CAPITAL UNIVERSITY OF SCIENCE AND TECHNOLOGY, ISLAMABAD

Relation Theoretic Coincidence and Common Fixed Point Results under $(F_w, \mathcal{R})_g$-Contractions

by

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I dedicate my dissertation to

*my beloved **family** specially*

***my mother**.*

A special feeling of gratitude is for

***my father**,*

who had been an inspiration throughout my life.

CERTIFICATE OF APPROVAL

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Abstract

The concept of fixed point in metric space under specific contraction mappings is demonstrated by many researchers. In the present dissertation, we discuss the notion of $(F_w, \mathcal{R})_g$ contractions and obtain coincidence points, fixed points, unique fixed point and common fixed point results for such contractions in the setting of metric spaces. We also prove some consequences in ordered metric space using the same idea. To elaborate the theorem we also prove an example. Our results will be valuable in metric space using $(F_w, \mathcal{R})_g$ contractions.

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Symbols

d	distance function
$\mathcal{R}$	binary relation
$\mathbb{N}$	Natural number
$\mathbb{R}$	Real number
$\mathbb{F}$	Family of functions
$\Rightarrow$	implies that
$\in$	Belongs to
τ	tau
$\notin$	Does not belongs to
$\forall$	for all
F_w	F-weak contraction
∞	Infinity

Chapter 1

Introduction

Mathematics has an important role in scientific knowledge that's why it is called mother of sciences. Mathematics has a lot of applications for humans in every field of life. Mathematics is divided into many branches and each branch has its significance. One of the important branch of mathematics is known as functional analysis. Fixed point theory is an important concept in functional analysis. Fixed point theory provides sufficient conditions for the existence of solution of a problem. Fixed point theory has countless appeals in the area of numerical analysis, polynomial interpolation, error estimation and finite difference methods.

Banach Contraction Principle (BCP) [1] is to be considered the most valuable outcome in metric fixed point theory.

Poincare [2] worked on fixed point theory first time. Later on Brouwer [3] considered the equation $f(a) = a$ and established the solution of this equation by proving a fixed point theorem in 1912. He also worked to prove fixed point result for the shapes like square and sphere etc. He has also established fixed point results in various dimensions [3].

In 1922, a notable mathematician Banach [4] demonstrated a significant fixed point result in the area of functional analysis acknowledge as BCP. This result is declared to be the most fundamental consequence in the area of fixed point theory. BCP is stated as: "A contraction mapping in a complete metric space has a unique fixed point." The two remarkable applications come from this principle. The first one is that it guarantees the uniqueness and existence of fixed point. The second one is that it provides an approach to determine the fixed point of mapping. Due to its extensive application potential, this concept has been observed in various forms over the year [5–8].

This theorem occupies a significant role in the area of mathematical analysis. A number of researchers in mathematics are interested in BCP by virtue of its description and generality. Presic [9] and Kannan [10] proved his contraction mapping concept. A lot of contractions have been established after the BCP, but we will discuss only those which are used in our work.

The concept of F-contractions was introduced by Wardowski [11]. He proved some new fixed point results for such kind of contractions. He build these results in a different way rather than traditional ways as done by many authors. Later on, fixed points for F-contractions were proved by Secelean [12] using iterated function. Abbas [13] extended the work of Wardowski and established various results of fixed points using F-contraction mapping.

The idea of $(F, \mathcal{R})$-contractions was established by Sawangsup et al. [14] and he used this idea to demonstrate some fixed point consequences using binary relation. The idea of $(F, \mathcal{R})$-contractions was demonstrated by Imdad et al. [15]. In present thesis, we studied the results presented in Alfaqih et al. [16] and we define $(F_w, \mathcal{R})_g$ -contractions and prove a theorem similar to Alfaqih et al. for $(F_w, \mathcal{R})_g$-contractions.

Following are the details of work, which I have done throughout this thesis.

• **Chapter 2**: This chapter is about the basic concepts and definitions of metric spaces and some examples which satisfy the properties of above spaces. Similarly we have discussed BCP and some examples to support it.

Finally, we talk about some basic tools for F-contractions.

• **Chapter 3**: The paper "Relation Theoretic Coincidence and Common Fixed Point Results under $(F, \mathcal{R})_g$- Contractions" is reviewed.

• **Chapter 4**: This chapter emphasizes on the idea of $(F_w, \mathcal{R})_g$ -contractions and focused on the extension of the results presented by Alfaqih et al. [16]. An example is given to verify our result. Conclusion is given in last section.

Chapter 2

Preliminaries

In this chapter, we will discuss about the fundamental definitions, results and examples which are used in subsequent chapters. The first section of this chapter covers some basics of metric spaces with few examples. The second section consists of Banach Contraction Principle. The third section consists of some basic tools for F-contraction.

2.1 Metric Space

In mathematics, Euclidean distance is a straight line distance. However, this distance can be other than the straight line like taxicab distance. In literature the term “metric” is useful to deduce the concept of distance and the space endowed with metric fulfilling few properties known as metric space. In 1906 Frachet [17] prescribed the scheme of metric space as follows.

2.1.1. Metric Space [18]

“A metric space is a pair (X, d), where X is a set and d is a metric on X (or distance function on X), that is, a function defined on $X \times X$ such that for all $x, y, z \in X$ we have:

1. d is real- valued, finite and nonnegative,

2. $d(x, y) = 0$ if and only if $\quad x = y$,

3. $d(x, y) = d(y, x)$ (Symmetric Property),

4. $d(x, y) \leq d(x, z) + d(z, y)$ (Triangle Inequality)."

Following are examples of metric spaces.

Example 2.1.1.

Let $K = \mathbb{R}$ defined by usual metric

$$d(k, l) = |k - l|,$$

is a metric space.

2.1.2. Continuous Mapping [19]

"Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be continuous at a point x_0 if for each $\epsilon > 0$ there exists $\delta > 0$ such that

$$d(Tx, Tx_0) \ \leq \ \epsilon \quad \text{whenever} \quad d(x, x_0) \ \leq \ \delta."$$

Example 2.1.2.

Let $K = \mathbb{R}$ with usual metric d as stated in (2.1.1). $Q : K \to K$ defined by

$$Q(\zeta) = \zeta^3 \ \text{ where } \ \zeta \in X.$$

Then Q satisfies the above definition.

2.1.3. Convergence of Sequence [18]

"A sequence $\{x_n\}$ in a metric space $X = (X, d)$ is said to converge or to be convergent if there is an $x \in X$ such that

$$\lim_{n \to \infty} d(x_n, x) = 0,$$

x is called the limit of $\{x_n\}$ and we write

$$\lim_{n \to \infty} x_n = x,$$

or simply

$$x_n \to x.$$

We say that x_n converges to x or has the limit x. If x_n is not convergent, it is said to be divergent."

Example 2.1.3.

Consider again the set $\mathbb{R}$ and $d(\eta, \omega) = |\eta - \omega|$, then the sequence $\{\eta_n = \frac{1}{n}\}$ in X is a convergent sequence.

2.1.4. Cauchy Sequence [18]

"A sequence $\{x_n\}$ in a metric space (X, d) is said to be a Cauchy sequence if for each $\epsilon > 0$ there exist $N \in \mathbb{N}$ such that

$$d(x_n, x_m) \ < \ \epsilon \quad \forall m, n > N."$$

2.1.5. Complete Metric Space [18]

"If every Cauchy sequence in a metric space (X, d) converges to a point $x \in X$ then X is called a complete metric space."

Example 2.1.4. [18]

Let $X = \mathbb{R}$ and closed interval $[0, 1]$ in $\mathbb{R}$ is a complete metric space with usual metric on $\mathbb{R}$.

2.2 Banach Contraction Principle

Stefan Banach Proved BCP in 1922 in his doctoral dissertation. BCP is considered as one of the basic result of fixed point theory. Many extensions has been made by many authors.

2.2.1 Contraction Mappings

2.2.1.1. Contraction [19]

"Let X be a metric space, a mapping $F : X \to X$ is called a contraction if there exists $k < 1$ such that for any $x, y \in X$,

$$d(Fx, Fy) \leq kd(x, y).$$

This contraction is also known as Banach contraction. Geometrically this means that any points x and y have images that are closer together than these points x and y; more precisely, the ratio $d(Fx, Fy)|d(x, y)$ does not exceed a constant α which is strictly less than 1. This contraction is also known as Banach contraction."

Example 2.2.1.1.

Let $K = [0, 1]$, $d(\zeta, \omega) = |\zeta - \omega|$ and $Q : K \to K$ given by

$$Q(\zeta) = \frac{1}{3 + \zeta},$$

then it is a contraction mapping.

Proof. Since

$$Q(\zeta) = \frac{1}{3 + \zeta},$$

then,

$$\begin{aligned} d(Q\zeta, Q\omega) &= d\left(\frac{1}{3+\zeta}, \frac{1}{3+\omega}\right), \\ &= \left|\frac{1}{3+\zeta} - \frac{1}{3+\omega}\right|, \\ &= \left|\frac{3+\omega-3-\zeta}{(3+\zeta)(3+\omega)}\right|, \\ &= \left|\frac{-(\zeta-\omega)}{(3+\zeta)(3+\omega)}\right|, \\ &\leq \frac{|\zeta-\omega|}{(3)(3)}, \\ &\leq \frac{1}{9}d(\zeta, \omega), \end{aligned}$$

therefore Q is a contraction with $\alpha = \dfrac{1}{9}$.

Example 2.2.1.2.

Let $K = \mathbb{R}$ and $d(\eta, \omega) = |\eta - \omega|$. Define $Q\colon K \to K$ by

$$Q(\eta) = \frac{\eta}{5} + 5,$$

then it is a contraction mapping.

Proof.

As

$$Q(\eta) = \frac{\eta}{5} + 5,$$

so,

$$\begin{aligned} d(Q\eta, Q\omega) =& d\left(\frac{\eta}{5} + 5, \frac{\omega}{5} + 5\right), \\ \leq& \left|\frac{\eta}{5} + 5 - (\frac{\omega}{5} + 5)\right|, \\ \leq& \left|\frac{\eta}{5} + 5 - \frac{\omega}{5} - 5\right|, \\ \leq& \left|\frac{\eta}{5} - \frac{\omega}{5}\right|, \\ \leq& \frac{1}{5}|\eta - \omega|, \\ =& \frac{1}{5}d(\eta, \omega), \end{aligned}$$

therefore contraction constant is $\alpha = \dfrac{1}{5}$.

2.2.1.2. Contractive Mapping [20]

"A self map $T : X \to X$ on a metric space is a contractive mapping if

$$d(Tx, Ty) < d(x, y),\ \forall\ x, y \in X,\ \ x \neq y.$$

Every contraction is contractive mapping but converse of statement is not true in general."

For instance look at example given below.

Example 2.2.1.3.

Consider $K = \mathbb{R}$ and (K, d) be a metric space. Q be a self-mapping on K. Define

$$Q(\eta) = \eta + \frac{1}{\eta}, \quad \forall\, \eta \in K.$$

Proof.

$$\begin{aligned}
d(Q(\zeta_1), Q(\zeta_2)) =& \left|\zeta_1 + \frac{1}{\zeta_1} - \zeta_2 - \frac{1}{\zeta_2}\right|, \\
=& \left|\zeta_1 - \zeta_2 + \left(\frac{1}{\zeta_1} - \frac{1}{\zeta_2}\right)\right|, \\
=& \left|(\zeta_1 - \zeta_2) + \left(\frac{\zeta_2 - \zeta_1}{\zeta_1\zeta_2}\right)\right|, \\
=& \left|(\zeta_1 - \zeta_2) - \left(\frac{\zeta_1 - \zeta_2}{\zeta_1\zeta_2}\right)\right|, \\
=& \left|(\zeta_1 - \zeta_2)\right|\left|1 - \left(\frac{1}{\zeta_1\zeta_2}\right)\right|, \\
<& |\zeta_1 - \zeta_2|, \\
=& d(\zeta_1, \zeta_2).
\end{aligned}$$

This shows that Q is contractive.

2.2.1.3. Non-Expansive [18]

"A self map $T : X \to X$ on a metric space is a non-expansive mapping if

$$d(Tx, Ty) \le d(x, y), \quad \forall\, x, y \in X, \ \ x \neq y."$$

Remark 2.2.1.1.

Note that Every contractive mapping is a non-expansive mapping but every non-expansive mapping is not contractive mapping and hence is not a contraction. For example, identity map is non-expansive but not a contraction.

2.2.1.4. Lipschitzian Mapping [19]

"Suppose that X is a metric space and F is a mapping from X to X. The mapping F is called a Lipschitz mapping if there exists a constant $k \ge 0$ such that

$$d(F(x), F(y)) \le kd(x, y) \ \ \forall\, x, y \in X.$$

The infimum over all such constants k is called the Lipschitz constant."

Example 2.2.1.4.

Consider a self-map on $X = \mathbb{R}$ defined as $T(\eta) = 5\eta \quad \forall\, \eta \in X$,

Proof.

$$\begin{aligned} d(T(\eta_1), T(\eta_2)) =& d(5\eta_1, 5\eta_2) \\ =& |5\eta_1 - 5\eta_2|, \\ =& 5|\eta_1 - \eta_2|, \\ =& 5d(\eta_1, \eta_2). \end{aligned}$$

Here $\lambda = 5$ is the lipschitzian constant.

Berinde has defined the weak contraction as follows.

2.2.1.5. Weak Contraction [21]

"Let (X, d) be a metric space, a self mapping $F : X \to X$ is said to be weak contraction if there exists a constant $\alpha \in (0, 1)$ and some $\beta \geqslant 0$ such that

$$d(Fx, Fy) \leqslant \alpha.d(x, y) + \beta.d(y, Fx)\ \forall\ x, y \in X.$$

Due to symmetry of distance, it includes following

$$d(Fx, Fy) \leqslant \alpha.d(x, y) + \beta.d(x, Fy)\ \forall\ x, y \in X."$$

2.2.2 Root Finding using Fixed Point Theory

A wide diversity of problems appearing in various fields of mathematics like differential equations, discrete and continuous system of dynamics can be demonstrated as fixed point problem. This portion is about the definition of fixed point and examples related to it.

2.2.2.1. Fixed Point [22]

"Let $T : X \to X$ be a mapping on a set X. A point $x \in X$ is said to be a fixed point of T if

$$Tx = x,$$

that is, a point is mapped onto itself.
Geometrically, if $y = f(x)$ is a real valued function on $\mathbb{R}$, then the fixed point of this function lies where the graph of the function f coincides with the real line $y = x$. Thus a function may or may not have fixed point. Furthermore, fixed point may or may not be unique."

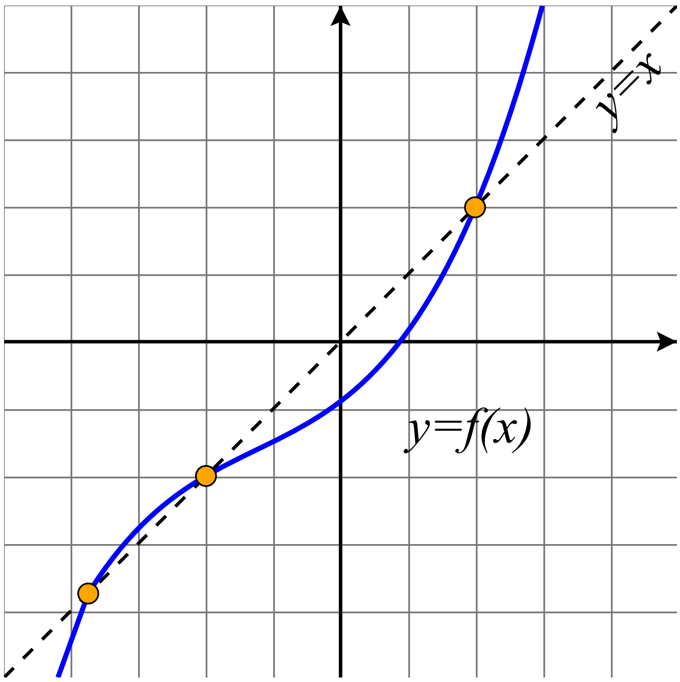

FIGURE 2.1: Three Fixed points

The graph mention above represents a function having three fixed points.

Example 2.2.2.1.

Consider $X = \mathbb{R}$ with the usual metric d. Suppose mapping $T : X \to X$ by

$$T(x) = x + 1 \quad \forall \;\; x \in X$$

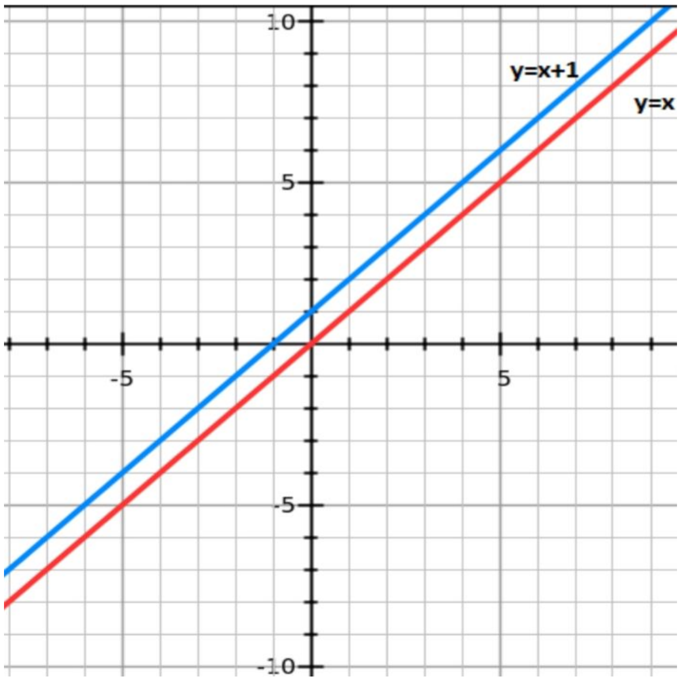

FIGURE 2.2: No Fixed Point

then T has no fixed point.

Example 2.2.2.2.

Let $X = \mathbb{R}$ be enriched with the usual metric d. Consider

$T : X \to X$ by

$$T(x) = 2x + 1 \quad \forall \;\; x \in X$$

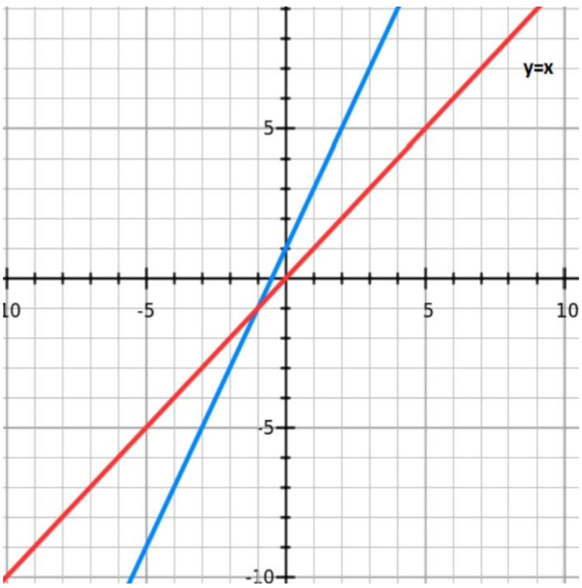

FIGURE 2.3: Unique Fixed Point

then $x = -1$ is a fixed point of T and it is unique.

Example 2.2.2.3.

Consider I be the identity map on $X = \mathbb{R}$ with usual metric d, that is

$$I(\eta) = \eta, \qquad \forall\ \eta \in X.$$

Then each and every point of X will be the fixed point of I.

2.2.2.2. Zeroes of a Function

Problem of finding zeroes of a real valued function $g(\eta)$ defined on an interval is equivalent to the problem of finding the fixed point of $f(\eta)$ where,

$$f(\eta) = \eta - g(\eta),$$

since, zeroes of $g(\eta)$ means η such that,

$$g(\eta) = 0,$$

$$\Rightarrow \eta - g(\eta) = \eta,$$

or

$$f(\eta) = \eta,$$

hence η is a fixed Point of $f(\eta)$.

Example 2.2.2.4.

Observe the quadratic equation

$$g(\eta) = \eta^2 + 5\eta + 4.$$

Then, zeroes of $g(\eta)$ are

$$\eta = -4, \quad \eta = -1,$$

which results $\quad g(\eta) = 0,$

$$\Rightarrow \eta^2 + 5\eta + 4 = 0,$$

$$\eta^2 + 4 = -5\eta,$$

$$\Rightarrow \eta = \frac{\eta^2 + 4}{-5} = f(\eta).$$

Clearly to find fixed point of $f(\eta)$ is identical to find the zeroes of $g(\eta)$.

2.2.3 Banach Contraction Principle

The basic and main result of fixed point theory is BCP. A polished mathematician Stefan Banach first present BCP in his Ph.D research during 1922. Many extensions and generalizations on BCP are made by many authors. [for instance see [23–25]].

Theorem 2.2.3.1. [24]

"Every contraction mapping on a complete metric space has a unique fixed point that is

if (X, d) is a complete metric space and $T : X \to X$ is a mapping such that $\forall\ x, y \in X,\ \exists\ \alpha \in [0, 1)$ such that

$$d(Tx, Ty) \leq \alpha\ d(x, y), \qquad x \neq y$$

then T has a unique fixed point $x_0 \in X$ that is $Tx_0 = x_0$."

In 1962, Edelstein [26] presented the subsequent well known result.

Theorem 2.2.3.2. [26]

"Let (X, d) be a compact metric space, and let T be a mapping on X. Assume $d(Tx, Ty) < d(x, y)$ for all $x, y \in X$ with $x \neq y$. Then T has a unique fixed point."

Example 2.2.3.1.

Consider $K = \mathbb{R}$ and $d(k, l) = |k - l|$. The self map Q is shown by $Q(k) = \frac{k}{f} + e$.

Proof :

Here $Q(k) = \frac{k}{f} + e$ and $Q(l) = \frac{l}{f} + e$, then

$$
\begin{aligned}
d(Qk, Ql) =& d(\frac{k}{f} + e, \frac{l}{f} + e) \ \forall \ k, l \in K, \\
=& |\frac{k}{f} + e - (\frac{l}{f} + e)|, \\
=& |\frac{k}{f} + e - \frac{l}{f} - e|, \\
=& |\frac{k}{f} - \frac{l}{f}|, \\
=& |\frac{k - l}{f}|, \\
=& \frac{1}{f} d(k, l).
\end{aligned}
$$

It is a contraction if $f > 1$.

We can find fixed point using the definition of fixed point.

$$
\begin{aligned}
Q(k) &= \frac{k}{f} + e, \\
k &= \frac{k}{f} + e, \\
k - \frac{k}{f} &= e, \\
kf - k &= ef, \\
k(f - 1) &= ef, \\
k &= \frac{ef}{f - 1}.
\end{aligned}
$$

It satisfies all properties of BCP.

2.3 Some Basic Tools for F-Contractions

2.3.1. Binary Relation [27]

"Let X be a nonempty set. A subset $\mathcal{R}$ of X^2 is called a binary relation on X. Notice that for each pair $x, y \in X$, one of the following conditions holds:

1. $x, y \in \mathcal{R}$; which amounts to saying that x is $\mathcal{R}$-related to y or x relates to y under $\mathcal{R}$. Sometimes we write $x\mathcal{R}y$ instead of $x, y \in \mathcal{R}$,

2. $x, y \notin \mathcal{R}$; which means that x is not $\mathcal{R}$-related to y or x does not relates to y under $\mathcal{R}$."

2.3.2. Partially Ordered Set [18]

"A partially ordered set is a set M on which there is defined partial ordering, that is, a binary relation which is written $\preceq$ and satisfies the conditions:

1. Reflexive; for each $a \in M$ we have $a \preceq a$.

2. Antisymmetric; If $a \preceq b$ and $b \preceq a \Rightarrow b = a$, $\forall\ \ a,\ b \in M$.

3. Transitive ; If $\ a\ \preceq\ b\ and\, b\ \preceq c\ \Rightarrow\ a \preceq c$, $\forall\ a,\ b,\ c \in M$.

Partially emphasizes that M may contain a and b for which neither $a \preceq$ nor $b \preceq a$ holds. Then a and b are called incompareable elements. In contrast, two elements a and b are called compareable elements if they satisfies $a \preceq b$ or $b \preceq a$ (or both)."

2.3.3. Totally Ordered Set [18]

"A totally ordered set or chain is a partially ordered set such that every two elements of the set are compareable. In other words, a chain is a partially orderd set that has no incompareable elements.

Every totally ordered set is partially ordered set but converse is not true."

2.3.4. Collection of F- Mappings [11]

"Let $\mathbb{F}$ be the family of all functions $F : (0, \infty) \to \mathbb{R}$ which satisfy the following conditions:

(F_1) F is strictly increasing.

(F_2) For every sequence $\beta_n \subset (0,\infty)$, $\lim_{n\to\infty} \beta_n = 0$ if only if $\lim_{n\to\infty} F(\beta_n) = -\infty$.

(F_3) There exists $k \in (0,1)$ such that $\lim_{\beta\to 0^+} \beta^k F(\beta) = 0$.

Throughout this work the family of all continuous functions which satisfy (F_2) is denoted by $\mathcal{F}$."

Example 2.3.1.

The subsequent functions $F : (0,\infty) \to \mathbb{R}$ satisfy codition F_2:

Here $F(\zeta) = \ln \zeta$,

Proof :

(i) $k < l$ such that $\ln k < \ln l \quad \forall k, l \in \mathbb{R}$.

Since natural log is an increasing function for base greaer than one.

(ii) Consider $\zeta_m \subseteq (0,\infty)$,

such that

$$\lim_{m\to\infty} \zeta_m = 0 \Leftrightarrow F(\zeta_m) = -\infty,$$

$$\Rightarrow \lim_{m\to\infty} \ln(\zeta_m) = \lim_{m\to\infty} \ln(0) = -\infty,$$

Natural logarithm is not defined for negative numbers because e can not be negative. Natural logarithm of zero would mean that e raise to power something zero, does not exist.

(iii) $\exists\ l \ \in (0,1)$ such that $\lim_{\zeta\to 0^+} \zeta^l \ln(\zeta) = 0$.

It fulfills all properties of above definition.

Example 2.3.2.

The subsequent functions $F : (0,\infty) \to \mathbb{R}$ satisfy codition F_2:

Here $F(\zeta) = \zeta - \frac{1}{\zeta}$, *Proof* :

(i) $\zeta < \eta$ such that $\zeta - \frac{1}{\zeta} < \eta - \frac{1}{\eta} \quad \forall\ \zeta, \eta \in \mathbb{R}$.

(ii) Consider $\zeta_m \subseteq (0, \infty)$,

such that

$$\lim_{m\to\infty} \zeta_m = 0 \Leftrightarrow F(\zeta_m) = -\infty,$$

$$\Rightarrow \lim_{m\to\infty} (\zeta_m - \frac{1}{\zeta_m}) = \lim_{m\to\infty} \ln(0) = -\infty,$$

(iii) $\exists\ l\ \in (0,1)$ such that $\lim_{\zeta\to 0^+} \zeta^l(\zeta - \frac{1}{\zeta}) = 0$.

It fulfills all properties of above definition.

Example 2.3.3.

The subsequent functions $F : (0, \infty) \to \mathbb{R}$ satisfy codition F_2:

Here $F(\zeta) = \ln(\frac{\zeta}{3} + sin\zeta)$,

Proof :

(i) $\zeta < \eta$ such that $\ln(\frac{\zeta}{3} + sin\zeta) < \ln(\frac{\eta}{3} + sin\eta) \quad \forall\ \zeta, \eta \in \mathbb{R}$.

(ii) Consider $\zeta_m \subseteq (0, \infty)$,

such that

$$\lim_{m\to\infty} \zeta_m = 0 \Leftrightarrow F(\zeta_m) = -\infty,$$

$$\Rightarrow \lim_{m\to\infty} (\frac{\zeta_m}{3} + sin\zeta_m) = \lim_{m\to\infty} \ln(0) = -\infty,$$

(iii) $\exists\ l\ \in (0,1)$ such that $\lim_{\zeta\to 0^+} \zeta^l(\frac{\zeta}{3} + sin\zeta) = 0$

It fulfills all properties of above definition.

In [11], Wardowski defined F-contractions as follows.

2.3.5. F-Contractions [11]

"Let (M, d) be a metric space. A mapping $T : M \to M$ is said to be an F-contractions if there exist $\tau > 0$ and $F \in \mathcal{F}$ such that

$$d(Tx, Ty) > 0 \textit{ implies } \quad \tau + F(d(Tx, Ty)) \leqslant F(d(x, y)) \ \forall \ x, y \ \in M."$$

Now we give some examples of F-contractions as follows.

Example 2.3.4.

Let $F : \mathbb{R}^+ \to \mathbb{R}$ and $F(k) = \ln k$. Then $F \in \mathcal{F}$ for any $l \in (0, 1)$,

$$\tau + \ln \Big(d(Qk, Ql) \Big) \leqslant \ln d(k, l) \ \forall \ k, l \in K,$$

$$\Rightarrow d(Qk, Ql) \leqslant \exp^{-\ \tau} d(k, l),$$

or

$$\frac{d(Qk, Ql)}{d(k, l)} \leqslant \exp^{-\ \tau}.$$

It satisfies all properties of above definition. Then it is an F contraction.

Example 2.3.5.

Let $F : \mathbb{R}^+ \to \mathbb{R}$ and $F(k) = \ln k + k$. Then $F \in \mathcal{F}$ for any $l \in (0, 1)$,

$$\tau + \ln \Big(d(Qk, Ql) \Big) + d(Qk, Ql) \leqslant \ln d(k, l) + d(k, l) \ \forall \ k, l \in K,$$

$$\Rightarrow \tau + \ln \Big(d(Qk, Ql) \Big) - \ln d(k, l) \leqslant d(k, l) - d(Qk, Ql),$$

or

$$\frac{d(Qk, Ql)}{d(k, l)} \exp^{d(Qk,Ql) - d(k,l)} \leqslant \exp^{-\ \tau} \ \forall \ k, l \in K \textit{ and } Qk \neq Ql.$$

It satisfies all properties of above definition. Then it is an F contraction.

Theorem 2.3.1. [11]

"Every F-contraction mapping T defined on a complete metric space (M, d) has a unique fixed point (say z). Moreover, for any $x \in M$ the sequence $\{T^n x\}$ converges to z."

In [11], Wardowski has extended the idea of F-contractions to $(F, \mathcal{R})$ contractions as follows.

2.3.6. $(F, \mathcal{R})$-Contractions [11]

"Let (M, d) be a metric space and $\mathcal{R}$ be a binary relation on M. A mapping $T : M \to M$ is said to be an $(F, \mathcal{R})$-contractions if there exist $\tau > 0$ and $F \in \mathcal{F}$ such that

$$\tau + F(d(Tx, Ty)) \leqslant F(d(x, y)), \quad \forall\, x, y \in M \text{ with } x\mathcal{R}^{\neq}y \text{ and } Tx\mathcal{R}^{\neq}Ty,$$

where $x\mathcal{R}^{\neq}y$ means $(x, y) \in \mathcal{R}$ and $x \neq y$".

Using the idea of [11], Wardowski [28] has defined F_w- contraction.

2.3.7. F**- Weak Contractions** [28]

"Let (M, d) be a metric space. A mapping $T : M \to M$ is said to be an F- weak contraction if there exist $\tau > 0$ and $F \in \mathbb{F}$ such that for all $x, y \in M$

$$\begin{aligned} &d(Tx, Ty) > 0 \text{ implies } \tau + F(d(Tx, Ty)) \leqslant F(m(x, y)), \\ &\text{where } m(x, y) =: \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\right\}." \end{aligned} \tag{2.1}$$

We can see from above definition that,

Remark 2.3.1. [28]

"Every F-contraction is an F-weak contraction but converse is not true."

Remark 2.3.2. [28]

"Let T be an F-contraction. Then $d(Tx, Ty) < d(x, y)$ for all $x, y \in X$ such that $Tx \neq Ty$. Also T is a continuous map."

An example of F_w contraction which is not an F-contraction is given here.

Example 2.3.6. Let $Q : [0,1] \to [0,1]$ defined by

$$Qk = \begin{cases} \frac{1}{2}, & if \quad k \in [0,1), \\ \frac{1}{4}, & if \quad k = 1. \end{cases}$$

By Remark 2.3.2, Q is not an F-contraction because Q is not continuous.

For $k \in [0,1)$ and $l = 1$, we have

$$d(Qk, Ql) = d\Big(\frac{1}{2}, \frac{1}{4}\Big) = \Big|\frac{1}{2} - \frac{1}{4}\Big| = \frac{1}{4} > 0$$

and

$$\max\Big\{d(k,1), d(k,Qk), d(1,Q1), \frac{d(k,Q1)+d(1,Qk)}{2} \geqslant d(1,Q1) = \frac{3}{4}\Big\}.$$

Here $\tau = \ln 3$ and $F(\eta) = \ln \eta$ and using the definition of F-contractions, we have

$$\begin{aligned} \ln 3 + \ln \frac{1}{4} &\leq \ln \frac{3}{4}, \\ \ln \frac{1}{4} &\leq \ln \frac{3}{4} - \ln 3, \\ &= \ln \frac{1}{4}. \end{aligned}$$

It satisfies all properties of above definition.

Next Theorem shows the existence of unique fixed point for F-weak contractions.

Theorem 2.3.2. [28]

"Let (M,d) be a complete metric space and $T : M \to M$ be an F-weak contraction. If F or T is continuous, then

(i) T has a unique fixed point (say $z \in M$),

(ii) $\lim_{n\to\infty} T^n x = z \quad \forall\, x \in M$."

2.3.8. Coincidence Point, Weakly Compatible [29]

"Let f and g be self-maps of a set X, that is $f, g : X \to X$. If $w = fx = gx$ for some $x \in X$, then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g.

Self-maps f and g are said to be weakly compatible if they commute at their coincidence point; that is, if $fx = gx$ for some $x \in X$, then $fgx = gfx$."

Example 2.3.7. Let $K = [0, 4]$ with $d(k, l) = |k-l|$. Define $p, q : [0, 4] \to [0, 4]$ by

$$p(k) = \begin{cases} k, & \text{if} \quad k \in [0, 1) \\ 4, & \text{if} \quad k \in [1, 4] \end{cases}$$

and

$$q(k) = \begin{cases} 4 - k, & \text{if} \quad k \in [0, 1) \\ 4, & \text{if} \quad k \in [1, 4] \end{cases}$$

p and q are weakly compatible maps on $[0, 4]$ as $pqk = qpk$ for any $k \in [1, 4]$.

2.3.9. Common Fixed Point [30]

"A common fixed point of a pair of self-mapping $K, L : X \to X$ is a point $x \in X$ for which $Kx = Lx = x$."

2.3.10. g-Continuous

Let f and g be self-maps of a set X. The mapping f is called g-continuous at $k \in X$ if $\forall \{k_m\} \subseteq X$, $gk_m \to gk$ implies $fk_m \to fk$.

Chapter 3

Relation Theoretic Concepts and Auxiliary Consequences

In this chapter we are going to discuss $(F, \mathcal{R})_g$ contractions and some coincidence and common fixed point results proved by Alfaqih et al. [16] and some consequences in order metric space are also produced. At the end we have discussed an example to support our result.

3.1 Relation Theoretic Definitions

3.1.1. Inverse Binary Relation [27]

"Let X be a nonempty set, and let $\mathcal{R}$ be a binary relation on X.

1. The inverse or transpose or dual relation of $\mathcal{R}$, denoted by $\mathcal{R}^{-1}$, is defined by
$$\mathcal{R}^{-1} = \{(x, y) \in X^2 : (y, x) \in \mathcal{R}\}.$$
2. The symmetric closure of $\mathcal{R}$, denoted by $\mathcal{R}^s$, is defined as the set $\mathcal{R} \cup \mathcal{R}^{-1}$, that is $\mathcal{R}^s := \mathcal{R} \cup \mathcal{R}^{-1}$. In fact , $\mathcal{R}^s$ is the smallest symmetric relation on X containing $\mathcal{R}$."

Notice that there is another binary relation $\mathcal{R}^{\neq} \subseteq \mathcal{R}$ on X which is defined as $k\mathcal{R}^{\neq}l$ whenever $k\mathcal{R}l$ and $k \neq l$.

3.1.2. Restriction of $\mathcal{R}$ to E [1]

"Let X be a nonempty set, let $E \subseteq X$, and let $\mathcal{R}$ be a binary relation on X. Then the restriction of $\mathcal{R}$ to E , denoted by $\mathcal{R}|E$, is defined as the set $\mathcal{R} \cap E^2$ that is $\mathcal{R}|E := \mathcal{R} \cap E^2$. In fact, $\mathcal{R}|E$ is a relation on E induced by $\mathcal{R}$."

3.1.3. Preorder [31]

"Consider a non-empty set X and a binary relation $\preceq$ on X. Then, $\preceq$ is a preorder, or quasiorder, if it is reflexive and transitive, that is, forall a, b and $c \in X$; we have that:

1. $a \preceq a$ (reflexivity),
2. If $a \preceq b$ and $b \preceq c$; then $a \preceq c$ (transitivity)."

3.1.4. $\mathcal{R}$- Preserving Sequence [32]

"Let M be a non-empty set and $\mathcal{R}$ be a binary relation on M. A sequence $\{x_n\} \subseteq M$ is said to be an $\mathcal{R}$-preserving sequence if

$$x_n \mathcal{R} x_{n+1} \quad \forall\, n \in \mathbb{N}_0."$$

3.1.5. T- Closed [32]

"Let M be a non-empty set and $T : M \to M$. A binary relation $\mathcal{R}$ on M is said to be T- closed if for all $x, y \in M$, $x\mathcal{R}y$ implies $Tx\mathcal{R}Ty$."

The following example of T-closed binary relation.

Example 3.1.1.

Let $M = \mathbb{R}$ and $d = |k - l|$, then (M, d) is a complete metric space. A binary relation on M is defined as

$$\mathcal{R} = \{(k, l) \in \mathbb{R}^2 : k - l \geqslant 0,\ k \in \mathbb{Q}\},$$

and $T : X \to X$ defined by

$$T(k) = 4 + \frac{1}{3}k.$$

$\mathcal{R}$ is T- closed.

Example 3.1.2.

Consider a complete metric space (M, d) and $M = [0, 2]$.

Define

$$\mathcal{R} = \{(0,0),(0,1),(1,0),(1,1),(0,2)\},$$

and $T : M \to M$ is defined by

$$T(k) = \begin{cases} 0, & \text{if} \quad 0 \leq k \leq 1 \\ 1, & \text{if} \quad 1 \leq k \leq 2 \end{cases}$$

Clearly T is not continuous and $\mathcal{R}$ is T- closed.

3.1.6. (T, g)- Closed [33]

"Let M be a non-empty set and $T, g : M \to M$. A binary relation $\mathcal{R}$ on M is said to be (T, g)- closed if for all $x, y \in M$, $gx\mathcal{R}gy$ implies $Tx\mathcal{R}Ty$."

3.1.7. $\mathcal{R}$-Complete [33]

"Let (M, d) be a metric space and $\mathcal{R}$ be a binary relation on M. We say that M is $\mathcal{R}$-complete if every $\mathcal{R}$-preserving Cauchy sequence in M converges to a limit in M."

Remark 3.1.1. [33]

"Every complete metric space is $\mathcal{R}$-complete, whatever the binary relation $\mathcal{R}$. Particularly, under the universal relation, the notion of $\mathcal{R}$-completeness coincides with the usual completeness."

3.1.8. $\mathcal{R}$-Continuous [33]

"Let (M, d) be a metric space and $\mathcal{R}$ be a binary relation on M, $T : M \to M$ and $x \in M$. We say that T is $\mathcal{R}$-continuous at x if for any $\mathcal{R}$-preserving sequence

$\{x_n\} \subseteq M$ such that $\{x_n\} \to x$, we have $\{Tx_n\} \to Tx$. Moreover T is called $\mathcal{R}$-continuous if it is $\mathcal{R}$-continuous at each point of M."

Remark 3.1.2. [33]

"Every continuous mapping is $\mathcal{R}$-continuous, whatever the binary relation $\mathcal{R}$. Particularly, under the universal relation, the notion of $\mathcal{R}$-continuity coincides with the usual continuity."

3.1.9. $(g,\mathcal{R})$-Continuous [33]

"Let (M,d) be a metric space and $\mathcal{R}$ be a binary relation on M and $T,g : M \to M$ and $x \in M$. We say that T is $(g,\mathcal{R})$-continuous at x if for any sequence $\{x_n\} \subseteq M$ such that $\{gx_n\}$ is $\mathcal{R}$-preserving and $\{gx_n\} \to gx$, we have $\{Tx_n\} \to Tx$. Moreover T is called $(g,\mathcal{R})$-continuous if it is $(g,\mathcal{R})$-continuous at each point of M."

Remark 3.1.3. [33]

"Every continuous mapping is $(g,\mathcal{R})$-continuous, whatever the binary relation $\mathcal{R}$. Particularly, under the universal relation, the notion of $(g,\mathcal{R})$-continuity coincides with the usual g-continuity."

3.1.10. $\mathcal{R}$-Compatible [33]

"Let (M,d) be a metric space and $\mathcal{R}$ be a binary relation on M and $T,g : M \to M$. We say that the pair (T,g) is $\mathcal{R}$-compatible if for any sequence $\{x_n\} \subseteq M$ such that $\{Tx_n\}$ and $\{gx_n\}$ are $\mathcal{R}$- preserving and $\lim\limits_{n\to\infty} gx_n = \lim\limits_{n\to\infty} Tx_n = x \in M$, we have

$$\lim_{n\to\infty} d(gTx_n, Tgx_n) = 0.$$

Remark 3.1.4. [33]

"Every compatible pair is $\mathcal{R}$-compatible, whatever the binary relation $\mathcal{R}$. Particularly, under the universal relation, the notion of $\mathcal{R}-$compatibility coincides with the usual compatibility."

3.1.11. d-Self Closed [32]

"Let (M,d) be a metric space, a binary relation $\mathcal{R}$ on M is said to be d-self closed if for any $\mathcal{R}$-preserving sequence $\{x_n\} \subseteq M$ such that $\{x_n\} \to x$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $[x_{n_k}, x] \in \mathcal{R}\ \forall\ k \in \mathbb{N}_0$."

Example 3.1.3.

Assume (M,d) be a complete metric space and $M=[0,2]$ endowed with $d=|k-l|$. Define

$$\mathcal{R}=\{(0,0),(0,1),(1,0),(1,1),(0,2)\}$$

on K and $T:M\to M$ is represented by

$$T(k)=\begin{cases}0, & if \quad 0\leq k\leq 1\\ 1, & if \quad 1\leq k\leq 2.\end{cases}$$

Consider $\{k_m\}$ be an $\mathcal{R}$ - preserving sequence such that

$k_m\to k$

so that $(k_m,k_{m+1})\in\mathcal{R}\ \ \forall\ m\in\mathbb{N}_0$,

observe that

$$(k_m,k_{m+1})\notin\{(0,2)\}$$

so that

$$(k_m,k_{m+1})\in\{(0,0),(0,1),(1,0),(1,1)\}\quad\forall\ m\in\mathbb{N}_0,$$

implies $\{k_m\}\subset\{0,1\}$.

Since $\{0,1\}$ is closed, so $[k_m,k]\in\mathcal{R}$. Hence $\mathcal{R}$ is d-self closed.

3.1.12. Path [1]

"For $x,y\in X$, a path of length p ($p\in\mathbb{N}$) in $\mathcal{R}$ from x to y is a finite sequence $\{u_0,u_1,......,u_p\}\subseteq X$ such that $u_0=x$, $u_p=y$, and $(u_i,u_{i+1})\in\mathcal{R}$ for each $i\in\{0,1,.....,p-1\}$."

3.1.13. $\mathcal{R}$-Connected [33]

"A subset $E\subseteq X$ is said to be $\mathcal{R}$- connected if, for each $x,y\in E$, there exists a path in $\mathcal{R}$ from x to y." [33]

Lemma 3.1.1. [34, 35]

Consider a metric space (X,d) and a sequence $\{k_m\}$ in X. If $\{k_m\}$ is not Cauchy in X, then $\exists\, \epsilon > 0$ and two subsequences $\{k_{m(j)}\}$ and $\{k_{t(j)}\}$ of $\{k_m\}$ such that

$$j \leq m(j) \leqslant t(j), \quad d(k_{m(j)}, k_{t(j)-1}) \leqslant \epsilon < d(k_{m(j)}, k_{t(j)})\ \forall j \in N_0$$

Moreover if $\{k_m\}$ is such that $\lim\limits_{m\to\infty} d(k_m, k_{m+1}) = 0$, then

$$\lim_{j\to\infty} d(k_{m(j)}, k_{t(j)}) = \lim_{j\to\infty} d(k_{m(j)-1}, k_{t(j)-1}) = \epsilon.$$

Proof.

If $\{k_m\}$ is not Cauchy in X, then $\exists\, \epsilon > 0$ and two subsequences $\{k_{m(j)}\}$ and $\{k_{t(j)}\}$ of $\{k_m\}$ such that

$$j \leqslant m(j) \leqslant t(j), \qquad \forall\ j \in N_0$$

and

$$d(k_{m(j)}, k_{t(j)-1}) \leqslant \epsilon \textit{ and } d(k_{m(j)}, k_{t(j)}) > \epsilon$$

then,

$$\epsilon < d(k_{m(j)}, k_{t(j)}) \leq d(k_{m(j)}, k_{t(j)-1}) + d(k_{t(j-1)}, k_{t(j)})$$

taking the $\lim_{j\to\infty}$ and using the assumption

$$\lim_{m\to\infty} d(k_m, k_{m+1}) = 0,$$

$$\Rightarrow \epsilon < d(k_{m(j)}, k_{t(j)}) \leq d(k_{m(j)}, k_{t(j)-1}) + 0$$

$$\Rightarrow \epsilon < d(k_{m(j)}, k_{t(j)}) \leq \epsilon$$

$$\Rightarrow d(k_{m(j)}, k_{t(j)}) = \epsilon,$$

similarly, we have

$$d(k_{m(j)-1}, k_{t(j)-1}) = \epsilon.$$

Lemma 3.1.2. [36]

"Let M be a non-empty set and $g : M \to M$. Then there exists a subset $E \subseteq M$ such that $g(E) = g(M)$ and $g : E \to E$ is one - one."

3.2 $(F, \mathcal{R})_g$-Contractions and Related Results

In [28], Alfaqih et al. has defined $(F, \mathcal{R})_g$-contractions and prove certain results.

3.2.1. $(F, \mathcal{R})_g$-Contractions [28]

"Let (M, d) be a metric space and $T, g : M \to M$. Then T is said to be an $(F, \mathcal{R})_g$-contraction if there exists $\tau > 0$ such that

$$\tau + F(d(Tx, Ty)) \leq F(d(gx, gy)) \tag{3.1}$$

for all $x, y \in M$ with $gx\mathcal{R}^{\#}gy$ and $Tx\mathcal{R}^{\#}Ty$, where $F : (0, \infty) \to \mathbb{R}$ is continuous mapping satisfying (F_2).

Due to the symmetricity of the metric d, the following proposition is immediate."

Proposition 3.2.1. [11]

"Let (M, d) be a metric space endowed with a transitive binary relation $\mathcal{R}$ and $T, g : M \to M$. Then for each continuous mapping $F : (0, \infty) \to \mathbb{R}$ satisfying (F_2), the following are equivalent:

1. for all $x, y \in M$ such that $(gx, gy) \in \mathcal{R}$ and $(Tx, Ty) \in \mathcal{R}$,

$$\tau + F(d(Tx, Ty)) \leq F(d(gx, gy)),$$

2. for all $x, y \in M$ such that either (gx, gy), $(Tx, Ty) \in \mathcal{R}$
   or
   (gy, gx), $(Ty, Tx) \in \mathcal{R}$,

$$\tau + F(d(Tx, Ty)) \leq F(d(gx, gy)).\text{"}$$

Theorem 3.2.2.

Consider a metric space (X, d) equipped with $\mathcal{R}$ where $\mathcal{R}$ is a transitive binary relation and $Q, g : X \to X$. Assume that the subsequent conditions are fulfilled:

(1) $\exists\ k_0 \in X$ such that $gk_0\mathcal{R}Qk_0$,

(2) $\mathcal{R}$ is (Q, g)-closed,

(3) Q is an $(F, \mathcal{R})_g$-contraction,

(4) (a) A subset K of X exists such that $Q(X) \subseteq K \subseteq g(X)$ and K is $\mathcal{R}$ -complete.

(b) One of the subsequent conditions is satisfied:

(i) Q is $(g, \mathcal{R})$-continuous, or

(ii) Q and g are continuous, or

(iii) $\mathcal{R}|K$ is d-self closed on condition that (3.1) holds for all $k, l \in X$ with $gk\mathcal{R}gl$ and $Qk\mathcal{R}^{\#}Ql$,

or on the other hand

(α) (α_1) $\exists$ a subset L of X such that $Q(X) \subseteq g(X) \subseteq L$ and L is $\mathcal{R}$- complete,

(α_2) (Q, g) is an $\mathcal{R}$- compatible pair,

(α_3) Q and g are $\mathcal{R}$-continuous.

Then (Q, g) has a coincidence point.

Proof. In the above two cases (4) and (α) we can see $Q(X) \subseteq g(X)$. Using assumption (1), we get $gk_0\mathcal{R}Qk_0$. If $Qk_0 = gk_0$, then coincidence point of (Q, g) is k_0 and it completes the proof.
Suppose that $Qk_0 \neq gk_0$, since $Q(X) \subseteq g(X)$, so there must exist $k_1 \in X$ such that $gk_1 = Qk_0$. Similarly, there is $k_2 \in X$ such that $gk_2 = Qk_1$. Proceeding in this way we can construct a sequence $\{k_m\} \subseteq X$ such that

$$gk_{m+1} = Qk_m \qquad \forall\ m \in \mathbb{N}_0. \tag{3.2}$$

Now we will prove an $\mathcal{R}$-preserving sequence $\{gk_m\}$, that is

$$gk_m\mathcal{R}gk_{m+1} \qquad \forall\ m \in \mathbb{N}_0. \tag{3.3}$$

By using induction we will prove this claim. If we put $m = 0$ in (3.2) and use condition (1), we get $gk_o\mathcal{R}gk_1$. Which implies that above statement holds for

$m = 0$. Suppose that (3.3) is accurate for $m = j \geqslant 1$, that is, $gk_j \mathcal{R} gk_{j+1}$. Since $\mathcal{R}$ is (Q, g)-closed, so we get $Qk_j \mathcal{R} Qk_{j+1}$, this yields that $gk_{j+1} \mathcal{R} gk_{j+2}$. Hence our claim is true $\forall\ m \in \mathbb{N}_0$.

By using (3.2) and (3.3), we can conclude that $\{Qk_m\}$ is also $\mathcal{R}$-preserving sequence, that is,

$$Qk_m \mathcal{R} Qk_{m+1} \qquad \forall\ m \in \mathbb{N}_0. \tag{3.4}$$

If $Qk_{m_0} = Qk_{m_{0+1}}$ for some $m_0 \in \mathbb{N}_0$, then we can conclude that k_{m_0} is a coincidence point of (Q, g).

Suppose to the contrary that $Qk_m \neq Qk_{m+1}\ \ \forall\ m \in \mathbb{N}_0$. With the help of (3.2), (3.3), (3.4) and condition (3), we can see that

$$\tau + F\Big(d(gk_m, gk_{m+1})\Big) = \tau + F\Big(d(Qk_{m-1}, Qk_m)\Big) \leq F\Big(d(gk_{m-1}, gk_m)\Big), \tag{3.5}$$

for all $m \in \mathbb{N}_0$.

Denote $\gamma_m = d(gk_m, gk_{m+1})$, with the help of equation (3.5) and condition (3) we obtain

$$F(\gamma_m) \leq F(\gamma_{m-1}) - \tau \leq F(\gamma_{m-2}) - 2\tau..... \leq F(\gamma_0) - m\tau \quad (\forall\ m \in \mathbb{N}).$$

Taking $m \to \infty$ in above inequality, we obtain

$$\lim_{m\to\infty} F(\gamma_m) = -\infty,$$

which together with (F_2) implies that

$$\lim_{m\to\infty} \gamma_m = 0. \tag{3.6}$$

Now, we will show that $\{gk_m\}$ is a Cauchy sequence. To do this assume to the contrary that $\{gk_m\}$ is not a Cauchy sequence. Using Lemma (3.1.1) and equation

(3.6) guarantees the existence of $\epsilon > 0$ and two subsequences $\{gk_{m_j}\}$ and $\{gk_{t_j}\}$ of $\{gk_m\}$ such that

$$d(gk_{m(j)}, gk_{t(j-1)}) \leq \epsilon < d(gk_{m(j)}, gk_{t(j)})$$

and

$$j \leq m(j) \leq t(j), \quad j \in \mathbb{N}_0$$

and

$$\lim_{j\to\infty} d\Big(gk_{m(j)}, gk_{t(j)}\Big) = d\Big(gk_{m(j)-1}, gk_{t(j)-1}\Big) = \epsilon \tag{3.7}$$

This implies that $\exists\ j_0 \in \mathbb{N}_0$ Such that $d(gk_{m(j)-1}, gk_{t(j)-1}) > 0\ \forall j \geq j_0$. Since $\mathcal{R}$ is transitive, so we have $gk_{m(j)-1}\mathcal{R}^{\sharp}gk_{t(j)-1}$ and $Qk_{m(j)-1}\mathcal{R}^{\sharp}Qk_{t(j)-1}\ \ \forall\ \ j \geq j_0$.

Using condition (3), we have

$$\tau + F\Big(d(Qk_{m(j)-1}, Qk_{t(j)-1})\Big) \leqslant F\Big(d(gk_{m(j)-1}, gk_{t(j)-1})\Big) \qquad \forall\ j \geqslant j_0. \tag{3.8}$$

Since F is continuous, let $j \to \infty$ in above equation and using (3.7), we obtain $\tau + F(\epsilon) \leqslant F(\epsilon)$, since $\tau > 0$ we have a contradiction to the fact that $\{gk_m\}$ is a cauchy sequence.

Suppose that condition (4) is true. With the help of (3.2) we obtain $\{gk_m\} \subseteq Q(X)$. Therefore, $\{gk_m\}$ is an $\mathcal{R}$- Preserving Cauchy sequence in K. By utilizing $\mathcal{R}$-completeness of K, $\exists\ l \in K$ Such that $\{gk_m\} \to l$.

As $K \subseteq g(X)$, $\exists\ v \in X$ such that $l = gv$. Hence by using (3.2), we acquired

$$\lim_{m\to\infty} gk_m = \lim_{m\to\infty} Qk_m = gv. \tag{3.9}$$

In order to prove that v is coincidence point of (Q, g), we will use three different cases of condition (b).

First of all, suppose that Q is $(g,\mathcal{R})$-continuous. By utilizing (3.3) and (3.9), we get

$$\lim_{m\to\infty} Qk_m = Qv. \tag{3.10}$$

By utilizing (3.9) and (3.10), we get

$$Qv = gv.$$

This shows that v is a coincidence point of (Q, g).

Now suppose second case of (b) that is Q and g are continuous. Since $X \neq 0$ and $g : X \to X$, then by using Lemma (3.1.2), $\exists\ B \subseteq X$ such that $g(B) = g(X)$ and $g : B \to B$ is one-one.

Define a mapping $f : g(B) \to g(X)$ by

$$f(gb) = Qb \qquad \forall\ gb \in g(B) \qquad \text{where } b \in B. \tag{3.11}$$

Since g is one-one and $Q(X) \subseteq g(X)$ implies that f is well- defined mapping. As Q and g are continuous implies that f is also continuous. Now utilizing the truth that $g(X) = g(B)$.

Because $g(X) = g(B)$ we can rewrite condition (a) as $Q(X) \subseteq K \subseteq g(B)$. So that, without loss of generality, we can select a sequence $\{k_m\}$ in B and $v \in B$. By using (3.9), (3.11) and continuity of f, we have

$$Q(v) = f(gv) = f(\lim_{m\to\infty} gk_m) = \lim_{m\to\infty} f(gk_m) = \lim_{m\to\infty} Qk_m = gv.$$

This implies that v is a coincidence point of (Q, g).

Finally suppose that (iii) of (b) holds which implies that $\mathcal{R}|K$ is d-self closed and (3.1) detain $\forall\ k, l \in X$ with $gk\mathcal{R}gl$ and $Qk\mathcal{R}^{\#}Ql$. As $\{gk_m\} \subseteq K$, $\{gk_m\}$ is $\mathcal{R}|K$ preserving due to (3.3) and with the help of (3.9) $\{gk_m\} \to gv$. So that $\exists$ a sub-sequence $\{gk_{m_j}\} \subseteq \{gk_m\}$ such that

$$[gk_{m_j}, gv] \in \mathcal{R}|K \subseteq \mathcal{R} \qquad \forall\ j \in \mathbb{N}_0 \tag{3.12}$$

Utilizing condition (2) and (3.12), we obtained

$$[Qk_{m_j}, Qv] \in \mathcal{R}|K \subseteq \mathcal{R} \qquad \forall\ j \in \mathbb{N}_0. \tag{3.13}$$

Now, let $q = \{j \in \mathbb{N} : Qk_{m_j} = Qv\}$. If q is infinite, then $\{Qk_{m_j}\}$ has a subsequence $\{Qk_{m_{j_p}}\}$, such that $Qk_{m_{j_p}} = Qv$. This implies that $\lim\limits_{p\to\infty} Qk_{m_{j_p}} = Qv\ \forall p \in \mathbb{N}$. By using (3.9), we have $\lim\limits_{m\to\infty} Qk_m = gv$. So we obtain $Qv = gv$.

If q is finite, then Qk_{m_j} has a subsequence $Qk_{m_{j_p}}$ such that $Qk_{m_{j_p}} \neq Qv\ \forall\ p \in \mathbb{N}$. With the help of (3.12), (3.13) and $Qk_{m_{j_p}} \neq Qv\ \forall\ p \in \mathbb{N}$, we have

$$[gk_{m_{j_p}}, gv] \in \mathcal{R} \setminus K \subseteq \mathcal{R} \qquad \forall\ p \in \mathbb{N}_0. \tag{3.14}$$

and

$$[Qk_{m_{j_p}}, Qv] \in \mathcal{R} \setminus K \subseteq \mathcal{R} \ \ \mathit{and} \ \ \ Qk_{m_{j_p}} \neq Qv \qquad \forall\ p \in \mathbb{N}_0. \tag{3.15}$$

Now with the help of (3.14), (3.15), proposition (3.2.1), and the fact that (3.1) satisfied, we obtain

$$F\Big(d(Qk_{m_{j_p}}, Qv)\Big) \leq F\Big(d(gk_{m_{j_p}}, gv)\Big) - \tau.$$

By using (3.9), (F_2) and taking $p \to \infty$, we get

$$\lim_{j\to\infty} Qk_{m_j} = Qv. \tag{3.16}$$

From (3.9) and (3.16), we obtain

$$Qv = gv.$$

Hence v is a coincidence point of (Q, g) in both cases either q is finite or infinite. Now if (α) holds then $\{gk_m\} \subseteq L$, and hence $\{gk_m\}$ is $\mathcal{R}$-preserving Cauchy sequence in L. Since L is $\mathcal{R}$- complete. This implies that $u \in L$ such that

$$\lim_{m\to\infty} gk_m = u. \tag{3.17}$$

Using equations (3.2) and (3.17, we obtain

$$\lim_{m\to\infty} Qk_m = u. \tag{3.18}$$

Now with the help of (3.3) ,(3.17) and continuity of g, we obtain

$$\lim_{m\to\infty} g(gk_m) = g(\lim_{m\to\infty} gk_m) = gu. \tag{3.19}$$

Utilizing (3.4), (3.18) and continuity of g to obtain

$$\lim_{m\to\infty} g(Qk_m) = g(\lim_{m\to\infty} Qk_m) = gu. \tag{3.20}$$

As Qk_m and gk_m are $\mathcal{R}$-Preserving due to (3.3), (3.4) and

$$\lim_{m\to\infty} Qk_m = \lim_{m\to\infty} gk_m = u.$$

Now using (3.18), (3.17) and condition (α_2),

$$\lim_{m\to\infty} d(gQk_m, Qgk_m) = 0. \tag{3.21}$$

Next, we will demonstrate that coincidence point of (Q,g) is u. Making use of (3.3), (3.17) and the $\mathcal{R}$- continuity of Q, we get

$$\lim_{m\to\infty} Q(gk_m) = Q(\lim_{m\to\infty} gk_m) = Qu. \tag{3.22}$$

With the use of (3.20), (3.21), (3.22), we obtain

$$\begin{aligned} d(gu, Qu) =& d(\lim_{m\to\infty} gQk_m, \lim_{m\to\infty} Qgk_m) \\ =& \lim_{m\to\infty} d(gQk_m, Qgk_m) = 0. \\ \Rightarrow& Qu = gu. \end{aligned}$$

This implies that u is a coincidence point of (Q, g).

Theorem (3.2.2) does not guarantees the uniqueness of coincidence point. The following theorem guarantees that coincidence point is unique.

Theorem 3.2.3. [16]
"If, in addition to hypothesis $(1-4)$ of theorem (3.2.2), we assume that for all distinct coincidence point $u, v \in$ coin (T, g), gu and gv are $\mathcal{R}$ - compareable and one of T and g is one-one, then (T, g) has a unique coincidence point."
Proof. The set coin (T, g) is nonempty, because of theorem (3.2.2). Consider two elements $u, v \in coin(T, g)$, then by definition of $coin(T, g)$, we have $[gv, gu] \in \mathcal{R}$ and $Tu = gu$, $Tv = gv$. This implies $[Tu, Tv] \in \mathcal{R}$.
Now if $gu = gv$, we obtain $Tv = gv = gu = Tu$, and hence $v = u$, since one of T and g is one-one.
By utilizing condition (3) and Proposition (3.2.1), we obtain

$$\tau + F\Big(d(Tu, Tv)\Big) \leqslant F\Big(d(gu, gv)\Big) = F\Big(d(Tu, Tv)\Big).$$

This is a contradiction as $\tau > 0$. Therefore a unique coincidence point of (Q, g) exists.

Next theorem guarantees the existence of unique common fixed point.

Theorem 3.2.4. [16]
"If in addition to hypothesis of above theorem, we assume that (T, g) is a weakly compatible pair, then the pair (T, g) has a unique common fixed point."
Proof. Above theorem assures that (T, g) has a unique coincidence point. Let v be the common coincidence point and suppose $z \in X$ be such that

$$z = Tv = gv.$$

Since T and g are weakly compatible, we acquired $Tz = Tgv = gTv = gz$. Which implies z is a coincidence point of T and g. Since v is unique implies $z = v$. Which implies uniqueness of common fixed point. Since all the assumptions of Theorem (3.1.2) are true implies the set coin (T, g) is nonempty.

Theorem 3.2.5. [16]

"Let (M, d) be a metric space endowed with a transitive binary relation $\mathcal{R}$ and $T : M \to M$. Assume that the following conditions are fulfilled:

(1) There exists $x_0 \in M$ such that $x_0 \mathcal{R} T x_0$,

(2) $\mathcal{R}$ is T-closed,

(3) T is an $(F, \mathcal{R})$ -contraction,

(4) (α) There exists a subset X of M such that $T(M) \subseteq X$ and X is $\mathcal{R}$ -complete,

(η) one of the following holds :

(i) T is $\mathcal{R}$-continuous; or

(ii) $\mathcal{R}|X$ is d-self closed provided Definition (3.1.1) holds for all $x, y \in M$ with $x\mathcal{R}y$ and $Tx\mathcal{R}^{\pi}Ty$. Then T has a fixed point. Moreover, if

(e) $[u, v] \in Fix(T)$ implies that $[u, v] \in \mathcal{R}$,

then T has a unique fixed point."

Theorem 3.2.6. [16]

"If condition (e) of above theorem is replaced by following:
(e*) $Fix(T)$ is $\mathcal{R}^s$-connected,
then the fixed point of T is unique."

Proof. Assume on contrary that T has more than one fixed points say u and v with $u \neq v$. Then there exist a path $\mathcal{R}^s \subseteq Fix(T)$. As it is from v to u of length q. Let us denote the path by $\{v_0, \cdots v_q\}$ such that $v_p \neq v_{p+1}$ for each p where $0 \leqslant p \leqslant j-1$. If $v = u$, it is a contradiction. so that

$$v_0 = v, \quad v_q = u \quad and \quad [v_p, v_{p+1}] \in \mathcal{R} \quad for\ each\ p\ (0 \leqslant p \leqslant q-1).$$

As $v_p \in Fix(T)$ implies that $Tv_p = v_p$ for each $p \in \{0, 1, ..., q\}$. With the help of condition (c), we have

$$\tau + F(v_p, v_{p+1}) \leqslant F(v_p, v_{p+1}) \quad \forall\ p\ (0 \leqslant p \leqslant j-1). \tag{3.23}$$

Since $\tau > 0$, it is a contradiction. Hence T has a unique fixed point.

Example 3.2.1.

Consider $X = (0, \infty)$ be enriched with the usual metric.

Define a sequence $\{\pi_n\} \subseteq X$ as follows

$$\pi_n = \frac{n(n+1)(n+2)}{3} \quad \forall\ n \geqslant 1.$$

Define a binary relation $\mathcal{R}$ on X by

$$\mathcal{R} = \{(\pi_1, \pi_1), (\pi_p, \pi_{p+1}) : p \geq 1\}.$$

Consider $Q : X \to X$ in the following manner

$$Qk = \begin{cases} k, & \text{if } 0 \leq k \leq \pi_1; \\ \pi_1, & \text{if } \pi_1 \leq k \leq \pi_2; \\ \pi_p + \Big(\frac{\pi_{p+1} - \pi_p}{\pi_{p+2} - \pi_{p+1}}\Big)\Big(k - \pi_{p+1}\Big), & \textit{if} \ \ \pi_{p+1} \leq k \leq \pi_{p+2},\ p = 1, 2, \cdots \end{cases}$$

and

Define $g : X \to X$ as

$$gk = \pi_p + \Big(\frac{\pi_{p+1} - \pi_p}{\pi_{p+2} - \pi_{p+1}}\Big)\Big(k - \pi_p\Big), \quad \textit{if} \quad \pi_p \leq k \leq \pi_{p+1}, p = 1, 2, \cdots$$

Proof.

We can see that mapping Q is continuous. Also we can observe that

$$Q\pi_{p+1} = \pi_p.$$

$$g\pi_p = \pi_p.$$

Combining above, we have

$$Q\pi_{p+1} = g\pi_p.$$

Define $F : (0, \infty) \to \mathbb{R}$ and $F \in \mathcal{F}$ by

$$F(\beta) = \beta - \frac{1}{\beta},$$

Notice that if $gk\mathcal{R}^{\shortparallel}gl$ and $Qk\mathcal{R}^{\shortparallel}Ql$, then $k = \pi_p$, $l = \pi_{p+1}$ for some $p \in \mathbb{N} - 1$.

Observe first

for $u > n = 1$, we have

$$|Q(\pi_u) - Q(\pi_1)| = |\pi_{u-1} - \pi_1| = \frac{2 \times 3 \times 4}{3} + ... + \frac{u(u-1)(u+1)}{3}.$$

$$|\pi_u - \pi_1| = \frac{2 \times 3 \times 4}{3} + ... + \frac{u(u+1)(u+2)}{3}.$$

Since $u > 1$, we have

$$\frac{-1}{\frac{2\times3\times4}{3} + ... + \frac{u(u-1)(u+1)}{3}} < \frac{-1}{\frac{2\times3\times4}{3} + ... + \frac{u(u+1)(u+2)}{3}}$$

we have

$$\begin{aligned}
&6 - \frac{1}{\frac{2\times3\times4}{3} + ... + \frac{u(u-1)(u+1)}{3}} < 6 - \frac{1}{\frac{2\times3\times4}{3} + ... + \frac{u(u+1)(u+2)}{3}},\\
&\Rightarrow 6 - \frac{1}{\frac{2\times3\times4}{3} + ... + \frac{u(u-1)(u+1)}{3}} + \frac{2 \times 3 \times 4}{3} + ... + \frac{u(u-1)(u+1)}{3}\\
&< 6 - \frac{1}{\frac{2\times3\times4}{3} + ... + \frac{u(u+1)(u+2)}{3}} + \frac{2 \times 3 \times 4}{3} + ... + \frac{u(u-1)(u+1)}{3},\\
&\leq \frac{-1}{\frac{2\times3\times4}{3} + ... + \frac{u(u+1)(u+2)}{3}} + \frac{2 \times 3 \times 4}{3} + ... + \frac{u(u-1)(u+1)}{3}\\
&+ \frac{u(u+1)(u+2)}{3},\\
&= \frac{1}{\frac{2\times3\times4}{3} + ... + \frac{u(u+1)(u+2)}{3}} + \frac{2 \times 3 \times 4}{3} + ... + \frac{u(u+1)(u+2)}{3},
\end{aligned}$$

so, it takes form

$$\begin{aligned}
6 - \frac{1}{|Q(\pi_u) - Q(\pi_1)|} + |Q(\pi_u) - Q(\pi_1)| = 6 + |\pi_{u-1} - \pi_1| - \frac{1}{|\pi_{u-1} - \pi_1|}\\
\leq |\pi_u - \pi_1| - \frac{1}{|\pi_u - \pi_1|}\\
= |g\pi_u - g\pi_1| - \frac{1}{|g\pi_u - g\pi_1|}.
\end{aligned}$$

For $u > n > 1$, we have

$$
\begin{aligned}
6 + |Q\pi_u - Q\pi_n| - \frac{1}{|Q\pi_u - Q\pi_n|} &= 6 + |\pi_{u-1} - \pi_{n-1}| - \frac{1}{|\pi_{u-1} - \pi_{n-1}|} \\
&\leq |\pi_u - \pi_n| - \frac{1}{|\pi_u - \pi_n|} \\
&= |g\pi_u - g\pi_n| - \frac{1}{|g\pi_u - g\pi_n|}.
\end{aligned}
$$

Consequently, $6 + F\Big(d(Qk, Ql)\Big) \leq F\Big(d(gk, gl)\Big)$ for all $k, l \in K$ such that $gk\mathcal{R}^{\sharp}gl$ and $Qk\mathcal{R}^{\sharp}Ql$. Hence It is prove that Q is $(F, \mathcal{R})_g$.

3.3 Some Consequences in Ordered Metric Spaces

3.3.1. Ordered Metric Space

Let (M, d) be a metric space and $(M, \preceq)$ an ordered set, then triplet $(M, d, \preceq)$ is known as an ordered metric space.

3.3.2. g-Increasing [37]

"Let $(M, \preceq)$ be an ordered set and $T, g : M \to M$. Then T is said to be g-increasing if, for any $x, y \in M$, $gx \preceq gy$ implies that $Tx \preceq Ty$."

Remark 3.3.1. [37]

"Observe that the notion of T is g-increasing is equivalent to saying that $\preceq$ is (T, g)-closed."

On setting $\mathcal{R} = \preceq$ in Theorem (3.2.2) to (3.2.4) and using Remark (3.3.1), we obtain the following result.

Corollary 1. [16]

"Let $(M, d, \preceq)$ an ordered metric space and $T, g : M \to M$. Assume that the following conditions are fulfilled:

(a) There exists $x_0 \in M$ such that $gx_0 \preceq Tx_0$,

(b) *T is* g-increasing,

(c) There exists $\tau > 0$ and a continuous function F satisfying (F_2) such that

$$\tau + F\Big(d(Tx,Ty)\Big) \leq F\Big(d(gx,gy)\Big)\ \forall\ x,y \in M \ \textit{with}\ gx \prec gy\ \textit{and}\ Tx \prec Ty.$$

(d) There exists a subset X of M such that $T(M) \subseteq X \subseteq g(M)$ and X is $\preceq$ -complete,

(e) either T is $(g,\preceq)$-continuous or T and g are continuous, then the pair (T,g) has a coincidence point. If in addition we assume that

(f) for all distinct coincidence points u,v $\in$coin(T,g), Tu and gv are $\preceq$ - compareable, then (T,g) has a unique coincidence point. Furthermore, if T and g are weakly compatible,then the pair (T,g) has a unique common fixed point.

On setting $\mathcal{R} =\preceq$ in Theorem (3.1.6) and using Remark 3.3.1, we deduce the following result."

Corollary 2. [16]
"Let $(M,d,\preceq)$ an ordered metric space and $T : M \to M$. Assume that the following conditions are fulfilled:

(a) There exists $x_0 \in M$ such that $x_0 \preceq Tx_0$.

(b) T is $\preceq$ -increasing.

(c) There exists $\tau > 0$ and a continuous function F satisfying (F_2) such that

$$\tau + F\Big(d(Tx,Ty)\Big) \leq F\Big(d(x,y)\Big)\ \forall\ x,y \in M\ \textit{with}\ x \prec y\ \textit{and}\ Tx \prec Ty.$$

(d) There exists subset K of X such that $Q(X) \subseteq K$ and K is $\preceq$ -complete,

(e) T is $\preceq$-continuous, then T has a common fixed point.
Moreover, if

(f) $u,v \in$ Fix (T) implies that $[u,v] \in\preceq$ then T has a unique fixed point."

Chapter 4

Relation-theoretic Coincidence and Common Fixed Point Results under $(F_w, \mathcal{R})_g$-Contractions

In current chapter we have introduced the notion of $(F_w, \mathcal{R})g$ -contractions and prove results of coincidence, common fixed points for such contraction using the idea of Alfaqih et al. [16]. We also prove some consequences in ordered metric space and give an example to explain our new notion.

4.1 F-weak Contraction with Binary Relation $\mathcal{R}$ under g

4.1.1. $(F_w, \mathcal{R})_g$**-contractions**

Consider a metric space (X, d) endowed with a transitive binary relation $\mathcal{R}$ on X. A self mapping $(Q, g) : X \to X$ is called an $(F_w, \mathcal{R})_g$ -contraction if $\exists\ \tau > 0$ such that

$$\tau + F\Big(d(Qk, Ql)\Big) \leqslant F\left(max\left\{d\Big(gk, gl\Big), d\Big(gk, Qk\Big), d\Big(gl, Ql\Big), \frac{d\Big(gk, Ql\Big) + d\Big(gl, Qk\Big)}{2}\right\}\right) \tag{4.1}$$

for all $k, l \in X$ with $gk\mathcal{R}^{\#}gl$ and $Qk\mathcal{R}^{\#}Ql$.

Where $F : (0, \infty) \to \mathbb{R}$ is a continuous mapping which satisfies (F_2).

Remark 4.1.1.

Every $(F, \mathcal{R})_g$ contraction is $(F_w, \mathcal{R})_g$ contraction, but converse of statement is not true.

Proposition 4.1.1.

Consider (X, d) be a metric space equipped with a transitive binary relation $\mathcal{R}$ and $Q, g : X \to X$.

Then for each continuous mapping $F : (0, \infty) \to \mathbb{R}$ which satisfies (F_2), the following are equivalent:

(a) $\forall\ k, l \in X$ such that $\Big(gk, gl\Big) \in \mathcal{R}$ and $\Big(Qk, Ql\Big) \in \mathcal{R}$,

$$\tau + F\Big(d(Qk, Ql)\Big) \leqslant F\left(max\left\{d\Big(gk, gl\Big), d\Big(gk, Qk\Big), \Big(gl, Ql\Big), \frac{d\Big(gk, Ql\Big) + d\Big(gl, Qk\Big)}{2}\right\}\right).$$

(b) $\forall\ k, l \in X$ such that either $\Big(gk, gl\Big)$, $\Big(Qk, Ql\Big) \in \mathcal{R}$

or

$\Big(gl, gk\Big)$, $\Big(Ql, Qk\Big) \in \mathcal{R}$,

$$\tau + F\Big(d(Qk, Ql)\Big) \leqslant F\left(max\left\{d\Big(gk, gl\Big), d\Big(gk, Qk\Big), \Big(gl, Ql\Big), \frac{d\Big(gk, Ql\Big) + d\Big(gl, Qk\Big)}{2}\right\}\right).$$

Theorem 4.1.2.

Consider a metric space (X, d) equipped with $\mathcal{R}$ (where $\mathcal{R}$ is a transitive binary relation) and $Q, g : X \to X$. Suppose that the subsequent conditions are fulfilled:

(1) $\exists\ k_0 \in X$ such that $gk_0 \mathcal{R} Qk_0$,

(2) $\mathcal{R}$ is (Q, g)-closed,

(3) Q is an $(F_w, \mathcal{R})_g$-contraction,

(4) (a) A subset K of X exists such that $Q(X) \subseteq K \subseteq g(X)$ and K is $\mathcal{R}$ - complete.

(b) One of the subsequent conditions is fulfilled:

(i) Q is $(g, \mathcal{R})$-continuous, or

(ii) Q and g are continuous, or

(iii) $\mathcal{R}|K$ is d-self closed on condition that (4.1) holds for all $k, l \in X$ with $gk\mathcal{R}gl$ and $Qk\mathcal{R}^{\neq}Ql$,

or on the other hand

(α) (α_1) $\exists$ a subset L of X such that $Q(X) \subseteq g(X) \subseteq L$ and L is $\mathcal{R}$- complete,

(α_2) (Q, g) is an $\mathcal{R}$- compatible pair,

(α_3) Q and g are $\mathcal{R}$-continuous,

then (Q, g) has a coincidence point.

Proof. In the above two cases (4) and (α) we can see $Q(X) \subseteq g(X)$. Using assumption (1), we get $gk_0 \mathcal{R} Qk_0$.
If $Qk_0 = gk_0$, then coincidence point of (Q, g) is k_0 and it completes the proof. Suppose that $Qk_0 \neq gk_0$, since $Q(X) \subseteq g(X)$, so there must exist $k_1 \in X$ such that $gk_1 = Qk_0$. Similarly, there is $k_2 \in X$ such that $gk_2 = Qk_1$. Proceeding in this way we can construct a sequence $\{k_m\} \subseteq X$ such that

$$gk_{m+1} = Qk_m \qquad \forall\ m \in \mathbb{N}_0. \tag{4.2}$$

Now we will prove $\{gk_m\}$ is an $\mathcal{R}$-preserving sequence, that is

$$gk_m \mathcal{R} gk_{m+1} \qquad \forall\ m \in \mathbb{N}_0. \tag{4.3}$$

By using induction we will prove this claim. If we put $m = 0$ in (4.2) and use condition (1), we get $gk_o \mathcal{R} gk_1$. Which implies that above statement holds for $m = 0$. Suppose that (4.3) is accurate for $m = j \geqslant 1$, that is, $gk_j \mathcal{R} gk_{j+1}$.
Since $\mathcal{R}$ is (Q, g)-closed, so we get $Qk_j \mathcal{R} Qk_{j+1}$, this yields that $gk_{j+1} \mathcal{R} gk_{j+2}$. Hence our claim is true $\forall\ m \in \mathbb{N}_0$.
By using (4.2) and (4.3), we can conclude that $\{Qk_m\}$ is also $\mathcal{R}$-preserving sequence, that is,

$$Qk_m \mathcal{R} Qk_{m+1} \qquad \forall\ m \in \mathbb{N}_0. \tag{4.4}$$

If $Qk_{m_0} = Qk_{m_{0+1}}$ for some $m_0 \in \mathbb{N}_0$, then we can conclude that k_{m_0} is a coincidence point of (Q, g).
Suppose to the contrary that $Qk_m \neq Qk_{m+1}\ \ \forall\ m \in \mathbb{N}_0$. With the help of (4.2), (4.3), (4.4) and condition (3), we can see that

$$\tau + F\Big(d(gk_m, gk_{m+1})\Big) = \tau + F\Big(d(Qk_{m-1}, Qk_m)\Big) \leq F\Big(max\Big\{d(gk_{m-1}, gk_m), \\ d(gk_{m-1}, Qk_{m-1}), d(gk_m, Qk_m), \frac{d(gk_{m-1}, Qk_m) + d(gk_m, Qk_{m-1})}{2}\Big\}\Big)\ \forall\ m \in \mathbb{N}_0. \tag{4.5}$$

Denote A as

$$d(gk_{m-1}, gk_m), d(gk_{m-1}, Qk_{m-1}), d(gk_m, Qk_m), \frac{d(gk_{m-1}, Qk_m) + d(gk_m, Qk_{m-1})}{2}$$

$$\text{If } \max\{A\} = d(gk_{m-1}, gk_m) \qquad \therefore gk_{m+1} = Qk_m$$

then

$$\tau + F\Big(d(Qk_{m-1}, Qk_m)\Big) \leq F\Big(d(gk_{m-1}, gk_m)\Big) = F\Big(d(Qk_{m-2}, Qk_{m-1})\Big)$$

$$\Rightarrow F\Big(d(Qk_{m-1}, Qk_m)\Big) \leq F\Big(d(Qk_{m-2}, Qk_{m-1})\Big) - \tau$$

$$
\begin{aligned}
&\leq F\Big(d(gk_{m-2}, gk_{m-1})\Big) - 2\ \tau \\
&= F\Big(d(Qk_{m-3}, Qk_{m-2})\Big) - 2\ \tau \\
&\leq F\Big(d(gk_{m-3}, gk_{m-2})\Big) - 3\ \tau \\
&\ \vdots \\
&\leq F\Big(d(gk_0, gk_1)\Big) - m\ \tau.
\end{aligned}
$$

Denote $\gamma_m = d(gk_m, gk_{m+1})$, with the help condition (3) we obtain

$$F(\gamma_m) \leq F(\gamma_{m-1}) - \tau \leq F(\gamma_{m-2}) - 2\tau..... \leq F(\gamma_0) - m\tau \quad (\forall\ m \in \mathbb{N}).$$

taking $m \to \infty$ in above inequality, we obtain

$$\lim_{m\to\infty} F(\gamma_m) = -\infty.$$

Which together with (F_2) implies that

$$\lim_{m\to\infty} \gamma_m = 0.$$

$$\text{If } \max\{A\} = d(gk_{m-1}, Qk_{m-1}) \qquad \therefore gk_{m+1} = Qk_m$$

then, we have

$$
\begin{aligned}
\tau + F\Big(d(gk_m, gk_{m+1})\Big) &\leq F\Big(d(gk_{m-1}, Qk_{m-1})\Big) \\
\Rightarrow\ \tau + F\Big(d(Qk_{m-1}, Qk_m)\Big) &\leq F\Big(d(Qk_{m-2}, Qk_{m-1})\Big) \\
F\Big(d(Qk_{m-1}, Qk_m)\Big) &\leq F\Big(d(Qk_{m-2}, Qk_{m-1})\Big) -\ \tau \\
&\leq F\Big(d(gk_{m-2}, gk_{m-1})\Big) - 2\ \tau \\
&= F\Big(d(Qk_{m-3}, Qk_{m-2})\Big) - 2\ \tau
\end{aligned}
$$

$$\leq F\Big(d(gk_{m-3}, gk_{m-2})\Big) - 3\ \tau$$

$$\vdots$$

$$\leq F\Big(d(gk_0, gk_1)\Big) - m\ \tau.$$

Taking $m \to \infty$ in above inequality, we obtain

$$\lim_{m\to\infty} F(\gamma_m) = -\infty.$$

By using (F_2), we have

$$\lim_{m\to\infty} \gamma_m = 0.$$

$$\text{If } \max\{A\} = d(gk_m, Qk_m) \qquad \therefore gk_{m+1} = Qk_m$$

then

$$\begin{aligned}
\tau + F\Big(d(gk_m, gk_{m+1})\Big) &\leq F\Big(d(gk_m, Qk_m)\Big) \\
\Rightarrow F\Big(d(gk_m, gk_{m+1})\Big) &\leq F\Big(d(gk_m, Qk_m)\Big) - \ \tau \\
&= F\Big(d(Qk_{m-1}, Qk_m)\Big) - \ \tau \\
&\leq F\Big(d(gk_{m-1}, gk_m)\Big) - 2\ \tau \\
&= F\Big(d(Qk_{m-2}, Qk_{m-1})\Big) - 2\ \tau \\
&\leq F\Big(d(gk_{m-2}, gk_{m-1})\Big) - 3\ \tau \\
&\ \ \vdots \\
&\leq F\Big(d(gk_1, gk_2)\Big) - m\ \tau.
\end{aligned}$$

Taking $m \to \infty$ in above inequality, we obtain

$$\lim_{m\to\infty} F(\gamma_m) = -\infty.$$

With the help of (F_2), we have

$$\lim_{m\to\infty} \gamma_m = 0.$$

$$\text{If } \max\{A\} = \frac{d(gk_{m-1}, Qk_m) + d(gk_m, Qk_{m-1})}{2},$$

then, we have

$$\begin{aligned}
\tau + F\Big(d(gk_m, gk_{m+1})\Big) &\leq F\left(\frac{d(gk_{m-1}, Qk_m) + d(gk_m, Qk_{m-1})}{2}\right)\\
\Rightarrow F\Big(d(gk_m, gk_{m+1})\Big) &\leq F\left(\frac{d(gk_{m-1}, Qk_m) + d(gk_m, Qk_{m-1})}{2}\right) - \ \tau\\
&= F\left(\frac{d(Qk_{m-2}, Qk_m) + d(Qk_{m-1}, Qk_{m-1})}{2}\right) - \ \tau\\
&\leq F\left(\frac{d(gk_{m-2}, gk_m) + d(gk_{m-1}, gk_{m-1})}{2}\right) - 2\ \tau\\
&= F\left(\frac{d(Qk_{m-3}, Qk_{m-1}) + d(Qk_{m-2}, Qk_{m-2})}{2}\right) - 2\ \tau\\
&\leq F\left(\frac{d(gk_{m-3}, gk_{m-1}) + d(gk_{m-2}, gk_{m-2})}{2}\right) - 3\ \tau\\
&\ \vdots\\
&\leq F\left(\frac{d(gk_0, gk_2) + d(gk_1, gk_1)}{2}\right) - m\ \tau.
\end{aligned}$$

Taking $m \to \infty$ in above inequality, we obtain

$$\lim_{m\to\infty} F(\gamma_m) = -\infty.$$

Using (F_2), we have

$$\lim_{m\to\infty} \gamma_m = 0. \tag{4.6}$$

Now, we will show that $\{gk_m\}$ is a Cauchy sequence. To do this assume to the contrary that $\{gk_m\}$ is not a Cauchy sequence. Using Lemma (3.1.1) and equation (4.6) guarantees the existence of $\epsilon > 0$ and two subsequences $\{gk_{m_j}\}$ and $\{gk_{t_j}\}$ of $\{gk_m\}$ such that

$$d(gk_{m(j)}, gk_{t(j-1)}) \leqslant \epsilon < d(gk_{m(j)}, gk_{t(j)})$$

and

$$j \leqslant m(j) \leqslant t(j), \qquad \forall\, j \in \mathbb{N}_0$$

and

$$\lim_{j\to\infty} d\Big(gk_{m(j)}, gk_{t(j)}\Big) = d\Big(gk_{m(j)-1}, gk_{t(j)-1}\Big) = \epsilon. \tag{4.7}$$

This implies that $\exists\ j_0 \in \mathbb{N}_0$ Such that $d(gk_{m(j)-1}, gk_{t(j)-1}) > 0 \quad \forall j \geqslant j_0$.
Since $\mathcal{R}$ is transitive, so we have

$$gk_{m(j)-1}\mathcal{R}^{\sharp}gk_{t(j)-1} \text{ and } Qk_{m(j)-1}\mathcal{R}^{\sharp}Qk_{t(j)-1}\ \forall\ j \geqslant j_0.$$

Using condition (3), we have for all $j \geqslant j_0$.

$$\tau + F\Big(d(Qk_{m(j)-1}, Qk_{t(j)-1})\Big) \ \leq F \max\Bigg(d(gk_{m(j)-1}, gk_{t(j)-1}), d(gk_{m(j)-1}, Qk_{m(j)-1}),$$
$$d(gk_{t(j)-1}, Qk_{t(j)-1}),\ \frac{d(gk_{m(j)-1}, Qk_{t(j)-1}) + d(gk_{t(j)-1}, Qk_{m(j)-1})}{2}\Bigg). \tag{4.8}$$

Denote

$$d(gk_{m(j)-1}, gk_{t(j)-1}), d(gk_{m(j)-1}, Qk_{m(j)-1}), d(gk_{t(j)-1}, Qk_{t(j)-1}),$$

$$\frac{d(gk_{m(j)-1}, Qk_{t(j)-1}) + d(gk_{t(j)-1}, Qk_{m(j)-1})}{2} = B$$

$$\text{If } \max\{B\} = d(gk_{m(j)-1}, gk_{t(j)-1})$$

then, we have

$$\tau + F\Big(d(Qk_{m(j)-1}, Qk_{t(j)-1})\Big) \le F\Big(d(gk_{m(j)-1}, gk_{t(j)-1})\Big).$$

Since F is continuous, let $j \to \infty$ in above equation and using (4.7), we obtained

$$\tau + F(\epsilon) \le F(\epsilon).$$

$$\text{If } \max\{B\} = d(gk_{m(j)-1}, Qk_{m(j)-1})$$

then, we obtain

$$\begin{aligned}\tau + F\Big(d(Qk_{m(j)-1}, Qk_{t(j)-1})\Big) &\le F\Big(d(gk_{m(j)-1}, Qk_{m(j)-1})\Big) \\ &= F\Big(d(gk_{m(j)-1}, gk_{m(j)})\Big).\end{aligned}$$

Since F is continuous, let $j \to \infty$ in above equation and using (4.7), we obtain

$$\tau + F(\epsilon) \le -\infty.$$

$$\text{If } \max\{B\} = d(gk_{t(j)-1}, Qk_{t(j)-1})$$

then

$$\begin{aligned}\tau + F\Big(d(Qk_{m(j)-1}, Qk_{t(j)-1})\Big) &\le F\Big((d(gk_{t(j)-1}, Qk_{t(j)-1})\Big) \\ &= F\Big(d(gk_{t(j)-1}, gk_{t(j)})\Big).\end{aligned}$$

Since F is continuous, let $j \to \infty$ in above equation and using (4.7), we obtained

$$\tau + F(\epsilon) \ \leq -\infty.$$

$$\text{If } \max\{B\} = \frac{d(gk_{m(j)-1}, Qk_{t(j)-1}) + d(gk_{t(j)-1}, Qk_{m(j)-1})}{2},$$

then, we have

$$\tau + F\Big(d(Qk_{m(j)-1}, Qk_{t(j)-1})\Big) \ \leq F\Big(\frac{d(gk_{m(j)-1}, Qk_{t(j)-1}) + d(gk_{t(j)-1}, Qk_{m(j)-1})}{2}\Big)$$

$$\tau + F\Big(d(Qk_{m(j)-1}, Qk_{t(j)-1})\Big) \ \leq F\Big(\frac{d(gk_{m(j)-1}, gk_{t(j)}) + d(gk_{t(j)-1}, gk_{m(j)})}{2}\Big).$$

Since F is continuous, let $j \to \infty$ in above equation and using (4.7), we obtained

$$\tau + F(\epsilon) \leq F\Big(\frac{\epsilon + \epsilon}{2}\Big) = F(\epsilon)$$

$$\Rightarrow \ \tau + F(\epsilon) \ \leq F(\epsilon).$$

Since F is continuous, let $j \to \infty$ in above equation and using (4.7), we obtain $\tau + F(\epsilon) \leqslant F(\epsilon)$, since $\tau > 0$ we have a contradiction to the fact that $\{gk_m\}$ is a Cauchy sequence.

Suppose that condition (4) is true. With the help of (4.2) we obtain $gk_m \subseteq Q(X)$. Therefore, gk_m is an $\mathcal{R}$- preserving Cauchy sequence in K. By utilizing $\mathcal{R}$-completeness of K, $\exists\ l \in K$ Such that $gk_m \to l$.

As $K \subseteq g(X)$, $\exists\ v \in X$ such that $l = gv$. Hence by using (4.2), we acquired

$$\lim_{m\to\infty} gk_m = \lim_{m\to\infty} Qk_m = gv. \tag{4.9}$$

In order to prove that v is coincidence point of (Q, g), we will use three different cases of condition (b).

First of all, suppose that Q is $(g, \mathcal{R})$-continuous. By utilizing (4.3) and (4.9), we get

$$\lim_{m\to\infty} Qk_m = Qv. \tag{4.10}$$

By utilizing (4.9) and (4.10), we get

$$Qv = gv.$$

This shows that v is a coincidence point of (Q, g).

Now suppose second case of (b) that is Q and g are continuous. Since $X \neq 0$ and $g : X \to X$, then by using Lemma (3.1.2), $\exists\ B \subseteq X$ such that $g(B) = g(X)$ and $g : B \to B$ is one-one.
Define a mapping $f : g(B) \to g(X)$ by

$$f(gb) = Q(b) \qquad \forall\ gb \in g(B) \qquad \textit{where } b \in B. \tag{4.11}$$

Since g is one-one and $Q(X) \subseteq g(X)$ implies that f is well- defined mapping. As Q and g are continuous implies that f is also continuous. Now utilizing the truth that $g(X) = g(B)$. Because $g(X) = g(B)$ we can rewrite condition (a) as $Q(X) \subseteq K \subseteq g(B)$, so that, without loss of generality, we can select a sequence $\{k_m\}$ in B and $v \in B$. By using (4.9), (4.11) and continuity of f, we have

$$Qv = f(gv) = f(\lim_{m\to\infty} gk_m) = \lim_{m\to\infty} f(gk_m) = \lim_{m\to\infty} Qk_m = gv.$$

Finally assume that condition (iii) of (b) holds which implies that $\mathcal{R}|K$ is d-self closed and (4.1) detain $\forall\ k, l \in X$ with $gk\mathcal{R}gl$ and $Qk\mathcal{R}^{\#}Ql$. As $gk_m \subseteq K$, gk_m is $\mathcal{R}|K$ preserving due to (4.3) and with the help of (4.9) $\{gk_m\} \to gv$.
So that $\exists$ a subsequence $\{gk_{m_j}\} \subseteq \{gk_m\}$ such that

$$[gk_{m_j}, gv] \in \mathcal{R}\,|K \subseteq \mathcal{R} \qquad \forall\ j \in \mathbb{N}_0. \tag{4.12}$$

Utilizing condition (b) and (4.12), we obtained

$$[Qk_{m_j}, Qv] \in \mathcal{R}|K \subseteq \mathcal{R} \qquad \forall\ j \in \mathbb{N}_0. \tag{4.13}$$

Now, let $q = \{j \in \mathbb{N} : Qk_{m_j} = Qv\}$. If q is infinite, then $\{Qk_{m_j}\}$ has a subsequence $\{Qk_{m_{j_p}}\}$, such that $Qk_{m_{j_p}} = Qv$. This implies that $\lim_{p\to\infty} Qk_{m_{j_p}} = Qv\ \forall p \in \mathbb{N}$. By using (4.9), we have $\lim\limits_{m\to\infty} Qk_m = gv$. So we obtain $Qv = gv$.
If q is finite, then Qk_{m_j} has a subsequence $Qk_{m_{j_p}}$ such that $Qk_{m_{j_p}} \neq Qv\ \forall\ p \in \mathbb{N}$. With the help of (4.12), (4.13) and $Qk_{m_{j_p}} \neq Qv\ \forall\ p \in \mathbb{N}$, we have

$$[gk_{m_{j_p}}, gv]\ \in \mathcal{R}\ |K\ \subseteq\ \mathcal{R} \qquad \forall\ p\ \in \mathbb{N}_0 \tag{4.14}$$

and

$$[Qk_{m_{j_p}}, Qv]\ \in\ \mathcal{R}\ |K\ \subseteq\ \mathcal{R}\ \ and\ \ Qk_{m_{j_p}}\ \neq Qv\ \forall\ p\ \in \mathbb{N}_0. \tag{4.15}$$

Now with the help of (4.14), (4.15), proposition (4.1.1), and the fact that (4.1) satisfied, we obtain

$$F\Big(d(Qk_{m_{j_p}}, Qv)\Big) \leq F\Big(max\Big\{d(gk_{m_{j_p}}, gv), d(gk_{m_{j_p}}, Qk_{m_{j_p}}), d(gv, Qv), \frac{d(gk_{m_{j_p}}, Qv) + d(gv, Qk_{m_{j_p}})}{2}\Big\}\Big) - \tau.$$

Denote

$$d(gk_{m_{j_p}}, gv), d(gk_{m_{j_p}}, Qk_{m_{j_p}}), d(gv, Qv), \frac{d(gk_{m_{j_p}}, Qv) + d(gv, Qk_{m_{j_p}})}{2} = C$$

$$\text{If } \max\{C\} = d(gk_{m_{j_p}}, gv)$$

then, we have

$$\tau + F\Big(d(Qk_{m_{j_p}}, Qv)\Big)\ \leq F\Big(d(gk_{m_{j_p}}, gv)\Big)$$

$$\begin{aligned}
\Rightarrow F\Big(d(Qk_{m_{j_p}}, Qv)\Big) \ &\leq F\Big(d(gk_{m_{j_p}}, gv)\Big) - \ \tau \\
&= F\Big(d(Qk_{m_{j_{p-1}}}, Qk_m)\Big) - \ \tau \\
&\leq F\Big(d(gk_{m_{j_{p-1}}}, gk_m)\Big) - 2\ \tau \\
&\ \vdots \\
&\leq F\Big(d(gk_{m_{j_{p-(m-1)}}}, gk_2)\Big) - m\ \tau.
\end{aligned}$$

By using (4.9), (F_2) and taking $p \to \infty$, we get

$$\lim_{j\to\infty} Qk_{m_j} = Qv.$$

$$\text{If } max\{C\} = d(gk_{m_{j_p}}, Qk_{m_{j_p}}),$$

then

$$\tau + F\Big(d(Qk_{m_{j_p}}, Qv)\Big) \ \leq F\Big(d(gk_{m_{j_p}}, Qk_{m_{j_p}})\Big)$$

$$\begin{aligned}
\Rightarrow F\Big(d(Qk_{m_{j_p}}, Qv)\Big) \ &\leq F\Big(d(gk_{m_{j_p}}, Qk_{m_{j_p}})\Big) - \ \tau \\
&= F\Big(d(Qk_{m_{j_{p-1}}}, Qk_{m_{j_p}})\Big) - \ \tau \\
&\leq F\Big(d(gk_{m_{j_{p-1}}}, gk_{m_{j_p}})\Big) - 2\ \tau \\
&= F\Big(d(Qk_{m_{j_{p-2}}}, Qk_{m_{j_{p-1}}})\Big) - 2\ \tau \\
&\leq F\Big(d(gk_{m_{j_{p-2}}}, gk_{m_{j_{p-1}}})\Big) - 3\ \tau \\
&\ \vdots \\
&\leq F\Big(d(gk_{m_{j_{p-(m-1)}}}, gk_{m_{j_{p-(m-2)}}})\Big) - m\ \tau
\end{aligned}$$

By using (4.9), (F_2) and taking $p \to \infty$, we get

$$\lim_{j\to\infty} Qk_{m_j} = Qv.$$

$$\text{If } max\{C\} = d(gv, Qv),$$

then, we have

$$\begin{aligned}
\tau + F\Big(d(Qk_{m_{j_p}}, Qv)\Big) \ &\leq F\Big(d(gv, Qv)\Big) \\
\Rightarrow F\Big(d(Qk_{m_{j_p}}, Qv)\Big) \ &\leq F\Big(d(gv, Qv)\Big) - \ \tau \\
&= F\Big(d(gk_m, Qk_{m_j})\Big) - \ \tau \\
&= F\Big(d(Qk_{m-1}, Qk_{m_j})\Big) - \ \tau \\
&\leq F\Big(d(gk_{m-1}, gkm_j)\Big) - 2\ \tau \\
&\ \vdots \\
&\leq F\Big(d(gk_{m-(m-1)}, gk_{m_{j-(m-2)}})\Big) - m\ \tau
\end{aligned}$$

By using (4.9), (F_2) and taking $p \to \infty$, we get

$$\lim_{j\to\infty} Qk_{m_j} = Qv.$$

$$\text{If } max\{C\} = \frac{d(gk_{m_{j_p}}, Qv) + d(gv, Qk_{m_{j_p}})}{2}$$

then, we obtain

$$\begin{aligned}
\tau + F\Big(d(Qk_{m_{j_p}}, Qv)\Big) &\leq F\left(\frac{d(gk_{m_{j_p}}, Qv) + d(gv, Qk_{m_{j_p}})}{2}\right) \\
\Rightarrow F\Big(d(Qk_{m_{j_p}}, Qv)\Big) \ &\leq F\left(\frac{d(Qk_{m_{j_{p-1}}}, Qk_{m_j}) + d(Qk_m, Qk_{m_{j_p}})}{2}\right) - \ \tau
\end{aligned}$$

$$\leq F\left(\frac{d(gk_{m_{j_{p-1}}}, gk_{m_j}) + d(gk_m, gk_{m_{j_p}})}{2}\right) - 2\,\tau$$

$$\leq F\left(\frac{d(Qk_{m_{j_{p-2}}}, Qk_{m_{j-1}}) + d(Qk_{m-1}, Qk_{m_{j_{p-1}}})}{2}\right) - 2\,\tau$$

$$\leq F\left(\frac{d(gk_{m_{j_{p-2}}}, gk_{m_{j-1}}) + d(gk_{m-1}, gk_{m_{j_{p-1}}})}{2}\right) - 3\,\tau$$

$$\vdots$$

$$\leq \left(\frac{d(gk_{m_{j_{p-(m-1)}}}, gk_{m_{j-(m-2)}}) + d(gk_2, gk_{m_{j_{p-(m-2)}}})}{2}\right) - m\,\tau.$$

By using (4.9), (F_2) and taking $p \to \infty$, we get

$$\lim_{j\to\infty} Q(k_{m_j}) = Qv. \tag{4.16}$$

From (4.9) and (4.16), we obtain

$$Qv = gv.$$

Hence v is a coincidence point of Q, g in both cases either q is finite or infinite. Now if (α) holds then $gk_m \subseteq L$, and hence gk_m is $\mathcal{R}$-preserving Cauchy sequence in L. Since L is $\mathcal{R}$- complete. This implies that $u \in L$ such that

$$\lim_{m\to\infty} gk_m = u. \tag{4.17}$$

Using equations (4.2) and (4.17), we obtain

$$\lim_{m\to\infty} Qk_m = u. \tag{4.18}$$

Now with the help of (4.3) ,(4.17) and continuity of g, we obtain

$$\lim_{m\to\infty} g(gk_m) = g(\lim_{m\to\infty} gk_m) = gu. \tag{4.19}$$

Utilizing (4.4), (4.18) and continuity of g to obtain

$$\lim_{m\to\infty} g(Qk_m) = g(\lim_{k\to\infty} Qk_m) = gu. \tag{4.20}$$

As Qk_m and gk_m are $\mathcal{R}$-Preserving due to (4.3), (4.4) and

$$\lim_{m\to\infty} Qk_m = \lim_{m\to\infty} gk_m = u.$$

Now using (4.18), (4.17) and condition (α_2),

$$\lim_{m\to\infty} d(gQk_m, Qgk_m) = 0. \tag{4.21}$$

Next, we will demonstrate that coincidence point of (Q, g) is u. Making use of (4.3), (4.17) and the $\mathcal{R}$- continuity of Q, we get

$$\lim_{m\to\infty} Q(gk_m) = Q(\lim_{m\to\infty} gk_m) = Qu. \tag{4.22}$$

With the use of (4.20), (4.21), (4.22), we obtained

$$d(gu, Qu) = d(\lim_{m\to\infty} gQk_m, \lim_{m\to\infty} Qgk_m)$$

$$= \lim_{m\to\infty} d(gQk_m, Qgk_m) = 0$$

$$\Rightarrow Qu = gu.$$

This implies that u is a coincidence point of (Q, g).

Example 4.1.1.

Consider $X = (0, \infty)$ be enriched with the usual metric.

Define a sequence $\{\pi_n\} \subseteq X$ as follows

$$\pi_n = \frac{n(n+1)(n+2)}{3} \quad \forall\ n \geqslant 1.$$

Define a binary relation $\mathcal{R}$ on X by

$$\mathcal{R} = \{(\pi_1, \pi_1), (\pi_p, \pi_{p+1}) : p \geq 1\}.$$

Consider $Q : X \to X$ in the following manner

$$Qk = \begin{cases} k, & if\ 0 \leq k \leq \pi_1; \\ \pi_1, & if\ \pi_1 \leq k \leq \pi_2; \\ \pi_p + \Big(\frac{\pi_{p+1}-\pi_p}{\pi_{p+2}-\pi_{p+1}}\Big)\Big(k - \pi_{p+1}\Big), & if\, \pi_{p+1} \leq k \leq \pi_{p+2}, p = 1, 2, \cdots \end{cases}$$

and

define $g : X \to X$ as

$$gk = \pi_p + \Big(\frac{\pi_{p+1} - \pi_p}{\pi_{p+2} - \pi_{p+1}}\Big)\Big(k - \pi_p\Big), \quad if \quad \pi_p \leq k \leq \pi_{p+1}, p = 1, 2, ...$$

Proof. We can see that mapping Q is continuous. Also we can observe that

$$Q(\pi_{p+1}) = \pi_p.$$

$$g\pi_p = \pi_p.$$

Combining above, we have

$$Q(\pi_{p+1}) = g\pi_p.$$

Define $F : (0, \infty) \to \mathbb{R}$ where $F \in \mathcal{F}$ by

$$F(\beta) = \beta + \frac{1}{\beta}.$$

Notice that if $gk\mathcal{R}^{\#}gl$ and $Qk\mathcal{R}^{\#}Ql$, then $k = \pi_p$, $l = \pi_{p+1}$ for some $p \in \mathbb{N} - 1$. Additionaly, $\forall\, n, t \in \mathbb{N}$ such that $t > n > 1$, by using definition of $(F_w, \mathcal{R})_g$, we get

$$\tau + F\Big(d(Q\pi_t, Q\pi_n)\Big) \leq F\Big(\max\Big\{d(g\pi_t, g\pi_n), d(g\pi_t, Q\pi_t), d(g\pi_n, Q\pi_n), \frac{d(g\pi_t, Q\pi_n) + d(g\pi_n, Q\pi_t)}{2}\Big\}\Big). \tag{4.23}$$

First of all, we will check it for maximum.

Let $t = 7$, $n = 5$ $\therefore \pi_n = \frac{n(n+1)(n+2)}{3}$

$\pi_7 = 168$,$\pi_5 = 70$,$\pi_2 = 8$,$\pi_6 = 112$,$\pi_4 = 40$

$d(g\pi_n, g\pi_t) = 98$

$d(g\pi_t, Q\pi_t) = 56$

$d(g\pi_n, Q\pi_t) = 30$

$\frac{d(\pi_t,\pi_{n-1})+d(\pi_n,\pi_{t-1})}{2} = 85.$

Denote

$$d(g\pi_t, g\pi_n), d(g\pi_t, Q\pi_t), d(g\pi_n, Q\pi_n), \frac{d(g\pi_t, Q\pi_n) + d(g\pi_n, Q\pi_t)}{2} = A. \tag{4.24}$$

Here

$$\max\{A\} = d(g\pi_n, g\pi_t) \tag{4.25}$$

so equation (4.27) takes the form

$$\begin{aligned} 7 + |Q\pi_t - Q\pi_n| + \frac{1}{|Q\pi_t - Q\pi_n}| =& 7 + |\pi_{t-1} - \pi_{n-1}| + \frac{1}{|\pi_{t-1} - \pi_{n-1}|} \\ \leq& |g\pi_t - g\pi_n| + \frac{1}{|g\pi_t - g\pi_n|} \end{aligned}$$

Using values in above inequality, we get

$$7 + 72 + \frac{1}{72} \leq 98 + \frac{1}{98}$$

$$\Rightarrow 79.014 \leq 98.010.$$

Consequently, $7 + F(d(Qk, Ql)) \leq F(d(gk, gl))$ *for all* $k, l \in K$ such that $gk\mathcal{R}^{\sharp}gl$ and $Qk\mathcal{R}^{\sharp}Ql$. As a result Q is an $(F_w, \mathcal{R})_g$- contraction.

Theorem (4.1.2) does not guarantees the uniqueness of coincidence point. The following theorem guarantees that coincidence point is unique.

Theorem 4.1.3.

Suppose all hypothesis of theorem (4.1.2) are true except (α) and assume that gu and gv are $\mathcal{R}$ - compareable for all distinct coincidence points $u, v \in$coin (Q, g), and one of Q and g is one-one, then a unique coincidence point of (Q, g) exists.

Proof.

The set coin (Q, g) is nonempty, because of above theorem.

Consider two elements $u, v \in$ coin (Q, g), then by definition of coin(Q, g), we have $[gv, gu] \in \mathcal{R}$ and $Qu = gu$, $Qv = gv$. This implies $[Qu, Qv] \in \mathcal{R}$.

Now if $gu = gv$, we obtain $Qv = gv = gu = Qu$, and hence $v = u$, since one of Q and g is one-one.

If $gu \neq gv$, then by utilizing condition (3) and Proposition (4.1.1), we obtain

$$\begin{aligned}
&\tau + F\Big(d(Qu, Qv)\Big) \\
\leqslant &F\Big(d(gu, gv), d(gu, Qv), d(gv, Qv), \frac{d(gu, Qv) + d(gv, Qv)}{2}\Big) \\
= &F\Big(d(Qu, Qv)\Big).
\end{aligned}$$

Since $\tau > 0$, so our assumption is false. Therefore a unique coincidence point of (Q, g) exists.

Next theorem guarantees the existence of unique common fixed point.

Theorem 4.1.4.

Consider above theorem and add a condition that (Q, g) is a weakly compatible pair, then a unique common fixed point of (Q, g) exists.

Proof.

Above theorem assures that the pair (Q, g) has a unique coincidence point. Let v be the common coincidence point and suppose $z \in X$ be such that

$z = Qv = gv.$

Since Q and g are weakly compatible, we acquired $Qz = Qgv = gQv = gz$. Which implies z is a coincidence point of Q and g. Since v is unique implies $z = v$. Which implies uniqueness of common fixed point. Since all the assumptions of Theorem (4.1.3) are true implies the set coin (Q, g) is nonempty.

On setting $g = I$ we obtain the following theorem.

Theorem 4.1.5.

Consider a self mapping $Q : X \to X$ and let (X, d) be a metric space with a transitive binary relation $\mathcal{R}$. Assume that the subsequent conditions are fulfilled:

(1) $\exists\ k_0 \in X$ such that $k_0 \mathcal{R} Q k_0$,

(2) $\mathcal{R}$ is Q-closed,

(3) Q is an $(F_w, \mathcal{R})$ -contraction,

(4) (α) $\exists$ a subset K of X such that $Q(X) \subseteq K$ and K is $\mathcal{R}$ - complete,

(η) one of these conditions holds:

(i) Q is $\mathcal{R}$-Continuous, or

(ii) $\mathcal{R}|K$ is d-self closed on condition that (2.1) with binary relation holds $\forall\ k, l \in X$ with $k\mathcal{R}l$ and $Qk\mathcal{R}^{\natural}Ql$.

Then a fixed point of Q exists. Furthermore, if

(e) $[u, v] \in Fix(Q) \Rightarrow [u, v] \in \mathcal{R}$.

Then a unique fixed point of Q exists.

Theorem 4.1.6.

Replace condition (e) of above theorem by following:

(e*) $Fix(Q)$ is $\mathcal{R}^s$-connected,

then Q has a unique fixed point.

Proof. Assume on contrary that Q has more than one fixed points say u and v with $u \neq v$. Then there exist a path $\mathcal{R}^s \subseteq Fix(Q)$. As it is from v to u of length q. Let us denote the path by $\{v_0, \cdots v_q\}$ such that $v_p \neq v_{p+1}$ for each p where $0 \leqslant p \leqslant j - 1$. If $v = u$, it is a contradiction. so that

$$v_0 = v, \quad v_q = u \quad \text{and} \quad [v_p, v_{p+1}] \in \mathcal{R} \quad \text{for each } p\ (0 \leqslant p \leqslant q - 1).$$

As $\upsilon_p \in Fix(Q)$ implies that $Q(\upsilon_p) = \upsilon_p$ for each $p \in \{0, 1, ..., q\}$. With the help of condition (c), we obtain

$$\tau + F\Big(\upsilon_p, \upsilon_{p+1}\Big) \leqslant F\Big(max\Big\{(\upsilon_p, \upsilon_{p+1}), (\upsilon_p, \upsilon_{p+1}), (\upsilon_{p+1}, \upsilon_{p+1}), \frac{(\upsilon_p, \upsilon_{p+1}) + (\upsilon_{p+1}, \upsilon_p)}{2}\Big\}\Big).$$

Since $\tau > 0$, our supposition is not true. Hence Q has a unique fixed point.

4.2 Some Consequences in Ordered Metric Spaces

4.2.1. Ordered Metric Space

Let (X, d) be a metric space and $(X, \preceq)$ is an ordered set, then triplet $(X, d, \preceq)$ is known as an ordered metric space.

4.2.2. g-Increasing

Consider a self mapping $Q, g : X \to X$ and an ordered set $(X, \preceq)$. If, for any $k, l \in X$, $gk \preceq gl$ implies that $Qk \preceq Ql$. Then Q is known as g-increasing.

Remark 4.2.1.

Notice that the notion of Q is g-increasing is equal to saying that $\preceq$ is (Q, g)-closed.

Taking $\mathcal{R} = \preceq$ in Theorem (4.1.2) to (4.1.4) and with the help of Remark (4.2.1), we obtain the result given below .

Corollary 3.

Consider a self mapping $Q, g : X \to X$ and an ordered metric space $(X, d, \preceq)$. Assume that the conditions given below are satisfied:

(a) $\exists\ k_0 \in X$ such that $gk_0 \preceq Qk_0$,

(b) Q *is* g-increasing,

(c) $\exists\ \tau > 0$ and $F \in \mathcal{F}$ such that

$$\tau + F\Big(d(Qk, Ql)\Big) \leqslant F\Big(max\Big\{d(gk, gl), d(gk, Qk), (gl, Ql), \frac{d(gk, Ql) + d(gl, Qk)}{2}\Big\}\Big).$$

(d) $\exists$ a subset K of X such that $Q(X) \subseteq K \subseteq g(X)$ and K is $\preceq$ -complete,

(e) either Q and g are continuous or Q is $(g, \preceq)$-continuous; then coincidence point of (Q, g) exists. Additionally we suppose that

(f) Qu and gv are $\preceq$-compareable for all distinct coincidence points $u, v \in$ coin (Q, g), then pair (Q, g) has a unique coincidence point.
Furthermore, if Q and g are weakly compatible,then a unique common fixed point of (Q, g) exists.

Taking $\mathcal{R} = \preceq$ in Theorem (4.1.5) and with the help of Remark (4.2.1), we conclude the result given below.

Corollary 4.
Consider an ordered metric space $(X, d, \preceq)$ and mapping $Q : X \to X$. Assume that conditions given below are fulfilled:

(a) $\exists\ k_0 \in X$ such that $k_0 \preceq Qk_0$,

(b) Q is $\preceq$ -increasing,

(c) $\exists\ \tau > 0$ and $F \in \mathcal{F}$ such that

$$\tau + F(d(Qk, Ql)) \leqslant F\Big(max\Big\{d(k, l), d(k, Qk), (l, Ql), \frac{d(k, Ql) + d(l, Qk)}{2}\Big\}\Big).$$

(d) $K \subseteq X$ exists and $Q(X) \subseteq K$ and K is $\preceq$ -complete,

(e) Q is $\preceq$-continuous. Then Q has a fixed point. Furthermore, if

(f) $u, v \in Fix(Q) \Rightarrow [u, v] \in \preceq$ then a unique fixed point of Q exists.

Chapter 5

Conclusion

The work of Alfaqih et al. [16] on "Relation-theoretic coincidence and common fixed point results under $(F, \mathcal{R})_g$ -contractions" is investigated in this thesis and also the brief description of their work and achievements.

The objective of this research was to extend the results established by Alfaqih et al.[16] in metric space. For this, the definition of F_w contraction with binary relation $\mathcal{R}$ under g is formulated in metric space and extended the coincidence and fixed point results using this definition. An example is also illustrated. We also proved some consequences in ordered metric space using this definition. These results might be valuable in solving particular results in fixed point theory using metric space with some binary relation.

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