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Heat Transfer and Boundary Layer Flow of Casson Fluid over a Non-Linear Stretching Sheet

by

Muhammad Asif Khan

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degree of Master of Philosophy

in the

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I dedicate my dissertation work to my family and friends. A special feeling of gratitude to my loving parents who have supported me in my studies.

CERTIFICATE OF APPROVAL

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Abstract

An effort is made to obtain the numerical solution of boundary layer flow and heat transfer of Casson fluid over a non-linear stretching sheet. The governing partial differential equations are reduced to system of non-linear ordinary differential equations using suitable transformations. The resulting non-linear coupled system subject to the boundary conditions is solved using the Shooting method. Graphical and numerical demonstrations of the convergence of the solutions are provided. Impact of various physical parameters on the dimensionless velocity and temperature profiles are presented and analyzed in the form of graphs. Numerical values of the skin friction coefficient and Nusselt number are also and discussed.

Contents

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Abbreviations

IVPs	Initial value problems
MHD	Magnetohydrodynamics
ODEs	Ordinary differential equations
PDEs	Partial differential equations
RK	Runge-Kutta

Symbols

μ	Viscosity
ν	Kinematic viscosity
τ	Stress tensor
κ	Thermal conductivity
α	Thermal diffusivity
u	x-component of fluid velocity
v	y-component of fluid velocity
b	The constant parameter
$T_w(x,t)$	Temperature of the sheet
T_∞	Temperature of the fluid faraway from sheet
ρ	Density
ν_f	Kinematic viscosity of the base fluid
T	Temperature
σ^*	Stefan Boltzmann constant
k^*	Absorption coefficient
v_w	Suction velocity
T_0, C_0	Positive refrence temperature and nanoparticles volume fraction
ψ	Stream function
C_f	Skin friction coefficient
Nu	Local Nusselt number
Re_x	Local Reynolds number
ϵ	Porosity of medium
Pr	Prandtl number

R	Thermal radiation parameter
We	Weissenberg number
n	Power-law index
m	Stretching parameter
Ec	Eckert number
β	Casson parameter

Chapter 1

Literature Review and Thesis Layout

1.1 Literature Review

During the past few decades, the boundary layer problems related to a stretching surface have attracted an extensive attention of researchers, because a number of applications related to this area are found in engineering and industrial manufacturing process. Actually the boundary layer has a meaningful concept in physics and fluid mechanics, which is introduced as the layer of the fluid in the region of a bounded area where the effect of viscosity is powerful. When the lubrication problems and polymer evolution is considered, the variation in the viscosity by two or three orders of magnitude is possible for some fluids and this cannot be avoided. So, the essential empirically achieved condition of the Newton's law of viscosity is that, the viscosity of fluid is allowed to alter with shear rate. This type of fluid is also known as generalized Newtonian fluids and described in [1]. The generalized form of Newtonion fluid is known as power-law constitutive relation. The power-law model has the restriction to the viscosity of fluids for smaller or very larger shear rates. The generalized Newtonian fluid called Carreau rheological model [2]. This model breaks the restriction of power-law model by introducing extended acknowledgement in engineering and industrial manufacturing processes. The Carreau fluid model can describe the rheology of distinct polymeric result, like 1% methylcellulose tylose in glycerol solution and 0.3% hydroxyethyl-cellulose Natrosol HHX in glycerol solution [3] and pure poly ethylene oxide [4]. Such polymers are mostly utilized in capillary electrophoresis to develop the solution in the partition of proteins [5] and DNAs [6]. As a result, a unique surface is introduced where the viscosity of the fluids and its shear rates have a uniform relationship. The Carreau model results a power-law surface. Moreover the power-law model assumes that viscosity and shear rate remain finite by approaching towards zero. As a result, the constitutive equation of Carreau fluid model is useful for free surface flows.

The Carreau model became popular among engineers and researchers due to its numerous applications. The flow of Carreau fluid over the spheres was studied by Chhabra and Uhlherr [7] and Bush and Phan-Thein [8]. Hsu and Yeh [9] observed the effect of drag on two rigid coaxial spheres revolving about a cylindrical axis containing Carreau fluid. Uddin et al. [10] studied squeeze flow of a Carreau fluid induced due to sphere impact. Shadid and Eckert [11] discussed the Carreau fluid flow around a cylinder. Tshehla [12] studied the Carreau fluid flow on an inclined plane. Olajuwon [13] examined the magnetohydrodynamics flow of Carreau fluid with transfer of mass and heat through a porous medium. Griffiths [14] studied the generalized Newtonian rotating disk flow. He also examined that while considering Carreau model rather than power-law model, the base flow similarity solution is still applicable.

The boundary layer flow problems due to extending surface became famous among many researchers. Such type of flow appears in extrusion process, such as crystal growing, glass fiber etc. In such cases, the fluid depends on the cooling rate and the stretching rate. Sakiadis [15] described the boundary layer flow problems on the solid surface moving with uniform velocity. Afterwards a number of researchers discussed various behaviors of such flows. Crane [16] and P. Gupta and A. Gupta [17] have discussed the transfer of mass and heat above an extending sheet having uniform temperature at the surface. Akbar [18] studied the numerical simulation of 2-D tangent hyperbolic flow of fluid through an extending sheet placed in a magnetic field. Nadeem et al. [19] explained the MHD flow of an incompressible Casson fluid on an exponentially stretching surface. Erickson et al. [20] explained the boundary layer incompressible fluid flow over an inextensible flat surface with uniform velocity. Akbar et al. [21] explained the dual nature of simulation of MHD fluid flow at stagnation point of Prandtl fluid model through a stretching surface. Vajravelu [22] discussed 2-D viscous flow on a non-linearly extending surface. Cortell [23] extended the problem by adding the impact of thermal radiation and viscous dissipation. The flow of MHD non-Newtonian Casson fluid over a pours stretching surface is discussed by Nadeem et al. [24].

1.2 Thesis Contribution

The major objective of this research work is to execute the impacts of heat transfer and boundary layer flow of Casson fluid over a non-linear stretching sheet. In this work, we convert a system of PDEs into non-linear ODEs using similarity transformations. A well shooting technique with fourth order RK method is used to obtained the numerical results. The numerically obtained results are computed by using MATLAB. The impact of significant parameters on velocity distribution $f'(\xi)$ and temperature distribution $\theta(\xi)$, skin friction Cf_x and Nusselt number Nu_x have been discussed in graphs and tables.

1.3 Layout of Thesis

A brief overview of the contents of the thesis is provided below.

Chapter 2 includes some basic definitions and terminologies, which are useful to understand the concepts discussed later on.

Chapter 3 contains the complete review of [25] boundary layer flow and heat transfer of Carreau fluid over a non-linear stretching sheet. The numerical results of the governing flow equations are derived by the shooting method.

Chapter 4 is an extension of the given model [25] by including the impacts of Casson fluid, thermal radiation and viscous dissipation variable. Numerical outcomes for skin friction coefficient and Nusselt number have also been discussed. Impacts of different physical parameters are analyzed through graphs and tables.

Chapter 5 provides the concluding remarks of the thesis.

References used in the thesis are mentioned in **Biblography**.

Chapter 2

Basic Terminologies and Governing Equations

In this chapter, we will discuss some basic definitions, terminologies, basic laws, and dimensionless numbers, which will be helpful in conducting the work for the next chapters. This section contains few essentials definitions and laws of the fluids, which will be used in the upcoming discussions.

2.1 Basic Terminologies

2.1.1 Fluid [26]

"A fluid is a substance that deforms continuously under the application of a shear (tangential) stress no matter how small the shear stress may be."

2.1.2 Fluid Mechanics [27]

"Fluid mechanics is that branch of science which deals with the behavior of the fluid (liquids or gases) at rest as well as in motion."

2.1.3 Fluid Statics [27]

"The study of fluid at rest is called fluid statics."

2.1.4 Fluid Dynamics [27]

"The study of fluid if the pressure forces are also considered for the fluids in motion, that branch of science is called fluid dynamics."

2.1.5 Viscosity [27]

"Viscosity is defined as the property of a fluid which offers resistance to the movement of one layer of fluid over another adjacent layer of the fluid. Mathematically,

$$\mu = \frac{\tau}{\frac{\partial u}{\partial y}},$$

where μ is viscosity coefficient, τ is shear stress and $\frac{\partial u}{\partial y}$ represents the velocity gradient."

2.1.6 Kinematic Viscosity [27]

"It is defined as the ratio between the dynamic viscosity and density of fluid. It is denoted by symbol ν called **'nu'.** Mathematically,

$$\nu = \frac{\mu}{\rho}."$$

2.1.7 Ideal Fluid [27]

"A fluid which is incompressible and has no viscosity, is known as an ideal fluid. Ideal fluid is only an imaginary fluid as all the fluids, which exist, have some viscosity."

2.1.8 Real Fluid [27]

"A fluid which possesses viscosity, is known as a real fluid. In actual practice, all the fluids are real fluids."

2.1.9 Newtonian Fluid [27]

"A real fluid in which the shear stress is directly proportional to the rate of shear strain (or velocity gradient), is known as a Newtonian fluid."

2.1.10 Non-Newtonian Fluid [27]

"A real fluid in which the shear stress is not directly proportional to the rate of shear strain (or velocity gradient), is known as a Non-Newtonian fluid."

2.1.11 Magnetohydrodynamics [28]

"Magnetohydrodynamics is concerned with the mutual interaction of fluid flow and magnetic fields. The fluids in question must be electrically conducting and non-magnetic, which limits us to liquid metals, hot ionised gases (plasmas) and strong electrolytes."

2.2 Types of Flow

2.2.1 Laminar Flow [27]

"Laminar flow is defined as that type of flow in which the fluid particles move along well-defined paths or stream line and all the stream-lines are straight and parallel."

2.2.2 Turbulent Flow [27]

"Turbulent flow is that type of flow in which the fluid particles move in a zig-zag way. Due to the movement of fluid particles in a zig-zag way."

2.2.3 Compressible Flow [27]

"Compressible flow is that type of flow in which the density of the fluid changes from point to point or in other words the density (ρ) is not constant for the fluid, Mathematically,

$$\rho \neq k,$$

where k is constant."

2.2.4 Incompressible Flow [27]

"Incompressible flow is that type of flow in which the density is constant for the fluid. Liquids are generally incompressible while gases are compressible, Mathematically,

$$\rho = k,$$

where k is constant."

2.2.5 Internal Flow [26]

"If the flows is completely bounded by a solid surfaces are called internal or duct flows."

2.2.6 External Flow [26]

"Flows over bodies immersed in an unbounded fluid are said to be an external flow."

2.2.7 Steady Flow [27]

"If the flow characteristics such as depth of flow, velocity of flow, rate of flow at any point in open channel flow donot change with respect to time, the flow is said to be steady flow. Mathematically,

$$\frac{\partial Q}{\partial t} = 0,$$

where Q is the rate of flow."

2.2.8 Unsteady Flow [27]

"If at any point in open channel flow, the velocity of flow, depth of flow or rate of flow changes with respect to time, the flow is said to be unsteady. Mathematically,

$$\frac{\partial Q}{\partial t} \neq 0,$$

where Q is the rate of flow."

2.3 Some Basic Definitions of Heat Transfer

2.3.1 Heat Transfer [29]

"Heat transfer is a branch of engineering that deals with the transfer of thermal energy from one point to another within a medium or from one medium to another due to the occurrence of a temperature difference."

2.3.2 Conduction [29]

"The transfer of heat within a medium due to a diffusion process is called conduction."

2.3.3 Convection [29]

"Convection heat transfer is usually defined as energy transport effected by the motion of a fluid. The convection heat transfer between two dissimilar media is governed by Newton's law of cooling."

2.3.4 Force Convection [30]

"Forced convection heat transfer is induced by forcing a liquid, or gas, over a hot body or surface."

2.3.5 Natural Convection [30]

"Natural convection is generated by the density difference induced by the temperature differences within a fluid system and the small density variations present in these types of flows."

2.3.6 Thermal Radiation [29]

"Thermal radiation is defined as radiant (electromagnetic) energy emitted by a medium and is solely to the temperature of the medium."

2.4 Thermal Conductivity and Diffusivity

2.4.1 Thermal Conductivity [29]

"The Fourier heat conduction law states that the heat flow is proportional to the temperature gradient. The coefficient of proportionality is a material parameter known as the thermal conductivity which may be a function of a number of variables."

2.4.2 Thermal Diffusivity [31]

"The rate at which heat diffuses by conducting through a material depends on the thermal diffusivity and can be defined as:

$$\alpha = \frac{\kappa}{\rho C_p},$$

where α is the thermal diffusivity, κ is the thermal conductivity, ρ is the density and C_p is the specific heat at constant pressure."

2.5 Dimensionless Numbers

2.5.1 Prandtl Number [32]

"The Prandtl number is the connecting link between the velocity field and the temperature field. The Prandtl number is dimensionless Mathematically,

$$\text{Pr} = \frac{\nu}{\alpha} = \frac{\mu/\rho}{\kappa/\rho C_p} = \frac{\mu C_p}{\kappa}.\text{"}$$

2.5.2 Skin Friction Coefficient [33]

"The steady flow of an incompressible gas or liquid in a long pipe of internal D. The mean velocity is denoted by u_w. The skin friction coefficient can be defined as

$$C_f = \frac{2\tau_0}{\rho u_w^2},$$

where τ_0 denotes the wall shear stress and ρ is the density."

2.5.3 Nusselt Number [30]

"The hot surface is cooled by a cold fluid stream. The heat from the hot surface, which is maintained at a constant temperature, is diffused through a boundary layer and convected away by the cold stream. Mathematically,

$$Nu = \frac{qL}{\kappa},$$

where q stands for the convection heat transfer, L for the characteristic length and κ stands for thermal conductivity."

2.5.4 Reynolds Number [27]

"It is defined as the ratio of inertia force of a flowing fluid and the viscous force of the fluid . Mathematically,

$$Re = \frac{LV}{\nu},$$

where V denotes the free stream velocity, L is the characteristics length and ν is kinematic viscosity."

2.6 Fundamental Equations of Flow

2.6.1 Continuity Equation [29]

"The principle of conservation of mass can be stated as the time rate of change of mass is fixed volume is equal to the net rate of flow of mass across the surface. Mathematically, it can be written as

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \mathbf{u}) = 0."$$

2.6.2 Conservation of Momentum [29]

"The momentum equation states that the time rate of change of linear momentum of a given set of particles is equal to the vector sum of all the external forces acting on the particles of the set, provided Newton's Third Law of action and reaction governs the internal forces. Mathematically, it can be written as:

$$\frac{\partial}{\partial t}(\rho \mathbf{u}) + \nabla.[(\rho \mathbf{u})\mathbf{u}] = \boldsymbol{\nabla}.\mathbf{T} + \rho g.\text{''} \tag{2.1}$$

2.6.3 Law of Conservation of Energy [29]

"The law of conservation of energy states that the time rate of change of the total energy is equal to the sum of the rate of work done by the applied forces and change of heat content per unit time.

$$\frac{\partial \rho}{\partial t} + \nabla.\rho \mathbf{u} = -\nabla.\mathbf{q} + Q + \phi, \tag{2.2}$$

where ϕ is a dissipation function."

Chapter 3

Flow and Heat Transfer of Carreau Fluid over a Non-Linear Stretching Sheet

3.1 Introduction

In this section the numerical analysis of a 2-D incompressible flow of Carreau fluid, along a stretchable surface has been discussed. The set of PDEs are convert into the dimensionless ODEs by an appropriate transformations. In order to solve ODEs, we used the shooting technique implemented in MATLAB. At the end of this chapter the numerical solution of various profiles has been discussed. The obtained numerical results are given in tables and graphs.

3.2 Mathematical Modeling

Assume a 2-D incompressible flow of a Carreau fluid due to the extending sheet with $y = 0$. The flow is considered along y-axis with $y > 0$. Here, T_w denotes the uniform temperature, T_∞ the ambient temperature and $u_w = bx^m$ is the non-linear velocity. The constants b and m are positive real numbers describe to the

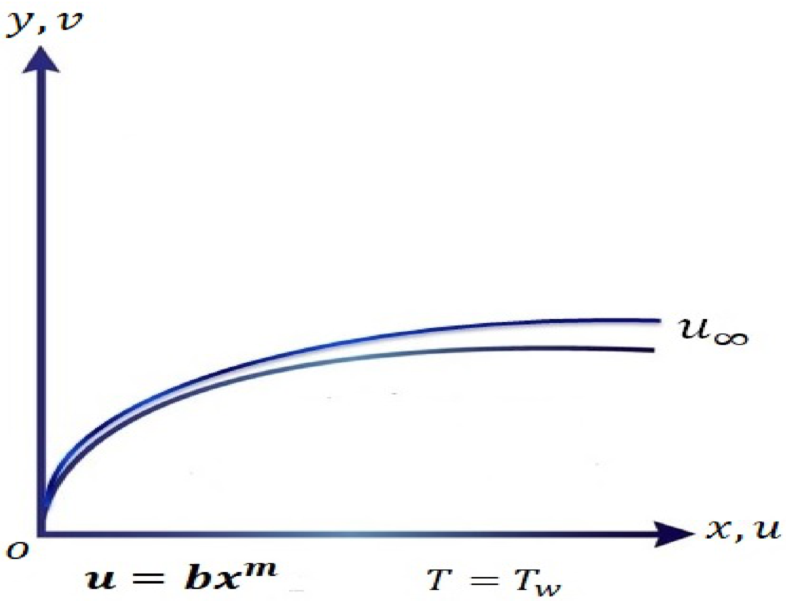

FIGURE 3.1: Systematic representation of physical model.

stretching speed.

The set of equations describing the flow are as follows.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{3.1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}\left[1 + \Gamma^2\left(\frac{\partial u}{\partial y}\right)^2\right]^{\frac{n-1}{2}} + \nu(n-1)\Gamma^2\frac{\partial^2 u}{\partial y^2}\left(\frac{\partial u}{\partial y}\right)^2\left[1 + \Gamma^2\left(\frac{\partial u}{\partial y}\right)^2\right]^{\frac{n-3}{2}}, \tag{3.2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2}. \tag{3.3}$$

The associated BCs have been taken as:

$$\left.\begin{aligned} &u = \boldsymbol{bx^m}, \quad v = 0, \quad T = T_w, \quad at \quad y = 0, \\ &u \to 0, \quad T \to T_\infty, \quad as \quad y \to \infty. \end{aligned}\right\} \tag{3.4}$$

In the above model, the power-law index is denoted by n, the kinematic viscosity by ν, Γ denotes the time dependant material constant, the temperature by T and the thermal diffusivity by $\alpha = \frac{k}{\rho c_p}$.

For the conversion of the mathematical model (3.1)-(3.3) into the system of ODEs, we introduce the following similarity transformation.

$$\left.\begin{aligned} &\psi(x,y) = \sqrt{\frac{2b\nu}{m+1}}x^{\frac{m+1}{2}}f(\xi), \qquad \theta(\xi) = \frac{T-T_\infty}{T_w-T_\infty}, \\ &\xi = x^{\frac{m-1}{2}}y\sqrt{\frac{(m+1)b}{2\nu}}. \end{aligned}\right\} \tag{3.5}$$

where ψ denotes the stream function.

The detailed procedure for the conversion of (3.1)-(3.3) into the dimensionless form has been described in the following paragraph.

Following are some important derivatives necessary for further derivation.

- $$u = \frac{\partial \psi}{\partial y},$$
$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right),$$
$$\frac{\partial \psi}{\partial y} = \frac{\partial}{\partial y}\left(\sqrt{\frac{2\nu b}{m+1}}x^{\frac{m+1}{2}}f(\xi)\right),$$
$$= \sqrt{\frac{2\nu b}{m+1}}x^{\frac{m+1}{2}}f'(\xi)\frac{\partial \xi}{\partial y},$$
$$= \sqrt{\frac{2\nu b}{m+1}}x^{\frac{m+1}{2}}f'(\xi)\sqrt{\frac{b(m+1)}{2\nu}}x^{\frac{m-1}{2}},$$
$$u = bf'(\xi)x^m. \tag{3.6}$$

- $$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(bf'(\xi)x^m\right),$$
$$= b\left(mx^{m-1}f'(\xi) + x^m f''(\xi)\frac{\partial \xi}{\partial x}\right),$$
$$= b\left(mx^{m-1}f'(\xi) + x^m f''(\xi)y\sqrt{\frac{b(m+1)}{2\nu}}\frac{m-1}{2}x^{\frac{m-3}{2}}\right),$$
$$= b\left(mx^{m-1}f'(\xi) + x^m f''(\xi)\xi\frac{m-1}{2x}\right),$$
$$= b\left(mx^{m-1}f'(\xi) + x^{m-1} f''(\xi)\xi\frac{m-1}{2}\right),$$
$$= bx^{m-1}\left(mf'(\xi) + \xi f''(\xi)\frac{m-1}{2}\right). \tag{3.7}$$

- $$v = -\frac{\partial \psi}{\partial x},$$
$$\frac{\partial v}{\partial y} = -\frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial x}\right),$$
$$\frac{\partial \psi}{\partial x} = -\frac{\partial}{\partial x}\left(\sqrt{\frac{2b\nu}{m+1}}x^{\frac{m+1}{2}}f(\xi)\right),$$

$$= -\sqrt{\frac{2\nu b}{m+1}}\left(\frac{m+1}{2}x^{\frac{m-1}{2}}f(\xi) + x^{\frac{m+1}{2}}f'(\xi)\frac{\partial \xi}{\partial x}\right),$$
$$= -\sqrt{\frac{2\nu b}{m+1}}\left(\frac{m+1}{2}x^{\frac{m-1}{2}}f(\xi) + x^{\frac{m+1}{2}}f'(\xi)y\sqrt{\frac{b(m+1)}{2\nu}}\frac{m-1}{2}x^{\frac{m-3}{2}}\right),$$
$$= -\sqrt{\frac{2\nu b}{m+1}}\left(\frac{m+1}{2}x^{\frac{m-1}{2}}f(\xi) + x^{\frac{m+1}{2}}f'(\xi)\xi\frac{m-1}{2}x^{-1}\right),$$
$$= -x^{\frac{m-1}{2}}\sqrt{\frac{2\nu b}{m+1}}\left(\frac{m+1}{2}f(\xi) + f'(\xi)\xi\frac{m-1}{2}\right),$$

Multiplying and dividing by $2(m+1)$ with right side of above equation

$$= -x^{\frac{m-1}{2}}\sqrt{\frac{4(m+1)\nu b}{2(m+1)^2}}\left(\frac{m+1}{2}f(\xi) + f'(\xi)\xi\frac{m-1}{2}\right),$$
$$= -x^{\frac{m-1}{2}}\sqrt{\frac{\nu b(m+1)}{2}}f(\xi) - x^{\frac{m-1}{2}}\sqrt{\frac{2(m+1)\nu b}{2}}f'(\xi)\xi\frac{m-1}{m+1},$$
$$v = -x^{\frac{m-1}{2}}\sqrt{\frac{\nu b(m+1)}{2}}\left(f(\xi) + \left(\frac{m-1}{m+1}\right)\xi f'(\xi)\right). \tag{3.8}$$

- $$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\left[-x^{\frac{m-1}{2}}\sqrt{\frac{(m+1)\nu b}{2}}\left(f(\xi) + \left(\frac{m-1}{m+1}\right)\xi f'(\xi)\right)\right],$$

$$= -x^{\frac{m-1}{2}}\sqrt{\frac{(m+1)\nu b}{2}}\left(f'(\xi)\frac{\partial \xi}{\partial y} + \frac{m-1}{m+1}\xi f''(\xi)\frac{\partial \xi}{\partial y} + \frac{m-1}{m+1}f'(\xi)\frac{\partial \xi}{\partial y}\right),$$
$$= -x^{\frac{m-1}{2}}\sqrt{\frac{(m+1)\nu b}{2}}\left(f'(\xi) + \frac{m-1}{m+1}\xi f''(\xi) + \frac{m-1}{m+1}f'(\xi)\right)$$
$$\sqrt{\frac{(m+1)b}{2\nu}}x^{\frac{m-1}{2}},$$
$$= -\frac{b}{2}x^{m-1}\left(f'(\xi)(m+1) + (m-1)\xi f''(\xi) + (m-1)f'(\xi)\right),$$
$$= -\frac{b}{2}x^{m-1}2mf'(\xi) - \frac{b}{2}x^{m-1}(m-1)\xi f''(\xi),$$
$$= -bx^{m-1}mf'(\xi) - bx^{m-1}\left(\frac{m-1}{2}\right)\xi f''(\xi). \tag{3.9}$$

Equation (3.1) is satisfied by using (3.7) and (3.9), as follows

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = bx^{m-1}mf'(\xi) + bx^{m-1}\left(\frac{m-1}{2}\right)\xi f''(\xi) - bx^{m-1}mf'(\xi)$$
$$- bx^{m-1}\left(\frac{m-1}{2}\right)\xi f''(\xi),$$
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{3.10}$$

Now, for the momentum equation (3.2), the following derivatives are required:

- $$\begin{aligned}\frac{\partial u}{\partial y} &= \frac{\partial}{\partial y}(bx^m f'(\xi)),\\ &= bx^m f''(\xi)\frac{\partial \xi}{\partial y},\\ \frac{\partial u}{\partial y} &= bx^m f''(\xi)\sqrt{\frac{b(m+1)}{2\nu}}x^{\frac{m-1}{2}}. \qquad (3.11)\end{aligned}$$
- $$\begin{aligned}\frac{\partial^2 u}{\partial y^2} &= bx^m f'''(\xi)\sqrt{\frac{b(m+1)}{2\nu}}x^{\frac{m-1}{2}}\frac{\partial \xi}{\partial y},\\ &= bf'''(\xi)\sqrt{\frac{b(m+1)}{2\nu}}x^{\frac{m-1}{2}}x^m\sqrt{\frac{b(m+1)}{2\nu}}x^{\frac{m-1}{2}},\\ \frac{\partial^2 u}{\partial y^2} &= b^2x^{2m-1}f'''(\xi)\left(\frac{m+1}{2\nu}\right). \qquad (3.12)\end{aligned}$$
- $$\begin{aligned}u\frac{\partial u}{\partial x} &= bx^m f'(\xi)\left(bx^{m-1}mf'(\xi) + bx^{m-1}\left(\frac{m-1}{2}\right)\xi f''(\xi)\right),\\ &= b^2x^{2m-1}mf'^2(\xi) + b^2x^{2m-1}\left(\frac{m-1}{2}\right)\xi f'(\xi)f''(\xi). \qquad (3.13)\end{aligned}$$
- $$\begin{aligned}v\frac{\partial u}{\partial y} &= -\sqrt{\frac{b\nu(m+1)}{2}}x^{\frac{m-1}{2}}\left[\xi f'(\xi)\left(\frac{m-1}{m+1}\right) + f(\xi)\right]\\ &\quad bx^m f''(\xi)\sqrt{\frac{b(m+1)}{2}}x^{\frac{m-1}{2}},\\ &= -\sqrt{\frac{b\nu(m+1)}{2}}x^{\frac{m-1}{2}}f'(\xi)\xi\frac{m-1}{m+1}bx^m f''(\xi)\sqrt{\frac{b(m+1)}{2\nu}}x^{\frac{m-1}{2}}\\ &\quad -\sqrt{\frac{b\nu(m+1)}{2}}x^{\frac{m-1}{2}}f(\xi)bx^m f''(\xi)\sqrt{\frac{b(m+1)}{2\nu}}x^{\frac{m-1}{2}},\\ &= -\frac{b^2(m+1)}{2}x^{2m-1}f'(\xi)f''(\xi)\xi\left(\frac{m-1}{m+1}\right)\\ &\quad -\frac{b^2(m+1)}{2}x^{2m-1}f(\xi)f''(\xi). \qquad (3.14)\end{aligned}$$

Using (3.13) and (3.14), the left side of (3.2) becomes

$$\begin{aligned}u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= bx^m f'(\xi)\left[bx^{m-1}mf'(\xi) + bx^{m-1}\left(\frac{m-1}{2}\right)\xi f''(\xi)\right]\\ &\quad -\sqrt{\frac{b\nu(m+1)}{2}}x^{\frac{m-1}{2}}\left[\xi f'(\xi)\left(\frac{m-1}{m+1}\right) + f(\xi)\right]\\ &\qquad bx^m f''(\xi)\sqrt{\frac{b(m+1)}{2}}x^{\frac{m-1}{2}},\\ &= b^2x^{2m-1}mf'^2(\xi) + b^2x^{2m-1}\left(\frac{m-1}{2}\right)\xi f'(\xi)f''(\xi)\end{aligned}$$

$$
\begin{aligned}
&- \frac{b^2(m+1)}{2} x^{2m-1} \xi f'(\xi) f''(\xi) \left(\frac{m-1}{m+1} \right) \\
&- \frac{b^2(m+1)}{2} x^{2m-1} f(\xi) f''(\xi), \\
&= b^2 x^{2m-1} m f'^2(\xi) - \frac{b^2(m+1)}{2} x^{2m-1} f(\xi) f''(\xi), \\
&= b^2 x^{2m-1} \left(m f'^2(\xi) - \left(\frac{m+1}{2} \right) f(\xi) f''(\xi) \right). \qquad (3.15)
\end{aligned}
$$

Using (3.11) and (3.12), the right side of (3.2) becomes

$$
\begin{aligned}
& v \frac{\partial^2 u}{\partial y^2} \left[1 + \Gamma^2 \left(\frac{\partial u}{\partial y} \right)^2 \right]^{\frac{n-1}{2}} + \nu(n-1)\Gamma^2 \frac{\partial^2 u}{\partial y^2} \left(\frac{\partial u}{\partial y} \right)^2 \left[1 + \Gamma^2 \left(\frac{\partial u}{\partial y} \right)^2 \right]^{\frac{n-3}{2}} \\
&= \left[1 + \Gamma^2 \left(\frac{\partial u}{\partial y} \right)^2 \right]^{\frac{n-3}{2}} \nu \frac{\partial^2 u}{\partial y^2} \left[1 + \Gamma^2 \left(\frac{\partial u}{\partial y} \right)^2 \right] + \nu(n-1)\Gamma^2 \frac{\partial^2 u}{\partial y^2} \left(\frac{\partial u}{\partial y} \right)^2, \\
&= \left[1 + \Gamma^2 \left(\frac{\partial u}{\partial y} \right)^2 \right]^{\frac{n-3}{2}} \left(\nu \frac{\partial^2 u}{\partial y^2} + \nu\Gamma^2 \left(\frac{\partial u}{\partial y} \right)^2 \right) \frac{\partial^2 u}{\partial y^2} + \nu n \Gamma^2 \frac{\partial^2 u}{\partial y^2} \left(\frac{\partial u}{\partial y} \right)^2 \\
&\quad - \nu\Gamma^2 \left(\frac{\partial u}{\partial y} \right)^2 \frac{\partial^2 u}{\partial y^2}, \\
&= \left[1 + \Gamma^2 \left(\frac{\partial u}{\partial y} \right)^2 \right]^{\frac{n-3}{2}} \nu \frac{\partial^2 u}{\partial y^2} \left(1 + \Gamma^2 \left(\frac{\partial u}{\partial y} \right)^2 + n\Gamma^2 \left(\frac{\partial u}{\partial y} \right)^2 - \Gamma^2 \left(\frac{\partial u}{\partial y} \right)^2 \right), \\
&= \left[1 + \Gamma^2 \left(b x^m f''(\xi) \sqrt{\frac{b(m+1)}{2\nu}} x^{\frac{m-1}{2}} \right) \right]^{\frac{n-3}{2}} \nu b^2 x^{2m-1} f'''(\xi) \left(\frac{m+1}{2\nu} \right) \\
&\quad \left[1 + n\Gamma^2 \left(b x^m f''(\xi) \sqrt{\frac{b(m+1)}{2\nu}} x^{\frac{m-1}{2}} \right)^2 \right], \\
&= \left[1 + \Gamma^2 b^3 x^{3m-1} f''^2(\xi) \left(\frac{m+1}{2\nu} \right) \right]^{\frac{n-3}{2}} b^2 x^{2m-1} f'''(\xi) \left(\frac{m+1}{2} \right) \\
&\quad \left[1 + n\Gamma^2 b^3 x^{3m-1} f''^2(\xi) \left(\frac{m+1}{2\nu} \right) \right], \\
&= \left[1 + \Gamma^2 b^3 x^{3m-1} f''^2(\xi) \left(\frac{m+1}{2\nu} \right) \right]^{\frac{n-3}{2}} \left[b^2 x^{2m-1} f'''(\xi) \left(\frac{m+1}{2} \right) \right] \\
&\quad \left[1 + n\Gamma^2 b^3 x^{3m-1} f''^2(\xi) \left(\frac{m+1}{2\nu} \right) \right], \\
&= \left[1 + We^2 f''^2(\xi) \right]^{\frac{n-3}{2}} \left[b^2 x^{2m-1} f'''(\xi) \left(\frac{m+1}{2} \right) \right] \\
&\quad \left[1 + nWe^2 f''^2(\xi) \right]. \qquad (3.16)
\end{aligned}
$$

Using (3.15) and (3.16), the dimensionless form of (3.2) can be seen as:

$$
\begin{aligned}
u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = v\frac{\partial^2 u}{\partial y^2}\left[1+\Gamma^2\left(\frac{\partial u}{\partial y}\right)^2\right]^{\frac{n-1}{2}} &+ \nu(n-1)\Gamma^2\frac{\partial^2 u}{\partial y^2}\left(\frac{\partial u}{\partial y}\right)^2 \\
&\left[1+\Gamma^2\left(\frac{\partial u}{\partial y}\right)^2\right]^{\frac{n-3}{2}},\\
\Rightarrow \qquad & b^2x^{2m-1}\left[mf'^2(\xi)-\left(\frac{m+1}{2}\right)f(\xi)f''(\xi)\right]\\
&=\left[1+We^2f''^2(\xi)\right]^{\frac{n-3}{2}}\left[b^2x^{2m-1}f'''(\xi)\frac{m+1}{2}\right]\\
&\left[1+nWe^2f''^2(\xi)\right],\\
\Rightarrow \qquad & f'''(\xi)\left(\frac{m+1}{2}\right)[1+nWe^2f''^2(\xi)][1+We^2f''^2(\xi)]^{\frac{n-3}{2}}\\
&-mf'^2(\xi)+\left(\frac{m+1}{2}\right)f(\xi)f''(\xi)=0.
\end{aligned} \tag{3.17}
$$

Multiplying both sides $\frac{2}{m+1}$ in (3.17), we get

$$
\begin{aligned}
& f'''(\xi)[1+nWe^2f''^2(\xi)][1+We^2f''^2(\xi)]^{\frac{n-3}{2}}\\
& -\left(\frac{2m}{m+1}\right)f'^2(\xi)+f(\xi)f''(\xi)=0,\\
\Rightarrow \qquad & [1+nWe^2f''^2][1+We^2f''^2]^{\frac{n-3}{2}}f'''+f''f-\left(\frac{2m}{m+1}\right)f'^2=0.
\end{aligned} \tag{3.18}
$$

Now, we include below the procedure for the conversion of (3.3) into the dimensionless form.

- $$
\begin{aligned}
&\theta(\xi)=\frac{T-T_\infty}{T_w-T_\infty},\\
&T=\theta(\xi)(T_w-T_\infty)+T_\infty,\\
&\frac{\partial T}{\partial x}=(T_w-T_\infty)\theta'(\xi)\frac{\partial \xi}{\partial x},\\
&\qquad=(T_w-T_\infty)y\sqrt{\frac{b(m+1)}{2\nu}}x^{\frac{m-3}{2}}\frac{m-1}{2}\theta'(\xi),\\
&\frac{\partial T}{\partial x}=(T_w-T_\infty)\left(\frac{m-1}{2x}\right)\xi\theta'(\xi).
\end{aligned} \tag{3.19}
$$

- $$\frac{\partial T}{\partial y}=(T_w-T_\infty)\theta'(\xi)\frac{\partial \xi}{\partial y},$$

$$= (T_w - T_\infty)\sqrt{\frac{b(m+1)}{2\nu}}x^{\frac{m-1}{2}}\theta'(\xi). \tag{3.20}$$

- $$\alpha\frac{\partial^2 T}{\partial y^2} = \alpha(T_w - T_\infty)\left(\frac{b(m+1)}{2\nu}\right)x^{m-1}\theta''(\xi). \tag{3.21}$$

- $$\begin{aligned} u\frac{\partial T}{\partial x} &= -x^{\frac{m-1}{2}}\sqrt{\frac{\nu b(m+1)}{2}}\left[\left(\frac{m-1}{m+1}\right)\xi f'(\xi) + f(\xi)\right], \\ &= bx^{m-1}(T_w - T_\infty)\left(\frac{m-1}{2}\right)\xi f'(\xi)\theta'(\xi). \end{aligned} \tag{3.22}$$

- $$\begin{aligned} v\frac{\partial T}{\partial y} &= -x^{\frac{m-1}{2}}\sqrt{\frac{\nu b(m+1)}{2}}\left[\left(\frac{m-1}{m+1}\right)\xi f'(\xi) + f(\xi)\right] \\ &\quad \left[(T_w - T_\infty)\sqrt{\frac{b(m+1)}{2\nu}}x^{\frac{m-1}{2}}\theta'(\xi)\right], \\ &= -\left(\frac{(m+1)b}{2}\right)x^{m-1}(T_w - T_\infty)\left(\frac{m-1}{m+1}\right)\xi\theta'(\xi)f'(\xi) \\ &\quad - \left(\frac{b(m+1)}{2}\right)x^{m-1}(T_w - T_\infty)f(\xi)\theta'(\xi). \end{aligned} \tag{3.23}$$

Using (3.22) and (3.23) in left side of (3.3), we get

$$\begin{aligned} u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} &= bx^m f'(\xi)\left[(T_w - T_\infty)\left(\frac{m-1}{2x}\right)\xi\theta'(\xi)\right] \\ &\quad - x^{\frac{m-1}{2}}\sqrt{\frac{\nu b(m+1)}{2}}\left[\left(\frac{m-1}{m+1}\right)\xi f'(\xi) + f(\xi)\right] \\ &\quad \left[(T_w - T_\infty)\sqrt{\frac{b(m+1)}{2\nu}}x^{\frac{m-1}{2}}\theta'(\xi)\right], \\ &= bx^{m-1}(T_w - T_\infty)\left(\frac{m-1}{2}\right)\xi f'(\xi)\theta'(\xi) \\ &\quad - \left(\frac{(m+1)b}{2}\right)x^{m-1}(T_w - T_\infty)\left(\frac{m-1}{m+1}\right)\xi\theta'(\xi)f'(\xi) \\ &\quad - \left(\frac{b(m+1)}{2}\right)x^{m-1}(T_w - T_\infty)f(\xi)\theta'(\xi), \\ &= bx^m\left(\frac{m-1}{2}\right)(T_w - T_\infty)\xi f'(\xi)\theta'(\xi) \\ &\quad - bx^m\left(\frac{m-1}{2}\right)(T_w - T_\infty)\xi f'(\xi)\theta'(\xi) \\ &\quad - bx^m\left(\frac{m+1}{2}\right)(T_w - T_\infty)f(\xi)\theta'(\xi), \\ u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} &= -bx^m\left[\left(\frac{m+1}{2}\right)(T_w - T_\infty)f(\xi)\theta'(\xi)\right]. \end{aligned} \tag{3.24}$$

Using (3.21) in the right side of (3.3), we get

$$\alpha\frac{\partial^2 T}{\partial y^2} = \alpha(T_w - T_\infty)\left(\frac{b(m+1)}{2\nu}\right)x^{m-1}\theta''(\xi). \tag{3.25}$$

With the help of (3.24) and (3.25), the dimensionless form of (3.3), is obtained.

$$\begin{aligned}
& u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2},\\
\Rightarrow\quad & -bx^m\left[\left(\frac{m+1}{2}\right)(T_w - T_\infty)f(\xi)\theta'(\xi)\right] = \alpha(T_w - T_\infty)\\
& \left(\frac{b(m+1)}{2\nu}\right)x^{m-1}\theta''(\xi),\\
\Rightarrow\quad & -\theta'(\xi)f(\xi) = \frac{\alpha}{\nu}\theta''(\xi),\\
\Rightarrow\quad & \theta'' + \frac{\nu}{\alpha}\theta' f = 0,\\
\Rightarrow\quad & \theta'' + Pr\theta' f = 0. \qquad (3.26)
\end{aligned}$$

The corresponding BCs are transformed into the non-dimensional form. The following step have been taken:

- $$\begin{aligned}
& u = U_w(x) = bx^m, \qquad at \quad y = 0\\
& u = bf'(\xi)x^m.\\
\Rightarrow\quad & bf'(\xi)x^m = bx^m.\\
\Rightarrow\quad & f'(\xi) = 1 \qquad at \quad \xi = 0.\\
\Rightarrow\quad & f'(0) = 1.
\end{aligned}$$

- $$\begin{aligned}
& v = 0, \qquad at \quad y = 0.\\
\Rightarrow\quad & -x^{\frac{m-1}{2}}\sqrt{\frac{(m+1)\nu b}{2}}\left(f(\xi) + \xi f'(\xi)\left(\frac{m-1}{m+1}\right)\right) = 0.\\
\Rightarrow\quad & -x^{\frac{m-1}{2}}\sqrt{\frac{b\nu(m+1)}{2}}f(0) = 0 \qquad at \quad \xi = 0\\
\Rightarrow\quad & f(0) = 0.
\end{aligned}$$

- $$\begin{aligned}
& T = T_w \qquad at \quad y = 0.\\
\Rightarrow\quad & \theta(\xi)(T_w - T_\infty) + T_\infty = T_w.\\
\Rightarrow\quad & \theta(\xi)(T_w - T_\infty) = (T_w - T_\infty).
\end{aligned}$$

$$\Rightarrow \quad \theta(\xi) = 1 \qquad at \quad \xi = 0.$$

$$\Rightarrow \quad \theta(0) = 1.$$

- $$u \to 0 \qquad as \quad y \to \infty.$$

$$\Rightarrow \quad bf'(\xi)x^m \to 0.$$

$$\Rightarrow \quad f'(\xi)bx^m \to 0.$$

$$\Rightarrow \quad f'(\xi) \to 0 \qquad as \quad \xi \to \infty.$$

$$\Rightarrow \quad f'(\xi) \to 0.$$

- $$T \to T_\infty \qquad as \quad y \to \infty.$$

$$\Rightarrow \quad \theta(\xi)(T_w - T_\infty) + T_\infty \to T_\infty.$$

$$\Rightarrow \quad \theta(\xi)(T_w - T_\infty) \to 0 \qquad as \quad \xi \to \infty.$$

$$\Rightarrow \quad \theta(\xi) \to 0 \qquad as \quad \xi \to \infty.$$

$$\Rightarrow \quad \theta(\infty) \to 0.$$

The final dimensionless form of the governing model, is

$$[1 + nWe^2 f''^2][1 + We^2 f''^2]^{\frac{n-3}{2}} f''' + f''f - \left(\frac{2m}{m+1}\right) f'^2 = 0. \tag{3.27}$$

$$\theta'' + Pr\theta' f = 0. \tag{3.28}$$

The associated BCs (3.4) in the dimensionless form are

$$\left.\begin{aligned} &f(0) = 0, \quad f'(0) = 1, \quad \theta(0) = 1. \\ &f'(\infty) \to 0, \quad \theta(\infty) \to 0. \end{aligned}\right\}$$

The skin friction coefficient, is given as follows

$$Cf_x = \frac{\tau_w\big|_{y=0}}{\rho U_w^2(x)}. \tag{3.29}$$

To achive the dimensionless form of Cf_x, the following step will be helpful.

$$\tau_w = \xi_0 \frac{\partial u}{\partial y}\left[1 + \Gamma^2 \left(\frac{\partial u}{\partial y}\right)^2\right]^{\frac{n-1}{2}}. \tag{3.30}$$

Using (3.29) and (3.30), we get the following form:

$$
\begin{aligned}
Cf_x &= \frac{1}{\rho U_w^2(x)}\xi_0 \frac{\partial u}{\partial y}\left[1+\Gamma^2\left(\frac{\partial u}{\partial y}\right)^2\right]^{\frac{n-1}{2}}, \\
&= \frac{1}{\rho U_w^2(x)}\xi_0 b x^m f''(\xi)\sqrt{\frac{(m+1)b}{2\nu}}x^{\frac{m-1}{2}} \\
&\quad \left[1+\Gamma^2 b^2 x^{2m} f''^2(\xi)\left(\frac{(m+1)b}{2\nu}\right)x^{m-1}\right]^{\frac{n-1}{2}}, \\
&= \frac{1}{\rho b^2 x^{2m}}\xi_0 b x^m f''(\xi)\sqrt{\frac{(m+1)b}{2\nu}}x^{\frac{m-1}{2}} \\
&\quad \left[1+\Gamma^2 b^3 x^{3m-1} f''^2(\xi)\left(\frac{(m+1)b}{2\nu}\right)\right]^{\frac{n-1}{2}}, \\
&= \frac{1}{\xi_0 b^2 x^{2m}}\nu\xi_0 b x^m f''(\xi)\sqrt{\frac{(m+1)b}{2\nu}}x^{\frac{m-1}{2}} \\
&\quad \left[1+\Gamma^2 b^3 x^{3m-1} f''^2(\xi)\left(\frac{(m+1)b}{2\nu}\right)\right]^{\frac{n-1}{2}}, \\
&= \frac{1}{\xi_0 b x^m}\nu\xi_0 f''(\xi)\sqrt{\frac{(m+1)b}{2\nu}}x^{\frac{m-1}{2}}\left[1+We^2 f''^2(\xi)\right]^{\frac{n-1}{2}}, \\
&= \frac{1}{\xi_0 b x^m}\nu\xi_0\sqrt{\frac{b}{\nu}}f''(\xi)\sqrt{\frac{(m+1)}{2}}x^{\frac{m-1}{2}}\left[1+We^2 f''^2(\xi)\right]^{\frac{n-1}{2}}, \\
&= \frac{1}{\nu^{\frac{1}{2}}\xi_0 b x^m}\xi_0 \nu b^{\frac{1}{2}} f''(\xi)\sqrt{\frac{(m+1)}{2}}x^{\frac{m-1}{2}}\left[1+We^2 f''^2(\xi)\right]^{\frac{n-1}{2}}, \\
&= \frac{1}{b^{\frac{1}{2}}x^{\frac{m+1}{2}}}\nu^{\frac{1}{2}}\sqrt{\frac{m+1}{2}}f''(\xi)\left[1+We^2 f''^2(\xi)\right]^{\frac{n-1}{2}}, \\
\frac{b^{\frac{1}{2}}x^{\frac{m+1}{2}}}{\nu^{\frac{1}{2}}} &= \sqrt{\frac{m+1}{2}}f''(\xi)\left[1+We^2 f''^2(\xi)\right]^{\frac{n-1}{2}}, \\
\sqrt{\frac{bx^{m+1}}{\nu}} &= \sqrt{\frac{m+1}{2}}f''(\xi)\left[1+We^2 f''^2(\xi)\right]^{\frac{n-1}{2}}, \\
Re^{\frac{1}{2}}Cf_x &= \sqrt{\frac{m+1}{2}}f''(\xi)\left[1+We^2 (f''(0))^2\right]^{\frac{n-1}{2}}. \\
Re^{\frac{1}{2}}Cf_x &= \sqrt{\frac{m+1}{2}}f''(0)\left[1+We^2 (f''(0))^2\right]^{\frac{n-1}{2}}.
\end{aligned} \tag{3.31}
$$

where Re denotes the Reynolds number defined as $Re = \frac{bx^{m+1}}{\nu}$.

Local Nusselt number is defined as follow:

$$
Nu_x = -\frac{x}{(T_w - T_\infty)}\left(\frac{\partial T}{\partial y}\right)|_y = 0. \tag{3.32}
$$

To achive the dimensionless form of Nu_x, the following steps will be helpful.

$$
\begin{aligned}
Nu_x &= -\frac{x}{(T_w - T_\infty)}\left(\frac{(m+1)b}{2\nu}\right) x^{\frac{m-1}{2}}(T_w - T_\infty)\theta'(0),\\
&= -\sqrt{\frac{b(m+1)}{2\nu}}x^{\frac{m+1}{2}}\theta'(0),\\
&= -\sqrt{\frac{m+1}{2}}\theta'(0)\sqrt{\frac{bx^{m+1}}{\nu}},\\
&= -\sqrt{\frac{m+1}{2}}\theta'(0)Re^{\frac{1}{2}},\\
Re^{\frac{-1}{2}}Nu_x &= -\sqrt{\frac{m+1}{2}}\theta'(0). \qquad (3.33)
\end{aligned}
$$

3.3 Numerical Method for Solution

The shooting method has been used to solve the ordinary differential equation system (3.27). Equation (3.27) is solved numerically and then its solution used in (3.28). The following notations have been considered:

$$
\begin{aligned}
&f = Z_1, \quad f' = Z_1' = Z_2, \qquad f'' = Z_1'' = Z_2' = Z_3, \quad f''' = Z_3'.\\
&x_1 = (1 + nWe^2 Z_3^2), \qquad x_2 = (1 + We^2 Z_3^2)^{\frac{n-3}{2}},\\
&x_4 = -Z_1 Z_3 + \frac{2m}{m+1}Z_2^2, \qquad x_3 = -Z_1 Z_6 - Z_3 Z_4 + \frac{4m}{m+1}Z_2 Z_5.
\end{aligned}
$$

By using the above notations in (3.27), the successive scheme of ODEs is obtained:

$$
\begin{aligned}
Z_1' &= Z_2, & Z_1(0) &= 0,\\
Z_2' &= Z_3, & Z_2(0) &= 1,\\
Z_3' &= \frac{1}{x_1 x_2}\left(-Z_1 Z_3 + \frac{2m}{m+1}Z_2^2\right), & Z_3(0) &= p.
\end{aligned}
$$

The above initial value problem will be numerically solved by RK technique. The above initial value problem, the missing condition p are to be chosen such that:

$$Z_2(\xi_\infty, p) = 0.$$

To solve the above equation, we apply Newton's method which has the following iterative scheme:

$$p^{k+1} = p^k - \frac{G(p)}{G'(p)}, \quad \textit{where} \quad G(p) = Z_2(\xi_\infty, p).$$

Let us now consider the following new notations:

$$\frac{\partial Z_1}{\partial p} = Z_4, \quad \frac{\partial Z_2}{\partial p} = Z_5, \quad \frac{\partial Z_3}{\partial p} = Z_6.$$

As a result, the following IVP is obtained:

$$\begin{aligned}
&Z_4' = Z_5, && Z_4(0) = 0.\\
&Z_5' = Z_6, && Z_5(0) = 0.\\
&Z_6' = \frac{1}{(x_1x_2)^2}[x_1x_2x_3 \\
&\quad - (x_4)\left((x_1)(n-3)(x_2)^{-1}(We^2Z_3Z_6) + (x_2)(2nWe^2Z_3Z_6)(x_4)\right)], \\
& && Z_6(0) = 1.
\end{aligned}$$

The equation (3.28) will be numerically solved by using shooting method by assuming f as a known function. For this we utilize the following notions have been defined:

$$\theta = Y_1, \quad \theta' = Y_2, \quad \theta'' = Y_2'.$$

By using the above notation in (3.28), the following ODEs is obtained:

$$\begin{aligned}
&Y_1' = Y_2, && Y_1(0) = 1.\\
&Y_2' = -PrfY_2, && Y_2(0) = q.
\end{aligned}$$

The above IVP will be numerically solved by RK technique of order four. In the above IVP, the missing condition q are to be chosen such that:

$$Y_1(\xi_\infty, q) = 0.$$

To incorporate Newton's method, we further apply the following notation:

$$\frac{\partial Y_1}{\partial q} = Y_3, \qquad \frac{\partial Y_2}{\partial q} = Y_4.$$

As a result of these new notations,

$$Y_3' = Y_4, \qquad Y_3(0) = 0.$$
$$Y_4' = -Prf Y_4, \qquad Y_4(0) = 1.$$

The above initial value problem will be numerically solved by RK technique. To refine the missing initial condition q, the equation $Y_1(\xi_\infty, q) = 0$ will be solved numerically by the Newton's method as follows:

$$\left(q^{k+1}\right) = \left(q^k\right) - \left(\frac{\partial Y_1}{\partial q}\right)^{-1} \left(Y_1, q^k\right),$$
$$\left(q^{k+1}\right) = \left(q^k\right) - \left(Y_3\right)^{-1} \left(Y_1, q^k\right).$$

The stopping criteria for the shooting method is set as

$$|Y_1(\xi_\infty)| < \epsilon,$$

where ϵ is an arbitrarily small positive number. Here ϵ is taken as $(10)^{-10}$.

3.4 Representation of Graphs and Tables

The physical impacts of significant parameters on the $f'(\xi)$ and $\theta(\xi)$ have been explained through graphs and tables. In the present survey, the shooting method has been opted for reproducing the solution of $f'(\xi)$ and $\theta(\xi)$. The results presented in Table 3.1 illustrate the impact of significant parameters on the skin friction coefficients $(Re)^{\frac{1}{2}} Cf_x$. For rising the value of n and m, the skin friction coefficient is increased. Furthermore, for the accelerating value of We, the skin friction coefficients decreases.

In Table 3.2, the impact of significant parameters on Nusselt number $(Re)^{\frac{-1}{2}} Nu_x$

has been discussed. The growing pattern is found in $(Re)^{\frac{-1}{2}}Nu_x$ due to the accelerating values of n, m and Pr, while for the accelerating value of We, the Nusselt number decreased.

Figures 3.2-3.3 illustrate the impact of power-law index n on the dimensionless velocity profile. It is clearly shown that the velocity profile is increased by enhancing the values of n. Figure 3.4 indicates the influence of power-law index on the dimensionless temperature distribution. It can be noted that the temperature distribution decreased by enhancing the values of n. Figure 3.5 illustrates the impact of power-law index on the dimensionless temperature distribution. This graph indicates that an increment in the values of n causes the decrement in temperature distribution. Figures 3.6-3.7 represents the influence of stretching parameter m on the dimensionless velocity profile. It can be noted that, the velocity profile is decreased by enhancing the values of m. Figure 3.8 represents the impact of stretching parameter m on the dimensionless temperature distribution. The temperature distribution expands by enhancing the values of m. Figure 3.9 delineated to show the impact of stretching parameter m on the dimensionless temperature distribution. It is clearly shown that the temperature distribution is increased by enhancing the values of m.

Figures 3.10-3.13 display the impact of velocity and temperature distributions by varying the values of We. By increasing We, the velocity of fluid velocity distribution decreases for the shear thinning fluid as displayed in Figures 3.10-3.11, while the velocity profile increases for shear thickening fluid. The impact of We on temperature distribution is shown in Figures 3.12-3.13. We see that temperature distribution increases by uplifting We for shear thinning fluid, while the impact of temperature distribution is opposite for shear thickening fluid. The effect of Pr on temperature distribution is shown in Figures 3.14-3.15 for different values of n. As we increase Pr, the temperature distribution decreases also, the thermal boundary layer thickness decreases for both shear thickening and thinning fluids. The decrease in temperature distribution because of the fluid viscosity occurs due to increase in Pr. Furthermore, for shear thinning fluid the thermal boundary layer thickness is larger while, for the shear thinning fluid it is smaller.

TABLE 3.1: Results of $-(Re)^{\frac{1}{2}}Cf_x$ for various parameters

m	We	n	$-(Re)^{\frac{1}{2}}Cf_x$
1	3	0.5	0.749993
1	3	1	1.000488
1	3	1.5	1.193961
1	3	2	1.356320
1	3	2.5	1.498823
1	3	0.5	0.749993
2	3	0.5	0.988643
3	3	0.5	1.179281
4	3	0.5	1.341682
5	3	0.5	1.486414
1	0.5	0.5	0.982634
1	1	0.5	0.938202
1	2	0.5	0.832767
1	4	0.5	0.689662
1	5	0.5	0.644163

TABLE 3.2: Results of $-(Re)^{\frac{-1}{2}}Nu_x$ for various parameters

n	We	m	Pr	$-(Re)^{\frac{-1}{2}}Nu_x$
0.5	3	1	1	0.524931
1	3	1	1	0.582900
1.5	3	1	1	0.617873
2	3	1	1	0.641953
2.5	3	1	1	0.659307
0.5	3	1	1	0.524931
0.5	3	2	1	0.623579
0.5	3	3	1	0.712281
0.5	3	4	1	0.789969
0.5	3	5	1	0.860833
0.5	0.5	1	1	0.578952
0.5	1	1	1	0.569701
0.5	2	1	1	0.545756
0.5	4	1	1	0.508219
0.5	5	1	1	0.496382
0.5	3	1	0.5	0.322305
0.5	3	1	1	0.524931
0.5	3	1	1.5	0.691959
0.5	3	1	2	0.835020
0.5	3	1	2.5	0.963313

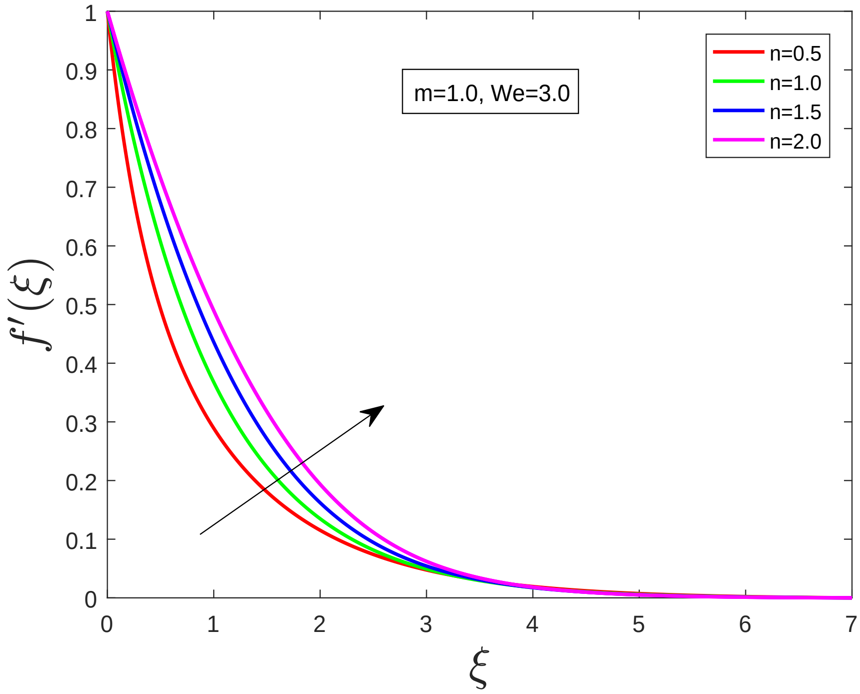

FIGURE 3.2: Impact of n on $f'(\xi)$ along with $m = 1.0$.

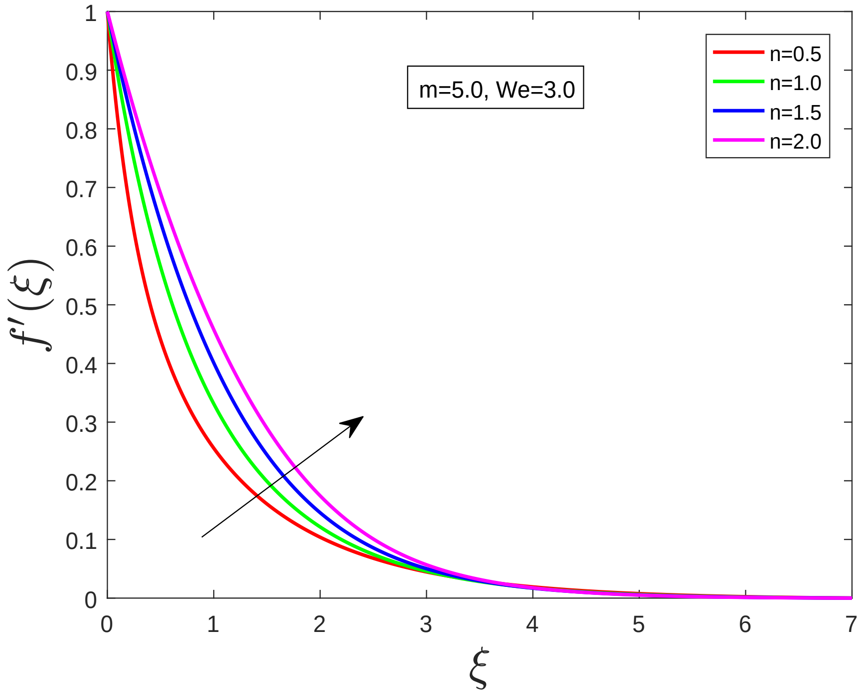

FIGURE 3.3: Impact of n on $f'(\xi)$ along with $m = 5.0$.

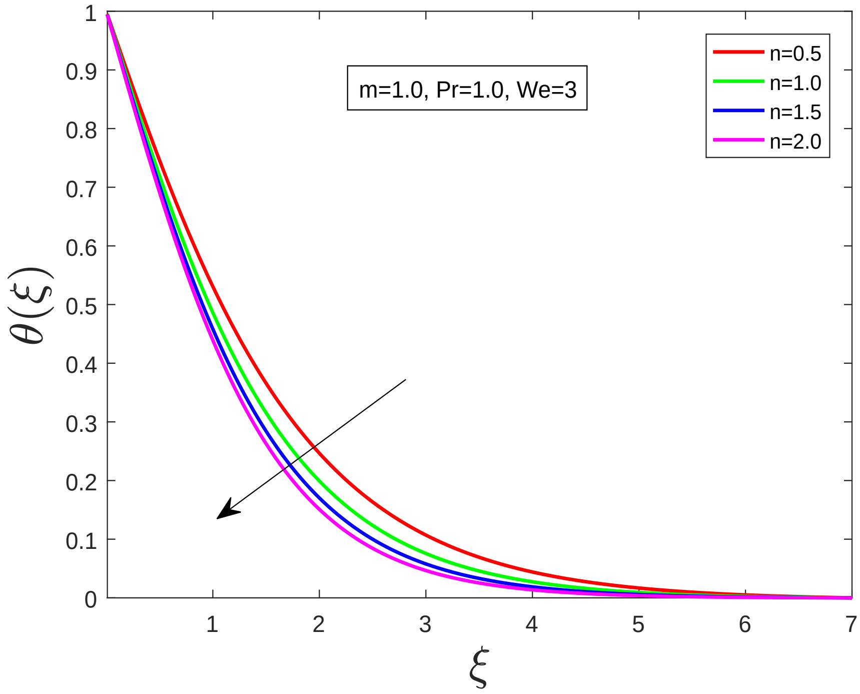

FIGURE 3.4: Impact of n on $\theta(\xi)$ along with $m = 1.0$.

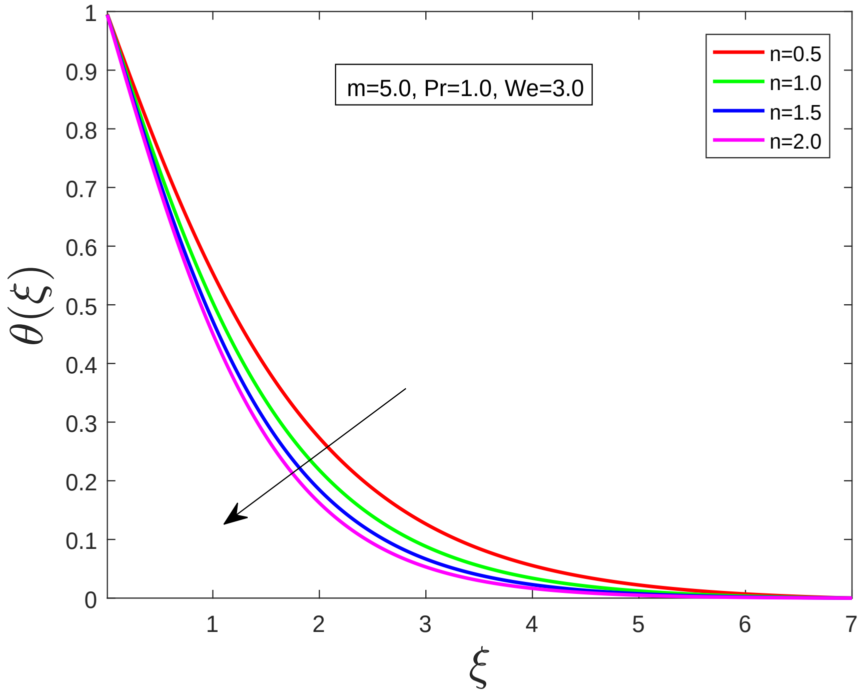

FIGURE 3.5: Impact of n on $\theta(\xi)$ along with $m = 5.0$.

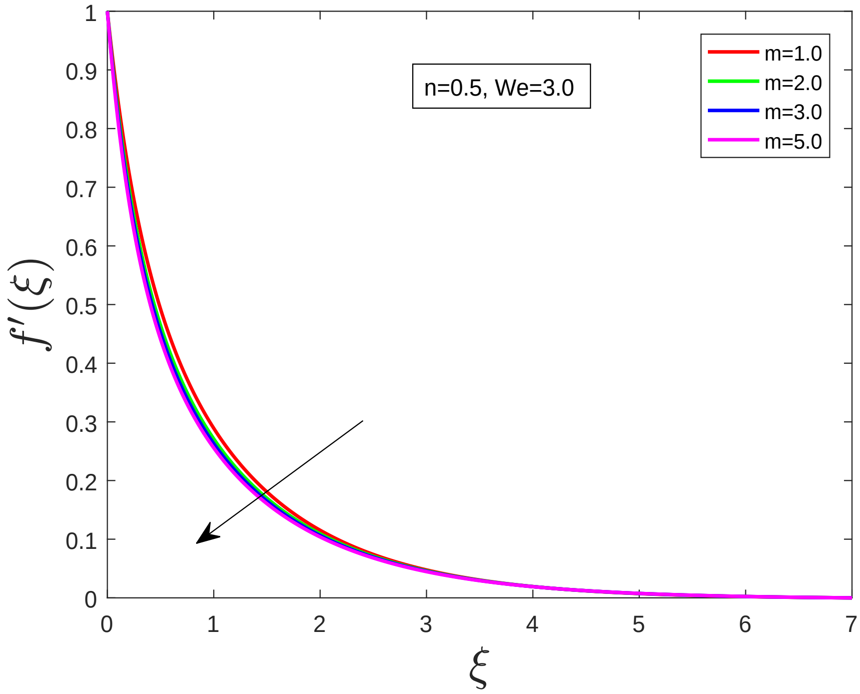

FIGURE 3.6: Impact of m on $f'(\xi)$ along with $n = 0.5$.

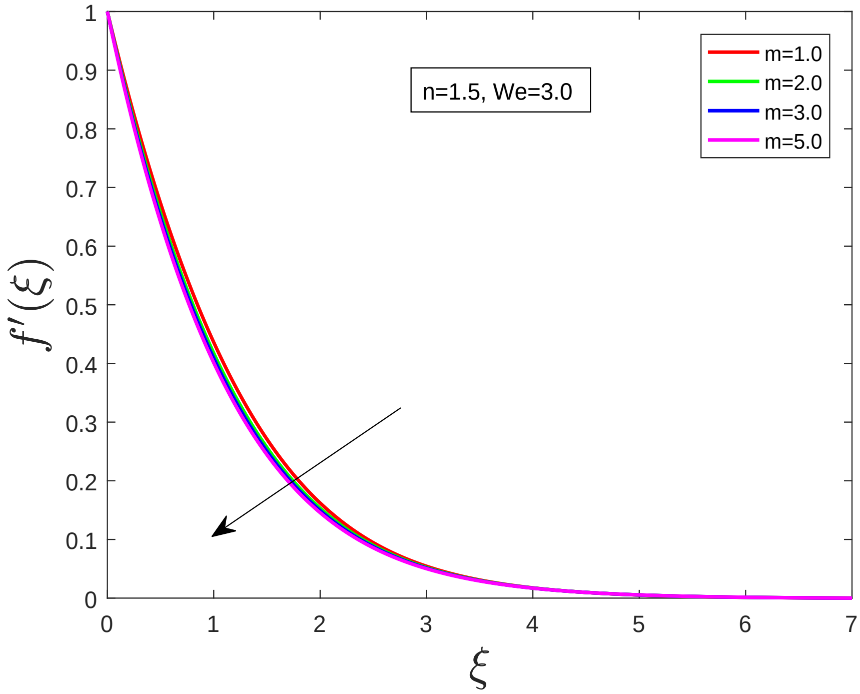

FIGURE 3.7: Impact of m on $f'(\xi)$ along with $n = 1.5$.

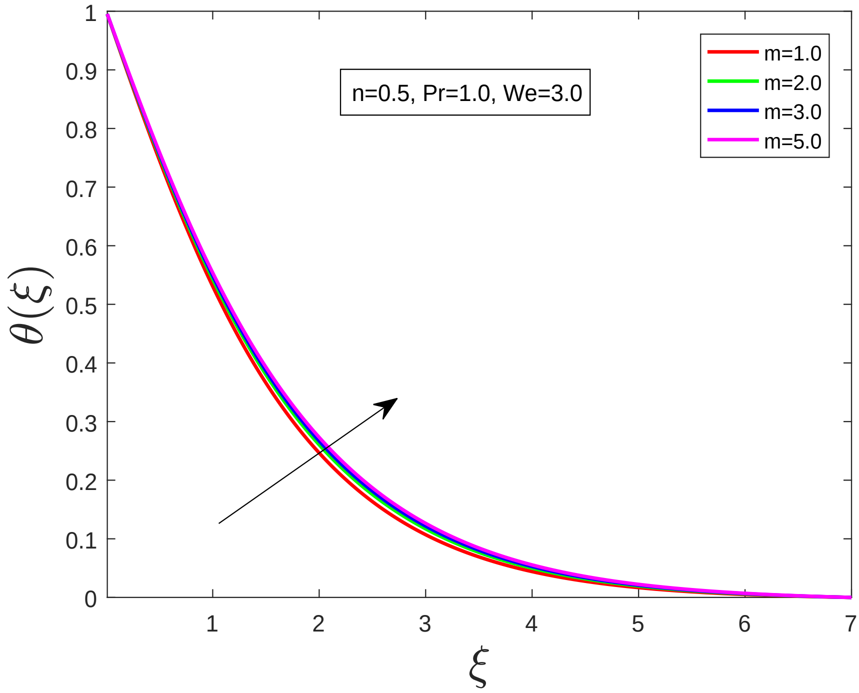

FIGURE 3.8: Impact of m on $\theta(\xi)$ along with $n = 0.5$.

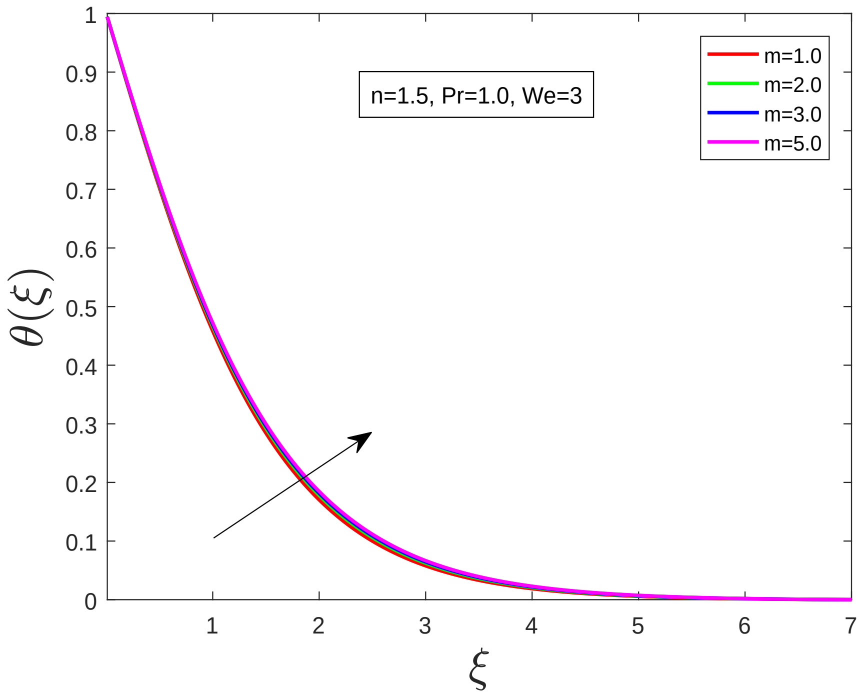

FIGURE 3.9: Impact of m on $\theta(\xi)$ along with $n = 1.5$.

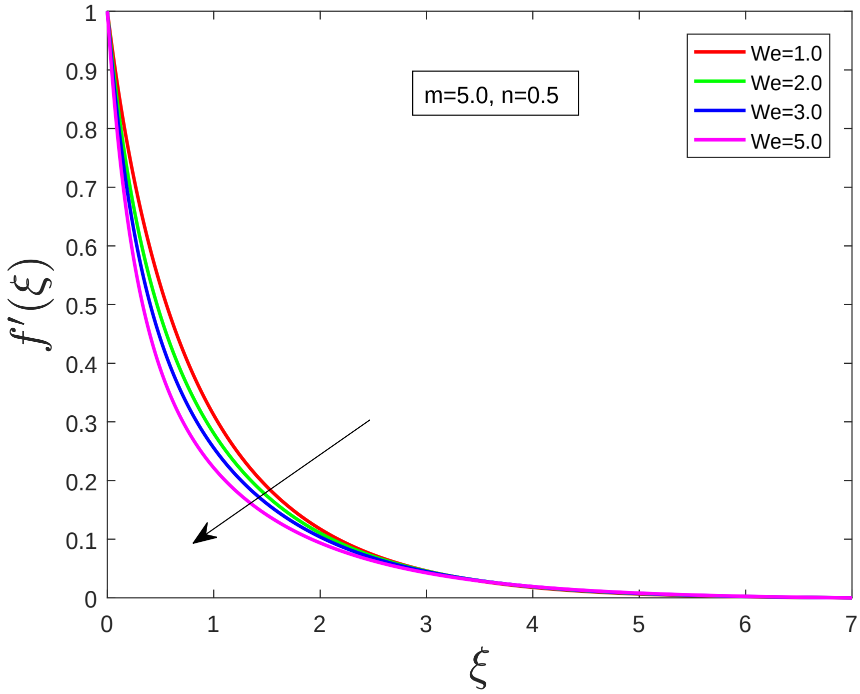

FIGURE 3.10: Impact of We on $f'(\xi)$ along with $n = 0.5$.

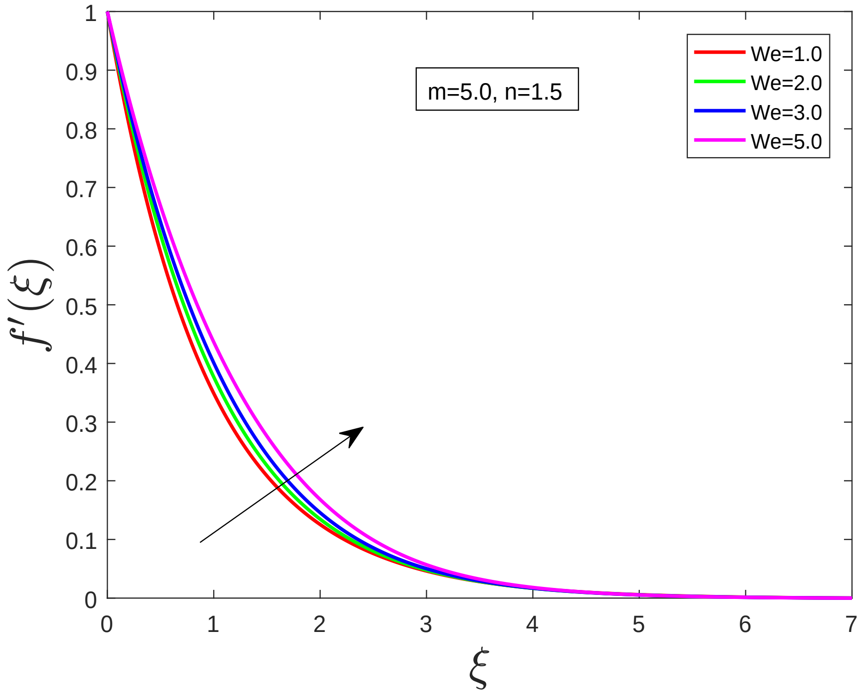

FIGURE 3.11: Impact of We on $f'(\xi)$ along with $n = 1.5$.

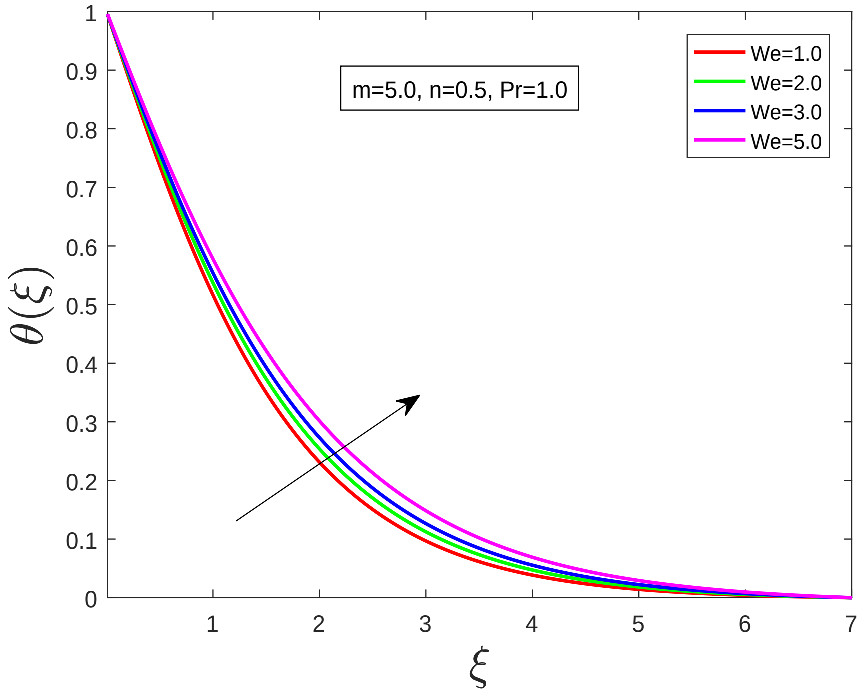

FIGURE 3.12: Impact of We on $\theta(\xi)$ along with $n = 0.5$.

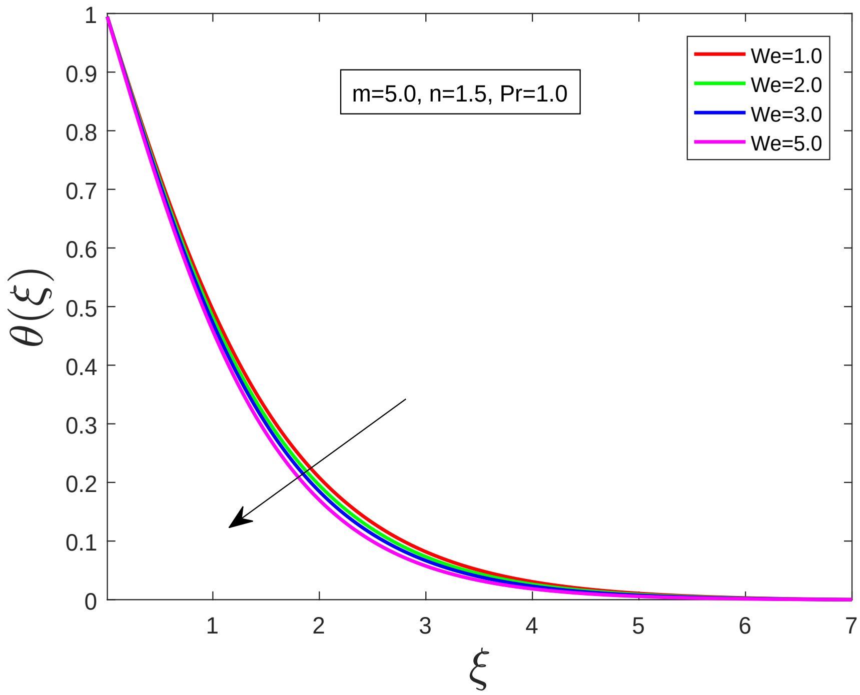

FIGURE 3.13: Impact of We on $\theta(\xi)$ along with $n = 1.5$.

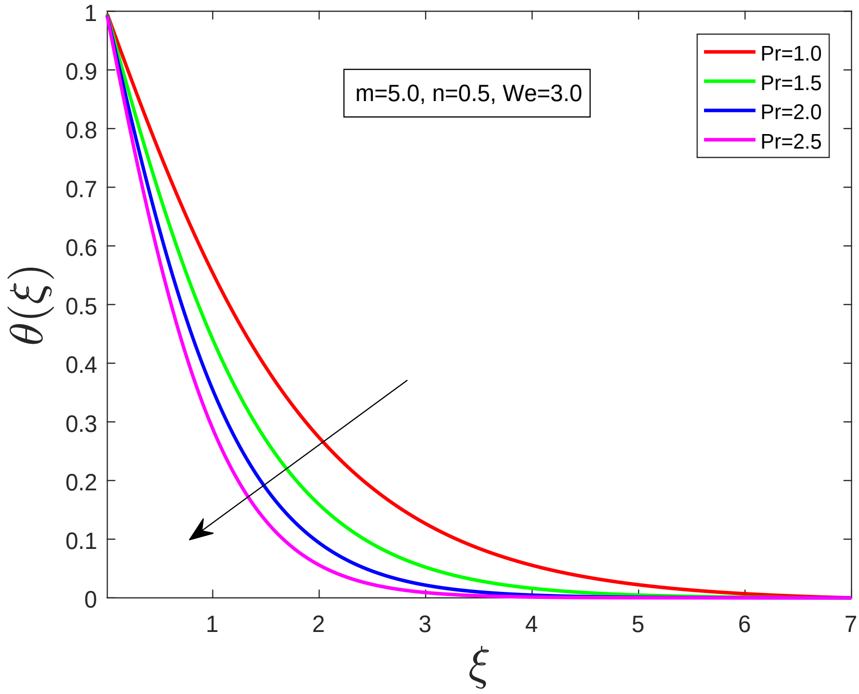

FIGURE 3.14: Impact of Pr on $\theta(\xi)$ along with $n = 0.5$.

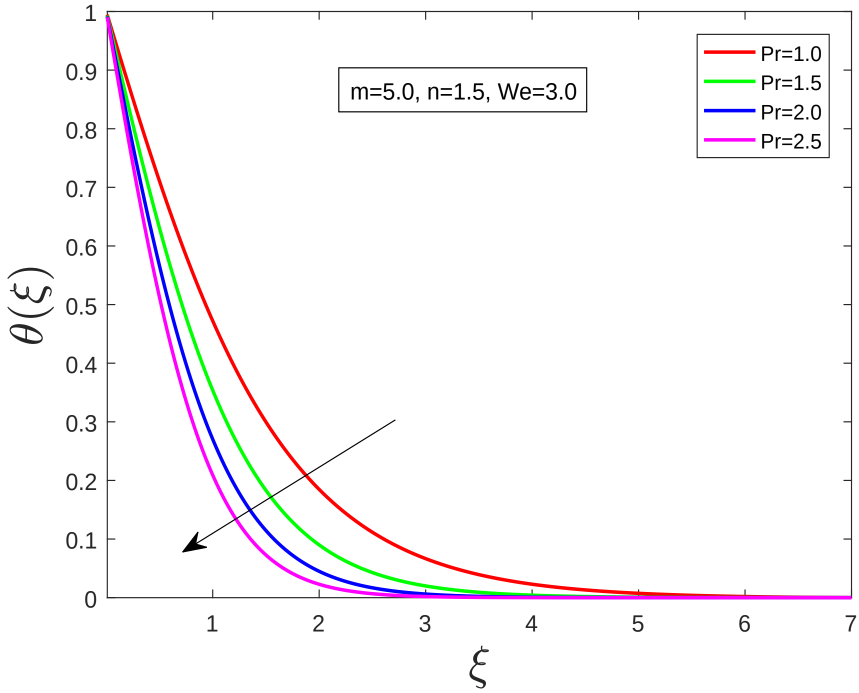

FIGURE 3.15: Impact of Pr on $\theta(\xi)$ along with $n = 1.5$.

Chapter 4

Flow and Heat Transfer of Casson Fluid over a Non-Linear Stretching Sheet

4.1 Introduction

This section contains the extension of the model [34] by considering Casson fluid in momentum equation. We also include thermal radiation and viscous dissipation in temperature equation. Furthermore, by using the similarity transformations, the non-linear PDEs are transformed into a system of ODEs. The numerical solution of ODEs is obtaind by applying numerical technique known as shooting method. At the end of this chapter, the final results are discussed for significant parameters that have impact on the $f'(\xi)$ and $\theta(\xi)$ which are shown in tables and graphs.

4.2 Mathematical Modeling

Assume a 2-D incompressible flow of a Casson fluid due to the extending surface with y=0. The flow is considered along y-axis with $y > 0$. Hence, T_w is the uniform temperature, T_∞ the ambient temperature of the fluid and $u_w = bx^m$ is the non-linear velocity. The constants b and m are positive real numbers describe to the stretching speed.

The system of equations describing the flow are given below.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{4.1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left(1 + \frac{1}{\beta}\right)\frac{\partial^2 u}{\partial y^2}, \tag{4.2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} + \frac{\partial q_r}{\partial y} + \frac{\nu}{C_p}\left(1 + \frac{1}{\beta}\right)\left(\frac{\partial u}{\partial y}\right)^2. \tag{4.3}$$

The associated BCs have been taken as:

$$\left.\begin{array}{l} u = \boldsymbol{bx^m}, \quad v = 0, \quad T = T_w \quad at \quad y = 0. \\ u \to 0, \quad T \to T_\infty \quad as \quad y \to \infty. \end{array}\right\} \tag{4.4}$$

The Casson parameter is denoted by β, density by ρ, the kinematic viscosity by ν, the thermal diffusivity by α.

For the conversion of the mathematical model (4.1)-(4.3) into the ODEs, we introduced the following similarity transformation.

$$\left.\begin{array}{l} \psi(x,y) = \sqrt{\dfrac{2\nu b}{m+1}}x^{\frac{m+1}{2}}f(\xi), \qquad \theta(\xi) = \dfrac{T - T_\infty}{T_w - T_\infty}, \\ \xi = y\sqrt{\dfrac{b(m+1)}{2\nu}}x^{\frac{m-1}{2}}. \end{array}\right\} \tag{4.5}$$

where ψ denotes the stream function.

The detailed procedure for the conversion of (4.1) has been discussed in Chapter 3.

Now, we include the below procedure for the conversion of (4.2) into the dimensionless form.

- $$u\frac{\partial u}{\partial x} = b^2x^{2m-1}mf'^2(\xi) + b^2x^{2m-1}\left(\frac{m-1}{2}\right)\xi f'(\xi)f''(\xi). \tag{4.6}$$
- $$v\frac{\partial u}{\partial y} = -\frac{b^2(m+1)}{2}x^{2m-1}f'(\xi)f''(\xi)\xi\left(\frac{m-1}{m+1}\right) - \frac{b^2(m+1)}{2}x^{2m-1}f(\xi)f''(\xi). \tag{4.7}$$

- $$\frac{\partial^2 u}{\partial y^2} = b^2 x^{2m-1} f'''(\xi)\left(\frac{m+1}{2\nu}\right). \tag{4.8}$$

Using (4.6)-(4.7) in left side of (4.2), we get

$$\begin{aligned}
& bf'(\xi)x^m bx^{m-1}\left[mf'(\xi)+\xi f''(\xi)\left(\frac{m-1}{2}\right)\right] \\
& -\sqrt{\frac{\nu b(m+1)}{2}}x^{\frac{m-1}{2}}\left[f(\xi)+\xi f''(\xi)\left(\frac{m-1}{m+1}\right)\right] \\
& \left[bx^m f''(\xi)\sqrt{\frac{b(m+1)}{2\nu}}x^{\frac{m-1}{2}}\right], \\
& = b^2 f'(\xi)x^{2m-1}mf'(\xi)+b^2 f'(\xi)x^{2m-1}\xi f''(\xi)\left(\frac{m-1}{2}\right) \\
& -\left[\sqrt{\frac{(m+1)\nu b}{2}}x^{\frac{m-1}{2}}f(\xi)-\sqrt{\frac{(m+1)\nu b}{2}}x^{\frac{m-1}{2}}\xi f'(\xi)\left(\frac{m-1}{m+1}\right)\right] \\
& \left[bx^m\sqrt{\frac{(m+1)b}{2}}x^{\frac{m-1}{2}}f''(\xi)\right], \\
& = b^2 f'(\xi)x^{2m-1}mf'(\xi)+b^2 f'(\xi)x^{2m-1}\xi f''(\xi)\left(\frac{m-1}{2}\right) \\
& -\sqrt{\frac{(m+1)\nu b}{2}}x^{\frac{m-1}{2}}f(\xi)bx^m\sqrt{\frac{(m+1)b}{2}}x^{\frac{m-1}{2}}f''(\xi) \\
& -\sqrt{\frac{(m+1)\nu b}{2}}x^{\frac{m-1}{2}}\xi f'(\xi)\left(\frac{m-1}{m+1}\right)bx^m\sqrt{\frac{(m+1)b}{2}}x^{\frac{m-1}{2}}f''(\xi), \\
& = b^2 f'(\xi)x^{2m-1}mf'(\xi)+b^2 f'(\xi)x^{2m-1}\xi f''(\xi)\left(\frac{m-1}{2}\right) \\
& -\left(\frac{(m+1)b^2}{2}\right)x^{2m-1}f(\xi)f''(\xi) \\
& -\left(\frac{(m+1)b^2}{2}\right)x^{2m-1}\xi f'(\xi)f''(\xi)\left(\frac{m-1}{m+1}\right), \\
& = b^2x^{2m-1}\left(f'(\xi)mf'(\xi)+f'(\xi)\xi f''(\xi)\left(\frac{m-1}{2}\right)\right) \\
& -\left(\frac{(m+1)b^2}{2}\right)x^{2m-1}f(\xi)f''(\xi) \\
& -\left(\frac{(m+1)b^2}{2}\right)x^{2m-1}\xi f'(\xi)f''(\xi)\left(\frac{m-1}{m+1}\right), \\
& = b^2x^{2m-1}\left(f'^2(\xi)m+\xi f'(\xi)f''(\xi)\left(\frac{m-1}{2}\right)-\left(\frac{m+1}{2}\right)f(\xi)f''(\xi)\right) \\
& \quad - b^2x^{2m-1}\left(\left(\frac{m+1}{2}\right)\xi f'(\xi)f''(\xi)\left(\frac{m-1}{m+1}\right)\right).
\end{aligned} \tag{4.9}$$

Using (4.8) in right side of (4.2), we get

$$
\begin{aligned}
&= -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left(1+\frac{1}{\beta}\right)\frac{\partial^2 u}{\partial y^2},\\
&= \nu\left(1+\frac{1}{\beta}\right)b^2x^{2m-1}f'''(\xi)\left(\frac{m+1}{2\nu}\right). \qquad (4.10)
\end{aligned}
$$

Comparing (4.9) and (4.10), the dimensionless form of (4.2) can be written as

$$
\begin{aligned}
& f'^2(\xi)m + \xi f'(\xi)f''(\xi)\left(\frac{m-1}{2}\right) - f(\xi)f''(\xi)\left(\frac{m+1}{2}\right)\\
& - \left(\frac{m+1}{2}\right)\xi f'(\xi)f''(\xi)\left(\frac{m-1}{m+1}\right) = \nu(1+\frac{1}{\beta})\left(\frac{m+1}{2\nu}\right)f'''(\xi),\\
\Rightarrow\quad & f'^2(\xi)\left(\frac{2m\nu}{m+1}\right) + f'(\xi)f''(\xi)\left(\frac{(m-1)\nu}{m+1}\right) - \nu f(\xi)f''(\xi)\\
& - f'(\xi)f''(\xi)\left(\frac{\nu(m-1)}{m+1}\right) = \nu f'''(\xi)\left(1+\frac{1}{\beta}\right),\\
\Rightarrow\quad & \nu\left(1+\frac{1}{\beta}\right)f'''(\xi) - f'^2(\xi)\left(\frac{2m\nu}{m+1}\right) + \nu f(\xi)f''(\xi) = 0,\\
\Rightarrow\quad & \left(1+\frac{1}{\beta}\right)f'''(\xi) - f'^2(\xi)\left(\frac{2m}{m+1}\right) + f(\xi)f''(\xi) = 0,\\
\Rightarrow\quad & \left(1+\frac{1}{\beta}\right)f''' - f'^2\left(\frac{2m}{m+1}\right) + ff'' = 0. \qquad (4.11)
\end{aligned}
$$

The following derivatives will help to convert the left side of (4.3) into the dimensionless form:

- $$
\begin{aligned}
\frac{\partial T}{\partial x} &= (T_w - T_\infty)\theta'(\xi)\frac{\partial \xi}{\partial x},\\
&= (T_w - T_\infty)y\sqrt{\frac{b(m+1)}{2\nu}}x^{\frac{m-3}{2}}\frac{m-1}{2}\theta'(\xi),\\
\frac{\partial T}{\partial x} &= (T_w - T_\infty)\left(\frac{m-1}{2x}\right)\xi\theta'(\xi). \qquad (4.12)
\end{aligned}
$$
- $$\frac{\partial^2 T}{\partial y^2} = (T_w - T_\infty)\left(\frac{b(m+1)}{2\nu}\right)x^{m-1}\theta''(\xi). \qquad (4.13)$$
- $$u\frac{\partial T}{\partial x} = bx^{m-1}(T_w - T_\infty)\left(\frac{m-1}{2}\right)\xi f'(\xi)\theta'(\xi). \qquad (4.14)$$
- $$
\begin{aligned}
v\frac{\partial T}{\partial y} &= -bx^{m-1}\left(\frac{m+1}{2}\right)\left(\frac{m-1}{m+1}\right)(T_w - T_\infty)\xi\theta'(\xi)f'(\xi)\\
&\quad - bx^{m-1}\left(\frac{m+1}{2}\right)(T_w - T_\infty)\theta'(\xi)f(\xi). \qquad (4.15)
\end{aligned}
$$

- $$\alpha\frac{\partial^2 T}{\partial y^2} = \alpha(T_w - T_\infty)\left(\frac{(m+1)b}{2\nu}\right)x^{m-1}\theta''(\xi). \tag{4.16}$$

- $$\begin{aligned}\frac{\partial q_r}{\partial y} + \frac{\nu}{C_p}\left(1+\frac{1}{\beta}\right)\left(\frac{\partial u}{\partial y}\right)^2 &= -\frac{16\sigma^\star T_0^3}{3k^\star}(T_w - T_\infty)\left(\frac{(m+1)b}{2\nu}\right)x^{m-1}\theta''(\xi)\\ &\quad + \frac{\nu}{C_p}\left(1+\frac{1}{\beta}\right)b^3x^{3m-1}f''^2(\xi)\left(\frac{m+1}{2\nu}\right),\\ &= bx^{m-1}\left(\frac{16\sigma^\star T_0^3}{3k^\star}(T_w - T_\infty)\left(\frac{m+1}{2\nu}\right)\theta''(\xi)\right)\\ &\quad + bx^{m-1}\left(\frac{\nu}{C_p}\left(1+\frac{1}{\beta}\right)b^2x^{2m}f''^2(\xi)\left(\frac{m+1}{2\nu}\right)\right).\end{aligned} \tag{4.17}$$

Using (4.14) and (4.15) in left side of (4.3), we get

$$\begin{aligned}
&= bf'(\xi)x^m(T_w - T_\infty)\left(\frac{m-1}{2x}\right)\xi\theta'(\xi)\\
&\quad - \sqrt{\frac{(m+1)\nu b}{2}}x^{\frac{m-1}{2}}\left[f(\xi) + \left(\frac{m-1}{m+1}\right)\xi f'(\xi)\right]\\
&\quad (T_w - T_\infty)\sqrt{\frac{(m+1)b}{2\nu}}x^{\frac{m-1}{2}}\theta'(\xi),\\
\Rightarrow\quad & bx^{m-1}(T_w - T_\infty)\left(\frac{m-1}{2}\right)\xi f'(\xi)\theta'(\xi)\\
&\quad - x^{\frac{m-1}{2}}\sqrt{\frac{(m-1)\nu b}{2}}\left(\frac{m-1}{m+1}\right)(T_w - T_\infty)\\
&\quad x^{\frac{m-1}{2}}\sqrt{\frac{b(m+1)}{2\nu}}\xi\theta'(\xi)\left(\frac{m-1}{m+1}\right)f'(\xi)\\
&\quad - x^{\frac{m-1}{2}}\sqrt{\frac{\nu b(m+1)}{2}}(T_w - T_\infty)\sqrt{\frac{(m+1)b}{2\nu}}x^{\frac{m-1}{2}}\theta'(\xi)f(\xi),\\
\Rightarrow\quad & bx^{m-1}(T_w - T_\infty)\left(\frac{m-1}{2}\right)\xi f'(\xi)\theta'(\xi)\\
&\quad - bx^{m-1}\left(\frac{m+1}{2}\right)\left(\frac{m-1}{m+1}\right)(T_w - T_\infty)\xi\theta'(\xi)f'(\xi)\\
&\quad - bx^{m-1}\left(\frac{m+1}{2}\right)(T_w - T_\infty)\theta'(\xi)f(\xi),\\
\Rightarrow\quad & bx^{m-1}\left[(T_w - T_\infty)\left(\frac{m-1}{2}\right)\xi f'(\xi)\theta'(\xi)\right]\\
&\quad - bx^{m-1}\left[\left(\frac{m-1}{2}\right)(T_w - T_\infty)\xi\theta'(\xi)f'(\xi)\right]\\
&\quad - bx^{m-1}\left[\left(\frac{m+1}{2}\right)(T_w - T_\infty)\theta'(\xi)f(\xi)\right],
\end{aligned}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = -bx^{m-1}\left(\frac{m+1}{2}\right)(T_w - T_\infty)\theta'(\xi)f(\xi), \tag{4.18}$$

Using (4.16) and (4.17) in right side of (4.3), we get

$$
\begin{aligned}
&\alpha\frac{\partial^2 T}{\partial y^2}+\frac{\partial q_r}{\partial y}+\frac{\nu}{C_p}\left(1+\frac{1}{\beta}\right)\left(\frac{\partial u}{\partial y}\right)^2,\\
\Rightarrow\quad &\alpha(T_w-T_\infty)\left(\frac{(m+1)b}{2\nu}\right)x^{m-1}\theta''(\xi)\\
&-\frac{16\sigma^\star T_0^3}{3k^\star}(T_w-T_\infty)\left(\frac{(m+1)b}{2\nu}\right)x^{m-1}\theta''(\xi)\\
&+\frac{\nu}{C_p}\left(1+\frac{1}{\beta}\right)b^3x^{3m-1}f''^2(\xi)\left(\frac{m+1}{2\nu}\right),\\
\Rightarrow\quad &bx^{m-1}\left[\alpha(T_w-T_\infty)\left(\frac{m+1}{2\nu}\right)\theta''(\xi)-\frac{16\sigma^\star T_0^3}{3k^\star}(T_w-T_\infty)\left(\frac{m+1}{2\nu}\right)\theta''(\xi)\right]\\
&+bx^{m-1}\left[\frac{\nu}{C_p}\left(1+\frac{1}{\beta}\right)b^2x^{2m}f''^2(\xi)\left(\frac{m+1}{2\nu}\right)\right],
\end{aligned}
\tag{4.19}
$$

With the help of (4.18) and (4.19), the dimensionless form of equation (4.3), can be seen as

$$
\begin{aligned}
\Rightarrow\quad &-bx^{m-1}\left(\frac{m+1}{2}\right)(T_w-T_\infty)\theta'(\xi)f(\xi)=\\
&bx^{m-1}\left[\alpha(T_w-T_\infty)\left(\frac{m+1}{2\nu}\right)\theta''(\xi)\right]\\
&-bx^{m-1}\left[\frac{16\sigma^\star T_0^3}{3k^\star}(T_w-T_\infty)\left(\frac{m+1}{2\nu}\right)\theta''(\xi)\right]\\
&+bx^{m-1}\left[\frac{\nu}{C_p}\left(1+\frac{1}{\beta}\right)b^2x^{2m}f''^2(\xi)\left(\frac{m+1}{2\nu}\right)\right],\\
\Rightarrow\quad &-(T_w-T_\infty)\theta'(\xi)f(\xi)=\alpha(T_w-T_\infty)\frac{1}{\nu}\theta''(\xi)\\
&-\frac{16\sigma^\star T_0^3}{3k^\star}(T_w-T_\infty)\frac{1}{\nu}\theta''(\xi)\\
&+\frac{\nu}{C_p}\left(1+\frac{1}{\beta}\right)b^2x^{2m}f''^2(\xi)\frac{1}{\nu},\\
\Rightarrow\quad &-f(\xi)\theta'(\xi)=\frac{\alpha}{\nu}\theta''(\xi)-\frac{16\sigma^\star T_0^3}{3K^\star\nu}\theta''(\xi)\\
&+\left(1+\frac{1}{\beta}\right)\frac{B^2x^{2m}}{C_p(T_w-T_\infty)}f''^2(\xi),\\
\Rightarrow\quad &-Pr\,f(\xi)\theta'(\xi)=\theta''(\xi)-\frac{16\sigma^\star T_0^3}{3K^\star\alpha}\theta''(\xi)\\
&+Pr\left(1+\frac{1}{\beta}\right)Ecf''^2(\xi),\\
\Rightarrow\quad &-Pr\,f\theta'=\left(1-\frac{4}{3}R\right)\theta''+Pr\left(1+\frac{1}{\beta}\right)Ecf''^2,
\end{aligned}
$$

$$\Rightarrow \qquad \left(1-\frac{4}{3}R\right)\theta''+Prf\theta'+Pr\left(1+\frac{1}{\beta}\right)Ecf''^2=0. \tag{4.20}$$

The final dimensionless form of the model is

$$\left(1+\frac{1}{\beta}\right)f'''+ff''-f'^2\left(\frac{2m}{m+1}\right)=0. \tag{4.21}$$

$$\left(1-\frac{4}{3}R\right)\theta''+Prf\theta'+Pr\left(1+\frac{1}{\beta}\right)Ecf''^2=0. \tag{4.22}$$

Different parameters used in (4.21)-(4.22) are explained as follows:

$$Pr=\frac{\nu}{\alpha}, \qquad R=\frac{4\sigma^\star T_0^3}{K^\star\alpha},$$
$$Ec=\frac{U_w^2}{C_p(T_w-T_\infty)}.$$

Dimensionless form of (4.4) is given as follows:

- $$u=U_w(x)=bx^m, \qquad at \quad y=0$$

$$\Rightarrow \quad u=bf'(\xi)x^m.$$

$$\Rightarrow \quad bf'(\xi)x^m=bx^m.$$

$$\Rightarrow \quad f'(\xi)=1 \qquad at \quad \xi=0.$$

$$\Rightarrow \quad f'(0)=1.$$

- $$v=0, \qquad at \quad y=0.$$

$$\Rightarrow \quad -x^{\frac{m-1}{2}}\sqrt{\frac{(m+1)\nu b}{2}}\left(f(\xi)+\xi f'(\xi)\left(\frac{m-1}{m+1}\right)\right)=0.$$

$$\Rightarrow \quad -x^{\frac{m-1}{2}}\sqrt{\frac{(m+1)\nu b}{2}}f(0)=0 \qquad at \quad \xi=0.$$

$$\Rightarrow \quad f(0)=0.$$

- $$T=T_w, \qquad at \quad y=0.$$

$$\Rightarrow \quad \theta(\xi)(T_w-T_\infty)+T_\infty=T_w.$$

$$\Rightarrow \quad \theta(\xi)(T_w-T_\infty)=(T_w-T_\infty)$$

$$\Rightarrow \quad \theta(\xi)=1 \qquad at \quad \xi=0$$

$$\Rightarrow \quad \theta(0)=1.$$

$$
\begin{array}{lll}
\bullet & u \to 0 & as \quad y \to \infty \\
\Rightarrow & bf'(\xi)x^m \to 0. & \\
\Rightarrow & f'(\xi) \to 0 & as \quad \xi \to \infty \\
\Rightarrow & f'(\xi) \to 0. & \\
\bullet & T \to T_\infty & as \quad y \to \infty \\
\Rightarrow & \theta(\xi)(T_w - T_\infty) + T_\infty \to T_\infty & \\
\Rightarrow & \theta(\xi)(T_w - T_\infty) \to 0. & \\
\Rightarrow & \theta(\xi) \to 0 & as \quad \xi \to \infty \\
\Rightarrow & \theta(\infty) \to 0. &
\end{array}
$$

The skin friction coefficient is defined as:

$$
Cf_x = \frac{\tau_w|_{y=0}}{\rho U_w^2(x)}. \tag{4.23}
$$

To achive the dimensionless form of of Cf_x, the following step will be helpful.

$$
\tau_w = \mu_B \left(1 + \frac{1}{\beta}\right) (\frac{\partial u}{\partial y})_{y=0} \tag{4.24}
$$

Using (4.23) and (4.24), we get the following form:

$$
\begin{aligned}
Cf_x &= \frac{bx^m \mu_B}{\rho U^2} \left(1 + \frac{1}{\beta}\right) \sqrt{\frac{b(m+1)}{2\nu}} x^{\frac{m-1}{2}} f''(0), \\
&= \frac{bx^m \mu_B}{\rho b^2 x^{2m}} \left(1 + \frac{1}{\beta}\right) \sqrt{\frac{b(m+1)}{2\nu}} x^{\frac{m-1}{2}} f''(0), \\
&= \frac{\mu_B}{\rho b x^m} \left(1 + \frac{1}{\beta}\right) \sqrt{\frac{b(m+1)}{2\nu}} x^{\frac{m-1}{2}} f''(0), \\
&= \frac{\mu_B \nu}{\mu_B b x^m} \left(1 + \frac{1}{\beta}\right) \sqrt{\frac{b(m+1)}{2\nu}} x^{\frac{m-1}{2}} f''(0), \\
&= \frac{\nu(\frac{b}{\nu})^{\frac{1}{2}}}{b x^m} \left(1 + \frac{1}{\beta}\right) \sqrt{\frac{m+1}{2}} x^{\frac{m-1}{2}} f''(0), \\
&= \frac{\nu^{\frac{1}{2}}}{b^{\frac{1}{2}} x^{\frac{m+1}{2}}} \left(1 + \frac{1}{\beta}\right) \sqrt{\frac{m+1}{2}} f''(0), \\
\sqrt{\frac{bx^{m+1}}{\nu}} &= \left(1 + \frac{1}{\beta}\right) \sqrt{\frac{m+1}{2}} f''(0),
\end{aligned}
$$

$$Re^{\frac{1}{2}}Cf_x = \left(1+\frac{1}{\beta}\right)\sqrt{\frac{m+1}{2}}f''(0). \tag{4.25}$$

To achive the dimensionless form of Nu_x, the following step will be helpful.

$$Re^{\frac{-1}{2}}Nu_x = -\sqrt{\frac{m+1}{2}}\theta'(0). \tag{4.26}$$

4.3 Method of Solution

The shooting method has been used to solved ODEs (4.21). Equation (4.21) is solved numerically and then its solution is used in (4.22). To solve the (4.21) independently by using shooting method, The following notations have been considered:

$$f = Z_1, \qquad f' = Z_1' = Z_2,$$
$$f'' = Z_1'' = Z_2' = Z_3, \quad f''' = Z_3'.$$

For simplification, the following notation have been defined.

$$A_2 = \left(1+\frac{1}{\beta}\right)$$

By using the above notations, the following system of ODEs is obtained:

$$\begin{aligned} Z_1' &= Z_2, & Z_1(0) &= 0,\\ Z_2' &= Z_3, & Z_2(0) &= 1,\\ Z_3' &= \frac{1}{A_2}\left(\frac{2m}{m+1}Z_2^2 - Z_1Z_3\right), & Z_3(0) &= p. \end{aligned}$$

The above initial value problem will be numerically solved by RK technique of order four. The above initial value problem, the missing condition p are to be chosen such that:

$$Z_2(\xi_\infty, p) = 0.$$

To solve the above algebric equation. We use the Newton's method which has the following iterative scheme:

$$p^{k+1} = p^k - \frac{G(p)}{G'(p)}, \quad \text{where} \quad G(p) = Z_2(\xi_\infty, p).$$

Let us now consider the following new notations:

$$\frac{\partial Z_1}{\partial p} = Z_4, \quad \frac{\partial Z_2}{\partial p} = Z_5, \quad \frac{\partial Z_3}{\partial p} = Z_6.$$

$$Z_4' = Z_5, \qquad Z_4(0) = 0,$$

$$Z_5' = Z_6, \qquad Z_5(0) = 0,$$

$$Z_6' = \frac{1}{A_2}\left(\frac{4m}{m+1}Z_2Z_5 - Z_1Z_6 - Z_3Z_4\right), \qquad Z_6(0) = 1.$$

Also, for equation (4.22), the following notation have been defined.

$$\theta = X_1, \quad \theta' = X_1' = X_2, \quad \theta'' = X_2'.$$

$$A_1 = \left(1 - \frac{4}{3}R\right), \quad A_2 = \left(1 + \frac{1}{\beta}\right).$$

The system of equation (4.22), can be written in the form of the following first order coupled ODEs.

$$X_1' = X_2, \qquad X_1(0) = 0,$$

$$X_2' = \frac{1}{A_1}(-PrX_2f - Pr(A_2)Ec(f'')^2), \qquad X_2(0) = q.$$

To incorporate Newton's method, we further apply the following notation:

$$\frac{\partial X_1}{\partial q} = X_3, \quad \frac{\partial X_2}{\partial q} = X_4.$$

As the result of these new notations,

$$X_3' = X_4, \qquad X_3(0) = 0,$$

$$X_4' = \frac{1}{A_1}(-PrX_4f), \qquad X_4(0) = 1.$$

The above IVP will be solved numerically by the RK method. To get the approximate solution.

$$X_1(\xi_\infty, q) = 0.$$

The above set of equations can be solved by using Newtons method with following iterative formula:

$$\left(q^{k+1}\right) = \left(q^k\right) - \left(\frac{\partial X_1}{\partial q}\right)^{-1} \left(X_1, q^k\right),$$
$$\left(q^{k+1}\right) = \left(q^k\right) - \left(X_3\right)^{-1} \left(X_1, q^k\right).$$

The RK method has been used to solve the IVP consisting of the above first order ODEs for some suitable choice of q. Stopping criteria for the shooting method is set as

$$|X_1(\xi_\infty)| < \epsilon,$$

where ϵ is an arbitrarily small positive number. Here ϵ is taken as $(10)^{-10}$.

4.4 Representation of Graphs and Tables

A thorough discussion on the graphs and tables has been conducted which contains the impact of velocity and temperature distribution. The impact of different factors like stretching parameter m, thermal radiation R, Prandtle number Pr, Casson parameter β and Eckert number Ec is observed graphically. Numerical results of the skin friction coefficient and Nusselt number for the distinct values of some fixed parameters are shown in Table 4.1-4.2.

Figure 4.1 represents the influence of Casson parameter β on the dimensionless velocity distribution. It can be noted that, the velocity profile is decreased by enhancing the values of β. Figure 4.2 delineated to show the impact of Casson parameter β on the dimensionless velocity distribution. It is clearly shown that the velocity profile is decreased by enhancing the value of β. Figures 4.3-4.4 delineated to show the impact of Casson parameter β on the dimensionless temperature distribution. It is clearly shown that the temperature distribution is increased by enhancing the values of β.

Figure 4.5 displays the impact of stretching parameter m on velocity distribution. By enhancing the values of m, the velocity distribution show the decreasing behavior. Figure 4.6 illustrates the impact of stretching parameter m on velocity distribution. This graph indicates that an increment in the values of m causes an decrement the velocity distribution. Figure 4.7 describes the impact of stretching parameter m on temperature distribution. By enhancing the values of m, the temperature distribution is increased. Figure 4.8 shows the impact of stretching parameter m on the dimensionless temperature distribution. This graph indicates that an increment in the values of m for shear thinning fluid.

Figures 4.9-4.10 is delineated to show the impact of Prandtl number Pr on temperature distribution. It is clearly shown that the temperature distibution is decreased by enhancing the values of Pr, the thermal boundary layer thickness decreases for both shear thickening and shear thinning fluids. Figure 4.11 describes the impact of Thermal radiation R on temperature distribution. By inhancing the value of R, the temperature distribution is decreased.

TABLE 4.1: Results of $-(Re)^{\frac{1}{2}}Cf_x$ for various parameters

m	β	$-(Re)^{\frac{1}{2}}Cf_x$
1	0.2	2.491782
2	0.2	3.346079
3	0.2	4.024429
1	0.4	1.881601
2	0.4	2.533696
3	0.4	3.050502
1	0.8	1.503013
2	0.8	2.025789
3	0.8	2.439841
1	2.5	1.183840
2	2.5	1.596175
1	5	1.095798
2	5	1.477537

TABLE 4.2: Results of $-(Re)^{\frac{-1}{2}}Nu_x$ for various parameters

m	β	Pr	Ec	R	$-(Re)^{\frac{-1}{2}}Nu_x$
1	0.2	1	0.2	1	2.391879
2	0.2	1	0.2	1	2.675449
3	0.2	1	0.2	1	2.989233
1	0.2	1	0.2	1	2.391879
1	0.4	1	0.2	1	1.198522
1	0.8	1	0.2	1	0.748903
1	0.2	0.5	0.2	1	1.564140
1	0.2	1	0.2	1	2.391879
1	0.2	1.5	0.2	1	2.906382
1	0.2	1	0	1	0.021453
1	0.2	1	0.2	1	2.391879
1	0.2	1	0.5	1	6.011878
1	0.2	1	0.2	0	-1.357192
1	0.2	1	0.2	0.5	-2.763368

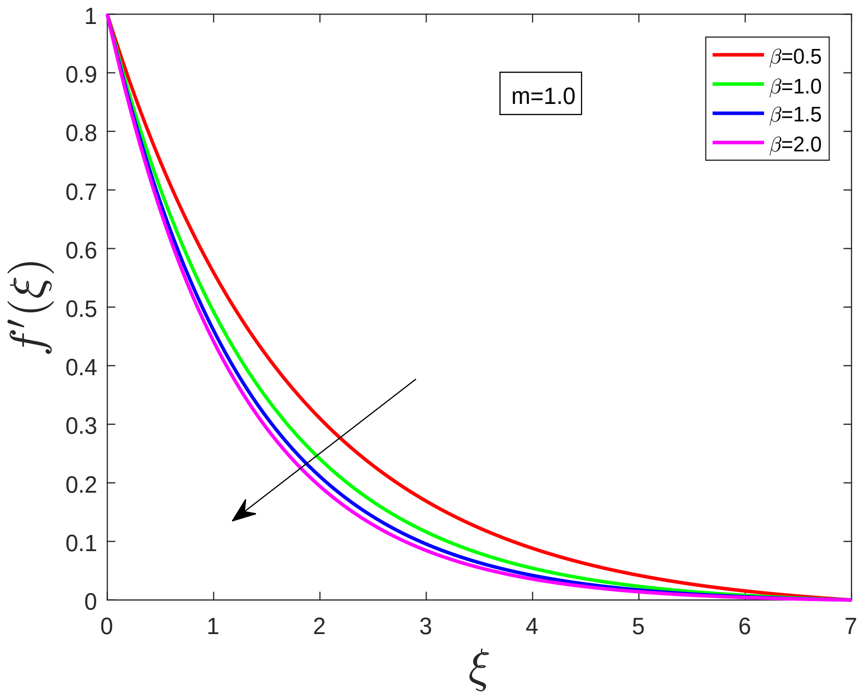

FIGURE 4.1: Effect of Casson Parameter β on Velocity Profile along with $m = 1.0$.

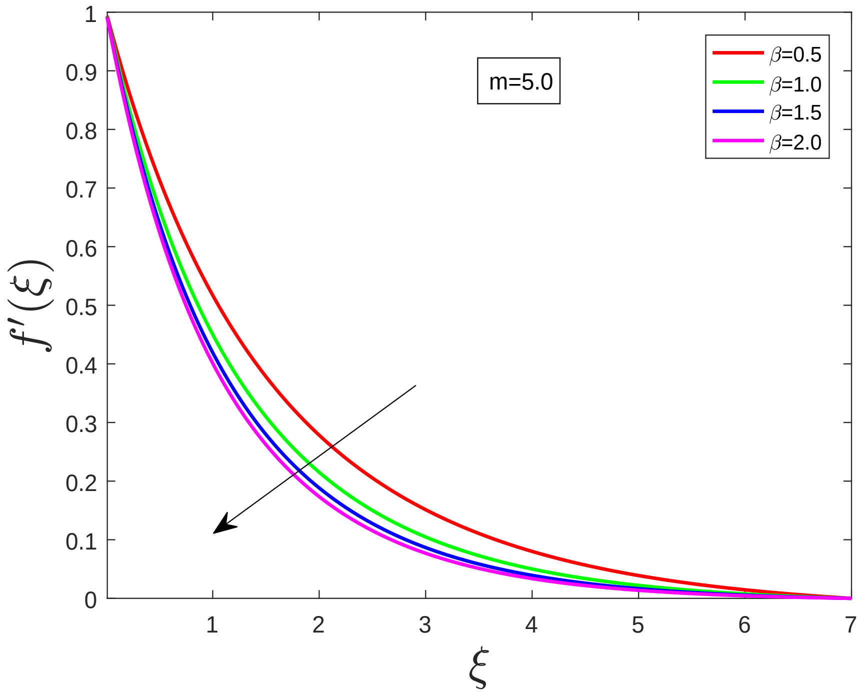

FIGURE 4.2: Effect of Casson Parameter β on Velocity Profile along with $m = 5.0$.

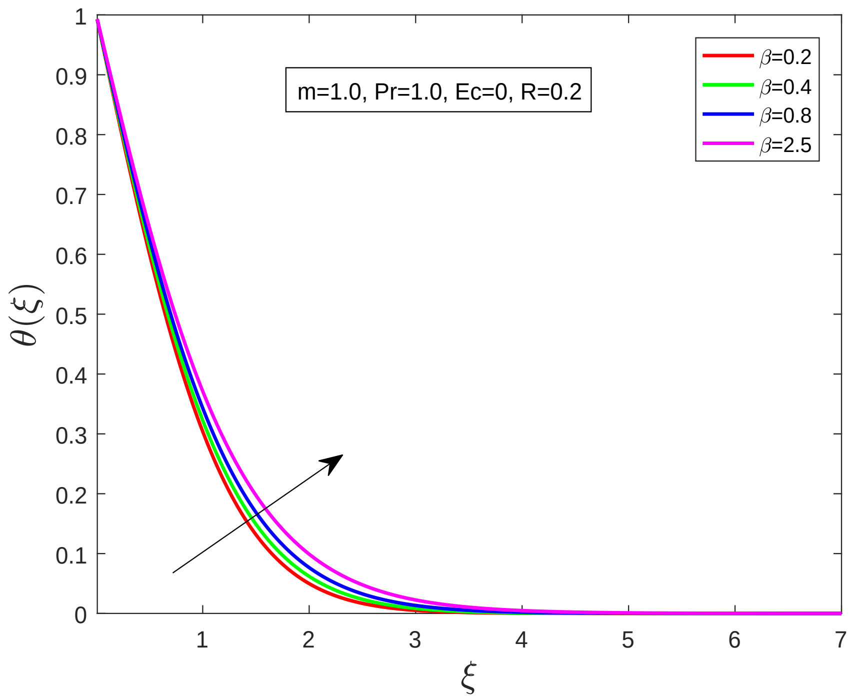

FIGURE 4.3: Effect of Casson Parameter β on Temperature Profile along with $m = 1.0$.

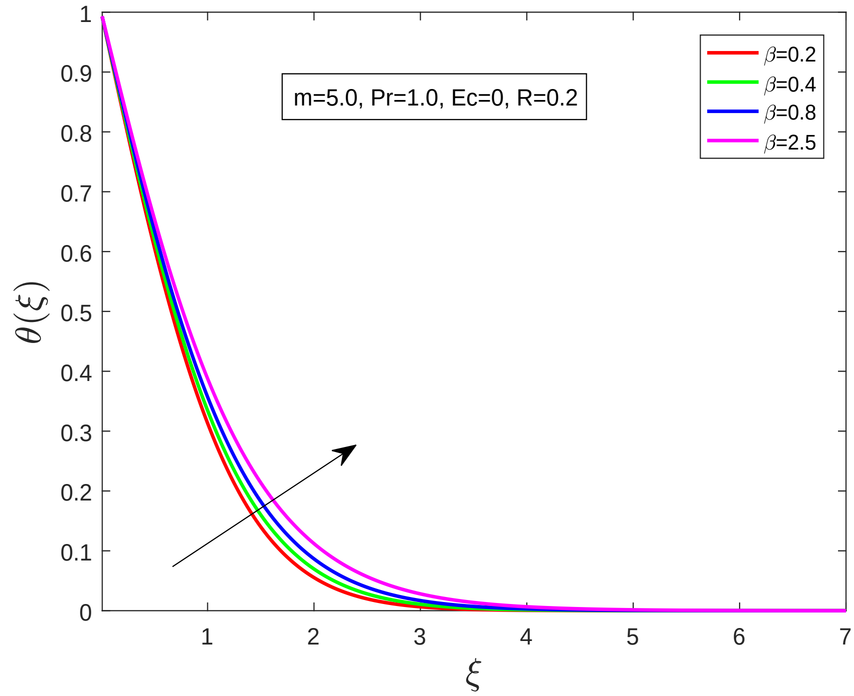

FIGURE 4.4: Effect of Casson Parameter β on Temperature Profile along with $m = 5.0$.

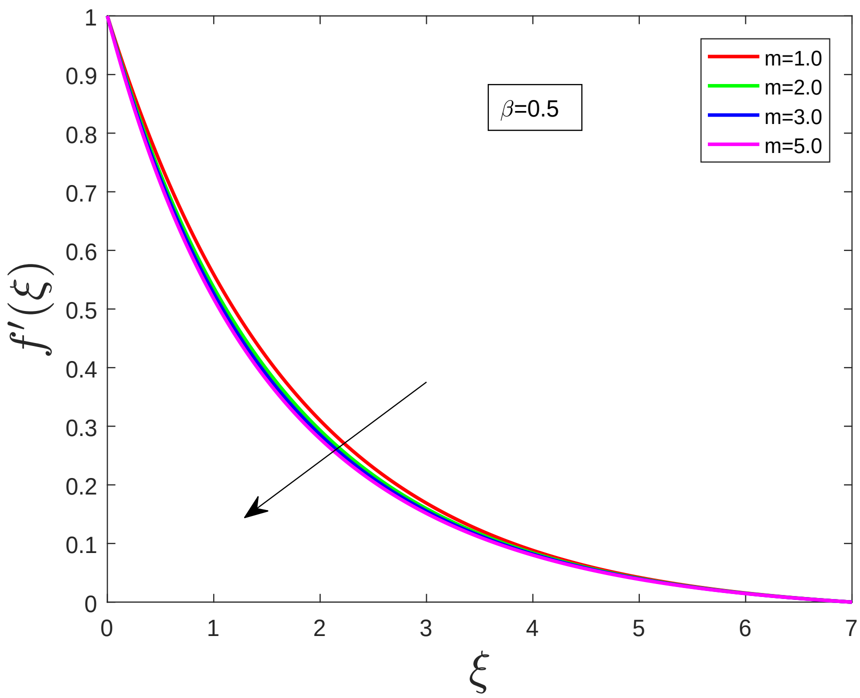

FIGURE 4.5: Effect of Stretching Parameter m on Velocity Profile along with $\beta = 0.5$.

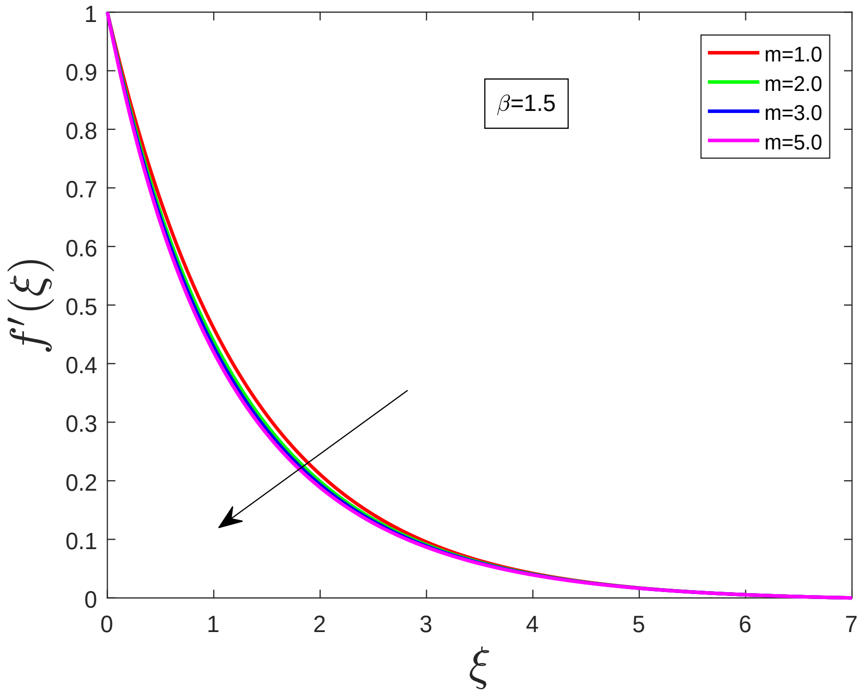

FIGURE 4.6: Effect of Stretching Parameter m on Velocity Profile along with $\beta = 1.5$.

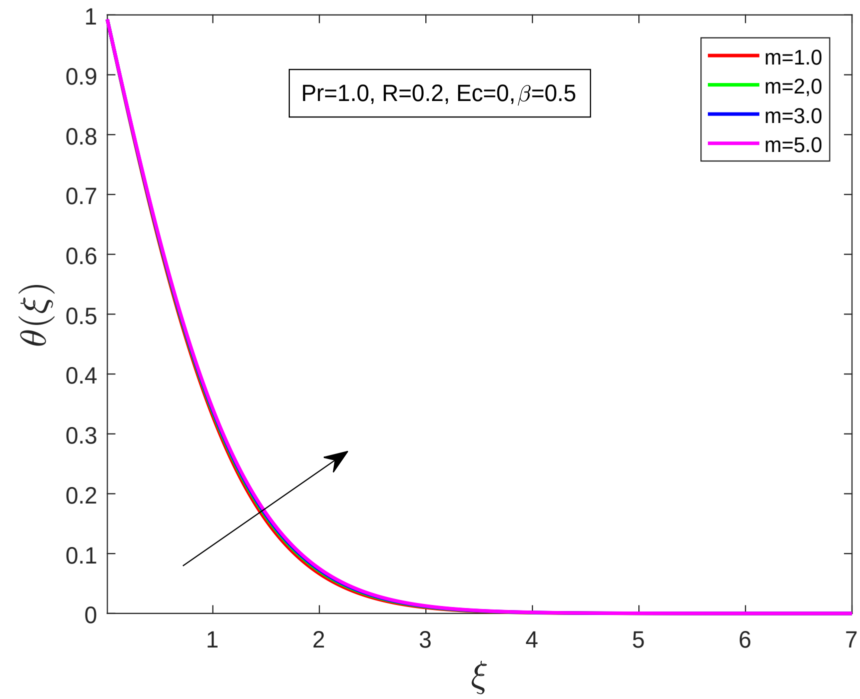

FIGURE 4.7: Effect of Stretching Parameter m on Temperature Profile along with $Pr = 1.0$.

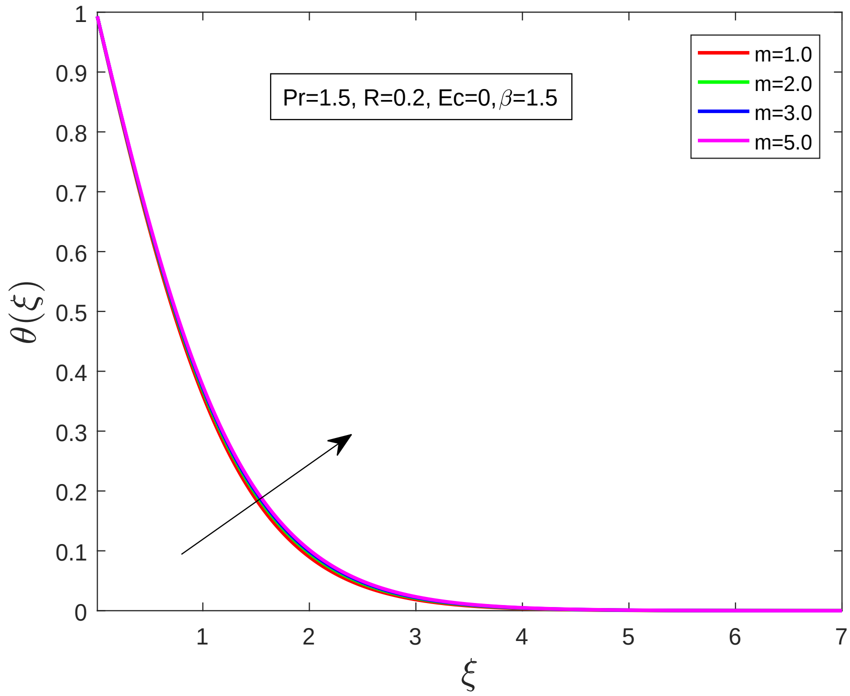

FIGURE 4.8: Effect of Stretching Parameter m on Temperature Profile along with $Pr = 1.5$.

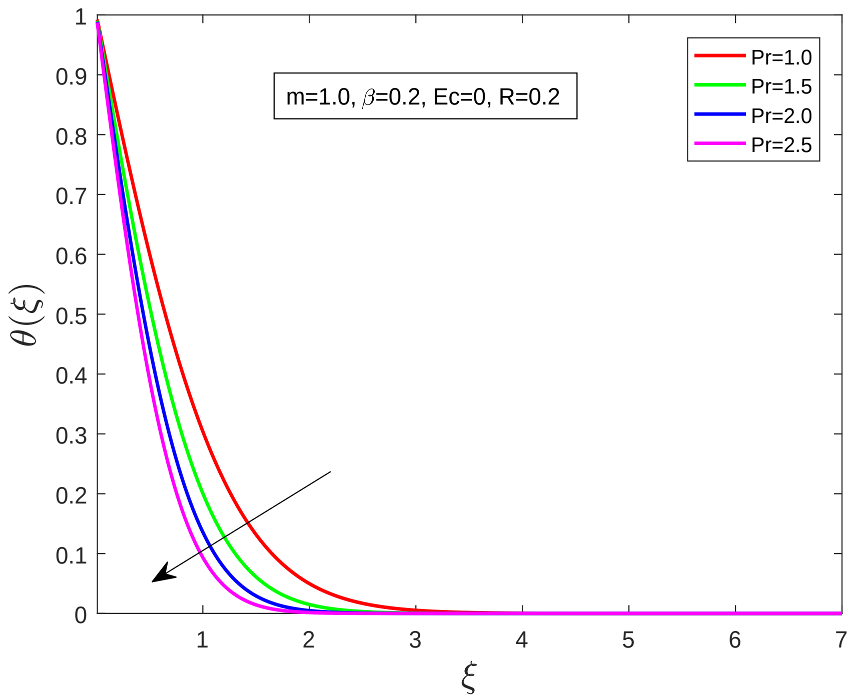

FIGURE 4.9: Effect of Prandtl number Pr on Temperature Profile along with $m = 1.0$.

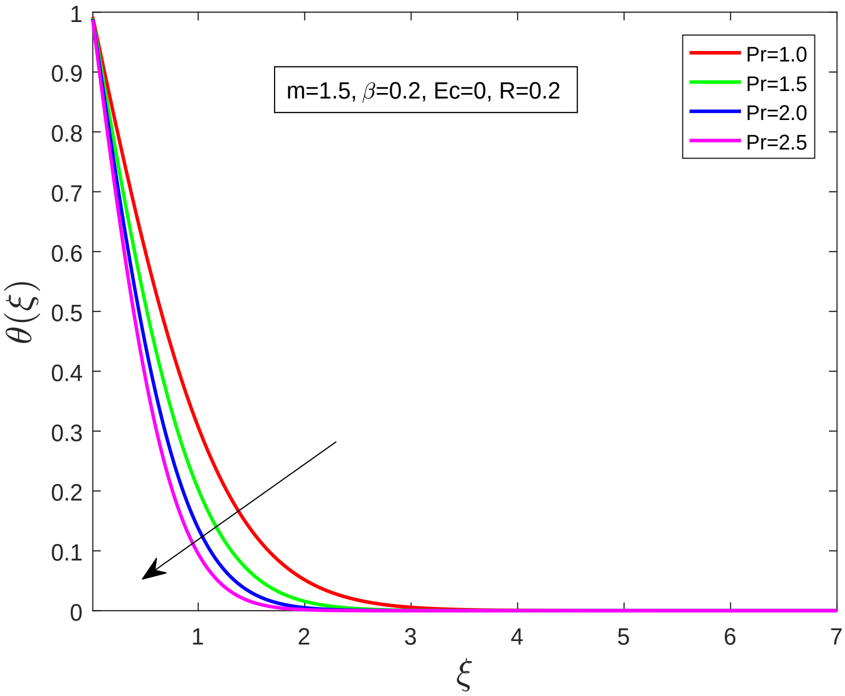

FIGURE 4.10: Effect of Prandtl number Pr on Temperature Profile along with $m = 1.5$.

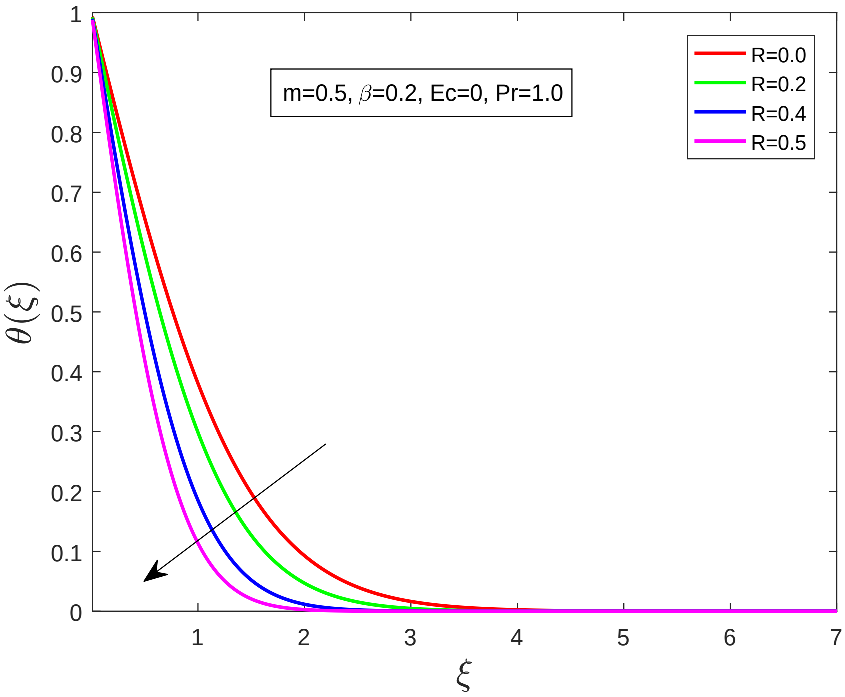

FIGURE 4.11: Effect of Thermal Radiation R on Temperature Profile along with $m = 0.5$.

Chapter 5

Conclusion

In this research work represents the 2-D incompressible flow of Casson and Carreau fluid along a boundary layer equation with the stretching parameter. Furthermore the impacts of the power-law index, Weissenberg number and stretching parameters are discussed. The obtained mathematical model contains the nonlinear PDEs of continuity equation, momentum and energy equations. Furthermore these PDEs are converted into a system of nonlinear ODEs by applying the similarity transformation. Meanwhile for the numerical results of ODEs, shooting technique has been utilized. The dimensionless velocity behavior, temperature distribution, skin friction coefficient and Nusselt number have been analyzed for various parameters. The numerical results are explained through different figures and tables. Some interesting findings have been listed below.

- By enhancing the value of n, we observed that the momentum boundary layer thickness is increased and thermal boundary layer thickness is decreased.
- An increment is noticed the parameter m, we observed that the momentum boundary layer thickness is decreased while the thermal boundary layer thickness is increased.
- By increasing the value of We, the magnitude of the velocity of fluid is decreased and increased for shear thickening fluid.

- A decrement is noticed the thermal boundary layer thickness and temperature distribution by enhancing the value of Pr.
- It is observed that a decrement is noticed in velocity distribution due to the accelerating values of β.
- The temperature profile is increased due to the rising value of β.
- The velocity distribution reduces for enhancing the value of m.
- An increment is noticed in temperature distribution due to the accelerating value of m.
- A decrement is noticed in temperature distribution due to the increasing value of Pr.

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