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MHD Radiative Nanofluid Flow with Cattaneo-Christov Heat Flux and Chemical Reaction

by

Mahzad Ahmed

A thesis submitted in partial fulfillment for the degree of Master of Philosophy

in the

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I dedicate my dissertation work to my ***family*** *and dignified* ***teachers****. A special feeling of gratitude to my loving parents who have supported me in my studies.*

CERTIFICATE OF APPROVAL

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(Mahzad Ahmed)

Abstract

This thises numerically investigates the influence of aligned magnetic field, Cattaeno-Christov heat flux and chemical reaction of the flow of an electrically conducting nanofluid past a nonlinear stretching sheet through a porous medium with frictional heating. The partial differential equtions governing the flow problems are converted to ordinary differential equtions via similarity variables. The reduced equtions are then solved numerically with the aid of shooting method. The influence of physical parameters such as nanoparticle volume fraction ϕ, permeability parameter K, nonlinear stretching sheet parameter n, magnetic field parameter M, heat generation parameter Q, Eckert number Ec, Prandtl number Pr, relaxation time parameter γ_1, thermophoresis parameter Nt, Brownain motion parameter Nb, Lewis number Le and chemical reaction parameter γ_2 on the velocity profile, temperature distribution, concentration profile, skin friction coefficient, Nusselt number and Sherwood number are studied and presented in graphical and tabular forms. The results obtained reveal that there is an enhancement in the rate of heat transfer with a rise in the nanoparticle volume fraction and permeability parameter. The temperature distribution is also influenced by the presence of relaxation time parameter γ_1, Brownain motion parameter Nb, thermal radiation R and nanoparticle volume fraction ϕ. This shows that the volume fraction of nanoparticles can be used in controlling the behaviours of heat transfer and nanofluid flows.

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Abbreviations

IVPs	Initial value problems
MHD	Magnetohydrodynamics
ODEs	Ordineary differential equations
PDEs	Partial differential equations
RK	Runge-Kutta

Symbols

μ	Viscosity
ρ	Density
ν	Kinematic viscosity
τ	Stress tensor
k	Thermal conductivity
α	Thermal diffisuitivity
σ	Electrical conductivity
u	x-component of fluid velocity
v	y-component of fluid velocity
B_0	Magnetic field constant
k_0	Permeability constant
a	Stretching constant
T_w	Temperature of the wall
T_∞	Ambient temperature of the nanofluid
T	Temperature
ρ_f	Density of the fluid
μ_f	Viscosity of the fluid
ν_f	Kinematic viscosity of the base fluid
ρ_{nf}	Density of the nanofluid
μ_{nf}	Viscosity of the nanofluid
q_r	Radiative heat flux
q	Heat generation constant
q_w	Heat flux

q_m	Mass flux
σ^*	Stefan Boltzmann constant
k^*	Absorption coefficient
ψ	Stream function
ξ	Similarity variable
C_f	Skin friction coefficient
Nu	Nusselt number
Nu_x	Local Nusselt number
Sh	Sherwood number
Sh_x	Local Sherwood number
Re	Reynolds number
Re_x	Local Reynolds number
ϕ	Nanoparticle volume fraction
R	Thermal radiation parameter
n	Stretching parameter
M	Magnetic parameter
K	Permeability parameter
Ec	Eckert number
Pr	Prandtl number
Q	Heat generation parameter
γ_1	Relaxation time parameter
Nb	Brownain motion parameter
Nt	Thermophoresis parameter
γ_2	Chemical reaction parameter
L_e	Lewis number
ρ_f	Density of the pure fluid
ρ_s	Density of nanoparticle
μ_f	Viscosity of the base fluid
$(\rho C_p)_f$	Heat capacitance of base fluid
$(\rho C_p)_s$	Heat capacitance of nanoparticle
σ_f	Electrical conductivity of the base fluid

σ_s	Electrical conductivity of the nanoparticle
k_f	Thermal conductivity of the base fluid
k_s	Thermal conductivity of the nanoparticle
f	Dimensionless velocity
θ	Dimensionless temperature
h	Dimensionless concentration
C_∞	Ambient concentration
C	Concentration
C_w	Nanoparticles concentration at the stretching surface

Chapter 1

Introduction

The efficiency of speration of heat transfer is dependant on the functioning of thermal conductivity of operating fluid, such as water, oil and ethyl glycol. If a little portion of nanoparticles (such as Cu, Ag, TiO_2 and Al_2O_3) is immersed into a conventional fluid, a new category of fluids is obtained which is called nanofluids [1]. Nanofluids paved a new pathway to innovations in the improvement of the characteristics of heat transfer. There is wide variety of nanoparticles which are categorised according to their size, shape, thermal and electrical conductivity and heat transfer abilities. They are made up of metals, carbides and oxides. Some are named as nanofibers, nanowires, nanotubes and nanosheets [2]. Nanofluid has various applications in industrial devices, heat exchanger [3], drug delivery, medicines, car radiators, cooling of heat exchanging equipments, transformer oil cooling, electronic cooling [4, 5]. The diameter of the suspended nanoparticle varies between 1 to 100nm. There appears a dramatic boost in the thermophysical properties of the conventional fluid when the nanoparticle are suspended in it.

On account of the point mentioned above, Choi [1] introduced solid nanoparticles into the operating conventional fluid with the target of forming a new class of fluids that will have high thermal conductivity in contrast to the customany conventional fluid. He designated the combination of nanoparticles and the conventional base fluid as nanofluid. Xuan and Li [6] analyzed the combination of Cu nanofluid and distilled water afterwards and mentioned that the thermal conductivity of the water-based Cu nanofluid is higher as compared with that of the distilled water in a ratio approximately 1.24 to 1.78. Moreover, Choi et al. [7] figured that a small inclusion of solid nanoparticles into conventional heat transfer liquid raises the thermal conductivity of the conventional liquid by a percentage of 200.

In 2006, Buongiorno [8] presented a detailed discussion on convective transport system in nanofluid. He encountered the fact that Brownian diffusion and thermophoresis are the primary mechanisms for the improvement of heat transfer and deduced that the immense fluctuations of temperature in the boundary layer zone result in noticeable reducetion in fluid viscosity which as a consequence leads to a rise in coefficient of heat transfer.

Tiwari and Das [9] in 2007, further devised a model for the examination of nanofluid and heat transfer within a two-sided lid-driven square cavity and analyzed the role of nanoparticle volume fraction. They emphasized on the prime role of nanoparticle volume fraction for evaluating the impact of nanoparticles in the fluid flow and rate of heat transfer. Yang et al. [10] mentioned that, the thermal conductivity of nanofluid relies highly on nanoparticles volume fraction and their different properties such as diameter and shape.

Khan and Pop [11] were the first to perform an experiment depicting the response of nanofluid flow over a stretching sheet using Buongiorno's configuration. They came out with a conclusion that the rate of heat transfer is reduced with an increase in the Brownain diffusion and thermophoresis parameters. With time, Rana and Bhargava [12] added slight modifications to Khan and Pop's [11] in original experiment. They focused on the steady viscous nanofluid flow over a nonlinear stretching sheet using finite element method (FEM). Their findings indicated that an increase in the Brownain motion and thermophoresis parameters cause an improvement in the thermal boundary layer thickness. Moreover, Hamad and Ferdows [13] following Tiwari and Das model addressed the similarity solution of viscous boundary layer flow of nanofluid over a nonlinear stretching surface. Soon it was made clear that the presence of nanoparticles in the base fluid is capable of bringing about change in the pattern and behaviour of fluid flow based on the impacts of nanoparticle and nonlinear stretching sheet parameter. The impact of radiation and variable wall temperature on nanofluid flow past a nonlinear stretching surface was investigated by Hady et al. [14]. According to them, the temperature of the nanofluid is reduced with a rise in the nonlinear stretching sheet and radiation parameters. In the presence of a partial condition effect, Das [15] again performed the same technique by taking account of the specified surface temperature. His main finding was that increase in the nonlinear stretching sheet parameter and slip parameter causes a fall in the nanofluid velocity and a rise in the thickness of the boundary layer. Khan et al. [16] observed a three dimensional nanofluid flow past a nonlinear stretching sheet depicting the 4th and 5th order Runge-kutta methods. Malvandi et al. [17] demonstrated a stagnation point nanofluid flow past a nonlinear stretching sheet with suction/injection. They demonstrate that with the increased suction parameter, the heat transfer rate rises and decreases with the increased blowing parameter. Khan and Shehzad [18] worked on the effect of thermophoresis and Brownain movement on third grade nanofluid and rate of heat transfer past an oscillatory dynamic sheet. Many authors [19–24] have contributed generously to the vastness of study of electrically conducting nanofluids covering the fields of engineering and technological process such the plasma studies, MHD pumps, MHD generators and bearings. Noteable considerations also include either the viscous dissipation, thermal radiation or heat generation responses on the boundary layer flow of nanofluid and the features of heat transfer rate embedded in porous medium. This method is commonly used in oil reservoirs and geothermal engineering. Ahmad et al. [25] analyzed the behaviour of MHD viscous flow over an exponentially stretching surface with radiative effect in a porous medium. In the presence of thermal radiation through a porous medium over a linear stretching sheet, Williamson fluid film flow and heat transfer were examined by Shah et al. [26]. In their study, they noted that an increase in the porosity parameter decreases the flow of thin films and that the Lorentz force affects the flow of liquid film. Research regarding MHD boundary layer flow of nanofluids in a porous medium were also put forth by Zeeshan et al. [27]. Pal and Mandal [28] demonstrated the impact of thermal radiation and heat generation on convective nanofluid flow through a stagnation point in a porous

medium. Hybrid approach to numerically dissect the effects of viscous dissipation on MHD boundary layer nanofluid flow over a nonlinear stretching sheet saturated in a porous medium was triumphantly used by Rama and Chandra [29]. Haroun et al. [30] devised the technique of spectral relaxation to examine the influence of chemical reaction, viscous dissipation and radiation on MHD nanofluid flow in a porous medium and found that velocity field is reduced with a rise in porosity parameter, while an increase in porosity parameter also increases the temperature distribution. On the same theme MHD nanofluid flow and rate of heat transfer between porous medium and stretching sheet was examined by Geng et al. [31]. Further adding to the list, Patel [32] throughly studied homotopy analysis, the influence of heat generation, nonlinear thermal diffusion, and cross-diffusion on an electrically conducting Casson fluid saturated in a porous medium. He concluded from his experience that with a decrease in the value of magnetic field, skin friction can be minimized.
The chemical reaction can further be classified heterogeneous and homogenous processes. In the case of the strong compound system, the reaction is heterogenous. In most of the cases of chemical reaction processes, the concentration rate depends upon the species itself as discussed by Magyari and Chamkha et al. [33]. Chamkha and Rashad [34] talked about the impact of chemical reaction on MHD flow in the presence of heat generation or absorption of uniform vertical permeable surface. Das [35] explained the impact of chemical reaction with radiation on the heat and mass exchange along the MHD flow.

1.1 Thesis Contributions

The present survey is focused on the numerical analysis of MHD radiative nanofluid flow with inclined magnetic field, Cattaneo-Christov heat flux, thermophoresis diffusion, Brownain motion and chemical reaction. The proposed nonlinear PDEs are converted into system of ODEs by applying similarity transformations. Further, for finding the numerical results of nonlinear ODEs, shooting method is utilized. The numerically obtained results are computed by using MATLAB. The impact of significant parameters on velocity distribution $f'(\xi)$, temperature distribution $\theta(\xi)$ and concentration distribution $h(\xi)$, skin friction coefficient Cf, local Nusselt number Nu_x and local Sherwood number Sh_x have been discussed in graphs and tables.

1.2 Layout of Thesis

A brief overview of the contents of the thesis is provided below.

Chapter 2 includes some basic definitions and terminologies, which are useful to understand the concepts discussed later on.

Chapter 3 provides the proposed analytical study of MHD radiative nanofluid flow in the porous medium induced by a nonlinear stretching sheet. The numerical results of the governing flow equations are derived by the shooting method.

Chapter 4 extends the proposed model flow discussed in Chapter 3 by including the impacts of inclined magnetic field, Cattaneo-Christov heat flux, thermophoresis diffusion, Brownain motion and chemical reaction.

Chapter 5 provides the concluding remarks of the thesis.

References used in the thesis are mentioned in **Biblography**.

Chapter 2

Preliminaries

This chapter contains some basic definitions and governing laws, which will be helpful in the subsequent chapters.

2.1 Some Basic Terminologies

Definition 2.1.1 (Fluid)

"A fluid is a substance that deforms continuously under the application of a shear (tangential) stress no matter how small the shear stress may be." [36]

Definition 2.1.2 (Fluid Mechanics)

"Fluid mechanics is that branch of science which deals with the behavior of the fluid (liquids or gases) at rest as well as in motion." [37]

Definition 2.1.3 (Fluid Dynamics)

"The study of fluid if the pressure forces are also considered for the fluids in motion, that branch of science is called fluid dynamics." [37]

Definition 2.1.4 (Fluid Statics)

"The study of fluid at rest is called fluid statics." [37]

Definition 2.1.5 (Viscosity)

"Viscosity is defined as the property of a fluid which offers resistance to the movement of one layer of fluid over another adjacent layer of the fluid. Mathematically,

$$\mu = \frac{\tau}{\frac{\partial u}{\partial y}},$$

where μ is viscosity coefficient, τ is shear stress and $\frac{\partial u}{\partial y}$ represents the velocity gradient." [37]

Definition 2.1.6 (Kinematic Viscosity)

"It is defined as the ratio between the dynamic viscosity and density of fluid. It is denoted by symbol ν called **nu**. Mathematically,

$$\nu = \frac{\mu}{\rho}.\text{"}\ [37]$$

Definition 2.1.7 (Thermal Conductivity)

"The Fourier heat conduction law states that the heat flow is proportional to the temperature gradient. The coefficient of proportionality is a material parameter known as the thermal conductivity which may be a function of a number of variables." [38]

Definition 2.1.8 (Thermal Diffusivity)

"The rate at which heat diffuses by conducting through a material depends on the thermal diffusivity and can be defined as,

$$\alpha = \frac{k}{\rho C_p},$$

where α is the thermal diffusivity, k is the thermal conductivity, ρ is the density and C_p is the specifc heat at constant pressure." [39]

2.2 Types of Fluid

Definition 2.2.1 (Ideal Fluid)
"A fluid, which is incompressible and has no viscosity, is known as an ideal fluid. Ideal fluid is only an imaginary fluid as all the fluids, which exist, have some viscosity." [37]

Definition 2.2.2 (Real Fluid)
"A fluid, which possesses viscosity, is known as a real fluid. In actual practice, all the fluids are real fluids." [37]

Definition 2.2.3 (Newtonian Fluid)
"A real fluid, in which the shear stress is directly proportional to the rate of shear strain (or velocity gradient), is known as a Newtonian fluid." [37]

Definition 2.2.4 (Non-Newtonian Fluid)
"A real fluid in which the shear stress is not directly proportional to the rate of shear strain (or velocity gradient), is known as a non-Newtonian fluid.

$$\tau_{xy} \propto \left(\frac{du}{dy}\right)^m, \qquad m \neq 1$$
$$\tau_{xy} = \mu\left(\frac{du}{dy}\right)^m."$$ [37]

Definition 2.2.5 (Magnetohydrodynamics)
"Magnetohydrodynamics(MHD) is concerned with the mutual interaction of fluid flow and magnetic fields. The fluids in question must be electrically conducting

and non-magnetic, which limits us to liquid metals, hot ionised gases (plasmas) and strong electrolytes." [40]

2.3 Types of Flow

Definition 2.3.1 (Rotational Flow)
"Rotational flow is that type of flow in which the fluid particles while flowing along stream-lines, also rotate about their own axis." [37]

Definition 2.3.2 (Irrotational Flow)
"Irrotational flow is that type of flow in which the fluid particles while flowing along stream-lines, do not rotate about their own axis then this type of flow is called irrotational flow." [37]

Definition 2.3.3 (Compressible Flow)
"Compressible flow is that type of flow in which the density of the fluid changes from point to point or in other words the density (ρ) is not constant for the fluid, Mathematically,

$$\rho \neq k,$$

where k is constant." [37]

Definition 2.3.4 (Incompressible Flow)
"Incompressible flow is that type of flow in which the density is constant for the fluid. Liquids are generally incompressible while gases are compressible, Mathematically,

$$\rho = k,$$

where k is constant." [37]

Definition 2.3.5 (Steady Flow)

"If the flow characteristics such as depth of flow, velocity of flow, rate of flow at any point in open channel flow do not change with respect to time, the flow is said to be steady flow. Mathematically,

$$\frac{\partial Q}{\partial t} = 0,$$

where Q is any fluid property." [37]

Definition 2.3.6 (Unsteady Flow)

"If at any point in open channel flow, the velocity of flow, depth of flow or rate of flow changes with respect to time, the flow is said to be unsteady. Mathematically,

$$\frac{\partial Q}{\partial t} \neq 0,$$

where Q is any fluid property." [37]

Definition 2.3.7 (Internal Flow)

"Flows completely bounded by a solid surfaces are called internal or duct flows." [36]

Definition 2.3.8 (External Flow)

"Flows over bodies immersed in an unbounded fluid are said to be an external flow." [36]

2.4 Modes of Heat Transfer

Definition 2.4.1 (Heat Transfer)

"Heat transfer is a branch of engineering that deals with the transfer of thermal energy from one point to another within a medium or from one medium to another due to the occurrence of a temperature difference." [38]

Definition 2.4.2 (Conduction)

"The transfer of heat within a medium due to a diffusion process is called conduction." [38]

Definition 2.4.3 (Convection)

"Convection heat transfer is usually defined as energy transport effected by the motion of a fluid. The convection heat transfer between two dissimilar media is governed by Newtons law of cooling." [38]

Definition 2.4.4 (Thermal Radiation)

"Thermal radiation is defined as radiant (electromagnetic) energy emitted by a medium and is solely to the temperature of the medium." [38]

2.5 Dimensionless Numbers

Definition 2.5.1 (Eckert Number)

"It is the dimensionless number used in continuum mechanics. It describes the relation between flows and the boundary layer enthalpy difference and it is used for characterized heat dissipation. Mathematically,

$$Ec = \frac{u^2}{C_p \nabla T}$$

where C_p denotes the specific heat." [36]

Definition 2.5.2 (Prandtl Number)

"It is the ratio between the momentum diffusivity ν and thermal diffusivity α.

Mathematically, it can be defined as

$$Pr = \frac{\nu}{\alpha} = \frac{\frac{\mu}{\rho}}{\frac{k}{C_p\rho}} = \frac{\mu C_p}{k}$$

where μ represents the dynamic viscosity, Cp denotes the specific heat and k stands for thermal conductivity. The relative thickness of thermal and momentum boundary layer is controlled by Prandtl number. For small Pr, heat distributed rapidly corresponds to the momentum." [36]

Definition 2.5.3 (Skin Friction Coefficient)

"The steady flow of an incompressible gas or liquid in a long pipe of internal D. The mean velocity is denoted by u_w. The skin friction coefficient can be defined as

$$C_f = \frac{2\tau_0}{\rho u_w^2}$$

where τ_0 denotes the wall shear stress and ρ is the density." [41]

Definition 2.5.4 (Nusselt Number)

"The hot surface is cooled by a cold fluid stream. The heat from the hot surface, which is maintained at a constant temperature, is diffused through a boundary layer and convected away by the cold stream. Mathematically,

$$Nu = \frac{qL}{k}$$

where q stands for the convection heat transfer, L for the characteristic length and k stands for thermal conductivity." [42]

Definition 2.5.5 (Sherwood Number)

"It is the nondimensional quantity which show the ratio of the mass transport by convection to the transfer of mass by diffusion. Mathematically:

$$Sh = \frac{kL}{D}$$

here L is characteristics length, D is the mass diffusivity and k is the mass transfer" coeffcient." [43]

Definition 2.5.6 (Reynolds Number)
"It is defined as the ratio of inertia force of a flowing fluid and the viscous force of the fluid. Mathematically,

$$Re = \frac{VL}{\nu},$$

where U denotes the free stream velocity, L is the characteristic length and ν stands for kinematic viscosity." [37]

2.6 Governing Laws

Definition 2.6.1 (Continuity Equation)
"The principle of conservation of mass can be stated as the time rate of change of mass is fixed volume is equal to the net rate of flow of mass across the surface. Mathematically, it can be written as

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \mathbf{u}) = 0." \text{ [38]}$$

Definition 2.6.2 (Momentum Equation)
"The momentum equation states that the time rate of change of linear momentum of a given set of particles is equal to the vector sum of all the external forces acting on the particles of the set, provided Newtons Third Law of action and reaction

governs the internal forces. Mathematically, it can be written as:

$$\frac{\partial}{\partial t}(\rho\mathbf{u}) + \nabla.[(\rho\mathbf{u})\mathbf{u}] = \nabla.\mathbf{T} + \rho g.\text{''} \ [38]$$

Definition 2.6.3 (Energy Equation)

"The law of conservation of energy states that the time rate of change of the total energy is equal to the sum of the rate of work done by the applied forces and change of heat content per unit time.

$$\frac{\partial \rho}{\partial t} + \nabla.\rho\mathbf{u} = -\nabla.\mathbf{q} + Q + \phi,$$

where ϕ is the dissipation function." [38]

2.7 Shooting Method

To elaborate the shooting method, consider the following nonlinear boundary value problem.

$$\left.\begin{aligned} &f''(x) = f(x)f'(x) + 2f^2(x) \\ &f(0) = 0, \qquad f(G) = J. \end{aligned}\right\} \tag{2.1}$$

To reduce the order of the above boundary value problem, introduce the following notations.

$$f = Y_1 \qquad f' = Y_1' = Y_2 \qquad f'' = Y_2'. \tag{2.2}$$

As a result, (2.1) is converted into the following system of first order ODEs.

$$Y_1' = Y_2, \qquad Y_1(0) = 0, \tag{2.3}$$

$$Y_2' = Y_1Y_2 + 2Y_1^2, \qquad Y_2(0) = w, \tag{2.4}$$

where w is the missing initial condition which will be guessed.

The above IVP will be numerically solved by the RK-4 method. The missing condition w is to be chosen such that.

$$Y_1(G, w) = J. \tag{2.5}$$

For convenience, now onward $Y_1(G, w)$ will be denoted by $Y_1(w)$.

Let us further denote $Y_1(w) - J$ by $H(w)$, so that

$$H(w) = 0. \tag{2.6}$$

The above equation can be solved by using Newton's method with the following iterative formula.

$$w^{n+1} = w^n - \frac{H(w^n)}{\frac{\partial H(w^n)}{\partial w}},$$

or

$$w^{n+1} = w^n - \frac{Y_1(w^n) - J}{\frac{\partial Y_1(w^n)}{\partial w}}. \tag{2.7}$$

To find $\frac{\partial Y_1(w^n)}{\partial w}$, introduce the following notations.

$$\frac{\partial Y_1}{\partial w} = Y_3, \quad \frac{\partial Y_2}{\partial w} = Y_4. \tag{2.8}$$

As a result of these new notations the Newton's iterative scheme, will then get the form.

$$w^{n+1} = w^n - \frac{Y_1(w) - J}{Y_3(w)}. \tag{2.9}$$

Now differentiating the system of two first order ODEs (2.3)-(2.4) with respect to w, we get another system of ODEs, as follows.

$$Y_3' = Y_4, \qquad Y_3(0) = 0. \tag{2.10}$$

$$Y_4' = Y_3Y_2 + Y_1Y_4 + 4Y_1Y_3, \qquad Y_4(0) = 1. \tag{2.11}$$

Writing all the four ODEs (2.3), (2.4), (2.10) and (2.11) together, we have the following initial value problem.

$$\begin{aligned}
Y_1' &= Y_2, & Y_1(0) &= 0.\\
Y_2' &= Y_1Y_2 + 2Y_1^2, & Y_2(0) &= w.\\
Y_3' &= Y_4, & Y_3(0) &= 0.\\
Y_4' &= Y_3Y_2 + Y_1Y_4 + 4Y_1Y_3, & Y_4(0) &= 1.
\end{aligned}$$

The above system together will be solved numerically by Runge-Kutta method of order four.

The stopping criteria for the Newton's technique is set as,

$$\mid Y_1(w) - J \mid < \epsilon,$$

where $\epsilon > 0$ is an arbitrarily small positive number.

Chapter 3

MHD Radiative Nanofluid Flow in the Porous Medium Induced by a Nonlinear Stretching Sheet

3.1 Introduction

In this chapter, consideration has been given to the numerical analysis of MHD nanofluid flow past nonlinear stretching sheet, saturated in a porous medium in the presence of magnetic field, heat generation and thermal radiation. The governing nonlinear PDEs are converted into a system of dimensionless ODEs by utilizing the appropriate transformations. In order to solve the ODEs, the shooting technique is implemented in MATLAB. At the end of this chapter the numerical solution for various parameters is discussed for the dimensionless velocity $f'(\xi)$ profile and temperature distribution $\theta(\xi)$. Investigation of obtained numerical results are given through tables and graphs. This chapter provides a detailed review of the work presented by Jafar et al. [44].

3.2 Mathematical Modeling

A $2D$ MHD flow of nanofluid past a nonlinear stretching sheet with $y = 0$ has been investigated. The flow is considered along y-axis with $y > 0$. It is assumed that the variable stretching velocity, the variable magnetic field and the variable permeability of the porous medium of the nanofluid flow are $U_w(x)=ax^n$, $B(x)=B_0x^{2n-1}$ and $k(x)=k_0x^{n-1}$ and respectively [29]. The fluid's layer along the stretching surface is maintained at a temperature of $T_w=T_\infty + bx^{2n-1}$, where n is the surface temperature parameter, and T_∞ is the nanofluid's ambient temperature.

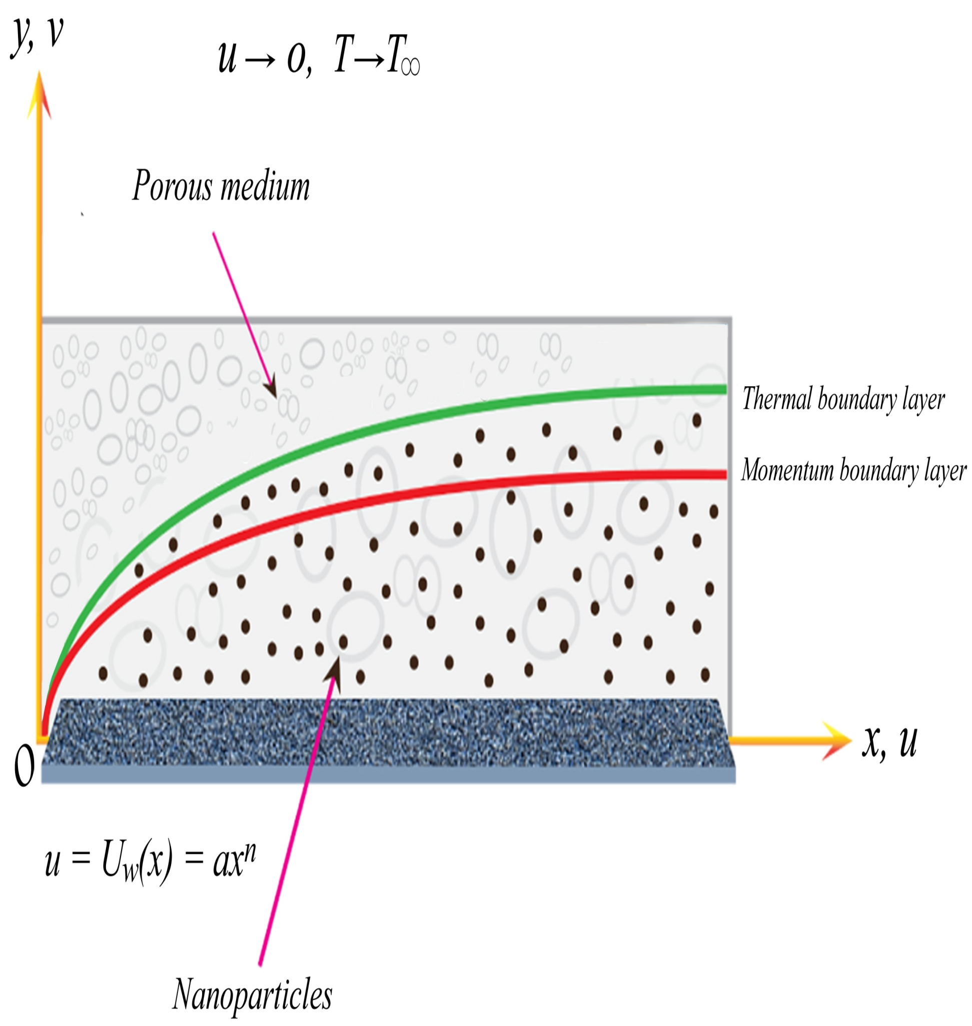

FIGURE 3.1: systematic representation of physical model.

The set of equations describing the flow are as follows.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{3.1}$$

$$\rho_{nf}\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \mu_{nf}\left(\frac{\partial^2 u}{\partial y^2}\right) - \frac{\mu_{nf}}{k(x)}u - \sigma_{nf}B^2(x)u, \tag{3.2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{\mu_{nf}}{(\rho C_p)_{nf}}\left(\frac{\partial u}{\partial y}\right)^2 + \alpha_{nf}\left(\frac{\partial^2 T}{\partial y^2}\right) - \frac{1}{(\rho C_p)_{nf}}\left(\frac{\partial q_r}{\partial y}\right) + \frac{q}{(\rho C_p)_{nf}}(T - T_\infty). \tag{3.3}$$

The associated BCs have been taken as.

$$\left.\begin{aligned} &u = U_w(x) = ax^n, \quad v = 0, \quad T = T_w = T_\infty + bx^{2n-1}, \quad at \quad y = 0, \\ &u \to 0, \quad T \to T_\infty, \quad as \quad y \to \infty. \end{aligned}\right\} \tag{3.4}$$

In the above model, x is the direction around the sheet, the direction perpendicular to the sheet is y, u and v are the xy-direction horizontal and vertical velocity. Where the radiative heat flux and heat generation constants are q_r and q.
The radiative heat flux is given by

$$q_r = -\frac{4\sigma^*}{3k^*}\frac{\partial T^4}{\partial y},$$

where σ^* is the Stefan-Boltzman constant and k^* is the absorption coefficient. If the temperature difference is very small, then the temperature T^4 can be expanded about T_∞ using Taylor series, as follows.

$$T^4 = T_\infty^4 + 4T_\infty^3(T - T_\infty) + 6T_\infty^2(T - T_\infty)^2 + ...$$

Ignoring the higher order terms, we have

$$\begin{aligned} T^4 &= T_\infty^4 + 4T_\infty^3(T - T_\infty), \\ T^4 &= T_\infty^4 + 4T_\infty^3 T - 4T_\infty^4, \\ T^4 &= -3T_\infty^4 + 4T_\infty^3 T, \\ T^4 &= 4T_\infty^3 T - 3T_\infty^4. \end{aligned}$$

The thermophysical properties of nanofluid are formulated as [9, 13, 45]:

$$\alpha_{nf} = \frac{k_f}{(\rho C_p)_{nf}},$$
$$\rho_{nf} = (1-\phi)\rho_f + \phi\rho_s,$$
$$\mu_{nf} = \frac{\mu_f}{(1-\phi)^{2.5}},$$
$$(\rho C_p)_{nf} = (1-\phi)(\rho C_p)_f + \phi(\rho C_p)_s,$$
$$\frac{k_{nf}}{k_f} = \frac{(k_s+2k_f)-2\phi(k_f-k_s)}{(k_s+2k_f)+\phi(k_f-k_s)},$$
$$\frac{\sigma_{nf}}{\sigma_f} = 1 + \frac{3\left(\frac{\sigma_s}{\sigma_f}-1\right)\phi}{\left(\frac{\sigma_s}{\sigma_f}+2\right)-\left(\frac{\sigma_s}{\sigma_f}-1\right)\phi}.$$

The following notations have been defined:

$$A_1 = (1-\phi)^{2.5},$$
$$A_2 = 1-\phi+\phi\frac{\rho_s}{\rho_f},$$
$$A_3 = 1 + \frac{3\left(\frac{\sigma_s}{\sigma_f}-1\right)\phi}{\left(\frac{\sigma_s}{\sigma_f}+2\right)-\left(\frac{\sigma_s}{\sigma_f}-1\right)\phi},$$
$$A_4 = \frac{k_{nf}}{k_f}\left(1+\frac{4}{3}R\right),$$
$$A_5 = \left(1-\phi+\phi\frac{(\rho C_p)_s}{(\rho C_p)_f}\right),$$
$$A_6 = \left(\frac{Ec}{(1-\phi)^{2.5}}\right).$$

For the conversion of the mathematical model (3.1)-(3.3) into the system of ODEs, the following similarity transformation was used by [44].

$$\left.\begin{aligned} &\psi(x,y) = \sqrt{\frac{2a\nu_f}{n+1}}x^{\frac{n+1}{2}}f(\xi), \qquad \theta(\xi) = \frac{T-T_\infty}{T_w-T_\infty}, \\ &\xi = x^{\frac{n-1}{2}}y\sqrt{\frac{(n+1)a}{2\nu_f}}, \end{aligned}\right\} \tag{3.5}$$

where ψ denotes the stream function.

The detailed procedure for the conversion of (3.1)-(3.3) into the dimensionless form has been discussed below.

$$
\begin{aligned}
u &= \frac{\partial \psi}{\partial y}, \\
\frac{\partial u}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right), \\
\frac{\partial \psi}{\partial y} &= \frac{\partial}{\partial y}\left(\sqrt{\frac{2\nu_f a}{n+1}} x^{\frac{n+1}{2}} f(\xi)\right), \\
&= \sqrt{\frac{2\nu_f a}{n+1}} x^{\frac{n+1}{2}} f'(\xi)\frac{\partial \xi}{\partial y}, \\
\frac{\partial \xi}{\partial y} &= \sqrt{\frac{a(n+1)}{2\nu_f}} x^{\frac{n-1}{2}}, \\
&= \sqrt{\frac{2\nu_f a}{n+1}} x^{\frac{n+1}{2}} f'(\xi)\sqrt{\frac{a(n+1)}{2\nu_f}} x^{\frac{n-1}{2}}, \\
\frac{\partial \psi}{\partial y} &= a x^n f'(\xi), \\
u &= a x^n f'(\xi). \qquad (3.6) \\
\frac{\partial u}{\partial x} &= \frac{\partial}{\partial x}\left(a f'(\xi) x^n\right), \\
&= a\frac{\partial}{\partial x}\left(f'(\xi) x^n\right), \\
&= a\left(n x^{n-1} f'(\xi) + x^n f''(\xi)\frac{\partial \xi}{\partial x}\right), \\
&= a\left(n x^{n-1} f'(\xi) + x^n f''(\xi) y\sqrt{\frac{a(n+1)}{2\nu_f}}\left(\frac{n-1}{2}\right) x^{\frac{n-3}{2}}\right), \\
&= a\left(n x^{n-1} f'(\xi) + x^{n-1} f''\xi\left(\frac{n-1}{2}\right)\right), \\
&= a x^{n-1}\left(n f'(\xi) + \xi f''(\xi)\left(\frac{n-1}{2}\right)\right). \qquad (3.7) \\
v &= -\frac{\partial \psi}{\partial x}, \\
\frac{\partial v}{\partial y} &= -\frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial x}\right), \\
\frac{\partial \psi}{\partial x} &= -\frac{\partial}{\partial x}\left(\sqrt{\frac{2a\nu_f}{n+1}} x^{\frac{n+1}{2}} f(\xi)\right), \\
&= -\left(\sqrt{\frac{2\nu_f a}{n+1}}\frac{n+1}{2} x^{\frac{n-1}{2}} f(\xi) + \sqrt{\frac{2\nu_f a}{n+1}} x^{\frac{n+1}{2}} f'(\xi)\frac{\partial \xi}{\partial x}\right), \\
&= -\sqrt{\frac{2\nu_f a}{n+1}}\left(\frac{n+1}{2} x^{\frac{n-1}{2}} f(\xi) + x^{\frac{n+1}{2}} f'(\xi)\frac{\partial \xi}{\partial x}\right),
\end{aligned}
$$

$$
\begin{aligned}
&= -\sqrt{\frac{2\nu_f a}{n+1}}\left(\frac{n+1}{2}x^{\frac{n-1}{2}}f(\xi) + x^{\frac{n+1}{2}}f'(\xi)y\sqrt{\frac{a(n+1)}{2\nu_f}}\left(\frac{n-1}{2}\right)x^{\frac{n-3}{2}}\right),\\
&= -\sqrt{\frac{2\nu_f a}{n+1}}\left(\frac{n+1}{2}x^{\frac{n-1}{2}}f(\xi) + x^{\frac{n+1}{2}}f'(\xi)\xi\left(\frac{n-1}{2}\right)x^{-1}\right),\\
&= -x^{\frac{n-1}{2}}\sqrt{\frac{2\nu_f a}{n+1}}\left(\frac{n+1}{2}f(\xi) + f'(\xi)\xi\left(\frac{n-1}{2}\right)\right),\\
&= -x^{\frac{n-1}{2}}\sqrt{\frac{4(n+1)\nu a}{2(n+1)^2}}\left(\frac{n+1}{2}f(\xi) + f'(\xi)\xi\left(\frac{n-1}{2}\right)\right),\\
&= -x^{\frac{n-1}{2}}\sqrt{\frac{\nu_f a(n+1)}{2}}f(\xi) - x^{\frac{n-1}{2}}\sqrt{\frac{2(n+1)\nu_f a}{2}}f'(\xi)\xi\left(\frac{n-1}{n+1}\right),
\end{aligned}
$$

$$
v = -x^{\frac{n-1}{2}}\sqrt{\frac{\nu_f a(n+1)}{2}}\left(f(\xi) + \left(\frac{n-1}{n+1}\right)\xi f'(\xi)\right). \tag{3.8}
$$

$$
\begin{aligned}
\frac{\partial v}{\partial y} &= \frac{\partial}{\partial y}\left[-x^{\frac{n-1}{2}}\sqrt{\frac{(n+1)\nu_f a}{2}}\left(f(\xi) + \left(\frac{n-1}{n+1}\right)\xi f'(\xi)\right)\right],\\
&= -x^{\frac{n-1}{2}}\sqrt{\frac{(n+1)\nu_f a}{2}}\left[f'(\xi)\frac{\partial\xi}{\partial y} + \left(\frac{n-1}{n+1}\right)\xi f''(\xi)\frac{\partial\xi}{\partial y} + \left(\frac{n-1}{n+1}\right)f'(\xi)\frac{\partial\xi}{\partial y}\right],\\
&= -x^{\frac{n-1}{2}}\sqrt{\frac{(n+1)\nu_f a}{2}}\left[f'(\xi) + \left(\frac{n-1}{n+1}\right)\xi f''(\xi)\right]\sqrt{\frac{(n+1)a}{2\nu_f}}x^{\frac{n-1}{2}}\\
&\quad - x^{\frac{n-1}{2}}\sqrt{\frac{(n+1)\nu_f a}{2}}\left(\left(\frac{n-1}{n+1}\right)f'(\xi)\right)\sqrt{\frac{(n+1)a}{2\nu_f}}x^{\frac{n-1}{2}},\\
&= -\frac{a}{2}x^{n-1}(n+1)\left(f'(\xi) + \left(\frac{n-1}{n+1}\right)\xi f''(\xi) + \left(\frac{n-1}{n+1}\right)f'(\xi)\right),\\
&= -\frac{a}{2}x^{n-1}\left(f'(\xi)(n+1) + (n-1)\xi f''(\xi) + (n-1)f'(\xi)\right),\\
&= -\frac{a}{2}x^{n-1}f'(\xi)(n+1+n-1) - \frac{a}{2}x^{n-1}(n-1)\xi f''(\xi),\\
&= -\frac{a}{2}x^{n-1}2nf'(\xi) - \frac{a}{2}x^{n-1}(n-1)\xi f''(\xi),\\
&= -ax^{n-1}nf'(\xi) - ax^{n-1}\left(\frac{n-1}{2}\right)\xi f''(\xi).
\end{aligned} \tag{3.9}
$$

Equation (3.1) is easily satisfied by using (3.7) and (3.9), as follows

$$
\begin{aligned}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= ax^{n-1}nf'(\xi) + ax^{n-1}\left(\frac{n-1}{2}\right)\xi f''(\xi) - ax^{n-1}nf'(\xi)\\
&\quad - ax^{n-1}\left(\frac{n-1}{2}\right)\xi f''(\xi),\\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0.
\end{aligned} \tag{3.10}
$$

Now, for the momentum equation (3.2) the following derivatives are required.

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(ax^n f'(\xi)),$$
$$= a\frac{\partial}{\partial y}(x^n f'(\xi)),$$
$$= ax^n f''(\xi)\frac{\partial \xi}{\partial y},$$
$$\frac{\partial u}{\partial y} = ax^n f''(\xi)\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{n-1}{2}}. \tag{3.11}$$
$$\frac{\partial^2 u}{\partial y^2} = ax^n f'''(\xi)\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{n-1}{2}}\frac{\partial \xi}{\partial y},$$
$$= af'''(\xi)\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{n-1}{2}}x^n\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{n-1}{2}},$$
$$\frac{\partial^2 u}{\partial y^2} = a^2x^{2n-1}f'''(\xi)\left(\frac{n+1}{2\nu_f}\right). \tag{3.12}$$
$$u\frac{\partial u}{\partial x} = ax^n f'(\xi)\left(ax^{n-1}nf'(\xi) + ax^{n-1}\left(\frac{n-1}{2}\right)\xi f''(\xi)\right),$$
$$= a^2x^{2n-1}nf'^2(\xi) + a^2x^{2n-1}\left(\frac{n-1}{2}\right)\xi f'(\xi)f''(\xi). \tag{3.13}$$
$$v\frac{\partial u}{\partial y} = -\sqrt{\frac{a\nu_f(n+1)}{2}}x^{\frac{n-1}{2}}\left(\xi f'(\xi)\left(\frac{n-1}{n+1}\right) + f(\xi)\right)\left(ax^n f''(\xi)\sqrt{\frac{a(n+1)}{2}}x^{\frac{n-1}{2}}\right),$$
$$= -\sqrt{\frac{a\nu_f(n+1)}{2}}x^{\frac{n-1}{2}}f'(\xi)\xi\frac{n-1}{n+1}ax^n f''(\xi)\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{n-1}{2}}$$
$$-\sqrt{\frac{a\nu_f(n+1)}{2}}x^{\frac{n-1}{2}}f(\xi)ax^n f''(\xi)\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{n-1}{2}},$$
$$= -\frac{a^2(n+1)}{2}x^{2n-1}f'(\xi)f''(\xi)\xi\left(\frac{n-1}{n+1}\right)$$
$$-\frac{a^2(n+1)}{2}x^{2n-1}f(\eta)f''(\xi). \tag{3.14}$$

Using (3.13) and (3.14) in the left side of (3.2) becomes

$$\rho_{nf}\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \rho_{nf}\left(a^2x^{2n-1}nf'^2(\xi) + a^2x^{2n-1}\left(\frac{n-1}{2}\right)\xi f'(\xi)f''(\xi)\right)$$
$$+\rho_{nf}\Big(-\frac{a^2(n+1)}{2}x^{2n-1}f'(\xi)f''(\xi)\xi\Big(\frac{n-1}{n+1}\Big)$$
$$-\frac{a^2(n+1)}{2}x^{2n-1}f(\xi)f''(\xi)\Big),$$

$$
\begin{aligned}
&= \rho_{nf} a^2 x^{2a-1} n f'^2(\xi) + \rho_{nf} a^2 x^{2n-1} \left(\frac{n-1}{2}\right) \xi f'(\xi) f''(\xi) \\
&\quad - \rho_{nf} \frac{a^2(n+1)}{2} x^{2n-1} \xi f'(\xi) f''(\xi) \left(\frac{n-1}{n+1}\right) \\
&\quad - \rho_{nf} \frac{a^2(n+1)}{2} x^{2n-1} f(\xi) f''(\xi), \\
&= \rho_{nf} \left(a^2 x^{2n-1} n f'^2(\xi) - \frac{a^2(n+1)}{2} x^{2n-1} f(\xi) f''(\xi) \right), \\
&= \rho_{nf} a^2 x^{2n-1} \left(n f'^2(\xi) - \left(\frac{n+1}{2}\right) f(\xi) f''(\xi) \right). \qquad (3.15)
\end{aligned}
$$

Using (3.6) and (3.12), in the right side of (3.2) becomes,

$$
\begin{aligned}
&\mu_{nf} \left(\frac{\partial^2 u}{\partial y^2}\right) - \frac{\mu_{nf}}{k(x)} u - \sigma_{nf} B^2(x) u = \mu_{nf} \left(a^2 x^{2n-1} f'''(\xi) \left(\frac{n+1}{2\nu_f}\right)\right) \\
&\quad - \frac{\mu_{nf}}{k(x)} a x^n f'(\xi) - \sigma_{nf} B^2(x) a x^{n-1} f'(\xi) \\
&= \mu_{nf} \left(a^2 x^{2n-1} f'''(\xi) \left(\frac{n+1}{2\nu_f}\right)\right) - \frac{\mu_{nf}}{k_0 x^{1-n}} a x^n f'(\xi) - \sigma_{nf} B_0^2 a x^{3n-1} f'(\xi). \qquad (3.16)
\end{aligned}
$$

Comparing (3.15) and (3.16), the dimensionless form of (3.2) can be written as.

$$
\begin{aligned}
&\rho_{nf} \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = \mu_{nf} \left(\frac{\partial^2 u}{\partial y^2}\right) - \frac{\mu_{nf}}{k(x)} u - \sigma_{nf} B^2(x) u, \\
&\rho_{nf} a^2 x^{2n-1} \left(n f'^2(\xi) - \left(\frac{n+1}{2}\right) f(\xi) f''(\xi) \right) = \mu_{nf} a^2 x^{2n-1} f'''(\xi) \left(\frac{n+1}{2\nu_f}\right) \\
&- \frac{\mu_{nf}}{k_0 x^{1-n}} a x^n f'(\xi) - \sigma_{nf} B_0^2 a x^{3n-1} f'(\xi), \\
&\nu_f \frac{\rho_{nf}}{\mu_{nf}} \left(\left(\frac{2n}{n+1}\right) f'^2(\xi) - f(\xi) f''(\xi) \right) = f'''(\xi) \\
&- \left(\frac{2}{n+1}\right) \left(\frac{\nu_f}{k_0 a} f'(\xi) + \frac{\sigma_{nf} \nu_f B_0^2 x^n}{\mu_{nf} a} f'(\xi) \right), \\
&f'''(\xi) + (1-\phi)^{2.5} \left(1 - \phi + \phi \frac{\rho_s}{\rho_f}\right) \left(f(\xi) f''(\xi) - \left(\frac{2n}{n+1}\right) f'^2(\xi) \right) \\
&- \left(\frac{2}{n-1}\right) \left[K + M(1-\phi)^{2.5} \left(1 + \frac{3\left(\frac{\sigma_s}{\sigma_f} - 1\right)\phi}{\left(\frac{\sigma_s}{\sigma_f} + 2\right) - \left(\frac{\sigma_s}{\sigma_f} - 1\right)\phi} \right) \right] f'(\xi) = 0, \\
&f'''(\xi) + A_1 A_2 \left(f(\xi) f''(\xi) - \left(\frac{2n}{n+1}\right) f'^2(\xi) \right) \\
&- \left(\frac{2}{n+1}\right) (K + M A_1 A_3) f'(\xi) = 0. \qquad (3.17)
\end{aligned}
$$

Now, for the conversion of energy equation (3.3) the following derivatives are required.

As in equation (3.5)

$$\theta(\xi) = \frac{T - T_\infty}{T_w - T_\infty},$$

$$T = \theta(\xi)(T_w - T_\infty) + T_\infty.$$

$$\frac{\partial T}{\partial x} = (T_w - T_\infty)\theta'(\xi)\frac{\partial \xi}{\partial x},$$

$$\frac{\partial \xi}{\partial x} = y\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{n-3}{2}}\left(\frac{n-1}{2}\right).$$

$$= (T_w - T_\infty)y\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{n-3}{2}}\left(\frac{n-1}{2}\right)\theta'(\xi). \tag{3.18}$$

$$\frac{\partial T}{\partial y} = (T_w - T_\infty)\theta'(\xi)\frac{\partial \xi}{\partial y},$$

$$= (T_w - T_\infty)\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{n-1}{2}}\theta'(\xi). \tag{3.19}$$

$$\frac{\partial^2 T}{\partial y^2} = (T_w - T_\infty)\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{n-1}{2}}\theta''(\xi)\frac{\partial \xi}{\partial y},$$

$$\frac{\partial^2 T}{\partial y^2} = (T_w - T_\infty)\left(\frac{a(n+1)}{2\nu_f}\right)x^{n-1}\theta''(\xi). \tag{3.20}$$

$$\left(\frac{\partial u}{\partial y}\right)^2 = \left(ax^{\frac{3n-1}{2}}\sqrt{\frac{a(n+1)}{2\nu_f}}f''(\xi)\right)^2,$$

$$\left(\frac{\partial u}{\partial y}\right)^2 = a^2x^{3n-1}\frac{a(n+1)}{2\nu_f}f''^2(\xi). \tag{3.21}$$

$$q_r = -\frac{4\sigma^*}{3k^*}\frac{\partial T^4}{\partial y},$$

$$q_r = -\frac{4\sigma^*}{3k^*}\frac{\partial}{\partial y}(4T_\infty^3 T - 3T_\infty^4).$$

$$q_r = -\frac{4\sigma^*}{3k^*}\frac{\partial}{\partial y}(4T_\infty^3 T).$$

$$q_r = -\frac{16\sigma^*}{3k^*}T_\infty^3\frac{\partial T}{\partial y}.$$

$$\frac{\partial q_r}{\partial y} = -\frac{16\sigma^*}{3k^*}T_\infty^3\frac{\partial^2 T}{\partial y^2},$$

$$\frac{\partial q_r}{\partial y} = -\frac{16\sigma^*}{3k^*}T_\infty^3 x^{n-1}\frac{a(n+1)}{2\nu_f}(T_w - T_\infty)\theta''(\xi). \tag{3.22}$$

$$(T - T_\infty) = (T_w - T_\infty)\theta(\xi). \tag{3.23}$$

Using (3.18) and (3.19) in the left side of (3.3), we get

$$
\begin{aligned}
u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y} &= ax^n f'(\xi)\left[(T_w-T_\infty)\left(\frac{n-1}{2x}\right)\xi\theta'(\xi)\right]+\left[-x^{\frac{n-1}{2}}\sqrt{\frac{a(n+1)}{2\nu_f}}\right.\\
&\quad\left.\left[\left(\frac{n-1}{n+1}\right)\xi f'(\xi)+f(\xi)\right]\right]\left[(T_w-T_\infty)\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{n-1}{2}}\theta'(\xi)\right],\\
&= ax^{n-1}(T_w-T_\infty)\left(\frac{n-1}{2}\right)\xi f'(\xi)\theta'(\xi)\\
&\quad-\left(\frac{(n+1)a}{2}\right)x^{n-1}(T_w-T_\infty)\left(\frac{n-1}{n+1}\right)\xi\theta'(\xi)f'(\xi)\\
&\quad-\left(\frac{a(n+1)}{2}\right)x^{n-1}(T_w-T_\infty)f(\xi)\theta'(\xi),\\
&= ax^{n-1}\left(\frac{n-1}{2}\right)(T_w-T_\infty)\xi f'(\xi)\theta'(\xi)\\
&\quad-ax^{n-1}\left(\frac{n-1}{2}\right)(T_w-T_\infty)\xi f'(\xi)\theta'(\xi)\\
&\quad-ax^{n-1}\left(\frac{n+1}{2}\right)(T_w-T_\infty)f(\xi)\theta'(\xi),\\
u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y} &= -ax^{n-1}\left(\frac{n+1}{2}\right)(T_w-T_\infty)f(\xi)\theta'(\xi). \qquad (3.24)
\end{aligned}
$$

Using (3.20)-(3.23) in the right side of (3.3), we get

$$
\begin{aligned}
&\alpha_{nf}\frac{\partial^2 T}{\partial y^2}+\frac{\mu_{nf}}{(\rho Cp)_{nf}}\left(\frac{\partial u}{\partial y}\right)^2-\frac{1}{(\rho Cp)_{nf}}\frac{\partial q_r}{\partial y}+\frac{q}{(\rho Cp)_{nf}}(T-T_\infty)\\
&=\alpha_{nf}\left(x^{n-1}\left(\frac{a(n+1)}{2\nu_f}\right)(T_w-T_\infty)\theta''(\xi)\right)\\
&\quad+\frac{\mu_{nf}}{(\rho Cp)_{nf}}\left(a^2x^{3n-1}\left(\frac{a(n+1)}{2\nu_f}\right)f''^2(\xi)\right)\\
&\quad+\frac{1}{(\rho Cp)_{nf}}\left(\frac{16\sigma^* T_\infty^3}{3k^*}(T_w-T_\infty)x^{n-1}\left(\frac{a(n+1)}{2\nu_f}\right)\theta''(\xi)\right)\\
&\quad+\frac{q}{(\rho Cp)_{nf}}(T_w-T_\infty)\theta(\xi). \qquad (3.25)
\end{aligned}
$$

With the help of (3.24) and (3.25), the dimensionless form of (3.3), is obtained.

$$
u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha_{nf}\frac{\partial^2 T}{\partial y^2}+\frac{\mu_{nf}}{(\rho Cp)_{nf}}\left(\frac{\partial u}{\partial y}\right)^2-\frac{1}{(\rho Cp)_{nf}}\frac{\partial q_r}{\partial y}+\frac{q}{(\rho Cp)_{nf}}(T-T_\infty),
$$

$$
\begin{aligned}
& - ax^{n-1}\left[\left(\frac{n+1}{2}\right)(T_w - T_\infty) f(\xi)\theta'(\xi)\right] = \alpha_{nf}\left[x^{n-1}\left(\frac{a(n+1)}{2\nu_f}\right)(T_w - T_\infty)\theta''(\xi)\right] \\
& + \frac{\mu_{nf}}{(\rho Cp)_{nf}}\left(a^2 x^{3n-1}\left(\frac{a(n+1)}{2\nu_f}\right) f''^2(\xi)\right) + \frac{q}{(\rho Cp)_{nf}}(T_w - T_\infty)\theta(\xi) \\
& + \frac{1}{(\rho Cp)_{nf}}\left(\frac{16\sigma^* T_\infty^3}{3k^*}(T_w - T_\infty)x^{n-1}\left(\frac{a(n+1)}{2\nu_f}\right)\theta''(\xi)\right), \\
& - \frac{\nu_f}{\alpha_{nf}} f(\xi)\theta'(\xi) = \theta''(\xi) + \frac{\mu_{nf}}{(\rho Cp)_{nf}\alpha_{nf}}\left(\frac{a^2 x^{2n}}{(T_w - T_\infty)}\right) f''^2(\xi) \\
& + \frac{1}{(\rho Cp)_{nf}\alpha_{nf}}\left(\frac{16\sigma^* T_\infty^3}{3k^*}\right)\theta''(\xi) + \frac{q}{(\rho Cp)_{nf}\alpha_{nf} a x^{n-1}}\left(\frac{2\nu_f}{n+1}\right)\theta(\xi), \\
& \left(1 + \frac{16\sigma^* T_\infty}{k_{nf} 3k^*}\right)\theta''(\xi) + \frac{(\rho Cp)_{nf}\nu_f}{k_n f} f(\xi)\theta'(\xi) + \frac{\mu_{nf} a^2 x^{2n}}{k_{nf}(T_w - T_\infty)} f''^2(\xi) \\
& + \frac{q}{k_{nf} a x^{n-1}}\left(\frac{2\nu_f}{n+1}\right)\theta(\xi) = 0, \\
& \frac{k_{nf}}{k_f}\left(1 + \frac{4}{3}R\right)\theta''(\xi) + \frac{(\rho Cp)_{nf}\nu_f}{k_n f} f(\xi)\theta'(\xi) + \frac{\mu_{nf} a^2 x^{2n}}{k_f(T_w - T_\infty)} f''^2(\xi) \\
& + \frac{q}{k_f a x^{n-1}}\left(\frac{2\nu_f}{n+1}\right)\theta(\xi) = 0, \\
& \frac{k_{nf}}{k_f}\left(1 + \frac{4}{3}R\right)\theta''(\xi) + \frac{\nu_f(\rho Cp)_f\left(1 - \phi + \phi\frac{(\rho Cp)_s}{(\rho Cp)_f}\right)}{\alpha_f(\rho Cp)_f} f(\xi)\theta'(\xi) \\
& + \frac{\mu_f a^2 x^{2n}}{(1-\phi)^{2.5}(\rho Cp)_f \alpha_f (T_w - T_\infty)} f''^2(\xi) + \frac{q x \nu_f}{(\rho Cp)_f \alpha_f a x^n}\left(\frac{2}{n+1}\right)\theta(\xi) = 0, \\
& \frac{k_{nf}}{k_f}\left(1 + \frac{4}{3}R\right)\theta''(\xi) + Pr\left(1 - \phi + \phi\frac{(\rho Cp)_s}{(\rho Cp)_f}\right) f(\xi)\theta'(\xi) \\
& + Pr\left(\frac{Ec}{(1-\phi)^{2.5}}\right) f''^2(\xi) + Pr\left(\frac{2}{n+1}\right) Q\theta(\xi) = 0,
\end{aligned}
$$

$$
A_4\theta''(\xi) + PrA_5 f(\xi)\theta'(\xi) + PrA_6 f''^2(\xi) + Pr\left(\frac{2}{n+1}\right)Q\theta(\xi) = 0. \tag{3.26}
$$

The corresponding BCs are transformed into the non-dimensional form through the following procedure.

$$
\begin{aligned}
& u = U_w(x) = ax^n, && at \quad y = 0. \\
\Rightarrow \quad & u = af'(\xi)x^n. && \\
\Rightarrow \quad & af'(\xi) = ax^n, && \\
\Rightarrow \quad & f'(\xi) = 1, && at \quad \xi = 0. \\
\Rightarrow \quad & f'(0) = 1. &&
\end{aligned}
$$

$$
\begin{aligned}
& v = 0, && at \quad y = 0.\\
\Rightarrow\quad & -x^{\frac{n-1}{2}}\sqrt{\frac{2\nu_f a}{n+1}}\left(\frac{n+1}{2}\right)f(\xi) - ax^{n-1}y\left(\frac{n-1}{2}\right)f'(\xi) = 0, &&\\
& && at \quad \xi = 0.\\
\Rightarrow\quad & -x^{\frac{n-1}{2}}\sqrt{\frac{a\nu_f(n+1)}{2}}f(0) = 0, && at \quad \xi = 0.\\
\Rightarrow\quad & f(0) = 0. &&\\
& T = T_w, && at \quad y = 0.\\
\Rightarrow\quad & \theta(\xi)(T_w - T_\infty) + T_\infty = T_w, &&\\
\Rightarrow\quad & \theta(\xi)(T_w - T_\infty) = (T_w - T_\infty), &&\\
\Rightarrow\quad & \theta(\xi) = 1, && at \quad \xi = 0.\\
\Rightarrow\quad & \theta(0) = 1. &&\\
& u \to (0), && as \quad y \to \infty.\\
\Rightarrow\quad & af'(\xi)x^n \to (0), &&\\
\Rightarrow\quad & ax^n f'(\xi) \to (0), &&\\
\Rightarrow\quad & f'(\xi) \to (0), && as \quad \xi \to \infty.\\
\Rightarrow\quad & f'(\infty) \to 0. &&\\
& T \to T_\infty, && as \quad y \to \infty.\\
\Rightarrow\quad & \theta(\xi)(T_w - T_\infty) + T_\infty \to T_\infty, &&\\
\Rightarrow\quad & \theta(\xi)(T_w - T_\infty) \to 0, && as \quad \xi \to \infty.\\
\Rightarrow\quad & \theta(\xi) \to 0, && as \quad \xi \to \infty.\\
\Rightarrow\quad & \theta(\infty) \to 0. &&
\end{aligned}
$$

The final dimensionless form of the governing model, is

$$
f'''(\xi) + A_1A_2\left(f(\xi)f''(\xi) - \left(\frac{2n}{n+1}\right)f'^2(\xi)\right) - \left(\frac{2}{n+1}\right)(K + MA_1A_3)f'(\xi) = 0. \tag{3.27}
$$

$$
A_4\theta''(\xi) + PrA_5f(\xi)\theta'(\xi) + PrA_6f''^2(\xi) + Pr\left(\frac{2}{n+1}\right)Q\theta(\xi) = 0. \tag{3.28}
$$

The associated BCs (3.4) in the dimensionless form are,

$$\left.\begin{aligned} &f(0)=0, \quad f'(0)=1, \quad \theta(0)=1. \\ &f'(\infty)\to 0, \quad \theta(\infty)\to 0. \end{aligned}\right\} \tag{3.29}$$

Different dimensionless parameters used in equations (3.27) and (3.28) are formulated as follows.

$$\begin{aligned} &M=\frac{\sigma_f B_0^2}{\rho_f a x^{-1}}, \quad K=\frac{\nu_f}{a k_0}, \quad R=\frac{4\sigma^* T_\infty^3}{k_{nf} k^*}, \\ &Pr=\frac{\nu_f}{\alpha_f}, \quad Ec=\frac{U_w^2}{(c_p)_f (T_w-T_\infty)}, \quad Q=\frac{qx}{(\rho c_p)_f U_w}. \end{aligned}$$

The skin friction coefficient, is given as follows.

$$C_f=\frac{\tau_w|_{y=0}}{\rho_f U_w^2(x)}. \tag{3.30}$$

To achive the dimensionless form of C_f the following steps will be helpful.
Since

$$\tau_w=\mu_{nf}\left(\frac{\partial u}{\partial y}\right)_{y=0}, \tag{3.31}$$

$$C_{fx}=\frac{1}{\rho_f U_w(x)^2}\mu_{nf}\left(\frac{\partial u}{\partial y}\right)_{y=0},$$

$$C_f=\frac{1}{\rho_f a^2 x^{2n}}\mu_{nf}\left(a x^n f''(\xi) x^{\frac{n-1}{2}}\sqrt{\frac{(n+1)a}{2\nu_f}}\right),$$

$$C_f=\frac{\mu_f}{\rho_f a^2 x^{2n}(1-\phi)^{2.5}}\left(a x^n f''(\xi) x^{\frac{n-1}{2}}\sqrt{\frac{(n+1)a}{2\nu_f}}\right),$$

$$C_f=\frac{\nu_f \rho_f a^{\frac{3}{2}} x^{\frac{3n-1}{2}}}{\rho_f a^2 x^{2n}(1-\phi)^{2.5}}\sqrt{\frac{(n+1)}{2\nu_f}}f''(\xi),$$

$$C_f=\frac{\nu_f}{a^{\frac{1}{2}} x^{\frac{n+1}{2}}(1-\phi)^{2.5}}\left(\frac{n+1}{2}\right)^{\frac{1}{2}} f''(\xi),$$

$$C_f=\frac{1}{Re_x^{\frac{1}{2}}(1-\phi)^{2.5}}\left(\frac{n+1}{2}\right)^{\frac{1}{2}} f''(\xi),$$

$$Re_x^{\frac{1}{2}} C_f=\frac{1}{(1-\phi)^{2.5}}\left(\frac{n+1}{2}\right)^{\frac{1}{2}} f''(\xi), \tag{3.32}$$

where Re denotes the Reynolds number defined as $Re = \frac{xu_x(x)}{\nu_f}$.

Local Nusselt number is defined as follow.

$$Nu_x = \frac{xq_w}{k_f(T_w - T_\infty)}. \tag{3.33}$$

To achive the dimensionless form of Nu_x, the following steps will be helpful.

Since

$$q_w = \left(-\left(k_{nf} + \frac{16\sigma^* T_\infty^3}{3k^*}\right)\left(\frac{\partial T}{\partial y}\right)\right)_{y=0}, \tag{3.34}$$

$$\begin{aligned}
Nu_x &= -\frac{x\left(k_{nf} + \frac{16\sigma^* T_\infty^3}{3k^*}\right)\left(\frac{\partial T}{\partial y}\right)_{y=0}}{k_f(T_w - T_\infty)},\\
Nu_x &= -\frac{x\left(k_{nf} + \frac{16\sigma^* T_\infty^3}{3k^*}\right)}{k_f(T_w - T_\infty)}\left(\frac{\partial T}{\partial y}\right)_{y=0},\\
Nu_x &= -\frac{x\left(k_{nf} + \frac{16\sigma^* T_\infty^3}{3k^*}\right)}{k_f(T_w - T_\infty)}\left(\frac{(n+1)a}{2\nu_f}\right)^{\frac{1}{2}} x^{\frac{n-1}{2}}(T_w - T_\infty)\theta'(\xi),\\
&= -\frac{k_{nf}}{k_f}\left(1 + \frac{4}{3}R\right)\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{n+1}{2}}\theta'(\xi),\\
&= -\frac{k_{nf}}{k_f}\left(1 + \frac{4}{3}R\right)\sqrt{\frac{n+1}{2}}\theta'(\xi)\sqrt{\frac{ax^{n+1}}{\nu_f}},\\
&= -\frac{k_{nf}}{k_f}\left(1 + \frac{4}{3}R\right)\sqrt{\frac{n+1}{2}}\theta'(\xi)Re_x^{\frac{1}{2}},
\end{aligned}$$

$$Re_x^{\frac{-1}{2}} Nu_x = -\frac{k_{nf}}{k_f}\left(1 + \frac{4}{3}R\right)\left(\frac{n+1}{2}\right)^{\frac{1}{2}}\theta'(\xi). \tag{3.35}$$

3.3 Numerical Method for Solution

The shooting method has been used to solve the ordinary differential equation (3.27). The following notations have been considered.

$$f = Z_1, \qquad f' = Z_1' = Z_2, \qquad f'' = Z_1'' = Z_2' = Z_3, \qquad f''' = Z_3'.$$

$$A_1 = (1-\phi)^{2.5}, \qquad A_2 = 1 - \phi + \phi\frac{\rho_s}{\rho_f}, \qquad A_3 = 1 + \frac{3\left(\frac{\sigma_s}{\sigma_f} - 1\right)\phi}{\left(\frac{\sigma_s}{\sigma_f} + 2\right) - \left(\frac{\sigma_s}{\sigma_f} - 1\right)\phi}.$$

As a result the momentum equation is converted into the following system of first order ODEs.

$$
\begin{aligned}
Z_1' &= Z_2, && Z_1(0) = 0.\\
Z_2' &= Z_3, && Z_2(0) = 1.\\
Z_3' &= -A_1A_2\left(Z_1Z_3 - \left(\frac{2n}{n+1}\right)Z_2^2\right) + \left(\frac{2}{n+1}\right)(K + MA_1A_3)Z_2, && Z_3(0) = p.
\end{aligned}
$$

The above IVP will be numerically solved by RK-4. The missing condition p is to be chosen such that.

$$Z_2(\xi_\infty, p) = 0.$$

Newton's method will be used to find p. This method has the following iterative scheme.

$$p^{n+1} = p^n - \frac{Z_2(\xi_\infty, p)}{\frac{\partial}{\partial p}(Z_2(\xi_\infty, p))}.$$

We further introduce the following notations,

$$\frac{\partial Z_1}{\partial p} = Z_4, \qquad \frac{\partial Z_2}{\partial p} = Z_5, \qquad \frac{\partial Z_3}{\partial p} = Z_6.$$

As a result of these new notations the Newton's iterative scheme gets the form

$$p^{n+1} = p^n - \frac{Z_2(\xi_\infty, p)}{Z_5(\xi_\infty, p)}.$$

Now differentiating the system of three first order ODEs with respect to p, we get another system of ODEs, as follows.

$$
\begin{aligned}
Z_4' &= Z_5, && Z_4(0) = 0.\\
Z_5' &= Z_6, && Z_5(0) = 0.\\
Z_6' &= -A_1A_2\left(Z_1Z_6 + Z_3Z_4 - \left(\frac{2n}{n+1}\right)2Z_2Z_5\right)\\
&\quad + \left(\frac{2}{n+1}\right)(K + MA_1A_3)Z_5, && Z_6(0) = 1.
\end{aligned}
$$

The stopping criteria for the Newton's technique is set as.

$$\mid Z_2(\xi_\infty, p) \mid < \epsilon,$$

where $\epsilon > 0$ is an arbitrarily small positive number. From now onward ϵ has been taken as 10^{-10}.

The equation (3.28) will be numerically solved by using shooting method by assuming f as a known function. For this we utilize the following notions.

$$\theta = Y_1, \quad \theta' = Y_2, \quad \theta'' = Y_2'.$$
$$A_4 = \frac{k_{nf}}{k_f}\left(1 + \frac{4}{3}R\right), \quad A_5 = 1 - \phi + \phi\frac{(\rho Cp)_s}{(\rho Cp)_f}, \quad A_6 = \frac{Ec}{(1-\phi)^{2.5}}.$$

As a result, the energy equation (3.28) is converted into the following system of first order ODEs.

$$Y_1' = Y_2, \qquad Y_1(0) = 1.$$
$$Y_2' = -\frac{Pr}{A_4}\left(A_5 f Y_2 + A_6 f''^2 + \left(\frac{2}{n+1}\right) Q Y_1\right), \qquad Y_2(0) = q.$$

The above initial value problem (IVP) will be numerically solved by RK-4 technique. In the above initial value problem, the missing condition q is to satisfy the following relation.

$$Y_1(\xi_\infty, q) = 0.$$

The above equation can be solved by using Newton's method with the following iterative formula.

$$q^{n+1} = q^n - \frac{Z_1(\xi_\infty, q)}{Z_1'(\xi_\infty, q)}.$$

We further introduce the following notations,

$$\frac{\partial Y_1}{\partial q} = Y_3, \qquad \frac{\partial Y_2}{\partial q} = Y_4.$$

Now differentiating the system of two first order ODEs with respect to q, we get another system of ODEs, as follows.

$$Y_3' = Y_4, \qquad Y_3(0) = 0.$$

$$Y_4' = -\frac{Pr}{A_4}\left(A_5 f Y_4 + \left(\frac{2}{n+1}\right) Q Y_3\right), \qquad Y_4(0) = 1.$$

The stopping criteria for the Newton's method is set as.

$$\mid Y_1(\xi_\infty, q) \mid < 10^{-10}.$$

3.4 Representation of Graphs and Tables

A thorough discussion on the graphs and tables has been conducted which contains the impact of dimensionless parameters on the skin friction coefficient $(Re_x)^{\frac{1}{2}} C_f$ and local Nusselt number $(Re_x)^{\frac{-1}{2}} Nu_x$. Table 3.1 explains the impact of nonlinear stretching parameter n, magnetic parameter M, nanoparticle volume fraction ϕ and permeability parameter K on $(Re_x)^{\frac{1}{2}} C_f$. For the rising values of ϕ, the skin fraction coefficient $(Re_x)^{\frac{1}{2}} C_f$ decreases. In Table 3.2, the effect of significant parameters on local Nusselt number $(Re_x)^{\frac{-1}{2}} Nu_x$ has been discussed. The rising pattern is found in $(Re_x)^{\frac{-1}{2}} Nu_x$ due to increasing values of n.

Figures 3.2-3.5 reflect the behaviour of the velocity profile $f'(\xi)$ and temperature profile $\theta(\xi)$ for different values of ϕ with and without M.

Figures 3.6 and 3.7 show the impact of K. For the rising values of K, the velocity profile $f'(\xi)$ decreases and the temperature profile $\theta(\xi)$ increases

Figures 3.8 and 3.9 represent the impact of n on $f'(\xi)$ and $\theta(\xi)$. It can be observed from Figure 3.8 that the velocity profile increases for larger values of n. This increment in the non-dimensional velocity of stretching is due to the greater value of n and helps to cause more liquid deformation. As the value of n increases, the momentum boundary layer becomes thicker, whereas with an increase in n, a reduction in the temperature profile is observed, leading to an increase in the heat transfer.

Figure 3.10 illustrates the impact of heat generation Q on $\theta(\xi)$. It is observed that for the rising values of Q, more heat is generated, because of this $\theta(\xi)$ and boundary layer thickness increases.

From Figure 3.11, it can be seen that by increasing the values of Eckert number Ec, the temperature profile also increases. Figure 3.12 shows the impact of thermal radiation R on $\theta(\xi)$. In this graph it is observed that on the rising values of R, the temperature profile $\theta(\xi)$ also increases. So, rate of heat transfer decreases with increase in thermal radiation R, because of that temperature profile $\theta(\xi)$ increases. Figure 4.13 displays the impact of M on the velocity distribution. By rising the values of M, the velocity distribution shows the decreasing behavior due to the presence of Lorentz force. Figure 4.14 describes the impact of M on the temperature distribution. The temperature distribution expands by rising the values of M. Figure 3.15 shows the influence of Prandtl number Pr, on $\theta(\xi)$. The rising values of Pr, the temperature profile $\theta(\xi)$ is decreased.

TABLE 3.1: Results of $(Re_x)^{\frac{1}{2}}C_f$ for various parameters

ϕ	n	M	K	$(Re_x)^{\frac{1}{2}}C_f$
0.0	2.0	2.0	1.0	-2.197435
0.1	2.0	2.0	1.0	-3.071673
0.2	2.0	2.0	1.0	-4.176089
0.1	1.0	2.0	1.0	-2.738772
0.1	3.0	2.0	1.0	-3.371334
0.1	7.0	2.0	1.0	-4.367344
0.1	2.0	1.0	1.0	-2.774333
0.1	2.0	3.0	1.0	-3.342529
0.1	2.0	4.0	1.0	-3.342529
0.1	2.0	2.0	0.0	-2.781840
0.1	2.0	2.0	2.0	-3.336290
0.1	2.0	2.0	3.0	-3.581328
0.1	2.0	2.0	4.0	-3.810591

TABLE 3.2: Results of $-(Re_x)^{\frac{-1}{2}} Nu_x$ some fixed parameters $\phi = 0.1$, $K = 1.0$ $R = 0.5$

n	M	Q	Ec	Pr	$-(Re_x)^{\frac{-1}{2}} Nu_x$
2.0	2.0	0.1	0.2	6.2	0.518692
3.0	2.0	0.1	0.2	6.2	0.928781
4.0	2.0	0.1	0.2	6.2	1.258709
5.0	2.0	0.1	0.2	6.2	1.540036
7.0	2.0	0.1	0.2	6.2	2.012070
8.0	2.0	0.1	0.2	6.2	2.217215
2.0	0.0	0.1	0.2	6.2	1.197220
2.0	1.0	0.1	0.2	6.2	0.837433
2.0	2.0	0.1	0.2	6.2	0.518692
2.0	3.0	0.1	0.2	6.2	0.231713
2.0	4.0	0.1	0.2	6.2	-0.029851
2.0	5.0	0.1	0.2	6.2	-0.270517
2.0	2.0	0.0	0.2	6.2	0.901739
2.0	2.0	0.1	0.2	6.2	0.518692
2.0	2.0	0.2	0.2	6.2	0.051935
2.0	2.0	0.3	0.2	6.2	-0.563865
2.0	2.0	0.4	0.2	6.2	-1.504916
2.0	2.0	0.1	0.0	6.2	2.168818
2.0	2.0	0.1	0.1	6.2	1.343755
2.0	2.0	0.1	0.2	6.2	0.518692
2.0	2.0	0.1	0.3	6.2	-0.306370
2.0	2.0	0.1	0.4	6.2	-1.131433
2.0	2.0	0.1	0.5	6.2	-1.956497
2.0	2.0	0.1	0.2	3.0	0.470148
2.0	2.0	0.1	0.2	4.0	0.474044
2.0	2.0	0.1	0.2	5.0	0.492361
2.0	2.0	0.1	0.2	7.0	0.533803
2.0	2.0	0.1	0.2	9.0	0.553315

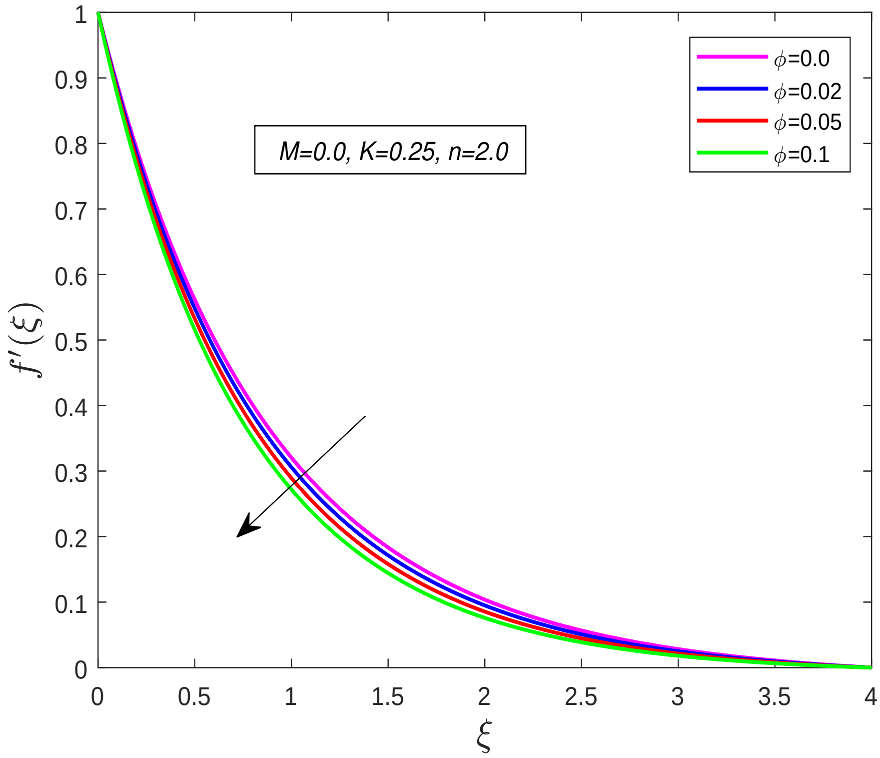

FIGURE 3.2: Impact of ϕ on $f'(\xi)$ for $M = 0$.

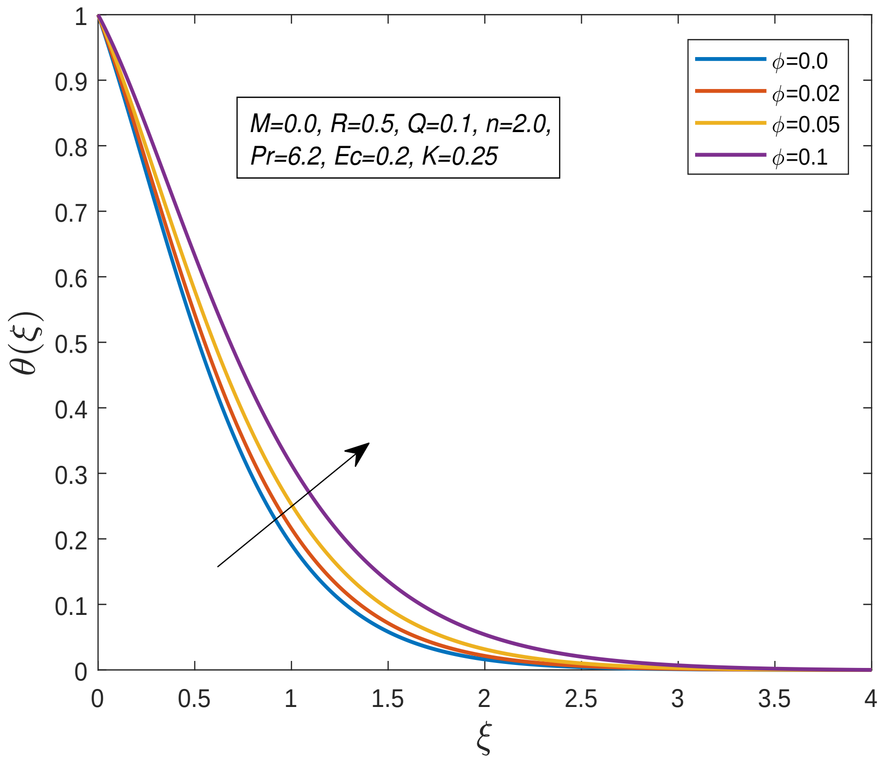

FIGURE 3.3: Impact of ϕ on $\theta(\xi)$ for $M = 0$.

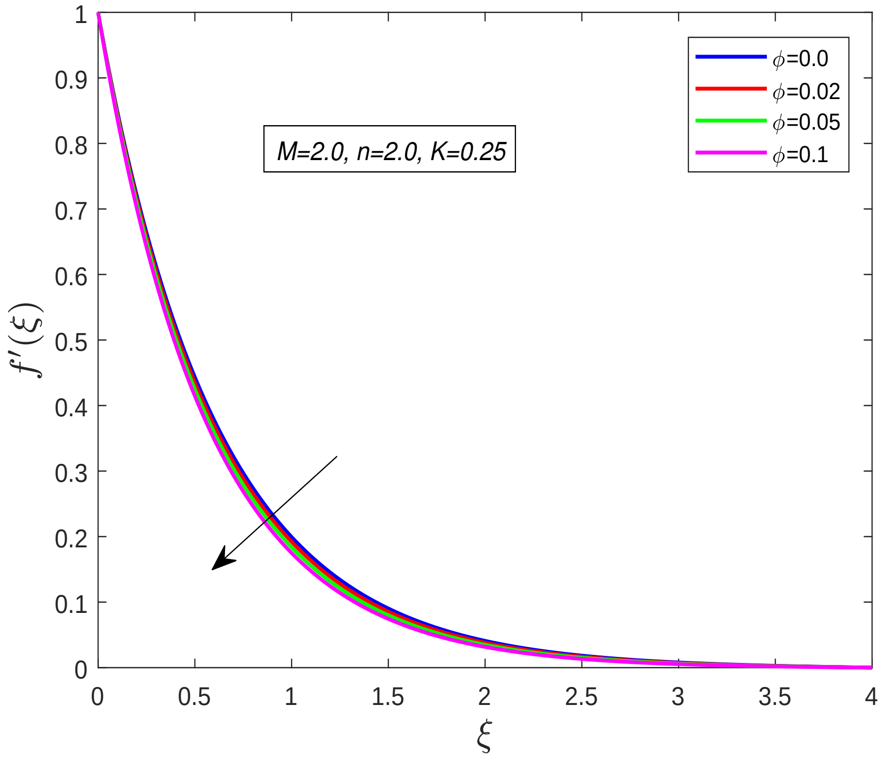

FIGURE 3.4: Impact of ϕ on $f'(\xi)$ for $M = 2$.

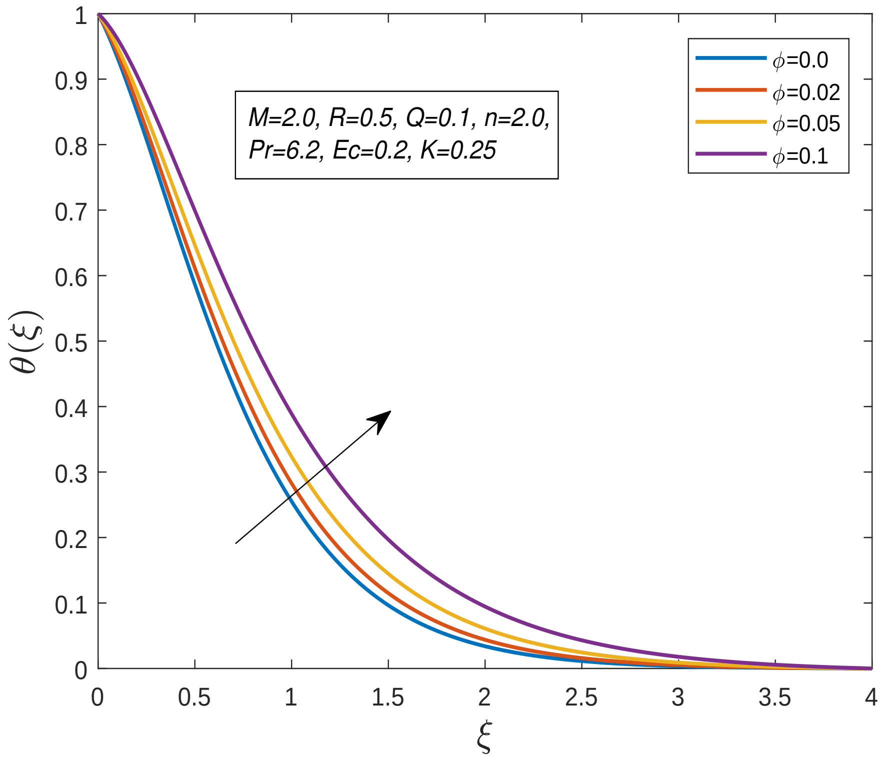

FIGURE 3.5: Impact of ϕ on $\theta(\xi)$ for $M = 2$.

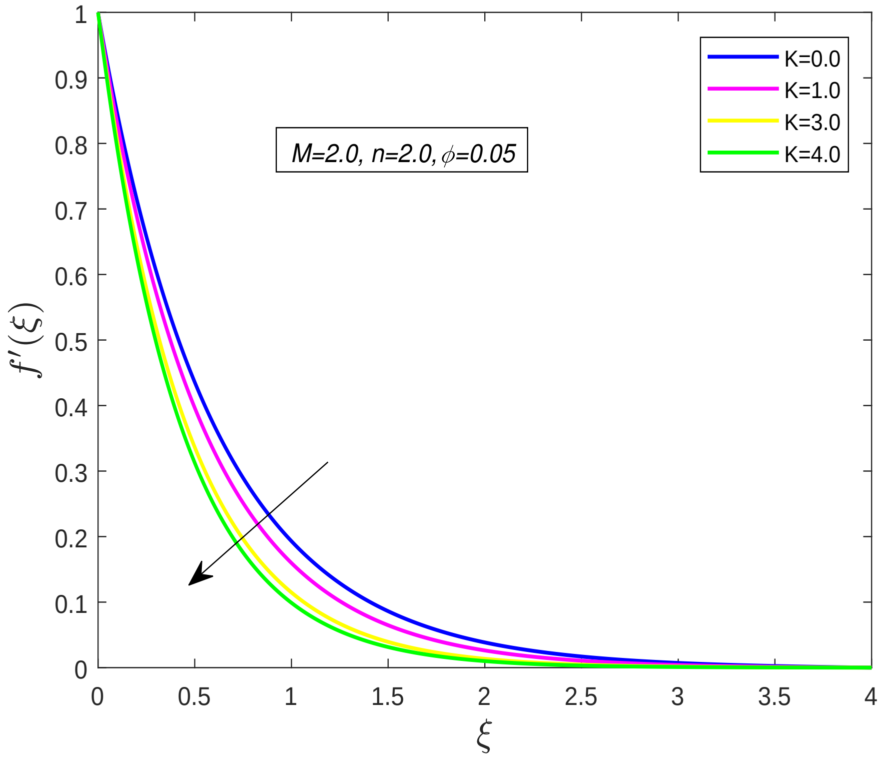

FIGURE 3.6: Impact of K on the velocity profile.

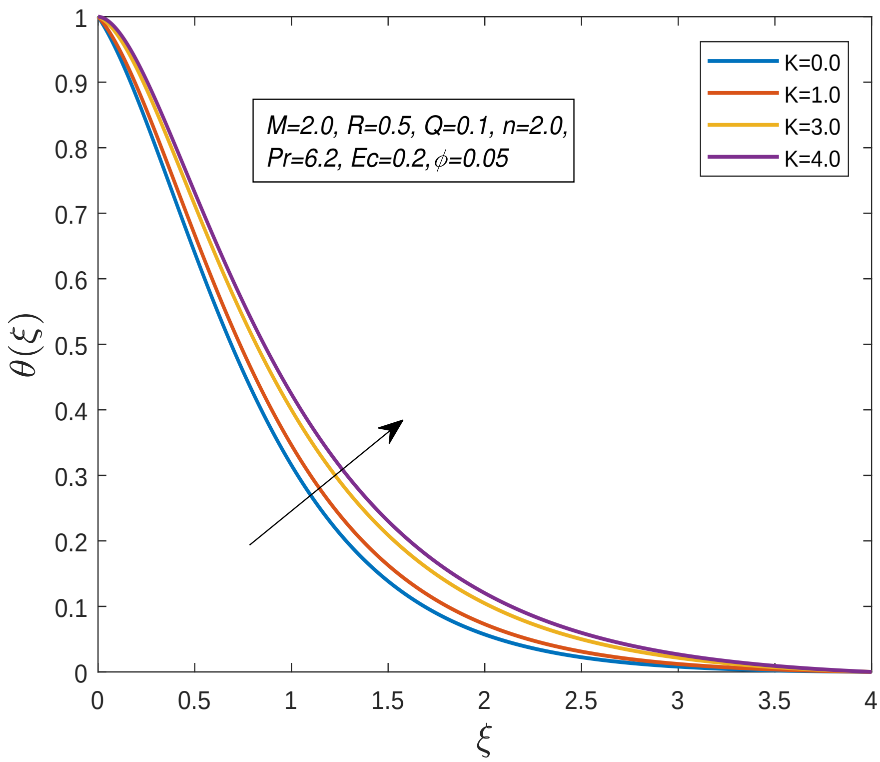

FIGURE 3.7: Impact of K on the temperature profile.

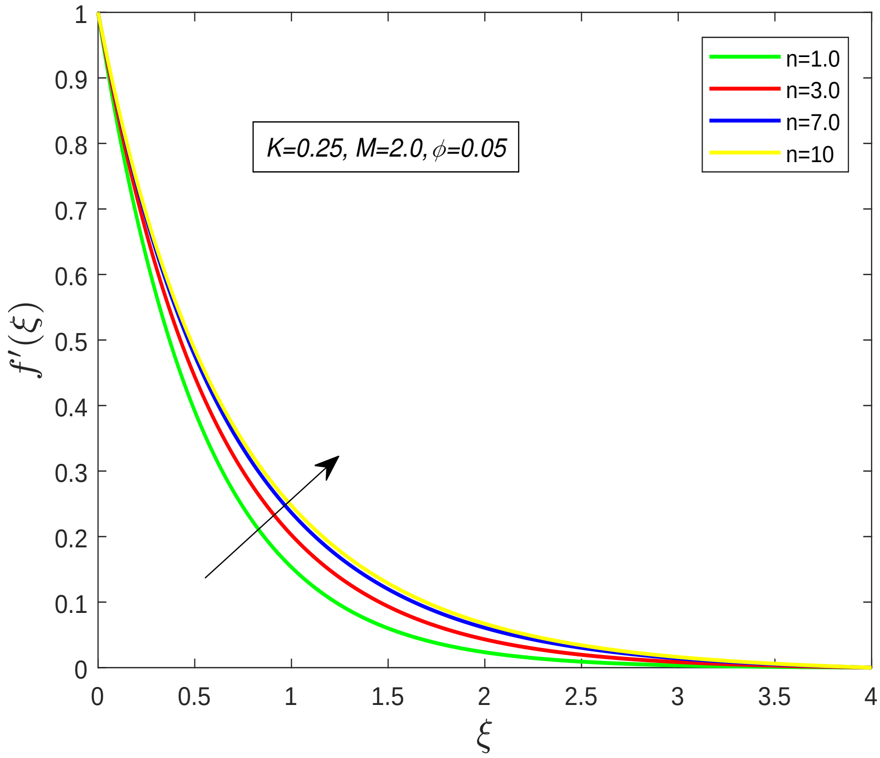

FIGURE 3.8: Impact of n on the velocity profile.

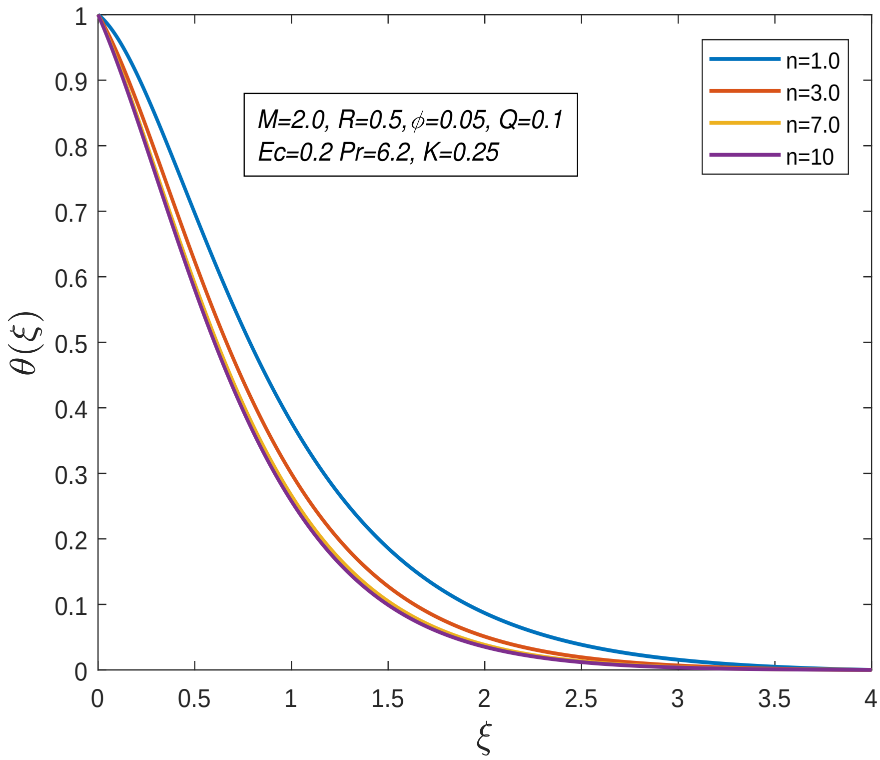

FIGURE 3.9: Impact of n on the temperature profile.

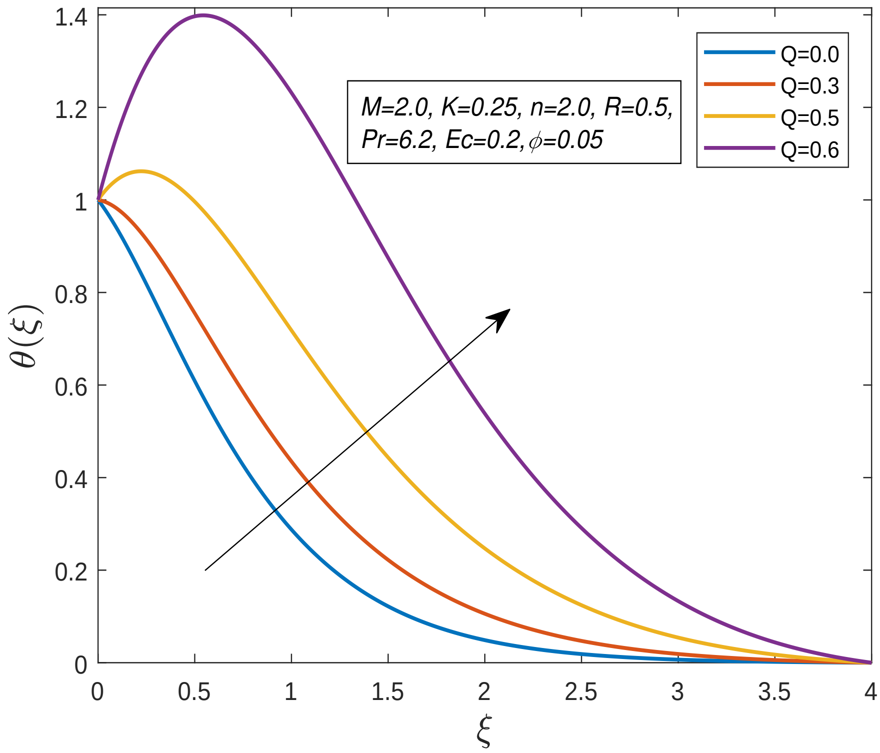

FIGURE 3.10: Impact of Q on the temperature profile.

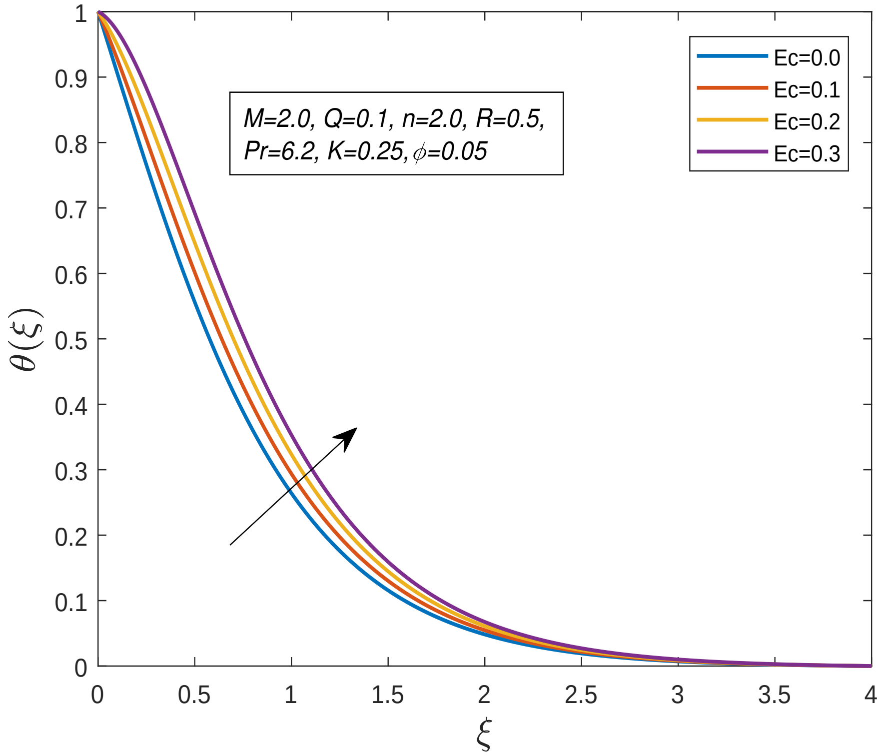

FIGURE 3.11: Impact of Ec on the temperature profile.

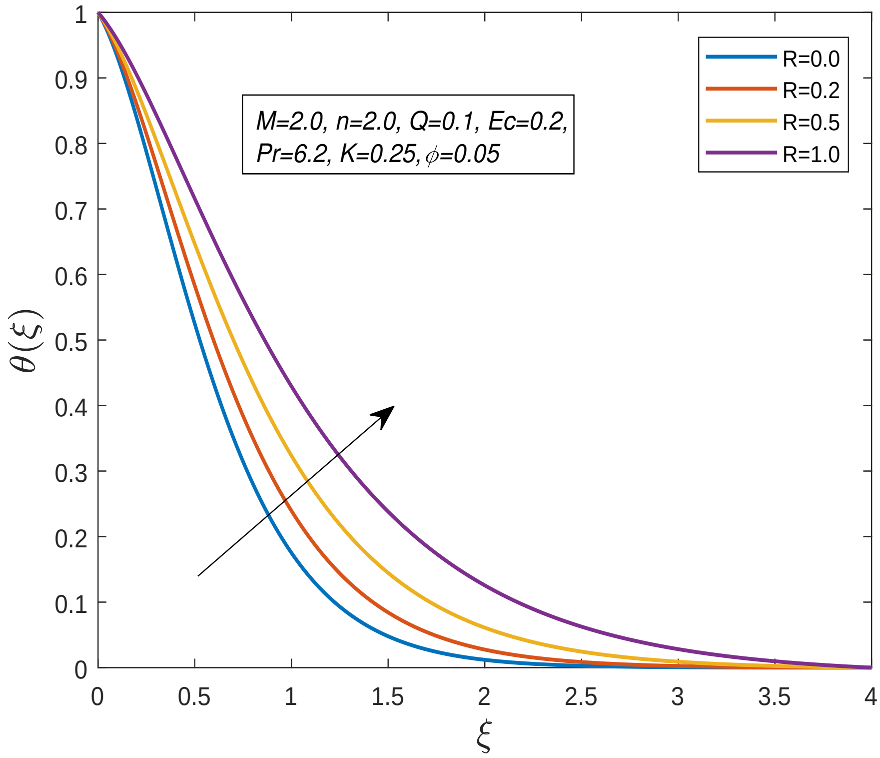

FIGURE 3.12: Impact of R on the temperature profile.

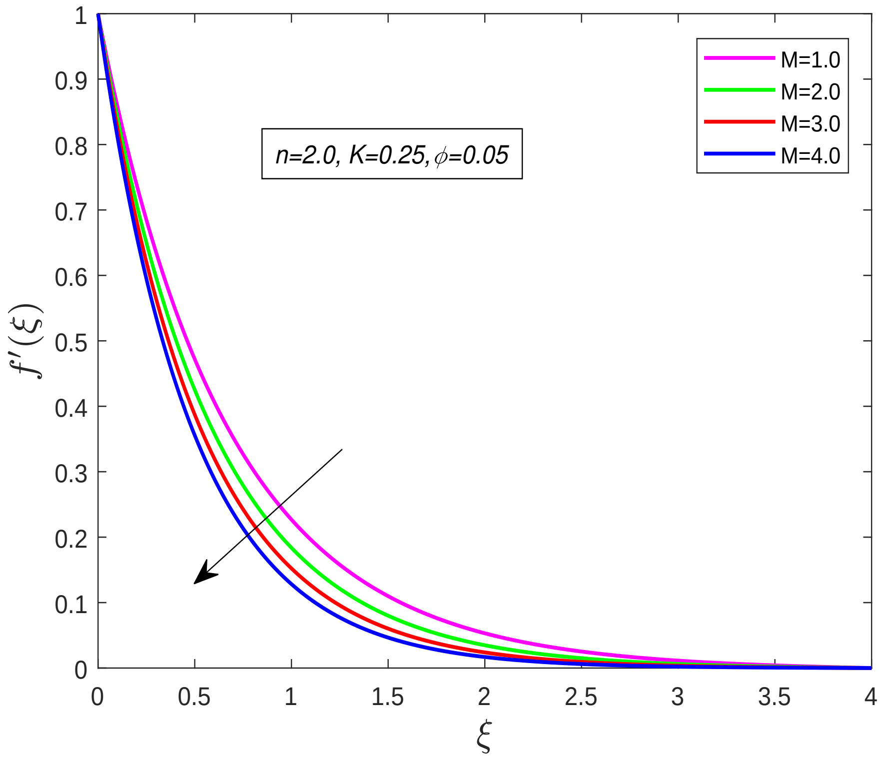

FIGURE 3.13: Impact of M on the velocity profile.

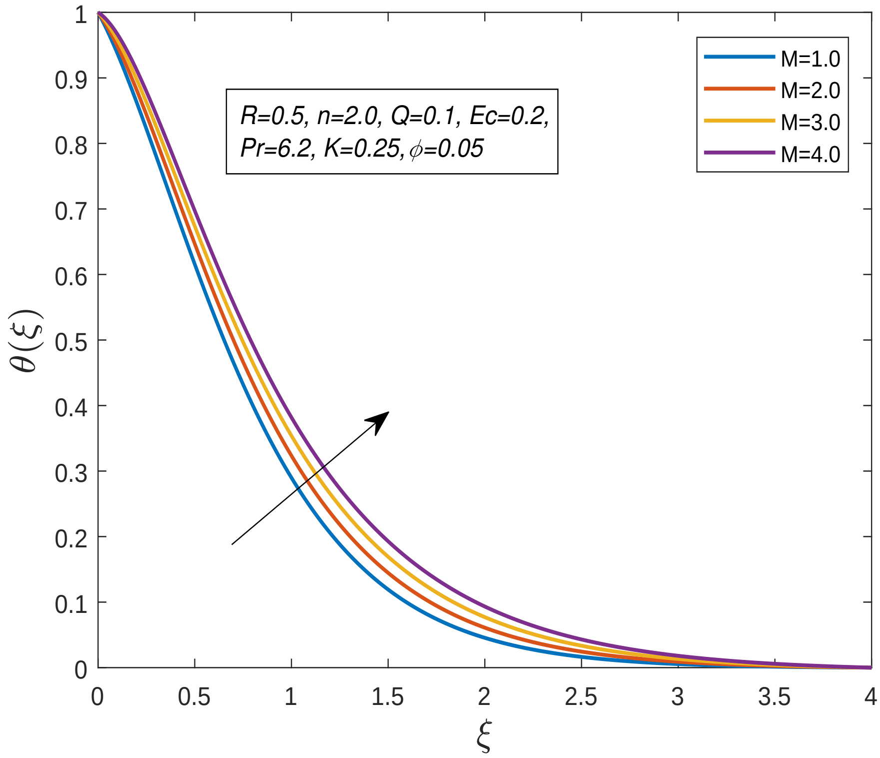

FIGURE 3.14: Impact of M on the temperature profile.

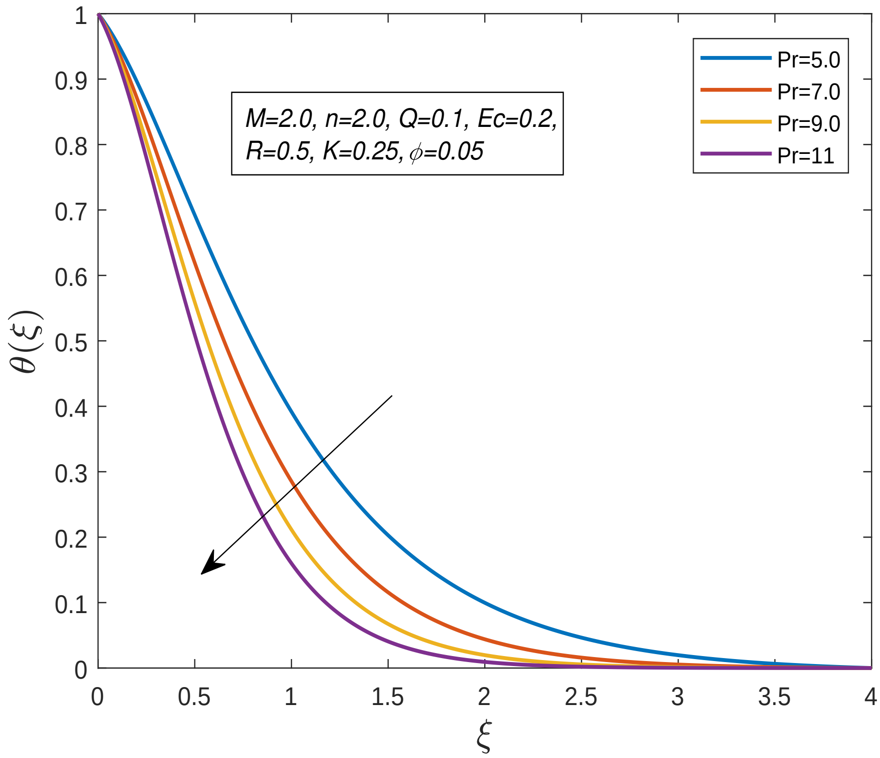

FIGURE 3.15: Impact of Pr on the temperature profile.

Chapter 4

MHD Radiative Nanofluid Flow with Cattaneo-Christov Heat Flux and Concentration with Chemical Reaction

4.1 Introduction

This chapter contains the extension of the model [44] by considering aligned magnetic field in momentum equation. The Cattaneo-Christov heat flux, thermophoresis diffusion and Brownian motion are also included in the temperature equation. Furthermore concentration equation is also taken into account along the with chemical reaction. The governing nonlinear PDEs are converted into a system of dimensionless ODEs by utilizing the similarity transformations. The numerical solution of ODEs is obtaind by applying numerical method known as shooting method. At the end of this chapter, the final results are discussed for significant parameters affecting $f'(\xi)$, $\theta(\xi)$ and $h(\xi)$ which are shown in tables and graphs.

4.2 Mathematical Modeling

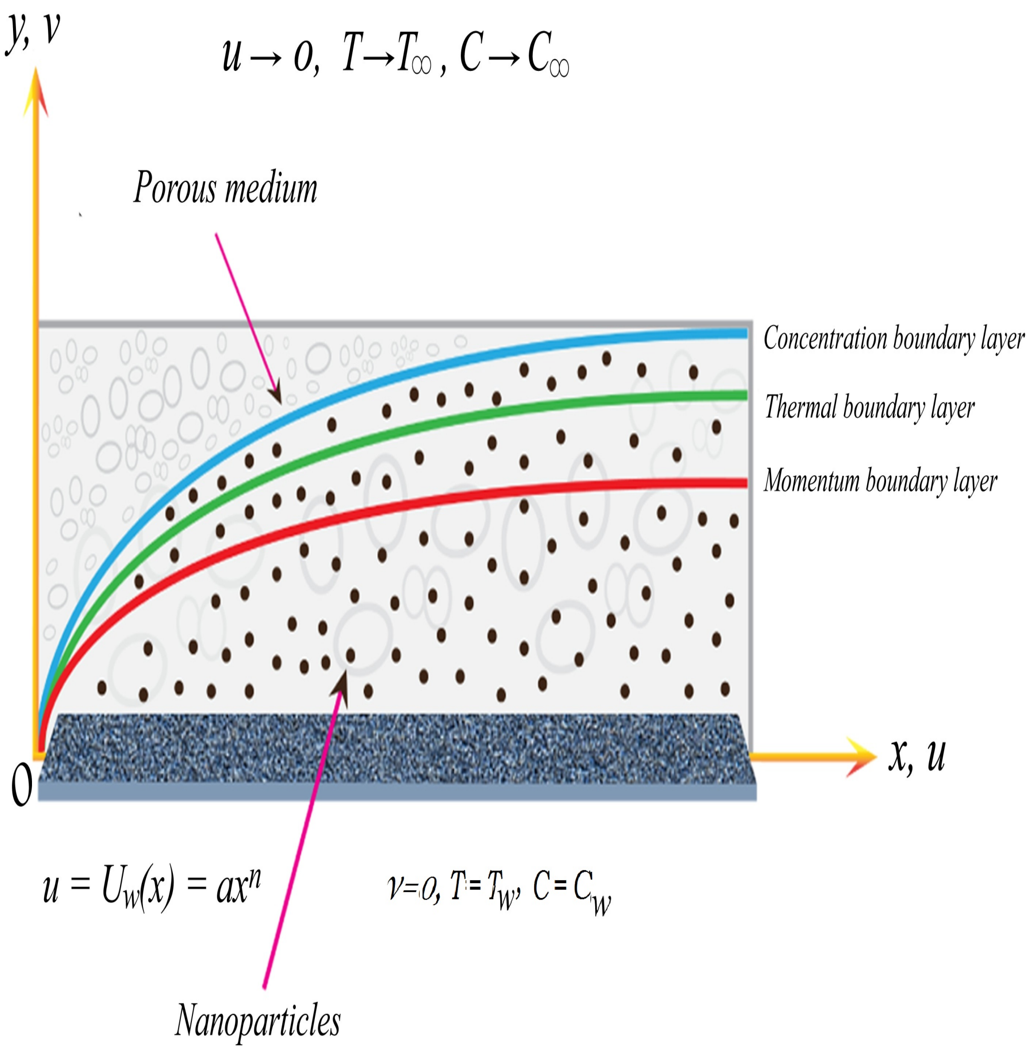

FIGURE 4.1: Geometry of physical model.

It is aimed to analyse the $2D$, MHD flow of nanofluid past a nonlinear stretching sheet and porous medium. The flow occupied the space $y > 0$. Magnetic field of strength B is applied with an inclination angle γ with the horizantal axis. Furthermore x-axis is taken in the direction of flow and y-axis normal to it. Energy transport analysis is also carried out in the presence of thermal radiation, viscous dissipation and Cattaneo-Christov heat flux. Moreover, the concentration of flow is discussed with the help of concentration equation under the effect of chemical reaction.

By considering the above assumptions, the governing PDEs are.

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{4.1}$$

$$\rho_{nf}\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=\mu_{nf}\left(\frac{\partial^2 u}{\partial y^2}\right)-\frac{\mu_{nf}}{k(x)}u-\sigma_{nf}B^2(x)\sin^2(\gamma)u, \tag{4.2}$$

$$\begin{aligned} &u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+\lambda\left[u\frac{\partial u}{\partial x}\frac{\partial T}{\partial x}+v\frac{\partial v}{\partial y}\frac{\partial T}{\partial y}+u\frac{\partial v}{\partial x}\frac{\partial T}{\partial y}+v\frac{\partial u}{\partial y}\frac{\partial T}{\partial x}+2uv\frac{\partial^2 T}{\partial x\partial y}\right.\\ &\left.+u^2\frac{\partial^2 T}{\partial x^2}+v^2\frac{\partial^2 T}{\partial y^2}\right]=\alpha_{nf}\left(\frac{\partial^2 T}{\partial y^2}\right)+\frac{\mu_{nf}}{(\rho C_p)_{nf}}\left(\frac{\partial u}{\partial y}\right)^2-\frac{1}{(\rho C_p)_{nf}}\left(\frac{\partial q_r}{\partial y}\right)\\ &+\frac{q}{(\rho C_p)_{nf}}(T-T_\infty)+\tau\left(D_B\frac{\partial C}{\partial y}\frac{\partial T}{\partial x}+\frac{D_T}{T_\infty}\left(\frac{\partial T}{\partial y}\right)^2\right), \end{aligned} \tag{4.3}$$

$$u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}=D_B\frac{\partial^2 C}{\partial y^2}+\left(\frac{D_T}{T_\infty}\right)\frac{\partial^2 T}{\partial y^2}-K_r(C-C_\infty). \tag{4.4}$$

The associated BCs have been taken as.

$$\left.\begin{aligned} &u=U_w(x)=ax^n, \quad v=0, \quad T=T_w, \quad C=C_w \quad at \quad y=0.\\ &u\to 0, \quad T\to T_\infty, \quad C\to C_\infty \quad as \quad y\to\infty. \end{aligned}\right\} \tag{4.5}$$

Following similarity transformation has been used to convert PDEs (4.1)-(4.4) into system of ODEs.

$$\left.\begin{aligned} &\psi(x,y)=\sqrt{\frac{2\nu_f a}{n+1}}x^{\frac{n+1}{2}}f(\xi),\\ &\xi=y\sqrt{\frac{a(n+1)}{2\nu_f}}x^{\frac{n-1}{2}},\\ &\theta(\xi)=\frac{T-T_\infty}{T_w-T_\infty},\\ &h(\xi)=\frac{C-C_\infty}{C_w-C_\infty}, \end{aligned}\right\} \tag{4.6}$$

where ψ stands for the stream function, ξ denotes the similarity variable, f, θ, and h are the dimensionless velocity, temperature and concentration.

The detailed procedure for the conversion of (4.1) has been discussed in chapter 3.

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0. \tag{4.7}$$

Now, I include the below procedure for the conversion of (4.2) into the dimensionless form.

$$
\begin{aligned}
u &= \frac{\partial \psi}{\partial y}, \\
u &= \frac{\partial}{\partial y}\left(\sqrt{\frac{2\nu_f a}{n+1}} x^{\frac{n+1}{2}} f(\xi)\right), \\
u &= a x^n f'(\xi). && (4.8) \\
v &= -\frac{\partial \psi}{\partial x}, \\
v &= -\frac{\partial}{\partial x}\left(\sqrt{\frac{2\nu_f a}{n+1}} x^{\frac{n+1}{2}} f(\xi)\right), \\
v &= -x^{n-1} a y \left(\frac{n-1}{2}\right) f'(\xi) - x^{\frac{n-1}{2}} \left(\frac{n+1}{2}\right)\sqrt{\frac{2\nu_f a}{n+1}} f(\xi). && (4.9) \\
u^2 &= (a x^n f'\xi)^2, \\
u^2 &= a^2 x^{2n} f'^2 \xi. && (4.10) \\
v^2 &= \left(-x^{n-1} a f'(\xi) y \left(\frac{n-1}{2}\right) - \sqrt{\frac{2\nu_f a}{n+1}}\left(\frac{n+1}{2}\right) x^{\frac{n-1}{2}} f(\xi)\right)^2, \\
v^2 &= a^2 x^{2n-2} y^2 \left(\frac{n-1}{2}\right)^2 f'^2(\xi) + x^{n-1}\left(\frac{n+1}{2}\right)^2\left(\frac{2\nu_f a}{n+1}\right) f^2(\xi) \\
&\quad + 2 a x^{3n-3} y \left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)\sqrt{\frac{2\nu_f a}{n+1}} f(\xi) f'(\xi). && (4.11)
\end{aligned}
$$

The complete procedure for the conversion of (4.2) discussed in chapter 3.

$$
\begin{aligned}
& f'''(\xi) + A_1 A_2 \left(f(\xi) f''(\xi) - \left(\frac{2n}{n+1}\right) f'^2(\xi)\right) \\
& - \left(\frac{2}{n-1}\right)(K + A_1 A_3 M \sin^2(\gamma)) f'(\xi) = 0. && (4.12)
\end{aligned}
$$

Now, we include below the procedure for the conversion of equation (4.3) into the dimensionless form. The (4.13)-(4.20) we have already derived in chapter 3.

$$
\frac{\partial u}{\partial x} = a x^{\frac{3n-3}{2}} y \left(\frac{n-1}{2}\right)\sqrt{\frac{(n+1)a}{2\nu_f}} f''(\xi) + n a x^{n-1} f'(\xi). \quad (4.13)
$$

$$
\frac{\partial u}{\partial y} = a x^{\frac{3n-1}{2}} \sqrt{\frac{(n+1)a}{2\nu_f}} f''(\xi). \quad (4.14)
$$

$$\frac{\partial T}{\partial x} = x^{\frac{n-3}{2}} y \left(\frac{n-1}{2}\right) \sqrt{\frac{(n+1)a}{2\nu_f}} (T_w - T_\infty)\theta'(\xi). \tag{4.15}$$

$$\begin{aligned}\frac{\partial^2 T}{\partial x^2} &= x^{n-3} y^2 \left(\frac{n-1}{2}\right)^2 \frac{a(n+1)}{2\nu_f} (T_w - T_\infty)\theta''(\xi) \\ &\quad + x^{\frac{n-5}{2}} y \left(\frac{n-1}{2}\right)\left(\frac{n-3}{2}\right) \sqrt{\frac{(n+1)a}{2\nu_f}} (T_w - T_\infty)\theta'(\xi).\end{aligned} \tag{4.16}$$

$$\frac{\partial T}{\partial y} = x^{\frac{n-1}{2}} \sqrt{\frac{(n+1)a}{2\nu_f}} (T_w - T_\infty)\theta'(\xi). \tag{4.17}$$

$$\frac{\partial^2 T}{\partial y^2} = x^{n-1} \frac{a(n+1)}{2\nu_f} (T_w - T_\infty)\theta''(\xi). \tag{4.18}$$

$$\begin{aligned}\frac{\partial^2 T}{\partial x \partial y} &= x^{\frac{n-3}{2}} \left(\frac{n-1}{2}\right) \sqrt{\frac{(n+1)a}{2\nu_f}} (T_w - T_\infty)\theta'(\xi) \\ &\quad + x^{n-2} y \left(\frac{n-1}{2}\right) \frac{(n+1)a}{2\nu_f} (T_w - T_\infty)\theta''(\xi).\end{aligned} \tag{4.19}$$

$$\frac{\partial v}{\partial y} = -nax^{n-1} f'(\xi) - ax^{\frac{3n-3}{2}} y \left(\frac{n-1}{2}\right) \sqrt{\frac{(n+1)a}{2\nu_f}} f''(\xi). \tag{4.20}$$

$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}\left[-x^{n-1} a f'(\xi) y \left(\frac{n-1}{2}\right) - \sqrt{\frac{2\nu_f a}{n+1}} \left(\frac{n+1}{2}\right) x^{\frac{n-1}{2}} f(\xi)\right],$$

$$\begin{aligned}\frac{\partial v}{\partial x} &= -ax^{\frac{3n-5}{2}} y^2 \left(\frac{n-1}{2}\right)^2 \sqrt{\frac{(n+1)a}{2\nu_f}} f''(\xi) - a(n-1)x^{n-2} y \left(\frac{n-1}{2}\right) f'(\xi) \\ &\quad - ax^{n-2} y \left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right) f'(\xi) - x^{\frac{n-3}{2}} \left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right) \sqrt{\frac{2\nu_f a}{n+1}} f(\xi).\end{aligned} \tag{4.21}$$

$$\begin{aligned}u\frac{\partial u}{\partial x}\frac{\partial T}{\partial x} &= ax^n f'(\xi) \left(ax^{\frac{3n-3}{2}} y \left(\frac{n-1}{2}\right) \sqrt{\frac{(n+1)a}{2\nu_f}} f''(\xi) + nax^{n-1} f'(\xi)\right) \\ &\qquad \left(x^{\frac{n-3}{2}} y \left(\frac{n-1}{2}\right) \sqrt{\frac{(n+1)a}{2\nu_f}} (T_w - T_\infty)\theta'(\xi)\right),\end{aligned}$$

$$\begin{aligned}u\frac{\partial u}{\partial x}\frac{\partial T}{\partial x} &= ax^n f'(\xi)\Bigg(ax^{2n-3} y^2 \left(\frac{n-1}{2}\right)^2 \frac{(n+1)a}{2\nu_f} f''(\xi)\theta'(\xi)(T_w - T_\infty) \\ &\quad + nax^{\frac{2n-5}{2}} y \left(\frac{n-1}{2}\right) \sqrt{\frac{(n+1)a}{2\nu_f}} f'(\xi)\theta'(\xi)(T_w - T_\infty)\Bigg),\end{aligned}$$

$$\begin{aligned}u\frac{\partial u}{\partial x}\frac{\partial T}{\partial x} &= a^2 x^{3n-3} y^2 \left(\frac{n-1}{2}\right)^2 \left(\frac{(n+1)a}{2\nu_f}\right) (T_w - T_\infty) f'(\xi) f''(\xi)\theta'(\xi) \\ &\quad + na^2 x^{\frac{5n-5}{2}} y \left(\frac{n-1}{2}\right) \sqrt{\frac{(n+1)a}{2\nu_f}} (T_w - T_\infty) f'^2(\xi)\theta'(\xi).\end{aligned} \tag{4.22}$$

$$
\begin{aligned}
v\frac{\partial v}{\partial y}\frac{\partial T}{\partial y} =& \left(-x^{n-1}ay\left(\frac{n-1}{2}\right)f'(\xi) - x^{\frac{n-1}{2}}\left(\frac{n+1}{2}\right)\sqrt{\frac{2\nu_f a}{n+1}}f(\xi)\right) \\
&\left(-nax^{n-1}f'(\xi) - ax^{\frac{3n-3}{2}}y\left(\frac{n-1}{2}\right)\sqrt{\frac{(n+1)a}{2\nu_f}}f''(\xi)\right) \\
&\left(x^{\frac{n-1}{2}}\sqrt{\frac{(n+1)a}{2\nu_f}}(T_w - T_\infty)\theta'(\xi)\right),
\end{aligned}
$$

$$
\begin{aligned}
v\frac{\partial v}{\partial y}\frac{\partial T}{\partial y} =& na^2x^{\frac{5n-5}{2}}y\left(\frac{n-1}{2}\right)\sqrt{\frac{(n+1)a}{2\nu_f}}(T_w - T_\infty)f'^2(\xi)\theta'(\xi) \\
&+ a^2x^{3n-3}y^2\left(\frac{n-1}{2}\right)^2\left(\frac{(n+1)a}{2\nu_f}\right)(T_w - T_\infty)f'(\xi)f''(\xi)\theta'(\xi) \\
&+ a^2x^{\frac{5n-5}{2}}y\left(\frac{n+1}{2}\right)\left(\frac{n-1}{2}\right)\sqrt{\frac{(n+1)a}{2\nu_f}}(T_w - T_\infty)f(\xi)f''(\xi)\theta'(\xi) \\
&+ na^2x^{2n-2}\left(\frac{n+1}{2}\right)(T_w - T_\infty)f(\eta)f'(\xi)\theta'(\xi). \qquad (4.23)
\end{aligned}
$$

$$
\begin{aligned}
u\frac{\partial v}{\partial x}\frac{\partial T}{\partial y} =& ax^nf'(\xi)\Bigg[-ax^{\frac{3n-5}{2}}y^2\left(\frac{n-1}{2}\right)^2\sqrt{\frac{(n+1)a}{2\nu_f}}f''(\xi) \\
&- (n-1)x^{n-2}ay\left(\frac{n-1}{2}\right)f'(\xi) - ax^{n-2}y\left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)f'(\xi) \\
&- x^{\frac{n-3}{2}}\left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)\sqrt{\frac{2\nu_f a}{n+1}}f(\xi)\Bigg]\left[x^{\frac{n-1}{2}}\sqrt{\frac{(n+1)a}{2\nu_f}}(T_w - T_\infty)\theta'(\xi)\right],
\end{aligned}
$$

$$
\begin{aligned}
u\frac{\partial v}{\partial x}\frac{\partial T}{\partial y} =& -a^2x^{3n-3}y^2\left(\frac{n-1}{2}\right)^2\left(\frac{(n+1)a}{2\nu_f}\right)(T_w - T_\infty)f'(\xi)f''(\xi)\theta'(\xi) \\
&- (n-1)a^2x^{\frac{5n-5}{2}}y\left(\frac{n-1}{2}\right)\sqrt{\frac{(n+1)a}{2\nu_f}}(T_w - T_\infty)f'^2(\xi)\theta'(\xi) \\
&- a^2x^{\frac{5n-5}{2}}y\left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)\sqrt{\frac{(n+1)a}{2\nu_f}}(T_w - T_\infty)f'^2(\xi)\theta'(\xi) \\
&- a^2x^{2n-2}\left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)(T_w - T_\infty)f(\xi)f'(\xi)\theta'(\xi). \qquad (4.24)
\end{aligned}
$$

$$
\begin{aligned}
v\frac{\partial u}{\partial y}\frac{\partial T}{\partial x} =& \left(-ax^{n-1}y\left(\frac{n-1}{2}\right)f'(\xi) - x^{\frac{n-1}{2}}\sqrt{\frac{2\nu_f a}{n+1}}f(\xi)\right)\left(ax^{\frac{3n-1}{2}}\sqrt{\frac{2\nu_f a}{n+1}}f''(\xi)\right) \\
&\left(x^{\frac{n-3}{2}}y\left(\frac{n-1}{2}\right)(T_w - T_\infty)\theta'(\xi)\right),
\end{aligned}
$$

$$
\begin{aligned}
v\frac{\partial u}{\partial y}\frac{\partial T}{\partial x} =& -a^2x^{3n-3}y^2\left(\frac{n-1}{2}\right)^2\left(\frac{(n+1)a}{2\nu_f}\right)(T_w - T_\infty)f'(\xi)f''(\xi)\theta'(\xi \\
&- a^2x^{\frac{5n-5}{2}}y\left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)\sqrt{\frac{(n+1)a}{2\nu_f}}(T_w - T_\infty)f(\xi)f''(\xi)\theta'(\xi).
\end{aligned}
$$

(4.25)

$$
\begin{aligned}
2uv\frac{\partial^2 T}{\partial x\partial y} &= 2ax^n f'(\xi)\Bigg(- x^{n-1}ay\left(\frac{n-1}{2}\right)f'(\xi) - x^{\frac{n-1}{2}}\left(\frac{n+1}{2}\right)\sqrt{\frac{2\nu_f a}{n+1}}f(\xi)\Bigg)\\
&\qquad\Bigg(x^{\frac{n-3}{2}}\frac{n-1}{2}\sqrt{\frac{(n+1)a}{2\nu_f}}(T_w - T_\infty)\theta'(\xi)\\
&\qquad + x^{n-2}y\frac{n-1}{2}\frac{(n+1)a}{2\nu_f}(T_w - T_\infty)\theta''(\xi)\Bigg),\\
2uv\frac{\partial^2 T}{\partial x\partial y} &= \Bigg(- 2a^2x^{2n-1}y\left(\frac{n-1}{2}\right)f'^2(\xi) - 2ax^{\frac{3n-1}{2}}\left(\frac{n+1}{2}\right)\sqrt{\frac{2\nu_f a}{n+1}}f(\xi)f'(\xi)\Bigg)\\
&\qquad\Bigg(x^{\frac{n-3}{2}}\sqrt{\frac{(n+1)a}{2\nu_f}}\theta'(\xi) + x^{n-2}y\frac{(n+1)a}{2\nu_f}\theta''(\xi)\Bigg)\left(\frac{n-1}{2}\right)(T_w - T_\infty),\\
2uv\frac{\partial^2 T}{\partial x\partial y} &= -2a^2x^{\frac{5n-5}{2}}y\left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)\sqrt{\frac{(n+1)a}{2\nu_f}}(T_w - T_\infty)f(\xi)f'(\xi)\theta''(\xi)\\
&\quad - 2a^2y^2x^{3n-3}\left(\frac{n-1}{2}\right)^2\frac{(n+1)a}{2\nu_f}(T_w - T_\infty)f'^2(\xi)\theta''(\xi)\\
&\quad - 2a^2x^{2n-2}\left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)(T_w - T_\infty)f(\xi)f'(\xi)\theta'(\xi)\\
&\quad - 2a^2x^{\frac{5n-5}{2}}y\left(\frac{n-1}{2}\right)^2\sqrt{\frac{(n+1)a}{2\nu_f}}(T_w - T_\infty)f'^2(\xi)\theta'(\xi). \qquad (4.26)
\end{aligned}
$$

$$
\begin{aligned}
u^2\frac{\partial^2 T}{\partial x^2} &= a^2x^{2n}f'^2(\xi)\Bigg(x^{n-3}y^2\left(\frac{n-1}{2}\right)^2\left(\frac{a(n+1)}{2\nu_f}\right)(T_w - T_\infty)\theta''(\xi)\\
&\quad + x^{\frac{n-5}{2}}y\left(\frac{n-1}{2}\right)\left(\frac{n-3}{2}\right)\sqrt{\frac{(n+1)a}{2\nu_f}}(T_w - T_\infty)\theta'(\xi)\Bigg),\\
u^2\frac{\partial^2 T}{\partial x^2} &= a^2x^{3n-3}y^2\left(\frac{n-1}{2}\right)^2\left(\frac{(n+1)a}{2\nu_f}\right)(T_w - T_\infty)f'^2(\xi)\theta''(\xi)\\
&\quad + a^2x^{\frac{5n-5}{2}}y\left(\frac{n-1}{2}\right)\left(\frac{n-3}{2}\right)\sqrt{\frac{(n+1)a}{2\nu_f}}(T_w - T_\infty)f'^2(\xi)\theta'(\xi). \qquad (4.27)
\end{aligned}
$$

$$
\begin{aligned}
v^2\frac{\partial^2 T}{\partial y^2} &= \left(-x^{n-1}ay\left(\frac{n-1}{2}\right)f'(\xi) - x^{\frac{n-1}{2}}\left(\frac{n+1}{2}\right)\sqrt{\frac{2\nu_f a}{n+1}}f(\xi)\right)^2\\
&\quad\left(x^{n-1}\frac{a(n+1)}{2\nu_f}(T_w - T_\infty)\theta''(\xi)\right),\\
v^2\frac{\partial^2 T}{\partial y^2} &= a^2x^{3n-3}y^2\left(\frac{n-1}{2}\right)^2\left(\frac{(n+1)a}{2\nu_f}\right)(T_w - T_\infty)f'^2(\xi)\theta''(\xi)\\
&\quad + 2a^2x^{\frac{5n-5}{2}}y\left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)\sqrt{\frac{(n+1)a}{2\nu_f}}(T_w - T_\infty)f(\xi)f'(\xi)\theta''(\xi)\\
&\quad + a^2x^{2n-2}\left(\frac{n+1}{2}\right)^2(T_w - T_\infty)f^2(\xi)\theta''(\xi). \qquad (4.28)
\end{aligned}
$$

Adding equations (4.22)-(4.28), we get.

$$
\begin{aligned}
&u\frac{\partial u}{\partial x}\frac{\partial T}{\partial x}+v\frac{\partial v}{\partial y}\frac{\partial T}{\partial y}+u\frac{\partial v}{\partial x}\frac{\partial T}{\partial y}+v\frac{\partial u}{\partial y}\frac{\partial T}{\partial x}+2uv\frac{\partial^2 T}{\partial x\partial y}+u^2\frac{\partial^2 T}{\partial x^2}+v^2\frac{\partial^2 T}{\partial y^2}\\
&=a^2x^{3n-3}y^2\left(\frac{n-1}{2}\right)^2\left(\frac{(n+1)a}{2\nu_f}\right)(T_w-T_\infty)f'(\xi)f''(\xi)\theta'(\xi)\\
&+na^2x^{\frac{5n-5}{2}}y\left(\frac{n-1}{2}\right)\sqrt{\frac{(n+1)a}{2\nu_f}}(T_w-T_\infty)f'^2(\xi)\theta'(\xi)\\
&+na^2x^{\frac{5n-5}{2}}y\left(\frac{n-1}{2}\right)\sqrt{\frac{(n+1)a}{2\nu_f}}(T_w-T_\infty)f'^2(\xi)\theta'(\xi)\\
&+a^2x^{3n-3}y^2\left(\frac{n-1}{2}\right)^2\left(\frac{(n+1)a}{2\nu_f}\right)(T_w-T_\infty)f'(\xi)f''(\xi)\theta'(\xi)\\
&+na^2x^{2n-2}\left(\frac{n+1}{2}\right)(T_w-T_\infty)f(\xi)f'(\xi)\theta'(\xi)\\
&+a^2x^{\frac{5n-5}{2}}y\left(\frac{n+1}{2}\right)\left(\frac{n-1}{2}\right)\sqrt{\frac{(n+1)a}{2\nu_f}}(T_w-T_\infty)f(\xi)f''(\xi)\theta'(\xi)\\
&-a^2x^{3n-3}y^2\left(\frac{n-1}{2}\right)^2\left(\frac{(n+1)a}{2\nu_f}\right)(T_w-T_\infty)f'(\xi)f''(\xi)\theta'(\xi)\\
&-(n-1)a^2x^{\frac{5n-5}{2}}y\left(\frac{n-1}{2}\right)\sqrt{\frac{(n+1)a}{2\nu_f}}(T_w-T_\infty)f'^2(\xi)\theta'(\xi)\\
&-a^2x^{\frac{5n-5}{2}}y\left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)\sqrt{\frac{(n+1)a}{2\nu_f}}(T_w-T_\infty)f'^2(\xi)\theta'(\xi)\\
&-a^2x^{2n-2}\left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)(T_w-T_\infty)f(\xi)f'(\xi)\theta'(\xi)\\
&-a^2x^{3n-3}y^2\left(\frac{n-1}{2}\right)^2\left(\frac{(n+1)a}{2\nu_f}\right)(T_w-T_\infty)f'(\xi)f''(\xi)\theta'(\xi)\\
&-a^2x^{\frac{5n-5}{2}}y\left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)\sqrt{\frac{(n+1)a}{2\nu_f}}(T_w-T_\infty)f(\xi)f''(\xi)\theta'(\xi)\\
&-2a^2x^{\frac{5n-5}{2}}y\left(\frac{n-1}{2}\right)^2\sqrt{\frac{(n+1)a}{2\nu_f}}(T_w-T_\infty)f'^2(\xi)\theta'(\xi)\\
&-2a^2y^2x^{3n-3}\left(\frac{n-1}{2}\right)^2\frac{(n+1)a}{2\nu_f}(T_w-T_\infty)f'^2(\xi)\theta''(\xi)\\
&-2a^2x^{2n-2}\left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)(T_w-T_\infty)f(\xi)f'(\xi)\theta'(\xi)\\
&-2a^2x^{\frac{5n-5}{2}}y\left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)\sqrt{\frac{(n+1)a}{2\nu_f}}(T_w-T_\infty)f(\xi)f'(\xi)\theta''(\xi)
\end{aligned}
$$

$$
\begin{aligned}
&+ a^2x^{3n-3}y^2\left(\frac{n-1}{2}\right)^2\left(\frac{(n+1)a}{2\nu_f}\right)(T_w-T_\infty)f'^2(\xi)\theta''(\xi)\\
&+ a^2x^{\frac{5n-5}{2}}y\left(\frac{n-1}{2}\right)\left(\frac{n-3}{2}\right)\sqrt{\frac{(n+1)a}{2\nu_f}}(T_w-T_\infty)f'^2(\xi)\theta'(\xi)\\
&+ a^2x^{3n-3}y^2\left(\frac{n-1}{2}\right)^2\left(\frac{(n+1)a}{2\nu_f}\right)(T_w-T_\infty)f'^2(\xi)\theta''(\xi)\\
&+ a^2x^{2n-2}\left(\frac{n+1}{2}\right)^2(T_w-T_\infty)f^2(\xi)\theta''(\xi)\\
&+ 2a^2x^{\frac{5n-5}{2}}y\left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)\sqrt{\frac{(n+1)a}{2\nu_f}}(T_w-T_\infty)f(\xi)f'(\xi)\theta''(\xi),\\
={}& a^2x^{2n-2}\left(\frac{n+1}{2}\right)^2(T_w-T_\infty)f^2(\xi)\theta''(\xi)\\
&+ a^2x^{2n-2}\left[\frac{n+1}{2}\right]\left[n-\left(\frac{n-1}{2}\right)-2\left(\frac{n-1}{2}\right)\right](T_w-T_\infty)f(\xi)f'(\xi)\theta'(\xi),\\
={}& a^2x^{2n-2}\left(\frac{n+1}{2}\right)^2(T_w-T_\infty)f^2(\xi)\theta''(\xi)\\
&+ a^2x^{2n-2}\left(\frac{n+1}{2}\right)\left(n-\left(\frac{n-1}{2}\right)-(n-1)\right)(T_w-T_\infty)f(\xi)f'(\xi)\theta'(\xi),\\
={}& a^2x^{2n-2}\left(\frac{n+1}{2}\right)^2(T_w-T_\infty)f^2(\xi)\theta''(\xi)\\
&+ a^2x^{2n-2}\left(\frac{n+1}{2}\right)\left(\frac{2n-n+1}{2}-n+1\right)(T_w-T_\infty)f(\xi)f'(\xi)\theta'(\xi),\\
={}& a^2x^{2n-2}\left(\frac{n+1}{2}\right)^2(T_w-T_\infty)f^2(\xi)\theta''(\xi)\\
&+ a^2x^{2n-2}\left(\frac{n+1}{2}\right)\left(\frac{2n-n+1-2n+2}{2}\right)(T_w-T_\infty)f(\xi)f'(\xi\theta'(\xi),\\
={}& a^2x^{2n-2}\left(\frac{n+1}{2}\right)^2(T_w-T_\infty)f^2(\xi)\theta''(\xi)\\
&+ a^2x^{2n-2}\left(\frac{n+1}{2}\right)\left(\frac{-n+1+2}{2}\right)(T_w-T_\infty)f(\xi)f'(\xi)\theta'(\xi),\\
={}& a^2x^{2n-2}\left(\frac{n+1}{2}\right)^2(T_w-T_\infty)f^2(\xi)\theta''(\xi)\\
&+ a^2x^{2n-2}\left(\frac{n+1}{2}\right)\left(\frac{-n+3}{2}\right)(T_w-T_\infty)f(\xi)f'(\xi)\theta'(\xi),\\
={}& a^2x^{2n-2}\left(\frac{n+1}{2}\right)^2(T_w-T_\infty)f^2(\xi)\theta''(\xi)\\
&- a^2x^{2n-2}\left(\frac{n+1}{2}\right)\left(\frac{n-3}{2}\right)(T_w-T_\infty)f(\xi)f'(\xi)\theta'(\xi). \qquad (4.29)
\end{aligned}
$$

$$\frac{\partial T}{\partial y} = x^{\frac{n-1}{2}}\sqrt{\frac{(n+1)a}{2\nu_f}}(T_w - T_\infty)\theta'(\xi). \tag{4.30}$$

$$\frac{\partial C}{\partial y} = x^{\frac{n-1}{2}}\sqrt{\frac{(n+1)a}{2\nu_f}}(C_w - C_\infty)h'(\xi). \tag{4.31}$$

$$\left(\frac{\partial T}{\partial y}\right)^2 = x^{n-1}\frac{(n+1)a}{2\nu_f}(T_w - T_\infty)^2\theta'^2(\xi). \tag{4.32}$$

$$\begin{aligned}
\tau\left(D_B\left(\frac{\partial T}{\partial y}\frac{\partial C}{\partial y}\right) + \frac{D_T}{T_\infty}\left(\frac{\partial T}{\partial y}\right)^2\right) &= \tau\left(\frac{D_T}{T_\infty}x^{n-1}\frac{(n+1)a}{2\nu_f}(T_w - T_\infty)^2\theta'^2(\xi)\right)\\
+\tau\left(D_B\left[x^{\frac{n-1}{2}}\sqrt{\frac{(n+1)a}{2\nu_f}}(T_w - T_\infty)\theta'(\xi)\right]\left[x^{\frac{n-1}{2}}\sqrt{\frac{(n+1)a}{2\nu_f}}(C_w - C_\infty)h'(\xi)\right]\right),\\
&= \tau D_B x^{n-1}\left(\frac{(n+1)a}{2\nu_f}\right)(T_w - T_\infty)(C_w - C_\infty)\theta'(\xi)h'(\xi)\\
&\quad + \tau\frac{D_T}{T_\infty}x^{n-1}\left(\frac{(n+1)a}{2\nu_f}\right)(T_w - T_\infty)^2\theta'^2(\xi),\\
&= \frac{\tau D_B(C_w - C_\infty)}{\nu_f}x^{n-1}\left(\frac{(n+1)a}{2}\right)(T_w - T_\infty)\theta'(\xi)h'(\xi)\\
&\quad + \frac{\tau D_T(T_w - T_\infty)}{T_\infty\nu_f}x^{n-1}\left(\frac{(n+1)a}{2}\right)(T_w - T_\infty)\theta'^2(\xi),\\
&= ax^{n-1}\left(\frac{n+1}{2}\right)Nb(T_w - T_\infty)\theta'(\xi)h'(\xi)\\
&\quad + ax^{n-1}\left(\frac{n+1}{2}\right)Nt(T_w - T_\infty)\theta'^2(\xi),\\
&= ax^{n-1}\left(\frac{n+1}{2}\right)(T_w - T_\infty)\left(Nb\theta'(\xi)h'(\xi) + Nt\theta'^2(\xi)\right).
\end{aligned} \tag{4.33}$$

Left hand side of (4.3)

$$\begin{aligned}
&= -ax^{n-1}\left(\frac{n+1}{2}\right)(T_w - T_\infty)f(\xi)\theta'(\xi) + \lambda a^2x^{2n-2}\left(\frac{n+1}{2}\right)^2(T_w - T_\infty)f^2(\xi)\theta''(\xi)\\
&\quad - \lambda a^2x^{2n-2}\left(\frac{n+1}{2}\right)\left(\frac{n-3}{2}\right)(T_w - T_\infty)f(\xi)f'(\xi)\theta'(\xi).
\end{aligned} \tag{4.34}$$

Right hand side of (4.3)

$$\begin{aligned}
&= \alpha_{nf}ax^{n-1}\left(\frac{n+1}{2\nu_f}\right)(T_w - T_\infty)\theta''(\xi) + \frac{\mu_{nf}}{(\rho Cp)_{nf}}a^3x^{3n-1}\left(\frac{n+1}{2\nu_f}\right)f''^2(\xi)\\
&\quad - \frac{1}{(\rho Cp)_{nf}}\frac{16\sigma^*T_\infty^3}{3k^*}ax^{n-1}(T_w - T_\infty)\left(\frac{n+1}{2\nu_f}\right)\theta''(\xi) + \frac{q}{(\rho Cp)_{nf}}(T_w - T_\infty)\theta(\xi)
\end{aligned}$$

$$+ax^{n-1}\left(\frac{n+1}{2}\right)(T_w-T_\infty)\left(Nb\theta'(\xi)h'(\xi)+Nt\theta'^2(\xi)\right). \tag{4.35}$$

Comparing (4.34) and (4.35)

$$\begin{aligned}
&-ax^{n-1}\left(\frac{n+1}{2}\right)(T_w-T_\infty)f(\xi)\theta'(\xi)\\
&+\lambda a^2x^{2n-2}\left(\frac{n+1}{2}\right)^2(T_w-T_\infty)f^2(\xi)\theta''(\xi)\\
&-\lambda a^2x^{2n-2}\left(\frac{n+1}{2}\right)\left(\frac{n-3}{2}\right)(T_w-T_\infty)f(\xi)f'(\xi)\theta'(\xi),\\
=&\,\alpha_{nf}ax^{n-1}\left(\frac{n+1}{2\nu_f}\right)(T_w-T_\infty)\theta''(\xi)+\frac{\mu_{nf}}{(\rho Cp)_{nf}}a^3x^{3n-1}\left(\frac{n+1}{2\nu_f}\right)f''^2(\xi)\\
&\frac{1}{(\rho Cp)_{nf}}\frac{16\sigma^*T_\infty^3}{3k^*}ax^{n-1}(T_w-T_\infty)\left(\frac{n+1}{2\nu_f}\right)\theta''(\xi)+\frac{q}{(\rho Cp)_{nf}}(T_w-T_\infty)\theta(\xi)\\
&+ax^{n-1}\left(\frac{n+1}{2}\right)(T_w-T_\infty)\left(Nb\theta'(\xi)h'(\xi)+Nt\theta'^2(\xi)\right),\\
&-\frac{\nu_f}{\alpha_{nf}}f(\xi)\theta'(\xi)+\frac{\nu_f}{\alpha_{nf}}\lambda ax^{n-1}\left[\left(\frac{n+1}{2}\right)f^2(\xi)\theta''(\xi)-\left(\frac{n-3}{2}\right)f(\xi)f'(\xi)\theta'(\xi)\right]\\
=&\,\theta''(\xi)+\frac{\mu_{nf}}{(\rho Cp)_{nf}\alpha_{nf}}\frac{a^2x^{2n}}{(T_w-T_\infty)}f''^2(\xi)+\frac{1}{(\rho Cp)_{nf}\alpha_{nf}}\frac{16\sigma^*T_\infty^3}{3k^*}\theta''(\xi)\\
&+\frac{q}{(\rho Cp)_{nf}\alpha_{nf}ax^{n-1}}\left(\frac{2\nu_f}{n+1}\right)\theta(\xi)+\frac{\nu_f}{\alpha_{nf}}((Nb\theta'(\xi)h'(\xi)+Nt\theta'^2(\xi)),\\
&\theta''(\xi)+\frac{\nu_f(\rho Cp)_{nf}}{k_{nf}}f(\xi)\theta'(\xi)+\frac{\mu_{nf}}{k_{nf}}\frac{a^2x^{2n}}{(T_w-T_\infty)}f''^2(\xi)+\frac{1}{k_{nf}}\frac{16\sigma^*T_\infty^3}{3k^*}\theta''(\xi)\\
&-\frac{\nu_f(\rho Cp)_{nf}}{k_{nf}}\lambda ax^{n-1}\left[\left(\frac{n+1}{2}\right)f^2(\xi)\theta''(\xi)-\left(\frac{n-3}{2}\right)f(\xi)f'(\xi)\theta'(\xi)\right]\\
&+\frac{q}{k_{nf}ax^{n-1}}\left(\frac{2\nu_f}{n+1}\right)\theta(\xi)+\frac{\nu_f(\rho Cp_{nf})}{k_{nf}}(Nb\theta'(\xi)h'(\xi)+Nt\theta'^2(\xi))=0. \qquad (4.36)\\
&\frac{k_{nf}}{k_f}\left(1+\frac{4}{3}R\right)\theta''(\xi)+\frac{\nu_f(\rho Cp)_{nf}}{k_f}f(\xi)\theta'(\xi)+\frac{\mu_{nf}}{k_f}\frac{a^2x^{2n}}{(T_w-T_\infty)}f''^2(\xi)\\
&-\frac{\nu_f(\rho Cp)_{nf}}{k_f}\lambda ax^{n-1}\left[\left(\frac{n+1}{2}\right)f^2(\xi)\theta''(\xi)-\left(\frac{n-3}{2}\right)f(\xi)f'(\xi)\theta'(\xi)\right]\\
&+\frac{\nu_f q}{k_fax^{n-1}}\left(\frac{2}{n+1}\right)\theta(\xi)+\frac{\nu_f(\rho Cp)_{nf}}{k_f}(Nb\theta'(\xi)h'(\xi)+Nt\theta'^2(\xi))=0,\\
&\frac{k_{nf}}{k_f}\left(1+\frac{4}{3}R\right)\theta''(\xi)+\frac{\nu_f(\rho Cp)_fA_5}{\alpha_f(\rho Cp)_f}f(\xi)\theta'(\xi)+\frac{\nu_f}{\alpha_f(1-\phi)^{2.5}}\frac{a^2x^{2n}}{(T_w-T_\infty)}f''^2(\xi)\\
&-\frac{\nu_f(\rho Cp)_fA_5}{\alpha_f(\rho Cp)_f}\lambda ax^{n-1}\left[\left(\frac{n+1}{2}\right)f^2(\xi)\theta''(\xi)-\left(\frac{n-3}{2}\right)f(\xi)f'(\xi)\theta'(\xi)\right]\\
&+\frac{\nu_f q}{\alpha_f(\rho Cp)_fax^{n-1}}\left(\frac{2}{n+1}\right)\theta(\xi)+\frac{\nu_f(\rho Cp)_fA_5}{\alpha_f(\rho Cp)_f}(Nb\theta'(\xi)h'(\xi)+Nt\theta'^2(\xi))=0,
\end{aligned}$$

$$
\begin{aligned}
&A_4\theta''(\xi) + A_5Prf(\xi)\theta'(\xi) + Pr\left(\frac{Ec}{(1-\phi)^{2.5}}\right)f''^2(\xi) + Pr\left(\frac{2}{n+1}\right)Q\theta(\xi) \\
&+ A_5Pr\gamma_1\left(\left(\frac{n-3}{2}\right)f(\xi)f'(\xi)\theta'(\xi) - \frac{n+1}{2}f^2(\xi)\theta''(\xi)\right) \\
&+ A_5Pr(Nb\theta'(\xi)h'(\xi) + Nt\theta'^2(\xi)) = 0. \qquad (4.37)
\end{aligned}
$$

Now, we include below the procedure for the conversion of equation (4.4) into the dimensionless form.

$$
\begin{aligned}
&h(\xi) = \frac{C - C_\infty}{C_w - C_\infty}, \\
&C = (C_w - C_\infty)h(\xi) + C_\infty. \\
&\frac{\partial C}{\partial x} = (C_w - C_\infty)h'(\xi)\frac{\partial \xi}{\partial x}, \\
&\frac{\partial C}{\partial x} = \left(\frac{n-1}{2}\right)x^{\frac{n-3}{2}}y\sqrt{\frac{a(n+1)}{2\nu_f}}(C_w - C_\infty)h'(\xi). \qquad (4.38) \\
&\frac{\partial C}{\partial y} = (C_w - C_\infty)h'(\xi)\frac{\partial \xi}{\partial y}, \\
&\frac{\partial C}{\partial y} = x^{\frac{n-1}{2}}\sqrt{\frac{a(n+1)}{2\nu_f}}(C_w - C_\infty)h'(\xi). \qquad (4.39) \\
&\frac{\partial^2 C}{\partial y^2} = x^{\frac{n-1}{2}}\sqrt{\frac{a(n+1)}{2\nu_f}}(C_w - C_\infty)h''(\xi)\frac{\partial \xi}{\partial y}, \\
&\frac{\partial^2 C}{\partial y^2} = x^{\frac{n-1}{2}}\sqrt{\frac{a(n+1)}{2\nu_f}}(C_w - C_\infty)h''(\xi)\left(x^{\frac{n-1}{2}}\sqrt{\frac{a(n+1)}{2\nu_f}}\right), \\
&\frac{\partial^2 C}{\partial y^2} = x^{n-1}\left(\sqrt{\frac{a(n+1)}{2\nu_f}}\right)^2(C_w - C_\infty)h''(\xi), \\
&\frac{\partial^2 C}{\partial y^2} = x^{n-1}\frac{a(n+1)}{2\nu_f}(C_w - C_\infty)h''(\xi). \qquad (4.40) \\
&\frac{\partial^2 T}{\partial y^2} = x^{n-1}\frac{a(n+1)}{2\nu_f}(T_w - T_\infty)\theta''(\xi). \qquad (4.41)
\end{aligned}
$$

Using (4.38) and (4.39) in left hand side of (4.4)

$$
\begin{aligned}
&u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} = ax^n f'(\xi)\left(\left(\frac{n-1}{2}\right)x^{\frac{n-3}{2}}y\sqrt{\frac{a(n+1)}{2\nu_f}}(C_w - C_\infty)h'(\xi)\right) \\
&+\left(\frac{n-1}{2}x^{n-1}yaf'(\xi) - x^{\frac{n-1}{2}}\frac{n+1}{2}\sqrt{\frac{2\nu_f a}{n+1}}f(\xi)\right)x^{\frac{n-1}{2}}\sqrt{\frac{a(n+1)}{2\nu_f}}(C_w - C_\infty)h'(\xi),
\end{aligned}
$$

$$
\begin{aligned}
u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y} &= ax^{\frac{3n-3}{2}}y\left(\frac{n-1}{2}\right)\sqrt{\frac{a(n+1)}{2\nu_f}}(C_w-C_\infty)f'(\xi)h'(\xi)\\
&\quad - x^{\frac{3n-3}{2}}y\left(\frac{n-1}{2}\right)\sqrt{\frac{a(n+1)}{2\nu_f}}(C_w-C_\infty)f'(\xi)h'(\xi)\\
&\quad - ax^{n-1}\left(\frac{n+1}{2}\right)(C_w-C_\infty)f(\xi)h'(\xi),\\
u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y} &= -ax^{n-1}\left(\frac{n+1}{2}\right)(C_w-C_\infty)f(\xi)h'(\xi). \qquad (4.42)
\end{aligned}
$$

Using (4.40) and (4.41) in right hand side of (4.4)

$$
\begin{aligned}
&D_B\frac{\partial^2 C}{\partial y^2}+\frac{D_T}{T_\infty}\frac{\partial^2 T}{\partial y^2}-K_r(C-C_\infty)=D_Bx^{n-1}\left(\frac{a(n+1)}{2\nu_f}\right)(C_w-C_\infty)h''(\xi)\\
&+\frac{D_T}{T_\infty}x^{n-1}\left(\frac{a(n+1)}{2\nu_f}\right)(T_w-T_\infty)\theta''(\xi)-K_r(C_w-C_\infty)h(\xi). \qquad (4.43)
\end{aligned}
$$

Comparing (4.32) and (4.33)

$$
\begin{aligned}
&-ax^{n-1}\left(\frac{n+1}{2}\right)(C_w-C_\infty)f(\xi)h'(\xi)=D_Bx^{n-1}\left(\frac{a(n+1)}{2\nu_f}\right)(C_w-C_\infty)h''(\xi)\\
&+\frac{D_T}{T_\infty}x^{n-1}\left(\frac{a(n+1)}{2\nu_f}\right)(T_w-T_\infty)\theta''(\xi)-K_r(C_w-C_\infty)h(\xi).
\end{aligned}
$$

Dividing both side $D_Bx^{n-1}a\left(\frac{n+1}{2\nu_f}\right)(C_w-C_\infty)$

$$
\begin{aligned}
&-\frac{\nu_f}{D_B}f(\xi)h'(\xi)=h''(\xi)+\frac{D_T(T_w-T_\infty)}{T_\infty D_B(C_w-C_\infty)}\theta''(\xi)-\frac{K_r2\nu_f}{D_B(n+1)ax^{n-1}}h(\xi),\\
&h''(\xi)+Lef(\xi)h'(\xi)+\frac{D_T\tau(T_w-T_\infty)\nu_f}{T_\infty\nu_fD_B\tau(C_w-C_\infty)}\theta''(\xi)-\frac{\nu_f}{D_B}\frac{2K_r}{(n+1)ax^{n-1}}h(\xi)=0,\\
&h''(\xi)+Lef(\xi)h'(\xi)+\frac{Nt}{Nb}\theta''(\xi)-\gamma_2Leh(\xi)=0. \qquad (4.44)
\end{aligned}
$$

Now discussing the procedure for conversion of boundary conditions into dimensionless form.

$$
\begin{aligned}
&u=U_w(x)=ax^n, \qquad at \quad y=0.\\
&u=af'(\xi)x^n,\\
\Rightarrow\quad &af'(\xi)x^n=ax^n,
\end{aligned}
$$

$$\Rightarrow \quad ax^n f'(\xi) = ax^n,$$

$$\Rightarrow \quad f'(\xi) = 1, \qquad at \quad \xi = 0.$$

$$\Rightarrow \quad f'(0) = 1.$$

$$v = 0, \qquad at \quad y = 0.$$

$$\Rightarrow \quad -x^{\frac{n-1}{2}}\sqrt{\frac{(n+1)\nu_f a}{2}}\left(f(\xi) + \xi f'(\xi)\left(\frac{n-1}{n+1}\right)\right) = 0, \qquad at \quad \xi = 0.$$

$$\Rightarrow \quad -x^{\frac{n-1}{2}}\sqrt{\frac{(n+1)\nu_f a}{2}} f(\xi) = 0, \qquad at \quad \xi = 0.$$

$$\Rightarrow \quad f(\xi) = 0,$$

$$\Rightarrow \quad f(0) = 0.$$

$$T = T_w, \qquad at \quad y = 0.$$

$$\Rightarrow \quad \theta(\xi)(T_w - T_\infty) + T_\infty = T_w,$$

$$\Rightarrow \quad \theta(\xi)(T_w - T_\infty) = (T_w - T_\infty),$$

$$\Rightarrow \quad \theta(\xi) = 1, \qquad at \quad \xi = 0.$$

$$\Rightarrow \quad \theta(0) = 1.$$

$$C = C_w, \qquad at \quad y = 0.$$

$$\Rightarrow \quad h(\xi)(C_w - C_\infty) + C_\infty = C_w,$$

$$\Rightarrow \quad h(\xi)(C_w - C_\infty) = (C_w - C_\infty),$$

$$\Rightarrow \quad h(\xi) = 1, \qquad at \quad \xi = 0.$$

$$\Rightarrow \quad h(0) = 1.$$

$$u \to (0), \qquad as \quad y \to \infty.$$

$$\Rightarrow \quad af'(\xi)x^n \to (0),$$

$$\Rightarrow \quad f'(\xi) \to (0), \qquad as \quad \xi \to \infty.$$

$$\Rightarrow \quad f'(\xi) \to 0.$$

$$T \to T_\infty, \qquad as \quad y \to \infty.$$

$$\Rightarrow \quad \theta(\xi)(T_w - T_\infty) + T_\infty \to T_\infty,$$

$$\Rightarrow \quad \theta(\xi)(T_w - T_\infty) \to 0,$$

$$\Rightarrow \quad \theta(\xi) \to 0, \qquad as \quad \xi \to \infty.$$

$$\Rightarrow \quad \theta(\infty) \to 0.$$

$$
\begin{aligned}
& C \to C_\infty, && as \quad y \to \infty. \\
\Rightarrow \quad & h(\xi)(C_w - C_\infty) + C_\infty \to C_\infty, && \\
\Rightarrow \quad & h(\xi)(C_w - C_\infty) \to 0, && \\
\Rightarrow \quad & h(\xi) \to 0, && as \quad \xi \to \infty. \\
\Rightarrow \quad & h(\infty) \to 0. &&
\end{aligned}
$$

The final dimensionless form of the governing model, is

$$
f'''(\xi) + A_1A_2\left(f(\xi)f''(\xi)) - \left(\frac{2n}{n+1}\right)f'^2(\xi)\right) - \left(\frac{2}{n+1}\right)(K + MA_1A_3\sin^2(\gamma))f'(\xi) = 0. \tag{4.45}
$$

$$
A_4\theta''(\xi) + PrA_5f(\xi)\theta'(\xi) + PrA_6f''^2(\xi) + Pr\left(\frac{2}{n+1}\right)Q\theta(\xi) + A_5Pr\gamma_1\left(\frac{n-3}{2}f(\xi)f'(\xi)\theta'(\xi) - \frac{n+1}{2}f^2(\xi)\theta''(\xi)\right) + A_5Pr(Nb\theta'(\xi)h'(\xi) + Nt\theta'^2(\xi)) = 0. \tag{4.46}
$$

$$
h''(\xi) + Lef(\xi)h'(\xi) + \frac{Nt}{Nb}\theta''(\xi) - \gamma_2 Leh(\xi) = 0. \tag{4.47}
$$

The associated BCs (4.5) in the dimensionless form are,

$$
\left.
\begin{aligned}
& f(0) = 0, \quad f'(0) = 1, \quad \theta(0) = 1, \quad h(0) = 1 \\
& f'(\infty) \to 0, \quad \theta(\infty) \to 0, \quad h(\infty) \to 0.
\end{aligned}
\right\} \tag{4.48}
$$

Different parameters used in equations (4.45)-(4.47) are formulated as follows.

$$
\begin{aligned}
& M = \frac{\sigma_f B_0^2}{\rho_f a x^{-1}}, \quad K = \frac{\nu_f}{ak_0}, \quad R = \frac{4\sigma^* T_\infty^3}{k_{nf}k^*}, \quad \gamma_1 = ax^{n-1}\lambda, \\
& Pr = \frac{\nu_f}{\alpha_f}, \quad Ec = \frac{U_w^2}{(c_p)_f(T_w - T_\infty)}, \quad Q = \frac{qx}{(\rho c_p)_f U_w}, \quad Le = \frac{\nu_f}{D_B} \\
& \gamma_2 = \frac{2k_r}{(n+1)ax^{n-1}} \quad Nb = \frac{\tau D_B(C_w - C_\infty)}{\nu_f}, \quad Nt = \frac{\tau D_T(T_w - T_\infty)}{T_\infty \nu_f}.
\end{aligned}
$$

The local Sherwood number are defined as

$$
Sh_x = \frac{xq_m}{D_B(C_w - C_\infty)}. \tag{4.49}
$$

To achive the dimensionless form of Sh_x, the following step will be helpful.

Since

$$q_m = -D_B \left(\frac{\partial C}{\partial y} \right)_{y=0}, \tag{4.50}$$

$$Sh_x = -\frac{x D_B}{D_B (C_w - C_\infty)} \left(\frac{\partial C}{\partial y} \right)_{y=0},$$

$$Sh_x = -\frac{x}{(C_w - C_\infty)} x^{\frac{n-1}{2}} \sqrt{\frac{a(n+1)}{2\nu_f}} (C_w - C_\infty) h'(\xi),$$

$$Sh_x = -x^{\frac{n+1}{2}} \sqrt{\frac{a(n+1)}{2\nu_f}} h'(\xi),$$

$$Sh_x = -\sqrt{\frac{a x^{n+1}}{\nu_f}} \left(\frac{n+1}{2} \right)^{\frac{1}{2}} h'(\xi),$$

$$Sh_x = -Re_x^{\frac{1}{2}} \left(\frac{n+1}{2} \right)^{\frac{1}{2}} h'(\xi),$$

$$\frac{Sh_x}{Re_x^{\frac{1}{2}}} = -\left(\frac{n+1}{2} \right)^{\frac{1}{2}} h'(\xi),$$

$$Re_x^{\frac{-1}{2}} Sh_x = -\left(\frac{n+1}{2} \right)^{\frac{1}{2}} h'(\xi), \tag{4.51}$$

where $\quad Re = \frac{x u_x(x)}{\nu_f}$.

4.3 Solution Methodology

In order to solve the system of ODEs (4.45) the shooting method has been used. The following notations have been cosidered.

$$f = Z_1, \quad f' = Z_1' = Z_2, \quad f'' = Z_1'' = Z_2' = Z_3, \quad f''' = Z_3'.$$

For simplification, the following notation have been defined.

$$A_1 = (1-\phi)^{2.5}, \quad A_2 = \left(1 - \phi + \phi \frac{\rho_s}{\rho_f} \right), \quad A_3 = 1 + \frac{3\left(\frac{\sigma_s}{\sigma_f} - 1\right)\phi}{\left(\frac{\sigma_s}{\sigma_f} + 2\right) - \left(\frac{\sigma_s}{\sigma_f} - 1\right)\phi}.$$

By using the notations, the equation (4.45) is converted into first order ODEs.

$$Z_1' = Z_2, \qquad Z_1(0) = 0.$$
$$Z_2' = Z_3, \qquad Z_2(0) = 0.$$
$$Z_3' = -A_1A_2\left(Z_1Z_3 - \frac{2n}{n+1}Z_2^2\right) + \left(\frac{2}{n+1}\right)(K + MA_1A_3\sin^2(\gamma))Z_2, \quad Z_3(0) = s.$$

The above initial value problem will be numerically solved by RK-4. The missing condition 's' assumed to satisfy the following relation.

$$Z_2(\xi_\infty)_s = 0.$$

To solve the above algebaric equations we use the Newton's method which has the following iterative scheme.

$$s^{n+1} = s^n - \frac{(Z_2(\xi_\infty))_{s=s^n}}{\left(\frac{\partial Z_2(\xi_\infty)}{\partial s}\right)_{s=s^n}}.$$

We further introduce the following notations,

$$\frac{\partial Z_1}{\partial s} = Z_4, \quad \frac{\partial Z_2}{\partial s} = Z_5, \quad \frac{\partial Z_3}{\partial s} = Z_6.$$

As a result of these new notations, the Newton's iterative scheme.

$$s^{n+1} = s^n - \frac{(Z_2(\xi_\infty))_{s=s^n}}{(Z_5(\xi_\infty))_{s=s^n}}.$$

Now differentiating system of three first order ODEs with respect to s, we get three more ODEs.

$$Z_4' = Z_5, \qquad Z_4(0) = 0.$$
$$Z_5' = Z_6, \qquad Z_5(0) = 0.$$
$$Z_6' = -A_1A_2\left(Z_1Z_6 + Z_3Z_4 - \left(\frac{2n}{n+1}\right)2Z_2Z_5\right)$$
$$+ \left(\frac{2}{n+1}\right)(K + MA_1A_3\sin^2(\gamma))Z_5, \qquad Z_6(0) = 1.$$

The missing condition s is updated by the Newton's method and process will be continued until the following criteria is met.

$$| (Z_2(\xi_\infty))_{s=s^n} | < \epsilon.$$

where ϵ is an arbitrarily small positive number. From now onward ϵ has been taken as 10^{-10}.

Also, for equations (4.46) and (4.47), the following notation have been used.

$$\begin{aligned}
&\theta = Y_1, \quad \theta' = Y_1' = Y_2, \quad \theta'' = Y_2'.\\
&h = Y_3, \quad h' = Y_3' = Y_4, \quad h'' = Y_4'.\\
&A_4 = \frac{k_{nf}}{k_f}\left(1 + \frac{4}{3}R\right), \qquad A_5 = \left(1 - \phi + \phi\frac{(\rho C_p)_s}{(\rho C_p)_f}\right),\\
&A_6 = Pr\left(\frac{Ec}{(1-\phi)^{2.5}}\right), \qquad A_7 = \left(A_4 - A_5 Pr\gamma_1\left(\frac{n+1}{2}\right)f^2\right).
\end{aligned}$$

The system of equations (4.46) and (4.47), can be written in the form of the following first order coupled ODEs.

$$\begin{aligned}
Y_1' &= Y_2, & Y_1(0) &= 1.\\
Y_2' &= -\frac{Pr}{A_7}\Bigg[A_5 f Y_2 + A_5\gamma_1\Big(\frac{n+1}{2}\Big)ff'Y_2 + A_6 f''^2\\
&\quad + \Big(\frac{2}{n+1}\Big)QY_1 + A_5 NbY_2Y_4 + A_5 NtY_2^2\Bigg], & Y_2(0) &= l.\\
Y_3' &= Y_4, & Y_3(0) &= 1.\\
Y_4' &= Le(-fY_4 + \gamma_2 Y_3) + \frac{Nt}{Nb}\Bigg[\frac{Pr}{A_7}\Big[A_5 fY_2 + A_5\gamma_1\Big(\frac{n+1}{2}\Big)ff'Y_2\\
&\quad + A_6 f''^2 + \Big(\frac{2}{n+1}\Big)QY_1 + A_5 NbY_2Y_4 + A_5 NtY_2^2\Big]\Bigg], & Y_4(0) &= m.
\end{aligned}$$

The RK-4 method has been taken into consideration for solving the above initial value problem. For the above system of equtions, the missing conditions are to be chosen such that.

$$(Y_1(l,m))_{\xi=\xi_\infty} = 0, \qquad (Y_3(l,m))_{\xi=\xi_\infty} = 0.$$

To solve the above algebaric equations, we apply the Newton's method which has the following scheme.

$$\begin{bmatrix} l^{n+1} \\ m^{n+1} \end{bmatrix} = \begin{bmatrix} l^{n} \\ m^{n} \end{bmatrix} - \begin{bmatrix} \frac{\partial Y_1}{\partial l} & \frac{\partial Y_1}{\partial m} \\ \frac{\partial Y_3}{\partial l} & \frac{\partial Y_3}{\partial m} \end{bmatrix}^{-1} \begin{bmatrix} Y_1 \\ Y_3 \end{bmatrix}$$

Now, introduce the following notations,

$$\frac{\partial Y_1}{\partial l} = Y_5, \qquad \frac{\partial Y_2}{\partial l} = Y_6, \qquad \frac{\partial Y_3}{\partial l} = Y_7, \qquad \frac{\partial Y_4}{\partial l} = Y_8.$$
$$\frac{\partial Y_1}{\partial m} = Y_9, \qquad \frac{\partial Y_2}{\partial m} = Y_{10}, \qquad \frac{\partial Y_3}{\partial m} = Y_{11}, \qquad \frac{\partial Y_4}{\partial m} = Y_{12}.$$

As the result of these new notations, the Newton's iterative scheme gets the form.

$$\begin{bmatrix} l^{n+1} \\ m^{n+1} \end{bmatrix} = \begin{bmatrix} l^{n} \\ m^{n} \end{bmatrix} - \begin{bmatrix} Y_5 & Y_9 \\ Y_7 & Y_{11} \end{bmatrix}^{-1} \begin{bmatrix} Y_1 \\ Y_3 \end{bmatrix}$$

Now differentiating the system of four first order ODEs with respect to l, and m we get another system of ODEs, as follows.

$$Y_5' = Y_6, \qquad Y_5(0) = 0.$$

$$Y_6' = -\frac{Pr}{A_7}\Bigg(A_5 f Y_6 + A_5\gamma_1\Big(\frac{n+1}{2}\Big) f f' Y_6 + \Big(\frac{2}{n+1}\Big) Q Y_5 + A_5 Nb(Y_6 Y_4 + Y_2 Y_8) + 2A_5 Nt Y_2 Y_6\Bigg), \qquad Y_6(0) = 1.$$

$$Y_7' = Y_8, \qquad Y_7(0) = 0.$$

$$Y_8' = Le(-fY_8 + \gamma_2 Y_7) + \frac{Nt}{Nb}\Bigg(\frac{Pr}{A_7}\Big(A_5 f Y_6 + A_5\gamma_1\Big(\frac{n+1}{2}\Big) f f' Y_6 + \Big(\frac{2}{n+1}\Big) Q Y_5 + A_5 Nb(Y_6 Y_4 + Y_2 Y_8) + 2A_5 Nt Y_2 Y_6\Big)\Bigg), \qquad Y_8(0) = 0.$$

$$Y_9' = Y_{10}, \qquad Y_9(0) = 0.$$

$$Y_{10}' = -\frac{Pr}{A_7}\Bigg(A_5 f Y_{10} + A_5\gamma_1\Big(\frac{n+1}{2}\Big) f f' Y_{10} + \Big(\frac{2}{n+1}\Big) Q Y_9 + A_5 Nb(Y_{10} Y_4 + Y_2 Y_{12}) + 2A_5 Nt Y_2 Y_{10}\Bigg), \qquad Y_{10}(0) = 0.$$

$$Y'_{11} = Y_{12}, \qquad Y_{11}(0) = 0.$$

$$Y'_{12} = Le(-fY_{12} + \gamma_2 Y_{11}) + \frac{Nt}{Nb}\Bigg[\frac{Pr}{A_7}\Bigg[A_5 f Y_{10} + A_5\gamma_1\Big(\frac{n+1}{2}\Big)ff'Y_{10} + \Big(\frac{2}{n+1}\Big)QY_9 + A_5 Nb(Y_{10}Y_4 + Y_2Y_{12}) + 2A_5 Nt Y_2 Y_{10}\Bigg]\Bigg], \qquad Y_{12}(0) = 1.$$

The stopping criteria for the Newton's method is set as.

$$max\{|Y_1(\xi_\infty)|, |Y_3(\xi_\infty)|\} < \epsilon.$$

4.4 Representation of Graphs and Tables

The principle object is about to examine the impact of different parameters against the velocity $f'(\xi)$, temperature $\theta(\xi)$ and concentration distribution $h(\xi)$. The impact of different factors like nonlinear stretching parameter n, magnetic parameter M, thermal radiation R and Lewis number Le is observed graphically. Numerical outcomes of the skin friction coefficient, local Nusselt number and local Sherwood number for the distinct values of some fixed parameters are shown in Tables 4.1-4.2.

Figures 4.2 and 4.3, show the effect of ϕ on velocity profile $f'(\xi)$ and temperature profile $\theta(\xi)$ respectively. By enhancing the values of ϕ, the velocity profile decreases and increases the boundary layer thickness. Reason behind this behavior is that, if we increases the ϕ effective viscosity will increase which provide more resistance to fluid particles.

Figures 4.4 and 4.5 show the impact of permeability parameter K. For the rising values of K, the velocity profile $f'(\xi)$ decreases and temperature profile $\theta(\xi)$ increases.

Figure 4.6 displays the impact of stretching parameter n on the velocity distribution. By rising the values of n, the velocity distribution show the increasing behavior. Figure 4.7 describes the impact of stretching parameter n on temperature distribution. By increasing the values of n, the temperature distribution is decreased.

The impact of heat generation Q on $\theta(\xi)$ can be seen in Figure 4.8. It is observed that for increasing values of Q more heat is generated, because of this $\theta(\xi)$ and thermal boundary layer thickness increases.

Figure 4.9 demonstrate the impact of Eckert number Ec, on $\theta(\xi)$. As Eckert number specify the ratio of kinetic energy and enthalpy change of flow. It is clearly obsreved that the $\theta(\xi)$ is increased by rising the values of Ec due to the decrement in heat transfer rate.

Figure 4.10 shows the impact of thermal radiation R on the temperature distribution $\theta(\xi)$. By enhancing the values of R, the temperature distribution $\theta(\xi)$ is increased.

Figure 4.11 displays the impact of M, on the velocity distribution. The higher values of M, shows decreasing behavior of velocity profile. As M specify as ratio of Lorentz forces to the viscous forces, so with an increment in M Lorentz force becomes dominant and it reduces the velocity of fiuid. Figure 4.12 describes the impact of M on $\theta(\xi)$. The temperature distribution expands by enhancing the values of M. Figure 4.13 describes the impact of M, on the concentration distribution. Rising the values of M, the concentration distribution $h(\xi)$ is increased due to the presence of Lorentz force.

Figure 4.14, shows the impact of Prandtl number Pr on the temperature distributions. Since Pr is directly proportionate to the viscous diffusion rate and inversely related to the thermal diffusivity, so the thermal diffusion rate suffers a reduction for the larger values of Pr and subsequently, the temperature of the fluid drops significantly. Moreover, a decrement in the thermal boundary layer thickness has been noted.

Figure 4.15 and Figure 4.16 indicate the impact of Nb on the dimensionless temperature and concentration distribution. The behavior of temperature distribution is increased and concentration profile is decreased due to the accelerating values of Nb.

Figure 4.17 shows the influence of relaxation time parameter γ_1 on the temperature proflie $\theta(\xi)$. An decrement is noticed in temperature distribution by rising the values of relaxation time parameter γ_1.

Figure 4.18-4.20 show the impact of inclined magnetic field γ on $f'(\xi)$, $\theta(\xi)$ and $h(\xi)$ distributions. Figure 4.18 displays the impact of inclination angle γ on the velocity profile. Physically an increase in the inclination angle we are actually increase the Lorentz force which are friction forces thus correspondingly the decreases velocity profile. Figure 4.19 increment is notced in the temperature distributions for the increasing values of γ. Physically by increasing the inclination angle increase the Lorentz force which generate more heat and thus increase the temperature profile. Figure 4.20 shows the impact of the inclination angle γ on concentration profile. Enhancing the values of γ the concentration profile is decreased

Figure 4.21 shows the relationship between Lewis numbers Le and the dimensional concentration distribution $h(\xi)$. Concentration profile decreasing for the rising values of Le and thus we have get a small molecular diffusivity and thermal boundary layer.

TABLE 4.1: Results of $(Re_x)^{\frac{1}{2}}C_f$ for fixed parameter $\gamma = \pi/3$

ϕ	n	M	K	$(Re_x)^{\frac{1}{2}}C_f$
0.0	2.0	2.0	1.0	-2.080434
0.1	2.0	2.0	1.0	-2.926805
0.2	2.0	2.0	1.0	-3.988520
0.1	1.0	2.0	1.0	-2.575523
0.1	3.0	2.0	1.0	-3.239700
0.1	7.0	2.0	1.0	-4.266206
0.1	2.0	1.0	1.0	-2.694844
0.1	2.0	3.0	1.0	-3.141588
0.1	2.0	4.0	1.0	-3.342529
0.1	2.0	2.0	0.0	-2.620896
0.1	2.0	2.0	2.0	-3.203476
0.1	2.0	2.0	3.0	-3.457976
0.1	2.0	2.0	4.0	-3.694925

TABLE 4.2: Results of $-(Re_x)^{\frac{-1}{2}}Nu_x$ and $-(Re_x)^{\frac{-1}{2}}Sh_x$ some fixed parameters $\gamma = \pi/3$, $n = 2.0$, $K = 1.0$, $Ec = 0.2$, $Q = 0.1$, $Nt = Nb = 0.1$

ϕ	M	R	Pr	γ_1	Le	γ_2	$-(Re_x)^{\frac{-1}{2}}Nu_x$	$-(Re_x)^{\frac{-1}{2}}Sh_x$
0.0	2.0	0.5	6.2	0.1	5.0	0.1	0.629182	2.056874
0.05	2.0	0.5	6.2	0.1	5.0	0.1	0.466804	2.116543
0.1	2.0	0.5	6.2	0.1	5.0	0.1	0.280950	2.173794
0.15	2.0	0.5	6.2	0.1	5.0	0.1	0.071398	2.228292
0.1	0.0	0.5	6.2	0.1	5.0	0.1	0.795658	2.085888
0.1	1.0	0.5	6.2	0.1	5.0	0.1	0.526569	2.129644
0.1	2.0	0.5	6.2	0.1	5.0	0.1	0.280950	2.173794
0.1	3.0	0.5	6.2	0.1	5.0	0.1	0.055493	2.217957
0.1	2.0	0.1	6.2	0.1	5.0	0.1	0.067574	2.305898
0.1	2.0	0.2	6.2	0.1	5.0	0.1	0.128470	2.260081
0.1	2.0	0.3	6.2	0.1	5.0	0.1	0.183624	2.224545
0.1	2.0	0.4	6.2	0.1	5.0	0.1	0.234113	2.196435
0.1	2.0	0.5	3.0	0.1	5.0	0.1	0.398620	2.055958
0.1	2.0	0.5	5.0	0.1	5.0	0.1	0.328947	2.126781
0.1	2.0	0.5	7.0	0.1	5.0	0.1	0.240460	2.207714
0.1	2.0	0.5	9.0	0.1	5.0	0.1	0.108374	2.300182
0.1	2.0	0.5	6.2	0.2	5.0	0.1	0.409990	2.140427
0.1	2.0	0.5	6.2	0.3	5.0	0.1	0.553534	2.103415
0.1	2.0	0.5	6.2	0.4	5.0	0.1	0.710646	2.062804
0.1	2.0	0.5	6.2	0.5	5.0	0.1	0.879991	2.018601
0.1	2.0	0.5	6.2	0.1	6.0	0.1	0.272986	2.409282
0.1	2.0	0.5	6.2	0.1	7.0	0.1	0.268145	2.624261
0.1	2.0	0.5	6.2	0.1	8.0	0.1	0.265270	2.823147
0.1	2.0	0.5	6.2	0.1	9.0	0.1	0.263679	3.009018
0.1	2.0	0.5	6.2	0.1	5.0	0.0	0.288737	1.871517
0.1	2.0	0.5	6.2	0.1	5.0	0.1	0.280950	2.173794
0.1	2.0	0.5	6.2	0.1	5.0	0.2	0.275834	2.423334
0.1	2.0	0.5	6.2	0.1	5.0	0.3	0.272352	2.638131

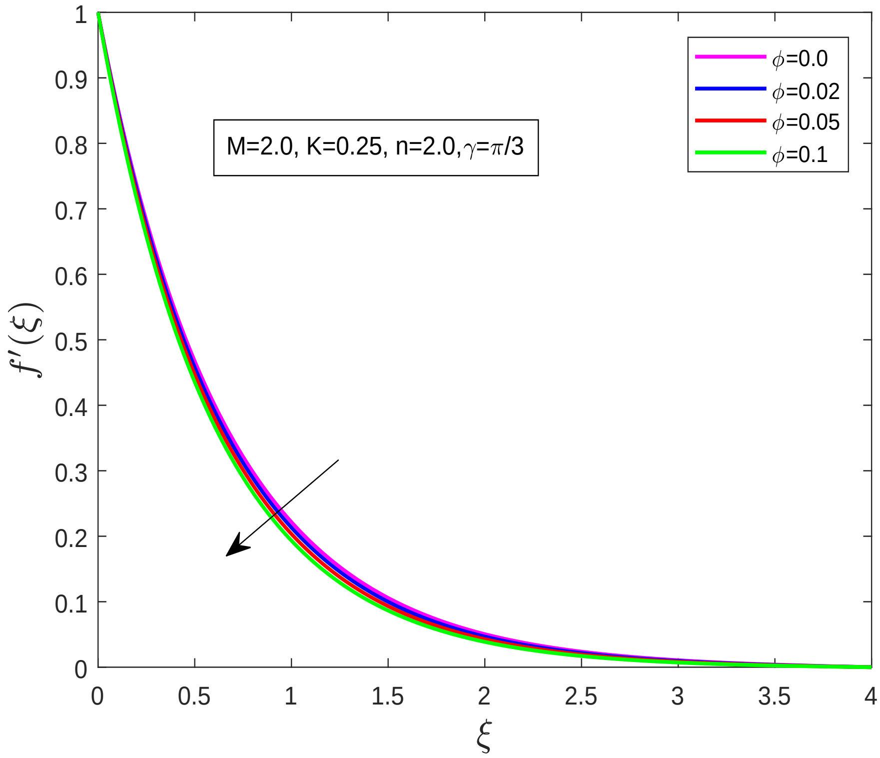

FIGURE 4.2: Impact of ϕ on the velocity profile.

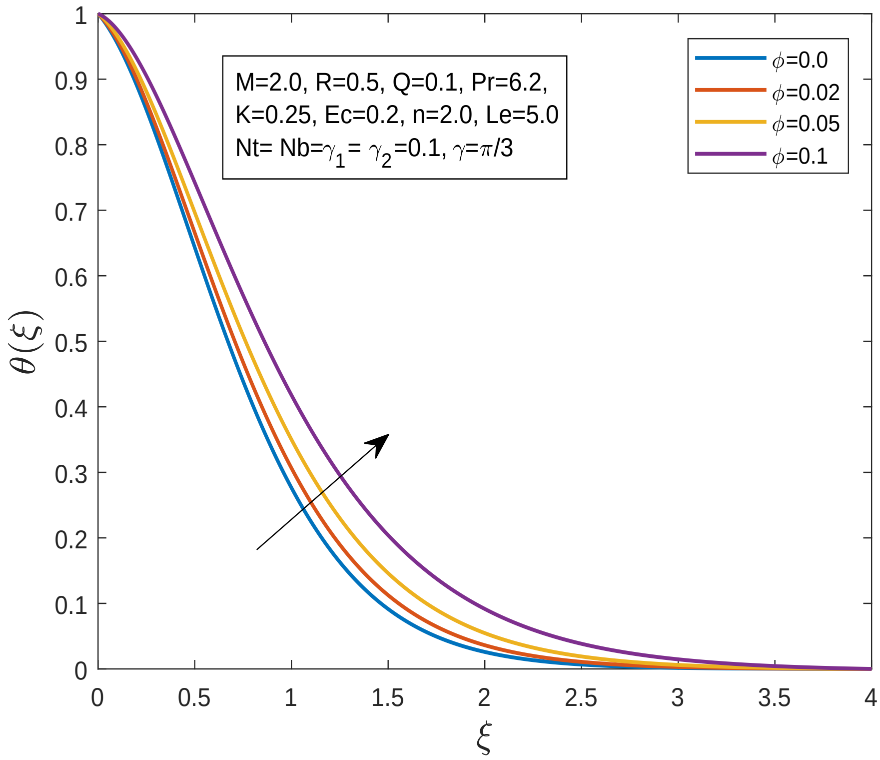

FIGURE 4.3: Impact of ϕ on the temperature profile.

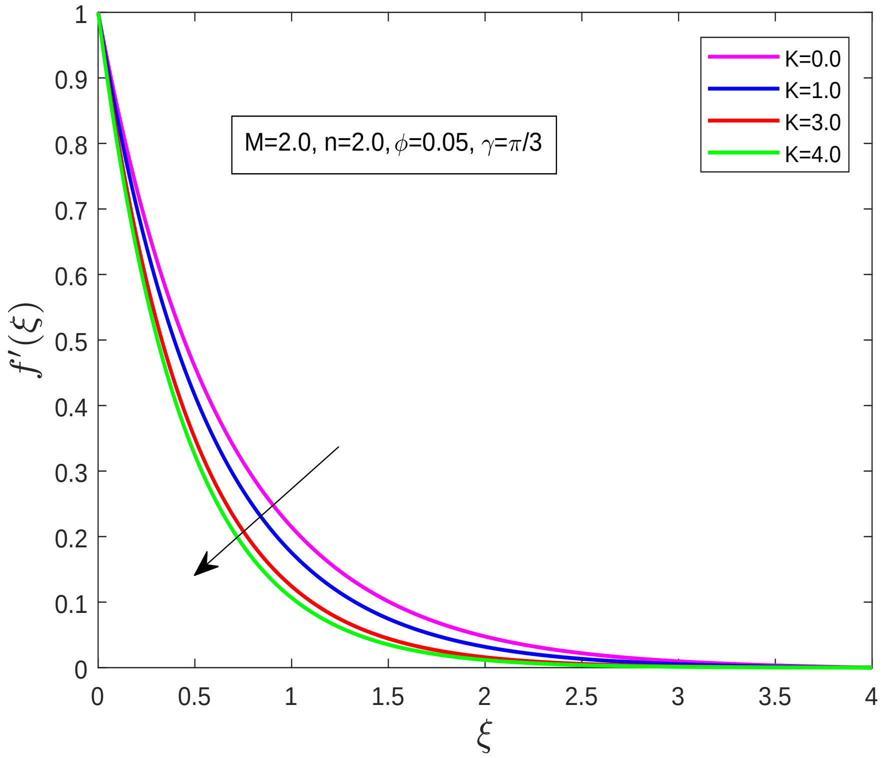

FIGURE 4.4: Impact of K on the velocity profile.

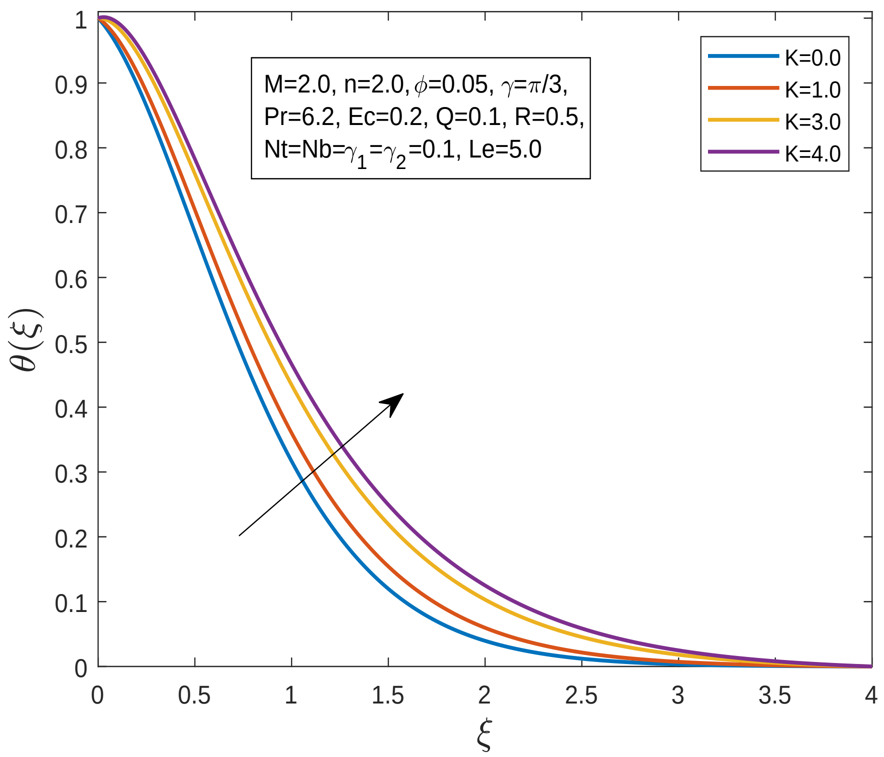

FIGURE 4.5: Impact of K on the temperature profile.

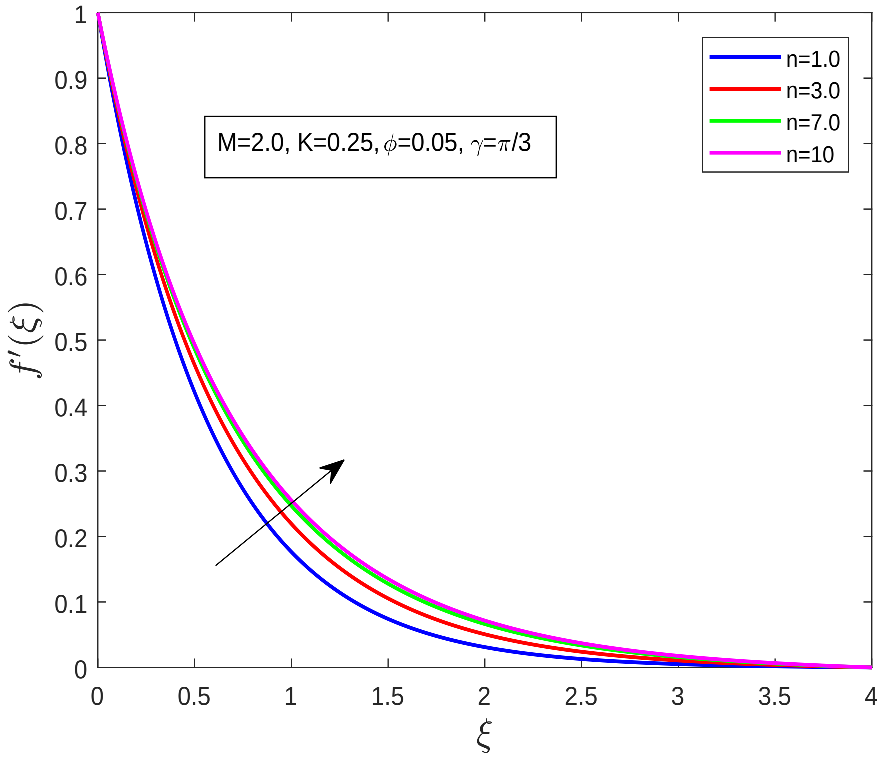

FIGURE 4.6: Impact of n on the velocity profile.

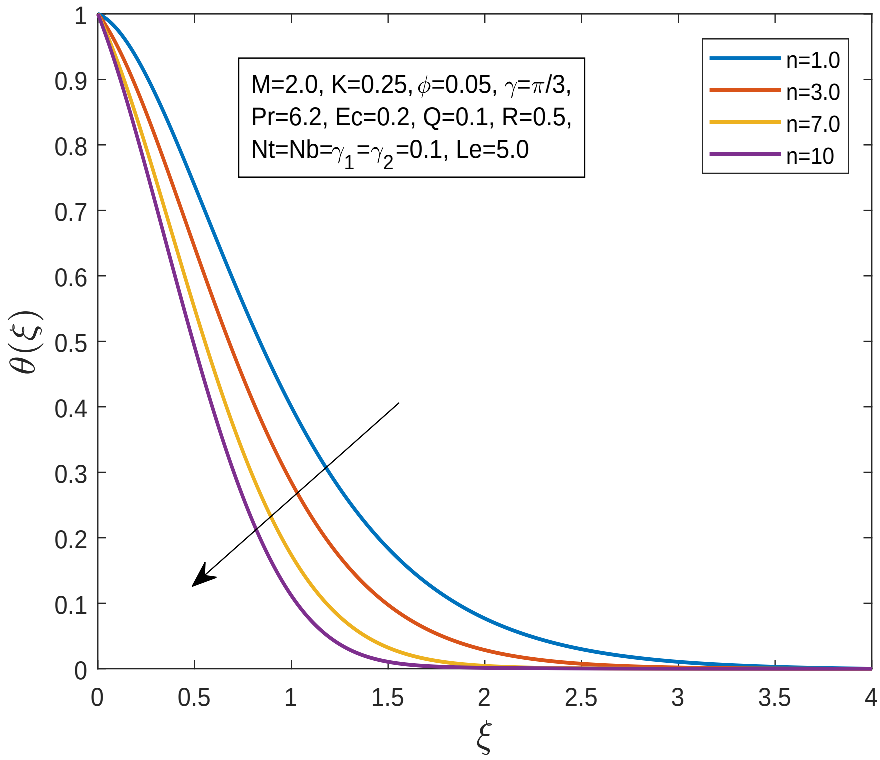

FIGURE 4.7: Impact of n on the temperature profile.

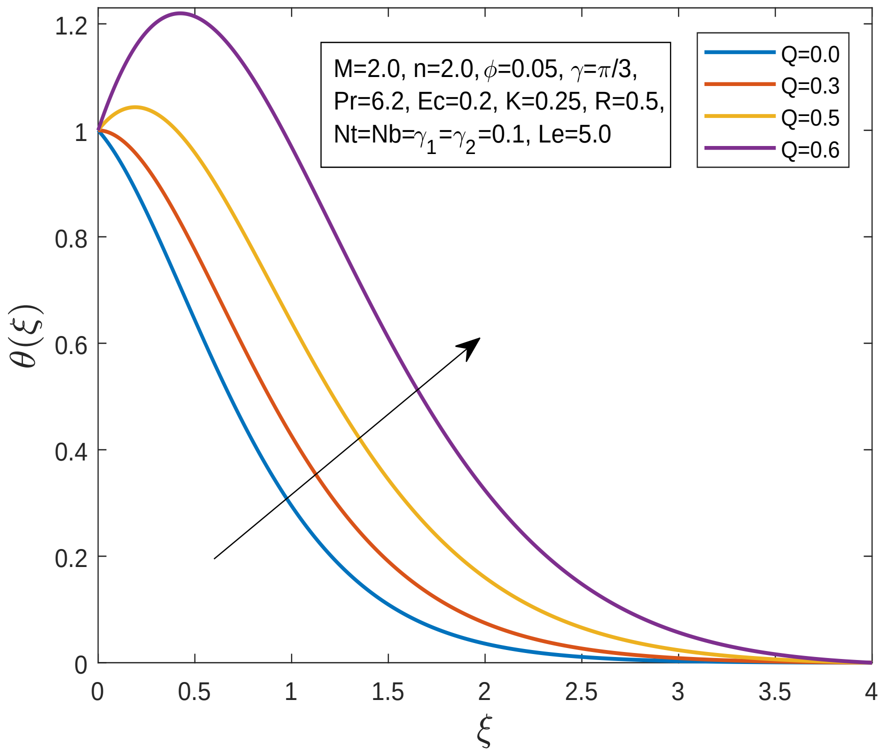

FIGURE 4.8: Impact of Q on the temperature profile.

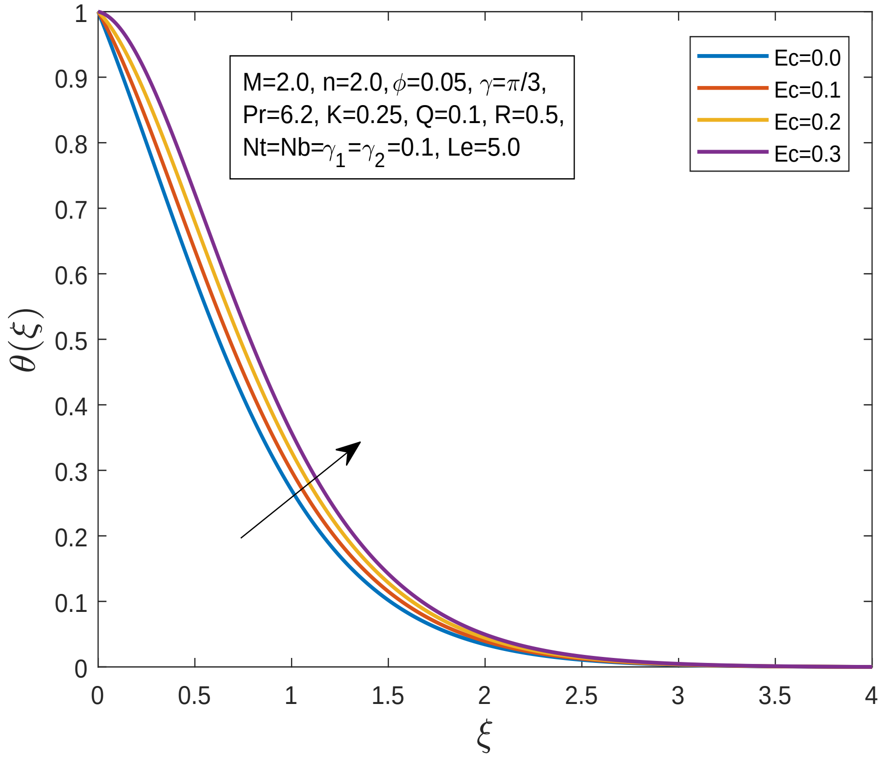

FIGURE 4.9: Impact of Ec on the temperature profile.

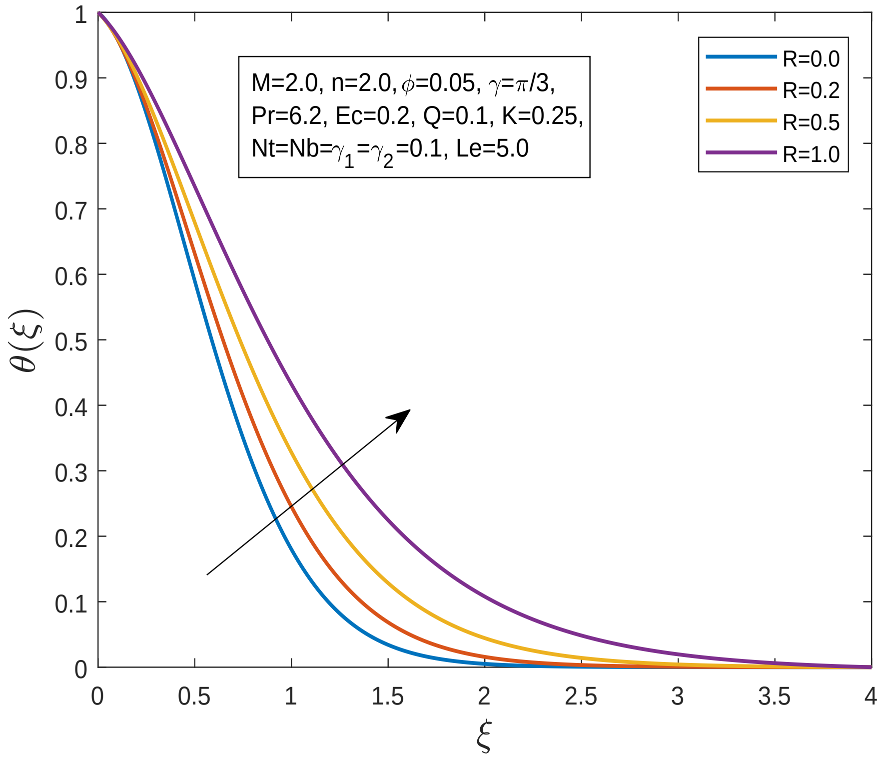

FIGURE 4.10: Impact of R on the temperature profile.

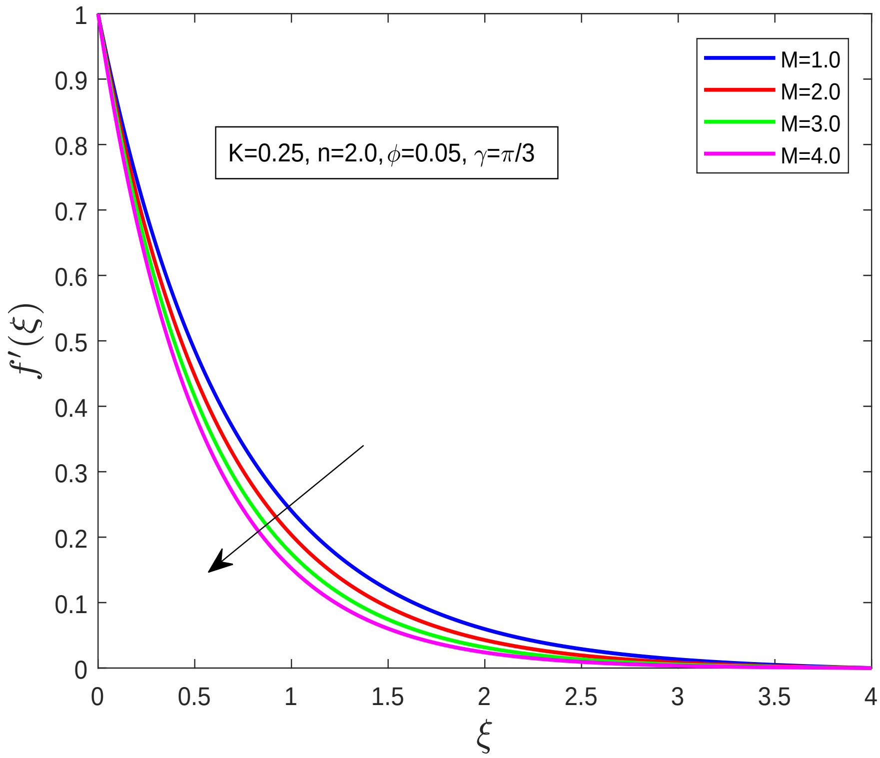

FIGURE 4.11: Impact of M on the velocity profile.

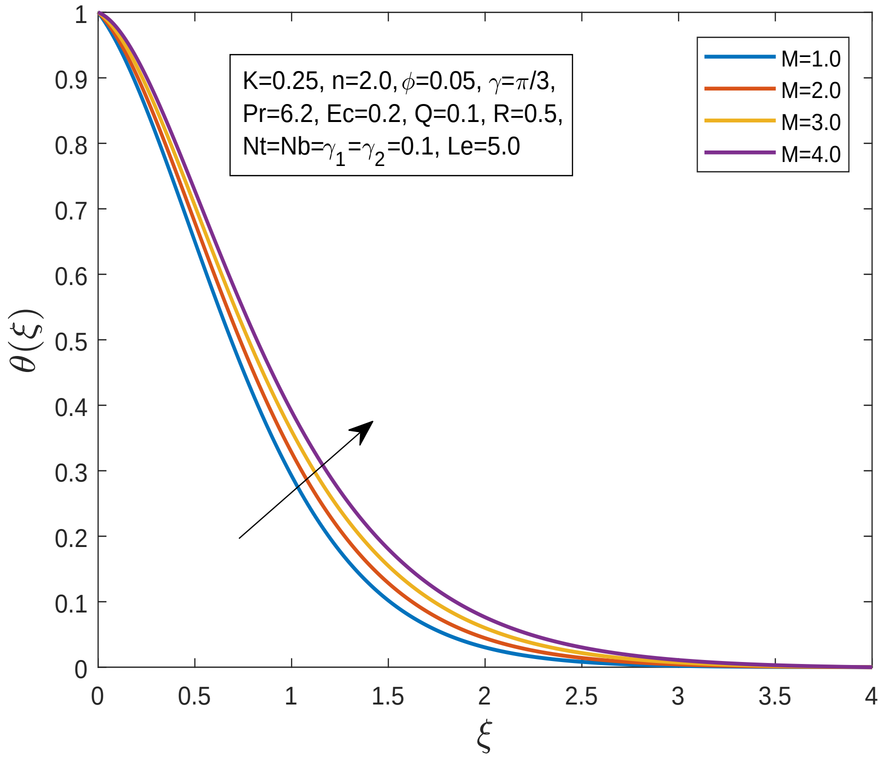

FIGURE 4.12: Impact of M on the temperature profile.

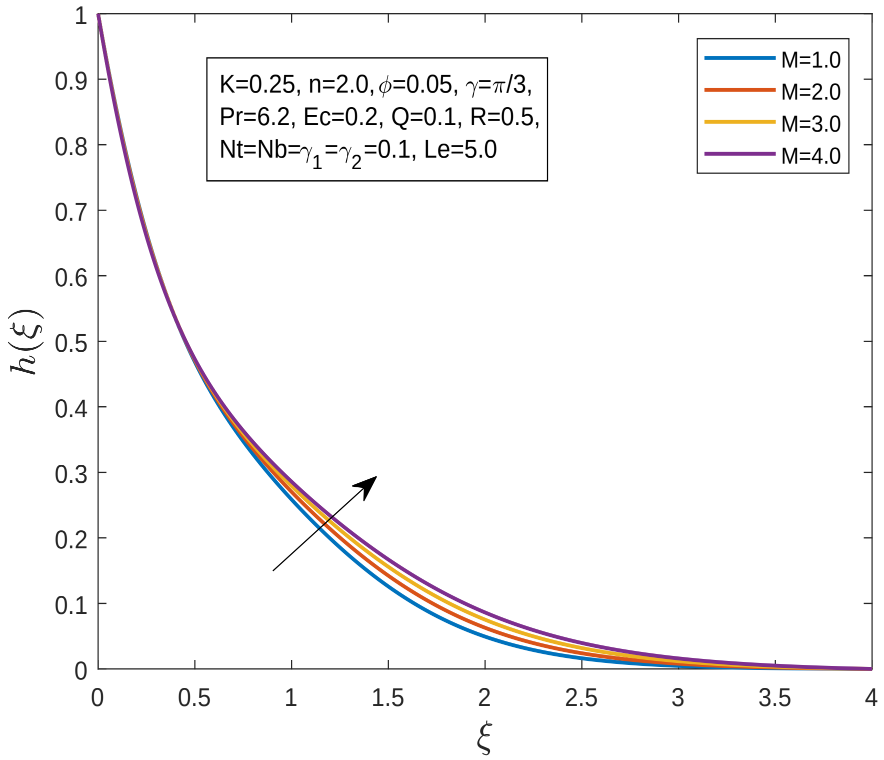

FIGURE 4.13: Impact of M on the concentration profile.

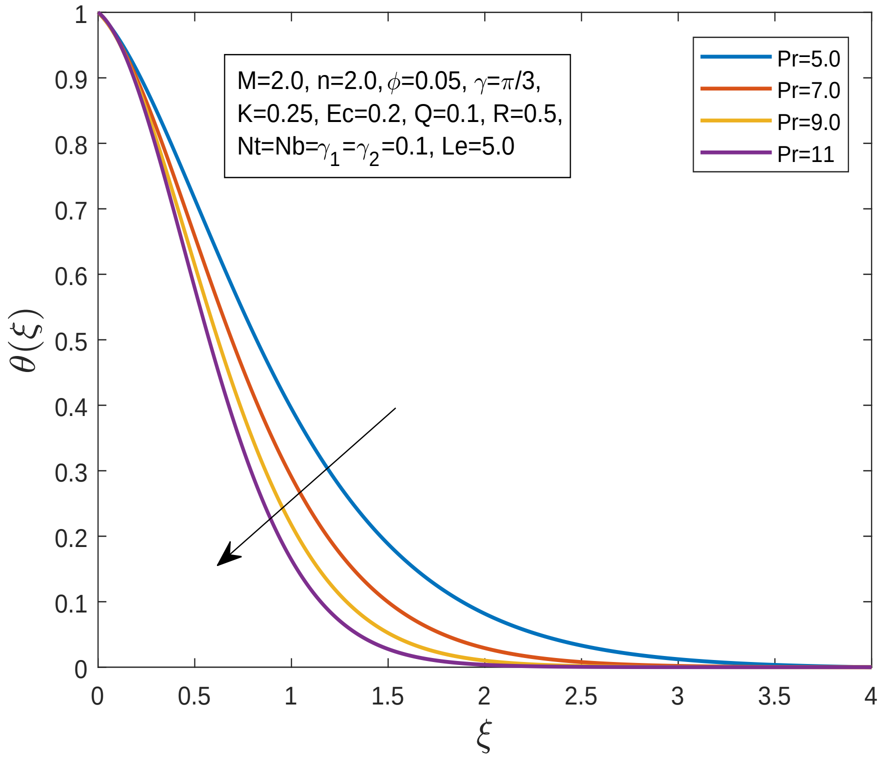

FIGURE 4.14: Impact of Pr on the temperature profile.

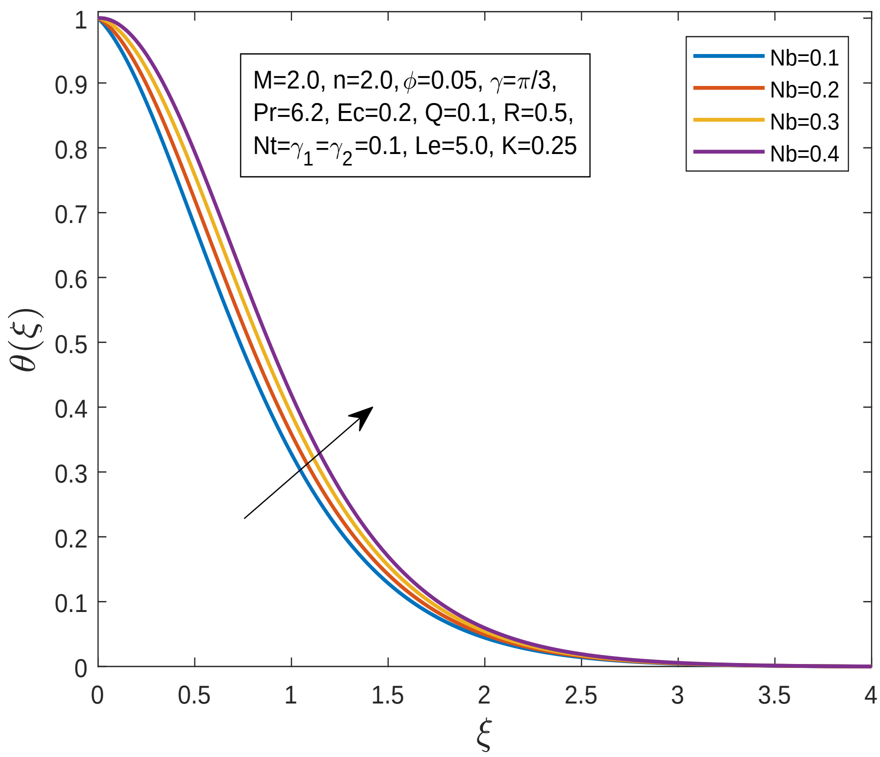

FIGURE 4.15: Impact of Nb on the temperature profile.

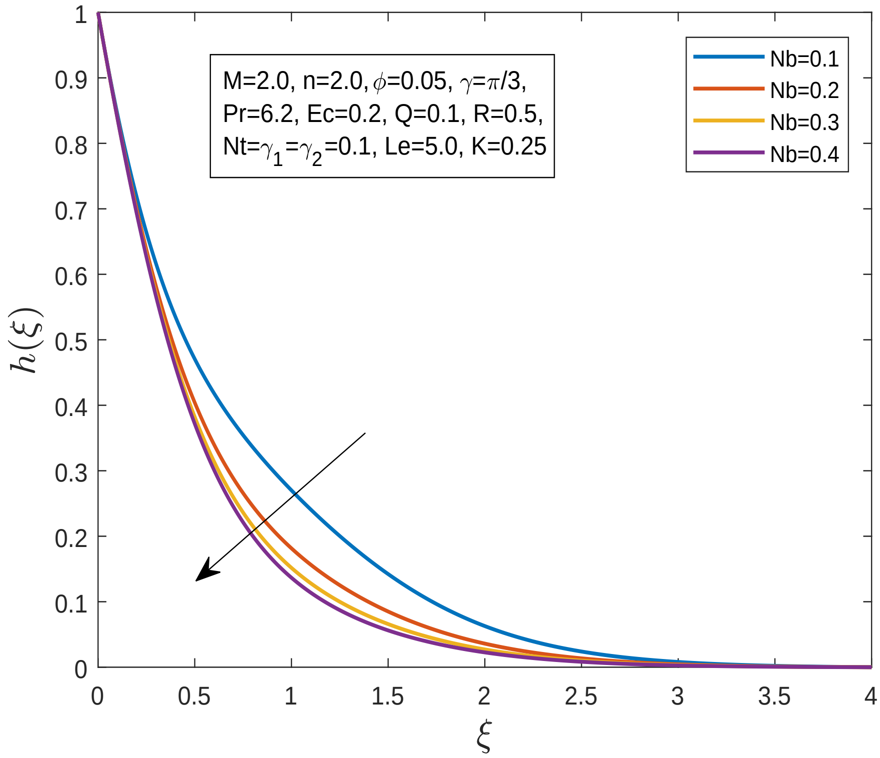

FIGURE 4.16: Impact of Nb on the concentration profile .

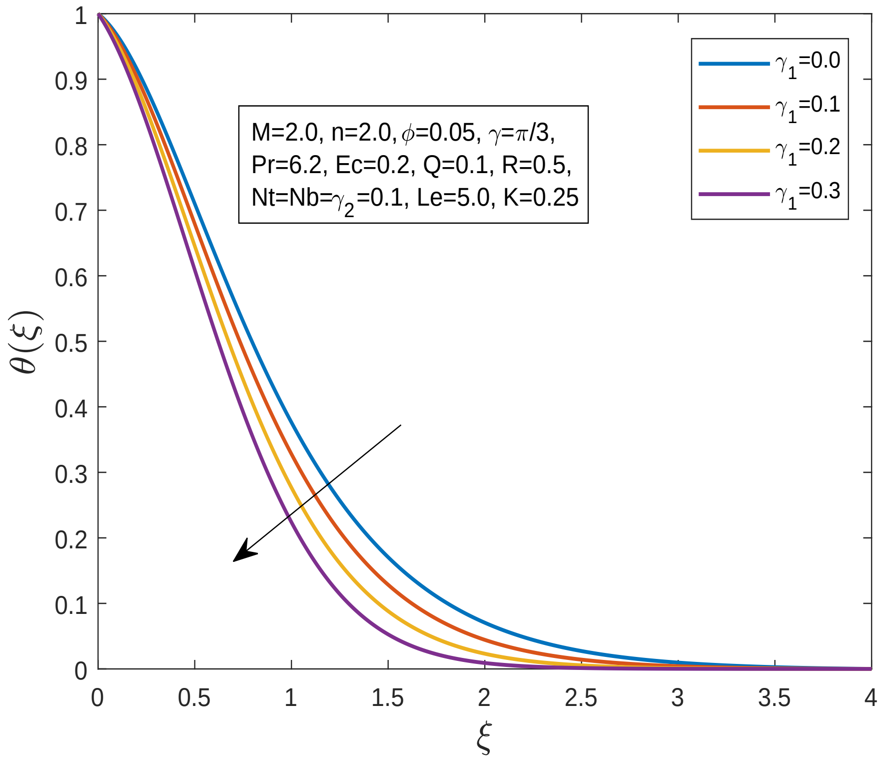

FIGURE 4.17: Impact of γ_1 on the temperature profile.

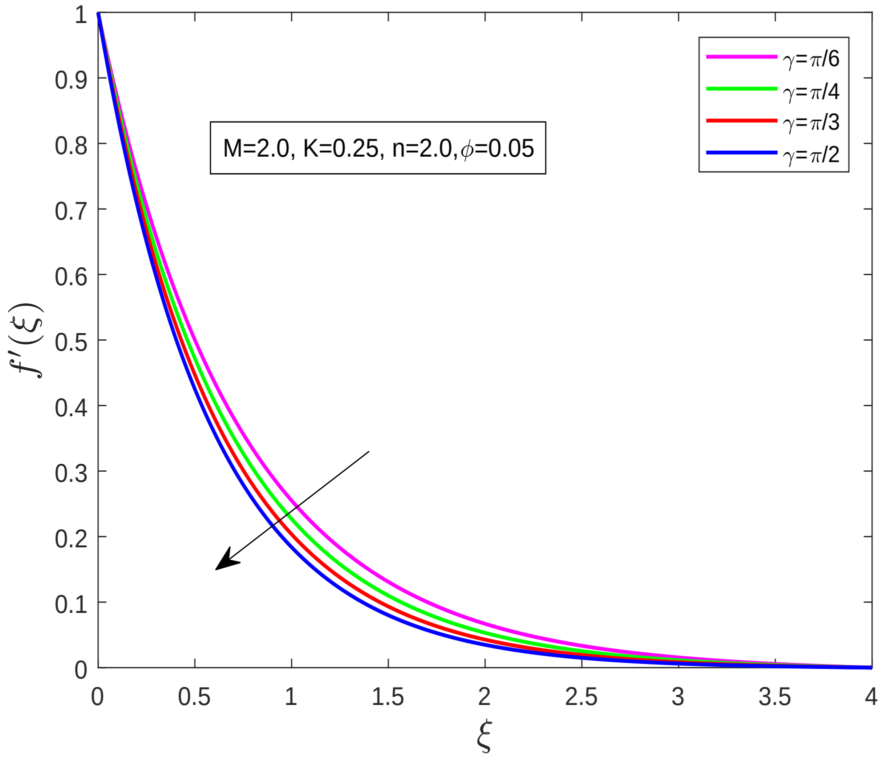

FIGURE 4.18: Impact of γ on the velocity profile.

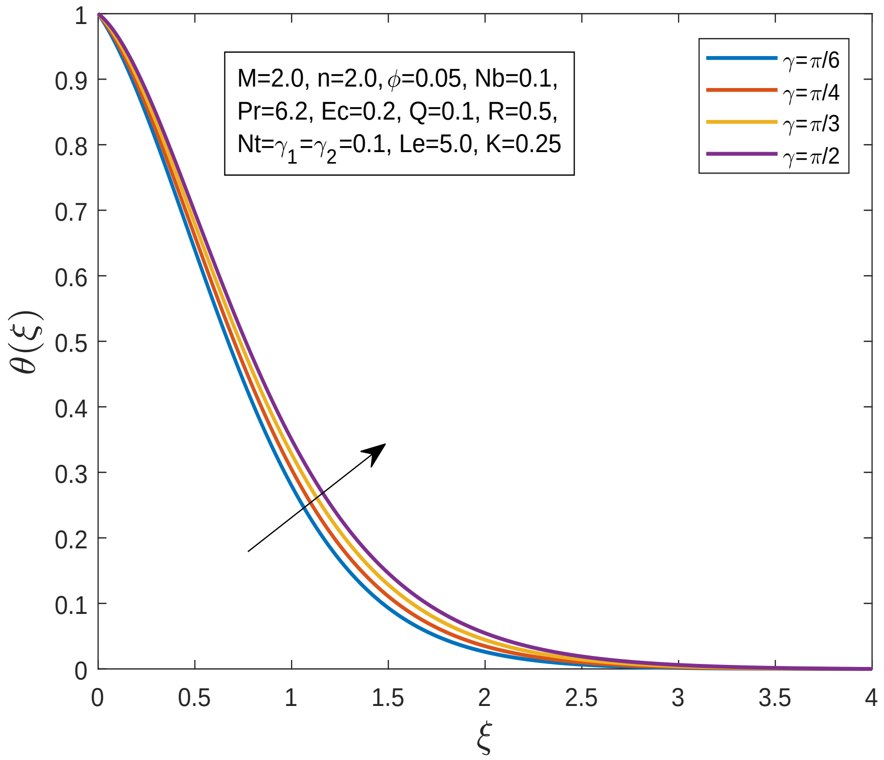

FIGURE 4.19: Impact of γ on the temperature profile.

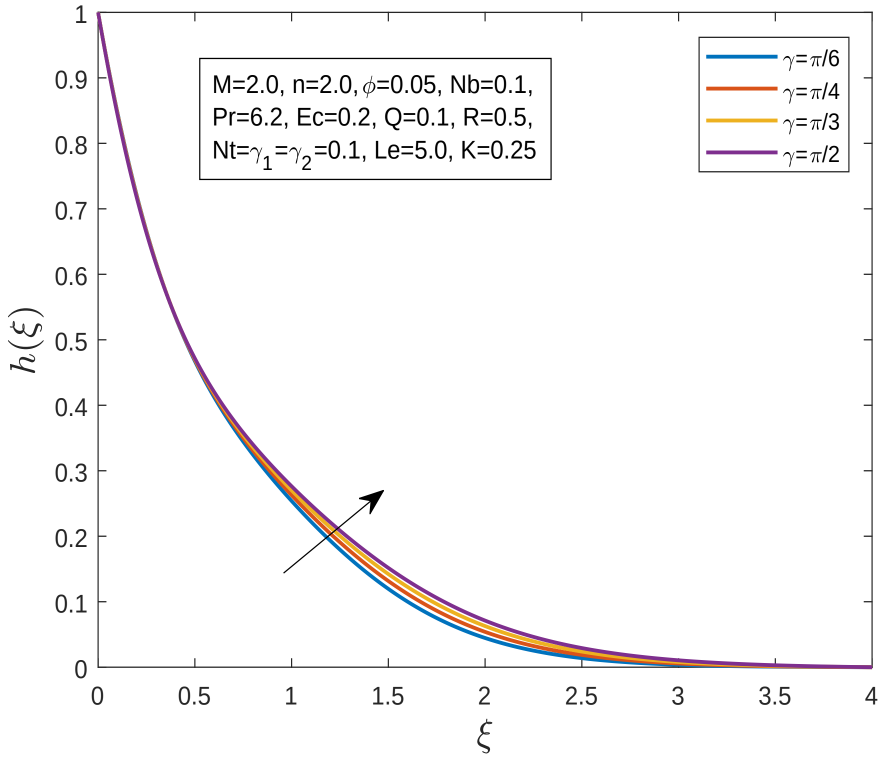

FIGURE 4.20: Impact of γ on the concentration profile.

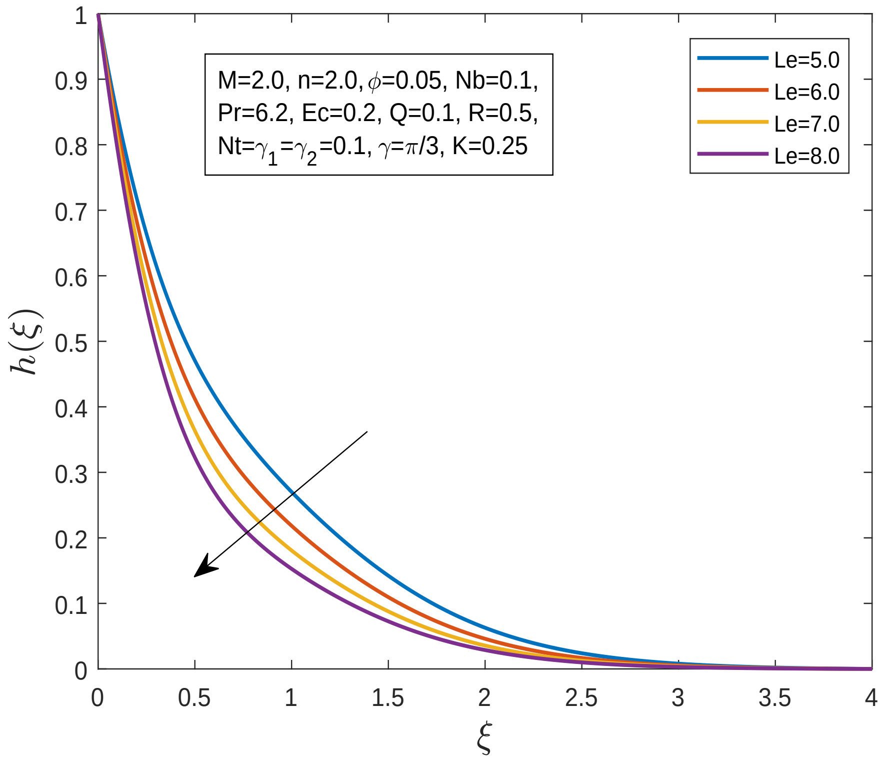

FIGURE 4.21: Impact of Le on the concentration profile.

Chapter 5

Conclusion

In this thesis, the work of Jafar et al. [44] is reviewed and extended with the effect of inclined magnetic field, Cattaneo-Christov heat flux, Brownain motion, thermophoresis diffusion and chemical reaction. First of all, momentum, energy and concentration equations are converted into the ODEs by using some similarity transformations. By using the shooting technique, numerical solution has been found for the transformed ODEs. Using different values of the governing physical parameters, the results are presented in the form of tables and graphs for velocity, temperature and concentration profiles. The achievements of the current research can be summarized as below:

- Increasing the values of ϕ, the velocity profile decreases while the temperature profile increases.
- For the enhancing values of R and Q, the temperature distribution is increased.
- The velocity profile is decreased due to the increasing values of the permeability paramater K.
- Rising the values of Prandtl number results in decrease the temperature profile.

- Increasing the magnetic parameter M results in a rise in the skin friction coefficent.
- A decrement is noticed in Nusselt number due to ascending values of Prandtl number.
- An increment is noticed in the temperature distribution by rising the values of Eckert number Ec.
- By increasing the values of M, the concentration profile increased.
- With a rise in Nb, the temperature profile increases.
- Due to the ascending values of Le, the numerical values of local Sh_x is increased.
- Due to the ascending values of the relaxation time parameter γ_1, the values of Nu_x are increased while Sh_x is decreased.

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