CAPITAL UNIVERSITY OF SCIENCE AND TECHNOLOGY, ISLAMABAD

Effects of Chemical Reaction on Nanofluid Flow Past a Stretching Sheet with Thermal Radiations

by

Iqra Zaib

A thesis submitted in partial fulfillment for the degree of Master of Philosophy

in the

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I am dedicating this heartfelt effort to my beloved family who supported me to conduct this research study and the respected teachers for helping and guiding me to make a final output.

CERTIFICATE OF APPROVAL

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Starting with the name of **ALLAH** Almighty who is most gracious and omnipresent, who makes the mankind and created this world to reveal what is veiled. Also, the **Prophet Muhammad (Peace Be Upon Him)** who is a guidance in every aspect of life for the betterment of Humanity.

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Abstract

A numerical investigation is performed for the two dimensional magnetohydrodynamics stagnation point past a stretching sheet in the presence of chemical reaction with convective boundary conditions and thermal radiation effects. Using suitable similarity transformations, the governing partial differential equations are transformed into a system of coupled non-linear ordinary differential equations. Shooting method is adopted for solving the set of transformed dimensionless ordinary differential equations, MATLAB software is utilized for the calculation of numerical results. The effects of respective flow on dimensionless temperature and concentration profiles are presented in graphs. It is noticeable, from the results that the heat and mass transfer rates escalates as chemical reaction parameter increases.

Contents

List of Figures

Abbreviations

IVP	Initial Value Problem
MHD	Magnetohydrodynamics
ODEs	Ordinary Differential Equations
PDEs	Partial Differential Equations
RK	Runge-Kutta
UCM	Upper Convected Maxwell

Symbols

λ	Stretching/Shrinking Constant
α_f	Thermal Diffusivity of Nanofluid
$u_e(x)$	Inviscid Flow Velocity
T_f	Constant Temperature at Bottom
h_f	Heat Transfer Coefficient
C_w	Wall Concentration
T_∞	Inviscid Fluid Temperature
C_∞	Inviscid Fluid Concentration
v_0	Constant Mass Velocity
B_0	Transfer Magnetic Field Strength
u	Velocity Component along x-axis
v	Velocity Component along y-axis
$\mathbf{V}$	Velocity Field
T	Nanofluid Temperature
C	Nanoparticle Volume Fraction
ν	Kinematic Viscosity
ρ	Density
k	Thermal Conductivity
k^*	Absorption Coefficient
σ	Electrical Conductivity
σ^*	Stefan-Boltzmann Constant
D_B/D_{B_∞}	Brownian Diffusion
D_T	Thermophoretic Diffusion

$(\rho c_p)_p$	Nanoparticle Heat Capacity
$(\rho c_p)_f$	Fluid Heat Capacity
Nr	Radiation Parameter
η	Dimensionless Variable
S	Surface Mass Transfer
Pr	Prandtl Number
Le	Lewis Number
Bi	Biot Number
M	Magnetic Parameter
Nb	Brownian Motion Parameter
Nt	Thermophoresis Parameter
C_f	Skin Friction
Nu_x	Local Nusselt Number
τ_w	Surface Shear Stress
q_w	Surface Heat Flux
q_r	Radiative Heat Flux
Re_x	Local Reynolds Number
$f'(\eta)$	Dimensionless Velocity
$\theta(\eta)$	Dimensionless Temperature
$\phi(\eta)$	Dimensionless Concenteration
γ	Chemical Reaction Parameter
ϵ	Species Diffusivity Parameter

Chapter 1

Introduction

Magnetohydrodynamics is concerned with the mathematical and physical framework that leads into the idea of magnetic-dynamics, in electrically conducting fluids for example in plasma studies, magnetohydrodynamics power generator, liquid metals system of fusion reactors and motion of earth's core etc. The practice of magnetohydrodynamics stagnation point flow in science and engineering are of great significance involving the effects on velocity, heat and mass transfer in the presence of chemical reaction. This phenomenon occurs frequently in petroleum industries and agriculture. The MHD factor plays key role in maintenance of the cooling rate and for achievement of the desired quality of the product as well.

1.1 Background

The MHD boundary layer flow of an electrically conducting fluid was first studied by Pavlov [1]. He investigated the flow of a conducting incompressible viscous fluid due to deformation of a plane elastic surface in a transverse magnetic field with the approximation of boundary-layer theory. Chakrabarti and Gupta [2] extended this study to include the temperature distribution over a stretching sheet in the presence of a uniform suction. Later, this problem was further extended to magnetohydrodynamic flow of an electrically conducting power-law fluid over a stretching sheet in the presence of a uniform transverse magnetic field by Andersson et. al. [3]. The influence of magnetohydrodynamics on two dimensional flow of an incompressible Eyring—Powell fluid towards a linear stretching sheet was investigated by Akbar et. al. [4].

Choi [5] was the first who introduced the theory of nanofluids with its applications. He investigated that by adding nanoparticles into the fluid, the thermal conductivity of the fluid enhanced. Further MHD stagnation point flow of a nanofluid towards a stretching sheet is discussed by Ibrahim et. al. [6]. The MHD flow of an electrically conducting fluid is important in modern metallurgy and metalworking process such as the process fusing of metals in an electrical furnace by applying a magnetic field and the process of cooling of the first wall inside a nuclear reactor containment vessel, where the hot plasma is isolated from the wall. Several authors have been studying the heat transfer behavior utilizing nanofluids. Bachok et. al. [7] studied the steady boundary layer flow of a nanofluid past a moving semi infinite flat plate in a uniform free stream. Khan and Pop [8] discussed the laminar fluid flow resulting from the stretching of a flat surface. In the recent years, for many research scientists the problem involving stagnation point flow is noticeable. Due to its remarkable properties, the study of flow close to a stagnation point over a stretching and shrinking sheet has a vast range of practical applications, for example, temperature reducing process of atomic reactors as well as electronic equipment, the layouts of thrust bearings, several hydrodynamics processes. Mahapatra [9] analyzed the flow nearby a stagnation point over a stretching sheet with heat transfer effects. Furthermore, Nazar et al. [10] extended the work by using a micropolar fluid. Later on, Mustafa et al. [11] investigated the steady boundary layer flow and heat transfer near the stagnation-point of a nanofluid towards a stretching sheet. Nandy and Pop [12] analyzed the steady two dimensional MHD stagnation-point flow and heat transfer of a power-law fluid over a continuously not stretching surface in the presence of thermal radiation. Nasir et al. [13] discussed the radiation effects on the MHD stagnation-point flow of a nanofluid over a stretching sheet with the convective boundary condition. This study is helpful for solving the problems of hypersonic flows around aircraft, rocket engines for plasma generators and detonation fronts and laser deflagration waves for various discharges.

The heat transfer phenomena that occurs between two or more bodies or within the same body due to a temperature difference is well–known for everyone. In various industrial and engineering processes, the characteristics of heat transfer have huge effects on microelectronics, transportation and fuel cells etc. The heat conduction law was suggested by Fourier [14], but it has a limitation that for the temperature field it generates a parabolic energy equation. To resolve this issue in the classical Fourier law of heat conduction, Cattaneo [15] added the thermal relaxation time. After that, the MHD 3D UCM fluid flow over a stretching sheet with Maxwell–Cattaneo heat flux model is discussed by Christov [16].

In engineering, heat and mass transfer problems with chemical reactions are part and parcel. Any chemical reaction can further be characterized with certain process including disappearance of evaporation, shifting of impetus and flow in a dessert cooler. A homogeneous reaction occurs with sole entity through specified region whereas a heterogeneous reaction occurs within confined region or space. The reaction in which rate and the concentration are directly proportional is regarded as first order reaction. The diffusion of species with chemical reaction has immense utilities regarding insulation, pollution studies, synthesis materials and oxidation. Das [17] considered the effects in MHD micropolar flow, heat and mass transfer with thermal radiation and chemical reaction.

Bhattacharyya [18] explored solutions for stagnation-point boundary layer flow with chemical reaction past a shrinking/stretching sheet. Khan et al. [19] studied the effects of uniform transverse magnetic field and chemical reaction on heat and mass transfer flow in an electrically conducting incompressible nanofluid past a continuously moving plate with variable surface temperature. Furthermore, Kumar and Verma [20] used Cattaneo–Chirstov heat flux model to examine the hydromagnetic boundary layer flow of a nanofluid over a stretching sheet with variable wall thickness. In this study, they considered the thermal conductivity of nanofluid and species molecular diffusion coefficient.

1.2 Thesis Contribution

The key focus of the current study is to perform the analysis for the MHD stagnation point flow past a stretching sheet in the presence of chemical reaction and species diffiuisivity variable. The effects of various parameters are examined on temperature and concentration profiles. The governing partial differential equations are formulated into set of coupled ordinary differential equations by employing suitable similarity transformations. Shooting method is utilized for the solution of coupled ODEs. Solutions are illustrated by means of graphs.

1.3 Thesis Layout

This thesis is further structured into four chapters.

Chapter 2 comprises of some basic definitions and fundamental governing laws of fluid dynamics that are useful for the understanding of the current study.

Chapter 3 is devoted to the detailed review of Nasir et al. [13]. It presents the two dimensional MHD stagnation point of nanofluid past a stretching sheet with radiation effects.

Chapter 4 focuses on the extention of Nasir et al. [13] by the addition of species diffusivity parameter and chemical reaction.

Chapter 5 summarizes the thesis and presents the main findings of the our work as a whole, and suggests recommendations for prospective studies.

All the references used in this study are provided in the **Bibliography**.

Chapter 2

Preliminaries

The purpose of this chapter is to introduce some fundamental definitions, terminologies and basic concepts of fluid. Some classical laws are also presented which are inevitable for the problems of fluid dynamics.

2.1 Basic Definitions

This section is devoted to some basic definitions related to fluid dynamics. These concepts are used to describe the flow, heat transfer and influence of thermophysical properties that are used in next chapters.

Definition 2.1.1. [21]
"Fluid is a substance exists in three primary phases: solid, liquid, and gas. (At very high temperatures, it also exists as plasma.) A substance in the liquid or gas phase is referred to as a fluid. Distinction between a solid and a fluid is made on the basis of the substances ability to resist an applied shear (or tangential) stress that tends to change its shape. A solid can resist an applied shear stress by deforming, whereas a fluid deforms continuously under the influence of shear stress, no matter how small. In solids stress is proportional to strain, but in fluids, stress is proportional to strain rate."

Definition 2.1.2. [21]

"Fluid mechanics is defined as the science that deals with the behavior of fluids at rest or in motion and the interaction of fluids with solid or other fluids at the boundaries."

Definition 2.1.3. [21]

"It is the study of the motion of liquids, gases and plasma from one place to another. Fluid dynamics has a wide range of applications like calculating force and moments on aircraft, mass flow rate of petroleum passing through pipelines, prediction of weather, etc."

Definition 2.1.4. [22]

"Fluid static is the part of fluid mechanics that deals with a fluid and its characteristics at the constant position."

Definition 2.1.5. [23]

"The study of the motion of fluids that are practically incompressible such as liquids, especially water and gases at low speeds, are usually referred as hydrodynamics."

Definition 2.1.6. [23]

"Magnetohydrodynamics (MHD) is concerned with the flow of electrically conducting fluids in the presence of magnetic fields, either externally applied or generated within the fluid by inductive action."

Definition 2.1.7. [23]

"A nanofluid is a fluid containing nanometer-sized particles, called nanoparticles. These fluids are engineered colloidal suspensions of nanoparticles in a base fluid. The nanoparticles used in nanofluids are typically made of metals, oxides, carbides, or carbon nanotubes."

Common base fluids include water, ethylene glycol and oil.

Definition 2.1.8. [24]

"It is a point in a flow field where the fluid velocity is zero. It exists at the surface of objects in the field is brought to rest by the object. Static pressure is an example of stagnation point."

2.2 Classification of Fluids

In this section, types of fluids are discussed which further help in understanding of fluids nature.

Definition 2.2.1. [22]

"Ideal or perfect fluids are those fluids having viscosity equal to zero, i.e., $\mu = 0$. The nature of such fluids are fictitious and do not have shear forces."

Definition 2.2.2. [22]

"Real or viscous fluids have non-zero viscosity, i.e., $\mu \neq 0$.

Such fluids always possess non-zero viscosity and are either compressible or incompressible in nature. For example kerosene, petrol and castor oil etc." Major real fluid classes are termed as Newtonian fluids and non-Newtonian fluids.

Definition 2.2.3. [22]

"It is relationship in which shear stress is directly and linearly proportional to the velocity gradient. Mathematically, it can be written as:

$$\tau_{xy} \propto \left(\frac{du}{dy}\right),$$

$$\tau_{xy} = \mu\left(\frac{du}{dy}\right),$$

where

- μ = Dynamic viscosity,
- τ_{xy} = Shear stress exerted by the fluid,
- $\frac{du}{dy}$ = Velocity gradient perpendicular to the direction of the shear."

Water, alcohol and glycerol etc, are the common examples of Newtonian fluid.

Definition 2.2.4. [22]

"The real fluids for which the shear stress of the fluid varies not linearly proportional to the deformation rate(velocity gradient), are called non-Newtonian fluids.

Mathematically it can be expressed as

$$\tau_{xy} \propto k\left(\frac{du}{dy}\right)^n, \ n \neq 1$$
$$\tau_{xy} = k\left(\frac{du}{dy}\right)^n,$$

where

- k = Flow consistency coefficient,
- $\frac{du}{dy}$ = Shear rate,
- n = Flow behavior index."

Paints, blood, biological fluids and polymer melts etc, are good examples of non-Newtonian fluids.

Definition 2.2.5. [25]

"It is a model of a non-Newtonian fluid that includes plasticity in addition to viscosity and it was first presented in 1944. Eyring and Powell did some fitting of measured data and came up with mathematical equation to represent the non-Newtonian behavior of some class of materials with time dependent behavior that depends on the rate of change of shear."

2.3 Types of Flow

In this section, types of flow are discussed depending upon fluid properties.

Definition 2.3.1. [22]

"Flow is the deformation of the material under the influence of different forces. If the deformation increase is continuous without any limit then the process is known as flow."

Definition 2.3.2. [22]

"When the velocity of flow does not change either in magnitude or in direction at any point in a flowing fluid, for a given time, it is said to be a uniform flow. In other words, it is the flow of a fluid in which each particle moves along its line of flow with constant speed and the cross section of each stream tube remains unchanged.

Mathematically, it is represented as:

$$\frac{\partial \ \mathbf{V}}{\partial L} = 0,$$

where $\mathbf{V}$ is velocity and L is length of cross sectional area."

Definition 2.3.3. [22]

"When there is change in velocity of the flow at different points in a flowing fluid, for a given time, it is said to be non-uniform flow. For example, the flow of liquids under pressure through long pipelines of varying diameter is referred as non-uniform flow.

Mathematically this can be written as:

$$\frac{\partial \mathbf{V}}{\partial L} \neq 0.$$

where $\mathbf{V}$ is velocity and L is the displacement."

Definition 2.3.4. [24]

"The flow bounded by a solid surface is known as an internal flow. An example of the internal flow is the flow in pipe or duct."

Definition 2.3.5. [24]

"The flow, which is not bounded by a solid surface, is known as an external flow. An example of the external flow is the water- flow in the river or in the ocean."

Definition 2.3.6. [22]

"The flow in which the material density varies during fluid flow is said to be compressible flow. Compressible fluid flow is used in high-speed jet engines, aircraft, rocket motors also in high-speed usage in a planetary atmosphere, gas pipelines and in commercial fields.

Mathematically, it is expressed as:

$$\rho(x,y,z,t) \neq k,$$

where ρ is the density and k is constant."

Definition 2.3.7. [22]

"A flow is said to be incompressible if the density remains nearly constant. Therefore, the volume of every portion of fluid remains unchanged over the course of its motion when the flow (or the fluid) is incompressible.

Mathematical notation is as follow:

$$\rho(x,y,z,t) = k.\text{"}$$

Definition 2.3.8. [22]

"A flow in which the property of fluid flowing per second is constant. In other words time independent flow is called steady flow.

Mathematical representation is given below:

$$\frac{\partial \tau}{\partial t} = 0,$$

where τ is any fluid property and t the time."

Definition 2.3.9. [22]

"A flow in which the property of fluid flowing per second is not constant. In other words time dependent flow is called unsteady flow.

Mathematical representation is given below:

$$\frac{\partial \tau}{\partial t} \neq 0,$$

where τ be any fluid property and t the time."

Definition 2.3.10. [21]

"Some flows are smooth and orderly while others are rather chaotic. The highly ordered fluid motion characterized by smooth layers of fluid is called laminar.

The word laminar comes from the movement of adjacent fluid particles together in laminates. The flow of high-viscosity fluids such as oils at low velocities is typically laminar."

Definition 2.3.11. [21]
"The highly disordered fluid motion that typically occurs at high velocities and is characterized by velocity fluctuations is called turbulent. The flow of low-viscosity fluids such as air at high velocities is typically turbulent."

Definition 2.3.12. [21]
"When two fluid layers move relative to each other, a friction force develops between them and the slower layer tries to slow down the faster layer. This internal resistance to flow is quantified by the fluid property viscosity, which is a measure of internal stickiness of the fluid. Viscosity is caused by cohesive forces between the molecules in liquids and by molecular collisions in gases. There is no fluid with zero viscosity, and thus all fluid flows involve viscous effects to some degree. Flows in which the frictional effects are significant are called viscous flows."

Definition 2.3.13. [21]
"In many flows of practical interest, there are regions (typically regions not close to solid surfaces) where viscous forces are negligibly small compared to inertial or pressure forces. Neglecting the viscous terms in such inviscid flow regions greatly simplifies the analysis without much loss in accuracy."

2.4 Properties of Fluid

Following are the properties of fluids, depending upon the type of fluid.

Definition 2.4.1. [26]
"Density is defined as mass per unit volume. It is represented as ρ.
Assuming m be a mass of a fluid and V be the volume, then density is given as:

$$\rho = \frac{m}{V}."$$

Definition 2.4.2. [26]

"In shear stress a force is tending to cause deformation in a material or fluid. The direction of force in this case is always parallel to the material.

Shear stress can be represented as η by following relation:

$$\eta = \frac{\mathbf{F}}{A}."$$

Definition 2.4.3. [26]

"Normal stress is the component of stress in which force acts perpendicular to the unit surface area."

Definition 2.4.4. [22]

"This is internal property of the fluid by virtue of which it offers resistance to the flow. Mathematically viscosity is described as the ratio of the shear stress to the rate of shear strain. i.e,

$$\mu = \frac{\text{Shear stress}}{\text{Rate of shear strain}}.$$

In above expression μ is called the co-efficient of viscosity. This is also known as the absolute viscosity or simply viscosity having dimensions $\left[\frac{M}{LT}\right]$.

Water is a thin fluid having low viscosity and on other hand honey is thick fluid carrying higher viscosity. Usually liquids have non-zero viscosity."

Definition 2.4.5. [22]

"Kinematic viscosity is the ratio of dynamic viscosity to density, a quantity in which no force is involved. It can be obtained by dividing the absolute viscosity of a fluid with the fluid mass density which can be mathematically expressed as:

$$\nu = \frac{\mu}{\rho},$$

where ρ denote density and μ denote dynamic viscosity respectively."

Definition 2.4.6. [26]

"A normal force $\mathbf{F}$ exerted by a fluid per unit area A is called pressure.

Formulated as:

$$p = \frac{\mathbf{F}}{A}.\text{”}$$

Definition 2.4.7. [26]

“Thermal conductivity k is the property of a material related to its ability to transfer heat. Mathematically , it is given by:

$$k = \frac{q^* \nabla L}{S \nabla T},$$

where q^* is the heat passing through a surface area S and the effect of a temperature difference ∇T over a distance is ∇L. Here L, S and ∇T are all assumed to be of unit measurement.

The SI unit of thermal conductivity is $\frac{W}{m.b}$ and its dimension is $[MLT^{-1}\theta^{-1}]$.”

Definition 2.4.8. [26]

“It is the ratio of the thermal conductivity of fluid or material to the specific heat capacity of fluid or material.

Mathematical formulation is:

$$\alpha = \frac{k}{\rho C_p},$$

where

- k = Thermal conductivity of material,
- ρ = Density,
- C_p = Specific heat capacity.”

2.5 Properties of Heat Transfer in Fluid

This section provides some properties of heat transfer in fluid.

Definition 2.5.1. [22]

“It is the energy transfer due to the temperature difference. At the point when there is a temperature contrast in a medium or between media, heat transfer must takes place. Heat transfer occurs when the temperature of objects are not equal to each other and refers to how this difference is changed to an equilibrium state."

Definition 2.5.2. [22]

"The flow of heat transfer through liquid or solid with rapid vibration between neighboring molecules and atoms is called conduction. In other words motion of free electrons moves from one atom to another is known as conduction. Mathematically, it can be written as

$$q = -kA\left(\frac{\triangle T}{\triangle n}\right),$$

where k denotes the constant of the thermal conductivity, A is the area and $\left(\frac{\triangle T}{\triangle n}\right)$ denotes gradient of temperature respectively." For example

- After a car is turned on, the engine is heated up. The hood will become warm as heat is conducted from the engine to the hood.
- Light bulbs gives off heat and if you touch one that is on, your hand will get burned.
- Picking up a hot cup of tea.

Definition 2.5.3. [27]

"It is a mechanism in which heat transfer occurs due to the motion of molecules within the fluid such as air and water. A mathematical expression for convection phenomena is

$$q = hA(T_f - T_\infty),$$

where q, h, A, T_f and T_∞ denote the the rate of convection heat transfer, heat transfer coefficient, the area, the temperature of the surface and the temperature away from the surface respectively." For example:

- If meat is still frozen when it's time to start cooking, it will that more quickly when placed under running water than if it is immersed in water.

The reason is the convection, or movement of the water and its heat circulation, will transfer heat more quickly into the frozen meat than if the meat sits immersed in water and has to absorb heat energy through conduction.

Convection is further categorized as free or natural, forced and mixed. An overview is as written below:

Definition 2.5.4. [25]
"It is the process, in which heat transfer is caused by the temperature differences. It effects the density of the fluids and the fluid motion is not developed by an external source. It occurs only in the presence of gravitational force and also known as free convection." For example:

- Natural convection can create a noticeable difference in temperature within a home. Often this becomes places where certain parts of the house are warmer and certain parts are cooler.

Definition 2.5.5. [25]
"It is the type of convection in which some external source is used too induced a force on the fluid's system for the transportation of heat. External source may be a pump, fan or a suction device."
For example:

- The sweat that our body produces is for effective heat transfer. So when the fan is of, the air around us absorbs the water vapor until its saturated. After that it stops and we start feeling more hot. So when we switch on the fan the air around us starts moving, so the air never gets saturated completely and hence the sweat keeps evaporating by absorbing our body heat and we feel cooler.

- Forced convection creates a more uniform and therefore comfortable temperature throughout the entire home. This reduces cold spots in the house, reducing the need to crank the thermostat to a higher temperature, or putting on sweaters.

Definition 2.5.6. [25]

"When both natural and forced convection affect the heat transfer process at the same time, then this mechanism is called mixed convection." For example

- A fan blowing upward on a hot plate. Since heat naturally rises, the air being forced upward over the plate adds to the heat transfer.

Definition 2.5.7. [27]

"Radiation is the energy transfer due to the release of photons or electromagnetic waves from a surface volume. Radiation does not require any medium to transfer heat. The energy produced by radiation is transformed by electromagnetic waves. Mathematical formulation for this phenomenon is:

$$\Bbbk = E\sigma A[\Delta T]^4,$$

where

- E is the emissivity of the scheme,
- σ is the constant of Stephan-Boltzmann $\left(5.670 \times 10^{-8}\frac{W}{m^2k^4}\right)$,
- ΔT is the variation of the temperature,
- A is the area,
- $\Bbbk$ is the amount of heat transferred."

Definition 2.5.8. [27]

"Mass transfer is the flow of molecules from one body to another when these bodies are in contact or within a system consisting of two components when the distribution of materials is not uniform. For example

- When copper plate is placed on steel plate, some molecules from either side will diffuse into the other side. When salt is placed in a glass and water poured over it,after sufficient time the salt molecules will diffuse into water body. Usually mass transfer takes place from a location where the particular component is proportionately low."

Definition 2.5.9. [27]

"The process by which heat is transferred from a body by virtue of its temperature, without the aid of any intervening medium, is called thermal radiation. Sometimes radiant energy is taken to be transported by electromagnetic waves while at other times it is supposed to be transported by particle like photons. Radiation is found to travel at the speed of light in vacuum. The term electromagnetic radiation encompasses many types of radiation such as:

- Short wave radiation like gamma rays, x-rays and microwave.
- Long wave radiation like radio wave and thermal radiation.

The cause for the emission of each type of radiation is different. Thermal radiation is emitted by a medium due to its temperature."

Definition 2.5.10. [28]

"A boundary layer is a thin layer of viscous fluid close to the solid surface of a wall in contact with a moving stream in which (within its thickness) the flow velocity varies from zero at the wall (where the flow *sticks* to the wall because of its viscosity) up to the start of free stream at the boundary. The fundamental concept of the boundary layer was suggested by *L.Prandtl* (1904).

In spite of its relative thinness, the boundary layer is very important for initiating processes of dynamic interaction between the flow and the body. The boundary layer determines the aerodynamic drag and lift of the flying vehicle, or the energy loss for fluid flow in channels (in this case, a hydrodynamic boundary layer because there is also a thermal boundary layer which determines the thermodynamic interaction of heat transfer)."

2.6 Generalized Governing Laws and Equations for Fluid Motion

In this section some basic laws are discussed which are necessary for the further discussion. In later part of this section generalized equations such as continuity equation, momentum equation and energy equation are presented.[21, 22]

"Several conservation laws such as the laws of conservation of mass, conservation of energy and conservation of momentum are of great use for the research community. Historically, the conservation laws were first applied to a fixed quantity of matter called a closed system or just a system, and then extended to regions in space called control volumes. The conservation relations are also called balance equations since any conserved quantity must balance during a process."

Definition 2.6.1. This law states that

"The mass inside a system is conserved and does not change. i.e.

$$\frac{Dm}{Dt} = 0,$$

where m is the mass of fluid flowing in system or control volume."

Definition 2.6.2. "In the conservation of mass of fluid entering and leaving the control volume, the resulting mass balance is called the equation of continuity. This equation reflects the fact that mass is conserved. For any fluid, conservation of mass is expressed by scalar equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0. \tag{2.1}$$

For the steady flow (2.1) can be written as:

$$\nabla \cdot (\rho \mathbf{V}) = 0. \tag{2.2}$$

For incompressible flow, (2.2) becomes:

$$\nabla \cdot \boldsymbol{V} = 0. \tag{2.3}$$

For incompressible and irrotational flow, the (2.3) is transformed in terms of velocity potential ϕ, which is given by:

$$\nabla^2 \phi = 0. \tag{2.4}$$

(2.4) is known as Laplace equation."

Definition 2.6.3. "The product of the mass and the velocity of a body is called the linear momentum or just the momentum of the body, and the momentum of a rigid body of mass m moving with a velocity $\overrightarrow{V}$ is $m\overrightarrow{V}$. Newtons second law states that the acceleration of a body is proportional to the net force acting on it and is inversely proportional to its mass, and that the rate of change of the momentum of a body is equal to the net force acting on the body. Therefore, the momentum of a system remains constant when the net force acting on it is zero, and thus the momentum of such systems is conserved. This is known as the conservation of momentum principle."

Definition 2.6.4. "Each particle of fluid obeys Newtons second law of motion which is at rest or in steady state or accelerated motion. This law states that the combination of all applied external forces acting on a body is equal to the time rate of change of linear momentum of the body. In vector notation this law can be written as:

$$\rho\frac{D\mathbf{V}}{Dt} = \nabla \cdot \tau + \rho b,$$

for Navier-Stokes equation

$$\tau = -\rho I + \mu A_1$$

where A_1 is the tensor and first time it was presented by Rivlin-Erickson.

$$A_1 = grad\mathbf{V} + (grad\mathbf{V})^t$$

In the above equations, $\frac{d}{dt}$ denote material time derivative or total derivative, ρ denote density, $\mathbf{V}$ denote velocity field, τ the Cauchy stress tensor, b the body forces, p the pressure, μ the dynamic viscosity.
The Cauchy stress tensor is expressed in the matrix form

$$\tau = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau zy & \sigma_{zz} \end{bmatrix}$$

where σ_{xx}, σ_{yy} and σ_{zz} are normal stresses, others wise the shear stresses. For two-dimensional flow, we have $\mathbf{V} = [u(x, y, 0), v(x, y, 0), 0]$ and thus

$$grad\mathbf{V} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & 0 \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$(grad\mathbf{V})^t = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & 0 \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence it can be easily seen that

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + v\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

Similarly, we repeat the above process for y-component as follows:

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + v\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right).\text{''}$$

Definition 2.6.5. First law of thermodynamics states that:
"The variation in internal energy E of a system during any transformation is equal to the amount of energy that system receives from the environment and the work done by the system.
Mathematically, it is written as:

$$\Delta E = Q - W,$$

where

- ΔE is change in internal energy,
- Q is heat added to the system,
- W is work done by the system."

Definition 2.6.6. "Energy can be transferred to or from a closed system by heat or work, and the conservation of energy principle requires that the net energy transfer to or from a system during a process be equal to the change in energy content of the system. control volume involves energy transfer via mass flow also, and the conservation of energy principle, also called energy balance, is expressed as:

$$E_{in} \; - \; E_{out} = \frac{dE_{CV}}{dt},$$

where E_{in} and E_{out} are the total rates of energy transfer into and out of the control volume, respectively, and dE_{CV}/dt is the rate of change of energy within the control volume boundaries. In fluid mechanics, we usually limit our consideration to mechanical forms of energy only."

2.7 Dimensionless Parameters

Definition 2.7.1. [29]

"The Prandtl number which is a dimensionless number, named after the German physicist Ludwig Prandtl, is defined as:

$$Pr = \frac{\nu}{\alpha} = \frac{\mu/\rho}{k/\rho C_p} = \frac{\mu C_p}{k},$$

where

- ν is the kinematic viscosity,
- α denotes the thermal diffusivity.

This number expresses the ratio of the momentum diffusivity (viscosity) to the thermal diffusivity. It characterizes the physical properties of a fluid with convective and diffusive heat transfers. It describes, for example, the phenomena connected with the energy transfer in a boundary layer. It expresses the degree of similarity between velocity and diffusive thermal fields or, alternatively, between hydrodynamic and thermal boundary layers."

Definition 2.7.2. [22]

"Skin friction coefficient occurs between the fluid and the solid surface which leads to slow down the motion of the fluid. The skin friction coefficient can be defined as:

$$C_f = \frac{2\tau_w}{\rho U^2},$$

where

- τ_w is the wall shear stress,
- ρ is the density,
- U is the free-stream velocity.

It expresses the dynamic friction resistance originating in viscous fluid flow around a fixed wall."

Definition 2.7.3. [29]

"The Lewis Number Le is defined as the ratio of the Schmidt Number Sc and the Prandtl Number Pr. The Lewis Number is also the ratio of thermal diffusivity and molecular diffusivity as is found from the definitions of Schmidt and Prandtl Number, as follows:

$$Le = \frac{Sc}{Pr}.$$

The Lewis Number is important in determining the relationship between mass and heat transfer coefficients."

Definition 2.7.4. [22]

"It is a dimensionless number which is used to clarify the different flow behaviors like turbulent or laminar flow. It helps to measure the ratio between inertial force and the viscous force.

Mathematically expressed as

$$R_e = \frac{\frac{\rho U^2}{L}}{\frac{\mu U}{L^2}} \Rightarrow R_e = \frac{LU}{\nu},$$

where U denotes the velocity of the fluid with respect to object, L the characteristics length. At low Reynolds number, laminar flow arises where the viscous forces are dominant. At high Reynolds number, Turbulent flow arises where the inertial forces are dominant."

Definition 2.7.5. [29]

"Biot number expresses the ratio of the heat flow transferred by convection on a body surface to the heat flow transferred by conduction in a body.The criterion was first introduced by French physicist, Jean-Baptiste Biot. Mathematically it can be expressed as

$$Bi = \frac{h_h L}{k},$$

where

- h_h is heat transfer coefficient,
- L denotes the characteristic length,
- k is the thermal conductivity.

Biot number shows how convection and conduction heat transfer phenomena are related. Small values of this number shows that the conduction is the main heat transfer method, while high values of this number indicates that the convection is the main heat transfer mechanism. Biot number arises when we use third kind of boundary condition (i.e convective heat transfer in presence of external fluid surface)."

Definition 2.7.6. [29]

"It is a dimensionless number, first introduced by a German engineer Ernst Kraft Wilhelm Nusselt and is defined as:

$$Nu = \frac{\alpha L}{k},$$

where

- α represents the heat transfer coefficient,
- L denotes the characteristic length,
- k is the thermal conductivity.

It expresses the ratio of the total heat transfer in a system to the heat transfer by conduction. In characterizes the heat transfer by convection between a fluid and the environment close to it or, alternatively, the connection between the heat transfer intensity and the temperature field in a flow boundary layer. It expresses the dimensionless thermal transference. The physical significance is based on the idea of a fluid boundary layer in which the heat is transferred by conduction. If it is not so, the criterion loses its significance."

Definition 2.7.7. [29]

"It is the ratio between viscosity υ and molecular diffusion D.It is denoted by S_c. Mathematical form can be written as:

$$S_c = \frac{\upsilon}{D},$$

where

- υ is the kinematic viscosity,
- D is the mass diffusivity."

Chapter 3

MHD Stagnation-point Flow of a Nanofluid Past a Stretching Sheet with Radiation Effects

3.1 Introduction

In this chapter, we discussed a detail review of Nasir et. al.[13]. Here we reinvestigated the governing laws and equations which are helpful in the analysis of forced heat convection over stretching sheet in the presence of magnetic field. Similarity transformations are used to reduce the governing partial differential equation into set of non-linear ordinary differential equation. These equations are then solved numerically using shooting method which followed by the application of RK-4 method and further by utilizing MATLAB tool. At last, the numerical results are discussed for different physical parameters causing impact on the heat and mass transfer of the flow. Graphs are represented to show the physical significance of distinct dimensionless quantities.

3.2 Mathematical Formulation

In the present chapter, we assume a steady two dimensional MHD boundary layer flow of a nanofluid past a stretching surface with the velocity $u_{w(x)} = \lambda U_{w(x)}$, where λ is the constant stretching parameter with $\lambda > 0$ for a stretching surface, as shown in FIGURE 3.1, stretching surface is measured along x-axis. The flow takes place at $y \geq 0$, where y is the coordinate measured normal to the stretching surface.

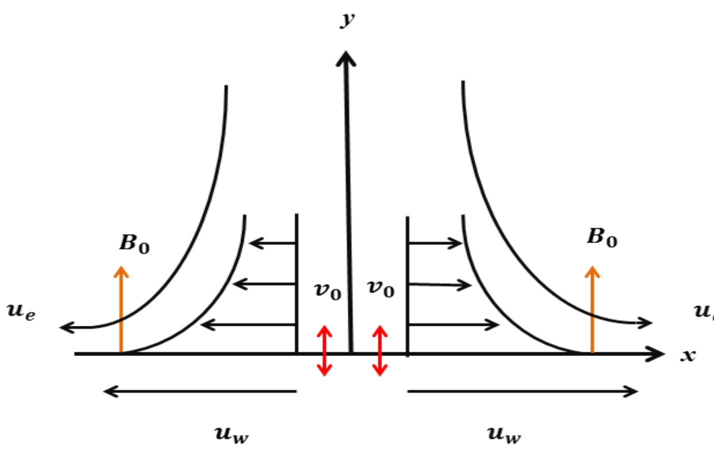

FIGURE 3.1: Geometry of Physical Model for Stretching Sheet.

It is assumed that $u_e(x)$ the velocity of the far field and the bottom surface of the sheet is heated by mean of convection heating, from a hot fluid at constant temperature T_f. T_∞ denotes the constant surface temperature and C_∞ represents concentration of the sheet. Further it is assumed that the constant mass velocity is v_0 with $v_0 < 0$ for suction and the flow is subjected to a constant transverse magnetic field of constant strength B_0, which is supposed to be applied in the positive y-direction, normal to the surface. The applied magnetic field is assumed to be greater as compared to the induced magnetic field, so induced magnetic field is neglected. The base fluid and suspended nanoparticles are in thermal equilibrium [30, 31].

Under the boundary layer assumptions, the basic unsteady equations can be written in the Cartesian coordinates as follows:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{3.1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = u_e\frac{du_e}{dx} + \nu\frac{\partial^2 u}{\partial y^2} + \frac{\sigma B_0^2}{\rho}(u_e - u), \tag{3.2}$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha_f\frac{\partial^2 T}{\partial y^2} + \beta\left[D_B\frac{\partial C}{\partial y}\frac{\partial T}{\partial y} + \frac{D_T}{T_\infty}\left(\frac{\partial T}{\partial y}\right)^2\right] - \frac{1}{\rho}\frac{\partial q_r}{\partial y}, \tag{3.3}$$

$$\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} = D_B\frac{\partial^2 C}{\partial y^2} + \frac{D_T}{T_\infty}\frac{\partial^2 T}{\partial y^2}, \tag{3.4}$$

along with the initial and boundary conditions

$$\left.\begin{aligned} & t < 0: \;\; u = 0, \;\; v = 0, \;\; T = T_\infty, \;\; C = C_\infty \text{ for any } x, y \\ & t \geq 0: \;\; u_w(x) = \lambda U_w(x), \;\; v = v_0, \;\; -k\frac{\partial T}{\partial y} = h_f(T_w - T), \\ & D_B\frac{\partial C}{\partial y} + \frac{D_T}{T_\infty}\frac{\partial T}{\partial y} = 0 \text{ at } y = 0, \\ & u \to u_e(x), \;\; T \to T_\infty, C \to C_\infty \text{ as } y \to \infty, \end{aligned}\right\} \tag{3.5}$$

where u and v are the velocity components along x and y axis, respectively, t is the time, T is the nanofluid temperature, C is the nanoparticle volume fraction, α_f is the thermal diffusivity of the nanofluid, ν is the kinematic viscosity of the fluid, ρ is the density, k is the thermal conductivity, σ is the electrical conductivity, D_B is the Brownian diffusion coefficient, D_T is the thermophoretic diffusion coefficient, β is defined as $\beta = \frac{(\rho c_p)_p}{(\rho c_p)_f}$, where $(\rho c_p)_p$ is the effective heat capacity of the nanoparticle material and $(\rho c_p)_f$ is the heat capacity of fluid and we also assumes that $U_w(x) = ax$ and $u_e(x) = ax$ with a a positive constant. The boundary condition

$$D_B\frac{\partial C}{\partial y} + \left(\frac{D_T}{T_\infty}\right)\frac{\partial T}{\partial y} = 0, \qquad \text{at } y = 0.$$

In (3.5), it is considered that the thermophoresis and the normal flux of nanoparticles is zero at the boundary [32].

Using Rosseland approximation for radiation [33], we have

$$q_r = -\frac{4\sigma^* T_\infty^4}{3k^*}\frac{\partial T}{\partial y} \tag{3.6}$$

For smaller value of temperature contrast, the temperature difference T^4 might be expanded about T_∞ using Taylor series, as follows:

$$T^4 = T_\infty^4 + \frac{4T_\infty^3}{1!}(T - T_\infty)^1 + \frac{12T_\infty^2}{2!}(T - T_\infty)^2 + \frac{24T_\infty}{3!}(T - T_\infty)^3 + \cdots,$$

excluding the higher order terms, we get:

$$T^4 = T_\infty^4 + \frac{4T_\infty^3}{1!}(T - T_\infty)^1.$$

Then,

$$\frac{\partial T^4}{\partial y} = 4T_\infty^3\frac{\partial T}{\partial y}. \tag{3.7}$$

Using (3.6) and (3.7), then differentiating with respect to y, we get

$$\frac{\partial q_r}{\partial y} = -\frac{16\sigma^* T_\infty^3}{3k^*}\frac{\partial^2 T}{\partial y^2}, \tag{3.8}$$

where σ^* represents the Stefan Boltzmann constant and k^* denotes the mean absorption coefficient. Thus (3.3) can be written as:

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha_f\left(1 + \frac{4}{3}Nr\right)\frac{\partial^2 T}{\partial y^2} + \beta\left[D_B\frac{\partial C}{\partial y}\frac{\partial T}{\partial y} + \frac{D_T}{T_\infty}\left(\frac{\partial T}{\partial y}\right)^2\right] \tag{3.9}$$

with $Nr = (1 + 4\sigma^* T_\infty^3/kk^*)$ being the radiation parameter. Following similarity transformations are being considered [8]

$$\left.\begin{aligned} &\bullet\ u = axf'(\eta), \qquad &&\bullet\ v = -\sqrt{a\alpha_f}f(\eta),\\ &\bullet\theta(\eta) = \frac{(T - T_\infty)}{(T_w - T_\infty)}, \qquad &&\bullet\ \phi(\eta) = \frac{(C - C_\infty)}{(C_w - C_\infty)},\\ &\bullet\ \eta = \sqrt{\frac{a}{\alpha_f}}\,y. && \end{aligned}\right\} \tag{3.10}$$

The detailed procedure of conversion of partial differential equations from (3.1)-(3.4) into non-dimensional ordinary differential equation is as follows:

- $$\begin{aligned}\frac{\partial \eta}{\partial x} &= \frac{\partial (\eta)}{\partial x}\\ &= \frac{\partial}{\partial x}\left(\sqrt{\frac{a}{\alpha_f}}\ y\right)\\ &= \sqrt{\frac{a}{\alpha_f}}\ \left(\frac{\partial y}{\partial x}\right)\\ &= \sqrt{\frac{a}{\alpha_f}}\ (0)\\ &= 0.\end{aligned}$$

- $$\begin{aligned}\frac{\partial \eta}{\partial y} &= \frac{\partial (\eta)}{\partial y}\\ &= \frac{\partial}{\partial y}\left(\sqrt{\frac{a}{\alpha_f}}\ y\right)\\ &= \sqrt{\frac{a}{\alpha_f}}\ \left(\frac{\partial y}{\partial y}\right)\\ &= \sqrt{\frac{a}{\alpha_f}}\ (1)\\ &= \sqrt{\frac{a}{\alpha_f}}.\end{aligned}$$

- $$\begin{aligned}\frac{\partial \eta}{\partial t} &= \frac{\partial (\eta)}{\partial x}\\ &= \frac{\partial}{\partial t}\left(\sqrt{\frac{a}{\alpha_f}}\ y\right)\\ &= \sqrt{\frac{a}{\alpha_f}}\ \left(\frac{\partial y}{\partial t}\right)\\ &= \sqrt{\frac{a}{\alpha_f}}\ (0)\\ &= 0.\end{aligned}$$

- $$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial}{\partial x}(ax\ f'(\eta))\\ &= a\left(\frac{\partial}{\partial x}(x\ f'(\eta))\right)\end{aligned}$$

$$= a\left(\frac{\partial x}{\partial x} f'(\eta) \ + \ x\frac{\partial f'(\eta)}{\partial x}\right)$$
$$= a\left((1)f'(\eta) \ + \ x \ f''(\eta) \cdot \frac{\partial \eta}{\partial x}\right)$$
$$= a\left(f'(\eta) \ + \ x \ f''(\eta) \cdot (0)\right)$$
$$= a \ f'(\eta).$$

- $$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(ax \ f'(\eta))$$
$$= a\left(\frac{\partial}{\partial y}(x \ f'(\eta))\right)$$
$$= a\left(\frac{\partial x}{\partial y} f'(\eta) \ + \ x\frac{\partial}{\partial y}(f'(\eta))\right)$$
$$= a\left((0)f'(\eta) \ + \ x \ f''(\eta) \cdot \frac{\partial \eta}{\partial y}\right)$$
$$= ax \ \sqrt{\frac{a}{\alpha_f}} f''(\eta).$$

- $$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x} \ (-\sqrt{a\alpha_f} \ f'(\eta))$$
$$= -\sqrt{a\alpha_f} \ \frac{\partial}{\partial x} \ (f'(\eta))$$
$$= -\sqrt{a\alpha_f} \ f''(\eta) \ \left(\frac{\partial \eta}{\partial x}\right)$$
$$= -\sqrt{a\alpha_f} \ f''(\eta) \ (0)$$
$$= 0.$$

- $$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}(-\sqrt{a\alpha_f} \ f'(\eta))$$
$$= -\sqrt{a\alpha_f} \ \frac{\partial}{\partial y} \ (f'(\eta))$$
$$= -\sqrt{a\alpha_f} \ f''(\eta) \ \left(\frac{\partial \eta}{\partial y}\right)$$
$$= -\sqrt{a\alpha_f} \ f'' \ \left(\sqrt{\frac{a}{\alpha_f}}\right)$$
$$= -\sqrt{a\alpha_f} \ \left(\sqrt{\frac{a}{\alpha_f}}\right) \ f''(\eta)$$
$$= -a \ f''(\eta).$$

The equation of continuity (3.1) is satisfied as follows:

$$\begin{aligned}\Rightarrow \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= af' - af' \\ &= 0.\end{aligned}$$

To convert (3.2) into dimensionless form, the following partial derivatives are converted into ordinary derivatives:

- $$\begin{aligned} u\frac{\partial u}{\partial x} &= (axf'(\eta))(af'(\eta)) \\ &= a^2xf'^2(\eta).\end{aligned}$$

- $$\begin{aligned} v\frac{\partial u}{\partial y} &= (-\sqrt{a\alpha_f}f)\left(ax\ \sqrt{\frac{a}{\alpha_f}}f''(\eta)\right) \\ &= -ax\ \left(\sqrt{\frac{a\cdot a\cdot \alpha_f}{\alpha_f}}\right)(f(\eta)\ f''(\eta)) \\ &= -ax\ (\sqrt{a^2})(f(\eta)\ f''(\eta)) \\ &= -a^2x\ f(\eta)f''(\eta).\end{aligned}$$

- $$u_e = ax.$$

- $$\begin{aligned} \frac{du_e}{dx} &= \frac{d}{dx}(ax) \\ &= a\ \frac{\partial x}{\partial x} \\ &= a\ (1) \\ &= a.\end{aligned}$$

- $$\begin{aligned} u_e\frac{du_e}{dx} &= (ax)(a) \\ &= a^2x.\end{aligned}$$

- $$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(ax\ \sqrt{\frac{a}{\alpha_f}}f''(\eta)\right)$$

$$
\begin{aligned}
&= a\sqrt{\frac{a}{\alpha_f}}\frac{\partial}{\partial y}(xf''(\eta)) \\
&= a\sqrt{\frac{a}{\alpha_f}}\left(\frac{\partial x}{\partial y}f''(\eta) \ + \ x\frac{\partial}{\partial y}(f''(\eta))\right) \\
&= a\sqrt{\frac{a}{\alpha_f}}\left((0)f''(\eta) \ + \ x \ f'''(\eta)\cdot\frac{\partial \eta}{\partial y}\right) \\
&= a\sqrt{\frac{a}{\alpha_f}}\left(x \ \sqrt{\frac{a}{\alpha_f}}f'''(\eta)\right) \\
&= ax\left(\frac{a}{\alpha_f}\right) \ f''' \\
&= \frac{a^2x}{\alpha_f}f'''(\eta).
\end{aligned}
$$

- $$\nu\frac{\partial^2 u}{\partial y^2} = \nu\frac{a^2x}{\alpha_f}f'''(\eta).$$

- $$
\begin{aligned}
(u_e - u) &= (ax - axf'(\eta)) \\
&= ax(1 - f'(\eta)).
\end{aligned}
$$

- $$\frac{\sigma B_0^2}{\rho}(u_e - u) = \frac{\sigma B_0^2}{\rho}(ax(1 - f'(\eta))).$$

By using above derivatives, dimensionless form of the L.H.S of (3.2) becomes:

$$
\begin{aligned}
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= (\ 0 \ ) + (\ a^2xf'^2(\eta) \ ) + (\ -a^2x \ f(\eta)f''(\eta) \ ) \\
&= a^2xf'^2(\eta) - a^2x \ f(\eta)f''(\eta) \\
&= a^2x(f'^2(\eta) - f(\eta)f''(\eta)).
\end{aligned}
$$

Likewise the R.H.S is as follows:

$$
\begin{aligned}
u_e\frac{du_e}{dx} \ + \ \nu\frac{\partial^2 u}{\partial y^2} \ + \ \frac{\sigma B_0^2}{\rho}(u_e - u) &= a^2x + \nu\frac{a^2x}{\alpha_f}f'''(\eta) + \frac{\sigma B_0^2}{\rho}(ax(1 - f'(\eta))) \\
&= a^2x\left(1 + \frac{\nu}{\alpha_f}f'''(\eta) + \frac{\sigma B_0^2}{a\rho}((1 - f'(\eta)))\right) \\
&= a^2x(1 + Prf'''(\eta) + M(1 - f'(\eta)))). \quad (3.11)
\end{aligned}
$$

Now finally the dimensionless form of (3.2) is:

$$\begin{aligned}
&\Rightarrow \quad a^2x(f'^2(\eta) - f(\eta)f''(\eta)) = a^2x(1 + Prf'''(\eta) + M(1 - f'(\eta))) \\
&\Rightarrow \quad f'^2(\eta) - f(\eta)f''(\eta) = 1 + Prf'''(\eta) + M(1 - f'(\eta)) \\
&\Rightarrow \quad Prf'''(\eta) + f(\eta)f''(\eta) + M(1 - f'(\eta)) + 1 - f'^2(\eta) = 0.
\end{aligned}$$

To convert the temperature equation (3.9) into an ordinary differential equation, we first calculate the following derivatives:

- $\theta(\eta) = \dfrac{(T - T_\infty)}{(T_w - T_\infty)}.$

- $T = T_\infty \ + \ (T_w \ + \ T_\infty)\ \theta(\eta).$

- $$\begin{aligned}
\frac{\partial T}{\partial t} &= \frac{\partial}{\partial t}\left(T_\infty \ + \ (T_w \ + \ T_\infty)\ \theta(\eta) \right) \\
&= \frac{\partial}{\partial t}\left(T_\infty \right) \ + \ \frac{\partial}{\partial t}\left((T_w \ + \ T_\infty)\ \theta(\eta) \right) \\
&= (0) \ + \ (T_w \ - \ T_\infty)\ \frac{\partial}{\partial t}\left(\theta(\eta)\right) \\
&= (T_w \ - \ T_\infty)\ \theta'(\eta) \left(\frac{\partial \eta}{\partial t}\right) \\
&= (T_w \ - \ T_\infty)\ \theta'(\eta)\ (0) \\
&= 0.
\end{aligned}$$

- $$\begin{aligned}
\frac{\partial T}{\partial x} &= \frac{\partial}{\partial x}\left(T_\infty \ + \ (T_w \ + \ T_\infty)\ \theta(\eta) \right) \\
&= \frac{\partial}{\partial x}\left(T_\infty \right) \ + \ \frac{\partial}{\partial x}\left((T_w \ + \ T_\infty)\ \theta(\eta) \right) \\
&= (0) \ + \ (T_w \ - \ T_\infty)\ \frac{\partial}{\partial x}\left(\theta(\eta)\right) \\
&= (T_w \ - \ T_\infty)\ \theta'(\eta) \left(\frac{\partial \eta}{\partial x}\right) \\
&= (T_w \ - \ T_\infty)\ \theta'(\eta)\ (0) \\
&= 0.
\end{aligned}$$

- $$\begin{aligned} u\frac{\partial T}{\partial x} &= (axf'(\eta))\frac{\partial}{\partial x}\ (\ T_\infty\ +\ (T_w\ +\ T_\infty)\ \theta(\eta)\ ) \\ &= (axf'(\eta))(0) \\ &= 0. \end{aligned}$$

- $$\begin{aligned} \frac{\partial T}{\partial y} &= \frac{\partial}{\partial y}\ (\ T_\infty\ +\ (T_w\ +\ T_\infty)\ \theta(\eta)\ ) \\ &= \frac{\partial}{\partial y}\ (\ T_\infty\ )\ +\ \frac{\partial}{\partial y}\ ((T_w\ +\ T_\infty)\ \theta(\eta)\ ) \\ &= (0)\ +\ (T_w\ -\ T_\infty)\ \frac{\partial}{\partial y}\ (\theta(\eta)\ ) \\ &= (T_w\ -\ T_\infty)\ \theta'(\eta)\ \left(\frac{\partial \eta}{\partial y}\right) \\ &= (T_w\ -\ T_\infty)\ \theta'(\eta)\ \left(\sqrt{\frac{a}{\alpha_f}}\right) \\ &= (T_w\ -\ T_\infty)\ \left(\sqrt{\frac{a}{\alpha_f}}\right)\ \theta'(\eta). \end{aligned}$$

- $$\begin{aligned} v\frac{\partial T}{\partial y} &= (-\sqrt{a\alpha_f}f(\eta))\frac{\partial}{\partial y}\ (\ T_\infty\ +\ (T_w\ +\ T_\infty)\ \theta(\eta)\ ) \\ &= -\sqrt{a\alpha_f}f(\eta)\left(\sqrt{\frac{a}{\alpha_f}}(T_w - T_\infty)\theta'(\eta)\right) \\ &= -\left(\sqrt{a\alpha_f}\ \sqrt{\frac{a}{\alpha_f}}\right)f(\eta)(T_w - T_\infty)\theta'(\eta) \\ &= \left(\sqrt{\frac{a^2\alpha_f}{\alpha_f}}\right)(T_w - T_\infty)f(\eta)\theta'(\eta) \\ &= -a(T_w - T_\infty)f(\eta)\theta'(\eta). \end{aligned}$$

- $$\begin{aligned} \frac{\partial^2 T}{\partial y^2} &= \frac{\partial}{\partial y}\ \left((T_w\ -\ T_\infty)\ \left(\sqrt{\frac{a}{\alpha_f}}\right)\ \theta'(\eta)\right) \\ &= (T_w\ -\ T_\infty)\ \left(\sqrt{\frac{a}{\alpha_f}}\right)\ \frac{\partial}{\partial y}\ (\theta'(\eta)\ ) \\ &= (T_w\ -\ T_\infty)\ \left(\sqrt{\frac{a}{\alpha_f}}\right)\ \theta''(\eta)\ \left(\frac{\partial \eta}{\partial y}\right) \\ &= (T_w\ -\ T_\infty)\ \left(\sqrt{\frac{a}{\alpha_f}}\right)\ \theta''(\eta)\ \left(\sqrt{\frac{a}{\alpha_f}}\right) \\ &= \frac{a}{\alpha_f}\ (T_w\ -\ T_\infty)\ \theta''. \end{aligned}$$

$$\bullet \left(\frac{\partial T}{\partial y}\right)^2 = \left((T_w - T_\infty)\left(\sqrt{\frac{a}{\alpha_f}}\right)\ \theta'(\eta)\right)^2$$
$$= (T_w - T_\infty)^2 \left(\sqrt{\frac{a}{\alpha_f}}\right)^2 (\theta'(\eta))^2$$
$$= \frac{a}{\alpha_f}(T_w - T_\infty)^2\,\theta'^2(\eta).$$

$$\bullet \quad \phi(\eta) = \frac{(C - C_\infty)}{(C_w - C_\infty)}.$$

$$\bullet \quad \frac{\partial C}{\partial y} = \frac{\partial}{\partial y}\left(C_\infty + (C_w + C_\infty)\,\phi(\eta) \right)$$
$$= \frac{\partial}{\partial y}(C_\infty) + \frac{\partial}{\partial y}((C_w + C_\infty)\,\phi(\eta))$$
$$= (0) + (C_w - C_\infty)\frac{\partial}{\partial y}(\phi(\eta))$$
$$= (C_w - C_\infty)\,\phi'(\eta)\left(\frac{\partial \eta}{\partial y}\right)$$
$$= (C_w - C_\infty)\,\phi'(\eta)\left(\sqrt{\frac{a}{\alpha_f}}\right)$$
$$= (C_w - C_\infty)\left(\sqrt{\frac{a}{\alpha_f}}\right)\phi'(\eta).$$

Now using the above derivatives, L.H.S of (3.9) is processed as follows:

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = (0) + (0) - \sqrt{a\alpha_f}f(\eta)\left(\sqrt{\frac{a}{\alpha_f}}(T_w - T_\infty)\theta'(\eta)\right)$$
$$= -a(T_w - T_\infty)f(\eta)\theta'(\eta). \tag{3.12}$$

Similarly R.H.S of (3.9), can be formulated as:

$$\alpha_f\left(1 + \frac{4}{3}Nr\right)\frac{\partial^2 T}{\partial y^2} + \beta\left[D_B\frac{\partial C}{\partial y}\frac{\partial T}{\partial y} + \frac{D_T}{T_\infty}\left(\frac{\partial T}{\partial y}\right)^2\right]$$
$$= \alpha_f\left(1 + \frac{4}{3}Nr\right)\left(\frac{a}{\alpha_f}\right)(T_w - T_\infty)\theta''(\eta) + \beta\left[D_B\left(\frac{a}{\alpha_f}\right)(T_w - T_\infty)(C_w - C_\infty)\phi'(\eta)\theta'(\eta) + \frac{D_T}{T_\infty}\left(\frac{a}{\alpha_f}\right)(T_w - T_\infty)^2\theta'^2(\eta)\right]$$
$$= a\left(1 + \frac{4}{3}Nr\right)(T_w - T_\infty)\theta''(\eta) + \beta\left(\frac{a}{\alpha_f}\right)\left[D_B(T_w - T_\infty)\right.$$

$$(C_w - C_\infty)\phi'(\eta)\theta'(\eta) + \frac{D_T}{T_\infty}(T_w - T_\infty)^2\theta'^2(\eta)\Bigg]$$
$$= a(T_w - T_\infty)\Bigg[\left(1 + \frac{4}{3}Nr\right)\theta''(\eta) + \frac{\beta D_B}{\alpha_f}(C_w - C_\infty)\theta'(\eta)\phi'(\eta)$$
$$+ \frac{\beta D_T}{T_\infty \alpha_f}(T_w - T_\infty)\theta'^2\Bigg]. \quad (3.13)$$

From (3.12) and(3.13) we get:

$$\Rightarrow \quad - a(T_w - T_\infty)f(\eta)\theta'(\eta) = a(T_w - T_\infty)\Bigg[\left(1 + \frac{4}{3}Nr\right)\theta''(\eta) + \frac{\beta D_B}{\alpha_f}(C_w - C_\infty)\theta'(\eta)\phi'(\eta) + \frac{\beta D_T}{T_\infty \alpha_f}(T_w - T_\infty)\theta'^2(\eta)\Bigg]$$
$$\Rightarrow \quad - f(\eta)\theta'(\eta) = \left(1 + \frac{4}{3}Nr\right)\theta''(\eta) + \frac{\beta D_B}{\alpha_f}(C_w - C_\infty)\theta'(\eta)\phi'(\eta) + \frac{\beta D_T}{T_\infty \alpha_f}(T_w - T_\infty)\theta'^2(\eta)$$
$$\Rightarrow \quad \left(1 + \frac{4}{3}Nr\right)\theta''(\eta) + Nb\theta'(\eta)\phi'(\eta) + Nt\theta'^2(\eta) + f(\eta)\theta'(\eta) = 0.$$

The dimensionless quantities used in the above equation are formulated as:

$$Nb = \frac{\beta D_B(C_w - C_\infty)}{\alpha_f},$$
$$Nt = \frac{\beta D_T(T_w - T_\infty)}{T_\infty \alpha_f}.$$

The following procedure elaborates the conversion of concentration equation (3.4) into the dimensionless form:

- $$\frac{\partial C}{\partial t} = \frac{\partial}{\partial t}\left(C_\infty + (C_w + C_\infty)\,\phi(\eta) \right)$$
$$= \frac{\partial}{\partial t}(C_\infty) + \frac{\partial}{\partial t}((C_w + C_\infty)\,\phi(\eta))$$
$$= (0) + (C_w - C_\infty)\frac{\partial}{\partial t}(\phi(\eta))$$
$$= (C_w - C_\infty)\,\phi'(\eta)\left(\frac{\partial \eta}{\partial t}\right)$$
$$= (C_w - C_\infty)\,\phi'(\eta)\,(0)$$
$$= 0.$$

- $$\begin{aligned}\frac{\partial C}{\partial x} &= \frac{\partial}{\partial x}\ (\ C_\infty\ +\ (C_w\ +\ C_\infty)\ \phi(\eta)\ ) \\ &= \frac{\partial}{\partial x}\ (\ C_\infty\ )\ +\ \frac{\partial}{\partial x}\ ((C_w\ +\ C_\infty)\ \phi(\eta)\ ) \\ &= (0)\ +\ (C_w\ -\ C_\infty)\ \frac{\partial}{\partial x}\ (\phi(\eta)\ ) \\ &= (C_w\ -\ C_\infty)\ \phi'(\eta)\ \left(\frac{\partial \eta}{\partial x}\right) \\ &= (C_w\ -\ C_\infty)\ \phi'(\eta)\ (0) \\ &= 0.\end{aligned}$$

- $$\begin{aligned}u\frac{\partial C}{\partial x} &= (axf'(\eta))\frac{\partial}{\partial x}\ (\ C_\infty\ +\ (C_w\ +\ C_\infty)\ \phi(\eta)\ ) \\ &= (axf'(\eta))(0) \\ &= 0.\end{aligned}$$

- $$\begin{aligned}v\frac{\partial C}{\partial y} &= (-\sqrt{a\alpha_f}f(\eta))\frac{\partial}{\partial y}\ (\ C_\infty\ +\ (C_w\ +\ C_\infty)\ \phi(\eta)\ ) \\ &= -\sqrt{a\alpha_f}f(\eta)\left(\sqrt{\frac{a}{\alpha_f}}(C_w - C_\infty)\phi'(\eta)\right) \\ &= -\left(\sqrt{a\alpha_f}\ \sqrt{\frac{a}{\alpha_f}}\right)f(\eta)(C_w - C_\infty)\phi'(\eta) \\ &= \left(\sqrt{\frac{a^2\alpha_f}{\alpha_f}}\right)(C_w - C_\infty)f(\eta)\phi'(\eta) \\ &= -a(C_w - C_\infty)f(\eta)\phi'(\eta).\end{aligned}$$

- $$\begin{aligned}\frac{\partial^2 C}{\partial y^2} &= \frac{\partial}{\partial y}\ \left((C_w\ -\ C_\infty)\ \left(\sqrt{\frac{a}{\alpha_f}}\right)\ \phi'(\eta)\right) \\ &= (C_w\ -\ C_\infty)\ \left(\sqrt{\frac{a}{\alpha_f}}\right)\ \frac{\partial}{\partial y}\ (\phi'(\eta)\ ) \\ &= (C_w\ -\ C_\infty)\ \left(\sqrt{\frac{a}{\alpha_f}}\right)\ \phi''(\eta)\ \left(\frac{\partial \eta}{\partial y}\right) \\ &= (C_w\ -\ C_\infty)\ \left(\sqrt{\frac{a}{\alpha_f}}\right)\ \phi''(\eta)\ \left(\sqrt{\frac{a}{\alpha_f}}\right) \\ &= \frac{a}{\alpha_f}\ (C_w\ -\ C_\infty)\ \phi''.\end{aligned}$$

Now, (3.4) can be reformulated by using above derivatives :

$$\begin{aligned}
\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} &= 0 + (axf'(\eta))(0) + (-a(C_w - C_\infty)f(\eta)\phi'(\eta)) \\
&= 0 - a(C_w - C_\infty)f(\eta)\phi'(\eta)) \\
&= -a(C_w - C_\infty)f(\eta)\phi'(\eta). \\
D_B\frac{\partial^2 C}{\partial y^2} + \frac{D_T}{T_\infty}\frac{\partial^2 T}{\partial y^2} &= \frac{D_B\ a}{\alpha_f}(C_w - C_\infty)\phi''(\eta) + \frac{D_T\ a}{T_\infty\ \alpha_f}(T_w - T_\infty)\theta''(\eta) \\
&= \frac{D_B\ a}{\alpha_f}(C_w - C_\infty)\left[\phi''(\eta) + \frac{D_T}{T_\infty\ D_B}\frac{(T_w - T_\infty)}{(C_w - C_\infty)}\theta''(\eta)\right] \\
&= \frac{D_B\ a}{\alpha_f}(C_w - C_\infty)\left[\phi''(\eta) + \frac{Nt}{Nb}\theta''(\eta)\right].
\end{aligned}$$

Combining the above L.H.S and R.H.S of (3.4):

$$\begin{aligned}
\Rightarrow \qquad -a(C_w - C_\infty)f(\eta)\phi'(\eta) &= \frac{D_B\ a}{\alpha_f}(C_w - C_\infty)\left[\phi''(\eta) + \frac{Nt}{Nb}\theta''(\eta)\right] \\
\Rightarrow \qquad -f(\eta)\phi'(\eta) &= \frac{D_B}{\alpha_f}\left[\phi''(\eta) + \frac{Nt}{Nb}\theta''(\eta)\right] \\
\Rightarrow \qquad -\frac{D_B}{\alpha_f}f(\eta)\phi'(\eta) &= \phi''(\eta) + \frac{Nt}{Nb}\theta''(\eta) \\
\Rightarrow \qquad -Lef(\eta)\phi'(\eta) &= \phi''(\eta) + \frac{Nt}{Nb}\theta''(\eta) \\
\Rightarrow \qquad \phi''(\eta) + Lef(\eta)\phi'(\eta) + \frac{Nt}{Nb}\theta''(\eta) &= 0.
\end{aligned}$$

Following parameters are used in expression of momentum equation and concentration equation:

$$Pr = \frac{\nu}{\alpha_f}, \qquad M = \frac{\sigma B_0^2}{a\rho}, \qquad Le = \frac{\alpha_f}{D_B}.$$

The procedure of converting boundary conditions into dimensionless form has been discussed below:

- $v(x,y) = v_0 \quad at \quad y = 0.$

$$\begin{aligned}
-\sqrt{a\alpha_f}f(\eta) &= v_0 \quad at \quad \eta = 0. \\
f(\eta) &= -\frac{v_0}{\sqrt{a\alpha_f}} \quad at \quad \eta = 0.
\end{aligned}$$

$$\Rightarrow \qquad f(\eta) = S.$$

- $$u(x,y) = u_w(x,y) = \lambda U_w(x,y) \quad at \quad y = 0.$$

$$ax\ f'(\eta) = \lambda\ ax \quad at \quad \eta = 0.$$

$$\Rightarrow \qquad f'(\eta) = \lambda.$$

- $$-k\frac{\partial T}{\partial y} = h_f(T_w - T) \quad at \quad y = 0.$$

$$-k\sqrt{\frac{a}{\alpha_f}}(T_w - T_\infty)\theta'(\eta) = h_f[T_w - (T_w - T_\infty)\theta(\eta) - T_\infty] \quad at \quad \eta = 0.$$

$$-k\sqrt{\frac{a}{\alpha_f}}(T_w - T_\infty)\theta'(\eta) = h_f(T_w - T_\infty)(1 - \theta(0))\ at\ \eta = 0.$$

$$-k\sqrt{\frac{a}{\alpha_f}}\theta'(\eta) = h_f[1 - \theta(0)] \quad at \quad \eta = 0.$$

$$\theta'(\eta) = -\frac{h_f}{k}\sqrt{\frac{\alpha_f}{a}}[1 - \theta(0)] \quad at \quad \eta = 0.$$

$$\Rightarrow \qquad \theta'(\eta) = -Bi[1 - \theta(0)]. \qquad \because \left(Bi = \frac{h_f}{k}\sqrt{\frac{\alpha_f}{a}}\right)$$

- $$D_B\frac{\partial C}{\partial y} + \frac{D_T}{T_\infty}\frac{\partial T}{\partial y} = 0 \quad at \quad y = 0.$$

$$D_B\sqrt{\frac{a}{\alpha_f}}(C_w - C_\infty)\phi'(\eta) + \frac{D_T}{T_\infty}\sqrt{\frac{a}{\alpha_f}}(T_w - T_\infty)\theta'(\eta) = 0 \quad at \quad \eta = 0.$$

Multiplying both sides by $\beta\sqrt{\frac{a}{\alpha_f}}$ then taking common α and divide on the right hand side, we get:

$$\beta D_B\sqrt{\frac{a}{\alpha_f}}^2(C_w - C_\infty)\phi'(\eta) + \beta\frac{D_T}{T_\infty}\sqrt{\frac{a}{\alpha_f}}^2(T_w - T_\infty)\theta'(\eta) = 0\left(\beta\sqrt{\frac{a}{\alpha_f}}\right) \quad at \quad \eta = 0.$$

$$\beta D_B\frac{a}{\alpha_f}(C_w - C_\infty)\phi'(\eta) + \beta\frac{D_T}{T_\infty}\frac{a}{\alpha_f}(T_w - T_\infty)\theta'(\eta) = 0 \quad at \quad \eta = 0.$$

$$a\left[\frac{\beta D_B(C_w - C_\infty)}{\alpha_f}\phi'(\eta) + \frac{\beta D_T(T_w - T_\infty)}{T_\infty\alpha_f}\theta'(\eta)\right] = 0 \quad at \quad \eta = 0.$$

$$\frac{\beta D_B(C_w - C_\infty)}{\alpha_f}\phi'(\eta) + \frac{\beta D_T(T_w - T_\infty)}{T_\infty\alpha_f}\theta'(\eta) = 0 \quad at \quad \eta = 0.$$

$$\Rightarrow \qquad Nb\phi'(\eta) + Nt\theta'(\eta) = 0.$$

- $u(x,y) \ \to \ u_e(x) \ \ as \ \ y \ \to \ \infty.$

  $ax \ f(\eta) \ \to \ ax \ \ as \ \ \eta \ \to \ \infty.$

$\Rightarrow \ \ f(\eta) \ \to \ 1.$

- $T \ \to \ T_\infty \ \ as \ \ y \ \to \ \infty.$

  $(T_w - T_\infty)\theta(\eta) + T_\infty \ \to \ T_\infty \ \ as \ \ \eta \ \to \ \infty.$

  $(T_w - T_\infty)\phi(\eta) \ \to \ T_\infty - T_\infty \ \ as \ \ \eta \ \to \ \infty.$

  $(T_w - T_\infty)\theta(\eta) \ \to \ 0 \ \ as \ \ \eta \ \to \ \infty.$

$\Rightarrow \ \ \theta(\eta) \ \to \ 0.$

- $C \ \to \ C_\infty \ \ as \ \ y \ \to \ \infty.$

  $(C_w - C_\infty)\phi(\eta) + T_\infty \ \to \ C_\infty \ \ as \ \ \eta \ \to \ \infty.$

  $(C_w - C_\infty)\phi(\eta) \ \to \ C_\infty - C_\infty \ \ as \ \ \eta \ \to \ \infty.$

  $(C_w - C_\infty)\phi(\eta) \ \to \ 0 \ \ as \ \ \eta \ \to \ \infty.$

$\Rightarrow \ \ \phi(\eta) \ \to \ 0.$

Thus finally, following set of dimensionless ODEs is obtained:

$$Prf'''(\eta) + f(\eta)f''(\eta) + M(1 - f'(\eta)) + 1 - f'^2(\eta) = 0. \tag{3.14}$$

$$\left(1 + \frac{4}{3}Nr\right)\theta''(\eta) + Nb\theta'(\eta)\phi'(\eta) + Nt\theta'^2(\eta) + f(\eta)\theta'(\eta) = 0. \tag{3.15}$$

$$\phi''(\eta) + Lef(\eta)\phi'(\eta) + \frac{Nt}{Nb}\theta''(\eta) = 0. \tag{3.16}$$

The transformed boundary conditions are as follows:

$$\left.\begin{array}{l} \text{for } \eta = 0 \\ f(\eta) = S, \quad f'(\eta) = \lambda, \\ \theta'(\eta) = -Bi[1 - \theta(0)], \quad Nb\phi'(\eta) + Nt\theta'(\eta) = 0, \\ f(\eta) \ \to \ 1, \ \ \theta(\eta) \ \to \ 0, \ \ \phi(\eta) \ \to \ 0 \ \ as \ \ \eta \ \to \ \infty. \end{array}\right\} \tag{3.17}$$

The physical quantities of practical interest are the skin friction coefficient C_f and local Nusselt number Nu_x, which are defined as

$$\left.\begin{aligned} C_f &= \frac{\tau_w}{\rho U_w^2}, \\ Nu_x &= \frac{xq_w}{k(T_f - T_\infty)}, \end{aligned}\right\} \tag{3.18}$$

where τ_w is the surface shear stress and q_w is the surface heat flux, which are repectively given by:

$$\left.\begin{aligned} \tau_w &= \mu\left(\frac{\partial u}{\partial y}\right)_{y=0}, \\ q_w &= -k\left(\frac{\partial T}{\partial y}\right)_{y=0}. \end{aligned}\right\} \tag{3.19}$$

Following steps are carried out for the conversion of (3.18) into dimensionless form, by using the transformation (3.19)

- $$\begin{aligned} \tau_w &= \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} \\ &= \mu\left(ax\sqrt{\frac{a}{\alpha_f}}\, f''(\eta)\right)_{y=0} \\ \Rightarrow \quad &= \mu\left(ax\sqrt{\frac{a}{\alpha_f}}\, f''(0)\right) \end{aligned} \tag{3.20}$$

- $$\begin{aligned} q_w &= -k\left(\frac{\partial T}{\partial y}\right)_{y=0} \\ &= -k\left(\sqrt{\frac{a}{\alpha_f}}(T_f - T_\infty)\theta'(\eta)\right)_{y=0} \\ &= -k\left(\sqrt{\frac{a}{\alpha_f}}(T_f - T_\infty)\theta'(0)\right) \end{aligned} \tag{3.21}$$

Using (3.20) and (3.21) in (3.18), we gets:

$$\bullet \quad C_f = \frac{\tau_w}{\rho U_w^2}$$
$$= \frac{\mu\left(ax\sqrt{\frac{a}{\alpha_f}}\, f''(0)\right)}{\rho U_w^2}$$
$$= \frac{\mu a x}{\rho U_w^2}\sqrt{\frac{a}{\alpha_f}}\; f''(0)$$
$$= \frac{\mu}{\rho U_w}\sqrt{\frac{a}{\alpha_f}}\; f''(0)$$
$$= \frac{\mu}{\rho a x}\sqrt{\frac{a}{\alpha_f}}\; f''(0)$$
$$= \frac{\nu}{a}\sqrt{\frac{\alpha_f}{U_w(x)x}}\; f''(0)$$
$$= Re_x{}^{-\frac{1}{2}} Pr\; f''(0)$$
$$\Rightarrow \quad Re_x^{\frac{1}{2}} Pr^{-1}\, C_f = f''(0). \tag{3.22}$$

$$\bullet \quad Nu_x = \frac{x q_w}{k(T_f - T_\infty)}$$
$$= \frac{-xk\left(\sqrt{\frac{a}{\alpha_f}}(T_f - T_\infty)\theta'(0)\right)}{k(T_f - T_\infty)}$$
$$= -x\sqrt{\frac{a}{\alpha_f}}\;\theta'(\eta)$$
$$= -Re_x{}^{\frac{1}{2}}\;\theta'(\eta)$$
$$\Rightarrow \quad Re_x{}^{-\frac{1}{2}}\; Nu_x = -\theta'(\eta). \tag{3.23}$$

Where

$Re_x = U_w(x)x/\alpha_f$ is a local Reynolds number,

$U_w(x) = a\ x.$

3.3 Solution Methodology

To obtain the numerical solution for the system of ordinary differential equations (3.14) − (3.16) subject to boundary conditions (3.17), shooting method is used.

Initially, the momentum equation (3.14) is solved independently then the results of f are utilized in coupled equations (3.15) and (3.16). Following notations have been considered for further procedure:

$$\begin{aligned} f &= y_1, \\ f' &= y_1' = y_2, \\ f'' &= y_1'' = y_2' = y_3, \\ f''' &= y_1''' = y_2'' = y_3'. \end{aligned}$$

Following system of ODE's is obtained by using above notations in equation(3.14):

$$\left.\begin{aligned} y_1' &= y_2, & y_1(0) &= S, \\ y_2' &= y_3, & y_2(0) &= \lambda, \\ y_3' &= \frac{1}{Pr}\Big[y_2^2 - y_1\, y_3 - M(1-y_2) - 1\Big], & y_3(0) &= r. \end{aligned}\right\} \tag{3.24}$$

In order to achieve approximate numerical results, (3.24) is solved by RK-4 method. The domain of our problem is considered to be bounded i.e. $[0, \eta_\infty]$, where η_∞ is a positive number and for which the variation in the solution is negligible after $\eta = \eta_\infty$. r is assumed as missing condition for the solution of (3.24) such that:

$$\begin{aligned} y_2(\eta, r) &= 1, \\ y_2(\eta, r) - 1 &= 0. \end{aligned} \tag{3.25}$$

Newton's method is applied to get the refined initial guess for the missing condition r which then further utilized for the solution of the algebraic equation (3.25), following iterative scheme is purposed:

$$r^{(i+1)} = r^{(i)} - \left(\left(\frac{\partial y_2}{\partial r}\right)^{-1} (y_2(\eta, r) - 1)\right)_{(\eta^{(i)}, r^{(i)}, \eta_\infty)}$$

To incorporate Newton's method, we further use the following notations:

$$\left.\frac{\partial y_1}{\partial r} = y_4, \ \frac{\partial y_2}{\partial r} = y_5, \ \frac{\partial y_3}{\partial r} = y_6 \quad \right\}" \tag{3.26}$$

As a result of these new notations, the Newton's iterative scheme gets the form:

$$r^{(i+1)} = r^{(i)} - \left(\left(\frac{\partial y_5}{\partial r}\right)^{-1} (y_2(\eta, r) - 1)\right)_{(\eta^{(i)}, r^{(i)}, \eta_\infty)},$$

where i is the number of iterations ($i = 0, 1, 2, 3...$).

Now differentiating the system of three first order ODEs (3.24) with respect to r, we get another system of ODEs, of first order. Hence following system of IVPs is obtained:

$$\begin{aligned}
y_1' &= y_2, & y_1(0) &= S,\\
y_2' &= y_3, & y_2(0) &= \lambda,\\
y_3' &= \frac{1}{Pr}\left(y_2^2 - y_1 y_3 - M(1 - y_2) - 1\right), & y_3(0) &= r,\\
y_4' &= y_5, & y_4(0) &= 0,\\
y_5' &= y_6, & y_5(0) &= 0,\\
y_6' &= -\frac{1}{Pr}\left(2y_2 y_5 - y_1 y_6 - y_4 y_3 + M y_5\right), & y(0) &= 1.
\end{aligned}$$

The required stopping criteria for shooting method is set as follows

$$|y_2(\eta, r) - 1| < \epsilon,$$

where ϵ is finitely small positive number up to 10^-10.

For the numerical solution of (3.15) and (3.16), the missing initial condition for $\theta(0)$ is denoted by l and for $\phi(0)$ is represented by m. Through the following notations have been taken into account.

$$\left.\begin{aligned} &\theta = h_1, \quad \theta' = h_1' = h_2, \quad \theta'' = h_2', \\ &\phi = h_3, \quad \phi' = h_3' = h_4, \quad \phi'' = h_4', \\ &\frac{\partial h_1}{\partial l} = h_5, \quad \frac{\partial h_2}{\partial l} = h_6, \quad \frac{\partial h_3}{\partial l} = h_7, \frac{\partial h_4}{\partial l} = h_8, \\ &\frac{\partial h_1}{\partial m} = h_9, \quad \frac{\partial h_2}{\partial m} = h_{10}, \quad \frac{\partial h_3}{\partial m} = h_{11}, \frac{\partial h_4}{\partial m} = h_{12}. \end{aligned}\right\} \tag{3.27}$$

Incorporating the set of notations (3.27), a system of first order ODEs is achieved that is stated below.

$$\begin{aligned}
&h_1' = h_2, && h_1(0) = l, \\
&h_2' = \frac{-3}{3+4Nr}(y_1 h_2 + Nb h_2 h_6 + Nt h_2^2), && h_2(0) = -Bi(1-l), \\
&h_3' = h_4, && h_3(0) = m, \\
&h_4' = -Le y_1 h_4 - \left(\frac{-3}{3+4Nr}\right)\left(\frac{Nt}{Nb}\right)(y_1 h_2 + Nb h_2 h_6 \\
&\quad + Nt h_2^2), && h_4(0) = \frac{Nt}{Nb} Bi(1-l), \\
&h_5' = h_6, && h_5(0) = 1, \\
&h_6' = \left(\frac{-3}{3+4Nr}\right)(y_1 h_6 + Nb(h_2 h_8 + h_6 h_4) + 2Nt h_2 h_6), && h_6(0) = Bi, \\
&h_7' = h_8, && h_7(0) = 0, \\
&h_8' = -Le y_1 h_8 - \left(\frac{-3Nt}{(3+4Nr)Nb}\right)(y_1 h_6 + Nb(h_2 h_8 + h_6 h_4) \\
&\quad + 2Nt h_2 h_6), && h_8(0) = -\frac{Nt}{Nb} Bi, \\
&h_9' = h_{10}, && h_9(0) = 0, \\
&h_{10}' = \left(\frac{-3}{3+4Nr}\right)(y_1 h_{10} + Nb(h_2 h_{12} + h_{10} h_4) + 2Nt h_2 h_{10}), && h_{10}(0) = 0, \\
&h_{11}' = h_{12}, && h_{11}(0) = 1, \\
&h_{12}' = -Le y_1 h_{12} - \left(\frac{-3}{3+4Nr}\right)\left(\frac{Nt}{Nb}\right)(y_1 h_{10} + Nb(h_2 h_{12} + h_{10} h_4) \\
&\quad + 2Nt h_2 h_{10}), && h_{12}(0) = 0.
\end{aligned}$$

Runge Kutta method of order four is applied for the solution of above initial value problems and the missing conditions are chosen such that

$$\left.\begin{aligned} (h_1(l,m))_{\eta=\eta_\infty} &= 0, \\ (h_3(l,m))_{\eta=\eta_\infty} &= 0. \end{aligned}\right\} \tag{3.28}$$

In order to get the closest initial guess, Newton's method is directed by the iterative formula for the solution of (3.28) as mentioned below:

$$\begin{pmatrix} l^{(i+1)} \\ m^{(i+1)} \end{pmatrix} = \begin{pmatrix} l^{(i)} \\ m^{(i)} \end{pmatrix} - \left(\begin{pmatrix} \frac{\partial h_1}{\partial l} & \frac{\partial h_3}{\partial l} \\ \frac{\partial h_1}{\partial m} & \frac{\partial h_3}{\partial m} \end{pmatrix}^{-1} \begin{pmatrix} h_1 \\ h_3 \end{pmatrix} \right)_{(l^{(i)}, m^{(i)}, \eta_\infty)}.$$

As per new notations (3.27), the Newton's iterative scheme takes the form:

$$\begin{pmatrix} l^{(i+1)} \\ m^{(i+1)} \end{pmatrix} = \begin{pmatrix} l^{(i)} \\ m^{(i)} \end{pmatrix} - \left(\begin{pmatrix} h_5 & h_7 \\ h_9 & h_{11} \end{pmatrix}^{-1} \begin{pmatrix} h_1 \\ h_3 \end{pmatrix} \right)_{(l^{(i)}, m^{(i)}, \eta_\infty)}.$$

The stopping criteria for the shooting method is set as:

$$max\{|h_1(\eta_\infty)|, |h_3(\eta_\infty)|\} < \epsilon,$$

for some very small positive number ϵ. The value of ϵ has been taken as 10^{-10} through out the calculation.

3.4 Results with Discussion

This section is dedicated to elaborate the numerical solution of [13]. To reinvestigate the impacts of various physical parameters on skin-friction coefficient and Nusselt number. The results are illustrated graphically.
For the validation of MATLAB code, the results of $f'(\eta)$, $\theta(\eta)$ and $\phi(\eta)$ are reproduced for the problem discussed by [13]. The set of transformed equations are solved for some values of the governing parameters, namely suction S,

Lewis number Le, magnetic parameter M, Brownian motion parameter Nb, thermophoresis parameter Nt and radiation parameter Nr while Prandlt number Pr and the Biot number Bi are fixed to 6.8 and 0.1 respectively.

In FIGURE 3.2 the effect of magnetic field parameter M on the velocity $f'(\eta)$ of the fluid can be seen. As $\eta \to \infty$, the velocity decreases with increase in M. The impact of strong magnetic field resulting into depressing the motion of fluid and thus decrements into momentum boundary layer caused by the opposing Lorentz force.

In FIGURE 3.3, the influence of the radiation parameter Nr on the temperature profile $\theta(\eta)$ is shown. It is clearly seen that by the increment of Nr, temperature inside the boundary layer increases. The radiation parameter Nr determines the relative contribution of conduction heat transfer to thermal radiation transfer. It is evident that by raising the radiation parameter results in increase of temperature within the boundary layer.

In FIGURE 3.4, the dimensionless nanoparticle fraction profile $\phi(\eta)$ with the influence of Brownian motion parameter Nb is represented. The concentration of fluid shows decreasing behavior as value of Nb raises.

In FIGURE 3.5, the variation of the skin friction coefficient $Re_x^{1/2}Pr^{-1}C_f$ is depicted with λ for various values of suction strength, S. For $0 \leq \lambda < 1$, we can see the increment of $Re_x^{1/2}Pr^{-1}C_f$ as the strength of S is increased. Furthermore, when $\lambda = 1$ for all values of S, the skin-friction coefficient is zero since $f(\eta) = \eta$ is the solution of (3.14) subject to boundary condition (3.17). As for $\lambda > 1$, the value of $Re_x^{1/2}Pr^{-1}C_f$ is decreasing as the S strength is increased.

In FIGURE 3.6, the relationship between of $Re_x^{1/2}Pr^{-1}C_f$ with S is displayed. When the domain for S is increased, the $Re_x^{1/2}Pr^{-1}C_f$ is decreasing as the value of stretching is decreasing.

FIGURE 3.7 represents the variation of $Re_x^{-1/2}Nu_x$ with λ for different values of S. The graph indicates that as the S is increased, the $Re_x^{-1/2}Nu_x$ also increases. The solution domain for $Re_x^{-1/2}Nu_x$ is getting bigger as the S is increased and the value of λ increases. As a result, when the S increases the friction at the fluid-split interface ,the heat transfer rate at the surface increases too.

FIGURE 3.8 depicts the variation of $Re_x^{-1/2}Nu_x$ with S for various values of λ. As the domain for S increased, the $Re_x^{-1/2}Nu_x$ is increasing as the value of λ increased.

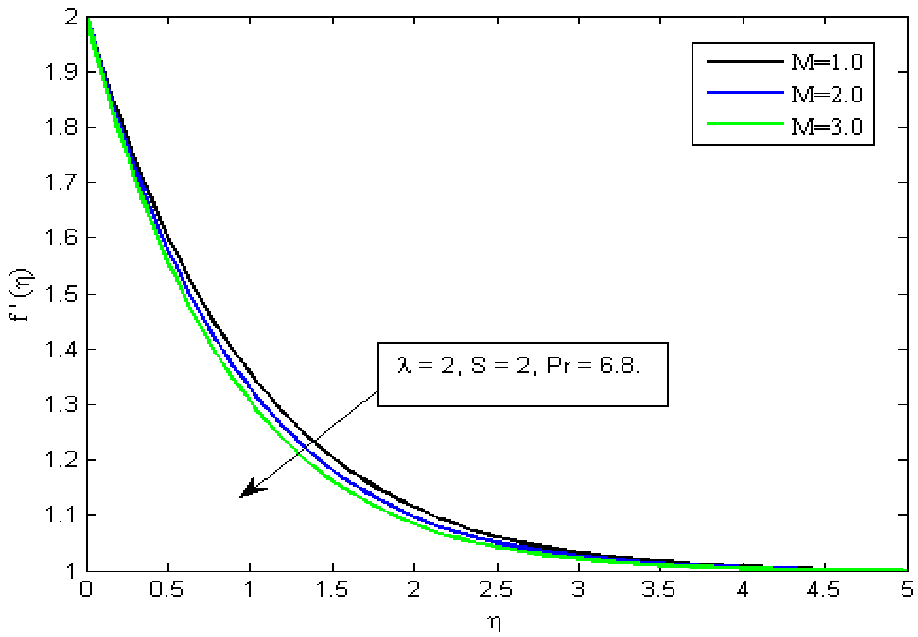

FIGURE 3.2: Velocity Profile $f'(\eta)$ for Various Values of M.

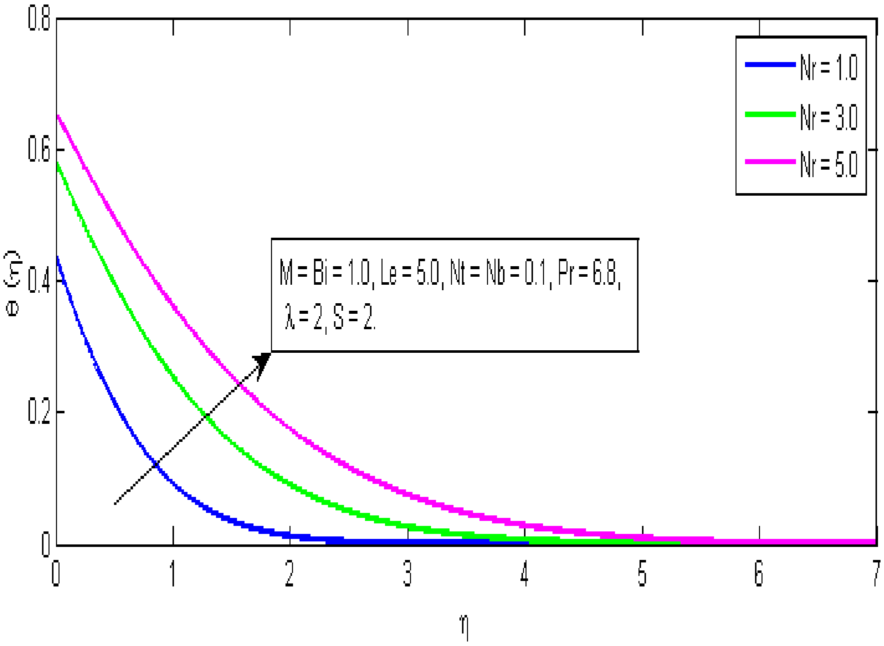

FIGURE 3.3: Temperature Profile $\theta(\eta)$ for Various Values of Nr.

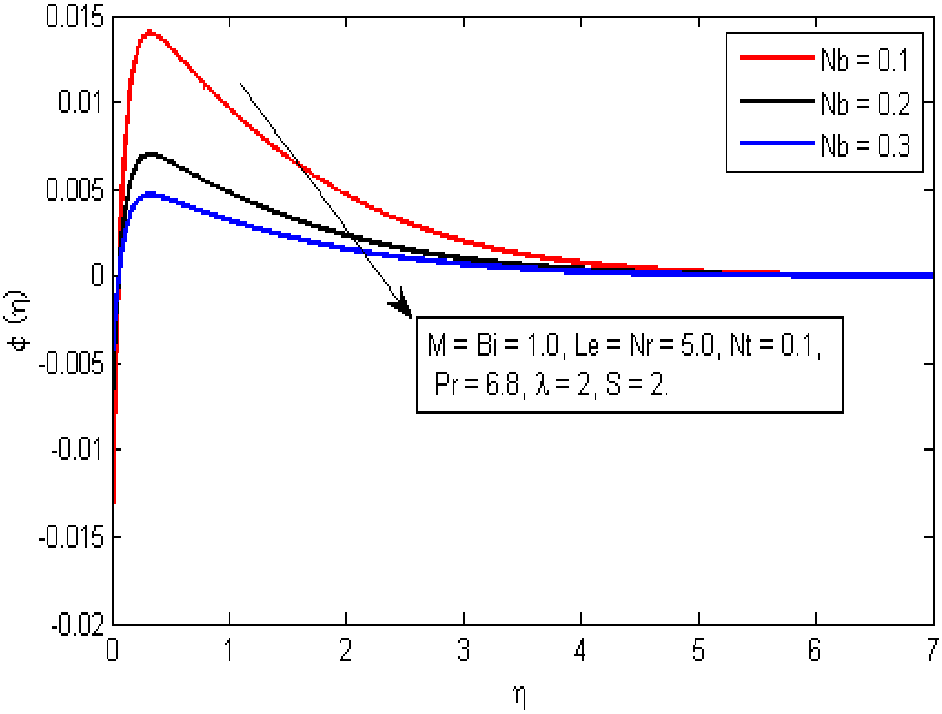

FIGURE 3.4: Nanoparticle Fraction $\phi(\eta)$ for Various Values of Nb.

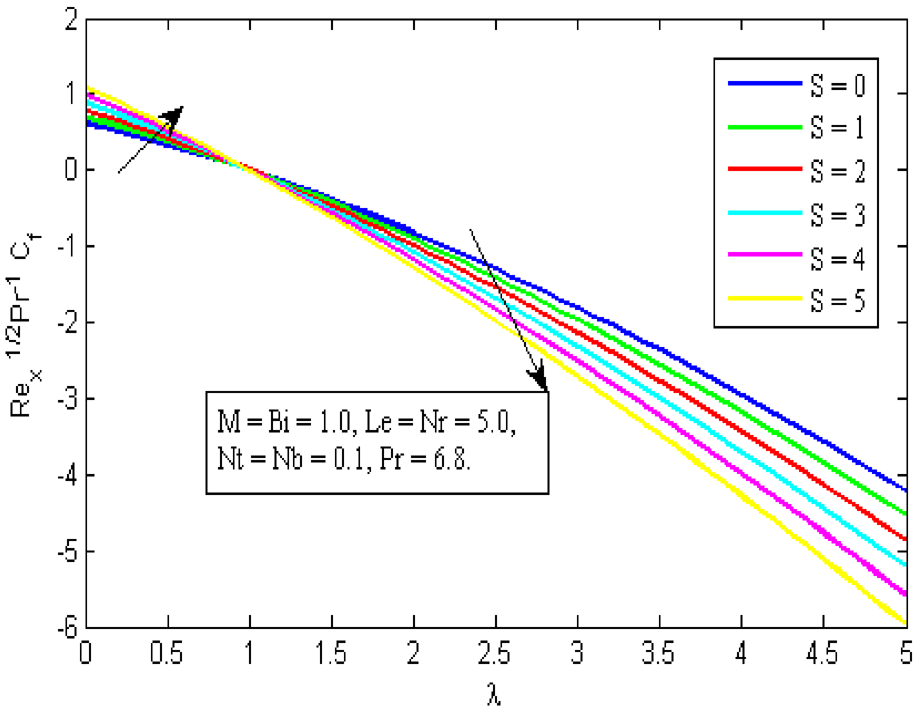

FIGURE 3.5: Variation of the $Re_x^{1/2}Pr^{-1}C_f$ with λ .

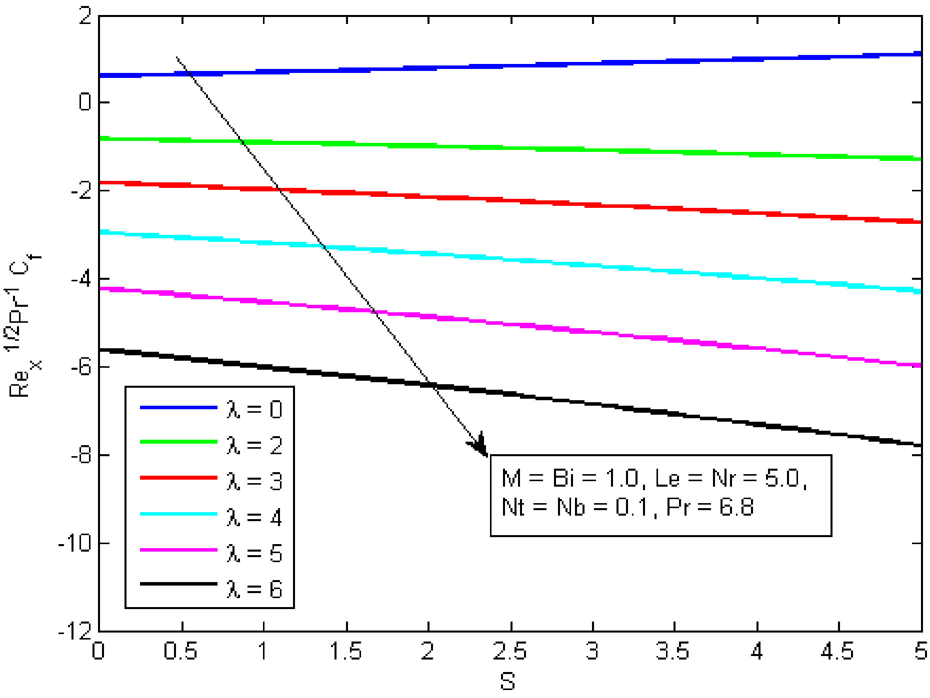

FIGURE 3.6: Variation of the $Re_x^{1/2}Pr^{-1}C_f$ with S .

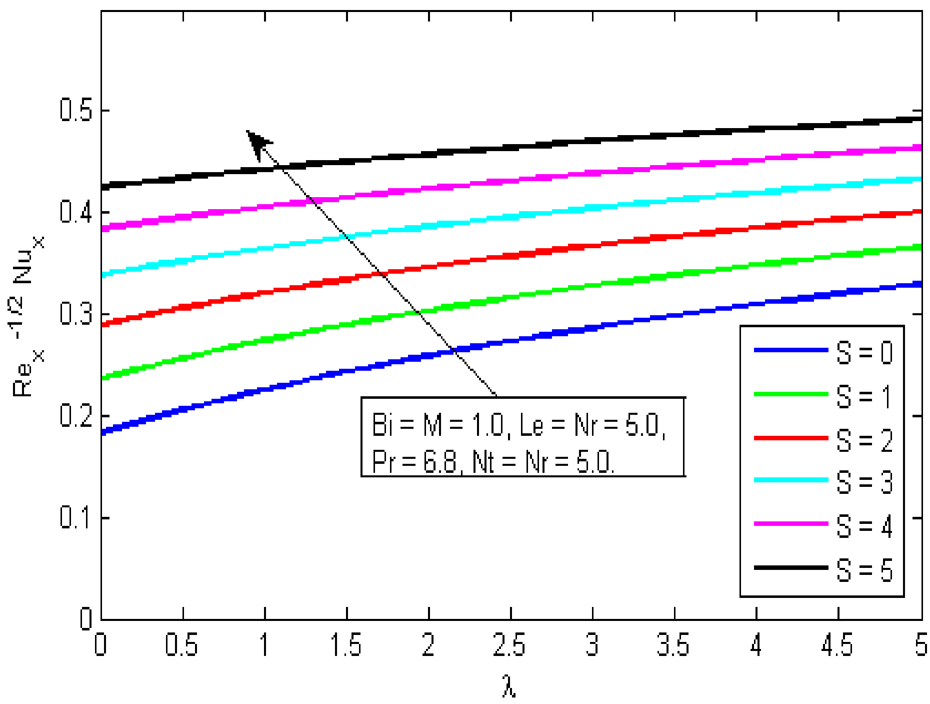

FIGURE 3.7: Variation of the $Re_x^{-1/2}Nu_x$ with λ .

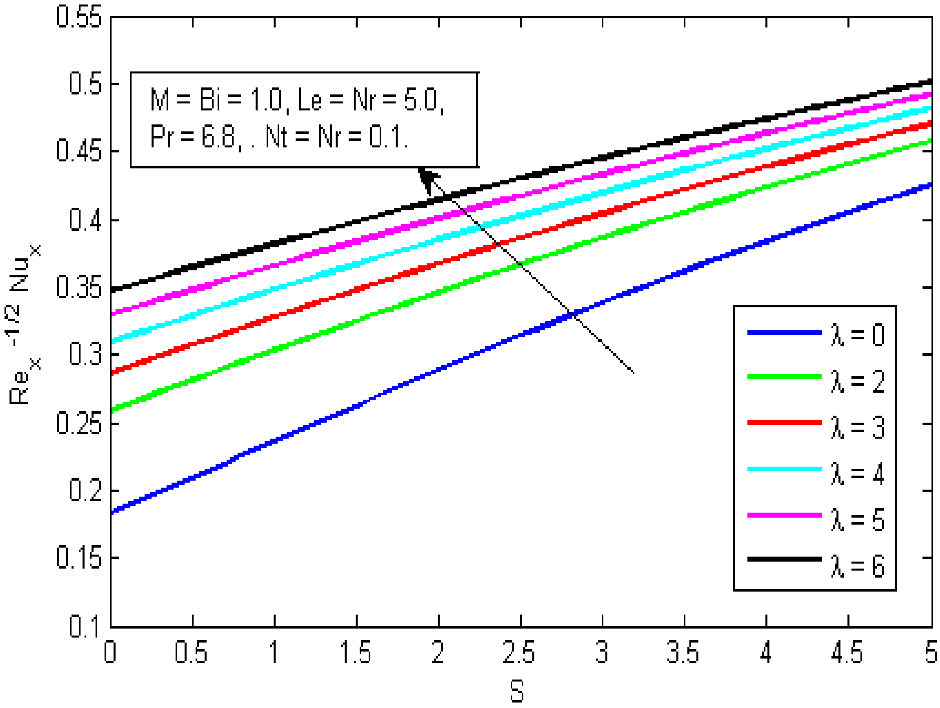

FIGURE 3.8: Variation of the $Re_x^{-1/2}Nu_x$ with S .

Chapter 4

Effects of Chemical Reaction on Nanofluid Flow Past a Stretching Sheet with Thermal Radiations

4.1 Introduction

In this chapter, the work of Nasir et al.[13] is extended by considering MHD stagnation point over a stretching sheet in the presence of chemical reaction with convective boundary conditions and thermal radiation effects. The non-linear partial differential equations of temperature and concentration profiles, are converted into a set of ordinary differential equations by employing helpful similarity transformations. By performing the shooting technique, the numerical solution of transformed governing ordinary differential equations is obtained. Utilizing MATLAB tool, the temperature and concentration profiles are analyzed for pertinent variables. Through graphs the dynamics of various variables of suction parameter, stretching parameter, species diffusivity coefficient and chemical reaction parameter are displayed.

4.2 Mathematical Formulation

We consider a steady two dimensional MHD boundary layer flow of a nanofluid past a stretching surface with the velocity of $u_w(x) = \lambda U_w(x)$, where λ is the stretching parameter, as displayed in the FIGURE 4.1. The flow takes place at $y \geq 0$, that is normal to the stretching surface and along x-axis the stretching surface is measured and a chemical reaction is also taking place as well. It is assumed that $u_e(x)$ is the velocity of the far field. The flow is subjected to a constant transverse, induced magnetic field B_0.

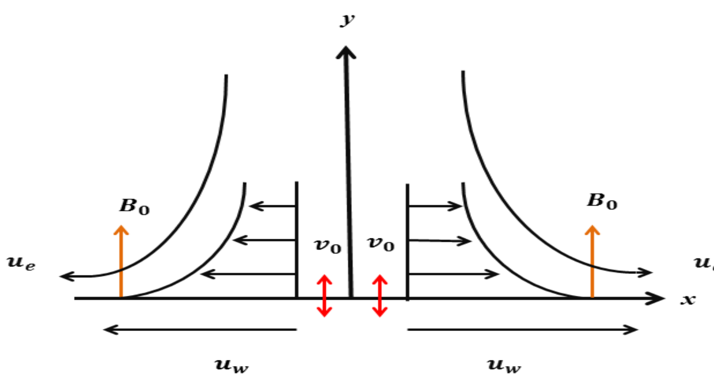

FIGURE 4.1: Geometry of Physical Model for Stretching Sheet.

The basic unsteady equations with the assumed boundary conditions can be written as [19, 20]:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{4.1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = u_e\frac{du_e}{dx} + \nu\frac{\partial^2 u}{\partial y^2} + \frac{\sigma B_0^2}{\rho}(u_e - u), \tag{4.2}$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha_f\frac{\partial^2 T}{\partial y^2} + \beta\left[D_B(C)\frac{\partial C}{\partial y}\frac{\partial T}{\partial y} + \frac{D_T}{T_\infty}\left(\frac{\partial T}{\partial y}\right)^2\right] - \frac{1}{\rho}\frac{\partial q_r}{\partial y}, \tag{4.3}$$

$$\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} = D_B(C)\frac{\partial^2 C}{\partial y^2} + \frac{D_T}{T_\infty}\frac{\partial^2 T}{\partial y^2} + K_r\,(C - C_\infty), \tag{4.4}$$

along with the initial and boundary conditions

$$\left.\begin{array}{l} t<0: \; u = 0, \; v = 0, \; T = T_\infty, \; C = C_\infty \text{ for any } x, y \\ t \geq 0: \; u_w(x) = \lambda U_w(x), \; v = v_0, \; -k\dfrac{\partial T}{\partial y} = h_f(T_w - T), \\ D_B(C)\dfrac{\partial C}{\partial y} + \dfrac{D_T}{T_\infty}\dfrac{\partial T}{\partial y} = 0 \text{ at } y = 0, \\ u \to u_e(x), \; T \to T_\infty, C \to C_\infty \text{ as } y \to \infty. \end{array}\right\} \tag{4.5}$$

The Rosseland approximation can be considered for radiative heat flux. Using Taylor series, we might expand the temperature difference T^4 about T_∞, for smaller values of temperature contrast. The formula for radiative heat flux is as follows:

$$q_r = -\frac{16\sigma^* T_\infty^3}{3k^*}\frac{\partial^2 T}{\partial y^2}. \tag{4.6}$$

Using (4.6), the energy equation (4.3) yields

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha_f\left(1 + \frac{4}{3}Nr\right)\frac{\partial^2 T}{\partial y^2} + \beta\left[D_B(C)\frac{\partial C}{\partial y}\frac{\partial T}{\partial y} + \frac{D_T}{T_\infty}\left(\frac{\partial T}{\partial y}\right)^2\right],$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha_f\left(1 + \frac{4}{3}Nr\right)\frac{\partial^2 T}{\partial y^2} + \beta\left[D_{B_\infty}(1+\epsilon\phi)\frac{\partial C}{\partial y}\frac{\partial T}{\partial y} + \frac{D_T}{T_\infty}\left(\frac{\partial T}{\partial y}\right)^2\right], \tag{4.7}$$

here $Nr = (1 + 4\sigma^* T_\infty^3/kk^*)$ being the radiation parameter. The variable molecular diffusivity is taken in the form as $D_B(C) = D_{B_\infty}(1 + \epsilon\phi(\eta))$, where D_{B_∞} is Brownian diffusion coefficient and ϵ is variable species diffusivity parameter, (see [20]).

Following similarity transformations are used to convert the energy equation (4.7) and concentration equation (4.4) into dimensionless form

$$\left.\begin{array}{ll} u = axf'(\eta), & \theta(\eta) = \dfrac{(T - T_\infty)}{(T_w - T_\infty)}, \\ v = -\sqrt{a\alpha_f}f(\eta), & \phi(\eta) = \dfrac{(C - C_\infty)}{(C_w - C_\infty)}, \\ \eta = \sqrt{\dfrac{a}{\alpha_f}}\,y. & \end{array}\right\} \tag{4.8}$$

The detailed conversion of continuity equation (4.1) and momentum equation (4.2) are already discussed in Chapter 3.

The conversion of (4.7) is presented below. To achieve this goal first L.H.S of (4.7) is transformed as follows:

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = -a(T_w - T_\infty)f(\eta)\theta'(\eta). \tag{4.9}$$

The right hand side of (4.7) is formulated as:

$$\begin{aligned}
\alpha_f\left(1 + \frac{4}{3}Nr\right)\frac{\partial^2 T}{\partial y^2} + \beta\left[D_{B_\infty}(1+\epsilon\phi(\eta))\frac{\partial C}{\partial y}\frac{\partial T}{\partial y} + \frac{D_T}{T_\infty}\left(\frac{\partial T}{\partial y}\right)^2\right] &= \alpha_f\left(1 + \frac{4}{3}Nr\right) \\
&\left(\frac{a}{\alpha_f}\right)(T_w - T_\infty)\theta''(\eta) + \beta\left[D_{B_\infty}(1+\epsilon\phi(\eta))\left(\frac{a}{\alpha_f}\right)(T_w - T_\infty)\right. \\
&\left.(C_w - C_\infty)\phi'(\eta)\theta'\ (\eta) + \frac{D_T}{T_\infty}\left(\frac{a}{\alpha_f}\right)(T_w - T_\infty)^2\theta'^2(\eta)\right] \\
&= a\left(1 + \frac{4}{3}Nr\right)(T_w - T_\infty)\theta''(\eta) + \beta\left(\frac{a}{\alpha_f}\right)\left[D_{B_\infty}(1+\epsilon\phi(\eta))\right. \\
&\left.(T_w - T_\infty)(C_w - C_\infty)\phi'(\eta)\theta'(\eta) + \frac{D_T}{T_\infty}(T_w - T_\infty)^2\theta'^2(\eta)\right] \\
&= a(T_w - T_\infty)\left[\left(1 + \frac{4}{3}Nr\right)\theta''(\eta) + \frac{\beta D_{B_\infty}(1+\epsilon\phi(\eta))}{\alpha_f}\right. \\
&\left.(C_w - C_\infty)\theta'(\eta)\phi'(\eta) + \frac{\beta D_T}{T_\infty \alpha_f}(T_w - T_\infty)\theta'^2\right].
\end{aligned} \tag{4.10}$$

Combining (4.9) and (4.10) we get

$$\begin{aligned}
-a(T_w - T_\infty)f(\eta)\theta'(\eta) &= a(T_w - T_\infty)\left[\left(1 + \frac{4}{3}Nr\right)\theta''(\eta) + \frac{\beta D_{B_\infty}(1+\epsilon\phi(\eta))}{\alpha_f}\right. \\
&\left.(C_w - C_\infty)\theta'(\eta)\phi'(\eta) + \frac{\beta D_T}{T_\infty\alpha_f}(T_w - T_\infty)\theta'^2(\eta)\right] \\
\Rightarrow \quad -f(\eta)\theta'(\eta) &= \left(1 + \frac{4}{3}Nr\right)\theta''(\eta) + \frac{\beta D_{B_\infty}(1+\epsilon\phi(\eta))}{\alpha_f}(C_w - C_\infty)\theta'(\eta) \\
&\phi'(\eta) + \frac{\beta D_T}{T_\infty\alpha_f}(T_w - T_\infty)\theta'^2(\eta) \\
\Rightarrow \quad \left(1 + \frac{4}{3}Nr\right)\theta''(\eta) &+ Nb(1+\epsilon\phi(\eta))\theta'(\eta)\phi'(\eta) + Nt\theta'^2(\eta) + f(\eta)\theta'(\eta) = 0.
\end{aligned}$$

The necessary steps to convert the concentration equation (4.4) into dimensionless form are as follows. The detailed conversion of L.H.S of concentration equation is already discussed in chapter 3.

$$\begin{aligned}\frac{\partial C}{\partial t}+u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}&=0-a(C_w-C_\infty)f(\eta)\phi'(\eta))\\&=-a(C_w-C_\infty)f(\eta)\phi'(\eta).\end{aligned}\tag{4.11}$$

Furtehrmore, the R.H.S of (4.4) is transformed into dimensionless form as shown below.

$$\begin{aligned}D_B(C)\frac{\partial^2 C}{\partial y^2}+\frac{D_T}{T_\infty}\frac{\partial^2 T}{\partial y^2}+K_r(C-C_\infty)&=\frac{D_{B_\infty}(1+\epsilon\phi(\eta))a}{\alpha_f}\,(C_w-C_\infty)\,\phi''(\eta)+\\&\quad\frac{D_T}{T_\infty}\frac{a}{\alpha_f}(T_w-T_\infty)\theta''(\eta)+K_r\,(C-C_\infty)\\&=\frac{D_{B_\infty}\,a}{\alpha_f}(C_w-C_\infty)\Bigg[(1+\epsilon\phi(\eta))\phi''(\eta)+\frac{D_T}{T_\infty\,D_{B_\infty}}\\&\quad\frac{(T_w-T_\infty)}{(C_w-C_\infty)}\theta''(\eta)\Bigg]+\frac{K_r}{D_{B_\infty}}\frac{\alpha_f}{a}\phi(\eta)\\&=\frac{D_{B_\infty}\,a}{\alpha_f}(C_w-C_\infty)\Bigg[(1+\epsilon\phi(\eta))\phi''(\eta)+\frac{Nt}{Nb}\theta''(\eta)\Bigg]+\\&\quad\frac{K_r\alpha_f}{D_{B_\infty}a}\phi(\eta).\end{aligned}\tag{4.12}$$

Now comparing (4.11) and (4.12) we get the following form:

$$\begin{aligned}\Rightarrow\quad -a(C_w-C_\infty)f(\eta)\phi'(\eta)&=\frac{D_{B_\infty}\,a}{\alpha_f}(C_w-C_\infty)\Bigg[(1+\epsilon\phi(\eta))\phi''(\eta)+\frac{Nt}{Nb}\theta''(\eta)\Bigg]\\&\quad+\frac{K_r\alpha_f}{D_{B_\infty}a}\phi(\eta)\\\Rightarrow\quad -\frac{\alpha_f}{D_{B_\infty}}f(\eta)\phi'(\eta)&=(1+\epsilon\phi(\eta))\phi''(\eta)+\frac{Nt}{Nb}\theta''(\eta)+\gamma LeR_{e_x}\phi(\eta)\\\Rightarrow\quad &(1+\epsilon\phi(\eta))\phi''(\eta)+\gamma LeR_{e_x}\phi(\eta)+Lef(\eta)\phi'(\eta)+\frac{Nt}{Nb}\theta''(\eta)=0.\\\Rightarrow\quad &(1+\epsilon\phi(\eta))\phi''(\eta)+\gamma LeR_{e_x}\phi(\eta)+Lef(\eta)\phi'(\eta)+\frac{Nt}{Nb}\left(\frac{-3}{1+4Nr}\right)\\&\Bigg[Nb(1+\epsilon\phi(\eta))\theta'(\eta)\phi'(\eta)+Nt\theta'^2(\eta)+f(\eta)\theta'(\eta)\Bigg]=0.\end{aligned}$$

The dimensionless form of boundary conditions are as follows:

- $v(x,y) = v_0$ at $y = 0$.

$\Rightarrow$ $f(\eta) = S$.

- $u(x,y) = u_w(x,y) = \lambda U_w(x,y)$ at $y = 0$.

$\Rightarrow$ $f'(\eta) = \lambda$.

- $-k\dfrac{\partial T}{\partial y} = h_f(T_w - T)$ at $y = 0$.

$\Rightarrow$ $\theta'(\eta) = -Bi[1 - \theta(0)]$.

- $D_{B_\infty}(1+\epsilon\phi(\eta))\dfrac{\partial C}{\partial y} + \dfrac{D_T}{T_\infty}\dfrac{\partial T}{\partial y} = 0$ at $y = 0$.

$$D_{B_\infty}(1+\epsilon\phi(\eta))\sqrt{\frac{a}{\alpha_f}}(C_w - C_\infty)\phi'(\eta) + \frac{D_T}{T_\infty}\sqrt{\frac{a}{\alpha_f}}(T_w - T_\infty)\theta'(\eta) = 0.$$

$$\beta D_{B_\infty}(1+\epsilon\phi(\eta))\frac{a}{\alpha_f}(C_w - C_\infty)\phi'(\eta) + \beta\frac{D_T}{T_\infty}\frac{a}{\alpha_f}(T_w - T_\infty)\theta'(\eta) = 0$$

$$a\left[\frac{\beta D_{B_\infty}(1+\epsilon\phi(\eta))(C_w - C_\infty)}{\alpha_f}\phi'(\eta) + \frac{\beta D_T(T_w - T_\infty)}{T_\infty\alpha_f}\theta'(\eta)\right] = 0$$

$$\frac{\beta D_{B_\infty}(1+\epsilon\phi(\eta))(C_w - C_\infty)}{\alpha_f}\phi'(\eta) + \frac{\beta D_T(T_w - T_\infty)}{T_\infty\alpha_f}\theta'(\eta) = 0$$

$\Rightarrow$ $$Nb(1+\epsilon\phi(\eta))\phi'(\eta) + Nt\theta'(\eta) = 0 \quad \text{at} \quad \eta = 0.$$

- $u(x,y) \rightarrow u_e(x)$ as $y \rightarrow \infty$.

$\Rightarrow$ $f(\eta) \rightarrow 1$.

- $T \rightarrow T_\infty$ as $y \rightarrow \infty$.

$\Rightarrow$ $\theta(\eta) \rightarrow 0$.

- $C \rightarrow C_\infty$ as $y \rightarrow \infty$.

$\Rightarrow$ $\phi(\eta) \rightarrow 0$.

Hence finally the following two dimensionless ordinary differential equations (4.7) and (4.4) are achieved

$$\left(1+\frac{4}{3}Nr\right)\theta''(\eta)+Nb(1+\epsilon\phi(\eta))\theta'(\eta)\phi'(\eta)+Nt\theta'^2(\eta)+f(\eta)\theta'(\eta)=0. \quad (4.13)$$

$$(1+\epsilon\phi(\eta))\phi''(\eta)+\gamma LeR_{e_x}\phi(\eta)+Lef(\eta)\phi'(\eta)+\left(-\frac{3}{1+4Nr}\right)$$
$$\left(\frac{Nt}{Nb}\right)\left[Nb(1+\epsilon\phi(\eta))\theta'(\eta)\phi'(\eta)+Nt\theta'^2(\eta)+f(\eta)\theta'(\eta)\right]=0. \quad (4.14)$$

Where the notation ($'$) denotes derivative with respect to η and the dimensionless quantities used in (4.13) and (4.14) are given below

$$Nb=\frac{\beta D_{B_\infty}(C_w-C_\infty)}{\alpha_f}, \qquad \text{(Brownian motion parameter)},$$

$$Nt=\frac{\beta D_T(T_w-T_\infty)}{T_\infty\alpha_f}, \qquad \text{(Thermophoresis paramter)},$$

$$R_{e_x}=\frac{U_w(x)x}{\alpha_f}, \qquad \text{(Local Reynolds number)},$$

$$Le=\frac{\alpha_f}{D_{B_\infty}}, \qquad \text{(Lewis number)},$$

$$\gamma=\frac{\alpha_f K_r}{a^2x^2}, \qquad \text{(Chemical reaction parameter)}.$$

The transformed boundary conditions are as follows:

$$\left.\begin{aligned}
&\text{At } \eta=0\\
&\quad f(\eta)=S,\quad f'(\eta)=\lambda,\\
&\quad \theta'(\eta)=-Bi[1-\theta(0)],\quad Nb(1+\epsilon\phi(\eta))\phi'(\eta)+Nt\theta'(\eta)=0,\\
&\quad f(\eta)\ \rightarrow\ 1,\quad \theta(\eta)\ \rightarrow\ 0,\quad \phi(\eta)\ \rightarrow\ 0\ \ as\ \ \eta\ \rightarrow\ \infty.
\end{aligned}\right\} \quad (4.15)$$

The physical quantities skin-friction and Nusselt number are calculated in the similar manner as that of Chapter 3.

- $C_f = \dfrac{\tau_w}{\rho U_w^2}$

$\Rightarrow \quad Re_x{}^{\frac{1}{2}} Pr^{-1}\, C_f = \; f''(0).$

- $Nu_x = \dfrac{x q_w}{k(T_f - T_\infty)}$

$\Rightarrow \quad Re_x{}^{-\frac{1}{2}}\; Nu_x = -\theta'(\eta),$

where

$Re_x = U_w(x)x/\alpha_f$ is a local Reynolds number,

$U_w(x) = a\, x.$

4.3 Solution Methodology

Shooting method is used for the solution of system of coupled ODEs (4.13) and (4.14) subjected to boundary conditions (4.15). In Chapter 3 results of y_1 were already calculated, then we utilize the computed value of y_1 in (4.13) and (4.14). The missing initial conditions are denoted by r and s for the numerical solution of our extended coupled ODEs. Following notation are being considered for further work

$$\left.\begin{aligned}
&\theta = Y_1, \quad \theta' = Y_1' = Y_2, \quad \theta'' = Y_2', \\
&\phi = Y_3, \quad \phi' = Y_3' = Y_4, \quad \phi'' = Y_4', \\
&\frac{\partial Y_1}{\partial r} = Y_5, \quad \frac{\partial Y_2}{\partial r} = Y_6, \quad \frac{\partial Y_3}{\partial r} = Y_7, \quad \frac{\partial Y_4}{\partial r} = Y_8, \\
&\frac{\partial Y_1}{\partial s} = Y_9, \quad \frac{\partial Y_2}{\partial s} = Y_{10}, \quad \frac{\partial Y_3}{\partial s} = Y_{11}, \quad \frac{\partial Y_4}{\partial s} = Y_{12}.
\end{aligned}\right\} \tag{4.16}$$

Using the notations (4.16), (4.13) and (4.14) can be converted into system of twelve first order ODEs as written below

$$
\begin{aligned}
&Y_1' = Y_2, && Y_1(0) = r,\\
&Y_2' = \frac{-3}{3+4Nr}(y_1Y_2 + Nb(1+\epsilon Y_3)Y_2Y_6 + NtY_2^2), && Y_2(0) = -Bi(1-r),\\
&Y_3' = Y_4, && Y_3(0) = s,\\
&Y_4' = \left(\frac{-1}{1+\epsilon Y_3}\right)\Bigg[Ley_1Y_4 + \gamma R_{e_x}LeY_3 + \left(\frac{-3}{3+4Nr}\right)\\
&\quad\left(\frac{Nt}{Nb}\right)(y_1Y_2 + Nb(1+\epsilon Y_3)Y_2Y_4 + NtY_2^2)\Bigg], && Y_4(0) = \frac{Nt}{Nb}Bi(1-r),\\
&Y_5' = Y_6, && Y_5(0) = 1,\\
&Y_6' = \left(\frac{-3}{3+4Nr}\right)(y_1Y_6 + Nb((1+\epsilon Y_3)(Y_2Y_8 + Y_6Y_4)+\\
&\quad\epsilon Y_2Y_4Y_7) + 2NtY_2Y_6), && Y_6(0) = Bi,\\
&Y_7' = Y_8, && Y_7(0) = 0,\\
&Y_8' = \left(\frac{\epsilon Y_7}{(1+\epsilon Y_3)^2}\right)\Bigg[Ley_1Y_4 + \gamma R_{e_x}LeY_3 + \left(\frac{-3}{3+4Nr}\right)\\
&\quad\left(\frac{Nt}{Nb}\right)(y_1Y_2 + Nb(1+\epsilon Y_3)Y_2Y_4 + NtY_2^2)\Bigg] - \left(\frac{1}{1+\epsilon Y_3}\right)\\
&\quad\Bigg[Ley_1Y_8 + \gamma R_{e_x}LeY_7 + \left(\frac{-3}{3+4Nr}\right)\left(\frac{Nt}{Nb}\right)(y_1Y_6+\\
&\quad Nb((1+\epsilon Y_3)(Y_2Y_8 + Y_6Y_4) + \epsilon Y_2Y_4Y_7) + 2NtY_2Y_6), && Y_8(0) = \frac{-Nt}{Nb}Bi,\\
&Y_9' = Y_10, && Y_9(0) = 0,\\
&Y_{10}' = \left(\frac{-3}{3+4Nr}\right)(y_1Y_{10} + Nb((1+\epsilon Y_3)(Y_2Y_{12} + Y_{10}Y_4)+\\
&\quad\epsilon Y_2Y_4Y_{11}) + 2NtY_2Y_{10}), && Y_{10}(0) = 0,\\
&Y_{11}' = Y_{12}, && Y_{11}(0) = 1,\\
&Y_{12}' = \left(\frac{\epsilon Y_{11}}{(1+\epsilon Y_3)^2}\right)\Bigg[Ley_1Y_4 + \gamma R_{e_x}LeY_3 + \left(\frac{-3}{3+4Nr}\frac{Nt}{Nb}\right)\\
&\quad(y_1Y_2 + Nb(1+\epsilon Y_3)Y_2Y_4 + NtY_2^2)\Bigg] - \left(\frac{1}{1+\epsilon Y_3}\right)\Bigg[Ley_1Y_{12}\\
&\quad+\gamma R_{e_x}LeY_{11} + \left(\frac{-3}{3+4Nr}\right)\left(\frac{Nt}{Nb}\right)(y_1Y_{10} + Nb((1+\epsilon Y_3)\\
&\quad(Y_2Y_{12} + Y_{10}Y_4) + \epsilon Y_2Y_4Y_{11}) + 2NtY_2Y_{10}), && Y_{12}(0) = 0.
\end{aligned}
$$

Runge Kutta method of order four is utilized for the solution of the above initial value problems. Missing conditions are chosen for the above IVPs, such that

$$\left.\begin{aligned} (Y_1(r,s))_{\eta=\eta_\infty} &= 0, \\ (Y_3(r,s))_{\eta=\eta_\infty} &= 0. \end{aligned}\right\} \tag{4.17}$$

The set of algebraic equation (4.17) are solved by using Newton's method which is governed by the iterative formula as given below:

$$\begin{pmatrix} r^{(i+1)} \\ s^{(i+1)} \end{pmatrix} = \begin{pmatrix} r^{(i)} \\ s^{(i)} \end{pmatrix} - \left(\begin{pmatrix} \frac{\partial Y_1}{\partial r} & \frac{\partial Y_3}{\partial r} \\ \frac{\partial Y_1}{\partial s} & \frac{\partial Y_3}{\partial s} \end{pmatrix}^{-1} \begin{pmatrix} Y_1 \\ Y_3 \end{pmatrix}\right)_{(l^{(i)},s^{(i)},\eta_\infty)}.$$

As per the notations (4.16), the Newton's iterative scheme takes the following form for $i = 1, 2, 3...$:

$$\begin{pmatrix} r^{(i+1)} \\ s^{(i+1)} \end{pmatrix} = \begin{pmatrix} r^{(i)} \\ s^{(i)} \end{pmatrix} - \left(\begin{pmatrix} Y_5 & Y_7 \\ Y_9 & Y_{11} \end{pmatrix}^{-1} \begin{pmatrix} Y_1 \\ Y_3 \end{pmatrix}\right)_{(r^{(i)},s^{(i)},\eta_\infty)}.$$

The required stopping criterion for the shooting method is set as:

$$max\{|Y_1(\eta_\infty)|, |Y_3(\eta_\infty)|\} < \epsilon,$$

for some very small positive number ϵ. The value of ϵ has been taken as 10^{-10}.

4.4 Results and Discussion

The key objective of this section, is to investigate the effect of various dimensionless parameters on temperature profile $\theta(\eta)$ and concentration profile $\phi(\eta)$ of the flow. The transformed ordinary differential equations (4.13) and (4.14) along with the boundary conditions (4.15) are numerically solved by using shooting method. By assuming different values for distinct physical parameters, the numerical solutions of skin-friction and Nusselt number are illustrated by graphs.

The MATLAB code is varified by the results of $\theta(\eta)$ and $\phi(\eta)$. The coupled equations along with extended terms are solved for some values of the governing parameters, namely suction S, Lewis number Le, magnetic parameter M, Brownian motion parameter Nb, chemical reaction parameter γ, species diffusitivity parameter ϵ, thermophoresis parameter Nt and radiation parameter Nr while Prandlt number Pr and the Biot number Bi are fixed.

FIGURE 4.2 depicts the behavior of concentration profile $\theta(\eta)$ while varying the radiation parameter parameter Nr. It is clearly visible that by the increment of Nr, the temperature inside the boundary layer increases. The radiation parameter Nr helps to determine the relative contribution of conduction heat transfer to thermal radiation transfer. It is noticeable that an increase in the radiation parameter results in increasing temperature within the boundary layer.

In FIGURE 4.3, the dimensionless nanoparticle fraction profile $\phi(\eta)$ with the influence of Brownian motion parameter Nb is represented. The concentration of fluid shows decreasing behavior as value of Nb raises.

FIGURE 4.4 represents the variation of $Re_x^{-1/2}Nu_x$ with λ for different values of S. The graph indicates that as the suction parameter S is increased, the $Re_x^{-1/2}Nu_x$ also increases. The solution domain for $Re_x^{-1/2}Nu_x$ is getting bigger as the S is increased and the value of λ increases. As a result, when the S increases the friction at the fluid-splid interface , the rate of heat transfer at the surface increases as well. FIGURE 4.5 depicts the variation of $Re_x^{-1/2}Nu_x$ with S for various values of λ. As the domain for S increased, the $Re_x^{-1/2}Nu_x$ is increasing as the value of λ increased.

FIGURE 4.6 and FIGURE 4.7 reveals the variation of $\theta(\eta)$ and $\phi(\eta)$ with γ, respectively. By increasing the chemical reaction parameter their is efficient increase in both temperature and concentration profile.

FIGURE 4.8 and FIGURE 4.9 depicts the changing of temperature profile and concentration profile with species diusivity parameter. By raising the value of ϵ their is slight decrease in both of the profiles.

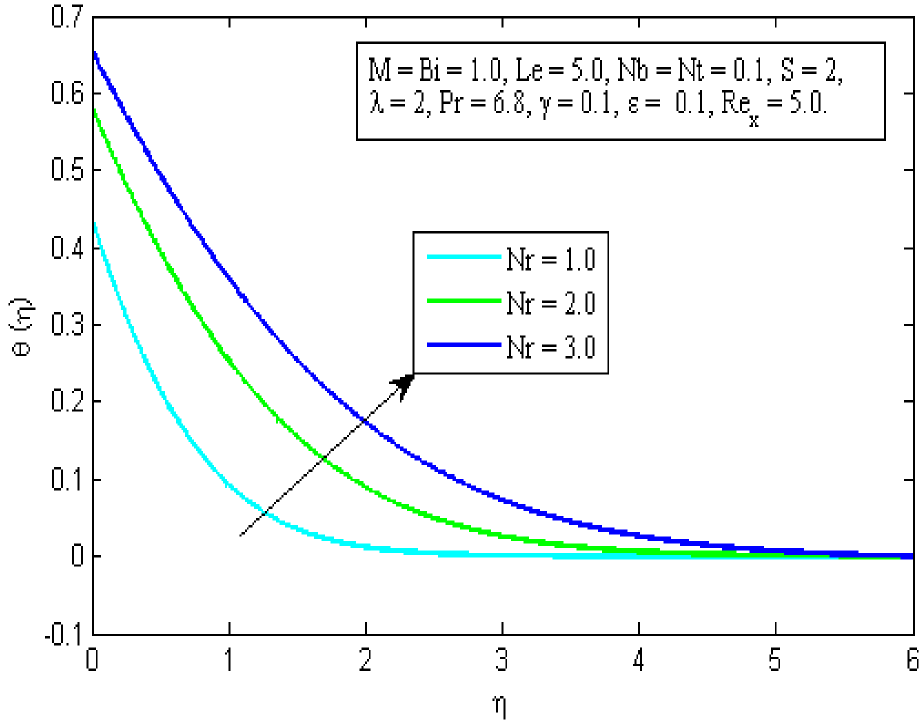

FIGURE 4.2: Effects of Nr on Temperature Profile $\theta(\eta)$.

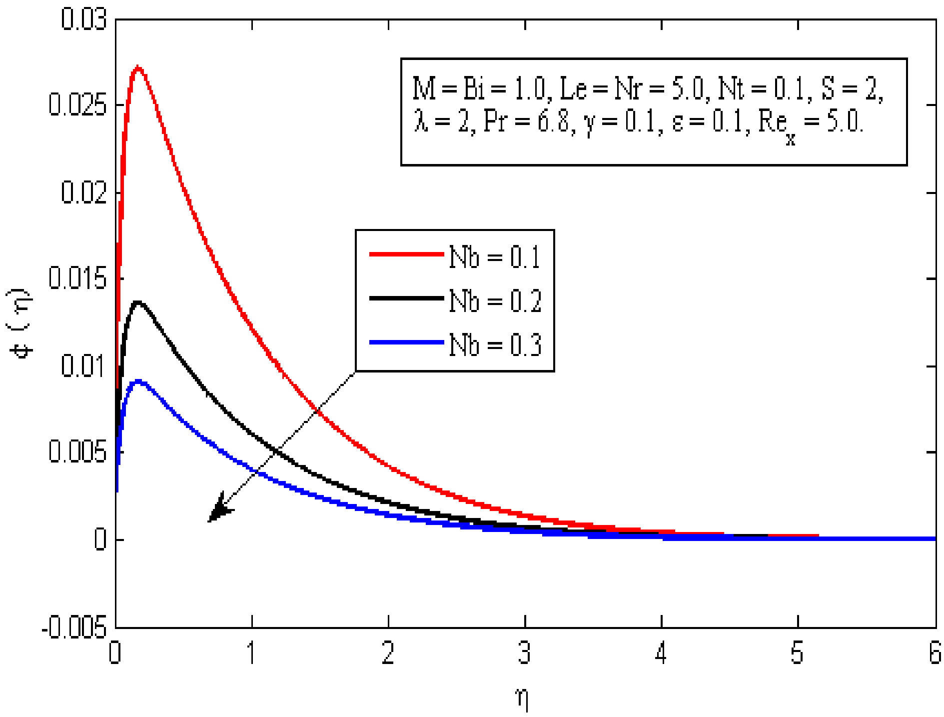

FIGURE 4.3: Effects of Nb on Nanoparticle Fraction $\phi(\eta)$.

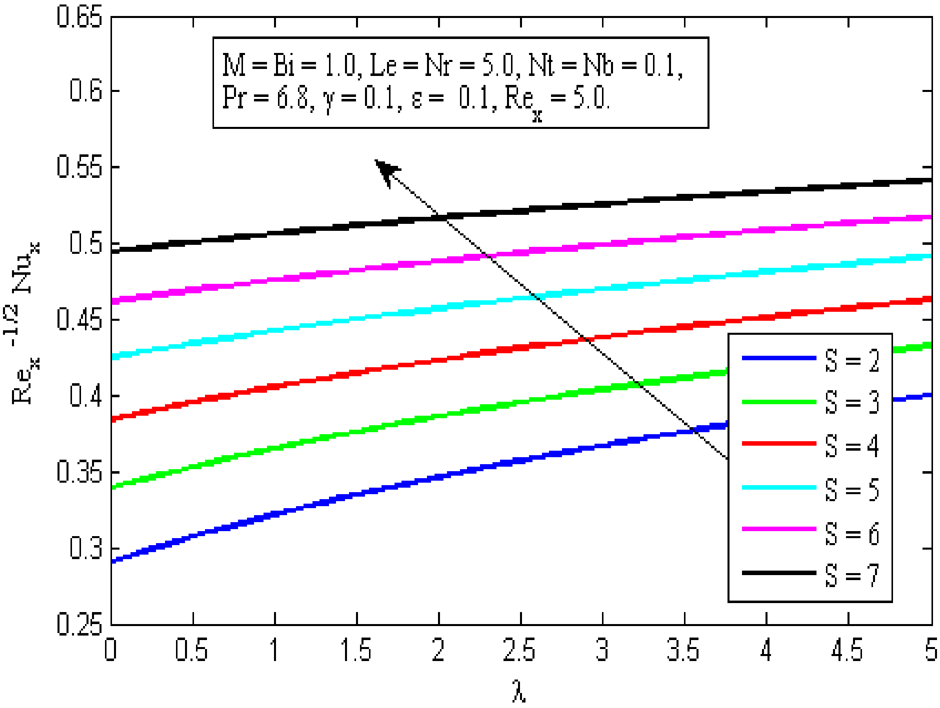

FIGURE 4.4: Variation of $Re_x^{-1/2} Nu_x$ with λ for Several Values of S.

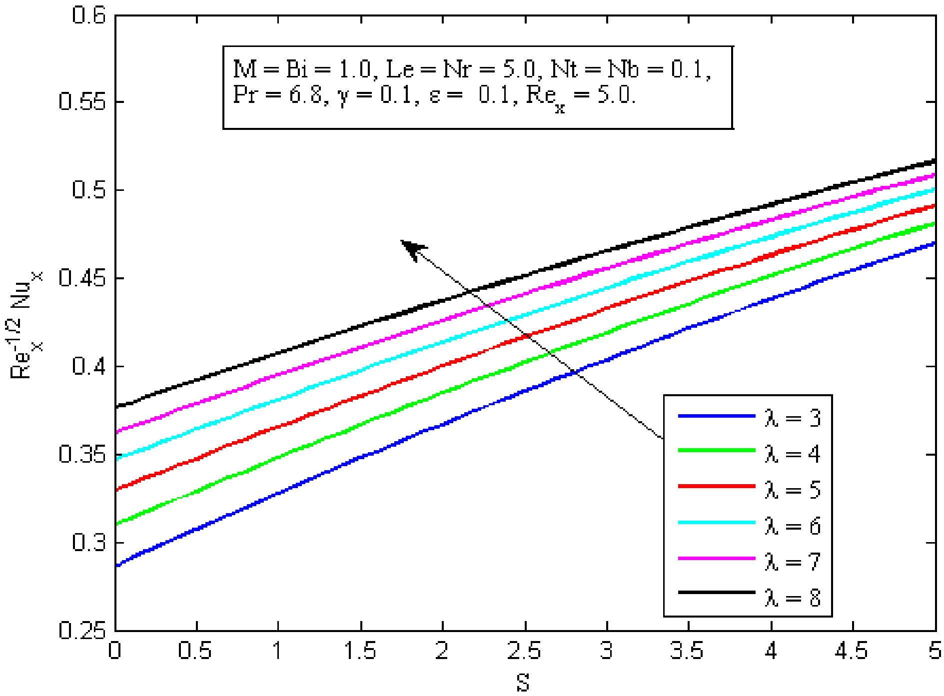

FIGURE 4.5: Variation of $Re_x^{-1/2} Nu_x$ with S for Several Values of λ.

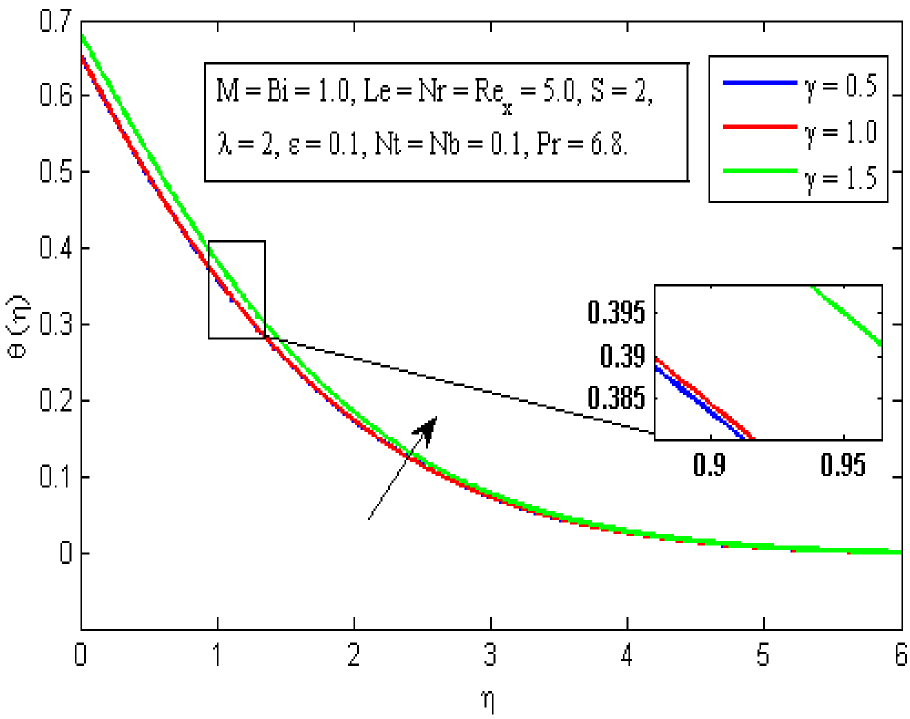

FIGURE 4.6: Temperature Profile $\theta(\eta)$ with Variation of γ.

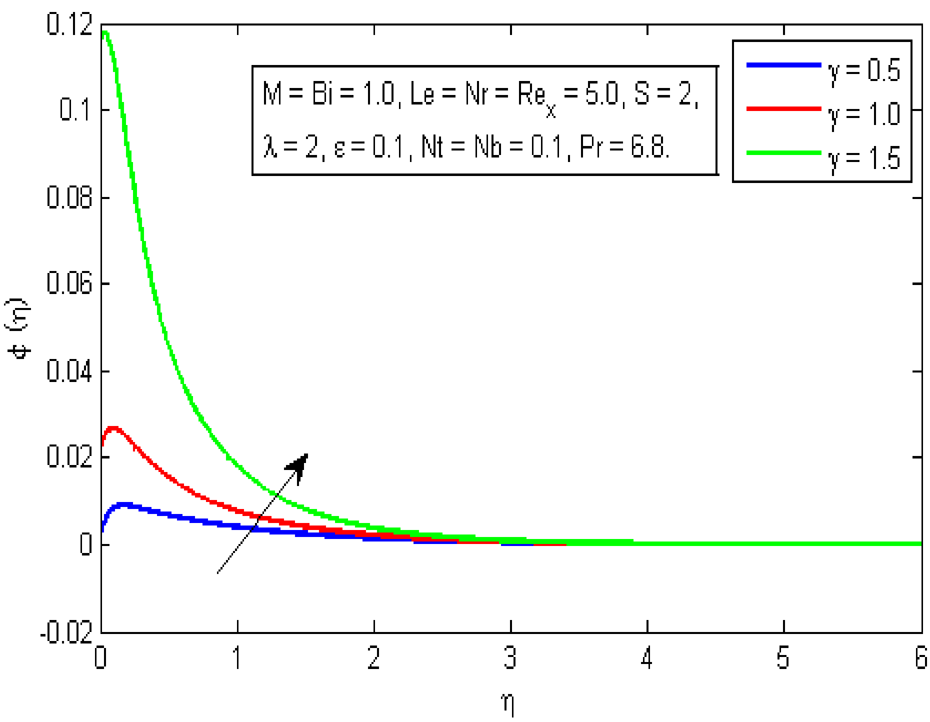

FIGURE 4.7: Nanoparticle Fraction $\phi(\eta)$ with Variation of γ.

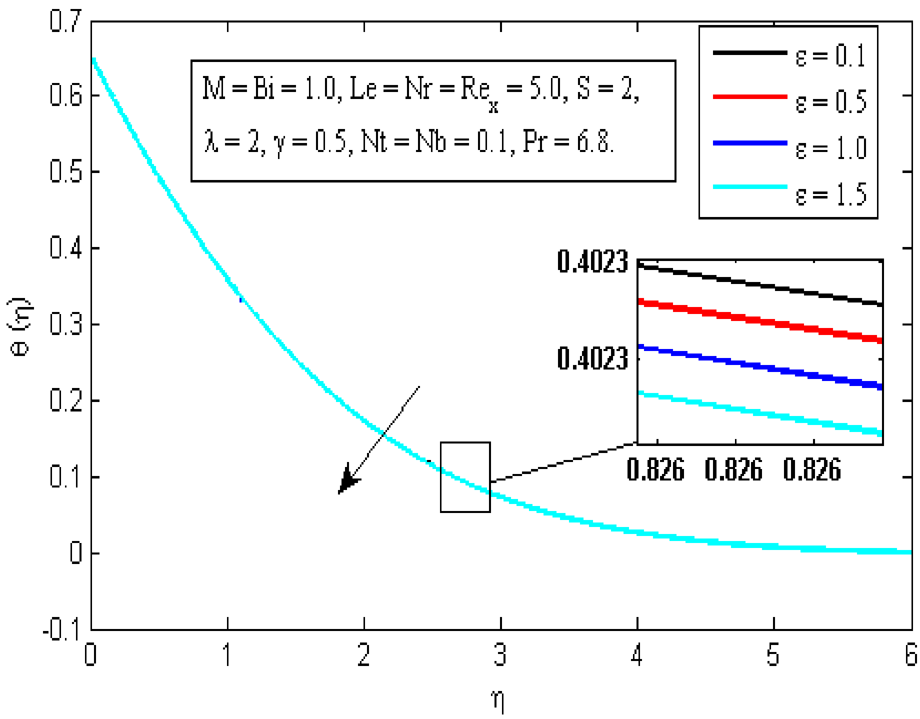

FIGURE 4.8: Temperature Profile $\theta(\eta)$ with Different Values of ϵ.

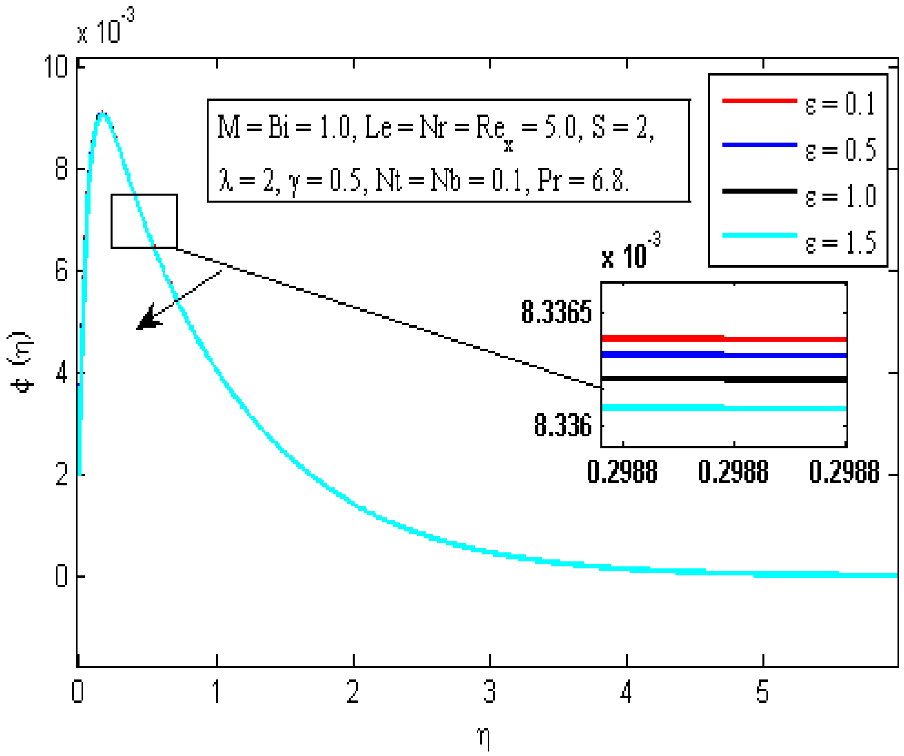

FIGURE 4.9: Nanoparticle Fraction $\phi(\eta)$ with Different Values of ϵ.

Chapter 5

Conclusion

Summary of this study represents the two-dimensional MHD stagnation-point flow of a nanofluid past a stretching sheet in the presence of chemical reaction with variable species diffiusivity and radiation effects. The conversion of non-linear PDEs, describing the proposed flow problem to a set of coupled ODEs which is carried out by applying appropriate similarity transformations. Numerical solution of the mathematical model is achieved by using the shooting technique. The impacts of pertinent flow parameters on the dimensionless energy and concentration profiles are illustrated in the graphical forms. The conclusions drawn from the numerical results are summarized below.

Concluding Remarks

The significant explorations of the current research can be concluded as below:

- The heat transfer rate escalates by an increment in the radiation parameter.

- The Brownian motion decreases the nanoparticle fraction profile but has no effect on temperature profile.

- An increase in the suction strength, the stretching increases the skin friction and Nusselt number.

- The heat and mass transfer rates climb as the chemical reaction parameter is escalated.

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