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Regions of Central Configurations in a Symmetric 4+1 Body Problem

by

Irtiza Ul Hassan

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To my parent, teachers, wife, friends and daughter Hoorain Fatima.

CERTIFICATE OF APPROVAL

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Abstract

In this review thesis, we model five body problem where four of the masses are placed at the vertices of an isosceles trapezoid and fifth mass is placed at the center of the system. We obtained the expression for m_0 and M as a function of α and β which are distance parameters. The regions of the central configurations where positive masses can be selected are derived analytically as well as numerically. Also it is shown that no central configurations are possible in the complement of these regions.

Contents

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Abbreviations

2BP	Two-Body Ptoblem
3BP	Three-Body Problem
4BP	Four-Body Problem
5BP	Five-Body Problem
CC	Central Configuration
D	Denominator
Ms	Mass of the Sun
N	Numerator
NBP	N-Body Problem
SI	System International

Physical Constants

Symbol	Name	Unit
G	Universal gravitational constant	$m^3kg^{-1}s^{-2}$
F	Gravitational force	Newton
r	Distance	Meter
P	Linear momentum	$kgms^{-1}$
L	Angular momentum	kgm^2s^{-1}
m_i	Point masses	kg
μ	Ratio of the masses	
$\mathbb{R}$	Real number	
$\ni$	Such that	
$\forall$	For all	
$\in$	Belongs to	

Chapter 1

Introduction

In classical mechanics, the 2-body problem (2BP) is to predict the motion of two massive objects which are abstractly viewed as point particles. The 2BP is most common in the case of a gravity that occurs in astronomy to determine orbits of objects such as satellites, planets and stars. Newton solved 2BP by his fundamental law of gravity. Newtonian mechanics is a mathematical model whose purpose is to account for the motions of the various objects in the universe. The basic concept of this model were first enunciated by Sir Isaac Newton in a work entitled "***Philosophiae Naturalis Principia Mathematica***"[1]. This work, which was published in 1687. The problem has no significant solution if $n \geq 3$. Although we have a restricted 3-body problems (3BP) [2] it may provide us with a particular solution. The 3BP is the problem of taking the initial positions and velocities of three point masses and solving for their subsequent motion according to Newton's laws of motion and Newton's law of universal gravitation. The 3BP is a special case of the N-body problem (NBP). The 3BP is one of the oldest problems in classical dynamics that continues to throw up surprises. It has challenged scientists from Newtons time to the present. It arose in an attempt to understand the Suns effect on the motion of the Moon around the Earth. NBP also known as many body problem [3]. The many body problem was first formulated precisely by Newton. In its form where the object involve point masses: "it may be stated as given at any time the position and velocities of three or more massive particles moving under their mutual gravitational forces, the mass also being known, calculated their positions and velocities at any other time". The NBP which predicts the individual motion of a system of celestial bodies that gravitationally attract with each other. The statement of the problem is "what would be the orbit, if we are given n celestial objects interacting with each others under the gravitational forces." Mathematicians and astronomers continued to work on the NBP over the last four centuries. First of all, Kepler in his planetary motion laws defining the elliptical trajectories of planets around the sun. Most important works in science history in which Newton derived and formulated Kepler's law [4]. Newton turned his attention to comparatively more difficult systems, after the justification of Kepler's laws. Although, he was unable to achieve any breakthrough in 3BP throughout his life after much struggle. Alexis Clairaut was able to present an approximation for the 3BP after twenty years of Newton's death. After some small adjustment, his work accounted for the moon's perigee, which was Newton's aim. In the 19th century, many famous astronomers and mathematicians worked on NBP [5].

1.1 Central Configuration (CC)

The idea of dynamics represented by complete masses collision or the rotating equilibrium, we are led to the idea of a CC. In CC, "*the acceleration of the ith mass must be proportional to its position (relative to the center of mass of the system)*"; thus, $\ddot{\mathbf{r}}_i = \lambda \mathbf{r}_i \ \forall \ i = 1, 2, 3, ..., n$. CC is common and basic concept in the study of NBP. Consequently, for years the question of few bodies in CC and general has fascinated considerable attention (see for example Albouy and Llibre [6] and Shoaib and Faye [7]).

Moulton first published linear solutions to the NBP [8]. Palmore [10] proposed many theorems in the study of points of equilibrium in the planar NBP. Papadakis and Kanavos [11] studied the restricted photogravitational 5BP, they investigated the movement of a massless object on a sphere. Kulesza et al. [12] have more recently examined a restricted rhomboidal 5BP. The masses are arranged in the same plane as the 5^{th} point is massless and the other masses on the vertices of the rhombus. Ollongren [13] studied a restricted 5BP with three bodies of equal mass m of the equilateral triangle placed on the vertices; rotating in circular orbits in triangular plane under the mutual gravitational attraction around its gravitational center. Under the gravitational attraction of other bodies a 5^{th} body with negligible mass as opposed to m moves in the plane.

Other notable studies are Kalvouridis [14] and Markellos et al. [15] on the restricted 5BP. Another restrictive approach used to investigate 5BP is some sort of symmetries added. For example, on a particular case of the 5BP, Roberts [16] addressed relative equilibria. He investigated a CC which consists of five bodies, four bodies are situated at the vertices of the rhombus and the 5^{th} body is in the middle. Mioc and Blaga [17] explain the similar problem but in the post Newtonian field of Manev. Existence of mono parametric families of relative equilibrium was proved by them for the primaries $(m_0, 1, m, 1, m)$, where m_0 is the central mass, and proved the problem of the Manev square.

The CC of the 5BP were addressed by Albouy and Llibre [6]. They studied on a sphere with a larger 5^{th} mass at its center they considered four equal masses. More recent studies on the symmetrically restricted 5BP include Shoaib et al. [18, 19]. Lee and Santoprete [20] also studies on the symmetrically restricted 5BP. Similarly Gidea and Llibre [21], and Marchesin and Vidal [22] discussed on the symmetrically restricted 5BP. As yet, in the non-collinear general four and 5BP, the basic interest has been on the same question: Is there a fixed arrangement of bodies and unique CC for a given set of masses?

Ouyang and Xie [23] investigated about a four-body collinear problem and Mello and Fernandes [24] discussed a rhomboidal 4BP and 5BP.

1.2 Thesis Contribution

In this thesis [9] we investigates four point masses at the vertices of an isosceles trapezoid with masses $m_1 = m_4$ at the locations $(\mp 0.5, r_B)$, $m_2 = m_3$ at the locations $(\mp\alpha/2, r_A)$, and m_0 at the center of mass(c.o.m). In phase space we derive regions of CC, both numerically and analytically, where positive masses can be chosen.

1.3 Dissertation Outlines

We divide this dissertation into five chapters.

In **Chapter 1** introduction of the problem and aim of this research is briefly discussed. First of all we discussed the basics concept and history of the 2BP, 3BP and NBP. The 2BP is the only problem in celestial mechanics that have analytical solution.

In **Chapter 2** contains some basic definitions related to celestial mechanics, Newton's laws of motion, Newton law of gravitation and Kepler's laws of planetary motion. In the last portion of this chapter, the 2BP and the solution to the 2BP is briefly discussed with the help of radial and transverse component.

In **Chapter 3** the classical equations of motion for the trapezoidal 5BP are solved with the help of diagram, the CC regions are studied using analytical as well as numerical techniques for a specific case of the trapezoidal 5BP, where on the trapezoid vertices four of the masses are same. the graphs are also shown at the end of this chapter.

In **Chapter 4** the isosceles trapezoid 5BP is investigated for the regions of the CC in its most basic form. We also discussed the special case for ($\alpha = \beta$). The CC regions are given numerically as well as analytically.

In **Chapter 5** we summarizes the whole study with concluding remarks.
References used in the thesis are mentioned in **Biblography**

Chapter 2

Preliminaries

This chapter contains some important definitions, concepts, governing laws which are essential to understand the work presented in next chapters.

2.1 Basics Definitions

2.1.1 Motion [25]

"Motion is the action used to change the location or position of an object with respect to the surroundings over time".

2.1.2 Mechanics [25]

"Mechanics is a branch of physics concerned with motion or change in position of physical objects. It is sometimes further subdivided into:

1. **Kinematics**, which is concerned with the geometry of the motion,
2. **Dynamics**, which is concerned with the physical causes of the motion,
3. **Statics**, which is concerned with conditions under which no motion is apparent".

2.1.3 Scalar [25]

"Various quantities of physics, such as length, mass and time, requires for their specification a single real number (apart from units of measurement which are decided upon in advance). Such quantities are called **Scalars** and the real number is called the magnitude of the quantity".

2.1.4 Vector [25]

"Other quantities of physics, such as displacement, velocity, momentum, force etc require for their specification a direction as well as magnitude. Such quantities are called **Vectors**".

2.1.5 Field [25]

"A field is a physical quantity associated with every point of spacetime. The physical quantity may be either in vector form, scalar form or tensor form".

2.1.6 Scalar Field [25]

"If at every point in a region, a scalar function has a defined value, the region is called a scalar field. i.e.,

$$f : \mathbb{R}^3 \longrightarrow \mathbb{R},$$

e.g., Temperature and pressure fields around the earth".

2.1.7 Vector Field [25]

"If at every point in a region, a vector function has a defined value, the region is called a vector field,

$$V : I\!R^3 \longrightarrow I\!R^3,$$

e.g., tangent vector around a smooth curve".

2.1.8 Conservative Vector Field [25]

"A vector field $\mathbf{V}$ is conservative if and only if there exists a contentiously differentiable scalar field f such that $\mathbf{V}$ = -∇f or equivalently if and only if,

$$\nabla \times \mathbf{V} = curl\mathbf{V} = \mathbf{0}\text{''}.$$

2.1.9 Uniform Force Field [25]

"A force field which has constant magnitude and direction is called a uniform or constant force field. If the direction of the field is taken as negative z direction and magnitude is constant $F_0 > 0$, then the force field is given by:

$$\mathbf{F} = -F_0\hat{\mathbf{k}}\text{''}.$$

2.1.10 Central Force [25]

"Suppose that a force acting on a particle of mass m such that
(a) it is always directed from m towards or away from a fixed point O,
(b) its magnitude depends only on the distance r from O.
Then we call the force a central force or central force field with the O as the center of the force field. Mathematically, $\mathbf{F}$ is central force if and only if

$$\mathbf{F} = f(r)\mathbf{r}_1 = f(r)\frac{\mathbf{r}}{r},$$

where $\mathbf{r}_1 = \frac{\mathbf{r}}{r}$ is a unit vector in the direction of $\mathbf{r}$. The central force is one of attraction towards O or repulsion from O according as $f(r) < 0$ or $f(r) > 0$ respectively".

2.1.11 Degree of Freedom [25]

"The number of coordinates required to specify the position of a system of one or more particles is called number of degree of freedom of the system.
Example: A particle moving freely in space requires 3 coordinates, e.g. (x, y, z), to specify its position. Thus the number of degree of freedom is 3".

2.1.12 Center of Mass [25]

"Let $r_1, r_2, ..., r_n$ be the position vector of a system of n particles of masses $m_1, m_2, ..., m_n$ respectively. The center of mass or centroid of the system of particles is defined as that point having position vector,

$$\hat{\mathbf{r}} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + ... + m_n\mathbf{r}_n}{m_1 + m_2 .... + m_n} = \frac{1}{M}\sum_{i=1}^{n} m_i\mathbf{r}_i,$$

where

$$M = \sum_{i=1}^{n} m_i,$$

is the total mass of the system".

2.1.13 Center of Gravity [25]

"If a system of particles is in a uniform gravitational field, the center of mass is sometimes called the center of gravity".

2.1.14 Torque [25]

"If a particle with a position vector r moves in a force field **F**, we define τ as torque or moment of the force as

$$\tau = \mathbf{r} \times \mathbf{F}.$$

The magnitude of τ is

$$\tau = rF \sin\theta.$$

The magnitude of torque is a measure of the turning effect produced on the particle by the force".

2.1.15 Momentum [25]

"The linear momentum **p** of an object with mass m and velocity **v** is defined as:

$$\mathbf{P} = m\mathbf{v}.$$

Under certain circumstances the linear momentum of a system is conserved. The linear momentum of a particle is related to the net force acting on that object:

$$\mathbf{F} = m\mathbf{a} = m\frac{d\mathbf{v}}{dt} = \frac{d}{dt}(m\mathbf{v}) = \frac{d\mathbf{P}}{dt}.$$

The rate of change of linear momentum of a particle is equal to the net force acting on the object, and is pointed in the direction of the force. If the net force acting on an object is zero, its linear momentum is constant (conservation of linear momentum). The total linear momentum **p** of a system of particles is defined as the vector sum of the individual linear momentum:

$$\mathbf{P} = \sum_{i=1}^{n} \mathbf{P}_i".$$

2.1.16 Point-like Particle [25]

"A point-like particle is an idealization of particles mostly used in different fields of physics. Its defining features is the lacks of spatial extension:being zero-dimensional, it does not take up space. A point-like particle is an appropriate representation of an object whose structure, size and shape is irrelevant in a given context. e.g., from far away, a finite-size mass (object) will look like a point-like particle".

2.1.17 Angular Momentum [25]

"Angular momentum for a point-like particle of mass m with linear momentum **p** about a point O, defined by the equation,

$$\mathbf{L} = \mathbf{r} \times \mathbf{P},$$

where **r** is the vector from the point O to the particle. The torque about the point O acting on the particle is equal to the rate of change of the angular momentum about the point O of the particle i.e.,

$$\tau = \frac{d\mathbf{L}}{dt}".$$

2.1.18 Lorentz Transformation [25]

"Lorentz transformation is the relationship between two different coordinate frames that move at a constant velocity and are relative to each other. The name of the transformation comes from a Dutch physicist Hendrik Lorentz. There are two frames of reference, which are"

2.1.18.1 Inertial Frame of Reference [25]

"A frame of reference that remains at rest or moves with constant velocity with respect to other frames of reference is called inertial frame of reference. Actually, an unaccelerated frame of reference is an inertial frame of reference. In this frame of reference a body does not acted upon by external forces. Newton's laws of motion are valid in all inertial frames of reference. All inertial frames of reference are equivalent. A frame which is not inertial is called non inertial frame".

2.1.19 Lagrange Points [25]

"A point in space where a small body with negligible mass under the gravitational influence of two large bodies will remain at rest relative to the larger ones. These points are locations in an orbital arrangement of two large bodies where a third smaller body, affected solely by gravity, is capable of maintaining a stable position relative to the two larger bodies. A lagrange point is also known as a equilibrium point and Liberation point named after a French mathematician and atronomer Joseph-Louis Lagrange. He was first to find these equlibrium points for the earth, sun, and moon system. He found five points out of these three are collinear".

2.1.20 Equilibrium Solution [25]

"The Equilibrium solution can guide us through the behavior of the equation that represents the problem without actually solving it. These solutions can be found only if we meet the sufficient condition of all rates equal to zero. If we have two variables then,

$$\dot{x} = \dot{y} = \ddot{x} = \ddot{y} = ... = x^{(n)} = y^{(n)} = 0.$$

These solutions may be stable or unstable. The stable solutions regarding in celestial Mechanics helps us find parking spaces where if a satellite or any object placed, it will remain there for ever. These type of places are also found along the Jupiters orbital path where bodies called trojan are present. These equilibrium points with respect to Celestial Mechanics are also called Lagrange points named after a French mathematician and astronomer Joseph-Louis Lagrange. He was first to find these equilibrium points for the Sun-Earth system. He found that three of these five points were collinear".

2.1.21 Holonomic and Non Holonomic Constraints [25]

"In classical mechanics, a constraint on a system is a parameter that the system must obey. The limitation on the motion are often called constraints. If the constraints condition can be expressed as an equation,

$$\phi(r_1, r_2, ..., r_n, t) = 0,$$

connecting the position vector of the particles and the time, then the constraints are called holonomic, otherwise non-holonomic".

2.1.22 Galilean Transformation [25]

"In physics, a Galilean transformation is used to transform between the coordinates of two reference frames which differ only by constant relative motion within the constructs of Newtonian physics. These transformations together with spatial rotations and translations in space and time form the inhomogeneous Galilean group (assumed throughout below). Without the translations in space and time the group is the homogeneous Galilean group. The Galilean group is the group of motions of Galilean relativity acting on the four dimensions of space and time, forming the Galilean geometry. Galilean transformations, also called Newtonian transformations, set of equations in classical physics that relate the space and time coordinates of two systems moving at a constant velocity relative to each other."

2.1.23 Celestial Mechanics [25]

“Celestial mechanics is the branch of astronomy that deals with the motions of objects in outer space. Historically, celestial mechanics applies principles of physics (classical mechanics) to astronomical objects, such as stars and planets, to produce ephemeris data. Actually celestial mechanics is the science devoted to the study of the motion of the celestial bodies on the basis of the laws of gravitation. It was founded by Newton and it is the oldest of the chapters of Physical Astronomy. The story of the mathematical representation of celestial motions starts in the antiquity and, notwithstanding the prevalent wrong ideas placing the Earth at the center of the universe, the prediction of the planetary motions were very accurate allowing, for instance, to forecast eclipses and to keep calendars synchronized with the motion of the Earth around the Sun. The epicycles, introduced by Apollonius of Perga around 200 BC, allowed the observed motions to be represented by series of circular functions. They were used to predict celestial motions for almost two millennia. Their long life was certainly related to the stagnation that prevailed in the western world during the dark ages between the end of the Helenic civilization and the Renaissance. In the 16th century, the Copernican revolution put the Sun in center of the Universe. However, the breakthrough in our knowledge of celestial motions was rather related to Tycho Brahe and Johannes Kepler. Tycho, in his Uraniborg observatory, accurately measured the position of the planets in the sky for more than 20 years. The work of Kepler is a monument to the human genius. First of all, Tychos data on Mars could not be fitted to a heliocentric uniform motion. With respect to a uniform motion, sometimes Mars was in advance, sometimes in retard. Kepler decided to tackle the problem from scratch! Remember that mathematics had remained stagnant since antiquity and the tools inherited from the Greeks, geometry and arithmetic, were the only available. Kepler considered as working hypotheses that the Earth was uniformly moving on a circle and that the motion of Mars was periodic and coplanar with the motion of the Earth. Then he used Tychos observations to determine the orbit of Mars. Tychos observations were apparent positions of the planets on the celestial sphere.Newtons theory of universal gravitation resulted from experimental and observational facts. The observational facts were those encompassed in the three Kepler laws. The experimental facts were those reported by Galileo in his book Discorsi intorno due nuove scienze (Discourses Relating to Two New Sciences, which should not be confounded with his most celebrated Dialogue Concerning the Two Chief World Systems). The basis of Newton theory arose from the perception that the force keeping the Moon in orbit around the Earth is the same that, on Earth, commands the fall of the bodies. This law inaugurated the Celestial Mechanics. Newton initially studied the problem of the motion followed by two bodies in mutual attraction. He showed that under ideal conditions, the relative motion obeys laws which, in some sense, include the first two laws of Kepler."

2.1.24 Kepler's Laws of Planetary Motion [25]

"Kepler's three laws of planetary motion can be described as follows:

1. Keplers first law states that every planet moves along an ellipse, with the Sun located at a focus of the ellipse. An ellipse is defined as the set of all points such that the sum of the distance from each point to two foci is a constant.

2. Keplers second law states that a planet moves in its ellipse so that the line between it and the Sun placed at a focus sweeps out equal areas in equal times.

3. The cube of the semi major axis of the planetary orbits are proportional to the square of the planets periods of revolution. Mathematically, Kepler's third law can be written as:

$$T^2 = \left(\frac{4\pi^2}{GM_s}\right) r^3,$$

where T is the time period, r is the semi major axis, M_s is the mass of sun and G is the universal gravitational constant".

2.1.25 Newton's Laws of Motion [25]

"The following three laws of motion given by Newton are considered the axioms of mechanics:

1. First law of motion

Every particle persists in a state of rest or of uniform motion in a straight line unless acted upon by a force.

2. Second law of motion

If $\mathbf{F}$ is the external force acting on a particle of mass m which as a reaction is moving with velocity $\mathbf{v}$, then

$$\mathbf{F} = \frac{d}{dt}(m\mathbf{v}) = \frac{d\mathbf{P}}{dt}.$$

If m is independent of time this becomes

$$\mathbf{F} = m\frac{d}{dt}(\mathbf{v}) = m\mathbf{a},$$

where $\mathbf{a}$ is the acceleration of the particle

2. Third law of motion

For every action, there is an equal and opposite reaction".

2.1.26 Newton's Universal Law of Gravitation [25]

"Every particle of matter in the universe attracts every other particle of matter with a force which is directly proportional to the product of the masses and inversely proportional to the square of the distance between them. Hence, for any Preliminaries 12 two particles separated by a distance r, the magnitude of the gravitational force $\mathbf{F}$ is:

$$\mathbf{F} = G\frac{m_1 m_2}{r^2}\mathbf{r},$$

where G is universal gravitational constant. Its numerical value in SI units is $6.67408 \times 10^{-11} m^3 kg^{-1} s^{-2}$."

2.2 Two Body Problem [26]

The two-body problem in classical mechanics is to predict the motion of two large objects, which are abstractly known as point particles. The problem assumes that the two objects communicate only with one another; the only force influencing each object comes from the other one, and all other objects are ignored.

2.3 The Solution to the Two-Body Problem [26]

Newtons universal gravitational law is the governing law for the two bodies:

$$\mathbf{F} = G\frac{m_1 m_2}{r^3}\mathbf{d}, \tag{2.1}$$

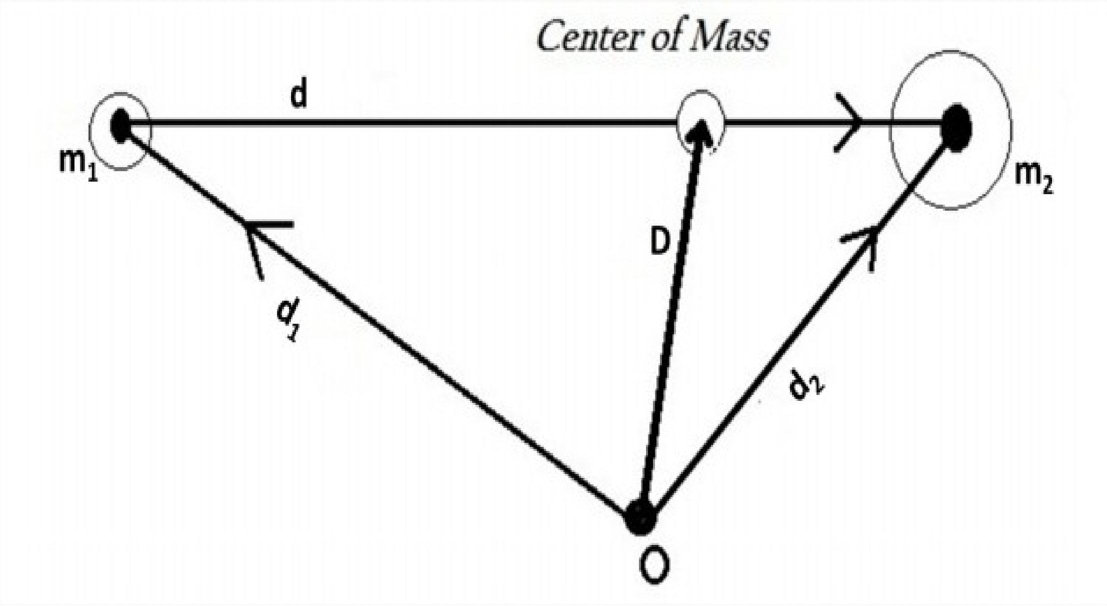

FIGURE 2.1: Center of mass of two body system.

For two masses, m_1 and m_2 are separated by a $\mathbf{d}$ distance, and the universal gravitational constant is G. The purpose here is to decide if the initial locations and velocities are known, the direction of the particles for some time t. The force of attraction $\mathbf{F}_{12}$ in Figure 2.1 is directed towards m_1 along $\mathbf{d}$, while the force $\mathbf{F}_{21}$ on m_2 is directed in the opposite direction. According to Newtons third law of motion,

$$\mathbf{F}_1 = -\mathbf{F}_2. \tag{2.2}$$

From "Figure 2.1,

$$\mathbf{F}_{12} = G\frac{m_1 m_2}{d^3}\mathbf{d}, \tag{2.3}$$

the equation of motion of the particles under their mutual gravitational attractions is given by equations (2.1) and (2.2) using Newtons second law of motion and by equations (2.1) and (2.2).

$$m\ddot{\mathbf{d}}_1 = G\frac{m_1 m_2}{d^3}\mathbf{d}, \tag{2.4}$$

$$m\ddot{\mathbf{d}}_2 = G\frac{m_1 m_2}{d^3}\mathbf{d}, \tag{2.5}$$

where the location vectors $\mathbf{d}_1$ and $\mathbf{d}_2$ are from the reference O, as shown in Figure 2.1. When the equations (2.4) and (2.5) are applied, we get:

$$m_1\ddot{\mathbf{d}}_1 + m_2\ddot{\mathbf{d}}_2 = \mathbf{0}. \tag{2.6}$$

The integration of the equations above yields:

$$m_1\dot{\mathbf{d}}_1 + m_2\dot{\mathbf{d}}_2 = \mathbf{k}_1. \tag{2.7}$$

The total linear momentum of the system is a constant, i.e., $m_1\mathbf{v}_{m1}+m_2\mathbf{v}_{m2} = \mathbf{k}_1$. Again integrating equation (2.7) implies that:

$$m\mathbf{d}_1 + m_2\mathbf{d}_2 = \mathbf{k}_1 t + \mathbf{k}_2, \tag{2.8}$$

where $\mathbf{k}_1$ and $\mathbf{k}_2$ represent the constant of integration. Using 2BPs description of the centre of mass, $\mathbf{D}$ is defined as $\mathbf{D}$:

$$\begin{aligned} (m_1 + m_2)\mathbf{D} &= m_1\mathbf{d}_1 + m_2\mathbf{d}_2, \\ m_t\mathbf{D} &= m_1\mathbf{d}_1 + m_2\mathbf{d}_2, \end{aligned} \tag{2.9}$$

where $m_t = m_1 + m_2$. We get the derivative of the (2.9) equation and compare it with the (2.7) equation.

$$m_t\dot{\mathbf{D}} = \mathbf{k}_1 \qquad \Rightarrow \qquad \dot{\mathbf{D}} = \frac{\mathbf{k}_1}{m_t} = constant,$$

show that $\dot{\mathbf{D}} = \mathbf{v}_c$ is constant.
Subtracting (2.6) from (2.5) from the equations gives:

$$\ddot{\mathbf{d}}_1 - \ddot{\mathbf{d}}_2 = G\frac{m_2}{d^3}\mathbf{d} + G\frac{m_1}{d^3}\mathbf{d}, \tag{2.10}$$

$$\begin{aligned} \ddot{\mathbf{d}}_1 - \ddot{\mathbf{d}}_2 &= G(m_1 + m_2)\frac{\mathbf{d}}{d^3}, \\ \Rightarrow \ddot{\mathbf{d}} &= \beta\frac{\mathbf{d}}{d^3}, \\ \Rightarrow \ddot{\mathbf{d}} - \beta\frac{\mathbf{d}}{d^3} &= \mathbf{0}, \end{aligned} \tag{2.11}$$

where $\beta = \mathrm{G}(m_1 + m_2)$ is defined as reduced mass and $\mathbf{d}_1 - \mathbf{d}_2 = -\mathbf{d}$, see Figure 2.1. Taking the cross product of $\mathbf{d}$ with equation (2.11) we obtain:

$$\mathbf{d} \times \beta\ddot{\mathbf{d}} + \frac{\beta^2}{d^3}\mathbf{d} \times \mathbf{d} = \mathbf{0},$$

$$\Rightarrow \mathbf{d} \times \ddot{\mathbf{d}} = \mathbf{0}, \tag{2.12}$$

integrating above equation yields:

$$\mathbf{d} \times \dot{\mathbf{d}} = \mathbf{H}, \tag{2.13}$$

where **H** is a constant vector. We should write equation (2.12),

$$\Rightarrow \mathbf{d} \times \beta\ddot{\mathbf{d}} = \mathbf{0}, \tag{2.14}$$

$$\Rightarrow \mathbf{d} \times \mathbf{F} = \mathbf{0}, \tag{2.15}$$

where $\mathbf{F} = \beta\ddot{\mathbf{d}}$.

The description of torque and angular momentum is taken from Chapter 2":

$$\tau = \frac{d\mathbf{H}}{dt} = \mathbf{d} \times \mathbf{F}. \tag{2.16}$$

Comparing equations (2.16) and (2.17), we get:

$$\tau = \frac{d\mathbf{H}}{dt} = \mathbf{d} \times \mathbf{F} = \mathbf{0}, \tag{2.17}$$

$$\frac{d\mathbf{H}}{dt} = \mathbf{0}, \tag{2.18}$$

$$\mathbf{H} = constant,$$

i.e. angular momentum of the system is conserved.

2.3.1 Radial and Transverse Components of Velocity and Acceleration [26]

The velocity components along and perpendicular to the radius vector joining m_1 to m_2 are $\dot{d}$ and $d\dot{\theta}$ if the polar co-ordinates d and θ are taken in this plane as shown in "Figure 2.2, then,

$$\dot{\mathbf{d}} = \dot{d}\mathbf{i} + d\dot{\theta}\mathbf{j}, \tag{2.19}$$

where the unit vectors **i** and **j** are located along and perpendicular to the vector radius. Thus, by means of equations (2.13) and (2.16),

$$\mathbf{d} \times (\dot{\mathbf{d}}\mathbf{i} + d\dot{\theta}\mathbf{j}) = d^2\dot{\theta}\mathbf{k} = \mathbf{H}\mathbf{k}, \tag{2.20}$$

where the constant **H** is shown to be twice the radius vector definition rate of the field. This is a mathematical version of the second law of Kepler. Now, if we use the scalar product $\dot{d}$ with the equation (2.11), we obtain equation (2.11) are as under.

$$\dot{\mathbf{d}}.\frac{d^2\mathbf{d}}{dt^2} + \beta\frac{\dot{\mathbf{d}}.\mathbf{d}}{d^3} = 0,$$

after integrated we have get,

$$\frac{1}{2}\dot{\mathbf{d}}.\dot{\mathbf{d}} - \frac{\beta}{d} = C, \tag{2.21}$$

$$\frac{1}{2}v^2 - \frac{\beta}{d} = C, \tag{2.22}$$

where C is a constant of integration. This is the type of energy conservation in the system. The C quantity is not absulate energy $1\beta^2/2$ is associated with kinetic energy, and $-mu/r$ is associated with the potential energy of the system, i.e. total energy of the system is conserved. Recall the components of the acceleration vector along and perpendicular to the radius vector from celestial mechanics:

$$\mathbf{a} = (\ddot{d} - d\dot{\theta}^2)\mathbf{i} + \frac{1}{d}\frac{d}{dt}(d^2\dot{\theta})\mathbf{j},$$

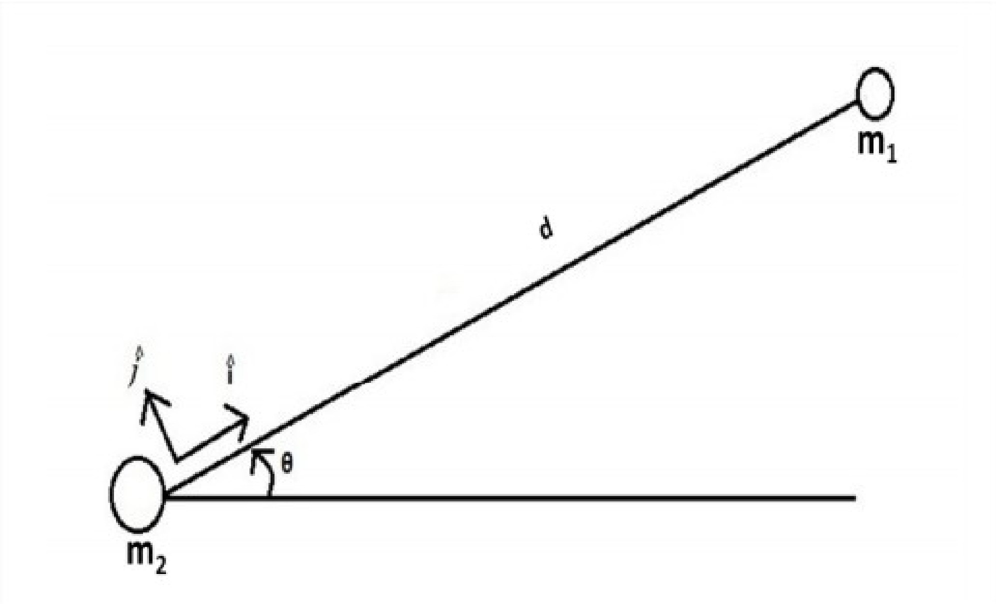

FIGURE 2.2: Center of mass of two body system.

apply the above equation in (2.11), we get

$$(\ddot{d} - d\dot{\theta}^2) = -\frac{\beta}{d^2}, \tag{2.23}$$

$$\frac{1}{d}\frac{d}{dt}(d^2\dot{\theta}) = 0. \tag{2.24}$$

After further integrating equation (2.21), we get the following angular momentum integral:

$$d^2\dot{\theta} = H, \tag{2.25}$$

under such type of substitution,

$$u = \frac{1}{d}. \tag{2.26}$$

The exclusion of the time between the equations (2.20) and (2.22) means that:

$$\frac{d^2u}{d\theta^2} + u = \frac{\beta}{H^2}. \tag{2.27}$$

The general solution of above equation is:

$$u = \frac{\beta}{H^2} + B\cos(\theta - \theta_0), \tag{2.28}$$

where B and θ_0 are two constants of integration. Substitute $u = \frac{1}{d}$ in above equation:

$$\frac{1}{d} = \frac{\beta}{H^2} + B\cos(\theta - \theta_0), \tag{2.29}$$

$$\Rightarrow \qquad d = \frac{\frac{H^2}{\beta}}{1 + \frac{H^2 B}{\beta}\cos(\theta - \theta_0)},$$

$$\Rightarrow \qquad d = \frac{p}{1 + e\cos(\theta - \theta_0)},$$

where

$$p = \frac{H^2}{\beta},$$

$$e = \frac{H^2 B}{\beta}.$$

The direction of one celestial body around another is defined by eccentricity e. Thus,

(i) If $0 < e < 1$ then the orbit is elliptical,

(ii) If $e = 1$ then the orbit is a parabolic,

(iii) If $e > 1$ then the orbit is a hyperbolic.

Therefore, a conic is the solution to the two-body problem, including the first law of Kepler as a special case. Mathematically can be defined as, $e = c/a$,

c is the distance from focus to the center and,

a represent the semi major axis,

b represent the minor axis.

where

$a^2 = b^2 + c^2$

$c^2 = a^2 - b^2$.

2.4 N-Body Problem

N-body problem also known as many body problem. The many body problem was 1st formulated precisely by Newton. In its form where the object involve a point masses it may be stated as follow:
Given at any time the position and velocities of three or more massive particles moving under their mutual gravitational forces, the mass also being known, calculated their positions and velocities at any other time.

2.4.1 The Equations of Motion in the NBP

The 2BP deals much of the important work in astrodynamics, but sometimes we need to model the real world by including other bodies. The next logical step, then, is to drive formulas for 3BP. A further generalization of three body problem is n-body problem. In general, solving general differential equations of motions in n-body problem requires a fixed number of integration constants.

Consider a simple gravity problem in which we have constant acceleration over time, $a(t) = a_0$. If we integrate this equation, we obtain the velocity, $v(t) = a_0 t + v_0$. Integrating once more provides, $r(t) = r_0 + v_0 t + \frac{1}{2} a_0 t^2$. To complete the solution, we must know the initial conditions. This example is a straight froward analytical solution using the initial values, or a function of the time and constants of integration, called integrals of the motion. Unfortunately, this isn't always the simple case. When initial conditions alone dont provide a solution, integrals of the motion can reduce the order of differential equations, also called the degrees of freedom of the dynamical system. Ideally, if the number of integrals equals the order of differential equations, we can reduce it to order zero. These integrals are constant functions of the initial conditions, as well as the position and velocity of at any time, hence the term constants of the motion.

For the n-body problem, a system of 3n second order differential equations, we need 6n integrals of motion for a complete solution. Conservation of linear momentum provides six, conservation of energy one, and conservation of total angular momentum three, for a total of ten. There are no laws analogous to Keplers first two laws to obtain additional constants, thus we are left with a system of order $6n - 10$ for $n \geq 3$.

These equations for n bodies $n \geq 3$, deny all attempts at closed-form solutions. H. Brun, in 1887, showed that there were no other algebraic integrals. Although Poincareé later generalized Brun's work, we still have only the ten known integrals. They give us insight into the motions within the three body and n-body problems. Conservation of total linear momentum assumes no external forces are on the system.

First, here we set up the equations of motions of n massive particles of masses $m_i (i = 1, 2, 3, ...n)$" whose radius vectors from an un accelerated point O are $\mathbf{r}_i$ while their mutual radius vectors are given by r_{ij} where,

$$\mathbf{r}_{ij} = \mathbf{r}_j - \mathbf{r}_i. \tag{2.30}$$

From Newton's laws of motion and the law of gravitation,

$$m_i \ddot{\mathbf{r}}_i = G \sum_{j=1, j \neq i}^{n} \frac{m_i m_j}{r_{ij}^3} \mathbf{r}_{ij}, \tag{2.31}$$

here we note that $\mathbf{r}_{ij}$ implies that the vector between m_i and m_j is directed for m_i to m_j, thus

$$\mathbf{r}_{ij} = -\mathbf{r}_{ji}. \tag{2.32}$$

where G is universal constant.

Chapter 3

Central Configuration Regions in Varying Central Mass With Four Equal Masses

3.1 Equation of Motion

The classical equation of motion for the NBP has the following form:

$$m_i \frac{d^2\mathbf{r}_i}{dt^2} = \sum_{j\neq i} \frac{m_i m_j(\mathbf{r}_j - \mathbf{r}_i)}{|\mathbf{r}_j - \mathbf{r}_i|^3} \quad , \qquad i = 1, 2, 3, ..., n, \tag{3.1}$$

where the units are chosen such that value of gravitational constant is one, $\mathbf{r}_i$ is the location vector of the i-th body. "A central configuration(CC) is a particular configuration of the N-body where the acceleration vector of each body is proportional to its position vector and the proportionality constant is same for the N-bodies," therefore,

$$\sum_{j=1, j\neq i}^{n} \frac{m_j(\mathbf{r}_j - \mathbf{r}_i)}{|\mathbf{r}_j - \mathbf{r}_i|^3} = -\lambda(\mathbf{r}_i - \mathbf{c}) \quad , \qquad i = 1, 2, 3, ..., n, \tag{3.2}$$

$$\mathbf{c} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + m_3\mathbf{r}_3 + ... + m_n\mathbf{r}_n}{M_t}, \tag{3.3}$$

where $\mathbf{c}$ is the center of mass of the system and M_t is the total mass of the system.

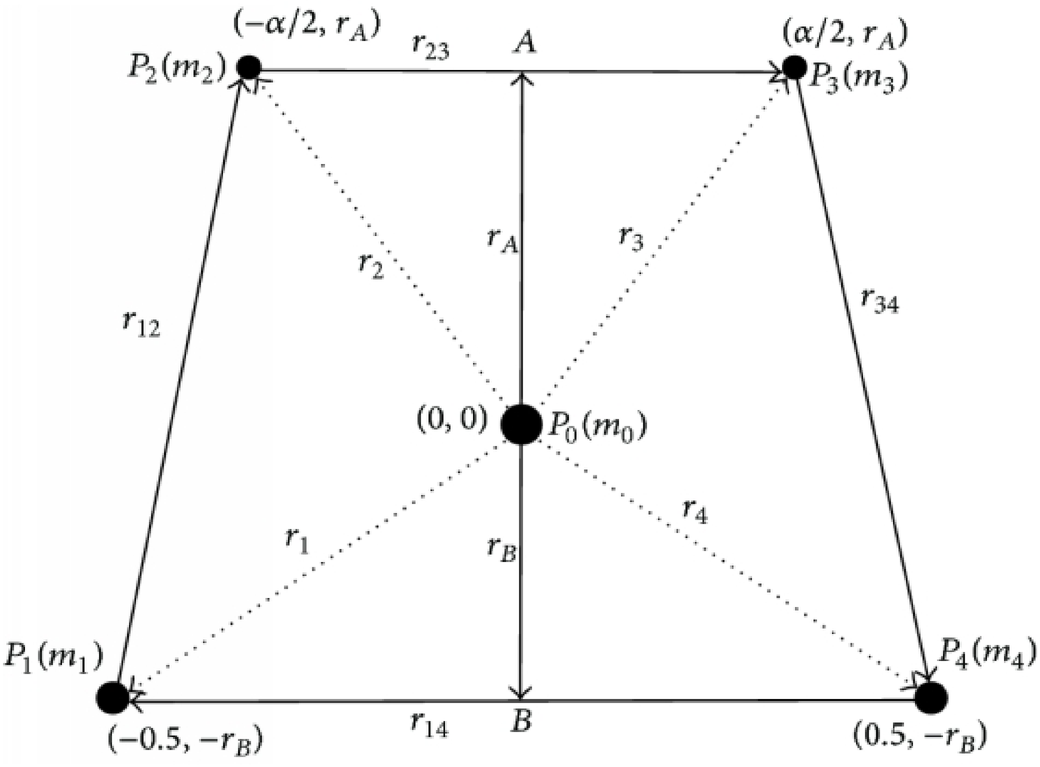

FIGURE 3.1: Configurations of four and five body trapezoidal

Let us consider the masses of five bodies m_0, m_1, m_2, m_3 and m_4. In Figure 3.1 m_0 is at rest at center of mass (c.o.m) of the system and m_1, m_2, m_3 and m_4 are placed at the vertices of an isosceles trapezoid. The geometry of the problem is symmetric about the y-axis. From the Figure 3.1, the c.o.m of m_2 and m_3 is at A, and the c.o.m of m_1 and m_4 is at B. Let $m_2 = m_3 = m$ and $m_1 = m_4 = M$. For the five bodies, coordinates selected are given as:

$$\begin{aligned}
\mathbf{r}_0 &= (0,0),\\
\mathbf{r}_1 &= \Big(-\frac{1}{2}, -r_B\Big),\\
\mathbf{r}_2 &= \Big(-\frac{\alpha}{2}, r_A\Big),\\
\mathbf{r}_3 &= \Big(\frac{\alpha}{2}, r_A\Big),\\
\mathbf{r}_4 &= \Big(\frac{1}{2}, -r_B\Big),
\end{aligned} \tag{3.4}$$

where the distance from the center of mass of the system to the center of mass of m_2 and m_3 is at r_A and the distance from the center of mass of the system to the center of mass of m_1 and m_4 is at r_B. Without loss of generality, suppose that $\mathbf{r}_{23}$ = - $\alpha\mathbf{r}_{41}$ and $r_{BA} = |\mathbf{r}_A$ - $\mathbf{r}_B| = \beta r_{41}$.

To find the vectors $\mathbf{r}_A$ and $\mathbf{r}_B$, we will use the center of mass of the system defined as:

$$\mathbf{c} = \frac{m\mathbf{r}_A + M\mathbf{r}_B}{m+M},$$

because the c.o.m is at origin i.e, $\mathbf{c} = \mathbf{0}$, therefore above equation will simplify as,

$$\begin{aligned}
& m\mathbf{r}_A + M\mathbf{r}_B = \mathbf{0},\\
\Rightarrow \quad & \mathbf{r}_A = -\frac{M}{m}\mathbf{r}_B \quad \text{or} \quad \mathbf{r}_B = -\frac{m}{M}\mathbf{r}_A,\\
\text{or} \quad & r_A = \frac{M}{m}r_B \quad \text{or} \quad r_B = \frac{m}{M}r_A,
\end{aligned}$$

where

$$|\mathbf{r}_A| = r_A \quad and \quad |\mathbf{r}_B| = r_B.$$

We know that $\mathbf{r}_{BA} = \mathbf{r}_A$ - $\mathbf{r}_B = \mathbf{r}$,

$$
\begin{aligned}
\mathbf{r}_B &= \mathbf{r}_A - \mathbf{r}, \\
\mathbf{r}_B &= -\frac{M}{m}\mathbf{r}_B - \mathbf{r}, \qquad \because \ \mathbf{r}_A = -\frac{M}{m}\mathbf{r}_B \\
\mathbf{r}_B + \frac{M}{m}\mathbf{r}_B &= -\mathbf{r}, \\
\left(\frac{m+M}{m}\right)\mathbf{r}_B &= -\mathbf{r}, \\
\mathbf{r}_B &= -\left(\frac{m}{m+M}\right)\mathbf{r}, \\
|\mathbf{r}_B| &= \left(\frac{m}{m+M}\right)|\mathbf{r}|,
\end{aligned}
$$

or

$$
r_B = \left(\frac{m}{m+M}\right)r, \tag{3.5}
$$

where $|\mathbf{r}_B| = r_B \ \ and \ \ |\mathbf{r}| = r$. Similarly

$$
r_A = \left(\frac{M}{m+M}\right)r.
$$

Using equation (3.5) in 2nd expression of equation (3.4), we get:

$$
\mathbf{r}_1 = \left(-\frac{1}{2}, -\frac{m}{m+M}r\right). \tag{3.6}
$$

Following the same procedure, we can get the value of $\mathbf{r}_2$, $\mathbf{r}_3$ and $\mathbf{r}_4$ as:

$$
\mathbf{r}_2 = \left(-\frac{\alpha}{2}, \frac{M}{m+M}r\right), \tag{3.7}
$$

$$\mathbf{r}_3 = \left(\frac{\alpha}{2}, \frac{M}{m+M}r\right), \tag{3.8}$$

$$\mathbf{r}_4 = \left(\frac{1}{2}, -\frac{m}{m+M}r\right). \tag{3.9}$$

Taking the magnitude of equation (3.6), we obtained:

$$\begin{aligned}
|\mathbf{r}_1| \;=\; r_1 \;&=\; \left(\frac{1}{4} \;+\; \frac{m^2}{(m+M)^2}r^2\right)^{1/2}, \\
r_1 \;&=\; \left(\frac{1}{4} \;+\; \frac{m^2}{(m+M)^2}\beta^2 {r^2}_{41}\right)^{1/2}, \qquad (r \;=\; \beta r_{41}) \\
r_1 \;&=\; \left(\frac{1}{4} \;+\; \frac{m^2\beta^2}{(m+M)^2}\right)^{1/2}, \qquad\qquad r_{41} = 1 \;\; \text{(see Figure 3.1)} \\
r_1^3 \;&=\; \left(\frac{1}{4} \;+\; \frac{m^2\beta^2}{(m+M)^2}\right)^{3/2}.
\end{aligned}$$

Similarly, we find

$$\begin{aligned}
r_2^3 \;&=\; \left(\frac{\alpha^2}{4} \;+\; \frac{M^2\beta^2}{(m+M)^2}\right)^{3/2}, \\
r_3^3 \;&=\; \left(\frac{\alpha^2}{4} \;+\; \frac{M^2\beta^2}{(m+M)^2}\right)^{3/2}, \\
r_4^3 \;&=\; \left(\frac{1}{4} \;+\; \frac{m^2\beta^2}{(m+M)^2}\right)^{3/2}, \\
r_{12}^3 \;&=\; \left(\left(\frac{1}{2} - \frac{\alpha}{2}\right)^2 \;+\; \beta^2\right)^{3/2}, \\
r_{13}^3 \;&=\; \left(\left(\frac{1}{2} + \frac{\alpha}{2}\right)^2 \;+\; \beta^2\right)^{3/2}, \\
r_{24}^3 \;&=\; \left(\left(\frac{1}{2} + \frac{\alpha}{2}\right)^2 \;+\; \beta^2\right)^{3/2}, \\
r_{34}^3 \;&=\; \left(\left(\frac{1}{2} - \frac{\alpha}{2}\right)^2 \;+\; \beta^2\right)^{3/2}.
\end{aligned}$$

Equation (3.1) becomes for $i = 1$,

$$\begin{aligned}
\ddot{\mathbf{r}}_1 &= \sum_{j\neq 1, j=0}^{4} \frac{m_j(\mathbf{r}_j - \mathbf{r}_1)}{|\mathbf{r}_j - \mathbf{r}_1|^3}, \\
\ddot{\mathbf{r}}_1 &= \frac{m_0(\mathbf{r}_0 - \mathbf{r}_1)}{|\mathbf{r}_0 - \mathbf{r}_1|^3} + \frac{m_2(\mathbf{r}_2 - \mathbf{r}_1)}{|\mathbf{r}_2 - \mathbf{r}_1|^3} + \frac{m_3(\mathbf{r}_3 - \mathbf{r}_1)}{|\mathbf{r}_3 - \mathbf{r}_1|^3} + \frac{m_4(\mathbf{r}_4 - \mathbf{r}_1)}{|\mathbf{r}_4 - \mathbf{r}_1|^3}, \\
\ddot{\mathbf{r}}_1 &= \frac{-m_0(\mathbf{r}_1)}{r_1^3} + \frac{m(\mathbf{r}_{12})}{r_{12}^3} + \frac{m(\mathbf{r}_{13})}{r_{13}^3} + \frac{M(\mathbf{r}_{14})}{r_{14}^3}. \qquad (3.10)
\end{aligned}$$

Similarly for $i = 2, 3$ and 4,

$$\ddot{\mathbf{r}}_2 = \frac{-m_0(\mathbf{r}_2)}{r_2^3} + \frac{M(\mathbf{r}_{21})}{r_{21}^3} + \frac{m(\mathbf{r}_{23})}{r_{23}^3} + \frac{M(\mathbf{r}_{24})}{r_{24}^3}, \qquad (3.11)$$

$$\ddot{\mathbf{r}}_3 = \frac{-m_0(\mathbf{r}_1)}{r_3^3} + \frac{M(\mathbf{r}_{31})}{r_{31}^3} + \frac{m(\mathbf{r}_{32})}{r_{32}^3} + \frac{M(\mathbf{r}_{34})}{\mathbf{r}_{34}^3}, \qquad (3.12)$$

$$\ddot{\mathbf{r}}_4 = \frac{-m_0(\mathbf{r}_4)}{r_4^3} + \frac{M(\mathbf{r}_{41})}{r_{41}^3} + \frac{m(\mathbf{r}_{42})}{r_{42}^3} + \frac{M(\mathbf{r}_{43})}{r_{43}^3}. \qquad (3.13)$$

Now we put the value of $r_1,\ r_{12},\ r_{13}$ and r_{14} in equation (3.10), equation (3.10) becomes,

$$\ddot{\mathbf{r}}_1 = \frac{-m_0(\mathbf{r}_1)}{d} + \frac{m(\mathbf{r}_{12})}{a} + \frac{m(\mathbf{r}_{13})}{b} + M(\mathbf{r}_{14}). \qquad (3.14)$$

Similarly equations (3.11)—(3.13) take the following form,

$$\ddot{\mathbf{r}}_2 = \frac{-m_0(\mathbf{r}_2)}{c} + \frac{M(\mathbf{r}_{21})}{a} + \frac{m(\mathbf{r}_{23})}{\alpha^3} + \frac{M(\mathbf{r}_{24})}{b}, \qquad (3.15)$$

$$\ddot{\mathbf{r}}_3 = \frac{-m_0(\mathbf{r}_1)}{c} + \frac{M(\mathbf{r}_{31})}{b} + \frac{m(\mathbf{r}_{32})}{\alpha^3} + \frac{M(\mathbf{r}_{34})}{a}, \qquad (3.16)$$

$$\ddot{\mathbf{r}}_4 = \frac{-m_0(\mathbf{r}_4)}{d} + M(\mathbf{r}_{41}) + \frac{m(\mathbf{r}_{42})}{b} + \frac{M(\mathbf{r}_{43})}{a}, \qquad (3.17)$$

where

$$
\begin{aligned}
a &= \left(\left(\frac{1}{2}-\frac{\alpha}{2}\right)^2 + \beta^2\right)^{3/2}, \\
b &= \left(\left(\frac{1}{2}+\frac{\alpha}{2}\right)^2 + \beta^2\right)^{3/2}, \\
c &= \left(\frac{\alpha^2}{4} + \frac{M^2\beta^2}{(m+M)^2}\right)^{3/2}, \\
d &= \left(\frac{1}{4} + \frac{m^2\beta^2}{(m+M)^2}\right)^{3/2}.
\end{aligned}
$$

As m_0 is at rest at c.o.m so $\ddot{\mathbf{r}}_0 = \mathbf{0}$. The c.o.m is at origin; therefore, $\mathbf{r}_0 = \mathbf{0}$. Hence the CC equation $\ddot{\mathbf{r}}_0 = -\lambda\mathbf{r}_0$ will simplify be $0 = 0$ form, therefore we don't have fifth equation. Let $\mathbf{r} = \mathbf{r}_A - \mathbf{r}_B$ (see Figure 3.1) and then using the geometry of our problem we get the relationships between $\mathbf{r}_i$, ($i = 1-4$), and $\mathbf{r}$ and $\mathbf{r}_{41}$ as:

$$
z \left.
\begin{aligned}
\mathbf{r}_1 &= -\frac{m}{M+m}\mathbf{r} + \frac{1}{2}\mathbf{r}_{41}, \\
\mathbf{r}_2 &= \frac{M}{M+m}\mathbf{r} + \frac{1}{2}\alpha\mathbf{r}_{41}, \\
\mathbf{r}_3 &= \frac{M}{M+m}\mathbf{r} - \frac{1}{2}\alpha\mathbf{r}_{41}, \\
\mathbf{r}_4 &= -\frac{m}{M+m}\mathbf{r} - \frac{1}{2}\mathbf{r}_{41}.
\end{aligned}
\right\} \tag{3.18}
$$

Using above expression in equation (3.14), we obtain

$$
\frac{1}{2}\ddot{\mathbf{r}}_{41} - \frac{m}{M+m}\ddot{\mathbf{r}} = -\frac{m_0}{d}\left(\left(-\frac{m}{M+m}\right)\mathbf{r} + \frac{1}{2}\mathbf{r}_{41}\right) + \frac{m}{\alpha}\left(\left(\frac{\alpha-1}{2}\right)\mathbf{r}_{41} + \mathbf{r}\right)
$$
$$
+M(-\mathbf{r}_{41}) + \frac{m}{b}\left(\mathbf{r} - \left(\frac{\alpha+1}{2}\right)\mathbf{r}_{41}\right).
$$

Simplifying the above expression, we get

$$
\begin{aligned}
\frac{1}{2}\ddot{\mathbf{r}}_{41} = & -\frac{m_0}{d}\Big(\frac{1}{2}\mathbf{r}_{41}\Big) + \frac{m}{\alpha}\Big(\Big(\frac{\alpha-1}{2}\Big)\mathbf{r}_{41}\Big) - M(\mathbf{r}_{41}) - \frac{m}{b}\Big(\Big(\frac{\alpha+1}{2}\Big)\mathbf{r}_{41}\Big), \\
\ddot{\mathbf{r}}_{41} = & \Big(-\frac{m_0}{d} + \frac{m}{\alpha}(\alpha-1) - 2M - \frac{m}{b}(\alpha+1)\Big)\mathbf{r}_{41}. \qquad (3.19)
\end{aligned}
$$

Similarly equation (3.15) in terms of $\mathbf{r}$ and $\mathbf{r}_{41}$ yields,

$$
\begin{aligned}
\frac{\alpha}{2}\ddot{\mathbf{r}}_{41} + \frac{M}{m+M}\ddot{\mathbf{r}} = -\frac{m_0}{c}\Big(\Big(\frac{\alpha}{2}\Big)\mathbf{r}_{41} + \frac{M}{m+M}\mathbf{r}\Big) + \frac{M}{a}\Big(\Big(\frac{1-\alpha}{2}\Big)\mathbf{r}_{41} - \mathbf{r}\Big) \\
+\frac{m}{\alpha^3}\Big(-\alpha\mathbf{r}_{41}\Big) + \frac{M}{b}\Big(\Big(-\frac{1}{2} - \frac{\alpha}{2}\Big)\mathbf{r}_{41} - \mathbf{r}\Big).
\end{aligned}
$$

Comparing the coefficient of $\mathbf{r}$ on both sides,

$$
\begin{aligned}
\frac{M}{m+M}\ddot{\mathbf{r}} = & -\frac{m_0}{c}\Big(\frac{M}{m+M}\Big)\mathbf{r} - \frac{M}{a}\mathbf{r} - \frac{M}{b}\mathbf{r}, \\
\frac{M}{m+M}\ddot{\mathbf{r}} = & -\Big(\frac{m_0}{c}\Big(\frac{M}{m+M}\Big) - \frac{M}{a} - \frac{M}{b}\Big)\mathbf{r}, \\
\ddot{\mathbf{r}} = & -\frac{m+M}{M}\Big(\frac{m_0}{c}\Big(\frac{M}{m+M}\Big) - \frac{M}{a} - \frac{M}{b}\Big)\mathbf{r}, \\
\ddot{\mathbf{r}} = & -\Big(\frac{m_0}{c} + \frac{m+M}{a} + \frac{m+M}{b}\Big)\mathbf{r}. \qquad (3.20)
\end{aligned}
$$

We know that c.o.m is at origin, the conditions for equation of CC for the trapezoidal 5BP are,

$$
\ddot{\mathbf{r}}_i = -\lambda\mathbf{r}_i, \qquad i = 1,\ 2,\ 3, ..., n \qquad (3.21)
$$

Using (3.21) in equation (3.19), we get

$$
\Big(-\frac{m_0}{d} + \frac{m}{a}(\alpha-1) - 2M - \frac{m}{b}(\alpha+1)\Big)\mathbf{r}_{41} = \ -\lambda\mathbf{r}_{41},
$$

$$\begin{aligned}\Big(-\frac{m_0}{d}+\frac{m}{a}(\alpha-1)-2M-\frac{m}{b}(\alpha+1)\Big)\mathbf{r}_{41}+\lambda\mathbf{r}_{41} &= \mathbf{0},\\ \Big(-\frac{m_0}{d}+\frac{m}{a}(\alpha-1)-2M-\frac{m}{b}(\alpha+1)+\lambda\Big)\mathbf{r}_{41} &= \mathbf{0},,\\ \Rightarrow \qquad -\frac{m_0}{d}+\frac{m}{a}(\alpha-1)-2M-\frac{m}{b}(\alpha+1)+\lambda &= 0,\end{aligned}$$

$$\frac{m_0}{d}-\frac{m}{a}(\alpha-1)+2M+\frac{m}{b}(\alpha+1) = \lambda. \tag{3.22}$$

Similarly equation (3.20) becomes,

$$\frac{m_0}{c}+\frac{m+M}{a}+\frac{m+M}{b} = \lambda. \tag{3.23}$$

Equation (3.16) is similar to equation (3.14) and reduces to equation (3.22). Also equation (3.17) is similar to equation (3.15) and becomes equation (3.23). Out of four equations (3.14) to (3.17), we have left only two equations (3.22) and (3.23), therefore in the next section we will solve these equations for m_0, m and M, which explains the region of CC of 5BP.

3.2 Varying Central Mass with Four Equal Masses

Let us suppose the four bodies having equal masses located at the vertices such that $m_i = M,\ i = 1-4$ and $\lambda = 1$. The CC's equations (3.22) and (3.23) become,

$$\frac{m_0}{d}-\frac{M}{a}(\alpha-1)+2M+\frac{M}{b}(\alpha+1) = 1, \tag{3.24}$$

$$\frac{m_0}{c}+\frac{M+M}{a}+\frac{M+M}{b} = 1, \tag{3.25}$$

where

$$
\begin{aligned}
d = &\ \left(\frac{1}{4} + \frac{M^2\beta^2}{(M+M)^2}\right)^{3/2}, \\
d = &\ \left(\frac{1}{4} + \frac{M^2\beta^2}{(4M^2)}\right)^{3/2} = \left(\frac{1}{4} + \frac{\beta^2}{4}\right)^{3/2} = e. \\
c = &\ \left(\frac{\alpha^2}{4} + \frac{M^2\beta^2}{(M+M)^2}\right)^{3/2}, \\
c = &\ \left(\frac{\alpha^2}{4} + \frac{M^2\beta^2}{(4M^2)}\right)^{3/2} = \left(\frac{\alpha^2}{4} + \frac{\beta^2}{4}\right)^{3/2} = f.
\end{aligned}
$$

Simplifying equations (3.24) and (3.25),

$$\frac{m_0}{e} + M\left(2 - \frac{1}{a}(\alpha - 1) + \frac{1}{b}(\alpha + 1)\right) = 1, \tag{3.26}$$

$$\frac{m_0}{f} + 2M\left(\frac{1}{a} + \frac{1}{b}\right) = 1. \tag{3.27}$$

We need to solve equations (3.26) and (3.27) for M and m_0. First of all we take equation (3.27) and find the value of m_0 in terms of M. Solving equation (3.27) for m_0, we get

$$m_0 = f\left(1 - M\left(\frac{2}{a} + \frac{2}{b}\right)\right). \tag{3.28}$$

Using equation (3.28) in (3.26) and simplifying, we obtain

$$\frac{f}{e}\left(1 - M\left(\frac{2}{a} + \frac{2}{b}\right)\right) + M\left(2 - \frac{1}{a}(\alpha - 1) + \frac{1}{b}(\alpha + 1)\right) = 1,$$

$$\begin{aligned}\frac{f}{e}-\frac{f}{e}M\left(\frac{2}{a}+\frac{2}{b}\right)+M\left(2-\frac{1}{a}(\alpha-1)+\frac{1}{b}(\alpha+1)\right) &= 1,\\ -\frac{f}{e}M\left(\frac{2}{a}+\frac{2}{b}\right)+M\left(2-\frac{1}{a}(\alpha-1)+\frac{1}{b}(\alpha+1)\right) &= 1-\frac{f}{e}.\end{aligned}$$

Simplifying the above expression, we obtain

$$M = \frac{ab(f-e)}{g(\alpha,\beta)}, \tag{3.29}$$

where

$$g(\alpha,\beta) = 2af + 2bf + eb(\alpha-1) - ea(\alpha+1) - 2eab.$$

Now using equation (3.29) in (3.28), we get

$$\frac{m_0}{f} + 2\frac{ab(f-e)}{g(\alpha,\beta)}\left(\frac{1}{a}+\frac{1}{b}\right) = 1,$$

$$m_0 = \frac{efh(\alpha,\beta)}{g(\alpha,\beta)}, \tag{3.30}$$

where

$$h(\alpha,\beta) = a + b - \alpha(a-b) - 2ab.$$

3.2.1 Positivity analysis of M and m_0

There are three factors in the expressions of M and m_0 (i.e, $(f\ -\ e)$, $g(\alpha,\beta)$ and $h(\alpha,\beta)$) which can make M and m_0 negative. The sign analysis of $(f-e)$, $g(\alpha,\beta)$

and $h(\alpha, \beta)$ is required to obtain the regions of CC in $\alpha\beta$-plane where $M > 0$ and $m_0 > 0$.

(i) As $\alpha \in (0, 1)$, $\alpha^2 < 1$; so $e > f$ $\forall$ α and β, then $(f - e) < 0$.
(ii) For sign analysis of $h(\alpha, \beta)$, we obtain the following region where $h(\alpha, \beta) < 0$;

$$\begin{aligned}
&h(\alpha, \beta) < 0, \\
&a + b - \alpha(a - b) - 2ab < 0, \\
&a + b - \alpha a + \alpha b - 2ab < 0, \\
&b + \alpha b - 2ab < \alpha a - a, \\
&b(1 + \alpha - 2a) < a(\alpha - 1), \\
&b < \frac{a(\alpha - 1)}{1 + \alpha - 2a}.
\end{aligned}$$

The region R_1 for which $f - e < 0$ and $h(\alpha, \beta) < 0$ is given below,

$$R_1 = \left\{ (\alpha, \beta) | 0 < \alpha < 1, \beta > 0, b < \frac{a(\alpha - 1)}{1 + \alpha - 2a} \right\}. \tag{3.31}$$

Ideally we need to get a region that's obviously described by the α or β function. It is not possible to get solution of $h(\alpha, \beta) < 0$ in terms of α or β in closed form because of the involvement of radicals. Therefore, we approximate it by a polynomial of degree 2 in α as shown below.

$$\begin{aligned}
h_{app} \approx &- 2\beta^6 - 1.5\beta^4 + 2(j - 0.19)\beta^2 + 0.5j + 0.03(1.5\beta^4 \\
&- (2.25\beta^2 + 0.75)j^{-1} - 0.09)\alpha^2 + O(\alpha^3),
\end{aligned} \tag{3.32}$$

where $j = \sqrt{\beta^2 + 0.25}$. The equation $h(\alpha, \beta)_{app} \approx 0$ shows α in term of β which gets a boundary between $h(\alpha, \beta) < 0$ and $h(\alpha, \beta) > 0$:

$$\alpha_1(\beta) \approx 1.15\sqrt{\frac{j(-\beta^6 - 0.75\beta^4 + (j - 0.18)\beta^2 + 0.25j - 0.02)}{j\beta^4 - 1.5\beta^2 - 0.07j - 0.5}}$$
$$= K_1(\beta). \tag{3.33}$$

Therefore, region R_1 can now be rewritten as below:

$$R_1 = \Big\{(\alpha, \beta) \mid \beta > 0, \alpha(\beta) < K_1(\beta)\Big\}. \tag{3.34}$$

It is numerically satisfied that $h(\alpha,\beta) = 0$ and $h(\alpha,\ \beta)_{app} \approx 0$ are nearly same graphs for all α and β and R_1 is shown in figure (3.2).

(iii) For sign analysis of $g(\alpha,\ \beta)$, we obtain the following region where $g(\alpha,\ \beta) < 0$;

$$\begin{aligned}
&g(\alpha, \beta) < 0,\\
&2af + 2bf + eb(\alpha - 1) - ea(\alpha + 1) - 2eab < 0,\\
&a(2f - e(\alpha + 1)) + 2bf + eb(\alpha - 1) - 2eab < 0,\\
&a(2f - e(\alpha + 1)) + 2bf + eb(\alpha - 1) - 2eab < 0,\\
&2bf + eb(\alpha - 1) - 2eab < -a(2f - e(\alpha + 1)),\\
&b(2f + e(\alpha - 1) - 2ea) < -a(2f - e(\alpha + 1)),\\
&b < \frac{a(e(\alpha + 1) - 2f)}{e(\alpha - 1) + 2f - 2ea}.
\end{aligned}$$

The region R_2 for which $f - e < 0$ and $g(\alpha, \beta) < 0$ are given below,

$$R_2 = \Big\{(\alpha, \beta)|0 < \alpha < 1, \beta > 0, b < \frac{a(-2f + e(\alpha + 1))}{2f + e(\alpha - 1) - 2ea}\Big\}. \tag{3.35}$$

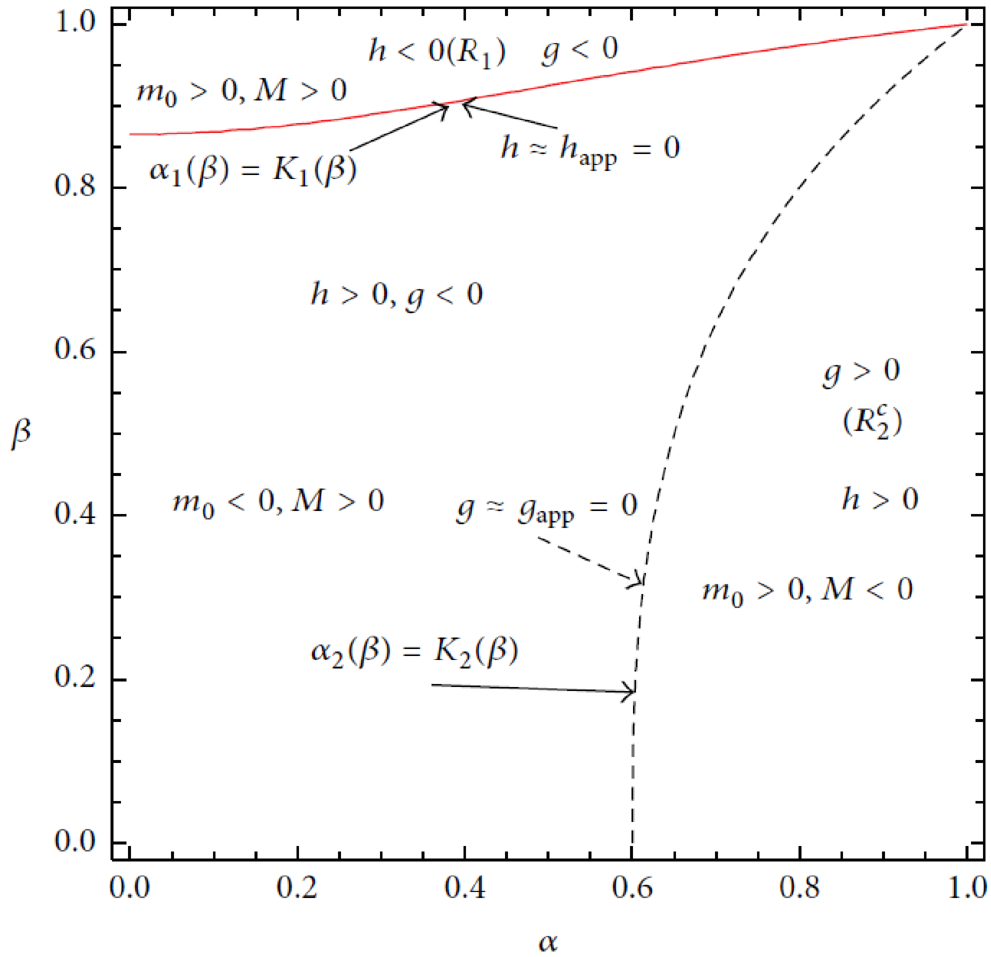

FIGURE 3.2: CC regions in trapezoidal 5BP varying central mass with four same masses.

Ideally we want to obtain a region that's obviously described by the α or β function. It is not possible to get a solution of $g(\alpha, \beta) < 0$ in terms of α or β in closed form because of the involvement of radicals. Therefore, we approximate it by a polynomial of degree 2 in α as shown below.

$$\begin{aligned} g(\alpha, \beta)_{app} \approx &- 0.77\beta^6 + 0.38\beta^5 - 0.74\beta^4 + 0.06\beta^3 \\ &- 0.3\beta^2 - 0.04j^{-1}(0.08\beta + 1.5\beta^5 + 0.75\beta^3 \\ &- 0.3j\sqrt{\beta^2+1}(\beta^6 + \beta^4 - 0.36))\alpha^2. \end{aligned} \tag{3.36}$$

The real valued function $g(\alpha, \beta)_{app} \approx 0$ shows α in terms of β which gets a boundary between $g(\alpha, \beta) < 0$ and $g(\alpha, \beta) > 0$:

$$\alpha_2 \approx \Bigg((j(0.77\beta^6 - 0.38\beta^5 + 0.74\beta^4 - 0.06\beta^3 + 0.3\beta^2 + 0.04)). \Big(0.08\beta + 1.5\beta^5 + 0.75\beta^3 - 0.3j\sqrt{\beta^2+1}(\beta^6 + \beta^4 - 0.36) \Big)^{-1} \Bigg)^{1/2} = K_2(\beta). \quad (3.37)$$

It is numerically satisfied that $g(\alpha, \beta) = 0$ and $g(\alpha, \beta)_{app} = 0$ have almost same graphs for all α and β in Figure 3.2, so R_2 becomes,

$$R_2 = \Big\{ (\alpha, \beta) \mid \beta > 0, \alpha_2(\beta) < K_2(\beta) \Big\}. \quad (3.38)$$

Numerically, region R_2 is shown in Figure 3.2. As the numerator of M is negative for all α and β, so R_2 gives a CC region in $\alpha\beta$-plane where $M > 0$. Similarly $m_0 > 0$ in $(R_1 \cap R_2) \cup (R_1^c \cap R_2^c)$. As $R_2^c = \emptyset$ and $R_1 \subset R_2$, therefore

$$R_3 = (R_1 \cap R_2) = R_1, \quad (3.39)$$

gives the region of CC for this specific system of 5BP where all masses are positive. Numerically, R_1, R_2 and R_3 are shown in Figure 3.2. The CC regions where $M > 0$ or $m_0 > 0$ and $M > 0$ and $m_0 > 0$ are shown in Figure 3.3—3.5 respectively. This Figures shows that the positive region of masses m_0 and M which means the shaded region of our region gives the values of α and β. This α and β put the value the value of a, b, c and d. The value of a, b, c and d put in m_0 and M to find the positive value of m_0 and M. The α and β are distance parameter it means we change the value of α and β than the value of m_0 and M also change. The positivity of m_0 and M depends upon the value of α and β. If we choose α and β in the shaded region of graph than m_0 and M must be positive.

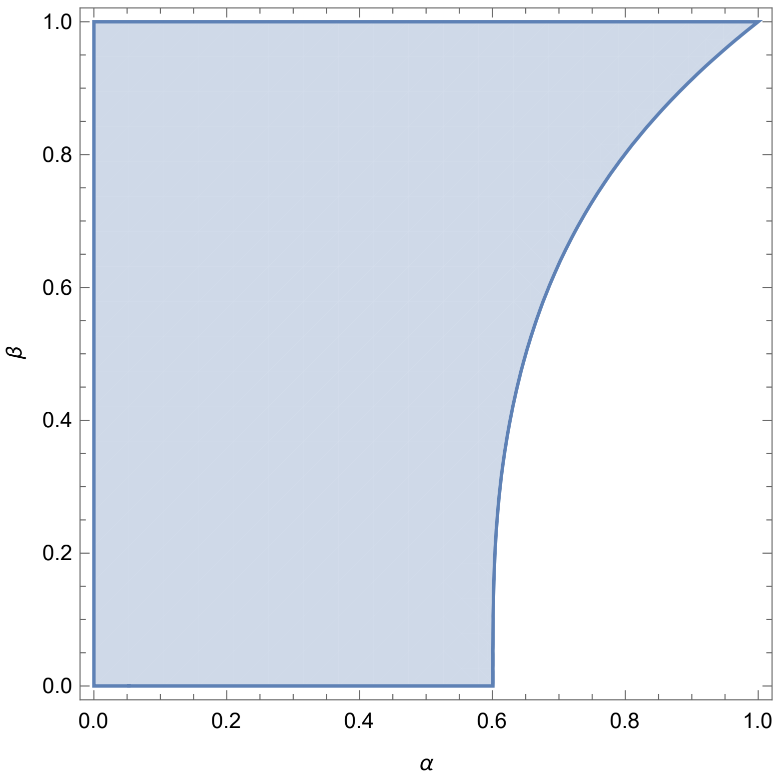

FIGURE 3.3: $M > 0$ (Shaded Region)

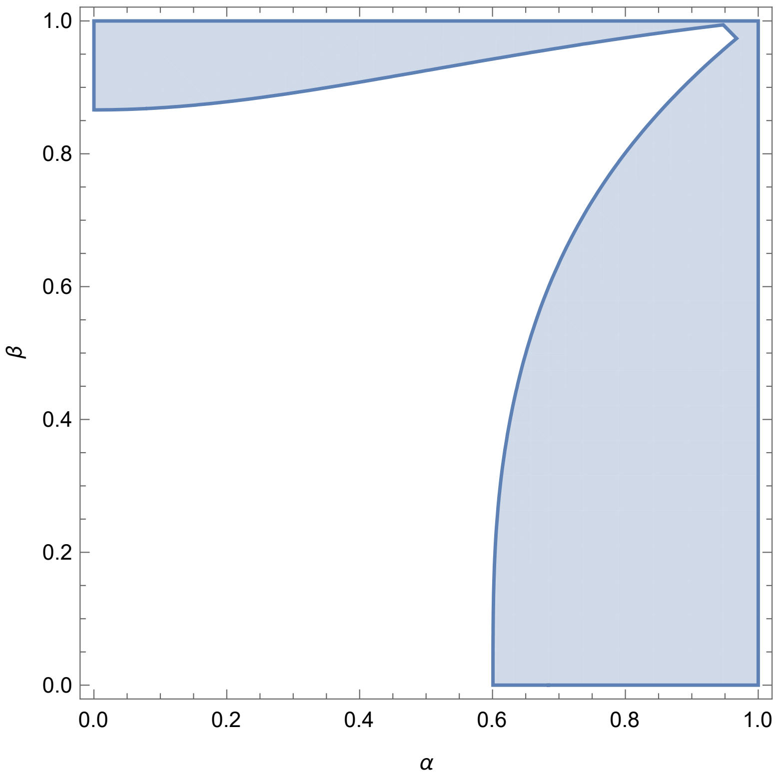

FIGURE 3.4: $m_0 > 0$ (Shaded Region)

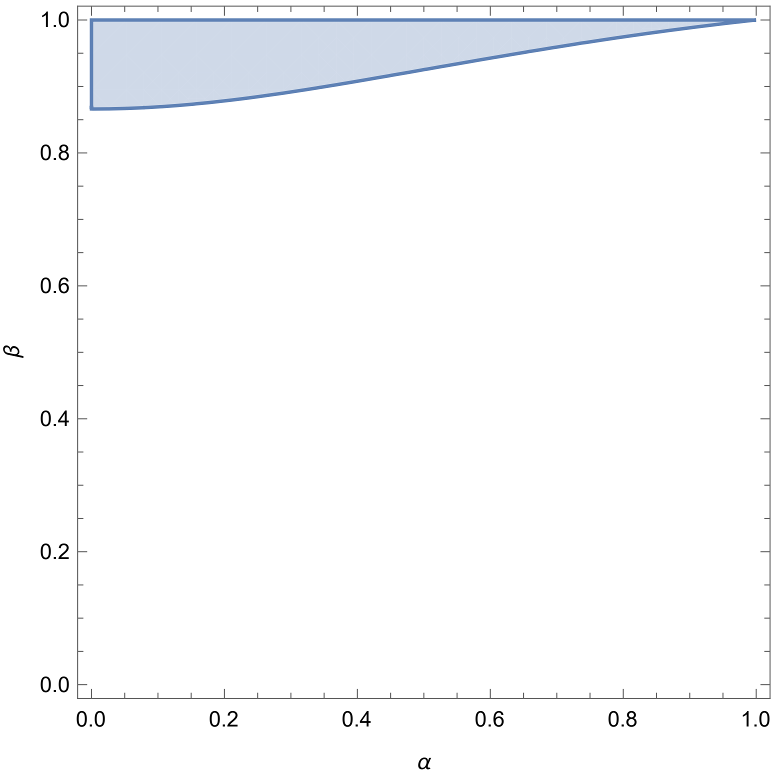

FIGURE 3.5: $M > 0$ *and* $m_0 > 0$ (Shaded Region)

Chapter 4

Central Configuration Regions in Central Mass and Two Pairs of Same Masses

The geometry in Figure 3.1 is same about the line AB, where A is the c.o.m of m_2 and m_3, and B is the center of mass of m_1 and m_4. Let $m_1 = m_4 = M$ and $m_2 = m_3 = m$. We suppose that $m \leq M$ and $m = CM$, where $0 < C \leq 1$. With these assumptions, equations (3.22) and (3.23) reduce to the following equations,

$$\frac{m_0}{e^*} - \frac{CM}{a}(\alpha - 1) + 2M + \frac{CM}{b}(\alpha + 1) = 1, \tag{4.1}$$

$$\frac{m_0}{f^*} + \frac{CM + M}{a} + \frac{CM + M}{b} = 1, \tag{4.2}$$

where

$$d = \left(\frac{1}{4} + \frac{C^2M^2\beta^2}{(CM+M)^2}\right)^{3/2},$$

$$d = \left(\frac{1}{4} + \frac{C^2M^2\beta^2}{(CM+M)^2}\right)^{3/2} = \left(\frac{1}{4} + \frac{C^2\beta^2}{(1+C)^2}\right)^{3/2} = e^*.$$

$$
\begin{aligned}
c = & \left(\frac{\alpha^2}{4} + \frac{M^2\beta^2}{(CM+M)^2}\right)^{3/2}, \\
c = & \left(\frac{\alpha^2}{4} + \frac{M^2\beta^2}{M^2(1+C)^2}\right)^{3/2} = \left(\frac{\alpha^2}{4} + \frac{\beta^2}{(1+C)^2}\right)^{3/2} = f^*.
\end{aligned}
$$

$$
m_0 = f^*\left(1 - \left(\frac{1+C}{a} + \frac{1+C}{b}\right)M\right). \tag{4.3}
$$

Using (4.3) in equation (4.1) and simplifying, we obtain

$$
\begin{aligned}
-f^*\left(1 - \left(\frac{1+C}{a} + \frac{1+C}{b}\right)M\right)/e^* + \frac{CM}{a}(\alpha-1) - 2M - \frac{CM}{b}(\alpha+1) = & \ -1, \\
-\frac{f^*}{e^*} + \frac{f^*}{e^*}\left(\frac{1+C}{a} + \frac{1+C}{b}\right)M + \frac{CM}{a}(\alpha-1) - 2M - \frac{CM}{b}(\alpha+1) = & \ -1, \\
M\left(2e^* - f^*\left(\frac{1+C}{a} + \frac{1+C}{b}\right) - \frac{e^*C}{a}(\alpha-1) + \frac{e^*C}{b}(\alpha+1)\right) = & \ e^* - f^*, \\
M\left(2e^* - f^*\left(\frac{(1+C)b + (1+C)a}{ab}\right) - \frac{e^*C}{a}(\alpha-1) + \frac{e^*C}{b}(\alpha+1)\right) = & \ e^* - f^*, \\
M\left(\frac{2e^*ab - C(\alpha-1)be^* + C(\alpha+1)ae^* - f^*(1+C)b - f^*(1+C)a}{ab}\right) = & \ e^* - f^*.
\end{aligned}
$$

$$
M = \frac{(e^* - f^*)ab}{D(\alpha,\beta,C)}, \tag{4.4}
$$

where

$$
D(\alpha,\beta,C) = 2e^*ab + \alpha Ce^*(a-b) + (a+b)(Ce^* - f^*(C+1)).
$$

Now using equation (4.4) in (4.3), we get

$$
m_0 = \frac{e^*f^*N(\alpha,\beta,C)}{D(\alpha,\beta,C)}, \tag{4.5}
$$

where

$$N(\alpha,\beta,C) = 2ab+\alpha C(a-b)-a-b,$$
$$D(\alpha,\beta,C) = 2e^{*}ab+\alpha Ce^{*}(a-b)+(a+b)(Ce^{*}-f^{*}(C+1)).$$

The value of M and m_0 represents regions of CC for 5BP in $\alpha\beta$C-plane. e.g,
(i) $\alpha = 0.2$, $\beta = 0.8$ and C = 0.3, the value of $M = 0.26$ and the value of $m_0 = 0.04$.
(ii) $\alpha = 0.4$, $\beta = 0.7$ and C = 0.5, the value of $M =$ -0.5 and the value of $m_0 = 0.4$.
As can be seen in number (ii), for particular values of α and β either of the masses becomes negative, which is unrealistic. So, we need to find closed regions in CC where no mass can be negative. The sign investigates of $(e^{*}-f^{*})$, $D(\alpha,\beta,C)$, and $N(\alpha,\beta,C)$ is required which is given below.

(i) $e^{*}-f^{*}>0$: it is simple to prove that $e^{*}-f^{*}>0$ in $R_4 = R_{4a}+R_{4b}+R_{4c}$,

where

$$R_{4a} = \left\{(\alpha,\beta,C)|0<C<0.6, 0<\alpha<1, 0.5\sqrt{\frac{\alpha^2(1+C)-C-1}{C-1}}<\beta<1\right\},$$
$$R_{4b} = \left\{(\alpha,\beta,C)|0.6<C<1, 0<\alpha<\sqrt{\frac{5C-3}{C+1}}, 0<\beta<1\right\},$$

$$R_{4b} = \left\{(\alpha,\beta,C)|0.6<C<1, \sqrt{\frac{5C-3}{C+1}}<\alpha<1, 0<\beta<0.5\sqrt{\frac{\alpha^2(1+C)-1-C}{C-1}}\right\}.$$

Region R_4 is shown in Figure 4.1.

(ii) The sign analysis of $N(\alpha,\beta,C)$ is similar as $h(\alpha,\beta)$ as in previous but it is comparatively easier to write a closed form solution of $N(\alpha,\beta,C)=0$ as $C(\alpha,\beta) = (a+b-2ab)/(\alpha(a-b))$. The $h(\alpha,\beta)$ depends upon α and β and $N(\alpha,\beta,C)$ depends upon α, β and C. This is the only difference between $h(\alpha,\beta)$ and $N(\alpha,\beta,C)$. Therefore, $N(\alpha,\beta,C)$ is positive in the following region,

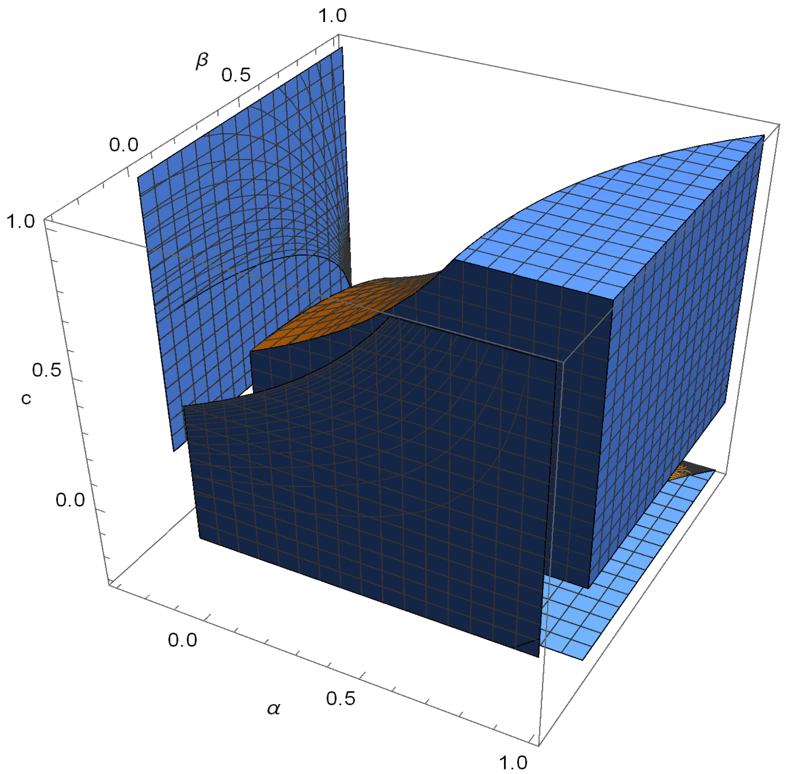

FIGURE 4.1: $(e^* - f^*)(\alpha,\beta,\text{C}) > 0$(colored).

$$
\begin{aligned}
&N(\alpha, \beta, C) > 0, \\
&2ab + \alpha C(a - b) - a - b > 0, \\
&\alpha C(a - b) > a + b - 2ab, \\
&C > \frac{a + b - 2ab}{\alpha(a - b)}.
\end{aligned}
$$

The region R_5 for which $N(\alpha, \beta, C) > 0$ is given below:

$$
R_5 = \left\{ (\alpha, \beta, C) | \frac{a + b - 2ab}{\alpha(a - b)} < C < 1, \beta > 0, 0 < \alpha < 1 \right\}. \tag{4.6}
$$

Similarly $N_{app}(\alpha, \beta, C)$ of as follows:

$$N_{app}(\alpha,\beta,C) \approx (1.5\beta^4 - \sqrt{j}(1.5C - 0.75\beta^2 j^{-1} - 0.375j^{-1}) - 0.1)\alpha^2$$
$$+(2\beta^6 + 1.5\beta^4 + 0.375\beta^2 - \sqrt{j}(2\beta^2 + 0.5) + 0.03). \quad (4.7)$$

Equation $N_{app}(\alpha,\beta,C) \approx 0$ shows α in terms of β and C which gets a boundary between the region where, $N_{app}(\alpha,\beta,C) < 0$ and $N_{app}(\alpha,\beta,C) > 0$:

$$\alpha_3(\beta,C) \approx \sqrt{\frac{-2\beta^6 - 1.5\beta^4 - 0.375\beta^2 + \sqrt{j}(2\beta^2 + 0.5) - 0.03}{1.5\beta^4 - \sqrt{j}(1.5C - 0.75\beta^2 j^{-1} - 0.375j^{-1})}}$$
$$= K_3(\beta,C). \quad (4.8)$$

Therefore, region R_5 can now be rewritten as below:

$$R_5 = \Big\{(\alpha,\beta,C)|0 < C < 1, \beta > 0, 0 < \alpha < K_3(\beta,C)\Big\}. \quad (4.9)$$

It is numerically satisfied that $N(\alpha,\beta,C)$ and $N_{app}(\alpha,\beta,C)$ are nearly same graphs for all α, β and C. R_5 is shown in Figure 4.2.

(iii) The sign analysis of $D(\alpha,\beta,C)$ is similar in nature to $g(\alpha,\beta)$ as in previous but it is comparatively easier to write a closed form solution of $D(\alpha,\beta,C) = 0$ as $C(\alpha,\beta) = (a+b)f^* - 2e^*ab/\alpha e^*(a-b) + (a+b)e^* - (a+b)f^*$. Therefore, $D(\alpha,\beta,C)$ is positive as follows,

$$D(\alpha,\beta,C) > 0,$$
$$2e^*ab + \alpha Ce^*(a-b) + (a+b)(Ce^* - f^*(C+1)) > 0,$$
$$2e^*ab + \alpha Ce^*(a-b) + (a+b)Ce^* - (a+b)f^*(C+1) > 0,$$

$$2e^*ab + \alpha Ce^*(a-b) + (a+b)Ce^* - (a+b)f^*C - (a+b)f^* > 0,$$
$$\alpha Ce^*(a-b) + (a+b)Ce^* - (a+b)f^*C > (a+b)f^* - 2e^*ab,$$
$$C\alpha e^*(a-b) + (a+b)e^* - (a+b)f^* > (a+b)f^* - 2e^*ab,$$

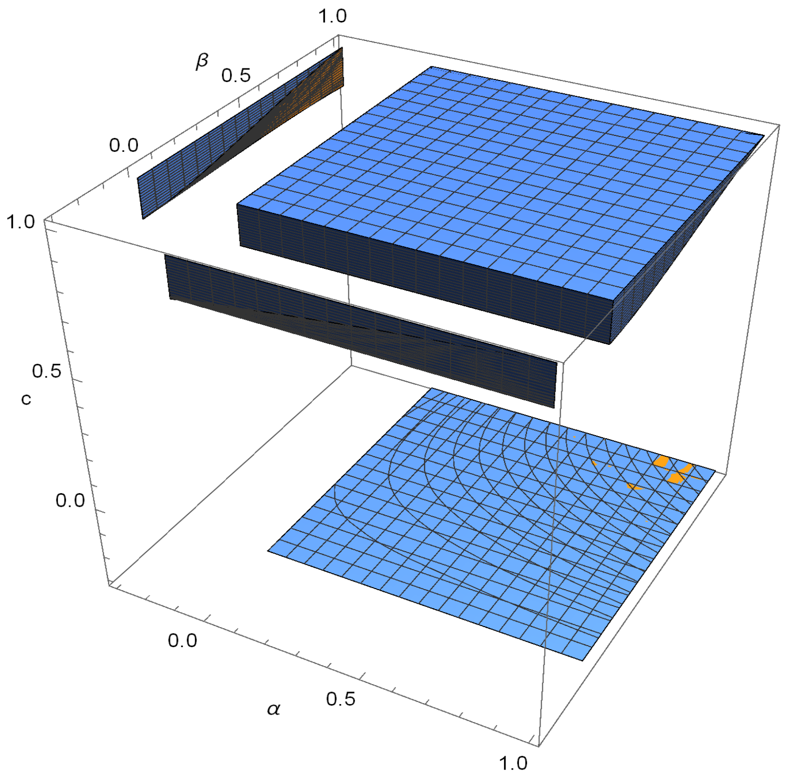

FIGURE 4.2: $N(\alpha,\beta,\text{C}) > 0$(colored).

$$C > \frac{(a+b)f^* - 2e^*ab}{\alpha e^*(a-b) + (a+b)e^* - (a+b)f^*}.$$

The region R_6 for which $D(\alpha,\beta,C)$ is positive as given below,

$$R_6 = \left\{(\alpha,\beta,C)|\frac{(a+b)f^* - 2e^*ab}{[\alpha e^*(a-b) + (a+b)e^* - (a+b)f^*]} < C < 1, \beta > 0, 0 < \alpha < 1\right\}. \tag{4.10}$$

Similarly the polynomial approximation of $D(\alpha,\beta,C)$ is given as below:

$$D_{app}(\alpha,\beta,C) \approx j^4(1.5Ce^*2e^*j^2 - \beta^2(-C^3\sqrt[3]{e^*}+\beta)(C+1)^{-2}) + 0.68C_2\alpha^2,$$

where

$$C_2 = (-1.5Ce^*j-\beta) + e^{*^{2/3}}(0.62+0.75C(1+Cj^2)).(-0.06+\beta^4)(1+6j^2Ce^{*^{1/3}})(C+1)^{-2}.$$

Equation $D_{app}(\alpha,\beta,C)\approx 0$ shows α in terms of β and C which gets a boundary between the region where, $D_{app}(\alpha,\beta,C)<0$ and $D_{app}(\alpha,\beta,C)>0$:

$$\alpha_3(\beta,C) \approx j^2\left(\frac{1}{C_2}(-1.5Ce^*2e^*j^2+\beta^2(-C^3\sqrt[3]{e^*}+\beta).(C+1)^{-2}) - 0.68C_2\alpha^2\right)^{1/2} = K_4(\beta,C) \tag{4.11}$$

Therefore, region R_6 can now be rewritten as,

$$R_6 = \left\{(\alpha,\beta,C)|0<C<1, \beta>0, 0<\alpha<K_4(\beta,C)\right\}.$$

R_6 is shown in Figure 4.3.

From the above investigation we summarize the region of CC where $M>0$ is,

$$R_7 = (R_4\cap R_6)\cup(R_4^c\cap R_6^c). \tag{4.12}$$

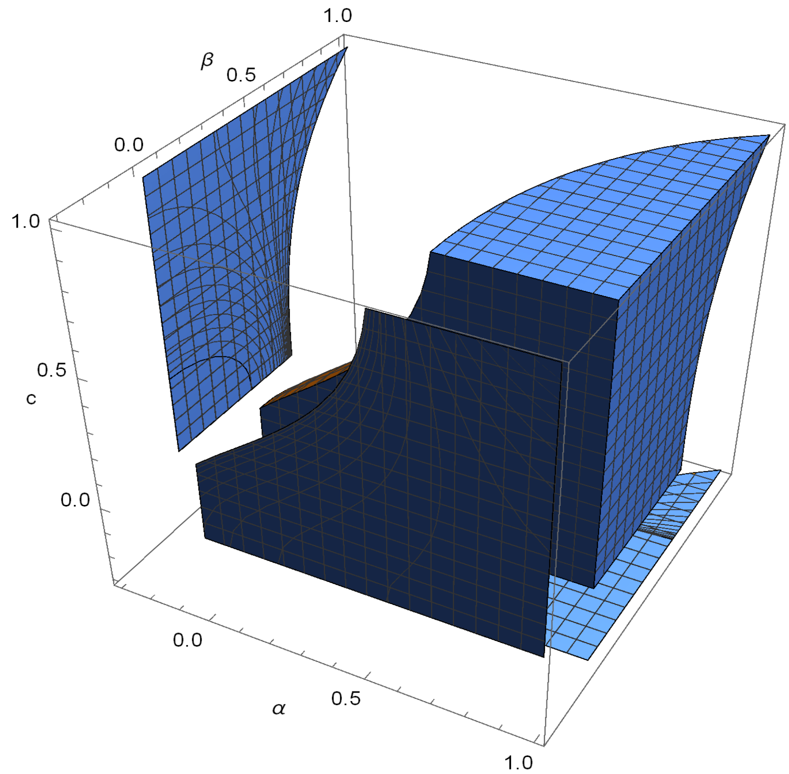

FIGURE 4.3: $D(\alpha,\beta,\mathrm{C})>0$(colored).

Similarly, the CC region where $m_0 > 0$ is,

$$R_8 = (R_5 \cap R_6) \cup (R_5^c \cap R_6^c). \tag{4.13}$$

This provides CC region for this particular setup of 5BP as,

$$R_9 = (R_7 \cap R_8) \tag{4.14}$$

R_9 is shown in Figure 4.4. In the complement of this region no CC is possible.

Hence, the CC region for 5BP with a stationary central mass and two pairs of masses is determined by R_7 and R_8 and is given by $R_9 = R_7 \cap R_8$. Numerically, in Figure 4.4 colored part shows R_9. To get the understanding of CC region R_9 its cross sections are shown in Figures 4.5—4.14 for different values of C.

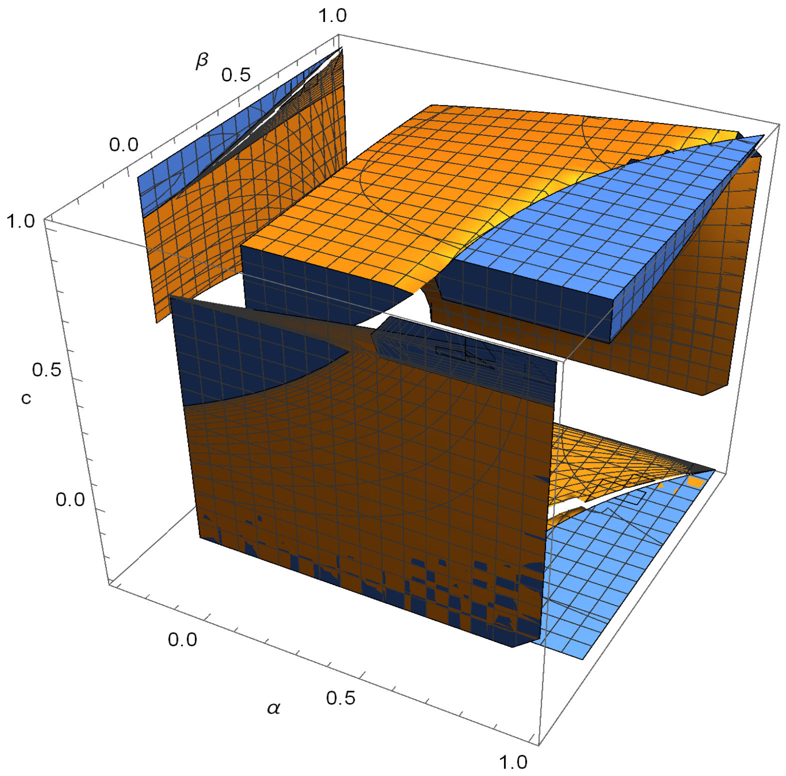

FIGURE 4.4: CC region in trapezoid $4 + 1$ body problem.

Figure 4.4 shows that the CC region in trapezoidal 5BP it mean we choose the value of α, β and C in the shaded region and get the positive masses m_0 and M. If we choose the value c = 0.1 and c= 0.2, the graph shown in Figure 4.5 and Figure 4.6. similarly we check the CC region if we put c = 0.3, c= 0.4, c = 0.5, c = 0.6, c = 0.7, c = 0.8, c = 0.9 and the last one we select c = 1 as shown in graph Figure 4.7 to 4.14. If we put c = 1 than we goes to the first case in which four of the masses are equal and situated at the vertices of and isosceles trapezoid. if we select α, β and C in the above Figure than our system which is two pair os masses are equal following the CC condition. otherwise our system does not the follow the CC condition. The CC condition states that " the acceleration is directly proportional to the position vector ". in our case we put the mass m_0 at the center of the system and this is the (c.o.m) of our system so we put **c** = 0. Generally **c** is not equal to zero.

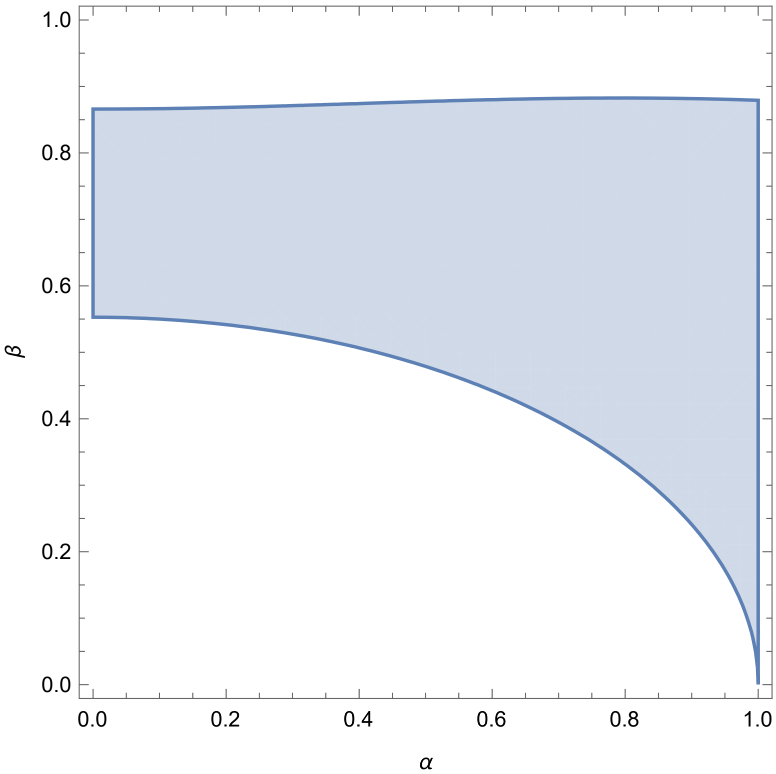

FIGURE 4.5: $M, m_0 > 0$ and $C = 0.1$

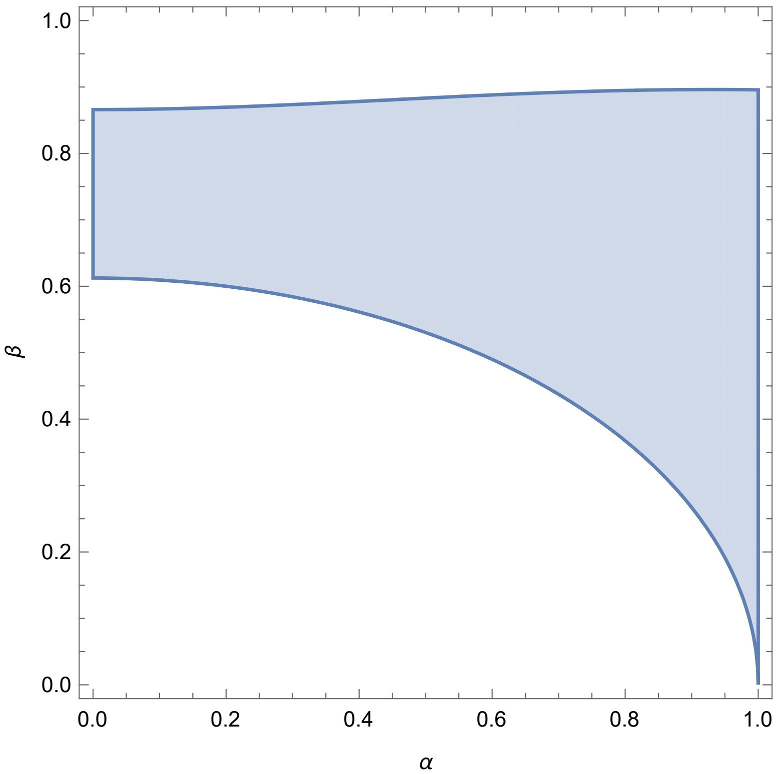

FIGURE 4.6: $M, m_0 > 0$ and $C = 0.2$

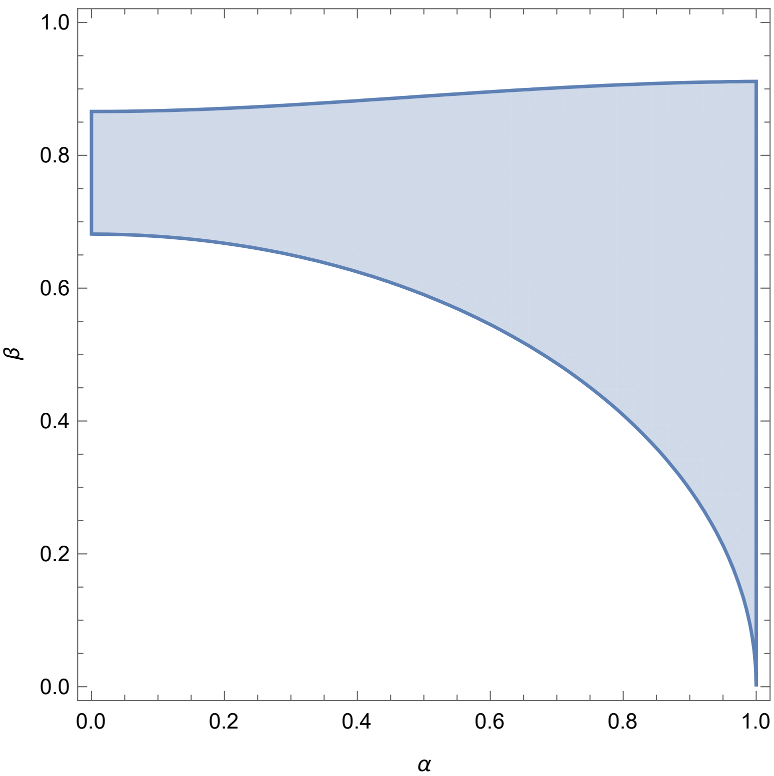

FIGURE 4.7: $M, m_0 > 0$ and $C = 0.3$

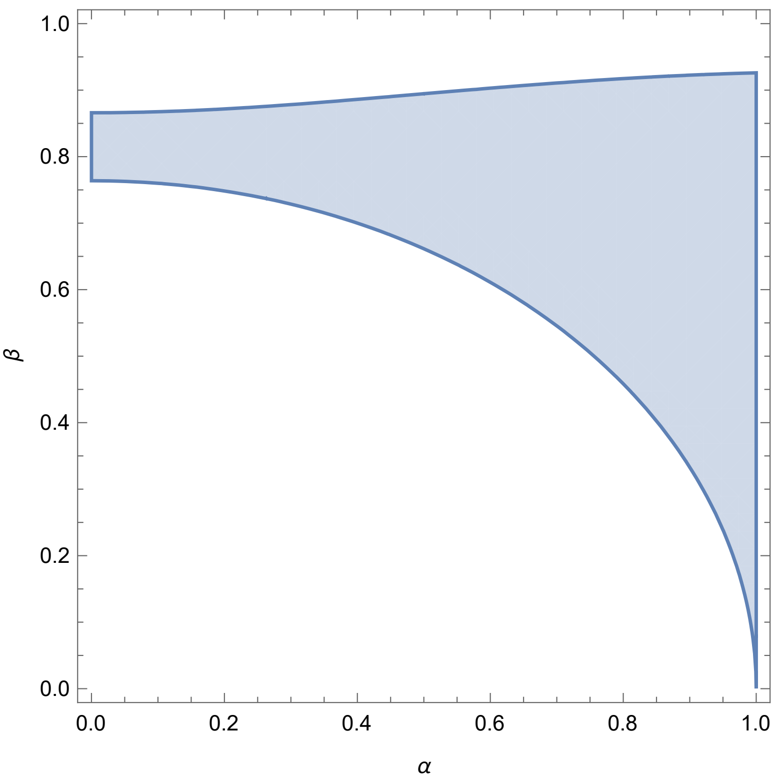

FIGURE 4.8: $M, m_0 > 0$ and $C = 0.4$.

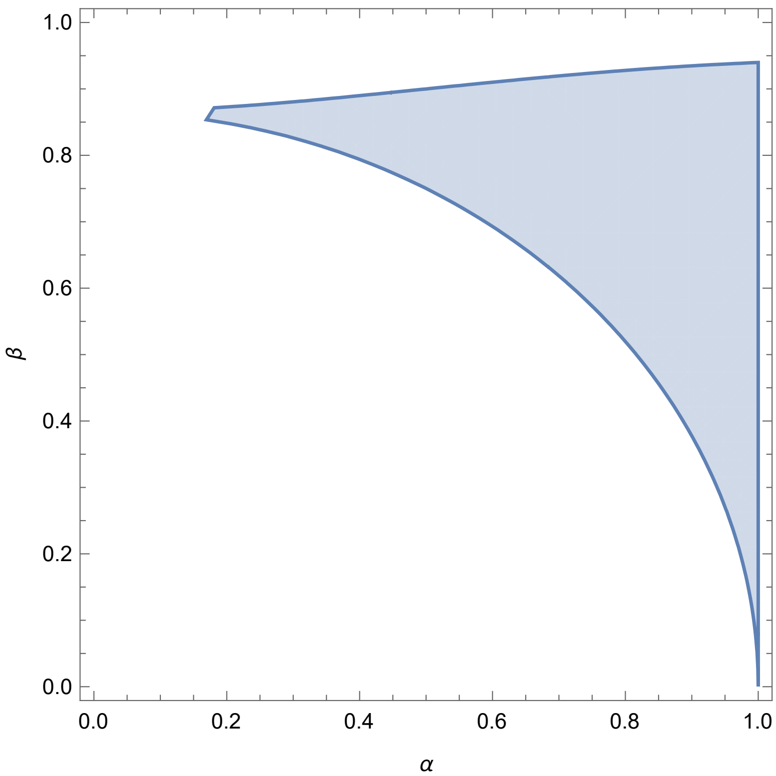

FIGURE 4.9: $M, m_0 > 0$ and $C = 0.5$

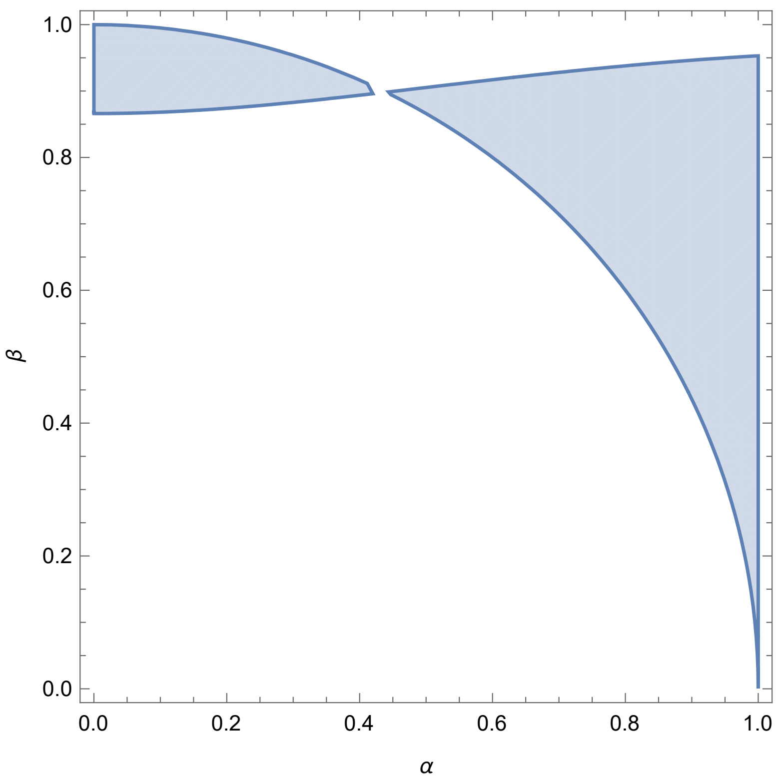

FIGURE 4.10: $M, m_0 > 0$ and $C = 0.6$

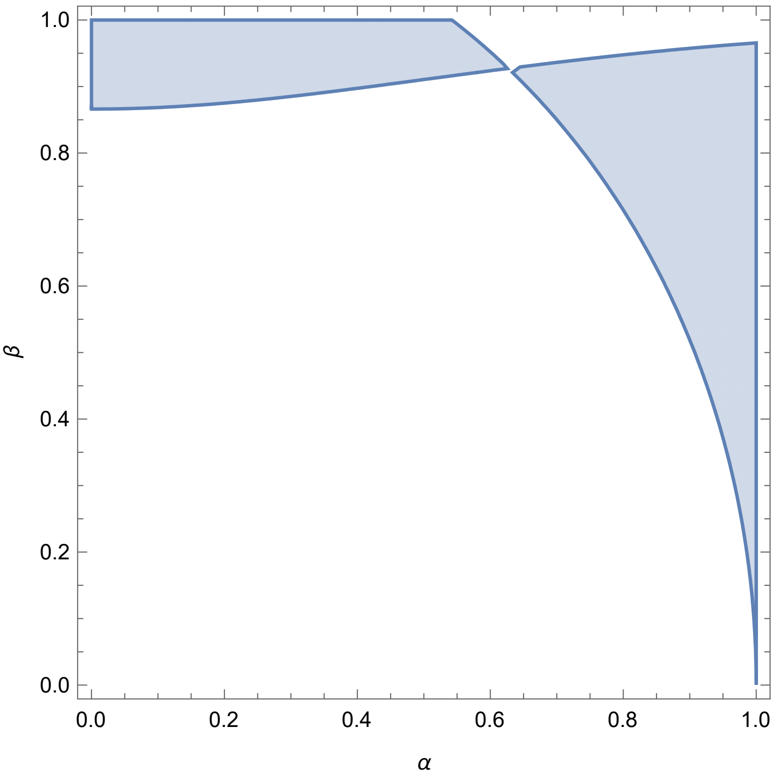

FIGURE 4.11: $M, m_0 > 0$ and $C = 0.7$

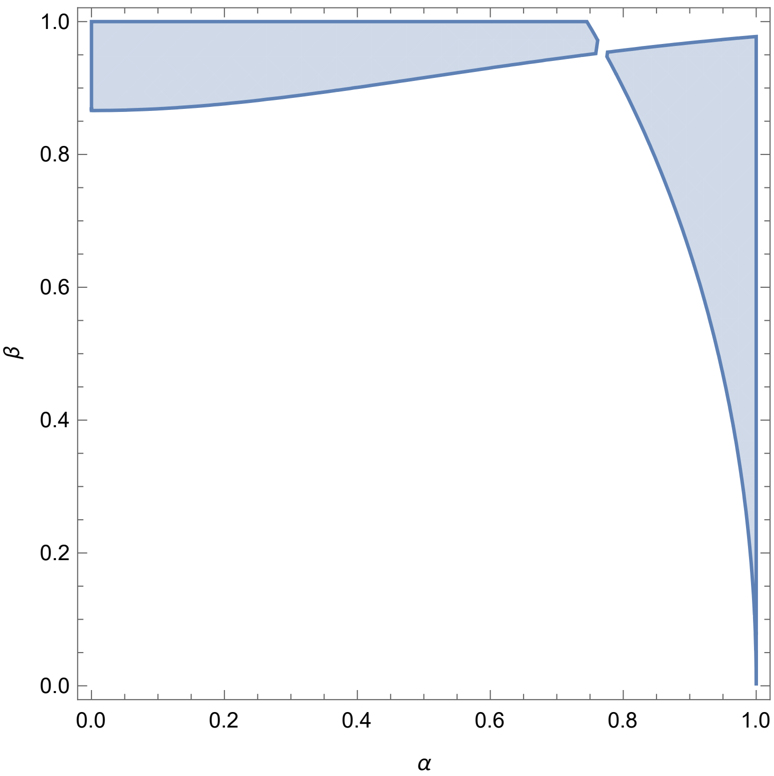

FIGURE 4.12: $M, m_0 > 0$ and $C = 0.8$

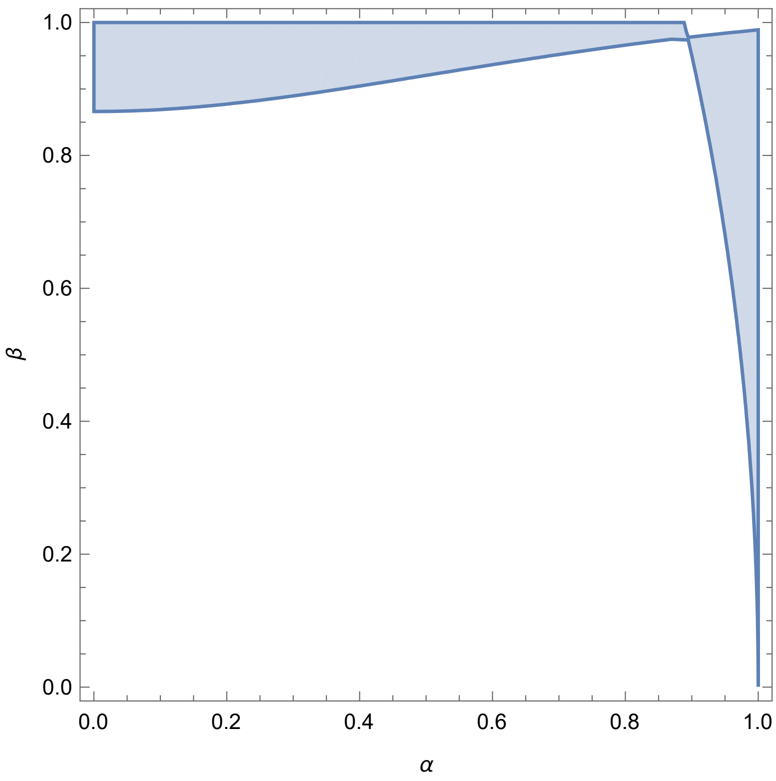

FIGURE 4.13: $M, m_0 > 0$ and $C = 0.9$

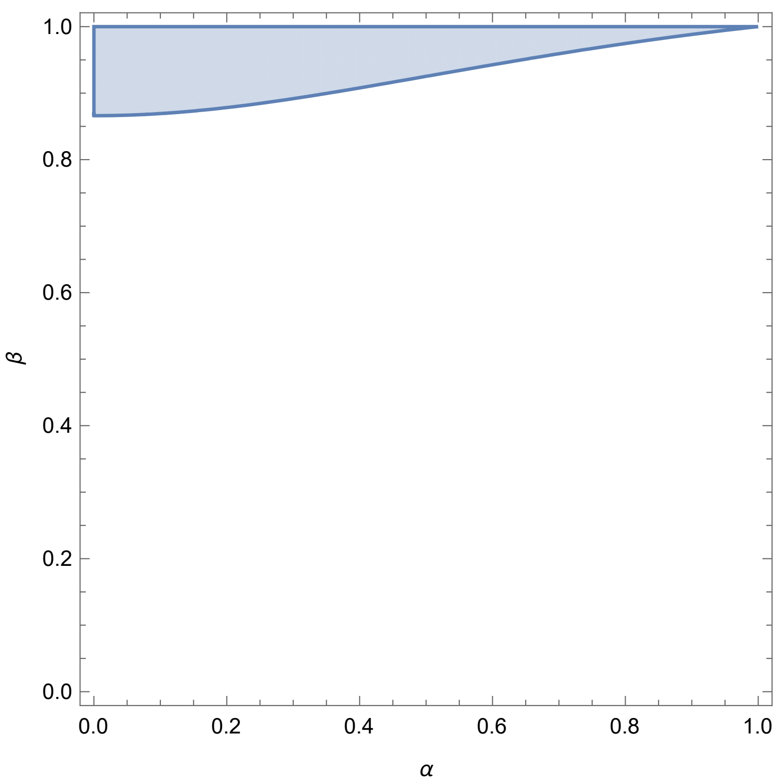

FIGURE 4.14: $M, m_0 > 0$ and $C = 1$

4.1 Special Case ($\alpha = \beta$)

To absolute understanding of five body case with masses of two pairs we study a special case where $\alpha = \beta$. This decreases the variables from three to two while we get sustain the basic geometry with masses of two pairs it means we draw the between β and C. Now we put the value α=β in equation (4.15) and (4.16) and shown in Figure 4.15, 4.16 and 4.17. It is simple to solve the analytical expressions by substituting $a,\ b,\ e^*,\ f^*$ with the following new values:

$$a^* = 1.4(0.2 - 0.4\beta + \beta^2)^{3/2},$$

$$b^* = 1.4(0.2 + 0.4\beta + \beta^2)^{3/2},$$

$$e_* = \left(\frac{\beta^2 C^2}{(C+1)^2} + 0.25\right)^{3/2},$$

$$f_* = \frac{\beta^3}{8}\left(\frac{4}{(C+1)^2} + 1\right)^{3/2}.$$

Using a^*, b^*, e_* *and* f_* in the given equations and find the value of m_0 and M in term of β and C,

$$\frac{m_0}{e_*} - \frac{CM}{a^*}(\alpha - 1) + 2M + \frac{CM}{b^*}(\alpha + 1) = 1, \tag{4.15}$$

$$\frac{m_0}{f_*} + \frac{CM + M}{a^*} + \frac{CM + M}{b^*} = 1, \tag{4.16}$$

CC regions are shown for $\alpha = \beta$ in Figure 4.15—4.17. It is obvious from Figure 4.17 there is no CC are valid for $\beta < 0.42$. For $\beta > 0.42$ there exists minimum one C s.t both m_0 and M are positive and form CC of five body trapezoid.

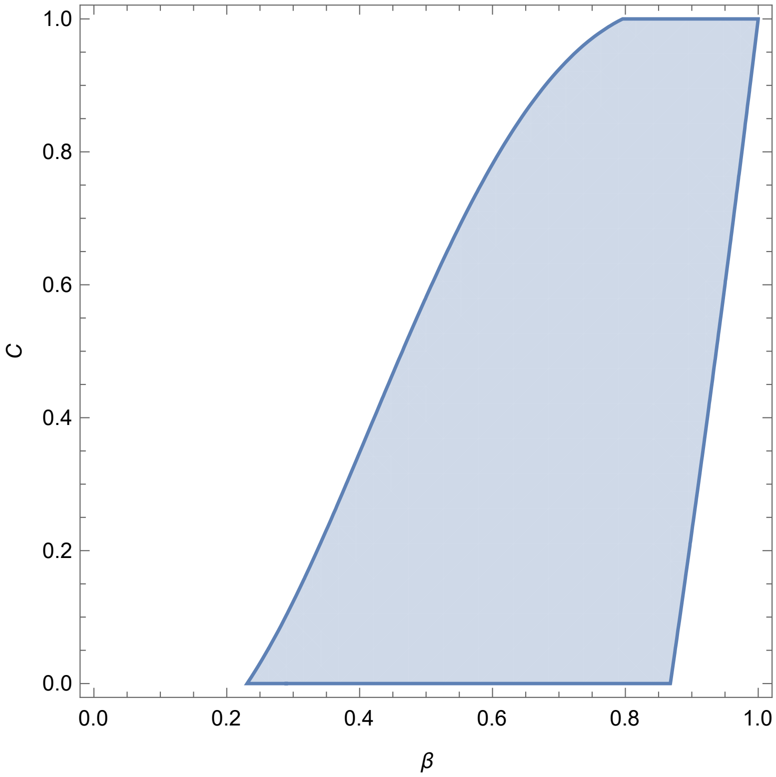

FIGURE 4.15: CC region in 5BP when
$\alpha = \beta, \quad m_0 > 0$

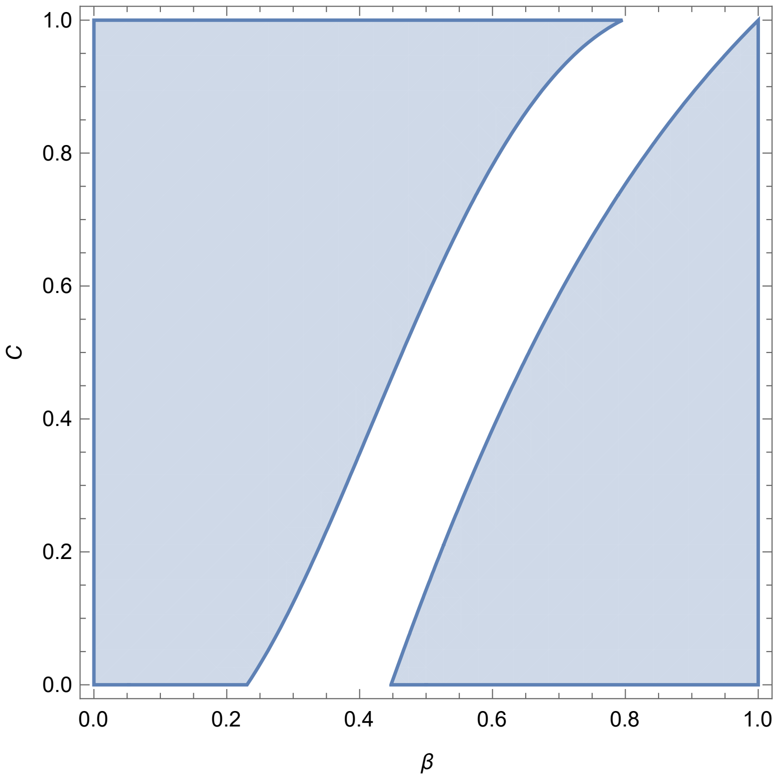

FIGURE 4.16: CC region in 5BP when
$\alpha = \beta, \quad M > 0$

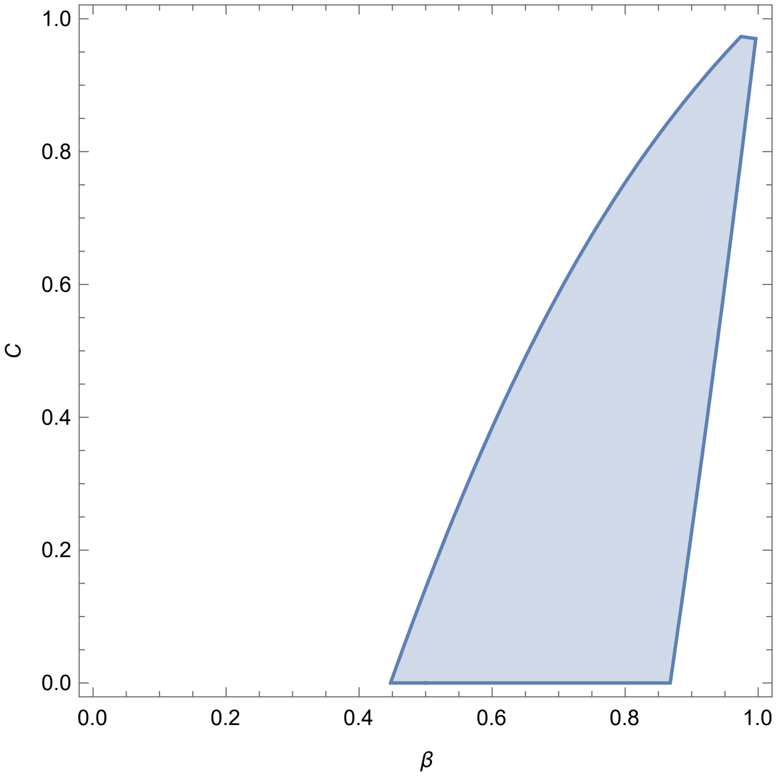

FIGURE 4.17: CC region in 5BP when $\alpha = \beta, \quad m_0 > 0$ and $M > 0$.

Chapter 5

Conclusion

In this review research work [9] CC's region in symmetric 5BP, we model 5BP where fifth mass is placed at the center of the system and other four masses are placed at the vertices of an isosceles trapezoid where m_0 is situated at the center of the system and m_1, m_2, m_3 and m_4 are situated at the vertices of an isosceles trapezoid.

Using the symmetry of the problem firstly we wrote the positions of all masses as a linear combination of vectors $\mathbf{r}$ and $\mathbf{r}_{41}$ (see equation (3.18)). This helped us to reduce the dimension of the problem to a manageable level. Secondly we studied a special case where all $m_i = M$, $(i = 1, 2, 3, 4)$ it means all the masses situated at the vertices of isosceles trapezoid are same. We obtained the expression for m_0 and M as a function of α and β, where α and β are the distance parameter. Furthermore we discussed the CC's regions using the positivity of masses m_0 and M and identified CC, region over the α and β plane where m_0 and M are positive for 5BP.

In the second case we made expression for m_0 and M as function of three parameter α, β and C. In this case we considered two pair of equal masses and a varying central mass and CC's region were defined both numerically and analytically. We summarize this the special case in which we select $\alpha = \beta$ and our system depends upon two variable β and C. Lastly we also found regions in phase space where no CC's are possible for $m_i > 0$, $(i = 1, 2, 3, 4)$

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