CAPITAL UNIVERSITY OF SCIENCE AND TECHNOLOGY, ISLAMABAD

A Carreau Nanofluid Flow over a Non-Linearly Stretching Sheet

by

Muhammad Qayyum Khan

A thesis submitted in partial fulfillment for the degree of Master of Philosophy

in the

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This thesis is dedicated to my beloved ***Parents*** *and elegant* ***Teachers*** *whose devotions and contributions to my life are really worthless and whose deep consideration on part of my academic career, made me consolidated and inspired me as I am upto this grade now.*

CERTIFICATE OF APPROVAL

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Muhammad Qayyum Khan

(MMT183010)

THESIS EXAMINING COMMITTEE

S. No.	Examiner	Name	Organization
(a)	External Examiner	Dr. Muhammad Ashraf	UOS, Sargodha
(b)	Internal Examiner	Dr. Muhammad Afzal	CUST, Islamabad
(c)	Supervisor	Dr. Muhammad Sagheer	CUST, Islamabad

Dr. Muhammad Sagheer
Thesis Supervisor
October, 2020

Dr. Muhammad Sagheer
Head
Dept. of Mathematics
October, 2020

Dr. Muhammad Abdul Qadir
Dean
Faculty of Computing
October, 2020

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(Muhammad Qayyum Khan)

Registration No: MMT183010

Abstract

In this analysis, the boundary layer viscous flow of nanofluids and heat transfer over a non-linearly stretching sheet in the presence of a magnetic field is presented. Velocity and thermal slip conditions are considered instead of no slip conditions at the boundary. A similarity transformation set is used to transform the governing partial differential equations into non-linear ordinary differential equations. The reduced equations are solved numerically using the shooting method. The influence of governing parameters on the dimensionless velocity, temperature and nanoparticle concentration as well as the skin friction coefficient, Nusselt number and local Sherwood number are analyzed. It is found that as the velocity slip parameter increases, the velocity profile, the skin friction and heat transfer are decreased while the mass transfer is increased. Increasing the thermal slip parameters causes a decrement in the heat and mass transfer rates. The results are presented in both graphical and tabular forms.

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Abbreviations

IVP	Initial value problem
MHD	Magneto-hydrodynamics
ODE	Ordinary differential equation
PDE	Partial differential equation
RK	Runge-Kutta

Symbols

u, v	Velocity component in the $x - axis$ and $y - axis$(unit:ms^{-1})
u_w	Velocity of the wall along $x - axis$(unit:ms^{-1})
x, y	Cartesian coordinates measured along the stretching sheet(unit:m)
$B(x)$	Magnetic field strength(unit:$A.m^{-1}$)
C	Nanopartile concentration(unit:$mol.m^{-3}$)
Cf_x	Skin-friction coefficient(unit:pascal)
Nu_x	Nusselt number
Sh_x	Sherwood number
C_w	Nanopartical concentration at the stretching surface (unit:$mol.m^{-3}$)
C_∞	Nanopartcal concentration far from the sheet(unit:$mol.m^{-3}$)
c_p	Specific heat capacity at constant pressure(unit:$J.kg^{-1}.k$)
D_T	Brownian diffusion coefficient
D_B	Thermophoresis diffusion coefficient
Ec	Eckert number
n	Non-linear stretching parameter
F	Dimensionless stream function
Sc	Schmidt number
M	Magnetic parameter
Nb	Brownian motion parameter
Nt	Thermophoresis parameter
Pr	Prandtl number
N_1	Velocity slip parameter
D_1	Thermal slip parameter

T	Fluid temperature(unit:K)
σ	Electrical conductivity(unit:S.m^{-1})
ψ	Stream function
ζ	Dimensionless similarity variable
μ	Dynamic viscosity of the base fluid(unit:$Kg.m^{-1}s^{-1}$)
ν	Kinematic viscosity(unit:m^2s^{-1})
κ	Thermal conductivity
ρ_f	Density of the fluid(unit:$Kg.m^{-3}$)
ρc_f	Heat capacity of the fluid(unit:$Kg.m^{-1}s^{-2}$)
ρc_ν	Heat capacity of the nanoparticle(unit:$Kg.m^{-1}s^{-2}$)
θ	Dimensionless temperature(unit:K)
p	Pressure(unit:$N.m^{-2}$)
ϕ	Nanoparticle volume fraction
λ	Velocity slip parameter
δ	Thermal slip parameter
T_w	Temperature at the surface(unit:K)
T_∞	Temperature of the fluid far away from the stretching sheet(unit:K)
Re_x	Reynolds number
q_w	Surface heat flux(unit:$W.m^{-2}$)
q_m	Surface mass flux
∞	Ambient conditions
We	Weissenberg number
R	Thermal radiation parameter
Kr	Chemical reaction parameter

Chapter 1

Introduction

During the past few decades, the boundary layer problems related to a stretching surface have attracted an extensive thinking of scholars, since the number of applications related to this area are found in engineering and industrial manufacturing process. Sakiadis [1] started the investigation over axi-symmetric and $2D$ boundary layer flow problems. Different perspectives were focused by few scholars in the previous decades. In any case, all these observations are confined to direct stretching of the surface. Kumaran and Ramaniah [2] observed the flow on a quadratic stretching sheet. Cortell [3] dissected the heat transfer rate and incompressible viscid flow on a non-linear stretchable surface numerically. Hayat et al. [4] investigated the hybrid convective flow on a non-linear stretchable surface of micro-polar fluid by utilizing homotopy analysis technique. Rashidi [5] analyzed the facts on free convective heat transfer rate for magnetohydrodynamics flow on a penetrable stretched surface in the presence of radiation and buyoancy impacts by applying homotopy analysis process. Prasad et al. [6] examined fluid chereacteristics of magnetohydrodynamics flow on a stretchable surface by utilizing Keller-box technique. Rapits and Perdikis [7] observed the viscid flow on a non-linear stretchable sheet in the presence of magnetic field parameter by applying shooting technique. The magnetohydrodynamics fluid flow of an electrically conducting and visco-elastic fluid flow through a stretched surface was concentrated by Turkyilmazoglu [8]. Khan et al. [9] observed the impacts of thermo diffusion over stagnation point flow of nanofluid in the presence of stretchable sheet with applied magnetic field parameter by using similarity transformations.

Choi and Eastman [10] observed the impacts of thermal conductivity of fluids with nanoparticle concentrations over a non-linearly stretching sheet. Gorla et al. [11] examined the flow of heat transfer rate on a stretched circular cylinder in a nanofluid. Fakour et al. [12] performed a similar work on the evaluation of heat transfer rate of a nanofluid in penetrable medium in the presence of magnetic field parameter. Makinde and Aziz [13] discussed the flow of increasing surface boundary layer in nanofluids with convective boundary conditions using the RK technique. Mustafa et al. [14] examined the boundary layer fluid flow of a stretch surface by utilizing homotopy analysis process to calculate semi-analytical solutions. Hamad [15] used the RK method to find a solution that is comparable to heat transfer and viscid flow of nanofluid on a non-linear stretchable sheet. The continuous flow of steady boundary layer across a stretchable sheet and transfer of heat of fluid through a non-linear penetrable stretched surface was analytically presented by Bachok et al. [16] and the similar work on boundary layer fluid flow on a non-linear penetrable stretched surface in nanofluids was numerically observed by Zaimi et al. [17]. Aman et al. [18] investigated the impacts of incompressible viscous flow in the presence of magnetic parameter and uniform $2D$ stagnation point flow of a viscid fluid.

Uddin et al. [19] observed the two dimensional magnetohydrodynamics free convention and viscous flow from a convectively heated penetrable surface in the presence of chemical reaction parameter. A Similarity transformations are used for the conversion of the governing equations and the boundary conditions. Khan et al. [20] discussed the impacts of two dimensional velocity slip parameters and axisymmetric flow of water and kerosene based nanofluid over a non-linearly stretched surface. Turkyilmazoglu et al. [21] observed the flow of electrically conductive, non-Newtonian fluid in binary and ternary solutions for magnetohydrodynamics under the impact of flow conditions through the stretching surface. The relation between specific and standard fluid flow was analyzed by Bachok et al. [22] and as a result, a method of reshaping was suggested which greatly specifies the estimation of flow variables like heat transfer and skin friction. Turkyilmazoglu et al. [23] presented the impacts of nanoparticles around the condensate boundary surface, which allowed the separation of nanoparticle concentrations from the film due to the widely used single phase model.

1.1 Thesis Contributions

The major objective of this research work is to execute the impacts of velocity slip and thermal wall slip parameter on MHD viscid flow and heat transfer rate of a nanofluid on a non-linear stretchable surface. The set of non-linear partial differential equations are transformed into ODEs and numerically solved by utilizing shooting method. Impacts of distinct parameters on the velocity, temperature and concentration distributions are expressed in tables and graphs.

1.2 Thesis Outlines

This thesis is classified into four main chapters:

Chapter 2 consists of some basic definitions of fluid, nanofluid, flow, boundary layer, heat transfer, viscous flow, basic governing laws and similarity transformations.

Chapter 3 contains the complete review of [15] which considers the impact of velocity slip parameter on magnetohydrodynamics two dimensional viscid flow of a nanofluid on a non-linearly stretchable sheet.

Chapter 4 is an extension of the given model [15] by including the impacts of Carreau nanofluid, thermal radiation and chemical reaction variables. Numerical outcomes for skin friction coefficient and Nusselt number have also been discussed. Impacts of different physical parameters are analyzed through graphs and tables.

Chapter 5 includes the summary of the entire study.

All the references used in this thesis are listed in **Bibliography**.

Chapter 2

Preliminaries

In this chapter some basic definitions, basic laws, terminologies and basic concepts for analyzing the non-linear partial differential equations are explained, which would be used in next chapters. These concepts will be helpful to develop an understanding for the rest of the thesis.

2.1 Basic Definitions

This section contains few essentials definitions and laws of the fluids, which will be used in the upcoming discussions.

2.1.1 Fluid [24]

"Fluids are substances whose molecular structure offers no resistance to external shear forces: even the smallest force causes deformation of fluid particles."

2.1.2 Fluid Mechanics [25]

"The branch of science that deals with the behavior of fluids at rest or in motion and the interaction of fluids with solids or other fluids at the boundaries."

2.1.3 Fluid Dynamics [26]

"It is the study of the motion of liquids, gases and plasma from one place to another."

2.1.4 Viscosity [27]

"Viscosity is a quantitative measure of a fluid's resistance to flow. Mathematically, it can be written as

$$\mu = \frac{\tau}{\frac{\partial u}{\partial y}},$$

where μ is viscosity coefficient, τ is shear stress and $\frac{\partial u}{\partial y}$ represents the velocity gradient or rate of shear strain."

2.1.5 Kinematic Viscosity [28]

"It is defined as the ratio between the dynamic viscosity and density of fluid. It is denoted by the Greek symbol ν, thus mathematically,

$$\nu = \frac{viscosity}{density} = \frac{\mu}{\rho},$$

where the unit of kinematic viscosity is $\frac{L^2}{T}$."

2.1.6 Ideal Fluids [28]

"A fluid which is incompressible and has no viscosity is known as an ideal fluid."

2.1.7 Real Fluids [28]

"A fluid which possesses viscosity is known as a real fluid. All the fluids in actual practice are real fluids."

2.1.8 Newtonian Fluids [28]

"A real fluid in which shear stress is directly proportional to the rate of shear strain (or velocity gradient) is known as a Newtonian fluid."

2.1.9 Non-Newtonian Fluids [28]

"A real fluid in which the shear stress is not proportional to the rate of shear strain (or velocity gradient) is known as a Non-Newtonian fluid."

2.1.10 Megnetohydrodynamics [29]

"Magnetohydrodynamics (MHD) is concerned with the flow of electrically conducting fluids in the presence of magnetic fields, either externally applied or generated within the fluid by inductive action."

2.2 Properties of Fluid

2.2.1 Heat Transfer [30]

"The energy transfer due to temperature difference is called heat transfer. Heat transfer occurs through different mechanisms."

2.2.2 Conduction [30]

"Due to collision of molecules in the contact form, heat is transferred from one object to another object. This phenomenon is called conduction. Such type of heat transfer occurs in the solids."

2.2.3 Radiation [30]

"In the radiation process, heat is transferred through electromagnetic rays and waves. It takes place in liquids and gasses."

2.2.4 Thermal Conductivity [30]

"It is the property of a substance which measures the ability to transfer heat. Fourier's law of conduction which relates the flow rate of heat by conduction to the temperature gradient is

$$\frac{dQ}{dt} = -\kappa A \frac{dT}{dx},$$

where A, κ, $\frac{dQ}{dt}$ and $\frac{dT}{dx}$ are the area, the thermal conductivity, the power per unit area transported and the temperature gradient respectively. The SI unit of thermal conductivity is $\frac{kgm}{s^3}$ and the dimension of thermal conductivity is $[\frac{ML}{T^3}]$."

2.2.5 Thermal Diffusivity [30]

"The ratio of the unsteady heat conduction of a substance to the product of specific heat capacity c_p and density ρ is called thermal diffusivity. Mathematically, it can be written as

$$\alpha = \frac{\kappa}{\rho c_p}.$$

The unit and dimension of thermal diffusivity in SI system are $m^2 s^{-1}$ and $[LT^{-1}]$ respectively."

2.3 Types of Flow

2.3.1 Compressible Flow [30]

"A flow in which the density variation is not negligible is known as compressible flow."

2.3.2 Incompressible Flow [30]

"A flow in the density remains nearly constant throughout is known as incompressible."

2.3.3 Steady Flow [30]

"A flow in which the velocity of the fluid at a particular fixed point does not change with time is known as steady flow."

2.3.4 Unsteady Flow [30]

"If any physical property of the fluid at a specific point changes with time then such flow is called the unsteady flow."

2.3.5 Viscous Flow [30]

"A flow in which viscosity of the fluid is not zero are called viscous (viscid) flow."

2.3.6 Inviscous Flow [30]

"A flow in which viscosity of the fluid is equal to zero is known as inviscous (inviscid) flow."

2.4 Conservation Laws

2.4.1 Equation of Mass [31]

"The law of conservation of mass states that for any system closed to all transfers of matter and energy, the mass of the system remain constant over time, as the system's mass cannot change, so quantity can neither be added nor removed. For any fluid conservation of mass is expressed by the scalar equation

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho u) = 0." \tag{2.1}$$

2.4.2 Equation of Momentum [31]

" The momentum equation states that the acceleration of a particle following the motion is the result of a net force, expressed by the gradient of pressure, viscous and gravity forces. Mathematically, it can be written as

$$\frac{\partial}{\partial t}(\rho u) + \nabla.[(\rho u)u] - \nabla.T - \rho g = 0." \tag{2.2}$$

2.4.3 Law of Conservation of Energy [31]

"The law of conservation of energy states that energy can neither be created nor destroyed, only converted from one form of energy to another. Conservation of thermal energy is expressed by

$$\rho \left[\frac{\partial U}{\partial t}\right] = [\tau.\nabla u + p\nabla.u] + \nabla(\kappa \nabla T) + H_r, \tag{2.3}$$

where U is the internal energy per unit mass, and Hr is the heat of reaction."

2.4.4 Newton's Law of Viscosity [32]

"It states that the shear stress (τ) on a fluid element layer is proportional to the rate of shear strain. The constant of proportionally is called coefficient of viscosity. Mathematically, it is expressed as

$$\tau = \mu \frac{du}{dy}." \tag{2.4}$$

2.5 Dimensionless Parameters

2.5.1 Reynolds Number (Re) [32]

"It is the most significant dimensionless number which is used to identify the different flow behaviors like laminar or turbulent flow. Mathematically, it is expressed as

$$Re = \frac{\rho U^2}{L}\frac{L^2}{\mu U}, \quad \Rightarrow Re = \frac{LU}{\nu},$$

where U denotes the free stream velocity, L is the characteristic length and ν stands for kinematic viscosity."

2.5.2 Nusselt Number (Nu) [32]

"It is the relationship between the convective to the conductive heat transfer through the boundary of the surface. Mathematically, it is defined as

$$Nu = \frac{hL}{\kappa},$$

where h stands for convective heat transfer, L stands for characteristic length and κ stands for thermal conductivity."

2.5.3 Prandtl Number (Pr) [32]

"The ratio of kinematic diffusivity to heat the diffusivity is said to be Prandtl number. It is denoted by Pr. Mathematically, it can be written as

$$Pr = \frac{\nu}{\alpha} \quad \Rightarrow Pr = \frac{\mu c_p}{\rho\kappa},$$

where μ and α denote the momentum diffusivity or kinetic diffusivity and thermal diffusivity respectively."

2.5.4 Weissenberg Number (We) [33]

"The Weissenberg number is typically defined as

$$We = \frac{\lambda u}{L},$$

where λ is a velocity slip parameter, u and L are a characteristic velocity and length scale for the flow respectively."

2.5.5 Skin Friction Coefficient (Cf_x) [33]

"The skin friction coefficient is typically defined as

$$C_f = \frac{2\tau_w}{\rho w_\infty^2},$$

where τ_w is the local wall shear stress, ρ is the fluid density and w_∞ is the free stream velocity (usually taken outside the boundary layer or at the inlet)."

2.5.6 Sherwood Number (Sh_x) [33]

"It is a non-dimensional quantity which describes the ratio of the mass transport by convection to the transfer of mass by diffusion. Mathematically,

$$Sh_x = \frac{kL}{D},$$

here L is characteristics length, D is the mass diffusivity and k is the mass transfer coefficient."

2.5.7 Thermophoresis Parameter (Nt) [33]

"In a temperature gradient, small particles are pushed towards the lower temperature because of the asymmetry of molecular impacts."

2.5.8 Eckert Number (Ec) [33]

"It is a dimensionless number used in continuum mechanics. It describes the relation between flows and the boundary layer enthalpy difference and it is used for characterized heat dissipation. Mathematically,

$$Ec = \frac{u^2}{c_p \nabla T}."$$

2.5.9 Schmidt Number (Sc) [33]

"It is the ratio between viscosity ν and molecular diffusion D. Mathematically, we can write it as:

$$Sc = \frac{\nu}{D},$$

where ν is the kinematic viscosity and D is the mass diffusivity."

Chapter 3

Impacts of Thermal Wall Slip and Velocity Slip Parameter on Magnetohydrodynamics

3.1 Introduction

In this chapter, we accomplish the successive observation on MHD incompressible viscous laminar fluid flow on a non-linearly stretchable sheet with the impacts of velocity and thermal wall slip parameters. The set of equations for energy, momentum and concentration are attained by utilizing the boundary layer approximation. Furthermore, the governing coupled non-linear partial differential equations are transmuted into ODEs by using the appropriate transformations. A numerical technique based on the shooting method is used for the solution of first order ODEs. At the end of this chapter, the numerical solution for different parameters are considered which impact on the skin friction coefficients, Nusselt and sherwood number. The tables and graphs are shown which are obtained through this investigation.

3.2 Mathematical Modeling

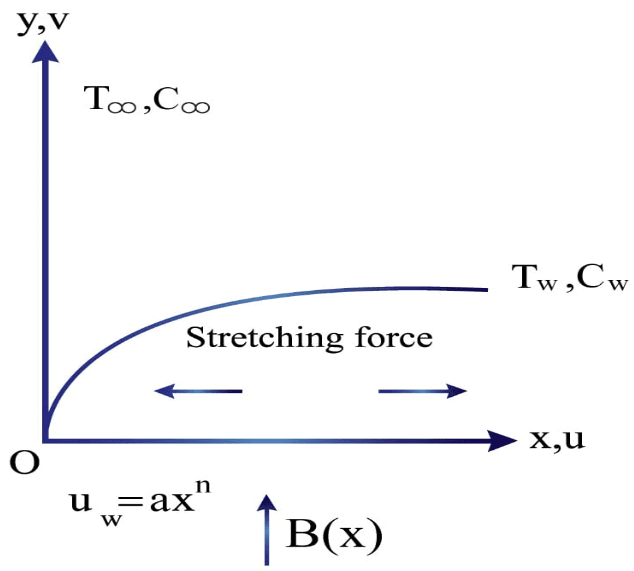

FIGURE 3.1: Systematic representation of physical model.

Assume a uniform 2D incompressible viscid flow of an electrically conducting fluid on a non-linearly stretchable sheet. Meanwhile, the plate has been stretched with the velocity $u_w = ax^n$ along $x - direction$. Here T_w is the wall temperature and C_w is the nanoparticle fraction, T_∞ is the ambient temperature and C_∞ is the nanoparticle fraction. The flow is explained by assuming the two dimensional governing equations containing the continuity, momentum and energy transfer:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{3.1}$$

$$u\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} - \frac{\sigma B^2}{\rho_f}u. \tag{3.2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} + \frac{\nu}{c_p}\left(\frac{\partial u}{\partial y}\right)^2 + \tau\left[D_B\frac{\partial T}{\partial y}\frac{\partial C}{\partial y} + \frac{D_T}{T_\infty}\left(\frac{\partial T}{\partial y}\right)^2\right]. \tag{3.3}$$

$$u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} = D_B\frac{\partial^2 C}{\partial y^2} + \frac{D_T}{T_\infty}\left(\frac{\partial^2 T}{\partial y^2}\right). \tag{3.4}$$

Here $\alpha = \frac{\kappa}{(\rho c)_f}(m^2.s^{-1})$ denotes the thermal diffusion, $\tau = \frac{(\rho c)_p}{(\rho c)_f}$ shows relation between the successive heat capacity and magnetic field equation $B(x) = B_0 x^{\frac{n-1}{2}}$.

The suitable BCs for the equations (3.1)-(3.4) are given below.

$$\left.\begin{aligned} &u = u_w - N\nu_f\left(\frac{\partial u}{\partial y}\right), \qquad v = 0,\\ &T = T_w + D\left(\frac{\partial T}{\partial y}\right), \quad C = C_w \quad as \quad y = 0.\\ &u \to 0,\ v \to 0,\ T \to T_\infty,\ C \to C_\infty \quad as \quad y \to \infty. \end{aligned}\right\} \tag{3.5}$$

Here, $u_w = ax^n$ denotes the stretching velocity, $T_w = T_\infty + bx^{2n}$ denotes the temperature of surface, a, b are constants parameters, $N = N_1 x^{\frac{-n-1}{2}}$ denotes the velocity slip parameter which changes with x, N_1 denotes the basic form of velocity slip parameter, $D = D_1 x^{\frac{-n-1}{2}}$ denotes the thermal slip parameter and D_1 denotes the basic form of thermal slip parameter, no-slip instance is retrieved for $N = 0$.

$$\left.\begin{aligned} &\zeta = y\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{n-1}{2}}, \quad u = ax^n F'(\zeta),\\ &v = -\sqrt{\frac{a(n+1)\nu_f}{2}}x^{\frac{n-1}{2}}\left[F(\zeta) + \frac{(n-1)}{(n+1)}\zeta F'(\zeta)\right],\\ &T = T_\infty + bx^{2n}\theta(\zeta), \quad \phi(\zeta) = \frac{(C - C_\infty)}{(C_w - C_\infty)}. \end{aligned}\right\} \tag{3.6}$$

The complete procedure for the conversion of equations (3.1)-(3.4) into the dimensionless form has been described in the upcoming discussion.

- $$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial}{\partial x}(ax^n F'(\zeta)),\\ &= ax^n\frac{\partial}{\partial x}(F'(\zeta)) + aF'(\zeta)\frac{\partial}{\partial x}(x^n),\\ &= anF'(\zeta)x^{n-1} + aF''(\zeta)\frac{\partial \zeta}{\partial x}x^n,\\ &= anF'(\zeta)x^{n-1} + aF''(\zeta)y\sqrt{\frac{a(n+1)}{2\nu_f}}\frac{(n-1)}{2}x^{\frac{n-3}{2}}x^n,\\ &= anF'(\zeta)x^{n-1} + \frac{a(n-1)}{2}\ \zeta\ F''(\zeta)x^{n-1}. \end{aligned} \tag{3.7}$$
- $$\begin{aligned} \frac{\partial v}{\partial y} &= -\sqrt{\frac{(n+1)a\nu_f}{2}}x^{\frac{n-1}{2}}\left[F'(\zeta)\frac{\partial \zeta}{\partial y} + \frac{(n-1)}{(n+1)}\left(\frac{\partial \zeta}{\partial y}F'(\zeta) + \zeta F''(\zeta)\frac{\partial \zeta}{\partial y}\right)\right],\\ &= -\sqrt{\frac{(n+1)a\nu_f}{2}}x^{\frac{n-1}{2}}\left[F'(\zeta)\frac{\zeta}{y} + \frac{(n-1)}{(n+1)}\left(\frac{\zeta}{y}F'(\zeta) + \zeta F''(\zeta)\frac{\zeta}{y}\right)\right], \end{aligned}$$

$$
\begin{aligned}
&= -\sqrt{\frac{(n+1)a\nu_f}{2}}x^{\frac{n-1}{2}}F'(\zeta)\frac{\zeta}{y} - \sqrt{\frac{(n+1)a\nu_f}{2}}x^{\frac{n-1}{2}}\frac{(n-1)}{(n+1)} \\
&\quad \left(\frac{\zeta}{y}F'(\zeta) + \zeta F''(\zeta)\frac{\zeta}{y}\right), \\
&= -\sqrt{\frac{(n+1)a\nu_f}{2}}x^{\frac{n-1}{2}}F'(\zeta)\frac{\zeta}{y} - \sqrt{\frac{(n+1)a\nu_f}{2}}x^{\frac{n-1}{2}}\frac{(n-1)}{(n+1)}\frac{\zeta}{y}F'(\zeta) - \\
&\quad \sqrt{\frac{(n+1)a\nu_f}{2}}F''(\zeta)\frac{\zeta^2}{y}x^{\frac{n-1}{2}}, \\
&= -\frac{a(n+1)}{2}F'(\zeta)x^{n-1} - \frac{a(n-1)}{2}F'(\zeta)x^{n-1} - \frac{a(n+1)}{2}\zeta F''(\zeta)x^{n-1}, \\
&= -aF'(\zeta)\left(\frac{(n-1)}{2} + \frac{(n+1)}{2}\right)x^{n-1} - \frac{a(n+1)}{2}\zeta F''(\zeta)x^{n-1}, \\
&= -aF'(\zeta)\left(\frac{(n-1+n+1)}{2}\right)x^{n-1} - \frac{a(n+1)}{2}\zeta F''(\zeta)x^{n-1}, \\
&= -anF'(\zeta)x^{n-1} - \frac{a(n-1)}{2}\zeta\ F''(\zeta)\ x^{n-1}.
\end{aligned} \tag{3.8}
$$

Equation (3.1) is very easily satisfied by using the (3.7)-(3.8) as follows:

$$
\begin{aligned}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= anF'(\zeta)x^{n-1} + \frac{a(n-1)}{2}\zeta F''(\zeta)x^{n-1} - anF'(\zeta)x^{n-1} - \\
&\quad \frac{a(n-1)}{2}\ \zeta\ F''(\zeta)x^{n-1} = 0
\end{aligned} \tag{3.9}
$$

For converting the equation (3.2), we proceed as follows:

- $$
\begin{aligned}
\frac{\partial u}{\partial y} &= \frac{\partial}{\partial y}(ax^n F'(\zeta)), \\
&= ax^n F''(\zeta)\frac{\partial \zeta}{\partial y}, \\
&= ax^n F''(\zeta)\frac{\zeta}{y}.
\end{aligned} \tag{3.10}
$$

- $$
\begin{aligned}
\frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial y}\left(ax^n F''(\zeta)\frac{\zeta}{y}\right), \\
&= ax^{\frac{3n-1}{2}}\sqrt{\frac{a(n+1)}{2\nu_f}}F'''(\zeta)\frac{\partial \zeta}{\partial y}, \\
&= \frac{a^2(n+1)}{2\nu_f}F'''(\zeta)x^{2n-2}.
\end{aligned} \tag{3.11}
$$

- $$u\frac{\partial u}{\partial x} = aF'(\zeta)x^n\frac{\partial}{\partial x}(aF'(\zeta)x^n),$$
$$= ax^nF'(\zeta)\left[\frac{a(n-1)}{2}y\sqrt{\frac{a(n+1)}{2\nu_f}}F''(\zeta)x^{\frac{3n-3}{2}} + anF'(\zeta)x^{n-1}\right],$$
$$= \frac{a^2(n-1)}{2}y\sqrt{\frac{a(n+1)}{2\nu_f}}F'(\zeta)F''(\zeta)x^{\frac{5n-3}{2}} + a^2n(F'(\zeta))^2x^{2n-1}. \quad (3.12)$$

- $$v\frac{\partial u}{\partial y} = -\sqrt{\frac{(n+1)a\nu_f}{2}}x^{\frac{n-1}{2}}\left[F(\zeta) + \frac{(n-1)}{(n+1)}(\zeta)F'(\zeta)\right]\frac{\partial}{\partial y}(ax^nF'(\zeta)),$$
$$= -\sqrt{\frac{(n+1)a\nu_f}{2}}x^{\frac{n-1}{2}}F(\zeta)F''(\zeta)a\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{3n-1}{2}} - \sqrt{\frac{(n+1)a\nu_f}{2}}$$
$$x^{\frac{n-1}{2}}\frac{(n-1)}{(n+1)}\zeta F'(\zeta)F''(\zeta)a\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{3n-1}{2}},$$
$$= -\frac{a^2(n+1)}{2}F(\zeta)F''(\zeta)x^{2n-1} - \frac{a^2(n-1)}{2}F'(\zeta)F''(\zeta)$$
$$y\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{5n-3}{2}}. \quad (3.13)$$

Using equations (3.12)-(3.13) in the left side of (3.2), we get

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{a^2(n-1)}{2}y\sqrt{\frac{a(n+1)}{2\nu_f}}F'(\zeta)F''(\zeta)x^{\frac{5n-3}{2}} + a^2n(F'(\zeta))^2x^{2n-1}$$
$$- \frac{a^2(n+1)}{2}F(\zeta)F''(\zeta)x^{2n-1} - \frac{a^2(n-1)}{2}F'(\zeta)F''(\zeta)$$
$$y\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{5n-3}{2}}. \quad (3.14)$$

For converting the right side of equation (3.2), following derivatives are as follows:

- $$\nu\frac{\partial^2 u}{\partial y^2} = \nu\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right),$$
$$= \nu\frac{\partial}{\partial y}\left(ax^nF''(\zeta)\frac{\zeta}{y}\right),$$
$$= \nu\frac{a^2(n+1)}{2\nu_f}F'''(\zeta)x^{2n-2}. \quad (3.15)$$

- $$\frac{\sigma B^2}{\rho_f}u = \frac{\sigma B^2}{\rho_f}(ax^nF'(\zeta)),$$
$$= \frac{\sigma B^2}{\rho_f}ax^nF'(\zeta). \quad (3.16)$$

Using (3.15)-(3.16) in the right side of (3.2), we get

$$\nu\frac{\partial^2 u}{\partial y^2}+\frac{\sigma B^2}{\rho_f}u=\nu\frac{a^2(n+1)}{2\nu_f}F'''(\zeta)x^{2n-2}+\frac{\sigma B^2}{\rho_f}ax^nF'(\zeta). \tag{3.17}$$

Comparing equations (3.14) and (3.17), the dimensionless form of (3.2) can be written as:

$$\begin{aligned}
& a^2n(F')^2(\zeta)x^{2n-1}+\frac{a^2(n-1)}{2}y\sqrt{\frac{a(n+1)}{2\nu_f}}F'(\zeta)F''(\zeta)x^{\frac{5n-3}{2}}\\
& -\frac{a^2(n+1)}{2}F(\zeta)F''(\zeta)x^{2n-1}-\frac{a^2(n-1)}{2}y\sqrt{\frac{a(n+1)}{2\nu_f}}F'(\zeta)F''(\zeta)x^{\frac{5n-3}{2}},\\
& =\nu F'''(\zeta)\frac{a^2(n+1)}{2\nu_f}x^{2n-1}-\frac{\sigma B^2}{\rho_f}ax^nF'(\zeta),\\
\Rightarrow\ & \nu F'''(\zeta)\frac{a^2(n+1)}{2\nu_f}x^{2n-1}+F''(\zeta)F'(\zeta)\frac{a^2(n+1)}{2}x^{2n-1}\\
& -a^2n(F'(\zeta))^2x^{2n-1}-\frac{\sigma B^2}{\rho_f}ax^nF'(\zeta)=0,\\
\Rightarrow\ & \left[\frac{\nu}{\nu_f}F'''(\zeta)+F(\zeta)F''(\zeta)-\frac{2n}{(n+1)}(F'(\zeta))^2-\frac{2\sigma B^2x^n}{\rho_f(n+1)ax^{2n-1}}F'(\zeta)\right]\\
& \frac{a^2(n+1)}{2}x^{2n-1}=0,\\
\Rightarrow\ & F'''(\zeta)+F(\zeta)F''(\zeta)-\frac{2n}{(n+1)}(F'(\zeta))^2-\frac{2\sigma B^2}{\rho_f(n+1)ax^{n-1}}F'(\zeta)=0,\\
\Rightarrow\ & F'''(\zeta)+F(\zeta)F''(\zeta)-\frac{2n}{(n+1)}(F'(\zeta))^2-\frac{2\sigma B_0^2}{\rho_f(n+1)a}F'(\zeta)=0,\\
\Rightarrow\ & F'''(\zeta)+F(\zeta)F''(\zeta)-\frac{2n}{(n+1)}(F'(\zeta))^2-MF'(\zeta)=0,\\
\Rightarrow\ & F'''+FF''-\frac{2n}{(n+1)}(F')^2-MF'=0.
\end{aligned} \tag{3.18}$$

The following derivatives will help to convert the equation (3.3) into the dimensionless form.

- $T=T_\infty+bx^{2n}\theta(\zeta).$
- $\frac{\partial T}{\partial x}=0+b\left[x^{2n}\theta'(\zeta)\frac{\partial\zeta}{\partial x}+2nx^{2n-1}\theta(\zeta)\right],$

$$=b\theta'(\zeta)y\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{5n-3}{2}}\frac{(n-1)}{2}+2bnx^{2n-1}\theta(\zeta). \tag{3.19}$$

- $$\frac{\partial T}{\partial y} = 0 + bx^{2n}\theta'(\zeta)\frac{\partial \zeta}{\partial y},$$
$$= bx^{2n}\theta'(\zeta)\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{n-1}{2}},$$
$$= b\theta'(\zeta)\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{5n-1}{2}}. \tag{3.20}$$
- $$\frac{\partial^2 T}{\partial y^2} = b\theta''(\zeta)\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{5n-1}{2}}\frac{\partial \zeta}{\partial y},$$
$$= b\theta''(\zeta)\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{5n-1}{2}}\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{n-1}{2}},$$
$$= \frac{ab(n+1)}{2\nu_f}\theta''(\zeta)x^{3n-1}. \tag{3.21}$$
- $$\phi(\zeta) = \frac{(C - C_\infty)}{(C_w - C_\infty)},$$
$$\Rightarrow \quad C = C_\infty + (C_w - C_\infty)\phi(\zeta).$$
- $$\frac{\partial C}{\partial y} = 0 + (C_w - C_\infty)\phi'(\zeta)\frac{\partial \zeta}{\partial y},$$
$$= (C_w - C_\infty)\phi'(\zeta)\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{n-1}{2}}. \tag{3.22}$$
- $$u\frac{\partial T}{\partial x} = ax^nF'(\zeta)\frac{\partial}{\partial x}(T_\infty + bx^{2n}\theta(\zeta)),$$
$$= \frac{ab(n-1)}{2}F'(\zeta)\theta'(\zeta)y\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{7n-3}{2}} + 2abnF'(\zeta)$$
$$\theta(\zeta)x^{3n-1}. \tag{3.23}$$
- $$v\frac{\partial T}{\partial y} = -\sqrt{\frac{(n+1)a\nu_f}{2}}x^{\frac{n-1}{2}}\left[F(\zeta) + \frac{(n-1)}{(n+1)}(\zeta)F'(\zeta)\right]\frac{\partial}{\partial y}(T_\infty + bx^{2n}\theta(\zeta)),$$
$$= -\sqrt{\frac{(n+1)a\nu_f}{2}}x^{\frac{n-1}{2}}F(\zeta)\theta'(\zeta)b\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{5n-1}{2}} - \sqrt{\frac{(n+1)a\nu_f}{2}}$$
$$x^{\frac{n-1}{2}}\frac{(n-1)}{(n+1)}\zeta F'(\zeta)\theta'(\zeta)b\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{5n-1}{2}},$$
$$= -\sqrt{\frac{(n+1)a\nu_f}{2}}F(\zeta)\theta'(\zeta)b\sqrt{\frac{a(n+1)}{2\nu_f}}x^{3n-1} - \sqrt{\frac{(n+1)a\nu_f}{2}}$$
$$\frac{(n-1)}{(n+1)}\zeta F'(\zeta)\theta'(\zeta)b\sqrt{\frac{a(n+1)}{2\nu_f}}x^{3n-1},$$
$$= -\frac{ab(n+1)}{2}F(\zeta)\theta'(\zeta)x^{3n-1} - \frac{ab(n-1)}{2}\zeta F'(\zeta)\theta'(\zeta)x^{3n-1}. \tag{3.24}$$

Using equations (3.23)-(3.24), the left side of (3.3) becomes:

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=2abnF'(\zeta)\theta(\zeta)x^{3n-1}+\frac{ab(n-1)}{2}F'(\zeta)\theta'(\zeta)\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{7n-3}{2}}y-\frac{ab(n+1)}{2}F'(\zeta)\theta'(\zeta)x^{3n-1}-\frac{ab(n-1)}{2}F'(\zeta)\theta'(\zeta)\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{7n-3}{2}}y. \tag{3.25}$$

The following derivatives will help to convert the right side of equation (3.3) into the dimensionless form:

- $$\alpha\frac{\partial^2 T}{\partial y^2}=\alpha\frac{\partial}{\partial y}\left(\frac{\partial T}{\partial y}\right),=\alpha\frac{\partial}{\partial y}\left(b\theta'(\zeta)\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{5n-1}{2}}\right),=\alpha\frac{ab(n+1)}{2\nu_f}\theta''(\zeta)x^{3n-1}. \tag{3.26}$$
- $$\frac{\nu}{c_p}\left(\frac{\partial u}{\partial y}\right)^2=\frac{\nu}{c_p}\left(ax^nF''(\zeta)\frac{\zeta}{y}\right)^2,=\frac{\nu}{c_p}a^2x^{2n}(F''(\zeta))^2\frac{\zeta^2}{y^2},=\frac{\nu}{c_p}\frac{a^3(n+1)}{2\nu_f}(F''(\zeta))^2x^{3n-1}. \tag{3.27}$$
- $$\tau\left(D_B\frac{\partial T}{\partial y}\frac{\partial C}{\partial y}+\frac{D_T}{T_\infty}\left(\frac{\partial T}{\partial y}\right)^2\right)=\tau D_B\left(b\theta'(\zeta)\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{5n-1}{2}}\right)\left((C_w-C_\infty)\phi'(\zeta)\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{n-1}{2}}\right)+\tau\frac{D_T}{T_\infty}\left(b\theta'(\zeta)\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{5n-1}{2}}\right)^2,=\tau D_B\frac{ab(n+1)}{2}(C_w-C_\infty)\theta'(\zeta)\phi'(\zeta)x^{3n-1}+\tau\frac{D_T(T_w-T_\infty)}{T_\infty}\frac{ab(n+1)}{2}(\theta'(\zeta))^2x^{3n-1}. \tag{3.28}$$

Using equations (3.26)-(3.28) in the right side of (3.3), we get

$$\alpha\frac{\partial^2 T}{\partial y^2}+\frac{\nu}{c_p}\left(\frac{\partial u}{\partial y}\right)^2+\tau\left(D_B\frac{\partial T}{\partial y}\frac{\partial C}{\partial y}+\frac{D_T}{T_\infty}\left(\frac{\partial T}{\partial y}\right)^2\right)=\alpha\frac{ab(n+1)}{2\nu_f}\theta''(\zeta)$$

$$x^{3n-1}+\frac{\nu}{c_p}\frac{a^3(n+1)}{2\nu_f}(F''(\zeta))^2x^{3n-1}+\tau D_B\frac{ab(n+1)}{2}(C_w-C_\infty)\theta'(\zeta)$$
$$\phi'(\zeta)x^{3n-1}+\tau\left(\frac{D_T(T_w-T_\infty)}{T_\infty}\frac{ab(n+1)}{2}\right)(\theta'(\zeta))^2x^{3n-1},$$
$$=\alpha\frac{ab(n+1)}{2\nu_f}\theta''(\zeta)x^{3n-1}+\frac{\nu}{c_p}\frac{a^3(n+1)}{2\nu_f}(F''(\zeta))^2x^{3n-1}+\tau D_B$$
$$\frac{ab(n+1)}{2}(C_w-C_\infty)\theta'(\zeta)\phi'(\zeta)x^{3n-1}+\tau\frac{D_T(T_w-T_\infty)}{T_\infty}$$
$$\frac{ab(n+1)}{2}(\theta'(\zeta))^2x^{3n-1}. \tag{3.29}$$

Comparing equations (3.25)-(3.29), the dimensionless form of equation (3.3) can be seen as:

$$2abnF'(\zeta)\theta(\zeta)x^{3n-1}+\frac{ab(n-1)}{2}F'(\zeta)\theta'(\zeta)\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{7n-3}{2}}y$$
$$-\frac{ab(n+1)}{2}F'(\zeta)\theta'(\zeta)x^{3n-1}-\frac{ab(n-1)}{2}F'(\zeta)\theta'(\zeta)\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{7n-3}{2}}y$$
$$=\alpha\frac{ab(n+1)}{2\nu_f}\theta''(\zeta)x^{3n-1}+\frac{\nu}{c_p}\frac{a^3(n+1)}{2\nu_f}(F''(\zeta))^2x^{3n-1}$$
$$+\tau D_B\frac{ab(n+1)}{2}(C_w-C_\infty)\theta'(\zeta)\phi'(\zeta)x^{3n-1}+\tau\frac{D_T(T_w-T_\infty)}{T_\infty}$$
$$\frac{ab(n+1)}{2}(\theta'(\zeta))^2x^{3n-1},$$
$$\Rightarrow\quad 2abnF'(\zeta)\theta(\zeta)x^{3n-1}+\frac{ab(n-1)}{2}F'(\zeta)\theta'(\zeta)\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{7n-3}{2}}y$$
$$-\frac{ab(n+1)}{2}F'(\zeta)\theta'(\zeta)x^{3n-1}-\frac{ab(n-1)}{2}F'(\zeta)\theta'(\zeta)\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{7n-3}{2}}y-$$
$$\alpha\frac{ab(n+1)}{2\nu_f}\theta''(\zeta)x^{3n-1}-\frac{\nu}{c_p}\frac{a^3(n+1)}{2\nu_f}(F''(\zeta))^2x^{3n-1}$$
$$-\tau D_B\frac{ab(n+1)}{2}(C_w-C_\infty)\theta'(\zeta)\phi'(\zeta)x^{3n-1}-\tau\frac{D_T(T_w-T_\infty)}{T_\infty}$$
$$\frac{ab(n+1)}{2}(\theta'(\zeta))^2x^{3n-1}=0,$$
$$\Rightarrow\quad \frac{\alpha}{\nu_f}\theta''(\zeta)+F(\zeta)\theta'(\zeta)-\frac{4n}{(n+1)}F'(\zeta)\theta(\zeta)+\tau D_B(C_w-C_\infty)\theta'(\zeta)\phi'(\zeta)$$
$$+\tau\left(\frac{D_T(T_w-T_\infty)}{T_\infty}\right)(\theta'(\zeta))^2+\frac{\nu}{c_p}\frac{a^2}{b\nu_f}(F''(\zeta))^2=0,$$

$$\Rightarrow \quad \frac{1}{Pr}\theta''(\zeta) + F(\zeta)\theta'(\zeta) - \frac{4n}{(n+1)}F'(\zeta)\theta(\zeta) + Nb\theta'(\zeta)\phi'(\zeta)$$
$$+ Nt(\theta'(\zeta))^2 + \frac{(u_w)^2}{c_p(T_w - T_\infty)}(F''(\zeta))^2 = 0,$$
$$\Rightarrow \quad \frac{1}{Pr}\theta''(\zeta) + F(\zeta)\theta'(\zeta) - \frac{4n}{(n+1)}F'(\zeta)\theta(\zeta) + Nb\theta'(\zeta)\phi'(\zeta)$$
$$+ Nt(\theta'(\zeta))^2 + Ec(F''(\zeta))^2 = 0,$$
$$\Rightarrow \quad \frac{1}{Pr}\theta'' + F\theta' - \frac{4n}{(n+1)}F'\theta + Nb\theta'\phi' + Nt(\theta')^2 + Ec(F'')^2 = 0. \tag{3.30}$$

To convert the equation (3.4) into ordinary differential form. We calculate the following derivatives:

- $\phi(\zeta) = \dfrac{(C - C_\infty)}{(C_w - C_\infty)},$

$$\Rightarrow \quad C = C_\infty + (C_w - C_\infty)\phi(\zeta).$$

- $\dfrac{\partial C}{\partial x} = 0 + (C_w - C_\infty)\phi'(\zeta)\dfrac{\partial \zeta}{\partial x},$

$$= (C_w - C_\infty)\frac{(n-1)}{2}\phi'(\zeta)y\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{n-3}{2}}. \tag{3.31}$$

- $\dfrac{\partial C}{\partial y} = 0 + (C_w - C_\infty)\phi'(\zeta)\dfrac{\partial \zeta}{\partial y},$

$$= (C_w - C_\infty)\phi'(\zeta)\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{n-1}{2}}. \tag{3.32}$$

- $\dfrac{\partial^2 C}{\partial y^2} = (C_w - C_\infty)\phi''(\zeta)\sqrt{\dfrac{a(n+1)}{2\nu_f}}x^{\frac{n-1}{2}}\dfrac{\partial \zeta}{\partial y},$

$$= (C_w - C_\infty)\phi''(\zeta)\frac{a(n+1)}{2\nu_f}x^{n-1}. \tag{3.33}$$

- $u\dfrac{\partial C}{\partial x} = ax^nF'(\zeta)\dfrac{\partial}{\partial x}(C_\infty + (C_w - C_\infty)\phi(\zeta)),$

$$= ax^nF'(\zeta)\frac{\partial}{\partial x}(0 + (C_w - C_\infty)\phi'(\zeta)\frac{\partial \zeta}{\partial x}),$$
$$= ax^nF'(\zeta)\frac{\partial}{\partial x}(C_w - C_\infty)\phi'(\zeta)\frac{\partial \zeta}{\partial x},$$
$$= ax^nF'(\zeta)(C_w - C_\infty)\frac{(n-1)}{2}\phi'(\zeta)y\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{n-3}{2}},$$
$$= F'(\zeta)(C_w - C_\infty)\frac{a(n-1)}{2}F'(\zeta)\phi'(\zeta)y\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{3n-3}{2}}. \tag{3.34}$$

- $$v\frac{\partial C}{\partial y} = -\sqrt{\frac{a(n+1)\nu_f}{2}}x^{\frac{n-1}{2}}\left[F(\zeta) + \frac{(n-1)}{(n+1)}\zeta F'(\zeta)\right]$$
$$\frac{\partial}{\partial y}((C_\infty + (C_w - C_\infty)\phi(\zeta)),$$
$$= -\sqrt{\frac{a(n+1)\nu_f}{2}}x^{n-1/2}\left[F(\zeta) + \frac{(n-1)}{(n+1)}\zeta F'(\zeta)\right](C_w - C_\infty)\phi'(\zeta)$$
$$\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{n-1}{2}},$$
$$= -\frac{a(n+1)}{2})(C_w - C_\infty)F(\zeta)\phi'(\zeta)x^{n-1} - \frac{a(n-1)}{2})(C_w - C_\infty)$$
$$F'(\zeta)\phi'(\zeta)y\sqrt{\frac{a(n+1)}{2}}x^{\frac{3n-3}{2}}. \tag{3.35}$$

Using equations (3.34)-(3.35) in the left side of (3.4), we get

$$u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} = F'(\zeta)(C_w - C_\infty)\frac{a(n-1)}{2}F'(\zeta)\phi'(\zeta)y\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{3n-3}{2}} -$$
$$\frac{a(n+1)}{2}(C_w - C_\infty)F(\zeta)\phi'(\zeta)x^{n-1} - \frac{a(n-1)}{2}(C_w - C_\infty)$$
$$F'(\zeta)\phi'(\zeta)y\sqrt{\frac{a(n+1)}{2}}x^{\frac{3n-3}{2}}. \tag{3.36}$$

The dimensionless form of the right side of (3.4) can be obtained as follows:

- $$D_B\frac{\partial^2 C}{\partial y^2} = D_B\frac{\partial}{\partial y}\left(\frac{\partial C}{\partial y}\right),$$
$$= D_B\frac{\partial}{\partial y}\left((C_w - C_\infty)\phi'(\zeta)\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{n-1}{2}}\right),$$
$$= D_B\frac{a(n+1)}{2\nu_f}\phi''(\zeta)(C_w - C_\infty)x^{n-1}. \tag{3.37}$$
- $$\frac{D_T}{T_\infty}\frac{\partial^2 T}{\partial y^2} = \frac{D_T}{T_\infty}\frac{\partial}{\partial y}\left(\frac{\partial T}{\partial y}\right),$$
$$= \frac{D_T}{T_\infty}\frac{\partial}{\partial y}\left(b\theta'(\zeta)\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{5n-1}{2}}\right),$$
$$= \frac{D_T}{T_\infty}\frac{\partial}{\partial y}\left(b\theta''(\zeta)\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{5n-1}{2}}\frac{\partial \zeta}{\partial y}\right),$$
$$= \frac{D_T}{T_\infty}\frac{a(n+1)}{2\nu_f}\theta''(\zeta)bx^{3n-1}. \tag{3.38}$$

Using equations (3.37) and (3.38) in the right side of (3.4), we get

$$D_B \frac{\partial^2 C}{\partial y^2} + \frac{D_T}{T_\infty}\frac{\partial^2 T}{\partial y^2} = D_B \frac{a(n+1)}{2\nu_f}\phi''(\zeta)(C_w - C_\infty)x^{n-1} + \frac{D_T}{T_\infty}\frac{a(n+1)}{2\nu_f}$$
$$\theta''(\zeta)bx^{3n-1}. \tag{3.39}$$

With the help of equations (3.36)-(3.39), the dimensionless form of (3.4) is obtained through the following steps:

$$\frac{a(n-1)}{2}(C_w - C_\infty)F'(\zeta)\phi'(\zeta)y\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{3n-3}{2}} - \frac{a(n+1)}{2}(C_w - C_\infty)$$
$$F(\zeta)\phi'(\zeta)x^{n-1} - \frac{a(n-1)}{2}(C_w - C_\infty)F'(\zeta)\phi'(\zeta)y\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{3n-3}{2}}$$
$$= D_B\frac{a(n+1)}{2\nu_f}\phi''(\zeta)(C_w - C_\infty)x^{n-1} + \frac{D_T a(n+1)}{2\nu_f T_\infty}\theta''(\zeta)bx^{3n-1},$$
$$\Rightarrow \quad \left[\phi''(\zeta) + \frac{\nu_f}{D_B}F(\zeta)\phi'(\zeta) + \frac{D_T}{T_\infty D_B(C_w - C_\infty)}\theta''(\zeta)bx^{2n}\right] D_B(C_w - C_\infty)$$
$$\frac{a(n+1)}{2\nu_f}x^{n-1} = 0,$$
$$\Rightarrow \quad \phi''(\zeta) + \frac{\nu_f}{D_B}F(\zeta)\phi'(\zeta) + \left(\frac{D_T(T_w - T_\infty)\tau\alpha}{T_\infty D_B(C_w - C_\infty)\tau\alpha}\right)\theta''(\zeta) = 0,$$
$$\Rightarrow \quad \phi''(\zeta) + \frac{\nu_f}{D_B}F(\zeta)\phi'(\zeta) + \left(\frac{\left(\frac{D_T(T_w - T_\infty)(\rho c)_f}{T_\infty \alpha(\rho c)_p}\right)}{\left(\frac{D_B(C_w - C_\infty)(\rho c)_f}{(\rho c)_p \alpha}\right)}\right)\theta''(\zeta) = 0,$$
$$\Rightarrow \quad \phi''(\zeta) + ScF(\zeta)\phi'(\zeta) + \frac{Nt}{Nb}\theta''(\zeta) = 0,$$
$$\Rightarrow \quad \phi'' + ScF\phi' + \frac{Nt}{Nb}\theta'' = 0. \tag{3.40}$$

Now for converting the associated boundary conditions into the dimensionless form, the following steps have been taken:

- $$u = u_w + N\nu_f\left(\frac{\partial u}{\partial y}\right) \quad at \quad y = 0.$$

$$\Rightarrow \quad ax^nF'(\zeta) = ax^n + N\nu_f\left(aF''(\zeta)\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{3n-1}{2}}\right) \quad at \quad \zeta = 0.$$
$$\Rightarrow \quad ax^nF'(\zeta) = ax^n + ax^nNF''(\zeta)\sqrt{\frac{a\nu_f(n+1)}{2\nu_f}}x^{\frac{n-1}{2}} \quad at \quad \zeta = 0.$$

$$\Rightarrow \quad F'(\zeta) = 1 + NF''(\zeta)\sqrt{\frac{a\nu_f(n+1)}{2\nu_f}}x^{\frac{n-1}{2}} \quad at \quad \zeta = 0.$$

$$\Rightarrow \quad F'(\zeta) = 1 + N_1\sqrt{\frac{a\nu_f(n+1)}{2\nu_f}}F''(\zeta) \quad at \quad \zeta = 0.$$

$$\Rightarrow \quad F' = 1 + \lambda F'' \quad at \quad \zeta = 0.$$

- $v = 0 \quad at \quad y = 0.$

$$\Rightarrow \quad -\sqrt{\frac{a(n+1)\nu_f}{2}}x^{\frac{n-1}{2}}\left[F(\zeta) + \frac{(n-1)}{(n+1)}\zeta F'(\zeta)\right] = 0 \quad at \quad \zeta = 0.$$

$$\Rightarrow \quad \left[F(\zeta) + \frac{(n-1)}{(n+1)}\zeta F'(\zeta)\right] = 0 \quad at \quad \zeta = 0.$$

$$\Rightarrow \quad F(\zeta) = 0 \quad at \quad \zeta = 0.$$

$$\Rightarrow \quad F = 0 \quad at \quad \zeta = 0.$$

- $T = T_w + D\left(\frac{\partial T}{\partial y}\right) \quad at \quad y = 0.$

$$\Rightarrow \quad T_\infty + bx^{2n}\theta(\zeta) = T_\infty + bx^{2n} + Dbx^{2n}\theta'(\zeta)\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{n-1}{2}} \quad at \quad \zeta = 0.$$

$$\Rightarrow \quad bx^{2n}\theta(\zeta) = bx^{2n} + Dbx^{2n}\theta'(\zeta)\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{n-1}{2}} \quad at \quad \zeta = 0.$$

$$\Rightarrow \quad \theta(\zeta) = 1 + D_1\theta'(\zeta)\sqrt{\frac{a(n+1)}{2\nu_f}} \quad at \quad \zeta = 0.$$

$$\Rightarrow \quad \theta(\zeta) = 1 + \delta\theta'(\zeta) \quad at \quad \zeta = 0.$$

$$\Rightarrow \quad \theta = 1 + \delta\theta' \quad at \quad \zeta = 0.$$

- $C = C_w \quad at \quad y = 0.$

$$\Rightarrow \quad C_\infty + (C_w - C_\infty)\phi(\zeta) = C_w \quad at \quad y = 0.$$

$$\Rightarrow \quad (C_w - C_\infty)\phi(\zeta) = (C_w - C_\infty) \quad at \quad \zeta = 0.$$

$$\Rightarrow \quad \phi(\zeta) = 1 \quad at \quad \zeta = 0.$$

$$\Rightarrow \quad \phi = 1 \quad at \quad \zeta = 0.$$

- $u \to 0 \quad as \quad y \to \infty.$

$$\Rightarrow \quad ax^nF'(\zeta) \to 0 \quad at \quad \zeta \to \infty.$$

$$\Rightarrow \quad F' \to 0.$$

- $v \to 0 \quad as \quad y \to \infty.$

- $T \to T_\infty \quad as \quad y \to \infty.$

$\Rightarrow \quad T_\infty + bx^{2n}\theta(\zeta) \to T_\infty \quad as \quad \zeta \to \infty.$

$\Rightarrow \quad \theta \to 0.$

- $C \to C_\infty \quad as \quad y \to \infty.$

$\Rightarrow \quad C_\infty + (C_w - C_\infty)\phi(\zeta) = C_\infty \quad at \quad \zeta \to \infty.$

$\Rightarrow \quad (C_w - C_\infty)\phi(\zeta) = 0 \quad at \quad \zeta \to \infty.$

$\Rightarrow \quad \phi \to 0 \quad at \quad \zeta \to \infty.$

The final dimensionless form of the governing model, is

$$F''' + FF'' - \frac{2n}{(n+1)}(F')^2 - MF' = 0. \tag{3.41}$$

$$\frac{1}{Pr}\theta'' + F\theta' - \frac{4n}{(n+1)}F'\theta + Nb\ \theta'\phi' + Nt\ (\theta')^2 + Ec\ (F'')^2 = 0. \tag{3.42}$$

$$\phi'' + ScF\phi' + \frac{Nt}{Nb}\ \theta'' = 0. \tag{3.43}$$

The associated boundary conditions (3.5) shown as:

$$\left.\begin{aligned} &\zeta = 0: \quad F' = 1 + \lambda F''(0), F = 0, \quad \theta = 1 + \delta\theta'(0), \phi = 1. \\ &\zeta \to \infty: \quad F' \to 0,\ F \to 0,\ \theta \to 0,\ \phi \to 0. \end{aligned}\right\} \tag{3.44}$$

Different parameters used in equations (3.41)-(3.43) are explained as follows:

$$\begin{aligned} &Pr = \frac{\nu}{\alpha}, Sc = \frac{\nu_f}{D_B}, Nb = \frac{D_B(C_w - C_\infty)(\rho C)_f}{(\rho C)_p \alpha}, \\ &Nt = \frac{D_T(T_w - T_\infty)(\rho c)_f}{(\rho c)_p T_\infty \alpha}, M = \frac{2\sigma B^2}{a\rho_f(n+1)}, \\ &\lambda = N_1\sqrt{\frac{a\nu_f(n+1)}{2}}, \delta = D_1\sqrt{\frac{a(n+1)}{a\nu_f}}, Ec = \frac{(u_w)^2}{c_p(T_w - T_\infty)}. \end{aligned} \tag{3.45}$$

The skin friction coefficient is defind as:

$$Cf_x = \frac{\mu_f}{\rho(u_w)^2}\left[\frac{\partial u}{\partial y}\right]_{y=0}. \tag{3.46}$$

To achieve the dimensionless form of Cf_x, the following steps will be found helpful.

- $$\frac{\partial u}{\partial y} = ax^{\frac{3n-1}{2}} F''(\zeta)\sqrt{\frac{a(n+1)}{2\nu_f}}.$$

$$\Rightarrow \left(\frac{\partial u}{\partial y}\right)_{y=0} = ax^{\frac{3n-1}{2}} F''(0)\sqrt{\frac{a(n+1)}{2\nu_f}}. \tag{3.47}$$

Using (3.47) in (3.46), we get in the following form:

$$Cf_x = \frac{\mu_f}{\rho(u_w)^2} ax^{\frac{3n-1}{2}} F''(0)\sqrt{\frac{a(n+1)}{2\nu_f}},$$

$$\Rightarrow Cf_x = \frac{\mu_f}{\rho(u_w)^2} ax^{\frac{3n-1}{2}} F''(0)\sqrt{\frac{a}{\nu_f}}\sqrt{\frac{(n+1)}{2}},$$

$$\Rightarrow Cf_x = \sqrt{\frac{(n+1)}{2}} F''(0)\sqrt{\frac{\nu_f}{ax^{n+1}}},$$

$$\Rightarrow Cf_x = \frac{1}{\sqrt{Re}}\sqrt{\frac{(n+1)}{2}} F''(0),$$

$$\Rightarrow Cf_x = (Re)^{\frac{-1}{2}}\sqrt{\frac{(n+1)}{2}} F''(0),$$

$$\Rightarrow (Re)^{\frac{1}{2}} Cf_x = \sqrt{\frac{(n+1)}{2}} F''(0). \tag{3.48}$$

The Nusselt number is defind as:

$$Nu_x = \frac{xq_w}{k(T_w - T_\infty)}. \tag{3.49}$$

To achieve the dimensionless form of Nu_x, the following steps will be found helpful.

- $$\frac{\partial T}{\partial y} = b\theta'(\zeta)\sqrt{\frac{a(n+1)}{2\nu_f}}\, x^{\frac{5n-1}{2}}.$$

$$\Rightarrow \left(\frac{\partial T}{\partial y}\right)_{y=0} = b\theta'(0)\sqrt{\frac{a(n+1)}{2\nu_f}}\, x^{\frac{5n-1}{2}}.$$

- $$q_w = -k\left[\frac{\partial T}{\partial y}\right]_{y=0},$$

$$\Rightarrow q_w = -kb\theta'(0)\sqrt{\frac{a(n+1)}{2\nu_f}}\, x^{\frac{5n-1}{2}}. \tag{3.50}$$

Using (3.50) in (3.49), we get the following form:

$$\begin{aligned}
Nu_x &= -\frac{xkb}{k(T_w - T_\infty)}\theta'(0)\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{5n-1}{2}},\\
\Rightarrow Nu_x &= -\frac{xb}{bx^{2n}}\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{5n-1}{2}}\theta'(0),\\
\Rightarrow Nu_x &= -\sqrt{\frac{ax^{n+1}}{\nu_f}}\theta'(0)\sqrt{\frac{(n+1)}{2}},\\
\Rightarrow Nu_x &= -\sqrt{Re}\theta'(0)\sqrt{\frac{(n+1)}{2}},\\
\Rightarrow (Re)^{-\frac{1}{2}}Nu_x &= -\sqrt{\frac{(n+1)}{2}}\theta'(0).
\end{aligned} \tag{3.51}$$

The local Sherwood number is defind as:

$$Sh_x = \frac{xq_m}{D_B(C_w - C_\infty)}. \tag{3.52}$$

To achieve the dimensionless form of Sh_x, the following steps will be found helpful.

- $$\frac{\partial C}{\partial y} = (C_w - C_\infty)\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{n-1}{2}}\phi'(\zeta).$$

$$\Rightarrow \left(\frac{\partial C}{\partial y}\right) = (C_w - C_\infty)\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{n-1}{2}}\phi'(0).$$

- $$q_m = -D_B\left[\frac{\partial C}{\partial y}\right]_{y=0},$$

$$\Rightarrow q_m = -D_B(C_w - C_\infty)\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{n-1}{2}}\phi'(0). \tag{3.53}$$

Using (3.53) in (3.52), we get

$$\begin{aligned}
Sh_x &= -\frac{xD_B(C_w - C_\infty)}{D_B(C_w - C_\infty)}\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{n-1}{2}}\phi'(0),\\
\Rightarrow \quad &= -x\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{n-1}{2}}\phi'(0),
\end{aligned}$$

$$\Rightarrow Sh_x = -\sqrt{\frac{ax^{n+1}}{\nu_f}}\sqrt{\frac{(n+1)}{2}}\phi'(0),$$
$$\Rightarrow Sh_x = -\sqrt{Re}\sqrt{\frac{(n+1)}{2}}\phi'(0),$$
$$\Rightarrow (Re)^{-\frac{1}{2}}Sh_x = -\sqrt{\frac{(n+1)}{2}}\phi'(0). \tag{3.54}$$

Here k is the thermal conductivity, the heat and mass fluxes q_w and q_m at the sheet respectively. Moreover, Re_x represents the Reynolds number defined as $Re_x = \frac{xu_w}{\nu}$.

3.3 Method of Solution

For the solution of ODE system (3.41)-(3.43), the shooting method has been used. Equation (3.41) is solved numerically and then its solution is used in equations (3.42)-(3.43). To solve the equation (3.41) independently by using shooting method, the following notations have been incorporated:

$$F = z_1, \quad F' = z_1' = z_2,$$
$$F'' = z_2' = z_3, \quad F''' = z_3'.$$

By using the above notations in equation (3.41), the successive scheme of ODEs is obtained:

$$z_1' = z_2, \qquad z_1(0) = 0.$$
$$z_2' = z_3, \qquad z_2(0) = 1 + \lambda s.$$
$$z_3' = \frac{2n}{(n+1)}z_2^2 + Mz_2 - z_1 z_3, \qquad z_3(0) = s.$$

The above IVP will be numerically solved by RK technique of order four. In the above initial value problem, the missing condition s satisfy the following relation:

$$F''(0) = \frac{1}{\lambda}(F'(0) - 1). \tag{3.55}$$

To solve the above algebric equation (3.55), we apply Newton's method which has the following iterative scheme:

$$S^{k+1} = S^k - \frac{G(\zeta)}{G'(\zeta)}, \quad \text{where} \quad G(\zeta) = y_2(\eta_\infty, \zeta).$$

To incorporate Newton's method, we further utilize the following notions:

$$\frac{\partial z_1}{\partial s} = z_4, \qquad \frac{\partial z_2}{\partial s} = z_5, \qquad \frac{\partial z_3}{\partial s} = z_6.$$

As a result, the following IVP is obtained:

$$z_4' = z_5, \qquad z_4(0) = 0.$$
$$z_5' = z_6, \qquad z_5(0) = \lambda.$$
$$z_6' = \frac{4n}{(n+1)} z_2 z_5 + M z_5 - z_1 z_6 - z_3 z_4, \qquad z_6(0) = 1.$$

Now, the coupled equations (3.42)-(3.43) will be treated similarly by considering F as a known function. For this, we use the following notations:

$$\theta = z_1, \quad \theta' = z_1' = z_2, \quad \theta'' = z_2'.$$
$$\phi = z_3, \quad \phi' = z_3' = z_4, \quad \phi'' = z_4'.$$

By using the above notations in equations (3.42)-(3.43), the following ODEs is obtained:

$$z_1' = z_2, \qquad z_1(0) = 1 + \delta u.$$
$$z_2' = Pr \left[\frac{4n}{(n+1)} F' z_1 - F z_2 - Nb z_2 z_4 - Nt z_2^2 - Ec(F'')^2 \right], \qquad z_2(0) = u.$$
$$z_3' = z_2, \qquad z_3(0) = 0.$$
$$z_4' = ScF z_4 - \frac{PrNt}{Nb} \left[\frac{4n}{(n+1)} F' z_1 - F z_2 - Nb z_2 z_4 - Nt z_2^2 - Ec(F'')^2 \right],$$
$$z_4(0) = v.$$

The above IVP will be numerically solved by RK technique of order four. In the above IVP, the missing conditions u and v are to be chosen such that:

$$\left(\frac{z_1(\zeta, u^{(k)}, v^{(k)}) - 1}{\delta}\right) = 0, \quad (z_2(\zeta, u^{(k)}, v^{(k)})) = 0. \tag{3.56}$$

To solve the above algebric equation (3.56), we apply the Newton's method which has the following scheme:

$$\begin{pmatrix} u^{(k+1)} \\ v^{(k+1)} \end{pmatrix} = \begin{pmatrix} u^{(k)} \\ v^{(k)} \end{pmatrix} - \begin{pmatrix} \frac{\partial z_1}{\partial u} & \frac{\partial z_2}{\partial v} \\ \frac{\partial z_1}{\partial u} & \frac{\partial z_2}{\partial v} \end{pmatrix}^{-1} \begin{pmatrix} \frac{z_1(\zeta, u^{(k)}, v^{(k)}) - 1}{\delta} \\ z_2(\zeta, u^{(k)}, v^{(k)}) \end{pmatrix}$$

To incorporate Newton's method, we further apply the following notions:

$$\frac{\partial z_1}{\partial u} = z_5, \quad \frac{\partial z_2}{\partial u} = z_6, \quad \frac{\partial z_3}{\partial u} = z_7, \quad \frac{\partial z_4}{\partial u} = z_8.$$
$$\frac{\partial z_1}{\partial v} = z_9, \quad \frac{\partial z_2}{\partial v} = z_{10}, \quad \frac{\partial z_3}{\partial v} = z_{11}, \quad \frac{\partial z_4}{\partial v} = z_{12}.$$

As the result of these new notations, the Newton's iterative scheme gets the form:

$$\begin{pmatrix} u^{(k+1)} \\ v^{(k+1)} \end{pmatrix} = \begin{pmatrix} u^{(k)} \\ v^{(k)} \end{pmatrix} - \begin{pmatrix} z_5 & z_6 \\ z_9 & z_{10} \end{pmatrix}^{-1} \begin{pmatrix} \frac{z_1(\zeta, u^{(k)}, v^{(k)}) - 1}{\delta} \\ z_2(\zeta, u^{(k)}, v^{(k)}) \end{pmatrix}$$

Here k is the number of iterations ($k = 0, 1, 2, 3, ...$). Now differentiating the above ODEs with respect to u and v, we have another scheme of eight ODEs of order one.

$$z_5' = z_6, \qquad z_5(0) = \delta.$$
$$z_6' = Pr\left[\frac{4n}{n+1}F' z_5 - F z_6 - Nb z_4 z_6 - Nb z_2 z_8 - 2Nt z_2 z_6\right], \qquad z_6(0) = 1.$$
$$z_7' = z_8, \qquad z_7(0) = 0.$$
$$z_8' = ScF z_8 - \frac{PrNt}{Nb}\left[\frac{4n}{n+1}F' z_5 - F z_6 - Nb z_4 z_6 - Nb z_2 z_8 - 2Nt z_2 z_6\right],$$
$$z_8(0) = 0.$$

$$z_9' = z_{10}, \qquad z_9(0) = 0.$$

$$z_{10}' = Pr\left[\frac{4n}{n+1}F'z_9 - Fz_{10} - Nbz_4z_{10} - Nbz_2z_{12} - 2Ntz_2z_{10}\right], \qquad z_{10}(0) = 0.$$

$$z_{11}' = z_{12}, \qquad z_{11}(0) = 0.$$

$$z_{12}' = ScFz_{12} - \frac{PrNt}{Nb}\left[\frac{4n}{n+1}F'z_9 - Fz_{10} - Nbz_4z_{10} - Nbz_2z_{12} - 2Ntz_2z_{10}\right], \qquad z_{12}(0) = 1.$$

The RK method has been used to solve the IVP consisting of the above twelve first order ODEs for some suitable choices of u and v. The iterative process is repeated until the following stopping criteria is met:

$$max\{|u^{(k+1)} - u^{(k)}|, |v^{(k+1)} - v^{(k)}|\} < \epsilon,$$

for an arbitrary small positive value of ϵ. Throughout this chapter ϵ has the value $(10)^{-6}$.

3.4 Results and Discussion

The main objective of this section is to analyze the impact of various parameters on the velocity, temperature and concentration distributions. The influence of different factors such as Schmidt number Sc, Brownian motion parameter Nb, thermophoresis parameter Nt, Prandtl number Pr, Eckert number Ec, velocity slip factor λ, thermal slip factor δ, non-linear stretching parameter n, magnetic field parameter M is observed graphically. Numerical results of the skin friction coefficient, Nusselt number and Sherwood number for the distinct values of some fixed parameters are shown in Tables 3.1-3.2.

Figure 3.2 shows the impact of non-linear stretching parameter n and magnetic parameter M on velocity distribution. By enhancing the values of M and n, the velocity distribution shows the decreasing behavior due to the presence of Lorentz force. Figure 3.3 exhibits the influence of magnetic parameter M and non-linear stretching parameter n on temperature distribution. The temperature distribution expands by enhancing the values of M and n. Figure 3.4 represents the impact of magnetic parameter M on the concentration distribution. By increasing the values of M, the nanoparticle concentration distribution is also escalated due to Lorentz force.

Figure 3.5 is delineated to show the impact of velocity slip factor λ on the dimensionless velocity profile. This graph reflects that by enhancing the estimations of λ, the velocity distribution is decreased because the velocity of the stretching sheet is not the same as the velocity of the flow. Figures 3.6-3.7 represents the influence of velocity slip parameter on the dimensionless velocity profile. It can be noted that, the temperature and concentration profiles are increased by enhancing the values of λ due to the presence of thermal slip and magnetic parameter. Figure 3.8 displays the impact of thermal slip parameter δ on the dimensionless temperature distribution. It can be noted that the temperature distribution reduces by enhancing the values of δ. Figure 3.9 illustrates the impact of thermal slip parameter δ on the dimensionless concentration profile. This graph indicates that an increment in the values of δ causes an increment in the nanoparticle concentration profile is increased. The thermal boundary layer thickness experiences a decrement by enhancing the values of δ.

Figure 3.10 is delineated to show the impact of Eckert number Ec on temperature distribution. It is clearly shown that the temperature distribution is increased by enhancing the values of Ec due to the decrement in heat transfer rate. Figure 3.11 indicates the influence of Brownian motion parameter Nb on concentration distribution. The behavior of concentration distribution is decreased due to the accelerating values of Nb. Figure 3.12 indicates the dynamics of the velocity distribution against the magnetic parameter M for rising values of non-linear stretched parameter n. By enhancing the values of n, the velocity distribution is declined due to the motion of fluid. Figure 3.13 represents the impact of thermal slip parameter δ on temperature distribution. It can be noted that the temperature distribution accelerates by enhancing the values of Pr. Figure 3.14 exhibits the impact of thermophoresis parameter Nt on heat transfer rate. It can be noted that, the temperature profile is declined by enhancing the values of Nt.

TABLE 3.1: Numerical outcomes for $(Re_x)^{\frac{1}{2}} Cf_x$ for various parameters.

n	Hammad [15]	Present result
0.1	0.6284	0.6284
0.2	0.7675	0.7676
0.5	0.8902	0.8902
1.0	1.0005	1.0006
3.0	1.1489	1.1489
10	1.2352	1.2352
20	1.2577	1.2578

TABLE 3.2: Numerical results for $(Re_x)^{\frac{1}{2}} Cf_x$, $(Re_x)^{\frac{-1}{2}} Nu_x$ and $(Re_x)^{\frac{1}{2}} Sh_x$ for various parameters.

M	λ	δ	$-F''(0)$	$-\theta'(0)$	$-\phi'(0)$
0.1	0.2	0.2	0.812078	1.486388	0.346328
0.3	0.2	0.2	0.875363	1.60938	0.236716
0.4	0.2	0.2	0.904627	1.651014	0.416661
0.5	0.2	0.2	0.932511	1.453194	0.394593
1.0	0.3	0.1	0.940762	1.081737	0.071086
1.0	0.4	0.1	0.850231	1.536613	0.619206
1.0	0.5	0.1	0.776596	1.490573	0.612050

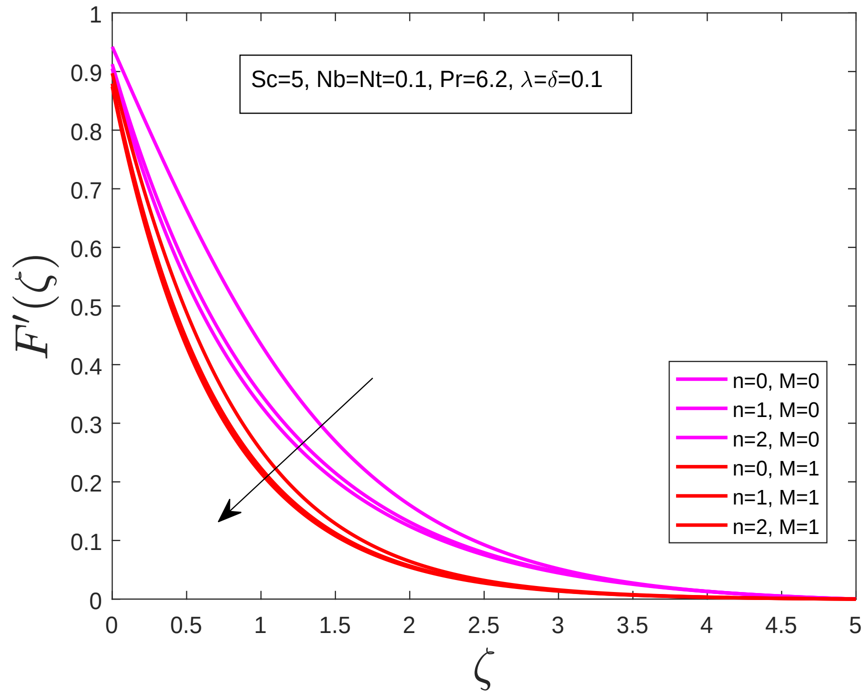

FIGURE 3.2: Impact of magnetic parameter with non-linear stretching parameter on the dimensionless velocity profile.

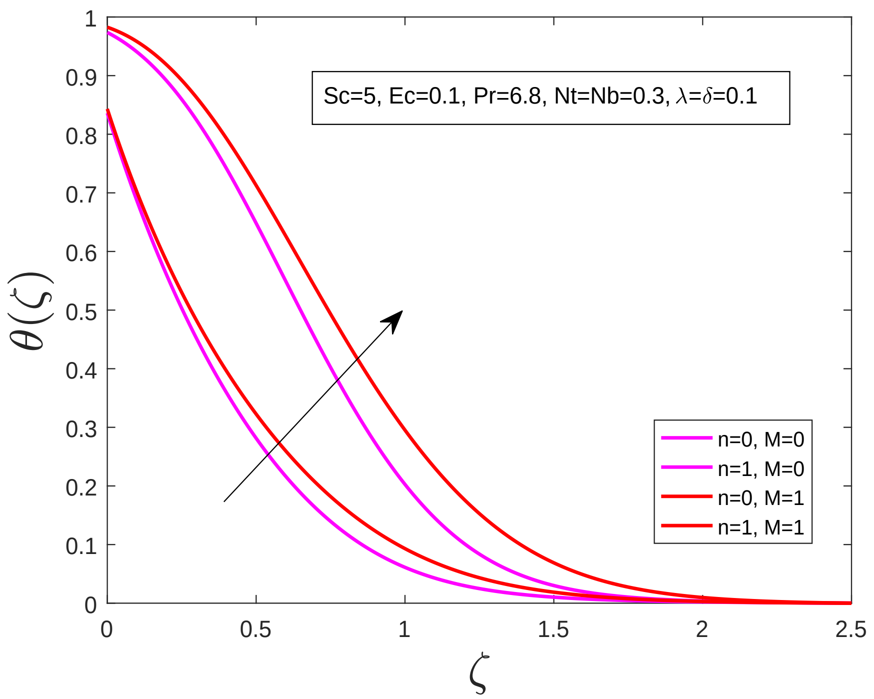

FIGURE 3.3: Impact of magnetic parameter with non-linearly stretched parameter on dimensionless temperature profile.

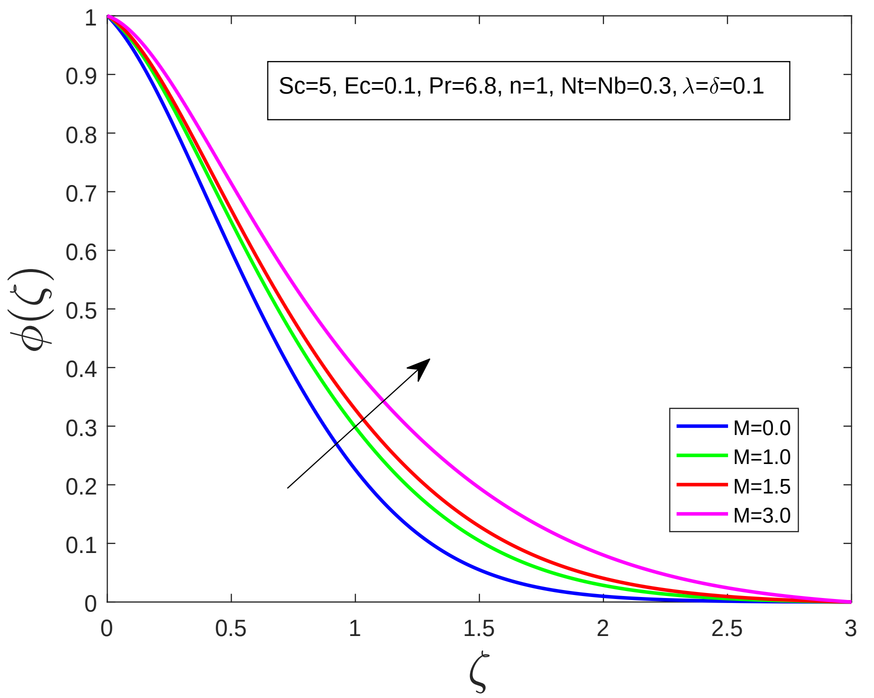

FIGURE 3.4: Influence of magnetic factor on concentration distribution.

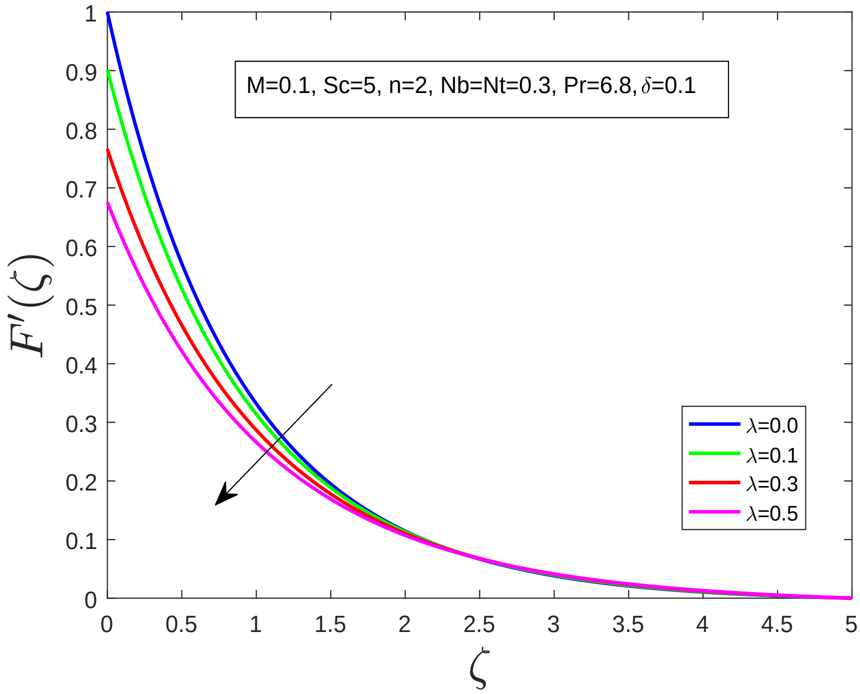

FIGURE 3.5: Impact of velocity slip factor on the dimensionless concentration profile.

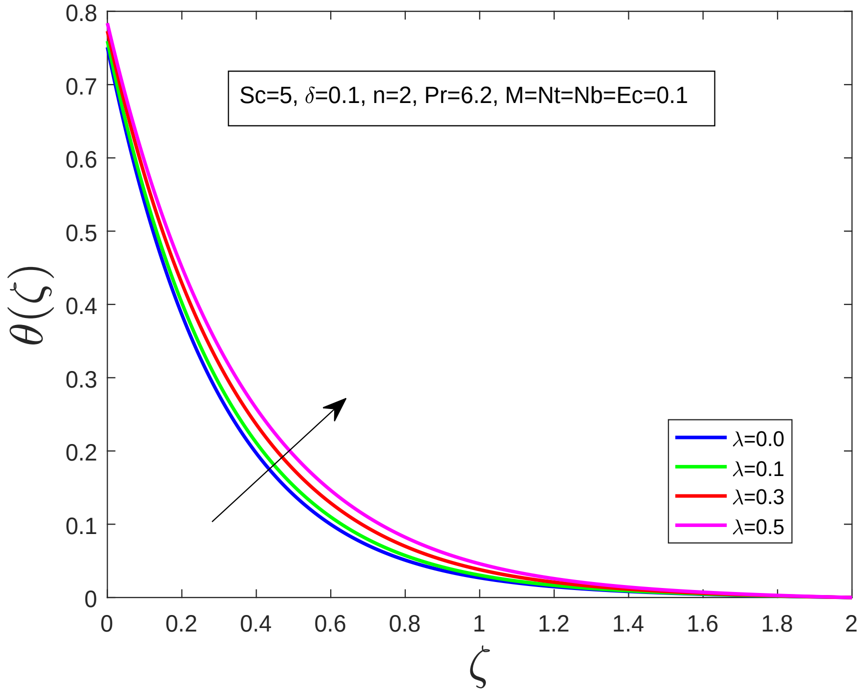

FIGURE 3.6: Impact of velocity slip factor on dimensionless temperature profile.

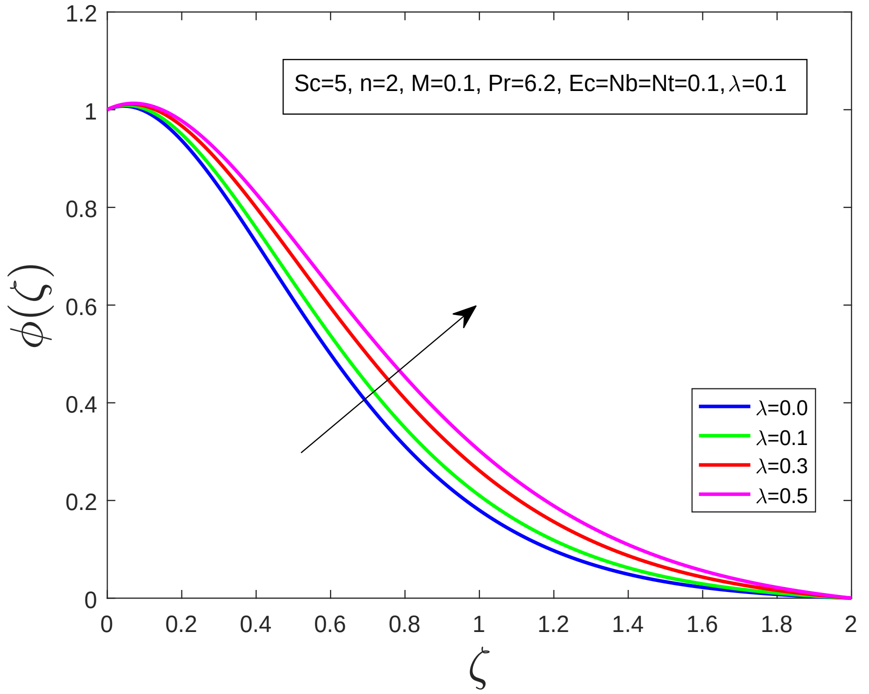

FIGURE 3.7: Influence of velocity slip factor on dimensionless concentration profile.

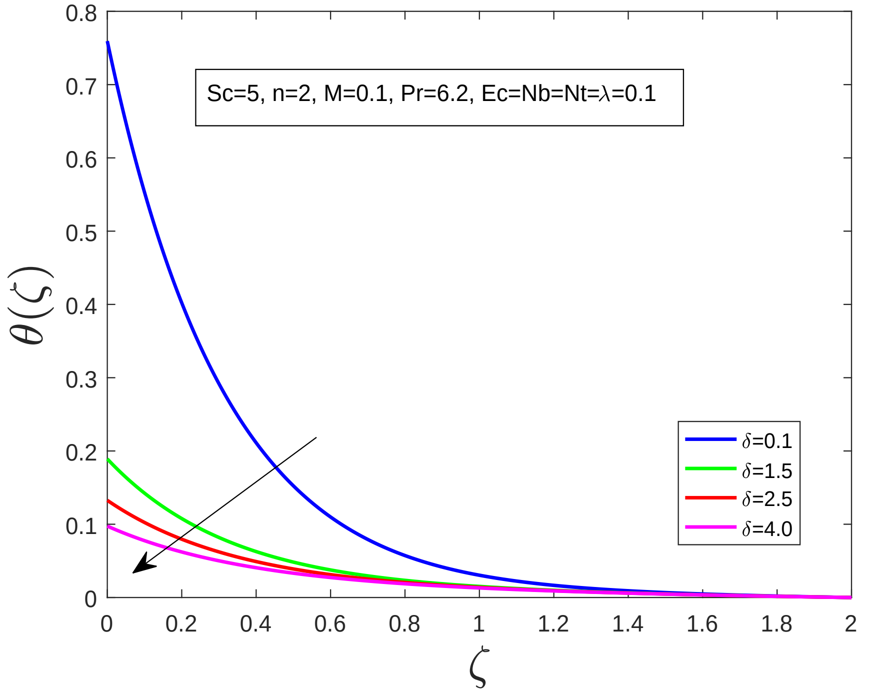

FIGURE 3.8: Impact of thermal slip parameter δ on temperature distribution.

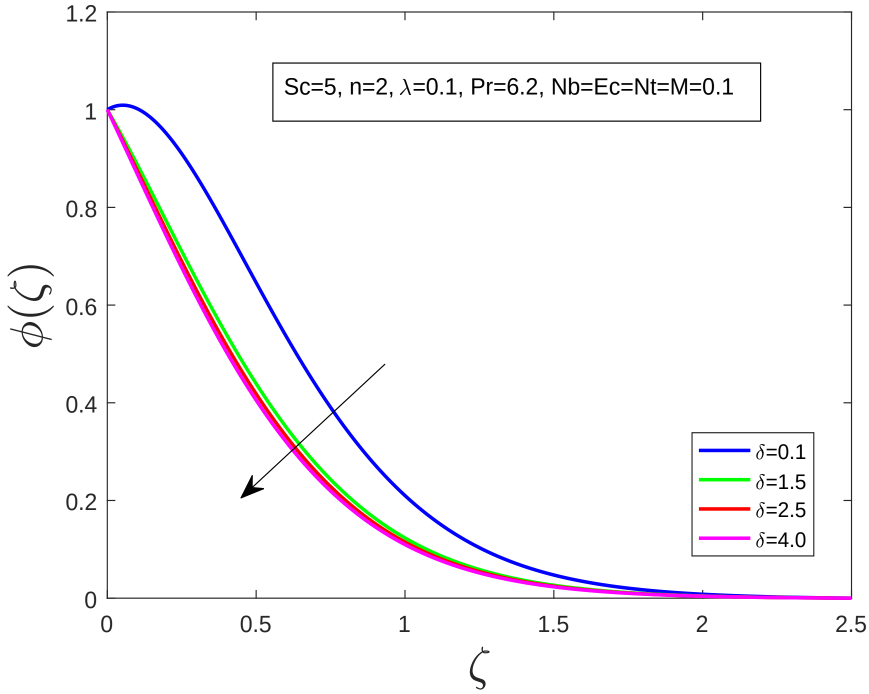

FIGURE 3.9: Influence of dimensionless similarity variable on the concentration distribution.

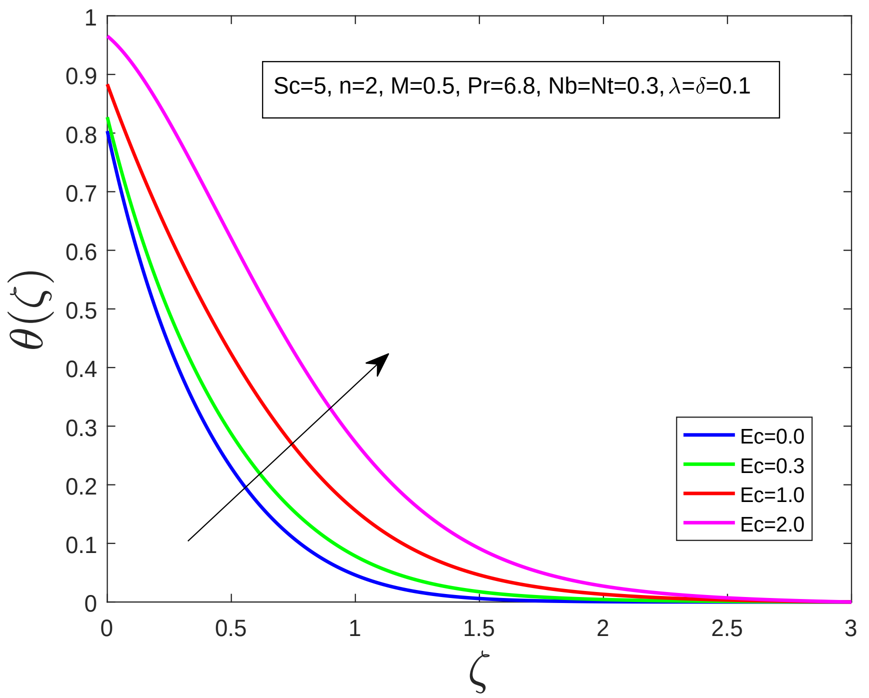

FIGURE 3.10: Impact of Ec on temperature distribution.

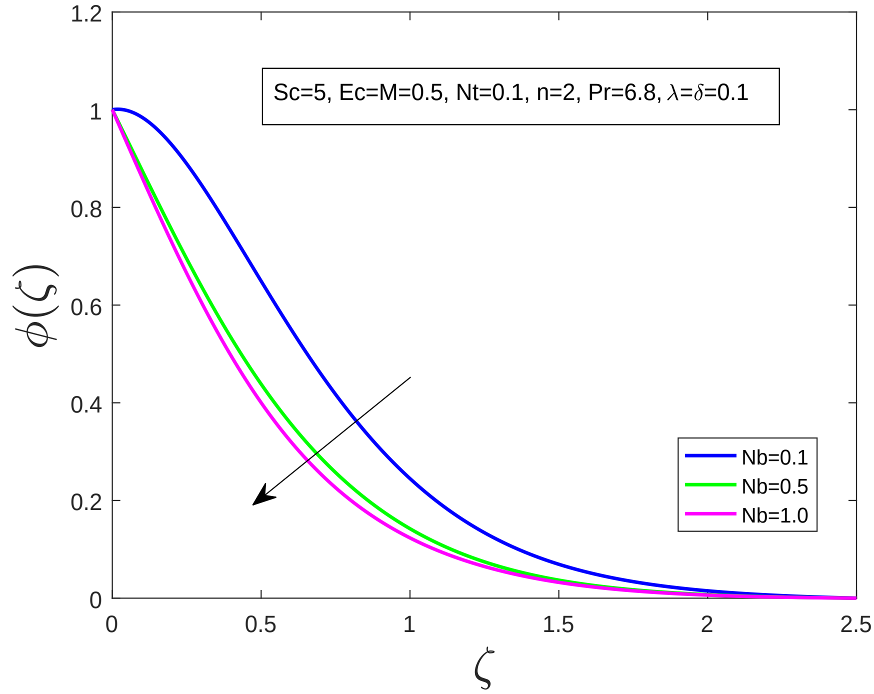

FIGURE 3.11: Influence of brownian motion parameter Nb on concentration profile.

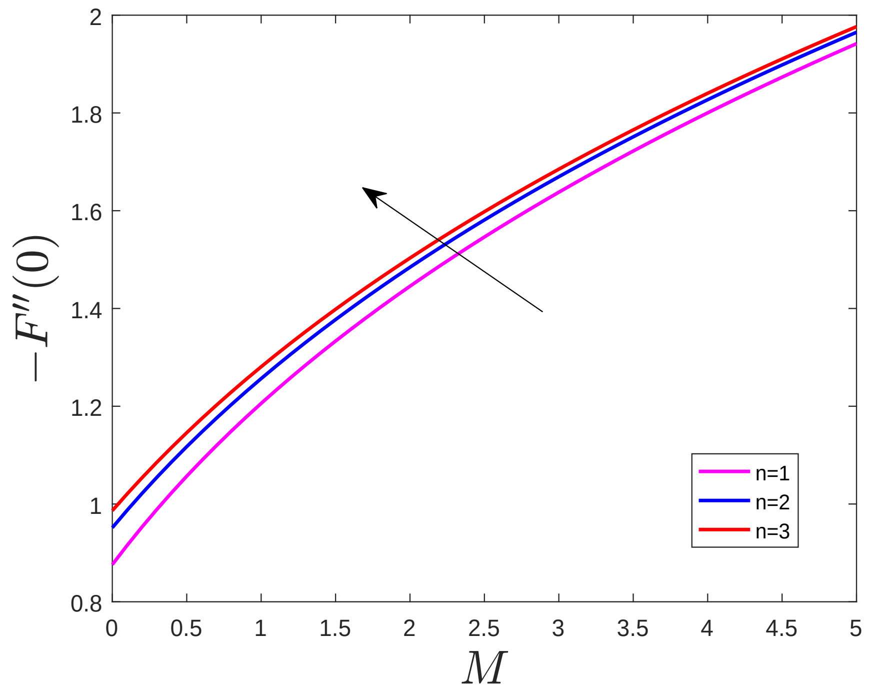

FIGURE 3.12: Influence of M with non-linear stretching parameter on the dimensionless profile.

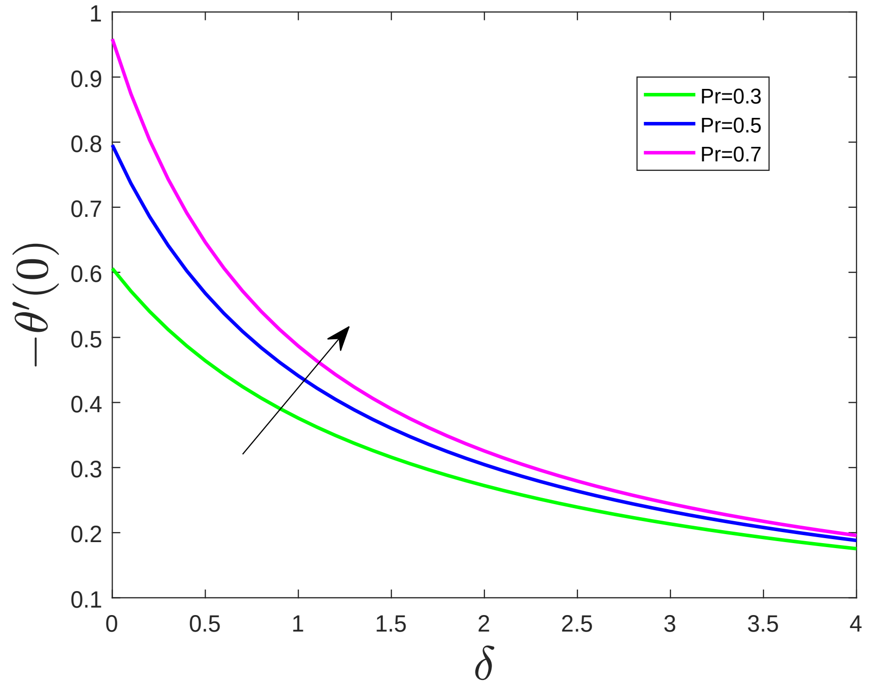

FIGURE 3.13: Influence of prandtl number Pr on temperature distribution.

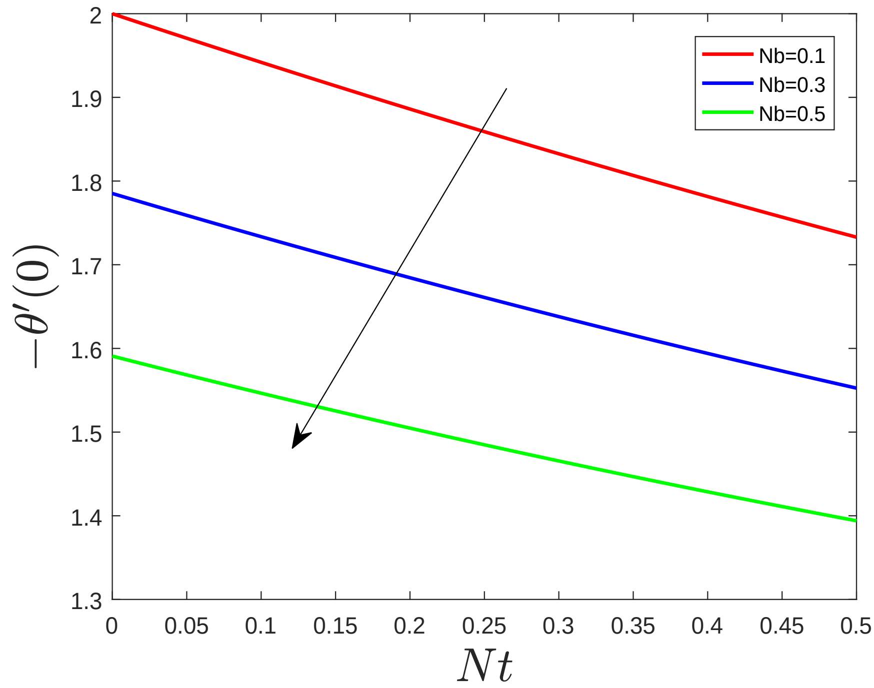

FIGURE 3.14: Impact of brownian motion parameter Nb on temperature distribution.

Chapter 4

A Carreau Nanofluid Flow on a Non-Linearly Stretchable Sheet

4.1 Introduction

In this chapter, we incorporate the impacts of velocity slip factor on MHD incompressible viscid laminar fluid flow with Carreau model, thermal radiation and chemical reaction parameters on a non-linear stretchable surface. The set of equations for energy, momentum and concentration are attained by utilizing the boundary layer approximation. Furthermore, the governing coupled non-linear partial differential equations are transformed into ODEs by using the appropriate transformations. A numerical technique based on the shooting method is used for the solutions of first order ODEs. At the end of this chapter, the numerical solution for different parameters like Weissenberg number We, thermal radiation R and chemical reaction are considered which impact on the skin friction coefficients, Nusselt and Sherwood number. The tables and graphs are shown which are obtained through this investigation.

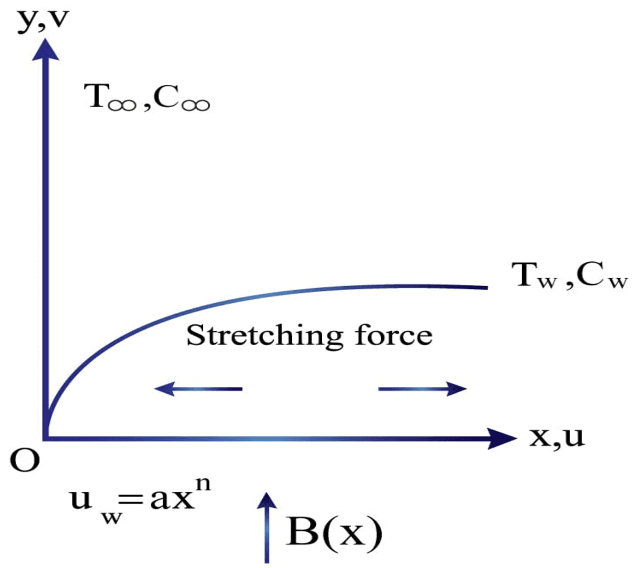

FIGURE 4.1: Systematic representation of physical model.

4.2 Mathematical Modeling

Assume a uniform 2D incompressible viscid flow of a Carreau model on a non-linearly stretchable sheet. Meanwhile, the plate has been stretched with the velocity $u_w = ax^n$ along $x - axis$. Here, T_w is the wall temperature and C_w is the nanoparticle fraction, T_∞ is the ambient temperature and C_∞ is the nanoparticle fraction. The flow is explained by assuming the two dimensional governing equations containing the continuity, momentum and energy transfer:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{4.1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}\left[1+\Gamma^2\left(\frac{\partial u}{\partial y}\right)^2\right]^{\frac{n-1}{2}} + \nu(n-1)\Gamma^2\frac{\partial^2 u}{\partial y^2}\left(\frac{\partial u}{\partial y}\right)^2$$
$$\left[1+\Gamma^2\left(\frac{\partial u}{\partial y}\right)^2\right]^{\frac{n-3}{2}} - \frac{\sigma B^2}{\rho_f}u. \tag{4.2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} + \frac{\nu}{c_p}\left(\frac{\partial u}{\partial y}\right)^2 + \tau\left[D_B\frac{\partial T}{\partial y}\frac{\partial C}{\partial y} + \frac{D_T}{T_\infty}\left(\frac{\partial T}{\partial y}\right)^2\right] + \frac{\partial q_r}{\partial y}. \tag{4.3}$$

$$u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} = D_B\frac{\partial^2 C}{\partial y^2} + \frac{D_T}{T_\infty}\left(\frac{\partial^2 T}{\partial y^2}\right) + Kr(C - C_\infty). \tag{4.4}$$

Here u and v are the elements of velocity. Furthermore, q_r denotes the radiative heat flux and Kr is chemical reaction constant. The associated BCs are:

$$\left.\begin{aligned} &u = u_w - N\nu_f\left(\frac{\partial u}{\partial y}\right), \quad v = 0. \\ &T = T_w + D\left(\frac{\partial T}{\partial y}\right), \quad C = C_w \quad as \quad y = 0. \\ &u \to 0,\ v \to 0,\ T = T_\infty,\ C = C_\infty \quad as \quad y \to \infty. \end{aligned}\right\} \tag{4.5}$$

Now, we introduce the appropriate variables, which are useful in transforming the partial differential equations (4.1)-(4.4) into the ODEs along with the boundary conditions (4.5).

$$\left.\begin{aligned} &\zeta = y\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{n-1}{2}}, \quad u = ax^n F'(\zeta), \\ &v = -\sqrt{\frac{a(n+1)\nu_f}{2}}x^{\frac{n-1}{2}}\left[F(\zeta) + \frac{(n-1)}{(n+1)}\zeta F'(\zeta)\right], \\ &T = T_\infty + bx^{2n}\theta(\zeta), \quad \phi(\zeta) = \frac{(C - C_\infty)}{(C_w - C_\infty)}. \end{aligned}\right\} \tag{4.6}$$

The complete procedure for the verification of continuity equation (4.1) has been discussed in Chapter 3.

To convert the equation (4.2) into the dimensionless form, following steps will be helpful. The left side of (4.2) can be shown as:

$$\begin{aligned} &= \frac{a^2(n-1)}{2}y\sqrt{\frac{a(n+1)}{2\nu_f}}F'(\zeta)F''(\zeta)x^{\frac{5n-3}{2}} + a^2n(F'(\zeta))^2x^{2n-1} - \\ &\frac{a^2(n+1)}{2}F(\zeta)F''(\zeta)x^{2n-1} - \frac{a^2(n-1)}{2}F'(\zeta)F''(\zeta)y\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{5n-3}{2}}. \end{aligned} \tag{4.7}$$

To convert the right side of equation (4.2) into the dimensionless form, we proceed as follows:

- $$\nu\frac{\partial^2 u}{\partial y^2}\left[1 + n\Gamma^2\left(\frac{\partial u}{\partial y}\right)^2\right]\left[1 + \Gamma^2\left(\frac{\partial u}{\partial y}\right)^2\right]^{\frac{n-3}{2}}$$

$$= \nu_f F'''(\zeta)\frac{a^2(n+1)}{2\nu_f}x^{2n-1}\left[1+n\Gamma^2a^2(F''(\zeta))^2\frac{a(n+1)}{2\nu_f}x^{3n-1}\right]$$
$$\left[1+\Gamma^2a^2(F''(\zeta))^2\frac{a(n+1)}{2\nu_f}x^{3n-1}\right]^{\frac{n-3}{2}},$$
$$= F'''(\zeta)\frac{a^2(n+1)}{2\nu_f}x^{2n-1}[1+n(F''(\zeta))^2We^2][1+(F''(\zeta))^2We^2]^{\frac{n-3}{2}}. \tag{4.8}$$

Using (4.8), the right side of equation (4.2) becomes:

$$= F'''(\zeta)\frac{a^2(n+1)}{2\nu_f}x^{2n-1}[1+n(F''(\zeta))^2We^2][1+(F''(\zeta))^2We^2]^{\frac{n-3}{2}}-$$
$$\frac{a\sigma B_0^2}{\rho_f}x^{2n-1}F'(\zeta). \tag{4.9}$$

Using (4.7) and (4.9), the dimensionless form of equation (4.2) can be seen as:

$$\frac{a^2(n-1)}{2}y\sqrt{\frac{a(n+1)}{2\nu_f}}F'(\zeta)F''(\zeta)x^{\frac{5n-3}{2}}+a^2n(F'(\zeta))^2x^{2n-1}-\frac{a^2(n+1)}{2}$$
$$F(\zeta)F''(\zeta)x^{2n-1}-\frac{a^2(n-1)}{2}F'(\zeta)F''(\zeta)y\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{5n-3}{2}}=F'''(\zeta)x^{2n-1}$$
$$\frac{a^2(n+1)}{2\nu_f}[1+n(F''(\zeta))^2We^2][1+(F''(\zeta))^2We^2]^{\frac{n-3}{2}}-\frac{a\sigma B_0^2}{\rho_f}x^{2n-1}F'(\zeta),$$
$$\Rightarrow \quad F'''(\zeta)[1+n(F''(\zeta))^2We^2][1+(F''(\zeta))^2We^2]^{\frac{n-3}{2}}+F(\zeta)F''(\zeta)-\frac{2n}{(n+1)}$$
$$(F'(\zeta))^2-MF'(\zeta)=0,$$
$$\Rightarrow \quad F'''[1+n(F'')^2We^2][1+(F'')^2We^2]^{\frac{n-3}{2}}+FF''-\frac{2n}{(n+1)}(F')^2-MF'=0. \tag{4.10}$$

To convert the equation (4.3) into ordinary differential form, following calculations are attained. The left side of (4.3) are:

$$=2abnF'(\zeta)\theta(\zeta)x^{3n-1}+\frac{ab(n-1)}{2}F'(\zeta)\theta'(\zeta)\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{7n-3}{2}}y-$$
$$\frac{ab(n+1)}{2}F'(\zeta)\theta'(\zeta)x^{3n-1}-\frac{ab(n-1)}{2}F'(\zeta)\theta'(\zeta)\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{7n-3}{2}}y. \tag{4.11}$$

The dimensionless form of the right side of (4.3) can be obtained as:

- $$\frac{\partial q_r}{\partial y} = \frac{\partial}{\partial y}\left(\frac{-4\sigma^*}{3K^*}\frac{\partial T^4}{\partial y}\right),$$

By using Taylor series, we have

$$\begin{aligned}
&\Rightarrow \quad T^4 = T_\infty^4 + 4T_\infty^3(T - T_\infty) + ...,\\
&\Rightarrow \quad T^4 = 4TT_\infty^3 - 3T_\infty^4,\\
&\Rightarrow \quad \frac{\partial q_r}{\partial y} = \frac{-16\sigma^* T_\infty^3}{3K^*}\frac{\partial^2 T}{\partial y^2}.
\end{aligned} \tag{4.12}$$

Here σ^* denotes the Stefan-Boltzman constant and the coefficient of absorption is K^*. Using (4.12) in the right side of (4.3), we get

$$\begin{aligned}
&\alpha\frac{\partial^2 T}{\partial y^2} + \frac{\nu}{c_p}\left(\frac{\partial u}{\partial y}\right)^2 + \tau\left[D_B\frac{\partial T}{\partial y}\frac{\partial C}{\partial y} + \frac{D_T}{T_\infty}\left(\frac{\partial T}{\partial y}\right)^2\right] + \frac{\partial q_r}{\partial y}\\
&= \alpha\frac{ab(n+1)}{2\nu_f}\theta''(\zeta)x^{3n-1} + \frac{\nu}{c_p}\frac{a^3(n+1)}{2\nu_f}(F''(\zeta))^2x^{3n-1} + \tau D_B\\
&\frac{ab(n+1)}{2}(C_w - C_\infty)\theta'(\zeta)\phi'(\zeta)x^{3n-1} + \tau\frac{D_T(T_w - T_\infty)}{T_\infty}\\
&\frac{ab(n+1)}{2}(\theta'(\zeta))^2x^{3n-1} - \frac{16\sigma^* T_\infty^3}{3K^*}\frac{\partial^2 T}{\partial y^2},\\
\Rightarrow \quad &= \alpha\frac{ab(n+1)}{2\nu_f}\theta''(\zeta)x^{3n-1} + \frac{\nu}{c_p}\frac{a^3(n+1)}{2\nu_f}(F''(\zeta))^2x^{3n-1} + \tau D_B\\
&\frac{ab(n+1)}{2}(C_w - C_\infty)\theta'(\zeta)\phi'(\zeta)x^{3n-1} + \tau\frac{D_T(T_w - T_\infty)}{T_\infty}\\
&\frac{ab(n+1)}{2}(\theta'(\zeta))^2x^{3n-1} - \frac{16\sigma^* T_\infty^3}{3K^*}\frac{ab(n+1)}{2\nu_f}\theta''(\zeta)x^{3n-1}.
\end{aligned} \tag{4.13}$$

Comparing equations (4.11) and (4.13), the dimensionless form of (4.3) can be written as:

$$\begin{aligned}
&2abnF'(\zeta)\theta(\zeta)x^{3n-1} + \frac{ab(n-1)}{2}F'(\zeta)\theta'(\zeta)\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{7n-3}{2}}y -\\
&\frac{ab(n+1)}{2}F'(\zeta)\theta'(\zeta)x^{3n-1} - \frac{ab(n-1)}{2}F'(\zeta)\theta'(\zeta)\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{7n-3}{2}}y
\end{aligned}$$

$$= \alpha\frac{ab(n+1)}{2\nu_f}\theta''(\zeta)x^{3n-1} + \frac{\nu}{c_p}\frac{a^3(n+1)}{2\nu_f}(F''(\zeta))^2x^{3n-1} + \tau D_B$$
$$\frac{ab(n+1)}{2}(C_w - C_\infty)\theta'(\zeta)\phi'(\zeta)x^{3n-1} + \tau\frac{D_T(T_w - T_\infty)}{T_\infty}$$
$$\frac{ab(n+1)}{2}(\theta'(\zeta))^2x^{3n-1} - \frac{16\sigma^*T_\infty^3}{3K^*}\frac{ab(n+1)}{2\nu_f}\theta''(\zeta)x^{3n-1},$$
$$\Rightarrow \quad \frac{1}{Pr}\left(1 - \frac{4}{3}R\right)\theta''(\zeta) + F(\zeta)\theta'(\zeta) - \frac{4n}{(n+1)}F'(\zeta)\theta(\zeta) + Nb\theta'(\zeta)\phi'(\zeta)$$
$$+ Nt(\theta'(\zeta))^2 + Ec(F''(\zeta))^2 = 0,$$
$$\Rightarrow \quad \frac{1}{Pr}\left(1 - \frac{4}{3}R\right)\theta'' + F\theta' - \frac{4n}{(n+1)}F'\theta + Nb\ \theta'\phi' + Nt\ (\theta')^2 + Ec\ (F'')^2 = 0. \tag{4.14}$$

To convert the equation (4.4) into ordinary differential form. We proceed as follows. The left side of (4.4) can be shown as:

$$= F'(\zeta)(C_w - C_\infty)\frac{a(n-1)}{2}F'(\zeta)\phi'(\zeta)y\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{3n-3}{2}} -$$
$$\frac{a(n+1)}{2}(C_w - C_\infty)F(\zeta)\phi'(\zeta)x^{n-1} - \frac{a(n-1)}{2}(C_w - C_\infty)$$
$$F'(\zeta)\phi'(\zeta)y\sqrt{\frac{a(n+1)}{2}}x^{\frac{3n-3}{2}}. \tag{4.15}$$

The right side of (4.4) can be attained as:

- $$K_r(C - C_\infty) = K_r(C_w - C_\infty)\phi(\zeta). \tag{4.16}$$

Using (4.16) in the right side of (4.4), we get

$$D_B\frac{\partial^2 C}{\partial y^2} + \frac{D_T}{T_\infty}\left(\frac{\partial^2 T}{\partial y^2}\right) + K_r(C - C_\infty)\phi(\zeta) = D_B\frac{a(n+1)}{2\nu_f}$$
$$\phi''(\zeta)(C_w - C_\infty)x^{n-1} + \frac{D_T}{T_\infty}\frac{a(n+1)}{2\nu_f}\theta''(\zeta)bx^{3n-1} +$$
$$K_r(C_w - C_\infty)\phi(\zeta),$$
$$= D_B\frac{a(n+1)}{2\nu_f}\phi''(\zeta)(C_w - C_\infty)x^{n-1} + \frac{D_T}{T_\infty}\frac{a(n+1)}{2\nu_f}\theta''(\zeta)bx^{3n-1} +$$
$$K_r(C_w - C_\infty)\phi(\zeta). \tag{4.17}$$

With the help of equations (4.15) and (4.17), the dimensionless form of (4.4) can be seen as:

$$
\begin{aligned}
& F'(\zeta)(C_w - C_\infty)\frac{a(n-1)}{2}F'(\zeta)\phi'(\zeta)y\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{3n-3}{2}} - \frac{a(n+1)}{2}(C_w - C_\infty) \\
& F(\zeta)\phi'(\zeta)x^{n-1} - \frac{a(n-1)}{2}(C_w - C_\infty)F'(\zeta)\phi'(\zeta)y\sqrt{\frac{a(n+1)}{2}}x^{\frac{3n-3}{2}} \\
& = D_B\frac{a(n+1)}{2\nu_f}\phi''(\zeta)(C_w - C_\infty)x^{n-1} + \frac{D_T}{T_\infty}\frac{a(n+1)}{2\nu_f}\theta''(\zeta)bx^{3n-1} \\
& + K_r(C_w - C_\infty)\phi(\zeta), \\
\Rightarrow \quad & D_B\frac{a(n+1)}{2\nu_f}\phi''(\zeta)(C_w - C_\infty)x^{n-1} + \frac{D_T}{T_\infty}\frac{a(n+1)}{2\nu_f}\theta''(\zeta)bx^{3n-1} \\
& + K_r(C_w - C_\infty)\phi(\zeta) + \frac{a(n+1)}{2}(C_w - C_\infty)F(\zeta)\phi'(\zeta)x^{n-1} = 0, \\
\Rightarrow \quad & \phi''(\zeta) + \frac{\nu_f}{D_B}F(\zeta)\phi'(\zeta) + \frac{Nt}{Nb}\theta''(\zeta) + K_r\frac{\nu_f}{D_B}\phi(\zeta) = 0, \\
\Rightarrow \quad & \phi''(\zeta) + ScF(\zeta)\phi'(\zeta) + \frac{Nt}{Nb}\theta''(\zeta) + K_r\frac{\nu_f}{D_B}\phi(\zeta) = 0, \\
\Rightarrow \quad & \phi''(\zeta) + ScF(\zeta)\phi'(\zeta) + \frac{Nt}{Nb}\theta''(\zeta) + K_r\ Sc\ \phi(\zeta) = 0, \\
\Rightarrow \quad & \phi'' + ScF\phi' + \frac{Nt}{Nb}\theta'' + K_r\ Sc\ \phi = 0.
\end{aligned} \tag{4.18}
$$

The final dimensionless form of the governing model is

$$
F'''[1 + n(F'')^2We^2][1 + (F'')^2We^2]^{\frac{n-3}{2}} + FF'' - \frac{2n}{(n+1)}(F')^2 - MF' = 0. \tag{4.19}
$$

$$
\frac{1}{Pr}\left(1 - \frac{4}{3}R\right)\theta'' + F\theta' - \frac{4n}{(n+1)}F'\theta + Nb\ \theta'\phi' + Nt\ (\theta')^2 + Ec\ (F'')^2 = 0. \tag{4.20}
$$

$$
\phi'' + Sc\ F\ \phi' + \frac{Nt}{Nb}\ \theta'' + K_r\ Sc\ \phi = 0. \tag{4.21}
$$

The corresponding dimensionless boundary conditions of (4.5) are:

$$
\left.
\begin{aligned}
\zeta = 0: \quad & F' = 1 + \lambda F''(0), F = 0, \theta = 1 + \delta\theta'(0), \phi = 1. \\
\zeta \to \infty: \quad & F' \to 0,\ F \to 0,\ \theta \to 0,\ \phi \to 0.
\end{aligned}
\right\} \tag{4.22}
$$

Different parameters used in equations (4.19)-(4.21) are:

$$
\begin{aligned}
&Pr=\frac{\nu}{\alpha}, Sc=\frac{\nu_f}{D_B}, Nb=\frac{D_B(C_w-C_\infty)(\rho C)_f}{(\rho C)_p\alpha},\\
&Nt=\frac{D_T(T_w-T_\infty)(\rho c)_f}{(\rho c)_p T_\infty\alpha}, M=\frac{2\sigma B^2}{a\rho_f(n+1)},\\
&\lambda=N_1\sqrt{\frac{a\nu_f(n+1)}{2}}, \delta=D_1\sqrt{\frac{a(n+1)}{a\nu_f}}, Ec=\frac{(u_w)^2}{c_p(T_w-T_\infty)},\\
&We=\sqrt{\Gamma^2\frac{a^3(n+1)}{2\nu_f}x^{3n-1}}, R=\frac{4\sigma^* T_\infty^3}{k^*\alpha}, Kr=\frac{2Kr'}{a(n+1)x^{n-1}}.
\end{aligned}
\tag{4.23}
$$

The skin friction is characterized as:

$$
Cf_x=\frac{\tau_w|_{\zeta=0}}{\rho u_w^2(x)}. \tag{4.24}
$$

To achieve the dimensionless form of Cf_x, we proceed as follows:

- $$\tau_w=\zeta_0\frac{\partial u}{\partial y}\left[1+\Gamma^2\left(\frac{\partial u}{\partial y}\right)^2\right]^{\frac{n-1}{2}},$$

$$
\begin{aligned}
\Rightarrow\quad &=\zeta_0 a x^{\frac{3n-1}{2}}F''(\zeta)\sqrt{\frac{a(n+1)}{2\nu_f}}\left[1+\Gamma^2\left(ax^{\frac{3n-1}{2}}(F''(\zeta))^2\sqrt{\frac{a(n+1)}{2\nu_f}}\right)^2\right]^{\frac{n-1}{2}},\\
\Rightarrow\quad &=\zeta_0 a x^{\frac{3n-1}{2}}F''(\zeta)\sqrt{\frac{a(n+1)}{2\nu_f}}\left[1+We^2(F''(\zeta))^2\right]^{\frac{n-1}{2}},\\
\Rightarrow\quad &\tau_w|_{\zeta=0}=\zeta_0 a x^{\frac{3n-1}{2}}F''(0)\sqrt{\frac{a(n+1)}{2\nu_f}}[1+We^2(F''(0))^2]^{\frac{n-1}{2}}.
\end{aligned}
\tag{4.25}
$$

Using (4.25) in (4.24), we get the following form:

$$
\begin{aligned}
&Cf_x=\frac{\zeta_0\nu}{\zeta_0 u_w^2(x)}ax^{\frac{3n-1}{2}}F''(0)\sqrt{\frac{a(n+1)}{2\nu_f}}[1+We^2(F''(0))^2]^{\frac{n-1}{2}},\\
\Rightarrow\ &Cf_x=\frac{\mu_f}{\rho(u_w)^2}ax^{\frac{3n-1}{2}}F''(0)\sqrt{\frac{a}{\nu_f}}\sqrt{\frac{(n+1)}{2}}[1+We^2(F''(0))^2]^{\frac{n-1}{2}},\\
\Rightarrow\ &Cf_x=\frac{\mu_f}{\rho(u_w)^2}ax^{\frac{3n-1}{2}}F''(0)\sqrt{\frac{a(n+1)}{2\nu_f}}[1+We^2(F''(0))^2]^{\frac{n-1}{2}},
\end{aligned}
$$

$$\Rightarrow Cf_x = \sqrt{\frac{(n+1)}{2}}F''(0)\sqrt{\frac{\nu_f}{ax^{n+1}}}[1+We^2(F''(0))^2]^{\frac{n-1}{2}},$$

$$\Rightarrow Cf_x = \frac{1}{\sqrt{Re}}\sqrt{\frac{(n+1)}{2}}F''(0)[1+We^2(F''(0))^2]^{\frac{n-1}{2}},$$

$$\Rightarrow (Re)^{\frac{1}{2}}Cf_x = \sqrt{\frac{(n+1)}{2}}F''(0)[1+We^2(F''(0))^2]^{\frac{n-1}{2}}. \tag{4.26}$$

The Nusselt number is defined by:

$$Nu_x = \frac{xq_w}{k(T_w - T_\infty)}. \tag{4.27}$$

To achieve the dimensionless form of Nu_x, we proceed as follows:

- $$\frac{\partial T}{\partial y} = b\theta'(\zeta)\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{5n-1}{2}}.$$

$$\Rightarrow \left(\frac{\partial T}{\partial y}\right)_{y=0} = b\theta'(0)\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{5n-1}{2}}.$$

- $$q_w = -k\left[\frac{\partial T}{\partial y}\right]_{y=0},$$

$$\Rightarrow q_w = -kb\theta'(0)\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{5n-1}{2}}. \tag{4.28}$$

Using (4.28) in (4.27), we get the following form:

$$Nu_x = -\frac{xkb}{k(T_w - T_\infty)}\theta'(0)\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{5n-1}{2}},$$

$$\Rightarrow Nu_x = -\frac{xb}{bx^{2n}}\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{5n-1}{2}}\theta'(0),$$

$$\Rightarrow Nu_x = -\frac{1}{x^n}\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{5n-1}{2}}\theta'(0),$$

$$\Rightarrow Nu_x = -\sqrt{\frac{ax^{n+1}}{\nu_f}}\theta'(0)\sqrt{\frac{(n+1)}{2}},$$

$$\Rightarrow Nu_x = -\sqrt{Re}\theta'(0)\sqrt{\frac{(n+1)}{2}},$$

$$\Rightarrow (Re)^{-\frac{1}{2}}Nu_x = -\sqrt{\frac{(n+1)}{2}}\theta'(0). \tag{4.29}$$

The Sherwood number is characterized by:

$$Sh_x = \frac{xq_m}{D_B(C_w - C_\infty)}. \tag{4.30}$$

To achieve the dimensionless form of Sh_x, we proceed as follows:

- $$\frac{\partial C}{\partial y} = (C_w - C_\infty)\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{n-1}{2}}\phi'(\zeta).$$
  $$\Rightarrow \left(\frac{\partial C}{\partial y}\right) = (C_w - C_\infty)\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{n-1}{2}}\phi'(0).$$
- $$q_m = -D_B\left[\frac{\partial C}{\partial y}\right]_{y=0},$$
  $$\Rightarrow q_m = -D_B(C_w - C_\infty)\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{n-1}{2}}\phi'(0). \tag{4.31}$$

Using (4.31) in (4.30), we get

$$Sh_x = -\frac{xD_B(C_w - C_\infty)}{D_B(C_w - C_\infty)}\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{n-1}{2}}\phi'(0),$$
$$\Rightarrow Sh_x = -\sqrt{\frac{ax^{n+1}}{\nu_f}}\sqrt{\frac{(n+1)}{2}}\phi'(0),$$
$$\Rightarrow Sh_x = -\sqrt{Re}\sqrt{\frac{(n+1)}{2}}\phi'(0),$$
$$\Rightarrow (Re)^{-\frac{1}{2}}Sh_x = -\sqrt{\frac{(n+1)}{2}}\phi'(0). \tag{4.32}$$

Here k is the thermal conductivity, heat and mass flux q_w and q_m at the sheet respectively. Moreover, Re_x represents the Reynolds number defined as $Re_x = \frac{xu_w}{\nu}$.

4.3 Method of Solution

The shooting technique has been used to determine the ordinary differential equation system (4.19)-(4.21). Equation (4.19) is solved numerically and then its solution is used in equations (4.20)-(4.21). To solve the equations (4.19) independently by using the shooting method, the following notations have been used:

$$F = z_1, \quad F' = z_1' = z_2, \quad F'' = z_2' = z_3, \quad F''' = z_3'.$$
$$A_1 = (1 + nWe^2 z_3^2), \qquad A_2 = (1 + We^2 z_3^2)^{\frac{n-3}{2}}.$$
$$A_3 = \frac{2n}{(n+1)} z_2^2 + Mz_2 - z_1 z_3.$$
$$A_4 = \frac{4n}{(n+1)} z_2 z_5 + Mz_5 - z_3 z_4 - z_1 z_6.$$

By using the above notations in equation (4.19), the following system of ODEs is attained:

$$z_1' = z_2, \qquad z_1(0) = 0.$$
$$z_2' = z_3, \qquad z_2(0) = 1 + \lambda s.$$
$$z_3' = \frac{1}{A_1 A_3}\left[\frac{2n}{(n+1)} z_2^2 + Mz_2 - z_1 z_3\right], \qquad z_3(0) = s.$$

The above IVP will be numerically determined by RK technique of order four. In the above initial value problem, the missing condition s satisfy the following relation:

$$F''(0) = \frac{1}{\lambda}(F'(0) - 1). \tag{4.33}$$

To solve the above algebric equation (4.33), We use the Newton's method which has the following iterative scheme:

$$S^{k+1} = S^k - \frac{G(\zeta)}{G'(\zeta)}, \quad \text{where} \quad G(\zeta) = y_2(\eta_\infty, \zeta).$$

To incorporate Newton's method, we further apply the following notations:

$$\frac{\partial z_1}{\partial s} = z_4, \qquad \frac{\partial z_2}{\partial s} = z_5, \qquad \frac{\partial z_3}{\partial s} = z_6.$$
$$z_4' = z_5, \qquad z_4(0) = 0.$$
$$z_5' = z_6, \qquad z_5(0) = \lambda.$$

$$z_6' = \frac{1}{A_1 A_2}\left[A_1 A_2^{\frac{n-3}{2}} A_4 - A_1 A_2^{\frac{n-5}{2}} A_3 (n-3) We^2 z_3 z_6 + 2nWe^2 z_3 z_6 A_2^{\frac{n-3}{2}} A_3\right], \qquad z_6(0) = 1.$$

Now, the coupled equations (4.20)-(4.21) will be treated similarly by considering F as a known function. For this, we use the following notations:

$$\theta = z_1, \qquad \theta' = z_1' = z_2, \qquad \theta'' = z_2'.$$
$$\phi = z_3, \qquad \phi' = z_3' = z_4, \qquad \phi'' = z_4'.$$

By using the above notations in equations (4.20)-(4.21), the following scheme of ODEs is obtained:

$$
\begin{aligned}
z_1' &= z_2, & z_1(0) &= 1 + \delta u.\\
z_2' &= \frac{3Pr}{3-4R}\left[\frac{4n}{(n+1)}F' z_1 - F z_2 - Ec(F')^3 - Nb z_2 z_4 - Nt z_2^2 -\right], & z_2(0) &= u.\\
z_3' &= z_4, & z_3(0) &= 0.\\
z_4' &= -\frac{3PrNt}{Nb(3-4R)}\left[\frac{4n}{(n+1)}F' z_1 - F z_2 - Nb z_2 z_4 - Nt z_2^2 - Ec(F'')^2\right] - \\
&\quad ScF z_4 - ScKr z_3 & z_4(0) &= v.
\end{aligned}
$$

The above IVP will be numerically solved by RK technique of order four. In the above initial value problem, the missing conditions u and v are to be chosen such that:

$$\left(\frac{z_1(\zeta, u^{(k)}, v^{(k)}) - 1}{\delta}\right) = 0, \quad (z_2(\zeta, u^{(k)}, v^{(k)})) = 0. \tag{4.34}$$

To solve the above algebric equation (4.34), we apply the Newton's iterative method which has the following scheme:

$$\begin{pmatrix} u^{(k+1)} \\ v^{(k+1)} \end{pmatrix} = \begin{pmatrix} u^{(k)} \\ v^{(k)} \end{pmatrix} - \begin{pmatrix} \frac{\partial z_1}{\partial u} & \frac{\partial z_2}{\partial v} \\ \frac{\partial z_1}{\partial u} & \frac{\partial z_2}{\partial v} \end{pmatrix}^{-1} \begin{pmatrix} \frac{z_1(\zeta, u^{(k)}, v^{(k)}) - 1}{\delta} \\ z_2(\zeta, u^{(k)}, v^{(k)}) \end{pmatrix}$$

To incorporate Newton's method, we further apply the following notions:

$$\frac{\partial y_1}{\partial u} = z_5, \quad \frac{\partial z_2}{\partial u} = z_6, \quad \frac{\partial z_3}{\partial u} = z_7, \quad \frac{\partial z_4}{\partial u} = z_8.$$
$$\frac{\partial z_1}{\partial v} = z_9, \quad \frac{\partial z_2}{\partial v} = z_{10}, \quad \frac{\partial z_3}{\partial v} = z_{11}, \quad \frac{\partial z_4}{\partial v} = z_{12}.$$

As the result of these new notations, the Newton's iterative scheme gets the shape:

$$\Rightarrow \begin{pmatrix} u^{(k+1)} \\ v^{(k+1)} \end{pmatrix} = \begin{pmatrix} u^{(k)} \\ v^{(k)} \end{pmatrix} - \begin{pmatrix} z_5 & z_6 \\ z_9 & z_{10} \end{pmatrix}^{-1} \begin{pmatrix} \frac{z_1(\zeta,u^{(k)},v^{(k)})-1}{\delta} \\ z_2(\zeta, u^{(k)}, v^{(k)}) \end{pmatrix}$$

Here k is the number of iterations $(k = 0, 1, 2, 3, ...)$. Now differentiating the above ODEs with respect to u and v, we have another scheme of eight ODEs.

$$z_5' = z_6, \qquad z_5(0) = \delta.$$
$$z_6' = \frac{3Pr}{(3-4R)}\left[\frac{4n}{(n+1)}F'z_5 - Fz_6 - Nbz_4z_6 - Nbz_2z_8 - 2Ntz_2z_6\right], \qquad z_6(0) = 1.$$
$$z_7' = z_8, \qquad z_7(0) = 0.$$
$$z_8' = -\frac{3PrNt}{Nb(3-4R)}\left[\frac{4n}{(n+1)}F'z_5 - Fz_6 - Nbz_4z_6 - Nbz_2z_8 - 2Ntz_2z_6\right] - ScFz_8 - ScKrz_7, \qquad z_8(0) = 0.$$
$$z_9' = z_{10}, \qquad z_9(0) = 0.$$
$$z_{10}' = \frac{3Pr}{(3-4R)}\left[\frac{4n}{n+1}F'z_9 - Fz_{10} - Nbz_4z_{10} - Nbz_2z_{12} - 2Ntz_2z_{10}\right], \qquad z_{10}(0) = 0.$$
$$z_{11}' = z_{12}, \qquad z_{11}(0) = 0.$$
$$z_{12}' = -\frac{3PrNt}{Nb(3-4R)}\left[\frac{4n}{(n+1)}F'z_9 - Fz_{10} - Nbz_4z_{10} - Nbz_2z_{12} - 2Ntz_2z_{10}\right] - ScFz_{12} - ScKrz_{11}, \qquad z_{12}(0) = 1.$$

The RK method has been used to solve the IVP consisting of the above twelve first order ODEs for some suitable choices of u and v. The iterative process is repeated until the following stopping criteria is met:

$$max\{|u^{(k+1)} - u^{(k)}|, |v^{(k+1)} - v^{(k)}|\} < \epsilon,$$

for an arbitrary small positive value of ϵ. Throughout this chapter ϵ has the value $(10)^{-6}$.

4.4 Results and Discussion

The principle object is about to examine the impact of different parameters against the velocity, temperature and concentration distribution. The impact of different factors like non-linear stretching parameter n, magnetic parameter M, velocity slip parameter λ, thermal slip parameter δ, Schmidt number Sc, Weissenberg number We, thermal radiation R and chemical reaction Kr is observed graphically. Numerical outcomes of the Sherwood number, Nusselt number and skin friction for the distinct values of some fixed parameters are shown in Tables 4.1-4.2.

Figure 4.2 displays the impact of magnetic parameter M and non-linear stretched parameter n on velocity distribution. By enhancing the values of M and n, the velocity distribution shows the decreasing behavior due to the presence of Lorentz force. Figure 4.3 describes the impact of magnetic parameter M and non-linear stretching parameter n on temperature distribution. The temperature distribution expands by enhancing the values of M and n. Figure 4.4 describes the impact of magnetic parameter M on concentration distribution. By enhancing the values of M, the nanoparticle concentration distribution is escalated due to the presence Lorentz force.

Figure 4.5 is delineated to show the impact of velocity slip parameter λ on the dimensionless velocity distribution. This graph reflects that by enhancing the values of λ, the velocity distribution is decreased because the velocity of the stretching sheet is not the same as the velocity of the flow. Figures 4.6-4.7 represents the influence of velocity slip parameter λ on the dimensionless velocity distribution. It can be noted that, the temperature and concentration distributions are increased by enhancing the values of λ due to the presence of thermal slip and magnetic parameter. Figure 4.8 displays the impact of thermal slip parameter δ on temperature distribution. It can be noted that the temperature distribution reduces by enhancing the values of δ. Figure 4.9 illustrates the impact of thermal slip parameter δ on the concentration distribution. This graph indicates that an increment in the values of δ causes an increment in the nanoparticle concentration distribution. The thermal boundary layer thickness experiences a decrement by enhancing the values of δ.

Figure 4.10 is delineated to show the impact of Eckert number Ec on the dimensionless temperature distribution. It is clearly shown that the temperature distribution is increased by enhancing the values of Ec due to the decrement in heat transfer rate. Figure 4.11 indicates the influence of Brownian motion parameter Nb on the dimensionless concentration distribution. The behavior of concentration distribution is decreased due to the accelerating values of Nb. The impacts of Weissenberg number on velocity and temperature profiles are not so prominent. However, a marginally decreasing behavior in velocity profile and slightly increasing behavior in temperature profile are observed.

TABLE 4.1: Numerical outcomes of $(Re_x)^{\frac{1}{2}}Cf_x$ for different parameters.

n	M	λ	$-F''(0)$
2	0.1	0.2	0.853593
2	0.3	0.2	0.907929
2	0.4	0.2	0.933158
2	0.5	0.2	0.957252
2	1.0	0.3	0.950450
2	1.0	0.4	0.859735
2	1.0	0.5	0.869677

TABLE 4.2: Numerical outcomes of $(Re_x)^{\frac{1}{2}}Cf_x$, $(Re_x)^{\frac{-1}{2}}Nu_x$ and $(Re_x)^{\frac{1}{2}}Sh_x$ for various parameters.

M	λ	δ	$-F''(0)$	$-\theta'(0)$	$-\phi'(0)$
0.1	0.2	0.2	0.853593	1.952540	-1.411272
0.3	0.2	0.2	0.909729	1.891209	-1.010253
0.4	0.2	0.2	0.933158	1.863276	-1.075327
0.5	0.2	0.2	0.957252	1.837292	-1.101932
0.7	0.5	0.1	0.950450	1.792117	-0.400813
0.7	0.5	0.3	0.859735	1.536613	-0.619206
1.0	0.3	0.1	0.869677	1.490573	-0.612050

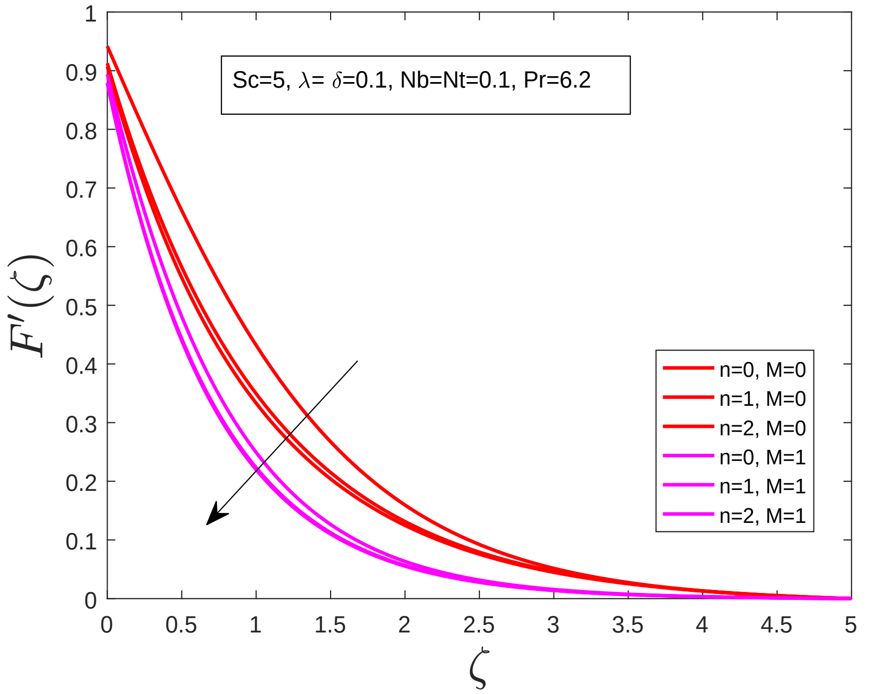

FIGURE 4.2: Influence of magnetic and non-linear stretching parameter on dimensionless velocity distribution.

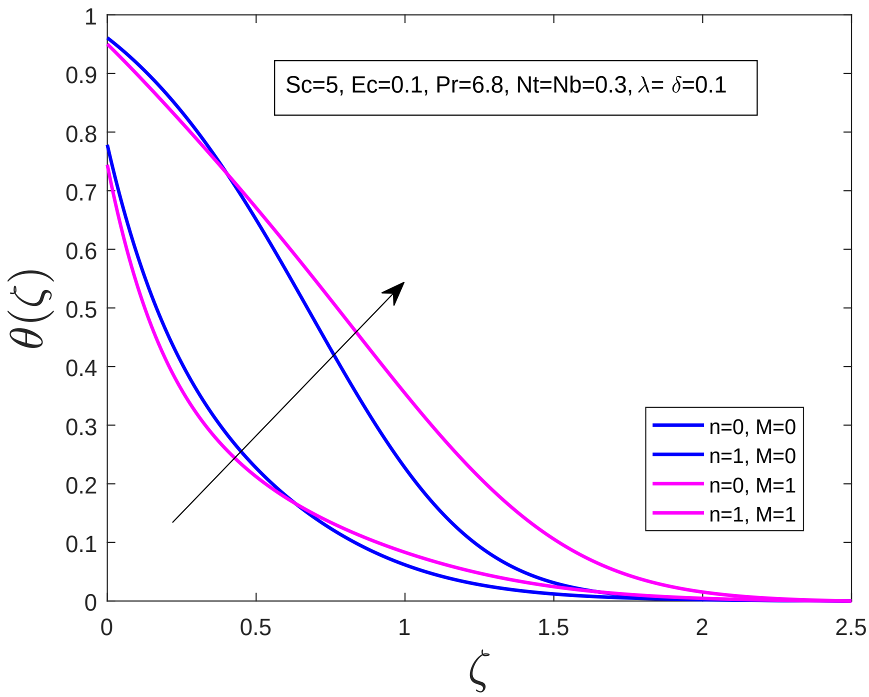

FIGURE 4.3: Impact of magnetic and non-linear stretching parameter on temperature distribution.

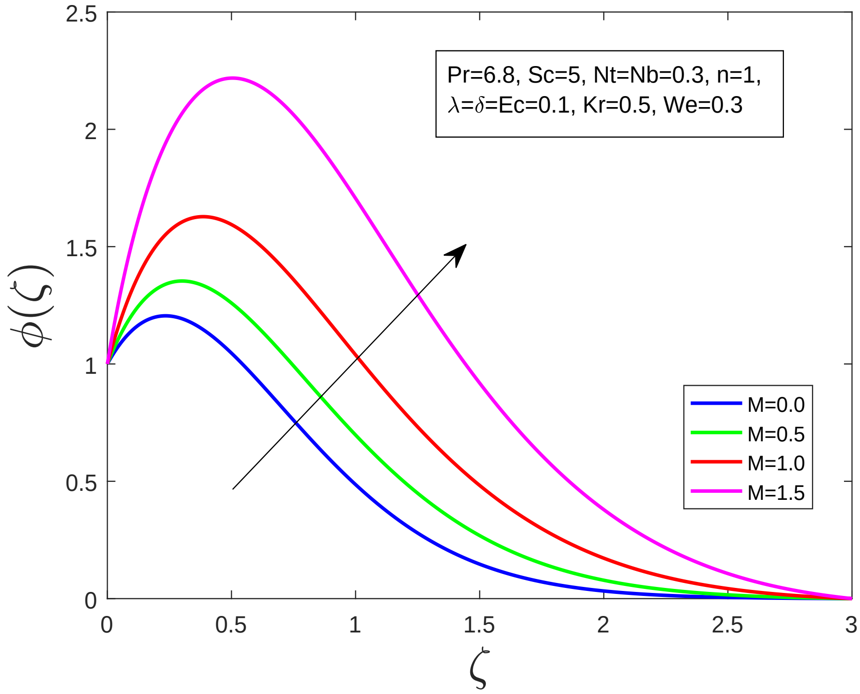

FIGURE 4.4: Influence of M on concentration distribution.

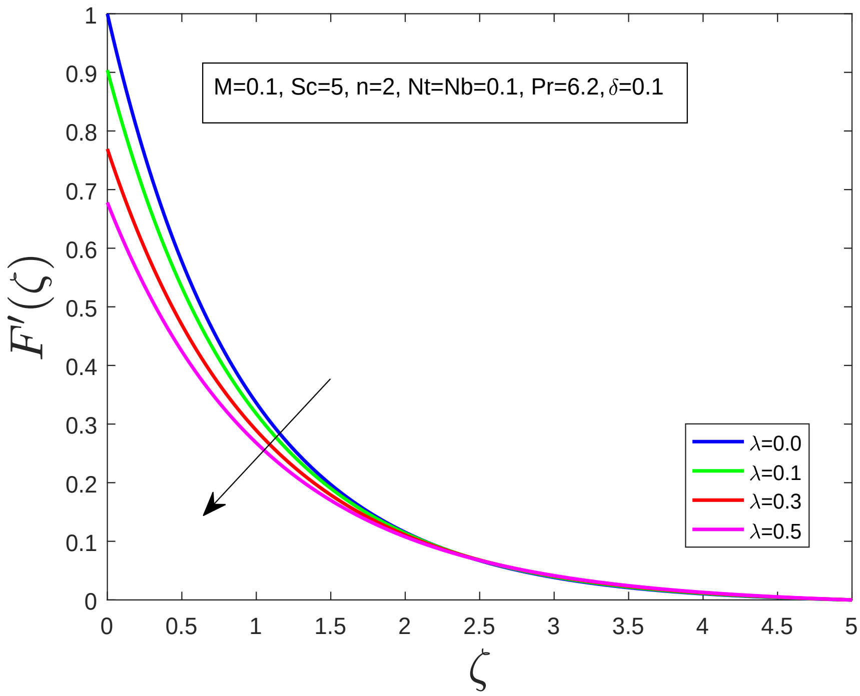

FIGURE 4.5: Influence of λ on velocity distribution.

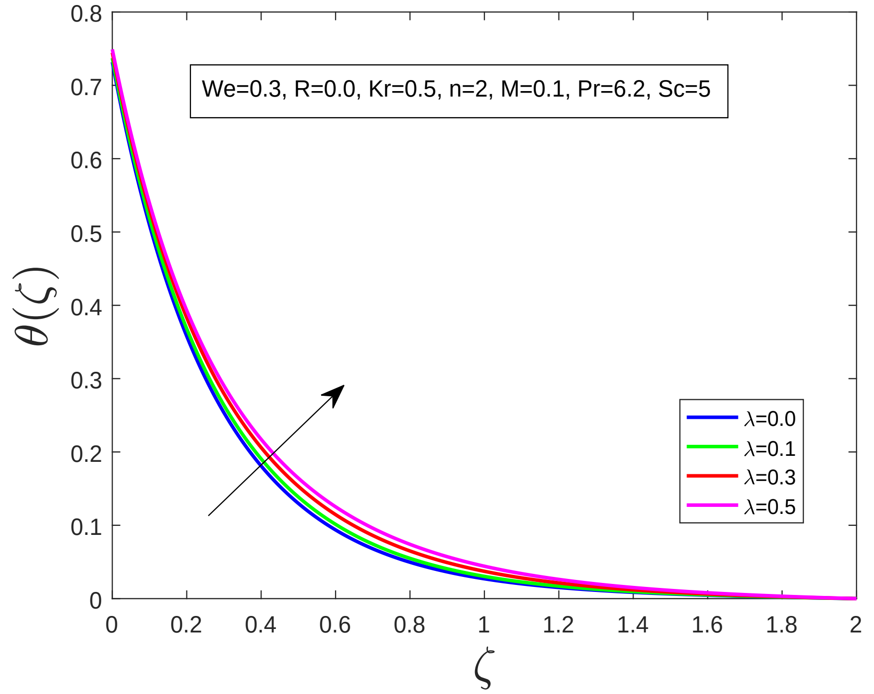

FIGURE 4.6: Impact of λ on temperature profile.

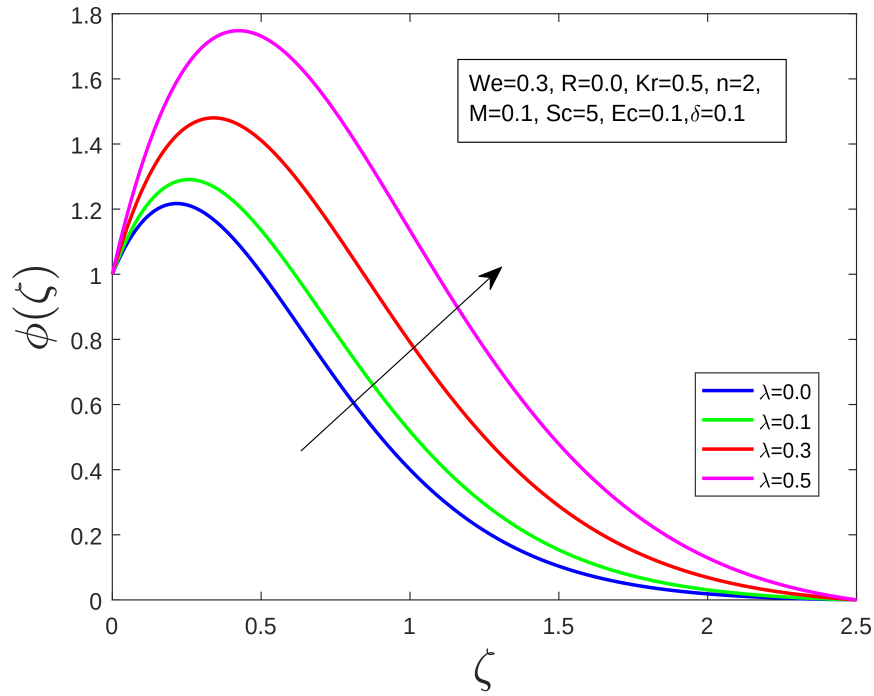

FIGURE 4.7: Influence of λ on concentration distribution.

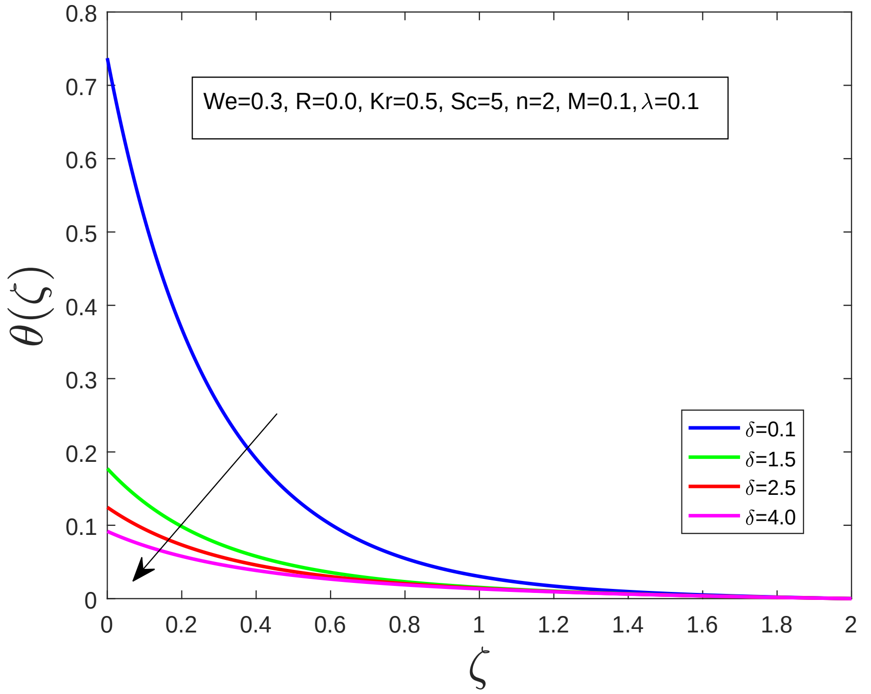

FIGURE 4.8: Impact of δ on the temperature profile.

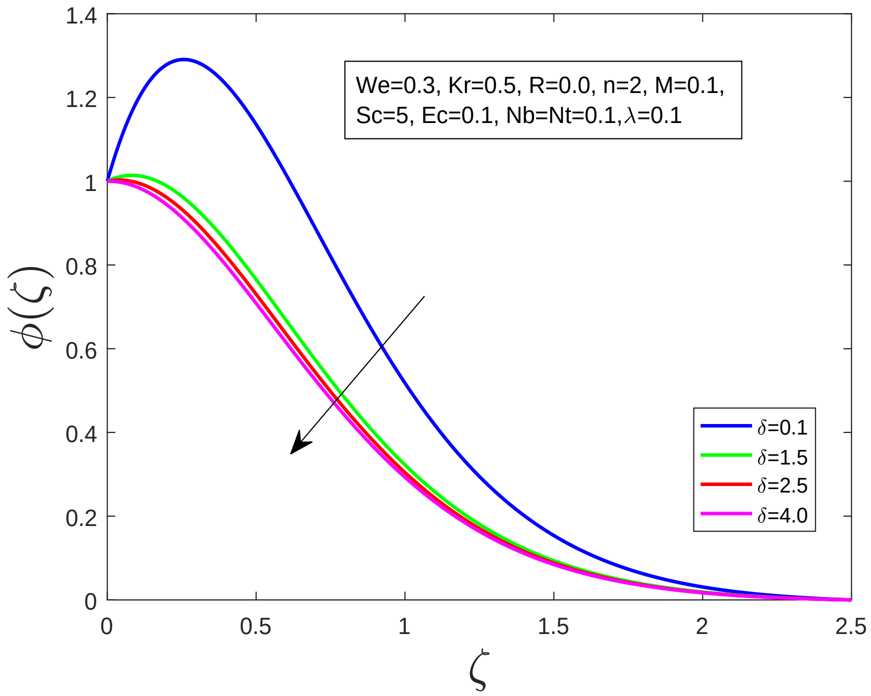

FIGURE 4.9: Influence of δ on concentration distribution.

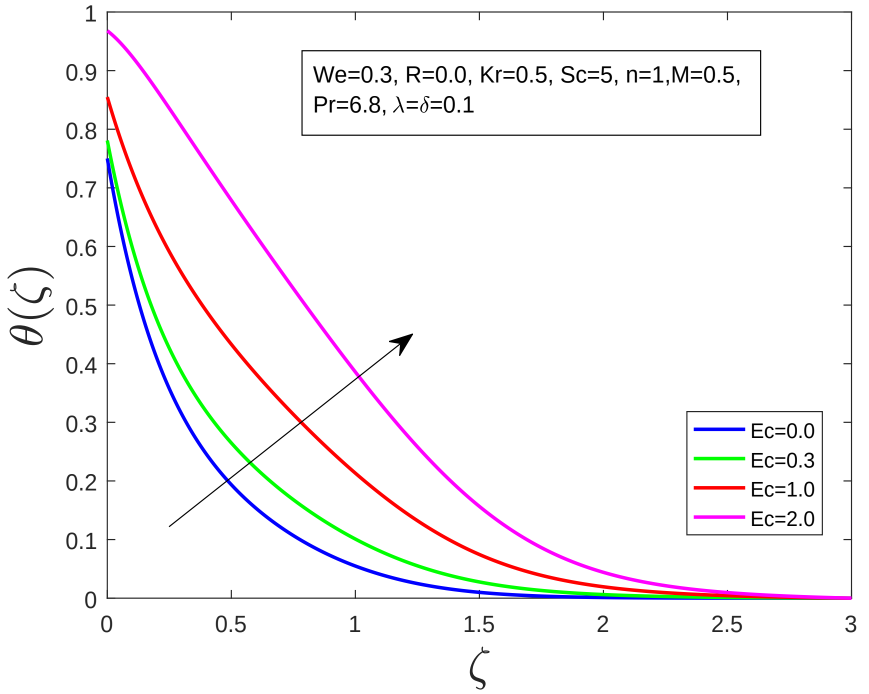

FIGURE 4.10: Impact of Ec on temperature distribution.

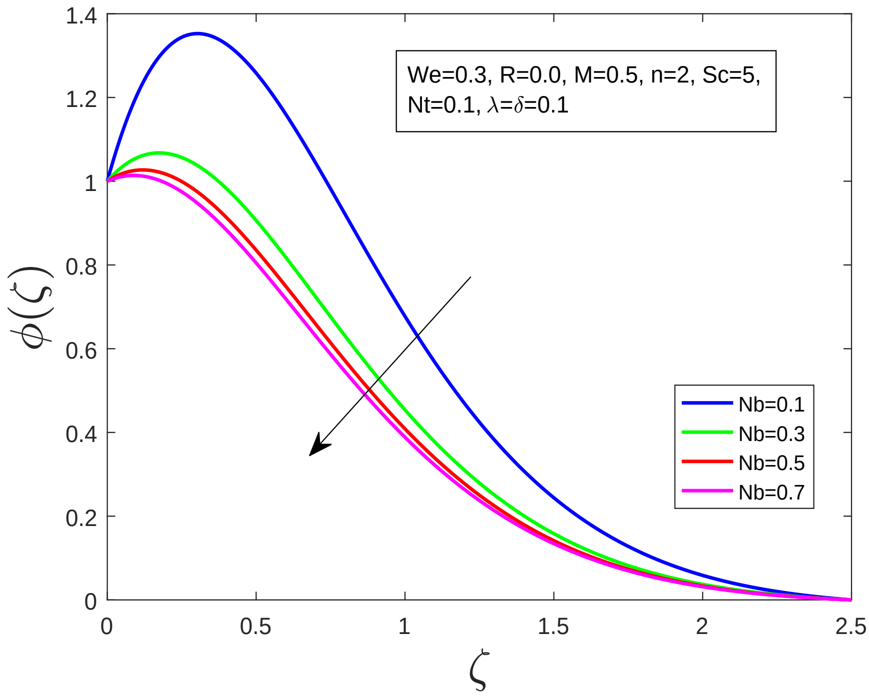

FIGURE 4.11: Impact of Nb on concentration distribution.

Chapter 5

Conclusion

In this dissertation, we first analyzed the MHD boundary layer and 2D incompressible viscous fluid flow on a non-linearly stretching sheet. Secondly, the flow is examined by considering the Carreau effect in momentum equation, thermal radiation effect in temperature equation and chemical reaction effect in the concentration equation. The set of non-linear energy, momentum and concentration equations are transmuted into ODEs by applying appropriate transformations. Numerical solutions are achieved by using shooting technique. The impacts of distinct parameters like Prandtl number Pr, Schmidt number Sc, Eckert number Ec, radiation parameter R, chemical reaction parameter Kr and Weissenberg number We on the velocity, temperature and the concentration profile are elaborated in the graphical and tabular form.

Some of the main conclusions are listed below:

- By enhancing the values of the magnetic parameter, the velocity behavior is declined while the temperature and concentration behaviors are increased.
- The velocity distribution reduces for enhancing the values of both magnetic and velocity slip parameter. On the other hand, the temperature and concentration distribution increase by a decrement in the magnetic parameter.
- An increment is noticed in temperature and concentration distributions due to the accelerating values of velocity slip parameter.

- A decrement is noticed in mass transfer and temperature behavior due to the boosting values of thermal slip parameter.
- The temperature profile decelerates due to the rising values of the thermal slip parameter.
- An increment is noticed in temperature distribution by enhancing the values of Eckert number.
- It is observed that a decrement is noticed in concentration distribution due to the accelerating values of Brownian motion parameter.
- An increment in the temperature profile is noted for the enhancing values of Prandtl number.
- The impacts of Weissenberg number, thermal radiation and chemical reaction parameters on the velocity and temperature profiles are not so prominent. However, a marginally decreasing behavior in velocity profile and slightly increasing behavior in temperature profile are observed.

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