

(b) Show that $(I - A)^2 = I - A$.

(c) Find $A\mathbf{v}$ and $(I - A)\mathbf{v}$ for $\mathbf{v} = (5, 0)$.

(d) Sketch the graphs of $\mathbf{u}$, $\mathbf{v}$, $A\mathbf{v}$, and $(I - A)\mathbf{v}$.

**61.** Let $S$ and $T$ be linear transformations from $V$ into $W$. Show that $S + T$ and $kT$ are both linear transformations, where $(S + T)(\mathbf{v}) = S(\mathbf{v}) + T(\mathbf{v})$ and $(kT)(\mathbf{v}) = kT(\mathbf{v})$.

**62.** **Proof** Let $T: R^2 \to R^2$ such that $T(\mathbf{v}) = A\mathbf{v} + \mathbf{b}$, where $A$ is a $2 \times 2$ matrix. (Such a transformation is called an **affine transformation.**) Prove that $T$ is a linear transformation if and only if $\mathbf{b} = \mathbf{0}$.

**Sum of Two Linear Transformations** **In Exercises 63 and 64, consider the sum $S + T$ of two linear transformations $S: V \to W$ and $T: V \to W$, defined as $(S + T)(\mathbf{v}) = S(\mathbf{v}) + T(\mathbf{v})$.**

**63.** **Proof** Prove that $\text{rank}(S + T) \le \text{rank}(S) + \text{rank}(T)$.

**64.** Give an example for each.

(a) $\text{Rank}(S + T) = \text{rank}(S) + \text{rank}(T)$

(b) $\text{Rank}(S + T) < \text{rank}(S) + \text{rank}(T)$

**65.** **Proof** Let $T: P_3 \to R$ such that

$T(a_0 + a_1x + a_2x^2 + a_3x^3) = a_0 + a_1 + a_2 + a_3$.

(a) Prove that $T$ is a linear transformation.

(b) Find the rank and nullity of $T$.

(c) Find a basis for the kernel of $T$.

**66.** **Proof** Let

$T: V \to U$ and $S: U \to W$

be linear transformations.

(a) Prove that $\ker(T)$ is contained in $\ker(S \circ T)$.

(b) Prove that if $S \circ T$ is onto, then so is $S$.

**67.** Let $V$ be an inner product space. For a fixed nonzero vector $\mathbf{v}_0$ in $V$, let $T$: $V \to R$ be the linear transformation $T(\mathbf{v}) = \langle \mathbf{v}, \mathbf{v}_0 \rangle$. Find the kernel, range, rank, and nullity of $T$.

**68. Calculus** Let $B = \{1, x, \sin x, \cos x\}$ be a basis for a subspace $W$ of the space of continuous functions, and let $D_x$ be the differential operator on $W$. Find the matrix for $D_x$ relative to the basis $B$. Find the range and kernel of $D_x$.

**69. Writing** Are the vector spaces $R^4$, $M_{2,2}$, and $M_{1,4}$ exactly the same? Describe their similarities and differences.

**70. Calculus** Define $T$: $P_3 \to P_3$ by $T(p) = p(x) + p'(x)$. Find the rank and nullity of $T$.

**Identifying and Representing a Transformation** **In Exercises 71–76, (a) identify the transformation, and (b) graphically represent the transformation for an arbitrary vector in $R^2$.**

**71.** $T(x, y) = (x, 2y)$

**72.** $T(x, y) = (x + y, y)$

**73.** $T(x, y) = (x, y + 3x)$

**74.** $T(x, y) = (5x, y)$

**75.** $T(x, y) = (x + 5y, y)$

**76.** $T(x, y) = \left(x, y + \frac{3}{2}x\right)$

**Sketching an Image of a Triangle** **In Exercises 77–80, sketch the image of the triangle with vertices (0, 0), (1, 0), and (0, 1) under the specified transformation.**

**77.** $T$ is a reflection in the $x$-axis.

**78.** $T$ is the expansion represented by $T(x, y) = (2x, y)$.

**79.** $T$ is the shear represented by $T(x, y) = (x + 3y, y)$.

**80.** $T$ is the shear represented by $T(x, y) = (x, y + 2x)$.

**Giving a Geometric Description** **In Exercises 81 and 82, give a geometric description of the linear transformation defined by the matrix product.**

**81.** $\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

**82.** $\begin{bmatrix} 1 & 0 \\ 6 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$

**Finding a Matrix to Produce a Rotation** **In Exercises 83–86, find the matrix that produces the given rotation. Then find the image of the vector (1, −1, 1).**

**83.** 45° about the $z$-axis

**84.** 90° about the $x$-axis

**85.** 60° about the $x$-axis

**86.** 30° about the $y$-axis

**Determining a Matrix to Produce a Pair of Rotations** **In Exercises 87–90, determine the matrix that produces the given pair of rotations.**

**87.** 60° about the $x$-axis followed by 30° about the $z$-axis

**88.** 120° about the $y$-axis followed by 45° about the $z$-axis

**89.** 30° about the $y$-axis followed by 45° about the $z$-axis

**90.** 60° about the $x$-axis followed by 60° about the $z$-axis

**Finding an Image of a Unit Cube** **In Exercises 91–94, find the image of the unit cube with vertices (0, 0, 0), (1, 0, 0), (1, 1, 0), (0, 1, 0), (0, 0, 1), (1, 0, 1), (1, 1, 1), and (0, 1, 1) when it rotates by the given angle.**

**91.** 45° about the $z$-axis

**92.** 90° about the $x$-axis

**93.** 30° about the $x$-axis

**94.** 120° about the $z$-axis

**True or False?** **In Exercises 95–98, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.**

**95.** (a) Linear transformations called reflections that map a point in the $xy$-plane to its mirror image across the line $y = x$ are defined by the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(b) Linear transformations called horizontal expansions or contractions are defined by the matrix $\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$.

(c) The matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & -\sqrt{3}/2 \\ 0 & \sqrt{3}/2 & 1/2 \end{bmatrix}$ would rotate a point 60° about the $x$-axis.

**96.** (a) Linear transformations called reflections that map a point in the $xy$-plane to its mirror image across the $x$-axis are defined by the matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

(b) Linear transformations called vertical expansions or contractions are defined by the matrix $\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$.

(c) The matrix $\begin{bmatrix} \sqrt{3}/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ -1/2 & 0 & \sqrt{3}/2 \end{bmatrix}$ would rotate a point 30° about the $y$-axis.

**97.** (a) In calculus, any linear function is also a linear transformation from $R^2$ to $R^2$.

(b) A linear transformation is said to be onto if and only if, for all $\mathbf{u}$ and $\mathbf{v}$ in $V$, $T(\mathbf{u}) = T(\mathbf{v})$ implies $\mathbf{u} = \mathbf{v}$.

(c) Because of the computational advantages, it is best to choose a basis for $V$ such that the transformation matrix is diagonal.

**98.** (a) For polynomials, the differential operator $D_x$ is a linear transformation from $P_n$ into $P_{n-1}$.

(b) The set of all vectors $\mathbf{v}$ in $V$ that satisfy $T(\mathbf{v}) = \mathbf{v}$ is called the kernel of $T$.

(c) The standard matrix $A$ of the composition of two linear transformations $T(\mathbf{v}) = T_2(T_1(\mathbf{v}))$ is the product of the standard matrix for $T_2$ and the standard matrix for $T_1$.

# 6 Projects

Let $\ell$ be the line $ax + by = 0$ in $R^2$. The linear transformation $L: R^2 \rightarrow R^2$ that maps a point $(x, y)$ to its mirror image with respect to $\ell$ is called the **reflection** in $\ell$. (See Figure 6.22.) The goal of these two projects is to find the matrix for this reflection relative to the standard basis.

## 1 Reflections in $R^2$ (I)

In this project, you will use transition matrices to determine the standard matrix for the reflection $L$ in the line $ax + by = 0$.

1. Find the standard matrix for $L$ for the line $x = 0$.
2. Find the standard matrix for $L$ for the line $y = 0$.
3. Find the standard matrix for $L$ for the line $x - y = 0$.
4. Consider the line $\ell$ represented by $x - 2y = 0$. Find a vector $\mathbf{v}$ parallel to $\ell$ and another vector $\mathbf{w}$ orthogonal to $\ell$. Determine the matrix $A$ for the reflection in $\ell$ relative to the ordered basis $\{\mathbf{v}, \mathbf{w}\}$. Finally, use the appropriate transition matrix to find the matrix for the reflection relative to the standard basis. Use this matrix to find the images of the points $(2, 1)$, $(-1, 2)$, and $(5, 0)$.
5. Consider the general line $\ell$ represented by $ax + by = 0$. Find a vector $\mathbf{v}$ parallel to $\ell$ and another vector $\mathbf{w}$ orthogonal to $\ell$. Determine the matrix $A$ for the reflection in $\ell$ relative to the ordered basis $\{\mathbf{v}, \mathbf{w}\}$. Finally, use the appropriate transition matrix to find the matrix for the reflection relative to the standard basis.
6. Find the standard matrix for the reflection in the line $3x + 4y = 0$. Use this matrix to find the images of the points $(3, 4)$, $(-4, 3)$, and $(0, 5)$.

## 2 Reflections in $R^2$ (II)

In this project, you will use projections to determine the standard matrix for the reflection $L$ in the line $ax + by = 0$. (See Figure 6.23.) Recall that the projection of the vector $\mathbf{u}$ onto the vector $\mathbf{v}$ is given by

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\mathbf{v}.$$

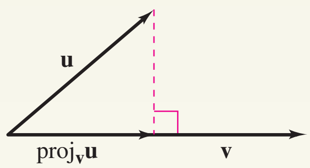



**Figure 6.23**

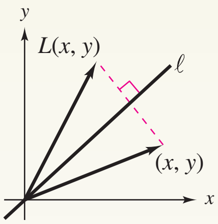



**Figure 6.22**

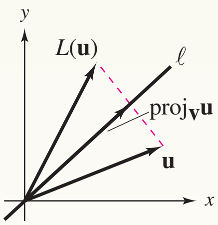



**Figure 6.24**

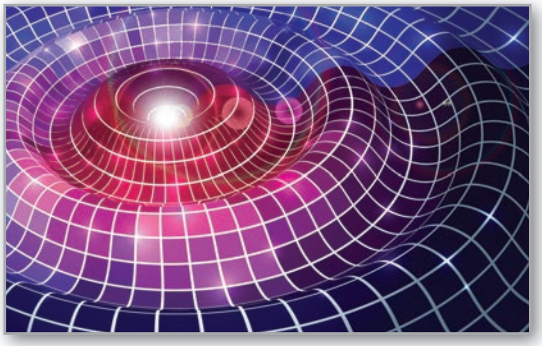
1. Find the standard matrix for the projection onto the $y$-axis. That is, find the standard matrix for $\text{proj}_{\mathbf{v}}\mathbf{u}$ when $\mathbf{v} = (0, 1)$.
2. Find the standard matrix for the projection onto the $x$-axis.
3. Consider the line $\ell$ represented by $x - 2y = 0$. Find a vector $\mathbf{v}$ parallel to $\ell$ and another vector $\mathbf{w}$ orthogonal to $\ell$. Determine the matrix $A$ for the projection onto $\ell$ relative to the ordered basis $\{\mathbf{v}, \mathbf{w}\}$. Finally, use the appropriate transition matrix to find the matrix for the projection relative to the standard basis. Use this matrix to find $\text{proj}_{\mathbf{v}}\mathbf{u}$ for $\mathbf{u} = (2, 1)$, $\mathbf{u} = (-1, 2)$, and $\mathbf{u} = (5, 0)$.
4. Consider the general line $\ell$ represented by $ax + by = 0$. Find a vector $\mathbf{v}$ parallel to $\ell$ and another vector $\mathbf{w}$ orthogonal to $\ell$. Determine the matrix $A$ for the projection onto $\ell$ relative to the ordered basis $\{\mathbf{v}, \mathbf{w}\}$. Finally, use the appropriate transition matrix to find the matrix for the projection relative to the standard basis.
5. Use Figure 6.24 to show that $\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{1}{2}(\mathbf{u} + L(\mathbf{u}))$, where $L$ is the reflection in the line $\ell$. Solve this equation for $L$ and compare your answer with the formula from the first project.

# 7 Eigenvalues and Eigenvectors



Population of Rabbits (p. 373)

Architecture (p. 382)



Relative Maxima and Minima (p. 369)

Genetics (p. 359)

Diffusion (p. 348)

# 7.1 Eigenvalues and Eigenvectors

- Verify eigenvalues and corresponding eigenvectors.
- Find eigenvalues and corresponding eigenspaces.
- Use the characteristic equation to find eigenvalues and eigenvectors, and find the eigenvalues and eigenvectors of a triangular matrix.
- Find the eigenvalues and eigenvectors of a linear transformation.

## THE EIGENVALUE PROBLEM

This section presents one of the most important problems in linear algebra, the **eigenvalue problem.** Its central question is as follows. When $A$ is an $n \times n$ matrix, do nonzero vectors $\mathbf{x}$ in $R^n$ exist such that $A\mathbf{x}$ is a scalar multiple of $\mathbf{x}$? The scalar, denoted by the Greek letter lambda $(\lambda)$, is called an **eigenvalue** of the matrix $A$, and the nonzero vector $\mathbf{x}$ is called an **eigenvector** of $A$ corresponding to $\lambda$. The origins of the terms *eigenvalue* and *eigenvector* are from the German word *Eigenwert,* meaning "proper value." So, you have

$$\overset{\boxed{\text{Eigenvalue}}}{\phantom{x}} \qquad A\mathbf{x} = \lambda\mathbf{x}. \qquad \underset{\boxed{\text{Eigenvector}}}{\phantom{x}}$$

Eigenvalues and eigenvectors have many important applications, many of which are discussed throughout this chapter. For now, you will consider a geometric interpretation of the problem in $R^2$. If $\lambda$ is an eigenvalue of a matrix $A$ and $\mathbf{x}$ is an eigenvector of $A$ corresponding to $\lambda$, then multiplication of $\mathbf{x}$ by the matrix $A$ produces a vector $\lambda\mathbf{x}$ that is parallel to $\mathbf{x}$, as shown in Figure 7.1.

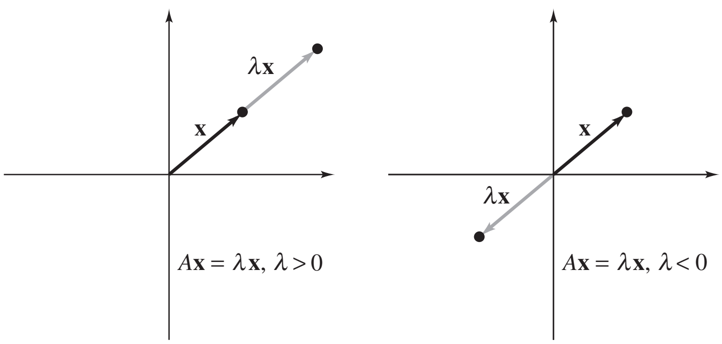



**Figure 7.1**

**REMARK**

Only real eigenvalues are presented in this chapter.

> **Definitions of Eigenvalue and Eigenvector**
>
> Let $A$ be an $n \times n$ matrix. The scalar $\lambda$ is called an **eigenvalue** of $A$ when there is a *nonzero* vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$. The vector $\mathbf{x}$ is called an **eigenvector** of $A$ corresponding to $\lambda$.

Note that an eigen*vector* cannot be zero. Allowing $\mathbf{x}$ to be the zero vector would render the definition meaningless, because $A\mathbf{0} = \lambda\mathbf{0}$ is true for all real values of $\lambda$. An eigen*value* of $\lambda = 0$, however, is possible. (See Example 2.)

A matrix can have more than one eigenvalue, as demonstrated in Examples 1 and 2.

## EXAMPLE 1 Verifying Eigenvalues and Eigenvectors

For the matrix

$$A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

verify that $\mathbf{x}_1 = (1, 0)$ is an eigenvector of $A$ corresponding to the eigenvalue $\lambda_1 = 2$, and that $\mathbf{x}_2 = (0, 1)$ is an eigenvector of $A$ corresponding to the eigenvalue $\lambda_2 = -1$.

### SOLUTION

Multiplying $\mathbf{x}_1$ on the left by $A$ produces

$$A\mathbf{x}_1 = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \underset{\text{Eigenvalue}}{2}\underset{\text{Eigenvector}}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}.$$

So, $\mathbf{x}_1 = (1, 0)$ is an eigenvector of $A$ corresponding to the eigenvalue $\lambda_1 = 2$. Similarly, multiplying $\mathbf{x}_2$ on the left by $A$ produces

$$A\mathbf{x}_2 = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} = -1\begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

So, $\mathbf{x}_2 = (0, 1)$ is an eigenvector of $A$ corresponding to the eigenvalue $\lambda_2 = -1$.

## EXAMPLE 2 Verifying Eigenvalues and Eigenvectors

For the matrix

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

verify that

$$\mathbf{x}_1 = (-3, -1, 1) \quad \text{and} \quad \mathbf{x}_2 = (1, 0, 0)$$

are eigenvectors of $A$ and find their corresponding eigenvalues.

### SOLUTION

Multiplying $\mathbf{x}_1$ on the left by $A$ produces

$$A\mathbf{x}_1 = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0\begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}.$$

So, $\mathbf{x}_1 = (-3, -1, 1)$ is an eigenvector of $A$ corresponding to the eigenvalue $\lambda_1 = 0$. Similarly, multiplying $\mathbf{x}_2$ on the left by $A$ produces

$$A\mathbf{x}_2 = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

So, $\mathbf{x}_2 = (1, 0, 0)$ is an eigenvector of $A$ corresponding to the eigenvalue $\lambda_2 = 1$.

### DISCOVERY

1. In Example 2, $\lambda_2 = 1$ is an eigenvalue of the matrix $A$. Calculate the determinant of the matrix $\lambda_2 I - A$, where $I$ is the $3 \times 3$ identity matrix.

2. Repeat this experiment for the other eigenvalue, $\lambda_1 = 0$.

3. In general, when $\lambda$ is an eigenvalue of the matrix $A$, what is the value of $|\lambda I - A|$?

## EIGENSPACES

Although Examples 1 and 2 list only one eigenvector for each eigenvalue, each of the four eigenvalues in Examples 1 and 2 has infinitely many eigenvectors. For instance, in Example 1, the vectors $(2, 0)$ and $(-3, 0)$ are eigenvectors of $A$ corresponding to the eigenvalue 2. In fact, if $A$ is an $n \times n$ matrix with an eigenvalue $\lambda$ and a corresponding eigenvector $\mathbf{x}$, then every nonzero scalar multiple of $\mathbf{x}$ is also an eigenvector of $A$. To see this, let $c$ be a nonzero scalar, which then produces

$$A(c\mathbf{x}) = c(A\mathbf{x}) = c(\lambda\mathbf{x}) = \lambda(c\mathbf{x}).$$

It is also true that if $\mathbf{x}_1$ and $\mathbf{x}_2$ are eigenvectors corresponding to the *same* eigenvalue $\lambda$, then their sum is also an eigenvector corresponding to $\lambda$, because

$$A(\mathbf{x}_1 + \mathbf{x}_2) = A\mathbf{x}_1 + A\mathbf{x}_2 = \lambda\mathbf{x}_1 + \lambda\mathbf{x}_2 = \lambda(\mathbf{x}_1 + \mathbf{x}_2).$$

In other words, the set of all eigenvectors of a given eigenvalue $\lambda$, together with the zero vector, is a subspace of $R^n$. This special subspace of $R^n$ is called the **eigenspace** of $\lambda$.

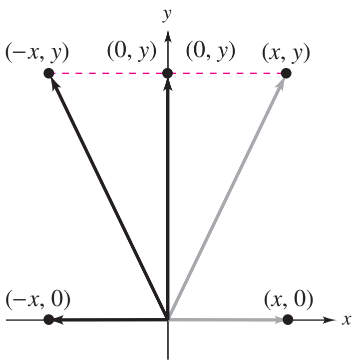
### THEOREM 7.1 Eigenvectors of $\lambda$ Form a Subspace

If $A$ is an $n \times n$ matrix with an eigenvalue $\lambda$, then the set of all eigenvectors of $\lambda$, together with the zero vector

$$\{\mathbf{x}: \mathbf{x} \text{ is an eigenvector of } \lambda\} \cup \{\mathbf{0}\}$$

is a subspace of $R^n$. This subspace is called the **eigenspace** of $\lambda$.

Determining the eigenvalues and corresponding eigenspaces of a matrix can be difficult. Occasionally, however, you can find eigenvalues and eigenspaces by inspection, as demonstrated in Example 3.




*A* reflects vectors in the *y*-axis.

**Figure 7.2**

### EXAMPLE 3 Finding Eigenspaces in $R^2$ Geometrically

Find the eigenvalues and corresponding eigenspaces of $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.

#### SOLUTION

Geometrically, multiplying a vector $(x, y)$ in $R^2$ by the matrix $A$ corresponds to a reflection in the $y$-axis. That is, if $\mathbf{v} = (x, y)$, then

$$A\mathbf{v} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}.$$

Figure 7.2 illustrates that the only vectors reflected onto scalar multiples of themselves are those lying on either the $x$-axis or the $y$-axis.

For a vector on the $x$-axis

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} -x \\ 0 \end{bmatrix} = -1\begin{bmatrix} x \\ 0 \end{bmatrix}$$

Eigenvalue is $\lambda_1 = -1$.

For a vector on the $y$-axis

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix} = 1\begin{bmatrix} 0 \\ y \end{bmatrix}$$

Eigenvalue is $\lambda_2 = 1$.

So, the eigenvectors corresponding to $\lambda_1 = -1$ are the nonzero vectors on the $x$-axis, and the eigenvectors corresponding to $\lambda_2 = 1$ are the nonzero vectors on the $y$-axis. This implies that the eigenspace corresponding to $\lambda_1 = -1$ is the $x$-axis, and that the eigenspace corresponding to $\lambda_2 = 1$ is the $y$-axis.

**REMARK**

The geometric solution in Example 3 is not typical of the general eigenvalue problem. A more general approach follows.

## FINDING EIGENVALUES AND EIGENVECTORS

To find the eigenvalues and eigenvectors of an $n \times n$ matrix $A$, let $I$ be the $n \times n$ identity matrix. Writing the equation $A\mathbf{x} = \lambda\mathbf{x}$ in the form $\lambda I\mathbf{x} = A\mathbf{x}$ then produces $(\lambda I - A)\mathbf{x} = \mathbf{0}$. This homogeneous system of equations has nonzero solutions if and only if the coefficient matrix $(\lambda I - A)$ is *not* invertible—that is, if and only if its determinant is zero. The next theorem formally states this.

**REMARK**

Because the characteristic polynomial of $A$ is of degree $n$, $A$ can have at most $n$ distinct eigenvalues. The Fundamental Theorem of Algebra states that an $n$th-degree polynomial has precisely $n$ roots. These $n$ roots, however, include both repeated and complex roots. In this chapter, you will find only the real roots of characteristic polynomials—that is, real eigenvalues.

### THEOREM 7.2 Eigenvalues and Eigenvectors of a Matrix

Let $A$ be an $n \times n$ matrix.

1. An eigenvalue of $A$ is a scalar $\lambda$ such that $\det(\lambda I - A) = 0$.
2. The eigenvectors of $A$ corresponding to $\lambda$ are the nonzero solutions of $(\lambda I - A)\mathbf{x} = \mathbf{0}$.

The equation $\det(\lambda I - A) = 0$ is called the **characteristic equation** of $A$. Moreover, when expanded to polynomial form, the polynomial

$$|\lambda I - A| = \lambda^n + c_{n-1}\lambda^{n-1} + \cdot \cdot \cdot + c_1\lambda + c_0$$

is called the **characteristic polynomial** of $A$. This definition tells you that the eigenvalues of an $n \times n$ matrix $A$ correspond to the roots of the characteristic polynomial of $A$.

### EXAMPLE 4 Finding Eigenvalues and Eigenvectors

Find the eigenvalues and corresponding eigenvectors of $A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}$.

**SOLUTION**

The characteristic polynomial of $A$ is

$$|\lambda I - A| = \begin{vmatrix} \lambda - 2 & 12 \\ -1 & \lambda + 5 \end{vmatrix} = \lambda^2 + 3\lambda - 10 + 12 = (\lambda + 1)(\lambda + 2).$$

So, the characteristic equation is $(\lambda + 1)(\lambda + 2) = 0$, which gives $\lambda_1 = -1$ and $\lambda_2 = -2$ as the eigenvalues of $A$. To find the corresponding eigenvectors, use Gauss-Jordan elimination to solve the homogeneous linear system represented by $(\lambda I - A)\mathbf{x} = \mathbf{0}$ twice: first for $\lambda = \lambda_1 = -1$, and then for $\lambda = \lambda_2 = -2$. For $\lambda_1 = -1$, the coefficient matrix is

$$(-1)I - A = \begin{bmatrix} -1 - 2 & 12 \\ -1 & -1 + 5 \end{bmatrix} = \begin{bmatrix} -3 & 12 \\ -1 & 4 \end{bmatrix}$$

which row reduces to $\begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix}$, showing that $x_1 - 4x_2 = 0$. Letting $x_2 = t$, you can conclude that every eigenvector of $\lambda_1$ is of the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4t \\ t \end{bmatrix} = t\begin{bmatrix} 4 \\ 1 \end{bmatrix}, \quad t \neq 0.$$

For $\lambda_2 = -2$, you have

$$(-2)I - A = \begin{bmatrix} -2 - 2 & 12 \\ -1 & -2 + 5 \end{bmatrix} = \begin{bmatrix} -4 & 12 \\ -1 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}.$$

**REMARK**

Try checking $A\mathbf{x} = \lambda_i\mathbf{x}$ for the eigenvalues and eigenvectors in this example.

Letting $x_2 = t$, you can conclude that every eigenvector of $\lambda_2$ is of the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3t \\ t \end{bmatrix} = t\begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad t \neq 0.$$

The homogeneous systems that arise when you are finding eigenvectors will always row reduce to a matrix having at least one row of zeros, because the systems must have nontrivial solutions. The following summarizes the steps used to find the eigenvalues and corresponding eigenvectors of a matrix.

### Finding Eigenvalues and Eigenvectors

Let $A$ be an $n \times n$ matrix.

1. Form the characteristic equation $|\lambda I - A| = 0$. It will be a polynomial equation of degree $n$ in the variable $\lambda$.
2. Find the real roots of the characteristic equation. These are the eigenvalues of $A$.
3. For each eigenvalue $\lambda_i$, find the eigenvectors corresponding to $\lambda_i$ by solving the homogeneous system $(\lambda_i I - A)\mathbf{x} = \mathbf{0}$. This requires row reducing an $n \times n$ matrix. The resulting reduced row-echelon form must have at least one row of zeros.

Finding the eigenvalues of an $n \times n$ matrix involves the factorization of an $n$th-degree polynomial. Once you have found an eigenvalue, finding the corresponding eigenvectors involves an application of Gauss-Jordan elimination.

## EXAMPLE 5 Finding Eigenvalues and Eigenvectors

Find the eigenvalues and corresponding eigenvectors of

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

What is the dimension of the eigenspace of each eigenvalue?

**SOLUTION**

The characteristic polynomial of $A$ is

$$\begin{aligned} |\lambda I - A| &= \begin{vmatrix} \lambda - 2 & -1 & 0 \\ 0 & \lambda - 2 & 0 \\ 0 & 0 & \lambda - 2 \end{vmatrix} \\ &= (\lambda - 2)^3. \end{aligned}$$

So, the characteristic equation is $(\lambda - 2)^3 = 0$, and the only eigenvalue is $\lambda = 2$. To find the eigenvectors of $\lambda = 2$, solve the homogeneous linear system represented by $(2I - A)\mathbf{x} = \mathbf{0}$.

$$2I - A = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This implies that $x_2 = 0$. Using the parameters $s = x_1$ and $t = x_3$, you can conclude that the eigenvectors of $\lambda = 2$ are of the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s \\ 0 \\ t \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad s \text{ and } t \text{ not both zero.}$$

Because $\lambda = 2$ has two linearly independent eigenvectors, the dimension of its eigenspace is 2.

If an eigenvalue $\lambda_i$ occurs as a *multiple root* ($k$ times) of the characteristic polynomial, then $\lambda_i$ has **multiplicity** $k$. This implies that $(\lambda - \lambda_i)^k$ is a factor of the characteristic polynomial and $(\lambda - \lambda_i)^{k+1}$ is not a factor of the characteristic polynomial. For instance, in Example 5, the eigenvalue $\lambda = 2$ has a multiplicity of 3.

Also note that in Example 5, the dimension of the eigenspace of $\lambda = 2$ is 2. In general, the multiplicity of an eigenvalue is greater than or equal to the dimension of its eigenspace. (In Exercise 61, you are asked to prove this.)

## EXAMPLE 6 Finding Eigenvalues and Eigenvectors

Find the eigenvalues of

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & -10 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}$$

and find a basis for each of the corresponding eigenspaces.

### SOLUTION

The characteristic polynomial of $A$ is

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & 0 & 0 & 0 \\ 0 & \lambda - 1 & -5 & 10 \\ -1 & 0 & \lambda - 2 & 0 \\ -1 & 0 & 0 & \lambda - 3 \end{vmatrix}$$
$$= (\lambda - 1)^2(\lambda - 2)(\lambda - 3).$$

So, the characteristic equation is $(\lambda - 1)^2(\lambda - 2)(\lambda - 3) = 0$ and the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = 3$. (Note that $\lambda_1 = 1$ has a multiplicity of 2.)

You can find a basis for the eigenspace of $\lambda_1 = 1$ as follows.

$$(1)I - A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & 10 \\ -1 & 0 & -1 & 0 \\ -1 & 0 & 0 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Letting $s = x_2$ and $t = x_4$ produces

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0s - 2t \\ s + 0t \\ 0s + 2t \\ 0s + t \end{bmatrix} = s\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \end{bmatrix}.$$

A basis for the eigenspace corresponding to $\lambda_1 = 1$ is

$B_1 = \{(0, 1, 0, 0), (-2, 0, 2, 1)\}$. Basis for $\lambda_1 = 1$

For $\lambda_2 = 2$ and $\lambda_3 = 3$, use the same procedure to obtain the eigenspace bases

$B_2 = \{(0, 5, 1, 0)\}$ Basis for $\lambda_2 = 2$

$B_3 = \{(0, -5, 0, 1)\}$. Basis for $\lambda_3 = 3$

### TECHNOLOGY

Many graphing utilities and software programs can find the eigenvalues and eigenvectors of an $n \times n$ matrix. Try using a graphing utility or a software program to find the eigenvalues and eigenvectors in Example 6. When finding the eigenvectors, your graphing utility or software program may produce a matrix in which the columns are scalar multiples of the eigenvectors you would obtain by hand calculations. The **Online Technology Guide,** available at *www.cengagebrain.com,* provides syntax for programs applicable to Example 6.

Finding eigenvalues and eigenvectors of matrices of order $n \geq 4$ can be tedious. Moreover, using the procedure followed in Example 6 on a computer can introduce roundoff errors. Consequently, it can be more efficient to use numerical methods of approximating the eigenvalues of large matrices. These numerical methods appear in texts on advanced linear algebra and numerical analysis.

The eigenvalues of some matrices are relatively easy to find. The next theorem states that the eigenvalues of an $n \times n$ triangular matrix are the entries on the main diagonal. The proof of this theorem follows from the fact that the determinant of a triangular matrix is the product of its main diagonal entries.

**THEOREM 7.3 Eigenvalues of Triangular Matrices**

If $A$ is an $n \times n$ triangular matrix, then its eigenvalues are the entries on its main diagonal.

**EXAMPLE 7 Finding Eigenvalues of Triangular and Diagonal Matrices**

Find the eigenvalues of each matrix.

**a.** $A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ 5 & 3 & -3 \end{bmatrix}$

**b.** $A = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$

**SOLUTION**

**a.** Without using Theorem 7.3, you can find that

$$|\lambda I - A| = \begin{vmatrix} \lambda - 2 & 0 & 0 \\ 1 & \lambda - 1 & 0 \\ -5 & -3 & \lambda + 3 \end{vmatrix} = (\lambda - 2)(\lambda - 1)(\lambda + 3).$$

So, the eigenvalues are $\lambda_1 = 2$, $\lambda_2 = 1$, and $\lambda_3 = -3$, which are the main diagonal entries of $A$.

**b.** In this case, use Theorem 7.3 to conclude that the eigenvalues are the main diagonal entries $\lambda_1 = -1$, $\lambda_2 = 2$, $\lambda_3 = 0$, $\lambda_4 = -4$, and $\lambda_5 = 3$.

**LINEAR ALGEBRA APPLIED**

Eigenvalues and eigenvectors are useful for modeling real-life phenomena. For instance, suppose that in an experiment to determine the diffusion of a fluid from one flask to another through a permeable membrane and then out of the second flask, researchers determine that the flow rate between flasks is twice the volume of fluid in the first flask and the flow rate out of the second flask is three times the volume of fluid in the second flask. The following system of linear differential equations, where $y_i$ represents the volume of fluid in flask $i$, models this situation.

$$\begin{aligned} y_1' &= -2y_1 \\ y_2' &= \phantom{-}2y_1 - 3y_2 \end{aligned}$$

In Section 7.4, you will use eigenvalues and eigenvectors to solve such systems of linear differential equations. For now, verify that the solution of this system is

$$\begin{aligned} y_1 &= C_1 e^{-2t} \\ y_2 &= 2C_1 e^{-2t} + C_2 e^{-3t}. \end{aligned}$$

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## EIGENVALUES AND EIGENVECTORS OF LINEAR TRANSFORMATIONS

This section began with definitions of eigenvalues and eigenvectors in terms of matrices. Eigenvalues and eigenvectors can also be defined in terms of linear transformations. A number $\lambda$ is called an **eigenvalue** of a linear transformation $T: V \rightarrow V$ when there is a nonzero vector $\mathbf{x}$ such that $T(\mathbf{x}) = \lambda\mathbf{x}$. The vector $\mathbf{x}$ is called an **eigenvector** of $T$ corresponding to $\lambda$, and the set of all eigenvectors of $\lambda$ (with the zero vector) is called the **eigenspace** of $\lambda$.

Consider $T: R^3 \rightarrow R^3$, whose matrix relative to the standard basis is

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

Standard basis:
$B = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$

In Example 5 in Section 6.4, you found that the matrix of $T$ relative to the basis $B' = \{(1, 1, 0), (1, -1, 0), (0, 0, 1)\}$ is the diagonal matrix

$$A' = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

Nonstandard basis:
$B' = \{(1, 1, 0), (1, -1, 0), (0, 0, 1)\}$

For a given linear transformation $T$, can you find a basis $B'$ whose corresponding matrix is diagonal? The next example gives an indication of the answer.

### EXAMPLE 8 Finding Eigenvalues and Eigenspaces

Find the eigenvalues and a basis for each corresponding eigenspace of

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

**SOLUTION**

Because

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & -3 & 0 \\ -3 & \lambda - 1 & 0 \\ 0 & 0 & \lambda + 2 \end{vmatrix} = [(\lambda - 1)^2 - 9](\lambda + 2) = (\lambda - 4)(\lambda + 2)^2$$

the eigenvalues of $A$ are $\lambda_1 = 4$ and $\lambda_2 = -2$. Bases for the eigenspaces are $B_1 = \{(1, 1, 0)\}$ and $B_2 = \{(1, -1, 0), (0, 0, 1)\}$, respectively (verify these).

Example 8 illustrates two results. If $T: R^3 \rightarrow R^3$ is the linear transformation whose standard matrix is $A$, and $B'$ is a basis for $R^3$ made up of the three linearly independent eigenvectors corresponding to the eigenvalues of $A$, then the results are as follows.

**1.** The matrix $A'$ for $T$ relative to the basis $B'$ is diagonal.

$$A' = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Nonstandard basis:
$B' = \{(1, 1, 0), (1, -1, 0), (0, 0, 1)\}$

Eigenvalues of $A$

Eigenvectors of $A$

**2.** The main diagonal entries of the matrix $A'$ are the eigenvalues of $A$.

The next section formalizes these two results and also characterizes linear transformations that can be represented by diagonal matrices.

# 7.1 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

**Verifying Eigenvalues and Eigenvectors** **In Exercises 1–8, verify that $\lambda_i$ is an eigenvalue of $A$ and that $\mathbf{x}_i$ is a corresponding eigenvector.**

1. $A = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$, $\lambda_1 = 2, \mathbf{x}_1 = (1, 0)$; $\lambda_2 = -2, \mathbf{x}_2 = (0, 1)$

2. $A = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}$, $\lambda_1 = -1, \mathbf{x}_1 = (1, 1)$; $\lambda_2 = 2, \mathbf{x}_2 = (5, 2)$

3. $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $\lambda_1 = 0, \mathbf{x}_1 = (1, -1)$; $\lambda_2 = 2, \mathbf{x}_2 = (1, 1)$

4. $A = \begin{bmatrix} -2 & 4 \\ 1 & 1 \end{bmatrix}$, $\lambda_1 = 2, \mathbf{x}_1 = (1, 1)$; $\lambda_2 = -3, \mathbf{x}_2 = (-4, 1)$

5. $A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$, $\lambda_1 = 2, \mathbf{x}_1 = (1, 0, 0)$; $\lambda_2 = -1, \mathbf{x}_2 = (1, -1, 0)$; $\lambda_3 = 3, \mathbf{x}_3 = (5, 1, 2)$

6. $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$, $\lambda_1 = 5, \mathbf{x}_1 = (1, 2, -1)$; $\lambda_2 = -3, \mathbf{x}_2 = (-2, 1, 0)$; $\lambda_3 = -3, \mathbf{x}_3 = (3, 0, 1)$

7. $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, $\lambda_1 = 1, \mathbf{x}_1 = (1, 1, 1)$

8. $A = \begin{bmatrix} 4 & -1 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$, $\lambda_1 = 4, \mathbf{x}_1 = (1, 0, 0)$; $\lambda_2 = 2, \mathbf{x}_2 = (1, 2, 0)$; $\lambda_3 = 3, \mathbf{x}_3 = (-2, 1, 1)$

9. Use $A$, $\lambda_i$, and $\mathbf{x}_i$ from Exercise 3 to show that
   (a) $A(c\mathbf{x}_1) = 0(c\mathbf{x}_1)$ for any real number $c$.
   (b) $A(c\mathbf{x}_2) = 2(c\mathbf{x}_2)$ for any real number $c$.

10. Use $A$, $\lambda_i$, and $\mathbf{x}_i$ from Exercise 5 to show that
   (a) $A(c\mathbf{x}_1) = 2(c\mathbf{x}_1)$ for any real number $c$.
   (b) $A(c\mathbf{x}_2) = -(c\mathbf{x}_2)$ for any real number $c$.
   (c) $A(c\mathbf{x}_3) = 3(c\mathbf{x}_3)$ for any real number $c$.

**Determining Eigenvectors** **In Exercises 11–14, determine whether x is an eigenvector of $A$.**

11. $A = \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix}$
   (a) $\mathbf{x} = (1, 2)$
   (b) $\mathbf{x} = (2, 1)$
   (c) $\mathbf{x} = (1, -2)$
   (d) $\mathbf{x} = (-1, 0)$

12. $A = \begin{bmatrix} -3 & 10 \\ 5 & 2 \end{bmatrix}$
   (a) $\mathbf{x} = (4, 4)$
   (b) $\mathbf{x} = (-8, 4)$
   (c) $\mathbf{x} = (-4, 8)$
   (d) $\mathbf{x} = (5, -3)$

13. $A = \begin{bmatrix} -1 & -1 & 1 \\ -2 & 0 & -2 \\ 3 & -3 & 1 \end{bmatrix}$
   (a) $\mathbf{x} = (2, -4, 6)$
   (b) $\mathbf{x} = (2, 0, 6)$
   (c) $\mathbf{x} = (2, 2, 0)$
   (d) $\mathbf{x} = (-1, 0, 1)$

14. $A = \begin{bmatrix} 1 & 0 & 5 \\ 0 & -2 & 4 \\ 1 & -2 & 9 \end{bmatrix}$
   (a) $\mathbf{x} = (1, 1, 0)$
   (b) $\mathbf{x} = (-5, 2, 1)$
   (c) $\mathbf{x} = (0, 0, 0)$
   (d) $\mathbf{x} = \left(2\sqrt{6} - 3, -2\sqrt{6} + 6, 3\right)$

**Finding Eigenspaces in $R^2$ Geometrically** **In Exercises 15 and 16, use the method shown in Example 3 to find the eigenvalue(s) and corresponding eigenspace(s) of $A$.**

15. $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

16. $A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$

**Characteristic Equation, Eigenvalues, and Eigenvectors** **In Exercises 17–28, find (a) the characteristic equation and (b) the eigenvalues (and corresponding eigenvectors) of the matrix.**

17. $\begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}$

18. $\begin{bmatrix} 1 & -4 \\ -2 & 8 \end{bmatrix}$

19. $\begin{bmatrix} 1 & -\frac{3}{2} \\ \frac{1}{2} & -1 \end{bmatrix}$

20. $\begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 \end{bmatrix}$

21. $\begin{bmatrix} 2 & -2 & 3 \\ 0 & 3 & -2 \\ 0 & -1 & 2 \end{bmatrix}$

22. $\begin{bmatrix} 3 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}$

23. $\begin{bmatrix} 1 & 2 & -2 \\ -2 & 5 & -2 \\ -6 & 6 & -3 \end{bmatrix}$

24. $\begin{bmatrix} 3 & 2 & -3 \\ -3 & -4 & 9 \\ -1 & -2 & 5 \end{bmatrix}$

25. $\begin{bmatrix} 0 & -3 & 5 \\ -4 & 4 & -10 \\ 0 & 0 & 4 \end{bmatrix}$

26. $\begin{bmatrix} 1 & -\frac{3}{2} & \frac{5}{2} \\ -2 & \frac{13}{2} & -10 \\ \frac{3}{2} & -\frac{9}{2} & 8 \end{bmatrix}$

27. $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 4 & 0 \end{bmatrix}$

28. $\begin{bmatrix} 3 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

**Finding Eigenvalues** **In Exercises 29–38, use a software program or a graphing utility with matrix capabilities to find the eigenvalues of the matrix.**

29. $\begin{bmatrix} -4 & 5 \\ -2 & 3 \end{bmatrix}$

30. $\begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$

31. $\begin{bmatrix} 0 & -\frac{1}{2} & 5 \\ -\frac{1}{3} & -\frac{1}{6} & -\frac{1}{4} \\ 0 & 0 & 4 \end{bmatrix}$

32. $\begin{bmatrix} \frac{1}{2} & 0 & 5 \\ -2 & \frac{1}{5} & \frac{1}{4} \\ 0 & 0 & 3 \end{bmatrix}$

33. $\begin{bmatrix} 2 & 4 & 2 \\ 1 & 0 & 1 \\ 1 & -4 & 5 \end{bmatrix}$

34. $\begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix}$

35. $\begin{bmatrix} 1 & 1 & 2 & 3 \\ 2 & 2 & 4 & 6 \\ 3 & 3 & 6 & 9 \\ 4 & 4 & 8 & 12 \end{bmatrix}$

36. $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 4 & 4 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 2 \end{bmatrix}$

37. $\begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 \\ -2 & 0 & 2 & -2 \\ 0 & 2 & 0 & 2 \end{bmatrix}$

38. $\begin{bmatrix} 1 & -3 & 3 & 3 \\ -1 & 4 & -3 & -3 \\ -2 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

**Eigenvalues of Triangular and Diagonal Matrices** **In Exercises 39–42, find the eigenvalues of the triangular or diagonal matrix.**

39. $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

40. $\begin{bmatrix} -5 & 0 & 0 \\ 3 & 7 & 0 \\ 4 & -2 & 3 \end{bmatrix}$

41. $\begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$

42. $\begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{5}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{3}{4} \end{bmatrix}$

**Eigenvalues and Eigenvectors of Linear Transformations** **In Exercises 43–46, consider the linear transformation $T: R^n \to R^n$ whose matrix $A$ relative to the standard basis is given. Find (a) the eigenvalues of $A$, (b) a basis for each of the corresponding eigenspaces, and (c) the matrix $A'$ for $T$ relative to the basis $B'$, where $B'$ is made up of the basis vectors found in part (b).**

43. $A = \begin{bmatrix} 2 & -2 \\ 1 & 5 \end{bmatrix}$

44. $A = \begin{bmatrix} -6 & 2 \\ 3 & -1 \end{bmatrix}$

45. $A = \begin{bmatrix} 0 & 2 & -1 \\ -1 & 3 & 1 \\ 0 & 0 & -1 \end{bmatrix}$

46. $A = \begin{bmatrix} 3 & 1 & 4 \\ 2 & 4 & 0 \\ 5 & 5 & 6 \end{bmatrix}$

**Cayley-Hamilton Theorem** **In Exercises 47–50, demonstrate the Cayley-Hamilton Theorem for the given matrix. The Cayley-Hamilton Theorem states that a matrix satisfies its characteristic equation. For example, the characteristic equation of**

$$A = \begin{bmatrix} 1 & -3 \\ 2 & 5 \end{bmatrix}$$

**is $\lambda^2 - 6\lambda + 11 = 0$, and by the theorem you have $A^2 - 6A + 11I_2 = O$.**

47. $\begin{bmatrix} 4 & 0 \\ -3 & 2 \end{bmatrix}$

48. $\begin{bmatrix} 6 & -1 \\ 1 & 5 \end{bmatrix}$

49. $\begin{bmatrix} 1 & 0 & -4 \\ 0 & 3 & 1 \\ 2 & 0 & 1 \end{bmatrix}$

50. $\begin{bmatrix} -3 & 1 & 0 \\ -1 & 3 & 2 \\ 0 & 4 & 3 \end{bmatrix}$

**51.** Perform the following computational checks on the eigenvalues found in Exercises 17–27 odd.

(a) The sum of the $n$ eigenvalues equals the trace of the matrix. (Recall that the **trace** of a matrix is the sum of the main diagonal entries of the matrix.)

(b) The product of the $n$ eigenvalues equals $|A|$.

(When $\lambda$ is an eigenvalue of multiplicity $k$, remember to enter it $k$ times in the sum or product of these checks.)

**52.** Perform the following computational checks on the eigenvalues found in Exercises 18–28 even.

(a) The sum of the $n$ eigenvalues equals the trace of the matrix. (Recall that the **trace** of a matrix is the sum of the main diagonal entries of the matrix.)

(b) The product of the $n$ eigenvalues equals $|A|$.

(When $\lambda$ is an eigenvalue of multiplicity $k$, remember to enter it $k$ times in the sum or product of these checks.)

**53.** Show that if $A$ is an $n \times n$ matrix whose $i$th row is identical to the $i$th row of $I$, then 1 is an eigenvalue of $A$.

**54.** **Proof** Prove that $\lambda = 0$ is an eigenvalue of $A$ if and only if $A$ is singular.

**55.** **Proof** For an invertible matrix $A$, prove that $A$ and $A^{-1}$ have the same eigenvectors. How are the eigenvalues of $A$ related to the eigenvalues of $A^{-1}$?

**56.** **Proof** Prove that $A$ and $A^T$ have the same eigenvalues. Are the eigenspaces the same?

**57.** **Proof** Prove that the constant term of the characteristic polynomial is $\pm|A|$.

**58.** Define $T: R^2 \to R^2$ by $T(\mathbf{v}) = \text{proj}_{\mathbf{u}}\mathbf{v}$, where $\mathbf{u}$ is a fixed vector in $R^2$. Show that the eigenvalues of $A$ (the standard matrix of $T$) are 0 and 1.

**59.** **Guided Proof** Prove that a triangular matrix is nonsingular if and only if its eigenvalues are real and nonzero.

Getting Started: Because this is an "if and only if" statement, you must prove that the statement is true in both directions. Review Theorems 3.2 and 3.7.

(i) To prove the statement in one direction, assume that the triangular matrix $A$ is nonsingular. Use your knowledge of nonsingular and triangular matrices and determinants to conclude that the entries on the main diagonal of $A$ are nonzero.

(ii) Because $A$ is triangular, use Theorem 7.3 and part (i) to conclude that the eigenvalues are real and nonzero.

(iii) To prove the statement in the other direction, assume that the eigenvalues of the triangular matrix $A$ are real and nonzero. Repeat parts (i) and (ii) in reverse order to prove that $A$ is nonsingular.

**60.** **Guided Proof** Prove that if $A^2 = O$, then 0 is the only eigenvalue of $A$.

Getting Started: You need to show that if there exists a nonzero vector $\mathbf{x}$ and a real number $\lambda$ such that $A\mathbf{x} = \lambda\mathbf{x}$, then if $A^2 = O$, $\lambda$ must be zero.

(i) Because $A^2 = A \cdot A$, you can write $A^2\mathbf{x}$ as $A(A\mathbf{x})$.

(ii) Use the fact that $A\mathbf{x} = \lambda\mathbf{x}$ and the properties of matrix multiplication to conclude that $A^2\mathbf{x} = \lambda^2\mathbf{x}$.

(iii) Because $A^2$ is a zero matrix, you can conclude that $\lambda$ must be zero.

**61.** **Proof** Prove that the multiplicity of an eigenvalue is greater than or equal to the dimension of its eigenspace.

**62.** **CAPSTONE** An $n \times n$ matrix $A$ has the characteristic equation

$$|\lambda I - A| = (\lambda + 2)(\lambda - 1)(\lambda - 3)^2 = 0.$$

(a) What are the eigenvalues of $A$?

(b) What is the order of $A$? Explain.

(c) Is $\lambda I - A$ singular? Explain.

(d) Is $A$ singular? Explain. (*Hint:* Use the result of Exercise 54.)

**63.** When the eigenvalues of

$$A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$$

are $\lambda_1 = 0$ and $\lambda_2 = 1$, what are the possible values of $a$ and $d$?

**64.** Show that

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

has no real eigenvalues.

**True or False?** **In Exercises 65 and 66, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.**

**65.** (a) The scalar $\lambda$ is an eigenvalue of an $n \times n$ matrix $A$ when there exists a vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$.

(b) To find the eigenvalue(s) of an $n \times n$ matrix $A$, you can solve the characteristic equation $\det(\lambda I - A) = 0$.

**66.** (a) Geometrically, if $\lambda$ is an eigenvalue of a matrix $A$ and $\mathbf{x}$ is an eigenvector of $A$ corresponding to $\lambda$, then multiplying $\mathbf{x}$ by $A$ produces a vector $\lambda\mathbf{x}$ parallel to $\mathbf{x}$.

(b) If $A$ is an $n \times n$ matrix with an eigenvalue $\lambda$, then the set of all eigenvectors of $\lambda$ is a subspace of $R^n$.

**Finding the Dimension of an Eigenspace** **In Exercises 67–70, find the dimension of the eigenspace corresponding to the eigenvalue $\lambda = 3$.**

**67.** $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ **68.** $A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

**69.** $A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$ **70.** $A = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

**71.** **Calculus** Let $T: C'[0, 1] \to C[0, 1]$ be given by

$T(f) = f'$.

Show that $\lambda = 1$ is an eigenvalue of $T$ with corresponding eigenvector $f(x) = e^x$.

**72.** **Calculus** For the linear transformation given in Exercise 71, find the eigenvalue corresponding to the eigenvector $f(x) = e^{-2x}$.

**73.** Define $T: P_2 \to P_2$ by

$$T(a_0 + a_1x + a_2x^2) = (-3a_1 + 5a_2) + (-4a_0 + 4a_1 - 10a_2)x + 4a_2x^2.$$

Find the eigenvalues and the eigenvectors of $T$ relative to the standard basis $\{1, x, x^2\}$.

**74.** Define $T: P_2 \to P_2$ by

$$T(a_0 + a_1x + a_2x^2) = (2a_0 + a_1 - a_2) + (-a_1 + 2a_2)x - a_2x.$$

Find the eigenvalues and eigenvectors of $T$ relative to the standard basis $\{1, x, x^2\}$.

**75.** Define $T: M_{2,2} \to M_{2,2}$ by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a - c + d & b + d \\ -2a + 2c - 2d & 2b + 2d \end{bmatrix}.$$

Find the eigenvalues and eigenvectors of $T$ relative to the standard basis

$$B = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}.$$

**76.** Find all values of the angle $\theta$ for which the matrix

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

has real eigenvalues. Interpret your answer geometrically.

**77.** What are the possible eigenvalues of an idempotent matrix? (Recall that a square matrix $A$ is **idempotent** when $A^2 = A$.)

**78.** What are the possible eigenvalues of a nilpotent matrix? (Recall that a square matrix $A$ is **nilpotent** when there exists a positive integer $k$ such that $A^k = 0$.)

**79.** **Proof** Let $A$ be an $n \times n$ matrix such that the sum of the entries in each row is a fixed constant $r$. Prove that $r$ is an eigenvalue of $A$. Illustrate this result with a specific example.

# 7.2 Diagonalization

- Find the eigenvalues of similar matrices, determine whether a matrix $A$ is diagonalizable, and find a matrix $P$ such that $P^{-1}AP$ is diagonal.
- Find, for a linear transformation $T: V \to V$, a basis $B$ for $V$ such that the matrix for $T$ relative to $B$ is diagonal.

## THE DIAGONALIZATION PROBLEM

The preceding section discussed the eigenvalue problem. In this section, you will look at another classic problem in linear algebra called the **diagonalization problem.** Expressed in terms of matrices*, the problem is as follows. For a square matrix $A$, does there exist an invertible matrix $P$ such that $P^{-1}AP$ is diagonal?

Recall from Section 6.4 that two square matrices $A$ and $B$ are called **similar** when there exists an invertible matrix $P$ such that $B = P^{-1}AP$.

Matrices that are similar to diagonal matrices are called **diagonalizable.**

### Definition of a Diagonalizable Matrix

An $n \times n$ matrix $A$ is **diagonalizable** when $A$ is similar to a diagonal matrix. That is, $A$ is diagonalizable when there exists an invertible matrix $P$ such that $P^{-1}AP$ is a diagonal matrix.

Given this definition, the diagonalization problem can be stated as "which square matrices are diagonalizable?" Clearly, every diagonal matrix $D$ is diagonalizable, because the identity matrix $I$ can play the role of $P$ to yield $D = I^{-1}DI$. Example 1 shows another example of a diagonalizable matrix.

### EXAMPLE 1 A Diagonalizable Matrix

The matrix from Example 5 in Section 6.4

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

is diagonalizable because

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

has the property

$$P^{-1}AP = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

As indicated in Example 8 in the preceding section, the eigenvalue problem is related closely to the diagonalization problem. The next two theorems shed more light on this relationship. The first theorem tells you that similar matrices must have the same eigenvalues.

*At the end of this section, the diagonalization problem will be expressed in terms of linear transformations.

**THEOREM 7.4 Similar Matrices Have the Same Eigenvalues**

If $A$ and $B$ are similar $n \times n$ matrices, then they have the same eigenvalues.

**PROOF**

Because $A$ and $B$ are similar, there exists an invertible matrix $P$ such that $B = P^{-1}AP$. By the properties of determinants, it follows that

$$\begin{aligned} |\lambda I - B| &= |\lambda I - P^{-1}AP| \\ &= |P^{-1}\lambda IP - P^{-1}AP| \\ &= |P^{-1}(\lambda I - A)P| \\ &= |P^{-1}||\lambda I - A||P| \\ &= |P^{-1}P||\lambda I - A| \\ &= |\lambda I - A|. \end{aligned}$$

This means that $A$ and $B$ have the same characteristic polynomial. So, they must have the same eigenvalues.

**REMARK**

This example states that matrices $A$ and $D$ are similar. Try checking $D = P^{-1}AP$ using the matrices

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

and

$$P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

In fact, the columns of $P$ are precisely the eigenvectors of $A$ corresponding to the eigenvalues 1, 2, and 3.

**EXAMPLE 2 Finding Eigenvalues of Similar Matrices**

The matrices $A$ and $D$ are similar.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ -1 & -2 & 4 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Use Theorem 7.4 to find the eigenvalues of $A$.

**SOLUTION**

Because $D$ is a diagonal matrix, its eigenvalues are the entries on its main diagonal—that is, $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = 3$. Because $A$ is said to be similar to $D$, you know from Theorem 7.4 that $A$ has the same eigenvalues. Check this by showing that the characteristic polynomial of $A$ is $|\lambda I - A| = (\lambda - 1)(\lambda - 2)(\lambda - 3)$.

The two diagonalizable matrices in Examples 1 and 2 provide a clue to the diagonalization problem. Each of these matrices has a set of three linearly independent eigenvectors. (See Example 3.) This is characteristic of diagonalizable matrices, as stated in Theorem 7.5.

**THEOREM 7.5 Condition for Diagonalization**

An $n \times n$ matrix $A$ is diagonalizable if and only if it has $n$ linearly independent eigenvectors.

**PROOF**

First, assume $A$ is diagonalizable. Then there exists an invertible matrix $P$ such that $P^{-1}AP = D$ is diagonal. Letting the column vectors of $P$ be $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$, and the main diagonal entries of $D$ be $\lambda_1, \lambda_2, \ldots, \lambda_n$, produces

$$PD = [\mathbf{p}_1 \quad \mathbf{p}_2 \quad \cdots \quad \mathbf{p}_n]\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = [\lambda_1\mathbf{p}_1 \quad \lambda_2\mathbf{p}_2 \quad \cdots \quad \lambda_n\mathbf{p}_n].$$

Because $P^{-1}AP = D$, $AP = PD$, which implies

$$[A\mathbf{p}_1 \quad A\mathbf{p}_2 \quad \cdots \quad A\mathbf{p}_n] = [\lambda_1\mathbf{p}_1 \quad \lambda_2\mathbf{p}_2 \quad \cdots \quad \lambda_n\mathbf{p}_n].$$

In other words, $A\mathbf{p}_i = \lambda_i\mathbf{p}_i$ for each column vector $\mathbf{p}_i$. This means that the column vectors $\mathbf{p}_i$ of $P$ are eigenvectors of $A$. Moreover, because $P$ is invertible, its column vectors are linearly independent. So, $A$ has $n$ linearly independent eigenvectors.

Conversely, assume $A$ has $n$ linearly independent eigenvectors $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$ with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Let $P$ be the matrix whose columns are these $n$ eigenvectors. That is, $P = [\mathbf{p}_1 \quad \mathbf{p}_2 \quad \cdots \quad \mathbf{p}_n]$. Because each $\mathbf{p}_i$ is an eigenvector of $A$, you have $A\mathbf{p}_i = \lambda_i\mathbf{p}_i$ and

$$AP = A[\mathbf{p}_1 \quad \mathbf{p}_2 \quad \cdots \quad \mathbf{p}_n] = [\lambda_1\mathbf{p}_1 \quad \lambda_2\mathbf{p}_2 \quad \cdots \quad \lambda_n\mathbf{p}_n].$$

The right-hand matrix in this equation can be written as the following matrix product.

$$AP = [\mathbf{p}_1 \quad \mathbf{p}_2 \quad \cdots \quad \mathbf{p}_n]\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = PD$$

Finally, because the vectors $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$ are linearly independent, $P$ is invertible and you can write the equation $AP = PD$ as $P^{-1}AP = D$, which means that $A$ is diagonalizable.

A key result of this proof is the fact that for diagonalizable matrices, *the columns of $P$ consist of the $n$ linearly independent eigenvectors.* Example 3 verifies this important property for the matrices in Examples 1 and 2.

## EXAMPLE 3 Diagonalizable Matrices

**a.** The matrix $A$ in Example 1 has the following eigenvalues and corresponding eigenvectors.

$$\lambda_1 = 4, \mathbf{p}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}; \quad \lambda_2 = -2, \mathbf{p}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}; \quad \lambda_3 = -2, \mathbf{p}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The matrix $P$ whose columns correspond to these eigenvectors is

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Moreover, because $P$ is row-equivalent to the identity matrix, the eigenvectors $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ are linearly independent.

**b.** The matrix $A$ in Example 2 has the following eigenvalues and corresponding eigenvectors.

$$\lambda_1 = 1, \mathbf{p}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}; \quad \lambda_2 = 2, \mathbf{p}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}; \quad \lambda_3 = 3, \mathbf{p}_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

The matrix $P$ whose columns correspond to these eigenvectors is

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

Again, because $P$ is row-equivalent to the identity matrix, the eigenvectors $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ are linearly independent.

The second part of the proof of Theorem 7.5 and Example 3 suggest the following steps for diagonalizing a matrix.

### Steps for Diagonalizing an $n \times n$ Square Matrix

Let $A$ be an $n \times n$ matrix.

1. Find $n$ linearly independent eigenvectors $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$ for $A$ (if possible) with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. If $n$ linearly independent eigenvectors do not exist, then $A$ is not diagonalizable.
2. Let $P$ be the $n \times n$ matrix whose columns consist of these eigenvectors. That is, $P = [\mathbf{p}_1 \quad \mathbf{p}_2 \quad \cdots \quad \mathbf{p}_n]$.
3. The diagonal matrix $D = P^{-1}AP$ will have the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ on its main diagonal (and zeros elsewhere). Note that the order of the eigenvectors used to form $P$ will determine the order in which the eigenvalues appear on the main diagonal of $D$.

## EXAMPLE 4 Diagonalizing a Matrix

Show that the matrix $A$ is diagonalizable.

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix}$$

Then find a matrix $P$ such that $P^{-1}AP$ is diagonal.

### SOLUTION

The characteristic polynomial of $A$ is $|\lambda I - A| = (\lambda - 2)(\lambda + 2)(\lambda - 3)$. (Verify this.) So, the eigenvalues of $A$ are $\lambda_1 = 2$, $\lambda_2 = -2$, and $\lambda_3 = 3$. From these eigenvalues, you obtain the following reduced row-echelon forms and corresponding eigenvectors.

Eigenvector

$$2I - A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 3 & -1 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$-2I - A = \begin{bmatrix} -3 & 1 & 1 \\ -1 & -5 & -1 \\ 3 & -1 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$$

$$3I - A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 0 & -1 \\ 3 & -1 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Form the matrix $P$ whose columns are the eigenvectors just obtained.

$$P = \begin{bmatrix} -1 & 1 & -1 \\ 0 & -1 & 1 \\ 1 & 4 & 1 \end{bmatrix}$$

This matrix is nonsingular (check this), which implies that the eigenvectors are linearly independent and $A$ is diagonalizable. So, it follows that

$$P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

## EXAMPLE 5 Diagonalizing a Matrix

Show that the matrix $A$ is diagonalizable.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & -10 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}$$

Then find a matrix $P$ such that $P^{-1}AP$ is diagonal.

### SOLUTION

In Example 6 in Section 7.1, you found that the three eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = 3$ have the following eigenvectors.

$$\lambda_1: \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \end{bmatrix} \qquad \lambda_2: \begin{bmatrix} 0 \\ 5 \\ 1 \\ 0 \end{bmatrix} \qquad \lambda_3: \begin{bmatrix} 0 \\ -5 \\ 0 \\ 1 \end{bmatrix}$$

The matrix whose columns consist of these eigenvectors is

$$P = \begin{bmatrix} 0 & -2 & 0 & 0 \\ 1 & 0 & 5 & -5 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

Because $P$ is invertible (check this), its column vectors form a linearly independent set. So, $A$ is diagonalizable, and you have

$$P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

## EXAMPLE 6 A Matrix That Is Not Diagonalizable

Show that the matrix $A$ is not diagonalizable.

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

### SOLUTION

Because $A$ is triangular, the eigenvalues are the entries on the main diagonal. So, the only eigenvalue is $\lambda = 1$. The matrix $(I - A)$ has the following reduced row-echelon form.

$$I - A = \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

This implies that $x_2 = 0$, and letting $x_1 = t$, you can find that every eigenvector of $A$ has the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ 0 \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

So, $A$ does not have two linearly independent eigenvectors, and you can conclude that $A$ is not diagonalizable.

**REMARK**

The condition in Theorem 7.6 is sufficient but not necessary for diagonalization, as demonstrated in Example 5. In other words, a diagonalizable matrix need not have distinct eigenvalues.

For a square matrix $A$ of order $n$ to be diagonalizable, the sum of the dimensions of the eigenspaces must be equal to $n$. One way this can happen is when $A$ has $n$ distinct eigenvalues. So, you have the next theorem.

**THEOREM 7.6 Sufficient Condition for Diagonalization**

If an $n \times n$ matrix $A$ has $n$ *distinct* eigenvalues, then the corresponding eigenvectors are linearly independent and $A$ is diagonalizable.

**PROOF**

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be $n$ distinct eigenvalues of $A$ corresponding to the eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$. To begin, assume the set of eigenvectors is linearly dependent. Moreover, consider the eigenvectors to be ordered so that the first $m$ eigenvectors are linearly independent, but the first $m + 1$ are dependent, where $m < n$. Then $\mathbf{x}_{m+1}$ can be written as a linear combination of the first $m$ eigenvectors:

$$\mathbf{x}_{m+1} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_m\mathbf{x}_m \qquad \text{Equation 1}$$

where the $c_i$'s are not all zero. Multiplication of both sides of Equation 1 by $A$ yields

$$A\mathbf{x}_{m+1} = Ac_1\mathbf{x}_1 + Ac_2\mathbf{x}_2 + \cdots + Ac_m\mathbf{x}_m.$$

Because $A\mathbf{x}_i = \lambda_i\mathbf{x}_i$, $i = 1, 2, \ldots, m + 1$, you have

$$\lambda_{m+1}\mathbf{x}_{m+1} = c_1\lambda_1\mathbf{x}_1 + c_2\lambda_2\mathbf{x}_2 + \cdots + c_m\lambda_m\mathbf{x}_m. \qquad \text{Equation 2}$$

Multiplication of Equation 1 by $\lambda_{m+1}$ yields

$$\lambda_{m+1}\mathbf{x}_{m+1} = c_1\lambda_{m+1}\mathbf{x}_1 + c_2\lambda_{m+1}\mathbf{x}_2 + \cdots + c_m\lambda_{m+1}\mathbf{x}_m. \qquad \text{Equation 3}$$

Now, subtracting Equation 2 from Equation 3 produces

$$c_1(\lambda_{m+1} - \lambda_1)\mathbf{x}_1 + c_2(\lambda_{m+1} - \lambda_2)\mathbf{x}_2 + \cdots + c_m(\lambda_{m+1} - \lambda_m)\mathbf{x}_m = \mathbf{0}$$

and, using the fact that the first $m$ eigenvectors are linearly independent, you can conclude that all coefficients of this equation must be zero. That is,

$$c_1(\lambda_{m+1} - \lambda_1) = c_2(\lambda_{m+1} - \lambda_2) = \cdots = c_m(\lambda_{m+1} - \lambda_m) = 0.$$

Because all the eigenvalues are distinct, it follows that $c_i = 0$, $i = 1, 2, \ldots, m$. But this result contradicts our assumption that $\mathbf{x}_{m+1}$ can be written as a linear combination of the first $m$ eigenvectors. So, the set of eigenvectors is linearly independent, and from Theorem 7.5, you can conclude that $A$ is diagonalizable.

**EXAMPLE 7 Determining Whether a Matrix Is Diagonalizable**

Determine whether the matrix $A$ is diagonalizable.

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -3 \end{bmatrix}$$

**SOLUTION**

Because $A$ is a triangular matrix, its eigenvalues are the main diagonal entries

$$\lambda_1 = 1, \qquad \lambda_2 = 0, \qquad \lambda_3 = -3.$$

Moreover, because these three values are distinct, you can conclude from Theorem 7.6 that $A$ is diagonalizable.

## DIAGONALIZATION AND LINEAR TRANSFORMATIONS

So far in this section, the diagonalization problem has been in terms of matrices. In terms of linear transformations, the diagonalization problem can be stated as follows. For a linear transformation

$$T: V \to V$$

does there exist a basis $B$ for $V$ such that the matrix for $T$ relative to $B$ is diagonal? The answer is "yes," provided the standard matrix for $T$ is diagonalizable.

**EXAMPLE 8** **Finding a Basis**

Let $T: R^3 \to R^3$ be the linear transformation represented by

$$T(x_1, x_2, x_3) = (x_1 - x_2 - x_3, x_1 + 3x_2 + x_3, -3x_1 + x_2 - x_3).$$

If possible, find a basis $B$ for $R^3$ such that the matrix for $T$ relative to $B$ is diagonal.

**SOLUTION**

The standard matrix for $T$ is

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix}.$$

From Example 4, you know that $A$ is diagonalizable. So, the three linearly independent eigenvectors found in Example 4 can be used to form the basis $B$. That is,

$$B = \{(-1, 0, 1), (1, -1, 4), (-1, 1, 1)\}.$$

The matrix for $T$ relative to this basis is

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

**LINEAR ALGEBRA APPLIED**

Genetics is the science of heredity. A mixture of chemistry and biology, genetics attempts to explain hereditary evolution and gene movement between generations based on the deoxyribonucleic acid (DNA) of a species. Research in the area of genetics called *population genetics,* which focuses on genetic structures of specific populations, is especially popular today. Such research has led to a better understanding of the types of genetic inheritance. For instance, in humans, one type of genetic inheritance is called *X-linked inheritance* (or *sex-linked inheritance*), which refers to recessive genes on the *X* chromosome. Males have one *X* and one *Y* chromosome, and females have two *X* chromosomes. If a male has a defective gene on the *X* chromosome, then its corresponding trait will be expressed because there is not a normal gene on the *Y* chromosome to suppress its activity. With females, the trait will not be expressed unless it is present on both *X* chromosomes, which is rare. This is why inherited diseases or conditions are usually found in males, hence the term *sex-linked inheritance*. Some of these include hemophilia A, Duchenne muscular dystrophy, red-green color blindness, and hereditary baldness. Matrix eigenvalues and diagonalization can be useful for coming up with mathematical models to describe *X*-linked inheritance in a given population.

## 7.2 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

**Diagonalizable Matrices and Eigenvalues** **In Exercises 1–6, (a) verify that $A$ is diagonalizable by computing $P^{-1}AP$, and (b) use the result of part (a) and Theorem 7.4 to find the eigenvalues of $A$.**

**1.** $A = \begin{bmatrix} -11 & 36 \\ -3 & 10 \end{bmatrix}$, $P = \begin{bmatrix} -3 & -4 \\ -1 & -1 \end{bmatrix}$

**2.** $A = \begin{bmatrix} 1 & 3 \\ -1 & 5 \end{bmatrix}$, $P = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$

**3.** $A = \begin{bmatrix} -2 & 4 \\ 1 & 1 \end{bmatrix}$, $P = \begin{bmatrix} 1 & -4 \\ 1 & 1 \end{bmatrix}$

**4.** $A = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}$, $P = \begin{bmatrix} 1 & 5 \\ 1 & 2 \end{bmatrix}$

**5.** $A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 3 & 0 \\ 4 & -2 & 5 \end{bmatrix}$, $P = \begin{bmatrix} 0 & 1 & -3 \\ 0 & 4 & 0 \\ 1 & 2 & 2 \end{bmatrix}$

**6.** $A = \begin{bmatrix} 0.80 & 0.10 & 0.05 & 0.05 \\ 0.10 & 0.80 & 0.05 & 0.05 \\ 0.05 & 0.05 & 0.80 & 0.10 \\ 0.05 & 0.05 & 0.10 & 0.80 \end{bmatrix}$,

$P = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \end{bmatrix}$

**Diagonalizing a Matrix** **In Exercises 7–14, for each matrix $A$, find (if possible) a nonsingular matrix $P$ such that $P^{-1}AP$ is diagonal. Verify that $P^{-1}AP$ is a diagonal matrix with the eigenvalues on the main diagonal.**

**7.** $A = \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}$

(See Exercise 17, Section 7.1.)

**8.** $A = \begin{bmatrix} 1 & -\frac{3}{2} \\ \frac{1}{2} & -1 \end{bmatrix}$

(See Exercise 19, Section 7.1.)

**9.** $A = \begin{bmatrix} 2 & -2 & 3 \\ 0 & 3 & -2 \\ 0 & -1 & 2 \end{bmatrix}$

(See Exercise 21, Section 7.1.)

**10.** $A = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}$

(See Exercise 22, Section 7.1.)

**11.** $A = \begin{bmatrix} 1 & 2 & -2 \\ -2 & 5 & -2 \\ -6 & 6 & -3 \end{bmatrix}$

(See Exercise 23, Section 7.1.)

**12.** $A = \begin{bmatrix} 3 & 2 & -3 \\ -3 & -4 & 9 \\ -1 & -2 & 5 \end{bmatrix}$

(See Exercise 24, Section 7.1.)

**13.** $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 0 & 2 \end{bmatrix}$

**14.** $A = \begin{bmatrix} 4 & 0 & 0 \\ 2 & 2 & 0 \\ 0 & 2 & 2 \end{bmatrix}$

**Showing That a Matrix Is Not Diagonalizable** **In Exercises 15–22, show that the matrix is not diagonalizable.**

**15.** $\begin{bmatrix} 0 & 0 \\ 5 & 0 \end{bmatrix}$

**16.** $\begin{bmatrix} 1 & \frac{1}{2} \\ -2 & -1 \end{bmatrix}$

**17.** $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

**18.** $\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$

**19.** $\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix}$

**20.** $\begin{bmatrix} 2 & 1 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{bmatrix}$

**21.** $\begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 \\ -2 & 0 & 2 & -2 \\ 0 & 2 & 0 & 2 \end{bmatrix}$

(See Exercise 37, Section 7.1.)

**22.** $\begin{bmatrix} 1 & -3 & 3 & 3 \\ -1 & 4 & -3 & -3 \\ -2 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

(See Exercise 38, Section 7.1.)

**Determining a Sufficient Condition for Diagonalization** **In Exercises 23–26, find the eigenvalues of the matrix and determine whether there is a sufficient number to guarantee that the matrix is diagonalizable. (Recall that the matrix may be diagonalizable even though it is not guaranteed to be diagonalizable by Theorem 7.6.)**

**23.** $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

**24.** $\begin{bmatrix} 2 & 0 \\ 5 & 2 \end{bmatrix}$

**25.** $\begin{bmatrix} -3 & -2 & 3 \\ 3 & 4 & -9 \\ 1 & 2 & -5 \end{bmatrix}$

**26.** $\begin{bmatrix} 4 & 3 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix}$

**Finding a Basis** **In Exercises 27–30, find a basis $B$ for the domain of $T$ such that the matrix for $T$ relative to $B$ is diagonal.**

**27.** $T: R^2 \to R^2$: $T(x, y) = (x + y, x + y)$

**28.** $T: R^3 \to R^3$: $T(x, y, z) = (-2x + 2y - 3z, 2x + y - 6z, -x - 2y)$

**29.** $T: P_1 \to P_1$: $T(a + bx) = a + (a + 2b)x$

**30.** $T: P_2 \to P_2$:
$T(c + bx + ax^2) = (2c + a) + (3b + 4a)x + ax^2$

**31. Proof** Let $A$ be a diagonalizable $n \times n$ matrix and let $P$ be an invertible $n \times n$ matrix such that $B = P^{-1}AP$ is the diagonal form of $A$. Prove that $A^k = PB^kP^{-1}$, where $k$ is a positive integer.

**32.** Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be $n$ distinct eigenvalues of the $n \times n$ matrix $A$. Use the result of Exercise 31 to find the eigenvalues of $A^k$.

**Finding a Power of a Matrix** **In Exercises 33–36, use the result of Exercise 31 to find the indicated power of $A$.**

**33.** $A = \begin{bmatrix} 10 & 18 \\ -6 & -11 \end{bmatrix}, A^6$ **34.** $A = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}, A^7$

**35.** $A = \begin{bmatrix} 3 & 2 & -3 \\ -3 & -4 & 9 \\ -1 & -2 & 5 \end{bmatrix}, A^8$

**36.** $A = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 2 & -2 \\ 3 & 0 & -3 \end{bmatrix}, A^5$

**True or False?** **In Exercises 37 and 38, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.**

**37.** (a) If $A$ and $B$ are similar $n \times n$ matrices, then they always have the same characteristic polynomial equation.

(b) The fact that an $n \times n$ matrix $A$ has $n$ distinct eigenvalues does not guarantee that $A$ is diagonalizable.

**38.** (a) If $A$ is a diagonalizable matrix, then it has $n$ linearly independent eigenvectors.

(b) If an $n \times n$ matrix $A$ is diagonalizable, then it must have $n$ distinct eigenvalues.

**39.** Are the two matrices similar? If so, find a matrix $P$ such that $B = P^{-1}AP$.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**40.** **Calculus** If $x$ is a real number, then you can define $e^x$ by the series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots.$$

In a similar way, if $X$ is a square matrix, then you can define $e^X$ by the series

$$e^X = I + X + \frac{1}{2!}X^2 + \frac{1}{3!}X^3 + \frac{1}{4!}X^4 + \cdots.$$

Evaluate $e^X$, where $X$ is the indicated square matrix.

(a) $X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (b) $X = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

(c) $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (d) $X = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$

**41.** **Writing** Can a matrix be similar to two different diagonal matrices? Explain your answer.

**42.** **Proof** Prove that if $A$ is diagonalizable, then $A^T$ is diagonalizable.

**43.** **Proof** Prove that if $A$ is diagonalizable with $n$ real eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then $|A| = \lambda_1\lambda_2 \cdot \cdots \cdot \lambda_n$.

**44.** **Proof** Prove that the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is diagonalizable when $-4bc < (a - d)^2$ and is not diagonalizable when $-4bc > (a - d)^2$.

**45.** **Guided Proof** Prove that if the eigenvalues of a diagonalizable matrix $A$ are all $\pm 1$, then the matrix is equal to its inverse.

Getting Started: To show that the matrix is equal to its inverse, use the fact that there exists an invertible matrix $P$ such that $D = P^{-1}AP$, where $D$ is a diagonal matrix with $\pm 1$ along its main diagonal.

(i) Let $D = P^{-1}AP$, where $D$ is a diagonal matrix with $\pm 1$ along its main diagonal.

(ii) Find $A$ in terms of $P$, $P^{-1}$, and $D$.

(iii) Use the properties of the inverse of a product of matrices and the fact that $D$ is diagonal to expand to find $A^{-1}$.

(iv) Conclude that $A^{-1} = A$.

**46.** **Guided Proof** Prove that nonzero nilpotent matrices are not diagonalizable.

Getting Started: From Exercise 78 in Section 7.1, you know that 0 is the only eigenvalue of the nilpotent matrix $A$. Show that it is impossible for $A$ to be diagonalizable.

(i) Assume $A$ is diagonalizable, so there exists an invertible matrix $P$ such that $P^{-1}AP = D$, where $D$ is the zero matrix.

(ii) Find $A$ in terms of $P$, $P^{-1}$, and $D$.

(iii) Find a contradiction and conclude that nonzero nilpotent matrices are not diagonalizable.

**47.** **Proof** Prove that if $A$ is a nonsingular diagonalizable matrix, then $A^{-1}$ is also diagonalizable.

**48.** **CAPSTONE** Explain how to determine whether an $n \times n$ matrix $A$ is diagonalizable using (a) similar matrices, (b) eigenvectors, and (c) distinct eigenvalues.

**Showing That a Matrix Is Not Diagonalizable** **In Exercises 49 and 50, show that the matrix is not diagonalizable.**

**49.** $\begin{bmatrix} 3 & k \\ 0 & 3 \end{bmatrix}, k \neq 0$ **50.** $\begin{bmatrix} 0 & 0 \\ k & 0 \end{bmatrix}, k \neq 0$

# 7.3 Symmetric Matrices and Orthogonal Diagonalization

- Recognize, and apply properties of, symmetric matrices.
- Recognize, and apply properties of, orthogonal matrices.
- Find an orthogonal matrix $P$ that orthogonally diagonalizes a symmetric matrix $A$.

## SYMMETRIC MATRICES

For most matrices, you must go through much of the diagonalization process before determining whether diagonalization is possible. One exception is with a triangular matrix that has distinct entries on the main diagonal. Such a matrix can be recognized as diagonalizable by inspection. In this section, you will study another type of matrix that is guaranteed to be diagonalizable: a **symmetric** matrix.

### Definition of Symmetric Matrix

A square matrix $A$ is **symmetric** when it is equal to its transpose: $A = A^T$.

**DISCOVERY**

1. Pick an arbitrary nonsymmetric square matrix and calculate its eigenvalues.
2. Can you find a nonsymmetric matrix for which the eigenvalues are not real?
3. Now pick an arbitrary symmetric matrix and calculate its eigenvalues.
4. Can you find a symmetric matrix for which the eigenvalues are not real?
5. What can you conclude about the eigenvalues of a symmetric matrix?

### EXAMPLE 1 Symmetric Matrices and Nonsymmetric Matrices

The matrices $A$ and $B$ are symmetric, but the matrix $C$ is not.

$$A = \begin{bmatrix} 0 & 1 & -2 \\ 1 & 3 & 0 \\ -2 & 0 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 3 \\ 3 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 3 & 2 & 1 \\ 1 & -4 & 0 \\ 1 & 0 & 5 \end{bmatrix}$$

Nonsymmetric matrices have the following properties that are not exhibited by symmetric matrices.

1. A nonsymmetric matrix may not be diagonalizable.
2. A nonsymmetric matrix can have eigenvalues that are not real. For instance, the matrix

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

has a characteristic equation of $\lambda^2 + 1 = 0$. So, its eigenvalues are the imaginary numbers $\lambda_1 = i$ and $\lambda_2 = -i$.

3. For a nonsymmetric matrix, the number of linearly independent eigenvectors corresponding to an eigenvalue can be less than the multiplicity of the eigenvalue. (See Example 6, Section 7.2.)

Symmetric matrices do not exhibit these three properties.

**REMARK**

Theorem 7.7 is called the **Real Spectral Theorem,** and the set of eigenvalues of $A$ is called the **spectrum** of $A$.

### THEOREM 7.7 Properties of Symmetric Matrices

If $A$ is an $n \times n$ symmetric matrix, then the following properties are true.

1. $A$ is diagonalizable.
2. All eigenvalues of $A$ are real.
3. If $\lambda$ is an eigenvalue of $A$ with multiplicity $k$, then $\lambda$ has $k$ linearly independent eigenvectors. That is, the eigenspace of $\lambda$ has dimension $k$.

A proof of Theorem 7.7 is beyond the scope of this text. The next example verifies that every $2 \times 2$ symmetric matrix is diagonalizable.

## EXAMPLE 2 The Eigenvalues and Eigenvectors of a 2 × 2 Symmetric Matrix

Prove that a symmetric matrix

$$A = \begin{bmatrix} a & c \\ c & b \end{bmatrix}$$

is diagonalizable.

**SOLUTION**

The characteristic polynomial of $A$ is

$$\begin{aligned} |\lambda I - A| &= \begin{vmatrix} \lambda - a & -c \\ -c & \lambda - b \end{vmatrix} \\ &= \lambda^2 - (a + b)\lambda + ab - c^2. \end{aligned}$$

As a quadratic in $\lambda$, this polynomial has a discriminant of

$$\begin{aligned} (a + b)^2 - 4(ab - c^2) &= a^2 + 2ab + b^2 - 4ab + 4c^2 \\ &= a^2 - 2ab + b^2 + 4c^2 \\ &= (a - b)^2 + 4c^2. \end{aligned}$$

Because this discriminant is the sum of two squares, it must be either zero or positive. If $(a - b)^2 + 4c^2 = 0$, then $a = b$ and $c = 0$, so $A$ is already diagonal. That is,

$$A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}.$$

On the other hand, if $(a - b)^2 + 4c^2 > 0$, then by the Quadratic Formula the characteristic polynomial of $A$ has two distinct real roots, which implies that $A$ has two distinct real eigenvalues. So, $A$ is diagonalizable in this case as well.

## EXAMPLE 3 Dimensions of the Eigenspaces of a Symmetric Matrix

Find the eigenvalues of the symmetric matrix

$$A = \begin{bmatrix} 1 & -2 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -2 & 1 \end{bmatrix}$$

and determine the dimensions of the corresponding eigenspaces.

**SOLUTION**

The characteristic polynomial of $A$ is

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & 2 & 0 & 0 \\ 2 & \lambda - 1 & 0 & 0 \\ 0 & 0 & \lambda - 1 & 2 \\ 0 & 0 & 2 & \lambda - 1 \end{vmatrix} = (\lambda + 1)^2(\lambda - 3)^2.$$

So, the eigenvalues of $A$ are $\lambda_1 = -1$ and $\lambda_2 = 3$. Because each of these eigenvalues has a multiplicity of 2, you know from Theorem 7.7 that the corresponding eigenspaces also have dimension 2. Specifically, the eigenspace of $\lambda_1 = -1$ has a basis of $B_1 = \{(1, 1, 0, 0), (0, 0, 1, 1)\}$ and the eigenspace of $\lambda_2 = 3$ has a basis of $B_2 = \{(1, -1, 0, 0), (0, 0, 1, -1)\}$.

## ORTHOGONAL MATRICES

To diagonalize a square matrix $A$, you need to find an *invertible* matrix $P$ such that $P^{-1}AP$ is diagonal. For symmetric matrices, the matrix $P$ can be chosen to have the special property that $P^{-1} = P^T$. This unusual matrix property is defined as follows.

### Definition of an Orthogonal Matrix

A square matrix $P$ is called **orthogonal** when it is invertible and when

$$P^{-1} = P^T.$$

### EXAMPLE 4 Orthogonal Matrices

**a.** The matrix $P = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is orthogonal because $P^{-1} = P^T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

**b.** The matrix

$$P = \begin{bmatrix} \frac{3}{5} & 0 & -\frac{4}{5} \\ 0 & 1 & 0 \\ \frac{4}{5} & 0 & \frac{3}{5} \end{bmatrix}$$

is orthogonal because

$$P^{-1} = P^T = \begin{bmatrix} \frac{3}{5} & 0 & \frac{4}{5} \\ 0 & 1 & 0 \\ -\frac{4}{5} & 0 & \frac{3}{5} \end{bmatrix}.$$

In parts (a) and (b) of Example 4, the columns of the matrices $P$ form orthonormal sets in $R^2$ and $R^3$, respectively. This suggests the next theorem.

### THEOREM 7.8 Property of Orthogonal Matrices

An $n \times n$ matrix $P$ is orthogonal if and only if its column vectors form an orthonormal set.

### PROOF

Suppose the column vectors of $P$ form an orthonormal set:

$$\begin{aligned} P &= [\mathbf{p}_1 \quad \mathbf{p}_2 \quad \cdots \quad \mathbf{p}_n] \\ &= \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix}. \end{aligned}$$

Then the product $P^TP$ has the form

$$P^TP = \begin{bmatrix} \mathbf{p}_1 \cdot \mathbf{p}_1 & \mathbf{p}_1 \cdot \mathbf{p}_2 & \cdots & \mathbf{p}_1 \cdot \mathbf{p}_n \\ \mathbf{p}_2 \cdot \mathbf{p}_1 & \mathbf{p}_2 \cdot \mathbf{p}_2 & \cdots & \mathbf{p}_2 \cdot \mathbf{p}_n \\ \vdots & \vdots & & \vdots \\ \mathbf{p}_n \cdot \mathbf{p}_1 & \mathbf{p}_n \cdot \mathbf{p}_2 & \cdots & \mathbf{p}_n \cdot \mathbf{p}_n \end{bmatrix}.$$

Because the set

$$\{\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n\}$$

is orthonormal, you have

$$\mathbf{p}_i \cdot \mathbf{p}_j = 0,\ i \neq j \quad \text{and} \quad \mathbf{p}_i \cdot \mathbf{p}_i = \|\mathbf{p}_i\|^2 = 1.$$

So, the matrix composed of dot products has the form

$$P^TP = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = I_n.$$

This implies that $P^T = P^{-1}$, so $P$ is orthogonal.

Conversely, if $P$ is orthogonal, then you can reverse the steps above to verify that the column vectors of $P$ form an orthonormal set.

## EXAMPLE 5 An Orthogonal Matrix

Show that

$$P = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ -\frac{2}{3\sqrt{5}} & -\frac{4}{3\sqrt{5}} & \frac{5}{3\sqrt{5}} \end{bmatrix}$$

is orthogonal by showing that $PP^T = I$. Then show that the column vectors of $P$ form an orthonormal set.

### SOLUTION

Because

$$PP^T = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ -\frac{2}{3\sqrt{5}} & -\frac{4}{3\sqrt{5}} & \frac{5}{3\sqrt{5}} \end{bmatrix}\begin{bmatrix} \frac{1}{3} & -\frac{2}{\sqrt{5}} & -\frac{2}{3\sqrt{5}} \\ \frac{2}{3} & \frac{1}{\sqrt{5}} & -\frac{4}{3\sqrt{5}} \\ \frac{2}{3} & 0 & \frac{5}{3\sqrt{5}} \end{bmatrix} = I_3$$

it follows that $P^T = P^{-1}$, so $P$ is orthogonal. Moreover, letting

$$\mathbf{p}_1 = \begin{bmatrix} \frac{1}{3} \\ -\frac{2}{\sqrt{5}} \\ -\frac{2}{3\sqrt{5}} \end{bmatrix}, \mathbf{p}_2 = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{\sqrt{5}} \\ -\frac{4}{3\sqrt{5}} \end{bmatrix}, \quad \text{and} \quad \mathbf{p}_3 = \begin{bmatrix} \frac{2}{3} \\ 0 \\ \frac{5}{3\sqrt{5}} \end{bmatrix}$$

produces

$$\mathbf{p}_1 \cdot \mathbf{p}_2 = \mathbf{p}_1 \cdot \mathbf{p}_3 = \mathbf{p}_2 \cdot \mathbf{p}_3 = 0$$

and

$$\|\mathbf{p}_1\| = \|\mathbf{p}_2\| = \|\mathbf{p}_3\| = 1.$$

So, $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ is an orthonormal set, as guaranteed by Theorem 7.8.

It can be shown that for a symmetric matrix, the eigenvectors corresponding to distinct eigenvalues are orthogonal. The next theorem states this property.

**THEOREM 7.9 Property of Symmetric Matrices**

Let $A$ be an $n \times n$ symmetric matrix. If $\lambda_1$ and $\lambda_2$ are distinct eigenvalues of $A$, then their corresponding eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$ are orthogonal.

**PROOF**

Let $\lambda_1$ and $\lambda_2$ be distinct eigenvalues of $A$ with corresponding eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$. So, $A\mathbf{x}_1 = \lambda_1\mathbf{x}_1$ and $A\mathbf{x}_2 = \lambda_2\mathbf{x}_2$. To prove the theorem, use the matrix form of the dot product, $\mathbf{x}_1 \cdot \mathbf{x}_2 = \mathbf{x}_1^T\mathbf{x}_2$. (See Section 5.1.) Now you can write

$$\begin{aligned}
\lambda_1(\mathbf{x}_1 \cdot \mathbf{x}_2) &= (\lambda_1\mathbf{x}_1) \cdot \mathbf{x}_2 \\
&= (A\mathbf{x}_1) \cdot \mathbf{x}_2 \\
&= (A\mathbf{x}_1)^T\mathbf{x}_2 \\
&= (\mathbf{x}_1^TA^T)\mathbf{x}_2 \\
&= (\mathbf{x}_1^TA)\mathbf{x}_2 && \text{Because } A \text{ is symmetric, } A = A^T. \\
&= \mathbf{x}_1^T(A\mathbf{x}_2) \\
&= \mathbf{x}_1^T(\lambda_2\mathbf{x}_2) \\
&= \mathbf{x}_1 \cdot (\lambda_2\mathbf{x}_2) \\
&= \lambda_2(\mathbf{x}_1 \cdot \mathbf{x}_2).
\end{aligned}$$

This implies that $(\lambda_1 - \lambda_2)(\mathbf{x}_1 \cdot \mathbf{x}_2) = 0$, and because $\lambda_1 \neq \lambda_2$ it follows that $\mathbf{x}_1 \cdot \mathbf{x}_2 = 0$. So, $\mathbf{x}_1$ and $\mathbf{x}_2$ are orthogonal.

## EXAMPLE 6 Eigenvectors of a Symmetric Matrix

Show that any two eigenvectors of

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

corresponding to distinct eigenvalues are orthogonal.

**SOLUTION**

The characteristic polynomial of $A$ is

$$|\lambda I - A| = \begin{vmatrix} \lambda - 3 & -1 \\ -1 & \lambda - 3 \end{vmatrix} = (\lambda - 2)(\lambda - 4)$$

which implies that the eigenvalues of $A$ are $\lambda_1 = 2$ and $\lambda_2 = 4$. Every eigenvector corresponding to $\lambda_1 = 2$ is of the form

$$\mathbf{x}_1 = \begin{bmatrix} s \\ -s \end{bmatrix}, \quad s \neq 0$$

and every eigenvector corresponding to $\lambda_2 = 4$ is of the form

$$\mathbf{x}_2 = \begin{bmatrix} t \\ t \end{bmatrix}, \quad t \neq 0.$$

So,

$$\mathbf{x}_1 \cdot \mathbf{x}_2 = st - st = 0$$

and you can conclude that $\mathbf{x}_1$ and $\mathbf{x}_2$ are orthogonal.

## ORTHOGONAL DIAGONALIZATION

A matrix $A$ is **orthogonally diagonalizable** when there exists an orthogonal matrix $P$ such that $P^{-1}AP = D$ is diagonal. The following important theorem states that the set of orthogonally diagonalizable matrices is precisely the set of symmetric matrices.

**THEOREM 7.10 Fundamental Theorem of Symmetric Matrices**

Let $A$ be an $n \times n$ matrix. Then $A$ is orthogonally diagonalizable and has real eigenvalues if and only if $A$ is symmetric.

### PROOF

The proof of the theorem in one direction is fairly straightforward. That is, if you assume $A$ is orthogonally diagonalizable, then there exists an orthogonal matrix $P$ such that $D = P^{-1}AP$ is diagonal. Moreover, because $P^{-1} = P^T$, you have

$$\begin{aligned} A &= PDP^{-1} \\ &= PDP^T \end{aligned}$$

which implies that

$$\begin{aligned} A^T &= (PDP^T)^T \\ &= (P^T)^T D^T P^T \\ &= PDP^T \\ &= A. \end{aligned}$$

So, $A$ is symmetric.

The proof of the theorem in the other direction is more involved, but it is important because it is constructive. Assume $A$ is symmetric. If $A$ has an eigenvalue $\lambda$ of multiplicity $k$, then by Theorem 7.7, $\lambda$ has $k$ linearly independent eigenvectors. Through the Gram-Schmidt orthonormalization process, use this set of $k$ vectors to form an orthonormal basis of eigenvectors for the eigenspace corresponding to $\lambda$. Repeat this procedure for each eigenvalue of $A$. The collection of all resulting eigenvectors is orthogonal by Theorem 7.9, and you know from the orthonormalization process that the collection is also orthonormal. Now let $P$ be the matrix whose columns consist of these $n$ orthonormal eigenvectors. By Theorem 7.8, $P$ is an orthogonal matrix. Finally, by Theorem 7.5, you can conclude that $P^{-1}AP$ is diagonal. So, $A$ is orthogonally diagonalizable.

### EXAMPLE 7 Determining Whether a Matrix Is Orthogonally Diagonalizable

Which matrices are orthogonally diagonalizable?

$$A_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 5 & 2 & 1 \\ 2 & 1 & 8 \\ -1 & 8 & 0 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix} \qquad A_4 = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}$$

### SOLUTION

By Theorem 7.10, the orthogonally diagonalizable matrices are the symmetric ones: $A_1$ and $A_4$.

As mentioned above, the second part of the proof of Theorem 7.10 is *constructive*. That is, it gives you steps to follow to diagonalize a symmetric matrix orthogonally. The following summarizes these steps.

## Orthogonal Diagonalization of a Symmetric Matrix

Let $A$ be an $n \times n$ symmetric matrix.

1. Find all eigenvalues of $A$ and determine the multiplicity of each.
2. For *each* eigenvalue of multiplicity 1, find a unit eigenvector. (Find any eigenvector and then normalize it.)
3. For each eigenvalue of multiplicity $k \geq 2$, find a set of $k$ linearly independent eigenvectors. (You know from Theorem 7.7 that this is possible.) If this set is not orthonormal, then apply the Gram-Schmidt orthonormalization process.
4. The results of Steps 2 and 3 produce an orthonormal set of $n$ eigenvectors. Use these eigenvectors to form the columns of $P$. The matrix $P^{-1}AP = P^TAP = D$ will be diagonal. (The main diagonal entries of $D$ are the eigenvalues of $A$.)

### EXAMPLE 8 Orthogonal Diagonalization

Find a matrix $P$ that orthogonally diagonalizes $A = \begin{bmatrix} -2 & 2 \\ 2 & 1 \end{bmatrix}$.

**SOLUTION**

1. The characteristic polynomial of $A$ is

$$|\lambda I - A| = \begin{vmatrix} \lambda + 2 & -2 \\ -2 & \lambda - 1 \end{vmatrix} = (\lambda + 3)(\lambda - 2).$$

So the eigenvalues are $\lambda_1 = -3$ and $\lambda_2 = 2$.

2. For each eigenvalue, find an eigenvector by converting the matrix $\lambda I - A$ to reduced row-echelon form.

Eigenvector

$$-3I - A = \begin{bmatrix} -1 & -2 \\ -2 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$2I - A = \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

The eigenvectors $(-2, 1)$ and $(1, 2)$ form an *orthogonal* basis for $R^2$. Normalize each of these eigenvectors to produce an *orthonormal* basis.

$$\mathbf{p}_1 = \frac{(-2, 1)}{\|(-2, 1)\|} = \left(-\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right) \qquad \mathbf{p}_2 = \frac{(1, 2)}{\|(1, 2)\|} = \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$$

3. Because each eigenvalue has a multiplicity of 1, go directly to step 4.

4. Using $\mathbf{p}_1$ and $\mathbf{p}_2$ as column vectors, construct the matrix $P$.

$$P = \begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$$

Verify that $P$ orthogonally diagonalizes $A$ by computing $P^{-1}AP = P^TAP$.

$$P^TAP = \begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} -2 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}$$

**EXAMPLE 9** **Orthogonal Diagonalization**

Find a matrix $P$ that orthogonally diagonalizes

$$A = \begin{bmatrix} 2 & 2 & -2 \\ 2 & -1 & 4 \\ -2 & 4 & -1 \end{bmatrix}.$$

**SOLUTION**

**1.** The characteristic polynomial of $A$, $|\lambda I - A| = (\lambda + 6)(\lambda - 3)^2$, yields the eigenvalues $\lambda_1 = -6$ and $\lambda_2 = 3$. The eigenvalue $\lambda_1$ has a multiplicity of 1 and the eigenvalue $\lambda_2$ has a multiplicity of 2.

**2.** An eigenvector for $\lambda_1$ is $\mathbf{v}_1 = (1, -2, 2)$, which normalizes to

$$\mathbf{u}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \left(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right).$$

**3.** Two eigenvectors for $\lambda_2$ are $\mathbf{v}_2 = (2, 1, 0)$ and $\mathbf{v}_3 = (-2, 0, 1)$. Note that $\mathbf{v}_1$ is orthogonal to $\mathbf{v}_2$ and $\mathbf{v}_3$ by Theorem 7.9. The eigenvectors $\mathbf{v}_2$ and $\mathbf{v}_3$, however, are not orthogonal to each other. To find two orthonormal eigenvectors for $\lambda_2$, use the Gram-Schmidt process as follows.

$$\mathbf{w}_2 = \mathbf{v}_2 = (2, 1, 0)$$

$$\mathbf{w}_3 = \mathbf{v}_3 - \left(\frac{\mathbf{v}_3 \cdot \mathbf{w}_2}{\mathbf{w}_2 \cdot \mathbf{w}_2}\right)\mathbf{w}_2 = \left(-\frac{2}{5}, \frac{4}{5}, 1\right)$$

These vectors normalize to

$$\mathbf{u}_2 = \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|} = \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right)$$

$$\mathbf{u}_3 = \frac{\mathbf{w}_3}{\|\mathbf{w}_3\|} = \left(-\frac{2}{3\sqrt{5}}, \frac{4}{3\sqrt{5}}, \frac{5}{3\sqrt{5}}\right).$$

**4.** The matrix $P$ has $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ as its column vectors.

$$P = \begin{bmatrix} \frac{1}{3} & \frac{2}{\sqrt{5}} & -\frac{2}{3\sqrt{5}} \\ -\frac{2}{3} & \frac{1}{\sqrt{5}} & \frac{4}{3\sqrt{5}} \\ \frac{2}{3} & 0 & \frac{5}{3\sqrt{5}} \end{bmatrix}$$

A check shows that $P^{-1}AP = P^TAP = \begin{bmatrix} -6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

**LINEAR ALGEBRA APPLIED**

The *Hessian matrix* is a symmetric matrix that can be helpful in finding relative maxima and minima of functions of several variables. For a function $f$ of two variables $x$ and $y$—that is, a surface in $R^3$—the Hessian matrix has the form

$$\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}.$$

The determinant of this matrix, evaluated at a point for which $f_x$ and $f_y$ are zero, is the expression used in the Second Partials Test for relative extrema.

# 7.3 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

**Determining Whether a Matrix Is Symmetric** In Exercises 1–6, determine whether the matrix is symmetric.

**1.** $\begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix}$ **2.** $\begin{bmatrix} 6 & -2 \\ -2 & 1 \end{bmatrix}$

**3.** $\begin{bmatrix} 4 & -2 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$ **4.** $\begin{bmatrix} 1 & -5 & 4 \\ -5 & 3 & 6 \\ -4 & 6 & 2 \end{bmatrix}$

**5.** $\begin{bmatrix} 0 & 1 & 2 & -1 \\ 1 & 0 & -3 & 2 \\ 2 & -3 & 0 & 1 \\ -1 & 2 & 1 & -2 \end{bmatrix}$ **6.** $\begin{bmatrix} 2 & 0 & 3 & 5 \\ 0 & 11 & 0 & -2 \\ 3 & 0 & 5 & 0 \\ 5 & -2 & 0 & 1 \end{bmatrix}$

**Proof** In Exercises 7–10, prove that the symmetric matrix is diagonalizable.

**7.** $A = \begin{bmatrix} 0 & 0 & a \\ 0 & a & 0 \\ a & 0 & 0 \end{bmatrix}$ **8.** $A = \begin{bmatrix} 0 & a & 0 \\ a & 0 & a \\ 0 & a & 0 \end{bmatrix}$

**9.** $A = \begin{bmatrix} a & 0 & a \\ 0 & a & 0 \\ a & 0 & a \end{bmatrix}$ **10.** $A = \begin{bmatrix} a & a & a \\ a & a & a \\ a & a & a \end{bmatrix}$

**Finding Eigenvalues and Dimensions of Eigenspaces** In Exercises 11–22, find the eigenvalues of the symmetric matrix. For each eigenvalue, find the dimension of the corresponding eigenspace.

**11.** $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ **12.** $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

**13.** $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ **14.** $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

**15.** $\begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix}$ **16.** $\begin{bmatrix} 0 & 4 & 4 \\ 4 & 2 & 0 \\ 4 & 0 & -2 \end{bmatrix}$

**17.** $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ **18.** $\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

**19.** $\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 5 & 3 \end{bmatrix}$ **20.** $\begin{bmatrix} -1 & 2 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 2 & -1 \end{bmatrix}$

**21.** $\begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$

**22.** $\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}$

**Determining Whether a Matrix Is Orthogonal** In Exercises 23–32, determine whether the matrix is orthogonal. If the matrix is orthogonal, then show that the column vectors of the matrix form an orthonormal set.

**23.** $\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$ **24.** $\begin{bmatrix} \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$

**25.** $\begin{bmatrix} \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix}$ **26.** $\begin{bmatrix} -\frac{4}{5} & 0 & \frac{3}{5} \\ 0 & 1 & 0 \\ \frac{3}{5} & 0 & \frac{4}{5} \end{bmatrix}$

**27.** $\begin{bmatrix} -4 & 0 & 3 \\ 0 & 1 & 0 \\ 3 & 0 & 4 \end{bmatrix}$ **28.** $\begin{bmatrix} 4 & -1 & -4 \\ -1 & 0 & -17 \\ 1 & 4 & -1 \end{bmatrix}$

**29.** $\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{6}}{6} & \frac{\sqrt{3}}{3} \\ 0 & \frac{\sqrt{6}}{3} & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{6} & -\frac{\sqrt{3}}{3} \end{bmatrix}$

**30.** $\begin{bmatrix} \frac{\sqrt{2}}{3} & 0 & \frac{\sqrt{5}}{2} \\ 0 & \frac{2\sqrt{5}}{5} & 0 \\ -\frac{\sqrt{2}}{6} & -\frac{\sqrt{5}}{5} & \frac{1}{2} \end{bmatrix}$

**31.** $\begin{bmatrix} \frac{1}{8} & 0 & 0 & \frac{3\sqrt{7}}{8} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{3\sqrt{7}}{8} & 0 & 0 & \frac{1}{8} \end{bmatrix}$

**32.** $\begin{bmatrix} \frac{1}{10}\sqrt{10} & 0 & 0 & -\frac{3}{10}\sqrt{10} \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{3}{10}\sqrt{10} & 0 & 0 & \frac{1}{10}\sqrt{10} \end{bmatrix}$

**Eigenvectors of a Symmetric Matrix** **In Exercises 33–38, show that any two eigenvectors of the symmetric matrix corresponding to distinct eigenvalues are orthogonal.**

**33.** $\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}$ **34.** $\begin{bmatrix} -1 & -2 \\ -2 & 2 \end{bmatrix}$

**35.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ **36.** $\begin{bmatrix} 3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

**37.** $\begin{bmatrix} 0 & \sqrt{3} & 0 \\ \sqrt{3} & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$ **38.** $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$

**Orthogonally Diagonalizable Matrices** **In Exercises 39–42, determine whether the matrix is orthogonally diagonalizable.**

**39.** $\begin{bmatrix} 4 & 5 \\ 0 & 1 \end{bmatrix}$ **40.** $\begin{bmatrix} 3 & 2 & -3 \\ -2 & -1 & 2 \\ -3 & 2 & 3 \end{bmatrix}$

**41.** $\begin{bmatrix} 5 & -3 & 8 \\ -3 & -3 & -3 \\ 8 & -3 & 8 \end{bmatrix}$ **42.** $\begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ -1 & 0 & -1 & 0 \end{bmatrix}$

**Orthogonal Diagonalization** **In Exercises 43–52, find a matrix $P$ such that $P^TAP$ orthogonally diagonalizes $A$. Verify that $P^TAP$ gives the proper diagonal form.**

**43.** $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

**44.** $A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$

**45.** $A = \begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix}$

**46.** $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

**47.** $A = \begin{bmatrix} 0 & 10 & 10 \\ 10 & 5 & 0 \\ 10 & 0 & -5 \end{bmatrix}$

**48.** $A = \begin{bmatrix} 0 & 3 & 0 \\ 3 & 0 & 4 \\ 0 & 4 & 0 \end{bmatrix}$

**49.** $A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & 2 \\ 2 & 2 & 2 \end{bmatrix}$

**50.** $A = \begin{bmatrix} -2 & 2 & 4 \\ 2 & -2 & 4 \\ 4 & 4 & 4 \end{bmatrix}$

**51.** $A = \begin{bmatrix} 4 & 2 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 2 & 4 \end{bmatrix}$

**52.** $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

**True or False?** **In Exercises 53 and 54, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.**

**53.** (a) Let $A$ be an $n \times n$ matrix. Then $A$ is symmetric if and only if $A$ is orthogonally diagonalizable.

(b) The eigenvectors corresponding to distinct eigenvalues are orthogonal for symmetric matrices.

**54.** (a) A square matrix $P$ is orthogonal when it is invertible—that is, when $P^{-1} = P^T$.

(b) If $A$ is an $n \times n$ symmetric matrix, then $A$ has real eigenvalues.

**55.** **Proof** Prove that if $A$ and $B$ are $n \times n$ orthogonal matrices, then $AB$ and $BA$ are orthogonal.

**56.** **Proof** Prove that if a symmetric matrix $A$ has only one eigenvalue $\lambda$, then $A = \lambda I$.

**57.** **Proof** Prove that if $A$ is an orthogonal matrix, then so are $A^T$ and $A^{-1}$.

**58.** **CAPSTONE** Consider the following matrix.

$$A = \begin{bmatrix} -3 & 2 & -5 \\ 2 & 4 & 3 \\ -5 & 3 & 5 \end{bmatrix}$$

(a) Is $A$ symmetric? Explain.

(b) Is $A$ diagonalizable? Explain.

(c) Are the eigenvalues of $A$ real? Explain.

(d) The eigenvalues of $A$ are distinct. What are the dimensions of the corresponding eigenspaces? Explain.

(e) Is $A$ orthogonal? Explain.

(f) For the eigenvalues of $A$, are the corresponding eigenvectors orthogonal? Explain.

(g) Is $A$ orthogonally diagonalizable? Explain.

**59.** Find $A^TA$ and $AA^T$ for the following matrix. What do you observe?

$$A = \begin{bmatrix} 1 & -3 & 2 \\ 4 & -6 & 1 \end{bmatrix}$$

# 7.4 Applications of Eigenvalues and Eigenvectors

- Model population growth using an age transition matrix and an age distribution vector, and find a stable age distribution vector.
- Use a matrix equation to solve a system of first-order linear differential equations.
- Find the matrix of a quadratic form and use the Principal Axes Theorem to perform a rotation of axes for a conic and a quadric surface.

## POPULATION GROWTH

Matrices can be used to form models for population growth. The first step in this process is to group the population into age classes of equal duration. For instance, if the maximum life span of a member is $L$ years, then the following $n$ intervals represent the age classes.

$$\left[0, \frac{L}{n}\right) \qquad \text{First age class}$$

$$\left[\frac{L}{n}, \frac{2L}{n}\right) \qquad \text{Second age class}$$

$$\vdots$$

$$\left[\frac{(n-1)L}{n}, L\right] \qquad n\text{th age class}$$

The **age distribution vector x** represents the number of population members in each age class, where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}. \qquad \begin{array}{l} \text{Number in first age class} \\ \text{Number in second age class} \\ \vdots \\ \text{Number in } n\text{th age class} \end{array}$$

Over a period of $L/n$ years, the *probability* that a member of the $i$th age class will survive to become a member of the $(i+1)$th age class is given by $p_i$, where

$$0 \leq p_i \leq 1, \quad i = 1, 2, \ldots, n-1.$$

The *average number* of offspring produced by a member of the $i$th age class is given by $b_i$, where

$$0 \leq b_i, \quad i = 1, 2, \ldots, n.$$

These numbers can be written in matrix form, as follows.

$$A = \begin{bmatrix} b_1 & b_2 & b_3 & \cdots & b_{n-1} & b_n \\ p_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & p_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & p_{n-1} & 0 \end{bmatrix}$$

Multiplying this **age transition matrix** by the age distribution vector for a specific time period produces the age distribution vector for the next time period. That is,

$$A\mathbf{x}_i = \mathbf{x}_{i+1}.$$

Example 1 illustrates this procedure.

## EXAMPLE 1 A Population Growth Model

A population of rabbits has the following characteristics.

**a.** Half of the rabbits survive their first year. Of those, half survive their second year. The maximum life span is 3 years.

**b.** During the first year, the rabbits produce no offspring. The average number of offspring is 6 during the second year and 8 during the third year.

The population now consists of 24 rabbits in the first age class, 24 in the second, and 20 in the third. How many rabbits will there be in each age class in 1 year?

### SOLUTION

The current age distribution vector is

$$\mathbf{x}_1 = \begin{bmatrix} 24 \\ 24 \\ 20 \end{bmatrix} \qquad \begin{array}{l} 0 \le \text{age} < 1 \\ 1 \le \text{age} < 2 \\ 2 \le \text{age} \le 3 \end{array}$$

and the age transition matrix is

$$A = \begin{bmatrix} 0 & 6 & 8 \\ 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix}.$$

After 1 year, the age distribution vector will be

$$\mathbf{x}_2 = A\mathbf{x}_1 = \begin{bmatrix} 0 & 6 & 8 \\ 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix}\begin{bmatrix} 24 \\ 24 \\ 20 \end{bmatrix} = \begin{bmatrix} 304 \\ 12 \\ 12 \end{bmatrix}. \qquad \begin{array}{l} 0 \le \text{age} < 1 \\ 1 \le \text{age} < 2 \\ 2 \le \text{age} \le 3 \end{array}$$

### REMARK

If the pattern of growth in Example 1 continued for another year, then the rabbit population would be

$$\mathbf{x}_3 = A\mathbf{x}_2 = \begin{bmatrix} 168 \\ 152 \\ 6 \end{bmatrix}.$$

From the age distribution vectors $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$, you can see that the percent of rabbits in each of the three age classes changes each year. To obtain a stable growth pattern, one in which the percent in each age class remains the same each year, the $(n + 1)$th age distribution vector must be a scalar multiple of the $n$th age distribution vector. That is, $\mathbf{x}_{n+1} = A\mathbf{x}_n = \lambda\mathbf{x}_n$. Example 2 shows how to solve this problem.

## EXAMPLE 2 Finding a Stable Age Distribution Vector

Find a stable age distribution vector for the population in Example 1.

### SOLUTION

To solve this problem, find an eigenvalue $\lambda$ and a corresponding eigenvector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$. The characteristic polynomial of $A$ is

$$|\lambda I - A| = (\lambda + 1)^2(\lambda - 2)$$

(check this), which implies that the eigenvalues are $-1$ and 2. Choosing the positive value, let $\lambda = 2$. Verify that the corresponding eigenvectors are of the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 16t \\ 4t \\ t \end{bmatrix} = t\begin{bmatrix} 16 \\ 4 \\ 1 \end{bmatrix}.$$

For instance, if $t = 2$, then the initial age distribution vector would be

$$\mathbf{x}_1 = \begin{bmatrix} 32 \\ 8 \\ 2 \end{bmatrix} \qquad \begin{array}{l} 0 \le \text{age} < 1 \\ 1 \le \text{age} < 2 \\ 2 \le \text{age} \le 3 \end{array}$$

and the age distribution vector for the next year would be

$$\mathbf{x}_2 = A\mathbf{x}_1 = \begin{bmatrix} 0 & 6 & 8 \\ 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix}\begin{bmatrix} 32 \\ 8 \\ 2 \end{bmatrix} = \begin{bmatrix} 64 \\ 16 \\ 4 \end{bmatrix}. \qquad \begin{array}{l} 0 \le \text{age} < 1 \\ 1 \le \text{age} < 2 \\ 2 \le \text{age} \le 3 \end{array}$$

Notice that the ratio of the three age classes is still 16 : 4 : 1, and so the percent of the population in each age class remains the same.

***Simulation***

Explore this concept further with an electronic simulation available at *www.cengagebrain.com*.

## SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS (CALCULUS)

A **system of first-order linear differential equations** has the form

$$\begin{aligned}
y_1' &= a_{11}y_1 + a_{12}y_2 + \cdot \cdot \cdot + a_{1n}y_n \\
y_2' &= a_{21}y_1 + a_{22}y_2 + \cdot \cdot \cdot + a_{2n}y_n \\
&\vdots \\
y_n' &= a_{n1}y_1 + a_{n2}y_2 + \cdot \cdot \cdot + a_{nn}y_n
\end{aligned}$$

where each $y_i$ is a function of $t$ and $y_i' = \dfrac{dy_i}{dt}$. If you let

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \quad \text{and} \quad \mathbf{y}' = \begin{bmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{bmatrix}$$

then the system can be written in matrix form as

$$\mathbf{y}' = A\mathbf{y}.$$

### EXAMPLE 3 Solving a System of Linear Differential Equations

Solve the system of linear differential equations.

$$\begin{aligned}
y_1' &= 4y_1 \\
y_2' &= -y_2 \\
y_3' &= 2y_3
\end{aligned}$$

**SOLUTION**

From calculus, you know that the solution of the differential equation $y' = ky$ is

$$y = Ce^{kt}.$$

So, the solution of the system is

$$\begin{aligned}
y_1 &= C_1e^{4t} \\
y_2 &= C_2e^{-t} \\
y_3 &= C_3e^{2t}.
\end{aligned}$$

The matrix form of the system of linear differential equations in Example 3 is $\mathbf{y}' = A\mathbf{y}$, or

$$\begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}.$$

So, the coefficients of $t$ in the solutions $y_i = C_i e^{\lambda_i t}$ are given by the *eigenvalues* of the matrix $A$.

If $A$ is a *diagonal* matrix, then the solution of

$$\mathbf{y}' = A\mathbf{y}$$

can be obtained immediately, as in Example 3. If $A$ is *not* diagonal, then the solution requires more work. First, attempt to find a matrix $P$ that diagonalizes $A$. Then, the change of variables $\mathbf{y} = P\mathbf{w}$ and $\mathbf{y}' = P\mathbf{w}'$ produces

$$P\mathbf{w}' = \mathbf{y}' = A\mathbf{y} = AP\mathbf{w} \quad \longrightarrow \quad \mathbf{w}' = P^{-1}AP\mathbf{w}$$

where $P^{-1}AP$ is a diagonal matrix. Example 4 demonstrates this procedure.

## EXAMPLE 4 Solving a System of Linear Differential Equations

Solve the system of linear differential equations.

$$\begin{aligned} y_1{}' &= 3y_1 + 2y_2 \\ y_2{}' &= 6y_1 - \phantom{2}y_2 \end{aligned}$$

### SOLUTION

First, find a matrix $P$ that diagonalizes $A = \begin{bmatrix} 3 & 2 \\ 6 & -1 \end{bmatrix}$. The eigenvalues of $A$ are $\lambda_1 = -3$ and $\lambda_2 = 5$, with corresponding eigenvectors $\mathbf{p}_1 = [1 \;\; -3]^T$ and $\mathbf{p}_2 = [1 \;\; 1]^T$. Diagonalize $A$ using the matrix $P$ whose columns consist of $\mathbf{p}_1$ and $\mathbf{p}_2$ to obtain

$$P = \begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} \frac{1}{4} & -\frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} \end{bmatrix}, \quad \text{and} \quad P^{-1}AP = \begin{bmatrix} -3 & 0 \\ 0 & 5 \end{bmatrix}.$$

The system $\mathbf{w}' = P^{-1}AP\mathbf{w}$ has the following form.

$$\begin{bmatrix} w_1{}' \\ w_2{}' \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \quad \longrightarrow \quad \begin{aligned} w_1{}' &= -3w_1 \\ w_2{}' &= \phantom{-}5w_2 \end{aligned}$$

The solution of this system of equations is

$$\begin{aligned} w_1 &= C_1 e^{-3t} \\ w_2 &= C_2 e^{5t}. \end{aligned}$$

To return to the original variables $y_1$ and $y_2$, use the substitution $\mathbf{y} = P\mathbf{w}$ and write

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

which implies that the solution is

$$\begin{aligned} y_1 &= \phantom{-3}w_1 + w_2 = \phantom{-3}C_1 e^{-3t} + C_2 e^{5t} \\ y_2 &= -3w_1 + w_2 = -3C_1 e^{-3t} + C_2 e^{5t}. \end{aligned}$$

If $A$ has eigenvalues with multiplicity greater than 1 or if $A$ has complex eigenvalues, then the technique for solving the system must be modified.

**1.** *Eigenvalues with multiplicity greater than 1:* The coefficient matrix of the system

$$\begin{aligned} y_1{}' &= \phantom{-4y_1 + 4}y_2 \\ y_2{}' &= -4y_1 + 4y_2 \end{aligned} \quad \text{is} \quad A = \begin{bmatrix} 0 & 1 \\ -4 & 4 \end{bmatrix}.$$

The only eigenvalue of $A$ is $\lambda = 2$, and the solution of the system is

$$\begin{aligned} y_1 &= \phantom{(2C_1 + C_2)}C_1 e^{2t} + \phantom{2}C_2 te^{2t} \\ y_2 &= (2C_1 + C_2)e^{2t} + 2C_2 te^{2t}. \end{aligned}$$

**2.** *Complex eigenvalues:* The coefficient matrix of the system

$$\begin{aligned} y_1{}' &= -y_2 \\ y_2{}' &= \phantom{-}y_1 \end{aligned} \quad \text{is} \quad A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

The eigenvalues of $A$ are $\lambda_1 = i$ and $\lambda_2 = -i$, and the solution of the system is

$$\begin{aligned} y_1 &= \phantom{-}C_1 \cos t + C_2 \sin t \\ y_2 &= -C_2 \cos t + C_1 \sin t. \end{aligned}$$

Try checking these solutions by differentiating and substituting into the original systems of equations.

## QUADRATIC FORMS

Eigenvalues and eigenvectors can be used to solve the rotation of axes problem introduced in Section 4.8. Recall that classifying the graph of the quadratic equation

$$ax^2 + bxy + cy^2 + dx + ey + f = 0 \qquad \text{Quadratic equation}$$

is fairly straightforward as long as the equation has no $xy$-term (that is, $b = 0$). If the equation has an $xy$-term, however, then the classification is accomplished most easily by first performing a rotation of axes that eliminates the $xy$-term. The resulting equation (relative to the new $x'y'$-axes) will then be of the form

$$a'(x')^2 + c'(y')^2 + d'x' + e'y' + f' = 0.$$

You will see that the coefficients $a'$ and $c'$ are eigenvalues of the matrix

$$A = \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}.$$

The expression

$$ax^2 + bxy + cy^2 \qquad \text{Quadratic form}$$

is called the **quadratic form** associated with the quadratic equation

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

and the matrix $A$ is called the **matrix of the quadratic form.** Note that the matrix $A$ *is symmetric*. Moreover, the matrix $A$ will be diagonal if and only if its corresponding quadratic form has no $xy$-term, as illustrated in Example 5.

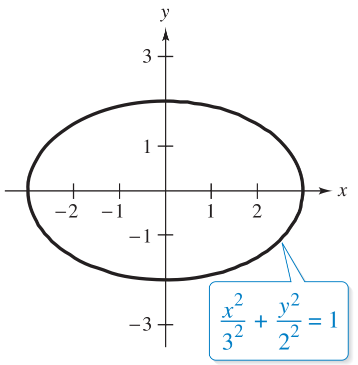



**Figure 7.3**

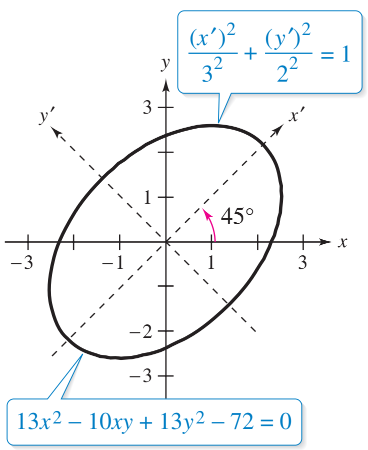



**Figure 7.4**

### EXAMPLE 5 Finding the Matrix of a Quadratic Form

Find the matrix of the quadratic form associated with each quadratic equation.

**a.** $4x^2 + 9y^2 - 36 = 0$ **b.** $13x^2 - 10xy + 13y^2 - 72 = 0$

### SOLUTION

**a.** Because $a = 4$, $b = 0$, and $c = 9$, the matrix is

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}. \qquad \text{Diagonal matrix (no } xy\text{-term)}$$

**b.** Because $a = 13$, $b = -10$, and $c = 13$, the matrix is

$$A = \begin{bmatrix} 13 & -5 \\ -5 & 13 \end{bmatrix}. \qquad \text{Nondiagonal matrix (} xy\text{-term)}$$

In standard form, the equation $4x^2 + 9y^2 - 36 = 0$ is

$$\frac{x^2}{3^2} + \frac{y^2}{2^2} = 1$$

which is the equation of the ellipse shown in Figure 7.3. Although it is not apparent by inspection, the graph of the equation $13x^2 - 10xy + 13y^2 - 72 = 0$ is similar. In fact, when you rotate the $x$- and $y$-axes counterclockwise 45° to form a new $x'y'$-coordinate system, this equation takes the form

$$\frac{(x')^2}{3^2} + \frac{(y')^2}{2^2} = 1$$

which is the equation of the ellipse shown in Figure 7.4.

To see how to use the matrix of a quadratic form to perform a rotation of axes, let

$$X = \begin{bmatrix} x \\ y \end{bmatrix}.$$

Then the quadratic expression $ax^2 + bxy + cy^2 + dx + ey + f$ can be written in matrix form as follows.

$$X^TAX + [d \quad e]X + f = [x \quad y]\begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + [d \quad e]\begin{bmatrix} x \\ y \end{bmatrix} + f$$
$$= ax^2 + bxy + cy^2 + dx + ey + f$$

If $b = 0$, then no rotation is necessary. But if $b \neq 0$, then because $A$ is symmetric, you can apply Theorem 7.10 to conclude that there exists an orthogonal matrix $P$ such that $P^TAP = D$ is diagonal. So, if you let

$$P^TX = X' = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

then it follows that $X = PX'$, and $X^TAX = (PX')^TA(PX') = (X')^TP^TAPX' = (X')^TDX'$.

The choice of the matrix $P$ must be made with care. Because $P$ is orthogonal, its determinant will be $\pm 1$. It can be shown (see Exercise 65) that if $P$ is chosen so that $|P| = 1$, then $P$ will be of the form

$$P = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

where $\theta$ gives the angle of rotation of the conic measured from the positive $x$-axis to the positive $x'$-axis. This leads to the **Principal Axes Theorem.**

**REMARK**

Note that the matrix product $[d \quad e]PX'$ has the form

$$(d\cos\theta + e\sin\theta)x' + (-d\sin\theta + e\cos\theta)y'.$$

### Principal Axes Theorem

For a conic whose equation is $ax^2 + bxy + cy^2 + dx + ey + f = 0$, the rotation given by $X = PX'$ eliminates the $xy$-term when $P$ is an orthogonal matrix, with $|P| = 1$, that diagonalizes $A$. That is,

$$P^TAP = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

where $\lambda_1$ and $\lambda_2$ are eigenvalues of $A$. The equation of the rotated conic is given by

$$\lambda_1(x')^2 + \lambda_2(y')^2 + [d \quad e]PX' + f = 0.$$

## EXAMPLE 6 Rotation of a Conic

Perform a rotation of axes to eliminate the $xy$-term in the quadratic equation

$$13x^2 - 10xy + 13y^2 - 72 = 0.$$

**SOLUTION**

The matrix of the quadratic form associated with this equation is

$$A = \begin{bmatrix} 13 & -5 \\ -5 & 13 \end{bmatrix}.$$

Because the characteristic polynomial of $A$ is $(\lambda - 8)(\lambda - 18)$ (check this), it follows that the eigenvalues of $A$ are $\lambda_1 = 8$ and $\lambda_2 = 18$. So, the equation of the rotated conic is

$$8(x')^2 + 18(y')^2 - 72 = 0$$

which, when written in the standard form

$$\frac{(x')^2}{3^2} + \frac{(y')^2}{2^2} = 1$$

is the equation of an ellipse. (See Figure 7.4.)

In Example 6, the eigenvectors of the matrix $A$ are

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

which you can normalize to form the columns of $P$, as follows.

$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Note first that $|P| = 1$, which implies that $P$ is a rotation. Moreover, because $\cos 45° = 1/\sqrt{2} = \sin 45°$, the angle of rotation is $45°$, as shown in Figure 7.4.

The orthogonal matrix $P$ specified in the Principal Axes Theorem is not unique. Its entries depend on the ordering of the eigenvalues $\lambda_1$ and $\lambda_2$ *and* on the subsequent choice of eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$. For instance, in the solution of Example 6, any of the following choices of $P$ would have worked.

$$\begin{array}{cc} \mathbf{x}_1 & \mathbf{x}_2 \\ \end{array} \qquad\qquad \begin{array}{cc} \mathbf{x}_1 & \mathbf{x}_2 \\ \end{array} \qquad\qquad \begin{array}{cc} \mathbf{x}_1 & \mathbf{x}_2 \\ \end{array}$$

$$\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \qquad \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \qquad \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\lambda_1 = 8,\ \lambda_2 = 18 \qquad \lambda_1 = 18,\ \lambda_2 = 8 \qquad \lambda_1 = 18,\ \lambda_2 = 8$$

$$\theta = 225° \qquad\qquad \theta = 135° \qquad\qquad \theta = 315°$$

For any of these choices of $P$, the graph of the rotated conic will, of course, be the same. (See Figure 7.5.)

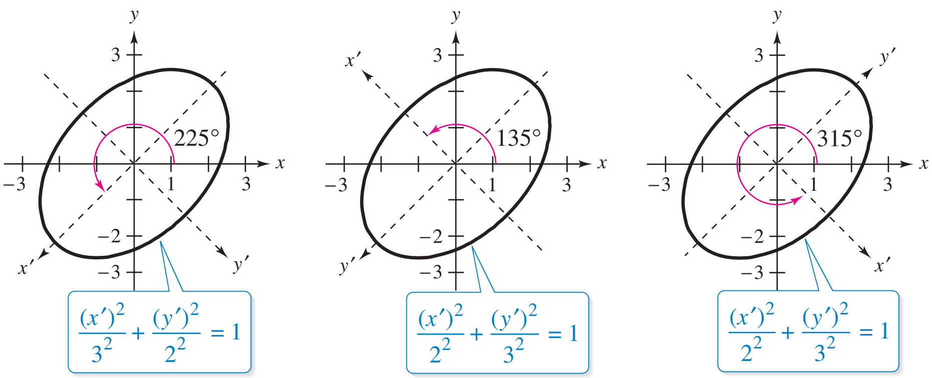



**Figure 7.5**

The following summarizes the steps used to apply the Principal Axes Theorem.

1. Form the matrix $A$ and find its eigenvalues $\lambda_1$ and $\lambda_2$.
2. Find eigenvectors corresponding to $\lambda_1$ and $\lambda_2$. Normalize these eigenvectors to form the columns of $P$.
3. If $|P| = -1$, then multiply one of the columns of $P$ by $-1$ to obtain a matrix of the form
$$P = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$
4. The angle $\theta$ represents the angle of rotation of the conic.
5. The equation of the rotated conic is $\lambda_1(x')^2 + \lambda_2(y')^2 + [d \quad e]PX' + f = 0$.

Example 7 shows how to apply the Principal Axes Theorem to rotate a conic whose center has been translated away from the origin.

### EXAMPLE 7 Rotation of a Conic

Perform a rotation of axes to eliminate the $xy$-term in the quadratic equation

$$3x^2 - 10xy + 3y^2 + 16\sqrt{2}x - 32 = 0.$$

**SOLUTION**

The matrix of the quadratic form associated with this equation is

$$A = \begin{bmatrix} 3 & -5 \\ -5 & 3 \end{bmatrix}.$$

The eigenvalues of $A$ are

$$\lambda_1 = 8 \quad \text{and} \quad \lambda_2 = -2$$

with corresponding eigenvectors of

$$\mathbf{x}_1 = (-1, 1) \quad \text{and} \quad \mathbf{x}_2 = (-1, -1).$$

This implies that the matrix $P$ is

$$P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \text{ where } |P| = 1.$$

Because $\cos 135° = -1/\sqrt{2}$ and $\sin 135° = 1/\sqrt{2}$, the angle of rotation is 135°. Finally, from the matrix product

$$[d \quad e]PX' = [16\sqrt{2} \quad 0]\begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$= -16x' - 16y'$$

the equation of the rotated conic is

$$8(x')^2 - 2(y')^2 - 16x' - 16y' - 32 = 0.$$

In standard form, the equation

$$\frac{(x'-1)^2}{1^2} - \frac{(y'+4)^2}{2^2} = 1$$

is the equation of a hyperbola. Its graph is shown in Figure 7.6.

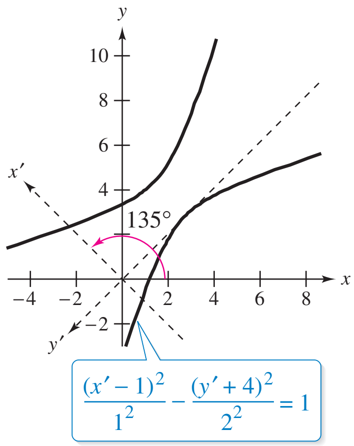
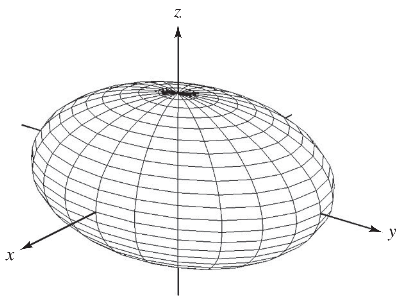



**Figure 7.6**

Quadratic forms can also be used to analyze equations of quadric surfaces in $R^3$, which are the three-dimensional analogs of conic sections. The equation of a quadric surface in $R^3$ is a second-degree polynomial of the form

$$ax^2 + by^2 + cz^2 + dxy + exz + fyz + gx + hy + iz + j = 0.$$

There are six basic types of quadric surfaces: ellipsoids, hyperboloids of one sheet, hyperboloids of two sheets, elliptic cones, elliptic paraboloids, and hyperbolic paraboloids. The intersection of a surface with a plane, called the **trace** of the surface in the plane, is useful to help visualize the graph of the surface in $R^3$. The six basic types of quadric surfaces, together with their traces, are shown on the next two pages.

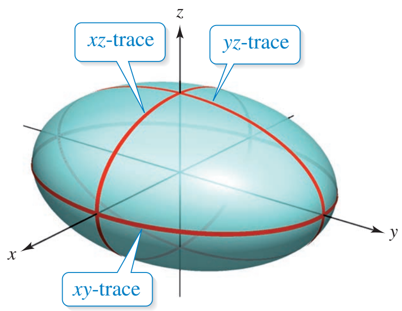


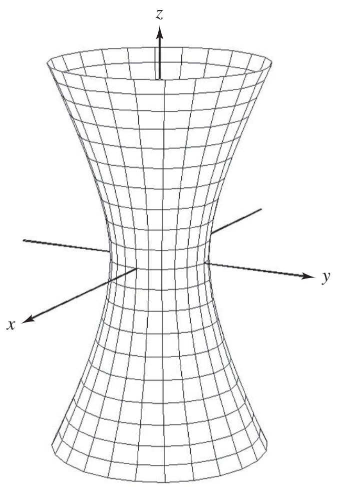

***Ellipsoid***

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

| *Trace* | *Plane* |
|---|---|
| Ellipse | Parallel to $xy$-plane |
| Ellipse | Parallel to $xz$-plane |
| Ellipse | Parallel to $yz$-plane |

The surface is a sphere when $a = b = c \neq 0$.



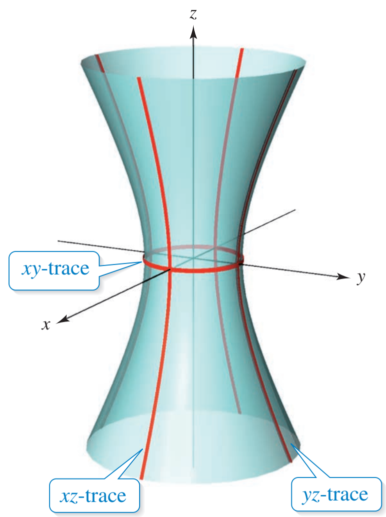



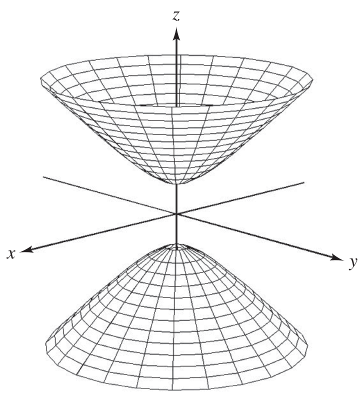

***Hyperboloid of One Sheet***

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

| *Trace* | *Plane* |
|---|---|
| Ellipse | Parallel to $xy$-plane |
| Hyperbola | Parallel to $xz$-plane |
| Hyperbola | Parallel to $yz$-plane |

The axis of the hyperboloid corresponds to the variable whose coefficient is negative.



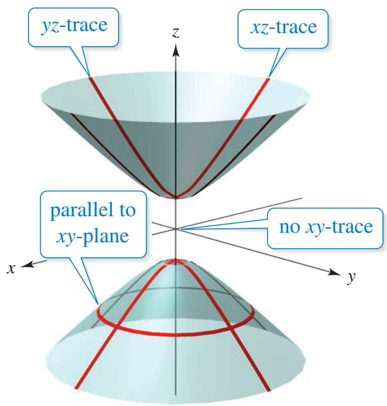



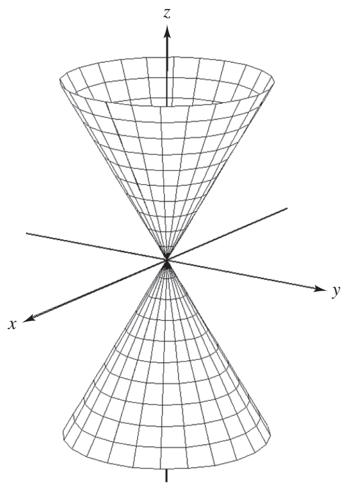

***Hyperboloid of Two Sheets***

$$\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

| *Trace* | *Plane* |
|---|---|
| Ellipse | Parallel to $xy$-plane |
| Hyperbola | Parallel to $xz$-plane |
| Hyperbola | Parallel to $yz$-plane |

The axis of the hyperboloid corresponds to the variable whose coefficient is positive. There is no trace in the coordinate plane perpendicular to this axis.



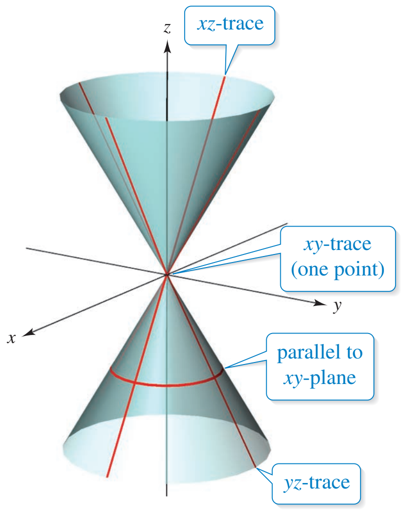


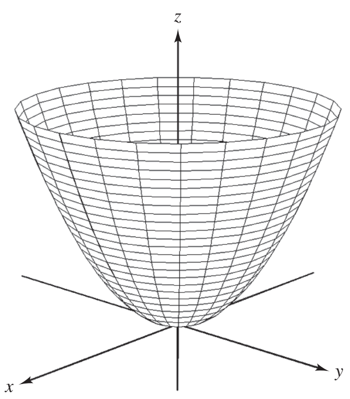

***Elliptic Cone***

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$$

| *Trace* | *Plane* |
|---|---|
| Ellipse | Parallel to $xy$-plane |
| Hyperbola | Parallel to $xz$-plane |
| Hyperbola | Parallel to $yz$-plane |

The axis of the cone corresponds to the variable whose coefficient is negative. The traces in the coordinate planes parallel to this axis are intersecting lines.



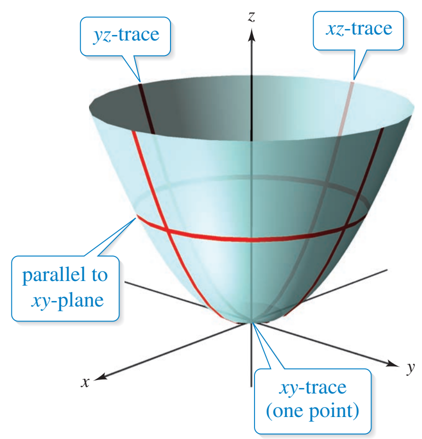



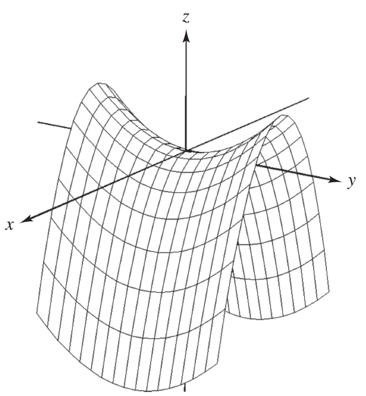

***Elliptic Paraboloid***

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

| *Trace* | *Plane* |
|---|---|
| Ellipse | Parallel to $xy$-plane |
| Parabola | Parallel to $xz$-plane |
| Parabola | Parallel to $yz$-plane |

The axis of the paraboloid corresponds to the variable raised to the first power.



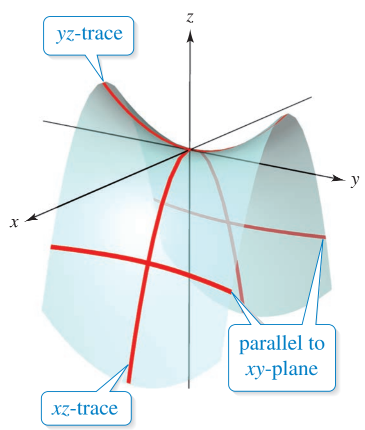




***Hyperbolic Paraboloid***

$$z = \frac{y^2}{b^2} - \frac{x^2}{a^2}$$

| *Trace* | *Plane* |
|---|---|
| Hyperbola | Parallel to $xy$-plane |
| Parabola | Parallel to $xz$-plane |
| Parabola | Parallel to $yz$-plane |

The axis of the paraboloid corresponds to the variable raised to the first power.



**LINEAR ALGEBRA APPLIED**

Some of the world's most unusual architecture makes use of quadric surfaces. For instance, *Catedral Metropolitana Nossa Senhora Aparecida,* a cathedral located in Brasilia, Brazil, is in the shape of a hyperboloid of one sheet. It was designed by Pritzker Prize winning architect Oscar Niemeyer, and dedicated in 1970. The sixteen identical curved steel columns, weighing 90 tons each, are intended to represent two hands reaching up to the sky. Pieced together between the columns, in the 10-meter-wide and 30-meter-high triangular gaps formed by the columns, is semitransparent stained glass, which allows light inside for nearly the entire height of the columns.

The quadratic form of the equation

$$ax^2 + by^2 + cz^2 + dxy + exz + fyz + gx + hy + iz + j = 0 \qquad \text{Quadric surface}$$

is defined as

$$ax^2 + by^2 + cz^2 + dxy + exz + fyz. \qquad \text{Quadratic form}$$

The corresponding matrix is

$$A = \begin{bmatrix} a & \frac{d}{2} & \frac{e}{2} \\ \frac{d}{2} & b & \frac{f}{2} \\ \frac{e}{2} & \frac{f}{2} & c \end{bmatrix}.$$

In its three-dimensional version, the Principal Axes Theorem relates the eigenvalues and eigenvectors of $A$ to the equation of the rotated surface, as shown in Example 8.

## EXAMPLE 8 Rotation of a Quadric Surface

Perform a rotation of axes to eliminate the $xz$-term in the quadratic equation

$$5x^2 + 4y^2 + 5z^2 + 8xz - 36 = 0.$$

### SOLUTION

The matrix $A$ associated with this quadratic equation is

$$A = \begin{bmatrix} 5 & 0 & 4 \\ 0 & 4 & 0 \\ 4 & 0 & 5 \end{bmatrix}$$

which has eigenvalues of $\lambda_1 = 1$, $\lambda_2 = 4$, and $\lambda_3 = 9$. So, in the rotated $x'y'z'$-system, the quadratic equation is $(x')^2 + 4(y')^2 + 9(z')^2 - 36 = 0$, which in standard form is

$$\frac{(x')^2}{6^2} + \frac{(y')^2}{3^2} + \frac{(z')^2}{2^2} = 1.$$

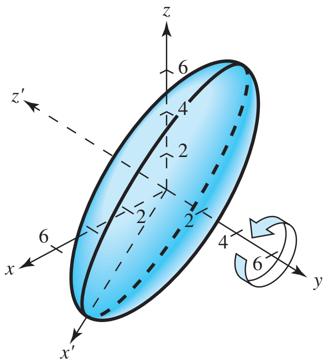



**Figure 7.7**

The graph of this equation is an ellipsoid. As shown in Figure 7.7, the $x'y'z'$-axes represent a counterclockwise rotation of 45° about the $y$-axis. Moreover, the orthogonal matrix

$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

whose columns are the eigenvectors of $A$, has the property that $P^TAP$ is diagonal.

# 7.4 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

**Finding Age Distribution Vectors** **In Exercises 1–6, use the age transition matrix $A$ and age distribution vector $\mathbf{x}_1$ to find the age distribution vectors $\mathbf{x}_2$ and $\mathbf{x}_3$.**

**1.** $A = \begin{bmatrix} 0 & 2 \\ \frac{1}{2} & 0 \end{bmatrix}, \mathbf{x}_1 = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$

**2.** $A = \begin{bmatrix} 0 & 4 \\ \frac{1}{16} & 0 \end{bmatrix}, \mathbf{x}_1 = \begin{bmatrix} 160 \\ 160 \end{bmatrix}$

**3.** $A = \begin{bmatrix} 0 & 3 & 4 \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{bmatrix}, \mathbf{x}_1 = \begin{bmatrix} 12 \\ 12 \\ 12 \end{bmatrix}$

**4.** $A = \begin{bmatrix} 0 & 2 & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{bmatrix}, \mathbf{x}_1 = \begin{bmatrix} 8 \\ 8 \\ 8 \end{bmatrix}$

**5.** $A = \begin{bmatrix} 0 & 2 & 2 & 0 \\ \frac{1}{4} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \end{bmatrix}, \mathbf{x}_1 = \begin{bmatrix} 100 \\ 100 \\ 100 \\ 100 \end{bmatrix}$

**6.** $A = \begin{bmatrix} 0 & 6 & 4 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 \end{bmatrix}, \mathbf{x}_1 = \begin{bmatrix} 24 \\ 24 \\ 24 \\ 24 \\ 24 \end{bmatrix}$

**7.** Find a stable age distribution vector for the age transition matrix in Exercise 1.

**8.** Find a stable age distribution vector for the age transition matrix in Exercise 2.

**9.** Find a stable age distribution vector for the age transition matrix in Exercise 3.

**10.** Find a stable age distribution vector for the age transition matrix in Exercise 4.

**11.** Find a stable age distribution vector for the age transition matrix in Exercise 5.

**12.** Find a stable age distribution vector for the age transition matrix in Exercise 6.

**13.** **Population Growth Model** A population has the following characteristics.

(a) A total of 75% of the population survives the first year. Of that 75%, 25% survives the second year. The maximum life span is 3 years.

(b) The average number of offspring for each member of the population is 2 the first year, 4 the second year, and 2 the third year.

The population now consists of 160 members in each of the three age classes. How many members will there be in each age class in 1 year? in 2 years?

**14.** **Population Growth Model** A population has the following characteristics.

(a) A total of 80% of the population survives the first year. Of that 80%, 25% survives the second year. The maximum life span is 3 years.

(b) The average number of offspring for each member of the population is 3 the first year, 6 the second year, and 3 the third year.

The population now consists of 120 members in each of the three age classes. How many members will there be in each age class in 1 year? in 2 years?

**15.** **Population Growth Model** A population has the following characteristics.

(a) A total of 60% of the population survives the first year. Of that 60%, 50% survives the second year. The maximum life span is 3 years.

(b) The average number of offspring for each member of the population is 2 the first year, 5 the second year, and 2 the third year.

The population now consists of 100 members in each of the three age classes. How many members will there be in each age class in 1 year? in 2 years?

**16.** Find the limit (if it exists) of $A^n\mathbf{x}_1$ as $n$ approaches infinity for the following matrices.

$$A = \begin{bmatrix} 0 & 2 \\ \frac{1}{2} & 0 \end{bmatrix} \text{ and } \mathbf{x}_1 = \begin{bmatrix} a \\ a \end{bmatrix}$$

**Solving a System of Linear Differential Equations** **In Exercises 17–28, solve the system of first-order linear differential equations.**

**17.** $\begin{aligned} y_1' &= 2y_1 \\ y_2' &= y_2 \end{aligned}$

**18.** $\begin{aligned} y_1' &= -5y_1 \\ y_2' &= 4y_2 \end{aligned}$

**19.** $\begin{aligned} y_1' &= -4y_1 \\ y_2' &= -\tfrac{1}{2}y_2 \end{aligned}$

**20.** $\begin{aligned} y_1' &= \tfrac{1}{2}y_1 \\ y_2' &= \tfrac{1}{8}y_2 \end{aligned}$

**21.** $\begin{aligned} y_1' &= -y_1 \\ y_2' &= 6y_2 \\ y_3' &= y_3 \end{aligned}$

**22.** $\begin{aligned} y_1' &= 5y_1 \\ y_2' &= -2y_2 \\ y_3' &= -3y_3 \end{aligned}$

**23.** $\begin{aligned} y_1' &= -12y_1 \\ y_2' &= -6y_2 \\ y_3' &= 7y_3 \end{aligned}$

**24.** $\begin{aligned} y_1' &= -\tfrac{2}{3}y_1 \\ y_2' &= -\tfrac{3}{5}y_2 \\ y_3' &= -8y_3 \end{aligned}$

**25.** $\begin{aligned} y_1' &= -0.3y_1 \\ y_2' &= 0.4y_2 \\ y_3' &= -0.6y_3 \end{aligned}$

**26.** $\begin{aligned} y_1' &= \pi y_1 \\ y_2' &= -\pi y_2 \\ y_3' &= \pi^2 y_3 \end{aligned}$

**27.** $\begin{aligned} y_1' &= 7y_1 \\ y_2' &= 9y_2 \\ y_3' &= -7y_3 \\ y_4' &= -9y_4 \end{aligned}$

**28.** $\begin{aligned} y_1' &= -0.1y_1 \\ y_2' &= -\tfrac{7}{4}y_2 \\ y_3' &= -2\pi y_3 \\ y_4' &= \sqrt{5}y_4 \end{aligned}$

**Solving a System of Linear Differential Equations** In Exercises 29–36, solve the system of first-order linear differential equations.

**29.** $\begin{aligned} y_1' &= y_1 - 4y_2 \\ y_2' &= 2y_2 \end{aligned}$

**30.** $\begin{aligned} y_1' &= y_1 - 4y_2 \\ y_2' &= -2y_1 + 8y_2 \end{aligned}$

**31.** $\begin{aligned} y_1' &= y_1 + 2y_2 \\ y_2' &= 2y_1 + y_2 \end{aligned}$

**32.** $\begin{aligned} y_1' &= y_1 - y_2 \\ y_2' &= 2y_1 + 4y_2 \end{aligned}$

**33.** $\begin{aligned} y_1' &= -3y_2 + 5y_3 \\ y_2' &= -4y_1 + 4y_2 - 10y_3 \\ y_3' &= 4y_3 \end{aligned}$

**34.** $\begin{aligned} y_1' &= -2y_1 + y_3 \\ y_2' &= 3y_2 + 4y_3 \\ y_3' &= y_3 \end{aligned}$

**35.** $\begin{aligned} y_1' &= y_1 - 2y_2 + y_3 \\ y_2' &= 2y_2 + 4y_3 \\ y_3' &= 3y_3 \end{aligned}$

**36.** $\begin{aligned} y_1' &= 2y_1 + y_2 + y_3 \\ y_2' &= y_1 + y_2 \\ y_3' &= y_1 + y_3 \end{aligned}$

**Writing a System and Verifying the General Solution** In Exercises 37–40, write the system of first-order linear differential equations represented by the matrix equation $\mathbf{y}' = A\mathbf{y}$. Then verify the given general solution.

**37.** $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $\begin{aligned} y_1 &= C_1e^t + C_2te^t \\ y_2 &= C_2e^t \end{aligned}$

**38.** $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, $\begin{aligned} y_1 &= C_1e^t \cos t + C_2e^t \sin t \\ y_2 &= -C_2e^t \cos t + C_1e^t \sin t \end{aligned}$

**39.** $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -4 & 0 \end{bmatrix}$,

$$\begin{aligned} y_1 &= C_1 + C_2 \cos 2t + C_3 \sin 2t \\ y_2 &= 2C_3 \cos 2t - 2C_2 \sin 2t \\ y_3 &= -4C_2 \cos 2t - 4C_3 \sin 2t \end{aligned}$$

**40.** $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix}$,

$$\begin{aligned} y_1 &= C_1e^t + C_2te^t + C_3t^2e^t \\ y_2 &= (C_1 + C_2)e^t + (C_2 + 2C_3)te^t + C_3t^2e^t \\ y_3 &= (C_1 + 2C_2 + 2C_3)e^t + (C_2 + 4C_3)te^t + C_3t^2e^t \end{aligned}$$

**Finding the Matrix of a Quadratic Form** In Exercises 41–46, find the matrix $A$ of the quadratic form associated with the equation.

**41.** $x^2 + y^2 - 4 = 0$

**42.** $x^2 - 4xy + y^2 - 4 = 0$

**43.** $9x^2 + 10xy - 4y^2 - 36 = 0$

**44.** $12x^2 - 5xy - x + 2y - 20 = 0$

**45.** $10xy - 10y^2 + 4x - 48 = 0$

**46.** $16x^2 - 4xy + 20y^2 - 72 = 0$

**Finding the Matrix of a Quadratic Form** In Exercises 47–52, find the matrix $A$ of the quadratic form associated with the equation. Then find the eigenvalues of $A$ and an orthogonal matrix $P$ such that $P^TAP$ is diagonal.

**47.** $2x^2 - 3xy - 2y^2 + 10 = 0$

**48.** $5x^2 - 2xy + 5y^2 + 10x - 17 = 0$

**49.** $13x^2 + 6\sqrt{3}xy + 7y^2 - 16 = 0$

**50.** $3x^2 - 2\sqrt{3}xy + y^2 + 2x + 2\sqrt{3}y = 0$

**51.** $16x^2 - 24xy + 9y^2 - 60x - 80y + 100 = 0$

**52.** $17x^2 + 32xy - 7y^2 - 75 = 0$

**Rotation of a Conic** In Exercises 53–60, use the Principal Axes Theorem to perform a rotation of axes to eliminate the $xy$-term in the quadratic equation. Identify the resulting rotated conic and give its equation in the new coordinate system.

**53.** $13x^2 - 8xy + 7y^2 - 45 = 0$

**54.** $x^2 + 4xy + y^2 - 9 = 0$

**55.** $2x^2 - 4xy + 5y^2 - 36 = 0$

**56.** $7x^2 + 32xy - 17y^2 - 50 = 0$

**57.** $2x^2 + 4xy + 2y^2 + 6\sqrt{2}x + 2\sqrt{2}y + 4 = 0$

**58.** $8x^2 + 8xy + 8y^2 + 10\sqrt{2}x + 26\sqrt{2}y + 31 = 0$

**59.** $xy + x - 2y + 3 = 0$

**60.** $5x^2 - 2xy + 5y^2 + 10\sqrt{2}x = 0$

**Rotation of a Quadric Surface** In Exercises 61–64, find the matrix $A$ of the quadratic form associated with the equation. Then find the equation of the quadric surface in the rotated $x'y'z'$-system.

**61.** $3x^2 - 2xy + 3y^2 + 8z^2 - 16 = 0$

**62.** $2x^2 + 2y^2 + 2z^2 + 2xy + 2xz + 2yz - 1 = 0$

**63.** $x^2 + 2y^2 + 2z^2 + 2yz - 1 = 0$

**64.** $x^2 + y^2 + z^2 + 2xy - 8 = 0$

**65.** Let $P$ be a $2 \times 2$ orthogonal matrix such that $|P| = 1$. Show that there exists a number $\theta$, $0 \le \theta < 2\pi$, such that

$$P = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

**66.** **CAPSTONE** Explain each of the following.

(a) How to model population growth using an age transition matrix and an age distribution vector, and how to find a stable age distribution vector

(b) How to use a matrix equation to solve a system of first-order linear differential equations

(c) How to use the Principal Axes Theorem to perform a rotation of axes for a conic and a quadric surface

# 7 Review Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

**Characteristic Equation, Eigenvalues, and Basis** In Exercises 1–6, find (a) the characteristic equation of $A$, (b) the eigenvalues of $A$, and (c) a basis for the eigenspace corresponding to each eigenvalue.

**1.** $A = \begin{bmatrix} 2 & 1 \\ 5 & -2 \end{bmatrix}$

**2.** $A = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}$

**3.** $A = \begin{bmatrix} 9 & 4 & -3 \\ -2 & 0 & 6 \\ -1 & -4 & 11 \end{bmatrix}$

**4.** $A = \begin{bmatrix} -4 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

**5.** $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

**6.** $A = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -2 \\ 1 & 0 & -2 \end{bmatrix}$

**Characteristic Equation, Eigenvalues, and Basis** In Exercises 7 and 8, use a software program or a graphing utility with matrix capabilities to find (a) the characteristic equation of $A$, (b) the eigenvalues of $A$, and (c) a basis for the eigenspace corresponding to each eigenvalue.

**7.** $A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

**8.** $A = \begin{bmatrix} 3 & 0 & 2 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

**Determining Whether a Matrix Is Diagonalizable** In Exercises 9–14, determine whether $A$ is diagonalizable. If it is, then find a nonsingular matrix $P$ such that $P^{-1}AP$ is diagonal.

**9.** $A = \begin{bmatrix} 1 & -4 \\ -2 & 8 \end{bmatrix}$

**10.** $A = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 \end{bmatrix}$

**11.** $A = \begin{bmatrix} -2 & -1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

**12.** $A = \begin{bmatrix} 3 & -2 & 2 \\ -2 & 0 & -1 \\ 2 & -1 & 0 \end{bmatrix}$

**13.** $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$

**14.** $A = \begin{bmatrix} 2 & -1 & 1 \\ -2 & 3 & -2 \\ -1 & 1 & 0 \end{bmatrix}$

**15.** For what value(s) of $a$ does the matrix

$$A = \begin{bmatrix} 0 & 1 \\ a & 1 \end{bmatrix}$$

have the following characteristics?

(a) $A$ has an eigenvalue of multiplicity 2.

(b) $A$ has $-1$ and 2 as eigenvalues.

(c) $A$ has real eigenvalues.

**16.** Show that if $0 < \theta < \pi$, then the transformation for a counterclockwise rotation through an angle $\theta$ has no real eigenvalues.

**Writing** In Exercises 17–20, explain why the matrix is not diagonalizable.

**17.** $A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$

**18.** $A = \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix}$

**19.** $A = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

**20.** $A = \begin{bmatrix} -2 & 3 & 1 \\ 0 & 4 & 3 \\ 0 & 0 & -2 \end{bmatrix}$

**Determining Whether Two Matrices Are Similar** In Exercises 21–24, determine whether the matrices are similar. If they are, then find a matrix $P$ such that $A = P^{-1}BP$.

**21.** $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

**22.** $A = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}, B = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$

**23.** $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

**24.** $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}, B = \begin{bmatrix} 1 & -3 & -3 \\ 3 & -5 & -3 \\ -3 & 3 & 1 \end{bmatrix}$

**Determining Symmetric and Orthogonal Matrices** In Exercises 25–30, determine whether the matrix is symmetric, orthogonal, both, or neither.

**25.** $A = \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$

**26.** $A = \begin{bmatrix} \frac{2\sqrt{5}}{5} & \frac{\sqrt{5}}{5} \\ \frac{\sqrt{5}}{5} & -\frac{2\sqrt{5}}{5} \end{bmatrix}$

**27.** $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

**28.** $A = \begin{bmatrix} \frac{4}{5} & 0 & \frac{3}{5} \\ 0 & 1 & 0 \\ -\frac{3}{5} & 0 & \frac{4}{5} \end{bmatrix}$

**29.** $A = \begin{bmatrix} -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{bmatrix}$

**30.** $A = \begin{bmatrix} \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} & \frac{2\sqrt{3}}{3} & 0 \\ \frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{3}}{3} \end{bmatrix}$

**Eigenvectors of a Symmetric Matrix** **In Exercises 31–34, show that any two eigenvectors of the symmetric matrix corresponding to distinct eigenvalues are orthogonal.**

**31.** $\begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$

**32.** $\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$

**33.** $\begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

**34.** $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

**Orthogonally Diagonalizable Matrices** **In Exercises 35 and 36, determine whether the matrix is orthogonally diagonalizable.**

**35.** $\begin{bmatrix} -3 & -1 \\ -1 & -2 \end{bmatrix}$

**36.** $\begin{bmatrix} 4 & 1 & 2 \\ 0 & -1 & 0 \\ 2 & 1 & -5 \end{bmatrix}$

**Orthogonal Diagonalization** **In Exercises 37–42, find a matrix $P$ that orthogonally diagonalizes $A$. Verify that $P^TAP$ gives the proper diagonal form.**

**37.** $A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$

**38.** $A = \begin{bmatrix} 8 & 15 \\ 15 & -8 \end{bmatrix}$

**39.** $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

**40.** $A = \begin{bmatrix} 3 & 0 & -3 \\ 0 & -3 & 0 \\ -3 & 0 & 3 \end{bmatrix}$

**41.** $A = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 2 \end{bmatrix}$

**42.** $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

**Steady State Probability Vector** **In Exercises 43–50, find the steady state probability vector for the matrix. An eigenvector v of an $n \times n$ matrix $A$ is called a steady state probability vector when $A\mathbf{v} = \mathbf{v}$ and the components of v add up to 1.**

**43.** $A = \begin{bmatrix} \frac{2}{3} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} \end{bmatrix}$

**44.** $A = \begin{bmatrix} \frac{1}{2} & 1 \\ \frac{1}{2} & 0 \end{bmatrix}$

**45.** $A = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$

**46.** $A = \begin{bmatrix} 0.4 & 0.2 \\ 0.6 & 0.8 \end{bmatrix}$

**47.** $A = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$

**48.** $A = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} \end{bmatrix}$

**49.** $A = \begin{bmatrix} 0.7 & 0.1 & 0.1 \\ 0.2 & 0.7 & 0.1 \\ 0.1 & 0.2 & 0.8 \end{bmatrix}$

**50.** $A = \begin{bmatrix} 0.3 & 0.1 & 0.4 \\ 0.2 & 0.4 & 0.0 \\ 0.5 & 0.5 & 0.6 \end{bmatrix}$

**51. Proof** Prove that if $A$ is an $n \times n$ symmetric matrix, then $P^TAP$ is symmetric for any $n \times n$ matrix $P$.

**52.** Show that the characteristic equation of

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -\frac{a_0}{a_n} & -\frac{a_1}{a_n} & -\frac{a_2}{a_n} & -\frac{a_3}{a_n} & \cdots & -\frac{a_{n-1}}{a_n} \end{bmatrix}$$

$a_n \neq 0$, is $p(\lambda) = a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0$. Matrix $A$ is called the **companion matrix** of the polynomial $p$.

**Finding the Companion Matrix and Eigenvalues** **In Exercises 53 and 54, use the result of Exercise 52 to find the companion matrix $A$ of the polynomial and find the eigenvalues of $A$.**

**53.** $p(\lambda) = -9\lambda + 4\lambda^2$

**54.** $p(\lambda) = 189 - 120\lambda - 7\lambda^2 + 2\lambda^3$

**55.** The characteristic equation of the matrix

$$A = \begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}$$

is $\lambda^2 - 10\lambda + 24 = 0$. Because $A^2 - 10A + 24I_2 = O$, you can find powers of $A$ by the following process.

$$A^2 = 10A - 24I_2, \quad A^3 = 10A^2 - 24A,$$
$$A^4 = 10A^3 - 24A^2, \ldots$$

Use this process to find the matrices $A^2$ and $A^3$.

**56.** Repeat Exercise 55 for the matrix

$$A = \begin{bmatrix} 9 & 4 & -3 \\ -2 & 0 & 6 \\ -1 & -4 & 11 \end{bmatrix}.$$

**57. Proof** Let $A$ be an $n \times n$ matrix.

(a) Prove or disprove that an eigenvector of $A$ is also an eigenvector of $A^2$.

(b) Prove or disprove that an eigenvector of $A^2$ is also an eigenvector of $A$.

**58. Proof** Let $A$ be an $n \times n$ matrix. Prove that if $A\mathbf{x} = \lambda\mathbf{x}$, then $\mathbf{x}$ is an eigenvector of $(A + cI)$. What is the corresponding eigenvalue?

**59. Proof** Let $A$ and $B$ be $n \times n$ matrices. Prove that if $A$ is nonsingular, then $AB$ is similar to $BA$.

**60. Proof**

(a) Find a symmetric matrix $B$ such that $B^2 = A$ for the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

(b) Generalize the result of part (a) by proving that if $A$ is an $n \times n$ symmetric matrix with positive eigenvalues, then there exists a symmetric matrix $B$ such that $B^2 = A$.

**61.** Find an orthogonal matrix $P$ such that $P^{-1}AP$ is diagonal for the matrix

$$A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}.$$

**62. Writing** Let $A$ be an $n \times n$ idempotent matrix (that is, $A^2 = A$). Describe the eigenvalues of $A$.

**63. Writing** The following matrix has an eigenvalue $\lambda = 2$ of multiplicity 4.

$$A = \begin{bmatrix} 2 & a & 0 & 0 \\ 0 & 2 & b & 0 \\ 0 & 0 & 2 & c \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

(a) Under what conditions is $A$ diagonalizable?

(b) Under what conditions does the eigenspace of $\lambda = 2$ have dimension 1? 2? 3?

**64.** Determine all $n \times n$ symmetric matrices that have 0 as their only eigenvalue.

**True or False?** **In Exercises 65 and 66, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.**

**65.** (a) An eigenvector of an $n \times n$ matrix $A$ is a nonzero vector $\mathbf{x}$ in $R^n$ such that $A\mathbf{x}$ is a scalar multiple of $\mathbf{x}$.

(b) Similar matrices may or may not have the same eigenvalues.

(c) To diagonalize a square matrix $A$, you need to find an invertible matrix $P$ such that $P^{-1}AP$ is diagonal.

**66.** (a) An eigenvalue of a matrix $A$ is a scalar $\lambda$ such that $\det(\lambda I - A) = 0$.

(b) An eigenvector may be the zero vector $\mathbf{0}$.

(c) A matrix $A$ is orthogonally diagonalizable when there exists an orthogonal matrix $P$ such that $P^{-1}AP = D$ is diagonal.

**Finding Age Distribution Vectors** **In Exercises 67–70, use the age transition matrix $A$ and the age distribution vector $\mathbf{x}_1$ to find the age distribution vectors $\mathbf{x}_2$ and $\mathbf{x}_3$. Then find a stable age distribution vector for the population.**

**67.** $A = \begin{bmatrix} 0 & 1 \\ \frac{1}{4} & 0 \end{bmatrix}, \mathbf{x}_1 = \begin{bmatrix} 100 \\ 100 \end{bmatrix}$

**68.** $A = \begin{bmatrix} 0 & 1 \\ \frac{3}{4} & 0 \end{bmatrix}, \mathbf{x}_1 = \begin{bmatrix} 32 \\ 32 \end{bmatrix}$

**69.** $A = \begin{bmatrix} 0 & 3 & 12 \\ 1 & 0 & 0 \\ 0 & \frac{1}{6} & 0 \end{bmatrix}, \mathbf{x}_1 = \begin{bmatrix} 300 \\ 300 \\ 300 \end{bmatrix}$

**70.** $A = \begin{bmatrix} 0 & 2 & 2 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \mathbf{x}_1 = \begin{bmatrix} 240 \\ 240 \\ 240 \end{bmatrix}$

**71. Population Growth Model** A population has the following characteristics.

(a) A total of 90% of the population survives the first year. Of that 90%, 75% survives the second year. The maximum life span is 3 years.

(b) The average number of offspring for each member of the population is 4 the first year, 6 the second year, and 2 the third year.

The population now consists of 120 members in each of the three age classes. How many members will there be in each age class in 1 year? in 2 years?

**72. Population Growth Model** A population has the following characteristics.

(a) A total of 75% of the population survives the first year. Of that 75%, 60% survives the second year. The maximum life span is 3 years.

(b) The average number of offspring for each member of the population is 4 the first year, 8 the second year, and 2 the third year.

The population now consists of 120 members in each of the three age classes. How many members will there be in each age class in 1 year? in 2 years?

**Solving a System of Linear Differential Equations** **In Exercises 73–78, solve the system of first-order linear differential equations.**

**73.** $y_1' = 3y_1$
$y_2' = 8y_2$
$y_3' = -8y_3$

**74.** $y_1' = 10y_1$
$y_2' = -0.1y_2$
$y_3' = \sqrt{2}y_3$
$y_4' = \frac{3}{4}y_4$

**75.** $y_1' = y_1 + 2y_2$
$y_2' = 0$

**76.** $y_1' = 3y_1$
$y_2' = y_1 - y_2$

**77.** $y_1' = y_2$
$y_2' = y_1$
$y_3' = 0$

**78.** $y_1' = 6y_1 - y_2 + 2y_3$
$y_2' = 3y_2 - y_3$
$y_3' = y_3$

**Rotation of a Conic** **In Exercises 79–82, (a) find the matrix $A$ of the quadratic form associated with the equation, (b) find an orthogonal matrix $P$ such that $P^TAP$ is diagonal, (c) use the Principal Axes Theorem to perform a rotation of axes to eliminate the $xy$-term in the quadratic equation, and (d) sketch the graph of each equation.**

**79.** $x^2 + 3xy + y^2 - 3 = 0$

**80.** $x^2 - \sqrt{3}xy + 2y^2 - 10 = 0$

**81.** $xy - 2 = 0$

**82.** $9x^2 - 24xy + 16y^2 - 400x - 300y = 0$

# 7 Projects


## 1 Population Growth and Dynamical Systems (I)

Systems of differential equations often arise in biological applications of population growth of various species of animals. These equations are called **dynamical systems** because they describe the changes in a system as functions of time. Suppose that over time $t$, you are studying the populations of predator sharks $y_1(t)$ and their small fish prey $y_2(t)$. One model for the relative growths of these populations is

$$y_1'(t) = ay_1(t) + by_2(t) \qquad \text{Predator}$$

$$y_2'(t) = cy_1(t) + dy_2(t) \qquad \text{Prey}$$

where $a$, $b$, $c$, and $d$ are constants. The constants $a$ and $d$ are positive, reflecting the growth rates of the species. In a predator-prey relationship, $b > 0$ and $c < 0$, which indicates that an increase in prey fish $y_2$ would cause an increase in predator sharks $y_1$, whereas an increase in $y_1$ would cause a decrease in $y_2$.

Suppose the following system of linear differential equations models the populations of sharks $y_1(t)$ and prey fish $y_2(t)$ with the given initial populations at time $t = 0$.

$$y_1'(t) = 0.5y_1(t) + 0.6y_2(t) \qquad y_1(0) = 36$$

$$y_2'(t) = -0.4y_1(t) + 3.0y_2(t) \qquad y_2(0) = 121$$

**1.** Use the diagonalization techniques of this chapter to find the populations $y_1(t)$ and $y_2(t)$ at any time $t > 0$.

**2.** Interpret the solutions in terms of the long-term population trends for the two species. Does one species ultimately disappear? Why or why not?

**3.** Graph the solutions $y_1(t)$ and $y_2(t)$ over the domain $0 \le t \le 3$.

**4.** Explain why the quotient $y_2(t)/y_1(t)$ approaches a limit as $t$ increases.

## 2 The Fibonacci Sequence

The **Fibonacci sequence** is named after the Italian mathematician Leonard Fibonacci of Pisa (1170–1250). To form this sequence, define the first two terms as $x_1 = 1$ and $x_2 = 1$, and then define the $n$th term as the sum of its two immediate predecessors. That is, $x_n = x_{n-1} + x_{n-2}$. So, the third term is $2 = 1 + 1$, the fourth term is $3 = 2 + 1$, and so on. The formula $x_n = x_{n-1} + x_{n-2}$ is called *recursive* because the first $n - 1$ terms must be calculated before the $n$th term can be calculated. In this project, you will use eigenvalues and diagonalization to derive an explicit formula for the $n$th term of the Fibonacci sequence.

**1.** Calculate the first 12 terms of the Fibonacci sequence.

**2.** Explain how the matrix identity $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_{n-1} \\ x_{n-2} \end{bmatrix} = \begin{bmatrix} x_{n-1} + x_{n-2} \\ x_{n-1} \end{bmatrix}$ can be used to generate the Fibonacci sequence recursively.

**3.** Starting with $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, show that $A^{n-2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix}$, where $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

**4.** Find a matrix $P$ that diagonalizes $A$.

**5.** Derive an explicit formula for the $n$th term of the Fibonacci sequence. Use this formula to calculate $x_1$, $x_2$, and $x_3$.

**6.** Determine the limit of $x_n/x_{n-1}$ as $n$ approaches infinity. Do you recognize this number?

**REMARK**

You can learn more about dynamical systems and population modeling in most books on differential equations. You can learn more about Fibonacci numbers in most books on number theory. You might find it interesting to look at the *Fibonacci Quarterly*, the official journal of the Fibonacci Association.

Roger Jegg - Fotodesign-Jegg.de/www.shutterstock.com

# 6 and 7 Cumulative Test

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

**Take this test to review the material in Chapters 6 and 7. After you are finished, check your work against the answers in the back of the book.**

**In Exercises 1 and 2, determine whether the function is a linear transformation.**

1. $T: R^3 \to R^2, T(x, y, z) = (2x, x + y)$
2. $T: M_{2,2} \to R, T(A) = |A + A^T|$
3. Let $T: R^n \to R^m$ be the linear transformation defined by $T(\mathbf{v}) = A\mathbf{v}$, where
$$A = \begin{bmatrix} 0 & 2 & 0 & 2 & 0 \\ 2 & 0 & 1 & 0 & 2 \end{bmatrix}.$$
Find the dimensions of $R^n$ and $R^m$.
4. Let $T: R^2 \to R^3$ be the linear transformation defined by $T(\mathbf{v}) = A\mathbf{v}$, where
$$A = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 0 \end{bmatrix}.$$
Find (a) $T(1, -2)$ and (b) the preimage of $(5, -5, 0)$.
5. Find the kernel of the linear transformation
$T: R^4 \to R^4, T(x_1, x_2, x_3, x_4) = (x_1 - x_2, x_2 - x_1, 0, x_3 + x_4)$.
6. Let $T: R^4 \to R^2$ be the linear transformation defined by $T(\mathbf{v}) = A\mathbf{v}$, where
$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix}.$$
Find a basis for (a) the kernel of $T$ and (b) the range of $T$. (c) Determine the rank and nullity of $T$.

**In Exercises 7–10, find the standard matrix for the linear transformation $T$.**

7. $T(x, y) = (3x + 2y, 2y - x)$
8. $T(x, y, z) = (x + y, y + z, x - z)$
9. $T(x, y, z) = (3z - 2y, 4x + 11z)$
10. $T(x_1, x_2, x_3) = (0, 0, 0)$

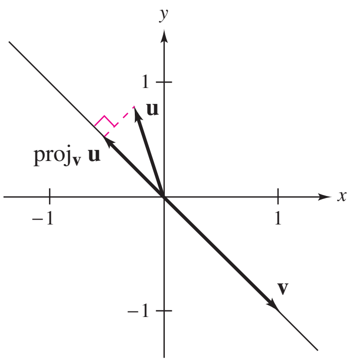



**Figure for 11**

11. Find the standard matrix $A$ for the linear transformation $\text{proj}_{\mathbf{v}}\mathbf{u}: R^2 \to R^2$ that projects an arbitrary vector $\mathbf{u}$ onto the vector $\mathbf{v} = [1 \quad -1]^T$, as shown in the figure. Use this matrix to find the images of the vectors $(1, 1)$ and $(-2, 2)$.
12. Let $T: R^2 \to R^2$ be the linear transformation defined by a counterclockwise rotation of $30°$ in $R^2$.
(a) Find the standard matrix $A$ for the linear transformation.
(b) Use $A$ to find the image of the vector $\mathbf{v} = (1, 2)$.
(c) Sketch the graph of $\mathbf{v}$ and its image.

**In Exercises 13 and 14, find the standard matrices for $T = T_2 \circ T_1$ and $T' = T_1 \circ T_2$.**

13. $T_1: R^2 \to R^2, T_1(x, y) = (x - 2y, 2x + 3y)$
$T_2: R^2 \to R^2, T_2(x, y) = (2x, x - y)$
14. $T_1: R^3 \to R^3, T_1(x, y, z) = (x + 2y, y - z, -2x + y + 2z)$
$T_2: R^3 \to R^3, T_2(x, y, z) = (y + z, x + z, 2y - 2z)$
15. Find the inverse of the linear transformation $T: R^2 \to R^2$ defined by $T(x, y) = (x - y, 2x + y)$. Verify that $(T^{-1} \circ T)(3, -2) = (3, -2)$.
16. Determine whether the linear transformation $T: R^3 \to R^3$ defined by $T(x_1, x_2, x_3) = (x_1 + x_2, x_2 + x_3, x_1 + x_3)$ is invertible. If it is, then find its inverse.

**17.** Find the matrix of the linear transformation $T(x, y) = (y, 2x, x + y)$ relative to the bases $B = \{(1, 1), (1, 0)\}$ for $R^2$ and $B' = \{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}$ for $R^3$. Use this matrix to find the image of the vector $(0, 1)$.

**18.** Let $B = \{(1, 0), (0, 1)\}$ and $B' = \{(1, 1), (1, 2)\}$ be bases for $R^2$.

(a) Find the matrix $A$ of $T: R^2 \rightarrow R^2$, $T(x, y) = (x - 2y, x + 4y)$, relative to the basis $B$.

(b) Find the transition matrix $P$ from $B'$ to $B$.

(c) Find the matrix $A'$ of $T$ relative to the basis $B'$.

(d) Find $[T(\mathbf{v})]_{B'}$ when $[\mathbf{v}]_{B'} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

(e) Verify your answer in part (d) by finding $[\mathbf{v}]_B$ and $[T(\mathbf{v})]_B$.

**In Exercises 19–22, find the eigenvalues and the corresponding eigenvectors of the matrix.**

**19.** $\begin{bmatrix} 7 & 2 \\ -2 & 3 \end{bmatrix}$

**20.** $\begin{bmatrix} 4 & 1 \\ -2 & 1 \end{bmatrix}$

**21.** $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 0 & -3 & -1 \end{bmatrix}$

**22.** $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

**In Exercises 23 and 24, find a nonsingular matrix $P$ such that $P^{-1}AP$ is diagonal.**

**23.** $A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$

**24.** $A = \begin{bmatrix} 0 & -3 & 5 \\ -4 & 4 & -10 \\ 0 & 0 & 4 \end{bmatrix}$

**25.** Find a basis $B$ for $R^3$ such that the matrix for $T: R^3 \rightarrow R^3$, $T(x, y, z) = (2x - 2z, 2y - 2z, 3x - 3z)$, relative to $B$ is diagonal.

**26.** Find an orthogonal matrix $P$ such that $P^TAP$ diagonalizes the symmetric matrix

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}.$$

**27.** Use the Gram-Schmidt orthonormalization process to find an orthogonal matrix $P$ such that $P^TAP$ diagonalizes the symmetric matrix

$$A = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix}.$$

**28.** Solve the system of differential equations.

$$y_1' = y_1$$
$$y_2' = 3y_2$$

**29.** Find the matrix of the quadratic form associated with the quadratic equation

$$4x^2 - 8xy + 4y^2 - 1 = 0.$$

**30.** A population has the following characteristics.

(a) A total of 80% of the population survives the first year. Of that 80%, 40% survives the second year. The maximum life span is 3 years.

(b) The average number of offspring for each member of the population is 3 the first year, 6 the second year, and 3 the third year.

The population now consists of 150 members in each of the three age classes. How many members will there be in each age class in 1 year? in 2 years?

**31.** Define an *orthogonal matrix.*

**32.** Prove that if $A$ is similar to $B$ and $A$ is diagonalizable, then $B$ is diagonalizable.

# Appendix Mathematical Induction and Other Forms of Proofs

- Use the Principle of Mathematical Induction to prove statements involving a positive integer *n*.
- Prove by contradiction that a mathematical statement is true.
- Use a counterexample to show that a mathematical statement is false.

## MATHEMATICAL INDUCTION

In this appendix, you will study some basic strategies for writing mathematical proofs—mathematical induction, proof by contradiction, and the use of counterexamples.

Example 1 illustrates the logical need for using mathematical induction.

**EXAMPLE 1** **Sum of Odd Integers**

Use the pattern to propose a formula for the sum of the first $n$ odd integers.

$$1 = 1$$
$$1 + 3 = 4$$
$$1 + 3 + 5 = 9$$
$$1 + 3 + 5 + 7 = 16$$
$$1 + 3 + 5 + 7 + 9 = 25$$

**SOLUTION**

Notice that the sums on the right are equal to the squares $1^2$, $2^2$, $3^2$, $4^2$, and $5^2$. Judging from this pattern, it appears that the sum $S_n$ of the first $n$ odd integers is

$$S_n = 1 + 3 + 5 + 7 + \cdot \cdot \cdot + (2n - 1) = n^2.$$

Although this particular formula *is* valid, it is important for you to see that recognizing a pattern and then simply *jumping to the conclusion* that the pattern must be true for all values of $n$ is not a logically valid method of proof. There are many examples in which a pattern appears to be developing for small values of $n$ and then at some point the pattern fails. One of the most famous cases of this was the conjecture by the French mathematician Pierre de Fermat (1601–1665), who speculated that all numbers of the form

$$F_n = 2^{2^n} + 1, \quad n = 0, 1, 2, . . .$$

are prime. For $n = 0, 1, 2, 3$, and 4, the conjecture is true.

$$F_0 = 3 \qquad F_1 = 5 \qquad F_2 = 17 \qquad F_3 = 257 \qquad F_4 = 65{,}537$$

The size of the next Fermat number $(F_5 = 4{,}294{,}967{,}297)$ is so great that it was difficult for Fermat to determine whether it was prime or not. However, another well-known mathematician, Leonhard Euler (1707–1783), later found the factorization

$$F_5 = 4{,}294{,}967{,}297 = (641)(6{,}700{,}417)$$

which proved that $F_5$ is not prime and Fermat's conjecture was false.

Just because a rule, pattern, or formula seems to work for several values of $n$, you cannot simply decide that it is valid for all values of $n$ without going through a *legitimate proof.* One legitimate method of proof for such conjectures is the **Principle of Mathematical Induction.**

### The Principle of Mathematical Induction

Let $P_n$ be a statement involving the positive integer $n$. If

**1.** $P_1$ is true, and
**2.** for every positive integer $k$, the truth of $P_k$ implies the truth of $P_{k+1}$

then the statement $P_n$ must be true for all positive integers $n$.

The next example uses the Principle of Mathematical Induction to prove the conjecture from Example 1.

## EXAMPLE 2 Using Mathematical Induction

Use mathematical induction to prove the following formula.

$$S_n = 1 + 3 + 5 + 7 + \cdot \cdot \cdot + (2n - 1) = n^2$$

**SOLUTION**

Mathematical induction consists of two distinct parts. First, you must show that the formula is true when $n = 1$.

**1.** When $n = 1$, the formula is valid because $S_1 = 1 = 1^2$.

The second part of mathematical induction has two steps. The first step is to *assume* that the formula is valid for some integer $k$ (the **induction hypothesis**). The second step is to use this assumption to prove that the formula is valid for the *next* integer, $k + 1$.

**2.** Assuming that the formula

$$S_k = 1 + 3 + 5 + 7 + \cdot \cdot \cdot + (2k - 1) = k^2$$

is true, you must show that the formula $S_{k+1} = (k + 1)^2$ is true.

$$\begin{aligned} S_{k+1} &= 1 + 3 + 5 + 7 + \cdot \cdot \cdot + (2k - 1) + [2(k + 1) - 1] \\ &= [1 + 3 + 5 + 7 + \cdot \cdot \cdot + (2k - 1)] + (2k + 2 - 1) \\ &= S_k + (2k + 1) && \text{Group terms to form } S_k. \\ &= k^2 + 2k + 1 && \text{Substitute } k^2 \text{ for } S_k. \\ &= (k + 1)^2 \end{aligned}$$

Combining the results of parts (1) and (2), you can conclude by mathematical induction that the formula is valid for *all* positive integers $n$.

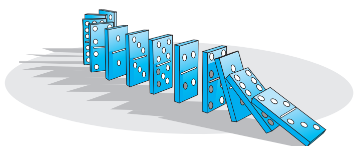


**Figure A.1**

A well-known illustration used to explain why The Principle of Mathematical Induction works is the unending line of dominoes shown in Figure A.1. If the line actually contains infinitely many dominoes, then it is clear that you could not knock down the entire line by knocking down only *one domino* at a time. However, suppose it were true that each domino would knock down the next one as it fell. Then you could knock them all down simply by pushing the first one and starting a chain reaction.

Mathematical induction works in the same way. If the truth of $P_k$ implies the truth of $P_{k+1}$ and if $P_1$ is true, then the chain reaction proceeds as follows:

$P_1$ implies $P_2$

$P_2$ implies $P_3$

$P_3$ implies $P_4$, and so on.

In the next example, you will see the proof of a formula that is often used in calculus.

## EXAMPLE 3 Using Mathematical Induction

Use mathematical induction to prove the formula for the sum of the first $n$ squares.

$$S_n = 1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

**SOLUTION**

**1.** When $n = 1$, the formula is valid, because

$$S_1 = 1^2 = \frac{1(1+1)[2(1)+1]}{6} = \frac{1(2)(3)}{6} = 1.$$

**2.** Assuming the formula is true for $k$,

$$S_k = 1^2 + 2^2 + 3^2 + 4^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

you must show that it is true for $k + 1$,

$$S_{k+1} = \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6} = \frac{(k+1)(k+2)(2k+3)}{6}.$$

To do this, write $S_{k+1}$ as the sum of $S_k$ and the $(k+1)$st term, $(k+1)^2$, as follows.

$$\begin{aligned} S_{k+1} &= (1^2 + 2^2 + 3^2 + 4^2 + \cdots + k^2) + (k+1)^2 \\ &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 && \text{Induction hypothesis} \\ &= \frac{(k+1)(2k^2+7k+6)}{6} && \text{Combine fractions and simplify.} \\ &= \frac{(k+1)(k+2)(2k+3)}{6} && S_k \text{ implies } S_{k+1}. \end{aligned}$$

Combining the results of parts (1) and (2), you can conclude by mathematical induction that the formula is valid for *all* positive integers $n$.

Many of the proofs in linear algebra use mathematical induction. Here is an example from Chapter 2.

## EXAMPLE 4 Using Mathematical Induction in Linear Algebra

If $A_1, A_2, \ldots, A_n$ are invertible matrices, then prove the generalization of Theorem 2.9.

$$(A_1A_2A_3 \cdots A_n)^{-1} = A_n^{-1} \cdots A_3^{-1}A_2^{-1}A_1^{-1}$$

**SOLUTION**

**1.** The formula is valid trivially when $n = 1$ because $A_1^{-1} = A_1^{-1}$.

**2.** Assuming the formula is valid for $k$, $(A_1A_2A_3 \cdots A_k)^{-1} = A_k^{-1} \cdots A_3^{-1}A_2^{-1}A_1^{-1}$, you must show that it is valid for $k + 1$. To do this, use Theorem 2.9, which states that the inverse of a product of two invertible matrices is the product of their inverses in reverse order.

$$\begin{aligned} (A_1A_2A_3 \cdots A_kA_{k+1})^{-1} &= [(A_1A_2A_3 \cdots A_k)A_{k+1}]^{-1} \\ &= A_{k+1}^{-1}(A_1A_2A_3 \cdots A_k)^{-1} && \text{Theorem 2.9} \\ &= A_{k+1}^{-1}(A_k^{-1} \cdots A_3^{-1}A_2^{-1}A_1^{-1}) && \text{Induction hypothesis} \\ &= A_{k+1}^{-1}A_k^{-1} \cdots A_3^{-1}A_2^{-1}A_1^{-1} && S_k \text{ implies } S_{k+1}. \end{aligned}$$

Combining the results of parts (1) and (2), you can conclude by mathematical induction that the formula is valid for *all* positive integers $n$.

## PROOF BY CONTRADICTION

Another basic strategy for writing a proof is *proof by contradiction*. In mathematical logic, proof by contradiction is described by the following equivalence.

$p$ implies $q$ if and only if not $q$ implies not $p$.

One way to prove that $q$ is a true statement is to assume that $q$ is not true. If this leads you to a statement that you know is false, then you have proved that $q$ must be true.

Example 5 shows how to use proof by contradiction to prove that $\sqrt{2}$ is irrational.

### EXAMPLE 5 Using Proof by Contradiction

Prove that $\sqrt{2}$ is an irrational number.

**SOLUTION**

Begin by assuming that $\sqrt{2}$ is *not* an irrational number. Then $\sqrt{2}$ is rational and can be written as the quotient of two integers $a$ and $b$ $(b \neq 0)$ that have no common factors.

$$\sqrt{2} = \frac{a}{b} \qquad \text{Assume that } \sqrt{2} \text{ is a rational number.}$$

$$2b^2 = a^2 \qquad \text{Square each side and multiply by } b^2.$$

This implies that 2 is a factor of $a^2$. So, 2 is also a factor of $a$. Let $a = 2c$.

$$2b^2 = (2c)^2 \qquad \text{Substitute } 2c \text{ for } a.$$

$$b^2 = 2c^2 \qquad \text{Simplify and divide each side by 2.}$$

This implies that 2 is a factor of $b^2$, and it is also a factor of $b$. So, 2 is a factor of both $a$ and $b$. But this is impossible because $a$ and $b$ have no common factors. It must be impossible that $\sqrt{2}$ is a rational number. You can conclude that $\sqrt{2}$ is an irrational number.

### EXAMPLE 6 Using Proof by Contradiction

**REMARK**

Proof by contradiction is not a new technique. Euclid came up with the proof in Example 6 around 300 B.C.

A positive integer greater than 1 is a *prime* when its only positive factors are 1 and itself. Prove that there are infinitely many prime numbers.

**SOLUTION**

Assume there are only finitely many prime numbers, $p_1, p_2, \ldots, p_n$. Consider the number $N = p_1p_2 \cdots p_n + 1$. This number is either prime or composite. If it is composite, then it can be factored as a product of primes. But, none of the primes $(p_1, p_2, \ldots, p_n)$ divide evenly into $N$. So, $N$ is itself prime, and you have found a new prime number, which contradicts the assumption that there are only $n$ prime numbers.

It follows that there are infinitely many prime numbers.

You can use proof by contradiction to prove many theorems in linear algebra.

### EXAMPLE 7 Using Proof by Contradiction in Linear Algebra

Let $A$ and $B$ be $n \times n$ matrices such that $AB$ is singular. Prove that either $A$ or $B$ is singular.

**SOLUTION**

Assume that neither $A$ nor $B$ is singular. Because you know that a matrix is singular if and only if its determinant is zero, $\det(A)$ and $\det(B)$ are both nonzero real numbers. By Theorem 3.5, $\det(AB) = \det(A)\det(B)$. So, $\det(AB)$ is not zero because it is a product of two nonzero real numbers. But this contradicts that $AB$ is a singular matrix. So, you can conclude that the assumption was wrong and that either $A$ or $B$ is singular.

## USING COUNTEREXAMPLES

Often you can disprove a statement using a *counterexample*. For instance, when Euler disproved Fermat's conjecture about prime numbers of the form $F_n = 2^{2^n} + 1$, $n = 0, 1, 2, \ldots$, he used the counterexample $F_5 = 4{,}294{,}967{,}297$, which is not prime.

### EXAMPLE 8 Using a Counterexample

Use a counterexample to show that the statement is false.

Every odd number is prime.

**SOLUTION**

Certainly, you can list many odd numbers that are prime (3, 5, 7, 11), but the statement above is not true, because 9 is odd but it is not a prime number. The number 9 is a counterexample.

Counterexamples can be used to disprove statements in linear algebra, as shown in the next two examples.

### EXAMPLE 9 Using a Counterexample in Linear Algebra

Use a counterexample to show that the statement is false.

If $A$ and $B$ are square singular matrices of order $n$, then $A + B$ is a singular matrix of order $n$.

**SOLUTION**

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Both $A$ and $B$ are singular of order 2, but

$$A + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is the identity matrix of order 2, which is not singular.

### EXAMPLE 10 Using a Counterexample in Linear Algebra

Use a counterexample to show that the statement is false.

The set of all $2 \times 2$ matrices of the form

$$\begin{bmatrix} 1 & b \\ c & d \end{bmatrix}$$

with the standard operations is a vector space.

**REMARK**

Recall that in order for a set to be a vector space, it must satisfy *each* of the ten axioms in the definition of a vector space. (See Section 4.2.)

**SOLUTION**

To show that the set of matrices of the given form is not a vector space, let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 5 \\ 6 & 7 \end{bmatrix}.$$

Both $A$ and $B$ are of the given form, but the sum of these matrices,

$$A + B = \begin{bmatrix} 2 & 7 \\ 9 & 11 \end{bmatrix}$$

is not. This means that the set does not have closure under addition, so it does not satisfy the first axiom in the definition.

# Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

**Using Mathematical Induction** **In Exercises 1–4, use mathematical induction to prove the formula for every positive integer $n$.**

1. $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$
2. $1^3 + 2^3 + 3^3 + \cdots + n^3 = \dfrac{n^2(n+1)^2}{4}$
3. $3 + 7 + 11 + \cdots + (4n - 1) = n(2n + 1)$
4. $\left(1 + \dfrac{1}{1}\right)\left(1 + \dfrac{1}{2}\right)\left(1 + \dfrac{1}{3}\right) \cdots \left(1 + \dfrac{1}{n}\right) = n + 1$

**Proposing a Formula and Using Mathematical Induction** **In Exercises 5 and 6, propose a formula for the sum of the first $n$ terms of the sequence. Then use mathematical induction to prove the formula.**

5. $2^1, 2^2, 2^3, \ldots$
6. $\dfrac{1}{1 \cdot 2}, \dfrac{1}{2 \cdot 3}, \dfrac{1}{3 \cdot 4}, \ldots$

**Using Mathematical Induction with Inequalities** **In Exercises 7 and 8, use mathematical induction to prove the inequality for the indicated integer values of $n$.**

7. $n! > 2^n, \quad n \geq 4$
8. $\dfrac{1}{\sqrt{1}} + \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} + \cdots + \dfrac{1}{\sqrt{n}} > \sqrt{n}, \quad n \geq 2$
9. **Using Mathematical Induction** Use mathematical induction to prove that for all integers $n > 0$,
$$a^0 + a^1 + a^2 + \cdots + a^n = \frac{1 - a^{n+1}}{1 - a}, \quad a \neq 1.$$
10. **Using Mathematical Induction in Linear Algebra** (From Chapter 2) Use mathematical induction to prove that $(A_1A_2A_3 \cdots A_n)^T = A_n^T \cdots A_3^TA_2^TA_1^T$, assuming that $A_1, A_2, A_3, \ldots, A_n$ are matrices with sizes such that the multiplications are defined.
11. **Using Mathematical Induction in Linear Algebra** (From Chapter 3) Use mathematical induction to prove that $|A_1A_2A_3 \cdots A_n| = |A_1||A_2||A_3| \cdots |A_n|$, where $A_1, A_2, A_3, \ldots, A_n$ are square matrices of the same size.
12. **Using Mathematical Induction in Linear Algebra** (From Chapter 6) Use mathematical induction to prove that, if the standard matrices of the linear transformations $T_1, T_2, T_3, \ldots, T_n$ are $A_1, A_2, A_3, \ldots, A_n$ respectively, then the standard matrix for the composition
$$T(\mathbf{v}) = T_n(T_{n-1} \cdots (T_3(T_2(T_1(\mathbf{v})))) \cdots)$$
is represented by
$$A = A_nA_{n-1} \cdots A_3A_2A_1.$$

**Using Proof by Contradiction** **In Exercises 13–19, use proof by contradiction to prove the statement.**

13. If $p$ is an integer and $p^2$ is odd, then $p$ is odd. (*Hint:* An odd number can be written as $2n + 1$, where $n$ is an integer.)
14. If $a$ and $b$ are real numbers and $a \leq b$, then $a + c \leq b + c$.
15. If $a$, $b$, and $c$ are real numbers such that $ac \geq bc$ and $c > 0$, then $a \geq b$.
16. If $a$ and $b$ are real numbers and $1 < a < b$, then $\dfrac{1}{a} > \dfrac{1}{b}$.
17. If $a$ and $b$ are real numbers and $(a + b)^2 = a^2 + b^2$, then $a = 0$ or $b = 0$ or $a = b = 0$.
18. If $a$ is a real number and $0 < a < 1$, then $a^2 < a$.
19. The sum of a rational number and an irrational number is irrational.
20. **Using Proof by Contradiction in Linear Algebra** (From Chapter 3) Use proof by contradiction to prove that, if $A$ and $B$ are square matrices of order $n$ such that $\det(AB) = 1$, then both $A$ and $B$ are nonsingular.
21. **Using Proof by Contradiction in Linear Algebra** (From Chapter 4) Use proof by contradiction to prove that in a given vector space, the zero vector is unique.
22. **Using Proof by Contradiction in Linear Algebra** (From Chapter 4) Let $S = \{\mathbf{u}, \mathbf{v}\}$ be a linearly independent set. Use proof by contradiction to prove that the set $\{\mathbf{u} - \mathbf{v}, \mathbf{u} + \mathbf{v}\}$ is linearly independent.

**Using a Counterexample** **In Exercises 23–30, use a counterexample to show that the statement is false.**

23. If $a$ and $b$ are real numbers and $a < b$, then $a^2 < b^2$.
24. The product of two irrational numbers is irrational.
25. If $a$ and $b$ are real numbers such that $a \neq 0$ and $b \neq 0$, then $(a + b)^3 = a^3 + b^3$.
26. If $f$ is a polynomial function and $f(a) = f(b)$, then $a = b$.
27. If $f$ and $g$ are differentiable functions and $y = f(x)g(x)$, then $y' = f'(x)g'(x)$.
28. (From Chapter 2) If $A$, $B$, and $C$ are matrices and $AC = BC$, then $A = B$.
29. (From Chapter 3) If $A$ is a matrix, then $\det(A^{-1}) = \dfrac{1}{\det A}$.
30. (From Chapter 4) The set of all $3 \times 3$ matrices of the form
$$\begin{bmatrix} 0 & a & b \\ c & 2 & d \\ e & f & 0 \end{bmatrix}$$
with the standard operations is a vector space.

## Answers to Selected Odd-Numbered Exercises

# Chapter 1

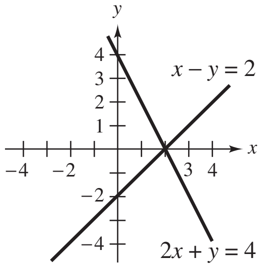
## Section 1.1 *(page 10)*

**1.** Linear **3.** Not linear **5.** Not linear

**7.** $x = 2t$
$y = t$

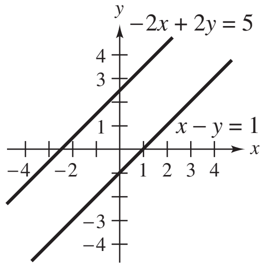
**9.** $x = 1 - s - t$
$y = s$
$z = t$

**11.**



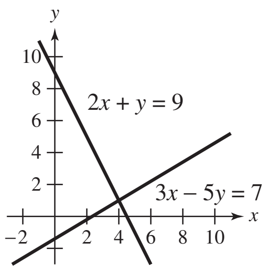

$x = 2$
$y = 0$

**13.**



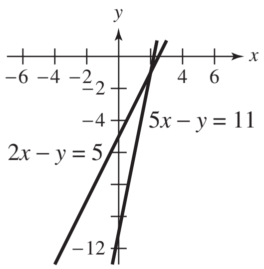

No solution

**15.**



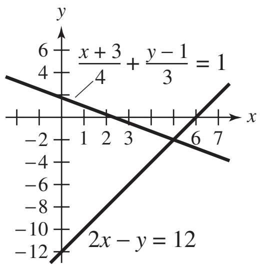

$x = 4$
$y = 1$

**17.**



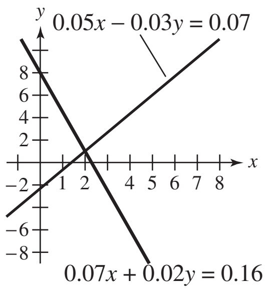

$x = 2$
$y = -1$

**19.**



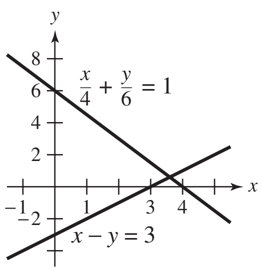

$x = 5$
$y = -2$

**21.**




$x = 2$
$y = 1$

**23.**



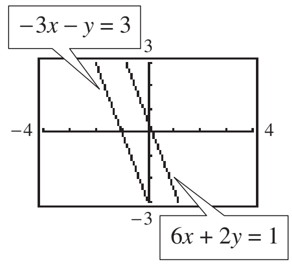

$x = \frac{18}{5}$
$y = \frac{3}{5}$

**25.** $x_1 = 5$
$x_2 = 3$

**27.** $x = \frac{3}{2}$
$y = \frac{3}{2}$
$z = 0$

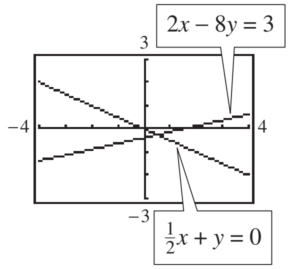
**29.** $x_1 = -t$
$x_2 = 2t$
$x_3 = t$

**31.** (a)



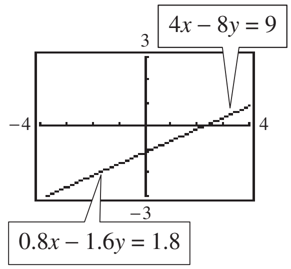

(b) Inconsistent

**33.** (a)




(b) Consistent
(c) $x = \frac{1}{2}$
$y = -\frac{1}{4}$
(d) $x = \frac{1}{2}$
$y = -\frac{1}{4}$
(e) The solutions are the same.

**35.** (a)




(b) Consistent
(c) There are infinitely many solutions.
(d) $x = \frac{9}{4} + 2t$
$y = t$
(e) The solutions are consistent.

**37.** $x_1 = -1$
$x_2 = -1$

**39.** $u = 40$
$v = 40$

**41.** $x = -\frac{1}{3}$
$y = -\frac{2}{3}$

**43.** $x = 14$
$y = -2$

**45.** $x_1 = 8$
$x_2 = 7$

**47.** $x = 1$
$y = 2$
$z = 3$

**49.** No solution

**51.** $x_1 = \frac{5}{2} - \frac{1}{2}t$
$x_2 = 4t - 1$
$x_3 = t$

**53.** No solution

**55.** $x = 1$
$y = 0$
$z = 3$
$w = 2$

**57.** $x_1 = -15$
$x_2 = 40$
$x_3 = 45$
$x_4 = -75$

**59.** $x_1 = \frac{1}{5}$
$x_2 = -\frac{4}{5}$
$x_3 = \frac{1}{2}$

**61.** This system must have at least one solution because $x = y = z = 0$ is an obvious solution.
Solution: $x = 0$
$y = 0$
$z = 0$
This system has exactly one solution.

**63.** This system must have at least one solution because $x = y = z = 0$ is an obvious solution.
Solution: $x = -\frac{3}{5}t$
$y = \frac{4}{5}t$
$z = t$
This system has an infinite number of solutions.

**65.** Apple juice: 95.5 mg
Orange juice: 81.9 mg

**67.** (a) True. You can describe the entire solution set using parametric representation.

$ax + by = c$

Choosing $y = t$ as the free variable, the solution is

$x = \frac{c}{a} - \frac{b}{a}t,\ y = t$, where $t$ is any real number.

(b) False. For example, consider the system

$x_1 + x_2 + x_3 = 1$
$x_1 + x_2 + x_3 = 2$

which is an inconsistent system.

(c) False. A consistent system may have only one solution.

**69.** $3x_1 - x_2 = 4$
$-3x_1 + x_2 = -4$
(The answer is not unique.)

**71.** $x = 3$
$y = -4$

**73.** $x = \dfrac{2}{5 - t}$
$y = \dfrac{1}{4t - 1}$
$z = \dfrac{1}{t}$, where $t \neq 5, \frac{1}{4}, 0$

**75.** $x = \cos\theta$
$y = \sin\theta$

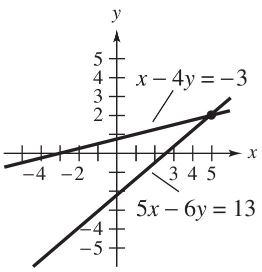
**77.** $k = -2$

**79.** All $k \neq \pm 1$ **81.** $k = \frac{8}{3}$ **83.** $k = 1, -2$

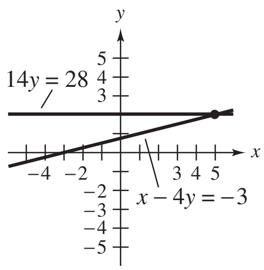
**85.** (a) Three lines intersecting at one point
(b) Three coincident lines
(c) Three lines having no common point

**87.** Answers will vary. (*Hint:* Choose three different values of $x$ and solve the resulting system of linear equations in the variables $a$, $b$, and $c$.)

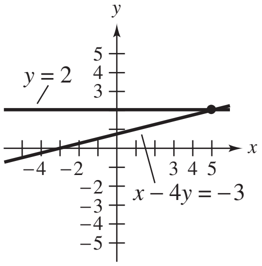
**89.** $x - 4y = -3$
$5x - 6y = 13$



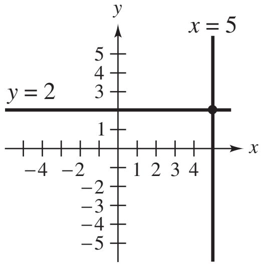

$x - 4y = -3$
$14y = 28$




$x - 4y = -3$
$y = 2$




$x = 5$
$y = 2$




The intersection points are all the same.

**91.** $x = 39{,}600$
$y = 398$
The graphs are misleading because, while they appear parallel, when the equations are solved for $y$, they have slightly different slopes.

## Section 1.2 *(page 22)*

**1.** $3 \times 3$ **3.** $2 \times 4$ **5.** $4 \times 5$

**7.** Add 5 times the second row to the first row.

**9.** Add 4 times the second row to the third row. Interchange the first and second rows.

**11.** $x_1 = 0$
$x_2 = 2$

**13.** $x_1 = 2$
$x_2 = -1$
$x_3 = -1$

**15.** $x_1 = 1$
$x_2 = 1$
$x_3 = 0$

**17.** $x_1 = -26$
$x_2 = 13$
$x_3 = -7$
$x_4 = 4$

**19.** Reduced row-echelon form

**21.** Not in row-echelon form

**23.** Not in row-echelon form

**25.** $x = 3$
$y = 2$

**27.** No solution

**29.** $x = 4$
$y = -2$

**31.** $x_1 = 4$
$x_2 = -3$
$x_3 = 2$

**33.** No solution

**35.** $x = 100 + 96t - 3s$
$y = s$
$z = 54 + 52t$
$w = t$

**37.** $x = 0$
$y = 2 - 4t$
$z = t$

**39.** $x_1 = 2$
$x_2 = -2$
$x_3 = 3$
$x_4 = -5$
$x_5 = 1$

**41.** $x_1 = 0$
$x_2 = -t$
$x_3 = t$

**43.** $x_1 = -t$
$x_2 = s$
$x_3 = 0$
$x_4 = t$

**45.** \$100,000 at 9%
\$250,000 at 10%
\$150,000 at 12%

**47.** Augmented
(a) Two equations in two variables
(b) All real $k \neq -\frac{4}{3}$
Coefficient
(a) Two equations in three variables
(b) All real $k$

**49.** (a) $a + b + c = 0$
(b) $a + b + c \neq 0$
(c) Not possible

**51.** (a) $x = \frac{8}{3} - \frac{5}{6}t$
$y = -\frac{8}{3} + \frac{5}{6}t$
$z = t$

(b) $x = \frac{18}{7} - \frac{11}{14}t$
$y = -\frac{20}{7} + \frac{13}{14}t$
$z = t$

(c) $x = 3 - t$
$y = -3 + t$
$z = t$

(d) Each system has an infinite number of solutions.

**53.** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

**55.** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & k \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

**57.** (a) True. In the notation $m \times n$, $m$ is the number of rows of the matrix. So, a $6 \times 3$ matrix has six rows.

(b) True. On page 16, the sentence reads, "Every matrix is row-equivalent to a matrix in row-echelon form."

(c) False. Consider the row-echelon form

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}$$

which gives the solution $x_1 = 0$, $x_2 = 1$, $x_3 = 2$, and $x_4 = 3$.

(d) True. Theorem 1.1 states that if a homogeneous system has fewer equations than variables, then it must have an infinite number of solutions.

**59.** Yes, it is possible:

$x_1 + x_2 + x_3 = 0$
$x_1 + x_2 + x_3 = 1$

**61.** $ad - bc \neq 0$ **63.** $\lambda = 1, 3$

**65.** The rows have been interchanged. The first elementary row operation is redundant, so you can just use the second and third elementary row operations.

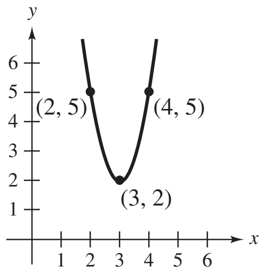
**67.** (a) An inconsistent matrix in row-echelon form would have a row consisting of all zeros except for the last entry.

(b) A matrix for a system with infinitely many solutions would have a row of all zeros or more than one column with no leading 1's.

## Section 1.3 *(page 32)*

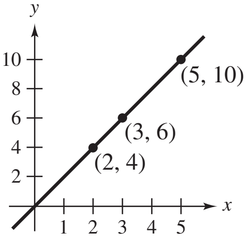
**1.** (a) $p(x) = 29 - 18x + 3x^2$

(b)



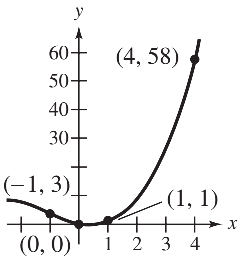

**3.** (a) $p(x) = 2x$

(b)



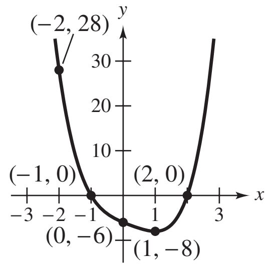

**5.** (a) $p(x) = -\frac{3}{2}x + 2x^2 + \frac{1}{2}x^3$

(b)




**7.** (a) $p(x) = -6 - 3x + x^2 - x^3 + x^4$

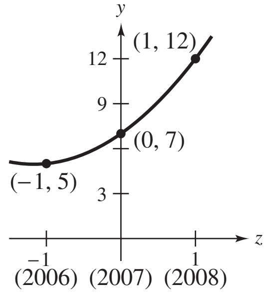
(b)




**9.** (a) Let $z = x - 2007$.

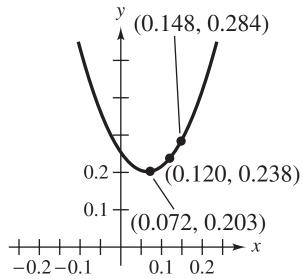
$p(z) = 7 + \frac{7}{2}z + \frac{3}{2}z^2$
$p(x) = 7 + \frac{7}{2}(x - 2007) + \frac{3}{2}(x - 2007)^2$

(b)




**11.** (a) $p(x) = 0.254 - 1.579x + 12.022x^2$

(b)




**13.** $p(x) = -\dfrac{4}{\pi^2}x^2 + \dfrac{4}{\pi}x$

$\sin\dfrac{\pi}{3} \approx \dfrac{8}{9} \approx 0.889$

(Actual value is $\sqrt{3}/2 \approx 0.866$.)

**15.** $(x - 5) + (y - 10)^2 = 65$

**17.** $281 + 3(x - 2000) - 0.02(x - 2000)^2$; 2020: 333 million; 2030: 353 million

**19.** (a) Using $z = x - 2000$,

$$\begin{aligned} a_0 + 3a_1 + 9a_2 + 27a_3 &= 10{,}526 \\ a_0 + 4a_1 + 16a_2 + 64a_3 &= 11{,}330 \\ a_0 + 5a_1 + 25a_2 + 125a_3 &= 12{,}715 \\ a_0 + 6a_1 + 36a_2 + 216a_3 &= 12{,}599 \end{aligned}$$

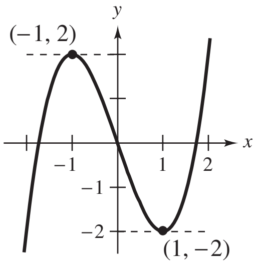
(b) $32{,}420 - 17{,}538.5(x - 2000) + 4454.5(x - 2000)^2 - 347(x - 2000)^3$

No. Answers will vary. Sample answer: The model predicts that profits will go down over the next few years and will be negative by 2008.

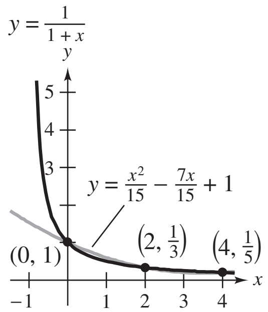
**21.** Solve the system:

$$\begin{aligned} p(-1) &= a_0 - a_1 + a_2 = 0 \\ p(0) &= a_0 \qquad\qquad\ \ = 0 \\ p(1) &= a_0 + a_1 + a_2 = 0 \\ a_0 &= a_1 = a_2 = 0 \end{aligned}$$

**23.** $p(x) = -3x + x^3$




**25.** (a) $p(x) = 1 - \frac{7}{15}x + \frac{1}{15}x^2$
(b) $p(x) = 1 + x$




**27.** (a) $x_1 = s$, $x_2 = t$, $x_3 = 600 - s$, $x_4 = s - t$, $x_5 = 500 - t$, $x_6 = s$, $x_7 = t$
(b) $x_1 = 0$, $x_2 = 0$, $x_3 = 600$, $x_4 = 0$, $x_5 = 500$, $x_6 = 0$, $x_7 = 0$
(c) $x_1 = 0$, $x_2 = -500$, $x_3 = 600$, $x_4 = 500$, $x_5 = 1000$, $x_6 = 0$, $x_7 = -500$

**29.** (a) $x_1 = 100 + t$, $x_2 = -100 + t$, $x_3 = 200 + t$, $x_4 = t$
(b) $x_1 = 100$, $x_2 = -100$, $x_3 = 200$, $x_4 = 0$
(c) $x_1 = 200$, $x_2 = 0$, $x_3 = 300$, $x_4 = 100$

**31.** $I_1 = 0$
$I_2 = 1$
$I_3 = 1$

**33.** (a) $I_1 = 1$, $I_2 = 2$, $I_3 = 1$  (b) $I_1 = 0$, $I_2 = 1$, $I_3 = 1$

**35.** $T_1 = 37.5°$, $T_2 = 45°$, $T_3 = 25°$, $T_4 = 32.5°$

**37.** $\dfrac{1}{x-1} + \dfrac{3}{x+1} - \dfrac{2}{(x+1)^2}$

**39.** $x = 2$
$y = 2$
$\lambda = -4$

## Review Exercises *(page 35)*

**1.** Not linear **3.** Linear **5.** Not linear

**7.** $x = -\frac{1}{4} + \frac{1}{2}s - \frac{3}{2}t$
$y = s$
$z = t$

**9.** $x = \frac{1}{2}$, $y = \frac{3}{2}$ **11.** $x = -12$, $y = -8$ **13.** $x = 0$, $y = 0$

**15.** No solution

**17.** $x = 0$, $y = 0$ **19.** $x_1 = -\frac{1}{2}$, $x_2 = \frac{4}{5}$ **21.** $2 \times 3$

**23.** $x_1 = -2t$
$x_2 = t$
$x_3 = 0$

**25.** Row-echelon form (not reduced)

**27.** Not in row-echelon form

**29.** $x = 2$, $y = -3$, $z = 3$ **31.** $x = \frac{1}{2}$, $y = -\frac{1}{3}$, $z = 1$ **33.** $x = 4 + 3t$, $y = 5 + 2t$, $z = t$

**35.** No solution **37.** $x_1 = 1$, $x_2 = 4$, $x_3 = -3$, $x_4 = -2$ **39.** $x = 0$, $y = 2 - 4t$, $z = t$

**41.** $x = 1$, $y = 0$, $z = 4$, $w = -2$ **43.** $x_1 = 2t$, $x_2 = -3t$, $x_3 = t$ **45.** $x_1 = -4t$, $x_2 = -\frac{1}{2}t$, $x_3 = t$

**47.** $k = \pm 1$

**49.** (a) $b = 2a$ and $a \neq -3$
(b) $b \neq 2a$
(c) $a = -3$ and $b = -6$

**51.** Use an elimination method to get both matrices in reduced row-echelon form. The two matrices are row-equivalent because each is row-equivalent to
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

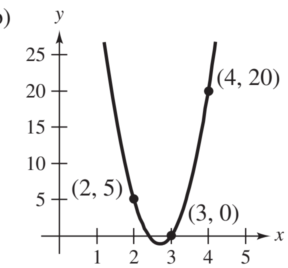
**53.** $$\begin{bmatrix} 1 & 0 & -1 & -2 & \cdots & 2-n \\ 0 & 1 & 2 & 3 & \cdots & n-1 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

**55.** (a) False. See page 3, following Example 2.
(b) True. See page 5, Example 4(b).

**57.** 6 touchdowns, 6 extra-point kicks, 1 field goal

**59.** $$\begin{aligned} A + B \phantom{{}+C} &= 8 \\ -2A \phantom{{}+B} + C &= 0 \\ A - B + C &= 0 \end{aligned}$$
$A = 2$, $B = 6$, $C = 4$

**61.** (a) $p(x) = 90 - \frac{135}{2}x + \frac{25}{2}x^2$
(b)




**63.** $p(x) = 50 + \frac{15}{2}x + \frac{5}{2}x^2$
(First year is represented by $x = 0$.)
Fourth-year sales: $p(3) = 95$

**65.** (a) $$\begin{aligned} a_0 \phantom{{}+80a_1+6400a_2} &= 80 \\ a_0 + 4a_1 + 16a_2 &= 68 \\ a_0 + 80a_1 + 6400a_2 &= 30 \end{aligned}$$

(b) and (c) $a_0 = 80$

$a_1 = -\frac{25}{8}$

$a_2 = \frac{1}{32}$

So, $y = \frac{1}{32}x^2 - \frac{25}{8}x + 80$.

(d) The results of parts (b) and (c) are the same.

(e) There is precisely one polynomial function of degree $n - 1$ (or less) that fits $n$ distinct points.

**67.** $I_1 = \frac{5}{13}$

$I_2 = \frac{6}{13}$

$I_3 = \frac{1}{13}$

# Chapter 2

## Section 2.1 *(page 48)*

**1.** $x = -4, y = 22$

**3.** $x = 2, y = 3$

**5.** (a) $\begin{bmatrix} 3 & -2 \\ 1 & 7 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & 0 \\ 3 & -9 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & -2 \\ 4 & -2 \end{bmatrix}$

(d) $\begin{bmatrix} 0 & -1 \\ 5 & -10 \end{bmatrix}$ (e) $\begin{bmatrix} \frac{5}{2} & -\frac{3}{2} \\ 0 & \frac{15}{2} \end{bmatrix}$

**7.** (a) $\begin{bmatrix} 7 & 3 \\ 1 & 9 \\ -2 & 15 \end{bmatrix}$ (b) $\begin{bmatrix} 5 & -5 \\ 3 & -1 \\ -4 & -5 \end{bmatrix}$ (c) $\begin{bmatrix} 12 & -2 \\ 4 & 8 \\ -6 & 10 \end{bmatrix}$

(d) $\begin{bmatrix} 11 & -6 \\ 5 & 3 \\ -7 & 0 \end{bmatrix}$ (e) $\begin{bmatrix} 4 & \frac{7}{2} \\ 0 & 7 \\ -\frac{1}{2} & \frac{25}{2} \end{bmatrix}$

**9.** (a) $\begin{bmatrix} 3 & 4 & 0 \\ 7 & 8 & 7 \\ 2 & 2 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 0 & -2 \\ -3 & 0 & 3 \\ -2 & 0 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 6 & 4 & -2 \\ 4 & 8 & 10 \\ 0 & 2 & 4 \end{bmatrix}$ (d) $\begin{bmatrix} 6 & 2 & -3 \\ -1 & 4 & 8 \\ -2 & 1 & 4 \end{bmatrix}$

(e) $\begin{bmatrix} \frac{3}{2} & 3 & \frac{1}{2} \\ 6 & 6 & \frac{9}{2} \\ 2 & \frac{3}{2} & 1 \end{bmatrix}$

**11.** (a), (b), (d), and (e) Not possible

(c) $\begin{bmatrix} 12 & 0 & 6 \\ -2 & -8 & 0 \end{bmatrix}$

**13.** (a) $c_{21} = -6$ (b) $c_{13} = 29$

**15.** $x = 3,\ y = 2,\ z = 1$

**17.** (a) $\begin{bmatrix} 0 & 15 \\ 6 & 12 \end{bmatrix}$ (b) $\begin{bmatrix} -2 & 2 \\ 31 & 14 \end{bmatrix}$

**19.** (a) $\begin{bmatrix} -8 & -2 & -5 \\ 4 & 8 & 17 \\ -20 & 1 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 9 & 5 & 4 \\ 3 & 11 & -5 \\ -17 & -1 & -16 \end{bmatrix}$

**21.** (a) Not possible (b) $\begin{bmatrix} 3 & -4 \\ 10 & 16 \\ 26 & 46 \end{bmatrix}$

**23.** (a) $[12]$ (b) $\begin{bmatrix} 6 & 4 & 2 \\ 9 & 6 & 3 \\ 0 & 0 & 0 \end{bmatrix}$

**25.** (a) $\begin{bmatrix} -1 & 19 \\ 4 & -27 \\ 0 & 14 \end{bmatrix}$ (b) Not possible

**27.** (a) $\begin{bmatrix} 3 \\ 10 \\ 26 \end{bmatrix}$ (b) Not possible

**29.** (a) $\begin{bmatrix} 60 & 72 \\ -20 & -24 \\ 10 & 12 \\ 60 & 72 \end{bmatrix}$ (b) Not possible

**31.** $3 \times 4$ **33.** $4 \times 2$ **35.** $3 \times 2$

**37.** Not possible, sizes do not match.

**39.** $x_1 = t, x_2 = \frac{5}{4}t, x_3 = \frac{3}{4}t$

**41.** $\begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$

**43.** $\begin{bmatrix} -2 & -3 \\ 6 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -4 \\ -36 \end{bmatrix}$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -7 \\ 6 \end{bmatrix}$

**45.** $\begin{bmatrix} 1 & -2 & 3 \\ -1 & 3 & -1 \\ 2 & -5 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ -6 \\ 17 \end{bmatrix}$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

**47.** $\begin{bmatrix} 1 & -5 & 2 \\ -3 & 1 & -1 \\ 0 & -2 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -20 \\ 8 \\ -16 \end{bmatrix}$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}$

**49.** $\mathbf{b} = 3\begin{bmatrix} 1 \\ 3 \end{bmatrix} + 0\begin{bmatrix} -1 \\ -3 \end{bmatrix} - 2\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \end{bmatrix}$

(The answer is not unique.)

**51.** $\mathbf{b} = 1\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + 0\begin{bmatrix} -5 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$

**53.** $\begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$ **55.** $a = 7, b = -4, c = -\frac{1}{2}, d = \frac{7}{2}$

**57.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}$

**59.** $AB = \begin{bmatrix} -10 & 0 \\ 0 & -12 \end{bmatrix}$

$BA = \begin{bmatrix} -10 & 0 \\ 0 & -12 \end{bmatrix}$

**61.** Proof **63.** 2 **65.** 4 **67.** Proof

**69.** $w = z, x = -y$

**71.** Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$.

Then the given matrix equation expands to

$\begin{bmatrix} a_{11} + a_{21} & a_{12} + a_{22} \\ a_{11} + a_{21} & a_{12} + a_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Because $a_{11} + a_{21} = 1$ and $a_{11} + a_{21} = 0$ cannot both be true, you can conclude that there is no solution.

**73.** (a) $A^2 = \begin{bmatrix} i^2 & 0 \\ 0 & i^2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

$A^3 = \begin{bmatrix} i^3 & 0 \\ 0 & i^3 \end{bmatrix} = \begin{bmatrix} -i & 0 \\ 0 & -i \end{bmatrix}$

$A^4 = \begin{bmatrix} i^4 & 0 \\ 0 & i^4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(b) $B^2 = \begin{bmatrix} -i^2 & 0 \\ 0 & -i^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

**75.** Proof **77.** Proof

**79.** [\$1037.50 \$1400.00 \$1012.50]

Each entry represents the total profit at each outlet.

**81.** $\begin{bmatrix} 0.40 & 0.15 & 0.15 \\ 0.28 & 0.53 & 0.17 \\ 0.32 & 0.32 & 0.68 \end{bmatrix}$

$P^2$ gives the proportions of the voting population that changed parties or remained loyal to their parties from the first election to the third.

**83.** $\left[\begin{array}{cc|c} -1 & 4 & 0 \\ -1 & 1 & 0 \\ \hline 0 & 0 & 5 \end{array}\right]$

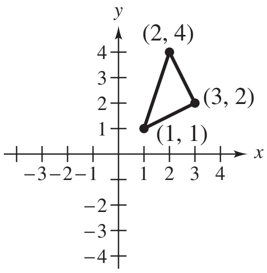
**85.** (a) True. On page 43, ". . . for the product of two matrices to be defined, the number of columns of the first matrix must equal the number of rows of the second matrix."

(b) True. On page 46, ". . . the system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ can be expressed as . . . a linear combination, where the coefficients of the linear combination are a solution of the system."

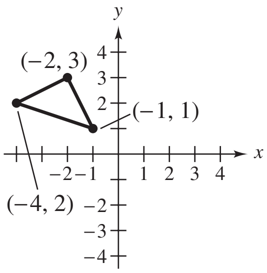
**87.** (a) $AT = \begin{bmatrix} -1 & -4 & -2 \\ 1 & 2 & 3 \end{bmatrix}$

$AAT = \begin{bmatrix} -1 & -2 & -3 \\ -1 & -4 & -2 \end{bmatrix}$

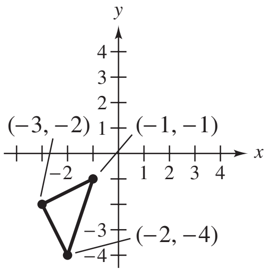
Triangle associated with $T$




Triangle associated with $AT$




Triangle associated with $AAT$




The transformation matrix $A$ rotates the triangle 90° counterclockwise about the origin.

(b) Given the triangle associated with $AAT$, the transformation that would produce the triangle associated with $AT$ would be a 90° clockwise rotation about the origin. Another such rotation would produce the triangle associated with $T$.

## Section 2.2 *(page 59)*

**1.** $\begin{bmatrix} -8 & -7 \\ 15 & -1 \end{bmatrix}$ **3.** $\begin{bmatrix} -24 & -4 & 12 \\ -12 & 32 & 12 \end{bmatrix}$ **5.** $\begin{bmatrix} 10 & 8 \\ -59 & 9 \end{bmatrix}$

**7.** $\begin{bmatrix} 3 & 2 \\ 13 & 4 \end{bmatrix}$ **9.** $\begin{bmatrix} 0 & -12 \\ 12 & -24 \end{bmatrix}$ **11.** $\begin{bmatrix} 7 & 7 \\ 28 & 14 \end{bmatrix}$

**13.** (a) $\begin{bmatrix} 3 & \frac{2}{3} \\ -\frac{4}{3} & \frac{11}{3} \\ \frac{10}{3} & 0 \end{bmatrix}$ (b) $\begin{bmatrix} -\frac{13}{3} & -\frac{10}{3} \\ 4 & -5 \\ -\frac{26}{3} & -\frac{16}{3} \end{bmatrix}$

(c) $\begin{bmatrix} -14 & -4 \\ 7 & -17 \\ -17 & -2 \end{bmatrix}$ (d) $\begin{bmatrix} -\frac{13}{6} & 1 \\ -\frac{1}{3} & -\frac{17}{6} \\ 0 & \frac{10}{3} \end{bmatrix}$

**15.** $\begin{bmatrix} -3 & -5 & -10 \\ -2 & -5 & -5 \end{bmatrix}$ **17.** $\begin{bmatrix} 1 & 6 & -1 \\ -2 & -2 & -8 \end{bmatrix}$

**19.** $\begin{bmatrix} 12 & -4 \\ 8 & 4 \end{bmatrix}$ **21.** (a) $\begin{bmatrix} 12 & 7 \\ 24 & 15 \end{bmatrix}$ (b) $\begin{bmatrix} 12 & 7 \\ 24 & 15 \end{bmatrix}$

**23.** $AB = \begin{bmatrix} -9 & 2 \\ -3 & 6 \end{bmatrix}$, $BA = \begin{bmatrix} -8 & 4 \\ 2 & 5 \end{bmatrix}$

**25.** $AC = BC = \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}$ **27.** Proof

**29.** $\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$ **31.** $\begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}$ **33.** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

**35.** $(A + B)(A - B) = A^2 + BA - AB - B^2$, which is not necessarily equal to $A^2 - B^2$ because $AB$ is not necessarily equal to $BA$.

**37.** $\begin{bmatrix} 1 & -3 & 5 \\ -2 & 4 & -1 \end{bmatrix}$ **39.** $(AB)^T = B^TA^T = \begin{bmatrix} 2 & -5 \\ 4 & -1 \end{bmatrix}$

**41.** $(AB)^T = B^TA^T = \begin{bmatrix} 4 & 0 & -4 \\ 10 & 4 & -2 \\ 1 & -1 & -3 \end{bmatrix}$

**43.** (a) $\begin{bmatrix} 16 & 8 & 4 \\ 8 & 8 & 0 \\ 4 & 0 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 21 & 3 \\ 3 & 5 \end{bmatrix}$

**45.** (a) $\begin{bmatrix} 68 & 26 & -10 & 6 \\ 26 & 41 & 3 & -1 \\ -10 & 3 & 43 & 5 \\ 6 & -1 & 5 & 10 \end{bmatrix}$ (b) $\begin{bmatrix} 29 & -14 & 5 & -5 \\ -14 & 81 & -3 & 2 \\ 5 & -3 & 39 & -13 \\ -5 & 2 & -13 & 13 \end{bmatrix}$

**47.** (a) True. See Theorem 2.1, part 1.
(b) True. See Theorem 2.3, part 1.
(c) False. See Theorem 2.6, part 4, or Example 9.
(d) True. See Example 10.

**49.** (a) $a = 3$ and $b = -1$

(b) $\begin{aligned} a + b &= 1 \\ b &= 1 \\ a \quad &= 1 \end{aligned}$

No solution

(c) $\begin{aligned} a + b + c &= 0 \\ b + c &= 0 \\ a \quad + c &= 0 \end{aligned}$

$a = -c \rightarrow b = 0 \rightarrow c = 0 \rightarrow a = 0$

(d) $a = -3t$
$b = t$
$c = t$
Let $t = 1$: $a = -3$, $b = 1$, $c = 1$

**51.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **53.** $\begin{bmatrix} \pm 3 & 0 \\ 0 & \pm 2 \end{bmatrix}$ **55.** $\begin{bmatrix} -4 & 0 \\ 8 & 2 \end{bmatrix}$

**57–65.** Proofs **67.** Skew-symmetric

**69.** Symmetric **71.** Proof

**73.** (a) $\frac{1}{2}(A + A^T)$

$$= \frac{1}{2}\left(\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix}\right)$$

$$= \frac{1}{2}\begin{bmatrix} 2a_{11} & a_{12} + a_{21} & \cdots & a_{1n} + a_{n1} \\ a_{21} + a_{12} & 2a_{22} & \cdots & a_{2n} + a_{n2} \\ \vdots & \vdots & & \vdots \\ a_{n1} + a_{1n} & a_{n2} + a_{2n} & \cdots & 2a_{nn} \end{bmatrix}$$

(b) $\frac{1}{2}(A - A^T)$

$$= \frac{1}{2}\left(\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} - \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix}\right)$$

$$= \frac{1}{2}\begin{bmatrix} 0 & a_{12} - a_{21} & \cdots & a_{1n} - a_{n1} \\ a_{21} - a_{12} & 0 & \cdots & a_{2n} - a_{n2} \\ \vdots & \vdots & & \vdots \\ a_{n1} - a_{1n} & a_{n2} - a_{2n} & \cdots & 0 \end{bmatrix}$$

(c) Proof

(d) $A = \frac{1}{2}(A - A^T) + \frac{1}{2}(A + A^T)$

$$= \underbrace{\begin{bmatrix} 0 & 4 & -\frac{1}{2} \\ -4 & 0 & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}}_{\text{Skew-symmetric}} + \underbrace{\begin{bmatrix} 2 & 1 & \frac{7}{2} \\ 1 & 6 & \frac{1}{2} \\ \frac{7}{2} & \frac{1}{2} & 1 \end{bmatrix}}_{\text{Symmetric}}$$

**75.** Sample answers:

(a) An example of a $2 \times 2$ matrix of the given form is

$A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

An example of a $4 \times 4$ matrix of the given form is

$A_3 = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$.

(b) $A_2^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$A_3^2 = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $A_3^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(c) The conjecture is that if $A$ is a $4 \times 4$ matrix of the given form, then $A^4$ is the $4 \times 4$ zero matrix. A graphing utility shows this to be true.

(d) If $A$ is an $n \times n$ matrix of the given form, then $A^n$ is the $n \times n$ zero matrix.

## Section 2.3 *(page 71)*

**1.** $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = BA$ **3.** $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = BA$

**5.** $AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = BA$ **7.** $\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$

**9.** $\begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix}$ **11.** $\begin{bmatrix} -19 & -33 \\ -4 & -7 \end{bmatrix}$ **13.** $\begin{bmatrix} 1 & 1 & -1 \\ -3 & 2 & -1 \\ 3 & -3 & 2 \end{bmatrix}$

**15.** Singular **17.** $\begin{bmatrix} -\frac{3}{2} & \frac{3}{2} & 1 \\ \frac{9}{2} & -\frac{7}{2} & -3 \\ -1 & 1 & 1 \end{bmatrix}$ **19.** $\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{5} \end{bmatrix}$

**21.** $\begin{bmatrix} 3.75 & 0 & -1.25 \\ 3.458\overline{3} & -1 & -1.375 \\ 4.1\overline{6} & 0 & -2.5 \end{bmatrix}$ **23.** $\begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{4} & \frac{1}{4} & 0 \\ \frac{7}{20} & -\frac{1}{4} & \frac{1}{5} \end{bmatrix}$

**25.** Singular **27.** $\begin{bmatrix} -24 & 7 & 1 & -2 \\ -10 & 3 & 0 & -1 \\ -29 & 7 & 3 & -2 \\ 12 & -3 & -1 & 1 \end{bmatrix}$ **29.** Singular

**31.** $\begin{bmatrix} \frac{5}{13} & -\frac{3}{13} \\ \frac{1}{13} & \frac{2}{13} \end{bmatrix}$ **33.** Does not exist **35.** $\begin{bmatrix} \frac{16}{59} & \frac{15}{59} \\ -\frac{4}{59} & \frac{70}{59} \end{bmatrix}$

**37.** $\begin{bmatrix} \frac{11}{4} & \frac{3}{2} \\ \frac{3}{4} & \frac{1}{2} \end{bmatrix}$ **39.** $\begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{9} \end{bmatrix}$

**41.** (a) $\begin{bmatrix} 35 & 17 \\ 4 & 10 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & -7 \\ 5 & 6 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & \frac{5}{2} \\ -\frac{7}{2} & 3 \end{bmatrix}$

**43.** (a) $\frac{1}{16}\begin{bmatrix} 138 & 56 & -84 \\ 37 & 26 & -71 \\ 24 & 34 & 3 \end{bmatrix}$ (b) $\frac{1}{4}\begin{bmatrix} 4 & 6 & 1 \\ -2 & 2 & 4 \\ 3 & -8 & 2 \end{bmatrix}$

(c) $\frac{1}{8}\begin{bmatrix} 4 & -2 & 3 \\ 6 & 2 & -8 \\ 1 & 4 & 2 \end{bmatrix}$

**45.** (a) $x = 1$, $y = -1$ (b) $x = 2$, $y = 4$

**47.** (a) $x_1 = 1$, $x_2 = 1$, $x_3 = -1$ (b) $x_1 = 0$, $x_2 = 1$, $x_3 = -1$

**49.** $x_1 = 0$, $x_2 = 1$, $x_3 = 2$, $x_4 = -1$, $x_5 = 0$

**51.** $x_1 = 1$, $x_2 = -2$, $x_3 = 3$, $x_4 = 0$, $x_5 = 1$, $x_6 = -2$

**53.** $x = 4$ **55.** $x = 6$

**57.** $\begin{bmatrix} -1 & \frac{1}{2} \\ \frac{3}{4} & -\frac{1}{4} \end{bmatrix}$ **59.** Proof; $A^{-1} = \begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$

**61.** $F^{-1} = \begin{bmatrix} 188.24 & -117.65 & -11.76 \\ -117.65 & 323.53 & -117.65 \\ -11.76 & -117.65 & 188.24 \end{bmatrix}$; $\mathbf{w} = \begin{bmatrix} 25 \\ 40 \\ 75 \end{bmatrix}$

**63.** (a) True. See Theorem 2.10, part 1.

(b) False. See Theorem 2.9.

(c) True. See "Finding the Inverse of a Matrix by Gauss-Jordan Elimination," part 2, page 64.

**65–71.** Proofs

**73.** The sum of two invertible matrices is not necessarily invertible. For example, let

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

**75.** (a) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

**77.** (a) Proof (b) $H = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

**79.** $A = PDP^{-1}$

No, $A$ is not necessarily equal to $D$.

**81.** Answers will vary. Sample answer: For an $n \times n$ matrix $A$, set up the matrix $[A \quad I]$ and row reduce it until you have $[I \quad A^{-1}]$. If this is not possible or if $A$ is not square, then $A$ has no inverse. If it is possible, then the inverse is $A^{-1}$.

**83.** Answers will vary. Sample answer: For the system of equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

write as the matrix equation

$$AX = B$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

If $A$ is invertible, then the solution is $X = A^{-1}B$.

## Section 2.4 *(page 82)*

**1.** Elementary, multiply Row 2 by 2.

**3.** Elementary, add 2 times Row 1 to Row 2.

**5.** Not elementary

**7.** Elementary, add $-5$ times Row 2 to Row 3.

**9.** $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ **11.** $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

**13.** $\begin{bmatrix} \frac{1}{5} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 7 \\ 5 & 10 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 7 \end{bmatrix}$

**15.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 6 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 & -1 & 0 \\ 0 & 4 & 8 & -4 \\ -6 & 12 & 8 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & -2 & -1 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & \frac{1}{2} \end{bmatrix}$

**17.** $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ **19.** $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

**21.** $\begin{bmatrix} \dfrac{1}{k} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, k \neq 0$

**23.** $\begin{bmatrix} 0 & 1 \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix}$ **25.** $\begin{bmatrix} 1 & 0 & \frac{1}{4} \\ 0 & \frac{1}{6} & \frac{1}{24} \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$

**27.** $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$

(The answer is not unique.)

**29.** $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

(The answer is not unique.)

**31.** $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(The answer is not unique.)

**33.** $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(The answer is not unique.)

**35.** (a) True. See “Remark” next to “Definition of an Elementary Matrix,” page 74.

(b) False. Multiplication of a matrix by a scalar is not a single elementary row operation, so it cannot be represented by a corresponding elementary matrix.

(c) True. See Theorem 2.13.

**37.** No. For example, $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$.

**39.** $A^{-1} = \begin{bmatrix} 1 & -a & 0 \\ -b & ab + 1 & 0 \\ 0 & 0 & \dfrac{1}{c} \end{bmatrix}$

**41.** $\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(The answer is not unique.)

**43.** $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}$

(The answer is not unique.)

**45.** (a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{bmatrix}$

(The answer is not unique.)

(b) $\mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix}$ (c) $\mathbf{x} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ -\frac{5}{3} \end{bmatrix}$

**47.** First, factor the matrix $A = LU$. Then, for each right-hand side $\mathbf{b}_i$, solve $L\mathbf{y} = \mathbf{b}_i$ and $U\mathbf{x} = \mathbf{y}$.

**49.** Idempotent **51.** Not idempotent

**53.** *Case 1:* $b = 1, a = 0$

*Case 2:* $b = 0, a =$ any real number

**55–59.** Proofs

## Section 2.5 *(page 95)*

**1.** Not stochastic **3.** Stochastic

**5.** (a) 350 (b) 475

**7.**

| | Nonsmokers | Smokers of 1 pack/day or less | Smokers of more than 1 pack/day |
|---|---|---|---|
| (a) | 5025 | 2500 | 2475 |
| (b) | 5047 | 2499 | 2454 |

**9.**

| | Brand A | Brand B | Neither |
|---|---|---|---|
| (a) | 24,500 | 34,000 | 41,500 |
| (b) | 27,625 | 36,625 | 35,750 |
| (c) | 29,788 | 38,356 | 31,856 |

**11.** Uncoded: $[19 \quad 5 \quad 12], [12 \quad 0 \quad 3], [15 \quad 14 \quad 19],$
$[15 \quad 12 \quad 9], [4 \quad 1 \quad 20], [5 \quad 4 \quad 0]$

Encoded: $-48, 5, 31, -6, -6, 9, -85, 23, 43,$
$-27, 3, 15, -115, 36, 59, 9, -5, -4$

**13.** Uncoded: $[3 \quad 15], [13 \quad 5], [0 \quad 8], [15 \quad 13], [5 \quad 0],$
$[19 \quad 15], [15 \quad 14]$

Encoded: 48, 81, 28, 51, 24, 40, 54, 95, 5, 10, 64, 113, 57, 100

**15.** HAPPY_NEW_YEAR **17.** ICEBERG_DEAD_AHEAD

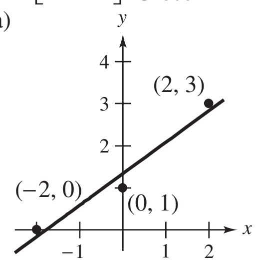
**19.** MEET_ME_TONIGHT_RON

**21.** _SEPTEMBER_THE_ELEVENTH_WE_WILL_ALWAYS_REMEMBER

**23.** $D = \begin{array}{c} \begin{matrix} \text{Coal} & \text{Steel} \end{matrix} \\ \begin{bmatrix} 0.1 & 0.2 \\ 0.8 & 0.1 \end{bmatrix} \end{array} \begin{matrix} \text{Coal} \\ \text{Steel} \end{matrix} \qquad X = \begin{bmatrix} 20{,}000 \\ 40{,}000 \end{bmatrix} \begin{matrix} \text{Coal} \\ \text{Steel} \end{matrix}$

**25.** $X = \begin{bmatrix} 8622.0 \\ 4685.0 \\ 3661.4 \end{bmatrix} \begin{matrix} \text{Farmer} \\ \text{Baker} \\ \text{Grocer} \end{matrix}$

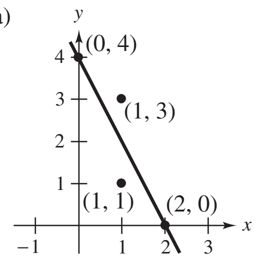
**27.** (a)




(b) $y = \frac{4}{3} + \frac{3}{4}x$

(c) $\frac{1}{6}$

**29.** (a)




(b) $y = 4 - 2x$

(c) 2

**31.** $y = -\frac{1}{3} + 2x$ **33.** $y = 1.3 + 0.6x$

**35.** $y = 0.412x + 3$ **37.** $y = -0.5x + 7.5$

**39.** (a) $y = 11{,}650 - 2400x$ (b) 3490 gallons

**41.** (a) and (b) $y = 2.81t + 226.76$

**43.** Answers will vary. **45.** Proof

## Review Exercises *(page 98)*

**1.** $\begin{bmatrix} -13 & -8 & 18 \\ 0 & 11 & -19 \end{bmatrix}$ **3.** $\begin{bmatrix} 14 & -2 & 8 \\ 14 & -10 & 40 \\ 36 & -12 & 48 \end{bmatrix}$

**5.** $\begin{bmatrix} 4 & 6 & 3 \\ 0 & 6 & -10 \\ 0 & 0 & 6 \end{bmatrix}$ **7.** $\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -8 \\ -4 \end{bmatrix}, x = \begin{bmatrix} -4 \\ 0 \end{bmatrix}$

**9.** $\begin{bmatrix} -3 & -1 & 1 \\ 2 & 4 & -5 \\ 1 & -2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}, x = \begin{bmatrix} \frac{2}{3} \\ -\frac{17}{3} \\ -\frac{11}{3} \end{bmatrix}$

**11.** $A^T = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ -3 & 2 \end{bmatrix}, A^TA = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 5 & -4 \\ -3 & -4 & 13 \end{bmatrix},$

$AA^T = \begin{bmatrix} 14 & -4 \\ -4 & 5 \end{bmatrix}$

**13.** $A^T = [1 \quad 3 \quad -1], A^TA = [11]$

$AA^T = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 9 & -3 \\ -1 & -3 & 1 \end{bmatrix}$

**15.** $\begin{bmatrix} 1 & -1 \\ 2 & -3 \end{bmatrix}$ **17.** $\begin{bmatrix} \frac{3}{20} & \frac{3}{20} & \frac{1}{10} \\ \frac{3}{10} & -\frac{1}{30} & -\frac{2}{15} \\ -\frac{1}{5} & -\frac{1}{5} & \frac{1}{5} \end{bmatrix}$ **19.** $\begin{bmatrix} 1 \\ -5 \end{bmatrix}$

**21.** $\begin{bmatrix} 0 \\ -\frac{1}{7} \\ \frac{3}{7} \end{bmatrix}$ **23.** $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 10 \\ -12 \end{bmatrix}$ **25.** $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix}$

**27.** $\begin{bmatrix} \frac{1}{14} & \frac{1}{42} \\ -\frac{1}{21} & \frac{2}{21} \end{bmatrix}$ **29.** $x \neq -3$ **31.** $\begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

**33.** $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

(The answer is not unique.)

**35.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(The answer is not unique.)

**37.** $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(The answer is not unique.)

**39.** $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$ and $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

(The answer is not unique.)

**41.** (a) $a = -1$ (b) and (c) Proofs
$b = -1$
$c = 1$

**43.** $\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ 0 & -1 \end{bmatrix}$

(The answer is not unique.)

**45.** $x = 4, y = 1, z = -1$

**47.** (a) False. See Theorem 2.1, part 1, page 52.
(b) True. See Theorem 2.6, part 2, page 57.

**49.** (a) False. The matrix $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is not invertible.
(b) False. See Exercise 65, page 61.

**51.** $\begin{bmatrix} 110 & 99 & 77 & 33 \\ 44 & 22 & 66 & 66 \end{bmatrix}$

**53.** (a) $\begin{bmatrix} 5455 & 128.2 \\ 3551 & 77.6 \\ 7591 & 178.6 \end{bmatrix}$

The first column of the matrix gives the total sales for gas on each day and the second column gives the total profit for each day.

(b) \$384.40

**55.** $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ **57.** Not stochastic

**59.** $PX = \begin{bmatrix} 80 \\ 112 \end{bmatrix}, P^2X = \begin{bmatrix} 68 \\ 124 \end{bmatrix}, P^3X = \begin{bmatrix} 65 \\ 127 \end{bmatrix}$

**61.** (a) $\begin{bmatrix} 110{,}000 \\ 100{,}000 \\ 90{,}000 \end{bmatrix} \begin{matrix} \text{Region 1} \\ \text{Region 2} \\ \text{Region 3} \end{matrix}$ (b) $\begin{bmatrix} 123{,}125 \\ 100{,}000 \\ 76{,}875 \end{bmatrix} \begin{matrix} \text{Region 1} \\ \text{Region 2} \\ \text{Region 3} \end{matrix}$

**63.** Uncoded: [15 14] [5 0] [9 6] [0 2] [25 0] [12 1] [14 4]

Encoded: 103 44 25 10 57 24 4 2 125 50 62 25 78 32

**65.** $A^{-1} = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}$; ALL_SYSTEMS_GO

**67.** $A^{-1} = \begin{bmatrix} -2 & -1 & 0 \\ 0 & 1 & 1 \\ -5 & -3 & -3 \end{bmatrix}$; INVASION_AT_DAWN

**69.** _CAN_YOU_HEAR_ME_NOW

**71.** $D = \begin{bmatrix} 0.20 & 0.50 \\ 0.30 & 0.10 \end{bmatrix}, X \approx \begin{bmatrix} 133{,}333 \\ 133{,}333 \end{bmatrix}$

**73.** $y = \frac{20}{3} - \frac{3}{2}x$ **75.** $y = 2.5x$

**77.** (a) $y = 19 + 14x$ (b) 41.4 kilograms per square kilometer

**79.** (a) $y = 0.13x + 2.01$

(b) $y = 0.13x + 2.01$

The models are the same.

(c)

| *Year* | 2005 | 2006 | 2007 | 2008 | 2009 | 2010 |
|---|---|---|---|---|---|---|
| *Actual* | 2.6 | 2.9 | 2.9 | 3.2 | 3.2 | 3.3 |
| *Estimated* | 2.7 | 2.8 | 2.9 | 3.1 | 3.2 | 3.3 |

The estimated values are close to the actual values.

# Chapter 3

## Section 3.1 *(page 110)*

**1.** 1 **3.** 5 **5.** 27 **7.** $-24$ **9.** 0

**11.** $\lambda^2 - 4\lambda - 5$

**13.** (a) $M_{11} = 4$, $M_{12} = 3$, $M_{21} = 2$, $M_{22} = 1$

(b) $C_{11} = 4$, $C_{12} = -3$, $C_{21} = -2$, $C_{22} = 1$

**15.** (a) $M_{11} = 23$ $M_{12} = -8$ $M_{13} = -22$

$M_{21} = 5$ $M_{22} = -5$ $M_{23} = 5$

$M_{31} = 7$ $M_{32} = -22$ $M_{33} = -23$

(b) $C_{11} = 23$ $C_{12} = 8$ $C_{13} = -22$

$C_{21} = -5$ $C_{22} = -5$ $C_{23} = -5$

$C_{31} = 7$ $C_{32} = 22$ $C_{33} = -23$

**17.** (a) $4(-5) + 5(-5) + 6(-5) = -75$

(b) $2(8) + 5(-5) - 3(22) = -75$

**19.** $-58$ **21.** $-30$ **23.** 0.002 **25.** $4x - 2y - 2$

**27.** 0 **29.** $65{,}644w + 62{,}256x + 12{,}294y - 24{,}672z$

**31.** $-100$ **33.** 14 **35.** $-0.175$ **37.** 19

**39.** $-24$ **41.** 0

**43.** (a) False. See "Definition of the Determinant of a $2 \times 2$ Matrix," page 104.

(b) True. See "Remark," page 106.

(c) False. See "Minors and Cofactors of a Square Matrix," page 105.

**45.** $x = -1, -4$ **47.** $x = -1, 4$ **49.** $\lambda = -1 \pm \sqrt{3}$

**51.** $\lambda = -2, 0$, or 1 **53.** $8uv - 1$ **55.** $e^{5x}$

**57.** $1 - \ln x$

**59.** Expanding along the first row, the determinant of a $4 \times 4$ matrix involves four $3 \times 3$ determinants. Each of these $3 \times 3$ determinants requires six triple products. So, there are $4(6) = 24$ quadruple products.

**61.** $wz - xy$ **63.** $wz - xy$

**65.** $xy^2 - xz^2 + yz^2 - x^2y + x^2z - y^2z$

**67.** (a) Proof

(b) $\begin{vmatrix} x & 0 & 0 & d \\ -1 & x & 0 & c \\ 0 & -1 & x & b \\ 0 & 0 & -1 & a \end{vmatrix}$

**69.** Proof

## Section 3.2 *(page 118)*

**1.** The first row is 2 times the second row. If one row of a matrix is a multiple of another row, then the determinant of the matrix is zero.

**3.** The second row consists entirely of zeros. If one row of a matrix consists entirely of zeros, then the determinant of the matrix is zero.

**5.** The second and third columns are interchanged. If two columns of a matrix are interchanged, then the determinant of the matrix changes sign.

**7.** The first row of the matrix is multiplied by 5. If a row in a matrix is multiplied by a scalar, then the determinant of the matrix is multiplied by that scalar.

**9.** A 4 is factored out of the second column and a 3 is factored out of the third column. If a column of a matrix is multiplied by a scalar, then the determinant of the matrix is multiplied by that scalar.

**11.** The matrix is multiplied by 5. If an $n \times n$ matrix is multiplied by a scalar $c$, then the determinant of the matrix is multiplied by $c^n$.

**13.** $-4$ times the first row is added to the second row. If a scalar multiple of one row of a matrix is added to another row, then the determinant of the matrix is unchanged.

**15.** A multiple of the first row is added to the second row. If a scalar multiple of one row is added to another row, then the determinants are equal.

**17.** The second row of the matrix is multiplied by $-1$. If a row of a matrix is multiplied by a scalar, then the determinant is multiplied by that scalar.

**19.** The fifth column is 2 times the first column. If one column of a matrix is a multiple of another column, then the determinant of the matrix is zero.

**21.** $-1$ **23.** 19 **25.** 28 **27.** 0 **29.** $-60$

**31.** $-1344$ **33.** 136 **35.** $-1100$

**37.** (a) True. See Theorem 3.3, part 1, page 113.
(b) True. See Theorem 3.3, part 3, page 113.
(c) True. See Theorem 3.4, part 2, page 115.

**39.** $k$ **41.** 1 **43.** Proof

**45.** (a) $\cos^2\theta + \sin^2\theta = 1$ (b) $\sin^2\theta - 1 = -\cos^2\theta$

**47.** Proof

## Section 3.3 *(page 125)*

**1.** (a) 0 (b) $-1$ (c) $\begin{bmatrix} -2 & -3 \\ 4 & 6 \end{bmatrix}$ (d) 0

**3.** (a) 2 (b) $-6$ (c) $\begin{bmatrix} 1 & 4 & 3 \\ -1 & 0 & 3 \\ 0 & 2 & 0 \end{bmatrix}$ (d) $-12$

**5.** (a) 3 (b) 6 (c) $\begin{bmatrix} 6 & 3 & -2 & 2 \\ 2 & 1 & 0 & -1 \\ 9 & 4 & -3 & 8 \\ 8 & 5 & -4 & 5 \end{bmatrix}$ (d) 18

**7.** $-44$ **9.** 54 **11.** 0 **13.** (a) $-2$ (b) $-2$ (c) 0

**15.** (a) 0 (b) $-1$ (c) $-15$

**17.** Singular **19.** Nonsingular **21.** Nonsingular

**23.** Singular **25.** $\frac{1}{5}$ **27.** $-\frac{1}{2}$ **29.** $\frac{1}{24}$

**31.** The solution is unique because the determinant of the coefficient matrix is nonzero.

**33.** The solution is not unique because the determinant of the coefficient matrix is zero.

**35.** The solution is unique because the determinant of the coefficient matrix is nonzero.

**37.** (a) 14 (b) 196 (c) 196 (d) 56 (e) $\frac{1}{14}$

**39.** (a) $-30$ (b) 900 (c) 900 (d) $-240$ (e) $-\frac{1}{30}$

**41.** (a) 29 (b) 841 (c) 841 (d) 232 (e) $\frac{1}{29}$

**43.** (a) $-24$ (b) 576 (c) 576 (d) $-384$ (e) $-\frac{1}{24}$

**45.** (a) 22 (b) 22 (c) 484 (d) 88 (e) $\frac{1}{22}$

**47.** (a) $-26$ (b) $-26$ (c) 676 (d) $-208$ (e) $-\frac{1}{26}$

**49.** (a) $-115$ (b) $-115$ (c) 13,225 (d) $-1840$ (e) $-\frac{1}{115}$

**51.** (a) 25 (b) 9 (c) $-125$ (d) 81

**53.** $k = -1, 4$ **55.** $k = 24$ **57.** Proof

**59.** $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
(The answer is not unique.)

**61.** 0

**63.** Proof

**65.** (a) False. See Theorem 3.6, page 121.
(b) True. See Theorem 3.8, page 122.
(c) True. See "Equivalent Conditions for a Nonsingular Matrix," parts 1 and 2, page 123.

**67.** No; in general, $P^{-1}AP \neq A$. For example, let

$P = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$, $P^{-1} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$, and $A = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$.

Then you have

$$P^{-1}AP = \begin{bmatrix} -27 & -49 \\ 16 & 29 \end{bmatrix} \neq A.$$

The equation $|P^{-1}AP| = |A|$ is true in general because

$$\begin{aligned} |P^{-1}AP| &= |P^{-1}||A||P| \\ &= |P^{-1}||P||A| = \frac{1}{|P|}|P||A| = |A|. \end{aligned}$$

**69.** Proof **71.** Orthogonal **73.** Not orthogonal

**75.** Orthogonal **77.** Proof

**79.** (a) $\begin{bmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{bmatrix}$ (b) $\begin{bmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{bmatrix}$ (c) 1
$A$ is orthogonal.

**81.** Proof

## Section 3.4 *(page 136)*

**1.** $\operatorname{adj}(A) = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$, $A^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$

**3.** $\operatorname{adj}(A) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -12 & -6 \\ 0 & 4 & 2 \end{bmatrix}$, $A^{-1}$ does not exist.

**5.** $\operatorname{adj}(A) = \begin{bmatrix} -7 & -12 & 13 \\ 2 & 3 & -5 \\ 2 & 3 & -2 \end{bmatrix}$, $A^{-1} = \begin{bmatrix} \frac{7}{3} & 4 & -\frac{13}{3} \\ -\frac{2}{3} & -1 & \frac{5}{3} \\ -\frac{2}{3} & -1 & \frac{2}{3} \end{bmatrix}$

**7.** $\operatorname{adj}(A) = \begin{bmatrix} 7 & 1 & 9 & -13 \\ 7 & 1 & 0 & -4 \\ -4 & 2 & -9 & 10 \\ 2 & -1 & 9 & -5 \end{bmatrix}$,

$$A^{-1} = \begin{bmatrix} \frac{7}{9} & \frac{1}{9} & 1 & -\frac{13}{9} \\ \frac{7}{9} & \frac{1}{9} & 0 & -\frac{4}{9} \\ -\frac{4}{9} & \frac{2}{9} & -1 & \frac{10}{9} \\ \frac{2}{9} & -\frac{1}{9} & 1 & -\frac{5}{9} \end{bmatrix}$$

**9.** Proof **11.** Proof

**13.** $|\operatorname{adj}(A)| = \begin{vmatrix} -2 & 0 \\ -1 & 1 \end{vmatrix} = -2,$

$|A|^{2-1} = \begin{vmatrix} 1 & 0 \\ 1 & -2 \end{vmatrix}^{2-1} = -2$

**15.** Proof

**17.** $x_1 = 1$, $x_2 = 2$ **19.** $x_1 = 2$, $x_2 = -2$ **21.** $x_1 = \frac{3}{4}$, $x_2 = -\frac{1}{2}$

**23.** Cramer's Rule does not apply because the coefficient matrix has a determinant of zero.

**25.** $x_1 = 1$, $x_2 = 1$, $x_3 = 2$ **27.** $x_1 = 1$, $x_2 = \frac{1}{2}$, $x_3 = \frac{3}{2}$

**29.** $x_1 = -1, x_2 = 3, x_3 = 2$ **31.** $x_1 = -12, x_2 = 10$

**33.** $x_1 = 5, x_2 = -3, x_3 = 2, x_4 = -1$

**35.** $x = \dfrac{4k-3}{2k-1}$, $y = \dfrac{4k-1}{2k-1}$
The system will be inconsistent if $k = \frac{1}{2}$.

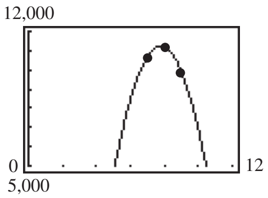
**37.** 3 **39.** 3 **41.** Collinear **43.** Not collinear

**45.** $3y - 4x = 0$ **47.** $x = -2$ **49.** $\frac{1}{3}$ **51.** 2

**53.** Not coplanar **55.** Coplanar

**57.** $4x - 10y + 3z = 27$ **59.** $x + y + z = 0$

**61.** Incorrect. The numerator and denominator should be interchanged.

**63.** (a) $49a + 7b + c = 10{,}697$
$64a + 8b + c = 11{,}162$
$81a + 9b + c = 9891$
(b) $a = -868, b = 13{,}485, c = -41{,}166$
(c)


(d) The polynomial fits the data exactly.

## Review Exercises *(page 138)*

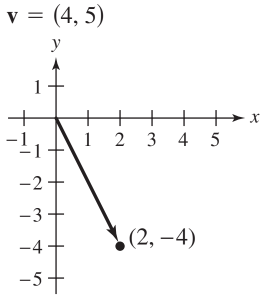
**1.** 10 **3.** 0 **5.** 0 **7.** $-6$ **9.** 1620 **11.** 82
**13.** $-64$ **15.** $-1$ **17.** $-1$
**19.** Because the second row is a multiple of the first row, the determinant is zero.
**21.** A $-4$ has been factored out of the second column and a 3 has been factored out of the third column. If a column of a matrix is multiplied by a scalar, then the determinant of the matrix is also multiplied by that scalar.
**23.** (a) $-1$ (b) $-5$ (c) $\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$ (d) 5
**25.** (a) $-12$ (b) $-1728$ (c) 144 (d) $-300$
**27.** (a) $-20$ (b) $-\frac{1}{20}$ **29.** $\frac{1}{6}$ **31.** $-\frac{1}{10}$
**33.** $x_1 = 0$, $x_2 = -\frac{1}{2}$, $x_3 = \frac{1}{2}$ **35.** $x_1 = -3$, $x_2 = -1$, $x_3 = 2$
**37.** Unique solution **39.** Unique solution
**41.** Not a unique solution
**43.** (a) 8 (b) 4 (c) 64 (d) 8 (e) $\frac{1}{2}$
**45.** Proof **47.** 0 **49.** $-\frac{1}{2}$ **51.** $-uv$
**53.** Row reduction is generally preferred for matrices with few zeros. For a matrix with many zeros, it is often easier to expand along a row or column having many zeros.
**55.** $x = \pi/4 + n\pi/2$, where $n$ is an integer. **57.** $\begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$
**59.** Unique solution: $x = 0.6$, $y = 0.5$
**61.** Unique solution: $x_1 = \frac{1}{2}$, $x_2 = -\frac{1}{3}$, $x_3 = 1$
**63.** $x_1 = 6, x_2 = -2$ **65.** 16 **67.** $x - 2y = -4$
**69.** $9x + 4y - 3z = 0$
**71.** Incorrect. In the numerator, the column of constants,
$\begin{bmatrix} -1 \\ 6 \\ 1 \end{bmatrix}$
should replace the third column of the coefficient matrix, not the first column.
**73.** (a) False. See "Minors and Cofactors of a Square Matrix," page 105.
(b) False. See Theorem 3.3, part 1, page 113.
(c) True. See Theorem 3.4, part 3, page 115.
(d) False. See Theorem 3.9, page 124.
**75.** (a) False. See Theorem 3.11, page 131.
(b) False. See "Test for Collinear Points in the *xy*-Plane," page 133.

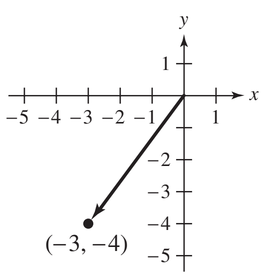
## Cumulative Test Chapters 1–3 *(page 143)*

**1.** Not linear **2.** Linear **3.** $x = 1, y = -2$
**4.** $x_1 = 2,\ x_2 = -3,\ x_3 = -2$
**5.** $x = 10, y = -20, z = 40, w = -12$
**6.** $x_1 = s - 2t,\ x_2 = 2 + t,\ x_3 = t,\ x_4 = s$
**7.** $x_1 = -2s,\ x_2 = s,\ x_3 = 2t,\ x_4 = t$ **8.** $k = 12$
**9.** $x = -3,\ y = 4$
**10.** $A^TA = \begin{bmatrix} 17 & 22 & 27 \\ 22 & 29 & 36 \\ 27 & 36 & 45 \end{bmatrix}$ **11.** $\begin{bmatrix} -\frac{1}{4} & \frac{1}{8} \\ \frac{1}{6} & \frac{1}{12} \end{bmatrix}$
**12.** $\begin{bmatrix} -\frac{2}{7} & \frac{1}{7} \\ \frac{1}{7} & \frac{2}{21} \end{bmatrix}$ **13.** $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$ **14.** $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ \frac{3}{5} & \frac{1}{5} & -\frac{9}{5} \end{bmatrix}$
**15.** $x = -\frac{3}{2}, y = -\frac{3}{4}$ **16.** $x = 4, y = 2$
**17.** $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -4 \end{bmatrix}$
(The answer is not unique.)
**18.** $-34$
**19.** (a) 14 (b) $-10$ (c) $\begin{bmatrix} -2 & -14 \\ -8 & 14 \end{bmatrix}$ (d) $-140$
**20.** (a) 84 (b) $\frac{1}{84}$
**21.** (a) 567 (b) 7 (c) $\frac{1}{7}$ (d) 343
**22.** $\begin{bmatrix} \frac{4}{11} & -\frac{10}{11} & \frac{7}{11} \\ -\frac{1}{11} & -\frac{3}{11} & \frac{1}{11} \\ -\frac{2}{11} & \frac{5}{11} & \frac{2}{11} \end{bmatrix}$
**23.** $a = 1,\ b = 0,\ c = 2$
(The answer is not unique.)
**24.** $y = \frac{7}{6}x^2 + \frac{1}{6}x + 1$ **25.** $3x + 2y = 11$ **26.** 16
**27.** $I_1 = 3, I_2 = 4, I_3 = 1$
**28.** $BA = [13{,}275.00 \quad 15{,}500.00]$
The entries represent the total values (in dollars) of the products sent to the two warehouses.
**29.** No; proof

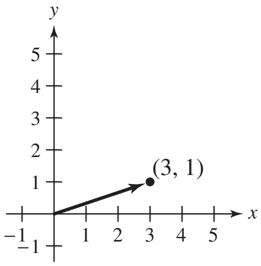
# Chapter 4

## Section 4.1 *(page 153)*

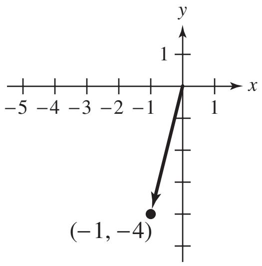
**1.** $\mathbf{v} = (4, 5)$
**3.**


**5.**


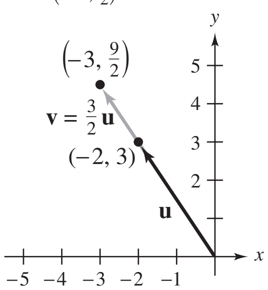
**7.** $\mathbf{u} + \mathbf{v} = (3, 1)$



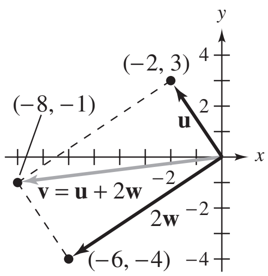

**9.** $\mathbf{u} + \mathbf{v} = (-1, -4)$



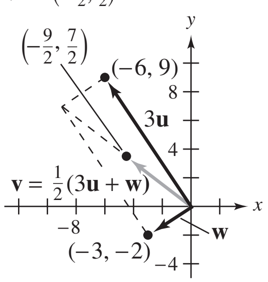

**11.** $\mathbf{v} = \left(-3, \frac{9}{2}\right)$



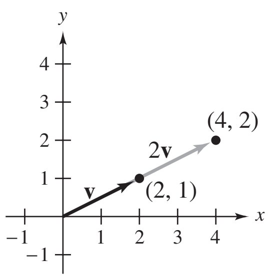

**13.** $\mathbf{v} = (-8, -1)$



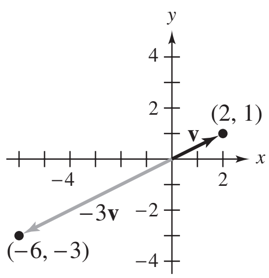

**15.** $\mathbf{v} = \left(-\frac{9}{2}, \frac{7}{2}\right)$



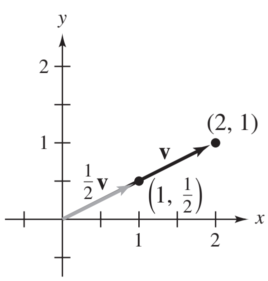

**17.** (a)




(b)



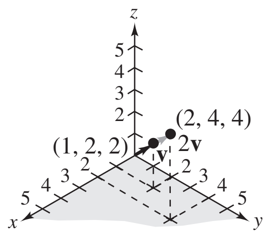

(c)



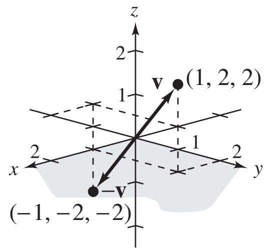

**19.** $\mathbf{u} - \mathbf{v} = (-1, 0, 4)$
$\mathbf{v} - \mathbf{u} = (1, 0, -4)$

**21.** $(6, 12, 6)$ **23.** $\left(\frac{7}{2}, 3, \frac{5}{2}\right)$

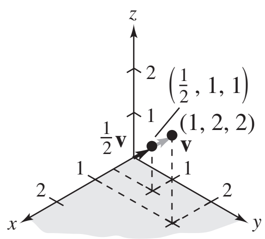
**25.** (a)




(b)




(c)




**27.** (a)

**29.** (a) $(4, -2, -8, 1)$ (b) $(8, 12, 24, 34)$
(c) $(-4, 4, 13, 3)$

**31.** (a) $(1, 6, -5, -3)$ (b) $(-1, -8, 10, 0)$
(c) $\left(-\frac{3}{2}, 11, -\frac{13}{2}, -\frac{21}{2}\right)$

**33.** $\left(\frac{1}{2}, -\frac{7}{2}, -\frac{9}{2}, 2\right)$ **35.** $(4, 8, 18, -2)$ **37.** $\left(-1, \frac{5}{3}, 6, \frac{2}{3}\right)$

**39.** $\mathbf{v} = \mathbf{u} + \mathbf{w}$ **41.** $\mathbf{v} = \mathbf{u} + 2\mathbf{w}$ **43.** $\mathbf{v} = -\mathbf{u}$

**45.** $\mathbf{v} = \mathbf{u}_1 + 2\mathbf{u}_2 - 3\mathbf{u}_3$

**47.** It is not possible to write **v** as a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$.

**49.** $\mathbf{v} = 2\mathbf{u}_1 + \mathbf{u}_2 - 2\mathbf{u}_3 + \mathbf{u}_4 - \mathbf{u}_5$

**51.** (a) True. Two vectors in $R^n$ are equal if and only if their corresponding components are equal, that is, $\mathbf{u} = \mathbf{v}$ if and only if $\mathbf{u}_1 = \mathbf{v}_1, \mathbf{u}_2 = \mathbf{v}_2, \ldots, \mathbf{u}_n = \mathbf{v}_n$.
(b) False. The vector $-\mathbf{v}$ is called the additive inverse of the vector **v**.

**53.** No **55.** Answers will vary. **57.** Proof

**59.** If $\mathbf{b} = x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n$ is a linear combination of the columns of $A$, then a solution to $A\mathbf{x} = \mathbf{b}$ is

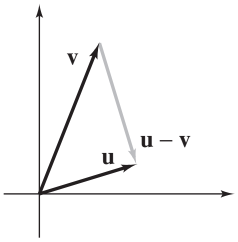
$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

The system $A\mathbf{x} = \mathbf{b}$ is inconsistent if **b** is not a linear combination of the columns of $A$.

**61.** (a) Additive identity
(b) Distributive property
(c) Add $-0\mathbf{v}$ to both sides.
(d) Additive inverse and associative property
(e) Additive inverse
(f) Additive identity

**63.** (a) Multiply both sides by $c^{-1}$.
(b) Associative property and Theorem 4.3, property 4
(c) Multiplicative inverse
(d) Multiplicative identity

**65.** You could describe vector subtraction as follows:




or write subtraction in terms of addition, $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-1)\mathbf{v}$.

## Section 4.2 *(page 160)*

**1.** $(0, 0, 0, 0)$ **3.** $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

**5.** $0 + 0x + 0x^2 + 0x^3$

**7.** $-(v_1, v_2, v_3, v_4) = (-v_1, -v_2, -v_3, -v_4)$

**9.** $-\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} -a_{11} & -a_{12} & -a_{13} \\ -a_{21} & -a_{22} & -a_{23} \end{bmatrix}$

**11.** $-(a_0 + a_1x + a_2x^2 + a_3x^3) = -a_0 - a_1x - a_2x^2 - a_3x^3$

**13.** The set is a vector space.

**15.** The set is not a vector space. Axiom 1 fails because $x^3 + (-x^3 + 1) = 1$, which is not a third-degree polynomial. (Axioms 4, 5, and 6 also fail.)

**17.** The set is not a vector space. Axiom 4 fails.

**19.** The set is a vector space.

**21.** The set is not a vector space. Axiom 6 fails because $(-1)(x, y) = (-x, -y)$, which is not in the set when $x \neq 0$.

**23.** The set is a vector space.

**25.** The set is a vector space.

**27.** The set is a vector space.

**29.** The set is not a vector space. Axiom 1 fails because
$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,
which is not singular.

**31.** The set is a vector space.

**33.** The set is a vector space.

**35.** (a) The set is not a vector space. Axiom 8 fails because
$(1 + 2)(1, 1) = 3(1, 1) = (3, 1)$
$1(1, 1) + 2(1, 1) = (1, 1) + (2, 1) = (3, 2)$.
(b) The set is not a vector space. Axiom 2 fails because
$(1, 2) + (2, 1) = (1, 0)$
$(2, 1) + (1, 2) = (2, 0)$.
(Axioms 4, 5, and 8 also fail.)
(c) The set is not a vector space. Axiom 6 fails because $(-1)(1, 1) = (\sqrt{-1}, \sqrt{-1})$, which is not in $R^2$. (Axioms 8 and 9 also fail.)

**37.** Proof

**39.** The set is not a vector space. Axiom 5 fails because $(1, 1)$ is the additive identity so $(0, 0)$ has no additive inverse. (Axioms 7 and 8 also fail.)

**41.** Yes, the set is a vector space. **43.** Proof

**45.** (a) True. See page 155.
(b) False. See Example 6, page 159.
(c) False. With standard operations on $R^3$, the additive inverse axiom is not satisfied.

**47.** Proof

## Section 4.3 *(page 167)*

**1.** Because $W$ is nonempty and $W \subset R^4$, you need only check that $W$ is closed under addition and scalar multiplication. Given
$(x_1, x_2, x_3, 0) \in W$ and $(y_1, y_2, y_3, 0) \in W$
it follows that
$(x_1, x_2, x_3, 0) + (y_1, y_2, y_3, 0)$
$= (x_1 + y_1, x_2 + y_2, x_3 + y_3, 0) \in W$.
Also, for any real number $c$ and $(x_1, x_2, x_3, 0) \in W$, it follows that
$c(x_1, x_2, x_3, 0) = (cx_1, cx_2, cx_3, 0) \in W$.

**3.** Because $W$ is nonempty and $W \subset M_{2,2}$, you need only check that $W$ is closed under addition and scalar multiplication. Given
$\begin{bmatrix} 0 & a_1 \\ b_1 & 0 \end{bmatrix} \in W$ and $\begin{bmatrix} 0 & a_2 \\ b_2 & 0 \end{bmatrix} \in W$
it follows that
$\begin{bmatrix} 0 & a_1 \\ b_1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & a_2 \\ b_2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a_1 + a_2 \\ b_1 + b_2 & 0 \end{bmatrix} \in W$.
Also, for any real number $c$ and
$\begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} \in W$, it follows that
$c\begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} = \begin{bmatrix} 0 & ca \\ cb & 0 \end{bmatrix} \in W$.

**5.** Recall from calculus that continuity implies integrability; $W \subset V$. So, because $W$ is nonempty, you need only check that $W$ is closed under addition and scalar multiplication. Given continuous functions $f, g \in W$, it follows that $f + g$ is continuous and $f + g \in W$. Also, for any real number $c$ and for a continuous function $f \in W$, $cf$ is continuous. So, $cf \in W$.

**7.** Not closed under addition:
$(0, 0, -1) + (0, 0, -1) = (0, 0, -2)$
Not closed under scalar multiplication:
$2(0, 0, -1) = (0, 0, -2)$

**9.** Not closed under scalar multiplication:
$\sqrt{2}(1, 1) = (\sqrt{2}, \sqrt{2})$

**11.** Not closed under scalar multiplication: $(-1)e^x = -e^x$

**13.** Not closed under scalar multiplication:
$(-2)(1, 1, 1) = (-2, -2, -2)$

**15.** Not closed under scalar multiplication:
$2\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

**17.** Not closed under addition:
$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

**19.** Not closed under addition:
$(2, 8) + (3, 27) = (5, 35)$
Not closed under scalar multiplication:
$2(3, 27) = (6, 54)$

**21.** Not a subspace **23.** Subspace **25.** Subspace
**27.** Subspace **29.** Subspace **31.** Not a subspace
**33.** Not a subspace **35.** Subspace
**37.** $W$ is a subspace of $R^3$. ($W$ is nonempty and closed under addition and scalar multiplication.)
**39.** $W$ is a subspace of $R^3$. ($W$ is nonempty and closed under addition and scalar multiplication.)
**41.** $W$ is not a subspace of $R^3$.
Not closed under addition: $(1, 1, 1) + (1, 1, 1) = (2, 2, 2)$
Not closed under scalar multiplication: $2(1, 1, 1) = (2, 2, 2)$
**43.** (a) True. See top of page 163.
(b) True. See Theorem 4.6, page 164.
(c) False. There may be elements of $W$ that are not elements of $U$, or vice-versa.
**45–59.** Proofs

## Section 4.4 *(page 178)*

**1.** (a) $\mathbf{z} = 2(2, -1, 3) - (5, 0, 4)$
(b) $\mathbf{v} = \frac{1}{4}(2, -1, 3) + \frac{3}{2}(5, 0, 4)$
(c) $\mathbf{w} = 8(2, -1, 3) - 3(5, 0, 4)$
(d) $\mathbf{u}$ cannot be written as a linear combination of the given vectors.
**3.** (a) $\mathbf{u} = -\frac{7}{4}(2, 0, 7) + \frac{5}{4}(2, 4, 5) + 0(2, -12, 13)$
(b) $\mathbf{v}$ cannot be written as a linear combination of the given vectors.
(c) $\mathbf{w} = -\frac{1}{6}(2, 0, 7) + \frac{1}{3}(2, 4, 5) + 0(2, -12, 13)$
(d) $\mathbf{z} = -4(2, 0, 7) + 5(2, 4, 5) + 0(2, -12, 13)$
**5.** $\begin{bmatrix} 6 & -19 \\ 10 & 7 \end{bmatrix} = 3A - 2B$
**7.** $\begin{bmatrix} -2 & 28 \\ 1 & -11 \end{bmatrix} = -A + 5B$
**9.** $S$ spans $R^2$. **11.** $S$ spans $R^2$.
**13.** $S$ does not span $R^2$. (It spans a line in $R^2$.)
**15.** $S$ does not span $R^2$. (It spans a line in $R^2$.)
**17.** $S$ does not span $R^2$. (It spans a line in $R^2$.)
**19.** $S$ spans $R^2$. **21.** $S$ spans $R^3$.
**23.** $S$ does not span $R^3$. (It spans a plane in $R^3$.)
**25.** $S$ does not span $R^3$. (It spans a plane in $R^3$.)
**27.** $S$ does not span $P_2$.
**29.** Linearly independent **31.** Linearly dependent
**33.** Linearly independent **35.** Linearly dependent
**37.** Linearly independent **39.** Linearly independent
**41.** Linearly dependent **43.** Linearly independent
**45.** Linearly dependent **47.** Linearly independent
**49.** $(3, 4) - 4(-1, 1) - \frac{7}{2}(2, 0) = (0, 0)$,
$(3, 4) = 4(-1, 1) + \frac{7}{2}(2, 0)$
(The answer is not unique.)
**51.** $(1, 1, 1) - (1, 1, 0) - (0, 0, 1) - 0(0, 1, 1) = (0, 0, 0)$
$(1, 1, 1) = (1, 1, 0) + (0, 0, 1) - 0(0, 1, 1)$
(The answer is not unique.)
**53.** (a) All $t \neq 1, -2$ (b) All $t \neq \frac{1}{2}$
**55.** Proof
**57.** Because the matrix $\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 2 & 5 & -1 \end{bmatrix}$ row reduces to $\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} -2 & -6 & 0 \\ 1 & 1 & -2 \end{bmatrix}$ row reduces to $\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 1 \end{bmatrix}$, $S_1$ and $S_2$ span the same subspace.
**59.** (a) False. See "Definition of Linear Dependence and Linear Independence," page 173.
(b) True. Any vector $\mathbf{u} = (u_1, u_2, u_3, u_4)$ in $R^4$ can be written as
$$\mathbf{u} = u_1(1, 0, 0, 0) - u_2(0, -1, 0, 0) + u_3(0, 0, 1, 0) + u_4(0, 0, 0, 1).$$
**61–73.** Proofs

## Section 4.5 *(page 187)*

**1.** $R^6$: $\{(1, 0, 0, 0, 0, 0), (0, 1, 0, 0, 0, 0), (0, 0, 1, 0, 0, 0), (0, 0, 0, 1, 0, 0), (0, 0, 0, 0, 1, 0), (0, 0, 0, 0, 0, 1)\}$
**3.** $M_{2,4}$: $\left\{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right\}$
**5.** $P_4$: $\{1, x, x^2, x^3, x^4\}$ **7.** $S$ is linearly dependent.
**9.** $S$ is linearly dependent and does not span $R^2$.
**11.** $S$ is linearly dependent and does not span $R^2$.
**13.** $S$ does not span $R^2$.
**15.** $S$ is linearly dependent and does not span $R^3$.
**17.** $S$ does not span $R^3$.
**19.** $S$ is linearly dependent and does not span $R^3$.
**21.** $S$ is linearly dependent.
**23.** $S$ is linearly dependent and does not span $P_2$.
**25.** $S$ does not span $M_{2,2}$.
**27.** $S$ is linearly dependent and does not span $M_{2,2}$.
**29.** The set is a basis for $R^2$. **31.** The set is not a basis for $R^2$.
**33.** $S$ is a basis for $R^2$. **35.** $S$ is a basis for $R^3$.
**37.** $S$ is not a basis for $R^3$. **39.** $S$ is a basis for $R^4$.
**41.** $S$ is a basis for $P_3$. **43.** $S$ is not a basis for $P_3$.
**45.** $S$ is a basis for $M_{2,2}$.
**47.** $S$ is a basis for $R^3$.
$(8, 3, 8) = 2(4, 3, 2) - (0, 3, 2) + 3(0, 0, 2)$
**49.** $S$ is not a basis for $R^3$. **51.** 6 **53.** 8 **55.** 6
**57.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
The dimension is 3.
**59.** $\{(1, 0), (0, 1)\}, \{(1, 0), (1, 1)\}, \{(0, 1), (1, 1)\}$
**61.** $\{(2, 2), (1, 0)\}$
**63.** (a) Line through the origin (b) $\{(2, 1)\}$ (c) 1
**65.** (a) Line through the origin (b) $\{(2, 1, -1)\}$ (c) 1
**67.** (a) $\{(2, 1, 0, 1), (-1, 0, 1, 0)\}$ (b) 2
**69.** (a) $\{(0, 6, 1, -1)\}$ (b) 1

**71.** (a) False. If the dimension of $V$ is $n$, then every spanning set of $V$ must have at least $n$ vectors.
(b) True. Find a set of $n$ basis vectors in $V$ that will span $V$ and add any other vector.

**73–77.** Proofs

## Section 4.6 *(page 199)*

**1.** (a) $(0, -2), (1, -3)$ (b) $\begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}-2\\-3\end{bmatrix}$

**3.** (a) $(4, 3, 1), (1, -4, 0)$ (b) $\begin{bmatrix}4\\1\end{bmatrix}, \begin{bmatrix}3\\-4\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}$

**5.** (a) $\{(1, 0), (0, 1)\}$ (b) 2

**7.** (a) $\left\{\left(1, 0, \frac{1}{2}\right), \left(0, 1, -\frac{1}{2}\right)\right\}$ (b) 2

**9.** (a) $\{(1, 2, -2, 0), (0, 0, 0, 1)\}$ (b) 2

**11.** $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ **13.** $\{(1, 1, 0), (0, 0, 1)\}$

**15.** $\{(1, 0, -1, 0), (0, 1, 0, 0), (0, 0, 0, 1)\}$

**17.** $\{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}$

**19.** (a) $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$ (b) 2 **21.** (a) $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$ (b) 2

**23.** (a) $\left\{\begin{bmatrix}1\\0\\\frac{5}{9}\\\frac{2}{9}\end{bmatrix}, \begin{bmatrix}0\\1\\-\frac{4}{9}\\\frac{2}{9}\end{bmatrix}\right\}$ (b) 2

**25.** $\{(1, 2)\}$ **27.** $\{(-2, 1, 0), (-3, 0, 1)\}$

**29.** $\{(-3, 0, 1)\}$ **31.** $\{(-1, 2, 1)\}$

**33.** $\{(2, -2, 0, 1), (-1, 1, 1, 0)\}$ **35.** $\{(0, 0, 0, 0)\}$

**37.** (a) $\{(4, 1)\}$ (b) 1 **39.** (a) $\{(-1, -3, 2)\}$ (b) 1

**41.** (a) $\{(-3, 0, 1), (2, 1, 0)\}$ (b) 2

**43.** (a) $\left\{(-4, -1, 1, 0), \left(-3, -\frac{2}{3}, 0, 1\right)\right\}$ (b) 2

**45.** (a) $\{(8, -9, -6, 6)\}$ (b) 1

**47.** (a) $\operatorname{rank}(A) = 3$
$\operatorname{nullity}(A) = 2$
(b) $\left\{\begin{bmatrix}-3\\1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}4\\-2\\0\\2\\1\end{bmatrix}\right\}$
(c) $\{(1, 0, 3, 0, -4), (0, 1, -1, 0, 2), (0, 0, 0, 1, -2)\}$
(d) $\{(1, 2, 3, 4), (2, 5, 7, 9), (0, 1, 2, -1)\}$
(e) Linearly dependent (f) (i) and (iii)

**49.** (a) Consistent (b) $\mathbf{x} = t(2, -4, 1) + (3, 5, 0)$

**51.** (a) Inconsistent (b) Not applicable

**53.** (a) Consistent
(b) $\mathbf{x} = t(5, 0, -6, -4, 1) + s(-2, 1, 0, 0, 0) + (1, 0, 2, -3, 0)$

**55.** $\begin{bmatrix}-1\\4\end{bmatrix} + 2\begin{bmatrix}2\\0\end{bmatrix} = \begin{bmatrix}3\\4\end{bmatrix}$

**57.** $\mathbf{b}$ is not in the column space of $A$. **59.** Proof

**61.** (a) $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$ (b) $\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}$
(c) $\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$

**63.** (a) $m$ (b) $r$ (c) $r$ (d) $R^n$ (e) $R^m$

**65.** Answers will vary.

**67.** (a) Proof (b) Proof (c) Proof

**69.** (a) False. The nullspace of $A$ is also called the solution space of the system $A\mathbf{x} = \mathbf{0}$.
(b) True. The nullspace of $A$ is the solution space of the homogeneous system $A\mathbf{x} = \mathbf{0}$.

**71.** (a) False. See "Remark," page 190.
(b) False. See Theorem 4.19, page 198.
(c) True. The columns of $A$ become the rows of $A^T$, so the columns of $A$ span the same space as the rows of $A^T$.

**73.** (a) $0, n$ (b) Proof **75.** Proof

## Section 4.7 *(page 210)*

**1.** $\begin{bmatrix}5\\-2\end{bmatrix}$ **3.** $\begin{bmatrix}7\\-4\\-1\\2\end{bmatrix}$ **5.** $\begin{bmatrix}8\\-3\end{bmatrix}$ **7.** $\begin{bmatrix}5\\4\\3\end{bmatrix}$ **9.** $\begin{bmatrix}-1\\2\\0\\1\end{bmatrix}$

**11.** $\begin{bmatrix}3\\2\end{bmatrix}$ **13.** $\begin{bmatrix}1\\-1\\2\end{bmatrix}$ **15.** $\begin{bmatrix}0\\-1\\2\end{bmatrix}$ **17.** $\begin{bmatrix}\frac{3}{2} & -\frac{1}{2}\\-2 & 1\end{bmatrix}$

**19.** $\begin{bmatrix}2 & -1\\4 & 3\end{bmatrix}$ **21.** $\begin{bmatrix}1 & 2 & -\frac{1}{2}\\0 & \frac{1}{2} & 0\\0 & -\frac{1}{3} & \frac{1}{12}\end{bmatrix}$ **23.** $\begin{bmatrix}3 & -2 & 1\\4 & -1 & 0\\0 & 1 & -3\end{bmatrix}$

**25.** $\begin{bmatrix}\frac{9}{5} & \frac{4}{5}\\\frac{8}{5} & \frac{3}{5}\end{bmatrix}$ **27.** $\begin{bmatrix}1 & 1 & -1\\-3 & 2 & -1\\3 & -3 & 2\end{bmatrix}$

**29.** $\begin{bmatrix}-7 & 3 & 10\\5 & -1 & -6\\11 & -3 & -10\end{bmatrix}$ **31.** $\begin{bmatrix}-24 & 7 & 1 & -2\\-10 & 3 & 0 & -1\\-29 & 7 & 3 & -2\\12 & -3 & -1 & 1\end{bmatrix}$

**33.** $\begin{bmatrix}1 & -\frac{3}{11} & \frac{5}{11} & 0 & -\frac{7}{11}\\0 & -\frac{2}{11} & \frac{3}{22} & 0 & -\frac{1}{11}\\-\frac{5}{4} & \frac{9}{22} & -\frac{19}{44} & -\frac{1}{4} & \frac{21}{22}\\-\frac{3}{4} & \frac{1}{2} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{2}\\0 & -\frac{1}{11} & -\frac{2}{11} & 0 & \frac{5}{11}\end{bmatrix}$

**35.** (a) $\begin{bmatrix}-\frac{1}{3} & \frac{1}{3}\\\frac{3}{4} & -\frac{1}{2}\end{bmatrix}$ (b) $\begin{bmatrix}6 & 4\\9 & 4\end{bmatrix}$ (c) Verify. (d) $\begin{bmatrix}6\\3\end{bmatrix}$

**37.** (a) $\begin{bmatrix}4 & 5 & 1\\-7 & -10 & -1\\-2 & -2 & 0\end{bmatrix}$ (b) $\begin{bmatrix}\frac{1}{2} & \frac{1}{2} & -\frac{5}{4}\\-\frac{1}{2} & -\frac{1}{2} & \frac{3}{4}\\\frac{3}{2} & \frac{1}{2} & \frac{5}{4}\end{bmatrix}$
(c) Verify. (d) $\begin{bmatrix}\frac{11}{4}\\-\frac{9}{4}\\\frac{5}{4}\end{bmatrix}$

**39.** (a) $\begin{bmatrix} -\frac{48}{5} & -24 & \frac{4}{5} \\ 4 & 10 & \frac{1}{2} \\ -\frac{6}{5} & -5 & -\frac{2}{5} \end{bmatrix}$ (b) $\begin{bmatrix} \frac{3}{32} & \frac{17}{20} & \frac{5}{4} \\ -\frac{1}{16} & -\frac{3}{10} & -\frac{1}{2} \\ \frac{1}{2} & \frac{6}{5} & 0 \end{bmatrix}$

(c) Verify. (d) $\begin{bmatrix} \frac{279}{160} \\ -\frac{61}{80} \\ -\frac{7}{10} \end{bmatrix}$

**41.** (a) $\begin{bmatrix} \frac{19}{39} & \frac{-9}{13} & \frac{44}{39} \\ -\frac{3}{13} & -\frac{6}{13} & -\frac{9}{13} \\ -\frac{23}{39} & \frac{2}{13} & -\frac{4}{39} \end{bmatrix}$ (b) $\begin{bmatrix} -\frac{2}{7} & -\frac{4}{21} & -\frac{13}{7} \\ -\frac{5}{7} & -\frac{8}{7} & -\frac{1}{7} \\ \frac{4}{7} & -\frac{13}{21} & \frac{5}{7} \end{bmatrix}$

(c) Verify. (d) $\begin{bmatrix} \frac{22}{7} \\ \frac{6}{7} \\ \frac{19}{7} \end{bmatrix}$

**43.** $\begin{bmatrix} 4 \\ 11 \\ 1 \\ 2 \end{bmatrix}$ **45.** $\begin{bmatrix} 1 \\ 5 \\ -2 \\ 1 \end{bmatrix}$ **47.** $\begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$ **49.** $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

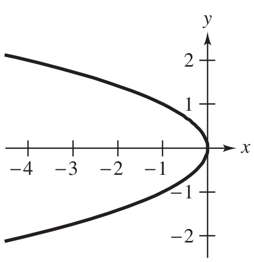
**51.** (a) False. See Theorem 4.20, page 204.
(b) True. See paragraph before Example 1, page 202.

**53.** $QP$

## Section 4.8 *(page 219)*

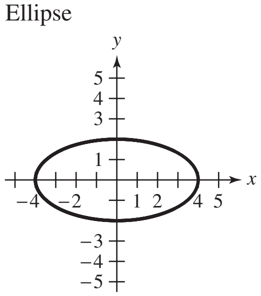
**1.** (b), (c), and (d) **3.** (c) **5.** (a), (b), and (d)
**7.** (b) **9.** (c) **11.** (b) **13.** $-(x \sin x + \cos x)$
**15.** $-2$ **17.** $-x$ **19.** 0 **21.** $2e^{3x}$ **23.** 12
**25.** $e^{-x}(\cos x - \sin x)$
**27.** (a) Verify. (b) Linearly independent
(c) $y = C_1 \sin 4x + C_2 \cos 4x$
**29.** (a) Verify. (b) Linearly dependent (c) Not applicable
**31.** (a) Verify. (b) Linearly independent
(c) $y = C_1 + C_2 \sin 2x + C_3 \cos 2x$
**33.** (a) Verify. (b) Linearly dependent
(c) Not applicable
**35.** (a) Verify.
(b) $\theta(t) = C_1 \sin \sqrt{\dfrac{g}{L}}t + C_2 \cos \sqrt{\dfrac{g}{L}}t$; proof
**37.** Proof **39.** Proof
**41.** No. For instance, consider $y'' = 1$. Two solutions are $y = \dfrac{x^2}{2}$ and $y = \dfrac{x^2}{2} + 1$. Their sum is not a solution.

**43.** Parabola



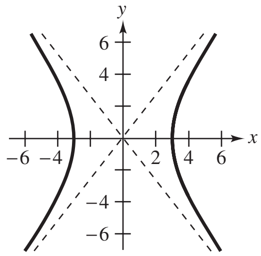
$y^2 + x = 0$

**45.** Ellipse



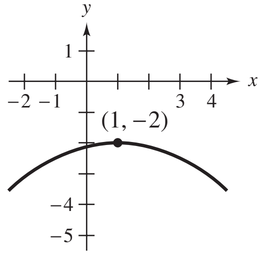
$x^2 + 4y^2 - 16 = 0$

**47.** Hyperbola



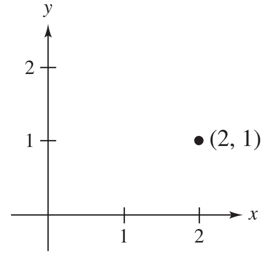
$\dfrac{x^2}{9} - \dfrac{y^2}{16} - 1 = 0$

**49.** Parabola



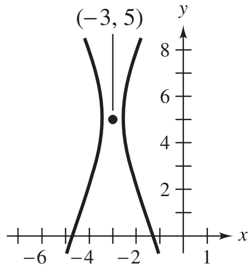

$x^2 - 2x + 8y + 17 = 0$

**51.** Point



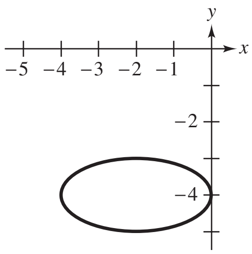

$9x^2 + 25y^2 - 36x - 50y + 61 = 0$

**53.** Hyperbola



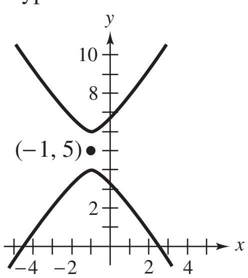

$9x^2 - y^2 + 54x + 10y + 55 = 0$

**55.** Ellipse



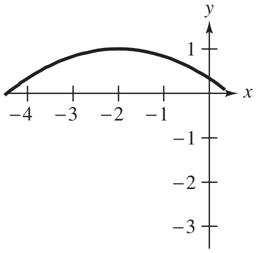
$x^2 + 4y^2 + 4x + 32y + 64 = 0$

**57.** Hyperbola




$2x^2 - y^2 + 4x + 10y - 22 = 0$

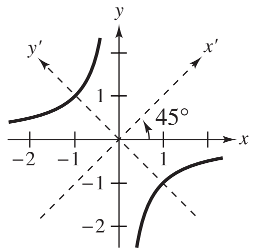
**59.** Parabola



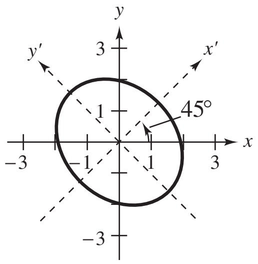

$x^2 + 4x + 6y - 2 = 0$

**61.** c **62.** b **63.** a **64.** d

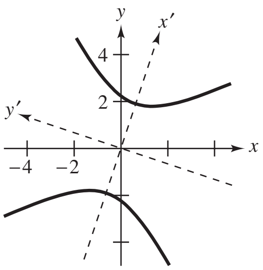
**65.** $\dfrac{(y')^2}{2} - \dfrac{(x')^2}{2} = 1$



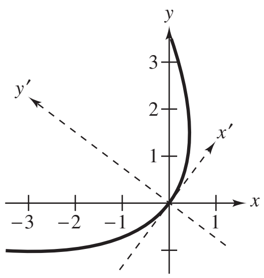

**67.** $\dfrac{(x')^2}{3} + \dfrac{(y')^2}{5} = 1$



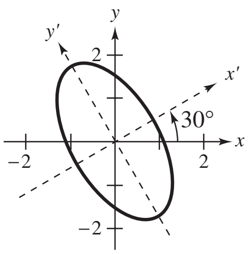

**69.** $\dfrac{(x')^2}{4} - \dfrac{(y')^2}{4} = 1$



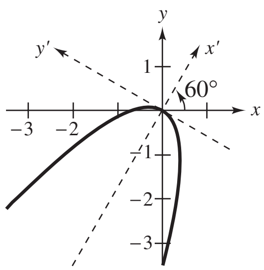

**71.** $y' = \dfrac{1}{4}(x')^2$



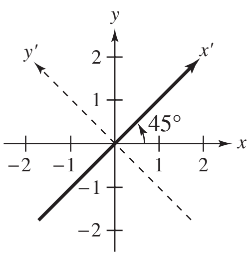

**73.** $(x')^2 + \dfrac{(y')^2}{4} = 1$



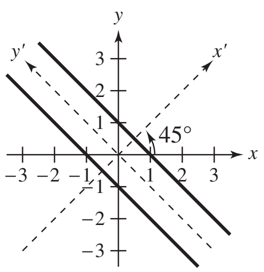

**75.** $x' = -(y')^2$




**77.** $y' = 0$




**79.** $x' = \pm\dfrac{\sqrt{2}}{2}$




**81.** Proof **83.** (a) Proof (b) Proof

## Review Exercises *(page 221)*

**1.** (a) $(0, 2, 5)$ (b) $(2, 0, 4)$ (c) $(-2, 2, 1)$ (d) $(-5, 6, 5)$

**3.** (a) $(3, 1, 4, 4)$ (b) $(0, 4, 4, 2)$ (c) $(3, -3, 0, 2)$ (d) $(9, -7, 2, 7)$

**5.** $\left(\frac{1}{2}, -4, -4\right)$ **7.** $\left(\frac{5}{2}, -6, 0\right)$

**9.** $\mathbf{v} = 2\mathbf{u}_1 - \mathbf{u}_2 + 3\mathbf{u}_3$ **11.** $\mathbf{v} = \frac{9}{8}\mathbf{u}_1 + \frac{1}{8}\mathbf{u}_2 + 0\mathbf{u}_3$

**13.** $O_{3,4} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$,

$-A = \begin{bmatrix} -a_{11} & -a_{12} & -a_{13} & -a_{14} \\ -a_{21} & -a_{22} & -a_{23} & -a_{24} \\ -a_{31} & -a_{32} & -a_{33} & -a_{34} \end{bmatrix}$

**15.** $O = (0, 0, 0)$
$-A = (-a_1, -a_2, -a_3)$

**17.** $W$ is a subspace of $R^2$. **19.** $W$ is not a subspace of $R^2$.

**21.** $W$ is a subspace of $R^3$.

**23.** $W$ is not a subspace of $C[-1, 1]$.

**25.** (a) $W$ is a subspace of $R^3$. (b) $W$ is not a subspace of $R^3$.

**27.** (a) Yes (b) Yes (c) Yes

**29.** (a) No (b) No (c) No

**31.** (a) Yes (b) No (c) No

**33.** $S$ is a basis for $P_3$. **35.** The set is not a basis for $M_{2,2}$.

**37.** (a) $\{(8, 5)\}$ (b) 1 (c) 1

**39.** (a) $\{(3, 0, 1, 0), (-1, -2, 0, 1)\}$ (b) 2 (c) 2

**41.** (a) $\{(4, -2, 1)\}$ (b) 1 (c) 2

**43.** (a) $\{(-3, 0, 4, 1), (-2, 1, 0, 0)\}$ (b) 2

**45.** (a) $\{(2, 3, 7, 0), (-1, 0, 0, 1)\}$ (b) 2

**47.** (a) $\{(1, 0), (0, 1)\}$ (b) 2

**49.** (a) $\{(1, -4, 0, 4)\}$ (b) 1

**51.** (a) $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ (b) 3

**53.** $\begin{bmatrix} -2 \\ 8 \end{bmatrix}$ **55.** $\begin{bmatrix} \frac{3}{4} \\ \frac{1}{4} \end{bmatrix}$ **57.** $\begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$ **59.** $\begin{bmatrix} \frac{2}{5} \\ -\frac{1}{4} \end{bmatrix}$

**61.** $\begin{bmatrix} -1 \\ 4 \\ \frac{3}{2} \end{bmatrix}$ **63.** $\begin{bmatrix} 3 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ **65.** $\begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix}$

**67.** $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

**69.** (a) $\begin{bmatrix} -1 & 3 \\ 1 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} \frac{1}{2} & \frac{3}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

(c) Verify. (d) $\begin{bmatrix} -12 \\ 6 \end{bmatrix}$

**71.** (a) $\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

(c) Verify. (d) $\begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix}$

**73.** Basis for $W$: $\{x, x^2, x^3\}$
Basis for $U$: $\{(x-1), x(x-1), x^2(x-1)\}$
Basis for $W \cap U$: $\{x(x-1), x^2(x-1)\}$

**75.** No. For example, the set $\{x^2 + x, x^2 - x, 1\}$ is a basis for $P_2$.

**77.** Yes, $W$ is a subspace of $V$. **79.** Proof

**81.** Answers will vary.

**83.** (a) True. See discussion above "Definitions of Vector Addition and Scalar Multiplication in $R^n$," page 149.
(b) False. See Theorem 4.3, part 2, page 151.
(c) True. See "Definition of a Vector Space" and the discussion following, page 155.

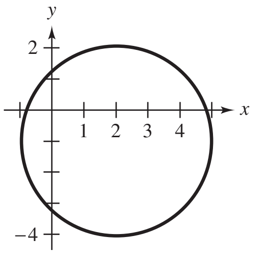
**85.** (a) True. See discussion under "Vectors in $R^n$," page 149.
(b) False. See "Definition of a Vector Space," part 4, page 155.
(c) True. See discussion following "Summary of Important Vector Spaces," page 157.

**87.** (a) and (d) **89.** (a) **91.** $e^x$ **93.** $-8$

**95.** (a) Verify. (b) Linearly independent
(c) $y(t) = C_1e^{-3x} + C_2xe^{-3x}$

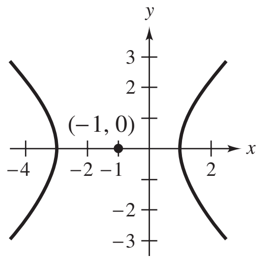
**97.** (a) Verify. (b) Linearly dependent (c) Not applicable

**99.** Circle



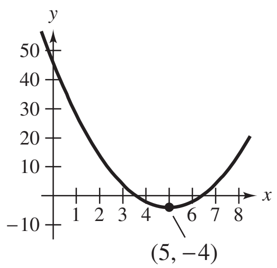

$x^2 + y^2 - 4x + 2y - 4 = 0$

**101.** Hyperbola



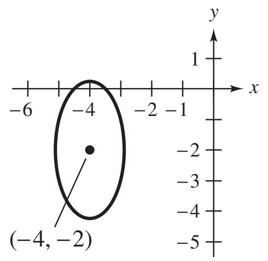

$x^2 - y^2 + 2x - 3 = 0$

**103.** Parabola



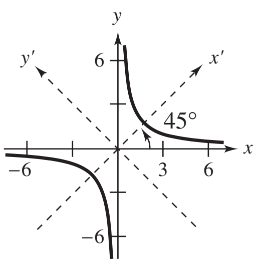

$2x^2 - 20x - y + 46 = 0$

**105.** Ellipse

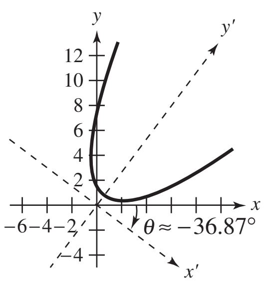



$4x^2 + y^2 + 32x + 4y + 63 = 0$

**107.** $\dfrac{(x')^2}{6} - \dfrac{(y')^2}{6} = 1$




**109.** $(x')^2 = 4(y' - 1)$




# Chapter 5

## Section 5.1 *(page 235)*

**1.** 5 **3.** 3 **5.** (a) $\dfrac{\sqrt{17}}{4}$ (b) $\dfrac{5\sqrt{41}}{8}$ (c) $\dfrac{\sqrt{577}}{8}$

**7.** (a) $\sqrt{6}$ (b) $2\sqrt{2}$ (c) $3\sqrt{2}$

**9.** (a) $\left(-\frac{5}{13}, \frac{12}{13}\right)$ (b) $\left(\frac{5}{13}, -\frac{12}{13}\right)$

**11.** (a) $\left(\dfrac{3}{\sqrt{38}}, \dfrac{2}{\sqrt{38}}, -\dfrac{5}{\sqrt{38}}\right)$ (b) $\left(-\dfrac{3}{\sqrt{38}}, -\dfrac{2}{\sqrt{38}}, \dfrac{5}{\sqrt{38}}\right)$

**13.** $\left(2\sqrt{2}, 2\sqrt{2}\right)$ **15.** $\left(1, \sqrt{3}, 0\right)$

**17.** (a) $\left(-\frac{1}{2}, \frac{3}{2}, 0, 2\right)$ (b) $(2, -6, 0, -8)$ **19.** $2\sqrt{2}$

**21.** 3

**23.** (a) $-6$ (b) 13 (c) 25 (d) $(-12, 18)$ (e) $-30$

**25.** (a) 0 (b) 14 (c) 6 (d) $\mathbf{0}$ (e) 0 **27.** $-7$

**29.** (a) $\|\mathbf{u}\| = 1.0843$, $\|\mathbf{v}\| = 0.3202$ (b) $(0, 0.7809, 0.6247)$
(c) $(-0.9223, -0.1153, -0.3689)$ (d) 0.1113
(e) 1.1756 (f) 0.1025

**31.** (a) $\|\mathbf{u}\| = 1.7321$, $\|\mathbf{v}\| = 2$ (b) $(-0.5, 0.7071, -0.5)$
(c) $(0, -0.5774, -0.8165)$ (d) 0 (e) 3 (f) 4

**33.** $|(3, 4) \cdot (2, -3)| \leq \|(3, 4)\| \, \|(2, -3)\|$
$6 \leq 5\sqrt{13}$

**35.** $|(1, 1, -2) \cdot (1, -3, -2)| \leq \|(1, 1, -2)\| \, \|(1, -3, -2)\|$
$2 \leq 2\sqrt{21}$

**37.** 1.713 radians (98.13°) **39.** $\dfrac{7\pi}{12}$ (105°)

**41.** 1.080 radians (61.87°) **43.** $\dfrac{\pi}{4}$ **45.** Orthogonal

**47.** Parallel **49.** Neither **51.** Neither **53.** $\mathbf{v} = (t, 0)$

**55.** $\mathbf{v} = (t, s, -2t + s)$

**57.** $\|(5, 1)\| \leq \|(4, 0)\| + \|(1, 1)\|$
$\sqrt{26} \leq 4 + \sqrt{2}$

**59.** $\|(1, 2, -1)\| \leq \|(1, 1, 1)\| + \|(0, 1, -2)\|$
$\sqrt{6} \leq \sqrt{3} + \sqrt{5}$

**61.** $\|(2, 0)\|^2 = \|(1, -1)\|^2 + \|(1, 1)\|^2$
$4 = \left(\sqrt{2}\right)^2 + \left(\sqrt{2}\right)^2$

**63.** $\|(7, 1, -2)\|^2 = \|(3, 4, -2)\|^2 + \|(4, -3, 0)\|^2$
$54 = \left(\sqrt{29}\right)^2 + 5^2$

**65.** (a) $-6$ (b) 13 (c) 25 (d) $\begin{bmatrix} -12 \\ 18 \end{bmatrix}$ (e) $-30$

**67.** (a) 0 (b) 14 (c) 6 (d) $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ (e) 0

**69.** Orthogonal; $\mathbf{u} \cdot \mathbf{v} = 0$

**71.** (a) False. See "Definition of Length of a Vector in $R^n$," page 226.
(b) False. See "Definition of Dot Product in $R^n$," page 229.

**73.** (a) $(\mathbf{u} \cdot \mathbf{v}) - \mathbf{v}$ is meaningless because $\mathbf{u} \cdot \mathbf{v}$ is a scalar and $\mathbf{v}$ is a vector.
(b) $\mathbf{u} + (\mathbf{u} \cdot \mathbf{v})$ is meaningless because $\mathbf{u}$ is a vector and $\mathbf{u} \cdot \mathbf{v}$ is a scalar.

**75.** $\left(-\frac{5}{13}, \frac{12}{13}\right), \left(\frac{5}{13}, -\frac{12}{13}\right)$

**77.** \$11,877.50
This value gives the total revenue earned from selling the hamburgers and hot dogs.

**79.** 54.7° **81–85.** Proofs

**87.** $A\mathbf{x} = \mathbf{0}$ means that the dot product of each row of $A$ with the column vector $\mathbf{x}$ is zero. So, $\mathbf{x}$ is orthogonal to the row vectors of $A$.

## Section 5.2 *(page 245)*

**1–7.** Proofs

**9.** Axiom 4 fails. $\langle(0, 1), (0, 1)\rangle = 0$, but $(0, 1) \neq \mathbf{0}$.

**11.** Axiom 4 fails. $\langle(1, 1), (1, 1)\rangle = 0$, but $(1, 1) \neq \mathbf{0}$.

**13.** Axiom 1 fails. If $\mathbf{u} = (1, 1, 1)$ and $\mathbf{v} = (1, 0, 0)$, $\langle\mathbf{u}, \mathbf{v}\rangle = 1$ and $\langle\mathbf{v}, \mathbf{u}\rangle = 0$.

**15.** Axiom 3 fails. If $\mathbf{u} = (1, 1, 1)$, $\mathbf{v} = (1, 0, 0)$, and $c = 2$, $c\langle\mathbf{u}, \mathbf{v}\rangle = 2$ and $\langle c\mathbf{u}, \mathbf{v}\rangle = 4$.

**17.** (a) $-33$ (b) 5 (c) 13 (d) $2\sqrt{65}$

**19.** (a) 15 (b) $\sqrt{57}$ (c) 5 (d) $2\sqrt{13}$

**21.** (a) $-34$ (b) $\sqrt{97}$ (c) $\sqrt{101}$ (d) $\sqrt{266}$

**23.** (a) 0 (b) $8\sqrt{3}$ (c) $\sqrt{411}$ (d) $3\sqrt{67}$

**25.** (a) 3 (b) $\sqrt{6}$ (c) 3 (d) 3 **27.** Proof

**29.** (a) $-6$ (b) $\sqrt{35}$ (c) $\sqrt{7}$ (d) $3\sqrt{6}$

**31.** (a) $-5$ (b) $\sqrt{39}$ (c) $\sqrt{5}$ (d) $3\sqrt{6}$ **33.** Proof

**35.** (a) $-4$ (b) $\sqrt{11}$ (c) $\sqrt{2}$ (d) $\sqrt{21}$

**37.** (a) 0 (b) $\sqrt{2}$ (c) $\sqrt{2}$ (d) 2

**39.** (a) 0 (b) $\sqrt{2}$ (c) $\dfrac{2\sqrt{2}}{\sqrt{5}}$ (d) $\dfrac{3\sqrt{2}}{\sqrt{5}}$

**41.** (a) $\dfrac{2}{e} \approx 0.736$ (b) $\dfrac{\sqrt{6}}{3} \approx 0.816$

(c) $\sqrt{\dfrac{e^2}{2} - \dfrac{1}{2e^2}} \approx 1.904$

(d) $\sqrt{\dfrac{e^2}{2} + \dfrac{2}{3} - \dfrac{1}{2e^2} - \dfrac{4}{e}} \approx 1.680$

**43.** 2.103 radians (120.5°) **45.** 1.16 radians (66.59°)

**47.** $\dfrac{\pi}{2}$ **49.** 1.23 radians (70.53°) **51.** $\dfrac{\pi}{2}$

**53.** (a) $|\langle(5, 12), (3, 4)\rangle| \leq \|(5, 12)\|\,\|(3, 4)\|$
$$63 \leq (13)(5)$$
(b) $\|(5, 12) + (3, 4)\| \leq \|(5, 12)\| + \|(3, 4)\|$
$$8\sqrt{5} \leq 13 + 5$$

**55.** (a) $|(1, 0, 4) \cdot (-5, 4, 1)| \leq \sqrt{17}\sqrt{42}$
$$1 \leq \sqrt{714}$$
(b) $\|(-4, 4, 5)\| \leq \sqrt{17} + \sqrt{42}$
$$\sqrt{57} \leq \sqrt{17} + \sqrt{42}$$

**57.** (a) $|\langle 2x, 3x^2 + 1\rangle| \leq \|2x\|\,\|3x^2 + 1\|$
$$0 \leq (2)\left(\sqrt{10}\right)$$
(b) $\|2x + 3x^2 + 1\| \leq \|2x\| + \|3x^2 + 1\|$
$$\sqrt{14} \leq 2 + \sqrt{10}$$

**59.** (a) $|0(-3) + 3(1) + 2(4) + 1(3)| \leq \sqrt{14}\sqrt{35}$
$$14 \leq \sqrt{14}\sqrt{35}$$
(b) $\left\|\begin{bmatrix} -3 & 4 \\ 6 & 4 \end{bmatrix}\right\| \leq \sqrt{14} + \sqrt{35}$
$$\sqrt{77} \leq \sqrt{14} + \sqrt{35}$$

**61.** (a) $|\langle \sin x, \cos x\rangle| \leq \|\sin x\|\|\cos x\|$
$$\frac{1}{4} \leq \left(\sqrt{\frac{\pi}{8} - \frac{1}{4}}\right)\left(\sqrt{\frac{\pi}{8} + \frac{1}{4}}\right)$$
(b) $\|\sin x + \cos x\| \leq \|\sin x\| + \|\cos x\|$
$$\sqrt{\frac{\pi}{4} + \frac{1}{2}} \leq \sqrt{\frac{\pi}{8} - \frac{1}{4}} + \sqrt{\frac{\pi}{8} + \frac{1}{4}}$$

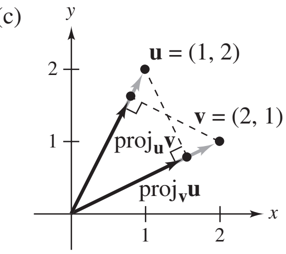
**63.** (a) $|\langle x, e^x\rangle| \leq \|x\|\,\|e^x\|$
$$1 \leq \sqrt{\tfrac{1}{3}} \cdot \sqrt{\tfrac{1}{2}e^2 - \tfrac{1}{2}}$$
(b) $\|x + e^x\| \leq \|x\| + \|e^x\|$
$$\sqrt{\tfrac{11}{6} + \tfrac{1}{2}e^2} \leq \sqrt{\tfrac{1}{3}} + \sqrt{\tfrac{1}{2}e^2 - \tfrac{1}{2}}$$

**65.** Because
$$\langle f, g\rangle = \int_{-\pi/2}^{\pi/2} \cos x \sin x \, dx = \frac{1}{2}\sin^2 x\Big]_{-\pi/2}^{\pi/2} = 0$$
$f$ and $g$ are orthogonal.

**67.** The functions $f(x) = x$ and $g(x) = \frac{1}{2}(5x^3 - 3x)$ are orthogonal because
$$\langle f, g\rangle = \int_{-1}^{1} x\frac{1}{2}(5x^3 - 3x)\, dx = \frac{1}{2}\int_{-1}^{1}(5x^4 - 3x^2)\, dx = \frac{1}{2}(x^5 - x^3)\Big]_{-1}^{1} = 0.$$

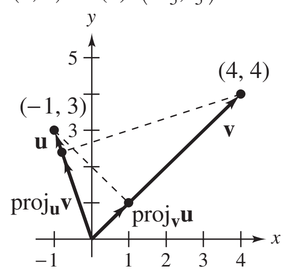
**69.** (a) $\left(\frac{8}{5}, \frac{4}{5}\right)$ (b) $\left(\frac{4}{5}, \frac{8}{5}\right)$

(c)




**71.** (a) $(1, 1)$ (b) $\left(-\frac{4}{5}, \frac{12}{5}\right)$

(c)




**73.** (a) $\left(0, \frac{5}{2}, -\frac{5}{2}\right)$ (b) $\left(-\frac{5}{14}, -\frac{15}{14}, \frac{5}{7}\right)$

**75.** (a) $\left(\frac{1}{2}, -\frac{1}{2}, -1, -1\right)$ (b) $\left(0, -\frac{5}{46}, -\frac{15}{46}, \frac{15}{23}\right)$

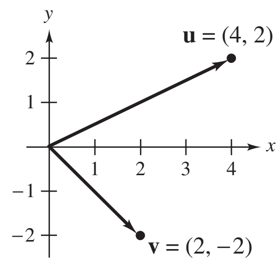
**77.** $\text{proj}_g f = \mathbf{0}$ **79.** $\text{proj}_g f = \dfrac{2e^x}{e^2 - 1}$

**81.** $\text{proj}_g f = \mathbf{0}$ **83.** $\text{proj}_g f = -\sin 2x$

**85.** (a) False. See the introduction to this section, page 237.
(b) False. $\|\mathbf{v}\| = \mathbf{0}$ if and only if $\mathbf{v} = \mathbf{0}$.

**87.** (a) $\langle\mathbf{u}, \mathbf{v}\rangle = 4(2) + 2(2)(-2) = 0 \implies$ **u** and **v** are orthogonal.

(b)




Not orthogonal in the Euclidean sense

**89–95.** Proofs **97.** $c_1 = \frac{1}{4}, c_2 = 1$

**99.** $c_1 = \frac{1}{4}, c_2 = \frac{1}{16}$ **101.** Proof

## Section 5.3 *(page 257)*

**1.** (a) Yes (b) No (c) Yes
**3.** (a) No (b) No (c) Yes
**5.** (a) Yes (b) Yes (c) Yes
**7.** (a) Yes (b) No (c) Yes
**9.** (a) Yes (b) Yes (c) Yes
**11.** (a) Yes (b) No (c) No
**13.** (a) Yes (b) Yes (c) No

**15.** (a) Proof (b) $\left(-\frac{1}{\sqrt{17}}, \frac{4}{\sqrt{17}}\right), \left(\frac{4}{\sqrt{17}}, \frac{1}{\sqrt{17}}\right)$

**17.** (a) Proof (b) $\left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right), \left(-\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right)$

**19.** The set $\{1, x, x^2, x^3\}$ is orthogonal because
$\langle 1, x\rangle = 0, \langle 1, x^2\rangle = 0, \langle 1, x^3\rangle = 0, \langle x, x^2\rangle = 0,$
$\langle x, x^3\rangle = 0, \langle x^2, x^3\rangle = 0.$
Furthermore, the set is orthonormal because
$\|1\| = 1, \|x\| = 1, \|x^2\| = 1$, and $\|x^3\| = 1$.
So, $\{1, x, x^2, x^3\}$ is an orthonormal basis for $P_3$.

**21.** $\begin{bmatrix} \frac{4\sqrt{13}}{13} \\ \frac{7\sqrt{13}}{13} \end{bmatrix}$ **23.** $\begin{bmatrix} \frac{\sqrt{10}}{2} \\ -2 \\ -\frac{\sqrt{10}}{2} \end{bmatrix}$ **25.** $\begin{bmatrix} 11 \\ 2 \\ 15 \end{bmatrix}$

**27.** $\left\{\left(\frac{3}{5}, \frac{4}{5}\right), \left(\frac{4}{5}, -\frac{3}{5}\right)\right\}$ **29.** $\{(0, 1), (1, 0)\}$

**31.** $\left\{\left(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right), \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right), \left(\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}\right)\right\}$

**33.** $\left\{\left(\frac{4}{5}, -\frac{3}{5}, 0\right), \left(\frac{3}{5}, \frac{4}{5}, 0\right), (0, 0, 1)\right\}$

**35.** $\left\{\left(0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right), \left(\frac{\sqrt{6}}{3}, \frac{\sqrt{6}}{6}, -\frac{\sqrt{6}}{6}\right), \left(\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)\right\}$

**37.** $\left\{\left(-\frac{4\sqrt{2}}{7}, \frac{3\sqrt{2}}{14}, \frac{5\sqrt{2}}{14}\right)\right\}$

**39.** $\left\{\left(\frac{3}{5}, \frac{4}{5}, 0\right), \left(\frac{4}{5}, -\frac{3}{5}, 0\right)\right\}$

**41.** $\left\{\left(\frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{3}, -\frac{\sqrt{6}}{6}, 0\right), \left(\frac{\sqrt{3}}{3}, 0, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right), \left(\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}, 0\right)\right\}$

**43.** $\left\{\left(\frac{2}{3}, -\frac{1}{3}\right), \left(\frac{\sqrt{2}}{6}, \frac{2\sqrt{2}}{3}\right)\right\}$

**45.** $\langle x, 1\rangle = \int_{-1}^{1} x\,dx = \frac{x^2}{2}\Big]_{-1}^{1} = 0$

**47.** $\langle x^2, 1\rangle = \int_{-1}^{1} x^2\,dx = \frac{x^3}{3}\Big]_{-1}^{1} = \frac{2}{3}$

**49.** $\langle x, x\rangle = \int_{-1}^{1} x^2\,dx = \frac{x^3}{3}\Big]_{-1}^{1} = \frac{2}{3}$

**51.** $\left\{\left(\frac{3\sqrt{10}}{10}, 0, \frac{\sqrt{10}}{10}, 0\right), \left(0, -\frac{2\sqrt{5}}{5}, 0, \frac{\sqrt{5}}{5}\right)\right\}$

**53.** $\left\{\left(-\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}, 0\right), \left(-\frac{\sqrt{6}}{6}, 0, \frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{3}\right)\right\}$

**55.** $\left\{\left(\frac{2\sqrt{5}}{5}, \frac{\sqrt{5}}{5}, 0\right), \left(-\frac{\sqrt{30}}{30}, \frac{\sqrt{30}}{15}, \frac{\sqrt{30}}{6}\right)\right\}$

**57.** (a) True. See "Definitions of Orthogonal and Orthonormal Sets," page 248.
(b) False. See "Remark," page 254.

**59.** Orthonormal **61.** $\{x^2, x, 1\}$ **63.** Orthonormal
**65.** Proof **67.** Proof

**69.** $N(A)$ basis: $\{(3, -1, 2)\}$ **71.** Proof
$N(A^T)$ basis: $\{(-1, -1, 1)\}$
$R(A)$ basis: $\{(1, 0, 1), (1, 2, 3)\}$
$R(A^T)$ basis: $\{(1, 1, -1), (0, 2, 1)\}$

## Section 5.4 *(page 269)*

**1.** $y = 1 + 2x$ **3.** Not collinear **5.** Not orthogonal

**7.** Orthogonal **9.** (a) span $\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$ (b) $R^3$

**11.** (a) span $\left\{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}\right\}$ (b) $R^4$ **13.** span $\left\{\begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}\right\}$

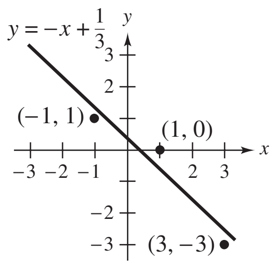
**15.** $\begin{bmatrix} 0 \\ \frac{2}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}$ **17.** $\begin{bmatrix} \frac{5}{3} \\ \frac{8}{3} \\ \frac{13}{3} \end{bmatrix}$

**19.** $N(A)$ basis: $\{(-3, 0, 1)\}$
$N(A^T) = \{(0, 0)\}$
$R(A)$ basis: $\{(1, 0), (2, 1)\} = R^2$
$R(A^T)$ basis: $\{(1, 2, 3), (0, 1, 0)\}$

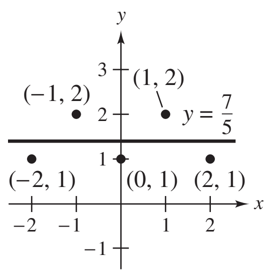
**21.** $N(A)$ basis: $\{(-1, -1, 0, 1), (0, -1, 1, 0)\}$
$N(A^T)$ basis: $\{(-1, -1, 1, 0), (-1, -2, 0, 1)\}$
$R(A)$ basis: $\{(1, 0, 1, 1), (0, 1, 1, 2)\}$
$R(A^T)$ basis: $\{(1, 0, 0, 1), (0, 1, 1, 1)\}$

**23.** $\mathbf{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ **25.** $\mathbf{x} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ **27.** $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

**29.**



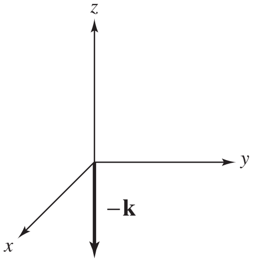

**31.**



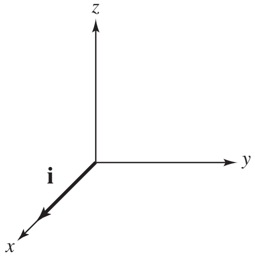

**33.** $y = x^2 - x$ **35.** $y = \frac{3}{7}x^2 + \frac{6}{5}x + \frac{26}{35}$
**37.** $y = 33.68 + 3.78x$; 90,380
**39.** $\ln y = -0.14 \ln x + 5.7$ or $y = 298.9x^{-0.14}$
**41.** (a) False. The orthogonal complement of $R^n$ is $\{\mathbf{0}\}$.
(b) True. See "Definition of Direct Sum," page 261.
**43.** Proof **45.** Proof

## Section 5.5 *(page 282)*

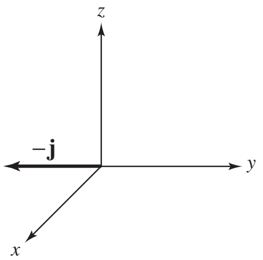
**1.** $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$




**3.** $\mathbf{j} \times \mathbf{k} = \mathbf{i}$




**5.** $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$



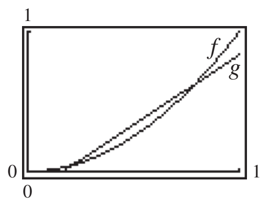

**7.** (a) $-\mathbf{i} - \mathbf{j} + \mathbf{k}$
(b) $\mathbf{i} + \mathbf{j} - \mathbf{k}$
(c) $\mathbf{0}$

**9.** (a) $5\mathbf{i} - 3\mathbf{j} - \mathbf{k}$
(b) $-5\mathbf{i} + 3\mathbf{j} + \mathbf{k}$
(c) $\mathbf{0}$

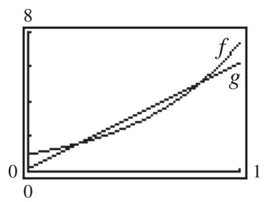
**11.** (a) $-14\mathbf{i} + 13\mathbf{j} + 17\mathbf{k}$
(b) $14\mathbf{i} - 13\mathbf{j} - 17\mathbf{k}$
(c) $\mathbf{0}$

**13.** $(-2, -2, -1)$ **15.** $(-8, -14, 54)$ **17.** $(-1, -1, -1)$
**19.** $(-1, 12, -2)$ **21.** $(-2, 3, -1)$ **23.** $(5, -4, -3)$
**25.** $(2, -1, -1)$ **27.** $(1, -1, -3)$ **29.** $(1, -5, -3)$
**31.** $\left(\frac{1}{3}, -\frac{2}{3}, -\frac{2}{3}\right)$ **33.** $\frac{1}{\sqrt{19}}\mathbf{i} - \frac{3}{\sqrt{19}}\mathbf{j} + \frac{3}{\sqrt{19}}\mathbf{k}$
**35.** $-\frac{71}{\sqrt{7602}}\mathbf{i} - \frac{44}{\sqrt{7602}}\mathbf{j} + \frac{25}{\sqrt{7602}}\mathbf{k}$ **37.** $\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}$
**39.** 1 **41.** $6\sqrt{5}$ **43.** $2\sqrt{83}$ **45.** 1 **47.** $-1$
**49.** Proof **51.** $\frac{9\sqrt{6}}{2}$ **53–61.** Proofs

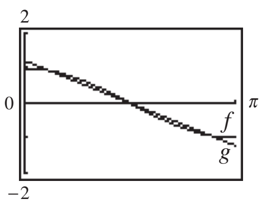
**63.** (a) $g(x) = x - \frac{1}{6}$
(b)



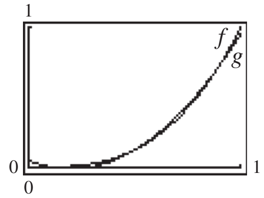

**65.** (a) $g(x) = 6x + \frac{1}{2}(e^2 - 7)$
(b)



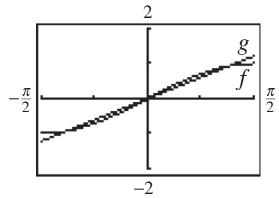

**67.** (a) $g(x) = \frac{12}{\pi^3}(\pi - 2x)$
(b)




**69.** (a) $g(x) = 1.5x^2 - 0.6x + 0.05$
(b)




**71.** (a) $g(x) = \frac{24}{\pi^3}x$
(b)




**73.** $g(x) = 2 \sin x + \sin 2x + \frac{2}{3} \sin 3x$

**75.** $g(x) = \frac{\pi^2}{3} + 4 \cos x + \cos 2x + \frac{4}{9} \cos 3x$

**77.** $g(x) = \frac{1}{2\pi}(1 - e^{-2\pi})(1 + \cos x + \sin x)$

**79.** $g(x) = \frac{1 - e^{-4\pi}}{20\pi}(5 + 8 \cos x + 4 \sin x)$

**81.** $g(x) = (1 + \pi) - 2 \sin x - \sin 2x - \frac{2}{3} \sin 3x$

**83.** $g(x) = \sin 2x$

**85.** $g(x) = 2\left(\sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots + \frac{\sin nx}{n}\right)$

**87.** $\frac{1 - e^{-2\pi}}{2\pi} + \frac{1 - e^{-2\pi}}{\pi}\sum_{j=1}^{n}\left(\frac{1}{j^2 + 1} \cos jx + \frac{j}{j^2 + 1} \sin jx\right)$

## Review Exercises *(page 284)*

**1.** (a) $\sqrt{5}$ (b) $\sqrt{17}$ (c) 6 (d) $\sqrt{10}$
**3.** (a) $\sqrt{6}$ (b) $\sqrt{14}$ (c) 7 (d) $\sqrt{6}$
**5.** (a) $\sqrt{6}$ (b) $\sqrt{3}$ (c) $-1$ (d) $\sqrt{11}$
**7.** (a) $\sqrt{7}$ (b) $\sqrt{7}$ (c) 6 (d) $\sqrt{2}$

**9.** $\|\mathbf{v}\| = \sqrt{38}$; $\mathbf{u} = \left(\frac{5}{\sqrt{38}}, \frac{3}{\sqrt{38}}, -\frac{2}{\sqrt{38}}\right)$

**11.** $\|\mathbf{v}\| = \sqrt{6}$; $\mathbf{u} = \left(-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)$

**13.** (a) $(4, 4, 3)$ (b) $\left(-2, -2, -\frac{3}{2}\right)$ (c) $(-16, -16, -12)$

**15.** $\frac{\pi}{2}$ **17.** $\frac{\pi}{12}$ **19.** $\pi$ **21.** $(s, 3t, 4t)$

**23.** $(2r - 2s - t, r, s, t)$ **25.** (a) $-2$ (b) $\frac{3\sqrt{11}}{2}$

**27.** Triangle Inequality:
$$\left\|\left(2, -\frac{1}{2}, 1\right) + \left(\frac{3}{2}, 2, -1\right)\right\| \le \left\|\left(2, -\frac{1}{2}, 1\right)\right\| + \left\|\left(\frac{3}{2}, 2, -1\right)\right\|$$
$$\frac{\sqrt{67}}{2} \le \sqrt{\frac{15}{2}} + \sqrt{\frac{53}{4}}$$
Cauchy-Schwarz Inequality:
$$\left|\left\langle\left(2, -\frac{1}{2}, 1\right), \left(\frac{3}{2}, 2, -1\right)\right\rangle\right| \le \left\|\left(2, -\frac{1}{2}, 1\right)\right\| \left\|\left(\frac{3}{2}, 2, -1\right)\right\|$$
$$2 \le \sqrt{\frac{15}{2}}\sqrt{\frac{53}{4}} \approx 9.969$$

**29.** (a) 0 (b) Orthogonal
(c) Because $\langle f, g\rangle = 0$, it follows that $|\langle f, g\rangle| \le \|f\|\|g\|$.

**31.** $\left(-\frac{9}{13}, \frac{45}{13}\right)$ **33.** $\left(\frac{24}{29}, \frac{60}{29}\right)$ **35.** $\left(\frac{18}{29}, \frac{12}{29}, \frac{24}{29}\right)$

**37.** $\left\{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right\}$

**39.** $\left\{\left(0, \frac{3}{5}, \frac{4}{5}\right), (1, 0, 0), \left(0, \frac{4}{5}, -\frac{3}{5}\right)\right\}$

**41.** (a) $(-1, 4, -2) = 2(0, 2, -2) - (1, 0, -2)$

(b) $\left\{\left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)\right\}$

(c) $(-1, 4, -2) = 3\sqrt{2}\left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) - \sqrt{3}\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$

**43.** $\langle f, g\rangle = \int_0^{\pi} \sin x \cos x\, dx = \frac{1}{2}\sin^2 x\Big]_0^{\pi} = 0$

**45.** (a) $\frac{1}{5}$ (b) $\frac{1}{\sqrt{7}}$ (c) $\frac{2\sqrt{2}}{\sqrt{105}}$ (d) $\left\{\sqrt{3}x, \frac{\sqrt{7}}{2}(5x^3 - 3x)\right\}$

**47.** $\left\{\left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right)\right\}$

(The answer is not unique.)

**49–57.** Proofs **59.** span $\left\{\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}\right\}$

**61.** $N(A)$ basis: $\{(1, 0, -1)\}$

$N(A^T) = \{(3, 1, 0)\}$

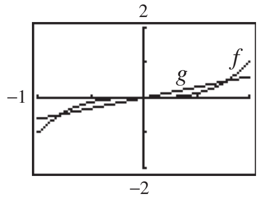
$R(A)$ basis: $\{(0, 0, 1), (1, -3, 0)\}$

$R(A^T)$ basis: $\{(0, 1, 0), (1, 0, 1)\}$

**63.** $y = 13.74x + 387.4$; 593.5 quadrillion Btu

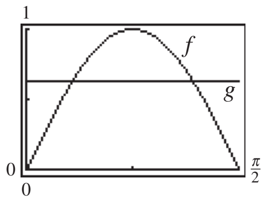
**65.** $(0, 1, -1)$ **67.** $13\mathbf{i} + 6\mathbf{j} - \mathbf{k}$ **69.** 1 **71.** 6 **73.** 7

**75.** (a) $g(x) = \frac{3}{5}x$ (b)



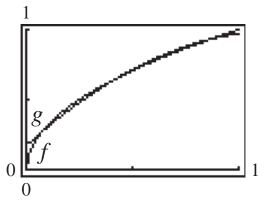

**77.** (a) $g(x) = \frac{2}{\pi}$

(b)




**79.** (a) $g(x) = \frac{2}{35}(-10x^2 + 24x + 3)$

(b)




**81.** $g(x) = \frac{\pi^2}{3} - 4\cos x$

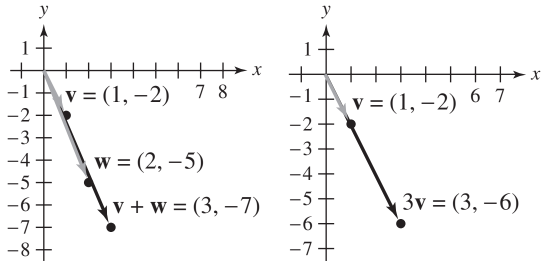
**83.** (a) True. See Theorem 5.18, page 273.

(b) False. See Theorem 5.17, page 272.

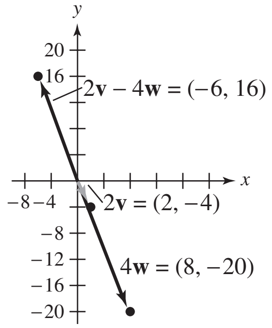
(c) True. See discussion before Theorem 5.19, page 277.

## Cumulative Test for Chapters 4 and 5 *(page 289)*

**1.** (a) $(3, -7)$ (b) $(3, -6)$




(c) $(-6, 16)$




**2.** $\mathbf{w} = 3\mathbf{v}_1 + \mathbf{v}_2 - \frac{2}{3}\mathbf{v}_3$ **3.** Not possible

**4.** $\mathbf{v} = 5\mathbf{u}_1 - \mathbf{u}_2 + \mathbf{u}_3 + 2\mathbf{u}_4 - 5\mathbf{u}_5 + 3\mathbf{u}_6$ **5.** Proof

**6.** Yes **7.** No **8.** Yes

**9.** (a) A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is linearly independent if the vector equation $c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$ has only the trivial solution.

(b) Linearly dependent

**10.** (a) A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ in a vector space $V$ is a basis for $V$ if the set is linearly independent and spans $V$.

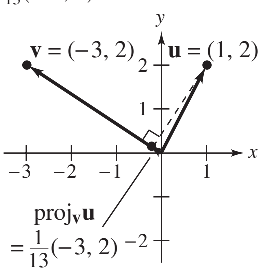
(b) Yes (c) Yes

**11.** $\left\{\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}\right\}$ **12.** $\begin{bmatrix} -4 \\ 6 \\ -5 \end{bmatrix}$ **13.** $\begin{bmatrix} 0 & 1 & -1 \\ 2 & 0 & 1 \\ -1 & -1 & 1 \end{bmatrix}$

**14.** (a) $\sqrt{5}$ (b) $\sqrt{11}$ (c) 4 (d) 1.0723 radians (61.44°)

**15.** $\frac{11}{12}$ **16.** $\left\{(1, 0, 0), \left(0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right), \left(0, -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)\right\}$

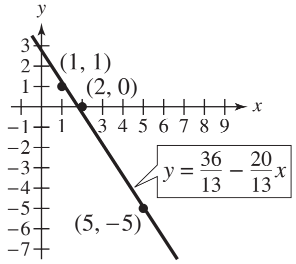
**17.** $\frac{1}{13}(-3, 2)$



**18.** $N(A)$ basis: $\{(0, 1, -1, 0)\}$
$N(A^T) = \{(0, 0, 0)\}$
$R(A) = R^3$
$R(A^T)$ basis: $\{(0, 1, 1, 0), (-1, 0, 0, 1), (1, 1, 1, 1)\}$

**19.** $\operatorname{span}\left\{\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}\right\}$ **20.** Proof

**21.** $y = \frac{36}{13} - \frac{20}{13}x$




**22.** (a) 3 (b) One basis consists of the first three rows of $A$.
(c) One basis consists of columns 1, 3, and 4 of $A$.
(d) $\left\{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 5 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 3 \\ -7 \\ 0 \\ 1 \end{bmatrix}\right\}$
(e) No (f) No (g) Yes (h) No

**23.** No. Two planes can intersect in a line, but not in a single point.

**24.** Proof

# Chapter 6

## Section 6.1 *(page 300)*

**1.** (a) $(-1, 7)$ (b) $(11, -8)$
**3.** (a) $(1, 5, 4)$ (b) $(5, -6, t)$
**5.** (a) $(-14, -7)$ (b) $(1, 1, t)$
**7.** (a) $(0, 2, 1)$ (b) $(-6, 4)$
**9.** Not linear **11.** Linear **13.** Not linear
**15.** Not linear **17.** Linear **19.** Linear **21.** Linear
**23.** $T(1, 4) = (-3, 5)$ **25.** $(3, 11, -8)$
$T(-2, 1) = (-3, -1)$
**27.** $(0, -6, 8)$ **29.** $(5, 0, 1)$ **31.** $\left(2, \frac{5}{2}, 2\right)$
**33.** $T: R^2 \to R^2$ **35.** $T: R^4 \to R^4$ **37.** $T: R^4 \to R^3$
**39.** (a) $(-1, -1)$ (b) $(-1, -1)$ (c) $(0, 0)$
**41.** (a) $(-1, 1, 2, 1)$ (b) $\left(-1, 1, \frac{1}{2}, 1\right)$
**43.** (a) $(-1, 9, 9)$ (b) $(-4t, -t, 0, t)$
**45.** (a) $\left(0, 4\sqrt{2}\right)$ (b) $\left(2\sqrt{3} - 2, 2\sqrt{3} + 2\right)$
(c) $\left(-\frac{5}{2}, \frac{5\sqrt{3}}{2}\right)$
**47.** $A^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$; clockwise rotation through $\theta$
**49.** Projection onto the $xz$-plane
**51.** Not a linear transformation **53.** Linear transformation
**55.** $x^2 - 3x - 5$
**57.** True. $D_x$ is a linear transformation and preserves addition and scalar multiplication.
**59.** False, because $\sin 2x \neq 2 \sin x$.
**61.** $g(x) = x^2 + x + C$ **63.** $g(x) = -\cos x + C$
**65.** (a) $-1$ (b) $\frac{1}{12}$ (c) $-4$
**67.** (a) False, because $\cos(x_1 + x_2) \neq \cos x_1 + \cos x_2$.
(b) True. See discussion following Example 10, page 299.
**69.** (a) $(x, 0)$ (b) Projection onto the $x$-axis
**71.** (a) $\left(\frac{1}{2}(x + y), \frac{1}{2}(x + y)\right)$ (b) $\left(\frac{5}{2}, \frac{5}{2}\right)$ (c) Proof
**73.** $A\mathbf{u} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x + \frac{1}{2}y \\ \frac{1}{2}x + \frac{1}{2}y \end{bmatrix} = T(\mathbf{u})$
**75.** (a) Proof (b) Proof (c) $(t, 0)$ (d) $(t, t)$
**77–83.** Proofs

## Section 6.2 *(page 312)*

**1.** $R^3$ **3.** $\{(0, 0, 0, 0)\}$
**5.** $\{a_1x + a_2x^2 + a_3x^3: a_1, a_2, a_3 \text{ are real}\}$
**7.** $\{a_0: a_0 \text{ is real}\}$ **9.** $\{(0, 0)\}$
**11.** (a) $\{(0, 0)\}$ (b) $\{(1, 0), (0, 1)\}$
**13.** (a) $\{(-4, -2, 1)\}$ (b) $\{(1, 0), (0, 1)\}$
**15.** (a) $\{(0, 0)\}$ (b) $\{(1, -1, 0), (0, 0, 1)\}$
**17.** (a) $\{(-1, 1, 1, 0)\}$
(b) $\{(1, 0, -1, 0), (0, 1, -1, 0), (0, 0, 0, 1)\}$
**19.** (a) $\{(0, 0)\}$ (b) 0 (c) $R^2$ (d) 2
**21.** (a) $\{(0, 0)\}$ (b) 0
(c) $\{(4s, 4t, s - t): s \text{ and } t \text{ are real}\}$ (d) 2
**23.** (a) $\{(t, -3t): t \text{ is real}\}$ (b) 1
(c) $\{(3t, t): t \text{ is real}\}$ (d) 1
**25.** (a) $\{(s + t, s, -2t): s \text{ and } t \text{ are real}\}$ (b) 2
(c) $\{(2t, -2t, t): t \text{ is real}\}$ (d) 1
**27.** (a) $\{(-11t, 6t, 4t): t \text{ is real}\}$ (b) 1 (c) $R^2$ (d) 2
**29.** (a) $\{(2s - t, t, 4s, -5s, s): s \text{ and } t \text{ are real}\}$ (b) 2
(c) $\{(7r, 7s, 7t, 8r + 20s + 2t): r, s, \text{ and } t \text{ are real}\}$ (d) 3
**31.** Nullity $= 1$
Kernel: a line
Range: a plane
**33.** Nullity $= 3$
Kernel: $R^3$
Range: $\{(0, 0, 0)\}$
**35.** Nullity $= 0$
Kernel: $\{(0, 0, 0)\}$
Range: $R^3$
**37.** Nullity $= 2$
Kernel: $\{(x, y, z): x + 2y + 2z = 0\}$ (plane)
Range: $\{(t, 2t, 2t), t \text{ is real}\}$ (line)
**39.** 2 **41.** 4
**43.** Because $|A| = -1 \neq 0$, the homogeneous equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. So, $\ker(T) = \{(0, 0)\}$ and $T$ is one-to-one (by Theorem 6.6). Furthermore, because $\operatorname{rank}(T) = \dim(R^2) - \operatorname{nullity}(T) = 2 - 0 = 2 = \dim(R^2)$, $T$ is onto (by Theorem 6.7).
**45.** Because $|A| = -1 \neq 0$, the homogeneous equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. So, $\ker(T) = \{(0, 0, 0)\}$ and $T$ is one-to-one (by Theorem 6.6). Furthermore, because $\operatorname{rank}(T) = \dim(R^3) - \operatorname{nullity}(T) = 3 - 0 = 3 = \dim(R^3)$, $T$ is onto (by Theorem 6.7).
**47.** One-to-one and onto **49.** One-to-one

| | | Zero | Standard Basis |
|---|---|---|---|
| **51.** | (a) | $(0, 0, 0, 0)$ | $\{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}$ |
| | (b) | $\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$ | $\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\}$ |
| | (c) | $\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$ | $\left\{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right\}$ |
| | (d) | $p(x) = 0$ | $\{1, x, x^2, x^3\}$ |
| | (e) | $(0, 0, 0, 0, 0)$ | $\{(1, 0, 0, 0, 0), (0, 1, 0, 0, 0), (0, 0, 1, 0, 0), (0, 0, 0, 1, 0)\}$ |

**53.** The set of constant functions: $p(x) = a_0$

**55.** (a) Rank $= 1$, nullity $= 2$ (b) $\{(1, 0, -2), (1, 2, 0)\}$

**57.** (a) Rank $= n$ (b) Rank $< n$

**59.** $T(A) = \mathbf{0} \implies A - A^T = \mathbf{0} \implies A = A^T$
So, $\ker(T) = \{A\colon A = A^T\}$.

**61.** (a) False. See "Definition of Kernel of a Linear Transformation," page 303.
(b) False. See Theorem 6.4, page 306.
(c) True. See discussion before "Definition of Isomorphism," page 311.

**63.** Proof **65.** Proof

**67.** If $T$ is onto, then $m \geq n$.
If $T$ is one-to-one, then $m \leq n$.

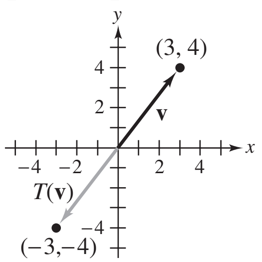
## Section 6.3 *(page 322)*

**1.** $\begin{bmatrix}1 & 2\\1 & -2\end{bmatrix}$ **3.** $\begin{bmatrix}1 & 1 & 0\\1 & -1 & 0\\-1 & 0 & 1\end{bmatrix}$ **5.** $\begin{bmatrix}3 & 0 & -2\\0 & 2 & -1\end{bmatrix}$

**7.** $(1, 4)$ **9.** $(4, -2, -2)$

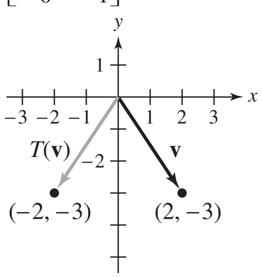
**11.** (a) $\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}$ (b) $(-3, -4)$

(c)



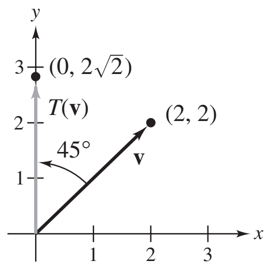

**13.** (a) $\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}$ (b) $(-2, -3)$

(c)



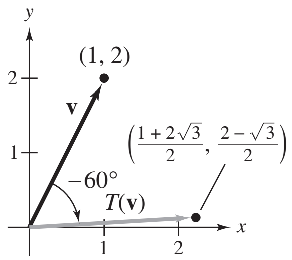

**15.** (a) $\begin{bmatrix}\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\end{bmatrix}$ (b) $(0, 2\sqrt{2})$

(c)



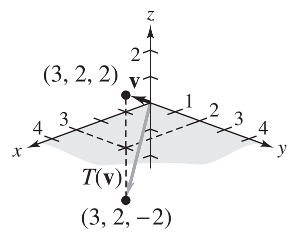

**17.** (a) $\begin{bmatrix}\frac{1}{2} & \frac{\sqrt{3}}{2}\\ -\frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix}$ (b) $\left(\frac{1}{2} + \sqrt{3}, 1 - \frac{\sqrt{3}}{2}\right)$

(c)



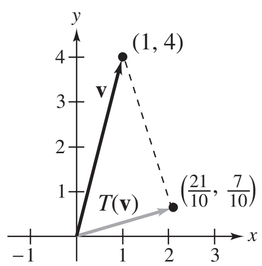

**19.** (a) $\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & -1\end{bmatrix}$ (b) $(3, 2, -2)$

(c)




**21.** (a) $\begin{bmatrix}\frac{9}{10} & \frac{3}{10}\\ \frac{3}{10} & \frac{1}{10}\end{bmatrix}$ (b) $\left(\frac{21}{10}, \frac{7}{10}\right)$

(c)




**23.** (a) $\begin{bmatrix}2 & 3 & -1\\3 & 0 & -2\\2 & -1 & 1\end{bmatrix}$ (b) $\begin{bmatrix}9\\5\\-1\end{bmatrix}$

**25.** (a) $\begin{bmatrix}1 & -1 & 0 & 0\\0 & 0 & 1 & 0\\1 & 2 & 0 & -1\\0 & 0 & 0 & 1\end{bmatrix}$ (b) $\begin{bmatrix}1\\1\\2\\-1\end{bmatrix}$

**27.** $A = \begin{bmatrix}2 & 3\\0 & 0\end{bmatrix}, A' = \begin{bmatrix}0 & 1\\0 & 2\end{bmatrix}$

**29.** $A = \begin{bmatrix}-4 & -1\\-2 & 5\end{bmatrix}, A' = \begin{bmatrix}5 & 3 & 2\\4 & -3 & 1\\-2 & -3 & -1\end{bmatrix}$

**31.** $T^{-1}(x, y) = \left(-\frac{1}{2}x, \frac{1}{2}y\right)$ **33.** $T$ is not invertible.

**35.** $T^{-1}(x_1, x_2, x_3) = (x_1, -x_1 + x_2, -x_2 + x_3)$

**37.** (a) and (b) $(9, 5, 4)$ **39.** (a) and (b) $(2, -4, -3, 3)$

**41.** (a) and (b) $(9, 16, -20)$

**43.** $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **45.** $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

**47.** $3 - 2e^x - 2xe^x$

**49.** (a) $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}$ (b) $6x - x^2 + \frac{3}{4}x^4$

**51.** (a) $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ (b) Proof

(c) $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

**53.** (a) True. See Theorem 6.10 on page 314.
(b) False. See sentence after "Definition of Inverse Linear Transformation," page 318.

**55.** Proof

**57.** Sometimes it is preferable to use a nonstandard basis. For example, some linear transformations have diagonal matrix representations relative to a nonstandard basis.

## Section 6.4 *(page 328)*

**1.** $A' = \begin{bmatrix} 4 & -3 \\ \frac{5}{3} & -1 \end{bmatrix}$ **3.** $A' = \begin{bmatrix} -\frac{1}{3} & \frac{4}{3} \\ -\frac{13}{3} & \frac{16}{3} \end{bmatrix}$

**5.** $A' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **7.** $A' = \begin{bmatrix} \frac{7}{3} & \frac{10}{3} & -\frac{1}{3} \\ -\frac{1}{6} & \frac{4}{3} & \frac{8}{3} \\ \frac{2}{3} & -\frac{4}{3} & -\frac{2}{3} \end{bmatrix}$

**9.** (a) $\begin{bmatrix} 6 & 4 \\ 9 & 4 \end{bmatrix}$ (b) $[\mathbf{v}]_B = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, [T(\mathbf{v})]_B = \begin{bmatrix} 4 \\ -4 \end{bmatrix}$

(c) $A' = \begin{bmatrix} 0 & -\frac{4}{3} \\ 9 & 7 \end{bmatrix}, P^{-1} = \begin{bmatrix} -\frac{1}{3} & \frac{1}{3} \\ \frac{3}{4} & -\frac{1}{2} \end{bmatrix}$ (d) $\begin{bmatrix} -\frac{8}{3} \\ 5 \end{bmatrix}$

**11.** (a) $\begin{bmatrix} 5 & 2 \\ 9 & 2 \end{bmatrix}$ (b) $[\mathbf{v}]_B = \begin{bmatrix} 3 \\ -1 \end{bmatrix}, [T(\mathbf{v})]_B = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

(c) $A' = \begin{bmatrix} -7 & -2 \\ 27 & 8 \end{bmatrix}, P^{-1} = \begin{bmatrix} -\frac{1}{4} & \frac{1}{4} \\ \frac{9}{8} & -\frac{5}{8} \end{bmatrix}$ (d) $\begin{bmatrix} -1 \\ 5 \end{bmatrix}$

**13.** (a) $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ (b) $[\mathbf{v}]_B = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, [T(\mathbf{v})]_B = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$

(c) $A' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, P^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$

**15.** $\begin{bmatrix} 1 & -2 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 12 & 7 \\ -20 & -11 \end{bmatrix}\begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix}$

**17.** $\begin{bmatrix} 5 & 8 & 0 \\ 10 & 4 & 0 \\ 0 & 12 & 6 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 5 & 10 & 0 \\ 8 & 4 & 0 \\ 0 & 9 & 6 \end{bmatrix}\begin{bmatrix} 5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

**19.** $\begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

**21.** Proof **23.** Proof **25.** $I_n$ **27–33.** Proof

**35.** The matrix for $I$ relative to $B$, or relative to $B'$, is the identity matrix. The matrix for $I$ relative to $B$ and $B'$ is the square matrix whose columns are the coordinates of $\mathbf{v}_1, \ldots, \mathbf{v}_n$ relative to the standard basis.

**37.** (a) True. See discussion before Example 1, page 324.
(b) False. See sentence following the proof of Theorem 6.13, page 326.

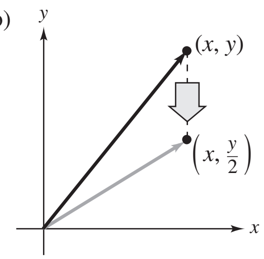
## Section 6.5 *(page 335)*

**1.** (a) $(3, -5)$ (b) $(2, 1)$ (c) $(a, 0)$
(d) $(0, -b)$ (e) $(-c, -d)$ (f) $(f, g)$

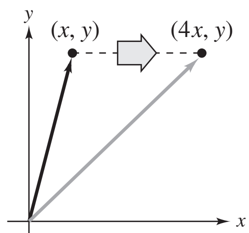
**3.** (a) $(1, 0)$ (b) $(3, -1)$ (c) $(0, a)$
(d) $(b, 0)$ (e) $(d, -c)$ (f) $(-g, f)$

**5.** (a) $(2x, y)$ (b) Horizontal expansion

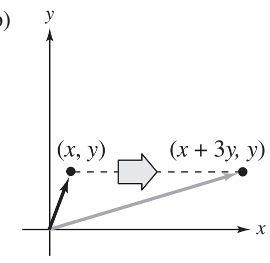
**7.** (a) Vertical contraction
(b)



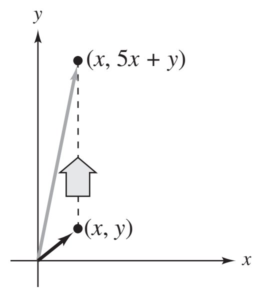

**9.** (a) Horizontal expansion
(b)




**11.** (a) Horizontal shear
(b)



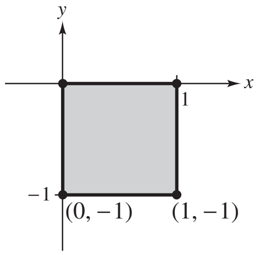

**13.** (a) Vertical shear
(b)



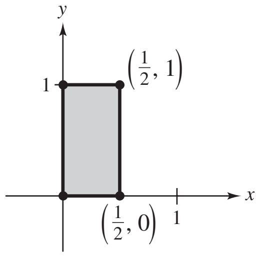

**15.** $\{(0, t): t \text{ is real}\}$ **17.** $\{(t, t): t \text{ is real}\}$

**19.** $\{(t, 0): t \text{ is real}\}$ **21.** $\{(t, 0): t \text{ is real}\}$

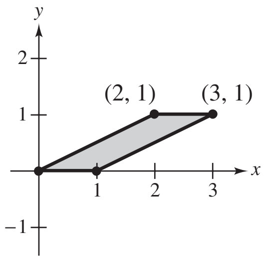
**23.**



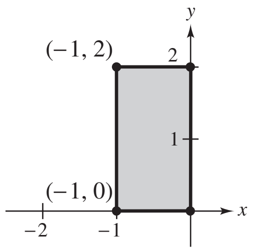

**25.**



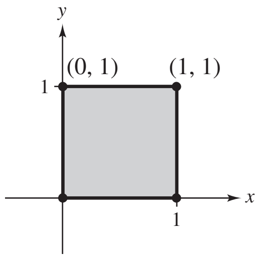

**27.**



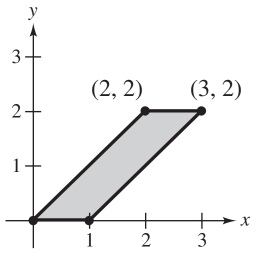

**29.**



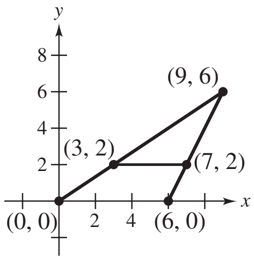

**31.**



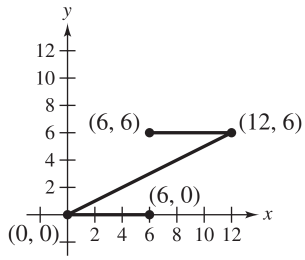

**33.**



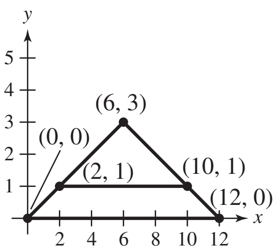

**35.** (a)



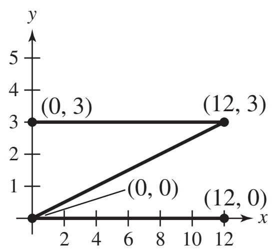

(b)



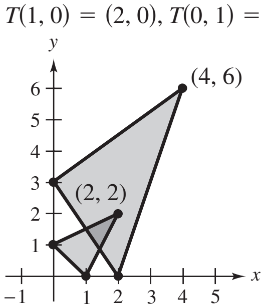

**37.** (a)




(b)




**39.** $T(1, 0) = (2, 0)$, $T(0, 1) = (0, 3)$, $T(2, 2) = (4, 6)$




**41.** Horizontal expansion **43.** Horizontal shear

**45.** Reflection in the $x$-axis and a vertical expansion (in either order)

**47.** Vertical shear followed by a horizontal expansion

**49.** $\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **51.** $\begin{bmatrix} \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{bmatrix}$

**53.** $\left(\left(\sqrt{3} - 1\right)/2, \left(\sqrt{3} + 1\right)/2, 1\right)$

**55.** $\left(\left(1 + \sqrt{3}\right)/2, 1, \left(1 - \sqrt{3}\right)/2\right)$

**57.** 90° about the $x$-axis **59.** 180° about the $y$-axis

**61.** 90° about the $z$-axis

**63.** $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}$; $(1, -1, -1)$

**65.** $\begin{bmatrix} \frac{\sqrt{3}}{4} & -\frac{1}{4} & \frac{\sqrt{3}}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{3}{4} & \frac{\sqrt{3}}{4} & \frac{1}{2} \end{bmatrix}$; $\left(\frac{3\sqrt{3} - 1}{4}, \frac{1 + \sqrt{3}}{2}, \frac{\sqrt{3} - 1}{4}\right)$

**67.** $\begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{4} & \frac{\sqrt{6}}{4} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{4} & \frac{\sqrt{6}}{4} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$; $\left(\frac{\sqrt{6} - \sqrt{2}}{4}, \frac{3\sqrt{2} + \sqrt{6}}{4}, \frac{\sqrt{3} - 1}{2}\right)$

## Review Exercises *(page 337)*

**1.** (a) $(2, -4)$ (b) $(4, 4)$

**3.** (a) $(0, -1, 7)$ (b) $\{(t - 3, 5 - t, t)\colon t \text{ is real}\}$

**5.** Linear, $A = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$ **7.** Linear, $A = \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix}$

**9.** Not linear **11.** Linear, $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$

**13.** $T(1, 1) = \left(\frac{3}{2}, \frac{3}{2}\right)$, $T(0, 1) = (1, 1)$

**15.** $T(0, -1) = (-1, -2)$

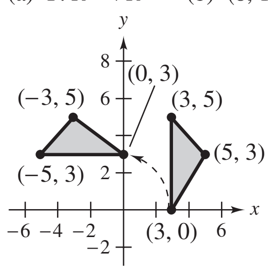
**17.** (a) $T\colon R^3 \to R^2$ (b) $(3, -12)$
(c) $\left\{\left(-\frac{5}{2}, 3 - 2t, t\right)\colon t \text{ is real}\right\}$

**19.** (a) $T\colon R^2 \to R^1$ (b) 5 (c) $\{(4 - t, t)\colon t \text{ is real}\}$

**21.** (a) $T\colon R^3 \to R^3$ (b) $(-2, -4, -5)$ (c) $(2, 2, 2)$

**23.** (a) $T\colon R^2 \to R^3$ (b) $(8, 10, 4)$ (c) $(1, -1)$

**25.**




**27.** (a) $\{(-2, 1, 0, 0), (2, 0, 1, -2)\}$ (b) $\{(5, 0, 4), (0, 5, 8)\}$

**29.** (a) $\{\mathbf{0}\}$ (b) $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$

**31.** (a) $\{(0, 0)\}$ (b) 0 (c) $\text{span}\left\{\left(1, 0, \frac{1}{2}\right), \left(0, 1, -\frac{1}{2}\right)\right\}$ (d) 2

**33.** (a) $\{(-3t, 3t, t)\}$ (b) 1
(c) $\text{span}\{(1, 0, -1), (0, 1, 2)\}$ (d) 2

**35.** 3 **37.** 2 **39.** $A^2 = I$ **41.** $A^3 = \begin{bmatrix} \cos 3\theta & -\sin 3\theta \\ \sin 3\theta & \cos 3\theta \end{bmatrix}$

**43.** $A' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, A = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$

**45.** $T$ is not invertible. **47.** $T^{-1}(x, y) = (x, -y)$

**49.** (a) One-to-one (b) Onto (c) Invertible

**51.** (a) Not one-to-one (b) Onto (c) Not invertible

**53.** (a) and (b) $(0, 1, 1)$ **55.** $A' = \begin{bmatrix} 3 & -1 \\ 1 & -1 \end{bmatrix}$

**57.** $\begin{bmatrix} 5 & -3 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} \frac{4}{7} & -\frac{5}{7} \\ \frac{1}{7} & -\frac{3}{7} \end{bmatrix} \begin{bmatrix} 18 & -19 \\ 11 & -12 \end{bmatrix} \begin{bmatrix} 3 & -5 \\ 1 & -4 \end{bmatrix}$

**59.** (a) $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{5} & \frac{2}{5} \\ 0 & \frac{2}{5} & \frac{4}{5} \end{bmatrix}$ (b) Answers will vary.

(c) Answers will vary.

**61.** Proof **63.** Proof

**65.** (a) Proof (b) Rank $= 1$, nullity $= 3$

(c) $\{1 - x, 1 - x^2, 1 - x^3\}$

**67.** $\mathrm{Ker}(T) = \{\mathbf{v}: \langle \mathbf{v}, \mathbf{v}_0 \rangle = 0\}$

Range $= R$

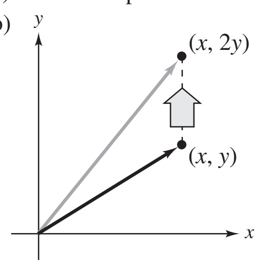
Rank $= 1$

Nullity $= \dim(V) - 1$

**69.** Although they are not the same, they have the same dimension (4) and are isomorphic.

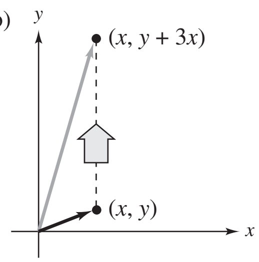
**71.** (a) Vertical expansion

(b)



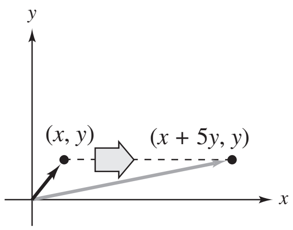

**73.** (a) Vertical shear

(b)

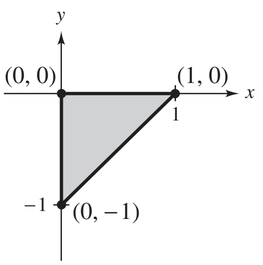



**75.** (a) Horizontal shear

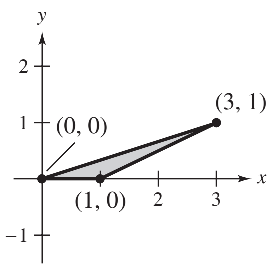
(b)




**77.**




**79.**




**81.** Reflection in the line $y = x$ followed by a horizontal expansion

**83.** $\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, (\sqrt{2}, 0, 1)$

**85.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}, \left(1, \frac{-1 - \sqrt{3}}{2}, \frac{-\sqrt{3} + 1}{2}\right)$

**87.** $\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{4} & \frac{\sqrt{3}}{4} \\ \frac{1}{2} & \frac{\sqrt{3}}{4} & -\frac{3}{4} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$ **89.** $\begin{bmatrix} \frac{\sqrt{6}}{4} & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{4} \\ \frac{\sqrt{6}}{4} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{4} \\ -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \end{bmatrix}$

**91.** $(0, 0, 0), \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right), (0, \sqrt{2}, 0),$
$\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right), (0, 0, 1), \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 1\right),$
$(0, \sqrt{2}, 1), \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 1\right)$

**93.** $(0, 0, 0), (1, 0, 0), \left(1, \frac{\sqrt{3}}{2}, \frac{1}{2}\right), \left(0, \frac{\sqrt{3}}{2}, \frac{1}{2}\right),$
$\left(0, -\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \left(1, -\frac{1}{2}, \frac{\sqrt{3}}{2}\right),$
$\left(1, \frac{-1 + \sqrt{3}}{2}, \frac{1 + \sqrt{3}}{2}\right), \left(0, \frac{-1 + \sqrt{3}}{2}, \frac{1 + \sqrt{3}}{2}\right)$

**95.** (a) False. See "Elementary Matrices for Linear Transformations in $R^2$," page 330.

(b) True. See "Elementary Matrices for Linear Transformations in $R^2$," page 330.

(c) True. See discussion following Example 4, page 334.

**97.** (a) False. See "Remark," page 294.

(b) False. See Theorem 6.7, page 310.

(c) True. See discussion following Example 5, page 327.

# Chapter 7

## Section 7.1 *(page 350)*

**1.** $\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

**3.** $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

**5.** $\begin{bmatrix} 2 & 3 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$

$\begin{bmatrix} 2 & 3 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = -1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix},$

$\begin{bmatrix} 2 & 3 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix} = 3 \begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix}$

**7.** $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

**9.** (a) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c \\ -c \end{bmatrix} = 0\begin{bmatrix} c \\ -c \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c \\ c \end{bmatrix} = \begin{bmatrix} 2c \\ 2c \end{bmatrix} = 2\begin{bmatrix} c \\ c \end{bmatrix}$

**11.** (a) No (b) Yes (c) Yes (d) No

**13.** (a) Yes (b) No (c) Yes (d) Yes

**15.** $\lambda = 1, (t, 0); \lambda = -1, (0, t)$

**17.** (a) $\lambda(\lambda - 7) = 0$ (b) $\lambda = 0, (1, 2); \lambda = 7, (3, -1)$

**19.** (a) $\lambda^2 - \frac{1}{4} = 0$ (b) $\lambda = -\frac{1}{2}, (1, 1); \lambda = \frac{1}{2}, (3, 1)$

**21.** (a) $(\lambda - 2)(\lambda - 4)(\lambda - 1) = 0$

(b) $\lambda = 4, (7, -4, 2); \lambda = 2, (1, 0, 0); \lambda = 1, (-1, 1, 1)$

**23.** (a) $(\lambda + 3)(\lambda - 3)^2 = 0$

(b) $\lambda = -3, (1, 1, 3); \lambda = 3, (1, 0, -1), (1, 1, 0)$

**25.** (a) $(\lambda - 4)(\lambda - 6)(\lambda + 2) = 0$

(b) $\lambda = -2, (3, 2, 0); \lambda = 4, (5, -10, -2);$
$\lambda = 6, (1, -2, 0)$

**27.** (a) $(\lambda - 2)^2(\lambda - 4)(\lambda + 1) = 0$

(b) $\lambda = 2, (1, 0, 0, 0), (0, 1, 0, 0); \lambda = 4, (0, 0, 1, 1);$
$\lambda = -1, (0, 0, 1, -4)$

**29.** $\lambda = -2, 1$ **31.** $\lambda = 4, -\frac{1}{2}, \frac{1}{3}$ **33.** $\lambda = -1, 4, 4$

**35.** $\lambda = 0, 0, 0, 21$ **37.** $\lambda = 0, 0, 3, 3$ **39.** $\lambda = 2, 3, 1$

**41.** $\lambda = -2, 4, -3, -3$

**43.** (a) $\lambda_1 = 3, \lambda_2 = 4$

(b) $B_1 = \{(2, -1)\}, B_2 = \{(1, -1)\}$

(c) $\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$

**45.** (a) $\lambda_1 = -1, \lambda_2 = 1, \lambda_3 = 2$

(b) $B_1 = \{(1, 0, 1)\}, B_2 = \{(2, 1, 0)\}, B_3 = \{(1, 1, 0)\}$

(c) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

**47.** $\lambda^2 - 6\lambda + 8$ **49.** $\lambda^3 - 5\lambda^2 + 15\lambda - 27$

**51.**

| Exercise | (a) Trace of $A$ | (b) Determinant of $A$ |
|---|---|---|
| 17 | 7 | 0 |
| 19 | 0 | $-\frac{1}{4}$ |
| 21 | 7 | 8 |
| 23 | 3 | $-27$ |
| 25 | 8 | $-48$ |
| 27 | 7 | $-16$ |

**53–61.** Proofs **63.** $a = 0,\ d = 1$ or $a = 1,\ d = 0$

**65.** (a) False. $\mathbf{x}$ must be nonzero.

(b) True. See Theorem 7.2, page 345.

**67.** Dim = 3 **69.** Dim = 1

**71.** $T(e^x) = \dfrac{d}{dx}[e^x] = e^x = 1(e^x)$

**73.** $\lambda = -2, 3 + 2x; \lambda = 4, -5 + 10x + 2x^2; \lambda = 6, -1 + 2x$

**75.** $\lambda = 0, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}; \lambda = 3, \begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix}$

**77.** $\lambda = 0, 1$ **79.** Proof

## Section 7.2 *(page 360)*

**1.** (a) $P^{-1} = \begin{bmatrix} 1 & -4 \\ -1 & 3 \end{bmatrix}, P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$ (b) $\lambda = 1, -2$

**3.** (a) $P^{-1} = \begin{bmatrix} \frac{1}{5} & \frac{4}{5} \\ -\frac{1}{5} & \frac{1}{5} \end{bmatrix}, P^{-1}AP = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$

(b) $\lambda = 2, -3$

**5.** (a) $P^{-1} = \begin{bmatrix} \frac{2}{3} & -\frac{2}{3} & 1 \\ 0 & \frac{1}{4} & 0 \\ -\frac{1}{3} & \frac{1}{12} & 0 \end{bmatrix}, P^{-1}AP = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

(b) $\lambda = 5, 3, -1$

**7.** $P = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$ (The answer is not unique.)

**9.** $P = \begin{bmatrix} 7 & 1 & -1 \\ -4 & 0 & 1 \\ 2 & 0 & 1 \end{bmatrix}$ (The answer is not unique.)

**11.** $P = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix}$ (The answer is not unique.)

**13.** $A$ is not diagonalizable.

**15.** There is only one eigenvalue, $\lambda = 0$, and the dimension of its eigenspace is 1. The matrix is not diagonalizable.

**17.** There is only one eigenvalue, $\lambda = 1$, and the dimension of its eigenspace is 1. The matrix is not diagonalizable.

**19.** There are two eigenvalues, 1 and 2. The dimension of the eigenspace for the repeated eigenvalue 1 is 1. The matrix is not diagonalizable.

**21.** There are two repeated eigenvalues, 0 and 3. The eigenspace associated with 3 is of dimension 1. The matrix is not diagonalizable.

**23.** $\lambda = 0, 2$ The matrix is diagonalizable.

**25.** $\lambda = 0, -2$

Insufficient number of eigenvalues to guarantee diagonalization

**27.** $\{(1, -1), (1, 1)\}$ **29.** $\{(-1 + x), x\}$

**31.** Proof **33.** $\begin{bmatrix} -188 & -378 \\ 126 & 253 \end{bmatrix}$

**35.** $\begin{bmatrix} 384 & 256 & -384 \\ -384 & -512 & 1152 \\ -128 & -256 & 640 \end{bmatrix}$

**37.** (a) True. See the proof of Theorem 7.4, page 354.

(b) False. See Theorem 7.6, page 358.

**39.** Yes. $P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

**41.** Yes, the order of elements on the main diagonal may change.

**43–47.** Proofs

**49.** Since $\lambda = 3$ is the only eigenvalue, and a basis for the eigenspace is $\{(1, 0)\}$, the matrix does not have two linearly independent eigenvectors. By Theorem 7.5, the matrix is not diagonalizable.

## Section 7.3 *(page 370)*

**1.** Symmetric **3.** Not symmetric **5.** Symmetric

**7.** $P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}, \quad P^{-1}AP = \begin{bmatrix} -a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$

**9.** $P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$, $P^{-1}AP = \begin{bmatrix} 0 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 2a \end{bmatrix}$

**11.** $\lambda = 1, \dim = 1$; $\lambda = 3, \dim = 1$

**13.** $\lambda = 2, \dim = 2$; $\lambda = 3, \dim = 1$

**15.** $\lambda = -2, \dim = 2$; $\lambda = 4, \dim = 1$

**17.** $\lambda = -1, \dim = 1$; $\lambda = 1 + \sqrt{2}, \dim = 1$; $\lambda = 1 - \sqrt{2}, \dim = 1$

**19.** $\lambda = -2, \dim = 1$; $\lambda = 3, \dim = 2$; $\lambda = 8, \dim = 1$

**21.** $\lambda = 1, \dim = 1$; $\lambda = 2, \dim = 3$; $\lambda = 3, \dim = 1$

**23.** Orthogonal **25.** Orthogonal **27.** Not orthogonal

**29.** Orthogonal **31.** Not orthogonal **33–37.** Proofs

**39.** Not orthogonally diagonalizable

**41.** Orthogonally diagonalizable

**43.** $P = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$ (The answer is not unique.)

**45.** $P = \begin{bmatrix} \sqrt{3}/3 & \sqrt{6}/3 \\ -\sqrt{6}/3 & \sqrt{3}/3 \end{bmatrix}$ (The answer is not unique.)

**47.** $P = \begin{bmatrix} -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$ (The answer is not unique.)

**49.** $\begin{bmatrix} -\sqrt{3}/3 & -\sqrt{2}/2 & \sqrt{6}/6 \\ -\sqrt{3}/3 & \sqrt{2}/2 & \sqrt{6}/6 \\ \sqrt{3}/3 & 0 & \sqrt{6}/3 \end{bmatrix}$

(The answer is not unique.)

**51.** $P = \begin{bmatrix} \sqrt{2}/2 & 0 & \sqrt{2}/2 & 0 \\ -\sqrt{2}/2 & 0 & \sqrt{2}/2 & 0 \\ 0 & \sqrt{2}/2 & 0 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & 0 & \sqrt{2}/2 \end{bmatrix}$

(The answer is not unique.)

**53.** (a) True. See Theorem 7.10, page 367.
(b) True. See Theorem 7.9, page 366.

**55.** Proof **57.** Proof

**59.** $A^TA = \begin{bmatrix} 17 & -27 & 6 \\ -27 & 45 & -12 \\ 6 & -12 & 5 \end{bmatrix}$, $AA^T = \begin{bmatrix} 14 & 24 \\ 24 & 53 \end{bmatrix}$

Both products are symmetric.

## Section 7.4 *(page 383)*

**1.** $\mathbf{x}_2 = \begin{bmatrix} 20 \\ 5 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$

**3.** $\mathbf{x}_2 = \begin{bmatrix} 84 \\ 12 \\ 6 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 60 \\ 84 \\ 6 \end{bmatrix}$

**5.** $\mathbf{x}_2 = \begin{bmatrix} 400 \\ 25 \\ 100 \\ 50 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 250 \\ 100 \\ 25 \\ 50 \end{bmatrix}$

**7.** $\mathbf{x} = t\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

**9.** $\mathbf{x} = t\begin{bmatrix} 8 \\ 4 \\ 1 \end{bmatrix}$

**11.** $\mathbf{x} = t\begin{bmatrix} 8 \\ 2 \\ 2 \\ 1 \end{bmatrix}$

**13.** $\mathbf{x}_2 = \begin{bmatrix} 1280 \\ 120 \\ 40 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 3120 \\ 960 \\ 30 \end{bmatrix}$

**15.** $\mathbf{x}_2 = \begin{bmatrix} 900 \\ 60 \\ 50 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 2200 \\ 540 \\ 30 \end{bmatrix}$

**17.** $y_1 = C_1e^{2t}$, $y_2 = C_2e^{t}$

**19.** $y_1 = C_1e^{-4t}$, $y_2 = C_2e^{t/2}$

**21.** $y_1 = C_1e^{-t}$, $y_2 = C_2e^{6t}$, $y_3 = C_3e^{t}$

**23.** $y_1 = C_1e^{-12t}$, $y_2 = C_2e^{-6t}$, $y_3 = C_3e^{7t}$

**25.** $y_1 = C_1e^{-0.3t}$, $y_2 = C_2e^{0.4t}$, $y_3 = C_3e^{-0.6t}$

**27.** $y_1 = C_1e^{7t}$, $y_2 = C_2e^{9t}$, $y_3 = C_3e^{-7t}$, $y_4 = C_4e^{-9t}$

**29.** $y_1 = C_1e^{t} - 4C_2e^{2t}$, $y_2 = C_2e^{2t}$

**31.** $y_1 = C_1e^{-t} + C_2e^{3t}$, $y_2 = -C_1e^{-t} + C_2e^{3t}$

**33.** $y_1 = 3C_1e^{-2t} - 5C_2e^{4t} - C_3e^{6t}$
$y_2 = 2C_1e^{-2t} + 10C_2e^{4t} + 2C_3e^{6t}$
$y_3 = 2C_2e^{4t}$

**35.** $y_1 = C_1e^{t} - 2C_2e^{2t} - 7C_3e^{3t}$
$y_2 = C_2e^{2t} + 8C_3e^{3t}$
$y_3 = 2C_3e^{3t}$

**37.** $y_1{}' = y_1 + y_2$, $y_2{}' = y_2$

**39.** $y_1{}' = y_2$, $y_2{}' = y_3$, $y_3{}' = -4y_2$

**41.** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

**43.** $\begin{bmatrix} 9 & 5 \\ 5 & -4 \end{bmatrix}$

**45.** $\begin{bmatrix} 0 & 5 \\ 5 & -10 \end{bmatrix}$

**47.** $A = \begin{bmatrix} 2 & -\frac{3}{2} \\ -\frac{3}{2} & -2 \end{bmatrix}, \lambda_1 = -\frac{5}{2}, \lambda_2 = \frac{5}{2}, P = \begin{bmatrix} \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{bmatrix}$

**49.** $A = \begin{bmatrix} 13 & 3\sqrt{3} \\ 3\sqrt{3} & 7 \end{bmatrix}, \lambda_1 = 4, \lambda_2 = 16, P = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$

**51.** $A = \begin{bmatrix} 16 & -12 \\ -12 & 9 \end{bmatrix}, \lambda_1 = 0, \lambda_2 = 25, P = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$

**53.** Ellipse, $5(x')^2 + 15(y')^2 - 45 = 0$

**55.** Ellipse, $(x')^2 + 6(y')^2 - 36 = 0$

**57.** Parabola, $4(y')^2 + 4x' + 8y' + 4 = 0$

**59.** Hyperbola, $\frac{1}{2}\left[-(x')^2 + (y')^2 - 3\sqrt{2}x' - \sqrt{2}y' + 6\right] = 0$

**61.** $A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 8 \end{bmatrix}, 2(x')^2 + 4(y')^2 + 8(z')^2 - 16 = 0$

**63.** $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}, (x')^2 + (y')^2 + 3(z')^2 - 1 = 0$

**65.** Let $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a $2 \times 2$ orthogonal matrix such that $|P| = 1$. Define $\theta \in (0, 2\pi)$ as follows.
(i) If $a = 1$, then $c = 0$, $b = 0$, and $d = 1$, so let $\theta = 0$.
(ii) If $a = -1$, then $c = 0$, $b = 0$, and $d = -1$, so let $\theta = \pi$.
(iii) If $a \geq 0$ and $c > 0$, let $\theta = \arccos(a)$, $0 < \theta \leq \pi/2$.
(iv) If $a \geq 0$ and $c < 0$, let $\theta = 2\pi - \arccos(a)$, $3\pi/2 \leq \theta < 2\pi$.
(v) If $a \leq 0$ and $c > 0$, let $\theta = \arccos(a)$, $\pi/2 \leq \theta < \pi$.
(vi) If $a \leq 0$ and $c < 0$, let $\theta = 2\pi - \arccos(a)$, $\pi < \theta \leq 3\pi/2$.
In each of these cases, confirm that

$$P = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

## Review Exercises *(page 385)*

**1.** (a) $\lambda^2 - 9 = 0$ (b) $\lambda = -3, \lambda = 3$
(c) A basis for $\lambda = -3$ is $\{(1, -5)\}$ and a basis for $\lambda = 3$ is $\{(1, 1)\}$.

**3.** (a) $(\lambda - 4)(\lambda - 8)^2 = 0$ (b) $\lambda = 4, \lambda = 8$
(c) A basis for $\lambda = 4$ is $\{(1, -2, -1)\}$ and a basis for $\lambda = 8$ is $\{(4, -1, 0), (3, 0, 1)\}$.

**5.** (a) $(\lambda - 2)(\lambda - 3)(\lambda - 1) = 0$
(b) $\lambda = 1, \lambda = 2, \lambda = 3$
(c) A basis for $\lambda = 1$ is $\{(1, 2, -1)\}$, a basis for $\lambda = 2$ is $\{(1, 0, 0)\}$, and a basis for $\lambda = 3$ is $\{(0, 1, 0)\}$.

**7.** (a) $(\lambda - 1)^2(\lambda - 3)^2 = 0$ (b) $\lambda = 1, \lambda = 3$
(c) A basis for $\lambda = 1$ is $\{(1, -1, 0, 0), (0, 0, 1, -1)\}$ and a basis for $\lambda = 3$ is $\{(1, 1, 0, 0), (0, 0, 1, 1)\}$.

**9.** $P = \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix}$ (The answer is not unique.)

**11.** Not diagonalizable

**13.** $P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$ (The answer is not unique.)

**15.** (a) $a = -\frac{1}{4}$ (b) $a = 2$ (c) $a \geq -\frac{1}{4}$

**17.** $A$ has only one eigenvalue, $\lambda = 0$, and the dimension of its eigenspace is 1. So, the matrix is not diagonalizable.

**19.** $A$ has only one eigenvalue, $\lambda = 3$, and the dimension of its eigenspace is 2. So, the matrix is not diagonalizable.

**21.** $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

**23.** Because the eigenspace corresponding to $\lambda = 1$ of matrix $A$ has dimension 1, while that of matrix $B$ has dimension 2, the matrices are not similar.

**25.** Both symmetric and orthogonal **27.** Symmetric

**29.** Neither **31.** Proof **33.** Proof

**35.** Orthogonally diagonalizable

**37.** $P = \begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$ (The answer is not unique.)

**39.** $P = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \end{bmatrix}$ (The answer is not unique.)

**41.** $P = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$ (The answer is not unique.)

**43.** $\left(\frac{3}{5}, \frac{2}{5}\right)$ **45.** $\left(\frac{3}{5}, \frac{2}{5}\right)$ **47.** $\left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right)$ **49.** $\left(\frac{4}{16}, \frac{5}{16}, \frac{7}{16}\right)$

**51.** Proof **53.** $A = \begin{bmatrix} 0 & 1 \\ 0 & \frac{9}{4} \end{bmatrix}, \lambda_1 = 0, \ \lambda_2 = \frac{9}{4}$

**55.** $A^2 = \begin{bmatrix} 56 & -40 \\ 20 & -4 \end{bmatrix}, \ A^3 = \begin{bmatrix} 368 & -304 \\ 152 & -88 \end{bmatrix}$

**57.** (a) and (b) Proofs **59.** Proof

**61.** $P = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

**63.** (a) $a = b = c = 0$
(b) Dim $= 1$ if $a \neq 0, \ b \neq 0, \ c \neq 0$.
Dim $= 2$ if exactly one of the three unknowns is 0.
Dim $= 3$ if exactly two of the three unknowns are 0.

**65.** (a) True. See "Definitions of Eigenvalue and Eigenvector," page 342.
(b) False. See Theorem 7.4, page 354.
(c) True. See "Definition of a Diagonalizable Matrix," page 353.

**67.** $\mathbf{x}_2 = \begin{bmatrix} 100 \\ 25 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 25 \\ 25 \end{bmatrix}, \mathbf{x} = t\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

**69.** $\mathbf{x}_2 = \begin{bmatrix} 4500 \\ 300 \\ 50 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 1500 \\ 4500 \\ 50 \end{bmatrix}, \mathbf{x} = t\begin{bmatrix} 24 \\ 12 \\ 1 \end{bmatrix}$

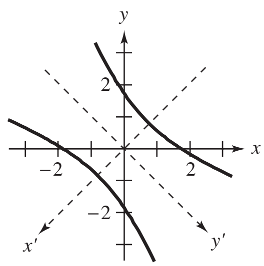
**71.** $\mathbf{x}_2 = \begin{bmatrix} 1440 \\ 108 \\ 90 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 6588 \\ 1296 \\ 81 \end{bmatrix}$

**73.** $y_1 = C_1e^{3t}$
$y_2 = C_2e^{8t}$
$y_3 = C_3e^{-8t}$

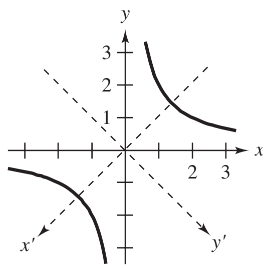
**75.** $y_1 = -2C_1 + C_2e^t$
$y_2 = C_1$

**77.** $y_1 = C_1e^t + C_2e^{-t}$
$y_2 = C_1e^t - C_2e^{-t}$
$y_3 = C_3$

**79.** (a) $A = \begin{bmatrix} 1 & \frac{3}{2} \\ \frac{3}{2} & 1 \end{bmatrix}$
(b) $P = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$
(c) $5(x')^2 - (y')^2 = 6$
(d)




**81.** (a) $A = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$
(b) $P = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$
(c) $(x')^2 - (y')^2 = 4$
(d)




## Cumulative Test for Chapters 6 and 7 *(page 389)*

**1.** Linear transformation **2.** Not a linear transformation

**3.** $\dim(R^n) = 5; \dim(R^m) = 2$

**4.** (a) $(1, -1, 0)$ (b) $(5, t)$ **5.** $\{(s, s, -t, t)\text{: } s, t \text{ are real}\}$

**6.** (a) $\text{span}\{(0, -1, 0, 1), (1, 0, -1, 0)\}$
(b) $\text{span}\{(1, 0), (0, 1)\}$ (c) Rank $= 2$, nullity $= 2$

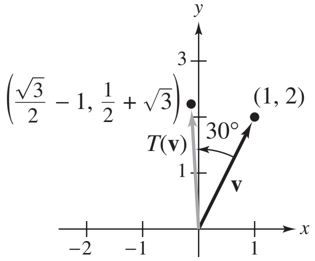
**7.** $\begin{bmatrix} 3 & 2 \\ -1 & 2 \end{bmatrix}$ **8.** $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$

**9.** $\begin{bmatrix} 0 & -2 & 3 \\ 4 & 0 & 11 \end{bmatrix}$ **10.** $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

**11.** $\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$, $T(1, 1) = (0, 0)$, $T(-2, 2) = (-2, 2)$

**12.** (a) $\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$ (b) $\begin{bmatrix} \frac{\sqrt{3}}{2} - 1 \\ \frac{1}{2} + \sqrt{3} \end{bmatrix}$

(c)




**13.** $T = \begin{bmatrix} 2 & -4 \\ -1 & -5 \end{bmatrix}$, $T' = \begin{bmatrix} 0 & 2 \\ 7 & -3 \end{bmatrix}$

**14.** $T = \begin{bmatrix} -2 & 2 & 1 \\ -1 & 3 & 2 \\ 4 & 0 & -6 \end{bmatrix}$, $T' = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -2 & 3 \\ 1 & 2 & -5 \end{bmatrix}$

**15.** $T^{-1}(x, y) = \left(\frac{1}{3}x + \frac{1}{3}y, -\frac{2}{3}x + \frac{1}{3}y\right)$

**16.** $T^{-1}(x_1, x_2, x_3) = \left(\frac{x_1 - x_2 + x_3}{2}, \frac{x_1 + x_2 - x_3}{2}, \frac{-x_1 + x_2 + x_3}{2}\right)$

**17.** $\begin{bmatrix} -1 & -2 \\ 0 & 1 \\ 2 & 1 \end{bmatrix}$, $T(0, 1) = (1, 0, 1)$

**18.** (a) $A = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}$ (b) $P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

(c) $A' = \begin{bmatrix} -7 & -15 \\ 6 & 12 \end{bmatrix}$ (d) $\begin{bmatrix} 9 \\ -6 \end{bmatrix}$

(e) $[\mathbf{v}]_B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $[T(\mathbf{v})]_B = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$

**19.** $\lambda = 5$ (repeated), $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

**20.** $\lambda = 3, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$; $\lambda = 2, \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

**21.** $\lambda = 1, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$; $\lambda = 0, \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}$; $\lambda = 2, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

**22.** $\lambda = 1$ (three times), $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

**23.** $P = \begin{bmatrix} 1 & 1 & 5 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ **24.** $P = \begin{bmatrix} 3 & -1 & -5 \\ 2 & 2 & 10 \\ 0 & 0 & 2 \end{bmatrix}$

**25.** $\{(0, 1, 0), (1, 1, 1), (2, 2, 3)\}$

**26.** $P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ **27.** $P = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}$

**28.** $y_1 = C_1e^t$
$y_2 = C_2e^{3t}$

**29.** $\begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix}$ **30.** $\mathbf{x}_2 = \begin{bmatrix} 1800 \\ 120 \\ 60 \end{bmatrix}$, $\mathbf{x}_3 = \begin{bmatrix} 6300 \\ 1440 \\ 48 \end{bmatrix}$

**31.** $P$ is orthogonal if $P^{-1} = P^T$. **32.** Proof

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## Properties of Matrix Addition and Scalar Multiplication

If $A$, $B$, and $C$ are $m \times n$ matrices and $c$ and $d$ are scalars, then the following properties are true.

1. $A + B = B + A$ **Commutative property of addition**
2. $A + (B + C) = (A + B) + C$ **Associative property of addition**
3. $(cd)A = c(dA)$ **Associative property of multiplication**
4. $1A = A$ **Multiplicative identity**
5. $c(A + B) = cA + cB$ **Distributive property**
6. $(c + d)A = cA + dA$ **Distributive property**

## Properties of Matrix Multiplication

If $A$, $B$, and $C$ are matrices (with orders such that the given matrix products are defined) and $c$ is a scalar, then the following properties are true.

1. $A(BC) = (AB)C$ **Associative property of multiplication**
2. $A(B + C) = AB + AC$ **Distributive property**
3. $(A + B)C = AC + BC$ **Distributive property**
4. $c(AB) = (cA)B = A(cB)$

## Properties of the Identity Matrix

If $A$ is a matrix of order $m \times n$, then the following properties are true.

1. $AI_n = A$
2. $I_mA = A$

## Properties of Vector Addition and Scalar Multiplication

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in $R^n$, and let $c$ and $d$ be scalars.

1. $\mathbf{u} + \mathbf{v}$ is a vector in $R^n$.
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
4. $\mathbf{u} + \mathbf{0} = \mathbf{u}$
5. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
6. $c\mathbf{u}$ is a vector in $R^n$.
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$
10. $1(\mathbf{u}) = \mathbf{u}$

## Summary of Equivalent Conditions for Square Matrices

If $A$ is an $n \times n$ matrix, then the following conditions are equivalent.

1. $A$ is invertible.
2. $A\mathbf{x} = \mathbf{b}$ has a unique solution for any $n \times 1$ matrix $\mathbf{b}$.
3. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
4. $A$ is row equivalent to $I_n$.
5. $|A| \neq 0$
6. $\text{Rank}(A) = n$
7. The $n$ row vectors of $A$ are linearly independent.
8. The $n$ column vectors of $A$ are linearly independent.

## Properties of the Dot Product

If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in $R^n$ and $c$ is a scalar, then the following properties are true.

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
3. $c(\mathbf{u} \cdot \mathbf{v}) = (c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v})$
4. $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$
5. $\mathbf{v} \cdot \mathbf{v} \geq 0$, and $\mathbf{v} \cdot \mathbf{v} = 0$ if and only if $\mathbf{v} = \mathbf{0}$.

## Properties of the Cross Product

If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in $R^3$ and $c$ is a scalar, then the following properties are true.

1. $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$
2. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$
3. $c(\mathbf{u} \times \mathbf{v}) = c\mathbf{u} \times \mathbf{v} = \mathbf{u} \times c\mathbf{v}$
4. $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$
5. $\mathbf{u} \times \mathbf{u} = \mathbf{0}$
6. $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$

## Types of Vector Spaces

$R$ = set of all real numbers
$R^2$ = set of all ordered pairs
$R^3$ = set of all ordered triples
$R^n$ = set of all $n$-tuples
$C(-\infty, \infty)$ = set of all continuous functions defined on the real line
$C[a, b]$ = set of all continuous functions defined on a closed interval $[a, b]$
$P$ = set of all polynomials
$P_n$ = set of all polynomials of degree $\leq n$
$M_{m,n}$ = set of all $m \times n$ matrices
$M_{n,n}$ = set of all $n \times n$ *square* matrices

## Finding Eigenvalues and Eigenvectors*

Let $A$ be an $n \times n$ matrix.

1. Form the characteristic equation $|\lambda I - A| = 0$. It will be a polynomial equation of degree $n$ in the variable $\lambda$.
2. Find the real roots of the characteristic equation. These are the eigenvalues of $A$.
3. For each eigenvalue $\lambda_i$, find the eigenvectors corresponding to $\lambda_i$ by solving the homogeneous system $(\lambda_i I - A)\mathbf{x} = \mathbf{0}$. This requires row-reducing an $n \times n$ matrix. The resulting reduced row-echelon form must have at least one row of zeros.

*For complicated problems, this process can be facilitated with the use of technology.