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Cone Extended b-Metric Space over Banach Algebra

by

Umair Ahmed Butt

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Dedicated to

My Mother

A determined and aristocratic embodiment who educate me to belief in Allah, believe in hard work and that so much could be done with little.

My Grandmother

For being my first beloved motivator in the world.

My Father

For my father I quote the remarkable words of Hadith,
"A father gives his child nothing better than an education".

CERTIFICATE OF APPROVAL

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Abstract

The main purpose of this thesis is to study and learn about the cone extended b-metric spaces over Banach algebra $\mathbb{A}$ with solid cone and to establish fixed points results for multivalued mappings in the setting of cone extended b-metric spaces. The target is achieved by first understanding the related fixed point results for multivalued mappings in cone b-metric spaces and then generalizing those results in the setting of cone extended b-metric spaces. Our results generalize and extend different results of cone metric, cone b-metric and cone extended b-metric spaces.

Contents

List of Figures

Abbreviations

BCP Banach Contraction Principle

Symbols

(M, d)	Metric space
d	Distance function
$\mathbb{A}$	Banach algebra
$\mathbb{E}$	Banach space
$\mathbb{R}$	Real number
$\mathbb{N}$	Natural number
$\mathbb{C}$	Complex number
ω	Cone
$CB(M)$	Closed and Bounded subset of M
C	Compact set
$Int.\omega$	Interior of cone ω
$N(M)$	Nonempty subset of M
s	Coefficient
$P(Q)$	Power set of Q
H	Hausdorff distance
Δ	Angle of cartesian product
D	Directed/controlled graph
$\mathbb{V}(D)$	Subset of vertices of directed graph D
$E(D)$	Edges of the graph D
$\tilde{D}$	Undirected/uncontrolled graph

Chapter 1

Introduction

1.1 Historical Background

Mathematics is one of the key branch of scientific knowledge in every field of life. It is further classified into many other branches. One of the most important branch of pure mathematics is fixed point theory. Fixed point theory is an interdisciplinary point which can be employed in different fields of mathematics and all the other disciplines of mathematical sciences like variational inequalities, approximation theory, mathematical economics, game theory and optimization theory. In the last $55-60$ years, fixed point theory become the greatest growing and very interesting research area for mathematicians. Poincare [1] was the first mathematician who worked in the field of fixed point theory in 1886. Later on, in 1922 Banach [2] established the existence of unique fixed point for contraction mappings in complete metric space.

One of the significant areas of fixed point theory was the metric fixed point theory and we accept that the more precious content in the expansion of non-linear analysis is metric fixed point theory. Historically the opening line in this field was well-defined by the establishment of Banach fixed point theorem which is stated as:

"Let (X, d) be a complete metric space and $T : X \to X$ be a contraction, that is, there exist $\alpha \in [0, 1)$ such that

$$d(Tx, Ty) \leq \alpha d(x, y) \qquad \text{for all} \quad x, y \in X.$$

Then T has a unique fixed point." It is accepted that Banach contraction principle (BCP) is fundamental outcome in the beginning of fixed point theory and plenty of different types of fixed point theorems where expressed into result. Currently, Banach contraction principle is extending and refining typically in these directions.

1. By extending the contraction conditions on the mapping G.
2. By extending the structure of the spaces on which G is defined.

In the above-stated first point, the massive quantity of literature is available on the extensions and generalizations of BCP. Presic [3] and Kannan [4] improved this contraction mapping principle in 1965 and 1968 respectively. In 1969 further development came from Mier and Keeler [5]. Later on Fomin [6] and Gupta [7] made further extension by means of rational expression in Banach contraction principle, afterwards Dolhore [8] extended this magnificent Banach contraction principle. Many contraction conditions can be seen in the very comprehensive work of Rhoades [9] and Collaco et al. [10]

In the above-stated second point, Nadler [11] also extended the Banach contraction principle from single valued to multivalued contraction maps, considering the metric defined on closed and bounded subsets of a nonempty set M. Moreover, by changing the wide structure of underlying space, the fixed point theory is widespreaded by introduces the notion of b-metric space [12, 13], b-metric-like space [14], partial metric space [15], quasi b-metric spaces [16–19], Extended b-metric spaces [20, 21], cone metric space [22–29], cone rectangular metric space [30–34], cone b-metric space [35] and many others.

The generalization of the concept of a metric space is initiated by Bakhtin [12] and thereafter used by Czerwick [13]. They established the idea of b-metric space and then used the same idea to set up some fixed point theorems for generalizing the Banach contraction principle. Thereafter, plenty of papers have been published in b-metric spaces for single valued functions and also on multivalued functions for instance see [36–40, 40–44]. The idea of extended b-metric space was explored by

Kamran et al. [20]. This new notion attracted the researchers in this domain and several fixed point results have been furnished for mappings defined on extended b-metric spaces. Applications of these results are discussed for G-contractions.

On the other hand, the notion of a cone metric spaces has also gain much attention of researchers in the community. Huang [45] was the first who introduced the cone metric space. A comparable explanation is also assumed by Rzepecki in [46]. Afterward carefully describing convergence and also completeness in cone metric spaces, the authors determined some fixed point results of contractive mappings. Recantly, Kutbi et al. [35] investigated the multivalued fixed point results in cone b-metric spaces over Banach algebra $\mathbb{A}$. Further fixed point results in cone metric spaces appeared in [22–29].

In this thesis, we introduced the notion of cone extended b-metric space over Banach algebra. Firstly fixed point results of Kutbi et al. [35] are presented and examples are employed. The next major goal of this work is to establish these fixed point results in the setting of cone extended b-metrics over Banach algebra $\mathbb{A}$. These new fixed point theorems generalize the results of Kutbi et al. [35].

The rest of the work is systematized as follows:

- **Chapter** 02, provides the basic ideas concerning metric space, b-metric space, extended b-metric space, fixed point, contraction, BCP, multivalued contraction and some other definitions and examples related to subsequent chapters.
- **Chapter** 03, emphasising on review of the paper in [35] i.e., multivalued fixed point theorems in cone b-metric spaces over Banach algebra $\mathbb{A}$ are discussed. Also some useful definitions and examples related to our work put into the chapter.
- **Chapter** 04, by generalizing the cone b-metric space we introduce the notion of cone extended b-metric space over Banach algebra $\mathbb{A}$ and established some new fixed point theorems by extending the results of cone b-metric space into cone extended b-metric space. At the end, we conclude our thesis.

Chapter 2

Preliminaries

In this chapter we will recall some basic definitions and examples by the study of cone extended b-metric spaces. The main purpose of the chapter is to present the basic results, definitions and examples that will be used in the next chapters.

2.1 Metric Space

Frechet [47] was the first person who introduced the concept of metric space in connection with a study of function spaces. Metric is a function that explains the concept of distance between any two nonempty sets.

Definition 2.1.1. [48](**Metric Space**)
"A metric space is a pair (X, d), where X is a set and d is a metric on X (or distance function on X), that is, a function defined on $X \times X$ such that for all $x, y, z \in X$ we have:

(M1) d is real-valued, finite and nonnegative;

(M2) $d(x, y) = 0$ if and only if $\quad x = y$;

(M3) $d(x, y) = d(y, x)$;

(M4) $d(x, y) \leq d(x, z) + d(z, y)$.

The symbol $\times$ denotes the Cartesian product of sets: $A\times B$ is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$. Hence $X \times X$ is the set of all ordered pairs of elements of X."

Example 2.1.1. [48]

Let $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a function defined by

$$d(\xi, \psi) = |\xi - \psi| \qquad \text{for all} \quad \xi, \psi \in \mathbb{R}.$$

Then d is metric on real numbers $\mathbb{R}$.

To show d is metric on $\mathbb{R}$, note that conditions (M_1) and (M_2) are obvious, we only show that condition (M_3) holds

$$\begin{aligned} d(\xi, \delta) =&\ |\xi - \delta|, \\ =&\ |\xi - \psi + \psi - \delta|, \\ \leq&\ |\xi - \psi| + |\psi - \delta|, \\ d(\xi, \delta) \leq&\ d(\xi, \psi) + d(\psi, \delta). \end{aligned}$$

This shows that d is metric on $\mathbb{R}$.

Example 2.1.2. [48]

Let $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be a function defined by

$$d(\xi, \psi) = |\xi_1 - \psi_1| + |\xi_2 - \psi_2| \qquad \text{for all} \quad \xi, \psi \in \mathbb{R}^2.$$

Then d is metric on $\mathbb{R}^2$.

To show that d is metric on $\mathbb{R}^2$, we note that the conditions (M_1) and (M_2) are trivially satisfied. Now for the last condition (M_3) for all $\xi, \psi, \delta \in \mathbb{R}^2$, we have

$$
\begin{aligned}
d(\xi,\delta) =\ & |\xi_1-\delta_1|+|\xi_2-\delta_2|,\\
=\ & |\xi_1-\psi_1+\psi_1-\delta_1|+|\xi_2-\psi_2+\psi_2-\delta_2|,\\
\leq\ & |\xi_1-\psi_1|+|\psi_1-\delta_1|+|\xi_2-\psi_2|+|\psi_2-\delta_2|,\\
=\ & (|\xi_1-\psi_1|+|\xi_2-\psi_2|)(|\psi_1-\delta_1|+|\psi_2-\delta_2|),\\
d(\xi,\delta) \leq\ & d(\xi,\psi)+d(\psi,\delta).
\end{aligned}
$$

This shows that d is metric on $\mathbb{R}^2$.

2.2 b-Metric Space

The concept of a b-metric space is initiated by Bakhtin [12] and thereafter used by Czerwick [13]. Czerwick introduced a condition which was weaker than the third property of metric space and formally defined a b-metric space. They established the idea of b-metric space and then used the same idea to establish some fixed point results for generalizing the Banach contraction principle.

Definition 2.2.1. [12, 13](**b-Metric Space**)
"Let X be a nonempty set and $d_b : X \times X \to [0,\infty)$ be a function satisfying the following conditions:

(b1) $d_b(x,y)=0$ if and only if $x=y$;

(b1) $d_b(x,y)=d_b(y,x)$ for all $x,y\in X$;

(b1) $d_b(x,y)\leq s[d_b(x,z)+d_b(z,y)]$ for all $x,y,z\in X$, where $s\geq 1$.

The function d_b is called a b-metric and the set (X,d_b) is called a b-metric space."

Remark 2.1.[13]
"The class of b-metric space is larger than the class of metric space. When $s=1$ the concept of b-metric space coincides with the concept of metric space."

Example 2.2.1. [48]

Let $d_b : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a function defined by

$$d_b(\xi, \psi) = (\xi - \psi)^2 \qquad \text{for all} \quad \xi, \psi \in \mathbb{R}.$$

Then d_b is b-metric on $\mathbb{R}$.

To show that d_b is a b-metric space, note that the conditions (b_1) and (b_2) are trivially satisfied. Now we check the last condition (b_3), for all $\xi, \psi, \delta \in \mathbb{R}$

$$\begin{aligned}
d_b(\xi, \delta) \leq\ & (\xi - \delta)^2, \\
=\ & (\xi - \psi + \psi - \delta)^2, \\
\leq\ & \{(\xi - \psi) + (\psi - \delta)\}^2, \\
=\ & (\xi - \psi)^2 + (\psi - \delta)^2 + 2(\xi - \psi)(\psi - \delta), \\
=\ & (\xi - \psi)^2 + (\psi - \delta)^2 + 2(\xi - \psi)(\psi - \delta) + (\xi - \psi)^2 + \\
& (\psi - \delta)^2 - (\xi - \psi)^2 - (\psi - \delta)^2, \\
=\ & 2(\xi - \psi)^2 + 2(\psi - \delta)^2 - (\xi - \psi)^2 - (\psi - \delta)^2 + \\
& 2(\xi - \psi)(\psi - \delta), \\
=\ & 2\{(\xi - \psi)^2 + (\psi - \delta)^2\} - \{(\xi - \psi)^2 + (\psi - \delta)^2 - \\
& 2(\xi - \psi)(\psi - \delta)\}, \\
=\ & 2\{(\xi - \psi)^2 + (\psi - \delta)^2\} - \{(\xi - \psi) - (\psi - \delta)\}^2, \\
\leq\ & 2\{(\xi - \psi)^2 + (\psi - \delta)^2\}, \\
\leq\ & 2d_b(\xi, \psi) + d_b(\psi, \delta).
\end{aligned}$$

Example 2.2.2. [49]

"Let $\mathrm{M} = \ell_p[0, 1]$ be the space of all real functions $\xi(t)$, $t \in [0, 1]$ such that

$$\int_0^1 |\xi(t)|^p \ < \ \infty \qquad\qquad \text{with} \quad 0 \ < \ p \ < 1,$$

Define $d_b : M \times M \to \mathbb{R}^+$ as:

$$d_b(\xi, \psi) = \left(\int_0^1 |\xi(t) - \psi(t)|^p dt \right)^{\frac{1}{p}},$$

Then d_b is b-metric space with $s = 2^{\frac{1}{p}}$."

Example 2.2.3. Consider $M = \mathbb{N} \cup \infty$ and suppose that $d_b : M \times M \to \mathbb{R}^+$ be defined by

$$d_b(\xi, \psi) \ = \ \begin{cases} 0 & \text{when} \quad m \ \geq \ \ell, \\ |\frac{1}{m} \ - \ \frac{1}{\ell}| & \text{when} \quad m\ell \ = \ \infty \text{ or } m\ell \text{ are even,} \\ 5 & \text{when } m \ \neq \ \ell \text{ and } m, \ell \text{ are odd,} \\ 2 & \text{otherwise,} \end{cases}$$

It may be checked that for all $\xi, \psi, \delta \in M$, we have

$$d_b(\xi, \psi) \ \leq \ \frac{5}{2}[d_b(\xi, \psi) \ + \ d_b(\psi, \delta)],$$

take $p_\ell = 2\ell$, for each $\ell \in \mathbb{N}$

$$\begin{aligned} \lim_{\ell \to \infty} d_b(p_\ell, \infty) \ &= \ \lim_{\ell \to \infty} d_b(2\ell, \infty), \\ &= \ |\frac{1}{2\ell} \ - \ \frac{1}{\infty}|, \\ &= \ \lim_{\ell \to \infty} \frac{1}{2\ell}, \\ &= \ 0. \end{aligned}$$

Moreover

$\lim_{\ell \to \infty} d_b(p_\ell, \infty) = 2 \neq 5 = d_b(\infty, 1)$. Hence (M, d_b) be a b-metric space with real number $b = \frac{5}{2}$, however it is not continuous.

Definition 2.2.2. [20](**Extended b-Metric Space**)

"Let X be a nonempty set and $\theta : X \times X \to [1, \infty)$. A function $d_\theta : X \times X \to [0, \infty)$ is called an extended b-metric if for all $x, y, z \in X$ it satisfies

$(d_\theta 1)$ $d_\theta(x, y) = 0$ if and only if $x = y$;

$(d_\theta 2)$ $d_\theta(x, y) = d_\theta(y, x)$;

$(d_\theta 3)$ $d_\theta(x, z) \leq \theta(x, z)[d_\theta(x, y) + d_\theta(y, z)]$;

Then the pair (X, d_θ) is called an extended b-metric space."

Kamran et al. [20] introduced the concept of extended b-metric space. Extended b-metric space is the generalization of metric space.

Remark 2.2.[20]

"If $\theta(\xi, \delta) = s$ for $s \geq 1$, then we obtain the definition of b-metric space."

Example 2.2.4. [20]

Consider $M = \{1, 3, 5\}$ and define $d_\theta : M \times M \to \mathbb{R}^+$ by $d_\theta(\xi, \psi) = (\xi - \psi)^2$. It is well known that d_θ is a b-metric with coefficient $b = 2$. Let us define the mapping

$$\theta : M \times M \to [1, \infty), \qquad \theta(\xi, \psi) = \xi + \psi + 1$$

Then it can be shown that (M, d_θ) is an extended b-metric space. As we know that the self distance between two points must be equal to zero, that is,

$$d_\theta(1, 1) = d_\theta(3, 3) = d_\theta(5, 5) = 0,$$

$$d_\theta(1, 3) = d_\theta(3, 1) = d_\theta(3, 5) = d_\theta(5, 3) = 4 \quad d_\theta(1, 5) = d_\theta(5, 1) = 16,$$

In addition,

$$\theta(1, 3) = \theta(3, 1) = 5, \quad \theta(3, 5) = \theta(5, 3) = 9, \quad \theta(1, 5) = \theta(5, 1) = 7,$$

To show that (M, d_θ) is an extended b-metric space, we note that the conditions $(d_\theta 1)$ and $(d_\theta 2)$ are trivially satisfied. Now for the last condition $(d_\theta 3)$ we have

$$d_\theta(\xi, \psi) \leq \theta(\xi, \psi)[d_\theta(\xi, \delta) + d_\theta(\delta, \psi)] \quad \text{for all} \quad \xi, \psi, \delta \in M.$$

For $\xi = 1$, $\psi = 3$, $\delta = 5$ the above inequality implies

$$d_\theta(1, 3) \leq \theta(1, 3)[d_\theta(1, 5) + d_\theta(5, 3)] \quad \text{for all} \quad \xi, \psi, \delta \in M,$$

using the above values we have

$$\begin{aligned} 4 \leq \;& 5[16 + 4] \quad \text{for all} \quad \xi, \psi, \delta \in M, \\ 4 \leq \;& 100 \quad \text{for all} \quad \xi, \psi, \delta \in M. \end{aligned}$$

Similarly for $d_\theta(3, 5)$ and $d_\theta(1, 5)$ we have

$$\begin{aligned} d_\theta(3, 5) \leq \;& \theta(3, 5)[d_\theta(3, 1) + d_\theta(1, 5)] \quad \text{for all} \quad \xi, \psi, \delta \in M, \\ 4 \leq \;& 9[4 + 16] \quad \text{for all} \quad \xi, \psi, \delta \in M, \\ 4 \leq \;& 180 \quad \text{for all} \quad \xi, \psi, \delta \in M. \end{aligned}$$

Now for $d_\theta(1, 5)$

$$\begin{aligned} d_\theta(1, 5) \leq \;& \theta(1, 5)[d_\theta(1, 3) + d_\theta(3, 5)] \quad \text{for all} \quad \xi, \psi, \delta \in M, \\ 16 \leq \;& 7[4 + 4] \quad \text{for all} \quad \xi, \psi, \delta \in M, \\ 4 \leq \;& 56 \quad \text{for all} \quad \xi, \psi, \delta \in M. \end{aligned}$$

Hence for all $\xi, \psi, \delta \in M$,

$$d_\theta(\xi, \psi) \leq \; \theta(\xi, \psi)[d_\theta(\xi, \delta) + d_\theta(\delta, \psi)].$$

Hence (M, d_θ) is an extended b-metric space.

Definition 2.2.3. [20](**Convergence**)

"Let (X, d_θ) be an extended b-metric space. A sequence $\{x_n\}$ in X is said to converge to $x \in X$, if for every $\epsilon > 0$ there exist $N = N(\epsilon) \in \mathbb{N}$ such that $d_\theta(x_n, x) < \epsilon$ for all $n \geq N$. In this case, we write $\lim_{n\to\infty} x_n = x$."

Definition 2.2.4. [20](**Cauchy Sequence**)

"Let (X, d_θ) be an extended b-metric space. A sequence $\{x_n\}$ in X is said to be Cauchy, if for every $\epsilon > 0$ there exists $N = N(\epsilon) \in \mathbb{N}$ such that $d_\theta(x_m, x_n) < \epsilon$ for all $n, m \geq N$."

Definition 2.2.5. [20](**Complete**)

"An extended b-metric space (X, d_θ) is complete if every Cauchy sequence in X is convergent in X."

2.3 Fixed Point

If f is a function then the fixed point of the function is an element from the domain of the function which is mapped to itself by the function. Fixed point theorems are extremely used to solve the different types of problems in various fields of mathematics especially in pure mathematics. A number of authors mainly Banach [2], Bhaskar [50], Khamsi [51] explain fixed point theorems onto complete metric space.

Definition 2.3.1. [48](**Fixed Point**)

"A fixed point of a mapping $G : M \to M$ of a set M into itself is an $\xi \in M$ which is mapped onto itself i.e.,

$$G\xi = \xi,$$

the image $G\xi$ coincides with ξ."

Example 2.3.1. [48]

Let $M = \mathbb{R}$ and $G : \mathbb{R} \to \mathbb{R}$ be a mapping defined as

$$G(\xi) = \frac{\xi}{3} + 2,$$

then, 3 is the fixed point of G.

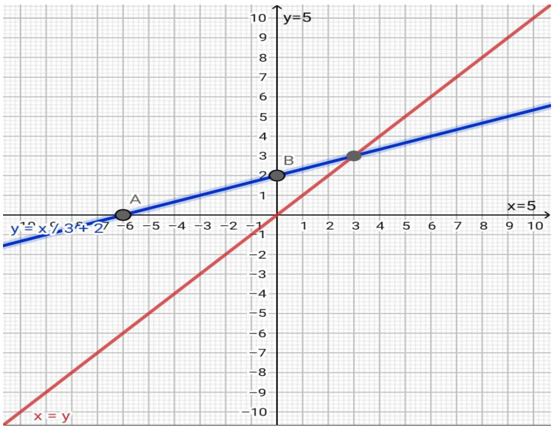

Figure 2.1: 1 Fixed Point.

Example 2.3.2. [48]

Let $M = \mathbb{R}$ and G maps M into M such that,

$$G(\xi) = \xi + 0.5,$$

then G has no fixed point, as $\xi + 0.5 = \xi$ has no solution.

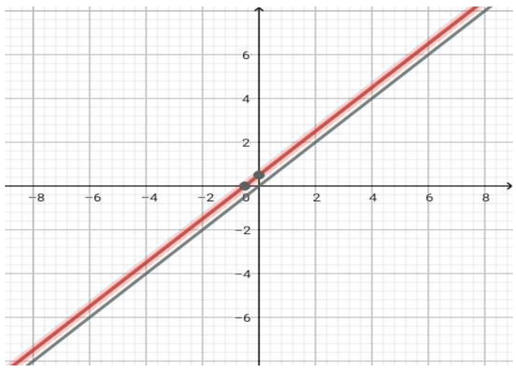

Figure 2.2: No Fixed Point.

Example 2.3.3. [48]

Consider $M = \mathbb{R}$ and G from M into M such that

$$G(\xi) = 2\xi + 0.5,$$

then, G has a unique fixed point of $\xi = -0.5$.

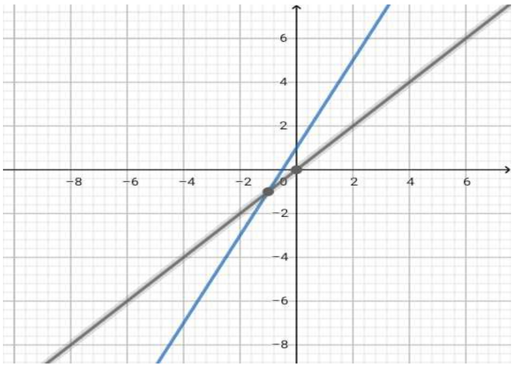

FIGURE 2.3: Unique Fixed Point.

Definition 2.3.2. [48](**Contraction**)

"Let (X, d) be a metric space. A mapping $T : X \to X$ is called a contraction on X if there is a positive real number $\alpha < 1$ such that for all $x, y \in X$

$$d(Tx, Ty) \leq \ \alpha d(x, y) \qquad\qquad \alpha < 1.$$

Geometrically this means that any point x and y have images that are closer together than those points x and y, more precisely the ratio $\frac{d(Tx,Ty)}{d(x,y)}$ does not exceed a constant α which is strictly less than 1."

Example 2.3.4. The function $G : \mathbb{R} \to \mathbb{R}$ defined by

$$G(\xi) = \cos(\cos \xi),$$

is a contraction.

As

$$G(\xi) = \cos(\cos\xi),$$

$$\begin{aligned} G'(\xi) &= -\sin(\cos\xi)[-\sin\xi], \\ &= \sin(\cos\xi)[\sin\xi], \end{aligned}$$

by using the Mean Value Theorem, we have

$$|G'(\xi)| = |\sin(\cos\xi)|\ |\sin\xi| < 1,$$

since

$$|\sin(\cos\xi)| \leq 1,$$

$$|\sin\xi| \leq 1.$$

Simultaneously both can not be equal to 1 at the same time. This implies that $G(\xi)$ is a contraction.

Example 2.3.5. Let (M, d) be a metric space and $d(\xi, \psi) = |\xi - \psi|$.

Then define a mapping $G : M \to M$ by

$$G(\xi) = \frac{\xi}{7} + 2,$$

is a contraction.

Example 2.3.6. Consider $M = [0, 1]$ with $d(\xi, \psi) = |\xi - \psi|$.

Then define a mapping $G : M \to M$ by

$$G(\xi) = \frac{1}{\xi + 2},$$

is a contraction.

As the mapping $G : M \to M$ by

$$G(\xi) = \frac{1}{\xi + 2},$$

$$
\begin{aligned}
d(G\xi, G\psi) \leq d\left(\frac{1}{\xi+2}, \frac{1}{\psi+2}\right), \\
&\leq \left|\frac{1}{\xi+2} - \frac{1}{\psi+2}\right|, \\
&\leq \left|\frac{\psi+2-\xi-2}{(\xi+2)(\psi+2)}\right|, \\
&\leq \left|\frac{-(\xi-\xi)}{(\xi+2)(\psi+2)}\right|, \\
&\leq \left|\frac{(\xi-\xi)}{(2)(2)}\right|, \\
&\leq \frac{1}{4} d(\xi, \psi),
\end{aligned}
$$

is contraction with $\alpha = \frac{1}{4}$

Theorem 2.3.1. [48](**Banach Contraction Principle**)

"Consider a metric space (X, d), where $X \neq \phi$. Suppose that X is complete and let $T : X \to X$ be a contraction on X. Then T has precisely one fixed point."

Definition 2.3.3. [48](**Partially Ordered Set**)

"A partially ordered set is a set M on which there is defined a partial ordering, i.e., a (binary) relation which is written as $\preceq$ and satisfies the conditions

(PO1) $a \preceq a$ for every $a \in M$;

(PO2) If $a \preceq b$ and $b \preceq a$, then $a = b$;

(PO3) If $a \preceq b$ and $b \preceq c$, then $a \preceq c$;

Partially emphasizes that M may contain elements a and b for which neither $a \preceq b$ nor $b \preceq a$ holds. Then a and b are called incomparable elements. In contrast, two elements a and b are called comparable elements if they satisfy $a \preceq b$ or $b \preceq a$."

Example 2.3.7. [48]

The following relation is partial order set

$$
R = \{(p, q) \mid p, q \in \mathbb{Z}, \ \frac{p}{q} \in \mathbb{Z}\}.
$$

Definition 2.3.4. [48](**Totally Ordered Set**)

"A totally ordered set or chain is a partially ordered set such that every two elements of the set are comparable. In other words, a chain is a partially ordered set that has no incomparable elements."

Example 2.3.8. [48]

Let $\mathbb{R}$ be the set of all real numbers and define a partial ordering as $\preceq$ let $\xi \leq \psi$. $\mathbb{R}$ is totally ordered. $\mathbb{R}$ has no maximal elements.

Remark 2.6.

Every total ordered set is a partially ordered set but the converse is not true.

Remark 2.7.

Total order gives better results as compare to the partial order.

2.4 Multivalued Mapping

Multivalued mapping plays a significant role in different fields of applied and pure mathematics because of its numerous significance, for example, in optimal control problems as well as in real and complex analysis. As the years have passed, this theory has enlarged its importance and consequently in literature a number of papers bring into focus on the multivalued mappings.

Definition 2.4.1. [52](**Multivalued Mapping**)

"Let X and Y be nonempty sets. T is said to be a multivalued mapping from X to Y if T is a function from X to the power set of Y. We denote a multivalued map by $T : X \to 2^Y$."

Definition 2.4.2. [11](**Multivalued Contraction**)

"Let (X, d) be a metric space. A map $f : X \to CB(X)$ is said to be multivalued contraction if there exists $0 \leq \lambda < 1$ such that

$$H(fx, fy) \leq \lambda d(x, y), \qquad \text{for all} \quad x, y \in X$$

where $CB(X)$ denotes the family of nonempty closed subsets of X and H is the Hausdorff distance."

Example 2.4.1. Consider $G : P \to A(Q)$, where G is called multivalued or set valued mapping. If

$P = \{f, g, h, i, j, k\}$,

$Q = \{1, 1.5, 2, 2.5, ..., 7\}$,

Then,

$G(f) = \{1, 1.5, 4.5\}$ $G(g) = \{2, 2.5, 3\}$,

$G(h) = \{4\}$ $G(i) = \{3\}$,

$G(j) = \{5, 5.5, 6\}$ $G(k) = \{2.5, 6.5, 7\}$,

Remark 2.5.

It is clear that $G : P \to Q$ is not a function, this is because the values of set P has multiple images in a set Q.

Theorem 2.4.1. [52]

"Let (X, d) be a complete metric space and let $S, T : X \to CB(X)$ be multivalued maps satisfying

$$H(Tx, Sy) \leq ad(x, Ty) + b(d(x, Sy) + d(Ty, Tx)),$$

where $0 < a + 2b < 1, a, b \geq 0$ for all $x, y \in X$. $F(X) = F(S) \neq \phi$ and $Tx = Sx = F(T)$, for all $x \in F(T)$."

Remark 2.8.[52]

"T is multivalued mapping if and only if for each $x \in X$, $Tx \subseteq Y$. Unless otherwise stated we always assume Tx is nonempty for each $x \in X$."

Definition 2.4.3. [48](**Bounded Set**)

"A subset M of a metric space X is a bounded set if its diameter

$$\delta(M) = \sup_{x,y \in M} d(x, y),$$

is finite."

Example 2.4.2. [48]

The set of real numbers $\mathbb{R}$ is said to be bounded if it consists of a finite interval.

Definition 2.4.4. [48](**Open Set**)

"A subset M of a metric space X is said to be open if it contains a ball about each of its points."

Definition 2.4.5. [48](**Closed Set**)

"A subset K of X is said to be closed if its compliment (in X) is open, that is, $K^c = X - K$ is open."

Example 2.4.3. [48]

Each singleton set $\{m\}$ is a closed subset of M.

Example 2.4.4. The closed interval $[0.1, 0.5]$ of real numbers $\mathbb{R}$ is closed.

Definition 2.4.6. [48](**Open Mapping**)

"Let X and Y be a metric spaces. Then $T : \mathfrak{D}(T) \to Y$ with domain $\mathfrak{D}(T) \subset X$ is called an open mapping if for every open set in $\mathfrak{D}(T)$ the image is an open set in Y."

Chapter 3

Multivalued Fixed Point Theorems in Cone b-Metric Spaces

In this chapter, focusing on review of the paper in [35]. Some multivalued fixed point theorems in cone b-metric spaces over Banach algebra $\mathbb{A}$ are proved and an example is presented these results. Some useful lemmas and definitions are given which will help to prove the subsequent theorems.

3.1 Cone Metric Space

Huang [45] was the first who introduced cone metric spaces. A comparable explanation was also assumed by Rzepecki in [46]. Afterward carefully describing convergence and also completeness in cone metric spaces, the authors determined some fixed point results of contractive mappings in such spaces. A short time ago, further fixed point theorems about cone metric spaces appeared in [22–24]

Definition 3.1.1. [53](**Banach algebra**)

"A Banach space $\mathbb{E}$ is called a Banach algebra if there exists a multiplication

$\mathbb{E} \times \mathbb{E} \to \mathbb{E}$ that has the following properties:
for all $\xi, \psi, \delta \in \mathbb{E}$, $\alpha \in \mathbb{R}$,

(1) $(\xi\psi)\delta = \xi(\psi\delta)$;

(2) $\xi(\psi + \delta) = \xi\psi + \psi\delta$ and $(\xi + \psi)\delta = \xi\delta + \psi\delta$;

(3) $\alpha(\xi\psi) = (\alpha\xi)\psi = \xi(\alpha\psi)$;

(4) there exists $e \in \mathbb{E}$ such that $\xi e = e\xi = \xi$;

(5) $\|e\| = 1$;

(6) $\|\xi\psi\| \leq \|\xi\|.\|\psi\|$.

An element $\xi \in \mathbb{E}$ is called invertible if there exists $\xi^{-1} \in \mathbb{E}$ such that $\xi\xi^{-1} = \xi^{-1}\xi = e$."

Example 3.1.1. The set of real numbers $\mathbb{R}$ and the set of complex numbers $\mathbb{C}$ with norm given with absolute values are Banach algebras.

Definition 3.1.2. [35](**Cone**)
"A subset ω of $\mathbb{A}$ is called a cone if and only if:

(C1) $\{e, \theta\} \subset \omega$;

(C2) $\omega^2 = \omega\omega \subset \omega, \omega \cap (-\omega) = \{\theta\}$;

(C3) $p, q \in \mathbb{R}$, $p, q \geq 0, p\omega + q\omega \subset \omega$.

For a given cone $\omega \subset \mathbb{A}$, we define a partial ordering $\preceq$ with respect to ω by $\xi \preceq \psi$ if and only if $\psi - \xi \in \omega$; $\xi \prec \psi$ will stand for $\xi \preceq \psi$ and $\xi \neq \psi$, while $\xi \ll \psi$ stand for $\psi - \xi \in int\omega$, $int\omega$ denotes the interior of ω. If $int\omega \neq \phi$, then ω is called a solid cone and θ is the zero element of Banach algebra $\mathbb{A}$. Write $\|.\|$ as the norm of $\mathbb{A}$. A cone ω is called normal if there is a number $C > 0$ such that for all $\xi, \psi \in \mathbb{A}$, we have

$$\theta \preceq \xi \preceq \psi \quad \Rightarrow \quad \|\xi\| \leq C\|\psi\|.$$

The least positive number C satisfying above is called the normal constant of ω."

Example 3.1.2. [54]

The set $\{\xi \in \mathbb{R}^2 | \ \xi_2 \geq 0, \ \xi_1 = 0\} \cup \{\xi \in \mathbb{R}^2 \ |\xi_1 \geq 0, \ \xi_2 = 0\}$, where $\xi = (\xi_1, \xi_2)$ is a cone.

Definition 3.1.3. [55](**Cone Metric Space**)

"Let M be a nonempty set. Suppose that the mapping $d : M \times M \to \mathbb{A}$ satisfies:

(C1) $\theta \prec d(\xi, \psi)$ for all $\xi, \psi \in M$ with $\xi \neq \psi$ and $d(\xi, \psi) = \theta$ if and only if $\xi = \psi$;

(C2) $d(\xi, \psi) = d(\psi, \xi)$ for all $\xi, \psi \in M$;

(C3) $d(\xi, \psi) \preceq d(\xi, \delta) + d(\delta, \psi)$ for all $\xi, \psi, \delta \in M$.

Then d is called a cone metric on M, and (M, d) is called a cone metric space over Banach algebra $\mathbb{A}$."

Example 3.1.3. [45]

"Let $\mathbb{E} = \mathbb{R}^2$, $\omega = \{(\xi, \psi) \in \mathbb{E} : \xi, \psi \geq 0\} \subset \mathbb{R}^2$, $M = \mathbb{R}$ and $d : M \times M \to \mathbb{E}$ such that $d(\xi, \psi) = (|\xi - \psi|, \alpha|\xi - \psi|)$, where $\alpha \geq 0$ is a constant. Then (M, d) is cone metric space."

Definition 3.1.4. [55](**Convergent Sequence**)

"Let (M, d) be a cone metric space over Banach algebra $\mathbb{A}$, $\xi \in M$ and $\{\xi_\ell\}$ a sequence in M converges to M whenever for every $c \gg \theta$ there is a natural number $\mathbb{N}$ such that

$$d(\xi_\ell, \xi) \ll c \qquad \text{for all} \quad \ell \geq \mathbb{N}$$

we denote this by $\lim_{\ell \to \infty} \xi_\ell = \xi$ or $\xi_\ell \to \xi \ (\ell \to \infty)$."

Definition 3.1.5. [55](**Cauchy Sequence**)

"Let (M, d) be a cone metric space over Banach algebra $\mathbb{A}$, $\xi \in M$ and $\{\xi_\ell\}$ a sequence in M whenever for every $c \gg \theta$ there is a natural number $\mathbb{N}$ such that

$$d(\xi_\ell, \xi_m) \ll c \qquad \text{for all} \quad \ell, m \geq \mathbb{N},$$

then the sequence is a Cauchy sequence in M."

Definition 3.1.6. [55](**Complete Space**)

"Let (M, d) be a cone metric space over Banach algebra $\mathbb{A}$ and $\xi \in M$, then (M, d) is complete if every Cauchy sequence is convergent in M."

Haung and Radenovic [56] enlarged the idea of cone metric space to cone b-metric space over Banach algebra $\mathbb{A}$.

Definition 3.1.7. [56](**Cone b-Metric Space**)

"Let M be a nonempty set and $s \geq 1$ be a real number. Suppose that the mapping $d : M \times M \to \mathbb{A}$ satisfies:

(Cb1) $\theta \prec d(\xi, \psi)$ for all $\xi, \psi \in M$ with $\xi \neq \psi$ and $d(\xi, \psi) = \theta$ if and only if $\xi = \psi$;

(Cb2) $d(\xi, \psi) = d(\psi, \xi)$ for all $\xi, \psi \in M$;

(Cb3) $d(\xi, \psi) \preceq s[d(\xi, \delta) + d(\delta, \psi)]$ for all $\xi, \psi, \delta \in M$.

Then d is called a cone metric on M, and (M, d) is called a cone b-metric space over Banach algebra $\mathbb{A}$."

Remark 3.1.[56]

"The class of cone b-metric space over Banach algebra is larger than the class of cone metric space over Banach algebra but the converse is not true"

Remark 3.2.[35]

"The notions of convergent sequence, Cauchy sequence and complete space in cone b-metric space over Banach algebra are similar to the notions in cone metric space over Banach algebra $\mathbb{A}$."

Example 3.1.4. [56]

"Let $\mathbb{A} = C[a, b]$ be the set of continuous functions on the interval $[a, b]$ with the supremum norm. Define multiplication in the usual way. Then $\mathbb{A}$ is a Banach algebra with a unit 1. Set $\omega = \{\xi \in \mathbb{A} : \xi(t) \geq 0, t \in [a, b]\}$ and $M = \mathbb{R}$. Define a mapping $d : M \times M \to \mathbb{A}$ by $d(\xi, \psi)(t) = |\xi - \psi|^p e^t$ for all $\xi, \psi \in M$, where $p > 1$ is a constant. This makes (M, d) into a cone b-metric space over Banach algebra

$\mathbb{A}$ with the coefficient $s = 2^{p-1}$, but it is not a cone metric over Banach algebra since the triangle inequality is not satisfied."

As $1 < p < \infty$, then by the convexity of function $f(\xi) = \xi^p$, where $\xi \geq 0$ implies

$$\left(\frac{o+\tau}{2}\right)^p \leq \frac{1}{2}(o^p + \varrho^p),$$

$$\frac{(o+\tau)^p}{(2)^p} \leq 2^{-1}(o^p + \varrho^p),$$

$$(o+\tau)^p \leq 2^p.2^{-1}(o^p + \varrho^p),$$

$$(o+\tau)^p \leq 2^{p-1}(o^p + \varrho^p).$$

Therefore for each $\xi, \psi, \delta \in M$, we get

$$d(\xi, \delta) = |\xi - \delta|^p e^t,$$

$$= |\xi - \psi + \psi - \delta|^p e^t,$$

$$\leq \{|\xi - \psi| + |\psi - \delta|\}^p e^t,$$

$$\leq 2^{p-1}\{|\xi - \psi|^p + |\psi - \delta|^p\}e^t,$$

$$d(\xi, \delta) \leq 2^{p-1}[d(\xi, \psi) + d(\psi, \delta)]e^t.$$

This shows (M, d) is a cone b-metric space over Banach algebra $\mathbb{A}$ with coefficient $s = 2^{p-1}$.

Following definition is taken from [35] in which $H(P, Q)$ is the Hausdorff distance in cone metric space, $CB(M)$ is the closed and bounded subsets of M and $h(P, Q)$ is the subset of Hausdorff distance.

Definition 3.1.8. [35](**Hausdorff Distance**)

"Let (M, d) be a cone metric space over Banach algebra $\mathbb{A}$, with the coefficient $b \geq 1$. If $\mathbb{A} = \mathbb{R}$ and $\omega = [0, +\infty)$, then (M, d) is a metric space. Define $H : CB(M) \times CB(M) \to \mathbb{A}$ as for any $P, Q \in CB(M)$, $H(P, Q) = \inf h(P, Q)$ is the Hausdorff distance introduced by d."

Example 3.1.5. If $M = \mathbb{R}$

Determine $H(P,Q)$ where $P = [2,4]$ and $Q = [3,7]$

Suppose that $p = 3$ and $q = 1$

$$\begin{aligned} Q \subset\ & N(p,P) = N(3,P) = [2-3, 4+3] = [-1,7], \\ P \subset\ & N(q,Q) = N(1,Q) = [3-1, 7+1] = [2,8], \end{aligned}$$

as we know that

$$E_{PQ} = \ \{\epsilon > 0; P \subseteq N(\epsilon, Q), Q \subseteq N(\epsilon, P)\},$$

then

$$E_{PQ} = [3,\ \infty)$$

$$H(P,Q) = \inf[3,\ \infty),$$

$$H(P,Q) = 3.$$

Definition 3.1.9. [57](**Interior Point, Closed and Bounded Sets**)

"Let (M,d) be a cone b-metric space and $Q \subseteq M$

(i) $q \in Q$ is called an interior point of Q whenever there is $0 \ll a$ such that $Q_0(q,a) \subseteq Q$, where $Q_0(q,a) = \{\psi \in M : d_\theta(\psi, q) \ll a\}$.

(ii) A subset $Q \subseteq M$ is called closed whenever each limit point of Q belongs to Q.

(iii) A subset $Q \subseteq M$ is said to be bounded whenever there exist $0 \ll c$ and $\xi_0 \in M$ such that $d_\theta(q, \xi_0) \ll c$ for all $q \in Q$."

Definition 3.1.10. [58](**Spectral Radius**)

"Let $\mathbb{A}$ be a Banach algebra with a unit e, then the spectral radius $\rho(k)$ of $k \in \mathbb{A}$ holds

$$\rho_{(k)} = \lim_{\ell \to \infty} \|k^\ell\|^{\frac{1}{\ell}} = \inf \|k^\ell\|^{\frac{1}{\ell}}.$$

If $\rho(k) < 1$, then $e - k$ is invertiable in $\mathbb{A}$ moreover, $(e-k)^{-1} = \sum_{i=0}^{\infty} k^i$."

Lemma 3.1.1. [35]

"If $\mathbb{E}$ is a real Banach space with a cone ω and if $p \preceq \lambda p$ with $p \in \omega$ and $0 \leq \lambda < 1$, then $p = \theta$."

Lemma 3.1.2. [35]

"If $\mathbb{E}$ is a real Banach space with a solid cone ω and if $\theta \preceq u \ll c$ for each $\theta \ll c$, then $u = \theta$."

Lemma 3.1.3. [35]

"If $\mathbb{E}$ is a real Banach space with a solid cone ω and if $\|\xi_\ell\| \to 0$ as $\ell \to \infty$, then for any $\theta \ll c$, there exist $\ell_0 \in \mathbb{N}$ such that, $\xi_\ell \ll c$ for all $\ell < \ell_0$."

Remark 3.3.[59]

"If $\rho(x) < 1$, then $\|x_\ell\| \to 0$ as $\ell \to \infty$."

Definition 3.1.11. [57](**Convergent, Cauchy, Complete**)

"Let (M, d) be a cone b-metric space over Banach algebra $\mathbb{A}$, $\xi \in M$, let $\{\xi_\ell\}$ be a sequence in M. Then

(i) $\{\xi_\ell\}$ converges to ξ whenever for every $c \in \mathbb{A}$ with $\theta \ll c$, there is a natural number ℓ_0 such that $d(\xi_\ell, \xi) \ll c$, for all $\ell \gg \ell_0$. We denote this by $\lim_{\ell\to\infty} \xi_\ell = \xi$;

(ii) $\{\xi_\ell\}$ is a Cauchy sequence whenever for every $c \in \mathbb{A}$ with $\theta \ll c$ there exist a natural number ℓ_0 such that $d(\xi_\ell, \xi_m) \ll c$ for all $\ell, m \gg \ell_0$;

(iii) (M, d) is a complete cone b-metric if every Cauchy sequence in M is convergent."

Hausdorff distance function on cone metric spaces was introduced by Cho et al. in [60], and established some fixed point theorems for multivalued mappings in cone Hausdorff metric space. Following the notion of Hausdorff distance function on cone b-metric spaces, in [35] Kutbi et al. extended it to cone b-metric spaces with a Banach algebra $\mathbb{A}$ as follows:

For a cone b-metric space (M, d) over Banach algebra $\mathbb{A}$, denote

$$C(M) = \{P : P \text{ is a nonempty compact subset of } M\}$$

$$N(M) = \{P : P \text{ is a nonempty subset of} M\}$$

$$h(t_1) = \{t_2 \in \mathbb{A} : t_1 \preceq t_2\} \qquad \text{for} \;\; t_1 \in \mathbb{A}$$

$$h(r_1, Q) = \bigcup_{r_2 \in Q} h(d(r_1, r_2)) = \bigcup_{r_2 \in Q} \{\xi \in \mathbb{A} : d(r_1, r_2) \preceq \xi\}$$

for $r_1 \in M$ and $Q \in N(M)$. For all $P, Q \in N(M)$, also denote

$$h(P, Q) = \left(\bigcap_{r_1 \in P} h(r_1, Q) \right) \bigcap \left(\bigcap_{r_2 \in Q} h(r_2, P) \right).$$

Example 3.1.6. If $M = \mathbb{R}$

Determine $H(P, Q)$ where $P = [1, 2.5]$ and $Q = [2, 6]$

Suppose that $r_1 = 3.5$ and $r_2 = 1$

$$\begin{aligned} Q \;\subset\; N(r_1, P) \;&=\; N(3.5, P) \;=\; [1 - 3.5, 2.5 + 3.5] \;=\; [-2.5, 6], \\ P \;\subset\; N(r_2, Q) \;&=\; N(1, Q) \;=\; [2 - 1, 6 + 1] \;=\; [1, 7], \end{aligned}$$

As we know that

$$H(P, Q) \;=\; \inf E_{PQ},$$

more generally,

$$H(P, Q) \;=\; \begin{cases} \inf E_{PQ} & \text{if} \quad E_{PQ} \;\neq\; \phi, \\ 0 & \text{if} \quad E_{PQ} \;=\; \phi, \end{cases}$$

where

$$E_{PQ} \;=\; \{\epsilon \;>\; 0 \;;\; P \;\subseteq\; N(\epsilon, Q),\; Q \;\subseteq\; N(\epsilon, P)\},$$

now

$$\begin{aligned} E_{PQ} \;&=\; [3.5, \infty) \qquad\qquad H(P, Q) = \inf[3.5, \infty), \\ H(P, Q) \;&=\; 3.5. \end{aligned}$$

Example 3.1.7. $M = \mathbb{R}^2$

Determine the Hausdorff distance $H(P, Q)$ where

$$P = \{(\xi, \psi) : 1 \leq \xi \leq 3, \ 1 \leq \psi \leq 3\},$$
$$Q = \{(\xi, \psi) : (\xi - 4)^2 + (\psi - 1)^2 \leq 1\},$$

Let us define the metric

$$d((\xi_1, \psi_1), (\xi_2, \psi_2)) = \sqrt{(\xi_1 - \xi_2)^2 + (\psi_1 - \psi_2)^2},$$

As the distance between two given points $(1, 3)$ and $(4, 1)$

$$\begin{aligned} d((1,3),(4,1)) &= \sqrt{(\xi_1 - \xi_2)^2 + (\psi_1 - \psi_2)^2}, \\ &= \sqrt{(1 - 4)^2 + (3 - 1)^2}, \\ &= \sqrt{(3)^2 + (2)^2}, \\ &= \sqrt{9 + 4}, \\ &= \sqrt{13} \\ r_1 &= \sqrt{13} - 1, \\ r_1 &= 2.6056, \end{aligned}$$

Now also the distance between two points $(3, 1)$ and $(5, 1)$

$$\begin{aligned} d((3,1),(5,1)) &= \sqrt{(\xi_1 - \xi_2)^2 + (\psi_1 - \psi_2)^2}, \\ &= \sqrt{(3 - 5)^2 + (1 - 1)^2}, \\ &= \sqrt{(2)^2 + (0)^2}, \\ &= \sqrt{4}, \\ q &= 2, \end{aligned}$$

As we know that

$$E_{PQ} = \{\epsilon > 0 \, ; \, P \subseteq N(\epsilon, Q), \ Q \subseteq N(\epsilon, P)\},$$

$$P \subset N(r_1, Q),$$
$$Q \subset N(r_2, P),$$

$$E_{PQ} = \{\epsilon > 0 \,;\, P \subseteq N(\epsilon, Q),\ Q \subseteq N(\epsilon, P)\},$$

now

$$E_{PQ} = [\sqrt{13} - 1, \infty)$$

$$H(P, Q) = \inf[2.6056, \infty),$$

$$H(P, Q) = 2.6056.$$

Lemma 3.1.4. [35] "Let (X, d) be a cone b-metric space over Banach algebra $\mathbb{A}$ with the coefficient $b \geq 1$ and let P be a solid cone.

(1) Let $p, q \in \mathbb{A}$. If $p \preceq q$, $h(q) \subset h(p)$;

(2) Let $x \in X$ and $A \in N(X)$. If $\theta \in h(x, A)$, then $x \in A$;

(3) Let $q \in P$ and let $A, B \in CB(X)$ and $a \in A$. If $q \in h(A, B)$, then $q \in h(a, B)$ for all $a \in A$ or $q \in h(A, b)$ for all $b \in B$;

(4) Let $q \in P$ and let $\lambda \geq 0$, then $\lambda\, h(q) \subseteq h(\lambda q)$."

3.2 α-admissible Mapping

The idea of α-admissible mapping is initiated 2012 by Samet et al. in [61]. They introduced the idea and set up certain fixed point results for self mappings. Following our notations, we start with the definition of multivalued α-admissible in [62] and α-generalized multivalued contraction from [35].

Definition 3.2.1. [62]**(Multivalued α-admissible)**
"Let (M) be a nonempty set, $G : M \rightarrow N(M)$ a multivalued mapping and $\alpha : M \times M \rightarrow [0, \infty)$. The mapping G is called multivalued α-admissible if for $\xi_0, \psi_0 \in M$,

$$\alpha(\xi_0, \psi_0) \geq 1 \quad \Rightarrow \quad \alpha(\xi_1, \psi_1) \geq 1,$$

where $\xi_1 \in G\xi_0$ and $\psi_1 \in G\psi_0$."

Definition 3.2.2. [35](**α-generalized multivalued contraction**)
"Let (M, d) be a cone b-metric space over a Banach algebra $\mathbb{A}$ with the coefficient $b \geq 1$ and ω be the underlying solid cone. Then the multivalued mapping $G : M \to C(M)$ is known as α-generalized multivalued contraction if there exist $\alpha : M \times M \to [0, +\infty)$ and $k \in \omega$ such that $\rho(k) < 1$ and

$$kd(\xi, \psi) \in \alpha(\xi, \psi)h(G\xi, G\psi) \tag{3.1}$$

for all $\xi, \psi \in M$."

Following theorems was proved by Kutbi et al. in [35].

Theorem 3.2.1. "Let (M, d) be a complete cone b-metric space over a Banach algebra $\mathbb{A}$ with the coefficient $b \geq 1$ and ω be the underlying solid cone. Suppose that the multivalued mapping G is α-admissible, α-generalized multivalued contraction and there exist $\xi_0 \in M$, $\xi_1 \in G\xi_0$ such that $\alpha(\xi_0, \xi_1) \geq 1$. If $\{\xi_\ell\}$ is a sequence in M such that $\alpha(\xi_\ell, \xi_{\ell+1}) \geq 1$ for all ℓ and $\xi_\ell \to \xi^*$ as $\ell \to \infty$ then by assumption, $\alpha(\xi_\ell, \xi^*) \geq 1$ for all ℓ. Then, G has a fixed point in M."

Proof. Suppose that ξ_0 be an arbitrary point in M, and $G\xi_0 \in C(M)$, so $G\xi_0 \neq \phi$. Consider $\xi_1 \in G\xi_0$. Then from Equation 3.1, we have

$$kd(\xi_0, \xi_1) \;\in\; \alpha(\xi_0, \xi_1)\; h(G\xi_0, G\xi_1).$$

As $\xi_1 \in G\xi_0$, by using Lemma 3.1.4, we have

$$kd(\xi_0, \xi_1) \;\in\; \alpha(\xi_0, \xi_1)\; h(\xi_1, G\xi_1).$$

By definition, we can take $\xi_2 \in G\xi_1$ such that

$$kd(\xi_0, \xi_1) \in \alpha(\xi_0, \xi_1)\ h(d(\xi_1, \xi_2)).$$

By using Lemma 3.1.4, we have

$$kd(\xi_0, \xi_1) \in \alpha(\xi_0, \xi_1)\ h(d(\xi_1, \xi_2)) \subseteq h(\alpha(\xi_0, \xi_1)d(\xi_1, \xi_2)).$$

So

$$\alpha(\xi_0, \xi_1)\ d(\xi_1, \xi_2) \preceq kd(\xi_0, \xi_1). \tag{3.2}$$

Now, as $\alpha(\xi_0, \xi_1) \geq 1$, hence it follows from Equation 3.2.

$$0 \prec d(\xi_1, \xi_2) \preceq \alpha(\xi_0, \xi_1)\ d(\xi_1, \xi_2) \preceq kd(\xi_0, \xi_1). \tag{3.3}$$

For $\xi_1 = \xi_2$, there is nothing to prove because in this case ξ_1 is our required fixed point. Therefore we assume that $\xi_1 \neq \xi_2$, then $\xi_2 \notin G\xi_2$ and so it follows from Equation 3.1.

$$kd(\xi_1, \xi_2) \in \alpha(\xi_1, \xi_2)\ h(G\xi_1, G\xi_2).$$

As $\xi_2 \in G\xi_1$, by using Lemma 3.1.4, we have

$$kd(\xi_1, \xi_2) \in \alpha(\xi_1, \xi_2)\ h(\xi_2, G\xi_2).$$

By definition, we can take $\xi_3 \in G\xi_2$ such that

$$kd(\xi_1, \xi_2) \in \alpha(\xi_1, \xi_2)\ h(d(\xi_2, \xi_3)).$$

Again by using Lemma 3.1.4.

$$kd(\xi_1, \xi_2) \in \alpha(\xi_1, \xi_2)\ h(d(\xi_2, \xi_3)) \subseteq h(\alpha(\xi_1, \xi_2)d(\xi_2, \xi_3)).$$

So

$$\alpha(\xi_1, \xi_2)\ d(\xi_2, \xi_3) \preceq kd(\xi_1, \xi_2). \tag{3.4}$$

As G is α-admissible on M, since $\alpha(\xi_1, \xi_2) \geq 1$ where $\xi_1 \in G\xi_0$ and $\xi_2 \in G\xi_1$. So from Equation 3.4, we have

$$0 \prec d(\xi_2, \xi_3) \preceq \alpha(\xi_1, \xi_2)\, d(\xi_2, \xi_3) \preceq kd(\xi_1, \xi_2). \tag{3.5}$$

For $\xi_2 = \xi_3$, there is nothing to prove because in this case ξ_2 is our required fixed point. Hence we assume that $\xi_2 \neq \xi_3$, then $\xi_3 \notin G\xi_3$ and it follows from Equation 3.1.

$$kd(\xi_2, \xi_3) \in \alpha(\xi_2, \xi_3)\, h(G\xi_2, G\xi_3).$$

As $\xi_3 \in G\xi_2$, by using Lemma 3.1.4, we have

$$kd(\xi_2, \xi_3) \in \alpha(\xi_2, \xi_3)\, h(\xi_3, G\xi_3).$$

By definition, we can take $\xi_4 \in G\xi_3$ such that

$$kd(\xi_2, \xi_3) \in \alpha(\xi_2, \xi_3)\, h(d(\xi_3, \xi_4)).$$

Again by using Lemma 3.1.4, we have

$$kd(\xi_2, \xi_3) \in \alpha(\xi_2, \xi_3)\, h(d(\xi_3, \xi_4)) \subseteq h(\alpha(\xi_2, \xi_3) d(\xi_3, \xi_4)).$$

So

$$\alpha(\xi_2, \xi_3)\, d(\xi_3, \xi_4) \preceq kd(\xi_2, \xi_3). \tag{3.6}$$

As G is α-admissible on M, since $\alpha(\xi_1, \xi_2) \geq 1$ where $\xi_1 \in G\xi_0$ and $\xi_2 \in G\xi_1$. Thus from Equation 3.6, we have

$$0 \prec d(\xi_3, \xi_4) \preceq \alpha(\xi_2, \xi_3)\, d(\xi_3, \xi_4) \preceq kd(\xi_2, \xi_3). \tag{3.7}$$

For $\xi_3 = \xi_4$, there is nothing to prove because in this case ξ_3 is our required fixed point. Hence we assume that $\xi_3 \neq \xi_4$, then $\xi_4 \notin G\xi_4$ and so it follows from Equation 3.1.

$$kd(\xi_3, \xi_4) \in \alpha(\xi_3, \xi_4)\, h(G\xi_3, G\xi_4).$$

As $\xi_3 \in G\xi_4$, by using Lemma 3.1.4, we have

$$kd(\xi_3, \xi_4) \ \in \ \alpha(\xi_3, \xi_4)\ h(\xi_4, G\xi_4).$$

By definition, we can take $\xi_5 \in G\xi_4$ such that

$$kd(\xi_3, \xi_4) \ \in \ \alpha(\xi_3, \xi_4)\ h(d(\xi_4, \xi_5)).$$

Again by using Lemma 3.1.4, such that

$$kd(\xi_3, \xi_4) \ \in \ \alpha(\xi_3, \xi_4)\ h(d(\xi_4, \xi_5)) \ \subseteq \ h(\alpha(\xi_3, \xi_4)d(\xi_4, \xi_5)).$$

So

$$\alpha(\xi_3, \xi_4)\ d(\xi_4, \xi_5) \ \preceq \ kd(\xi_3, \xi_4). \tag{3.8}$$

As G is α-admissible on M, since $\alpha(\xi_1, \xi_2) \geq 1$ where $\xi_1 \in G\xi_0$ and $\xi_2 \in G\xi_1$. So from Equation 3.8, we have

$$0 \ \prec \ d(\xi_4, \xi_5) \ \preceq \ \alpha(\xi_3, \xi_4)\ d(\xi_4, \xi_5) \ \preceq \ kd(\xi_3, \xi_4). \tag{3.9}$$

Similarly by adopting the same procedure, we acquire a sequence $\{\xi_\ell\}$ in M such that $\xi_{\ell+1} \in G\xi_\ell$, $\xi_{\ell+1} \neq \xi_\ell$, $\alpha(\xi_\ell, \xi_{\ell+1}) \geq 1$ and

$$\begin{aligned}
d(\xi_\ell, \xi_{\ell+1}) \ &\preceq \ kd(\xi_{\ell-1}, \xi_\ell), \\
&\preceq \ k.kd(\xi_{\ell-2}, \xi_{\ell-1}), \\
&= \ k^2 d(\xi_{\ell-2}, \xi_{\ell-1}), \\
&\preceq \ k^2.kd(\xi_{\ell-3}, \xi_{\ell-2}), \\
&= \ k^3 d(\xi_{\ell-3}, \xi_{\ell-2}) \ ... \ \preceq \ k^\ell d(\xi_0, \xi_1).
\end{aligned}$$

By adopting the same procedure we obtain

$$d(\xi_\ell, \xi_{\ell+1}) \preceq k^\ell d(\xi_0, \xi_1) \qquad \text{for all} \;\; \ell. \tag{3.10}$$

Assuming that $\ell < m$, we have

$$\begin{aligned}
d(\xi_\ell, \xi_\ell) &\preceq b[d(\xi_\ell, \xi_{\ell+1}) + d(\xi_{\ell+1}, \xi_m)], \\
&= bd(\xi_\ell, \xi_{\ell+1}) + bd(\xi_{\ell+1}, \xi_m), \\
&\preceq bd(\xi_\ell, \xi_{\ell+1}) + b^2[d(\xi_{\ell+1}, \xi_{\ell+2}) + d(\xi_{\ell+2}, \xi_m)], \\
&= bd(\xi_\ell, \xi_{\ell+1}) + b^2 d(\xi_{\ell+1}, \xi_{\ell+2}) + b^2 d(\xi_{\ell+2}, \xi_m), \\
&\preceq bd(\xi_\ell, \xi_{\ell+1}) + b^2 d(\xi_{\ell+1}, \xi_{\ell+2}) + \\
&\quad\; b^3[d(\xi_{\ell+2}, \xi_{\ell+3}) + d(\xi_{\ell+3}, \xi_m)], \\
&= bd(\xi_\ell, \xi_{\ell+1}) + b^2 d(\xi_{\ell+1}, \xi_{\ell+2}) + b^3 d(\xi_{\ell+2}, \xi_{\ell+3}) + \\
&\quad\; b^3 d(\xi_{\ell+3}, \xi_m), \\
d(\xi_\ell, \xi_m) &\preceq bd(\xi_\ell, \xi_{\ell+1}) + b^2 d(\xi_{\ell+1}, \xi_{\ell+2}) + b^3 d(\xi_{\ell+2}, \xi_{\ell+3}) + \ldots \\
&\quad + b^{m-l} d(\xi_{m-1}, \xi_m).
\end{aligned}$$

Now, by using Equation 3.10. we have

$$\begin{aligned}
d(\xi_\ell, \xi_m) &\preceq bk^\ell d(\xi_0, \xi_1) + b^2 k^{\ell+1} d(\xi_0, \xi_1) + b^3 k^{\ell+2} d(\xi_0, \xi_1) + \ldots \\
&\quad + b^{m-\ell} k^{m-1} d(\xi_0, \xi_1), \\
&\preceq bk^\ell [1 + (bk) + (bk)^2 + (bk)^3 + \ldots + (bk)^{m-\ell-1}] d(\xi_0, \xi_1).
\end{aligned}$$

Using the geometric series we have

$$a + ar + ar^2 + ar^3 + \ldots + ar^{\ell-1} = a\left(\frac{1 - r^\ell}{1 - r}\right),$$

$$= bk^{\ell}d(\xi_0,\xi_1)\left[\frac{1-(bk)^{m-\ell}}{1-(bk)}\right],$$

Thus

$$d(\xi_\ell,\xi_m) \preceq \left[\frac{bk^{\ell}}{1-(bk)}\right]d(\xi_0,\xi_1),$$

$$d(\xi_\ell,\xi_m) \preceq bk^{\ell}(1-bk)^{-1}d(\xi_0,\xi_1).$$

As $\rho(k)<1$, and by Lemma 3.1.3, we have $\|k^{\ell}\|\to 0$ since $\ell \to\infty$. As a result for every $c\in\mathbb{A}$ with $\theta\ll c$ there exists $\ell_0\in\mathbb{N}$ such that

$$d(\xi_\ell,\xi_m) \preceq bk^{\ell}(1-bk)^{-1}d(\xi_0,\xi_1) \ll c,$$

for all $\ell>\ell_0$. So we conclude that $\{\xi_\ell\}$ is a Cauchy sequence and hence there exists $\xi^*\in M$ such that $\xi_\ell\to\xi^*$ as $\ell\to\infty$. Since $\{\xi_\ell\}$ is a sequence in M such that $\alpha(\xi_\ell,\xi_{\ell+1})\geq 1$ for all ℓ and $d(\xi_\ell,\xi^*)$ as $\ell\to\infty$ then by assumption, $\alpha(\xi_\ell,\xi^*)\geq 1$ for all ℓ. From Equation 3.1, we have

$$kd(\xi_\ell,\xi^*) \in \alpha(\xi_\ell,\xi^*)\ h(G\xi_\ell,G\xi^*) \quad \text{for all} \quad \ell\in\mathbb{N}.$$

As $\xi_{\ell+1}\in G\xi_\ell$ and by using Lemma 3.1.4, we have

$$kd(\xi_\ell,\xi^*) \in \alpha(\xi_\ell,\xi^*)\ h(\xi_{\ell+1},G\xi^*).$$

By using Definition 3.2.1, choose $\upsilon_\ell\in G\xi^*$ such that

$$kd(\xi_\ell,\xi^*) \in \alpha(\xi_\ell,\xi^*)\ h(d(\xi_{\ell+1},\upsilon_\ell)).$$

By using Lemma 3.1.4, we have

$$kd(\xi_\ell,\xi^*) \in \alpha(\xi_\ell,\xi^*)\ h(d(\xi_{\ell+1},\upsilon_\ell)) \subseteq h(\alpha(\xi_\ell,\xi^*)d(\xi_{\ell+1},\upsilon_\ell)).$$

Thus

$$\alpha(\xi_\ell,\xi^*)d(\xi_{\ell+1},\upsilon_\ell) \preceq kd(\xi_\ell,\xi^*).$$

As $\alpha(\xi_\ell,\xi^*)\geq 1$ for all $\ell\in\mathbb{N}$, so

$$0 \prec d(\xi_{\ell+1},\upsilon_\ell) \preceq \alpha(\xi_\ell,\xi^*)\, d(\xi_{\ell+1},\upsilon_\ell) \preceq kd(\xi_\ell,\xi^*) \prec d(\xi_\ell,\xi^*),$$

Since $\xi_\ell\to\xi^*$ as $\ell\to+\infty$, so for $c\in Int\omega$, and also there exists $k\in\mathbb{N}$ such that $d(\xi_\ell,\xi^*)\ll\frac{c}{2b}$ and $d(\xi_{\ell+1},\xi^*)\ll\frac{c}{2b}$ for $k(c)=k\leq\ell$. Now by triangular inequality, we have

$$\begin{aligned}
d(\xi^*,\upsilon_\ell) &\preceq bd(\xi^*,\xi_{\ell+1}) + bd(\xi_{\ell+1},\upsilon_\ell),\\
&\preceq bd(\xi^*,\xi_{\ell+1}) + b\alpha(\xi_\ell,\xi^*)d(\xi_{\ell+1},\upsilon_\ell),\\
&\preceq bd(\xi^*,\xi_{\ell+1}) + bkd(\xi_\ell,\xi^*),\\
&\preceq bd(\xi^*,\xi_{\ell+1}) + bd(\xi_\ell,\xi^*),\\
&\ll b\left(\frac{c}{2b}\right) + b\left(\frac{c}{2b}\right) \qquad \text{for} \quad k(c) = k\leq\ell,\\
&\ll c, \qquad \text{for} \quad k(c) = k\leq\ell,\\
d(\xi^*,\upsilon_\ell) &\ll c.
\end{aligned}$$

Thus, $\lim_{\ell\to\infty}\upsilon_\ell=\xi^*$. In view of the fact that $G\xi^*$ is closed, $\xi^*\in G\xi^*$. This shows ξ^* is a fixed point of G. This completes the proof. □

Corollary. Let a complete cone metric space (M,d) over a Banach algebra $\mathbb{A}$ and ω be the underlying solid cone. Let G is α-generalized multivalued contraction and α-admissible such that there exist $\xi_0\in M$, $\xi_1\in G\xi_0$ and $\alpha(\xi_0,\xi_1)\geq 1$. If $\{\xi_\ell\}$ is a sequence in M as $\alpha(\xi_\ell,\xi_{\ell+1})\geq 1$ for all $\ell\in\mathbb{N}$ and $\xi_\ell\to\xi^*$ such that $\ell\to+\infty$ then $\alpha(\xi_\ell,\xi^*)\geq 1$ for all $\ell\in\mathbb{N}$, then there exist a point ξ^* in M such that $\xi^*\in G\xi^*$.

Proof. Proof follows immediately by taking $b=1$ in Theorem (3.2.1). □

Corollary. Let a complete cone b-metric space (M,d) over a Banach algebra $\mathbb{A}$ with a real number $b\geq 1$ and ω be the underlying solid cone. Let $G:M\to C(M)$

be a multivalued mapping and satisfies

$$kd(\xi,\psi) \ \in \ h(G\xi,G\psi) \qquad \text{for all} \quad \xi,\psi \ \in \ M.$$

Then, there exists a point $\xi^* \in M$ such that $\xi^* \in G\xi^*$.

Proof. Proof follows immediately by taking $\alpha(\xi,\psi) = 1$ in Theorem (3.2.1). □

Corollary. Let a complete b-metric space (M,d), $\alpha : M \times M \to [0,+\infty)$ be a function and $G : M \to CB(M)$ is α- admissible. If there exist $k \in [0,1)$ such that

$$\alpha(\xi,\psi)H(G\xi,G\psi) \ \leq \ kd(\xi,\psi) \qquad \text{for all } \xi,\psi \in M.$$

Let there exist $\xi_0 \in M$ as $\alpha(\xi_0,G\xi_0) \geq 1$. Assuming that if $\{\xi_\ell\}$ is a sequence in M such that $\alpha(\xi_\ell,G\xi_{\ell+1}) \geq 1$ for all ℓ and $\xi_\ell \to u$ as $\ell \to +\infty$ then $\alpha(\xi_\ell,u) \geq 1$ for all ℓ. Then there exist ξ^* in M such that $\xi^* \in G\xi^*$.

Corollary. Let a complete b-metric space (M,d) and $G : M \to CB(M)$ be a multivalued mapping. If there exists a constant $k \in [0,1)$ such that

$$H(G\xi,G\psi) \leq kd(\xi,\psi) \qquad \text{for all} \quad \xi,\psi \in M,$$

then there exist a point ξ^* contained in M such that $\xi^* \in G\xi^*$.

Rremark 3.4.[35] “Taking $b = 1$, in above results we can get various fixed point theorems in cone metric spaces and metric spaces including Nadler’s fixed point theorem.”

The following example is taken from [48]

Example 3.2.1. Consider $\mathbb{A} = \mathbb{R}^2$ with the norm

$$\|(\xi_1,\xi_2)\| \ = \ \sqrt{\xi_1^2 \ + \ \xi_2^2}.$$

Define the multiplication on $\mathbb{A}$ by $\xi\psi = (\xi_1\psi_1, \xi_1\psi_2 + \xi_2\psi_1)$ for all $\xi = (\xi_1,\xi_2)$, $\psi = (\psi_1,\psi_2) \in \mathbb{A}$. Then $\mathbb{A}$ is a Banach algebra with unit $e = (0,1)$.

Consider $\omega = \{(\xi_1, \xi_2) \in \mathbb{A} \mid \xi_1, \xi_2 \geq 0\}$. Then ω be a solid cone which is non normal. Consider $M = \mathbb{R}$ and define the mapping of cone metric $d : M \times M \to \mathbb{A}$ by

$$d(\xi, \psi) \ = \ (|\xi \ - \ \psi|^2, 0).$$

As $M = \mathbb{R}$ being the set of real numbers with above norm is complete. Let us define a mapping $G : M \to 2^M$ by

$$G\xi \ = \ \left[0, \frac{\xi}{10}\right] \qquad \text{for all} \quad \xi \ \in \ M,$$

and also we have

$$\alpha(\xi, \psi) \ = \ \begin{cases} 1 & \text{if} \quad \xi \ \neq \ \psi, \\ 0, & \text{otherwise,} \end{cases}$$

From the above expression it is clear that G is α-admissible. Now if we choose $k = (\frac{1}{9}, 0)$, then there must be $\rho(k) = \frac{1}{9} < 1$. We observe that the condition of the theorem for $\xi = \psi$ holds trivially, for $\xi < \psi$ we have

$$h(G\xi, G\psi) \ = \ h\left(\left|\frac{\xi}{10} - \frac{\psi}{10}\right|^2, 0\right),$$

and

$$G\xi \ = \left[0, \frac{\xi}{10}\right],$$

$$G\psi \ = \left[0, \frac{\psi}{10}\right],$$

also

$$d(\xi, \psi) \ = \ h\left(\left|\xi - \psi\right|^2, 0\right),$$

This implies that

$$\begin{aligned} \left(\frac{1}{9}, 0\right)\left(\left|\xi - \psi\right|^2, 0\right) \ &= \left(\frac{1}{9}|\xi - \psi|^2, 0\right), \\ &\succeq \frac{1}{100}\left(|\xi - \psi|^2, 0\right), \end{aligned}$$

$$= \left(\left| \frac{\xi}{10} - \frac{\psi}{10} \right|^2, 0 \right),$$

$$\left(\frac{1}{9}, 0\right)\left(|\xi - \psi|^2, 0\right) \in \left(\left| \frac{\xi}{10} - \frac{\psi}{10} \right|^2, 0 \right).$$

This shows that

$$kd(\xi, \psi) \in \alpha(\xi, \psi)\, h(G\xi, G\psi),$$

So this shows that G is α-generalized multivalued contraction on M. Hence all the axioms of main theorem are fulfilled and 0 is the fixed point of G.

Application

In 2008, Jachymski [63] established certain fixed point results in metric spaces endowed with a graph and generalized at the same time Banach contraction principle from metric and partially ordered metric spaces. Consistent with Jachymski, suppose that (M, d) be a metric space and Δ designate the diagonal of the Cartesian product $M \times M$. Suppose that a directed graph D such as the set $\mathbf{V}(D)$ of its vertices coincides with M and the set $E(D)$ of its edges holds all loops, that is, $\Delta \subseteq E(D)$. Also assuming that the graph D has no equivalent circumferences and consequently, one can distinguish D with the set $[\mathbf{V}(D), E(D)]$. Further more, we may present D as a weighted graph [63] by assigning to every edge the distance between its vertices. If ξ and ψ are vertices in a graph D, then a path in D from ξ to ψ of length ν ($\nu \in \mathbb{N}$) is a sequence $\{\xi_i\}_{i=0}^{\nu}$ of $\nu + 1$ vertices such that $\xi_0 = \xi$, $\xi_\nu = \psi$ and $(\xi_{\ell-1}, \xi_\ell) \in E(D)$ for every $i = 1, 2, 3, ..., \nu$.

Definition 3.2.3. [63](**Banach G-contraction**)
A mapping $G : M \to M$ is said to be a Banach G-contraction if

(i) G preserves edges of D; for each $\xi, \psi \in M$ with $(\xi, \psi) \in E(D)$, we have

$$(G\xi, G\psi) \in E(D);$$

(ii) G decreases weights of edges of D if there exist $\alpha \in [0, 1)$ such that for all $\xi, \psi \in M$ with $(\xi, \psi) \in E(D)$, we have

$$d(G\xi, G\psi) \leq \alpha d(\xi, \psi).$$

Example 3.2.2. [64]

"Let $M = (0, 1)$ and d be the usual metric defined as

$$d(\xi_1, \xi_2) = |\xi_1 - \xi_2|.$$

let a graph D with

$$\mathbf{V}(D) = M,$$

and

$$E(D) = \{(\xi, \xi) : \xi \in M\} \cup \{\left(\frac{1}{2n}, \frac{1}{2^{n+1}}\right)\} : n \in \mathbb{N},$$

then $G : M \to M$ defined as

$$G\xi = \frac{\xi}{2},$$

is a Banach G-contraction."

Definition 3.2.4. (**Orbitally Continuous**)

A mapping $G : M \to M$ is known as orbitally continuous if $\xi, \psi \in M$ and $\{\kappa_\ell\}_{\ell \in \mathbb{N}}$ be any sequence of positive integers

$$G^{\kappa_\ell}\xi \to \psi \quad \Rightarrow \quad G(G^{\frac{\kappa_\ell \kappa}{\xi}}) \to G\psi \quad \text{as} \quad \ell \to \infty.$$

Definition 3.2.5. [64](**G-Continuous**)

"A self map G on M is called G-Continuous if each $\xi \in M$ there exist a sequence $\{\xi_\ell\}_{\ell \in \mathbb{N}}$, such that
$\{\xi_\ell\}$ converges to ξ and $(\xi_\ell, \xi_{\ell+1})$ belongs to $E(D)$ for $\ell \in \mathbb{N} \Rightarrow G\xi_\ell = G\xi$."

It is observes that the graph D is connected if there be a connected path in the middle of any two vertices and is weakly connected if $\tilde{D}$ is connected, here $\tilde{D}$ designates uncontrolled graph acquires from D by neglecting the direction of edges. The graph acquired from D indicates as a D^{-1} by reversing the way of edges. Thus

$$\mathbf{V}(D^{-1}) = \mathbf{V}(D) \text{ and } E(D^{-1}) = \{(\xi,\psi) \in M \times M : (\psi,\xi) \in E(D)\}.$$

To a greater extent, it is appropriate to handle $\tilde{D}$ as a directed graph such as the set of its edges is symmetric, due to this we have

$$E(\tilde{D}) = E(D) \cup E(D^{-1}).$$

As a subgraph of D we mean a graph ß which fulfils $\mathbf{V}(ß) \subseteq \mathbf{V}(D)$ and $E(ß) \subseteq E(D)$ as $\mathbf{V}(ß)$ contains the vertices of all edges of $(Eß)$. When $E(D)$ is symmetric, then for $\xi \in \mathbf{V}(D)$, at that time the subgraph D_ξ containing all edges and vertices that are contained in some path in D starting at ξ is said to be the component of D incorporating ξ. In this situation, $\mathbf{V}(D_\xi) = [\xi]_D$, here $[\xi]_D$ indicates the equivalence class of R defined on $\mathbf{V}(D)$ by the method:

$$\psi R\delta \text{ when there is a way in D from } \psi \text{ to } \delta.$$

Clearly D_ξ is connected for all $\xi \in D$. We indicate by $\vartheta = \{D : D$ is a directed graph with $\mathbf{V}(D) = M$ and $\Delta \subseteq E(D)\}$. In this part, we present the existence of fixed point theorems for multivalued mappings in a cone metric space over Banach algebra $\mathbb{A}$ endowed with graph under the generalized Hausdorff distance. Before introducing our consequences, we will establish new definitions in extended cone b-metric space over Banach algebra $\mathbb{A}$ endowed with a graph.

Definition 3.2.6. (**Weakly Preserves Edges of** G)

Let (M,d) be an extended cone b-metric space over Banach algebra $\mathbb{A}$ with a real number $b \geq 1$ and ω be the underlying solid cone endowed with a graph G. Then $G : M \to C(M)$ is multivalued mapping and weakly preserves edges of D if for each $\xi \in M$, and $\psi \in G(\xi)$ with $(\xi,\psi) \in E(D)$ as $(\psi,\delta) \in E(D)$ for all $\delta \in G(\psi)$.

Definition 3.2.7. (**Generalized Multivalued G-contraction**)
Let (M, d) be an extended cone b-metric space over Banach algebra $\mathbb{A}$ with a real number $b \geq 1$ and ω be the underlying solid cone endowed with a graph G. Then $G : M \to C(M)$ is multivalued mapping and is called generalized multivalued G-contraction if there exist $k \in \omega$, such that $\rho(k) < 1$ and

$$kd(\xi, \psi) \in h(G\xi, G\psi) \qquad\qquad \xi, \psi \in M. \tag{3.11}$$

Example 3.2.3. Any operator $G : M \to C(M)$ stated as $G(\xi) = \{h\}$ where $h \in M$ is a multivalued G-contraction for any graph D with $\mathbf{V}(D) = M$.

Theorem 3.2.2. "Let (M, d) be a complete cone b-space over a Banach algebra $\mathbb{A}$ with the coefficient $b \geq 1$ and ω be the underlying solid cone endowed with a graph D. Suppose $G : M \to C(M)$ is generalized multivalued G-contraction and satisfy the following conditions:

(I) G weakly preserves edges of D;

(ii) there exist $\xi_0 \in M$ and $\xi_1 \in G\xi_0$ such that $(\xi_0, \xi_1) \in E(D)$;

(iii) M has the G-regular property.

Then there exist a point $\xi^* \in M$ such that $\xi^* \in G\xi^*$, that is, G has a fixed point in M."

Chapter 4

Multivalued Fixed Point Theorems in Cone Extended b-Metric Spaces

In this chapter, by generalizing the cone b-metric space we introduced the notion of cone extended b-metric space over Banach algebra $\mathbb{A}$ and established a new fixed point theorems by extending the results of cone b-metric space into cone extended b-metric space over Banach algebra $\mathbb{A}$.

4.1 Cone Extended b-Metric Space

We begin first by defining cone extended b-metric space as follows:

Definition 4.1.1. (**Cone Extended b-Metric Space**)
Let M be a nonempty set and $\mathbb{A}$ be a Banach algebra with $\preceq$ as partial order on $\mathbb{A}$ and $\theta : M \times M \to [1, \infty)$. Suppose that the mapping $d_\theta : M \times M \to \mathbb{A}$ is called cone extended b-metric if for all $\xi, \psi, \delta \in M$, it satisfies

1. $0_{\mathbb{A}} \prec d_\theta(\xi, \psi)$ with $\xi \neq \psi$ and $d_\theta(\xi, \psi) = 0_{\mathbb{A}} \Leftrightarrow \xi = \psi$;

2. $d_\theta(\xi, \psi) = d_\theta(\psi, \xi)$;

3. $d_\theta(\xi,\psi) \preceq \theta(\xi,\psi)[d_\theta(\xi,\delta)+d_\theta(\delta,\psi)]$.

where $0_{\mathbb{A}}$ is zero element of Banach algebra $\mathbb{A}$, and (M,d_θ) is called a cone extended b-metric space over Banach algebra $\mathbb{A}$.

Remark 4.1.
If $\theta(\xi,\psi)=b$ where $b \geq 1$, then the above definition corresponds with the definition of cone b-metric space.

Example 4.1.1. Let $M=\{1,2,3\}$ and define $d_\theta : M \times M \to \mathbb{A}$. Let $\mathbb{A}=\mathbb{R}$ with the partial order defined by $\xi \preceq \psi$ if and only if $\xi \leq \psi$ for all $\xi,\psi \in \mathbb{R}$ and $d_\theta(\xi,\psi)=(\xi-\psi)^2$. It is well known that d_θ is a b-metric space. Let us define the mapping

$$\theta : M \times M \to [1,\ \infty), \quad \text{for all} \quad \xi,\psi \in M \quad \theta(\xi,\psi)=\xi+\psi+1.$$

Then it can be shown that (M,d_θ) is cone extended b-metric space.
From the $d_\theta(\xi,\psi)=(\xi-\psi)^2$, we have

$$d_\theta(1,1)=d_\theta(2,2)=d_\theta(3,3)=0,$$

$$d_\theta(1,2)=d_\theta(2,1)=d_\theta(2,3)=d_\theta(3,2)=1 \quad d_\theta(1,3)=d_\theta(3,1)=4.$$

From the $\theta(\xi,\psi)=\xi+\psi+1$, we have

$$\theta(1,2)=\theta(2,1)=4, \quad \theta(2,3)=\theta(3,2)=6, \quad \theta(1,3)=\theta(3,1)=5.$$

To show that (M,d_θ) is a cone extended b-metric space, note that the axioms (1) and (2) are trivially satisfied. Now for the last axiom (3) we have

$$d_\theta(\xi,\psi) \preceq \theta(\xi,\psi)[d_\theta(\xi,\delta)+d_\theta(\delta,\psi)] \quad \text{for all} \quad \xi,\psi,\delta \in M.$$

For $\xi=1$, $\psi=2$, $\delta=3$ the above inequality implies

$$d_\theta(1,2) \preceq \theta(1,2)[d_\theta(1,3)+d_\theta(3,2)] \quad \text{for all} \quad \xi,\psi,\delta \in M,$$

using the above values we have

$$\begin{aligned}
1 &\preceq\ 4[4+1] \quad \text{for all} \quad \xi,\psi,\delta \in M,\\
1 &\preceq\ 20 \quad \text{for all} \quad \xi,\psi,\delta \in M,
\end{aligned}$$

also

$$\begin{aligned}
d_\theta(1,3) &\preceq\ \theta(1,3)[d_\theta(1,2)+d_\theta(2,3)] \quad \text{for all} \quad \xi,\psi,\delta \in M,\\
4 &\preceq\ 5[1+1] \quad \text{for all} \quad \xi,\psi,\delta \in M,\\
4 &\preceq\ 10 \quad \text{for all} \quad \xi,\psi,\delta \in M.
\end{aligned}$$

Similarly for $d_\theta(2,3)$ and $d_\theta(1,3)$ we have

$$\begin{aligned}
d_\theta(2,3) &\preceq \theta(2,3)[d_\theta(2,1)+d_\theta(1,3)] \quad \text{for all} \quad \xi,\psi,\delta \in M,\\
1 &\preceq\ 6[1+4] \quad \text{for all} \quad \xi,\psi,\delta \in M,\\
1 &\preceq\ 30 \quad \text{for all} \quad \xi,\psi,\delta \in M,
\end{aligned}$$

and also

$$\begin{aligned}
d_\theta(1,3) &\preceq \theta(1,3)[d_\theta(1,2)+d_\theta(2,3)] \quad \text{for all} \quad \xi,\psi,\delta \in M,\\
4 &\preceq\ 5[1+1] \quad \text{for all} \quad \xi,\psi,\delta \in M,\\
4 &\preceq\ 10 \quad \text{for all} \quad \xi,\psi,\delta \in M.
\end{aligned}$$

Hence for all $\xi,\psi,\delta \in M$,

$$d_\theta(\xi,\psi) \preceq \theta(\xi,\psi)[d_\theta(\xi,\delta)+d_\theta(\delta,\psi)].$$

Hence (M,d_θ) is a cone extended b-metric space.

Definition 4.1.2. (**Interior Point, Closed and Bounded Sets**)
Let (M,d_θ) be a cone extended b-metric space and $Q \subseteq M$

(i) $q \in Q$ is said to be an interior point of Q if there is $0 \ll a$ such that $Q_0(q,a) \subseteq Q$, where $Q_0(q,a) = \{\psi \in M : d_\theta(\psi,q) \ll a\}$.

(ii) An element $\xi \in M$ is said to be a limit point of Q if for every $0 \ll e, Q_0(\xi, e)\backslash(Q\backslash\{\xi\}) \neq \phi$. A subset $Q \subseteq M$ is called closed if each limit point of Q belongs to Q.

(iii) A subset $Q \subseteq M$ is said to be bounded if there exist $0 \ll c$ and $\xi_0 \in M$ such that $d_\theta(q, \xi_0) \ll c$ for all $q \in Q$.

Now we define the notions of convergent sequence, Cauchy sequence and completeness in the setting of cone extended b-metric space over Banach algebra $\mathbb{A}$.

Definition 4.1.3. Suppose that (M, d_θ) be a cone extended b-metric space over Banach algebra $\mathbb{A}$, $\xi \in M$, consider a sequence $\{\xi_\ell\}$ in M. Then

(i) The sequence $\{\xi_\ell\}$ converges to ξ if for every $\epsilon \in \mathbb{A}$ with $0_{\mathbb{A}} \preceq \epsilon$ there exist ℓ_0 such that

$$d_\theta(\xi_\ell,\ \xi) \preceq \epsilon \qquad\qquad \text{for all} \quad \ell \geq \ell_0.$$

(ii) The sequence $\{\xi_\ell\}$ in M is said to be Cauchy if for every $\epsilon \in \mathbb{A}$ with $0_{\mathbb{A}} \preceq \epsilon$ there exist ℓ_0 such that

$$d_\theta(\xi_\ell,\ \xi_m) \preceq \epsilon \qquad\qquad \text{for all} \quad \ell, m \geq \ell_0.$$

(iii) The space (M, d_θ) is complete cone extended b-metric if every Cauchy sequence converges in M.

Following the notion of Hausedorff distance function on cone b-metric space over Banach algebra $\mathbb{A}$, for a cone extended b-metric space (M, d_θ) over Banach algebra $\mathbb{A}$, denote

$$C(M) \;=\; \{P \;:\; P \text{ is a nonempty compact subset of } M\}$$

$$N(M) \;=\; \{P \;:\; P \text{ is a nonempty subset of } M\}$$

$$h(t_1) = \{t_2 \in \mathbb{E} : t_1 \preceq t_2\} \qquad \text{for} \;\; t_1 \in \mathbb{A}.$$

For $r_1 \in M$ and $Q \in N(M)$.

$$h(r_1, Q) = \bigcup_{r_2 \in Q} h(d_\theta(r_1, r_2)) = \bigcup_{r_2 \in Q} \{\xi \in \mathbb{A} : d_\theta(r_1, r_2) \preceq \xi\}$$

For $P, Q \in N(M)$ we denote

$$h(P,Q) = \left(\bigcap_{r_1 \in P} h(r_1, Q)\right) \bigcap \left(\bigcap_{r_2 \in Q} h(r_2, P)\right).$$

Lemma 4.1.1. Let (M, d_θ) be a cone extended b-metric space over Banach algebra $\mathbb{A}$ with $\theta : M \times M \to [1,\ \infty)$ and let ω be a solid cone.

(1) Let $r_1, r_2 \in \mathbb{A}$. Whenever $r_1 \preceq r_2$ then $h(r_2) \subset h(r_1)$.

(2) Let $\xi \in M$ and $A \in N(M)$. Whenever $\theta \in h(\xi, A)$, then $\xi \in A$.

(3) Let $r_2 \in \omega$ and let $P, Q \in CB(M)$ and $p \in P$. Whenever $r_2 \in h(P,Q)$, then $r_2 \in h(p,Q)$ for all $p \in P$ or $r_2 \in h(P,q)$ for all $q \in Q$.

(4) Let $r_2 \in \omega$ and consider $\gamma \geq 0$, then $\gamma\ h(r_2) \subseteq h(\gamma r_2)$.

Remark 4.2. Let (M, d_θ) be a cone extended b-metric space over Banach algebra $\mathbb{A}$ with real number $\theta(\xi,\psi) \geq 1$. If $\mathbb{A} = \mathbb{R}$ and $\omega = [0, +\infty)$, then (M, d_θ) is a metric space. Furthermore, define $H : CB(M) \times CB(M) \to \mathbb{A}$ for $P, Q \in CB(M)$, $H(P,Q) = \inf h(P,Q)$ is the Housdorff distance induced by d_θ.

Remark 4.3. For multivalued mapping $\theta : C(M) \times C(M) \to [1,\ \infty)$ also $\theta(P,Q) = \sup\{\theta(p,q), p \in P, q \in Q\}$.

Following the notion of α-generalized multivalued contraction for cone extended b-metric spaces over Banach algebra $\mathbb{A}$.

Definition 4.1.4. (**α-generalized multivalued contraction**)

Let (M, d_θ) be a cone extended b-metric space over a Banach algebra $\mathbb{A}$ with $\theta : M \times M \to [1,\ \infty)$ and ω be the suppressed solid cone. Then the multivalued mapping $G : M \to C(M)$ is said to be α-generalized multivalued contraction if there exist $\alpha : M \times M \to [0, +\infty)$ and $k \in \omega$ and $\rho(k) < 1$ and

$$k d_\theta(\xi,\psi) \in \alpha(\xi,\psi) h(G\xi, G\psi) \qquad \text{for all } \xi, \psi \in M. \tag{4.1}$$

Theorem 4.1.2. Consider (M, d_θ) be a complete cone extended b-metric space over a Banach algebra $\mathbb{A}$ with $\theta : M \times M \to [1,\ \infty)$ where $\lim_{\ell,m\to\infty} \theta(\xi_\ell, \xi_m) < \frac{1}{\alpha}$

and ω be the underlying solid cone. Let G is α-admissible, α-generalized multi-valued contraction and there exist $\xi_0 \in M$, $\xi_1 \in G\xi_0$ such that $\alpha(\xi_0, \xi_1) \geq 1$. If $\{\xi_\ell\}$ is a sequence in M such that $\alpha(\xi_\ell, \xi_{\ell+1}) \geq 1$ for all ℓ and $(\xi_\ell \to \xi^*)$ as $\ell \to \infty$ then $\alpha(\xi_\ell, \xi^*) \geq 1$ for all ℓ. Then, G has a fixed point in M.

Proof. Suppose that ξ_0 is an arbitrary point in M and $G\xi_0 \in C(M)$. Since $\xi_1 \in G\xi_0$. Then by Definition 4.1.4. of α-generalized multivalued contraction we have

$$kd_\theta(\xi_0, \xi_1) \ \in \ \alpha(\xi_0, \xi_1)\ h(G\xi_0, G\xi_1).$$

Since $\xi_1 \in G\xi_0$, by using Lemma 4.1.1, we have

$$kd_\theta(\xi_0, \xi_1) \ \in \ \alpha(\xi_0, \xi_1)\ h(\xi_1, G\xi_1).$$

By definition, we can take $\xi_2 \in G\xi_1$ such that

$$kd_\theta(\xi_0, \xi_1) \ \in \ \alpha(\xi_0, \xi_1)\ h(d_\theta(\xi_1, \xi_2)).$$

By using Lemma 4.1.1, we have

$$kd_\theta(\xi_0, \xi_1) \ \in \ \alpha(\xi_0, \xi_1)\ h(d_\theta(\xi_1, \xi_2)) \ \subseteq \ h(\alpha(\xi_0, \xi_1)d_\theta(\xi_1, \xi_2)).$$

So

$$\alpha(\xi_0, \xi_1)\ d_\theta(\xi_1, \xi_2) \ \preceq \ kd_\theta(\xi_0, \xi_1). \tag{4.2}$$

Now, as $\alpha(\xi_0, \xi_1) \geq 1$, then it follows from Equation 4.2.

$$0 \ \prec \ d_\theta(\xi_1, \xi_2) \ \preceq \ \alpha(\xi_0, \xi_1)\ d_\theta(\xi_1, \xi_2) \ \preceq \ kd_\theta(\xi_0, \xi_1). \tag{4.3}$$

For $\xi_1 = \xi_2$, there is nothing to prove because in this case ξ_1 is our required fixed point. Therefore we assume that $\xi_1 \neq \xi_2$, then $\xi_2 \notin G\xi_2$ and so it follows from Definition 4.1.4.

$$kd_\theta(\xi_1, \xi_2) \ \in \ \alpha(\xi_1, \xi_2)\ h(G\xi_1, G\xi_2),$$

Since $\xi_2 \in G\xi_1$, by using Lemma 4.1.1, we have

$$kd_\theta(\xi_1, \xi_2) \in \alpha(\xi_1, \xi_2)\ h(\xi_2, G\xi_2).$$

By definition, we can take $\xi_3 \in G\xi_2$ such that

$$kd_\theta(\xi_1, \xi_2) \in \alpha(\xi_1, \xi_2)\ h(d_\theta(\xi_2, \xi_3)).$$

Again by using Lemma 4.1.1.

$$kd_\theta(\xi_1, \xi_2) \in \alpha(\xi_1, \xi_2)\ h(d_\theta(\xi_2, \xi_3)) \subseteq h(\alpha(\xi_1, \xi_2) d_\theta(\xi_2, \xi_3)).$$

So

$$\alpha(\xi_1, \xi_2)\ d_\theta(\xi_2, \xi_3) \preceq kd_\theta(\xi_1, \xi_2). \tag{4.4}$$

As G is α-admissible on M, since $\alpha(\xi_1, \xi_2) \geq 1$ where $\xi_1 \in G\xi_0$ and $\xi_2 \in G\xi_1$. So from Equation 4.4, we have

$$0 \prec d_\theta(\xi_2, \xi_3) \preceq \alpha(\xi_1, \xi_2)\ d_\theta(\xi_2, \xi_3) \preceq kd_\theta(\xi_1, \xi_2). \tag{4.5}$$

For $\xi_2 = \xi_3$, there is nothing to prove because in this case ξ_2 is our required fixed point. Hence we assume that $\xi_2 \neq \xi_3$, then $\xi_3 \notin G\xi_3$ and it follows from Definition 4.1.4.

$$kd_\theta(\xi_2, \xi_3) \in \alpha(\xi_2, \xi_3)\ h(G\xi_2, G\xi_3).$$

Since $\xi_3 \in G\xi_2$, by using Lemma 4.1.1, we have

$$kd_\theta(\xi_2, \xi_3) \in \alpha(\xi_2, \xi_3)\ h(\xi_3, G\xi_3).$$

By definition, we can take $\xi_4 \in G\xi_3$ such that

$$kd_\theta(\xi_2, \xi_3) \in \alpha(\xi_2, \xi_3)\ h(d_\theta(\xi_3, \xi_4)).$$

Again by using Lemma 4.1.1, we have

$$kd_\theta(\xi_2, \xi_3) \;\in\; \alpha(\xi_2, \xi_3)\; h(d_\theta(\xi_3, \xi_4)) \;\subseteq\; h(\alpha(\xi_2, \xi_3) d_\theta(\xi_3, \xi_4)).$$

So

$$\alpha(\xi_2, \xi_3)\; d_\theta(\xi_3, \xi_4) \;\preceq\; kd_\theta(\xi_2, \xi_3). \tag{4.6}$$

As G is α-admissible on M, since $\alpha(\xi_1, \xi_2) \geq 1$ where $\xi_1 \in G\xi_0$ and $\xi_2 \in G\xi_1$. Thus from Equation 4.6, we have

$$0 \;\prec\; d_\theta(\xi_3, \xi_4) \;\preceq\; \alpha(\xi_2, \xi_3)\; d_\theta(\xi_3, \xi_4) \;\preceq\; kd_\theta(\xi_2, \xi_3). \tag{4.7}$$

For $\xi_3 = \xi_4$, there is nothing to prove because in this case ξ_3 is our required fixed point. Hence we assume that $\xi_3 \neq \xi_4$, then $\xi_4 \notin G\xi_4$ and so it follows from Definition 4.1.4.

$$kd_\theta(\xi_3, \xi_4) \;\in\; \alpha(\xi_3, \xi_4)\; h(G\xi_3, G\xi_4).$$

Since $\xi_4 \in G\xi_3$, by using Lemma 4.1.1, we have

$$kd_\theta(x\xi_3, \xi_4) \;\in\; \alpha(\xi_3, \xi_4)\; h(\xi_4, G\xi_4).$$

By definition, we can take $\xi_5 \in G\xi_4$ such that

$$kd_\theta(\xi_3, \xi_4) \;\in\; \alpha(\xi_3, \xi_4)\; h(d_\theta(\xi_4, \xi_5)).$$

Again by using Lemma 4.1.1, such that

$$kd_\theta(\xi_3, \xi_4) \;\in\; \alpha(\xi_3, \xi_4)\; h(d_\theta(\xi_4, \xi_5)) \;\subseteq\; h(\alpha(\xi_3, \xi_4) d_\theta(\xi_4, \xi_5)).$$

So

$$\alpha(\xi_3, \xi_4)\; d_\theta(\xi_4, \xi_5) \;\preceq\; kd_\theta(\xi_3, \xi_4). \tag{4.8}$$

As G is α-admissible on M, since $\alpha(\xi_1, \xi_2) \geq 1$ where $\xi_1 \in G\xi_0$ and $\xi_2 \in G\xi_1$. So from Equation 4.8, we have

$$0 \prec d_\theta(\xi_4, \xi_5) \preceq \alpha(\xi_3, \xi_4)\, d_\theta(\xi_4, \xi_5) \preceq k d_\theta(\xi_3, \xi_4). \tag{4.9}$$

Similarly by adopting the same procedure, we acquire a sequence $\{\xi_\ell\}$ in M such that $\xi_{\ell+1} \in G\xi_\ell$, $\xi_{\ell+1} \neq \xi_\ell$, $\alpha(\xi_\ell, \xi_{\ell+1}) \geq 1$ and

$$\begin{aligned}
d_\theta(\xi_\ell, \xi_{\ell+1}) &\preceq k d_\theta(\xi_{\ell-1}, \xi_\ell),\\
&\preceq k.k d_\theta(\xi_{\ell-2}, \xi_{\ell-1}),\\
&= k^2 d_\theta(\xi_{\ell-2}, \xi_{\ell-1}),\\
&\preceq k^2.k d_\theta(\xi_{\ell-3}, \xi_{\ell-2}),\\
&= k^3 d_\theta(\xi_{\ell-3}, \xi_{\ell-2}) \ \ldots \ \preceq k^\ell d_\theta(\xi_0, \xi_1).
\end{aligned}$$

This implies that

$$d_\theta(\xi_\ell, \xi_{\ell+1}) \preceq k^\ell d_\theta(\xi_0, \xi_1) \qquad \text{for all} \ \ \ell. \tag{4.10}$$

Assuming that $\ell < m$, by triangular inequality, we have

$$\begin{aligned}
d_\theta(\xi_\ell, \xi_m) &\preceq \theta(\xi_\ell, \xi_m)[d_\theta(\xi_\ell, \xi_{\ell+1}) + d_\theta(\xi_{\ell+1}, \xi_m)],\\
&= \theta(\xi_\ell, \xi_m) d_\theta(\xi_\ell, \xi_{\ell+1}) + \theta(\xi_\ell, \xi_m) d_\theta(\xi_{\ell+1}, \xi_m),\\
&\preceq \theta(\xi_\ell, \xi_m) d_\theta(\xi_\ell, \xi_{\ell+1}) +\\
&\quad\ \theta(\xi_\ell, \xi_m)\theta(\xi_{\ell+1}, \xi_m)[d_\theta(\xi_{\ell+1}, \xi_{\ell+2}) + d_\theta(\xi_{\ell+2}, \xi_m)],\\
&= \theta(\xi_\ell, \xi_m) d_\theta(\xi_\ell, \xi_{\ell+1}) + \theta(\xi_\ell, \xi_m)\theta(\xi_{\ell+1}, \xi_m) d_\theta(\xi_{\ell+1}, \xi_{\ell+2}) +\\
&\quad\ \theta(\xi_\ell, \xi_m)\theta(\xi_{\ell+1}, \xi_m) d(\xi_{\ell+2}, \xi_m),
\end{aligned}$$

$$
\begin{aligned}
&\preceq \theta(\xi_\ell, \xi_m)d_\theta(\xi_\ell, \xi_{\ell+1}) \ + \ \theta(\xi_\ell, \xi_m)\theta(\xi_{\ell+1}, \xi_m)d_\theta(\xi_{\ell+1}, \xi_{\ell+2}) + \\
&\quad \theta(\xi_\ell, \xi_m)\theta(\xi_{\ell+1}, \xi_m)\theta(\xi_{\ell+2}, \xi_m)[d_\theta(\xi_{\ell+2}, \xi_{\ell+3}) \ + \ d_\theta(\xi_{\ell+3}, \xi_m)], \\
&= \theta(\xi_\ell, \xi_m)d_\theta(\xi_\ell, \xi_{\ell+1}) \ + \ \theta(\xi_\ell, \xi_m)\theta(\xi_{\ell+1}, \xi_m)d_\theta(\xi_{\ell+1}, \xi_{\ell+2}) + \\
&\quad \theta(\xi_\ell, \xi_m)\theta(\xi_{\ell+1}, \xi_m)\theta(\xi_{\ell+2}, \xi_m)d_\theta(\xi_{\ell+2}, \xi_{\ell+3}) + \\
&\quad \theta(\xi_\ell, \xi_m)\theta(\xi_{\ell+1}, \xi_m)\theta(\xi_{\ell+2}, \xi_m)d_\theta(\xi_{\ell+3}, \xi_m)],
\end{aligned}
$$

$$
\begin{aligned}
d_\theta(\xi_\ell, \xi_m) &\preceq \theta(\xi_\ell, \xi_m)d_\theta(\xi_\ell, \xi_{\ell+1}) \ + \ \theta(\xi_\ell, \xi_m)\theta(\xi_{\ell+1}, \xi_m)d_\theta(\xi_{\ell+1}, \xi_{\ell+2}) + \\
&\quad \theta(\xi_\ell, \xi_m)\theta(\xi_{\ell+1}, \xi_m)\theta(\xi_{\ell+2}, \xi_m)d_\theta(\xi_{\ell+2}, \xi_{\ell+3}) + \\
&\quad \theta(\xi_\ell, \xi_m)\theta(\xi_{\ell+1}, \xi_m)\theta(\xi_{\ell+2}, \xi_m)\theta(\xi_{\ell+2}, \xi_{\ell+3}) \ ... \\
&\quad \theta(\xi_{m-1}, \xi_m) \ d(\xi_{m-1}, \xi_m).
\end{aligned}
$$

Now, by using Equation 4.10. we have

$$
\begin{aligned}
d_\theta(\xi_\ell, \xi_m) &\preceq \theta(\xi_\ell, \xi_m)k^\ell d_\theta(\xi_0, \xi_1) \ + \ \theta(\xi_\ell, \xi_m)\theta(\xi_{\ell+1}, \xi_m)k^{\ell+1}d_\theta(\xi_0, \xi_1) \ + \ ... \\
&+ \ \theta(\xi_\ell, \xi_m)\theta(\xi_{\ell+1}, \xi_m)\theta(\xi_{\ell+2}, \xi_m) \ ... \\
&\quad \theta(\xi_{m-2}, \xi_m)\theta(\xi_{m-1}, \xi_m)k^{m-1}d_\theta(\xi_0, \xi_1), \\
&\preceq d_\theta(\xi_0, \xi_m)[\theta(\xi_1, \xi_m)\theta(\xi_2, \xi_m)\theta(\xi_3, \xi_m) \ ... \ \theta(\xi_{\ell-1}, \xi_m)\theta(\xi_\ell, \xi_m)k^\ell \ + \\
&\quad \theta(\xi_1, \xi_m)\theta(\xi_2, \xi_m)\theta(\xi_3, \xi_m) \ ... \ \theta(\xi_\ell, \xi_m)\theta(\xi_{\ell+1}, \xi_m)k^{\ell+1} \ + \\
&\quad \theta(\xi_1, \xi_m)\theta(\xi_2, \xi_m)\theta(\xi_3, \xi_m) \ ... \ \theta(\xi_{\ell+1}, \xi_m)\theta(\xi_{\ell+2}, \xi_m)k^{\ell+2} \ + \ ... \ + \\
&\quad \theta(\xi_1, \xi_m)\theta(\xi_2, \xi_m)\theta(\xi_3, \xi_m) \ ... \ \theta(\xi_{\ell+1}, \xi_m)\theta(\xi_{\ell+2}, \xi_m)k^{\ell+1} \ + \ ... \ + \\
&\quad \theta(\xi_{m-2}, \xi_m)\theta(\xi_{m-1}, \xi_m)k^{m-1}].
\end{aligned}
$$

As, $\lim_{\ell,m\to\infty}\theta(\xi_\ell,\xi_m) < \frac{1}{\alpha}$ hence the series $\sum_{\ell=1}^{\infty} k^\ell \prod_{i=1}^{\ell}\theta(\xi_i,\xi_m)$ converges by "d-Alembert's Test" for every $m \in \mathbb{N}$. Suppose that

$$U = \sum_{\ell=1}^{\infty} k^\ell \prod_{i=1}^{\ell} \theta(\xi_i,\xi_m), \quad V_\ell = \sum_{j=1}^{\ell} k^j \prod_{i=1}^{j} \theta(\xi_i,\xi_m),$$

Hence for $\ell < m$ the above triangular inequality implies

$$d_\theta(\xi_\ell,\xi_m) \preceq d_\theta(\xi_0,\xi_1)[U_{m-1} - V_{\ell-1}].$$

letting $\ell \to \infty$ we get $\{\xi_\ell\}$ is a Cauchy sequence. As M is complete

$$\Rightarrow \quad \xi_\ell \to \xi \in M,$$

Since $\{\xi_\ell\}$ is a sequence in M such that $\alpha(\xi_\ell,\xi_{\ell+1}) \geq 1$ for all ℓ and $d(\xi_\ell,\xi^*)$ as $\ell \to \infty$ then by assumption, $\alpha(\xi_\ell,\xi^*) \geq 1$ for all ℓ. From Equation 4.1, we have

$$kd_\theta(\xi_\ell,\xi^*) \in \alpha(\xi_\ell,\xi^*)\ h(G\xi_\ell,G\xi^*) \quad \text{for all} \quad \ell \in \mathbb{N}.$$

As $\xi_{\ell+1} \in G\xi_\ell$ and by using Lemma 4.1.1, we have

$$kd_\theta(\xi_\ell,\xi^*) \in \alpha(\xi_\ell,\xi^*)\ h(\xi_{\ell+1},G\xi^*).$$

By definition, we can take $\upsilon_\ell \in G\xi^*$ such that

$$kd_\theta(\xi_\ell,\xi^*) \in \alpha(\xi_\ell,\xi^*)\ h(d_\theta(\xi_{\ell+1},\upsilon_\ell)).$$

By using Lemma 4.1.1, we have

$$kd_\theta(\xi_\ell,\xi^*) \in \alpha(\xi_\ell,\xi^*)\ h(d_\theta(\xi_{\ell+1},\upsilon_\ell)) \subseteq h(\alpha(\xi_\ell,\xi^*)d_\theta(\xi_{\ell+1},\upsilon_\ell)).$$

Thus

$$\alpha(\xi_\ell,\xi^*)d_\theta(\xi_{\ell+1},\upsilon_\ell) \preceq kd_\theta(\xi_\ell,\xi^*).$$

As $\alpha(\xi_\ell, \xi^*) \geq 1$ for all $\ell \in \mathbb{N}$, so

$$0 \prec d_\theta(\xi_{\ell+1}, \upsilon_\ell) \preceq \alpha(\xi_\ell, \xi^*)\, d_\theta(\xi_{\ell+1}, \upsilon_\ell) \preceq k d_\theta(\xi_\ell, \xi^*) \prec d_\theta(\xi_\ell, \xi^*),$$

Since $\xi_\ell \to \xi^*$ as $\ell \to +\infty$, so for $c \in int\omega$, and also there exists $k \in \mathbb{N}$ such that $d_\theta(\xi_\ell, \xi^*) \ll \frac{c}{2\theta(\xi^*, \upsilon_\ell)}$ and $d_\theta(\xi_{\ell+1}, \xi^*) \ll \frac{c}{2\theta(\xi^*, \upsilon_\ell)}$ for $k(c) = k \leq \ell$. Now by using the triangular inequality, we get

$$\begin{aligned}
d_\theta(\xi^*, \upsilon_\ell) &\preceq \theta(\xi^*, \upsilon_\ell) d_\theta(\xi^*, \xi_{\ell+1}) + \theta(\xi^*, \upsilon_\ell) d_\theta(\xi_{\ell+1}, \upsilon_\ell),\\
&\preceq \theta(\xi^*, \upsilon_\ell) d_\theta(\xi^*, \xi_{\ell+1}) + \theta(\xi^*, \upsilon_\ell) \alpha(\xi_\ell, \xi^*) d_\theta(\xi_{\ell+1}, \upsilon_\ell),\\
&\preceq \theta(\xi^*, \upsilon_\ell) d_\theta(\xi^*, \xi_{\ell+1}) + \theta(\xi^*, \upsilon_\ell) k d_\theta(\xi_\ell, \xi^*),\\
&\preceq \theta(\xi^*, \upsilon_\ell) d_\theta(\xi^*, \xi_{\ell+1}) + \theta(\xi^*, \upsilon_\ell) d_\theta(\xi_\ell, \xi^*),\\
&\ll \theta(\xi^*, \upsilon_\ell). \frac{c}{2\theta(\xi^*, \upsilon_\ell)} + \theta(\xi^*, \upsilon_\ell). \frac{c}{2\theta(\xi^*, \upsilon_\ell)} \quad \text{for} \quad k(c) = k \leq \ell,\\
&\ll c, \qquad \text{for} \quad k(c) = k \leq \ell,\\
d_\theta(\xi^*, \upsilon_\ell) &\ll c.
\end{aligned}$$

Thus, $\lim_{\ell \to \infty} \upsilon_\ell = \xi^*$. In view of the fact that $G\xi^*$ is closed, $\xi^* \in G\xi^*$. This shows ξ^* is a fixed point of G. This completes the proof. □

Corollary. Consider (M, d_θ) be a complete cone extended b-metric space over a Banach algebra $\mathbb{A}$ with $\theta(\xi, \psi) \geq 1$ and ω be the underlying solid cone. Let $G : M \to C(M)$ be multivalued mapping and satisfies

$$k d_\theta(\xi_0, \xi_1) \in h(G\xi_0, G\xi_1) \qquad \text{for all} \quad \xi, \psi \in M,$$

then, there exists a point $\xi \in M$ such that $\xi \in G\xi$.

Proof. Proof follows from previous Theorem 4.1.2 by taking $\theta(\xi, \psi) = 1$. □

Chapter 5

Conclusion

We bring to an end this thesis as follows:

• The work of Kutbi et al. on "Multivalued fixed point theorems in cone b-metric spaces over Banach algebra $\mathbb{A}$ with application" is investigated in this thesis.

• The objective of this research is to study about cone extended b-metric spaces over Banach algebra $\mathbb{A}$ and introduced certain fixed point results of multivalued mappings in the setting of cone extended b-metric spaces.

• Our results generalize and extend different fixed point results of cone b-metric space.

• We have generalized some fixed point results established in the paper of Kutbi et al. by introducing the idea of cone extended b-metric spaces over Banach algebra $\mathbb{A}$.

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