CAPITAL UNIVERSITY OF SCIENCE AND TECHNOLOGY, ISLAMABAD

Multivalued Fractals in Cone b-Metric Spaces

by

Usman Shehzad

A thesis submitted in partial fulfillment for the degree of Master of Philosophy

in the

Faculty of Computing
Department of Mathematics

2020

I dedicate my dissertation work to

my beloved ***family*** *specially*

my mother*,*

my wife*,*

and

my loving sister*.*

A special feeling of gratitude is for

my father*,*

the most unswerving man,

I ever know in this world.

CERTIFICATE OF APPROVAL

Multivalued Fractals in Cone b-Metric Spaces

by

Usman Shehzad

(MMT183001)

THESIS EXAMINING COMMITTEE

S. No.	Examiner	Name	Organization
(a)	External Examiner	Dr. Muhammad Usman Ali	COMSATS Attock Campus
(b)	Internal Examiner	Dr. Muhammad Sagheer	CUST, Islamabad
(c)	Supervisor	Dr. Samina Batul	CUST, Islamabad

Dr. Samina Batul
Thesis Supervisor
October, 2020

Dr. Muhammad Sagheer
Head
Dept. of Mathematics
October, 2020

Dr. Muhammad Abdul Qadir
Dean
Faculty of Computing
October, 2020

Author's Declaration

I, **Usman Shehzad** hereby state that my MPhil thesis titled "**Multivalued Fractals in Cone b-Metric Spaces**" is my own work and has not been submitted previously by me for taking any degree from Capital University of Science and Technology, Islamabad or anywhere else in the country/abroad.
At any time if my statement is found to be incorrect even after my graduation, the University has the right to withdraw my MPhil Degree.

(Usman Shehzad)

Registration No: MMT183001

Plagiarism Undertaking

I solemnly declare that research work presented in this thesis titled "**Multivalued Fractals in Cone b-Metric Spaces**" is solely my research work with no significant contribution from any other person. Small contribution/help wherever taken has been dully acknowledged and that complete thesis has been written by me.

I understand the zero tolerance policy of the HEC and Capital University of Science and Technology towards plagiarism. Therefore, I as an author of the above titled thesis declare that no portion of my thesis has been plagiarized and any material used as reference is properly referred/cited.

I undertake that if I am found guilty of any formal plagiarism in the above titled thesis even after award of MPhil Degree, the University reserves the right to withdraw/revoke my MPhil degree and that HEC and the University have the right to publish my name on the HEC/University website on which names of students are placed who submitted plagiarized work.

(Usman Shehzad)

Registration No: MMT183001

Acknowledgements

First and foremost, all praise and appreciation to **ALLAH** who is the most beneficent, merciful and the most kind. Countless respect and endurance for **Prophet Muhammad** (Peace Be Upon Him), the fortune of knowledge, who took the humanity out of ignorance and showed the right path.

I would like to express my warmest thanks to my kind nature supervisor **Dr. Samina Batul** for the supervision and constant support. Her invaluable help of constructive comments and suggestions throughout this work have contributed to the success of this research. This work would not have been possible without her guidance, support and encouragement.

In addition, I am thankful to the Mathematics department Capital University of Science and Technology **(CUST)** Islamabad specially the Head of the Department, **Dr. Muhammad Sagheer** for providing us a learning and creative environment. I should be thankful to my respected teacher **Dr. Rashid Ali** for the encouragement and suggestions. May God keep blessing you all.

A special word of gratitude to my family who were always here to support and accomplish my research work. Most importantly, none of this would have been possible with out the love and patience of my family. I am extremely grateful to parents for their love, care and sacrifices for educating and preparing me for my future. My father **Abdul Qayyum** sacrifices for me more than any other in life. My brother and sister have also a great part of my work. Finally, this thesis might not exist at all without the love and support of my wife **Kinza Eman**. I am thankful to her for support and encouragement.

Abstract

Motivated by the idea of cone b-metric space and multivalued fractals operators, the idea of multivalued fractals in the cone b-metric spaces has been introduced. Some fixed point results on multivalued fractals by using single valued and multivalued mappings have been established which generalize the already existing results in the b-metric spaces. The multivalued fractals in the cone b-metric spaces over the Banach algebra have been given the main focus.

Contents

List of Figures

Abbreviations

BCP Banch Contraction Principle

IMS Iterated Multifunction System

Symbols

(X, d)	Metric space
d	Distance function
$\mathbb{N}$	Set of natural numbers
$\mathbb{R}$	Set of real number
$\notin$	Does not belongs to
(X, d_b)	b-metric space
K	Cone
$\mathcal{A}$	Banach algebra
H	Hausdorff distance
T_F	Multivalued fractal operator
η	Picard operator
$\Rightarrow$	Implies that
$\in$	Belongs to
$\forall$	For all
$\sum$	Sigma
∞	Infinity
$\lim\limits_{x \to \infty}$	Limit
H	Hausdorff distance
H_{db}	H cone b-metric
$\preceq$	Partial order
T_F	Multivalued fractals

Chapter 1

Introduction

1.1 Background

Mathematics has an important role in scientific knowledge that is why it is called mother of sciences. Mathematics has a lot of applications for humans in every field of life. Mathematics is divided into many branches and each branch has its significance. One of the important branches of mathematics is known as functional analysis. In functional analysis, fixed point theory is a valuable and dominant theory. Fixed point theory provides sufficient conditions for the existence of solution of a problem. The concept of fixed point theory has a lot of applications in different fields of science, such as in the area of numerical analysis, polynomial interpolation, error estimation, optimization theory, mathematical economics, variational inequalities, approximation theory and finite difference methods.

Poincare [1] was the first mathematician who studied the field of fixed point theory in 1886 and substantiate various fixed point results. Later on Brouwer [2] considered the equation $T(\eta) = \eta$ and established the solution of this equation by proving a fixed point theorem in 1910. He also worked to prove fixed point results for the shapes like square and in a sphere etc. In 1922, a notable mathematician Stephan Banach [3] demonstrated a significant fixed point result in the field of functional analysis acknowledged as Banach contraction principle. This result is declared to be the most fundamental in the field of fixed point theory. The two remarkable applications come from this principle. The first one is that it guarantees the existence and uniqueness of fixed point of contraction mapping. The second and the very emotive one is that it developed an approach to determine the fixed point of such a contractive mapping. This principle occupies a significant part in the field of functional analysis. Afterwards, Banach contraction principle has been extending and refining typically in various directions. Different mathematicians used different approaches to extend this principle, by either replacing the contraction condition or taking the different spaces [4–7]. Kannan [8] proved Banach contraction principle. Nadler [9] also extended the Banach contraction principle from single valued to multivalued contraction mappings. On the other hand few authors used different spaces like pseudo metric space [10], metric like space [11], partially ordered space [12]. The b-metric space is one of the interesting generalization of the metric space which was initiated by Bakhtin [13], Czerwik [14]. They established the idea of b-metric space and then used the same idea to set up some fixed point theorems for generalizing the Banach contraction principle. Thereafter, a plenty of papers have been published in b-metric spaces for single valued functions and also on multivalued functions [14–16].

On the other hand, the notion of cone metric spaces has also gained too much importance for the researchers. In 2007, Huang and Zhang [17] introduced the idea of cone metric spaces. A comparable explanation of cone metric is also assumed by Rzepecki [18]. Kutbi et al. [19] found out the multivalued fixed point theorems in cone b-metric spaces over Banach algebra $\mathcal{A}$.

Gottfried Leibniz [20] was the first person who introduced the mathematics behind the fractals in the 17th century by considering recursive self-similarity. It was not possible until 1872 that the graph of a function would ever be considered as a fractal, when Karl Weierstrass [21] gave an example of a function of being everywhere continuous. In 1904, Helg Von Koch [22] dissatisfied with Karl idea, and gave a more geometric definition of similar functions named as Koch curves. French and American Mathematician Benoit Mandelbrot [23] was one of the main developers of fractals in 1975 by investigating a variety of self similar structure in nature, and used geometry to help to prove this theory of self-similarity.

Fractals and multivalud fractals have many application in graphics designing, dynamical system, astrophysics and geophysics etc. This thesis presents the detail study of results of [24]. This study leads to extension of multivalued fractals in cone b-metric spaces.

1.2 Thesis layout

1. **Chapter 2**:

   Chapter 2 consists of brief literature review of metric in fixed point theory. In Basic Tools, our focus is on basic notations, definitions and results regarding metric spaces. In Banach Contraction Principle, fixed point, cotractive mappings and related examples have been discussed. The Multivalued Mappings, contains some useful definitions and examples of different multivalued mappings related to our work. In b-Metric Space, definitions, examples and results of b-metric space have been collected. At the end of this chapter, the cone b-metric space, introduced by Huang and Zhang [17] and Radenovic [25] have been discussed.

2. **Chapter 3**:

   This chapter is regarding the review work of Monica Boriceanu [24]. This chapter also contains some useful definitions and examples which are used to prove results of Monica Boriceanu [24].

3. **Chapter 4**:

   In this chapter motivated with the idea of cone b-metric space and multivalued fractals operators, the idea of multivalued fractals in cone b-metric spaces has been introduced. Some fixed point theorems on multivalued fractals by using multivalued mappings have been established. These results generalizes the already existing results in b-metric spaces [24].

4. **Chapter 5**:

   The conclusion is given in this chapter.

Chapter 2

Preliminaries

Chapter 2 is about few basic definitions, results and examples which are used in the subsequent chapters. The first section of this chapter covers some basics of metric spaces with few examples. The second section deals with the multivalued mappings. The third section deals with the b-metric spaces and related examples. In section 4, Banach contraction principle (BCP) and fixed points in metric space and few significant fixed point theorems in metric spaces will be discussed. The end of this chapter deals with the basics of cone metric spaces and cone b-metric spaces and related examples.

2.1 Basic Tools

2.1.1. Partially Ordered Set [26]

"A partially ordered set is a set M on which there is defined partial ordering, that is, a binary relation which is written $\preceq$ and satisfies the conditions:

$(PO1)$ $a \preceq a$ for every $a \in M$. (Reflexivity)

$(PO2)$ $a \preceq b$ and $b \preceq a$, then $a = b$. (Antisymmetry)

$(PO3)$ $a \preceq b$ and $b \preceq c$, then $a \preceq c$. (Transitivity)

Partially emphasizes that M may contain a and b for which neither $a \preceq b$ nor $b \preceq a$ holds. Then a and b are called incomparable elements. In contrast, two elements a and b are called comparable elements if they satisfy $a \preceq b$ or $b \preceq a$ (or both)."

Example 2.1.

Consider the relation.

$$S = \{(s,t) \mid s,t \in \mathbb{Z}, \frac{s}{t} \in \mathbb{Z},\ s \neq 0\}.$$

Then this relation is partial ordered set.

2.1.2. Totally Ordered Set [26]

"A totally ordered set or chain is a partially ordered set such that every two elements of the set are comparable. In other words, a chain is a partially ordered set that has no incomparable elements."

Example 2.2.

Consider S having only real numbers and let $s \preceq t$. Then S is totally ordered, there are no maximal elements in S.

Remark 1. [26]

"Every totally ordered set is a partially ordered set but the converse is not true."

The idea of metric space was first introduced by Frechet [27] in connection with the study of function spaces. Metric space is the most important topic of pure as well as applied mathematics. This section is regarding the basic definitions and examples of metric spaces. Now a days topological spaces and metric spaces are frequently used in scientific researches.

2.1.3. Metric Space [26]

"A metric space is a pair (X,d), where X is a set and d is a metric on X (or distance function on X), that is a function defined on $X \times X$ such that for all $x, y, z \in X$
we have:

(M_1) d is real-valued, finite and non-negative.

(M_2) $d(x,y)=0$ if and only if $x=y$.

(M_3) $d(x,y)=d(y,x)$ (Symmetry).

(M_4) $d(x,y)\leq d(x,z)+d(z,y)$ (Triangular inequality)."

Example 2.3.

Let there be a function defined by $d:\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ such that:

$$d(\zeta,\xi)=|\zeta-\xi|.$$

Then d is a metric on real number. Since first three conditions are obiviously true now for the fourth property we proceed as follows:

$$\begin{aligned}
d(\zeta,\psi)&=|\zeta-\psi|,\\
d(\zeta,\psi)&=|\zeta-\xi+\xi-\psi|,\\
d(\zeta,\psi)&\leq|\zeta-\xi|+|\xi-\psi|,\\
d(\zeta,\psi)&=d(\zeta,\xi)+d(\xi,\psi).
\end{aligned}$$

$\Rightarrow$ d is metric on $\mathbb{R}$.

Example 2.4.

Let $d:\mathbb{R}\times\mathbb{R}\rightarrow\mathbb{R}$ be a function defined by $d(\zeta,\psi)=\sqrt{|\ \zeta-\psi\ |}$, for all $\zeta,\psi\in\mathbb{R}$. Then d is metric on $\mathbb{R}$.

By the definition of d, it is clear that $d(\zeta,\psi)$ is the non-negative square root of the modulus of the difference of the real numbers ζ and ψ, so $(M1)$ is satisfied and $(M2)$, $(M3)$ are obviously true.

for $(M4)$ we proceed as follows.

$(M4)$: Since

$$\begin{aligned}
&|\ s_1+s_2\ |\leq(|\ s_1\ |+|\ s_2\ |)\leq(|\ s_1\ |+|\ s_2\ |+2\sqrt{|\ s_1\ |}\sqrt{|\ s_2\ |}),\\
\Rightarrow\quad &|\ s_1+s_2\ |\leq(\sqrt{|\ s_1\ |}+\sqrt{|\ s_2\ |})^2,\\
&\sqrt{|\ s_1+s_2\ |}\leq\sqrt{|\ s_1\ |}+\sqrt{|\ s_2\ |}.
\end{aligned}$$

Putting $s_1 = \zeta - \psi$, $s_2 = \psi - \phi$ in above we have,

$$\sqrt{|\ \zeta - \phi\ |} \leq \sqrt{|\ \zeta - \psi\ |} + \sqrt{|\ \psi - \phi\ |},$$
$$d(\zeta, \phi) \leq d(\zeta, \psi) + d(\psi, \phi) \qquad \text{for all} \ \ \zeta, \psi, \phi \in \mathbb{R}.$$

This shows that d is metric on $\mathbb{R}$.

Example 2.5.

Let $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be a function defined by

$d(P_1, P_2) =|\ \zeta_1 - \psi_1\ | + |\ \zeta_2 - \psi_2\ |$, for all $P_1, P_2 \in \mathbb{R}^2$

where $P_1 = (\zeta_1, \zeta_2), P_2 = (\psi_1, \psi_2)$.

Then d is metric on $\mathbb{R}^2$.

To show that d is metric on $\mathbb{R}^2$, note that the $(M1)$, $(M2)$ and $(M3)$ are obviously satisfied.

Now for $(M4)$ let for all $P_1, P_2, P_3 \in \mathbb{R}^2$ (where $P_3 = (\phi_1, \phi_2)$).

$$\begin{aligned} d(P_1, P_3) &=|\ \zeta_1 - \phi_1\ | + |\ \zeta_2 - \phi_2\ |, \\ &=|\ \zeta_1 - \psi_1 + \psi_2 - \phi_1\ | + |\ \zeta_2 - \psi_2 + \psi_2 - \phi_2\ |, \\ &\leq|\ \zeta_1 - \psi_1\ | + |\ \psi_1 - \phi_1\ | + |\ \zeta_2 - \psi_2\ | + |\ \psi_2 - \phi_2\ |, \\ &\leq|\ \zeta_1 - \psi_1\ | + |\ \zeta_2 - \psi_2\ | + |\ \psi_1 - \phi_1\ | + |\ \psi_2 - \phi_2\ |, \\ d(P_1, P_3) &\leq d(P_1, P_2) + d(P_1, P_2). \end{aligned}$$

This shows that d is metric on $\mathbb{R}^2$.

2.1.4. Convergence of Sequence [26]

"Let (X, d) be a metric space, a sequence $\{x_n\}$ in a metric space $X = (X, d)$ is said to converge or to be convergent if there is an $x \in X$ such that

$$\lim_{n\to\infty} d(x_n, x) = 0,$$

x is called the limit of $\{x_n\}$ and we write

$$\lim_{n\to\infty} x_n = x,$$

or simply

$$x_n \to x.$$

We say that x_n converges to x or has the limit x. If x_n is not convergent, it is said to be divergent."

2.1.5. Cauchy Sequence [26]

"Let (X, d) be a metric space, a sequence $\{x_n\}$ in a metric space (X, d) is said to be a Cauchy sequence if for each $\epsilon > 0$ there exist $N \in N(\epsilon)$ such that

$$d(x_n, x_m) \ < \ \epsilon \quad \forall \ m, n > N." \tag{2.1}$$

2.1.6. Completeness [26]

"Let (X, d) be a metric space, if every Cauchy sequence in a metric space (X, d) converges to a point $x \in X$, then X is called a complete metric space."

Example 2.6.

Let us consider a closed interval $[0, 1] \in \mathbb{R}$. Then by using $d(\zeta, \xi) = |\zeta - \xi|$ above interval define a complete metric space.

2.1.7. Bounded set [26]

"A subset M of a metric space (X, d) is said to be bounded set if its diameter

$$\delta(M) = \sup_{x,y \in M} d(x, y),$$

is finite."

2.1.8. Open set [26]

"A subset M of a metric space (X, d) is said to be open if it contains a ball about each of its points."

2.1.9. Closed set [26]

"A subset K of a metric space (X, d) is said to be closed if its complement (in X) is open, that is $K^c = X - K$ is open."

Example 2.7.

The closed interval [1,2] of real numbers $\mathbb{R}$ is closed.

Example 2.8.

Let us consider (W, d) be a metric space then each single set $\{w\}$ is a closed subset of S.

2.1.10. Continuous mapping [26]

"Let $X = (X, d)$ and $Y = (Y, d)$ be metric spaces. A mapping $T : X \to Y$ is said to be continuous at a point $x_0 \in X$ if for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$d(Tx, Tx_0) \;\leq\; \epsilon \quad \text{for all } x \text{ satisfying} \quad d(x, x_0) \;\leq\; \delta.$$

T is said to be continuous if it is continuous at every point of X."

Example 2.9.

Consider a mapping $S : M \to M$ defined on a usual metric space (M, d) as follows:

$$T(\rho) \;=\; \rho^5 \quad \text{where} \;\; \rho \;\in\; M,$$

is a continuous mapping.

2.1.11. Hausdorff Distance [29]

"Let X be a non-empty set and d be a metric on X. Let $CB(X)$ be the collection of non-empty closed and bounded subsets of X.
Define the map $H : CB(X) \times CB(X) \longrightarrow \mathbb{R}$ as follows:

$$H(A, B) = \max\{\sup_{a \in A} d(a, B), \;\; \sup_{b \in B} d(b, A)\},$$

for each $A, B \in CB(X)$. Then H is a metric on $CB(X)$. Where

$$d(a, B) = \inf\{d(a, b) : b \in B\}.$$

This is called Hausdorff metric on $CB(X)$. This is also known as the Hausdorff distance between the sets in $CB(X)$ generated by the metric d on X. The pair $(CB(X), H)$ is called Hausdorff metric space."

Example 2.10.

Let $X = \mathbb{R}$

Find the value of $H(P,Q)$ where $P=[3,5]$ and $Q=[4,8]$

Let $s=4$ and $t=2$

$$Q\subset N(s,P)=N(4,P)=[3-4,5+4]=[-1,9],$$

$$P\subset N(t,Q)=N(1,Q)=[4-2,8+2]=[2,10],$$

as we know that

$$T_PQ=\{\epsilon>0;P\subseteq N(\epsilon,Q),\ \ Q\subseteq N(\epsilon,P)\},$$

then

$$T_PQ=[3,\infty),$$

$$H(P,Q)=\inf[3,\infty),$$

$$H(P,Q)=3.$$

2.2 Contractive Mappings

2.2.1. Contraction [26]

"Let $X=(X,d)$ be a metric space. A mapping $T:X\longrightarrow X$ is called a contraction on X if there is a positive real number $\alpha<1$ such that for all $x,y\in X$

$$d(Tx,Ty)\leq \alpha d(x,y) \qquad\qquad (\alpha<1).$$

Geometrically this means that any points x and y have images that are closer together than those points x and y, more precisely, the ratio $\dfrac{d(Tx,Ty)}{d(x,y)}$ does not exceed a constant α which is strictly less than 1."

Example 2.11.

Let (M,d) be a metric space and $d(\phi,\psi)=\mid\phi-\psi\mid$.

Then define a mapping $S:M\to M$ by

$$S(\phi)=\frac{\phi}{5}+3$$

$$\begin{aligned}
d(S\phi, S\psi) &= \mid (\frac{\phi}{5}+3)-(\frac{\psi}{5}+3) \mid, \\
d(S\phi, S\psi) &= \mid \frac{\phi}{5}-\frac{\psi}{5} \mid, \\
d(S\phi, S\psi) &= \frac{1}{5} \mid \phi-\psi \mid,
\end{aligned}$$

$\Rightarrow \alpha = \frac{1}{5} < 1$.

Then S is a contraction with $\alpha = \frac{1}{5} < 1$.

Example 2.12.

Let $M = [0,1]$ be a metric space and $d(\phi,\psi) = \mid \phi-\psi \mid$.

Then define a mapping $S : M \to M$ by

$$S(\phi) = \frac{1}{\phi+7} .$$

$$\begin{aligned}
d(S_\phi, S_\psi) &= \mid (\frac{1}{\phi+7})-(\frac{1}{\psi+7}) \mid, \\
d(S_\phi, S_\psi) &\leq \mid \frac{\psi+7-\phi-7}{(\phi+7)(\psi+7)} \mid, \\
&\leq \mid \frac{\psi-\phi}{(\phi+7)(\psi+7)} \mid, \\
&\leq \mid \frac{-(\phi-\psi)}{(\phi+7)(\psi+7)} \mid, \\
&\leq \mid \frac{\phi-\psi}{(\phi+7)(\psi+7)} \mid, \\
&\leq \mid \frac{\phi-\psi}{(\phi+7)(\psi+7)} \mid, \\
&\leq \mid \frac{\phi-\psi}{(7)(7)} \mid, \\
&\leq \frac{1}{49} \mid \phi-\psi \mid, \\
&\leq \frac{1}{49} d(\phi,\psi),
\end{aligned}$$

then S is a contraction with $\alpha = \frac{1}{49} < 1$.

2.2.2. Contraction by Derivative [30]

"If X is a Banach space and P maps a convex closed subset M of X into itself and if P has a derivative at every point of M, then

$$\sup_{x\in M} ||P'(x)|| = \alpha < 1$$

implies that P is a contraction on M."

Example 2.13.

The function $S : \mathbb{R} \longrightarrow \mathbb{R}$ define by

$$S(\eta) = \sin(\sin \eta),$$

is a contraction.

As

$$\begin{aligned} S(\eta) &= \sin(\sin \eta), \\ S'(\eta) &= \cos(\sin \eta)(\cos \eta), \\ |S'(\eta)| &= |\cos(\sin \eta)(\cos \eta)| \leq 1, \end{aligned}$$

since

$$|\cos(\sin \eta)| \leq 1, \; |\cos \eta| \leq 1.$$

Simultaneously both can not be equal to 1.

$\Rightarrow S(\eta)$ is a contraction.

2.2.3. Contractive mapping [31]

"A self map $T : X \to X$ on a metric space is a contractive mapping if

$$d(Tx, Ty) < d(x, y), \; \forall \; x, y \in X, \; x \neq y."$$

Remark 2.

Every contraction is contractive mapping but converse of statement is not true in general.

The above remark illustrate the following example.

Example 2.14.

Consider a metric space (M, d) defined on $\mathbb{R}$. Let S be the self-mapping on M defined by

$$T(\zeta) = \zeta + \frac{1}{\zeta} \quad , \quad \forall\, \zeta \in M. \tag{2.2}$$

Then T is contractive but not a contraction.

2.2.4. Multivalued Contraction [9]

"Let (X, d) be a metric space. A map $T : X \longrightarrow CB(X)$ is said to be multivalued contraction if there exist $0 \leq \lambda < 1$ such that

$$H(Tx, Ty) \leq \lambda d(x, y), \qquad \forall\, x, y \in X$$

where $CB(X)$ denotes the family of nonempty closed subsets of X and H is the Hausdorff distance."

2.3 Banach Contraction Principle (BCP)

Stefan Banach proved Banach contraction principle in 1922. BCP is known to be the basic outcomes in the field of function analysis. The Banach contraction principle (BCP) provides us with an unique fixed point. Fixed point theorems play an important role in both pure and applied mathematics.

2.3.1. Fixed Point [32]

"Let $T : X \to X$ be a mapping on a set X. A point $x \in X$ is said to be a fixed point of T if

$$Tx = x, \tag{2.3}$$

that is, a point is mapped onto itself.

Geometrically,

if $y = f(x)$ is a real valued function on $\mathbb{R}$, then the fixed point of this function lies where the graph of the function f coincides with the real line $y = x$. Thus a function may or may not have fixed point. Furthermore, fixed point may or may not be unique."

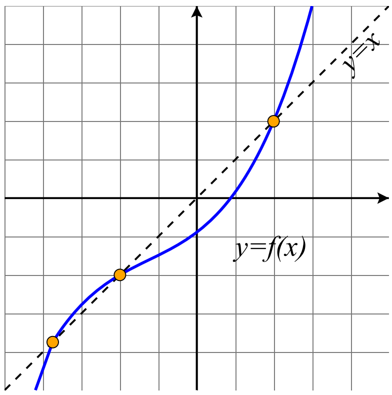

FIGURE 2.1: Three fixed points

The above graph represents a function having three fixed points.

Example 2.15.

Consider $X = \mathbb{R}$ and $S : X \to X$ be a mapping defined as

$$S(t) = \frac{t}{4} + 3.$$

S has a unique fixed point $t = 4$.

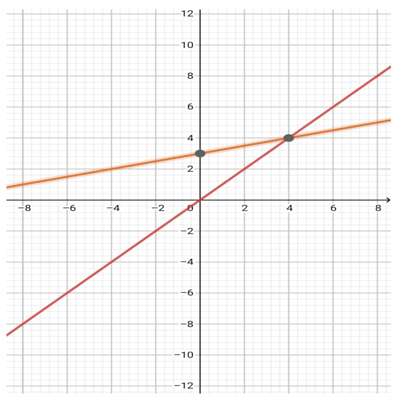

FIGURE 2.2: One fixed point

Example 2.16.

Consider $X = \mathbb{R}$ and $S : X \to X$ be a mapping defined as

$$S(t) = t + 3.$$

S has no fixed point.

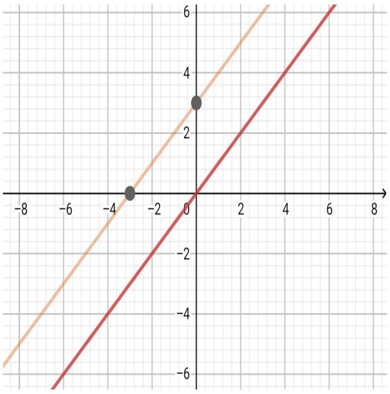

FIGURE 2.3: No fixed point

2.3.2. Banach Contraction Principle [26]

"Consider a metric space $X = (X, d)$, where $X \neq \emptyset$. Suppose that X is a complete and let $T : X \to X$ be a contraction on X. Then T has precisely one fixed point."

2.3.3. Compact Metric Space [26]

"A metric space X is said to be compact if every sequence in X has a convergent subsequence. A subset M of X is said to be compact if M is compact considered as a subspace of X, that is, if every sequence in M has a convergent subsequence whose limit is an element of M."

Banach [33] established the following fixed point result, popularly named as Banach contraction principle.

Theorem 2.3.4 [33]

"Let (X, d) be a compact metric space, and let T be a mapping on X. Assume $d(Tx, Ty) < d(x, y)$ for all $x, y \in X$ with $x \neq y$. Then T has a unique fixed point."

2.4 Multivalued Mappings

Multivalued mapping contributes much in pure mathematics as well as in applied mathematics. In pure mathematics its important role in real and complex analysis can not be denied. In applied mathematics such as in optimal control system it plays an important role.

2.4.1. Multivalued Mappings [35]

"Let X and Y be nonempty sets. T is said to be multivalued mapping from X to Y if T is a function from X to the power set of Y. We denote the multivalued mapping by $T : X \longrightarrow 2^Y$."

Example 2.17.

Consider $S = [0, 1]$ and $T(S) = \{P \subset S : P \neq \emptyset\}$.

Dene $G : S \longrightarrow T(S)$ and $L : S \longrightarrow T(S)$ by:

$$Gs = [0, s]$$

$$Ls = \begin{cases} [0,1] & \text{if} \quad s \neq \dfrac{1}{2}, \\ \left[\dfrac{1}{2}, 1\right] & \text{if} \quad s = \dfrac{1}{2}. \end{cases}$$

Both G and L are multivalued mappings.

Example 2.18.

Suppose $s, t \in \mathbb{R}$ satisfying the condition that $t > s$.

Dene $L : [s, t] \longrightarrow [s, t]$, by

$$L = \begin{cases} [v, t] & \text{if} \quad s < v < t, \\ [s, t] & \text{if} \quad v \in \{s, t\}. \end{cases}$$

Then L is a multivalued mapp.

2.5 b-Metric Spaces

Bakhtin [13] was the first person who introduced the concept of b metric spaces in 1989, which has been used to established a generalization of Banach contraction principle [36] in such spaces. Czerwick [14] introduced a condition which was weaker than the third property of metric space and formally defined a b-metric space and established a fixed point result for such space. Then this result was used to prove common fixed point satisfying ϕ contraction [37] in b metric spaces for single and multivalued mappings. Also some authors [24, 38, 39] established some fixed point theorems by using contractive mappings in ordered complete b-metric spaces.

2.5.1. b-Metric Space [39]

"Let X be a non-empty set and $(b \geq 1)$ be a real number. A function $d_b : X \times X \to [0, \infty)$ is called b-metric if it satisfies the following properties for each $x, y, z \in X$,

$(b1)$ $d_b(x, y) = 0 \Leftrightarrow x = y$,

$(b2)$ $d_b(x, y) = d_b(y, x)$, (symmetry),

$(b3)$ $d_b(x, y) \leq b[d_b(x, z) + d_b(z, y)]$, (triangular property),

the pair (X, d_b) is called a b-metric space."

Remark 3. [24]

"The class of b-metric space is larger than the class of metric space. When $b = 1$ then the concept of b-metric space coincides with concept of metric space."

Example 2.19.

Let there be a mapping $d_b : S \times S \to S$ (where $S = \mathbb{R}$) defined by

$d_b(\zeta, \psi) = (\phi - \psi)^2$ for all $\zeta, \psi \in \mathbb{R}$.

Since $(b1), (b2)$ are obiviously true now for the third property of b-metric we proceed as follows:

$$\begin{aligned}
d_b(\phi, \xi) &= (\phi - \xi)^2 \\
&= (\zeta - \psi + \psi - \xi)^2, \\
&= [(\zeta - \psi) + (\psi - \xi)]^2, \\
&= (\zeta - \psi)^2 + (\psi - \xi)^2 + 2(\zeta - \psi)(\psi - \xi), \\
&= (\zeta - \psi)^2 + (\psi - \xi)^2 + 2(\zeta - \psi)(\psi - \xi) + \\
&\quad (\zeta - \psi)^2 + (\psi - \xi)^2 - (\zeta - \psi)^2 - (\psi - \xi)^2, \\
&= 2(\zeta - \psi)^2 + 2(\psi - \xi)^2 + 2(\zeta - \psi)(\psi - \xi) - \\
&\quad (\zeta - \psi)^2 - (\psi - \xi)^2, \\
&= 2[(\zeta - \psi)^2 + 2(\psi - \xi)^2] - [(\zeta - \psi)^2 + (\psi - \xi)^2 - \\
&\quad 2(\zeta - \psi)(\psi - \xi)],
\end{aligned}$$

$$= 2[(\zeta-\psi)^2+(\psi-\xi)^2]-[(\zeta-\psi)+(\psi-\xi)]^2,$$
$$\leq 2[(\zeta-\psi)^2+(\psi-\xi)^2],$$
$$\Rightarrow d_b(\zeta,\xi) \leq 2[d_b(\zeta,\psi)+d_b(\psi-\xi)].$$

It is a b-metric with $b = 2$.

Example 2.20.

Consider $l_p(\mathbb{R})$ with $0 < p < 1$, also

$$l_p(\mathbb{R}) = [\{\phi_v\} \subseteq \mathbb{R} \mid \sum_{v=1}^{\infty} \mid \phi_v \mid^p < \infty]$$

with the mapping $d : l_p(\mathbb{R}) \times l_p(\mathbb{R}) \to \mathbb{R}^+$ defined by

$$d(\phi,\psi) = \langle \sum_{v=1}^{\infty} \mid \phi_v - \psi_v \mid^p \rangle^{\frac{1}{p}}$$

for each $\phi = \{\phi_v\}$, $\psi = \{\psi_v\} \in l_p(\mathbb{R})$, is a b- metric space with coefficient $b = 2^{\frac{1}{p}-1}$.

We will only prove the third condition of b-metric space.

Consider $\phi = \{\phi_v\}, \quad \eta = \{\eta_v\}, \quad \psi = \{\psi_v\} \in l_p(\mathbb{R})$, we shall show that,

$$\langle \sum_{v=1}^{\infty} \mid \phi_v - \eta_v \mid^p \rangle^{\frac{1}{p}} \leq 2^{\frac{1}{p}-1} \left\{ \langle \sum_{v=1}^{\infty} \mid \phi_v - \psi_v \mid^p \rangle^{\frac{1}{p}} + \langle \sum_{v=1}^{\infty} \mid \psi_v - \eta_v \mid^p \rangle^{\frac{1}{p}} \right\}. \tag{2.4}$$

Consider $t_v = \phi_v - \psi_v, \;\; s_v = \psi_v - \eta_v$ and $s_n + t_n = \phi_v - \eta_v$.

$$\langle \sum_{v=1}^{\infty} \mid s_v + t_v \mid^p \rangle^{\frac{1}{p}} \leq 2^{\frac{1}{p}-1} \left\{ \langle \sum_{v=1}^{\infty} \mid s \mid^p \rangle^{\frac{1}{p}} + \langle \sum_{v=1}^{\infty} \mid t \mid^p \rangle^{\frac{1}{p}} \right\}. \tag{2.5}$$

To prove (2.6), consider following inequality:

$$(s+t)^p \leq s^p + t^p \qquad (s, \quad t \geq 0, \quad 0 < p \leq 1),$$

$$(s+t)^p \leq 2^{p-1}(s^p + t^p) \qquad (s, \quad t \geq 0, \quad p \geq 1),$$

$$\Rightarrow \langle\sum_{v=1}^{\infty} | s_v + t_v |^p\rangle^{\frac{1}{p}} \leq \left\{\langle\sum_{v=1}^{\infty}(| s_v | + | t_v |)^p\rangle^{\frac{1}{p}}\right\},$$

$$\leq \left\{\langle\sum_{v=1}^{\infty} | s_v |^p + | t_v |^p\rangle^{\frac{1}{p}}\right\},$$

$$= \left\{\langle\sum_{v=1}^{\infty} | s_v |^p + \sum_{v=1}^{\infty} | t_v |^p\rangle^{\frac{1}{p}}\right\},$$

$$\leq 2^{\frac{1}{p}-1}\left\{\langle\sum_{v=1}^{\infty} | s |^p\rangle^{\frac{1}{p}} + \langle\sum_{v=1}^{\infty} | t |^p\rangle^{\frac{1}{p}}\right\}.$$

It is a b-metric with $b = 2^{\frac{1}{p}-1}$.

Remark 4. [38]

"Let (X, d_b) be a b-metric space. Then in general b-metric is not continuous."

Above example illustrates the above remark.

Example 2.21. [38]

Consider $M = \mathbb{N} \cup \{\infty\}$. A function $d_b : M \times M \to \mathbb{R}$ defined by:

$$d_b(m,n) = \begin{cases} 0 & \text{if} \quad m = n, \\ |\frac{1}{m} - \frac{1}{n}| & \text{if oneof } m,n \text{ is even and or } = \infty, \\ 5 & \text{if one of } m,n \text{ is odd } \ m \neq \ n \text{ or } \infty, \\ 2 & \text{if otherwise.} \end{cases}$$

This shows that for $m, n, t \in M$, we have

$$d_b(m,p) \leq \frac{5}{2}\ (d_b(m,n)\ +\ d_b(n,t)).$$

$\Rightarrow (M, d_b)$ is a b-metric space (with $b = \frac{5}{2}$). Let $x_n = 2n\ \forall\ n \in \mathbb{N}$,

$$\text{as} \quad n\ \to\ \infty,\ d_b(2n,\infty) = \frac{1}{2n}\ \to\ 0$$

we have $\{x_n\} \to \infty$, since $d(x_n, 1) = 2 \neq 5 = d(\infty, 1)\ as\ n \to \infty$.

2.5.2. Closure [24]

"Let (X, d) be a b-metric space. If Y is a nonempty subset of X, then the closure $\overline{Y}$ of Y is the set of limits of all convergent sequences of points in Y, that is, $\overline{Y} = \{x \in X : \text{there exists a sequence } (x_n)_{n\in\mathbb{N}} \text{ such that } \lim\limits_{n\longrightarrow\infty} x_n = x\}$."

2.5.3. Closed set [24]

"Let (X, d) be a b-metric space. If Y is a nonempty subset of X, is called closed if and only if for each sequence $(x_n)_{n\in\mathbb{N}} \in Y$ which converges to an element x, we have $x \in Y$ (that is $Y = \overline{Y}$)."

2.5.4. Compact Set [24]

"Let (X, d) be a b-metric space. If Y is a nonempty subset of X, is compact if and only if for every sequence of elements of Y there exists a subsequence that converges to an element of Y."

2.5.5. Bounded Set [24]

"Let (X, d) be a b-metric space. If Y is a nonempty subset of X, is called bounded if and only if,

$$\delta(Y) = \sup\{d(a, b)|\ a, b \in Y\} < \infty."$$

2.5.6. Convergent, Cauchy and Completeness [38]

"Let (X, d) be a b-metric space.

(i) A sequence $\{x_n\}$ in X is called b-convergent if and only if there exist $x \in X$ such that $d(x_n, x) \to 0$ as $n \to 0$.

(ii) A sequence $\{x_n\}$ in X is called b-Cauchy if and only if $d(x_n, x_m) \to 0$ as $m, n \to \infty$.

(iii) The b-metric space (X, d) is said to be b-complete if every b-Cauchy sequence in X is b-convergent."

Remark 5. [24]

"Let (X, d) be a b-metric space. Then a convergent sequence has a unique limit."

Following proof illustrate the above remark.

Proof.

Suppose a sequence $(\phi_n)_{n\in\mathbb{N}}$ in X

Let $\phi_n \to s$ as $n \to \infty$

$$\Rightarrow d(\phi_n, s) \to 0$$

Also consider $\phi_n \to t$ as $n \to \infty$

$$\Rightarrow d(\phi_n, t) \to 0$$

Then we have to prove that limit is unique or $s = t$.

Here we use the third property of b-metric space.

$$\begin{aligned}
0 \leq d(s,t) &\leq s\{d(s,\phi_n) + d(\phi_n,t)\}, \\
0 \leq d(s,t) &\leq s\{0+0\}, \\
0 &\leq d(s,t), \\
\Rightarrow s &= t.
\end{aligned}$$

Remark 6. [24]

"Let (X, d) be a b-metric space. Then each convergent sequence is Cauchy."

Following proof illustrate the above remark.

Proof.

Let there be a sequence $(\phi_n)_{n\in N}$ in X which converges to $t \in \mathbb{R}$.

Choose a natural number N and $\epsilon > 0$ so that if $n > N$ then

$$\begin{aligned}
&(\phi_n)_{n\in N} \in X, \\
&\phi_n \to a \quad \forall \quad n \to \infty, \\
&\phi_m \to a \quad \forall \quad m \to \infty.
\end{aligned}$$

Here we use the third property of b-metric space.

$$\begin{aligned}
0 \leq d(\phi_n,\phi_m) &\leq s\{d(\phi_n,a) + d(a,\phi_m)\}, \\
0 \leq d(\phi_n,\phi_m) &\leq s\{0+0\},
\end{aligned}$$

$$0 \leq d(\phi_n, \phi_m),$$

$$\Rightarrow d(\phi_n, \phi_m) \longrightarrow 0.$$

2.6 Cone Metric Spaces

This section will describe the idea of cone metric space. The idea of cone metric spaces was first introduced by Haung and Zhang [17]. Also Rzepecki [18] produced a comparable explanation of cone metric space.

2.6.1. $\mathcal{A}$ Real Banach Algebra Space [19]

"Let $\mathcal{A}$ be the real Banach Algebra, that is $\mathcal{A}$ is a real Banach Algebra space in which an operation of multiplication is defined subjected to following properties: for all $x, y, z \in \mathcal{A}$ and $a \in \mathbb{R}$.

(i) $x(yz) = (xy)z$;

(ii) $x(y+z) = xy + xz$ and $(x+y)z = xz + yz$;

(iii) $a(xy) = (ax)y = x(ay)$;

(iv) $||xy|| \leq ||x||||y||$."

2.6.2. Cone [19]

"A subset K of $\mathcal{A}$ is called a cone if and only if

(i) $\{e,\ \theta\} \subset K$;

(ii) $K^2 = KK \subset K,\ K \cap (-K) = \{\theta\}$;

(iii) $a, b \in \mathbb{R},\ a, b \geq 0,\ \Rightarrow aK + bK \subset K$.

For a given cone $K \subset \mathcal{A}$, we define the partial ordering $\preceq$ with respect to K by $\eta \preceq \phi$ if and only if $\phi - \eta \in K$, $\eta \prec \phi$ will stand for $\eta \preceq \phi$ and $\eta \neq \phi$, while $\eta \ll \phi$ stand for $\phi - \eta \in intK$, where $intK$ denotes the interior of K. If $intK \neq \emptyset$, then

K is called a solid cone. Write $||.||$ as the norm of $\mathcal{A}$. A cone K is called a normal cone if there exist a number $M > 0$ such that for all $\eta, \phi \in \mathcal{A}$, we have

$$\theta \preceq \eta \preceq \phi \Longrightarrow ||\eta|| \leq M||\phi||.$$

The least positive number satisfying above is called the normal constant of ν. Note that, for any normal cone K we have $M \geq 1$."

Remark 7. [19]
"We always suppose that $\mathcal{A}$ is real Banach algebra with the unit e, K is a solid cone and $\preceq$ is the partial ordering with respect to K."

2.6.3. Cone Metric Space [17]

"Let X be a nonempty set and $\mathcal{A}$ be a Banach algebra. A function $d : X \times X \to \mathcal{A}$ is said to be cone metric, if the following conditions hold:

$(C1)$ $\theta \preceq d(\eta, \phi)$ for all $\eta, \phi \in X$ and $d(\eta, \phi) = \theta$ if and only if $\eta = \phi$;

$(C2)$ $d(\eta, \phi) = d(\phi, \eta)$ for all $\eta, \phi \in X$;

$(C3)$ $d(\eta, \psi) \preceq d(\eta, \phi) + d(\phi, \psi)$ for all $\psi, \eta, \phi \in X$."

Example 2.22.
Let $\mathcal{A} = \mathbb{R}^2$, $K = \{(\psi, \xi) \in \mathcal{A}|\ \psi, \xi \geq 0\} \subset \mathcal{A}$, $M = \mathbb{R}$ and $d : M \times M \longrightarrow \mathcal{A}$ such that,

$$d(\psi, \xi) = (|\psi - \xi|,\ v|\psi - \xi|) \ \text{ where } \ v \geq 0 \text{ is a constant.}$$

The pair (M, d) is called a cone metric over Banach algebra $\mathcal{A}$.

2.6.4. Convergence, Cauchy and Completeness [19]

"Let (X, d) be a cone metric space over Banach algebra $\mathcal{A}$, $\eta \in X$,

(i) let η_n be a sequence in X. Then η_n converges to η whenever for every $c \in \mathcal{A}$ with $\theta \ll c$ there is a natural number n_0 such that $d(\eta_n, \eta) \ll c$ for all $n \geq n_0$

We denote this by

$$\lim_{n \longrightarrow \infty} \eta_n = \eta.$$

(ii) let η_n be a sequence in X. Then η_n is a Cauchy whenever for every $c \in \mathcal{A}$ with $\theta \ll c$ there is a natural number n_0 such that $d(\eta_m, \eta_n) \ll c$ for all $m, n \geq n_0$.

(iii) Let (X, d) be a cone metric space over Banach algebra $\mathcal{A}$, $\eta \in X$, let η_n be a sequence in X. Then (X, d) is complete cone metric if every Cauchy sequence in X is convergent."

Lemma 1. [19]
"If E is a real Banach space with a cone K and if $a \preceq \lambda a$ with $a \in K$ and $0 \leq \lambda < 1$, then $a = \theta$."

Lemma 2. [19]
"If E is a real Banach space with a solid cone K and if $||xn|| \to 0$ as $n \longrightarrow \infty$, then for any $\theta \ll c$, there exists $n_0 \in \mathbb{N}$ such that, $||xn|| \ll c$ for all $n < n_0$."

2.7 Cone b-Metric Spaces

Afterwards Huang and Zhang [17] and Radenovic [25] extended the concept of cone metric space over Banach algebra into cone b-metric space over Banach algebra in the following way.

2.7.1. Cone b-Metric Space [25]
"Let X be a nonempty set and $(b \geq 1)$ be a constant and $\mathcal{A}$ be a Banach algebra. A function $d : X \times X \to \mathcal{A}$ is said to be cone b-metric, if the following conditions are hold:

$(C1)$ $\theta \preceq d(\eta, \phi)$ for all $\eta, \phi \in X$ and $d(\eta, \phi) = \theta$ if and only if $\eta = \phi$;

$(C2)$ $d(\eta, \phi) = d(\phi, \eta)$ for all $\eta, \phi \in X$;

$(C3)$ $d(\eta,\psi) \preceq b\{d(\eta,\phi)+d(\phi,\psi)\}$ for all $\psi,\eta,\phi \in X$.

The pair (X,d) is then called a cone b-metric over Banach algebra $\mathcal{A}$."

Remark 8. [19]

"The class of cone b-metric space over Banach algebra $\mathcal{A}$ is larger than the class of cone metric space over Banach algebra since the latter must be the former, but the converse is not true."

Example 2.23.

Let $\mathcal{A} = C[s,t]$ be the set of continuous functions on the interval $[s,t]$ with the supremum norm having multiplication in the usual way. Then $\mathcal{A}$ is a Banach algebra with a unit 1.

Set $K = \{\xi \in \mathcal{A} \ : \ \xi(v) \geq 0, \ v \in [s,t]\}$ and $M = \mathbb{R}$. Consider a mapping $d : M \times M \to \mathcal{A}$ by $d(\xi,\psi) = |\xi-\psi|^p e^v$ for all $\xi,\psi \in M$, where $p > 1$ is a constant.

$\Rightarrow$ (M,d) into a cone b-metric space over Banach algebra $\mathcal{A}$ with the coefficient $b = 2^{p-1}$.

2.7.2. Convergence, Cauchy and Completeness [41]

"Let (X,d) be a cone b-metric space over Banach algebra $\mathcal{A}$,

(i) we say that η_n is a convergent sequence if for every c sequence. It means that η_n converges to η in E with $\theta \ll c$, there is an N such that for all $n > N$, $d(\eta_n,\eta) \ll c$ for some fixed $\eta \in X$. A cone b-metric space X is said to be complete if every Cauchy sequence in X is convergent in X.

(ii) we say that η_n is a Cauchy sequence if for every $c \in E$ with $\theta \ll c$, there is an N such that for all $n,m \geq N$, $d(\eta_n,\eta_m) \ll c$.

(iii) Let (X,d) be a cone b-metric space over Banach algebra $\mathcal{A}$, we say that (X,d) is a complete cone b-metric space, if every Cauchy sequence is convergent."

2.7.3. Interior Point [41]

"Let (W,d) be a cone b-metric space and $B \subseteq W$. Let $t \in B$ is called an interior point of B whenever there is $0 \ll p$ such that $B_0(t,p) \subseteq B$, where

$$B_0(t,p) = \{y \in W : d(y,t) \ll p\}.\text{"}$$

2.7.4. Open Set [41]

"Let (W,d) be a cone b-metric space and $B \subseteq W$. A subset $A \subseteq W$ is called open whenever each element of A is an interior point of A, that is, for any $a \in A$, there exists $c \in intP$ such that the open ball $B_0(a,c) \subseteq A$."

2.7.5. Closed Set [41]

"Let (W,d) be a cone b-metric space and $B \subseteq W$. For each $\theta \ll c$ and $w \in W$, the set $B(w,c) = \{y \in W : d(w,y) \preceq c\}$ is closed."

2.7.6. Compact Set [41]

"Let (W,d) be a cone b-metric space and $B \subseteq W$. A subset $B \subseteq W$ is called compact whenever every open cover of B has a finite sub-cover."

2.7.7. Bounded Set [41]

"Let (W,d) be a cone b-metric space and $B \subseteq W$ is called bounded whenever there exist $\theta \ll c$ and $w_0 \in W$ such that $d(t,w_0) \ll c$ for all $t \in B$."

Chapter 3

Multivalued Fractals in b-Metric Spaces

3.1 Introduction

Fractals and multivalued fractals have many application in graphics designing, dynamical system, astronomy, astrophysics and geophysics etc. Moreover, iterated function system has important consequence in applied science. A lot of examples of fractals and multivalued fractals are come from fixed point theory for single and multivalued operators. The most common study of fractals is in the case of complete and compact metric spaces.

3.2 Fractals

Fractals are define as any structure which is self similar. This means that if you view it at one scale it look very similar if you view it at much closer scale. For example equilateral triangle and take the mid point of all of sides of the triangle and connect them then we see a repeated process as shown in figure.

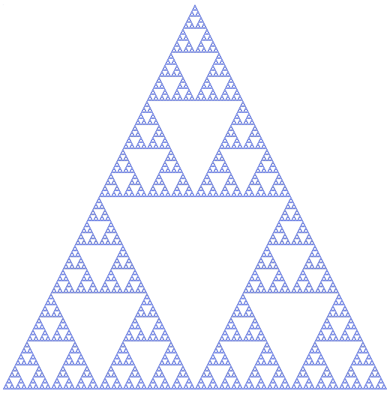

FIGURE 3.1: Fractals 1

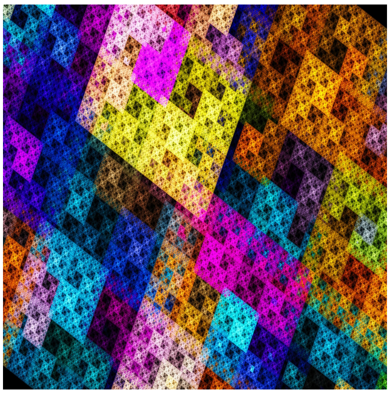

FIGURE 3.2: Fractals 2

FIGURE 3.3: Fractals in nature

3.3 Comparison Functions

3.3.1. Comparison Function [37]

“A mapping $\phi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is said to be a comparison function, if the following axioms are fulfilled

(*i*) ϕ is increasing;

(*ii*) $\lim_{n \longrightarrow \infty} \phi^n(t) \longrightarrow 0$, for all $t > 0$.

Clearly, if ϕ is a comparison function, then $\phi(t) < t$ for each $t > 0$, $\phi(0) = 0$, and ϕ is continuous at 0."

The following examples illustrate the above definition.

Example 3.1.
Consider a mapping $\phi : [0, \infty) \longrightarrow [0, \infty)$. Then the function

$$\phi(t) = ut \quad (u \in (0, 1)),$$

is a comparison function. It is also a b-comparison function. It is clear that $\phi(t) = at < t$ for $u \in (0, 1)$. Also $\phi(0) = 0$ and ϕ is continuous at 0 because

$$\lim_{t \longrightarrow 0} \phi(t) = 0.$$

Example 3.2.
Consider a mapping $\phi : [0, \infty) \longrightarrow [0, \infty)$. Then the function

$$\phi(t) = \frac{t}{1 + 2t} \ ,$$

is a comparison function. It is clear that $\phi(t) < \dfrac{t}{1 + 2t} < t$. Also $\phi(0) = 0$,

$$\phi(0) = \frac{0}{1 + 2(0)} = 0.$$

ϕ is continuous at 0,

$$\lim_{t \longrightarrow 0} \phi(t) = 0.$$

Example 3.3.

Consider a mapping $\phi : [0, \infty) \longrightarrow [0, \infty)$. Then the function

$$\phi(t) = \ln(1 + 2t),$$

is a comparison function. Consider $t = 1$, then clearly $\phi(t) < t$. Also $\phi(0) = 0$,

$$\phi(0) = \ln(1 + 2(0)) = 0.$$

ϕ is continuous at 0,

$$\lim_{t \longrightarrow 0} \phi(t) = 0.$$

3.3.2. b-Comparison Function [24]

"A function $\phi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is called b-comparison function with $(b \geq 1)$. If ϕ is increasing and there exist $s_0 \in \mathbb{N}$, $a \in (0, 1)$ and a convergent series of non-negative terms $\sum_{s=1}^{\infty} v_s$ such that

$$b^{s+1}\phi^{s+1}(t) \preceq ab^s\phi^s(t) + v_s \quad \text{for each } t \in \mathbb{R}_+, \text{ and each } s \geq s_0.$$

As consequence, if $\phi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a b-comparison function, then the series $\sum_{s=1}^{\infty} b^s\phi^s(t)$ converges for each $t \in \mathbb{R}_+$, and the function $e_b : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ defined by,

$$e_b(t) = \sum_{s=1}^{\infty} b^s\phi^s(t), \quad t \in \mathbb{R}_+,$$

is increasing and continuous at 0."

3.3.3. ϕ-Contractions in b-Metric Space [24]

"Let $(X, d), (Y, \rho)$ be b-metric spaces. An operator $T : X \longrightarrow Y$ is said to be a ϕ-contraction if $\phi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ and $\rho(T(x), T(y)) \leq \phi(d(x, y))$, for all $x, y \in X$."

3.3.4. Generalized ϕ-Contraction in b-Metric Spaces [42]

"Let (W, d) be a complete b-metric space. An operator $\eta : W \longrightarrow W$ is said to be generalized ϕ-contraction if $\phi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that

$$d(\eta(w_1), \eta(w_2)) \leq \phi(d(w_1, w_2)), \text{ for all } w_1, w_2 \in W."$$

3.4 Family of Subsets in b-Metric Spaces

3.4.1. Subsets in b-Metric Spaces [24]

"We know consider the following families of subset of a b-metric space (X, d):

(i) $P(X) = \{Y| \ Y \subset X\}; P(X) = Y \in P(X)| \ Y \neq \emptyset$;

(ii) $P_b(X) = \{Y \in P(X)| \ Y$ is bounded $\}$;
$P_{cp}(X) = \{Y \in P(X)| \ Y$ is compact $\}$;

(iii) $P_{cl}(X) = \{Y \in P(X)| \ Y$ is closed $\}$; $P_{b,cl}(X) = P_b(X) \cap P_{cl}(X)$."

Now we introduce the following generalized functions on b-metric space (X, d_b).

3.4.2. The Gap Functional [24]

"Let $D : P(X) \times P(X) \longrightarrow \mathbb{R}_+ \cup \{+\infty\}$ then

$$D(A, B) = \begin{cases} \inf\{d_b(a, b)| \ a \in A, b \in B\} & \text{if} \quad A \neq \emptyset \neq B; \\ 0 & \text{if} \quad A = \emptyset = B; \\ +\infty & \text{otherwise.} \end{cases}$$

In particular, if $x_0 \in X$ then $D(x_0, B) = D(\{x_0\}, B)$."

3.4.3. The Excess Generalized Functional [24]

"Let $\rho : P(X) \times P(X) \longrightarrow \mathbb{R}_+ \cup \{+\infty\}$ then

$$\rho(A, B) = \begin{cases} \sup\{D(a, B)| \ a \in A\} & \text{if} \quad A \neq \emptyset \neq B; \\ 0 & \text{if} \quad A = \emptyset; \\ +\infty & \text{if} \quad B = \emptyset \neq A. \end{cases}$$

Then this functional is called the excess generalized functional."

3.4.4. Pompeiu-Hausdorff Generalized Functional [24]

"Let $H : P(X) \times P(X) \longrightarrow \mathbb{R}_+ \cup \{+\infty\}$ then

$$H(A,B) = \begin{cases} \max\{\rho(A,B),\ \rho(B,A)\} & \text{if} \quad A \neq \emptyset \neq B, \\ 0 & \text{if} \quad A = \emptyset = B, \\ +\infty & \quad \text{otherwise.} \end{cases}$$

Then this functional is called Pompeiu-Hausdorff generalized functional. Then $(P_{cp}(X), H)$ is a complete b-metric space provided (X, d_b) is a complete b-metric space."

Lemma 3. [24]

"Let (X, d) be a b-metric space and $A, B \in P_{cp}(X)$. We assume that there exist a $\mu > 0$ such that

(i) for each $a \in A$ and $b \in B$ such that $d_b(a, b) \leq \mu$;

(ii) for each $b \in B$ and $a \in A$ such that $d_b(a, b) \leq \mu$.

Then $H(A, B) \leq \mu$."

Lemma 4. [24]

"Let (X, d) be a b-metric space then

$$D(x, A) \leq b[D(x, B) + H(A, B)], \quad \forall\ x \in X \quad \text{and} \quad A, B \in P(X).\text{"}$$

Lemma 5. [24]

"Let (X, d) be a b-metric space. Then for all $A, B, C \in P(X)$

$$H(A, C) \leq b[H(A, B) + H(B, C)].\text{"}$$

3.4.5. L-space [43]

"Let $(X, \longrightarrow)$ be an L-space. An operator $f : X \longrightarrow X$ is, by definition, a Picard operator if

(i) $Fixf = x^*$;

(ii) $f^n \to x^*$, as $n \longrightarrow \infty$, for all $x \in X$."

3.4.6. ϕ-Contraction [37]

"Let (X, d) be a complete metric space and $f : X \longrightarrow X$ is a ϕ-contraction, that is $\phi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a comparison function and $d(f(x), f(y)) \leq \phi(d(x, y))$, for all $x, y \in X$."

Theorem 3.4.7 [37]

"Let (X, d) be a complete metric space and $f : X \longrightarrow X$ is a ϕ-contraction, that is $\phi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a comparison function and $d(f(x), f(y)) \leq \phi(d(x, y))$, for all $x, y \in X$. Then the operator f is Picard."

3.5 Fractals in b-metric spaces

3.5.1. Multivalued Fractals [43]

"Let (X, d) be a metric space and $F_1, ..., F_m : X \longrightarrow P(X)$ be multivalued operators. The system $F = (F_1, ..., F_m)$ is called an iterated multifunction system(briefly IMS). If $F = (F_1, ..., F_m)$ is an iterated multifunction system such that $F_i : X \longrightarrow P_{cp}(X)$ is upper semi continuous, for $i \in \{1, ..., m\}$, then the operator

$$T_F : P_{cp}(X) \longrightarrow P_{cp}(X),$$
$$T_F(Y) = \bigcup_{i=1}^{m} F_i(Y),$$

is said to multi-fractal operator generated by the iterated multifunction system. A fixed point of T_F is, by definition, a multivalued fractal."

3.5.2. Fixed Point of a Multivalued Fractals [43]

"If (X, d) is a complete metric space and $F_i : X \longrightarrow P_{cp}(X)$ are multivalued α_i-contractions (for $i \in \{1, ..., m\}$), then the multi-fractal operator T_F is a single-valued α-contractions (where $\alpha = \max\limits_{1 \leq i \leq m} \alpha_i$) and hence it is a Picard operator. The unique fixed point $V_F^* \in P_{cp}(X)$ of T_F is a multivalued fractal. Moreover, because for each $V_0 \in P_{cp}(X)$ we have $T_F^n(V_0) \longrightarrow V_F^*$, as $n \longrightarrow \infty$, the set V_F^* is an attractor of the iterated multifunction system."

3.6 Results in b-Metric Spaces

The following results are proved by [24] in b-metric spaces

Theorem 3.6.1 [24]

"Let (W, d_b) be a complete b-metric space such that the b-metric is a continuous functional on $W \times W$. Also $\phi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ a b-comparison function and

$\eta : W \longrightarrow W$ is a ϕ-contraction. Then η is Picard operator with unique fixed point w^*"

Proof.

Let $w_0 \in W$ and $w_n = \eta(w_{n-1}),\ n \geq 1$. For $n \geq 1$ we have,

$$\begin{aligned} d_b(w_n, w_{n+1}) &= d_b(\eta(w_{n-1}), \eta(w_n)) \\ &\leq \phi(d_b(w_{n-1}, w_n)) \\ &\ \vdots \quad \vdots \qquad \vdots \\ &\leq \phi^n d_b(w_0, w_1). \end{aligned}$$

Since (W, d_b) is cone b-metric space hence for $n \geq 0$, on the other hand for each $m > n$, we have;

$$\begin{aligned} d_b(w_n, w_m) &\leq b[d_b(w_n, w_{n+1}) + d_b(w_{n+1}, w_m)], \\ &\leq bd_b(w_n, w_{n+1}) + b[b(d_b(w_{n+1}, w_{n+2}) + d_b(w_{n+2}, w_m)))], \\ &= bd_b(w_n, w_{n+1}) + b^2(d_b(w_{n+1}, w_{n+2}) + b^2 d_b(w_{n+2}, w_m)), \\ &\leq bd_b(w_n, w_{n+1}) + b^2(d_b(w_{n+1}, w_{n+2}) + b^2[b((d_b(w_{n+2}, w_{n+3}) \\ &\quad + d_b(w_{n+3}, w_m)))], \end{aligned}$$

$$\begin{aligned}
&= bd_b(w_n, w_{n+1}) + b^2(d_b(w_{n+1}, w_{n+2}) + b^3 d_b(w_{n+2}, w_{n+3}) \\
&\quad + b^3 d_b(w_{n+3}, w_m), \\
&\leq bd_b(w_n, w_{n+1}) + b^2(d_b(w_{n+1}, w_{n+2}) + b^3(d_b(w_{n+2}, w_{n+3}) \\
&\quad + b^3 d_b(w_{n+3}, w_{n+4})) + \cdots + b^m(d_b(w_{m-1}, w_m)).
\end{aligned}$$

Now by applying ϕ contraction on each term of above expression,

$$\begin{aligned}
d_b(w_n, w_m) \leq b(\phi^n)d_b(w_0, w_1) &+ b^2(\phi^{n+1})d_b(w_0, w_1) \\
&+ b^3(\phi^{n+2})(d_b(w_{n+2}, w_{n+3}) + b^3(\phi^{n+3})d_b(w_0, w_1) + \cdots \\
&+ b^m(\phi^{m-1})d_b(w_0, w_1).
\end{aligned}$$

Hence we obtain,

$$d_b(w_n, w_m) \leq b\sum_{s=1}^{m-1} b^s \phi^s d_b(w_0, w_1)$$

Replace $d_b(w_0, w_1) = t$ in above, we must have,

$$\Rightarrow d_b(w_n, w_m) \leq b\sum_{s=1}^{m-1} b^s \phi^s(t)$$

Since,

$$d_b(w_n, w_m) \leq b\sum_{s=1}^{m-1} b^s \phi^s(t) \leq b\sum_{s=1}^{\infty} b^s \phi^s(t). \tag{3.1}$$

Since $\phi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a b-comparison function for which there exist a convergent series of positive terms $\sum_{s=1}^{\infty} v_s$ and there exist $s_0 \in \mathbb{N}$ $a \in (0, 1)$.

$$b^{s+1}\phi^{s+1}(t) \leq ab^s\phi^s(t) + v_s \text{ for each } t \in \mathbb{R}_+, \text{ and each } s \geq s_0 \tag{3.2}$$

Using generalized ratio test then the series in (3.1) converges for each $t \in \mathbb{R}_+$ and its sum is monotone increasing and continuous at zero.

Thus for $m > n$, letting $n \longrightarrow \infty$ we conclude that $\{w_n\}$ is a Cauchy sequence in a complete b-metric space (W, d_b). So there is a $w^* \in W$ such that

$$\lim_{m \longrightarrow \infty} w_n = w^*.$$

To prove that η is a Picard operator we will show that w^* is a unique fixed point of η.
For $n \geq 0$

$$\begin{aligned} d_b(w_{n+1}, \eta(w^*)) =& d_b(\eta(w_n), \eta(w^*)) \\ \leq& \phi(d_b(w_n, w^*)). \end{aligned}$$

But d is continuous and by the definition of ϕ it is also continuous at 0.
Letting $n \longrightarrow \infty$ we obtain that,

$$d_b(w^*, \eta(w^*)) = 0.$$

that is w^* is the fixed point.
To prove uniqueness, let v^* is another fixed point then,

$$d_b(w^*, v^*) = d_b(\eta(w^*), \eta(v^*)) \leq \phi(d_b(w^*, v^*)) \leq d_b(w^*, v^*).$$

Hence η is a Picard operator.

Theorem 3.6.2 Abstract College Theorem [24]
"Let (W, d) be a complete b-metric space with $(b \geq 1)$ such that the b-metric is continuous functional on $W \times W$. Let $\phi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a b-comparison function and $\eta : W \longrightarrow W$ is a ϕ contraction.
If the function $\tau : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, defined by

$$\tau(s) = s - b\phi(s)$$

is strictly increasing and onto, then

$$d(w, w^*) \leq \tau^{-1}(bd(w, \eta(w))), \quad \text{for each} \quad w \in W."$$

Proof.

By Theorem 3.6.1 we know that η is a Picard operator.

Let $w \in W$ be an arbitrary then by using the triangular property of b-metric spaces,

$$d(w, \eta(w)) \leq b\{d(w, w^*) + d(w^*, \eta(w))\}. \tag{3.3}$$

By using the symmetry of b-metric spaces that is $d(w^*, \eta(w)) = d(\eta(w), w^*)$ in (3.3) we must have,

$$\Rightarrow d(w, \eta(w)) \leq b\{d(w, w^*) + d(\eta(w), w^*)\}. \tag{3.4}$$

Since η is a ϕ contraction and w^* is a unique fixed point.

$$\Rightarrow d(\eta(w), w^*) \leq \phi(d(w, w^*)), \text{ as we know that } \eta(w^*) = w^*.$$

Therefore (3.4) becomes

$$\Rightarrow d(w, \eta(w)) \leq b\{d(w, w^*) + \phi(d(w, w^*))\},$$

Now as given that,

$$\tau(s) = s - b\phi(s) \tag{3.5}$$

is strictly increasing and onto, then replace $s = d(w, w^*)$ in (3.5) we obtain,

$$\tau(d(w, w^*)) = d(w, w^*) - b\phi(d(w, w^*)). \tag{3.6}$$

Now by using the triangular inequality of b-metric space on $d(w, w^*)$ in (3.6) we have,

$$\tau(d(w, w^*)) \leq b\{d(w, \eta(w)) + d(\eta(w), w^*)\} - b\phi(d(w, w^*)),$$
$$\Rightarrow \tau(d(w, w^*)) \leq bd(w, \eta(w)) + bd(\eta(w), w^*) - b\phi(d(w, w^*)).$$

As $d(\eta(w), w^*) \leq \phi(d(w, w^*))$, therefore,

$$\begin{aligned}
\tau(d(w, w^*)) &\leq \{bd(w, \eta(w)) + bd(\eta(w), w^*) - bd(\eta(w), w^*)\},\\
\Rightarrow \tau(d(w, w^*)) &\leq \{bd(w, \eta(w))\},\\
\Rightarrow \quad (d(w, w^*)) &\leq \frac{1}{\tau}\{bd(w, \eta(w))\},\\
\Rightarrow \quad d(w, w^*) &\leq \tau^{-1}(bd(w, \eta(w))).
\end{aligned}$$

Theorem 3.6.3 Abstract Anti College Theorem [24]

"Let (W, d) be a complete b-metric space with $(b \geq 1)$ such that the b-metric is continuous functional on $W \times W$. Let $\phi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a b-comparison function and $\eta : W \longrightarrow W$ is a ϕ contraction. If the function $\zeta : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, defined by

$$\zeta(s) = s + \phi(s)$$

is strictly increasing and onto, then:

$$d(w, w^*) \geq \zeta^{-1}(\frac{1}{b}d(w, \eta(w))), \quad \text{for each} \quad w \in W."$$

Proof.

By Theorem 3.6.1 we know that η is a Picard operator.

Let $w \in W$ be an arbitrary then by using the triangular property of b-metric spaces,

$$d(w, \eta(w)) \leq b\{d(w, w^*) + d(w^*, \eta(w))\}. \tag{3.7}$$

By using the symmetry of b-metric spaces that is $d(w^*, \eta(w)) = d(\eta(w), w^*)$ in (3.7) we must have

$$\Rightarrow d(w, \eta(w)) \leq b\{d(w, w^*) + d(\eta(w), w^*)\}. \tag{3.8}$$

Since η is a ϕ contraction and w^* is a unique fixed point.

$$d(\eta(w), w^*) \leq \phi(d(w, w^*)), \text{ as we know that } \eta(w^*) = w^*.$$

Therefore (3.8) becomes,

$$\Rightarrow d(w,\eta(w)) \leq b\{d(w,w^*) + \phi(d(w,w^*))\}, \tag{3.9}$$

Now as given that,

$$\zeta(s) = s + \phi(s) \tag{3.10}$$

is strictly increasing and onto, then. Replace $s = d(w,w^*)$ in (3.10) we have,

$$\Rightarrow \zeta(d(w,w^*)) = d(w,w^*) + \phi(d(w,w^*)),$$

$$\Rightarrow \phi(d(w,w^*)) = \zeta(d(w,w^*)) - d(w,w^*). \tag{3.11}$$

using the value of $\phi(d(w,w^*))$ from equation (3.11) and replace in (3.9) we have,

$$\begin{aligned} d(w,\eta(w)) &\leq b\{d(w,w^*) + \zeta(d(w,w^*)) - d(w,w^*)\}, \\ \Rightarrow d(w,\eta(w)) &\leq b\{\zeta(d(w,w^*))\}. \end{aligned}$$

Since ζ is strictly increasing and bijection.

$$\begin{aligned} b\zeta(d(w,w^*)) &\geq d(w,\eta(w)), \\ \Rightarrow \quad \zeta(d(w,w^*)) &\geq \frac{1}{b}d(w,\eta(w)), \\ \Rightarrow \quad (d(w,w^*)) &\geq \zeta^{-1}\frac{1}{b}d(w,\eta(w)). \end{aligned}$$

Lemma 6. [24]
"Let (W,d) be a complete b-metric space with $(b \geq 1)$ and $F_i : W \longrightarrow P_{cp}(W)$ be a multivalued ϕ-contractive operator. Then for any $V \in P_{cp}(W)$ we have that $F(V) \in P_{cp}(W)$."

Proof.

Assume a sequence $\{v_n\}$ in $F(V)$. Since $\{v_n\} \subset F(V)$ there exist a sequence $\{z_n\} \subset V$ such that $v_n \in F(z_n),\ n \in \mathbb{N}$.

We may suppose that,

$$\{z_n\} \longrightarrow z, \text{ and } z_n \neq z, \text{ for each } n \in \mathbb{N}.$$

Then by using Lemma 3., for $v_n \in F(z_n)$ there exist $q_n \in F(z)$ such that the,

$$d_b(v_n, q) \leq bH(F(z_n), F(z)) \leq bd_b(z_n, z) \longrightarrow 0 \text{ as } n \longrightarrow +\infty.$$

Hence we have,

$$d_b(v_n, q_n) \longrightarrow 0 \text{ as } n \longrightarrow +\infty.$$

As $F(z)$ is a compact set, we obtain a sub-sequence by $\{q_n\}$ which converges to an element $q \in F(z)$. We denote this sub-sequence $\{q_n\}$ by itself. Then we have,

$$d_b(v_n, q) \leq b[d_b(v_n, q_n) + d_b(q_n, q)] \longrightarrow 0 \text{ as } n \longrightarrow +\infty.$$

Thus $\{v_n\} \longrightarrow q \in F(z) \subset F(V)$.
This complete the proof.

Theorem 3.6.4 [24]
"Let (W, d) be a complete b-metric space with $(b \geq 1)$ such that the cone b-metric is continuous functional on $W \times W$. Let $\phi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a b-comparison function. Also $F_i : W \longrightarrow P_{cp}(W)$ b a multivalued ϕ contractions. Then:

(1) $T_F : (P_{cp}(W), H_{d_b}) \longrightarrow (P_{cp}(W), H_{d_b})$;

(2) T_F is ϕ-contraction;

(3) T_F is a Picard operator having a unique fixed point $V^* \in P_{cp}(W)$ which is a multi-valued fractal and an attractor of $IMSF = (F_1, F_2, ...F_m)$."

Proof.

1) Let $\phi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ being a b-comparison function, also a comparison function.

So by using Lemma 3., we must have

$$H(F_i(w_1), F_i(w_2)) \leq \phi d_b(w_1, w_2) \leq d_b(w_1, w_2) \text{ for all } w_1, w_2 \in W, \ \phi(t) < t \text{ for } t > 0,$$

therefore by Lemma 6., we get that,

$$T_F : (P_{cp}(W), H_{d_b}) \longrightarrow (P_{cp}(W), H_{d_b})$$

2) We show that,

$$H(T_F(V_1), T_F(V_2)) \leq \phi H(V_1, V_2) \text{ for all } V_1, V_2 \in P_{cp}(W).$$

In order to prove above, consider $V_1, V_2 \in P_{cp}(W)$ and let $u_1 \in T_F(V_1)$. Then there exist $i \in \{1, 2, ...m\}$ such that $u_1 \in F_i(V_1)$. Moreover, there exist $a_1 \in V_1$ such that $u_1 \in F_i(V_1)$. Since V_1, V_2 are compact for $a_1 \in V_1$, there exist $b_1 \in V_2$ such that

$$d_b(a_1, b_1) \leq H(V_1, V_2). \tag{3.12}$$

So for $u_1 \in F_i(V_1)$, by Lemma 3.3.1, there exist $v_1 \in F_i(b_1)$ such that

$$d_b(u_1, v_1) \leq H(F_i(a_1), F_i(b_1)). \tag{3.13}$$

Thus by (3.12) and (3.13) we get for each $u_1 \in T_F(V_1)$ there exist $v_1 \in T_F(V_2)$ such that

$$d_b(u_1, v_1) \leq H(F_i(a_1), F_i(b_1)) \leq \phi d_b(a_1, b_1) \leq \phi H(V_1, V_2). \tag{3.14}$$

By similar procedure we obtain for each $v_1 \in T_F(V_2)$ there exist $u_1 \in T_F(V_1)$ such that

$$d_b(u_1, v_1) \leq \phi H(V_1, V_2). \tag{3.15}$$

By by Lemma 3.3.1, equation (3.14), (3.15), together imply

$$H(T_F(V_1), T_F(V_2)) \leq \phi H(V_1, V_2). \tag{3.16}$$

Thus we obtain that T_F is a self ϕ contraction on a complete metric space $(P_{cp}(W), H_{d_b})$. Now (3) can be obtain from Theorem 3.1.1.

Theorem 3.6.5 College Theorem [24]

"Let (W, d) be a complete b-metric space with $(b \geq 1)$ such that the b-metric is continuous functional on $W \times W$. Let $\phi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a b-comparison function and $F_i : W \longrightarrow P_{cp}(W)$ is a multivalued ϕ contraction.

If the function $\tau : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, defined by,

$$\tau(s) = s - b\phi(s)$$

is strictly increasing and onto, then:

$$H(V, V_F^*) \leq \tau^{-1}(bH(V, T_F(V))), \quad \text{for each} \quad V \in P_{cp}(W)."$$

Proof.

By Theorem 3.6.1 we know that T_F is a Picard operator.

Let $V \in P_{cp}(W)$ be an arbitrary then by using the triangular property of b-metric spaces,

$$H(V, V_F^*) \leq b\{H(V, T_F(V)) + H(T_F(V), V_F^*)\}. \tag{3.17}$$

Since T_F is a ϕ contraction and V_F^* is a unique fixed point,

$$H(T_F(V), V_F^*) \leq \phi(H(V, V_F^*)), \quad \text{also } T_F(V^*) = V_F^*.$$

Therefore (3.17) becomes,

$$\Rightarrow H(V, V_F^*) \leq b\{H(V, T_F(V)) + \phi(H(V, V_F^*))\}.$$

Now as given that,

$$\tau(s) = s - b\phi(s) \tag{3.18}$$

is strictly increasing and onto, replacing $s = H(V, V_F^*)$ in (3.18) we must have

$$\Rightarrow \tau(H(V, V_F^*)) = H(V, V_F^*) - b\phi(H(V, V_F^*)), \tag{3.19}$$

Now by using the triangular property of b-metric space on $H(V, V_F^*)$ in (3.19)

$$\begin{aligned} \tau(H(V, V_F^*)) &\leq b\{H(V, T_F(V)) + H(T_F(V), V_F^*)\} - b\phi(H(V, V_F^*)), \\ \Rightarrow \tau(H(V, V_F^*)) &\leq bH(V, T_F(V)) + bH(T_F(V), V_F^*) - b\phi(H(V, V_F^*)). \end{aligned}$$

Now replace $H(T_F(V), V_F^*) \leq \phi(H(V, V_F^*))$.

$$\begin{aligned} \tau(H(V, V_F^*)) &\leq \{bH(V, T_F(V)) + bH(T_F(V), V_F^*) - bH(T_F(V), V_F^*)\}, \\ \Rightarrow \tau(H(V, V_F^*)) &\leq \{bH(V, T_F(V))\}, \\ \Rightarrow \quad (H(V, V_F^*)) &\leq \frac{1}{\tau}\{bH(V, T_F(V))\}, \\ \Rightarrow \quad (H(V, V_F^*)) &\leq \tau^{-1}(bH(V, T_F(V))). \end{aligned}$$

Theorem 3.6.6 Anti-College Theorem [24]

"Let (W, d) be a complete b-metric space with $(b \geq 1)$ such that the b-metric is continuous functional on $W \times W$. Let $\phi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a b-comparison function and $F_i : W \longrightarrow P_{cp}(W)$ is a multivalued ϕ contraction.

If the function $\zeta : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$,

$$\zeta(s) = s + \phi(s)$$

is onto, then

$$H(V, V_F^*) \leq \zeta^{-1}(\frac{1}{b}H(V, T_F(V))), \quad \text{for each} \quad V \in P_{cp}(W)."$$

Proof.

By Theorem 3.6.1 we know that T_F is a Picard operator.

Let $V \in P_{cp}(W)$ be an arbitrary then by using the triangular property of b-metric spaces,

$$H(V, V_F^*) \leq b\{H(V, T_F(V)) + H(T_F(V), V_F^*)\}. \tag{3.20}$$

Since T_F is a ϕ contraction and V_F^* is a unique fixed point,

$$H(T_F(V), V_F^*) \leq \phi(H(V, V_F^*)), \quad \text{also } T_F(V^*) = V_F^*.$$

Therefore (3.20) becomes,

$$\Rightarrow H(V, V_F^*) \leq b\{H(V, T_F(V)) + \phi(H(V, V_F^*))\}.$$

Now as given that,

$$\zeta(s) = s + \phi(s) \tag{3.21}$$

is strictly increasing and onto, then replacing $s = H(V, V_F^*)$, in (3.21) we have,

$$\begin{aligned} \zeta(H(V, V_F^*)) &= (H(V, V_F^*)) + \phi(H(V, V_F^*)), \\ \Rightarrow \phi(H(V, V_F^*)) &= \zeta(H(V, V_F^*)) - (H(V, V_F^*)). \end{aligned}$$

Now putting the value of $\phi(H(V, V_F^*))$ from above into (3.20) we get

$$\begin{aligned} &\Rightarrow H(V, T_F(V)) \leq b\{(H(V, V_F^*)) + \zeta(H(V, V_F^*)) - (H(V, V_F^*))\}, \\ &\Rightarrow H(V, T_F(V)) \leq b\{\zeta(H(V, V_F^*))\}. \end{aligned}$$

As ζ is strictly increasing and bijection so we have,

$$\begin{aligned} b\zeta(H(V, V_F^*)) &\geq H(V, T_F(V)), \\ \Rightarrow \ \zeta(H(V, V_F^*)) &\geq \frac{1}{b} H(V, T_F(V)), \\ \Rightarrow \ (H(V, V_F^*)) &\geq \zeta^{-1} \frac{1}{b} H(V, T_F(V)). \end{aligned}$$

Chapter 4

Multivalued Fractals in Cone b-Metric Spaces

This chapter is the extention of the results presented in [24] in the setting of cone b-metric spaces over the Banach algebra $\mathcal{A}$. In the start of this chapter some definitions has been introduced which will be used in the main result.

4.1 Basic Tools

4.1.1. ϕ-Contractions in Cone b-Metric Spaces

Consider Banach algebra $\mathcal{A}$ with the solid cone K and (V, d), (S, ρ) be the two cone b-metric spaces. An operator $T : V \longrightarrow S$ is said to be cone b ϕ-contraction if $\phi : \mathcal{A}_+ \longrightarrow \mathcal{A}_+$ satisfies,

$$\rho(T(t), T(s)) \preceq \phi(d(t, s)), \text{ for all } t, s \in V.$$

4.1.2. Generalized ϕ-Contractions in Cone b-Metric Spaces

Consider Banach algebra $\mathcal{A}$ with the solid cone K and (W, d) be the cone b-metric space. An operator $\eta : W \longrightarrow W$ is said to be generalized cone b ϕ-contraction if

$\phi : \mathcal{A}_+ \longrightarrow \mathcal{A}_+$ satisfies,

$$d(\eta(w_1), \eta(w_2)) \preceq \phi(d(w_1, w_2)), \text{ for all } w_1, w_2 \in W.$$

4.1.3. Cone b-Comparison Function

Consider a solid cone K over Banach algebra $\mathcal{A}$. A mapping $\phi : \mathcal{A}_+ \longrightarrow \mathcal{A}_+$ is called cone b-comparison function over cone b-metric space ($b \geq 1$), if ϕ is increasing and there exist $s_0 \in \mathbb{N}$, $a \in (0,1)$ and a convergent series of non-negative terms $\sum_{s=1}^{\infty} v_s$ satisfies,

$$b^{s+1}\phi^{s+1}(t) \preceq ab^s\phi^s(t) + v_s \text{ for each } t \in \mathcal{A}_+, \text{ and each } s \geq s_0.$$

As consequence, if $\phi : \mathcal{A}_+ \longrightarrow \mathcal{A}_+$ is a b-comparison function, then the series $\sum_{s=1}^{\infty} b^s\phi^s(t)$ converges for each $t \in \mathcal{A}_+$, and the function $c_b : \mathcal{A}_+ \longrightarrow \mathcal{A}_+$ defined by,

$$c_b(t) = \sum_{s=1}^{\infty} b^s\phi^s(t), \quad t \in \mathcal{A}_+,$$

is increasing and continuous at θ.

Example 4.1.

Consider a mapping $\phi : \mathcal{A} \longrightarrow \mathcal{A}$. Then the function

$$\phi(t) = \nu t \quad (\nu \in (0,1)),$$

is a comparison function. It is also a cone b-comparison function. It is clear that $\phi(t) = at \ll t$ for $\nu \in (0,1)$. Also $\phi(\theta) = \theta$ (because θ is the zero element of space) and ϕ is continuous at θ because

$$\lim_{t \longrightarrow \theta} \phi(t) = \theta.$$

4.2 Results in Cone b-Metric Spaces

Theorem 4.2.1 Let (W, d_b) be a complete cone b-metric space over the Banach algebra $\mathcal{A}$ with K be the solid cone. Also $\phi : \mathcal{A}_+ \longrightarrow \mathcal{A}_+$ be a cone b-comparison function and $\eta : W \longrightarrow W$ is a cone b ϕ-contraction. Then η is Picard operator with unique fixed point w^*.

Proof

Let $w_0 \in W$ and $w_n = \eta(w_{n-1}),\ n \geq 1$. For $n \geq 1$ we have,

$$\begin{aligned}
d_b(w_n, w_{n+1}) &= d_b(\eta(w_{n-1}), \eta(w_n)) \\
&\preceq \phi(d_b(w_{n-1}, w_n)) \\
&\ \ \vdots \quad \vdots \qquad \vdots \\
&\preceq \phi^n d_b(w_0, w_1).
\end{aligned}$$

Since (W, d_b) is cone b-metric space hence for $n \geq 0$, on the other hand for each $m > n$, we have,

$$\begin{aligned}
d_b(w_n, w_m) &\preceq b[d_b(w_n, w_{n+1}) + d_b(w_{n+1}, w_m)], \\
&\preceq bd_b(w_n, w_{n+1}) + b[b(d_b(w_{n+1}, w_{n+2}) + d_b(w_{n+2}, w_m)))], \\
&= bd_b(w_n, w_{n+1}) + b^2(d_b(w_{n+1}, w_{n+2}) + b^2 d_b(w_{n+2}, w_m)), \\
&\preceq bd_b(w_n, w_{n+1}) + b^2(d_b(w_{n+1}, w_{n+2}) + b^2[b((d_b(w_{n+2}, w_{n+3}) \\
&\qquad + d_b(w_{n+3}, w_m)))], \\
&= bd_b(w_n, w_{n+1}) + b^2(d_b(w_{n+1}, w_{n+2}) + b^3 d_b(w_{n+2}, w_{n+3}) \\
&\qquad + b^3 d_b(w_{n+3}, w_m), \\
&\preceq bd_b(w_n, w_{n+1}) + b^2(d_b(w_{n+1}, w_{n+2}) + b^3(d_b(w_{n+2}, w_{n+3}) \\
&\qquad + b^3 d_b(w_{n+3}, w_{n+4})) + \cdots + b^m(d_b(w_{m-1}, w_m)).
\end{aligned}$$

Now by applying ϕ contraction on each term of above expression,

$$\begin{aligned} d_b(w_n, w_m) \preceq\ & b(\phi^n)d_b(w_0, w_1) + b^2(\phi^{n+1})d_b(w_0, w_1) \\ & + b^3(\phi^{n+2})(d_b(w_{n+2}, w_{n+3}) + b^3(\phi^{n+3})d_b(w_0, w_1) + \cdots \\ & + b^m(\phi^{m-1})d_b(w_0, w_1). \end{aligned}$$

Hence we obtain,

$$d_b(w_n, w_m) \preceq b\sum_{s=1}^{m-1} b^s \phi^s d_b(w_0, w_1)$$

Replace $d_b(w_0, w_1) = t$ in above, we must have,

$$\Rightarrow d_b(w_n, w_m) \preceq b\sum_{s=1}^{m-1} b^s \phi^s(t)$$

Since,

$$d_b(w_n, w_m) \preceq b\sum_{s=1}^{m-1} b^s \phi^s(t) \preceq b\sum_{s=1}^{\infty} b^s \phi^s(t). \tag{4.1}$$

Since $\phi : \mathcal{A}_+ \longrightarrow \mathcal{A}_+$ be a cone b-comparison function for which there exist a convergent series of positive terms $\sum_{s=1}^{\infty} v_s$ and a real number $a \in (0, 1)$.

$$b^{s+1}\phi^{s+1}(t) \preceq ab^s\phi^s(t) + v_s \;\text{ for each } t \in \mathcal{A}_+, \text{ and each } s \geq \mathbb{N} \tag{4.2}$$

By using generalized ratio test the series in (4.1) converges for each $t \in \mathcal{A}_+$ and its sum is monotone increasing and continuous at zero.
Thus for $m > n$, letting $n \longrightarrow \infty$ we conclude that $\{w_n\}$ is a Cauchy sequence in a complete cone b-metric space (W, d_b). So there is a $w^* \in W$ such that

$$\lim_{m \longrightarrow \infty} w_n = w^*.$$

To prove that η is a Picard operator we will show that w^* is a unique fixed point of η.

For $n \geq 0$

$$
\begin{aligned}
d_b(w_{n+1}, \eta(w^*)) &= d_b(\eta(w_n), \eta(w^*)) \\
&\preceq \phi(d_b(w_n, w^*)).
\end{aligned}
$$

But d is continuous and by the definition of ϕ it is also continuous at θ.
Letting $n \longrightarrow \infty$ we obtain that,

$$d_b(w^*, \eta(w^*)) = \theta.$$

that is w^* is the fixed point.
To prove uniqueness, let v^* is another fixed point then,

$$d_b(w^*, v^*) = d_b(\eta(w^*), \eta(v^*)) \preceq \phi(d_b(w^*, v^*)) \preceq d_b(w^*, v^*).$$

Hence η is a Picard operator.

Theorem 4.2.2 Abstract College Theorem

Consider Banach algebra $\mathcal{A}$ with K be the solid cone. Let (W, d) be a complete cone b-metric space with the base $(b \geq 1)$ such that the cone b-metric is continuous functional on $W \times W$. Let $\phi : \mathcal{A}_+ \longrightarrow \mathcal{A}_+$ be a cone b-comparison function and $\eta : W \longrightarrow W$ is a cone b ϕ-contraction.
If the function $\tau : \mathcal{A}_+ \longrightarrow \mathcal{A}_+$, defined by,

$$\tau(s) = s - b\phi(s)$$

is strictly increasing and onto, then:

$$d(w, w^*) \preceq \tau^{-1}(bd(w, \eta(w))), \quad \text{for each} \quad w \in W.$$

Proof.
By Theorem 4.2.1 we know that η is a Picard operator.
Let $w \in W$ be an arbitrary then by using the triangular property of cone b-metric

spaces, we have

$$d(w,\eta(w)) \preceq b\{d(w,w^*) + d(w^*,\eta(w))\}. \tag{4.3}$$

By using the symmetry of cone b-metric spaces that is $d(w^*,\eta(w)) = d(\eta(w),w^*)$ in (4.3) we must have,

$$\Rightarrow d(w,\eta(w)) \preceq b\{d(w,w^*) + d(\eta(w),w^*)\}. \tag{4.4}$$

Since η is a ϕ contractions and w^* is a unique fixed point.

$$\Rightarrow d(\eta(w),w^*) \preceq \phi(d(w,w^*)), \text{ as we know that } \eta(w^*) = w^*.$$

Therefore (4.4) becomes

$$\Rightarrow d(w,\eta(w)) \preceq b\{d(w,w^*) + \phi(d(w,w^*))\},$$

Now as given that,

$$\tau(s) = s - b\phi(s) \tag{4.5}$$

is strictly increasing and onto, replace $s = d(w,w^*)$ in (4.5) we obtain,

$$\tau(d(w,w^*)) = d(w,w^*) - b\phi(d(w,w^*)). \tag{4.6}$$

Now by using the triangular inequality of cone b-metric space on $d(w,w^*)$ in (4.6) we have,

$$\tau(d(w,w^*)) \preceq b\{d(w,\eta(w)) + d(\eta(w),w^*)\} - b\phi(d(w,w^*)),$$
$$\tau(d(w,w^*)) \preceq bd(w,\eta(w)) + bd(\eta(w),w^*) - b\phi(d(w,w^*)).$$

As $d(\eta(w), w^*) \preceq \phi(d(w, w^*))$, therefore,

$$\begin{aligned}
\tau(d(w, w^*)) &\preceq \{bd(w, \eta(w)) + bd(\eta(w), w^*) - bd(\eta(w), w^*)\}, \\
\Rightarrow \tau(d(w, w^*)) &\preceq \{bd(w, \eta(w))\}, \\
\Rightarrow \quad (d(w, w^*)) &\preceq \frac{1}{\tau}\{bd(w, \eta(w))\}, \\
\Rightarrow \qquad d(w, w^*) &\preceq \tau^{-1}(bd(w, \eta(w))).
\end{aligned}$$

Theorem 4.2.3 Abstract Anti College Theorem

Consider Banach algebra $\mathcal{A}$ with K be the solid cone. Let (W, d) be a complete cone b-metric space with the base $(b \geq 1)$ such that the cone b-metric is continuous functional on $W \times W$. Let $\phi : \mathcal{A}_+ \longrightarrow \mathcal{A}_+$ be a cone b-comparison function and $\eta : W \longrightarrow W$ is a cone b ϕ-contraction.

If the function $\zeta : \mathcal{A}_+ \longrightarrow \mathcal{A}_+$, defined by,

$$\zeta(s) = s + \phi(s)$$

is strictly increasing and onto, then:

$$d(w, w^*) \succeq \zeta^{-1}(\frac{1}{b}d(w, \eta(w))), \quad \text{for each} \quad w \in W.$$

Proof.

By using Theorem 4.2.1 we must have η is a Picard operator.

Let $w \in W$ an arbitrary then by using the triangular property of cone b-metric spaces,

$$d(w, \eta(w)) \preceq b\{d(w, w^*) + d(w^*, \eta(w))\}. \tag{4.7}$$

By using the symmetry of cone b-metric spaces that is $d(w^*, \eta(w)) = d(\eta(w), w^*)$ in (4.7) we must have

$$\Rightarrow d(w, \eta(w)) \preceq b\{d(w, w^*) + d(\eta(w), w^*)\}. \tag{4.8}$$

Since η is a ϕ contractions and w^* is a unique fixed point.

$$d(\eta(w), w^*) \preceq \phi(d(w, w^*)), \text{ as we know that } \eta(w^*) = w^*.$$

Therefore we get,

$$\Rightarrow d(w, \eta(w)) \preceq b\{d(w, w^*) + \phi(d(w, w^*))\}, \tag{4.9}$$

Now as given that,

$$\zeta(s) = s + \phi(s) \tag{4.10}$$

is strictly increasing and onto, replace $s = d(w, w^*)$ in (4.10) we have,

$$\Rightarrow \zeta(d(w, w^*)) = d(w, w^*) + \phi(d(w, w^*)),$$

$$\Rightarrow \phi(d(w, w^*)) = \zeta(d(w, w^*)) - d(w, w^*). \tag{4.11}$$

using the value of $\phi(d(w, w^*))$ from equation (4.11) and replace in (4.9) we have,

$$\begin{aligned} d(w, \eta(w)) &\preceq b\{d(w, w^*) + \zeta(d(w, w^*)) - d(w, w^*)\}, \\ \Rightarrow d(w, \eta(w)) &\preceq b\{\zeta(d(w, w^*))\}. \end{aligned}$$

Since ζ is strictly increasing and bijection, therefore

$$\begin{aligned} b\zeta(d(w, w^*)) &\succeq d(w, \eta(w)), \\ \Rightarrow \quad \zeta(d(w, w^*)) &\succeq \frac{1}{b} d(w, \eta(w)), \\ \Rightarrow \quad (d(w, w^*)) &\succeq \zeta^{-1} \frac{1}{b} d(w, \eta(w)). \end{aligned}$$

Remark 9. Let (W, d) be a complete cone b-metric space with$(b \geq 1)$ over the Banach algebra $\mathcal{A}$. Let K be the solid cone.

By taking $\mathcal{A} = \mathbb{R}$ and $K = [0, \infty)$, the cone b-metric space (W, d) becomes complete b-metric space (W, d) and the results of Boriceanu et al. [24] becomes special case of the above Theorems.

4.3 Cone b-Metric Spaces and Multi-Fractal Operators

4.3.1. Multivalued Fractal

Consider (W, d_b) be the cone b-metric space over Banach algebra $\mathcal{A}$ with solid cone K. Let $F_1, ..., F_p : W \longrightarrow P(W)$ be multivalued operators. The system $F = (F_1, ..., F_m)$ is called an iterated multifunction system. If $F = (F_1, ..., F_p)$ is an iterated multifunction system such that $F_i : W \longrightarrow P_{cp}(W)$ is upper semi continuous, for $i \in \{1, ..., p\}$, then the operator

$$T_F(Y) = \bigcup_{i=1}^{p} F_i(Y), \quad \text{for each } Y \in P_{cp}(W)$$

is said to multi-fractal operator generated by the iterated multifunction system. A fixed point of T_F is, by definition, a multivalued fractal.

4.3.2. Fixed Point of a Multivalued Fractals

Let a mapping $T_F : P_{cp}(W) \longrightarrow P_{cp}(W)$. A nonempty compact subset V^* of W is said to be multivalued fractal with respect to an iterated multifunction system $F = (F_1, ..., F_p)$ if and only if it is a fixed point of multi-fractal operator. In particular, if $F_i = \eta_i$ are continuous single valued then its fixed point is given by

$$\begin{aligned} T_\eta : P_{cp}(W) &\longrightarrow P_{cp}(W), \\ T_\eta(Y) &= \bigcup_{i=1}^{p} F_i(Y), \end{aligned}$$

generated by iterated function system $\eta = (\eta_1, ...\eta_p)$ is said to be a fractal.

4.3.3. Family of Subsets

Let (W, d_b) be a cone b-metric space and $S \subseteq W$ we have the following subsets.

(i) $P(W) = \{S|\ S \subset W\}$;

(ii) $P_b(W) = \{S \in P(W)|\ S \text{ is bounded }\}$;
$P_{cp}(W) = \{S \in P(W)|\ S \text{ is compact }\}$;

(iii) $P_{cl}(W) = \{S \in P(W)|\ S \text{ is closed }\}$; $P_{b,cl}(W) = P_b(W) \cap P_{cl}(W)$.

Now introduce the following generalized functions on cone b-metric space (W, d_b).

4.3.4. The Gap Functional in Cone b-Metric Spaces

Let $D : P(W) \times P(W) \longrightarrow \mathcal{A}_+ \cup \{+\infty\}$ then

$$D(V,S) = \begin{cases} \inf\{d_b(u,s)|\ u \in V, s \in S\} & \text{if} \quad V \neq \emptyset \neq S; \\ \theta & \text{if} \quad V = \emptyset = S; \\ +\infty & \text{otherwise.} \end{cases}$$

In particular, if $w_0 \in W$ then $D(w_0, S) = D(\{w_0\}, S)$.

4.3.5. The Excess Generalized Functional in Cone b-Metric Spaces

Let $\rho : P(W) \times P(W) \longrightarrow \mathcal{A}_+ \cup \{+\infty\}$ then

$$\varrho(V,S) = \begin{cases} \sup\{D(u,S)|\ u \in V\} & \text{if} \quad V \neq \emptyset \neq S; \\ \theta & \text{if} \quad V = \emptyset; \\ +\infty & \text{if} \quad S = \emptyset \neq V. \end{cases} .$$

4.3.6. Pompeiu-Hausdorff Generalized Functional in Cone b-Metric Spaces

Let $H : P(W) \times P(W) \longrightarrow \mathcal{A}_+ \cup \{+\infty\}$ then

$$H(V,S) = \begin{cases} \max\{\varrho(V,S),\ \varrho(S,V)\} & \text{if} \quad V \neq \emptyset \neq S, \\ \theta & \text{if} \quad V = \emptyset = S, \\ +\infty & \text{otherwise.} \end{cases}$$

Then $(P_{cp}(W), H)$ is a complete cone b-metric space provided (W, d_b) is a complete cone b-metric space.

Lemma 7.

Let (W, d) be a cone b-metric space and $V_1, V_2 \in P_{cp}(W)$. We assume that there exist a $\mu > 0$ such that

(i) for each $u_1 \in V_1$ and $v_1 \in V_2$ such that $d_b(u_1, v_1) \preceq \mu$;

(ii) for each $v_1 \in V_2$ and $u_1 \in V_1$ such that $d_b(u_1, v_1) \preceq \mu$.

Then $H(V_1, V_2) \preceq \mu$.

Lemma 8. Consider Banach algebra $\mathcal{A}$ with K be the solid cone. Let (W, d) be a complete cone b-metric space with the base $(b \geq 1)$ and $F_i : W \longrightarrow P_{cp}(W)$ be a multivalued ϕ contractive operators. Then for any $V \in P_{cp}(W)$ we have that $F(V) \in P_{cp}(W)$.

Proof.

Assume a sequence $\{v_n\}$ in $F(V)$. Since $\{v_n\} \subset F(V)$ there exist a sequence $\{z_n\} \subset V$ so that $v_n \in F(z_n),\ n \in \mathbb{N}$.

We may suppose that,

$$\{z_n\} \longrightarrow z, \text{ and } z_n \neq z, \text{ for each } n \in \mathbb{N}.$$

Then by using above Lemma 7., for $v_n \in F(z_n)$ there exist $q_n \in F(z)$ such that the,

$$d_b(v_n, q) \preceq bH(F(z_n), F(z)) \preceq bd_b(z_n, z) \longrightarrow \theta \text{ as } n \longrightarrow +\infty.$$

Hence we have,

$$d_b(v_n, q_n) \longrightarrow \theta \text{ as } n \longrightarrow +\infty.$$

As $F(z)$ is a compact set, we get a sub-sequence by $\{q_n\}$ which converges to a element $q \in F(z)$. We denote this sub-sequence by $\{q_n\}$ by itself. Then we have:

$$d_b(v_n, q) \preceq b[d_b(v_n, q_n) + d_b(q_n, q)] \longrightarrow \theta \quad \text{as} \quad n \longrightarrow +\infty.$$

Thus $\{v_n\} \longrightarrow q \in F(z) \subset F(V)$.
This complete the proof.

Proceed to prove the results of [24] in the setting of H cone b-metric spaces.

Theorem 4.3.7 Consider Banach algebra $\mathcal{A}$ with K be the solid cone. Let (W, d) be a complete cone b-metric space with $(b \geq 1)$ such that the cone b-metric is continuous functional on $W \times W$. Let $\phi : \mathcal{A}_+ \longrightarrow \mathcal{A}_+$ be a cone b-comparison function. Also $F_i : W \longrightarrow P_{cp}(W)$ b a multivalued ϕ-contractions.

(1) $T_F : (P_{cp}(W), H_{d_b}) \longrightarrow (P_{cp}(W), H_{d_b})$;

(2) T_F is ϕ-contraction;

(3) T_F is a Picard operator with a unique fixed point $V^* \in P_{cp}(W)$ which is a multi-valued fractal and an attractor of $IMSF = (F_1, F_2, ...F_m)$;

Proof.
1)Let $\phi : \mathcal{A}_+ \longrightarrow \mathcal{A}_+$ is a cone b-comparison function, the also a comparison function. So by using Lemma 7., we must have

$$H(F_i(w_1), F_i(w_2)) \preceq \phi(d_b(w_1, w_2)) \ll d_b(w_1, w_2) \quad \text{for all} \quad w_1, w_2 \in W, \ \phi(t) \gg t \text{ for } t > 0,$$

therefore by Lemma 8., we get that,

$$T_F : (P_{cp}(W), H_{d_b}) \longrightarrow (P_{cp}(W), H_{d_b})$$

2) We show that

$$H(T_F(V_1), T_F(V_2)) \preceq \phi H(V_1, V_2) \quad \text{for all} \quad V_1, V_2 \in P_{cp}(W).$$

In order to prove above, consider $V_1, V_2 \in P_{cp}(W)$ and let $u_1 \in T_F(V_1)$. Then there exist $i \in \{1, 2, ...m\}$ such that $u_1 \in F_i(V_1)$. Also we have $a_1 \in V_1$ such that $u_1 \in F_i(V_1)$. Since $V_1,\ V_2$ are compact for $a_1 \in V_1$, there exist $b_1 \in V_2$ such that

$$d_b(a_1, b_1) \preceq H(V_1, V_2). \tag{4.12}$$

So for $u_1 \in F_i(V_1)$, by Lemma 7., there exist $v_1 \in F_i(b_1)$ such that

$$d_b(u_1, v_1) \preceq H(F_i(a_1), F_i(b_1)). \tag{4.13}$$

Thus by (4.12) and (4.13) we get for each $u_1 \in T_F(V_1)$ there exist $v_1 \in T_F(V_2)$ such that

$$d_b(u_1, v_1) \preceq H(F_i(a_1), F_i(b_1)) \preceq \phi(d_b(a_1, b_1)) \preceq \phi(H(V_1, V_2)). \tag{4.14}$$

By similar procedure we obtain for each $v_1 \in T_F(V_2)$ there exist $u_1 \in T_F(V_1)$ such that

$$d_b(u_1, v_1) \preceq \phi H(V_1, V_2). \tag{4.15}$$

By by Lemma 4.2.1, (4.14), (4.15), together imply

$$H(T_F(V_1), T_F(V_2)) \preceq \phi H(V_1, V_2). \tag{4.16}$$

Thus we obtain that T_F is a self ϕ contraction on a complete metric space $(P_{cp}(W), H_{d_b})$. Now (3) to (5) can be obtained immediately from Theorem 4.2.1.

Theorem 4.3.8 College Theorem

Consider Banach algebra $\mathcal{A}$ with K be the solid cone. Let (W, d) be a complete cone b-metric space with $(b \geq 1)$ such that the cone b-metric is continuous functional on $W \times W$. Let $\phi : \mathcal{A}_+ \longrightarrow \mathcal{A}_+$ be a cone b-comparison function. Also $F_i : W \longrightarrow P_{cp}(W)$ be a multivalued ϕ contractions. If the function

$\tau : \mathcal{A}_+ \longrightarrow \mathcal{A}_+$, defined by,

$$\tau(s) = s - b\phi(s)$$

a strictly increasing and onto, then:

$$H(V, V_F^*) \preceq \tau^{-1}(bH(V, T_F(V)))$$

for each $V \in P_{cp}(W)$.

Proof.

By Theorem 4.2.1 we know that T_F is a Picard operator.

Let $V \in P_{cp}(W)$ be an arbitrary then by using the triangular property of cone b-metric spaces,

$$H(V, V_F^*) \preceq b\{H(V, T_F(V)) + H(T_F(V), V_F^*)\}. \tag{4.17}$$

Since T_F is a ϕ contractions and V_F^* is a unique fixed point,

$$H(T_F(V), V_F^*) \preceq \phi(H(V, V_F^*)), \quad \text{also } T_F(V^*) = V_F^*.$$

Therefore (4.17) becomes,

$$\Rightarrow H(V, V_F^*) \preceq b\{H(V, T_F(V)) + \phi(H(V, V_F^*))\}.$$

Now as given that,

$$\tau(s) = s - b\phi(s) \tag{4.18}$$

is strictly increasing and onto, replacing $s = H(V, V_F^*)$ in (4.18) we must have

$$\Rightarrow \tau(H(V, V_F^*)) = H(V, V_F^*) - b\phi(H(V, V_F^*)), \tag{4.19}$$

Now by using the triangular property of cone b-metric space on $H(V, V_F^*)$ in (4.19)

$$\Rightarrow \tau(H(V, V_F^*)) \preceq b\{H(V, T_F(V)) + H(T_F(V), V_F^*)\} - b\phi(H(V, V_F^*)),$$
$$\Rightarrow \tau(H(V, V_F^*)) \preceq bH(V, T_F(V)) + bH(T_F(V), V_F^*) - b\phi(H(V, V_F^*)).$$

Now replace $H(T_F(V), V_F^*) \preceq \phi(H(V, V_F^*))$.

$$\tau(H(V, V_F^*)) \preceq \{bH(V, T_F(V)) + bH(T_F(V), V_F^*) - bH(T_F(V), V_F^*)\},$$
$$\Rightarrow \tau(H(V, V_F^*)) \preceq \{bH(V, T_F(V))\},$$
$$\Rightarrow \quad (H(V, V_F^*)) \preceq \frac{1}{\tau}\{bH(V, T_F(V))\},$$
$$\Rightarrow \quad (H(V, V_F^*)) \preceq \tau^{-1}(bH(V, T_F(V))).$$

Theorem 4.3.9 Anti-College Theorem

Consider Banach algebra $\mathcal{A}$ with K be the solid cone. Let (W, d) be a complete cone b-metric space with $(b \geq 1)$ such that the cone b-metric is continuous functional on $W \times W$. Let $\phi : \mathcal{A}_+ \longrightarrow \mathcal{A}_+$ be a cone b-comparison function. Also $F_i : W \longrightarrow P_{cp}(W)$ be a multivalued ϕ contractions. If the function $\zeta : \mathcal{A}_+ \longrightarrow \mathcal{A}_+$, defined by

$$\zeta(s) = s + \phi(s)$$

is onto, then

$$H(V, V_F^*) \succeq \zeta^{-1}(\frac{1}{b}H(V, T_F(V))), \quad \text{for each} \quad V \in P_{cp}(W).$$

Proof.

By Theorem 4.2.1 we know that T_F is a Picard operator.

Let $V \in P_{cp}(W)$ be an arbitrary then proceeding same as in Theorem 4.2.5, we must have

$$\Rightarrow H(V, V_F^*) \preceq b\{H(V, T_F(V)) + \phi(H(V, V_F^*))\}. \tag{4.20}$$

Now as given that,

$$\zeta(s) = s + \phi(s) \tag{4.21}$$

is strictly increasing and onto, replacing $s = H(V, V_F^*)$, in (4.21) we have,

$$\Rightarrow \zeta(H(V, V_F^*)) = (H(V, V_F^*)) + \phi(H(V, V_F^*)),$$
$$\Rightarrow \phi(H(V, V_F^*)) = \zeta(H(V, V_F^*)) - (H(V, V_F^*)).$$

Now putting the value of $\phi(H(V, V_F^*))$ in (4.20)

$$\Rightarrow H(V, T_F(V)) \preceq b\{(H(V, V_F^*)) + \zeta(H(V, V_F^*)) - (H(V, V_F^*))\},$$
$$\Rightarrow H(V, T_F(V)) \preceq b\{\zeta(H(V, V_F^*))\}.$$

As ζ is strictly increasing and bijection so we have

$$b\zeta(H(V, V_F^*)) \succeq H(V, T_F(V)),$$
$$\Rightarrow \ \zeta(H(V, V_F^*)) \succeq \frac{1}{b}H(V, T_F(V)),$$
$$\Rightarrow \ (H(V, V_F^*)) \succeq \zeta^{-1}\frac{1}{b}H(V, T_F(V)).$$

Remark 10. Let (W, H_{d_b}) be a complete cone b-metric space with$(b \geq 1)$ over the Banach algebra $\mathcal{A}$ with K be the solid cone. By taking $\mathcal{A} = \mathbb{R}$ and $K = [0, \infty)$, the cone b-metric space (W, H_{d_b}) becomes a complete b-metric space (W, H_{d_b}) and the results of Monica Boriceanu [24] becomes special case of the above Theorems.

Chapter 5

Conclusion and Future Work

The work of Boriceanu et al. [24] "Multivalued fractals in b-metric spaces" is investigated in this thesis with detailed description. The idea of multivalued fractals in the sense of metric spaces, b-metric spaces under specific contraction mappings is demonstrated by many researchers. In this dissertation, we have proved some results in the setting of cone b-metric spaces. These results are the extensions of the results presented by Boriceanu et al. [24].

The definitions of cone b-comparison function and cone b-ϕ contraction have been established in this thesis. A result regarding Picard operator is also proved in the setting of cone b-metric spaces. Then the definition of multivalued fractals in the setting cone b-metric spaces is also presented. By using the definition of multivalued mappings and multivalued fractals, some results in the setting of H-cone b-metric space are also proved.

Bibliography

[1] H. Poincare, "Surless courbes define barles equations differentiate less," *J. de Math*, vol. 2, pp. 54–65, 1886.

[2] L. E. J. Brouwer, "Über abbildung von mannigfaltigkeiten," *Mathematische annalen*, vol. 71, no. 1, pp. 97–115, 1911.

[3] S. Banach, "Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales," *Fund. math*, vol. 3, no. 1, pp. 133–181, 1922.

[4] A. Arvanitakis, "A proof of the generalized Banach contraction conjecture," *Proceedings of the American Mathematical Society*, vol. 131, no. 12, pp. 3647–3656, 2003.

[5] D. W. Boyd and J. S. Wong, "On nonlinear contractions," *Proceedings of the American Mathematical Society*, vol. 20, no. 2, pp. 458–464, 1969.

[6] B. S. Choudhury and K. Das, "A new contraction principle in Menger spaces," *Acta Mathematica Sinica, English Series*, vol. 24, no. 8, p. 1379, 2008.

[7] J. Merryfield, B. Rothschild, and J. Stein Jr, "An application of Ramseys theorem to the Banach contraction principle," *Proceedings of the American Mathematical Society*, vol. 130, no. 4, pp. 927–933, 2002.

[8] R. Kannan, "Some results on fixed points," *Bull. Cal. Math. Soc.*, vol. 60, pp. 71–76, 1968.

[9] S. B. Nadler *et al.*, "Multi-valued contraction mappings." *Pacific Journal of Mathematics*, vol. 30, no. 2, pp. 475–488, 1969.

[10] A. Arkhangel'Skii and V. Fedorchuk, *General topology I: basic concepts and constructions dimension theory.* Springer Science & Business Media, 2012, vol. 17.

[11] A. Amini-Harandi, "Metric-like spaces, partial metric spaces and fixed points," *Fixed Point Theory and Applications*, vol. 2012, no. 1, p. 204, 2012.

[12] S. Oltra and O. Valero, "Banach's fixed point theorem for partial metric spaces," 2004.

[13] I. Bakhtin, "The contraction mapping principle in quasimetric spaces," *Func. An., Gos. Ped. Inst. Unianowsk*, vol. 30, pp. 26–37, 1989.

[14] S. Czerwik, "Contraction mappings in b-metric spaces," *Acta Mathematica et Informatica Universitatis Ostraviensis*, vol. 1, no. 1, pp. 5–11, 1993.

[15] M. Boriceanu, "Fixed point theory for multivalued generalized contraction on a set with two b-metrics." *Studia Universitatis Babes-Bolyai, Mathematica*, no. 3, 2009.

[16] M. Boriceanu, A. Petrusel, and I. Rus, "Fixed point theorems for some multivalued generalized contractions in b-metric spaces," *International Journal of Mathematics and Statistics*, vol. 6, no. S10, pp. 65–76, 2010.

[17] L.-G. Huang and X. Zhang, "Cone metric spaces and fixed point theorems of contractive mappings," *Journal of mathematical Analysis and Applications*, vol. 332, no. 2, pp. 1468–1476, 2007.

[18] B. Rzepecki, "On fixed point theorems of maia type," *Publications de lInstitut Mathematique*, vol. 28, no. 42, pp. 179–186, 1980.

[19] M. A. Kutbi, J. Ahmad, A. E. Al-Mazrooei, and N. Hussain, "Multivalued fixed point theorems in cone b-metric spaces over banach algebra with applications," *J. Math. Anal*, vol. 9, no. 1, pp. 52–64, 2018.

[20] T. Chengkun, "From experiments in visualizing fractal theory to rethinking social networks as moistmedia," *Technoetic Arts*, vol. 14, no. 3, pp. 263–273, 2016.

[21] K. Weierstrass, "On continuous functions of a real argument that do not have a well-defined differential quotient," *Mathematische werke*, vol. 2, pp. 71–74, 1895.

[22] E. W. Weisstein, "Koch snowflake," *https://mathworld. wolfram. com/*, 2008.

[23] B. B. Mandelbrot, "The fractal geometry of nature/revised and enlarged edition," *whf*, 1983.

[24] M. Boriceanu, M. Bota, and A. Petruşel, "Multivalued fractals in b-metric spaces," *Central European Journal of Mathematics*, vol. 8, no. 2, pp. 367–377, 2010.

[25] H. Huang and S. Radenovic, "Common fixed point theorems of generalized lipschitz mappings in cone b-metric spaces over banach algebras and applications," *J. Nonlinear Sci. Appl*, vol. 8, no. 5, pp. 787–799, 2015.

[26] E. Kreyszig, *Introductory Functional Analysis with Applications.* wiley New York, 1978, vol. 1.

[27] M. M. Fréchet, "Sur quelques points du calcul fonctionnel," *Rendiconti del Circolo Matematico di Palermo (1884-1940)*, vol. 22, no. 1, pp. 1–72, 1906.

[28] K. (https://math.stackexchange.com/users/128900/kida424), "Is $d(x, y) = \sqrt{|x-y|}$ a metric on r?" Mathematics Stack Exchange, uRL:https://math.stackexchange.com/q/707468 (version: 2017-10-25). [Online]. Available: https://math.stackexchange.com/q/707468

[29] S. Batul, "Fixed point theorems in operator-valued metric spaces," Ph.D. dissertation, Capital University, 2016.

[30] J. M. Holtzman, "Contraction maps and equivalent linearization," *Bell System Technical Journal*, vol. 46, no. 10, pp. 2405–2435, 1967.

[31] B. E. Rhoades, "A comparison of various definitions of contractive mappings," *Transactions of the American Mathematical Society*, vol. 226, pp. 257–290, 1977.

[32] C. Mongkolkeha, C. Kongban, and P. Kumam, "Existence and uniqueness of best proximity points for generalized almost contractions," in *Abstract and Applied Analysis*, vol. 2014. Hindawi, 2014.

[33] M. Edelstein, "On fixed and periodic points under contractive mappings," *Journal of the London Mathematical Society*, vol. 1, no. 1, pp. 74–79, 1962.

[34] A. Poniecki, "The Banach Contraction Principle," 2008.

[35] J. M. Joseph and E. Ramganesh, "Fixed point theorem on multi-valued mappings," *International Journal of Analysis and Applications*, vol. 1, no. 2, pp. 123–127, 2013.

[36] L. B. Ćirić, "A generalization of banachs contraction principle," *Proceedings of the American Mathematical society*, vol. 45, no. 2, pp. 267–273, 1974.

[37] I. A. Rus, *Generalized contractions and applications.* Cluj University Press, 2001.

[38] A. Latif, V. Parvaneh, P. Salimi, and A. Al-Mazrooei, "Various suzuki type theorems in b-metric spaces," *J. Nonlinear Sci. Appl*, vol. 8, no. 4, pp. 363–377, 2015.

[39] T. Kamran, M. Samreen, and Q. UL Ain, "A generalization of b-metric space and some fixed point theorems," *Mathematics*, vol. 5, no. 2, p. 19, 2017.

[40] H. Huang, S. Radenovi, and G.-T. Deng, "A sharp generalization on cone b-metric space over banach algebra," *The Journal of Nonlinear Sciences and Applications*, vol. 10, 01 2017.

[41] N. Hussain and M. H. Shah, "Kkm mappings in cone b-metric spaces," *Computers & Mathematics with Applications*, vol. 62, no. 4, pp. 1677–1684, 2011.

[42] V. Berinde, "Generalized contractions in quasimetric spaces," in *Seminar on Fixed Point Theory*, vol. 3, 1993, pp. 3–9.

[43] C. Chifu and A. Petruşel, "Multivalued fractals and generalized multivalued contractions," *Chaos, Solitons & Fractals*, vol. 36, no. 2, pp. 203–210, 2008.