CAPITAL UNIVERSITY OF SCIENCE AND TECHNOLOGY, ISLAMABAD

Restricted Trapezoid Five-Body Problem

by

Muti-Ur-Rahman

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First of all, I dedicate this research project to Allah Almighty, The most merciful and beneficent, creator and Sustainer of the earth

And

Dedicated to Prophet Muhammad (peace be upon him) whom, the world where we live and breathe owes its existence to his blessings

And

Dedicated to my parents and Siblings, who pray for me and always pave the way to success for me

And

Dedicated to my teachers, who are a persistent source of inspiration and encouragement for me

CERTIFICATE OF APPROVAL

Restricted Trapezoid Five-Body Problem

by

Muti-Ur-Rahman

MMT181002

THESIS EXAMINING COMMITTEE

S. No.	Examiner	Name	Organization
(a)	External Examiner	Dr. Ibrar Hussain	NUST, Islamabad
(b)	Internal Examiner	Dr. Shafqat Hussain	CUST, Islamabad
(c)	Supervisor	Dr. Abdul Rehman Kashif	CUST, Islamabad

Dr. Abdul Rehman Kashif
Thesis Supervisor
June, 2020

Dr. Muhammad Sagheer
Head
Dept. of Mathematics
June, 2020

Dr. Muhammad Abdul Qadir
Dean
Faculty of Computing
June, 2020

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Abstract

In this thesis, we discuss the central configuration in Restricted Five-Body Problem. Consider four point masses which are placed at the vertices of an isosceles trapezoid with two pairs of equal masses, and find the equation of motion of the fifth mass being negligible and not change the motion of four primaries. After evaluating the equations of motion of the fifth body, calculate the positions of equilibrium points for different intervals and check the stability of equilibrium points finding eigenvalues analysis using mathematica. In the end, explore the Newton basins of attractions relative to the equilibrium points as discussed above.

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Abbreviations

CCs	Central Configurations
2BP	Two-Body Problem
3BP	Three-Body Problem
4BP	Four-Body Problem
5BP	Five-Body Problem
R5BP	Restricted Five-Body Problem
RT5BP	Restricted Trapezoid Five-Body Problem
SI	System International
$\mathbf{M}_s$	Mass of the Sun
N	Numerator
D	Denominator
R	Region

Symbols

Symbol	Name	Unit
$\mathbf{G}$	Universal gravitational constant	$m^3kg^{-1}s^{-2}$
$\mathbf{F}$	Gravitational force	Newton
r	Distance	Meter
P	Linear momentum	$kg\ m\ s^{-1}$
L	Angular momentum	$kg\ m^2\ s^{-1}$
m_i	Point masses	kg
$\mathbb{R}$	Real number's	
$\ni$	Such that	
$\forall$	For all	
$\in$	Belongs to	

Chapter 1

Introduction

The n-body problem in mechanics is the problem of determining the individual motions of a group of celestial objects that interact gravitationally towards each other. The purpose behind resolving these sort of problems is to know about the motion of the moon, the sun, planets, visible stars etc. In the 17th century mathematicians and astronomers were attracted to n-body problem. The problem statement is "What would be the orbit, if we are given n celestial objects interacting with each other under the gravitational forces." Isaac Newton resolved two body-problem (2BP) through his laws of motions and the universal law of gravity. There's no significant way to solve the problem if $n \geq 3$, but if we have a restricted n-body problem it may provide a particular solution. Mathematicians and astronomers have continued working on the n-body problem during the last four centuries. First, in the 17th century, Kepler defined the elliptical trajectories of planets around the sun in his planetary laws of motion between 1609 and 1619 ***Philosophiae Naturalis Principia Mathematica*** [1], One of the most important work in the history of science, in which Isaac Newton derived Kepler's law and formulated it. He gave a more general explanation of the planetary motion by developing Newton's Laws of motion and Newton's Universal Law of Gravitation. In a special case, the law for two point particles as they interact by gravitational force with each other is,

$$\mathbf{F} = G\frac{m_1 m_2}{r^3}\mathbf{r}. \tag{1.1}$$

Where G is a universal gravitational constant and r is the distance of masses m_1 and m_2 from each other. Isaac Newton, after explaining Kepler's laws, turned his attention to comparatively more complex systems. Although, after a lot of struggle, he was unable throughout his life to get any breakthrough in three-body problem (3BP). Alexis Clairaut succeeded in providing an approximation for the 3BP after twenty years of Isaac Newton's death. After some small adjustment, his work accounted for the perigee of the moon, which was the aim of Newton. In 1752 he received the St. Petersburg Academy Award. As Halley's comet travelled through the earth in 1759, the value of his approximations was apply to demonstrate its motion. He himself take off the margin of error which he predicted in his equations, within a month.

In addition, Leonhard Euler and Henri Poincaré works on the 3BP. The extremely influential work of 3BP were ended the traditional work period. In the late 19th century King Oscar II of Sweden set up an award to solve the n-body problem (A more general type of n-body problem instead of 3 masses) Karl Weierstrass, Gsta Mittag-Leffler and Charles Hermite suggested. The statement is as follows: [2]

"*Given a system of arbitrarily many point masses that attract each other according to Newton's law, under the assumption that no two points ever collide, try to find a representation of the coordinates of each point as a series in a variable that is some known function of time and for all of whose values the series converges uniformly*".

Several eminent mathematicians and astronomers worked on it in the 19th century, such as Carl Gustav Jacob Jacobi, Lagrange and Euler. Until 1991, the general solution to the problem was remained unsolved, when a Professor in the University of Arizona, Qiudong Wang published "The global solution of n-body problem" [3]. However his work meets the requirements of King Oscars problem, Wang himself would have described his result as a simple and useless answer while praising the publications that Poincaré have already completed [4].

1.1 Central Configuration

A *Central Configuration*(CC) is a special arrangement of point masses interacting by Newton's law of gravitation with the following property "**the gravitational acceleration vector produced on each mass by all others should point toward the center of mass and proportional to the distance to the center of mass**". CCs play an important role in the study of the Newtonian n-body problem. For example, they lead to the only explicit solutions of the equations of motion, they govern the behaviour of solutions near collisions, and they influence the topology of the integral manifolds.

Approximately 67% of our galaxy stars are known to be included in the multi-stellar system, that is why it is very important to understand the four-body problem and the Five-Body Problem. CC is useful to understand the gravitational problems of n-body [5, 6]. There is a convex central configuration with a given order of the particles for any four positive masses [7, 8]. Moreover, the exact number of central four-body configurations is known only in the case of the four-masses [9]. Any four-body convex central configuration with perpendicular diagonals must be a kite configuration [10]. The convex central configurations with similar opposite masses were studied by Long and Sun [11] and proved that such CC must have a symmetry. M Corbera *et al.* showed the set of central four-body trapezoid configurations with positive masses that form a trapezoid rhombus and isosceles in [12]. They can also be helpful in finding equations of motion and periodic solutions [13]. Kashif *et al.* discussed the CC of the five-body trapezoid isosceles problem where four of the masses are located at the vertices of the trapezoid isosceles and the fifth body will take different position on the axis of symmetry in [14].

1.1.1 Restricted Few-Body Problem

First time Euler's solved the three-body problem for the motion of a particle that is influenced by the gravitational field of two other point masses fixed in space.

This problem is explicitly solvable and provides an approximate solution for moving particles in the gravitational fields. Lagrange points and their stability in a restricted four-body problem where three bodies are finite and fourth is infinitesimal, do not affect the movement of the three bodies moving in circles around their center of mass fixed at the origin explained in [15]. A systematic analysis of periodic orbits was done in the problem of the two-dimensional, elliptic, restricted three-body [16]. The position and stability of the five points of equilibrium in the planar, circular restricted three-body problem is investigated when a variety of studies of drag forces act on the third body [17]. For equal masses, Yan *et al.* studied the existence and linear stability of periodic orbits [18]. In the restricted three-body problem, the presence of transversal ejection-collision orbits discussed [19]. Conley *et al.* discussed new long periodic solutions in plane, of the restricted three-body problem [20]. Simmons and Bakker gave analysis (linear stability) of a rhomboidal 4BP and show that collisions (isolated binary) can be regularized at origin [21]. Prokopenya discussed the stability of the equilibrium solutions in the elliptic restricted many-body problem [22]. Planar central configurations of the Four-Body Problem with three equal masses discussed in [23]. Santos discussed each equilibrium solution must be defined by the primaries along a diagonal [24].

1.1.2 Newton's Basins Of Attraction

The iterative scheme Newton – Raphson uses intrinsic properties of the dynamic system to determine the basins of convergence associated with the Lagrange points. A collection of literature is available that discusses the basins of convergence associated with the Lagrange points in different kinds of dynamic system such as the restricted problem of three bodies [25, 26]. Suraj *et al.* recently studied the "The analysis of restricted five-body problem within frame of variable mass" and investigated the location, movement and stability of Lagrange point with respect to the perturbation parameter that is influenced by the variable mass of the fifth very small body [27]. They also used iterative Newton-Raphson bivariate scheme to investigate the convergence basins for the points of equilibrium. The basins of attraction demonstrate how the system's equilibrium points attract the initial conditions and lie on the configuration plane as nodes, which form the convergence domain of basins. The Newton-Raphson convergence basins, connected to the Lagrange points (which operate as attractors), were discussed in the planar circular restricted 5-body problem [28]. Bogdan *et al.* discussed attraction fractal basins associated with the damped Newton approach [29]. Suraj *et al.* recently studied the concave axisymmetric configuration numerically restricted five-body problems by using the iterative Newton-Raphson scheme to find the convergence of basins [30].

1.2 Thesis Contribution

Assume a restricted five-body problem, in which four positive masses, two pairs of equal masses at adjacent vertices, moving in such a way that their configuration is always a isosceles trapezoid and the small mass m_5 moving in the same plane under the influence of gravity of four primaries, does not influence the movement of four primaries. Calculate the positions of equilibrium points and check the stability of equilibrium points, also find the Newton's basins of attraction for equilibrium points.

1.3 Dissertation Outlines

We divide this dissertation into four chapters.

Chapter 1 introduction of the problem and aim of this research is briefly discussed.

Chapter 2 contains some basic definitions related to celestial mechanics, Newton's laws of motion and Kepler's laws of planetary motion. In the last portion of this chapter, the two-body problem and n-body problem is briefly discussed.

In **Chapter 3**, the paper [31] is comprehensively reviewed.

Chapter 4 summarizes the whole study with concluding remarks.

References used in the thesis are mentioned in **Biblography**.

Chapter 2

Preliminaries

This chapter includes the basic definitions and basic concepts that will help us better understanding of our objective research [32, 33].

2.1 Basic Definitions

Definition 2.1.1.(Motion)

"Motion is the action used to change the location or position of an object with respect to the surroundings over time."

Definition 2.1.2. (Mechanics)

"Mechanics is a branch of physics concerned with motion or change in position of physical objects. It is sometimes further subdivided into:

1. **Kinematics**, which is concerned with the geometry of the motion,

2. **Dynamics**, which is concerned with the physical causes of the motion,

3. **Statics**, which is concerned with conditions under which no motion is apparent."

Definition 2.1.3. (Scalar)

"Various quantities of physics, such as length, mass and time, requires for their specification a single real number (apart from units of measurement which are decided upon in advance). Such quantities are called **Scalars** and the real number is called the magnitude of the quantity."

Definition 2.1.4. (Vector)
"Other quantities of physics, such as displacement, velocity, momentum, force etc require for their specification a direction as well as magnitude. Such quantities are called **Vectors**."

Definition 2.1.5. (Field)
"A field is a physical quantity associated with every point of spacetime. The physical quantity may be either in vector form, scalar form or tensor form."

Definition 2.1.6. (Scalar Field)
"If at every point in a region, a scalar function has a defined value, the region is called a scalar field. i.e.,

$$f : \mathbb{R}^3 \to \mathbb{R},$$

e.g. temperature and pressure fields around the earth."

Definition 2.1.7. (Vector Field)
"If at every point in a region, a vector function has a defined value, the region is called a vector field.

$$V : \mathbb{R}^3 \to \mathbb{R}^3,$$

e.g. tangent vector around a smooth curve."

Definition 2.1.8. (Conservative Vector Field)
"A vector field $\mathbf{V}$ is conservative if and only if there exists a continuously differentiable scalar field f such that $\mathbf{V} = -\nabla f$ or equivalently if and only if

$$\nabla \times \mathbf{V} = \texttt{Curl}\mathbf{V} = \mathbf{0}.\text{''}$$

Definition 2.1.10. (Uniform Force Field)
"A force field which has constant magnitude and direction is called a uniform or constant force field. If the direction of the field is taken as negative z direction and magnitude is constant $F_0 > 0$, then the force field is given by

$$\mathbf{F} = -F_0\hat{\mathbf{k}}.\text{''}$$

Definition 2.1.11. (Central Force)
"Suppose that a force acting on a particle of mass m such that
(a) it is always directed from m toward or away from a fixed point O,
(b) its magnitude depends only on the distance r from O.
then we call the force a central force or central force field with O as the center of force. In symbols $\mathbf{F}$ is a central force if and only if

$$\mathbf{F} = f(r)\mathbf{r}_1 = f(r)\tfrac{\mathbf{r}}{r},$$

where $\mathbf{r}_1 = \frac{\mathbf{r}}{r}$ is a unit vector in the direction of $\mathbf{r}$. The central force is one of attraction towards O or repulsion from O according as *f(r)* < 0 or *f(r)* > 0 respectively."

Definition 2.1.12. (Degree of Freedom)
"The number of coordinates required to specify the position of a system of one or more particles is called number of degree of freedom of the system.
Example: A particle moving freely in space requires 3 coordinates, e.g. (x, y, z), to specify its position. Thus the number of degree of freedom is 3."

Definition 2.1.13. (Center of Mass)
"Let $r_1, r_2, ..., r_n$ be the position vector of a system of n particles of masses $m_1, m_2, ...m_n$ respectively. The center of mass or centroid of the system of particles is defined as that point having position vector

$$\hat{\mathbf{r}} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + ... + m_n\mathbf{r}_n}{m_1 + m_2 + ... + m_n} = \frac{1}{\mathbf{M}}\sum_{\nu=1}^{n} m_\nu \mathbf{r}_\nu,$$

where

$$\mathbf{M} = \sum_{\nu=1}^{n} m_\nu,$$

is the total mass of the system."

Definition 2.1.14. (Center of Gravity)

"If a system of particles is in a uniform gravitational field, the center of mass is sometimes called the center of gravity."

Definition 2.1.15. (Torque)

"If a particle with a position vector $\mathbf{r}$ moves in a force field $\mathbf{F}$, we define $\boldsymbol{\tau}$ as torque or moment of the force as

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}.$$

The magnitude of $\boldsymbol{\tau}$ is

$$\tau = rF\sin\theta.$$

The magnitude of torque is a measure of the turning effect produced on the particle by the force."

Definition 2.1.16. (Momentum)

"The linear momentum $\mathbf{p}$ of an object with mass m and velocity $\mathbf{v}$ is defined as:

$$\mathbf{p} = m\mathbf{v}.$$

Under certain circumstances the linear momentum of a system is conserved. The linear momentum of a particle is related to the net force acting on that object:

$$\mathbf{F} = m\mathbf{a} = m\frac{d\mathbf{v}}{dt} = \frac{d}{dt}(m\mathbf{v}) = \frac{d\mathbf{p}}{dt}.$$

The rate of change of linear momentum of a particle is equal to the net force acting on the object, and is pointed in the direction of the force. If the net force acting on an object is zero, its linear momentum is constant (conservation of linear momentum). The total linear momentum $\mathbf{p}$ of a system of particles is defined as the vector sum of the individual linear momentum.

$$\mathbf{p} = \sum_{1}^{n} \mathbf{p}_i.\text{''}$$

Definition 2.1.17. (Point-like Particle)
"A point-like particle is an idealization of particles mostly used in different fields of physics. Its defining features is the lacks of spatial extension:being zero-dimensional, it does not take up space. A point-like particle is an appropriate representation of an object whose structure, size and shape is irrelevant in a given context. e.g., from far away, a finite-size mass (object) will look like a point-like particle."

Definition 2.1.18. (Angular Momentum)
"Angular momentum for a point-like particle of mass m with linear momentum $\mathbf{p}$ about a point O, defined by the equation

$$\mathbf{L} = \mathbf{r} \times \mathbf{p},$$

where $\mathbf{r}$ is the vector from the point O to the particle. The torque about the point O acting on the particle is equal to the rate of change of the angular momentum about the point O of the particle i.e.,

$$\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt}.\text{''}$$

Definition 2.1.20. (Inertial Frame of Reference)
"A frame of reference that remains at rest or moves with constant velocity with respect to other frames of reference is called inertial frame of reference. Actually,

an unaccelerated frame of reference is an inertial frame of reference. In this frame of reference a body does not acted upon by external forces. Newton's laws of motion are valid in all inertial frames of reference. All inertial frames of reference are equivalent."

Definition 2.1.22. (Equilibrium Solution)

"The **Equilibrium solution** can guide us through the behaviour of the equation that represents the problem without actually solving it. These solutions can be found only if we meet the sufficient condition of all rates equal to zero. If we have two variables then

$$\dot{x} = \dot{y} = \ddot{x} = \ddot{y} = ... = x^{(n)} = y^{(n)} = 0.$$

These solutions may be stable or unstable. The stable solutions regarding in celestial Mechanics helps us find parking spaces where if a satellite or any object placed, it will remain there for ever. These type of places are also found along the Jupiter's orbital path where bodies called trojan are present. These equilibrium points with respect to Celestial Mechanics are also called Lagrange points named after a French mathematician and astronomer Joseph-Louis Lagrange. He was first to find these equilibrium points for the Sun-Earth system. He found that three of these five points were collinear.

Procedure for Stability Analysis and Equilibrium Points:

We need to follow the following steps to check the stability of equilibrium points.

1) Determine the equilibrium points, $\mathbf{x}^*$, solving $\Omega(\mathbf{x}^*) = \mathbf{0}$.

2) Construct the Jacobian matrix, $J(\mathbf{x}^*) = \frac{\partial \Omega}{\partial \mathbf{x}^*}$.

3) Compute eigenvalues of $\Omega(\mathbf{x}^*)$: $det|\Omega(\mathbf{x}^*) - \lambda\ I| = 0$.

4) Stability or instability of $\mathbf{x}^*$ based on the real parts of eigenvalues.

5) Point is stable, if all eigenvalues have real parts negative.

6) Unstable, If (at least) one of the eigenvalues have a positive real part.

7) Otherwise, there is no conclusion, (i.e, require an investigation of higher order terms)."

Definition 2.1.23. (Basin of Attraction)
"Newton method is used to find the roots of equations but Arthur Cayley [34] found that if the roots of a function are already known then Newton's method can guide to another problem that is which initial guesses iterate to which roots and the region of these initial guesses is called basins of attraction of the roots."

2.2 Kepler's Laws of Planetary Motion

"Kepler's three laws of planetary motion can be described as follows:

1. All planets are moving in an elliptical path with sun at one focus.
2. The radius vector drawn from the sun to a planet sweeps out equal areas in equal time intervals.
3. The cube of the semi major axis of the planetary orbits are proportional to the square of the planets periods of revolution. Mathematically, Kepler's third law can be written as:

$$T^2 = \left(\frac{4\pi^2}{GM_s}\right) r^3,$$

   where T is the time period, r is the semi major axis, M_s is the mass of sun and G is the universal gravitational constant."

2.3 Newton's Laws of Motion

"The following three laws of motion given by Newton are considered the axioms of mechanics:

1. **First law of Motion**
   Every particle persists in a state of rest or of uniform motion in a straight line unless acted upon by a force.

2. **Second law of Motion**

   If $\mathbf{F}$ is the external force acting on a particle of mass m which as a reaction is moving with velocity $\mathbf{v}$, then

$$\mathbf{F} = \frac{d}{dt}(m\mathbf{v}) = \frac{d\mathbf{P}}{dt}.$$

   If m is independent of time this becomes

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt} = m\mathbf{a},$$

   where $\mathbf{a}$ is the acceleration of the particle.

3. **Third law of Motion**

   For every action, there is an equal and opposite reaction."

2.3.1 Newton's Universal Law of Gravitation

"Every particle of matter in the universe attracts every other particle of matter with a force which is directly proportional to the product of the masses and inversely proportional to the square of the distance between them. Hence, for any two particles separated by a distance r, the magnitude of the gravitational force $\mathbf{F}$ is:

$$\mathbf{F} = G\frac{m_1 m_2}{r^3}\mathbf{r}$$

where G is universal gravitational constant. Its numerical value in SI units is $6.67408 \times 10^{-11} m^3 kg^{-1} s^{-2}$."

2.4 Two Body Problem

"The two-body problem , first studied and solved by Newton, states: Suppose that the positions and velocities of two massive bodies moving under their mutual gravitational force are given at any time t, then what should their position and velocities be for any other time t, if the masses are known? Example include a planet orbiting around a star (Earth-Sun, Moon-Earth), two stars orbiting around each other, satellite orbiting around orbit. The two-body problem is very important because of the following facts:

1. It is the only gravitational problem in celestial mechanics, apart from rather restricted solutions of three body problem, for which we have a complete and general solution.
2. A wide range of practical orbital motion problems can be treated as approximate two-body problems.
3. The two-body solution may be used to provide approximate orbital parameters and predictions or serve as a starting point for the generation of analytical solutions valid to higher orders of accuracy."

2.4.1 The Solution to the Two-Body Problem

"The governing law for the two-body is Newton's universal gravitational law:

$$\mathbf{F} = G\frac{m_1 m_2}{r^3}\mathbf{r}, \tag{2.1}$$

for two masses m_1 and m_2 separated by a distance of $\mathbf{r}$, and G the universal gravitational constant. The aim here is to determine the path of the particles for any time t, if the initial positions and velocities are known. In Figure 2.1, the force of attraction $\mathbf{F}_1$ is directed along r towards m, while the force $\mathbf{F}_2$ on M is in opposite direction. By Newton's third law,

$$\mathbf{F}_1 = -\mathbf{F}_2. \tag{2.2}$$

From Figure 2.1,

$$\mathbf{F}_1 = G\frac{mM}{r^3}\mathbf{r}. \tag{2.3}$$

Using Newton's second law of motion and by equation (2.1) and (2.2), the equation of motion of the particles under their mutual gravitational attractions are given by

$$m\ddot{\mathbf{r}}_1 = m\frac{d^2\mathbf{r}_1}{dt^2} = G\frac{mM}{r^3}\mathbf{r}, \tag{2.4}$$

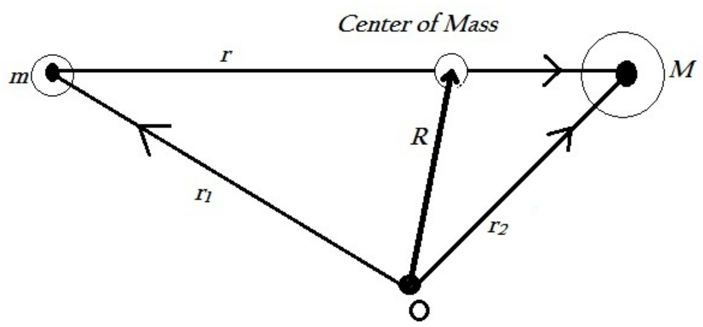


Figure 2.1: Center of mass of two body system

$$M\ddot{\mathbf{r}}_2 = M\frac{d^2\mathbf{r}_2}{dt^2} = -G\frac{mM}{r^3}\mathbf{r}, \tag{2.5}$$

where $\mathbf{r}_1$ and $\mathbf{r}_2$ be the position vectors from the reference O as shown in Figure 2.1. Adding equation (2.4) and (2.5), we get:

$$m\ddot{\mathbf{r}_1} + M\ddot{\mathbf{r}_2} = \mathbf{0}, \tag{2.6}$$

integrating above equations yields:

$$m\dot{\mathbf{r}_1} + M\dot{\mathbf{r}_2} = \mathbf{c}_1, \tag{2.7}$$

that the total linear momentum of the system i.e. $m\mathbf{v}_m + M\mathbf{v}_M = \mathbf{c}_1$ is a constant. Again integrating equation (2.7) implies:

$$m\mathbf{r}_1 + M\mathbf{r}_2 = \mathbf{c}_1 t + \mathbf{c}_2, \tag{2.8}$$

where $\mathbf{c}_1$ and $\mathbf{c}_2$ are constant vectors.

Using the definition of center of mass in 2BP, $\mathbf{R}$ is defined as:

$$\begin{aligned}(m+M)\mathbf{R} &= m\mathbf{r}_1 + M\mathbf{r}_2,\\ M_t\mathbf{R} &= m\mathbf{r}_1 + M\mathbf{r}_2, \end{aligned} \tag{2.9}$$

where $M_t = m + M$. Taking the derivative of equation (2.9) and comparing with equation (2.21), we get

$$M_t\dot{\mathbf{R}} = \mathbf{c}_1 \quad \Rightarrow \quad \dot{\mathbf{R}} = \frac{\mathbf{c}_1}{M_t} = constant$$

show that $\dot{\mathbf{R}} = \mathbf{v}_c$ (velocity of center of mass) is constant.

Subtracting the equations (2.4) and (2.5) gives:

$$\ddot{\mathbf{r}}_1 - \ddot{\mathbf{r}}_2 = \frac{GM}{r^3}r + \frac{Gm}{r^3}\mathbf{r}, \tag{2.10}$$

$$\ddot{\mathbf{r}}_1 - \ddot{\mathbf{r}}_2 = G(m+M)\frac{\mathbf{r}}{r^3}$$

$$\Rightarrow \ddot{\mathbf{r}} = \mu\frac{\mathbf{r}}{r^3}$$

$$\Rightarrow \ddot{\mathbf{r}} + \mu\frac{\mathbf{r}}{r^3} = \mathbf{0}, \tag{2.11}$$

where $\mu = G(m+M)$ is defined as reduced mass and $\mathbf{r}_1 - \mathbf{r}_2 = -\mathbf{r}$, see Figure 2.1. Taking the cross product of $\mathbf{r}$ with equation (2.11) we obtain:

$$\begin{aligned}\mathbf{r} \times \mu\ddot{\mathbf{r}} + \frac{\mu^2}{r^3}\mathbf{r} \times \mathbf{r} &= \mathbf{0}\\ \Rightarrow \mathbf{r} \times \ddot{\mathbf{r}} &= \mathbf{0}, \end{aligned} \tag{2.12}$$

integrating above equation yields:

$$\mathbf{r} \times \dot{\mathbf{r}} = \mathbf{L}, \tag{2.13}$$

where $\mathbf{L}$ is a constant vector. We may write equation (2.12),

$$\begin{aligned} &\Rightarrow \mathbf{r} \times \mu\ddot{\mathbf{r}} = \mathbf{0}, \\ &\Rightarrow \mathbf{r} \times \mathbf{F} = \mathbf{0}, \end{aligned} \tag{2.14}$$

where $\mathbf{F} = \mu\ddot{\mathbf{r}} = \mu\mathbf{a}$ (μ is reduced mass i.e. constant).
From the definition of torque and angular momentum:

$$\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt} = \mathbf{r} \times \mathbf{F}. \tag{2.15}$$

Comparing equations (2.14) and (2.15), we get:

$$\begin{aligned} &\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt} = \mathbf{r} \times \mathbf{F} = \mathbf{0}, \\ &\frac{d\mathbf{L}}{dt} = \mathbf{0} \end{aligned}$$

$$\Rightarrow \mathbf{L} = \text{constant},$$

i.e. angular momentum of the system is constant.

Radial and Transverse Components of Velocity and Acceleration:

If polar coordinates r and θ are taken in this plane as in Figure 2.2, the velocity components along and perpendicular to the radius vector joining m to M are $\dot{r}$ and $r\dot{\theta}$, then,

$$\dot{\mathbf{r}} = \frac{d\mathbf{r}}{dt} = \dot{r}\mathbf{i} + r\dot{\theta}\mathbf{j}, \tag{2.16}$$

where $\hat{\imath}$ and $\hat{\jmath}$ are unit vectors along and perpendicular to the radius vector. Hence, by equations (2.13) and (2.16),

$$\mathbf{r} \times (\dot{r}\hat{\imath} + r\dot{\theta}\hat{\jmath}) = r^2\dot{\theta}\hat{k} = L\hat{k}, \tag{2.17}$$

where $\hat{k}$ is a unit vector perpendicular to the plane of the orbit. We may then write

$$r^2\dot{\theta} = L, \tag{2.18}$$

where the constant L is seen to be twice the rate of description of area by the radius vector. This is the mathematical form of Kepler's second law.

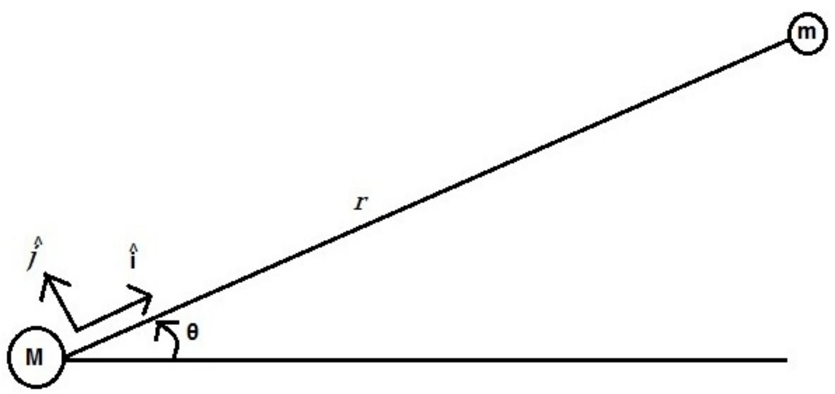


Figure 2.2: Radial and transverse components of velocity and acceleration

Now taking the scalar product of $\dot{\mathbf{r}}$ with equation (2.11), we get:

$$\dot{\mathbf{r}}.\frac{d^2\mathbf{r}}{dt^2} + \mu\frac{\dot{\mathbf{r}}.\mathbf{r}}{r^3} = 0,$$

which may be integrated to give:

$$\begin{aligned} \frac{1}{2}\dot{\mathbf{r}}.\dot{\mathbf{r}} - \frac{mu}{r} &= C, \\ \frac{1}{2}v^2 - \frac{\mu}{r} &= C, \end{aligned} \tag{2.19}$$

where C is a constant. This is the energy conservation form of the system. The quantity C is not the total energy; $\frac{1}{2}\mu^2$ is related to the kinetic energy and $\frac{-mu}{r}$

to the potential energy of the system i.e. total energy is conserved.

Recall that from celestial mechanics, components of acceleration vector along and perpendicular to the radius vector (see Figure 2.2):

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\hat{\imath} + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\hat{\jmath},$$

using above equation in (2.11), we get

$$\ddot{r} - r\dot{\theta}^2 = -\frac{\mu}{r^2}, \tag{2.20}$$

$$\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0. \tag{2.21}$$

Integrating equation (2.21) gives the angular momentum integral:

$$r^2\dot{\theta} = L, \tag{2.22}$$

making the usual substitution of

$$u = \frac{1}{r}, \tag{2.23}$$

and eliminating the time between equation (2.20) and (2.22), implies:

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{L^2}. \tag{2.24}$$

The general solution of above equation is :

$$u = \frac{\mu}{L^2} + A\cos(\theta - \theta_0), \tag{2.25}$$

where A and θ_0 are two constants of integration. Substitute $u = \frac{1}{r}$ in above equation:

$$\frac{1}{r} = \frac{\mu}{L^2} + A\cos(\theta - \theta_0)$$

$$\Rightarrow r = \frac{\frac{L^2}{\mu}}{1 + \frac{L^2 A}{\mu}\cos(\theta - \theta_0)},$$

is the polar form of the equation of the conic and may be written as:

$$r = \frac{p}{1 + e\cos(\theta - \theta_0)},$$

where

$$p = \frac{L^2}{\mu},$$
$$e = \frac{AL^2}{\mu}.$$

Eccentricity e classifies the trajectory of one celestial body around another. Thus:
(i) If $0 < e < 1$ then the orbit is elliptical,
(ii) If $e = 1$ then the orbit is a parabolic,
(iii) If $e > 1$ then the orbit is a hyperbolic.
Hence the solution of the two-body problem is a conic, includes Kepler's first law as a special case."

2.5 The Equations of Motion in the n-Body Problem

"The 2BP deals much of the important work in astrodynamics, but sometimes we need to model the real world by including other bodies. The next logical step, then, is to drive formulas for 3BP. A further generalization of three body problem is n-body problem. In general, solving general differential equations of motions in n-body problem requires a fixed number of integration constants. Consider a simple gravity problem in which we have constant acceleration over time, $a(t) = a_0$. If we integrate this equation, we obtain the velocity, $v(t) = a_0 t + v_0$. Integrating once more provides, $r(t) = r_0 + v_0 t + \frac{1}{2}a_0 t^2$. To complete the solution, we must know the initial conditions. This example is a straight froward analytical solution using the initial values, or a function of the time and constants of integration, called integrals of the motion. Unfortunately, this isn't always the simple case.

When initial conditions alone don't provide a solution, integrals of the motion can reduce the order of differential equations, also called the degrees of freedom of the dynamical system. Ideally, if the number of integrals equals the order of differential equations, we can reduce it to order zero. These integrals are constant functions of the initial conditions, as well as the position and velocity of at any time, hence the term constants of the motion.

For the n-body problem, a system of $3n$ second order differential equations, we need $6n$ integrals of motion for a complete solution. Conservation of linear momentum provides six, conservation of energy one, and conservation of total angular momentum three, for a total of ten. There are no laws analogous to Kepler's first two laws to obtain additional constants, thus we are left with a system of order $6n - 10$ for $n \geq 3$.

These equations for n bodies, $n \geq 3$, defy all attempts at closed-form solutions. H. Brun, in 1887, showed that there were no other algebraic integrals. Although Poincaré later generalized Brun's work, we still have only the ten known integrals. They give us insight into the motions within the three body and n-body problems. Conservation of total linear momentum assumes no external forces are on the system.

First, here we set up the equations of motions of n massive particles of masses $m_i(i = 1, 2...n)$ whose radius vectors from an unaccelerated point O are $\mathbf{r}_i$ while their mutual radius vectors are given by r_{ij} where

$$\mathbf{r}_{ij} = \mathbf{r}_j - \mathbf{r}_i \tag{2.26}$$

From Newton's laws of motion and the law of gravitation,

$$m_i\ddot{\mathbf{r}}_i = G \sum_{j=1 j\neq i}^{n} \frac{m_i m_j}{r_{ij}^3}\mathbf{r}_{ij}, \qquad (i = 1, ...n) \tag{2.27}$$

here we note that $\mathbf{r}_{ij}$ implies that the vector between m_i and m_j is directed for m_i to m_j, thus

$$\mathbf{r}_{ij} = -\mathbf{r}_{ji} \tag{2.28}$$

The set of equations (2.27) are the required equation of motion for n-body problem, G being the constant of gravitation."

Chapter 3

Restricted Trapezoid Five-Body Problem

3.1 Introduction

In this review research work, set up a restricted trapezoid five-body problem (RT5BP) [31], in which four positive masses are called primaries. The primary masses are m_1, m_2, m_3 and m_4 respectively. There are two pairs of equal masses at adjacent vertices, moving in such a way that their configuration is always a isosceles trapezoid. Assume that the mass m_5 is very small moving in the same plane of the primaries and does not influence the movement of the four primaries. First discuss, our four objects effectively make an isosceles trapezoid and the small mass m_5 moving in the same plane under the influence of gravity of four primaries (See Figure 3.1). Secondly, analyse the motion of mass m_5 and use the contour plot to find an equilibrium solution. Third, test the stability of equilibrium solutions. Finally, end up with the finding the Newton's basins of attraction for equilibrium points.

3.2 Characterization of the Trapezoid Configuration

Suppose that n point positive masses $(m_1, m_2, ..., m_n), m_i \in \mathbb{R}, i = 1, ...n$, and $\mathbf{r}_i \in \mathbb{R}^2, i = 1, ...n$ are n mass position vectors, and the Euclidean distance between any two masses are $\mathbf{r}_{ij}, i, j = 1, ...n$.
The classical equation of motion for n positive masses has the form

$$m_i \ddot{\mathbf{r}}_i = G \sum_{j=1 j \neq i}^{n} \frac{m_i m_j}{r_{ij}^3} \mathbf{r}_{ij}, \qquad i = 1, ...n, \tag{3.1}$$

$\mathbf{r}_i$ is position vector of the ith body, m_i is the mass of the ith body and G is the universal constant of gravitation. The *central configuration* of n-body system is obtained, if the position vector of each particle with respect to the center of mass is a common scalar multiple of its acceleration,
i.e.,

$$\ddot{\mathbf{r}}_i = -\lambda(\mathbf{r}_i - \mathbf{c}) \qquad i = 1, ...n,$$

where λ is a constant and $\lambda \neq 0$, $\mathbf{c}$ is the centre of mass of the system.

For four bodies, we put $n = 4$ in equation (3.1), we will get central configuration for general four-body problem (4BP) equations are given below:

$$m_2 \frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|^3} + m_3 \frac{\mathbf{r}_3 - \mathbf{r}_1}{|\mathbf{r}_3 - \mathbf{r}_1|^3} + m_4 \frac{\mathbf{r}_4 - \mathbf{r}_1}{|\mathbf{r}_4 - \mathbf{r}_1|^3} = -\lambda(\mathbf{r}_1 - \mathbf{c}), \tag{3.2}$$

$$m_1 \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3} + m_3 \frac{\mathbf{r}_3 - \mathbf{r}_2}{|\mathbf{r}_3 - \mathbf{r}_2|^3} + m_4 \frac{\mathbf{r}_4 - \mathbf{r}_2}{|\mathbf{r}_4 - \mathbf{r}_2|^3} = -\lambda(\mathbf{r}_2 - \mathbf{c}), \tag{3.3}$$

$$m_1 \frac{\mathbf{r}_1 - \mathbf{r}_3}{|\mathbf{r}_1 - \mathbf{r}_3|^3} + m_2 \frac{\mathbf{r}_2 - \mathbf{r}_3}{|\mathbf{r}_2 - \mathbf{r}_3|^3} + m_4 \frac{\mathbf{r}_4 - \mathbf{r}_3}{|\mathbf{r}_4 - \mathbf{r}_3|^3} = -\lambda(\mathbf{r}_3 - \mathbf{c}), \tag{3.4}$$

$$m_1 \frac{\mathbf{r}_1 - \mathbf{r}_4}{|\mathbf{r}_1 - \mathbf{r}_4|^3} + m_2 \frac{\mathbf{r}_2 - \mathbf{r}_4}{|\mathbf{r}_2 - \mathbf{r}_4|^3} + m_3 \frac{\mathbf{r}_3 - \mathbf{r}_4}{|\mathbf{r}_3 - \mathbf{r}_4|^3} = -\lambda(\mathbf{r}_4 - \mathbf{c}). \tag{3.5}$$

Now consider the four positive masses, the masses are m_1, m_2, m_3 and m_4 which

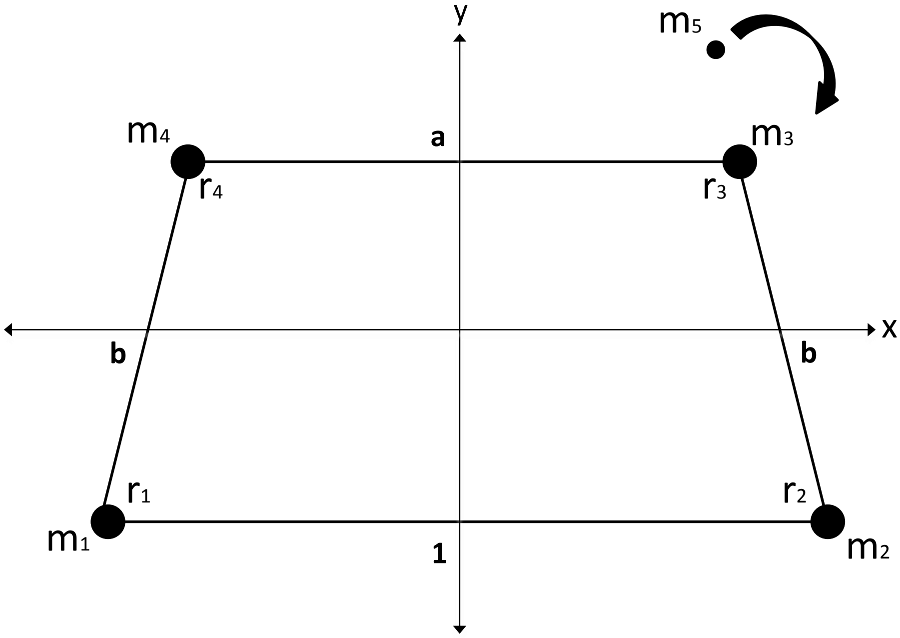


Figure 3.1: Restricted Trapezoid Five-Body Problem

are fixed at $\mathbf{r}_1 = \left(\frac{-1}{2}, 0\right)$, $\mathbf{r}_2 = \left(\frac{1}{2}, 0\right)$, $\mathbf{r}_3 = \left(\frac{a}{2}, h\right)$ and $\mathbf{r}_4 = \left(\frac{-a}{2}, h\right)$ respectively, where $h = \sqrt{b^2 - (\frac{1-a}{2})^2}$.
We consider $\mathbf{r} = (\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4)$ forms a isosceles trapezoid, and the side $[\mathbf{r}_1, \mathbf{r}_2]$ is parallel to the side $[\mathbf{r}_3, \mathbf{r}_4]$. We assume masses of isosceles trapezoid central configuration as,

$$m_1 = m_2 = 1 \quad \textit{and} \quad m_3 = m_4 = m.$$

After calculated the Euclidean distance between the masses from each other are:

$$r_{12} = 1, \quad r_{14} = r_{23} = b, \quad r_{34} = a, \quad r_{13} = r_{24} = \sqrt{b^2 + a}.$$

The centre of mass for four-bodies can be written as,

$$\mathbf{c} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + m_3\mathbf{r}_3 + m_4\mathbf{r}_4}{m_1 + m_2 + m_3 + m_4},$$

after using the values of $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$, $\mathbf{r}_4$ and $m_1 = m_2 = 1$, $m_3 = m_4 = m$, then equation of the centre of mass will becomes,

$$\mathbf{c} = \left(0, \frac{2mh}{2+2m}\right).$$

Now using the values of Euclidean distances, position vectors, masses $m_1 = m_2 = 1$ $m_3 = m_4 = m$ and centre of mass $\mathbf{c}$ in equations (3.2), (3.3), (3.4) and (3.5), reduces to the following four equations:

$$(1,0) + m\frac{(\frac{1+a}{2}, h)}{(a+b)^{\frac{3}{2}}} + m\frac{(\frac{1-a}{2}, h)}{b^3} + \lambda\left(\frac{-1}{2}, \frac{-mh}{1+m}\right) = 0, \tag{3.6}$$

$$(-1,0) + m\frac{(\frac{-1-a}{2}, h)}{(a+b)^{\frac{3}{2}}} + m\frac{(\frac{a-1}{2}, h)}{b^3} + \lambda\left(\frac{1}{2}, \frac{-mh}{1+m}\right) = 0, \tag{3.7}$$

$$\frac{(\frac{a-1}{2}, -h)}{b^3} + \frac{(\frac{a+1}{2}, -h)}{(a+b)^{\frac{3}{2}}} + m\frac{(a,0)}{a^3} + \lambda\left(\frac{-a}{2}, \frac{h}{1+m}\right) = 0, \tag{3.8}$$

$$\frac{(\frac{1-a}{2}, -h)}{b^3} + \frac{(\frac{-1-a}{2}, -h)}{(a+b)^{\frac{3}{2}}} + m\frac{(-a,0)}{a^3} + \lambda\left(\frac{a}{2}, \frac{h}{1+m}\right) = 0. \tag{3.9}$$

equations in xy coordinates are:

$$1 + m\frac{(1+a)}{2\ (a+b)^{\frac{3}{2}}} + m\frac{(1-a)}{2\ b^3} - \frac{\lambda}{2} = 0, \tag{3.10}$$

$$-1 - m\frac{(1+a)}{2\ (a+b)^{\frac{3}{2}}} + m\frac{(a-1)}{2\ b^3} + \frac{\lambda}{2} = 0, \tag{3.11}$$

$$\frac{(a-1)}{2\ b^3} + \frac{(a+1)}{2\ (a+b)^{\frac{3}{2}}} + \frac{m}{a^2} - \frac{\lambda\ a}{2} = 0, \tag{3.12}$$

$$\frac{(1-a)}{2\ b^3} - \frac{(1+a)}{2\ (a+b)^{\frac{3}{2}}} - \frac{m}{a^2} + \frac{\lambda\ a}{2} = 0, \tag{3.13}$$

$$\frac{m\ h}{(a+b)^{\frac{3}{2}}} + \frac{m\ h}{b^3} - \frac{\lambda\ m\ h}{1+m} = 0, \tag{3.14}$$

$$\frac{m\ h}{(a+b)^{\frac{3}{2}}} + \frac{m\ h}{b^3} - \frac{\lambda\ m\ h}{1+m} = 0, \tag{3.15}$$

$$-\frac{h}{b^3}-\frac{h}{(a+b)^{\frac{3}{2}}}+\frac{\lambda\, h}{1+m}=0, \tag{3.16}$$

$$-\frac{h}{b^3}-\frac{h}{(a+b)^{\frac{3}{2}}}+\frac{\lambda\, h}{1+m}=0. \tag{3.17}$$

By simplifying and rearranging, Some of these are similar in eight equations. Therefore, the central configuration equations are reduced to the following three equations:

$$2+m\frac{(a+1)}{X}+m\frac{(1-a)}{b^3}-\lambda=0, \tag{3.18}$$

$$\frac{a-1}{b^3}+\frac{a+1}{X}+\frac{2m}{a^2}-\lambda a=0, \tag{3.19}$$

$$\frac{1}{b^3}+\frac{1}{X}-\frac{\lambda}{1+m}=0, \tag{3.20}$$

where,

$$X=(a+b)^{\frac{3}{2}}.$$

Simplifying (3.20) equation, we obtain

$$\lambda=\frac{1+m}{b^3}+\frac{1+m}{X}. \tag{3.21}$$

Putting equation (3.21) in equation (3.19), we have

$$2+m\frac{(a+1)}{X}+m\frac{(1-a)}{b^3}-a\left[\frac{1+m}{b^3}+\frac{1+m}{X}\right]=0,$$

$$\frac{1}{X}\left[a+1-a(1+m)\right]+\frac{1}{b^3}\left[a-1-a(1+m)\right]+\frac{2m}{a^2}=0,$$

$$\frac{1}{X}(1-am)+\frac{1}{b^3}(-1-am)+\frac{2m}{a^2}=0,$$

$$m\left[\frac{2}{a^2}-\frac{a}{X}-\frac{a}{b^3}\right]=\frac{1}{b^3}-\frac{1}{X},$$

$$m\left[\frac{2b^3X-a^3b^3-a^3X}{a^2b^3X}\right]=\frac{X-b^3}{b^3X},$$

$$m\left[\frac{2b^3X-a^3b^3-a^3X}{a^2}\right]=X-b^3,$$

$$m=\frac{a^2(X-b^3)}{2b^3X-a^3b^3-a^3X},$$

using the value of X and some simplifications and rearrangements, we obtain

$$m=\frac{a^2(1-b^3)}{b^3-a^3}. \tag{3.22}$$

Now plugging Eq. (3.21) and (3.22) into Eq. (3.18), we get

$$g(a,b)=(b^3+a)^{\frac{3}{2}}(2b^3-a^3-1)-b^3-a^3b^3+2a^3=0. \tag{3.23}$$

The function $g(a,b)=0$ is a necessary condition for the central configurations to exist. It is not possible to analytically solve $g(a,b)=0$ for either a or b, therefore we use interpolation using mathematica to write $b=\phi(x\ ;\ a)$, where

$$b(x\ ;\ a)=a^3-2a^6+a^9+(3a^2+6a^5+3a^8)x^2+(3a+6a^4+3a^7)x^4+(-12a^2-12a^5)x^5+ \\ 4a^3x^6+(-12a-12a^4)x^7+12a^2x^8+(-4-4a^3)x^9+12ax^{10}+4x^{12}.$$

3.2.1 Proposition

If $a > 0$ $(a \neq 1)$ then for atmost one real root b of $\phi(x\ ;\ a)$, we have $m > 0$ and $g(a, b) = 0$.

3.3 Dynamics of 5^{th} Body

In this section we discuss the dynamic of 5^{th} particle in the plan moving according to the gravitational field formed by the attraction of four primaries moving in a planar isosceles trapezoid configuration as shown in the previous section. We call this problem to restricted trapezoidal five-body problem. Equation of motion describe the planer motion of restricted 5^{th} particle, mass m_5 written from equation (3.1) reads in inertial frame of reference,

$$\ddot{\mathbf{r}}_5 = m_1 \frac{\mathbf{r}_1 - \mathbf{r}_5}{|\mathbf{r}_1 - \mathbf{r}_5|^3} + m_2 \frac{\mathbf{r}_2 - \mathbf{r}_5}{|\mathbf{r}_2 - \mathbf{r}_5|^3} + m_3 \frac{\mathbf{r}_3 - \mathbf{r}_5}{|\mathbf{r}_3 - \mathbf{r}_5|^3} + m_4 \frac{\mathbf{r}_4 - \mathbf{r}_5}{|\mathbf{r}_4 - \mathbf{r}_5|^3}. \tag{3.24}$$

We now introduce a coordinate system that is rotating about the center of mass with uniform angular speed ω. Let (x, y) be the coordinates of m_5 in this new rotating frame (non-inertial frame). We can convert equation (3.24) from fixed inertial frame to the rotating coordinates system with the following orthogonal system,

$$\mathbf{e}_1 = e^{iwt}, \quad \mathbf{e}_2 = ie^{iwt},$$

where ω is angular speed and "t" represents time. The position vector of m_5 in the rotating frame is,

$$\mathbf{r}_5 = x(t)\ \mathbf{e}_1 + y(t)\ \mathbf{e}_2, \tag{3.25}$$

choosing ω, (without loss of generality) and taking first and second derivatives of equation (3.25) yield,

$$\begin{aligned}\dot{\mathbf{r}}_5 &= [(\dot{x}-y)+i(x+\dot{y})]e^{it},\\ \ddot{\mathbf{r}}_5 &= [(\ddot{x}-2\dot{y}-x)+i(\ddot{y}+2\dot{x}-y)]\,e^{it}.\end{aligned} \tag{3.26}$$

Using equation (3.26) in equation (3.24), the equations of motion of m_5 in rotating frame in component form are,

$$\ddot{x}-2\dot{y} = x - \left[\left(\frac{x+0.5}{r_{51}^3}+\frac{x-0.5}{r_{52}^3}\right)+m\left(\frac{x-0.5a}{r_{53}^3}+\frac{x+0.5a}{r_{54}^3}\right)\right], \tag{3.27}$$

$$\ddot{y}+2\dot{x} = y - \left[\left(\frac{1}{r_{51}^3}+\frac{1}{r_{52}^3}\right)y+m\left(\frac{1}{r_{53}^3}+\frac{1}{r_{54}^3}\right)(y-h)\right], \tag{3.28}$$

where mutual distances are described as,

$$\begin{aligned}r_{51} &= \sqrt{(x+0.5)^2+y^2},\\ r_{52} &= \sqrt{(x-0.5)^2+y^2},\\ r_{53} &= \sqrt{(x-0.5a)^2+(y-h)^2},\\ r_{54} &= \sqrt{(x+0.5a)^2+(y-h)^2}.\end{aligned}$$

The equation of motion of m_5 moving in the plane of primaries can also be written as,

$$U_x = \frac{\partial U}{\partial x} = \ddot{x}-2\dot{y}, \tag{3.29}$$

$$U_y = \frac{\partial U}{\partial y} = \ddot{y}+2\dot{x}, \tag{3.30}$$

where the effective potential $U(x,y)$ can pe expressed as,

$$U(x,y) = \frac{x^2+y^2}{2} + \left[\left(\frac{1}{r_{51}}+\frac{1}{r_{52}}\right)+m\left(\frac{1}{r_{53}}+\frac{1}{r_{54}}\right)\right]. \tag{3.31}$$

Comparing the equations (3.27), (3.28) and (3.29), (3.30), we may write equations of motion of m_5 as,

$$U_x(x,y) = x - \left[\left(\frac{x+0.5}{r_{51}^3} + \frac{x-0.5}{r_{52}^3}\right) + m\left(\frac{x-0.5a}{r_{53}^3} + \frac{x+0.5a}{r_{54}^3}\right)\right], \quad (3.32)$$

$$U_y(x,y) = y - \left[\left(\frac{1}{r_{51}^3} + \frac{1}{r_{52}^3}\right) y + m\left(\frac{1}{r_{53}^3} + \frac{1}{r_{54}^3}\right)(y-h)\right]. \quad (3.33)$$

3.4 Equilibrium Solutions

The equations (3.32) and (3.33), do not have an analytical solution of a closed form, we can use both equations to determine the location of the equilibrium points. These are the places in space where the infinitesimal mass m_5 would have zero velocity and acceleration,i.e., where m_5 appears at rest permanently relative to the primaries m_1,m_2,m_3 and m_4 respectively. When located at an equilibrium point (also called libration point / Lagrange point), a body will apparently stay there. These solutions can be found only if we meet the sufficient condition of all rates equal to zero,

$$\dot{x} = \dot{y} = \ddot{x} = \ddot{y} = 0.$$

Finally the equations (3.27) and (3.28) take the form,

$$x - \left[\left(\frac{x+0.5}{r_{51}^3} + \frac{x-0.5}{r_{52}^3}\right) + m\left(\frac{x-0.5a}{r_{53}^3} + \frac{x+0.5a}{r_{54}^3}\right)\right] = 0, \quad (3.34)$$

$$y - \left[\left(\frac{1}{r_{51}^3} + \frac{1}{r_{52}^3}\right) y + m\left(\frac{1}{r_{53}^3} + \frac{1}{r_{54}^3}\right)(y-h)\right] = 0. \quad (3.35)$$

Equations (3.34) and (3.35) are highly non linear coupled algebraic equations. To find the zero's (x, y) or equilibrium points / Lagrange point, we need to solve these equations numerically or drawing contour plot using mathematica. The classification of equilibrium points/ Lagrange points for RT5BP are,

3.5 When $a \in (0\ ,\ 1)$

i. When $a \in (0.00500\ ,\ 0.13967)$ there exist 5 equilibrium points.

ii. When $a \in (0.13967\ ,\ 0.15099)$ there exist 7 equilibrium points.

iii. When $a \in (0.15099\ ,\ 0.18274)$ there exist 5 equilibrium points.

iv. When $a \in (0.18274\ ,\ 0.43386)$ there exist 7 equilibrium points.

v. When $a \in (0.43386\ ,\ 0.60867)$ there exist 11 equilibrium points.

vi. When $a \in (0.60867\ ,\ 0.64166)$ there exist 13 equilibrium points.

vii. When $a \in (0.64166\ ,\ 0.64220)$ there exist 11 equilibrium points.

viii. When $a \in (0.64220\ ,\ 0.99999)$ there exist 9 equilibrium points.

In addition, these intervals can be discuss in cases compared to other similar equilibrium points. The intersections of the non-linear equations $U_x = 0$, and $U_y = 0$ define the positions of the equilibrium points. The intersections of $U_x = 0$ (blue) and $U_y = 0$ (orange), respectively. The red dots represent the position of the primary masses and black dots represent the position of equilibrium points (See figures 3.2 to 3.9 below)

3.5.1 Case 1: Five Equilibrium Points

We start our analysis with the first case for equilibrium points, where five equilibrium points are present in two different intervals "i" and "iii" respectively. I will choose any point, because the behaviour does not change of the equilibrium point in these intervals. So i will choose and draw contour plot one by one for these intervals.

3.5.1.1 Contour-Plot for $a \in (0.00500\ ,\ 0.13967)$

when $a \in (0.00500\ ,\ 0.13967)$, we take $a = 0.11984$ to be any point in the interval, the corresponding value of $b = 0.97188, h = 0.8644$ and $m = 0.001377$ respectively. Contour plot for these values shows that the L_1, L_2 and L_5 are collinear along x-axis, while L_3, L_4 and L_5 are collinear along y-axis.

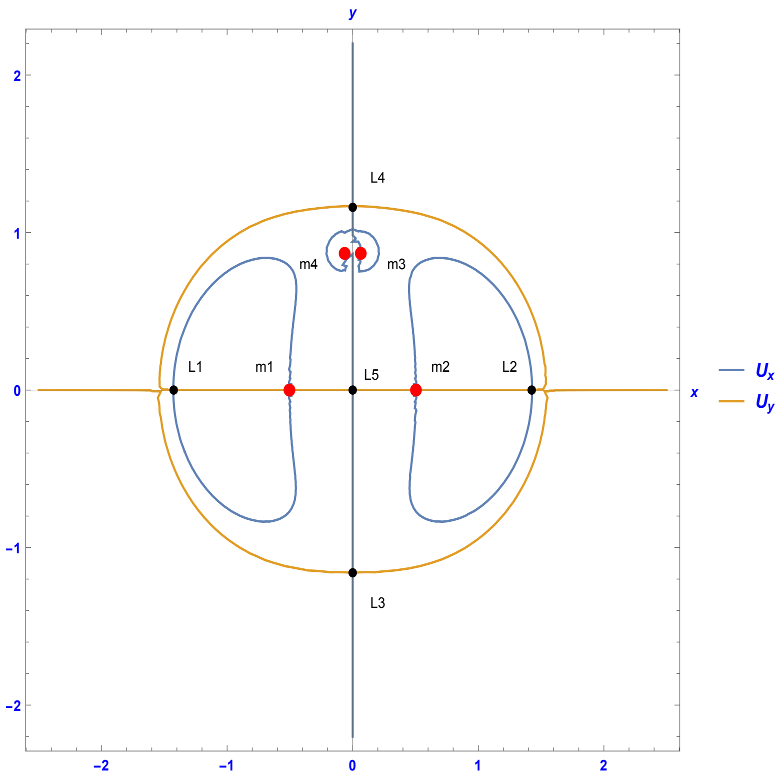


Figure 3.2: Case 1: Five equilibrium points for $0.00500 < a < 0.13967$; Positions (Black dots) and equilibrium points numbering $(L_i, i = 1, ..., 5)$ through the intersections of $U_x = 0$ (blue) and $U_y = 0$ (orange), when $a = 0.11984$, $b = 0.97188$, $m = 0.001377$ and $h = 0.8644$. The Red dots represent the centers $(m_i, i = 1, 2, 3, 4)$ of the primaries.

3.5.1.2 Contour-Plot for $a \in (0.15099\ ,\ 0.18274)$

When $a \in (0.15099\ ,\ 0.18274)$, we take $a = 0.16687$ to be any point in the interval, the corresponding value of $b = 0.96189, h = 0.86491$ and $m = 0.00365$ respectively. Contour plot for these values shows that the L_1, L_2 and L_5 are collinear along x-axis, while L_3, L_4 and L_5 are collinear along y-axis.

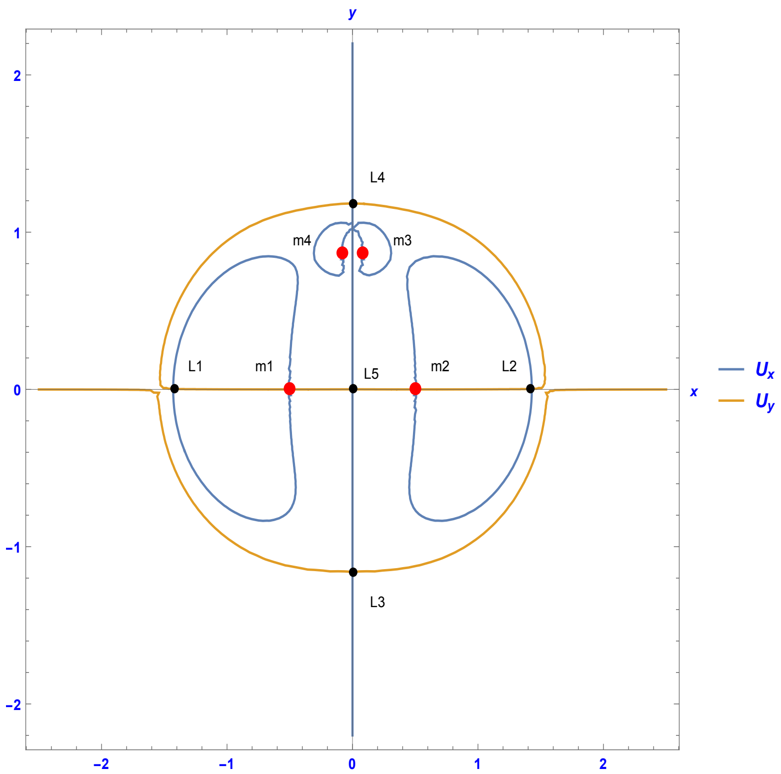


Figure 3.3: Case 1: Five equilibrium points for $0.15099 < a < 0.18274$; Positions (Black dots) and equilibrium points numbering $(L_i, i = 1, ..., 5)$ through the intersections of $U_x = 0$ (blue) and $U_y = 0$ (orange), when $a = 0.16687$, $b = 0.96189$, $m = 0.00365$ and $h = 0.86491$. The Red dots represent the centers $(m_i, i = 1, 2, 3, 4)$ of the primaries.

3.5.2 Case 2: Seven Equilibrium Points

We continue our analysis with the second case for equilibrium points, where seven equilibrium points are present in two different intervals "ii" and "iv" respectively.

I will choose any point, because the behaviour does not change of the equilibrium point in these intervals. So i will choose and draw contour plot one by one for these intervals.

3.5.2.1 Contour-Plot for $a \in (0.13967\ ,\ 0.15099)$

When $a \in (0.13967\ ,\ 0.15099)$, we take $a = 0.15001$ to be any point in the interval, the corresponding value of $b = 0.9654, h = 0.87194$ and $m = 0.00216$ respectively. Contour plot for these values shows that the L_1, L_2 and L_7 are collinear along x-axis, while L_3, L_4 and L_7 are collinear along y-axis, other points L_5 and L_6 are non-collinear.

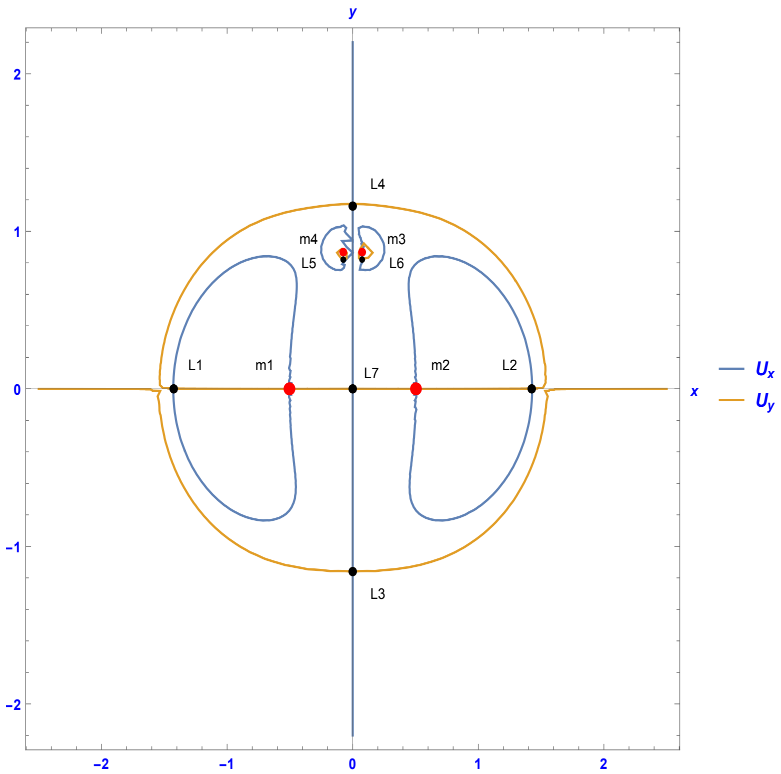


Figure 3.4: Case 2: Seven equilibrium points for $0.13967 < a < 0.15099$; Positions (Black dots) and equilibrium points numbering $(L_i, i = 1, ..., 7)$ through the intersections of $U_x = 0$ (blue) and $U_y = 0$ (orange), when $a = 0.15001$, $b = 0.9654$, $m = 0.00216$ and $h = 0.87194$. The Red dots represent the centers $(m_i, i = 1, 2, 3, 4)$ of the primaries.

3.5.2.2 Contour-Plot for $a \in (0.18274\ ,\ 0.43386)$

When $a \in (0.18274\ ,\ 0.43386)$, we take $a = 0.30834$ to be any point in the interval, the corresponding value of $b = 0.93569$, $h = 0.87406$ and $m = 0.02009$ respectively. Contour plot for these values shows that the L_3, L_4 and L_7 are collinear along y-axis, while other points L_1, L_2, L_5 and L_6 are non-collinear.

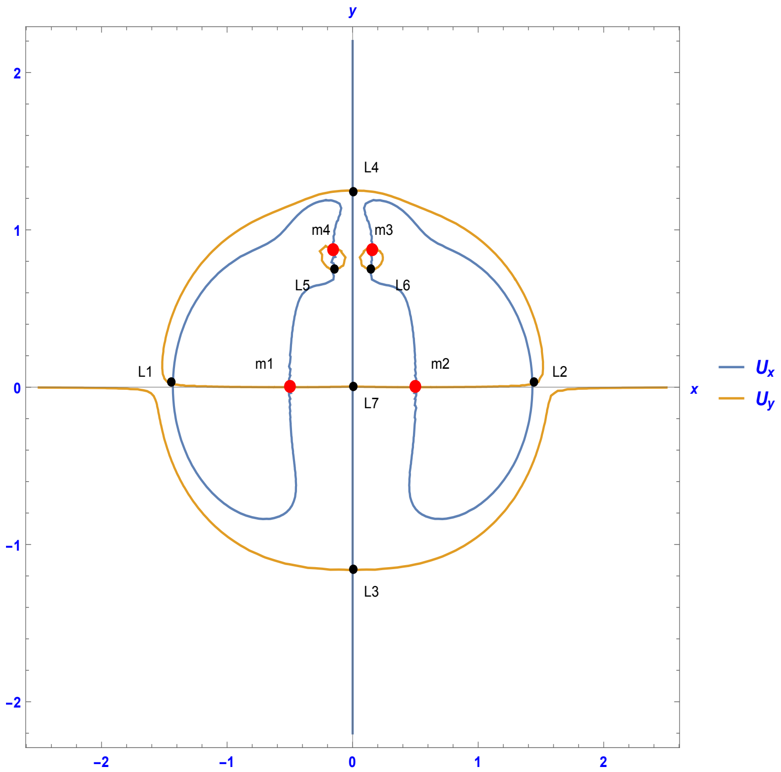


Figure 3.5: Case 2: Seven equilibrium points for $0.18274 < a < 0.43386$; Positions (Black dots) and equilibrium points numbering $(L_i, i = 1, ..., 7)$ through the intersections of $U_x = 0$ (blue) and $U_y = 0$ (orange), when $a = 0.30834$, $b = 0.93569$, $m = 0.02009$ and $h = 0.87406$. The Red dots represent the centers $(m_i, i = 1, 2, 3, 4)$ of the primaries.

3.5.3 Case 3: Nine Equilibrium Points

The present analysis with the third case for equilibrium points, where nine equilibrium points are present in only one interval "viii". I will choose any point, because the behaviour does not change of the equilibrium point in the interval. So i will choose and draw contour plot for this interval.

3.5.3.1 Contour-Plot for $a \in (0.64220\ ,\ 0.99999)$

When $a \in (0.64220\ ,\ 0.99999)$, we take $a = 0.80742$ to be any point in the interval, the corresponding value of $b = 0.92959$, $h = 0.92459$ and $m = 0.46305$ respectively. Contour plot for these values shows that the L_3, L_4, L_7, L_8 and L_9 are collinear along y-axis, while all other points L_1, L_2, L_5 and L_6 are non-collinear.

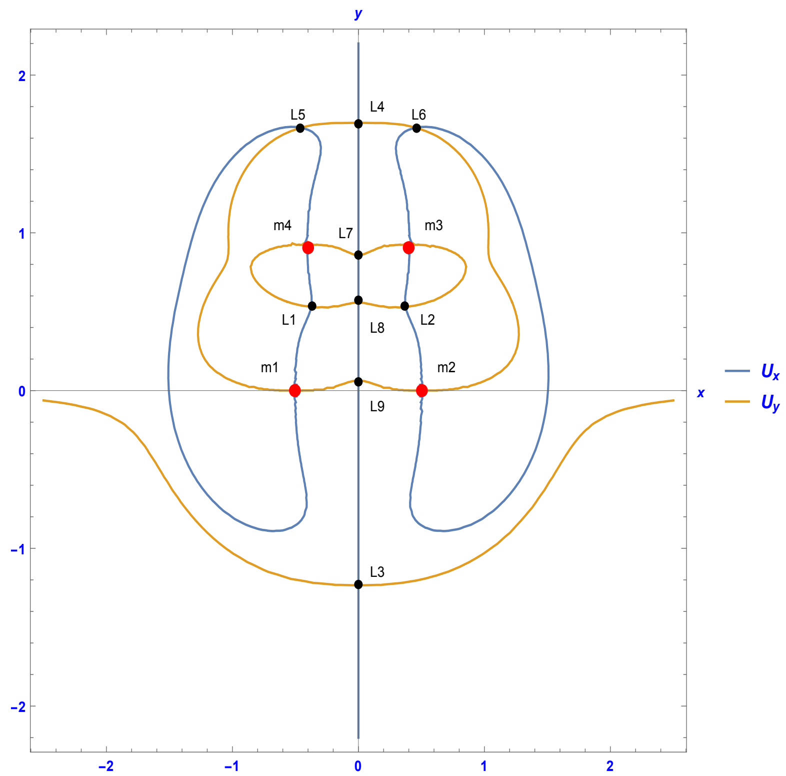


Figure 3.6: Case 3: Nine equilibrium points for $0.64220 < a < 0.99999$; Positions (Black dots) and equilibrium points numbering $(L_i, i = 1, ..., 9)$ through the intersections of $U_x = 0$ (blue) and $U_y = 0$ (orange), when $a = 0.80742$, $b = 0.92959$, $m = 0.46305$ and $h = 0.92459$. The Red dots represent the centers $(m_i, i = 1, 2, 3, 4)$ of the primaries.

3.5.4 Case 4: Eleven Equilibrium Points

We further continue our analysis with the fourth case for equilibrium points, where eleven equilibrium points are present in two different intervals "v" and "vii" respectively. I will choose any point, because the behaviour does not change of the equilibrium point in these intervals. So i will choose and draw contour plot one by one for these intervals.

3.5.4.1 Contour-Plot for $a \in (0.43386\ ,\ 0.60867)$

When $a \in (0.43386\ ,\ 0.60867)$, we take $a = 0.52134$ to be any point in the interval, the corresponding value of $b = 0.91067, h = 0.87796$ and $m = 0.10946$ respectively. Contour plot for these values shows that the L_3, L_4 and L_{11} are collinear along y-axis, while all other points L_1, L_2, L_5, L_6, L_7, L_8, L_9 and L_{10} are non-collinear.

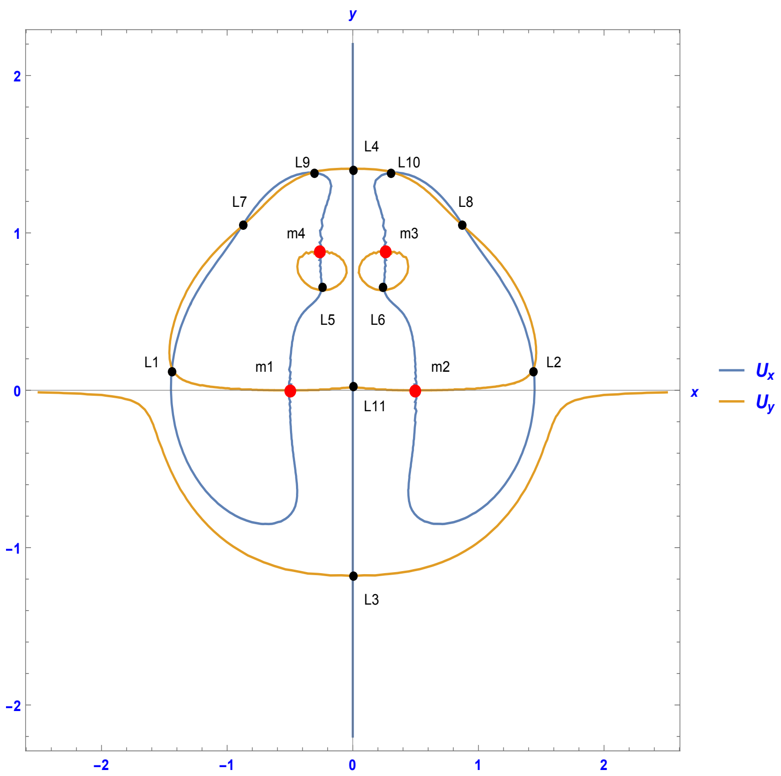


Figure 3.7: Case 4: Eleven equilibrium points for $0.43386 < a < 0.60867$; Positions (Black dots) and equilibrium points numbering $(L_i, i = 1, ..., 11)$ through the intersections of $U_x = 0$ (blue) and $U_y = 0$ (orange), when $a = 0.52134$, $b = 0.91067$, $m = 0.10946$ and $h = 0.87796$. The Red dots represent the centers $(m_i, i = 1, 2, 3, 4)$ of the primaries.

3.5.4.2 Contour-Plot for $a \in (0.64166\ ,\ 0.64220)$

When $a \in (0.64166\ ,\ 0.64220)$, we take $a = 0.64175$ to be any point in the interval, the corresponding value of $b = 0.90867$, $h = 0.89084$ and $m = 0.21163$ respectively. Contour plot for these values shows that the L_3, L_4, L_9, L_{10} and L_{11} are collinear along y-axis, while all other points L_1, L_2, L_5, L_6, L_7, and L_8 are non-collinear.

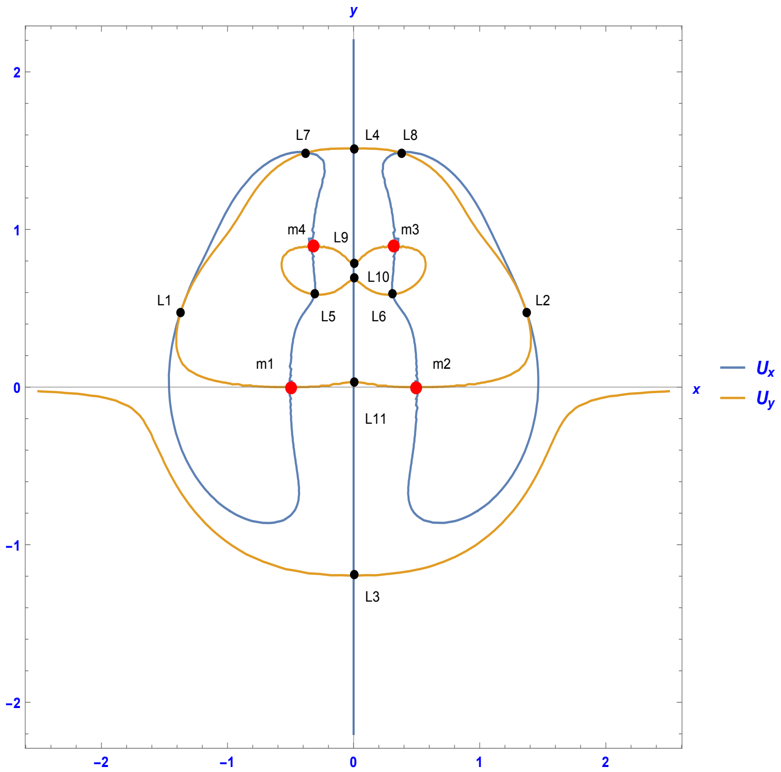


Figure 3.8: Case 4: Eleven equilibrium points for $0.64166 < a < 0.64220$; Positions (Black dots) and equilibrium points numbering $(L_i, i = 1, ..., 11)$ through the intersections of $U_x = 0$ (blue) and $U_y = 0$ (orange), when $a = 0.64175$, $b = 0.90867$, $m = 0.21163$ and $h = 0.89084$. The Red dots represent the centers $(m_i, i = 1, 2, 3, 4)$ of the primaries.

3.5.5 Case 5: Thirteen Equilibrium Points

In the last case, we start our analysis for equilibrium points, where thirteen equilibrium points are present in only one interval "vi". I will choose any point, because the behaviour does not change of the equilibrium point in the interval. So i will choose and draw contour plot for this interval.

3.5.5.1 Contour-Plot for $a \in (0.60867\ ,\ 0.64166)$

When $a \in (0.60867\ ,\ 0.64166)$, we take $a = 0.63333$ to be any point in the interval, the corresponding value of $b = 0.90842$, $h = 0.88972$ and $m = 0.20262$ respectively. Contour plot for these values shows that the L_3, L_4, L_{11}, L_{12} and L_{13} are collinear along y-axis, while all other points L_1, L_2, L_5, L_6, L_7, L_8, L_9 and L_{10} are non-collinear.

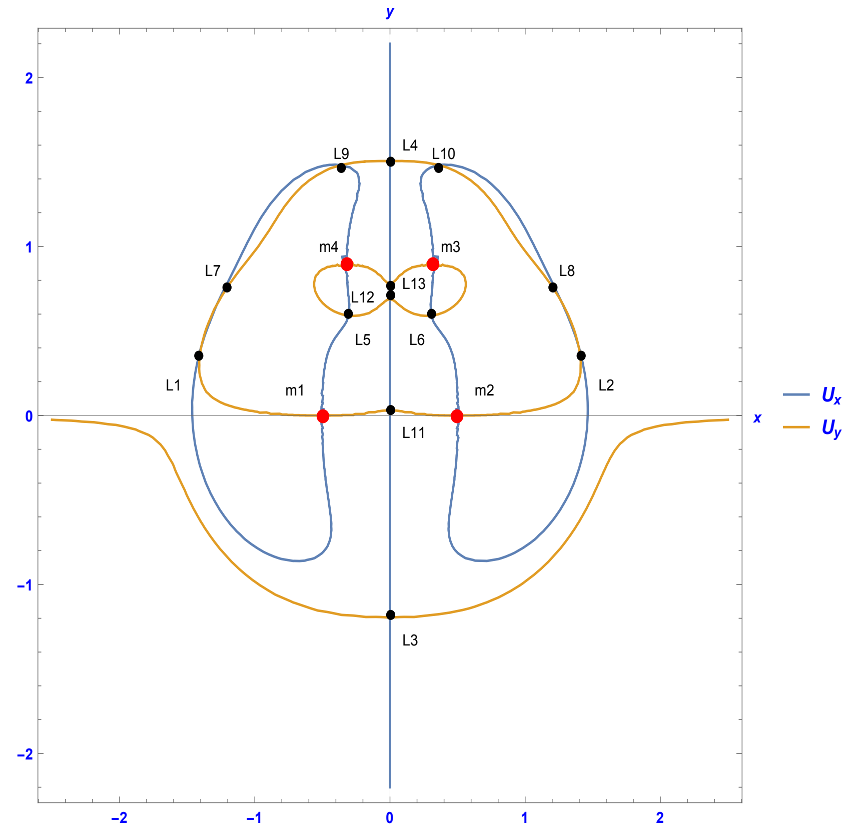


Figure 3.9: Case 5: Thirteen equilibrium points for $0.60867 < a < 0.64166$; Positions (Black dots) and equilibrium points numbering ($L_i, i = 1, ..., 13$) through the intersections of $U_x = 0$ (blue) and $U_y = 0$ (orange), when $a = 0.63333$, $b = 0.90842$, $m = 0.20262$ and $h = 0.88972$. The Red dots represent the centers ($m_i, i = 1, 2, 3, 4$) of the primaries.

3.6 Stability Analysis

This section is devoted to the mathematical analysis of the stability of equilibrium points in RT5BP. We check whether the equilibrium points are either stable or unstable. To check stability, we perform an individual eigenvalue analysis for each equilibrium point.

Eigenvalues of Case-1:

Choosing $a = 0.11984$ from (0.00500 , 0.13967) and the corresponding value of $b = 0.97188, h = 0.8644$, $m = 0.001377$ and $L_1(-1.42, 0)$ (see figure 3.2), we will follow the procedure given in chapter (2) at page 12 for the stability analysis, the Jacobian matrix form is

$$A = \begin{pmatrix} 3.851705199625337 & 0.000804070790164424 \\ 0.000804070790164424 & -0.42560628034999026 \end{pmatrix}.$$

The eigenvalues of matrix A are: $(3.85171, -0.425606)$. Likewise, all eigenvalues of equilibrium points L_i, where $i = 1, 2, 3, 4, 5$ in case-1 are given below in following Tables 3.1 and 3.2.

Equilibrium points	Eigenvalues
L_1(-1.42, 0)	(3.85171, -0.425606)
L_2(1.42, 0)	(3.85171, -0.425606)
L_3(0, -1.16)	(2.51884, 0.473792)
L_4(0, 1.16)	(2.70701, 0.385649)
L_5(0, 0.001)	(32.9954, -14.9913)

Table 3.1: Stability Analysis for Case 1: $a = 0.11984$, $b = 0.97188$, $m = 0.001377$, $h = 0.8644$.

Equilibrium points	Eigenvalues
L_1(-1.42, 0)	(3.86994, -0.428414)
L_2(1.42, 0)	(3.86994, -0.428414)
L_3(0, -1.16)	(2.53566, 0.465404)
L_4(0, 1.18)	(6.26458, -1.20836)
L_5(0, 0.002)	(32.8896, -14.7781)

Table 3.2: Stability Analysis for Case 1: $a = 0.16687$, $b = 0.96189$, $m = 0.00365$, $h = 0.86491$.

The same approach also applies to each equilibrium point is given in the Cases 2 to 5, the eigenvalues are given below:

Eigenvalues of Case-2:

Equilibrium points	Eigenvalues
L_1(-1.42, 0)	(3.85209, -0.425655)
L_2(1.42, 0)	(3.85209, -0.425655)
L_3(0, -1.16)	(2.51921, 0.47361)
L_4(0, 1.16)	(2.81473, 0.341456)
L_5(-0.07, 0.82)	(33.6278, -13.5616)
L_6(0.07, 0.82)	(33.6278, -13.5616)
L_7(0, 0.001)	(32.9933, -14.9869)

Table 3.3: Stability Analysis for Case 2: $a = 0.15001$, $b = 0.9654$, $m = 0.00216$, $h = 0.87194$.

Equilibrium points	Eigenvalues
L_1(-1.44, 0.03)	(3.78087, -0.352358)
L_2(1.44, 0.03)	(3.78087, -0.352358)
L_3(0, -1.16)	(2.61206, 0.427586)
L_4(0, 1.24)	(12.2615, -3.00334)
L_5(-0.142, 0.751)	(212.063, -95.0214)
L_6(0.142, 0.751)	(212.063, -95.0214)
L_7(0, 0.002)	(32.4724, -13.8942)

Table 3.4: Stability Analysis for Case 2: $a = 0.30834$, $b = 0.93569$, $m = 0.02009$, $h = 0.87406$.

Eigenvalues of Case-3:

Equilibrium points	Eigenvalues
L_1(-0.37, 0.54)	(27.2378, -9.70538)
L_2(0.37, 0.54)	(27.2378, -9.70538)
L_3(0, -1.23)	(2.51229, 0.430088)
L_4(0, 1.68)	(3.6175, 0.227726)
L_5(-0.46, 1.66)	(4.17228, -0.340726)
L_6(0.46, 1.66)	(4.17228, -0.340726)
L_7(0, 0.86)	(26.5962, -9.01314)
L_8(0, 0.57)	(6.5374, 6.02044)
L_9(0, 0.06)	(31.1606, -12.4341)

Table 3.5: Stability Analysis for Case 3: $a = 0.80742$, $b = 0.92959$, $m = 0.46305$, $h = 0.92459$.

Eigenvalues of Case-4:

Equilibrium points	Eigenvalues
L_1(-1.44, 0.12)	(3.67421, -0.306207)
L_2(1.44, 0.12)	(3.67421, -0.306207)
L_3(0, -1.18)	(2.51437, 0.460346)
L_4(0, 1.4)	(3.5548, 0.155627)
L_5(-0.24, 0.65)	(23.6733, -7.91968)
L_6(0.24, 0.65)	(23.6733, -7.91968)
L_7(-0.87, 1.05)	(3.05099, 0.372611)
L_8(0.87, 1.05)	(3.05099, 0.372611)
L_9(-0.31, 1.38)	(3.93311, -0.21745)
L_{10}(0.31, 1.38)	(3.93311, -0.21745)
L_{11}(0, 0.02)	(32.7816, -14.4952)

Table 3.6: Stability Analysis for Case 4: $a = 0.52134$, $b = 0.91067$, $m = 0.10946$, $h = 0.87796$.

Equilibrium points	Eigenvalues
L_1(-1.368, 0.475)	(3.41786, -0.0577455)
L_2(1.368, 0.475)	(3.41786, -0.0577455)
L_3(0, -1.192)	(2.52367, 0.447568)
L_4(0, 1.512)	(3.53834, 0.195379)
L_5(-0.311, 0.5902)	(23.6719, -8.08655)
L_6(0.311, 0.5902)	(23.6719, -8.08655)
L_7(-0.3798, 1.484)	(4.06275, -0.292254)
L_8(0.3798, 1.484)	(4.06275, -0.292254)
L_9(0, 0.7853)	(19.4133, -3.95287)
L_{10}(0, 0.689)	(10.0442, 2.96726)
L_{11}(0, 0.02)	(32.5073, -14.0161)

Table 3.7: Stability Analysis for Case 4: $a = 0.64175$, $b = 0.90867$, $m = 0.21163$, $h = 0.89084$.

Eigenvalues of case-5:

Equilibrium points	Eigenvalues
L_1(-1.41, 0.35)	(3.50858, -0.150676)
L_2(1.41, 0.35)	(3.50858, -0.150676)
L_3(0, -1.18)	(2.54957, 0.443643)
L_4(0, 1.15)	(3.52751, 0.203549)
L_5(-0.303, 0.599)	(23.4403, -7.94466)
L_6(0.303, 0.599)	(23.4403, -7.94466)
L_7(-1.21, 0.758)	(3.15824, 0.210705)
L_8(1.21, 0.758)	(3.15824, 0.210705)
L_9(-0.365, 1.465)	(4.15808, -0.324322)
L_{10}(0.365, 1.465)	(4.15808, -0.324322)
L_{11}(0, 0.02)	(32.524, -14.0656)
L_{12}(0, 0.710)	(11.2418, 1.95783)
L_{13}(0, 0.770)	(17.1172, -2.38383)

Table 3.8: Stability Analysis for Case 5: $a = 0.63333$, $b = 0.90842$, $m = 0.20262$, $h = 0.88972$.

Therefore all points of equilibrium are unstable in xy-plane motion.

3.7 Basins of Attraction

The Newton-Raphson method is one of the most well-known numerical methods for finding successive approximations to the roots of non-linear equations. We evaluate basins of attraction of equilibrium points with the help of the Newton-Raphson method. It is a good technique for finding trajectory convergence derived from a point of equilibrium neighbourhood. We present a fixed point basin attraction, which means that the set of points converges under successive iterations of certain transformation towards a equilibrium point or Lagrange point. This technique is applicable through the iterative scheme to multivariate function systems $f(x) = 0$ by an iterative scheme,

$$x_{n+1} = x_n - J^{-1} f(x_n),$$

where $f(x_n)$ denotes equations, whereas J^{-1} is the corresponding Jacobian inverse matrix. For each x and y coordinate, the above iterative scheme can be decomposed as follows:

$$x_{n+1} = x_n - \left(\frac{U_x U_{yy} - U_y U_{xy}}{U_{yy} U_{xx} - U_{xy}^2} \right)_{(x_n, y_n)}, \tag{3.36}$$

$$y_{n+1} = y_n + \left(\frac{U_x U_{yx} - U_y U_{xx}}{U_{yy} U_{xx} - U_{xy}^2} \right)_{(x_n, y_n)}. \tag{3.37}$$

Where x_n, y_n are the coordinate values of x and y at the n-th point of the iterative process, while the subscripts denote the corresponding first and second order partial derivatives of the effective potential function $U(x, y)$. The equation of effective potential (3.31) are:

$$U(x, y) = \frac{x^2 + y^2}{2} + \left[\left(\frac{1}{r_{51}} + \frac{1}{r_{52}} \right) + m \left(\frac{1}{r_{53}} + \frac{1}{r_{54}} \right) \right]. \tag{3.38}$$

Where r_{51}, r_{52}, r_{53} and r_{54} are the distances of the infinitesimal body to the primaries,

$$\begin{aligned} r_{51} &= \sqrt{(x + 0.5)^2 + y^2}, \\ r_{52} &= \sqrt{(x - 0.5)^2 + y^2}, \\ r_{53} &= \sqrt{(x - 0.5a)^2 + (y - h)^2}, \\ r_{54} &= \sqrt{(x + 0.5a)^2 + (y - h)^2}. \end{aligned}$$

The Newton-Raphson method is based on the following philosophy: An initial condition (x, y), which activates the code on the configuration plane. If the initial point converges quickly to one of the points of equilibrium then this point (x, y) will be a member of the convergence of the root. Once the successive approximation converges to an attractor, this process stops.

3.7.1 Case 1: Basins of Attraction for Five Equilibrium Points

We start our analysis with the first case, where five equilibrium points are present in two different intervals "i" and "iii" respectively.

3.7.1.1 Basins of Attraction for $a \in (0.00500\ ,\ 0.13967)$

3.7.1.2 Basins of Attraction for $a \in (0.15099\ ,\ 0.18274)$

3.7.2 Case 2: Basins of Attraction for Seven Equilibrium Points

We continue our analysis with the second case, where seven equilibrium points are present in two different intervals "ii" and "iv" respectively.

3.7.2.1 Basins of Attraction for $a \in (0.13967\ ,\ 0.15099)$

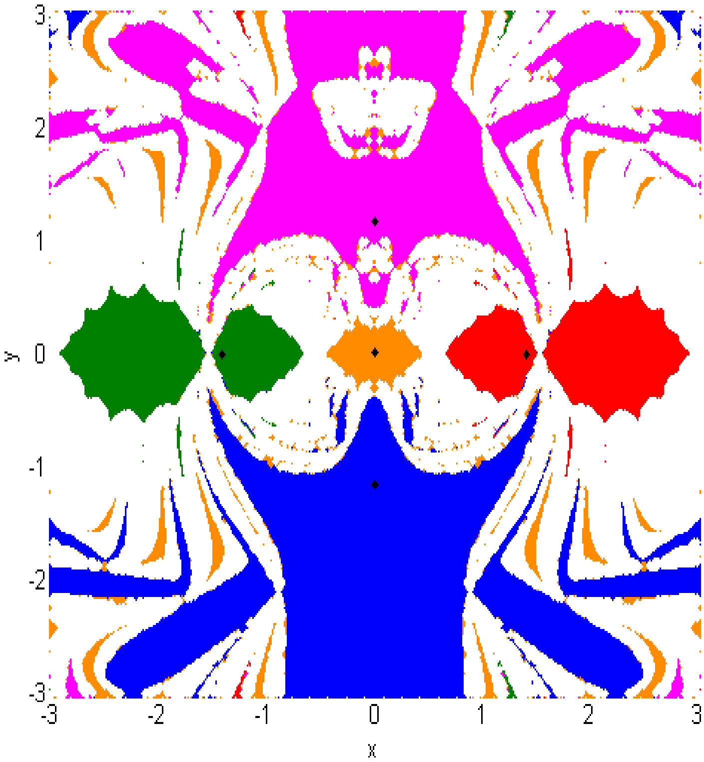


Figure 3.10: Case 1: The Newton-Raphson basins of attraction, in the xy configuration plane, where five equilibrium points are present. Here $a = 0.11984$, $b = 0.97188$, $m = 0.001377$ and $h = 0.8644$. The black dots indicate the position of five equilibrium points. The initial conditions that lead to certain points of equilibrium are marked using the following colour code: L_1 ('Green'); L_2 ('Red'); L_3 ('Blue'); L_4 ('Magenta'); L_5 ('Dark orange'); non converging points ('White').

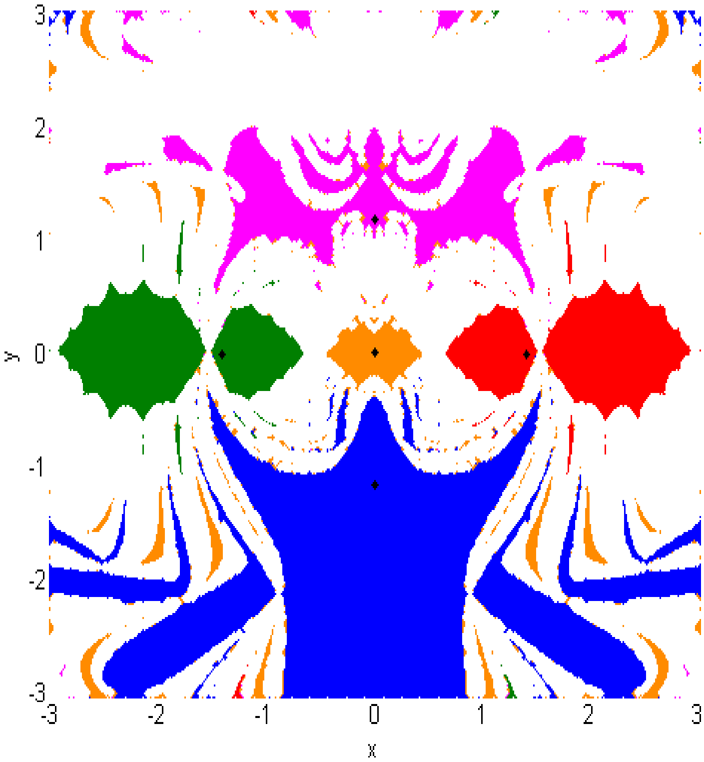
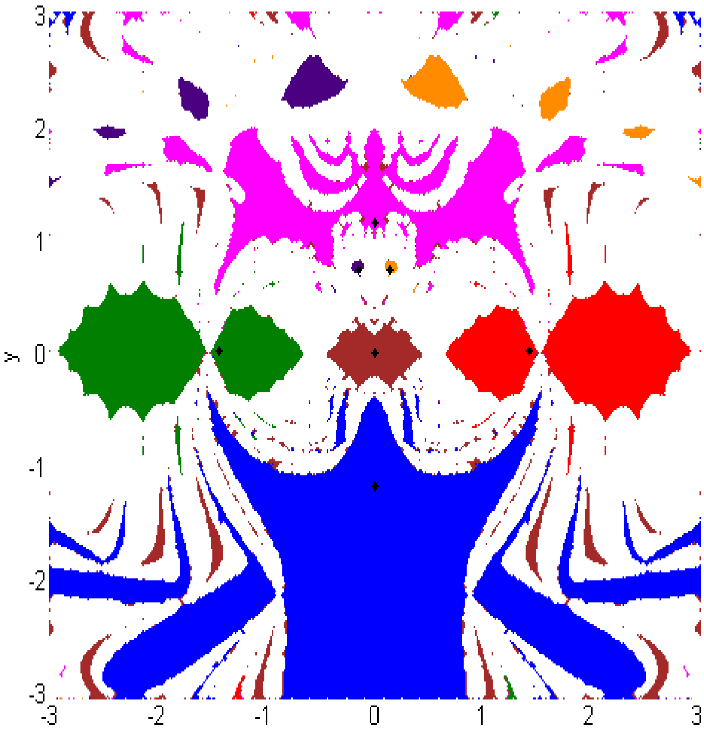


Figure 3.11: Case 1: The Newton-Raphson basins of attraction, in the xy configuration plane, where five equilibrium points are present. Here $a = 0.16687$, $b = 0.96189$, $m = 0.00365$ and $h = 0.86491$. The black dots indicate the position of five equilibrium points. The initial conditions that lead to certain points of equilibrium are marked using the following colour code L_1 ('Green'); L_2 ('Red'); L_3 ('Blue'); L_4 ('Magenta'); L_5 ('Dark orange'); non converging points ('White').

3.7.2.2 Basins of Attraction for $a \in (0.18274\ ,\ 0.43386)$



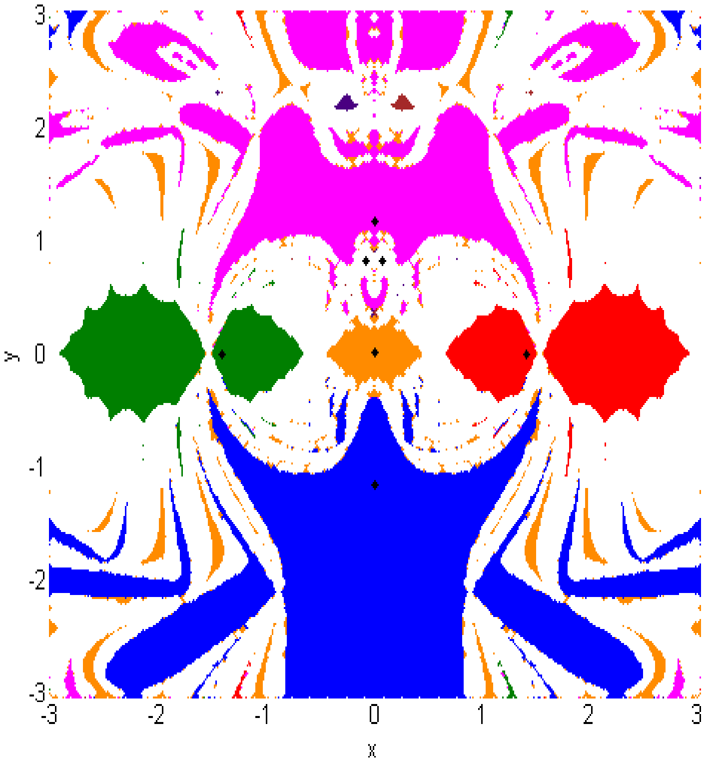


Figure 3.12: Case 2: The Newton-Raphson basins of attraction, in the xy configuration plane, where seven equilibrium points are present. Here $a = 0.15001$, $b = 0.9654$, $m = 0.00216$ and $h = 0.87194$. The black dots indicate the position of seven equilibrium points. The initial conditions that lead to certain points of equilibrium are marked using the following colour code: L_1 ('Green'); L_2 ('Red'); L_3 ('Blue'); L_4 ('Magenta'); L_5 ('Indigo'); L_6 ('Brown'); L_7 ('Dark orange'); non converging points ('White').

3.7.3 Case 3: Basins of Attraction for Nine Equilibrium Points

The present analysis with the third case for equilibrium points, where nine equilibrium points are present in only one interval "viii".

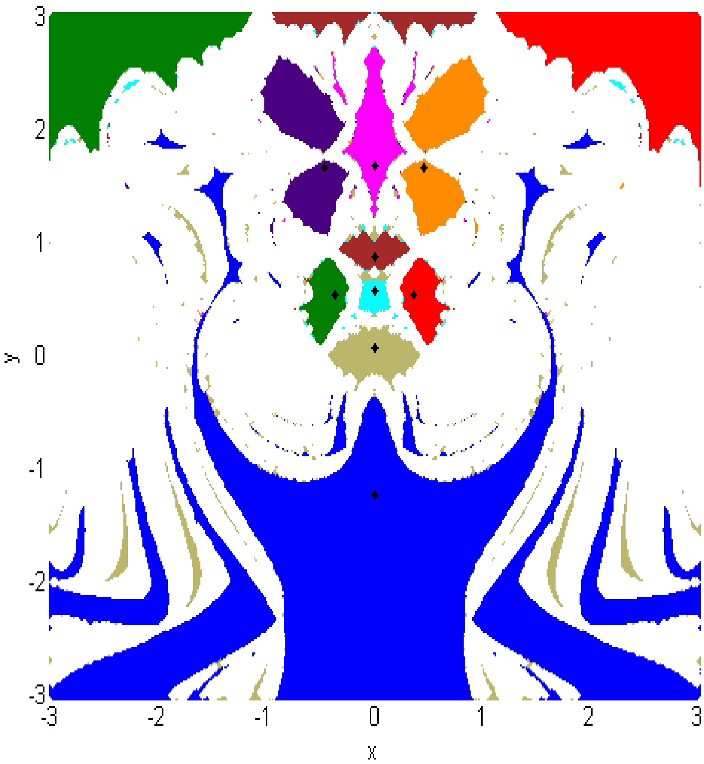


Figure 3.14: Case 3: The Newton-Raphson basins of attraction, in the xy configuration plane, where nine equilibrium points are present. Here $a = 0.80742$, $b = 0.92959$, $m = 0.46305$ and $h = 0.92459$. The black dots indicate the position of nine equilibrium points. The initial conditions that lead to certain points of equilibrium are marked using the following colour code: L_1 ('Green'); L_2 ('Red'); L_3 ('Blue'); L_4 ('Magenta'); L_5 ('Indigo'); L_6 ('Dark orange'); L_7 ('Brown'); L_8 ('Cyan'); L_9 ('Dark khaki'); non converging points ('White').

3.7.3.1 Basins of Attraction for $a \in (0.64220\ ,\ 0.99999)$

3.7.4 Case 4: Basins of Attraction for Eleven Equilibrium Points

We further continue our analysis with the fourth case, where eleven equilibrium points are present in two different intervals "v" and "vii" respectively.

3.7.4.1 Basins of Attraction for $a \in (0.43386\ ,\ 0.60867)$

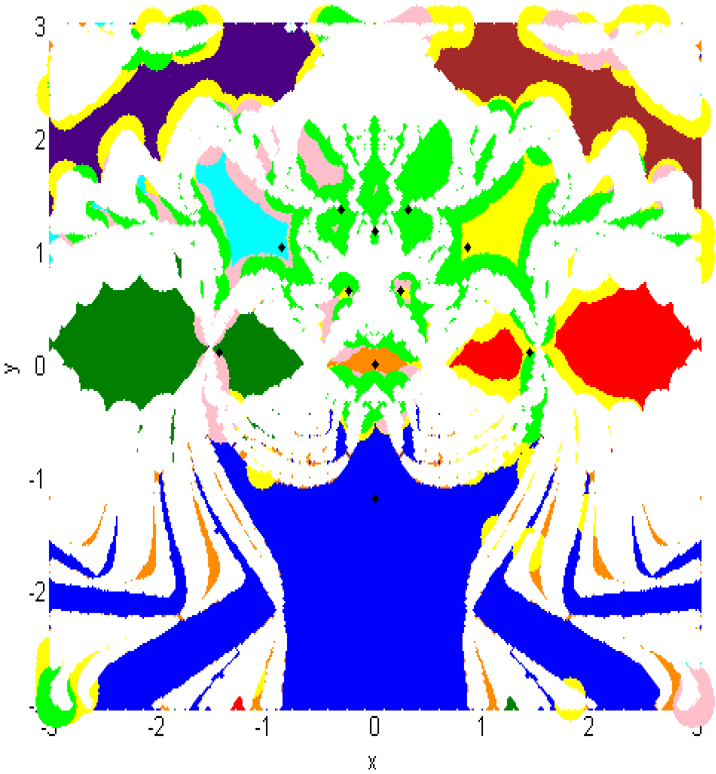


Figure 3.15: Case 4: The Newton-Raphson basins of attraction, in the xy configuration plane, where eleven equilibrium points are present. Here $a = 0.52134$, $b = 0.91067$, $m = 0.10946$ and $h = 0.87796$. The black dots indicate the position of eleven equilibrium points. The initial conditions that lead to certain points of equilibrium are marked using the following colour code: L_1 ('Green'); L_2 ('Red'); L_3 ('Blue'); L_4 ('Magenta'); L_5 ('Dark orange'); L_6 ('Indigo'); L_7 ('Brown'); L_8 ('Cyan'); L_9 ('Dark khaki'); L_{10} ('Pink'); L_{11} ('Lime'); non converging points ('White').

3.7.4.2 Basins of Attraction for $a \in (0.64166\ ,\ 0.64220)$

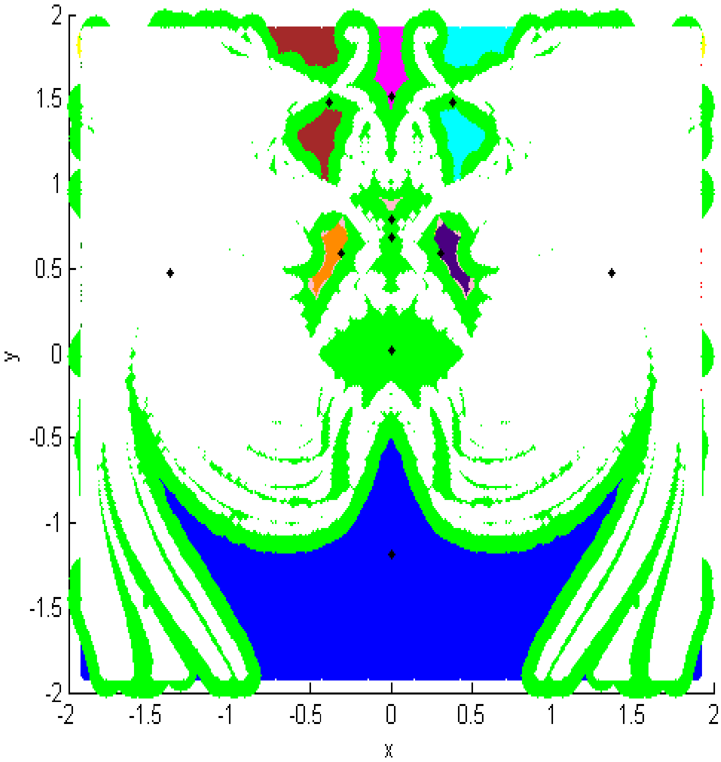


Figure 3.16: Case 4: The Newton-Raphson basins of attraction, in the xy configuration plane, where eleven equilibrium points are present. Here $a = 0.64175$, $b = 0.90867$, $m = 0.21163$ and $h = 0.89084$. The black dots indicate the position of eleven equilibrium points. The initial conditions that lead to certain points of equilibrium are marked using the following colour code: L_1 ('Green'); L_2 ('Red'); L_3 ('Blue'); L_4 ('Magenta'); L_5 ('Dark orange'); L_6 ('Indigo'); L_7 ('Brown'); L_8 ('Cyan'); L_9 ('Dark khaki'); L_{10} ('Pink'); L_{11} ('Lime'); non converging points ('White').

3.7.5 Case 5: Basins of Attraction for Thirteen Equilibrium Points

In the last case, we start our analysis, where thirteen equilibrium points are present in only one interval "vi".

3.7.5.1 Basins of Attraction for $a \in (0.60867\ ,\ 0.64166)$

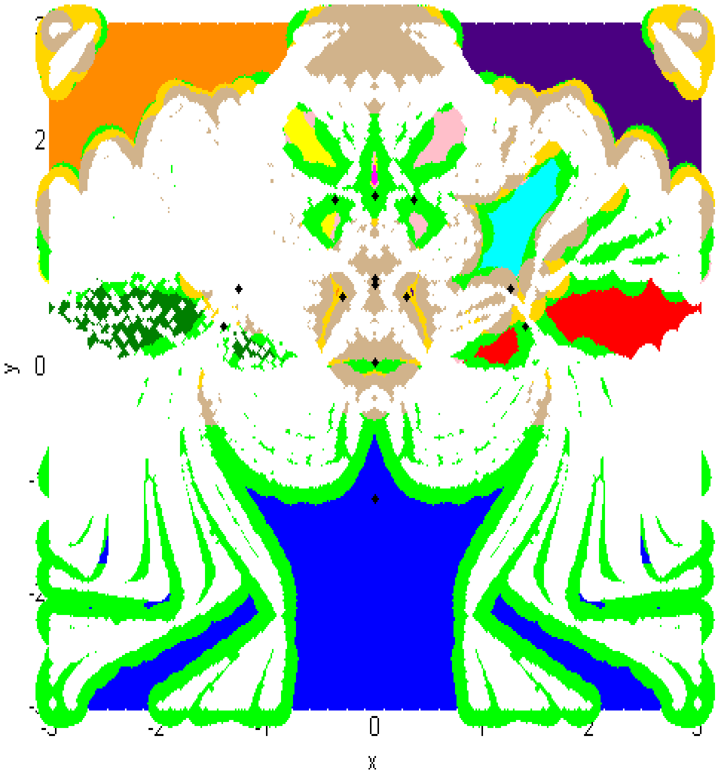


Figure 3.17: Case 5: The Newton-Raphson basins of attraction, in the xy configuration plane, where thirteen equilibrium points are present. Here $a = 0.63333$, $b = 0.90842$, $m = 0.20262$ and $h = 0.88972$. The black dots indicate the position of thirteen equilibrium points. The initial conditions that lead to certain points of equilibrium are marked using the following colour code: L_1 ('Green'); L_2 ('Red'); L_3 ('Blue'); L_4 ('Magenta'); L_5 ('Dark orange'); L_6 ('Indigo'); L_7 ('Brown'); L_8 ('Cyan'); L_9 ('Dark khaki'); L_{10} ('Pink'); L_{11} ('Lime'); L_{12} ('Gold'); L_{13} ('Tan'); non converging points ('White').

3.8 When $a > 1$

3.8.1 Case 6: Seven Equilibrium Points for $a > 1$

We continue our analysis with the first case for equilibrium points, when $a > 1$. Where seven equilibrium points are present, so i will choose different values greater than 1 and draw contour plot.

3.8.1.1 Contour-Plot for $a = 1.53421$

We take $a = 1.53421$ to be any point when $a > 1$, the corresponding value of $b = 1.39472$, $h = 1.36889$ and $m = 4.4887$ respectively. Contour plot for this value shows that the L_3, L_4 and L_7 are collinear along y-axis, while L_1, L_2, L_5 and L_6 are non-collinear.

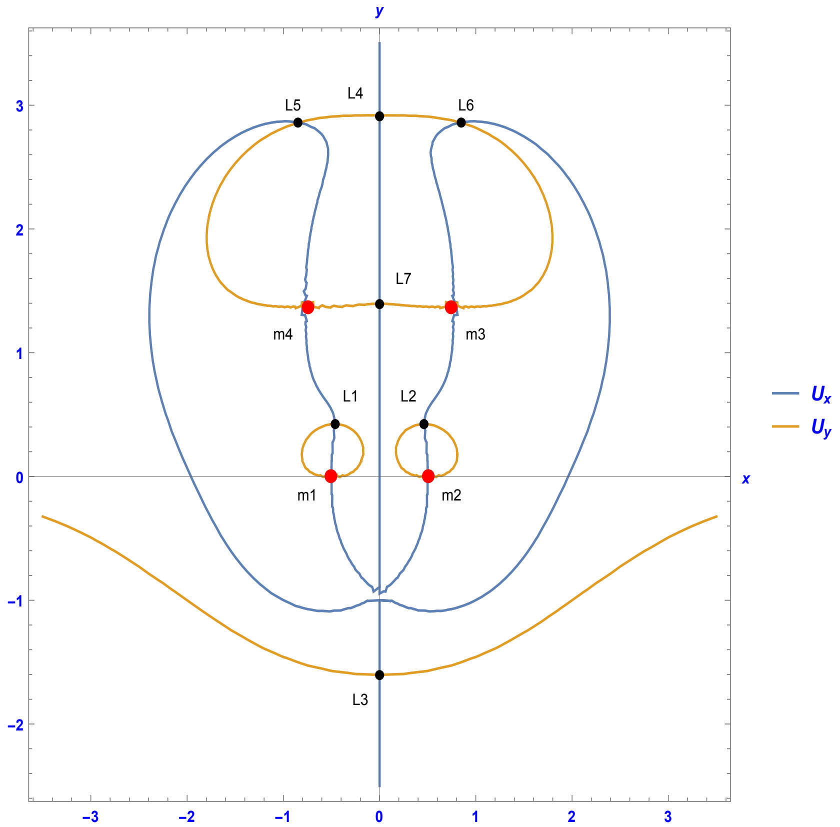


Figure 3.18: Case 6: Seven equilibrium points.
Positions (Black dots) and equilibrium points numbering ($L_i, i = 1, ..., 7$) through the intersections of $U_x = 0$ (blue) and $U_y = 0$ (orange), when $a = 1.53421$, $b = 1.39472$, $m = 4.4887$ and $h = 1.36889$. The Red dots represent the centers ($m_i, i = 1, 2, 3, 4$) of the primaries.

3.8.1.2 Contour-Plot for $a = 1.75343$

We take $a = 1.75343$ to be any point when $a > 1$, the corresponding value of $b = 1.59296$, $h = 1.54777$ and $m = 6.93437$ respectively. Contour plot for this value shows that the L_3, L_4 and L_7 are collinear along y-axis, while L_1, L_2, L_5 and L_6 are non-collinear.

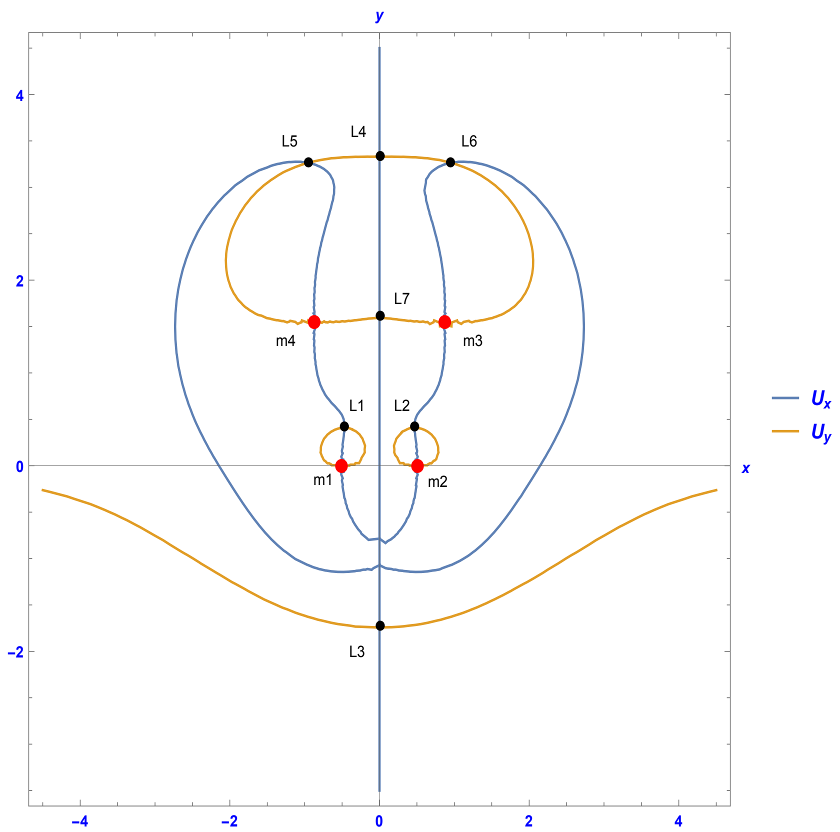


Figure 3.19: Case 6: Seven equilibrium points.
Positions (Black dots) and equilibrium points numbering ($L_i, i = 1, ..., 7$) through the intersections of $U_x = 0$ (blue) and $U_y = 0$ (orange), when $a = 1.75343$, $b = 1.59296$, $m = 6.93437$ and $h = 1.54777$. The Red dots represent the centers ($m_i, i = 1, 2, 3, 4$) of the primaries.

3.8.1.3 Contour-Plot for $a = 2.03512$

We take $a = 2.03512$ to be any point when $a > 1$, the corresponding value of $b = 1.85765$, $h = 1.78408$ and $m = 11.1018$ respectively. Contour plot for this value shows that the L_3, L_4 and L_7 are collinear along y-axis, while L_1, L_2, L_5 and L_6 are non-collinear.

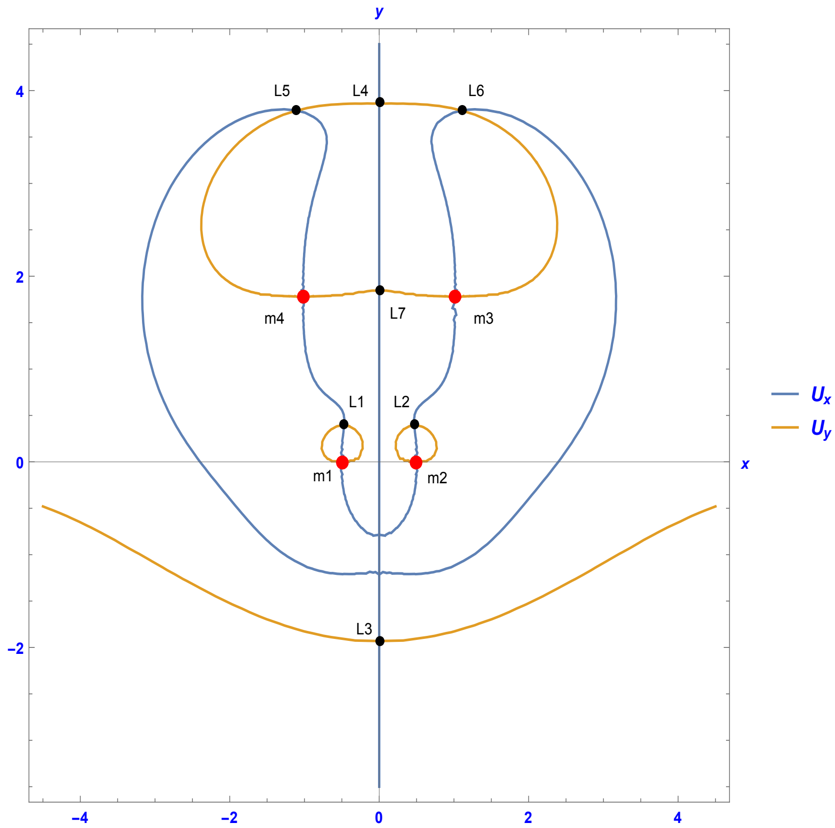


Figure 3.20: Case 6: Seven equilibrium points.
Positions (Black dots) and equilibrium points numbering $(L_i, i = 1, ..., 7)$ through the intersections of $U_x = 0$ (blue) and $U_y = 0$ (orange), when $a = 2.03512$, $b = 1.85765$, $m = 11.1018$ and $h = 1.78408$. The Red dots represent the centers $(m_i, i = 1, 2, 3, 4)$ of the primaries.

3.9 Stability Analysis

The mathematical analysis of the stability of equilibrium points in RT5BP, when $a > 1$. We check whether the equilibrium points are either stable or unstable. To check stability, similarly we perform an individual eigenvalue analysis for each equilibrium point.

Eigenvalues of Case-6:

Choosing $a = 1.53421$, and the corresponding value of $b = 1.39472$, $m = 4.4887$, $h = 1.36889$ and $L_1(-0.4674, 0.4199)$ (see figure 3.18), we will follow the procedure given in chapter (2) at page 12 for the stability analysis. The eigenvalues of L_1 are: $(35.0891, -13.1076)$. Likewise, all eigenvalues of equilibrium points L_i, where $i = 1, ...7$, in case-1 are given below in following tables 3.9, 3.10 and 3.11.

Equilibrium points	Eigenvalues
L_1(-0.4674,0.4199)	(35.0891, -13.1076)
L_2(0.4674,0.4199)	(35.0891, -13.1076)
L_3(0, -1.601)	(2.2856, 0.441247)
L_4(0, 2.913)	(3.61119, 0.217801)
L_5(-0.8413 , 2.859)	(4.16967, -0.318566)
L_6(0.8413 , 2.859)	(4.16967, -0.318566)
L_7(0, 1.399)	(40.1896, -17.7378)

Table 3.9: Stability Analysis for Case 6: $a = 1.53421$ $b = 1.39472$, $m = 4.4887$, $h = 1.36889$.

Equilibrium points	Eigenvalues
L_1(-0.4644,0.4226)	(33.4405, -12.1516)
L_2(0.4644,0.4226)	(33.4405, -12.1516)
L_3(0, -1.732)	(2.24195, 0.453873)
L_4(0, 3.338)	(3.58638, 0.216561)
L_5(-0.9578 , 3.273)	(4.12706, -0.297477)
L_6(-0.9578 , 3.273)	(4.12706, -0.297477)
L_7(0, 1.609)	(41.257, -18.4078)

Table 3.10: Stability Analysis for Case 6: $a = 1.75343$, $b = 1.59296$, $m = 6.93437$, $h = 1.54777$.

Equilibrium points	Eigenvalues
L_1(-0.4717,0.4017)	(36.8383, -13.9599)
L_2(0.4717,0.4017)	(36.8383, -13.9599)
L_3(0, -1.928)	(2.15548, 0.487005)
L_4(0, 3.87)	(3.5936, 0.216146)
L_5(-1.112 , 3.787)	(4.16428, -0.310835)
L_6(1.112 , 3.787)	(4.16428, -0.310835)
L_7(0, 1.851)	(42.3793, -19.1577)

Table 3.11: Stability Analysis for Case 6: $a = 2.03512$, $b = 1.85765$, $m = 11.1018$, $h = 1.78408$.

Therefore all equilibrium points, when $a > 1$ are unstable in xy-plane motion.

3.9.1 Case 6: Basins of Attraction for Seven Equilibrium Points

We continue our analysis with the six case when $a > 1$, where seven equilibrium points are present.

3.9.1.1 Basins of Attraction for $a = 1.53421$

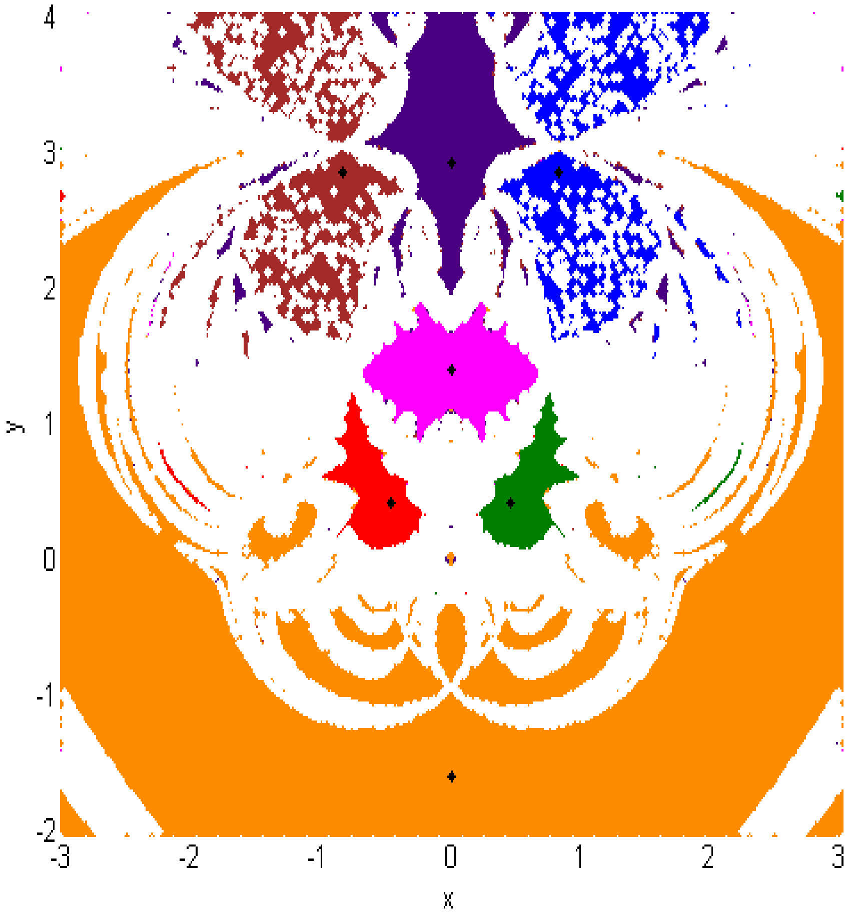


Figure 3.21: Case 6: The Newton-Raphson basins of attraction, in the xy configuration plane, where seven equilibrium points are present. Here $a = 1.53421$, $b = 1.39472$, $m = 4.4887$ and $h = 1.36889$. The black dots indicate the position of seven equilibrium points. The initial conditions that lead to certain points of equilibrium are marked using the following colour code: L_1 ('Red'); L_2 ('Green'); L_3 ('Dark orange'); L_4 ('Indigo'); L_5 ('Brown'); L_6 ('Blue'); L_7 ('Magenta'); non converging points ('White').

3.9.1.2 Basins of Attraction for $a = 1.75343$

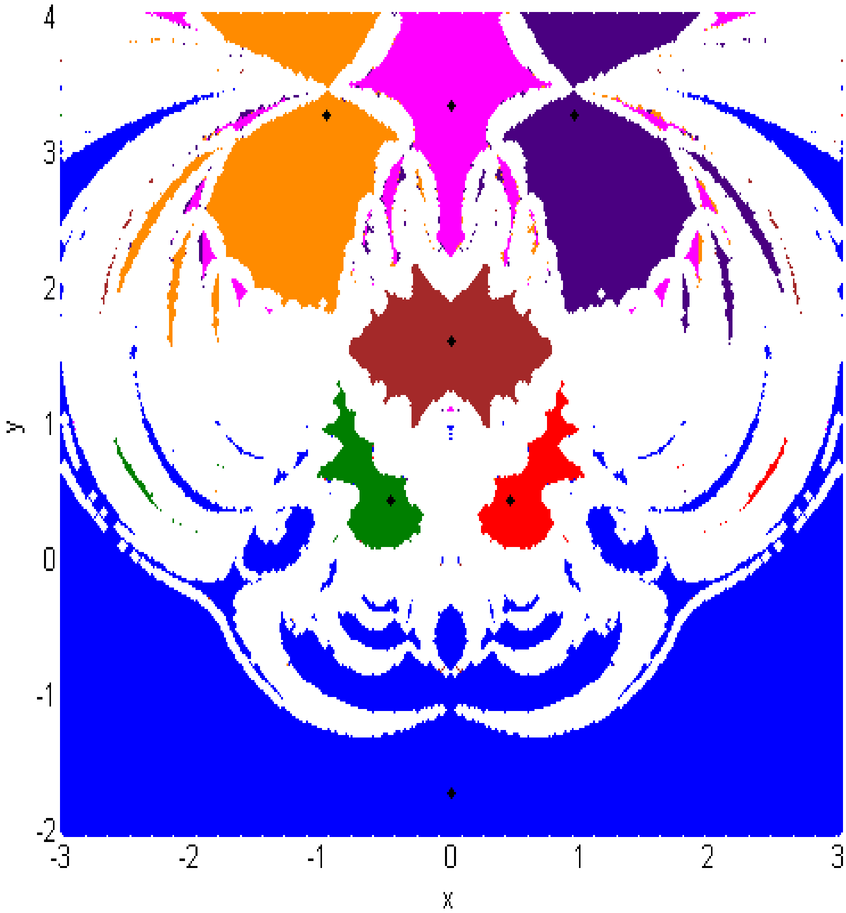


Figure 3.22: Case 6: The Newton-Raphson basins of attraction, in the xy configuration plane, where seven equilibrium points are present. Here $a = 1.75343$, $b = 1.59296$, $m = 6.93437$ and $h = 1.54777$. The black dots indicate the position of seven equilibrium points. The initial conditions that lead to certain points of equilibrium are marked using the following colour code: L_1 ('Green'); L_2 ('Red'); L_3 ('Blue'); L_4 ('Magenta'); L_5 ('Dark orange'); L_6 ('Indigo'); L_7 ('Brown'); non converging points ('White').

3.9.1.3 Basins of Attraction for $a = 2.03512$

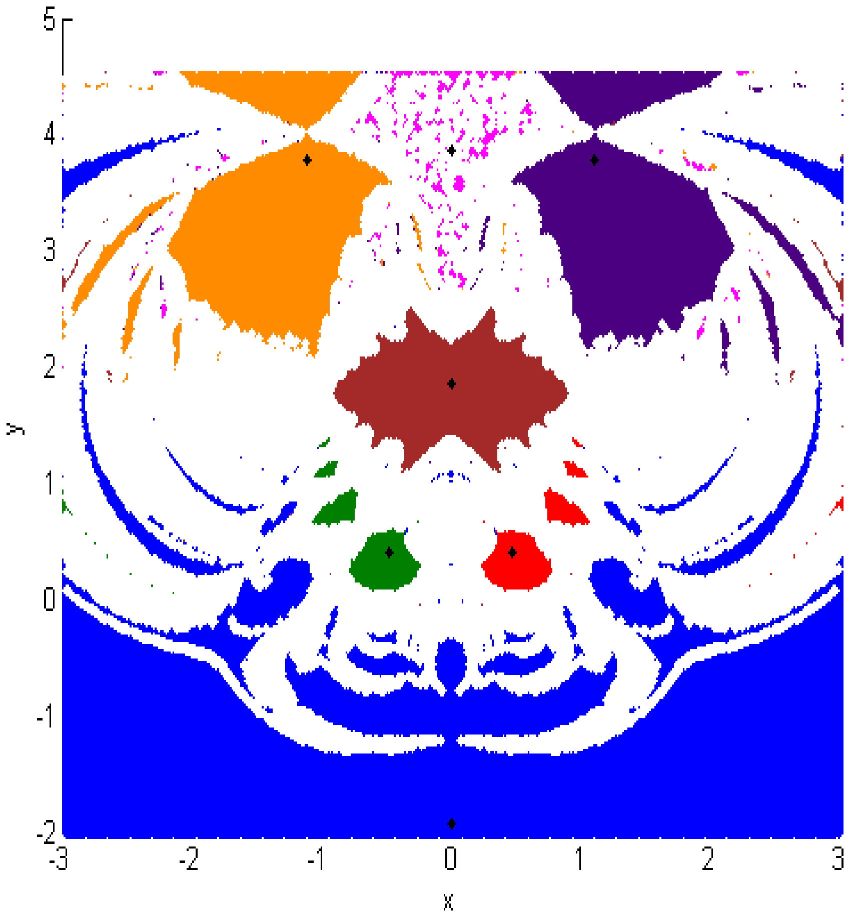


Figure 3.23: Case 6: The Newton-Raphson basins of attraction, in the xy configuration plane, where seven equilibrium points are present. Here $a = 2.03512$, $b = 1.85765$, $m = 11.1018$ and $h = 1.78408$. The black dots indicate the position of seven equilibrium points. The initial conditions that lead to certain points of equilibrium are marked using the following colour code: L_1 ('Green'); L_2 ('Red'); L_3 ('Blue'); L_4 ('Magenta'); L_5 ('Dark orange'); L_6 ('Indigo'); L_7 ('Brown'); non converging points ('White').

Chapter 4

Conclusions

In the present work, we studied the motion of an infinitesimal mass m_5 in the xy-plane under influence of the gravitational force of four primaries m_1, m_2, m_3 and m_4 respectively. There are two pairs of masses which are at the adjacent vertices of an isosceles trapezoid moving in such a way that their central configuration is always an isosceles trapezoid. After finding the equation of motion of m_5 of being negligible mass and not influence the motion of four primaries, we calculated the position of the equilibrium points for different intervals, and examined the stability of equilibrium points for finding eigenvalues using Mathematica. We investigated different intervals and found different equilibrium points in these intervals. There are five cases for $a \in (0\ ,\ 1)$ and one case for $a > 1$. It is seen that the involved parameters in the equation of motions of m_5 influenced the positions of the equilibrium points. The numerical investigation of these values revealed that all the equilibrium points lying in the xy-plane are unstable.
Another aspect of the present work is to study the multivariate version of the Newton-Raphson iterative method to study the basins of convergence to the liberation points of the dynamical system. The Newton-Raphson basin of attraction are plotted in the xy-plane. It is found that the shape, nature of the basin of attraction and the number of equilibrium points /liberation points varies drastically with the change in the position of the primaries. These attracting domains play an important role, since they explain how the system's liberation points attract

each point of the configuration plane, which act as attractors in a way.

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