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Some Fixed Point Theorems in Partial Extended b-Metric Spaces

by

Mehvish Sultan

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dedicated to my brothers Badar Sultan Baz and Babar Sultan Baz.

CERTIFICATE OF APPROVAL

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Mehvish Sultan

(MMT181005)

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(c)	Supervisor	Dr. Samina Batool	CUST, Islamabad

Dr. Samina Batool
Thesis Supervisor
May, 2020

Dr. Muhammad Sagheer
Head
Dept. of Mathematics
May, 2020

Dr. Muhammad Abdul Qadir
Dean
Faculty of Computing
May, 2020

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(Mehvish Sultan)

Registration No: MMT181005

Abstract

The concept of partial metric space initiated from the work of Matthews. The concept of extended b-metric initiated Kamran et al. inspired by the concept of extended b-metric space, and partial metric we have established the concept of partial extended b-metric space, and from the work of Aydi et al. we have introduced partial Hausdorff extended b-metric space. We also prove some fixed point theorems on such spaces. To elaborate the theorem we also established an example.

Contents

Chapter 1

Introduction

1.1 Background

Metric space played a very pivotal role in the evolution of fixed point theory. The concept of metric space was first initiated in 1905 by French Mathematician Frechet [1]. He is considered as founder of modern topology and contributed significantly in the field of set theory and functional theory. He made significant contributions in the topology of point sets and by solely introducing the general idea of entire metric space.

Stefan Banach proved Banach Contraction Principle in 1922. "The Banach fixed point theorem that states conditions which are sufficient for the existence of a fixed point and its uniqueness. More precisely, the Banach fixed point theorem states that if (X, d) is a complete metric space and $T : X \to X$ is a contraction on X, that is there exist a constant $0 \leq \alpha < 1$ such that

$$d(Tx, Ty) \leq \alpha d(x, y) \;\; \forall x \in X,$$

then T has a unique fixed point $x \in X$."

Different mathematicians used different approaches to extend Banach Contraction Principle, either replacing the contraction condition or taking the different spaces.

Kannan and Chatterjea [2] proved Banach Contraction Principle by replacing contraction condition with

$$“d(T(x), T(y)) \leq \alpha\{d(x, Tx) + d(y, Ty)\},$$

where $0 < \alpha < \frac{1}{2}$”.

On the other hand few authors used different spaces like pseudo- metric space [3], metric like spaces [4], partially ordered metric space [5–7] the cone metric spaces. The b-metric space is one of the interesting generalization of metric space which was initiated from the work of Bakhtin [8], Bourbaki [9], Czerwik[10], and Fagin [11].

Modern development in the field of computation pave the way for the introduction of new metric space known as partial metric space. The idea of partial metric spaces was first initiated by Matthews [12], it plays a significant role in computation theory. Much progress and work has been done in partial metric spaces by various mathematicians see for example [4, 13–16]. S. Oltra [6] proved Banach fixed point theorem on this space.

After amelioration in the axioms of metric space, different spaces came into existence. By taking order axioms into consideration, Ran and Reuring [17] in 2004 investigated the existence of fixed points in partial ordered metric spaces and proved results. Subsequently many authors proved some results in this space, see for example [18, 19].

Kamran et al. [20] extended the apprehension of b-metric spaces and originate extended b-metric spaces, and proved number of fixed point theorems in this space. For further details we can see for example [21–24].

By using the idea of b-metric and partial metric Shukla in [25] has established new results in partial b-metric spaces. Afterwards, many authors also proved results in this direction see for example [26–28].

The idea of partial Hausdorff metric was introduced by Aydi et al. [16].

By taking into consideration of the idea of extended b-metric space [20], we have introduced the idea of partial extended b-metric space and proved few results in these spaces. We also elaborated our results with an example.

Moreover, by using the idea of Aydi et al. [16] we have introduced partial Hausdorff extended b-metric space.

The rest of dissertation is organized as follows:

Chapter 2:

Chapter 2 is divided into three sections named as: Metric Space, b-Metric Space and Partial Metric Space. In Metric Space section we focused on basic notations, definitions and results regarding this space. In b-Metric Space, we collected definitions, examples and results of b-Metric Space. In the last section, we discussed about Partial Metric Space being introduced by Matthews [12].

Chapter 3:

This chapter is about the review work of Kamran et al. [20]. They also established results by using this space.

Chapter 4:

This chapter is about literature review of article Aydi et al. [16] . We are going to discuss an analogous of Nadler fixed point theorem in partial metric space.

Chapter 5:

In this chapter we introduced the idea of partial extended b-metric space and partial Hausdorff extended b-metric space by generalizeing the results presented [20] and [16] respectively. To varify our results we also established an example.

The conclusion is presented in last section.

Chapter 2

Preliminaries

In this chapter we defined some basic ideas, examples and results that will be used in the subsequent chapters. First section is concerning to Metric Space its definition and examples. Second section is about the concept of a b-Metric Space and its examples. The last section is pertaining to Partial Metric Space and some auxiliary lemmas that are used in the main results.

2.1 Metric Space

Concept of metric space goes back to 1906. This concept was first given by a French Mathematician Frechet [1].

Definition 2.1.1. [29] (**Metric Space**).
“ A metric space is a pair (X, d), where X is a set and d is a metric on X (or distance function on X), that is a function defined on $X \times X$ such that for all $x, y, z \in X$ we have:

1. d is real- valued, finite and nonnegative,

2. $d(x, y) = 0$ if and only if $x = y$,

3. $d(x, y) = d(y, x)$ (Symmetric Property),

4. $d(x, y) \leq d(x, z) + d(z, y)$ (Triangle Inequality)."

Following are examples of metric space.

Example 2.1.2. [29].
" This is the set of all real numbers, taken with the usual metric defined by $d(x, y) = |x - y|$."

Example 2.1.3. [29]
"Function space $C[a, b]$. As a set X we take the set of all real- valued functions $(x, y, \ldots)$ which are functions of an independent t real variable t and are defined and continuous on a given closed interval $J = [a, b]$.
Choosing the metric defined by

$$d(x, y) = \max_{t \in J} |x(t) - y(t)|,$$

where max denotes the maximum, we obtain a metric space which is denoted by $C[a, b]$. This is a function space because every point of $C[a, b]$ is a function."

Definition 2.1.4. [29] **(Cauchy Sequence in Metric Space)**
"A sequence $\{x_n\}$ in a metric space $X = (X, d)$ is said to be Cauchy (or fundamental) if for every $\epsilon > 0$ there is an $N = N(\epsilon)$ such that

$$d(x_m, x_n) < \epsilon."$$

Definition 2.1.5. [29] **(Convergence in Metric Space)**.
"A sequence $\{x_n\}$ in a metric space $X = (X, d)$ is said to converge or to be convergent if there is an $x \in X$ such that

$$\lim_{n \to \infty} d(x_n.x) = 0,$$

x is called the limit of $\{x_n\}$ and we write

$$\lim_{n \to \infty} x_n = x.$$

Simply,

$$x_n \to x.$$

We say that $\{x_n\}$ converges to x or has the limit x. If $\{x_n\}$ is not convergent, it is said to be divergent."

Definition 2.1.6. [29] (**Completeness in Metric Space**).
"The space X is said to be complete if every Cauchy sequene in X converges (that is, has a limit which is an element of X)."

Definition 2.1.7. [29] (**Fixed Point**)
"A fixed point of a mapping $T : X \to X$ of a set X into itself ia an $x \in X$ which is mapped onto itself (is "kept fixed"), that is,

$$Tx = x$$

the image Tx coincides with x."

A fixed point or invariant point in fixed point theory is an element that is mapped to itself under given mapping, that is it remains invariant under given mapping. Every function needs not to have a fixed point. And a fixed point may or may not be unique.
A function $f(\eta) = \eta + 1$ where the function lies in the set of real numbers $\mathbb{R}$ has no fixed point as there exist no real number that satisfies $\eta = \eta + 1$.

Example 2.1.8. .
Let us define a mapping T on set of real numbers $\mathbb{R}$, by

$$T(s) = s^2 - 3s + 4,$$

where $s \in \mathbb{R}$. 2 is a fixed point of mapping.

Definition 2.1.9. [29] (**Contraction Mapping**)
"Let $X = (X, d)$ be metric space. A mapping $T : X \to X$ is called a contraction on X if there is a positive real number $\alpha < 1$ such that for all $x, y \in X$

$$d(Tx, Ty) \leq \alpha d(x, y).$$

Geometrically this means that any points x and y have images that are closer together than those points x and y; more precisely, the ratio $d(Tx, Ty)/d(x, y)$ does not exceed a constant α which is strictly less than 1."

Stefan Banach proved Banach Contraction Principle in 1922 in his doctoral dissertation.

Theorem 2.1.10. [29] (**Banach Contraction Principle) or(BCP**)
"Consider a metric space $X = (X, d)$ where $X \neq \phi$. Suppose that X is complete and let $T : X \to X$ be a contraction on X. Then T has precisely one fixed point."

Multivalued mappings are extensively used in numerous application.

Definition 2.1.11. [30] (**Multivalued Mapping**)
"A mapping $T : X \to Y$ is said to be a multivalued, if for each element $x \in X$, Tx is a nonempty subset of Y. In other words, a multivalued map T from a set X to Y is a nonempty subset of the product set $X \times Y$. That is, if $T \subset X \times Y$ is a nonempty set, then T is said to be a multivalued map and the image of an element $x \in X$ under T is denoted by Tx and defined by

$$Tx = \{y \in Y | (x, y) \in T\} \subset Y.$$

Where X and Y are nonempty sets. The set Tx may be closed, compact, open, bounded, etc.
The inverses of hyperbolic, trignometric, exponential, integer power functions are all multivalued."

Example 2.1.12. [30]
"The inverse of single valued continuous function $f : X \to Y$ from X onto Y is a multivalued map $\psi_f : Y \to X$ defined by

$$\psi_f(y) = f^{-1}(y) = \{x \in X : f(x) = y,\ for\ y \in Y\}."$$

Example 2.1.13. [30]

"Suppose $a, b \in \mathbb{R}$ be such that $b > a$,

Define $T : [a, b] \to [a, b]$, by

$$T(x) = \begin{cases} [x, b] & \text{if} \quad a < x < b, \\ [a, b] & \text{if} \quad x \in \{a, b\}. \end{cases} \tag{2.1}$$

Then T is a multivalued map."

Example 2.1.14. [30]

"Take $X = [0, 1]$, and consider

$$N(X) = \left\{ A \subset X : A \neq \phi \right\}$$

Define $T : X \to N(X)$, and $S : X \to N(X)$ by :

$$Tx = [0, x]$$

$$S(x) = \begin{cases} [0, 1] & \text{if} \;\; x \neq \frac{1}{2}, \\ [\frac{1}{2}, 1] & \text{if} \;\; x = \frac{1}{2}, \end{cases} \tag{2.2}$$

T, S are examples of multivalued mappings."

Definition 2.1.15. [29] (**Open and Closed Set in Metric Space**)

"A subset M of a metric space X is said to be open if it contains a ball about each of its points. A subset K of X is said to be closed if its complements (in X) is open, that is $K^c = X - K$ is open."

Definition 2.1.16. [29] (**Hausdorff Metric**)

"Let (X, d) be a metric space and $CB(X)$ denotes the collection of all nonempty closed and bounded subsets of X. For $A, B \in CB(X)$, define

$$H(A, B) = \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right\},$$

where $d(x, A) = \inf\{d(x, a) : a \in A\}$ is the distance of point x to the set A. It is known that H is a metric on $CB(X)$, called the Hausdorff metric induced by the metric d."

Definition 2.1.17. [31] **(Multi-Valued Contraction Mapping)**
" Let $(X, d_1), (Y, d_2)$ are two metric spaces. A function $F : X \to CB(Y)$ is said to be a multi-valued Lipschitz mapping (abbreviated m.v.l.m.) of X into Y if and only if $H(F(x), F(z)) \leq \alpha d_1(x, z)$ for all $x, z \in X$ and $\alpha \geq 0$ is a fixed real number.
α is called Lipschitz constant for F. If F has a Lipschitz constant $\alpha < 1$, then F is called a multi-valued contraction mapping (abbreviated m.v.c.m.). A m.v.l.m. is continuous."

Theorem 2.1.18. [31] **(Multi-Valued Contraction Principle)**
"Let (X, d) be a complete metric space. If $F : X \to CB(X)$ is a multi value contraction mapping then it has unique fixed point."

2.2 b-Metric Space

The idea of b-metric was initiated from the work of [8, 9].

Definition 2.2.1. [27] (b-**Metric Space**)
"Let X be a nonempty set and let $s \geq 1$ be given real number. A function $d : X \times X \longrightarrow [0, \infty)$ is called b-metric if for all $x, y, z \in X$ the following conditions are satisfied:

1. $d(x, y) = 0$ if and only if $x = y$,
2. $d(x, y) = d(y, x)$,
3. $d(x, y) \leq s[d(x, z) + d(z, y)]$.

The pair (X, d) is called b-metric space. The number $s \geq 1$ is called the coefficient of (X, d)."

Definition 2.2.2. [20] (**Cauchy, Convergent, Completeness**)

"Let (X, d) be a complete b-metric space, a sequance $\{x_n\} \in X$ is said to be:

(i): Cauchy if and only if $d(x_m, x_m) \to 0$ as $m, n \to \infty$,

(ii): Convergent if and only if there exist $x \in X$, such that $d(x_n, x) \to 0$ as $n \to \infty$ and we write $\lim\limits_{n \to +\infty} x_n = x$,

(iii): The b-metric space (X, d) is complete if every Cauchy sequence is convergent."

Example 2.2.3. [20]

"Let $X = l_p(\mathbb{R})$ with $0 < p < 1$, where

$$l_p(\mathbb{R}) = \{\{x_n\} \subset \mathbb{R} : (\sum_{n=1}^{\infty} |x_n - y_n|^p)^{\frac{1}{p}} < \infty\}.$$

Defined $d : X \times X \to \mathbb{R}^+$ as,

$$d(x, y) = (\sum_{n=1}^{\infty} |x_n - y_n|^p)^{\frac{1}{p}}."$$

Example 2.2.4. [20]

"Let $X = L_p[0, 1]$ be the space of all real functions $x(t), t \in [0, 1]$ such that

$$\int_0^1 |x(t)|^p < \infty,$$

with $0 < p < 1$.

Defined $d : X \times X \to \mathbb{R}^+$

$$d(x, y) = (\int_0^1 |x(t) - y(t)|^p dt^{\frac{1}{p}}.$$

Then d is b-metric space with cofficient $s = 2^{\frac{1}{p}}$."

Remark 2.2.5. [20]

"The above examples show that the class of b-metric spaces is larger than the class of metric spaces. When $s = 1$ the concept of b-metric space coincides with the concept of metric space."

Theorem 2.2.6. [20]

"Let (X, d) be a complete b-metric space with constant $s \geq 1$, such that b-metric is a continuous functional. Let $T : X \to X$ be a contraction having contraction constant $k \in (0, 1)$ such that $sk < 1$. Then T has a unique fixed point."

Theorem 2.2.7. [20]

"Let (X, d_θ) be an extended b-metric space. If d_θ is continuous, then every convergent sequence has a unique limit."

Definition 2.2.8. [22] **(Hausdorff-Pompieu Metric)**

"Let (X, d_θ) be an extended b-metric space. $A, B \in H(X)$

$$H_\theta(A, B) = \left(\max \left\{ \sup_{a \in A} d_\theta(a, B), \sup_{b \in B} d_\theta(b, A) \right\} \right),$$

the mapping H is said to be the Pompieu-Hausdorff metric induced by d_θ."

Theorem 2.2.9.

Assume that (Q, d_ϕ) is an extended b-metric spaces, then $(CB(Q), H_\psi)$ is an extended Hausdorff- Pompieu b-metric spaces.

2.3 Partial Metric Space

Matthews [12] initiated the concept of partial metric space, and generalized the Banach Contraction Principal.

Definition 2.3.1. [13]

"A partial metric on a nonempty set X is a function
$p : X \times X \to \mathbb{R}^+$

(p_1) $x = y \Longleftrightarrow p(x, x) = p(y, y) = p(x, y)$,

(p_2) $p(x,x) \leq p(x,y)$,

(p_3) $p(x,y) = p(y,x)$,

(p_4) $p(x,y) \leq p(x,z) + p(y,z) - p(z,z)$.

A partial metric space is a pair (X,p) such that X is a nonempty set and p is a partial metric on X."

Example 2.3.2. [16]
"A mapping $p : X \times X \to \mathbb{R}^+$; $\forall (x,y) \in \mathbb{R}^+$, defined by

$$p(x,y) = \max\{x,y\},$$

is example of partial metric space."

Example 2.3.3. [32]
"Let $\mathbb{I}$ be the collection of nonempty closed bounded intervals in $\mathbb{R}$

$\mathbb{I} = \{[a,b] : a \leq b\}$. For $[a,b],[c,d] \in [I]$,

define $p([a,b],[c,d]) = \max(b,d) - \min(a,c)$.

Then it can be shown that p is a partial metric over $\mathbb{I}$."

Definition 2.3.4. [16] **(Open p Ball)**
"Each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family open p- balls

$$\Big\{B_p(x,\epsilon) : x \in X, \epsilon > 0\Big\},$$

where $B_p(x,\epsilon) = \Big\{y \in X : p(x,y) < p(x,x) + \epsilon\Big\}$, for all $x \in X$ and $\epsilon > 0$."

Definition 2.3.5. [16] **(Convergence in Partial Metric Space)**
"A sequence $\{x_n\}$ in a partial metric space (X,p) converges to a point $x \in X$,

with respect to τ_p, if and only if

$$p(x,x) = \lim_{n\to\infty} p(x_n, x).\text{''} \tag{2.3}$$

Definition 2.3.6. [28] **(Cauchy and Completeness in Partial Metric Space)**
"Let (X,p) be a partial metric space.

(1) A sequence $\{x_n\}$ in X is said to be a Cauchy sequence if $\lim\limits_{m,n\to\infty} p(x_n, x_m)$ exists and is finite.

(2) (X,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges with respect to τ_p to a point $x \in X$ such that $\lim\limits_{n\to+\infty} p(x,x_n) = p(x,x)$. In this case, we say that the partial metric p is complete."

Definition 2.3.7. [16]
"If p is a partial metric on X, then the function $p^s : X \times X \to \mathbb{R}^+$ given by $p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y)$, defines a metric on X. Furthermore, a sequence $\{x_n\}$ converges to $x \in X$ in (X,p^s) iff

$$\lim_{m,n\to\infty} p(x_m, x_n) = \lim_{m\to\infty} p(x_m, x) = p(x,x).\text{''} \tag{2.4}$$

Lemma 2.3.8. [23]

(1) " A sequence $\{x_n\}$ in X is a Cauchy sequence in $(X,p) \Longleftrightarrow$ it is a Cauchy sequence in (X,p^s).

(2) A partial metric space (X,p) is complete $\Longleftrightarrow$ metric space (X,p^s) is complete."

Partial b-metric space was being initiated by Shukla et al. [25] and proved fixed point result on this space.

Definition 2.3.9. [25] (**Partial b-Metric Space**)
"Let X be a nonempty set and $s \geq 1$ be a given real number.

A function $p_b : X \times X \longrightarrow [0, \infty)$ is called a partial b-metric if for all $x, y, z \in X$ the following conditions are satisfied:

(i) $p_b(x,x) = p_b(y,y) = p_b(x,y)$ if and only if $x = y$,

(ii) $p_b(x,x) \leq p_b(x,y)$,

(iii) $p_b(x,y) = p_b(y,x)$,

(iv) $p_b(x,y) \leq s\Big[p_b(x,z) + p_b(z,y) - p_b(z,z)\Big]$.

The (X, p_b) is called a partial b-metric space. The number $s \geq 1$ is called the coefficient of(X, p_b)."

Definition 2.3.10. [33] **(Iterative Sequence)**

"Let U be any set, $T : U \to CB(U)$ be a multi-valued map. For any $u_0 \in U'$, the sequence $\{u_n\}_{n=0}^{\infty}$ given by,

$$u_{n+1} \in Tu_n, \; n = 0, 1, 2, ... \tag{2.5}$$

Is called an iterative sequence with initial point u_0."

Remark 2.3.11. [27]

"The class of partial b-metric space (X, p_b) is effectively larger than the class of partial metric space. Since partial metric space is special case of partial b-metric space (X, p_b) when $s = 1$. Also the class of partial b-metric space $(X.p_b)$ is effectively larger than the class of b-metric space. Since a b-metric space is a special case of partial b-metric space (X, p_b) when the self distance $p(x,x) = 0$"

Chapter 3

Generalization of b-Metric Space

Kamran et al. [20] formulated the new space named as extended b-metric space. They also established some fixed point results for self mapping on extended b-metric space. Some necessary results and example are also mentioned. The first results extend Banach Contraction Principle and second results extend main fixed point results of Hicks and Rhoades with an example.

3.1 Generalization of b-Metric Space

Definition 3.1.1. [20]**(Extended b-Metric Space)**
"Let X be a nonempty set and $\theta : X \times X \to [1, \infty)$. A function
$d_\theta : X \times X \to [0, \infty)$ is called an extended b-metric space,
if for all $x, y, z \in X$ it satisfies:

$(d_\theta 1)$ $d_\theta(x, y) = 0 \Longleftrightarrow x = y$,

$(d_\theta 2)$ $d_\theta(x, y) = d_\theta(y, x)$,

$(d_\theta 3)$ $d_\theta(x, z) \leq, \theta(x, z)\Big[d_\theta(x, y) + d_\theta(y, z)\Big]$.

The pair (X, d_θ) is called an extended b-metric space."

Example 3.1.2.

Let $\mathbb{M} = \{1, 2, 3\}$, define $\Omega : \mathbb{M} \times \mathbb{M} \to \mathbb{R}^+$ and $d_\Omega : \mathbb{M} \times \mathbb{M} \to \mathbb{R}^+$, as:

$$\Omega(a, b) = a + b + 1,$$

$$d_\Omega(1, 1) = d_\Omega(2, 2) = d_\Omega(3, 3) = 0,$$

$$d_\Omega(1, 2) = d_\Omega(2, 1) = 80,$$

$$d_\Omega(1, 3) = d_\Omega(3, 1) = 1000,$$

$$d_\Omega(2, 3) = d_\Omega(3, 2) = 600.$$

Proof. First two conditions of extended b-metric space holds trivially, to prove triangular inequality we have:

$$\begin{aligned} d_\Omega(1, 2) &= 80, \\ \Omega(1, 2)[d_\Omega(1, 3) + d_\Omega(3, 2)] &= 4(1000 + 600) = 6400, \\ d_\Omega(1, 2) &\leq \Omega(1, 2)[d_\Omega(1, 3) + d_\Omega(3, 2)], \\ d_\Omega(1, 3) &= 1000, \\ \Omega(1, 3)[d_\Omega(1, 2) + d_\Omega(2, 3)] &= 5(80 + 600) = 3400, \\ d_\Omega(1, 3) &\leq \Omega(1, 3)[d_\Omega(1, 2) + d_\Omega(2, 3)]. \end{aligned}$$

Similarly, we can calculate other values and prove that

$$d_\Omega(a, c) \leq \Omega(a, c)[d_\Omega(a, b) + d_\Omega(b, c)].$$

Hence it is proved that (M, d_Ω) is an extended b-metric space. □

Example 3.1.3. [20]

"Let $X = C([a, b], \mathbb{R}$ be the space of all continuous real valued functions defined on $[a, b]$.

Note that X is complete extended b- metric space by considering,

$d_\theta(x, y) = \sup_{t \in [a,b]} |x(t) - y(t)|^2$ with

$\theta(x, y) = |x(t)| + |y(t)| + 2$, where

$\theta : X \times X \to [1, \infty)$."

Definition 3.1.4. [20] **(Convergence, Cauchy and Completeness)**

"Let (X, d_θ) be an extended b-metric space.

(i) A sequence $\{x_n\}$ in X is said to converge to $x \in X$, if for every $\epsilon > 0$ there exists $N = N(\epsilon) \in \mathbb{N}$ such that $d_\theta(x_n, x) < \epsilon$, for all $n \geq N$. In this case, we write $\lim\limits_{n \to +\infty} x_n = x$.

(ii) A sequence $\{x_n\}$ in X is said to be Cauchy, if for every $\epsilon > 0$ there exists $N = N(\epsilon) \in \mathbb{N}$ such that $d_\theta(x_m, x_n) < \epsilon$, for all $m, n \geq N$.

(iii) An extended b-metric space (X, d_θ) is complete if every Cauchy sequence in X is convergent."

Theorem 3.1.5.

"Let (M, d_Ω) be a complete extended b-metric space such that d_Ω is a continuous functional. Let $T : M \to M$ satisfy:

$$d_\Omega(Tp, Tq) \leq k d_\Omega(p, q) \;\; \forall\, p, q \in M$$

where $k \in [0, 1)$ be such that for each $p_0 \in M$,

$\lim\limits_{n,m \to +\infty} \Omega(p_m, p_n) < \frac{1}{k}$, here $p_m = T^m p_o$, $m = 1, 2, \ \ldots$.

Then T has precisely one fixed point. Moreover for each $\eta \in M, T^m\eta \to \eta$".

Proof. Let us suppose any $p_0 \in M$ be arbitrary, define a sequence $\{p_n\}$ by:

p_0,

$p_1 = Tp_0$,

$p_2 = T^2 p_0 \ldots p_m = T^m p_o$.

Let $p_1, p_2 \in M$

$$\begin{aligned} d_\Omega(Tp_1, Tp_2) &\leq r d_\Omega(p_1, p_2), \\ &\leq r d_\Omega(Tp_0, Tp_1), \\ &\leq r^2 d_\Omega(p_0, p_1) \ldots \leq r^m d_\Omega(p_0, p_1). \end{aligned}$$

By similar method we obtain,

$$d_\Omega(p_m, p_{m+1}) \leq r^m d_\Omega(p_0, p_1). \tag{3.1}$$

By using triangular inequality of extended b-metric space, we have

$$\begin{aligned}
d_\Omega(p_m, p_n) \leq& \Omega(p_{,m}, p_n)\Big[d_\Omega(p_m, p_{m+1} + d_\Omega(p_{m+1}, p_n)\Big],\\
\leq& \Omega(p_m, p_n)d_\Omega(p_m, p_{m+1}) + \Omega(p_m, p_n)d_\Omega(p_{m+1}, p_n),\\
\leq& \Omega(p_m, p_n)r^m d_\Omega(p_0, p_1)+\\
&\Omega(p_m, p_n)\Big[\Omega(p_{m+1}, p_n)d_\Omega(p_{m+1}, p_{m+2}) + d_\Omega(p_{m+2}, p_n)\Big],\\
\leq& \Omega(p_m, p_n)r^m d_\Omega(p_0, p_1) + \Omega(p_m, p_n)\Omega(p_{m+1}, p_n)d_\Omega(p_{m+1}, p_{m+2})+\\
&\Omega(p_m, p_n)\Omega(p_{m+1}, p_n)d_\Omega(p_{m+2}, p_n).
\end{aligned}$$

Therefor we can write,

$$\begin{aligned}
\leq& \Omega(p_m, p_n)r^m d_\Omega(p_0, p_1) + \Omega(p_m, p_n)\Omega(p_{m+1}, p_n)r^{m+1}d_\Omega(p_0, p_1)+\\
&\Omega(p_m, p_n)\Omega(p_{m+1}, p_n)\Big[\Omega(p_{m+2}, p_n)[d_\Omega(p_{m+2}, p_{m+3}) + d_\Omega(p_{m+3}, p_n)\Big],\\
\leq& \Omega(p_m, p_n)r^m d_\Omega(p_0, p_1) + \Omega(p_m, p_n)\Omega(p_{m+1}, p_n)r^{m+1}d_\Omega(p_0.p_1)+\\
&\Omega(p_n, p_m)\Omega(p_{m+1}, p_m)\Omega(p_{m+2}, p_n)d_\Omega(p_{m+2}, p_{m+3})+\\
&\Omega(p_n, p_m)\Omega(p_{m+1}p_n)\Omega(p_{m+2}, p_n)d_\Omega(p_{m+3}, p_n).
\end{aligned}$$

Proceeding this way we get,

$$\begin{aligned}
d_\Omega(p_m, p_n) \leq& \Omega(p_m, p_n)r^m d_\Omega(p_0, p_1) + \Omega(p_m, p_n)\Omega(p_{m+1}, p_n)r^{m+1}d_\Omega(p_0, p_1) + \ \ldots +\\
&\Omega(p_m, p_n)\Omega(p_{m+1}, p_n)\Omega(p_{m+2}, p_n)\ \ldots\ \Omega(p_{n-2}, p_n)\Omega(p_{n-1}, p_n)r^{n-1}d_\Omega(p_0.p_1),\\
\leq& d_\Omega(p_0, p_1)\Big[\Omega(p_1, p_n)\Omega(p_2, p_n)\ \ldots\ \Omega(p_{m-1}, p_n)\Omega(p_m, p_n)r^m+\\
&\Omega(p_1, p_n)\Omega(p_2, p_n)\ \ldots\ \Omega(p_m, p_n)\Omega(p_{m+1}, p_n)r^{m+1} + \ \ldots +\\
&\Omega(p_1, p_n)\Omega(p_2, p_n)\ \ldots\ \Omega(p_m, p_n)\Omega(p_{m+1}, p_n)\ \ldots\ \Omega(p_{n-2}, p_m)\Omega(p_{n-1}, p_n)r^{n-1}\Big].
\end{aligned}$$

Since, $\lim\limits_{m,n\to\infty} \Omega(p_m, p_n)k < 1$, this implies that $\sum_{m=1}^{\infty} r^n \prod_{i=1}^{m} \Omega(p_i, p_n)$ is convergent by ratio test for each $n \in \mathbb{N}$.

$$S = \sum_{m=1}^{\infty} r^m \prod_{i=1}^{m} \Omega(p_i, p_n), \ S_m = \sum_{j=1}^{m} r^j \prod_{i=1}^{j} \Omega(p_i, p_n)$$

and

$$S_m = \Omega(p_1, p_n)r + \Omega(p_1, p_n)\Omega(p_2, p_n)r^2 + \ldots + \Omega(p_1, p_n)\Omega(p_2, p_n) \ldots \Omega(p_n, p_n)r^m.$$

$S = \Omega(p_1, p_n)r + \Omega(p_1, p_n)\Omega(p_2, p_n)r^2 + \ldots + \Omega(p_1, p_n)\Omega(p_2, p_n) \ldots \Omega(p_m, p_n)r^m +$
$\Omega(p_1, p_n)\Omega(p_2, p_n) \ldots \Omega(p_m, p_n)\Omega(p_{m+1}, p_n)r^{m+1} + \ldots + \Omega(p_1, p_n)\Omega(p_2, p_n) \ldots$
$\Omega(p_m, p_n)\Omega(p_{m-1}, p_n \ldots$
$\ldots \Omega(p_{n-2}, p_n)\Omega(p_{n-1}, p_n)r^{n-1} + \ldots .$
For $n > m$, by using above inequality we get,

$$d_\Omega(p_m, p_n) \leq d_\Omega(p_0, p_1)\left[S_{n-1} - S_{m-1}\right].$$

Letting $m \to \infty$, we get a Cauchy sequence $\{p_m\}$, since M is complete consider $p_m \to \eta \in M$. To prove that T has a fixed point, suppose

$$\begin{aligned} d_\Omega(T\eta, \eta) \leq & \Omega(T\eta, \eta)\Big[d_\Omega(T\eta, p_m) + d_\Omega(p_m, \eta)\Big], \\ \leq & \Omega(T\eta, \eta)\Big[kd_\Omega(\eta, p_{m-1}) + d_\Omega(p_m, \eta)\Big], \\ d_\Omega(T\eta, \eta) \leq & 0, \quad as\ m \to \infty \\ \implies d_\Omega(T\eta, \eta) = & 0. \end{aligned}$$

Hence it is proved that η is a fixed point of T. Assume another fixed point γ, so that

$$d_\Omega(\eta, \gamma) \leq \Omega\Big[d_\Omega(\eta, p_m) + d_\Omega(p_m, \gamma)\Big],$$

$$d_\Omega(\eta, \gamma) \leq \Omega\Big[d_\Omega(\eta, p_m) + d_\Omega(p_m, \gamma)\Big],$$

letting $m \to \infty$, we get
$d_\Omega(\eta, \gamma) \leq 0,$

$\Longrightarrow \eta = \gamma.$ □

Definition 3.1.6.

"Let $T : M \to M$ for some $p_0 \in M$, $O(p_0) = \{p_0, Tp_0, T^2p_0, \ . \ . \ . \ \}$ be the orbit of p_0. A function $H : M \to \mathbb{R}$ is said to be T-orbitally lower semi-contiuous at $t \in M$ if $\{p_n\} \subset O(p_0)$ and $p_n \to t$, implies

$$H(t) \leq \liminf_{n\to\infty} H(x_n)."$$

Theorem 3.1.7.

"Let (M, d_Ω) be a complete extended b-metric space such that d_Ω is a continuous functional. Let $T : M \to M$ and there exists $m_0 \in M$ such that:

$$d_\Omega(Ty, T^2y) \leq rd_\Omega(y, Ty), \tag{3.2}$$

for each $y \in O(p_0)$

where $k \in [0, 1)$ be such that $y_0 \in M$,

$$\lim_{m,n\to\infty} \Omega(y_m, y_m) < \frac{1}{k},$$

$y_n = T^n y_0$, $n = 1, 2, \ . \ . \ .$ then $T^n y_0 \to \eta \in T$.

Furthermore η is a fixed point of T if and only if $H(y) = d(y, Ty)$ is T orbitally lower semi continuous at η."

Proof. Let $y_0 \in M$, define an iterative sequence $\{y_n\}$ by :

$y_0,$

$Ty_0 = y_1,$

$y_2 = Ty_1 = T(Ty_0) = T^2(y_0) \ . \ . \ . \ \ y_n = T^n y_0.$

By use of successive iteration of above Inequality (4.2)

$$\begin{aligned} d_\Omega(T^n y_0, T^{n+1} y_0) =& d_\Omega(y_n, y_{n+1}), \\ \leq& r d_\Omega(y_{n-1}, y_n), \\ \leq& r^2 d_\Omega(y_{n-2}, y_{n-1}) \ldots \leq r^n d_\Omega(y_0, y_1). \end{aligned}$$

Proceeding by following Theorem (3.4.5) we can prove that $\{y_n\}$ is a Cauchy sequence.
As M is complete then $y_n = T^n y_0 \to \eta \in M$.

$$d_\Omega(\eta, T\eta) \leq \liminf_{n\to\infty} d_\Omega(T^n y_0, T^{n+1} y_0), \tag{3.3}$$

$$\leq \liminf_{n\to\infty} r^n d_\Omega(y_0, y_1) = 0, \tag{3.4}$$

$$\Longrightarrow d_\Omega(\eta, T\eta) = 0,$$

$$T\eta = \eta,$$

$\Longrightarrow \eta$ is fixed point of T.
Conversely assume that η is fixed point of T, now we will prove that H is orbitally lower semi continuous.
Let $y_n \in O(y)$ with $y_n \to \eta$, then

$$\begin{aligned} H(\eta) &= d_\Omega(\eta, T\eta) = 0, \\ &\leq \liminf_{n\to\infty} H(y_n), \\ &= d_\Omega(T^n y_0, T^{n+1} y_0). \end{aligned}$$

□

Example 3.1.8.
"Let $M = [0, \infty)$, define $d_\Omega : M \times M \to \mathbb{R}^+$ and
$\Omega : M \times M \to [1, \infty)$ as

$$d_\Omega(m_1, m_2) = (m_1 - m_2)^2,\ \Omega(m_1, m_2) = (m_1 + m_2 + 2), \quad \forall\, m_1, m_2 \in M$$

then d_Ω is a complete extended b-metric space on M.

Define $T : M \to M$ by $Tm_1 = \frac{m_1}{2}$, then

$$\begin{aligned} d_\Omega(Tm_1, Tm_2) &= \left(\frac{m_1}{2} - \frac{m_2}{2}\right)^2, \\ &\leq \frac{1}{3}(m_1 - m_2)^2, \\ &= kd_\Omega(m_1, m_2). \end{aligned}$$

for each $m_1 \in M$ and $T^n m_1 = \frac{m_1}{2^n}$, thus we get

$$\lim_{m,n\to\infty} \Omega(T^m m_1, T^n m_1) = \lim_{m,n\to\infty} \left(\frac{m_1}{2^n} + \frac{m_1}{2^m} + 2\right) < 3.$$

Since all conditions of Theorem (4.0.9) satisfied.

Hence T has a unique fixed point."

Chapter 4

Partial Hausdorff Metric Space

In this chapter, the review of the work of Aydi et al. [16] is presented. They introduced the concept of partial Hausdorff metric and proved Nadler's fixed point theorem on this space. They also give example to varify main results. The theory of multivalued mapping has significant application in convex optimization, control theory.

4.1 Partial Hausdorff Metric Space

Before define partial Hausdorff metric, we start with some important ingredients like closedness and boundedness.

Definition 4.1.1. [16](**Closedness**)
"Let (M, p) be a partial metric space and $CB^p(M)$ be the family of all nonempty, closed and bounded subsets of the partial metric space (M, p). A is closed in (M, p) if and only if $A = \bar{A}$. Here A be any nonempty set in (M, p). "

Definition 4.1.2. [23](**Boundedness**)
"Let (M, p) be a partial metric space and $CB^p(M)$ be the family of all nonempty, closed and bounded subset of (M, p). A is bounded subset in (M, p) if there exist

$x_0 \in M$ and $K \geq 0$ such that for all $a \in A$, we have $a \in B_p(x_0, K)$ that is $p(x_0, a) < p(a, a) + K$."

Closeness can be elaborated by following example.

Example 4.1.3.

Suppose $M = \{0, 1, 4\}$ is a set and $p : M \times M \to \mathbb{R}^+$ is partial metric space defined by,

$$p(s,t) = \frac{1}{4}|s-t| + \frac{1}{2}\max\{s,t\},$$

for all $s, t \in M$.

As

$$p(1,1) = \frac{1}{2} \neq 0,$$

$$p(4,4) = 2 \neq 0,$$

As $p^s(s-t) = |s-t|$ so (M, p) is a complete.

$$\begin{aligned} s \in \{\bar{0}\} &\Longleftrightarrow p(s, \{0\}), \\ &\Longleftrightarrow \frac{3}{4}s = \frac{1}{2}s \Longleftrightarrow s = 0, \\ &\Longleftrightarrow s \in \{0\}. \end{aligned}$$

$$\begin{aligned} s \in \{0,\bar{}4\} &\Longleftrightarrow p(s, \{0,4\}) = p(s,s), \\ &\Longleftrightarrow \min\left\{\frac{3}{4}s, \frac{1}{4}|s-4| + \frac{1}{2}\max\{s,4\}\right\} = \frac{1}{2}s, \\ \frac{3}{4}s &= \frac{1}{2}s, \\ (\frac{3}{4} - \frac{1}{2})s &= 0, \\ &\Longrightarrow s = 0, \\ \frac{1}{4}|s-4| + \frac{1}{2}\max\{s,4\} &= \frac{1}{2}s. \end{aligned}$$

We have two cases here, when

$$
\begin{aligned}
s &> 4,\\
\frac{1}{4}(s-4)+\frac{1}{2}s &= \frac{1}{2}s,\\
\frac{1}{4}(s-4) &= 0,\\
s &= 4.\\
s &< 4,\\
-\frac{1}{4}(s-4)+2 &= \frac{1}{2}s,\\
-\frac{1}{4}s+1+2 &= \frac{1}{2}s,\\
3 &= \frac{3}{4}s,\\
s &= 4,\\
\iff s &\in \{0,4\}.
\end{aligned}
$$

It shows that $\{0,4\}$ is closed set in(M,p).

$$
\begin{aligned}
s \in \bar{\{0,1\}} &\iff p(s,\{0,1\}) = p(s,s),\\
&\iff \min\left\{\frac{3}{4}s, \frac{1}{4}|s-1| + \frac{1}{2}\max\{s,1\}\right\} = \frac{1}{2}s,\\
\frac{3}{4}s &= \frac{1}{2}s,\\
(\frac{3}{4}-\frac{1}{2})s &= 0,\\
&\iff s = 0,\\
\frac{1}{4}|s-1| + \frac{1}{2}\max\{s,1\} &= \frac{1}{2}s.
\end{aligned}
$$

here two cases arises, when

$$
\begin{aligned}
s &> 1,\\
\frac{1}{4}(s-1)+\frac{1}{2}s &= \frac{1}{2}s,\\
\frac{1}{4}(s-1) &= 0,\\
s &= 1.\\
s &< 1,\\
-\frac{1}{4}(s-1)+\frac{1}{2} &= \frac{1}{2}s,\\
-\frac{1}{4}s+\frac{1}{4}+\frac{1}{2} &= \frac{1}{2}s,\\
\frac{3}{4} &= \frac{3}{4}s,\\
s &= 1,\\
\iff s &\in \{0,1\}.
\end{aligned}
$$

"For $E, F \in CB^p(X)$ and $x \in X$, define
$p(x,E) = \inf\{p(x,a), a \in E\}$,
$\delta_p(E,F) = \sup\{p(a,F) : a \in E\}$,
and $\delta_p(F,E) = \sup\{p(b,E) : b \in F\}$."

Remark 4.1.4. [16]
"Let (M,p) be a partial metric space and A any nonempty set in (M,p), then

$$a \in \bar{A} \iff p(a,A) = p(a,a), \tag{4.1}$$

where $\bar{A}$ denotes the closure of A with respect to partial metric p. And is closed in (X,p) if and only if $A = \bar{A}$."

Now we have proposition that we will use in main results.

Proposition 4.1.5.
Let (M,p) is a partial metric space and $\mathcal{J},\mathcal{K},\mathcal{L} \in CB^p(M)$, then we have

(1) $\delta_p(\mathcal{J}, \mathcal{J}) = \sup \left\{ p(j, j) : j \in \mathcal{J} \right\}$,

(2) $\delta_p(\mathcal{J}, \mathcal{J}) \leq \delta_p(\mathcal{J}, \mathcal{K})$,

(3) $\delta_p(\mathcal{J}, \mathcal{K}) = 0 \Rightarrow \mathcal{J} \subseteq \mathcal{K}$,

(4) $\delta_p(\mathcal{J}, \mathcal{K}) \leq \delta_p(\mathcal{J}, \mathcal{L}) + \delta_p(\mathcal{L}, \mathcal{K}) - \inf_{l \in \mathcal{L}} p(\ell, \ell)$.

Proof. To prove (1) we proceed as follows.
If $\mathcal{J} \in CB^p(M)$, then $\forall\ j \in \mathcal{J}$,
we have, by using Equation (4.1)

$$p(j, \mathcal{J}) = p(j, j)\ as\ \bar{\mathcal{J}} = \mathcal{J}.$$

Therefore,

$$\delta_p(\mathcal{J}, \mathcal{J}) = \sup \{p(j, \mathcal{J}) : j \in J\},$$

$$= \sup \{p(j, j) : j \in \mathcal{J}\}. \tag{4.2}$$

To prove (2)
For $j \in \mathcal{J}$,
we have $p(j, j) \leq p(j, k)$ from the condition of partial metric space, for all $k \in \mathcal{K}$
$p(j, j) \leq p(j, \mathcal{K})$,

$$\begin{aligned} p(j, j) &\leq p(j, \mathcal{K}), \\ &\leq \sup\{p(j, \mathcal{K}) : j \in \mathcal{J}\}, \\ &= \delta_p(\mathcal{J}, \mathcal{K}). \end{aligned}$$

From Equation (4.2)

$$\delta_p(\mathcal{J}, \mathcal{J}) = \sup\{p(j, j) : j \in \mathcal{J}\},$$
$$\leq \delta_p(\mathcal{J}, \mathcal{K}).$$

In order to prove (3) we proceed as follows,

suppose that, $\delta_p(\mathcal{J}, \mathcal{K}) = 0$,

as a result we get,

$$p(j, \mathcal{K}) = 0 \ \forall j \in \mathcal{J},$$

because $\delta_p(\mathcal{J}, \mathcal{K}) = \sup\{p(j, \mathcal{K}) \ j \in \mathcal{J}\}$,

and $p(j, \mathcal{K}) \leq \delta_p(\mathcal{J}, \mathcal{K})$, then by using Proposition (3.1.6) we will get

$$p(j, j) \leq \delta_p(\mathcal{J}, \mathcal{K}) = 0 \ \forall j \in \mathcal{J},$$

$p(j, j) = 0$ for all $j \in \mathcal{J}$.

Hence we get

$$p(j, \mathcal{K}) = p(j, j).$$

As a result we get $j \in \bar{\mathcal{K}} = \mathcal{K}$ whenever $j \in \mathcal{J}$ so $\mathcal{J} \subset \mathcal{K}$.

We shall prove (4) as follows.

Suppose $j \in \mathcal{J}, k \in \mathcal{K} \ and \ \ell \in \mathcal{L}$,

by using Definition (2.3.1) of partial metric space,

$$p(j, k) \leq p(j, \ell) + p(\ell, k) - p(\ell, \ell),$$

Since k is an arbitrary element of $\mathcal{K}$, so we have

$$p(j, \mathcal{K}) \leq p(\ell, \ell) + p(\ell, \mathcal{K}) - p(\ell, \ell),$$

and for all $\ell \in \mathcal{L}$, we have

$$p(j, \mathcal{K}) + p(\ell, \ell) \leq p(j, \ell) + \delta_p(\mathcal{L}, \mathcal{K}),$$

As ℓ is an arbitrary element of $\mathcal{L}$, therefore

$$p(j,\mathcal{K}) + \inf_{\ell\in\mathcal{L}} p(\ell,\ell) \leq p(j,\mathcal{L}) + \delta_p(\mathcal{L},\mathcal{K}),$$

As $j \in \mathcal{J}$ is arbitrary element,

$$\delta_p(\mathcal{J},\mathcal{K}) \leq \delta_p(\mathcal{J},\mathcal{L}) + \delta_p(\mathcal{L},\mathcal{K}) - \inf_{\ell\in\mathcal{L}} p(\ell,\ell).$$

□

Consider (M,p) a partial metric space and let $S,T \in CB^p(M)$, define

$$H_p(\mathcal{J},\mathcal{K}) = \max\left\{\delta_p(\mathcal{J},\mathcal{K}), \delta_p(\mathcal{K},\mathcal{J})\right\}. \tag{4.3}$$

To give H_p a structure Aydi et al. [16] prove the following proposition. Let (M,p) be a partial metric space. For $\mathcal{J},\mathcal{K} \in CB^p(M)$, define

$$H_p\left(\mathcal{J},\mathcal{K}\right) = \max\left\{\delta_p(\mathcal{J},\mathcal{K}), \delta_p(\mathcal{K},\mathcal{J})\right\}. \tag{4.4}$$

Proposition 4.1.6.

Suppose that (M,p) be a partial metric space, and $\mathcal{J},\mathcal{K},\mathcal{L} \in CB^p(M)$, we have following inequalities

(1) $H_p(\mathcal{J},\mathcal{J}) \leq H_p(\mathcal{J},\mathcal{K})$,

(2) $H_p(\mathcal{J},\mathcal{K}) = H_p(\mathcal{K},\mathcal{J})$,

(3) $H_p(\mathcal{J},\mathcal{K}) \leq H_p(\mathcal{J},\mathcal{L}) + H_p(\mathcal{L},\mathcal{K}) - \inf_{\ell\in\mathcal{L}} p(\ell,\ell)$.

Proof. To prove $H_p(\mathcal{J},\mathcal{J}) \leq H_p(\mathcal{J},\mathcal{K})$, we proceed as follows, from the Definition (4.4)

$$H_p(\mathcal{J},\mathcal{J}) = \max\left\{\delta_p(\mathcal{J},\mathcal{J}), \delta_p(\mathcal{J},\mathcal{J})\right\} = \delta_p(\mathcal{J},\mathcal{J}),$$

$$H_p(\mathcal{J},\mathcal{J}) = \delta_p(\mathcal{J},\mathcal{J}).$$

Then from Proposition (3.1.4), we can write

$$H_p(\mathcal{J},\mathcal{J}) = \delta_p(\mathcal{J},\mathcal{J}) \leq \delta_p(\mathcal{J},\mathcal{K}) \leq H_p(\mathcal{J},\mathcal{K}),$$

$$\delta_p(\mathcal{J},\mathcal{K}) \leq H_p(\mathcal{J},\mathcal{K}).$$

So we can write it as follows,

$$\Rightarrow \quad H_p(\mathcal{J},\mathcal{J}) \leq H_p(\mathcal{J},\mathcal{K}),$$

for proving (2), we use the definition

$$H_p(\mathcal{J},\mathcal{K}) = \max\{\delta_p(\mathcal{J},\mathcal{K}), \delta_p(\mathcal{K},\mathcal{J})\},$$

similarly we can write,

$$H_p(\mathcal{J},\mathcal{K}) = \max\{\delta_p(\mathcal{K},\mathcal{J}), \delta_p(\mathcal{J},\mathcal{K})\},$$

$$H_p(\mathcal{J},\mathcal{K}) = H_p(\mathcal{K},\mathcal{J}).$$

We shall prove(3) as follows,

$$H_p(\mathcal{J},\mathcal{K}) = \max\{\delta_p(\mathcal{J},\mathcal{K}), \delta_p(\mathcal{K},\mathcal{J})\},$$

by using Proposition (3.1.4) part (4), we have

$$\begin{aligned}
&\leq \max\left\{\delta_p(\mathcal{J},\mathcal{L}) + \delta_p(\mathcal{L},\mathcal{K}) - \inf_{l\in\mathcal{L}} p(\ell,\ell), \delta_p(\mathcal{K},\mathcal{L}) + \delta_p(\mathcal{L},\mathcal{J}) - \inf_{\ell\in\mathcal{L}} p(\ell,\ell)\right\},\\
&\leq \max\left\{\delta_p(\mathcal{J},\mathcal{L}) + \delta_p(\mathcal{L},\mathcal{K}), \delta_p(\mathcal{K},\mathcal{L}) + \delta_p(L,\mathcal{J})\right\} - \inf_{\ell\in\mathcal{L}} p(\ell,\ell),\\
&\leq \max\left\{\delta_p(\mathcal{J},\mathcal{L}) + \delta_p(\mathcal{L},\mathcal{K})\right\} + \max\left\{\delta_p(\mathcal{K},\mathcal{L}) + \delta_p(\mathcal{L},\mathcal{J})\right\} - \inf_{\ell\in\mathcal{L}} p(\ell,\ell),
\end{aligned}$$

$$H_p(\mathcal{J},\mathcal{K}) \leq H_p(\mathcal{J},\mathcal{L}) + H_p(\mathcal{L},\mathcal{K}) - \inf_{\ell\in\mathcal{L}} p(\ell,\ell).$$

□

Corollary 4.1.7.

Let (M, p) is a partial metric space, $\mathcal{J},\ \mathcal{K} \in CB^p(M)$, then following assertions hold:

$$H_p(\mathcal{J}, \mathcal{K}) = 0 \Rightarrow \mathcal{J} = \mathcal{K}.$$

Proof. Suppose that

$$H_p(\mathcal{J}, \mathcal{K}) = 0,$$

then by using definition we will get,

$$H_p(\mathcal{J}, \mathcal{K}) = \delta_p(\mathcal{J}, \mathcal{K}) = \delta_p(\mathcal{K}, \mathcal{J}) = 0,$$

as a result we get,

$$\mathcal{J} \subset \mathcal{K}, \mathcal{K} \subset \mathcal{J},$$

$\Longrightarrow \mathcal{J} = \mathcal{K}.$

□

This corollary tells that partial Hausdorff distance $\mathcal{J},\ \mathcal{K}$ of two sets is zero then set must be equal to each other. But converse of this statement needs not to be true, as if sets are equal to each other then their Hausdorff distance needs not be zero.

Remark 4.1.8.

Converse of above corollary needs not be true.

Example 4.1.9.

"Suppose that $M = [0, 1]$, with partial metric $p : M \times M \to \mathbb{R}^+$

and $p(m, n) = \max\{m, n\}$.

By using definition, we have

$H_p(M, , M) = \delta_p(M, M)$

$= \sup \left\{ m : 0 \leq m \leq 1 \right\} = 1 \neq 0.$"

Remark 4.1.10.

"The above example show that any Hausdorff metric is a partial Hausdorff metric, but converse not exist."

4.2 Fixed Point Of Multi-Valued Contraction Mapping

Lemma 4.2.1.

Consider (M, p) partial metric space, $\mathcal{J},\ \mathcal{K} \in CB^p(M)$ and $h > 1$. For any $j \in \mathcal{J}$ there exist $y = y(j)$ such that

$$p(j, k) \leq hH_p(\mathcal{J}, \mathcal{K}). \tag{4.5}$$

Proof. To prove above lemma, two cases arises. The first case is when sets are equal to each other,

that is $\mathcal{J} = \mathcal{K}$ then by using (1) of Proposition (3.2.4) we have,

$H_p(\mathcal{J}, \mathcal{K}) = H_p(\mathcal{K}, \mathcal{J})$.

For $j \in \mathcal{J}$ we have,

$$\begin{aligned} p(j, j) &\leq \sup_{j \in \mathcal{J}} p(j, j), \\ &= H_p(\mathcal{J}, \mathcal{K}), \\ &\leq hH_p(\mathcal{J}, \mathcal{K}), \quad h > 1 \end{aligned}$$

Hence sets are same then we get our desired result.

When sets are distinct , suppose contradictory that

$$p(j, k) > hH_p(\mathcal{J}, \mathcal{K}),$$

for all $k \in \mathcal{K}$, from this we get,

$$\inf\{p(j,k) : k \in \mathcal{K}\} \geq hH_p(\mathcal{J},\mathcal{K}),$$
$$p(j,\mathcal{K}) \geq hH_p(\mathcal{J},\mathcal{K}).$$

Using Propositions (3.1.6) we have,

$$H_p(\mathcal{J},\mathcal{K}) \geq \delta_p(\mathcal{J},\mathcal{K}). \tag{4.6}$$

From Proposition (3.1.7) we can write

$$= \sup_{j\in\mathcal{J}} p(j,\mathcal{K}) \geq p(j,k) \geq hH_p(\mathcal{J},\mathcal{K}. \tag{4.7}$$

Combining Equations (4.6) and (4.7) we have,
$H_p(\mathcal{J},\mathcal{K}) \geq hH_p(\mathcal{J},\mathcal{K})$,
since $\mathcal{J} \neq \mathcal{K}$, from Corollary (3.1.8) we have $H_p(\mathcal{J},\mathcal{K}) \neq 0$, so we have $h \leq 1$ which is a contradiction. □

Definition 4.2.2. (Multivalued Partial Contraction).
"Let (M,p) be a partial metric space and $k \in (0,1)$. A mapping $T : M \to CB^p(M)$ is said to be multivalued partial contraction if

$$H_p(Tm_1,Tm_2) \leq kp(m_1,m_2), \quad \forall m_1,m_2 \in M\text{"}$$

Theorem 4.2.3.
"Let (M,p) be a complete partial metric space. If $T : M \to CB^p(M)$ is a multivalued mapping such that for all $m_1,,m_2 \in M$ we have

$$H_p(Tm_1,Tm_2) \leq kp(m_1,m_2), \tag{4.8}$$

where $k \in (0,1)$. Then T has a fixed point."

Proof. Suppose $m_0 \in M$ and $m_1 \in Tm_0$ from Lemma (3.2.1) with $h = \frac{1}{\sqrt{k}}$, there exist $m_2 \in Tm_1$, such that

$$p(m_1, m_2) \leq \frac{1}{\sqrt{k}} H_p(Tm_0, Tm_1).$$

Using contraction condition (4.8) we have,

$$H_p(Tm_0, Tm_1) \leq kp(m_0, m_1).$$

Combining above two inequalities we have,

$$p(m_1, m_2) \leq \sqrt{k} p(m_0, m_1).$$

For $m_2 \in Tm_1$there exist $m_3 \in Tm_2$ such that

$$\begin{aligned} p(m_2, m_3) &\leq \frac{1}{\sqrt{k}} H_p(Tm_1, Tm_2), \\ &\leq \sqrt{k} p(m_1, m_2), \\ &\leq \sqrt{k}.\sqrt{k} p(m_0, m_1), \\ &\leq kp(m_0, m_1), \\ p(m_3, m_4) &\leq k^{\frac{3}{2}} p(m_0, m_1). \end{aligned}$$

Similarly, continuing in this way we will get,

$$p(m_n, m_{n+1}) \leq \sqrt{k} p(m_n, m_{n-1}) \; \forall n \geq 1, \tag{4.9}$$

$$\leq \sqrt{k}\sqrt{k} p(m_{n-1}, m_{n-2}),$$

$$\leq k^2 p(m_{n-1}, m_{n-2}).$$

Continuing this process we obtain,

$$\leq (\sqrt{k})^n p(m_1, m_0),$$

$$p(m_n, m_{n+1}) \leq (\sqrt{k})^n p(m_0, m_1). \tag{4.10}$$

By using Definition (2.3.1) of partial metric space, and Equation (4.10) we can write

$$\begin{aligned} p(m_n, m_{n+m}) &\leq p(m_n, m_{n+1}) + p(m_{n+1}, m_{n+2}) + \dots\dots\dots + p(m_{n+m-1}, m_{n+m}), \\ &\leq (\sqrt{k})^n p(m_0, m_1) + (\sqrt{k})^{n+1} p(m_0, m_1) + \dots\dots\dots + (\sqrt{k})^{n+m-1} p(m_0, m_1), \\ &\leq (\sqrt{k})^n + (\sqrt{k})^{n+1} + \dots\dots + (\sqrt{k})^{n+m-1} p(m_0, m_1), \\ &\leq \frac{(\sqrt{k})^n}{1-\sqrt{k}} p(m_0, m_1) \to 0 \ as \ n \to \infty. \end{aligned}$$

$\{m_n\}$ is Cauchy sequence in partial metric space (M, p), using definition of p^s we have,

$$p^s(m_1, m_2) = 2p(m_1, m_2) - p(m_1, m_1) - p(m_2, m_2),$$

$$p^s(m_n, m_{n+m}) \leq 2p(m_n, m_{n+m}) \to 0 \ \ as \ n \to \infty. \tag{4.11}$$

This shows that $\{m_n\}$ is a Cauchy sequence in (M, p^S). By Lemma (2.3.8), it is also Cauchy sequence in (M, p). Hence $\{m_n\}$ converges to m^* in p^s. Hence there exist $m^* \in M$ such that, $\lim_{n\to+\infty} p^s(m_n, m^*) = 0$.
Using the Definition (2.3.7) for convergent we have,

$$p(m^*, m^*) = \lim_{n\to+\infty} p(m_n, m^*) = \lim_{n\to+\infty} p(m_n, m_n) = 0. \tag{4.12}$$

This implies $\{m_n\}$ converges m^* in (M, p), by using Equation (4.10) we have,

since $H_p(Tm_n, Tm^*) \leq kp(m_n, m^*)$, therefore we have

$$\lim_{n\to+\infty} H_p(Tm_n, Tm^*) = 0. \tag{4.13}$$

Since $m_{n+1} \in Tm_n$ implies that,

$$p(m_{n+1}, Tm^*) \leq \delta_p(Tm_n, Tm^*) \leq H_p(Tm_n, Tm^*),$$

by (4.13) $H_p(Tm_n, Tm^*) = 0$, this implies that

$$\lim_{n \to +\infty} p(m_{n+1}, Tm^*) = 0. \tag{4.14}$$

Consequently, by applying triangular inequality we get,

$$p(m^*, Tm^*) \leq p(m^*, m_{n+1}) + p(m_{n+1}, Tm^*),$$

taking limit $n \to +\infty$, and using Equations (4.12), (4.14), we get $p(m^*, Tm^*)$.
Again using of Equation (4.12), we have $p(m^*, m^*) = 0$.
$\Longrightarrow p(m^*, m^*) = p(m^*, Tm^*) = 0$.
$\Longrightarrow m^* \in \bar{Tm^*}$.
Tm^* is closed. Hence we have our desired result. □

Example 4.2.4.

Suppose $M = \{0, 1, 4\}$ is a set and $p : M \times M \to \mathbb{R}^+$ defined by

$$p(s, t) = \frac{1}{4}|s - t| + \frac{1}{2}\max\{s, t\} \quad \forall s, t \in M. \tag{4.15}$$

Observe that

$$p(1, 1) = \frac{1}{2} \neq 0,$$
$$p(4, 4) = 2 \neq 0.$$

As $p^s(s - t) = |s - t|$ to prove it we proceed as follows,

$$p^s(s, t) = 2p(s, t) - p(s, s) - p(t, t), \tag{4.16}$$

$$p(s, t) = \frac{1}{4}|s - t| + \frac{1}{2}\max\{s, t\},$$
$$p(s, s) = \frac{1}{2},$$
$$p(t, t) = \frac{1}{2}t,$$

Putting values in Equation (4.16) ,

$$p^s(s,t) = \frac{1}{2}|s-t| + \max\{s,t\} - \frac{1}{2}s - \frac{1}{2}t.$$

Case (1):

$$\begin{aligned}
&s \geq t,\\
p^s(s,t) &= \frac{1}{2}(s-t) + s - \frac{1}{2}s - \frac{1}{2}t,\\
&= \frac{1}{2}s - \frac{1}{2}t + s - \frac{1}{2}s - \frac{1}{2}t,\\
&= s - t.
\end{aligned}$$

Case (2):

$$\begin{aligned}
&t \geq s,\\
p^s(s,t) &= \frac{1}{2}(t-s) + t - \frac{1}{2}s - \frac{1}{2}t,\\
&= t - s.
\end{aligned}$$

From above two cases we conclude that, $p^s(s,t) = |s-t|$.

So (M,p) is complete partial metric space.

$\implies (M,p^s)$ is complete metric space, by using Lemma (2.3.8).

We can also prove that $\{0\},\{0,1\},\{0,4\}$ are closed set by continuing the same way as in Example (3.1.3).

Now we define a mapping $L : M \to CB^p(M)$ by

$L(0) = L(1) = \{0\}$ and $L(4) = \{0,1\}$,

In order to prove contraction condition let $s,t \in M$ consider the following two cases.

Case (1):

$s, t \in \{0, 1\}$,

$H_p(L(s), L(t)) = H_p(\{0\}, \{0\}) = 0$.

(4.8) is satisfied.

Case(2):

If $s \in \{0, 1\}$, $t = 4$,

$$\begin{aligned} H_p(L(0), L(4)) = H_p(L(1), L(4)) &= H_p(\{0\}, \{0, 1\}), \\ &= \max\{\delta_p(\mathcal{J}, \mathcal{K}), \delta_p(\mathcal{K}, \mathcal{J})\}, \\ &= \max\Big\{p(0, \{0, 1\}), \max\{p(0, 0), p(1, 0)\}\Big\}, \\ &= \frac{3}{4} \leq \frac{11}{8} = kp(1, 4) < \frac{3}{2} = kp(0, 4). \end{aligned}$$

$s = t = 4$, then we have

$$\begin{aligned} H_p(L(4), L(4)) &= H_p(\{0, 1\}, \{0, 1\}), \\ &= \sup\Big\{p(s, s) : s \in \{0, 1\}\Big\}, \\ &= \max\Big\{p(0, 0), p(1, 1)\Big\}, \\ &= \frac{1}{2}, \\ &\leq 1, \\ &= kp(4, 4). \end{aligned}$$

Thus all hypothesis satisfied.

Chapter 5

Partial Hausdorff Extended b-Metric Space

Matthews [12] generalized fixed point results in partial metric spaces. Motivated by his work, we initiated the idea of partial extended b-metric spaces and we extend the results of Kamran et al. [20] on setting of partial extended b-metric spaces. We also give some examples to explain our new notion.

5.1 Partial Extended b-Metric Space

In this section we first elaborate the idea of partial extended b-metric space and then with the help of example we explain our new definition.

Definition 5.1.1.
Let M be nonempty set and $\Omega : M \times M \to [1, +\infty)$ be a continuous function. A function p_Ω where, $p_\Omega : M \times M \to [1, +\infty)$ is called partial extended b-metric space if for all $r_1, r_2, r_3 \in M$ it satisfies the following.

(EPB1): $p_\Omega(r_1, r_1) = p_\Omega(r_2, r_2) = p_\Omega(r_1, r_2) \Longleftrightarrow r_1 = r_2$,

(EPB2): $p_\Omega(r_1, r_1) \leq p_\Omega(r_1, r_2)$, (EPB3): $p_\Omega(r_1, r_2) = p_\Omega(r_2, r_1)$,

(EPB4): $p_\Omega(r_1, r_3) \leq \Omega(r_1, r_3)\Big[p_\Omega(r_1, r_2) + p_\Omega(r_2, r_3) - p_\Omega(r_2, r_2)\Big]$.

Remark 5.1.2.

(1) If $\Omega(r_1, r_3) = s$ then partial extended b-metric coincide partial b- metric.

(2) If $\Omega(r_1, r_3) = 1$ then partial extended b-metric coincide partial metric.

Example 5.1.3.

Suppose $M = \{1, 2, 3\}$, defined as $\Omega : M \times M \to \mathbb{R}^+$ and $p_\Omega : M \times M \to \mathbb{R}^+$ as:

$$\Omega(m_1, m_2) = (2 + m_1 + m_2),$$

$p_\Omega(1, 1) = 1 = p_\Omega(2, 2) = 2 = p_\Omega(3, 3) = 3$,

$p_\Omega(1, 2) = p_\Omega(2, 1) = 82$,

$p_\Omega(1, 3) = p_\Omega(3, 1) = 1000$,

$p_\Omega(2, 3) = p_\Omega(3, 2) = 600$.

Proof. First three axioms of above definition holds, so there is need to check triangular inequality :

$p_\Omega(1, 2) = 82$,

$\Omega(1, 2)[p_\Omega(1, 3) + p_\Omega(3, 2) - p_\Omega(3, 3)] = 5(1597) = 7985$.

Similarly we can

find other values as well, hence for all $r_1, r_2, r_3 \in M$

$$p_\Omega(r_1, r_3) \leq \Omega(r_1, r_3)[p_\Omega(r_1, r_2) + p_\Omega(r_2, r_3) - p_\Omega(r_2, r_2)].$$

Hence (M, p_Ω) is an partial extended b-metric space. □

Definition 5.1.4. (**Cauchy, Completeness, Convergence in Partial Extended b-Metric Space**)

Let (M, p_Ω) be partial extended b-metric space.

(1): Sequence $\{h_n\}$ is said to be Cauchy sequence in partial extended b-metric space if $\lim\limits_{n,m\to\infty} p_\Omega(h_m, h_n)$ exist and finite.

(2): The space is said to be complete in (M, p_Ω) space iff it is complete in extended b-metric space.

More precisely, $\lim\limits_{m\to+\infty} p_\Omega(h, h_m) = 0 \Longleftrightarrow$
$\lim\limits_{m\to+\infty} p_\Omega(h, h_m) = p_\Omega(h, h) = \lim\limits_{n,m\to\infty} p_\Omega(h_m, h_n)$.
A sequence $\{h_n\}$ converges in (M, p_Ω) to $m \in M$ iff

$$\lim_{m\to+\infty} p_\Omega(h, h_m) = p_\Omega(h, h).$$

If p_Ω be partial extended b-metric space on M, then $d_\Omega : M \times M \to \mathbb{R}^+$, defined by
$d_\Omega(m, n) = 2p_\Omega(m, n) - p_\Omega(m, m) - p_\Omega(n, n)$,
is an extended b-metric on M.

Theorem 5.1.5.
"Let (M, p_Ω) is a complete partial extended b-metric space, such that p_Ω is a continuous functional.
Consider a self map $T : M \to M$ such that

$$p_\Omega(Tw, Tv) \leq kp_\Omega(w, v), \ \forall\, w, v \in M, \tag{5.1}$$

where $k \in [0, 1)$ such that for each $w_0 \in M$,
$\lim\limits_{m,n\to\infty} \Omega(w_n, w_m) < \frac{1}{k}$, here $w_n = T^n w_0$, $n = 1, 2, \ \ldots$.
Then η is unique fixed point of mapping. Moreover, for every $v \in M$, $T^n v \to \eta$."

Proof. Suppose that $w_0 \in M$ is an arbitrary element, iterative sequence $\{w_n\}$ defined by
w_0, $Tw_0 = w_1$,
$w_2 = Tw_1 = T(Tw_0) = T^2(w_0) \ldots w_n = T^n w_0$.
By applying above inequality we obtain,

$$\begin{aligned} p_\Omega(Tw_1, Tw_2) &\leq kp_\Omega(w_1, w_2), \\ &\leq kp_\Omega(Tw_0, Tw_1), \\ &\leq k^2 p_\Omega(w_0, w_1). \end{aligned}$$

Applying this inequality successively we obtain:

$$p_\Omega(w_n, w_{n+1}) \leq k^n p_\Omega(w_0, w_1), \tag{5.2}$$

by use of triangular inequality and Equation (5.2), for $m > n$ we have,

$$\begin{aligned}
p_\Omega(w_n, w_m) \leq & \Omega(w_n, w_m) \Big[p_\Omega(w_n, w_{n+1}) + p_\Omega(w_{n+1}, w_m) \\
& - p_\Omega(w_{n+1}, w_{n+1}) \Big], \\
\leq & \Omega(w_n, w_m) \Big[p_\Omega(w_n, w_{n+1}) \Big] + \Omega(w_n, w_m) \Big[p_\Omega(w_{n+1}, w_m) \\
& - p_\Omega(w_{n+1}, w_{n+1}) \Big], \\
\leq & \Omega(w_n, w_m) \Big[p_\Omega(w_n, w_{n+1}) \Big] + \Omega(w_n, w_m) \Big[p_\Omega(w_{n+1}, w_m) \Big], \\
\leq & \Omega(w_n, x_m) \Big[p_\Omega(w_n, w_{n+1}) \Big] + \Omega(w_n, w_m) \Omega(w_{n+1}, w_m), \\
& \Big[p_\Omega(w_{n+1}, w_{n+2}) + p_\Omega(w_{n+2}, w_m) - p_\Omega(w_{n+2}, w_{n+2}) \Big], \\
\leq & \Omega(w_n, w_m) \Big[p_\Omega(w_n, w_{n+1}) \Big] + \Omega(w_n, w_m) \Omega(w_{n+1}, w_m) \\
& \Big[p_\Omega(w_{n+1}, w_{n+2}) + p_\Omega(w_{n+2}, w_m) \Big].
\end{aligned}$$

Continuing this process we get,

$$\begin{aligned}
p_\Omega(w_n, w_m) \leq & \Omega(w_n, w_m) \Big[p_\Omega(w_n, w_{n+1}) \Big] + \\
& \Omega(w_n, w_m) \Omega(w_{n+1}, w_m) \Big[p_\Omega(w_{n+1}, w_{n+2}) \Big] + \\
& \Omega(w_n, w_m) \Omega(w_{n+1}, w_m) \Omega(w_{n+2}, w_m) \Big[p_\Omega(w_{n+2}, w_{n+3}) \Big] + \\
& \ldots + \Omega(w_n, w_m) \Omega(w_{n+1}, w_m) \ \ldots \Omega(w_{m-2}, w_m) \Omega(w_{m-1}, w_m) \\
& \Big[p_\Omega(w_{m-1}, w_m) \Big].
\end{aligned}$$

By using Equation (5.2),

$$\begin{aligned}
p_\Omega(w_n, w_m) \leq & \Omega\left(w_n, w_m\right) k^n p_\Omega(w_0, w_1) + \Omega(w_n, w_m)\Omega(w_{n+1}, w_m) + k^{n+1} p_\Omega(w_0, w_1) + \\
& \ldots + \Omega\left(w_{n+1}, w_m\right) \Omega\left(w_{n+2}, w_m\right) \ldots \\
& \Omega\left(w_{m-2}, w_m\right) \Omega\left(w_{m-1}, w_m\right) k^{m-1} p_\Omega(w_0, w_1).
\end{aligned}$$

Therefore,

$$\begin{aligned}
p_\Omega(w_n, w_m) \leq & p_\Omega(w_0, w_1) \Big[\Omega\left(w_n, w_m\right) k^n + \Omega\left(w_n, w_m\right) \Omega\left(w_{n+1}, w_m\right) k^{n+1} + \ldots + \\
& \Omega\left(w_n, w_m\right) \Omega\left(w_{n+1}, w_m\right) \Omega\left(w_{n+2}, w_m\right) \ldots \\
& \Omega\left(w_{m-2}, w_m\right) \Omega\left(w_{m-1}, w_m\right) k^{m-1} \Big], \\
& \leq p_\Omega(w_0, w_1) \Big[\Omega\left(w_1, w_m\right) \Omega\left(w_2, w_m\right) \ldots \Omega\left(w_{n-1}, w_m\right) \Omega\left(w_n, w_m\right) k^n \\
& + \Omega\left(w_1, w_m\right) \Omega\left(w_2, w_m\right) \ldots \Omega\left(w_{n-1}, w_m\right) \Omega\left(w_n, w_m\right) \Omega\left(w_{n+1}, w_m\right) k^{n+1} \\
& \ldots + \Omega\left(w_1, w_m\right) \Omega\left(w_2, w_m\right) \ldots \Omega\left(w_{n-1}, w_m\right) \Omega\left(w_n, w_m\right) \Omega\left(w_{n+1}, w_m\right) \\
& \Omega\left(w_{n+2}, w_m\right) \ldots \Omega\left(w_{m-2}, w_m\right) \Omega\left(w_{m-1}, w_m\right) k^{m-1} \Big].
\end{aligned}$$

Since, $\lim\limits_{m,n\to\infty} \Omega(w_n, w_m)k < 1$, this implies that $\sum_{n=1}^{\infty} k^n \prod_{i=1}^{n} \Omega(w_i, w_m)$ is convergent by ratio test for each $m \in \mathbb{N}$. Let

$$S = \sum_{n=1}^{\infty} k^n \prod_{i=1}^{n} \Omega(w_i, w_m), \;\; S_n = \sum_{j=1}^{n} k^j \prod_{i=1}^{j} \Omega(w_i, w_m).$$

$$\begin{aligned}
S_n &= \Omega(w_1, w_m)k + \Omega(w_1, w_m)\Omega(w_2, w_m)k^2 + \ldots + \Omega(w_1, w_m) \ldots \Omega(w_n, w_m)k^n \\
S &= \Omega(w_1, w_m)k + \Omega(w_1, w_m)\Omega(w_2, w_m)k^2 + \ldots + \Omega(w_1, w_m) \\
& \quad \ldots \Omega(w_n, w_m)k^n + \ldots + \Omega(w_1, w_m)\Omega(w_2, w_m) \ldots \Omega(w_n, w_m)\Omega(w_{n+1}, x_m) \\
& \quad \ldots \Omega(w_{m-2}, w_m)\Omega(w_{m-1}, w_m)k^{m-1} + \ldots .
\end{aligned}$$

For $m > n$, by using above inequality we get,

$$p_\Omega(w_n, w_m) \leq p_\Omega(w_0, w_1)\Bigg[S_{m-1} - S_{n-1}\Bigg],$$

taking $n \to \infty$ we obtain $\{w_n\}$ is a Cauchy sequence since M is complete $w_n \to \eta \in M$,

$$\begin{aligned} p_\Omega(T\eta, \eta) \leq& \Omega(T\eta, \eta)\Bigg[p_\Omega(T\eta, w_n) + p_\Omega(w_n, \eta) - p_\Omega(w_n, w_n)\Bigg], \\ \leq& \Omega(T\eta, \eta)\Bigg[p_\Omega(\eta, w_{n-1}) + p_\Omega(w_n, \eta)\Bigg], \\ p_\Omega(T\eta, \eta) \leq& 0, \quad n \to \infty \\ \implies p_\Omega(T\eta, \eta) =& 0. \end{aligned}$$

Hence this implies that η is fixed point of mapping. For uniqueness

$$p_\Omega(\eta, \gamma) = p_\Omega(T\eta, T\gamma) \leq kp_\Omega(\eta, \gamma)$$

here $k < 1$ so $p_\Omega(\eta, \gamma) = 0 \implies \eta = \gamma$, is a unique fixed point of mapping. □

Definition 5.1.6.

Let $T : M \to M$ for $w_0 \in M$, $O(w_0) = \{w_0, Tw_0, T^2w_0 \ldots\}$ as the orbit of w_0. A function $H : M \to \mathbb{R}$ is said to be T-orbitally lower semi-contiuous at $w \in M$, if $\Bigg\{w_n\Bigg\} \subset O(w_0)$ and $w_n \to w$, implies

$$H(w) \leq \liminf_{n\to\infty} H(w_n).$$

Theorem 5.1.7.

Let (M, p_Ω) be a complete partial extended b-metric space and p_Ω is a continuous functional. Let $T : M \to M$ and there is a $w_0 \in M$ such that

$$p_\Omega(Tw, T^2w) \leq kp_\Omega(w, Tw), \tag{5.3}$$

for each $w \in O(w_0)$, where $k \in [0, 1)$ such that for $w_0 \in M$, $\lim_{m,n\to\infty} \Omega(w_m, w_m) < \frac{1}{k}$, $w_n = T^n w_0$, $n = 1, 2, \ldots,$ then $T^n w_0 \to \eta \in M$. Further

η is a fixed point of T iff $H(w) = p_\Omega(w, Tw)$ is T-orbitally lower semi continuous at η.

Proof. Let $w_0 \in W$, defined by an iterative sequence $\{w_n\}$ by
w_0, $Tw_0 = w_1$,
$w_2 = Tw_1 = T(Tw_0) = T^2(w_0) \ldots \; w_n = T^n w_0$.
By use of successive iteration of above Inequality (5.3),

$$\begin{aligned} p_\Omega(T^n w_0, T^{n+1} w_0) =& p_\Omega(w_n, w_{n+1}), \\ \leq& k p_\Omega(w_{n-1}, w_n), \\ \leq& k^2 p_\Omega(w_{n-2}, w_{n-1}) \ldots \leq k^n p_\Omega(w_0, w_1). \end{aligned}$$

By proceeding as in Theorem (5.0.4) we can prove $\{w_n\}$ is a Cauchy sequence. As M is complete then $w_n = T^n w_0 \to \eta \in M$.
Since T is orbitally lower semi continuous at $\eta \in W$.

$$p_\Omega(w, Tw) \leq \liminf_{n\to\infty} p_\Omega(T^n w_0, T^{n+1} w_0), \tag{5.4}$$

$$\leq \liminf_{n\to\infty} k^n p_\Omega(w_0, w_1) = 0, \tag{5.5}$$

$$\implies p_\Omega(\eta, T\eta) = 0.$$

So η is fixed point.
Conversely, asssume that η is fixed point of T, now we will prove that H is orbitally lower semi continuous.
Let $w_n \in O(w)$ with $w_n \to \eta$, then

$$\begin{aligned} H(\eta) &= p_\Omega(\eta, T\eta) = 0, \\ &\leq \lim_{n\to\infty} \inf H(w_n), \\ &= p_\Omega(T^n w_0, T^{n+1} w_0). \end{aligned}$$

Hence T is orbitally lower semi continuous. $\square$

Example 5.1.8.

If $M = [0, \infty)$, define $p_\Omega : M \times M \to \mathbb{R}^+$ by $\Omega : M \times M \to [1, \infty)$ and

$$p_\Omega(w, v) = \max(w, v),\ \Omega(w, v) = (w + v + 1),$$

then p_Ω represent complete partial extended b-metric space on M.

Let $T : M \to M$ defined by

$$Tw = \frac{w}{2},$$

we have,

$$\begin{aligned} p_\Omega(Tw, T^2w) &= p_\Omega\left(\frac{w}{2}, \frac{w}{4}\right), \\ &= \frac{w}{2}, \\ &= kp_\Omega(w, Tw). \end{aligned}$$

For each $w \in M$, and $T^n w = \frac{w}{2^n}$. Thus we get,

$$\lim_{m,n\to\infty} \Omega(T^m w, T^n w) = \lim_{m,n\to\infty} \left(\frac{w}{2^m} + \frac{w}{2^n} + 1\right) < 2.$$

Since Theorem (5.1.6) are satisfied, hence mapping has unique fixed point.

5.2 Partial Hausdorff Extended b-Metric Space

Aydi et al. [16] introduced the notion of partial Hausdorff metric space. This is further used by Kanwal et al. [24] for establishing fixed point results on weak partial b-metric space. In this section we establish fixed point results on partial Hausdorff extended b-metric space. We first give some requiste definitions.

Definition 5.2.1. (**Closedness in partial extended b-metric space**).

Suppose (M, p_Ω) be a partial extended b-metric space. A be a nonempty set of (M, p_Ω), A is said to be closed in (M, p_Ω) if and only if $\bar{A} = A$.

Definition 5.2.2. (**Boundedness in Partial Extended b-Metric Space**)

Suppose (M, p_Ω) be a partial extended b-metric space. X is a bounded subset in(M, p_Ω), if $\exists\ n_0 \in M$ for $N \geq 0$ and $\forall\ s \in S$, we have $x \in B_{p_\Omega}(n_0, N)$,

$$p_\Omega(n_0, x) < p_\Omega(x, x) + N.$$

The Ω can be defined as
$\Omega : CB^{p_\Omega}(M) \times CB^{p_\Omega}(M) \to [1, +\infty)$,
here CB^{p_Ω} represents those subset of (M, p_Ω) which are closed and bounded.
$\Omega(P, Q) = \sup\left\{\Omega(m_1, m_2) : m_1 \in P, m_2 \in Q\right\}$.
For $P, Q \in CB^{p_\Omega}(M)$ and $m \in M$, we define
$p_\Omega(y, P) = \inf\left\{p_\Omega(y, m_1), m_1 \in P\right\}$,
$\delta_{p_\Omega}(P, Q) = \sup\left\{p_\Omega(m_1, Q) : m_1 \in P\right\}$,
$\delta_{p_\Omega}(Q, P) = \sup\left\{p_\Omega(m_2, P) : m_2 \in Q\right\}$,
where $\delta_{p_\Omega} : CB^{p_\Omega}(M) \times CB^{p_\Omega}(M) \to [0, \infty)$.

Remark 5.2.3.
Assume (M, p_Ω) be partial extended b-metric space and B be any non-empty set in (M, p_Ω) then:
$b \in \bar{B}$ if and only if

$$p_\Omega(b, B) = p_\Omega(b, b), \tag{5.6}$$

where $\bar{B}$ is closure of B. B is closed in (M, p_Ω) if and only if $B = \bar{B}$.

Proposition 5.2.4.
Assume (M, p_Ω) is partial extended b-metric space, for any $P, Q, R \in CB^{p_\Omega}(M)$, we have the following:

(1) $\delta_{p_\Omega}(P, P) = \sup\left\{p_\Omega(m_1, m_1) : m_1 \in P\right\}$,

(2) $\delta_{p_\Omega}(P, P) \leq \delta_{p_\Omega}(P, Q)$,

(3) $\delta_{p_\Omega}(P, Q) = 0 \Rightarrow P \subseteq Q$,

(4) $\delta_{p_\Omega}(P, Q) \leq \Omega(P, Q)\left[\delta_{p_\Omega}(P, R) + \delta_{p_\Omega}(R, Q) - \inf\limits_{m_3 \in R} p_\Omega(m_3, m_3)\right]$.

Proof. To prove (1) we proceed as follows,

If $P \in CB^{p_\Omega}(M)$, then for all $m_1 \in P$, we have by use of Equation (5.6)

$$p_\Omega(m_1, P) = p_\Omega(m_1, m_1).$$

Therefore,

$$\delta_{p_\Omega}(P, P) = \sup \Big\{ p_\Omega(m_1, P) : m_1 \in P \Big\},$$
$$\delta_{p_\Omega}(P, P) = \sup \Big\{ p_\Omega(m_1, m_1) : m_1 \in P \Big\}.$$

To prove (2),

Let $m_1 \in P$, and for all $m_2 \in Q$

$$p_\Omega(m_1, m_1) \leq p_\Omega(m_1, m_2), \tag{5.7}$$

by the definition of partial extended b-metric

also we know that,

$$\delta_{p_\Omega}(P, Q) = \sup\{p_\Omega(m_1, Q) : m_1 \in P\}. \tag{5.8}$$

Combining Equations (5.7) and (5.8) we have,

$$p_\Omega(m_1, m_1) \leq p_\Omega(m_1, Q) \leq \sup\{p_\Omega(m_1, Q) : m_1 \in P\},$$

$$= \delta_{p_\Omega}(P, Q),$$
$$\implies \delta_{p_\Omega}(P, P) = \sup \Big\{ p_\Omega(m_1, P) : m_1 \in P \Big\},$$
$$\leq \delta_{p_\Omega}(P, Q).$$

In order to prove (3) we proceed as follows,

Suppose that $\delta_{p_\Omega}(P, Q) = 0, \implies \Big\{ p_\Omega(m_1, Q) = 0, \text{ for all } m_1 \in P \Big\},$

$\delta_{p_\Omega}(P, Q) = \sup \Big\{ p_\Omega(m_1, Q) \colon m_1 \in P \Big\},$

$\implies \sup \Big\{ p_\Omega(m_1, Q) = 0 \text{ for all } m_1 \in P \Big\}.$

Since

$$\delta_{p_\Omega}(P,P) \leq \delta_{p_\Omega}(P,Q) = 0,$$
$$\Longrightarrow \sup\left\{p_\Omega(m_1,m_1) : m_1 \in P\right\} = 0.$$

Hence we get

$$p_\Omega(m_1,Q) = p_\Omega(m_1,m_1).$$

As a result we get $m_1 \in \bar{Q} = \mathcal{Q}$ whenever $m_1 \in P$ so $P \subseteq Q$.

We shall prove (4) as follows:

Consider $m_1 \in P$, $m_2 \in Q$ and $m_3 \in R$, and by using Definition (5.1.1)of partial extended b-metric space, we can write,

$$p_\Omega(m_1,m_2) \leq \Omega(m_1,m_2)\left[p_\Omega(m_1,m_3) + p_\Omega(m_3,m_2) - p_\Omega(m_3,m_3)\right].$$

Since m_2 is an arbitrary element of Q, so we have,

$$p_\Omega(m_1,Q) \leq \Omega(m_1,Q)\left[p_\Omega(m_1,m_3) + p_\Omega(m_3,Q) - p_\Omega(m_3,m_3)\right].$$

And $m_3 \in R$ so we have,

$$p_\Omega(m_1,Q) \leq \Omega(m_1,Q)\left[p_\Omega(m_1,m_3) + \delta_{p_\Omega}(R,Q) - p_\Omega(m_3,m_3)\right],$$

as $m_3 \in R$ is an arbitrary element of R, so

$$p_\Omega(m_1,Q) \leq \Omega(m_1,Q)\left[p_\Omega(m_1,R) + \delta_{p_\Omega}(R,Q) - \inf_{m_3 \in R} p_\Omega(m_3,m_3)\right], \tag{5.9}$$

where m_3 is an arbitrary element of R.

Suppose that (M, p_Ω) be partial extended b-metric space. For $P,Q \in CB^{p_\Omega}(M)$, define

$$H_{p_\Omega}(P,Q) = \max\left(\left[\delta_{p\Omega}(P,Q)], \delta_{p\Omega}(Q,P)\right]\right).$$

□

Proposition 5.2.5.

Suppose (M, p_Ω) be partial extended b-metric space, for any $P, Q, R \in CB^{p_\Omega}(M)$, we have,

(1) $H_{p_\Omega}(P, P) \leq H_{p_\Omega}(P, Q)$,

(2) $H_{p_\Omega}(P, Q) = H_{p_\Omega}(Q, P)$,

(3) $H_{p_\Omega}(P, Q) \leq \Omega(P, Q)\left[H_{p_\Omega}(P, R) + H_{p_\Omega}(R, Q) - \inf_{m_3 \in R} p(m_3, m_3)\right]$.

To prove $H_{p_\Omega}(P, P) \leq H_{p_\Omega}(P, Q)$ we proceed as follows,
Since

$$\begin{aligned} H_{p_\Omega}(P, P) =& \max\left[(\delta_{p_\Omega}(P, P), \delta_{p_\Omega}(P, P))\right], \\ =& \max\left[(\delta_{p_\Omega}(P, P), \delta_{p_\Omega}(P, P))\right], \\ =& \delta_{p_\Omega}(P, P). \end{aligned}$$

By Proposition (4.1.4)

$$\begin{aligned} \leq& \delta_{p_\Omega}(P, Q), \\ \leq& H_{p_\Omega}(P, Q). \end{aligned}$$

For (2) we use the definition

$$H_{p_\Omega}(P, Q) = \max\left(\left[\delta_{p_\Omega}(P, Q), \delta_{p_\Omega}(Q, P)\right]\right).$$

Similarly we can write,

$$H_{p_\Omega}(Q, P) = \max\left(\left[\delta_{p_\Omega}(Q, P), \delta_{p_\Omega}(P, Q)\right]\right),$$

$$H_{p_\Omega}(Q, P) = H_{p_\Omega}(P, Q).$$

To prove (3) we proceed as follows,

Since

$$H_{p_\Omega}(P,Q) = \max\left\{\delta_{p_\Omega}(P,Q), \delta_{p_\Omega}(Q,P)\right\}.$$

By using (4) of Proposition (4.1.4) , we can write,

$$\begin{aligned}
&\leq \max\left\{\Omega(P,Q)\left[\delta_{p_\Omega}(P,R) + \delta_{p_\Omega}(R,Q) - \inf_{m_3\in R} p(m_3,m_3)\right]\right\}, \\
&\Omega(Q,P)\left[(\delta_{p_\Omega}(Q,R) + \delta_{p_\Omega}(R,P) - \inf_{r_3\in R} p(m_3,m_3)\right], \\
&\leq \max\left\{\Omega(P,Q)\left[\delta_{p_\Omega}(P,R) + \delta_{p_\Omega}(R,Q), (\delta_{p_\Omega}(Q,R) + \delta_{p_\Omega}(R,P)\right]\right\} \\
&- \inf_{m_3\in R} p(m_3,m_3),
\end{aligned}$$

$$\begin{aligned}
&\leq \max\left\{\Omega(P,Q)\left[\delta_{p_\Omega}(P,R) + \delta_{p_\Omega}(R,Q), (\delta_{p_\Omega}(Q,R) + \delta_{p_\Omega}(R,P)\right]\right\} \\
&- \inf_{m_3\in R} p(m_3,m_3), \\
&\leq \Omega(P,Q)\left[\max\{(\delta_{p_\Omega}(P,R), \delta_{p_\Omega}(R,P)\} + \{\delta_{p_\Omega}(R,Q) + \delta_{p_\Omega}(Q,R)\}\right] \\
&- \inf_{m_3\in R} p(m_3,m_3), \\
H_{p_\Omega}(P,Q) &\leq \Omega(P,Q)\left\{H_{p_\Omega}(P,R) + H_{p_\Omega}(R,Q) - \inf_{m_3\in R} p(m_3,m_3)\right\}.
\end{aligned}$$

Corollary 5.2.6.

Assume (M, p_Ω) be partial Hausdorff extended b-metric space. For $P, Q \in CB^{p_\Omega}(M)$ then

$$H_{p_\Omega}(P,Q) = 0 \Rightarrow P = Q.$$

Proof. Let $H_{p_\Omega}(P,Q) = 0$ then $\delta_{p_\Omega}(P,Q) = \delta_{p_\Omega}(Q,P) = 0$ so by using Proposition (4.1.4), we get $P \subseteq Q$ and $Q \subseteq P$,

$\Longrightarrow P = Q.$ $\square$

This corollary tells that partial Hausdorff distance of two sets in extended b-metric is zero then set must be equal to each other. But converse of this statement need not to be true, as if sets are equal to each other then their Hausdorff distance in extended b-metric needs not to be zero.

5.3 Fixed Point Results On Partial Hausdorff Extended b-Metric Space

Lemma 5.3.1. Suppose that (M, p_Ω) be partial Hausdorff extended b-metric space, $P, Q \in CB^{p_\Omega}(M)$ and $h > 1$, for any $m_1 \in P$, there exists $m_2 = m_2(m_1) \in Q$ such that

$$p_\Omega(m_1, m_2) \leq hH_{p_\Omega}(P, Q). \tag{5.10}$$

Proof. We consider two cases to prove the result, the first one is when sets are equal to each other that is $P = Q$, then from Proposition (4.1.4), we have
$H_{p_\Omega}(P, Q) = H_{p_\Omega}(P, P) = \delta_{p_\Omega}(P, P) = \sup_{m_1 \in P} \Big\{ p_\Omega(m_1, m_1) \Big\}$.
Let $m_1 \in P$, since $h > 1$. Therefore we have

$$\begin{aligned} p_\Omega(m_1, m_1) &\leq \sup_{x \in P} \Big\{ p_\Omega(m_1, m_1) \Big\}, \\ &\leq H_{p_\Omega}(P, Q), \\ &\leq hH_{p_\Omega}(P, Q). \end{aligned}$$

Consequently $P = Q$ satisfies Equation (5.10) Hence when sets are same then we get our desired result.
Case (2):
When $P \neq Q$ suppose contradictory that $p_\Omega(m_1, m_2) > hH_{p_\Omega}(P, Q)$,
for all $m_2 \in Q$ and $m_1 \in P$.

This implies that

$$\inf \Big\{ p_\Omega(m_1, m_2) : \ m_2 \in Q \Big\} \geq h H_{p_\Omega}(P, Q).$$

By using Proposition (4.1.6) we can write,

$$H_{p_\Omega}(P, Q) \geq \delta_{p_\Omega}(P, Q). \tag{5.11}$$

Using Proposition (4.1.4) we can write,

$$= \sup\{p_\Omega(x, Q)\} \geq p_\Omega(m_1, Q) \geq h H_{p_\Omega}(P, Q). \tag{5.12}$$

Combining Equations (5.11) and (5.12) we have,
$H_{p_\Omega}(P, Q) \geq h H_{p_\Omega}(P, Q)$,
by Corollary (5.2.6) we have

$$H_{p_\Omega}(P, Q) \neq 0. \tag{5.13}$$

Since $P \neq Q$, and by Equation (5.13) $h \leq 1$, which is a contradiction. □

Theorem 5.3.2.
Let (M, p_Ω) be a complete partial Hausdorff extended b-metric space.
If $T : M \to CB^{p_\Omega}(M)$ is a multi-valued mapping such that for all $m_1, m_2 \in M$ we have,

$$H_{p_\Omega}(Tm_1, Tm_2) \leq k p_\Omega(m_1, m_2), \tag{5.14}$$

where $k \in (0, 1)$, $\lim\limits_{m,n\to\infty} \Omega(m_n, m_{m'}) < \frac{1}{k}$, and p_Ω is continuous functional. Then T has a fixed point.

Proof. Let $m_0 \in M$, $m_1 \in Tm_0$, $m_2 \in Tm_1$, Continuing this process, we obtain a sequence $\{m_n\}$ such that $m_{n+1} \in Tm_n$.
Let $h = \frac{1}{\sqrt{k}}$, and by use of previous lemma, we have

$$p_\Omega(m_1, m_2) \leq \frac{1}{\sqrt{k}} H_{p_\Omega}(Tm_0, Tm_1),$$

By using Equation (5.14) we get,

$$H_{p_\Omega}(Tm_0, Tm_1). \leq k p_\Omega(m_0, m_1),$$

so we have,

$$p_\Omega(m_1, m_2) \leq \sqrt{k} p_\Omega(m_0, m_1),$$

$$p_\Omega(m_2, m_3) \leq (\sqrt{k})^2 p_\Omega(m_1, m_2).$$

Similarly continuing this process, we have a sequence $\{m_n\}$ in M such that

$$p_\Omega(m_{n+1}, m_n) \leq \sqrt{k} p_\Omega(m_n, m_{n-1}), \quad \forall\, n \geq 1 \tag{5.15}$$

$$p_\Omega(m_{n+1}, m_n) \leq (\sqrt{k})^n p_\Omega(m_0, m_1), \quad \forall\, n \in \mathbb{N} \tag{5.16}$$

then by Definition (5.1.1) of partial extended b-metric space, we have,

$$\begin{aligned}
p_\Omega(m_n, m_{n+m'}) &\leq \Omega(m_n, m_{n+m'}) \Big[p_\Omega(m_n, m_{n+1}) + p_\Omega(m_{n+1}, m_{n+m'}) \\
&\quad - p_\Omega(m_{n+1}, m_{n+1}) \Big], \\
p_\Omega(m_n, m_{n+m'}) &\leq \Omega(m_n, m_{n+m'}) \Big[p_\Omega(m_n, m_{n+1}) + p_\Omega(m_{n+1}, m_{n+m'}) \Big], \\
p_\Omega(m_n, m_{n+m'}) &\leq \Omega(m_n, m_{n+m'}) \Big[k^n p_\Omega(m_0, m_1) + p_\Omega(m_{n+1}, m_{n+m'}) \Big],
\end{aligned}$$

by using Equation (5.16)

$$p_\Omega(m_n, m_{n+m'}) \leq \Omega(m_n, m_{n+m})k^n p_\Omega(m_0, m_1) + \Omega(m_n, m_{m'})\Omega(m_{n+1}, m_{m'})$$
$$k^{n+1}p_\Omega(m_0 + r_1) + \ \ldots$$
$$p_\Omega(m_n, m_{n+m'}) \leq p_\Omega(m_0, m_1)\Big[\Omega(m_1, m_m)\Omega(m_2, m_{m'}) \ldots \Omega(m_{n-1}, m_{m'})$$
$$\Omega(m_n, m_{m'})k^n + \Omega(m_1, m_m)\Omega(m_2, m_{m'}) \ldots \Omega(m_n, m_{m'})$$
$$\Omega(m_{n+1}, r_m)k^{n+1} + \ \ldots \ + \Omega(m_1, m_{m'})\Omega(m_2, m_{m'}) \ldots \Omega(m_n, m_{m'})$$
$$\ldots \Omega(m_{m-2}, m_{m'})\Omega(m_{m-1}, m_{m'})k^{m-1}\Big].$$

Since $\lim\limits_{m,n\to\infty} \Omega(m_{n+1}, m_{m'})k < 1$,
$\sum_{n=1}^{\infty} k^n \prod_{i=1}^{n} \Omega(m_i, m_{m'})$ is convergent by ratio test for every $m \in \mathbb{N}$.

$$S = \sum_{n=1}^{\infty} k^n \prod_{i=1}^{n} \Omega(m_i, m_m), \ S_n = \sum_{j=1}^{n} k^j \prod_{i=1}^{j} \Omega(m_i, m_{m'})$$

$$p_\Omega(m_n, m_{m'}) \leq p_\Omega(m_0, m_1)\Big[S_{m'-1} - S_{n-1}\Big],$$

taking $n \to \infty$ we conclude that $\{m_n\}$ is a Cauchy sequence in (M, p_Ω), it is also Cauchy in partial extended b-metric, since M is complete.
Therefore the sequence $\{m_n\}$ converges to some $m^* \in M$ with respect to the metric p_Ω, that is $\lim\limits_{n\to+\infty} p_\Omega(m_n, m^*) = 0$, we have

$$\lim_{n\to\infty} p_\Omega(m_n, m^*) = p_\Omega(m^*, m^*) = \lim_{m,n\to\infty} p_\Omega(m_n, m_{m'}) = 0. \tag{5.17}$$

$H_{p_\Omega}(Tm_n, Tm^*) \leq kp_\Omega(m_n, m^*)$, therefore

$$\lim_{n\to\infty} H_{p_\Omega}(Tm_n, Tm^*) = 0. \tag{5.18}$$

Now $m_{n+1} \in Tm_n$ gives that

$$p_\Omega(m_{n+1}, Tm^*) \leq \delta_{p_\Omega}(Tm_n, Tm^*) \leq H_{p_\Omega}(Tm_n, Tm^*). \tag{5.19}$$

By using Equation (5.19), we get

$$\lim_{n\to\infty} p_\Omega(m_{n+1}, Tm^*) = 0, \tag{5.20}$$

we have

$p_\Omega(m^*, Tm^*) \leq \Omega(m^*, Tm^*)\Big[p_\Omega(m^*, m_{n+1}) + p_\Omega(m_{n+1}, Tm^*) - p_\Omega(m_{n+1}, m_{n+1})\Big]$.

as limits $n \to \infty$ and using Equations (5.20) and (5.17), we obtain

$$p_\Omega(m^*, Tm^*) = 0.$$

Therefore by Equation 5.17 $p_\Omega(m^*, m^*) = 0$, we obtain

$$p_\Omega(m^*, m^*) = p_\Omega(m^*, Tm^*) = 0.$$

By Remark (5.2.3) we have $m^* \in Tm^*$. □

5.4 Conclusion

In this dissertation, we have proved some fixed point theorems in the setting of partial extended b-metric space. These results are the extensions of previous results presented by Matthews and Aydi et al. [16].

We start with metric space and some related fixed point results on metric space then elaborated recent work done in this field from different perspectives. The notion of b-metric space, partial metric and extended b-metric spaces are also discussed with examples and some related corollaries and remarks. The work of Aydi et al. [16] is elaborated and discussed to represent the complete review of the article.

Moreover main results of Kamran et al. [20] is also presented. In fact, Kamran et al. [20] after introducing the notion of an extended b-metric space established Banach Contraction Principle.

Finally, we extended the main results of Aydi et al. [16] by following the approach used by Kamran et al. [16]. For this we first formulated the notion of partial

Hausdorff extended b-metric space. Three main results are established in the setting of new space. Precisely, the main fixed point results of Aydi et al. [16] is proved by partial Hausdorff extended b- metric space, the other two results are related to work of Kamran et al. [20]. The first result extend Banach Contraction Principle and second results extend main fixed point results of Hicks and Rhoades in setting of partial extended b-metric space.

Bibliography

[1] M. Fréchet, "Sur quelques points du calcul fonctionnel," *Rendiconti del Circolo Matematico di Palermo (1884-1940)*, vol. 22, no. 1, pp. 1–72, 1906.

[2] S. K. Chatterjea, "Fixed point theorems theorems," *C. R. Acad. Bulgare Sci*, vol. 25, pp. 727–730, 1972.

[3] A. Arkhangelskiĭ and L. S. Pontriagin, *Basic Concepts and Constructions, Dimension Theory.* Springer-Verlag, 1990.

[4] A. Amini-Harandi, "Metric-like spaces, partial metric spaces and fixed points," *Fixed Point Theory and Applications*, vol. 2012, no. 1, p. 204, 2012.

[5] S. Oltra, S. Romaguera, and E. Sánchez-Pérez, "Bicompleting weightable quasi-metric spaces and partial metric spaces," *Rendiconti del Circolo Matematico di Palermo*, vol. 51, no. 1, pp. 151–162, 2002.

[6] S. Oltra and O. Valero, "Banach's fixed point theorem for partial metric spaces," *Università degli Studi di Trieste. Dipartimento di Matematica e Informatica*, 2004.

[7] S. G. Matthews, "Partial metric topology," *Annals of the New York Academy of Sciences-Paper Edition*, vol. 728, pp. 183–197, 1994.

[8] I. Bakhtin, "The contraction mapping principle in almost metric spaces. 30," *Functional Analysis*, pp. 26–37.

[9] N. Bourbaki, *Topologie générale: Chapitres 1 à 4.* Springer Science & Business Media, 2007.

[10] S. Czerwik, "Contraction mappings in b-metric spaces," *Acta mathematica et informatica universitatis ostraviensis*, vol. 1, no. 1, pp. 5–11, 1993.

[11] R. Fagin and L. Stockmeyer, "Relaxing the triangle inequality in pattern matching," *International Journal of Computer Vision*, vol. 30, no. 3, pp. 219–231, 1998.

[12] S. G. Matthews, "Partial metric topology," *Annals of the New York Academy of Sciences-Paper Edition*, vol. 728, pp. 183–197, 1994.

[13] L. Ćirić, B. Samet, H. Aydi, and C. Vetro, "Common fixed points of generalized contractions on partial metric spaces and an application," *Applied Mathematics and Computation*, vol. 218, no. 6, pp. 2398–2406, 2011.

[14] C. Di Bari and P. Vetro, "Fixed points for weak ϕ-contractions on partial metric spaces," *Int. J. Eng. Contemp. Math. Sci*, vol. 1, no. 1, pp. 5–13, 2011.

[15] D. Paesano and P. Vetro, "Suzukis type characterizations of completeness for partial metric spaces and fixed points for partially ordered metric spaces," *Topology and its Applications*, vol. 159, no. 3, pp. 911–920, 2012.

[16] H. Aydi, M. Abbas, and C. Vetro, "Partial hausdorff metric and nadlers fixed point theorem on partial metric spaces," *Topology and its Applications*, vol. 159, no. 14, pp. 3234–3242, 2012.

[17] A. C. Ran and M. C. Reurings, "A fixed point theorem in partially ordered sets and some applications to matrix equations," *proceedings of the American Mathematical Society*, pp. 1435–1443, 2004.

[18] J. J. Nieto and R. Rodríguez-López, "Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations," *Order*, vol. 22, no. 3, pp. 223–239, 2005.

[19] C. Djeraba, *Mathematical Tools For Data Mining: Set Theory, Partial Orders, Combinatorics. Advanced Information and Knowledge Processing.* Springer, 2008.

[20] T. Kamran, M. Samreen, and Q. UL Ain, "A generalization of b-metric space and some fixed point theorems," *Mathematics*, vol. 5, no. 2, p. 19, 2017.

[21] L. Subashi, "Some topological properties of extended b-metric space," in *Proceedings of the 5th International Virtual Conference on Advanced Scientific Results*, vol. 5, 2017, pp. 164–167.

[22] L. Subashi and N. Gjini, "Some results on extended b-metric spaces and pompeiu-hausdorff metric," *J. Progressive Res. Math*, vol. 12, pp. 2021–2029, 2017.

[23] I. Beg and H. K. Pathak, "A variant of nadlers theorem on weak partial metric spaces with application to a homotopy result," *Vietnam Journal of Mathematics*, vol. 46, no. 3, pp. 693–706, 2018.

[24] T. Kanwal, A. Hussain, P. Kumam, and E. Savas, "Weak partial b-metric spaces and nadlers theorem," *Mathematics*, vol. 7, no. 4, p. 332, 2019.

[25] S. Shukla, "Partial b-metric spaces and fixed point theorems," *Mediterranean Journal of Mathematics*, vol. 11, no. 2, pp. 703–711, 2014.

[26] Z. Mustafa, J. R. Roshan, V. Parvaneh, and Z. Kadelburg, "Some common fixed point results in ordered partial b-metric spaces," *Journal of Inequalities and Applications*, vol. 2013, no. 1, p. 562, 2013.

[27] A. Mukheimer, "α-ψ-ϕ-contractive mappings in ordered partial b-metric spaces," *J. Nonlinear Sci. Appl*, vol. 7, pp. 168–179, 2014.

[28] M. Abbas, T. Nazir, and S. Radenović, "Common fixed points of four maps in partially ordered metric spaces," *Applied Mathematics Letters*, vol. 24, no. 9, pp. 1520–1526, 2011.

[29] E. Kreyszig, *Introductory functional analysis with applications.* wiley New York, 1978, vol. 1.

[30] S. Batul, "Fixed point theorems in operator-valued metric spaces," Ph.D. dissertation, Capital University, 2016.

[31] S. B. Nadler *et al.*, "Multi-valued contraction mappings." *Pacific Journal of Mathematics*, vol. 30, no. 2, pp. 475–488, 1969.

[32] S. M. Michaek Bakhtin, Ralph Kopperman and H. Pajoohesh, "Partial metric spaces," *The Mathematical Association of America*, vol. 116, pp. 708–718, 2009.

[33] Q. Kiran, N. Alamgir, N. Mlaiki, and H. Aydi, "On some new fixed point results in complete extended b-metric spaces," *Mathematics*, vol. 7, no. 5, p. 476, 2019.