CAPITAL UNIVERSITY OF SCIENCE AND TECHNOLOGY, ISLAMABAD

Some Fixed Point Results in b-Metric Spaces via Graph Structure

by

Mariya Saeed

A thesis submitted in partial fulfillment for the degree of Master of Philosophy

in the

Faculty of Computing
Department of Mathematics

2023

Dedicated to My Parents

Whose ceaseless and profound love and unswerving support lie at the cornerstone of my academic accomplishment. Wouldn't have done without them.

My Husband

My cherished partner, my source of inspiration, through every triumph and challenge your steadfast presence has been my guiding light.

My Daughter

My precious and resilient little one, your innocent heart has taught me invaluable lessons about determination, patience, and the boundless capacity of a child's love.

CERTIFICATE OF APPROVAL

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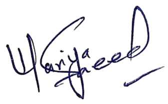

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Acknowledgement

All praise is due to Allah alone, who is the most beneficent and the most merciful. I would like to be thankful to all those who provided support and encouraged me during this work. First and foremost, I am immensely grateful to my supervisor Dr. Samina Batul for invaluable guidance, patience and expertise throughout the entire research journey. Her unwavering support, insightful feedback and encouragement have been instrumental in shaping this thesis. I would like to extend my gratitude to Dr. Dur-e-Shehwar Sagheer and Dr.Rashid Ali in this regard.

I am thankful from the core of my heart to Prof. Dr. Muhammad Sagheer Head of Department of Mathematics for his dedication to foster a stimulating academic environment and their commitment to excellence have greatly enriched my learning experience.

I am also thankful to my teachers Dr. Javed Ahsan, Dr.Sajida Kousar and many others in my whole educational period for their co-operation and guidance.

I would like to thank my parents, my husband, my friends and family for their unconditional support and encouragement. Their belief in me, even during moments of self-doubt, has been a constant source of inspiration. Their understanding and patience during this demanding period have been immeasurable.

While I have made every effort to acknowledge everyone who has contributed, I apologize if I inadvertently omitted any individuals or organization. Thank you all. May the Lord continue to bless you.

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Abstract

This research aimed to establish some fixed points results via graph structure in the setting of b-metric spaces. For this purpose, the work of Acar et al. is reviewed and notions of rational-type multivaled G-contractions and $\mathcal{F}$-contractions are established in b-metric spaces endowed with graph structure. To strengthen the validity of our results a supportive example is provided. Our results generalizes several existing results in literature.

Contents

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Abbreviations

bCF	b-Comparison function
BCP	Banach contraction principle
bMS	b-Metric space
(c)-CF	c-Comparison function
CF	Comparison function
CMS	Complete metric space
CS	Cauchy sequence
FP	Fixed point
MS	Metric space
TFP	Tarski fixed point
USC	Upper semi continuous
WGP	Weakly graph-preserving

Symbols

(X, d)	Metric space
(X, d_b)	b-Metric space
$CB(X)$	Closed and bounded subsets of X
$K(X)$	Compact subsets of X
$P(B)$	Power set of B
$\mathsf{E}(G)$	Set of edges of graph G
H	Hausdroff distance

Chapter 1

Introduction

1.1 Historical Background

Mathematics plays a pivotal role in various domains of life, serving as a fundamental branch of scientific knowledge. Within this vast discipline, mathematics is further divided into numerous subfields. Among these, fixed point (FP) theory stands out as a highly significant branch within pure mathematics. FP theory is a fundamental and influential area of mathematics that has applications in various fields. It focuses on the study of mathematical functions that possess at least one point that remains unchanged when the function is applied. This point is called a FP. FP theory has gained significant importance because it provides fundamental tools and concepts that are applicable to a wide range of mathematical problems and scientific disciplines. It offers insights into the existence, uniqueness and stability of solutions. It has practical applications in optimization, differential equations, economics, computer science and more.

In the late 19th century, Poincare [1] emerged as a trailblazing mathematician, making noteworthy advancements in the realm of FP theory. His influential work laid the foundation for this field. Then metric space (MS) was introduced by Frechet [2] in 1906. He defined a MS as a set of points equiped with a distance function satisfying certain axioms. Later, in 1922 Banach [3] further expanded the contribution of Poincar by proving the existence of FP within a complete metric space (CMS) for contraction mapping. The exploration of metric FP theory thus became a prominent domain within the broader realm of FP theory. The Banach FP theorem holds a vital position within metric FP theory, serving as a fundamental outcome. According to this theorem, if (X, d) be a CMS then for a contraction mapping $\Gamma : X \to X$ there is a unique FP. The mapping Γ is called a contraction mapping if the following condition is satisfied,

$$\mathsf{d}(\Gamma\psi, \Gamma\theta) \leq \alpha\ \mathsf{d}(\psi, \theta) \quad \text{forall } \psi,\ \theta \in X \text{ and } \alpha\ \in [0, 1). \tag{1.1}$$

It is known as Banach Contraction Principle (BCP). FP theory has been evolved particularly in two directions. Some authors applied different contraction and others have changed the space under consideration.

Initially the generalization is done by Edelstein's [4] by applying different contraction condition, in which condition (1.1) is eased by considering different points from X and taking $\alpha = 1$. Later, a new contraction condition was introduced by Rakotch [5], where the constant α of (1.1) is substituted by a funtion $\alpha : [0, \infty) \to [0, 1]$ that is decreasing monotonically. So,

$$\mathsf{d}(\Gamma\psi,\ \Gamma\theta)\ \leq\ \alpha(t)\mathsf{d}(\psi,\ \theta)) \quad \text{for all } \psi,\ \theta\ \in\ X. \tag{1.2}$$

Because every contraction is continuous. So it is questionable that if there are contraction conditions which does not imply the continuity of the mappings. Then in 1968 Kannan [6] provided the answer to such queries in which Kannan replaced the contraction condition with,

$$\mathsf{d}(\Gamma\psi, \Gamma\theta) \leq \alpha\ \mathsf{d}(\psi,\ \Gamma\psi) + \mathsf{d}(\theta,\ \Gamma\theta) \quad \forall\ \psi,\ \theta \in X, \text{and } \alpha\ \in [0, 1/2). \tag{1.3}$$

Further more generalizations of BCP were made by Mier and Keeler [8] by the expansion of contraction conditions. Then in 1975 Dass and Gupta [9] made an extension in BCP by introducing the rational contraction condition.

Then S.K.Chatterjea [7] proved a FP theorem for operators which satisfy the following condition

$$\mathsf{d}(\Gamma\psi,\Gamma\theta) \leq \alpha\ \mathsf{d}(\psi,\ \Gamma\theta) + \mathsf{d}(\theta,\ \Gamma\psi) \quad \forall\ \psi,\ \theta \in X \text{ where } \alpha\ \in\ (0,1). \tag{1.4}$$

In the second category of generalization of BCP, the space structure is considered on which the Γ is defined. In [10] Abbas and Jungck established the existence of coincidence points and common fixed points for mappings satisfying certain contraction conditions, without appealing to continuity, in a cone MS. In 1989 Bakhtin [11] introduced the concept of bMS by relaxing the triangular inequality and then replace the MS for proving several FP results to generalize BCP.
Nadler [12] extends the structure of the spaces in which the mapping Γ is defined. Specifically he extended the BCP from a single-valued contraction mapping to a mutivalued (M.valued) contraction mapping. Later, Batul and Kamran [13] generalize the notion of C^*-valued contraction mapping by weakening the contraction condition of Ma et al. [14] and established a FP theorem for such mapping.
According to BCP, Γ satisfies the contraction condition for every element of $X \times X$. Here question arises, whether it is possible to generalize BCP by imposing appropriate condition on ordered pairs from $X \times X$ s.t (1.1) holds on a subset of $X \times X$ and that the mapping still has a FP. The initiative in this direction is taken by Ran and Reurings [15]. They showed that, assuming Γ is contractive for the related pairs, the mapping Γ still has an FP subject to the completeness of the partially ordered set X. Later on, many authors like Bashkar and lakshmikanthm [16] and Neito and Roriguez [17] have made significant contribution in the FP theory on partially ordered MS.
In 2006 Espinola and Kirk [18] applied FP results in graph theory. Jachymski [19] developed this concept further by replacing the ordered structure with structure of graph on MS. Using ordered pairs in terms of their vertices and edges of a graph, he illustrated that Γ has a FP if contraction condition holds.
The concept of a graph can be traced back to the 18th century when the Swiss mathematician Leonhard Euler [20] introduced the Seven Bridges of Konigsberg problem in 1736. Euler's solution to this problem laid the foundation for graph theory. He represented the city of Konigsberg as a graph with land masses as

vertices and bridges as edges. Euler demonstrated that it was impossible to find a path that crossed each bridge exactly once, leading to the development of the theory of graphs.

Acar et al. [21] obtain several FP theorems in MS via graph structure for multi-valued mappings. In this paper the author introduced the new concept of rational type G-contraction and $\mathcal{F}$-contractions.

Influenced by the work of Acar et al. we bring to light some FP theorems in bMS via graph structures. The new FP theorems generalize the work of Acar et al. [21]. The remaining content can be summarized as follows

Chapter 2, gives the primary definitions of MS, bMS, Pompieu hausdroff MS, FP, mutivalued contraction mapping, basics on graph and some associated examples.

Chapter 3, provides the review of the article [21]. In this some FP results on MS endowed with graph structure are presented with new type of G-contraction and multivalued $\mathcal{F}$-contractions.

Chapter 4, is about the existence of FP results in bMS via graph structure. Some FP results are proved by using G-contraction and multivalued $\mathcal{F}$-contractions. In the end, an example is presented to show the validity of our obtained results.

Chapter 5, provides the conclusion of the thesis.

Chapter 2

Preliminaries

In this chapter fundamental definitions and examples are given. Presenting the fundamental findings, explanations and examples that will be utilized in the next chapters is the chapter's major goal.

2.1 Metric Space

MS introduced by Frechet [2] is a fundamental concept in mathematics that provides a framework for understanding distance and proximity between points.

Definition 2.1.1. "Let X be a non-empty set. A function $\mathsf{d} : X \times X \to \mathbb{R}$ is said to be a metric on X, if for all $\psi,\ \theta,\ z \in X$, it satisfies the following axioms:

M_1) $\mathsf{d}(\psi,\ \theta) \geq 0;$ (Non-negativity)

M_2) $\mathsf{d}(\psi,\ \theta) = 0 \Leftrightarrow \psi = \theta;$ (Reflexive property)

M_3) $\mathsf{d}(\psi,\ \theta) = \mathsf{d}(\theta,\ \psi);$ (Symmetric property)

M_4) $\mathsf{d}(\psi,\ z) \leq \mathsf{d}(\psi,\ \theta) + \mathsf{d}(\theta,\ z).$ (Triangle inequality)

The pair $(X,\ \mathsf{d})$ is called the MS. The set X is called the underlying set or the ground set. The elements of X are called the points of the MS. Instead of $(X,\ \mathsf{d})$, we may write X for a MS." [22]

Example 2.1.2. Consider the set $\mathbb{R}$, the set of real numbers. The function $\mathsf{d} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, defined as $\mathsf{d}(\psi, \theta) = |\psi - \theta| \quad \forall\ \psi, \theta \in \mathbb{R}$, satisfies the conditions of metric on $\mathbb{R}$. As $\mathsf{d}(\psi, z) = |\psi - z| = |(\psi - \theta) + (\theta - z)| \leq |\psi - \theta| + |\theta - z| = \mathsf{d}(\psi, \theta) + \mathsf{d}(\theta, z)$.

Example 2.1.3. The set $C[a, b]$ of all real-valued continuous functions on the the interval $[a, b]$ is a MS, where d is defined as

$$\mathsf{d}(f, g) = \int_a^b |f(\psi) - g(\psi)| d\psi,$$

2.1.1 Convergence, Cauchy Sequence and Completeness in Metric Space

The significance of sequences of real numbers in calculus cannot be overstated, as they serve as a fundamental tool for understanding the concept of convergence. This understanding is made possible by the fact that sequences of real numbers define a metric on $\mathbb{R}$. In MS the situation is quite similar, that is we consider a sequence $\{\psi_q\}$ of elements $\psi_1, \psi_2, \ldots$ of X and use the metric d to define convergence in a fashion analogous to that in Calculus.

Definition 2.1.4. "Let (X, d) be a MS, then

(a) A sequence $\{\psi_q\}$ in X is said to converge to $\psi \in X$, if for every $\epsilon > 0$ there exists $N = N(\epsilon) \in \mathbb{N}$ s.t $\mathsf{d}(\psi_q, \psi) < \epsilon$, for all $q \geq N$. Hence $\lim_{q \to \infty} \psi_q = \psi$.

(b) A sequence $\{\psi_q\}$ in X is said to be Cauchy, if for every $\epsilon > 0$ there exists $N = N(\epsilon) \in \mathbb{N}$ s.t $\mathsf{d}(\psi_p, \psi_q) < \epsilon$, for all $p, q \geq N$.

(c) A MS (X, d) is said to be complete if every Cauchy sequence (CS) in X converges." [22]

2.1.2 Banach Contraction Principle

The FP of a function Γ refers to an element within the function's domain that the function maps to itself. FP theorems have widespread applications in various areas of pure mathematics. Several notable authors such as Banach [3], Bhaskar [16] and Khamsi [23], provide explanations of FP theorems across the entire field of mathematical sciences.

Definition 2.1.5. "Consider a MS (X, d). A mapping $\Gamma : X \to X$ is referred to as a contraction on X if there exists a positive real number $\alpha < 1$, s.t

$$\mathsf{d}(\Gamma\psi, \Gamma\theta) \leq \alpha \mathsf{d}(\psi, \theta) \quad \forall \psi, \theta \in X.$$

This implies that for any given points ψ and θ, the images of ψ and θ under the mapping Γ are closer to each other than the original points ψ and θ. To be precise, the ratio $\frac{\mathsf{d}(\Gamma\psi,\Gamma\theta)}{\mathsf{d}(\psi,\theta)}$ is always less than or equal to a constant α, where α is a positive value strictly smaller than 1." [22]

Example 2.1.6. Consider the function $\Gamma : \mathbb{R} \to \mathbb{R}$ defined as follows:

$$\Gamma(\psi) = \cos(\cos\psi) \Rightarrow \Gamma'(\psi) = -\sin(\cos\psi)[-\sin\psi] = \sin(\cos\psi)\sin\psi.$$

Through the application of the Mean Value Theorem, we derive the following result:

$$|\Gamma'(\psi)| = |\sin(\cos\psi)||\sin\psi| < 1.$$

This inequality holds because:

$$|\sin(\cos\psi)| \leq 1$$
$$|\sin\psi| \leq 1$$

Both terms on the right-hand side are bounded by 1. It is impossible for both terms to simultaneously equal to 1, which implies that $\Gamma(\psi)$ is a contraction.

FP of a mapping is an element that maps to itself.

Definition 2.1.7. "A FP of a mapping $\Gamma : X \to X$ of a set Γ to itself, is an element $\psi \in X$ s.t,

$$\Gamma\psi = \psi,$$

the image $\Gamma\psi$ coincides with ψ." [22]

Example 2.1.8. Consider a mapping $\Gamma : \mathbb{R} \to \mathbb{R}$ defined by

$$\Gamma(\psi) = \frac{\psi}{4} + 3, \quad \text{has a unique FP } \psi = 4.$$

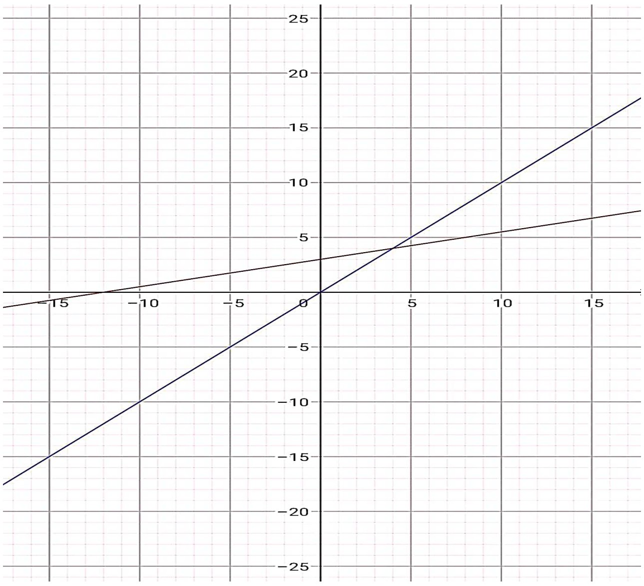

FIGURE 2.1: Graph of Function $\Gamma(\psi) = \frac{\psi}{4} + 3$.

Example 2.1.9. Consider a mapping $\Gamma : \mathbb{R} \to \mathbb{R}$ defined as

$$\Gamma(\psi) = \psi + 3, \quad \text{has no FP.}$$

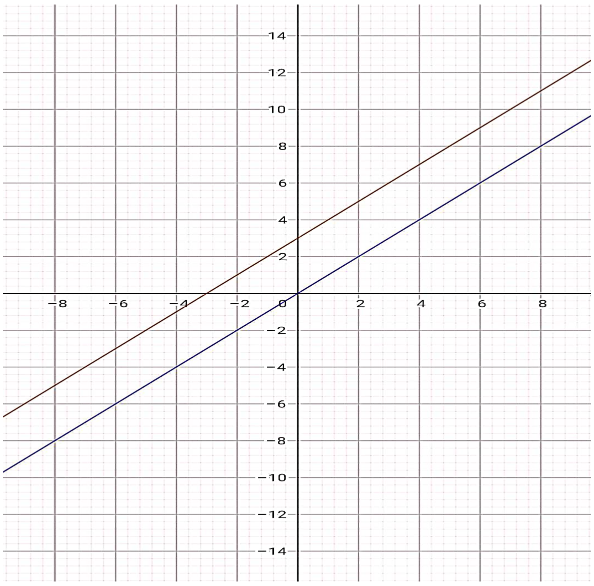

FIGURE 2.2: Graph of Function $\Gamma(\psi) = \psi + 3$.

Example 2.1.10. Consider a mapping $\Gamma : \mathbb{R} \to \mathbb{R}$ defined as

$$\Gamma(\psi) = \psi^2 - 3\psi + 3.$$

then $\psi = 1, 3$ are two FPs of Γ .

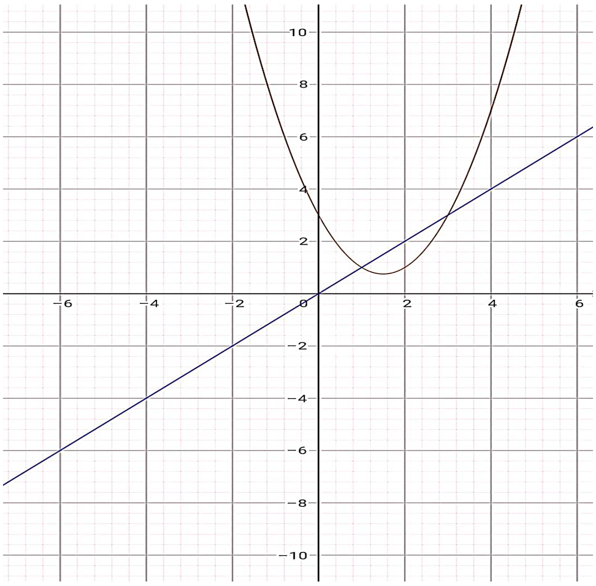

FIGURE 2.3: Graph of Function $\Gamma(\psi) = \psi^2 - 3\psi + 3$.

Example 2.1.11. Consider a mapping $\Gamma : \mathbb{R} \to \mathbb{R}$ defined as

$$\Gamma(\psi) = \psi + \sin\psi,$$

then Γ has infinite many FPs.

Theorem 2.1.12. "Assume (X, d) is a MS where $X \neq \phi$. Suppose that X is complete and let $\Gamma : X \to X$ be a contraction on X. Then Γ has precisely one FP." [22]

Example 2.1.13. Let $(\mathbb{R}, \mathsf{d})$ be a MS, where $\mathsf{d}(\psi, \theta) = |\psi - \theta|$. Let's define a mapping $\Gamma : \mathbb{R} \to \mathbb{R}$ as follows:

$$\Gamma(\psi) = \frac{\psi}{7} + 2.$$

So Γ is a contraction with $\alpha = \frac{1}{7}$. Then Γ has a only one FP i.e $\psi = \frac{7}{3}$.

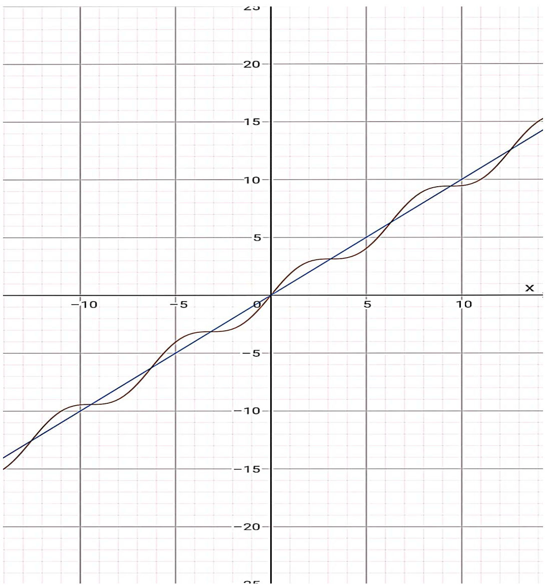

FIGURE 2.4: Graph of Function $\Gamma(\psi) = \psi + \sin\psi$.

2.2 Multivalued Mapping

multivalued mapping has many applications in real analysis, complex analysis, optimal control issues and other areas of practical and pure mathematics. multivalued mapping has a considerable impact in these areas. As the years have gone by, this theory's significance has grown and as a result, several publications have focused on multivalued mappings in the literature.

Definition 2.2.1. "Suppose A and B are non-empty sets. A multivalued mapping from A to $P(B)$ is denoted by $\Gamma : A \to 2^B$, where Γ is a function that maps elements from A to subsets of B." [24]

Example 2.2.2. Let

$$A = \{\psi_1,\ \psi_2,\ \psi_3,\ \psi_4,\ \psi_5,\ \psi_6\}$$
$$B = \{1,\ 1.5,\ 2,\ 2.5,\ \ldots,\ 7\}.$$

Define $\Gamma : A \to P(B)$, by

$$\Gamma(\psi_1) = \{1,\ 1.5,\ 4.5\}\ \ \Gamma(\psi_2) = \{2,\ 2.5,\ 3\}\ \ \Gamma(\psi_3) = \{4\}$$
$$\Gamma(\psi_4) = \{3\}\ \ \Gamma(\psi_5) = \{5,\ 5.5,\ 6\}\ \ \Gamma(\psi_6) = \{2.5,\ 6.5,\ 7\},$$

then Γ is a multivalued mapping.

Definition 2.2.3. "Let $(X,\ \mathsf{d})$ be a MS. We denote the family of all non-empty, closed and bounded subsets of X as $CB(X)$. The Pompeiu-Hausdorff metric $H : CB(X) \times CB(X) \to [0,\ \infty)$ is defined as follows:

$$H(A,\ B) = \max\left\{\sup_{\alpha\in A} D(\alpha,\ B),\ \ \sup_{\beta\in B} D(\beta,\ A)\right\}$$

where, A and B are elements of $CB(X)$ and $D(\alpha,\ B) = \inf_{\beta\in B} \mathsf{d}(\alpha,\ \beta)$". [25]

Lemma 2.2.4. "Consider $(X,\ \mathsf{d})$ a MS,. Let $A,\ B \subset X$ and let $q > 1$ be a constant. Then $\forall\ \ \psi\ \in A,\ \exists\ \theta \in B$ s.t the inequality

$$\mathsf{d}(\psi,\ \theta) \leq qH(A,\ B),$$

where H is a Pompeiu-Hausdorff metric."[26]

Definition 2.2.5. "Let (X, d) be a MS. A function $\Gamma : X \to CB(X)$ is defined as a multivalued contraction if there exists a constant $0 \leq \lambda < 1$ s.t

$$H(\Gamma\psi, \Gamma\theta) \leq \lambda \mathsf{d}(\psi, \theta), \quad \text{for all} \quad \psi,\ \theta \in X$$

In this context, $CB(X)$ represents the collection of non-empty closed and bounded subsets of X and H represents the Hausdorff distance." [27]

2.3 b-Metric Space

Bakhtin [11] is the first to introduce the concept of a bMS and Czerwick [28] is the next. Czerwick explicitly defined a bMS and proposed a condition that was weaker than the third feature of MS. They developed the concept of bMS and then applied the same concept to develop some FP findings for generalizing the BCP.

Definition 2.3.1. "Consider a non-empty set X and a function $\mathsf{d}_b : X \times X \to [0, \infty)$ that satisfies the following conditions:

M_{b1}) $\mathsf{d}_b(\psi,\theta)=0 \Leftrightarrow \psi=\theta$;

M_{b2}) $\mathsf{d}_b(\psi,\theta)=\mathsf{d}_b(\psi,\theta)\ \forall\ \psi,\theta\in X$;

M_{b3}) $\mathsf{d}_b(\psi,z)\leq s\,[\mathsf{d}_b(\psi,\theta)+\mathsf{d}_b(\theta,z)]\quad \forall\ \psi,\ \theta,\ z\in X,$ where $s\geq 1$.

The function d_b is referred to as a b-metric and the set (X,d_b) is denoted as a bMS." (Bakhtin [11], Czerwik [28])

Remark 2.3.2. The class of MS is smaller than of bMS. In the case of $s=1$, the notions of MS and bMS coincide.

Remark 2.3.3. The notion of Cauchyness, convergence and completeness in bMS can be generalized naturally as in MS.

Example 2.3.4. The function $\mathsf{d}_b:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ defined by $\mathsf{d}_b(\psi,\theta)=(\psi-\theta)^2$ is a bMS on $\mathbb{R}$ with $s=2$.

Example 2.3.5. Consider $X=\ell_r[0,1]$ as the set comprising of real functions $f(\psi)$, where $\psi\in[0,1]$, satisfying the condition that

$$\int_0^1 |\ f(\psi)|^r<\infty \quad \text{with} \quad 0<r<1$$

Let $\mathsf{d}_b: X\times X\to\mathbb{R}^+$ be defined as follows:

$$\mathsf{d}_b(f,g)=\left(\int_0^1 |f(\psi)-g(\psi)|^r d\psi\right)^{\frac{1}{r}}$$

Then d_b is bMS with $s=2^{\frac{1}{r}}$.

2.4 Graph

Graphs serve as mathematical structures employed to depict real-world scenarios by establishing connections between elements within specific domains..

Definition 2.4.1. "A graph is a pair of two sets that are the following:

(a) The set of vertices represented as $V(G)$, is a non-empty collection that includes all the vertices of the graph.

(b) The set of edges, denoted as $\mathsf{E}(G)$, is a binary operation applied to the set of vertices, $V(G)$.

The primary method of representing a graph, denoted as $G = (V(G), \mathsf{E}(G))$, is through a diagram where vertices are depicted as points and edges are depicted as line segments connecting the vertices." [29]

Example 2.4.2. For the graph in the accompanying figure:
$V(G) = \{1, 2, 3, 4, 5, 6\}$ and
$E(G) = \{(1, 2), (1, 4), (4, 2), (2, 5), (5, 4), (3, 5), (3, 6), (6, 6)\}$

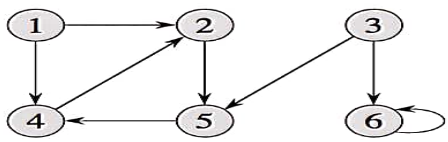

FIGURE 2.5: A directed graph.

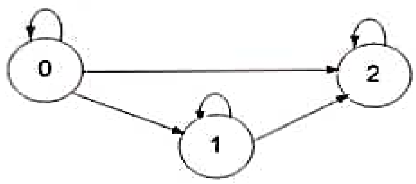

FIGURE 2.6: A reflexive graph with loops.

Definition 2.4.3. "Consider a non-empty set X and let Δ represent the diagonal of the Cartesian product $X \times X$. A directed graph or digraph G is characterized

by a non-empty set $V(G)$ and the set $\mathsf{E}(G) \subset V(G) \times V(G)$ of its directed edges. A digraph is reflexive if any vertex admits a loop.
For a given digraph $G = (V, E)$,

(a) If whenever $(\psi,\ \theta) \in \mathsf{E}(G) \Rightarrow (\theta,\ \psi) \notin \mathsf{E}(G)$, then the digraph G is called an oriented graph.

(b) A digraph G is transitive whenever $(\psi,\ \theta) \in \mathsf{E}(G)$ and $(\theta,\ z) \in \mathsf{E}(G) \Rightarrow (\psi,\ z) \in \mathsf{E}(G)$, for any $\psi,\ \theta,\ z \in V(G)$.

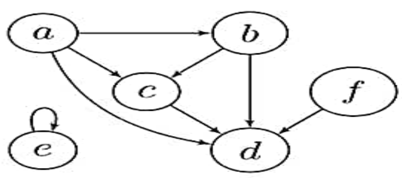

FIGURE 2.7: A transitive graph.

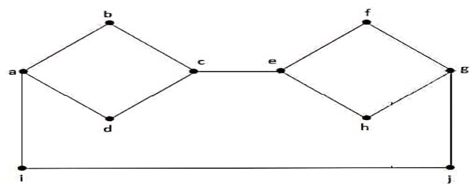

FIGURE 2.8: A connected graph.

(c) A path of G is a sequence $\psi_0,\ \psi_1,\ \psi_2,\ \ldots,\ \psi_n,\ \ldots$ with $(\psi_i,\ \psi_{i+1}) \in \mathsf{E}(G)$ for each $i \in \mathbb{N}$.

(d) G is connected if there is a path between every two vertices, and it is weakly connected if the corresponding undirected graph $\widetilde{G}$ is connected, where $\widetilde{G}$ is obtained from G by ignoring the direction of edges.

(e) G^{-1} be the graph obtained from G by reversing the direction of edges. Thus, $\mathsf{E}(G^{-1}) = \{(\psi,\ \theta) \in X \times X :\ \ (\theta,\ \psi) \in \mathsf{E}(G)\}$.

(f) $(V',\ \mathsf{E}')$ is called subgraph of G if $V' \subset V(G)$ and $\mathsf{E}' \subset \mathsf{E}(G)$ and for any edge $(\psi,\ \theta) \in \mathsf{E}',\ \psi, \theta \in V'$." [19]

In 2012, Wardowski [30] introduced a new type of contraction called F-contraction and proved a FP theorem concerning F-contraction.

Definition 2.4.4. "Let $\mathcal{F} : (0,\ \infty) \to \mathbb{R}$ be a function that satisfies the following conditions, as stated in:

(F1) For any $\alpha,\ \beta \in (0,\ \infty)$ s.t $\alpha < \beta$ then $\mathcal{F}(\alpha) < \mathcal{F}(\beta)$.

(F2) For any positive real sequence $\{\psi_q\}$,

$$\lim_{q\to\infty} \psi_q = 0 \text{ if and only if } \lim_{q\to\infty} \mathcal{F}(\psi_q) = -\infty.$$

(F3) There exists a constant $k \in (0,\ 1)$ s.t $\lim_{\alpha\to 0^+} \alpha^k \mathcal{F}(\alpha) = 0$.

(F4) For any subset $A \subset (0,\ \infty)$ with $\inf A > 0$, we possess $\mathcal{F}(\inf A) = \inf \mathcal{F}(A)$.

Throughout the thesis $\mathcal{J}$ represents the collection of functions that satisfy conditions (F1)-(F3) and $\mathcal{J}*$ as the collection of functions $\mathcal{F}$ that satisfy conditions (F1)-(F4)." [30]

In 2015, Definition (2.4.4) is extended by Cosentino et al. [31] for obtaining some FP results in bMS.

"Let (X, d_b) be a bMS and for $\mathcal{F} : (0,\ \infty) \to \mathbb{R}$ be a mapping and $s \geq 1$ be a real number.

If each sequence $\{\psi_q\}$ $q \in \mathbb{N}$ of positive numbers s.t $\gamma + \mathcal{F}(s\psi_q) \leq \mathcal{F}(\psi_{q-1})$ for all $q \in \mathbb{N}$ and some $\gamma > 0$, then

$$\gamma + \mathcal{F}(s^q \psi_q) \leq \mathcal{F}\left(s^{q-1}\psi_{q-1}\right) \qquad \text{for all } q \in \mathbb{N}." \tag{2.1}$$

Definition 2.4.5. "A function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ is considered a comparison function (CF) if it fulfills the following condition:

(a) Φ is strictly increasing.

(b) $\lim_{n\to\infty} \Phi^n(t) = 0$ for every $t \in \mathbb{R}^+$." [32]

In [33] Berinde has introduced the concept of c-comparison function (c)-CF by adding one more condition to comparison function.

Definition 2.4.6. "A function $\Phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is considered a (c)-CF if it satisfies the following conditions:

(a) Φ is monotonically increasing.

(b) $\lim_{n\to\infty} \Phi^n(t) = 0$ for every $t \in \mathbb{R}^+$.

(c) The series $\sum_{n=0}^{\infty} \Phi^n(t)$ is convergent for each $t \geq 0$." [33]

Chapter 3

Fixed Point Results in Metric Spaces via Graph Structure

This chapter centers around a comprehensive analysis of the paper [21], emphasizing the examination of variant multivalued mappings through a graph structure. To do this, rational-type multivalued G-contraction and multivalued $\mathcal{F}$-contractions in Ms endowed with graph [34] are introduced.

3.1 On Multivalued G-Contractions

Two significant outcomes in FP Theory include the BCP and the Tarski fixed point (TFP) theorem. Echenique [35] presented proof of the TFP theorem by employing a combination of FP techniques and graph theory. Subsequently, in [19], Jachymski introduced an alternative framework in the FP theory of MS by replacing order structures with graph structures on MS. In order to illustrate the relationship between ordered pairs of components in terms of their vertices and directed edges, FP theory and graph theory create an intersection between the theories of FP outcomes and graph.

Definition 3.1.1. [34] "Consider a MS (X, d) equipped with a graph G, where the vertex set $V(G)$ corresponds to X. Let $\Gamma : X \to CB(X)$ be a multivalued mapping. We say that Γ possesses the weakly graph-preserving (WGP) property if for every $\psi \in X$ and $\theta \in \Gamma x$ s.t $(\psi,\ \theta)$ belongs to the set of directed edges $\mathsf{E}(G)$, it follows that $(\theta,\ z)$ is an element of the set of directed edges $\mathsf{E}(G)$ for all $z \in \Gamma\theta$."

Lemma 3.1.2. Let $(X,\ \mathsf{d})$ be a MS and $\Gamma : X \to P(X)$ be an upper semi-continuous (USC) mapping s.t for every $r \in X$, the set Γr is closed. If $r_q \to r_0,\ t_q \to t_0$ and $t_q \in \Gamma r_q$, then $t_0 \in \Gamma r_0$. [34]

Next the notion of Multivalued G-contraction of type-I used by Acar et al. is defined then a FP theorem is proved.

Definition 3.1.3. Let $(X,\ \mathsf{d})$ be a CMS equipped with directed graph. Then $g : X \to CB(X)$ is called rational multivalued G-contraction of type-I if

$$H(g\psi, g\theta) \leq \Phi(N(\psi,\theta)), \quad \forall\ (\psi,\ \theta) \in \mathsf{E}(G), \tag{3.1}$$

where Φ is (c)-CF and,

$$N(\psi,\ \theta) = \max\left\{\mathsf{d}(\psi,\ \theta),\ \frac{D(\psi,\ g\psi) + D(\theta,\ g\theta)}{2},\ \frac{D(\psi,\ g\theta) + D(\theta,\ g\psi)}{2}\right.$$
$$\left.\frac{D(\psi,\ g\psi)D(\theta,\ g\theta)}{\mathsf{d}(\psi,\ \theta)},\ \frac{D(\theta,\ g\theta)[1 + D(\psi,\ g\psi)]}{1 + \mathsf{d}(\psi,\ \theta)}\right\}.$$

Theorem 3.1.4. Let $(X,\ \mathsf{d})$ be a CMS and $g : X \to CB(X)$ is USC and a weakly graph-preserving mapping satisfying the following conditions:

(a) g is rational multivalued G-contraction of type-I;

(b) $N_g = \{\psi \in X : (\psi,\ v) \in \mathsf{E}(G) \text{ for } v \in g\psi\}$ is non-empty,

then g has a FP.

Proof. Consider $\psi_0 \in N_g$, then $\exists\ \psi_1 \in g\psi_0$ s.t $(\psi_0,\ \psi_1) \in \mathsf{E}(G)$.
As g satisfies condition (a), therefore by (3.1.3)

$$\begin{aligned} D(\psi_1,\ g\psi_1) &\leq H(g\psi_0,\ g\psi_1) \\ &\leq \Phi(N(\psi_0,\ \psi_1)) \end{aligned}$$

$$
\begin{aligned}
&= \Phi \left(\max \left\{ \mathsf{d}(\psi_0, \psi_1), \frac{D(\psi_0, g\psi_0) + D(\psi_1, g\psi_1)}{2}, \right.\right. \\
&\quad \frac{D(\psi_0, g\psi_1) + D(\psi_1, g\psi_0)}{2}, \frac{D(\psi_0, g\psi_0) D(\psi_1, g\psi_1)}{\mathsf{d}(\psi_0, \psi_1)}, \\
&\quad \left.\left. \frac{D(\psi_0, g\psi_1)[1 + D(\psi_0, g\psi_0)]}{1 + \mathsf{d}(\psi_0, \psi_1)} \right\} \right) \\
&\leq \Phi \left(\max \left\{ \mathsf{d}(\psi_0, \psi_1), \frac{\mathsf{d}(\psi_0, \psi_1) + \mathsf{d}(\psi_1, \psi_2)}{2}, \right.\right. \\
&\quad \frac{\mathsf{d}(\psi_0, \psi_2) + \mathsf{d}(\psi_1, \psi_1)}{2}, \frac{\mathsf{d}(\psi_0, \psi_1) \cdot \mathsf{d}(\psi_1, \psi_2)}{\mathsf{d}(\psi_0, \psi_1)}, \\
&\quad \left.\left. \frac{\mathsf{d}(\psi_1, \psi_2)[1 + \mathsf{d}(\psi_0, \psi_1)]}{1 + \mathsf{d}(\psi_0, \psi_1)} \right\} \right) \\
&\leq \Phi \left(\max \left\{ \mathsf{d}(\psi_0, \psi_1), \frac{\mathsf{d}(\psi_0, \psi_1) + \mathsf{d}(\psi_1, \psi_2)}{2}, \mathsf{d}(\psi_1, \psi_2) \right\} \right) \\
&\leq \Phi \left(\max \left\{ \mathsf{d}(\psi_0, \psi_1), \mathsf{d}(\psi_1, \psi_2) \right\} \right) \\
&\leq \Phi \left(\mathsf{d}(\psi_0, \psi_1) \right)
\end{aligned}
$$

$$
\Rightarrow D(\psi_1, g\psi_1) \leq H(g\psi_0, g\psi_1) \leq \Phi(\mathsf{d}(\psi_0, \psi_1)).
$$

Let $\varrho > 1$ be an arbitrary constant so from Lemma (2.2.4) $\exists\ \psi_2 \in g\psi_1$ s.t,

$$
\mathsf{d}(\psi_1, \psi_2) \leq \sqrt{\varrho} H(g\psi_0, g\psi_1). \tag{3.2}
$$

So (3.2) can be written as

$$
\mathsf{d}(\psi_1, \psi_2) \leq \sqrt{\varrho}\Phi(\mathsf{d}(\psi_0, \psi_1)) < \varrho\Phi(\mathsf{d}(\psi_0, \psi_1)).
$$

Given that Φ is strictly increasing, it follows that

$$
0 < \Phi(\mathsf{d}(\psi_1, \psi_2)) < \Phi(\varrho\Phi(\mathsf{d}(\psi_0, \psi_1))).
$$

Consider $\varrho_1 = \dfrac{\Phi(\varrho\Phi(\mathsf{d}(\psi_0, \psi_1)))}{\Phi(\mathsf{d}(\psi_1, \psi_2))}$.
Now $\varrho_1 > 1$, applying the same procedure as above an iterative sequence can be obtained. Since $\psi_2 \in g\psi_1$, then by using WGP property, $(\psi_1, \psi_2) \in \mathsf{E}(G)$ so,

$$
\begin{aligned}
D(\psi_2, g\psi_2) &\leq H(g\psi_1, g\psi_2) \\
&\leq \Phi(N(\psi_1, \psi_2))
\end{aligned}
$$

$$
\begin{aligned}
&= \Phi\left(\max\left\{\mathsf{d}\left(\psi_1,\ \psi_2\right),\ \frac{D\left(\psi_1,\ g\psi_1\right)+D\left(\psi_2,\ g\psi_2\right)}{2},\right.\right.\\
&\qquad \frac{D\left(\psi_1,\ g\psi_2\right)+D\left(\psi_2,\ g\psi_1\right)}{2},\ \frac{D\left(\psi_1,\ g\psi_1\right)D\left(\psi_2,\ g\psi_2\right)}{\mathsf{d}\left(\psi_1,\ \psi_2\right)},\\
&\qquad \left.\left.\frac{D\left(\psi_2,\ g\psi_2\right)\left[1+D\left(\psi_1,\ g\psi_1\right)\right]}{1+\mathsf{d}\left(\psi_1,\ \psi_2\right)}\right\}\right)\\
&\le \Phi\left(\max\left\{\mathsf{d}\left(\psi_1,\ \psi_2\right),\ \frac{\mathsf{d}\left(\psi_1,\ \psi_2\right)+\mathsf{d}\left(\psi_2,\ \psi_3\right)}{2},\ \frac{\mathsf{d}\left(\psi_1,\ \psi_3\right)+\mathsf{d}\left(\psi_2,\ \psi_2\right)}{2},\right.\right.\\
&\qquad \left.\left.\frac{\mathsf{d}\left(\psi_1,\ \psi_2\right)\mathsf{d}\left(\psi_2,\ \psi_3\right)}{\mathsf{d}\left(\psi_1,\ \psi_2\right)},\ \frac{\mathsf{d}\left(\psi_2,\ \psi_3\right)\left[1+\mathsf{d}\left(\psi_1,\ \psi_2\right)\right]}{1+\mathsf{d}\left(\psi_1,\ \psi_2\right)}\right\}\right)\\
&= \Phi\left(\max\left\{\mathsf{d}\left(\psi_1,\ \psi_2\right),\ \frac{\mathsf{d}\left(\psi_1,\ \psi_2\right)+\mathsf{d}\left(\psi_2,\ \psi_3\right)}{2},\ \mathsf{d}\left(\psi_2,\ \psi_3\right)\right\}\right)\\
&= \Phi\left(\max\left\{\mathsf{d}\left(\psi_1,\ \psi_2\right),\ \mathsf{d}\left(\psi_2,\ \psi_3\right)\right\}\right)\\
&\le \Phi\left(\mathsf{d}\left(\psi_1,\ \psi_2\right)\right)\\
&< \sqrt{\varrho_1}\Phi\left(\mathsf{d}\left(\psi_1,\ \psi_2\right)\right).
\end{aligned}
$$

As, $\varrho_1 > 1$, so by Lemma (2.2.4), $\exists\ \ \psi_3 \in g\psi_2$ s.t

$$\mathsf{d}\left(\psi_2,\ \psi_3\right) \le \sqrt{\varrho_1}H\left(g\psi_1,\ g\psi_2\right) < \varrho_1\Phi\left(\mathsf{d}\left(\psi_2,\ \psi_1\right)\right) = \Phi\left(\varrho\Phi\left(\mathsf{d}\left(\psi_0,\ \psi_1\right)\right)\right).$$

Due to the strictly increasing property of Φ,

$$0 < \Phi\left(\mathsf{d}\left(\psi_2,\ \psi_3\right)\right) < \Phi^2\left(\varrho\Phi\left(\mathsf{d}\left(\psi_0,\ \psi_1\right)\right)\right).$$

Let

$$\varrho_2 = \frac{\Phi^2\left(\varrho\Phi\left(\mathsf{d}\left(\psi_0,\ \psi_1\right)\right)\right)}{\Phi\left(\mathsf{d}\left(\psi_2,\ \psi_3\right)\right)} > 1.$$

Continuing in the same way, a sequence $\{\psi_q\} \in X$ can be constructed so that $\psi_{q+1} \in g\psi_q$ s.t $(\psi_q,\ \psi_{q+1}) \in \mathsf{E}(G)$ and

$$\mathsf{d}\left(\psi_q,\ \psi_{q+1}\right) \le \Phi^q\left(\varrho\Phi\left(\mathsf{d}\left(\psi_0,\ \psi_1\right)\right)\right).$$

To prove that $\{\psi_q\}$ is a CS, take $p,\ q \in \mathbb{N}$ with $p > q$.

$$\mathsf{d}\left(\psi_q,\ \psi_p\right) \le \sum_{i=q}^{p-1}\mathsf{d}\left(\psi_i,\ \psi_{i+1}\right) \le \sum_{i=q}^{p-1}\Phi^i\left(\varrho\Phi\left(\mathsf{d}\left(\psi_0,\ \psi_1\right)\right)\right).$$

The R.H.S must be convergent because Φ is a (c)-CF therefore when $q,\ p \to \infty$ then

$$\mathsf{d}\left(\psi_q,\ \psi_p\right) \to 0.$$

Since $(X,\ \mathsf{d})$ is a complete MS, therefore,

$$\lim_{q\to\infty} \psi_q = \mu\ \in\ X.$$

As g is USC, so by using Lemma (3.1.2) $\mu \in g\mu$. Thus g has a FP. □

Consider the following property:

The (P)-property can be stated as follows: For any sequence $\{\psi_q\}$ in X, if ψ_q converges to ψ and $(\psi_q, \psi_{q+1}) \in \mathsf{E}(G)$, then there exists a subsequence $\{\psi_{q_k}\}$ s.t $(\psi_{q_k}, \psi) \in \mathsf{E}(G)$.

Definition 3.1.5. Let $(X,\ \mathsf{d})$ be a CMS equipped with directed graph G and Φ be a(c)-CF. Then $g : X \to CB(X)$ is called rational multivalued G-contraction of type-II if

$$H(g\psi, g\theta) \leq \Phi(N(\psi, \theta)), \quad \forall\ (\psi,\ \theta) \in \mathsf{E}(G), \tag{3.3}$$

$$N(\psi, \theta) = \max\left\{\mathsf{d}(\psi, \theta), \frac{D(\psi, g\psi) + D(\theta, g\theta)}{2}, \frac{D(\psi, g\theta) + D(\theta, g\psi)}{2}, \frac{D(\psi, g\psi)D(\theta, g\theta)}{1 + H(g\psi, g\theta)}\right\}$$

Theorem 3.1.6. Let $(X,\ \mathsf{d})$ be a CMS and $g : X \to CB(X)$ be a multivalued mapping satisfying the following conditions:

(a) g is rational multivalued G-contraction of type-II;

(b) $N_g = \{\psi \in X : (\psi,\ v) \in \mathsf{E}(G) \text{ for } v \in g\psi\}$ is non-empty;

(c) g satisfies the (P)-property;

(d) g is a weakly graph-preserving mapping.

Then g has a FP.

Proof. Consider $\psi_0 \in N_g$, there exists $\psi_1 \in g\psi_0$ s.t $(\psi_0,\ \psi_1) \in \mathsf{E}(G)$. According to given condition(a), for ψ_0 and ψ_1

$$\begin{aligned}
D(\psi_1,\ g\psi_1) &\leq H(g\psi_0,\ g\psi_1) \\
&\leq \Phi(M(\psi_0,\ \psi_1)) \\
&= \Phi\Bigg(\max\Bigg\{\mathsf{d}(\psi_0,\ \psi_1),\ \frac{D(\psi_0,\ g\psi_0)+D(\psi_1,\ g\psi_1)}{2}, \\
&\qquad \frac{D(\psi_0,\ g\psi_1)+D(\psi_1,\ g\psi_0)}{2},\ \frac{D(\psi_0,\ g\psi_0)\,D(\psi_1,\ g\psi_1)}{1+H(g\psi_0,\ g\psi_1)}\Bigg\}\Bigg) \\
&\leq \Phi\Bigg(\max\Bigg\{\mathsf{d}(\psi_0,\ \psi_1),\ \frac{\mathsf{d}(\psi_0,\ \psi_1)+\mathsf{d}(\psi_1,\ \psi_2)}{2}, \\
&\qquad \frac{\mathsf{d}(\psi_0,\ \psi_2)+\mathsf{d}(\psi_1,\ \psi_1)}{2},\ \frac{\mathsf{d}(\psi_0,\ \psi_1)\,\mathsf{d}(\psi_1,\ \psi_2)}{D(\psi_1,\ g\psi_1)}\Bigg\}\Bigg) \\
&\leq \Phi\Bigg(\max\Bigg\{\mathsf{d}(\psi_0,\ \psi_1),\ \frac{\mathsf{d}(\psi_0,\ \psi_1)+\mathsf{d}(\psi_1,\ \psi_2)}{2}, \\
&\qquad \frac{\mathsf{d}(\psi_0,\ \psi_2)}{2},\ \frac{\mathsf{d}(\psi_0,\ \psi_1)\,\mathsf{d}(\psi_1,\ \psi_2)}{\mathsf{d}(\psi_1,\ \psi_2)}\Bigg\}\Bigg) \\
&\leq \Phi\Bigg(\max\Bigg\{\mathsf{d}(\psi_0,\ \psi_1),\ \frac{\mathsf{d}(\psi_0,\ \psi_1)+\mathsf{d}(\psi_1,\ \psi_2)}{2},\ \mathsf{d}(\psi_1,\ \psi_2)\Bigg\}\Bigg) \\
&\leq \Phi(\max\{\mathsf{d}(\psi_0,\ \psi_1),\ \mathsf{d}(\psi_1,\ \psi_2)\}) \\
&\leq \Phi(\mathsf{d}(\psi_0,\ \psi_1)) \\
D(\psi_1,\ g\psi_1) &\leq H(g\psi_0,\ g\psi_1) \leq \Phi(\mathsf{d}(\psi_0,\ \psi_1)).
\end{aligned}$$

Suppose $\varrho > 1$ be an arbitrary constant so from Lemma (2.2.4) $\exists\ \psi_2 \in g\psi_1$ s.t

$$\mathsf{d}(\psi_1,\ \psi_2) \leq \sqrt{\varrho} H(g\psi_1,\ g\psi_2) < \varrho\Phi(\mathsf{d}(\psi_0,\ \psi_1)).$$

As Φ is strictly increasing, so

$$0 < \Phi(\mathsf{d}(\psi_1,\ \psi_2)) < \Phi(\varrho\Phi(\mathsf{d}(\psi_0,\ \psi_1))).$$

Take $\varrho_1 = \dfrac{\Phi(\varrho\Phi(\mathsf{d}(\psi_0,\ \psi_1)))}{\Phi(\mathsf{d}(\psi_1,\ \psi_2))} > 1.$
In view of $(\psi_0,\ \psi_1) \in \mathsf{E}(G)$, $\psi_1 \in g\psi_0$, $\psi_2 \in g\psi_1$ and by using *WGP*, $(\psi_1,\ \psi_2) \in$

$\mathsf{E}(G)$. Therefore,

$$
\begin{aligned}
D(\psi_2,\ g\psi_2) &\le H(g\psi_1,\ g\psi_2)\\
&\le \Phi(N(\psi_1,\ \psi_2))\\
&= \Phi\left(\max\left\{\mathsf{d}(\psi_1,\ \psi_2),\ \frac{D(\psi_1,\ g\psi_1)+D(\psi_2,\ g\psi_2)}{2},\right.\right.\\
&\qquad \left.\left.\frac{D(\psi_1,\ g\psi_2)+D(\psi_2,\ g\psi_1)}{2},\ \frac{D(\psi_1,\ g\psi_1)\,D(\psi_2,\ g\psi_2)}{1+H(g\psi_1,\ g\psi_2)}\right\}\right)\\
&\le \Phi\left(\max\left\{\mathsf{d}(\psi_1,\ \psi_2),\ \frac{\mathsf{d}(\psi_1,\ \psi_2)+\mathsf{d}(\psi_2,\ \psi_3)}{2},\right.\right.\\
&\qquad \left.\left.\frac{\mathsf{d}(\psi_1,\ \psi_3)+\mathsf{d}(\psi_2,\ \psi_2)}{2},\ \frac{\mathsf{d}(\psi_1,\ \psi_2)\,\mathsf{d}(\psi_2,\ \psi_3)}{D(\psi_2,\ g\psi_2)}\right\}\right)\\
&= \Phi\left(\max\left\{\mathsf{d}(\psi_1,\ \psi_2),\ \frac{\mathsf{d}(\psi_1,\ \psi_2)+\mathsf{d}(\psi_2,\ \psi_3)}{2},\ \mathsf{d}(\psi_2,\ \psi_3)\right\}\right)\\
&= \Phi(\max\{\mathsf{d}(\psi_1,\ \psi_2),\ \mathsf{d}(\psi_2,\ \psi_3)\})\\
&\le \Phi(\mathsf{d}(\psi_1,\ \psi_2))\\
&< \sqrt{\varrho_1}\Phi(\mathsf{d}(\psi_1,\ \psi_2)).
\end{aligned}
$$

From Lemma(2.2.4), $\exists\ \psi_3 \in g\psi_2$ s.t

$$
\begin{aligned}
\mathsf{d}(\psi_2,\ \psi_3) &\le \sqrt{\varrho_1}H(g\psi_1,\ g\psi_2) < \varrho_1\Phi(\mathsf{d}(\psi_2,\ \psi_1))\\
&= \Phi(\varrho\Phi(\mathsf{d}(\psi_0,\ \psi_1))).\\
\implies \mathsf{d}(\psi_2,\ \psi_3) &\le \Phi(\varrho\Phi(\mathsf{d}(\psi_0,\ \psi_1))).
\end{aligned}
$$

AS Φ is strictly increasing, so

$$0 < \Phi(\mathsf{d}(\psi_2,\ \psi_3)) < \Phi^2(\varrho\Phi(\mathsf{d}(\psi_0,\ \psi_1))).$$

Consider

$$\varrho_2 = \frac{\Phi^2(\varrho\Phi(\mathsf{d}(\psi_0,\ \psi_1)))}{\Phi(\mathsf{d}(\psi_2,\ \psi_3))} > 1.$$

Continuing similarly, we construct a sequence $\{\psi_q\} \in X$ s.t $\psi_{q+1} \in g\psi_q$ and $(\psi_q,\ \psi_{q+1}) \in \mathsf{E}(G)$,

$$\mathsf{d}(\psi_q,\ \psi_{q+1}) \le \Phi^q(\varrho\Phi(\mathsf{d}(\psi_0,\ \psi_1))).$$

Next, our goal is to show that $\{\psi_q\}$ is a CS. Consider p and q as natural numbers with $p > q$. By triangular inequality,

$$\mathsf{d}\left(\psi_q,\ \psi_p\right) \leq \sum_{i=q}^{p-1} \mathsf{d}\left(\psi_i,\ \psi_{i+1}\right) \leq \sum_{i=q}^{p-1} \Phi^i\left(\varrho\Phi\left(\mathsf{d}\left(\psi_0,\ \psi_1\right)\right)\right).$$

The R.H.S must be convergent because Φ is a (c)-CF therefore when $q,\ p \to \infty$ then $\mathsf{d}\left(\psi_q,\ \psi_p\right) \to 0$. Hence $\{\psi_q\}$ is a CS in the MS, which is complete. Therefore, $\lim_{q\to\infty} \psi_q = \mu$. As (P)-property is satisfied, so $\exists$ a subsequence $\{\psi_{q_k}\}$ of $\{\psi_q\}$ s.t $(\psi_{q_k},\ \mu) \in \mathsf{E}(G)$ for each $k \in \mathbb{N}$. Suppose $D(\mu,\ g\mu) > 0$, so that

$$\lim_{q\to\infty} D\left(\psi_{q_k},\ \psi_{q_{k+1}}\right) = 0,$$

$$\lim_{q\to\infty} D\left(\psi_{q_k},\ \mu\right) = 0.$$

$$D\left(\psi_{q_k},\ \psi_{q_{k+1}}\right) < \frac{1}{3} D(\mu,\ g\mu), \tag{3.4}$$

for $q_0 \in \mathbb{N}$ s.t $q_k > q_0$. Furthermore, there exists $q_1 \in \mathbb{N}$ s.t for any $q_k > q_1$

$$D\left(\psi_q,\ \mu\right) < \frac{1}{3} D(\mu,\ g\mu). \tag{3.5}$$

Consider $q_k > \max\{q_0,\ q_1\}$, so that

$$\begin{aligned}
D\left(\psi_{q_{k+1}},\ g\mu\right) &\leq H\left(g\psi_{q_k},\ g\mu\right) \\
&\leq \Phi(N\left(\psi_{q_k},\ \mu\right)) \\
&\leq \Phi\left(\max\left\{\mathsf{d}\left(\psi_{q_k},\ \mu\right),\ \frac{D\left(\psi_{q_k},\ g\psi_{q_k}\right) + D(\mu,\ g\mu)}{2},\right.\right. \\
&\qquad \left.\left.\frac{D\left(\psi_{q_k},\ g\mu\right) + D\left(\mu,\ g\psi_{q_k}\right)}{2},\ \frac{D\left(\psi_{q_k},\ g\psi_{q_k} D(\mu,\ g\mu)\right)}{1 + H\left(g\psi_{n_k},\ g\mu\right)}\right\}\right) \\
&\leq \Phi\left(\max\left\{\frac{1}{3} D(\mu,\ g\mu),\ \frac{\frac{1}{3} D(\mu,\ g\mu) + D(\mu,\ g\mu)}{2},\right.\right. \\
&\qquad \left.\left.\frac{D\left(\psi_{q_k},\ g\mu\right) + D\left(\mu,\ g\psi_{q_k}\right)}{2},\ \frac{\frac{1}{3} D(\mu,\ g\mu) D(\mu,\ g\mu)}{D\left(\psi_{q_{k+1}},\ g\mu\right)}\right\}\right)
\end{aligned}$$

Now, take $k \to \infty$, then $D(\mu,\ g\mu) \leq \Phi(D(\mu,\ g\mu)) < D(\mu,\ g\mu)$, which is a contradiction. So $D(\mu,\ g\mu) = 0$ and since $g\mu$ is closed, so $\mu \in g\mu$. Hence g admits a FP. □

Theorem 3.1.7. Suppose $(X,\ \mathsf{d})$ is a CMS with a directed graph G and a multivalued mapping $g : X \to K(X)$. Suppose g be a USC and a weakly graph-preserving mapping. Assume that

(a) There is a (c)-CF Φ s.t

$$H(g\psi,\ g\theta) \leq \Phi(N(\psi,\ \theta))\ \forall\ (\psi,\ \theta) \in \mathsf{E}(G).$$

where $N(\psi,\theta)$ is same as in Theorem (3.1.6)

(b) N_g is non-empty.

So that, g has a FP.

Proof. Suppose that $\psi_0 \in N_g,\ \ \psi_1 \in g\psi_0$ s.t $(\psi_0,\ \psi_1) \in \mathsf{E}(G)$. So, by condition (a)

$$\begin{aligned}
D(\psi_1,\ g\psi_1) &\leq H(g\psi_0,\ g\psi_1) \\
&\leq \Phi(N(\psi_0,\ \psi_1)) \\
&= \Phi\left(\max\left\{\mathsf{d}(\psi_0,\ \psi_1),\ \frac{D(\psi_0,\ g\psi_0) + D(\psi_1,\ g\psi_1)}{2},\right.\right. \\
&\qquad \left.\left.\frac{D(\psi_0,\ g\psi_1) + \mathsf{d}(\psi_1,\ g\psi_0)}{2},\ \frac{D(\psi_0,\ g\psi_0)\, D(\psi_1,\ g\psi_1)}{1 + H(g\psi_0,\ g\psi_1)}\right\}\right) \\
&\leq \Phi\left(\max\left\{\mathsf{d}(\psi_0,\ \psi_1),\ \frac{\mathsf{d}(\psi_0,\ \psi_1) + \mathsf{d}(\psi_1,\ \psi_2)}{2},\right.\right. \\
&\qquad \left.\left.\frac{\mathsf{d}(\psi_0,\ \psi_2) + \mathsf{d}(\psi_1,\ \psi_1)}{2},\ \frac{\mathsf{d}(\psi_0,\ \psi_1)\,\mathsf{d}(\psi_1,\ \psi_2)}{D(\psi_1,\ g\psi_1)}\right\}\right) \\
&\leq \Phi\left(\max\left\{\mathsf{d}(\psi_0,\ \psi_1),\ \frac{\mathsf{d}(\psi_0,\ \psi_1) + \mathsf{d}(\psi_1,\ \psi_2)}{2},\ \mathsf{d}(\psi_1;\psi_2)\right\}\right) \\
&\leq \Phi(\mathsf{d}(\psi_0,\ \psi_1)).
\end{aligned}$$

since $g\psi_1$ is compact then $\exists\ \psi_2 \in g\psi_1$ and $\mathsf{d}(\psi_1,\ \psi_2) = D(\psi_1,\ g\psi_1)$ so

$$\mathsf{d}(\psi_1,\ \psi_2) \leq \Phi(\mathsf{d}(\psi_0,\ \psi_1)).$$

Since $(\psi_0,\ \psi_1) \in E(G)$, $\psi_1 \in g\psi_0$ and $\psi_2 \in g\psi_1$, using weakly graph-preserving property, $(\psi_1,\ \psi_2) \in E(G)$. Then

$$\begin{aligned} D(\psi_2,\ g\psi_2) &\leq H(g\psi_1,\ g\psi_2) \\ &\leq \Phi(N(\psi_1,\ \psi_2)) \\ &\leq \Phi(\mathsf{d}(\psi_1,\ \psi_2)). \end{aligned}$$

Again by the compactness of $g\psi_2$, $\exists\ \psi_3 \in g\psi_2$ s.t $\mathsf{d}(\psi_2,\ \psi_3) = D(\psi_2,\ g\psi_2)$. Therefore

$$\mathsf{d}(\psi_2,\ \psi_3) \leq \Phi(\mathsf{d}(\psi_1,\ \psi_2)).$$

So a sequence $\{\psi_q\}$ in X can be constructed s.t $\psi_{q+1} \in g\psi_q,\ (\psi_q,\ \psi_{q+1}) \in \mathsf{E}(G)$, and

$$\begin{aligned} \mathsf{d}(\psi_q, \psi_{q+1}) &\leq \Phi(\mathsf{d}(\psi_{q-1},\ \psi_q)) \\ &\leq \Phi^2(\mathsf{d}(\psi_{q-2},\ \psi_{q-1})) \\ &\ \ \vdots \\ &\leq \Phi^q(\mathsf{d}(\psi_0,\ \psi_1)). \end{aligned}$$

Thus

$$\mathsf{d}(\psi_q,\ \psi_{q+1}) \leq \Phi^q(\mathsf{d}(\psi_0,\ \psi_1)).$$

To show that $\{\psi_q\}$ is a CS. Let $p,\ q \in \mathbb{N}$ and $p > q$. Then by triangular inequality,

$$\begin{aligned} \mathsf{d}(\psi_q,\ \psi_p) &\leq \sum_{i=q}^{p-1} \mathsf{d}(\psi_i,\ \psi_{i+1}) \\ &\leq \sum_{i=q}^{p-1} \Phi^q(\mathsf{d}(\psi_0,\ \psi_1)). \end{aligned}$$

The R.H.S must be convergent because Φ is a (c)-CF. Therefore, $\mathsf{d}(\psi_q,\ \psi_m) \to 0$ as $q,\ p \to \infty$.

So $\{\psi_q\}$ is a CS in $(X,\ \mathsf{d})$ which is a CMS. Therefore, $\lim_{q\to\infty} \psi_q = \mu \in X$.

As g is USC, so by using Lemma (3.1.2), it follows that $\mu \in g\mu$. In other words, g admits a FP. □

3.2 On $\mathcal{F}$-Contractions

In this section FP theorems are examined and elaborated for $\mathcal{F}$-contractions. For this, some sets are defined here.

Let $(X,\ \mathsf{d})$ be a MS and G be a directed graph on X and a mapping $\Gamma : X \to CB(X)$. Define

$$\Gamma_G \equiv \{(\psi,\ \theta) \in \mathsf{E}(G) : H(\Gamma\psi,\ \Gamma\theta) > 0\},$$
$$X_\Gamma = \{\psi \in X : (\psi,\ \theta) \in \mathsf{E}(G) \text{ for some } \theta \in \Gamma\psi\},$$

and

$$L(\psi,\ \theta) = \max\left\{ \begin{array}{c} \mathsf{d}(\psi,\ \theta),\ D(\psi,\ \Gamma\psi),\ D(\theta,\ \Gamma\theta), \\ \dfrac{D(\psi,\ \Gamma\psi)\,D(\psi,\ \Gamma\theta) + D(\theta,\ \Gamma\theta)\,D(\theta,\ \Gamma\psi)}{\max\{D(\psi,\ \Gamma\theta),\ D(\theta,\ \Gamma\psi)\}} \end{array} \right\},$$

with $\max\{D(\psi,\Gamma\theta), D(\theta,\Gamma\psi)\} \neq 0$.

Now here is the definition of $\mathcal{F}$-contraction.

Definition 3.2.1. [36] "Consider (X, d) be a MS and a mapping $\Gamma : X \to CB(X)$. Then Γ is called a multivalued $\mathcal{F}$-contraction if there exist $\mathcal{F} \in \mathcal{J}$ and $\gamma > 0$ s.t

$$\gamma + \mathcal{F}(H(\Gamma\psi,\ \Gamma\theta)) \leq \mathcal{F}(L(\psi,\ \theta)) \tag{3.6}$$

for all $\psi,\ \theta \in X$ with $(\psi,\ \theta) \in \Gamma_G$."

Theorem 3.2.2. Consider a multivalued $\mathcal{F}$-contraction $\Gamma : X \to K(X)$ on $(X,\ \mathsf{d})$ which is a complete MS with a directed graph G. If X_Γ is non-empty then Γ admits a FP.

Proof. To prove that Γ has a FP, we on contrary assume that Γ has no FP then, $D(\psi,\ \Gamma\psi) > 0\ \forall\ \psi \in X$. Consider $\psi_0 \in X_\Gamma$, then $(\psi_0,\ \psi_1) \in \mathsf{E}(G)$ for any $\psi_1 \in \Gamma\psi_0$, and

$$0 < D(\psi_1,\ \Gamma\psi_1) \leq H(\Gamma\psi_0,\ \Gamma\psi_1).$$

So $(\psi_0,\ \psi_1) \in \Gamma_G$. By using (3.6) for ψ_0 and ψ_1

$$\begin{aligned}
\mathcal{F}\left(D\left(\psi_1,\ \Gamma\psi_1\right)\right) &\leq \mathcal{F}\left(H\left(\Gamma\psi_0,\ \Gamma\psi_1\right)\right)\\
&\leq \mathcal{F}\left(L\left(\psi_0,\ \psi_1\right)\right) - \gamma\\
&= \mathcal{F}\Bigg(\max\Bigg\{\mathsf{d}\left(\psi_0,\ \psi_1\right),\ D\left(\psi_0,\ \Gamma\psi_0\right),\ D\left(\psi_1,\ \Gamma\psi_1\right),\\
&\qquad \frac{D\left(\psi_0,\ \Gamma\psi_0\right)D\left(\psi_0,\ \Gamma\psi_1\right)+D\left(\psi_1,\ \Gamma\psi_1\right)D\left(\psi_1,\ \Gamma\psi_0\right)}{\max\left\{D\left(\psi_0,\ \Gamma\psi_1\right),\ D\left(\psi_1,\ \Gamma\psi_0\right)\right\}}\Bigg\}\Bigg) - \gamma\\
&\leq \mathcal{F}\Bigg(\max\Bigg\{\mathsf{d}\left(\psi_0,\ \psi_1\right),\ \mathsf{d}\left(\psi_1,\ \psi_2\right),\\
&\qquad \frac{\mathsf{d}\left(\psi_0,\ \psi_1\right)\mathsf{d}\left(\psi_0,\ \psi_2\right)+\mathsf{d}\left(\psi_1,\ \psi_2\right)\mathsf{d}\left(\psi_1,\ \psi_1\right)}{\max\left\{\mathsf{d}\left(\psi_0,\ \psi_2\right),\ \mathsf{d}\left(\psi_1,\ \psi_1\right)\right\}}\Bigg\}\Bigg) - \gamma\\
&\leq \mathcal{F}\left(\max\left\{\mathsf{d}\left(\psi_0,\ \psi_1\right),\ \mathsf{d}\left(\psi_1,\ \psi_2\right)\right\}\right) - \gamma\\
&\leq \mathcal{F}\left(\mathsf{d}\left(\psi_0,\ \psi_1\right)\right) - \gamma.
\end{aligned}$$

By Compactness of $\Gamma\psi_1$, there is, $\psi_2 \in \Gamma\psi_1$ s.t $\mathsf{d}\left(\psi_1,\ \psi_2\right) = D\left(\psi_1,\ \Gamma\psi_1\right)$. Then

$$\mathcal{F}\left(\mathsf{d}\left(\psi_1,\ \psi_2\right)\right) \leq \mathcal{F}\left(\mathsf{d}\left(\psi_0,\ \psi_1\right)\right) - \gamma.$$

Since $(\psi_0,\ \psi_1) \in \mathsf{E}(G)$, $\psi_1 \in \Gamma\psi_0$ and $\psi_2 \in \Gamma\psi_1$, by using the property of weakly graph-preserving, $(\psi_1,\ \psi_2) \in \mathsf{E}(G)$, and $(\psi_1,\ \psi_2) \in \Gamma_G$. Now, proceeding similarly,

$$0 < D\left(\psi_2,\ \Gamma\psi_2\right) \leq H\left(\Gamma\psi_1,\ \Gamma\psi_2\right),$$

$$\begin{aligned}
\mathcal{F}\left(D\left(\psi_2,\ \Gamma\psi_2\right)\right) &\leq \mathcal{F}\left(H\left(\Gamma\psi_1,\ \Gamma\psi_2\right)\right)\\
&\leq \mathcal{F}\left(L\left(\psi_1,\ \psi_2\right)\right) - \gamma\\
&= \mathcal{F}\Bigg(\max\Bigg\{\mathsf{d}\left(\psi_1,\ \psi_2\right),\ D\left(\psi_1,\ \Gamma\psi_1\right),\ D\left(\psi_2,\ \Gamma\psi_2\right),\\
&\qquad \frac{D\left(\psi_1,\ \Gamma\psi_1\right)D\left(\psi_1,\ \Gamma\psi_2\right)+D\left(\psi_2,\ \Gamma\psi_2\right)D\left(\psi_2,\ \Gamma\psi_1\right)}{\max\left\{D\left(\psi_1,\ \Gamma\psi_2\right),\ D\left(\psi_2,\ \Gamma\psi_1\right)\right\}}\Bigg\}\Bigg) - \gamma\\
&\leq \mathcal{F}\Bigg(\max\Bigg\{\mathsf{d}\left(\psi_1,\ \psi_2\right),\ \mathsf{d}\left(\psi_2,\ \psi_3\right)\\
&\qquad \frac{\mathsf{d}\left(\psi_1,\ \psi_2\right)\mathsf{d}\left(\psi_1,\ \psi_3\right)+\mathsf{d}\left(\psi_2,\ \psi_3\right)\mathsf{d}\left(\psi_2,\ \psi_2\right)}{\max\left\{\mathsf{d}\left(\psi_1,\ \psi_3\right),\ \mathsf{d}\left(\psi_2,\ \psi_2\right)\right\}}\Bigg\}\Bigg) - \gamma\\
&\leq \mathcal{F}\left(\max\left\{\mathsf{d}\left(\psi_1,\ \psi_2\right),\ \mathsf{d}\left(\psi_2,\ \psi_3\right)\right\}\right) - \gamma\\
&\leq \mathcal{F}\left(\mathsf{d}\left(\psi_1,\ \psi_2\right)\right) - \gamma.
\end{aligned}$$

The compactness of $\Gamma\psi_2$ implies that $\psi_3 \in \Gamma\psi_2$ s.t $\mathsf{d}(\psi_2, \psi_3) = D(\psi_2, \Gamma\psi_2)$ so

$$\mathcal{F}(\mathsf{d}(\psi_2, \psi_3)) \leq \mathcal{F}(\mathsf{d}(\psi_1, \psi_2)) - \gamma. \tag{3.7}$$

So a sequence $\{\psi_q\} \in X$ can be constructed s.t $\psi_{q+1} \in \Gamma\psi_q$, $(\psi_q, \psi_{q+1}) \in \Gamma_G$ and

$$\mathcal{F}(\mathsf{d}(\psi_q, \psi_{q+1})) \leq \mathcal{F}(\mathsf{d}(\psi_{q-1}, \psi_q)) - \gamma \quad \forall\, q \in \mathbb{N}. \tag{3.8}$$

Let us assume that $\tau_q = \mathsf{d}(\psi_q, \psi_{q+1})$, where ψ_q and ψ_{q+1} are elements of a metric space. So $\tau_q > 0$ and $\{\tau_q\}$ is a decreasing sequence of real numbers, there exists $\omega \geqslant 0$ s.t $\lim_{q\to\infty} \tau_q = \omega$.

$$\begin{aligned}\mathcal{F}(\tau_q) &\leq \mathcal{F}(\tau_{q-1}) - \gamma \\ &\leq \mathcal{F}(\tau_{q-2}) - 2\gamma \\ &\ \ \vdots \\ &\leq \mathcal{F}(\tau_0) - q\gamma\end{aligned}$$

$$\mathcal{F}(\tau_q) \leq \mathcal{F}(\tau_0) - q\gamma. \tag{3.9}$$

Now $\lim_{q\to+\infty} \mathcal{F}(\tau_q) = -\infty$, therefore

$$\omega = \lim_{q\to\infty} \tau_q = 0.$$

Due to $(F3)$, there is a constant $k \in (0, 1)$ s.t, $\lim_{q\to\infty} \tau_q^k \mathcal{F}(\tau_q) = 0$. Then by (3.9)

$$\tau_q^k \mathcal{F}(\tau_q) - \tau_q^k \mathcal{F}(\tau_0) \leq -\tau_q^k q\gamma \leq 0. \tag{3.10}$$

Which is true $\forall\, q \in \mathbb{N}$. Suppose $q \to \infty$, then $\lim_{q\to\infty} q\tau_q^k = 0$. From (3.10), suppose there is a $q_1 \in \mathbb{N}$ then $q\tau_q^k \leq 1\ \forall\, q \geqslant q_1$. Thus

$$\tau_q \leq \frac{1}{q^{1/k}} \quad \forall\, q \geqslant q_1. \tag{3.11}$$

To show $\{\psi_q\}$ is a CS, suppose that $p,\ q \in \mathbb{N}$ and $p > q \geqslant q_1$. Then,

$$\begin{aligned}
\mathsf{d}\left(\psi_q,\ \psi_p\right) &\leq \mathsf{d}\left(\psi_q,\ \psi_{q+1}\right) + \mathsf{d}\left(\psi_{q+1},\ \psi_{q+2}\right) + \cdots + \mathsf{d}\left(\psi_{p-1},\ \psi_p\right) \\
&= \tau_q + \tau_{q+1} + \tau_{q+2} + \cdots + \tau_{p-1} \\
&= \sum_{i=q}^{p-1} \tau_i \\
&\leq \sum_{i=q}^{\infty} \tau_i \\
&\leq \sum_{i=q}^{\infty} \left(\frac{1}{i^{1/k}}\right) \\
\Longrightarrow \mathsf{d}\left(\psi_q,\ \psi_p\right) &\leq \sum_{i=q}^{\infty} \left(\frac{1}{i^{1/k}}\right).
\end{aligned}$$

As $k \in (0,\ 1)$, then $\sum_{i=1}^{\infty} \frac{1}{i^{1/k}}$ is convergent. So $\mathsf{d}\left(\psi_q,\ \psi_p\right) \to 0$ as $q,\ p \to \infty$, this implies that $\{\psi_q\}$ is a CS in the MS, which is complete. So it converges to some $\mu \in X$. Using the USC of Γ and from Lemma (2.2.4), it follows that $\mu \in \Gamma_\mu$. However, this contradicts our initial assumption. Hence, Γ has a FP. □

Theorem 3.2.3. Consider $(X,\ \mathsf{d})$ a complete MS, with a digraph G. Let $\Gamma : X \to K(X)$ be a multivalued $\mathcal{F}$-contraction, where $\mathcal{F} \in \mathcal{J}_*$. If the set X_Γ is non-empty, then Γ has a FP.

Proof. Let Γ does not have a FP, then $D(\psi,\ \Gamma\psi) > 0$, for every $\psi \in X$. Let ψ_0 be an element in X_Γ. Consequently, there exists $(\psi_0,\ \psi_1) \in \mathsf{E}(G)$ for some $\psi_1 \in \Gamma\psi_0$. Thus

$$0 < D\left(\psi_1,\ \Gamma\psi_1\right) \leq H\left(\Gamma\psi_0,\ \Gamma\psi_1\right).$$

Hence, $(\psi_0,\ \psi_1) \in \Gamma_G$. By $\mathcal{F}$-contraction condition (3.6)

$$\begin{aligned}
\mathcal{F}\left(D\left(\psi_1,\ \Gamma\psi_1\right)\right) &\leq \mathcal{F}\left(H\left(\Gamma\psi_0,\ \Gamma\psi_1,\ \right)\right) \\
&\leq \mathcal{F}\left(L\left(\psi_0,\ \psi_1\right)\right) - \frac{\gamma}{2} \\
&= \mathcal{F}\left(\max\left\{\mathsf{d}\left(\psi_0,\ \psi_1\right),\ D\left(\psi_0,\ \Gamma\psi_0\right),\ D\left(\psi_1,\ \Gamma\psi_1\right)\right\}\right. \\
&\qquad \left.\left.\frac{D\left(\psi_0,\ \Gamma\psi_0\right) D\left(\psi_0,\ \Gamma\psi_1\right) + D\left(\psi_1,\ \Gamma\psi_1\right) D\left(\psi_1,\ \Gamma\psi_0\right)}{\max\left\{D\left(\psi_0,\ \Gamma\psi_1\right),\ D\left(\psi_1,\ \Gamma\psi_0\right)\right\}}\right\}\right) - \frac{\gamma}{2}
\end{aligned}$$

$$\leq \mathcal{F}\left(\max\left\{\mathsf{d}\left(\psi_0,\ \psi_1\right),\ \mathsf{d}\left(\psi_1,\ \psi_2\right),\ \frac{\mathsf{d}\left(\psi_0,\ \psi_1\right)\mathsf{d}\left(\psi_0,\ \psi_2\right)+\mathsf{d}\left(\psi_1,\ \psi_2\right)\mathsf{d}\left(\psi_1,\ \psi_1\right)}{\max\left\{\mathsf{d}\left(\psi_0,\ \psi_1\right),\ \mathsf{d}\left(\psi_1,\ \psi_1\right)\right\}}\right\}\right)-\frac{\gamma}{2}$$
$$\leq \mathcal{F}\left(\max\left\{\mathsf{d}\left(\psi_0,\ \psi_1\right),\ \mathsf{d}\left(\psi_1,\ \psi_2\right)\right\}\right)-\frac{\gamma}{2}$$
$$\leq \mathcal{F}\left(\mathsf{d}\left(\psi_0,\ \psi_1\right)\right)-\frac{\gamma}{2}.$$

In view of (F_4),

$$\mathcal{F}\left(D\left(\psi_1,\ \Gamma\psi_1\right)\right)=\mathcal{F}\left(\inf\left\{\mathsf{d}\left(\psi_1,\ v\right): v\in\Gamma\psi_1\right\}\right)$$
$$=\inf\left\{\mathcal{F}\left(\mathsf{d}\left(\psi_1,\ v\right): v\in\Gamma\psi_1\right)\right\}\leq \mathcal{F}\left(\mathsf{d}\left(\psi_0,\ \psi_1\right)\right)-\frac{\gamma}{2}.$$

Due to the compactness of $\Gamma\psi_1$, there is $\psi_2\in\Gamma\psi_1$ s.t $\mathsf{d}\left(\psi_1,\ \psi_2\right)=D\left(\psi_1,\ \Gamma\psi_1\right)$. So

$$\mathcal{F}\left(\mathsf{d}\left(\psi_1,\ \psi_2\right)\right)\leq \mathcal{F}\left(\mathsf{d}\left(\psi_0,\ \psi_1\right)\right)-\frac{\gamma}{2}. \tag{3.12}$$

Since $(\psi_0,\ \psi_1)\in\mathsf{E}(G)$, $\psi_1\in\Gamma\psi_0$ and $\psi_2\in\Gamma\psi_1$. By *WGP* property $(\psi_1,\ \psi_2)\in\mathsf{E}(G)$, and $0<D\left(\psi_2,\ \Gamma\psi_2\right)\leq H\left(\Gamma\psi_1,\ \Gamma\psi_2\right)$. So $(\psi_1,\ \psi_2)$ belongs to Γ_G, then

$$\mathcal{F}\left(D\left(\psi_2,\ \Gamma\psi_2\right)\right)\leq \mathcal{F}\left(H\left(\Gamma\psi_1,\ \Gamma\psi_2\right)\right)$$
$$\leq \mathcal{F}\left(L\left(\psi_1,\ \psi_2\right)\right)-\frac{\gamma}{2}$$
$$=\mathcal{F}\left(\max\left\{\mathsf{d}\left(\psi_1,\ \psi_2\right),\ D\left(\psi_1,\ \Gamma\psi_1\right),\ D\left(\psi_2,\ \Gamma\psi_2\right),\ \frac{D\left(\psi_1,\ \Gamma\psi_1\right)D\left(\psi_1,\ \Gamma\psi_2\right)+D\left(\psi_2,\ \Gamma\psi_2\right)D\left(\psi_2,\ \psi_1\right)}{\max\left\{D\left(\psi_1,\ \Gamma\psi_2\right),\ D\left(\psi_2,\ \Gamma\psi_1\right)\right\}}\right\}\right)-\frac{\gamma}{2}$$
$$\leq \mathcal{F}\left(\max\left\{\mathsf{d}\left(\psi_1,\ \psi_2\right),\ \mathsf{d}\left(\psi_2,\ \psi_3\right),\ \frac{\mathsf{d}\left(\psi_1,\ \psi_2\right)\mathsf{d}\left(\psi_1,\ \psi_3\right)+\mathsf{d}\left(\psi_2,\ \psi_3\right)\mathsf{d}\left(\psi_2,\ \psi_2\right)}{\max\left\{\mathsf{d}\left(\psi_1,\ \psi_3\right),\ \mathsf{d}\left(\psi_2,\ \psi_2\right)\right.}\right\}\right)-\frac{\gamma}{2}$$
$$\leq \mathcal{F}\left(\max\left\{\mathsf{d}\left(\psi_1,\ \psi_2\right),\ \mathsf{d}\left(\psi_2,\ \psi_3\right)\right\}\right)-\frac{\gamma}{2}.$$

In view of $(F4)$

$$\mathcal{F}\left(D\left(\psi_2,\ \Gamma\psi_2\right)\right)=\mathcal{F}\left(\inf\left\{\left(\psi_2,\ \nu\right): \nu\in\Gamma\psi_2\right\}\right)$$
$$=\inf\left\{\mathcal{F}\left(\mathsf{d}\left(\psi_2,\ v\right): \nu\in\Gamma\psi_2\right)\right\}\leq \mathcal{F}\left(\mathsf{d}\left(\psi_1,\ \psi_2\right)\right)-\frac{\gamma}{2}.$$

Due to the Compactness of $\Gamma\psi_2$, there is $\psi_3 \in \Gamma\psi_2$, s.t $\mathsf{d}(\psi_2,\ \psi_3) = D(\psi_2,\ \Gamma\psi_2)$ so

$$\mathcal{F}(\mathsf{d}(\psi_2,\ \psi_3)) \leq \mathcal{F}(\mathsf{d}(\psi_1,\ \psi_2)) - \frac{\gamma}{2}. \tag{3.13}$$

By following a similar approach, construct a sequence $\{\psi_q\}$ in X s.t $\psi_{q+1} \in \Gamma\psi_q$ and $(\psi_q,\ \psi_{q+1}) \in \Gamma_G$, and

$$\mathcal{F}(\mathsf{d}(\psi_q,\ \psi_{q+1})) \leq \mathcal{F}(\mathsf{d}(\psi_{q-1},\ \psi_q)) - \frac{\gamma}{2}, \quad \forall\, q \in \mathbb{N} \tag{3.14}$$

Let $\tau_q = \mathsf{d}(\psi_q,\ \psi_{q+1})$, then $\tau_q > 0$ and from (3.14), $\{\tau_q\}$ is a decreasing sequence of real numbers, there exists a non-negative value $\omega \geqslant 0$ s.t $\lim_{q\to\infty} \tau_q = \omega$. Now

$$\mathcal{F}(\tau_q) \leq \mathcal{F}(\tau_{q-1}) - \frac{\gamma}{2} \leq \mathcal{F}(\tau_{q-2}) - 2\left(\frac{\gamma}{2}\right) \cdots \leq \mathcal{F}(\tau_0) - q\left(\frac{\gamma}{2}\right)$$

$$\implies \mathcal{F}(\tau_q) \leq \mathcal{F}(\tau_0) - q\left(\frac{\gamma}{2}\right). \tag{3.15}$$

The R.H.S of (3.15) goes to $-\infty$ when $q \to +\infty$. By utilizing (F2), $\omega = \lim_{q\to\infty} \tau_q = 0$. As a consequence of $(F3)$, there is $k \in (0,1)$ s.t $\lim_{q\to\infty} \tau_q^k \mathcal{F}(\tau_q) = 0$. By the inequality (3.15)

$$\tau_q^k \mathcal{F}(\tau_q) - \tau_q^k \mathcal{F}(a_0) \leq -\tau_q^k q\left(\frac{\gamma}{2}\right) \leq 0 \quad q \in \mathbb{N} \tag{3.16}$$

This condition holds for all $q \in \mathbb{N}$. Equation (3.16) implies, $\lim_{q\to\infty} q\tau_q^k = 0$. So there is $q_1 \in \mathbb{N}$ s.t $q\tau_q^k \leq 1$ for all $q \geqslant q_1$. Thus $\tau_q \leq \frac{1}{q^{1/k}}, \quad \forall\, q \geqslant q_1$. Now claim that $\{\tau_q\}$ is CS. For this, take $p, q \in \mathbb{N}$ with $p > q \geq q_1$. Hence,

$$\mathsf{d}(\psi_q,\ \psi_p) \leq \sum_{i=q}^{p-1} \mathsf{d}(\psi_i,\ \psi_{i+1}) = \sum_{i=q}^{p-1} \tau_i \leq \sum_{i=q}^{\infty} \tau_i \leq \sum_{i=q}^{\infty} \left(\frac{1}{i^{1/k}}\right).$$

As k belongs to the interval $(0,1)$, the series $\sum_{i=1}^{\infty} \left(\frac{1}{i^{1/k}}\right)$ converges. Consequently, as q and p approach infinity then $\mathsf{d}(\psi_q, \psi_m) \to 0$. In other words, the sequence

$\{\psi_q\}$ is a CS in $(X,\ \mathsf{d})$ which is a CMS. Therefore, $\{\psi_q\}$ converges to some $\mu \in X$. USC of Γ and the Lemma (3.1.2) implies $\mu \in \Gamma\mu$. So it contradicts our assumption. Thus Γ must admits a FP. □

Theorem 3.2.4. Consider a CMS $(X,\ \mathsf{d})$ equipped with a directed graph G that satisfies the following property:

For any $\{\psi_q\}$ in X, if ψ_q converges to ψ and $(\psi_q,\ \psi_{q+1}) \in \mathsf{E}(G)$, then there exists a subsequence $\{\psi_{q_k}\}$ with $(\psi_{q_k},\ \psi) \in \mathsf{E}(G)$.

Consider a multivalued mapping $\Gamma : X \to K(X)$, where Γ is also a $\mathcal{F}$-contraction. Suppose Γ is *WGP* mapping and the set X_Γ is non-empty. If $\mathcal{F}$ is a continuous function, then Γ must have a FP.

Proof. Assume that Γ has no FP then, $D(\psi,\ \Gamma\psi) > 0\ \forall\ \psi \in X$. Consider $\psi_0 \in X_\Gamma$. So $(\psi_0,\ \psi_1) \in \Gamma_G$. By using (3.6) for ψ_0 and ψ_1, $(\psi_0,\ \psi_1) \in \mathsf{E}(G)$ for any $\psi_1 \in \Gamma\psi_0$,

$$0 < D(\psi_1,\ \Gamma\psi_1) \leq H(\Gamma\psi_0,\ \Gamma\psi_1).$$

$$\begin{aligned}
\mathcal{F}(D(\psi_1,\ \Gamma\psi_1)) &\leq \mathcal{F}(H(\Gamma\psi_0,\ \Gamma\psi_1)) \\
&\leq \mathcal{F}(L(\psi_0,\ \psi_1)) - \gamma \\
&= \mathcal{F}\Bigg(\max\Bigg\{\mathsf{d}(\psi_0,\ \psi_1),\ D(\psi_0,\ \Gamma\psi_0),\ D(\psi_1,\ \Gamma\psi_1), \\
&\quad \frac{D(\psi_0,\ \Gamma\psi_0)\, D(\psi_0,\ \Gamma\psi_1) + D(\psi_1,\ \Gamma\psi_1)\, D(\psi_1,\ \Gamma\psi_0)}{\max\{D(\psi_0,\ \Gamma\psi_1),\ D(\psi_1,\ \Gamma\psi_0)\}}\Bigg\}\Bigg) - \gamma \\
&\leq \mathcal{F}\Bigg(\max\Bigg\{\mathsf{d}(\psi_0,\ \psi_1),\ \mathsf{d}(\psi_1,\ \psi_2), \\
&\quad \frac{\mathsf{d}(\psi_0,\ \psi_1)\, \mathsf{d}(\psi_0,\ \psi_2) + \mathsf{d}(\psi_1,\ \psi_2)\, \mathsf{d}(\psi_1,\ \psi_1)}{\max\{\mathsf{d}(\psi_0,\ \psi_2),\ \mathsf{d}(\psi_1,\ \psi_1)\}}\Bigg\}\Bigg) - \gamma \\
&\leq \mathcal{F}(\max\{\mathsf{d}(\psi_0,\ \psi_1),\ \mathsf{d}(\psi_1,\ \psi_2)\}) - \gamma \\
&\leq \mathcal{F}(\mathsf{d}(\psi_0,\ \psi_1)) - \gamma.
\end{aligned}$$

So

$$\mathcal{F}(D(\psi_1,\ \Gamma_{\psi_1})) \leq \mathcal{F}(\mathsf{d}(\psi_0,\ \psi_1)) - \gamma. \tag{3.17}$$

By Compactness of $\Gamma\psi_1$, there is, $\psi_2 \in \Gamma\psi_1$ s.t $\mathsf{d}(\psi_1, \psi_2) = D(\psi_1, \Gamma\psi_1)$. Then

$$\mathcal{F}(\mathsf{d}(\psi_1, \psi_2)) \leq \mathcal{F}(\mathsf{d}(\psi_0, \psi_1)) - \gamma.$$

Since $(\psi_0, \psi_1) \in \mathsf{E}(G)$, $\psi_1 \in \Gamma\psi_0$ and $\psi_2 \in \Gamma\psi_1$, by using the property of weakly graph-preserving, $(\psi_1, \psi_2) \in \mathsf{E}(G)$, and

$$0 < D(\psi_2, \Gamma\psi_2) \leq H(\Gamma\psi_1, \Gamma\psi_2),$$

so $(\psi_1, \psi_2) \in \Gamma_G$. Then apply the same procedure as above it can be shown that

$$\mathcal{F}(D(\psi_2, \Gamma\psi_2)) \leq \mathcal{F}(H(\Gamma\psi_1, \Gamma\psi_2)) \leq \mathcal{F}(L(\psi_1, \psi_2)) - \gamma. \tag{3.18}$$

So

$$\begin{aligned}
\mathcal{F}(D(\psi_2, \Gamma\psi_2)) &= \mathcal{F}\Bigg(\max\Bigg\{\mathsf{d}(\psi_1, \psi_2), D(\psi_1, \Gamma\psi_1), D(\psi_2, \Gamma\psi_2), \\
&\quad \frac{D(\psi_1, \Gamma\psi_1)D(\psi_1, \Gamma\psi_2) + D(\psi_2, \Gamma\psi_2)D(\psi_2, \Gamma\psi_1)}{\max\{D(\psi_1, \Gamma\psi_2), D(\psi_2, \Gamma\psi_1)\}}\Bigg\}\Bigg) - \gamma \\
&\leq \mathcal{F}\Bigg(\max\Bigg\{\mathsf{d}(\psi_1, \psi_2), \mathsf{d}(\psi_2, \psi_3) \\
&\quad \frac{\mathsf{d}(\psi_1, \psi_2)\mathsf{d}(\psi_1, \psi_3) + \mathsf{d}(\psi_2, \psi_3)\mathsf{d}(\psi_2, \psi_2)}{\max\{\mathsf{d}(\psi_1, \psi_3), \mathsf{d}(\psi_2, \psi_2)\}}\Bigg\}\Bigg) - \gamma \\
&\leq \mathcal{F}(\max\{\mathsf{d}(\psi_1, \psi_2), \mathsf{d}(\psi_2, \psi_3)\}) - \gamma \\
&\leq \mathcal{F}(\mathsf{d}(\psi_1, \psi_2)) - \gamma.
\end{aligned}$$

The compactness of $\Gamma\psi_2$ implies that $\psi_3 \in \Gamma\psi_2$ s.t $\mathsf{d}(\psi_2, \psi_3) = D(\psi_2, \Gamma\psi_2)$ so

$$\mathcal{F}(\mathsf{d}(\psi_2, \psi_3)) \leq \mathcal{F}(\mathsf{d}(\psi_1, \psi_2)) - \gamma. \tag{3.19}$$

So a sequence $\{\psi_q\} \in X$ can be constructed s.t $\psi_{q+1} \in \Gamma\psi_q$, $(\psi_q, \psi_{q+1}) \in \Gamma_G$ and

$$\mathcal{F}(\mathsf{d}(\psi_q, \psi_{q+1})) \leq \mathcal{F}(\mathsf{d}(\psi_{q-1}, \psi_q)) - \gamma \quad \forall\, q \in \mathbb{N}. \tag{3.20}$$

Let us assume that $\tau_q = \mathsf{d}(\psi_q,\ \psi_{q+1})$, where ψ_q and ψ_{q+1} are elements of a metric space. In this case, we can conclude that $\tau_q > 0$. Since $\{\tau_q\}$ is a decreasing sequence of real numbers, there exists a non-negative value $\omega \geqslant 0$ s.t $\lim\limits_{q\to\infty} \tau_q = \omega$.

$$\begin{aligned}\mathcal{F}(\tau_q) &\leq \mathcal{F}(\tau_{q-1}) - \gamma \\ &\leq \mathcal{F}(\tau_{q-2}) - 2\gamma \\ &\ \ \vdots \\ &\leq \mathcal{F}(\tau_0) - q\gamma\end{aligned}$$

$$\Rightarrow \mathcal{F}(\tau_q) \leq \mathcal{F}(\tau_0) - q\gamma. \tag{3.21}$$

Now $\lim\limits_{q\to+\infty} \mathcal{F}(\tau_q) = -\infty$, therefore

$$\omega = \lim_{q\to\infty} \tau_q = 0.$$

Due to $(F3)$ there is a constant $k \in (0,\ 1)$

$$\lim_{q\to\infty} \psi_q^k \mathcal{F}(\psi_q) = 0.$$

Then by (3.21)

$$\tau_q^k \mathcal{F}(\tau_q) - \tau_q^k \mathcal{F}(\tau_0) \leq -\tau_q^k q\gamma \leq 0. \tag{3.22}$$

Which is true $\forall\ q \in \mathbb{N}$. Suppose $q \to \infty$ then

$$\lim_{q\to\infty} q\tau_q^k = 0. \tag{3.23}$$

From (3.10), suppose there is a $q_1 \in \mathbb{N}$ then $q\psi_q^k \leq 1\ \forall\ q \geqslant q_1$. Thus $\tau_q \leq \dfrac{1}{q^{1/k}}$ $\forall\ q \geqslant q_1$.By claiming $\{\tau_q\}$ is CS, suppose that $p,\ q \in \mathbb{N}$ and $p > q \geqslant q_1$. Then,

$$\mathsf{d}(\psi_q,\ \psi_p) \leq \sum_{i=q}^{p-1} \mathsf{d}(\psi_i,\ \psi_{i+1}) = \sum_{i=q}^{p-1} \tau_i \leq \sum_{i=q}^{\infty} \tau_i \leq \sum_{i=q}^{\infty} \left(\frac{1}{i^{1/k}}\right).$$

As $k \in (\psi_0,\ 1)$, the series $\sum\limits_{i=1}^{\infty} \frac{1}{i^{1/k}}$ converges, so $\mathsf{d}\left(\psi_q,\ \psi_p\right) \to 0$ as $q,\ p \to \infty$, then this implies that $\{\psi_q\}$ is a CS in the MS, which is complete. So it converges to some $\mu \in X$. By the given property, there exists a subsequence $\{\psi_{q_k}\}$ of $\{\psi_q\}$ s.t $(\psi_{q_k},\ \mu)$ is an element of $\mathsf{E}(G)$ for every $k \in \mathbb{N}$. Since $\lim\limits_{q \to 0} \psi_{q_k} = \mu$ and $D(\mu,\ \Gamma\mu) > 0$, there is no natural number q_0 s.t $D(\psi_{q_{k+1}},\ \Gamma\mu) = 0$ for all $q_k \geq q_0$. Thus, for all $q_k \geqslant q_0$,

$$H\left(\Gamma\psi_{q_k},\ \Gamma\mu\right) > 0.$$

Thus $(\psi_{q_k},\ \mu) \in \Gamma_G$ for all $q_k \geqslant q_0$. Therefore, by $\mathcal{F}$-contraction condition and (F_1). for all$q_k \geq q_0$.

$$\begin{aligned}
\mathcal{F}\left(D\left(\psi_{q_k+1},\ \Gamma\mu\right)\right) &\leq \mathcal{F}\left(H\left(\Gamma\psi_q,\ \Gamma\mu\right)\right) - \gamma \\
&\leq \mathcal{F}\left(L\left(\psi_{q_k},\ \mu\right)\right) - \gamma \\
&\leq \mathcal{F}\Big(\max\Big\{\mathsf{d}\left(\psi_{q_k},\ \mu\right),\ D\left(\psi_{q_q},\ \Gamma\psi_{q_k}\right),\ D(\mu,\ \Gamma\mu) \\
&\qquad \frac{D\left(\psi_{q_k},\ \Gamma\psi_{q_k}\right) D\left(\psi_{q_k},\ \Gamma\mu\right) + D(\mu,\ \Gamma\mu) D\left(\mu,\ \Gamma\psi_{q_k}\right)}{\max\left\{D\left(\psi_{q_k},\ \Gamma\mu\right),\ D\left(\mu,\ \Gamma\psi_{q_k}\right)\right\}}\Big\}\Big) - \gamma \\
&\leq \mathcal{F}\Big(\max\Big\{\mathsf{d}\left(\psi_{q_k},\ \mu\right),\ \mathsf{d}\left(\psi_{q_k},\ \psi_{q_{k+1}}\right),\ D(\mu,\ \Gamma\mu), \\
&\qquad \frac{\mathsf{d}\left(\psi_{q_k},\ \psi_{q_{k+1}}\right) D\left(\psi_{q_k},\ \Gamma\mu\right) + D(\mu,\ \Gamma\mu)\mathsf{d}\left(\mu,\ \psi_{q_{k+1}}\right)}{\max\left\{D\left(\psi_{q_k},\ \Gamma\mu\right),\ D\left(\mu,\ \Gamma\psi_{q_k}\right)\right\}}\Big\}\Big) - \gamma.
\end{aligned}$$

Taking $k \to \infty$ and by the continuity of $\mathcal{F}$ this leads to a contradiction, so

$$\gamma + \mathcal{F}(D(\mu,\ \Gamma\mu)) \leq \mathcal{F}(D(\mu,\ \Gamma\mu)).$$

$\Rightarrow \Gamma$ has a FP. □

Now, the following corollaries are presented by changing the some conditions with WGP property. Then an example is elaborated by using the theorems results and how the main results will not hold if set of edges $\mathsf{E}(G)$ is not considered.

Corollary 3.2.5. Suppose $(X,\ \mathsf{d})$ is a CMS with a digraph G and a mapping $\Gamma : X \to K(X)$. Suppose for $\mathcal{F} \in \mathcal{J}$ and $\gamma > 0$ s.t

$$\gamma + \mathcal{F}(H(\Gamma v,\ \Gamma\psi)) \leq \mathcal{F}(\mathsf{d}(v,\ \psi))$$

$\forall\, \upsilon,\ \psi \in X$ with $(\upsilon,\ \psi) \in \Gamma_G$. If Γ is both USC and a *WGP* mapping and the set X_Γ is non-empty, then Γ has a FP.

Corollary 3.2.6. Let $(X,\ \mathsf{d})$ be a CMS endowed with a directed graph G, and $\Gamma : X \to CB(X)$ be a mapping. Let $\mathcal{F} \in \mathcal{J}_*$ and γ a positive constant s.t

$$\gamma + \mathcal{F}(H(\Gamma\upsilon,\ \Gamma\psi)) \leq \mathcal{F}(\mathsf{d}(\upsilon,\ \psi))$$

for $\upsilon, \psi \in X$, with $(\upsilon,\ \psi) \in \Gamma_G$ Assuming that Γ is USC and a *WGP* and the set ψ_Γ is non-empty, it can be concluded that Γ possesses a FP.

Example 3.2.7. Let $X = \left\{\omega_\kappa = \dfrac{\kappa(\kappa+1)}{2}; \kappa \geq 1,\ \ \kappa \text{ is an integer}\right\} \cup \{0\}$ and the $\mathsf{d}(\rho, \sigma) = |\rho - \sigma|$. Then $(X,\ \mathsf{d})$ is a CMS.
Now, define a mapping $\Gamma : X \to CB(X)$ by:

$$\Gamma(\rho) = \begin{cases} \{0\} & ,\ \text{if } \rho = 0 \\ \{\omega_1\} & ,\ \text{if } \rho = \omega_1 \\ \{\omega_1,\ \omega_2,\ \ldots,\ \omega_{\kappa-1}\} & ,\ \text{if } \rho = \omega_\kappa,\ \kappa \geq 2 \end{cases}$$

and a graph on X by $V(G) = X$ and

$$\mathsf{E}(G) = \{(\rho,\ \sigma) \mid \rho = \sigma \text{ or } \rho = \omega_\kappa,\ \sigma = \omega_p,\ p < \kappa\}.$$

Then Γ is USC and a *WGP* mapping. To show that Γ is a multivalued $\mathcal{F}$-contraction, where $\mathcal{F}(\rho) = \rho + \ln \rho$ and $\gamma = 1$. Let $(\rho,\ \sigma) \in \mathsf{E}(G)$ be s.t $\Gamma(\rho) \neq \Gamma(\kappa)$. We will consider two cases:
Case-1. If $\rho = \omega_\kappa,\ \kappa \geq 2$ and $\sigma = \omega_1$, then
As $\Gamma\omega_1 = \omega_1$, so
$H(\Gamma(\rho),\ \Gamma(\sigma)) = \max\{D(\rho, \Gamma\sigma), D(\sigma, \Gamma\sigma)\} = \max\{D(\omega_\kappa, \Gamma\omega_1), D(\omega_1, \Gamma\omega_1)\}$.
$\Rightarrow H(\Gamma(\rho),\ \Gamma(\sigma)) = |\omega_{\kappa-1} - \omega_1|$.
Also $N(\rho,\ \sigma) = \mathsf{d}(\omega_\kappa, \omega_1) = |\omega_\kappa - \omega_1|$

$$\frac{H\left(\Gamma(\rho),\ \Gamma(\sigma)\right)}{N(\rho,\ \sigma)} e^{H(\Gamma(\rho),\ \Gamma(\sigma)) - N(\rho,\ \sigma)} = \frac{\omega_{\kappa-1} - 1}{\omega_\kappa - 1} e^{\omega_{\kappa-1} - \omega_\kappa} < e^{-1}.$$

Case-2. If $\rho = \omega_\kappa, \sigma = \omega_p,\ \kappa > p > 1$, then
$H(\Gamma(\rho),\ \Gamma(\sigma)) = \max\{D(\rho, \Gamma\sigma), D(\sigma, \Gamma\sigma)\} = \max\{D(\omega_\kappa, \Gamma\omega_p), D(\omega_p, \Gamma\omega_p)\}$.
$\Rightarrow H(\Gamma(\rho),\ \Gamma(\sigma)) = \kappa + p - 1$.
Also $N(\rho,\ \sigma) = \mathsf{d}(\omega_\kappa, \omega_p) = \kappa + p + 1$

$$\frac{H\left(\Gamma(\rho),\ \Gamma(\sigma)\right)}{N(\rho,\ \sigma)} e^{H(\Gamma(\rho),\ \Gamma(\sigma)) - N(\rho,\ \sigma)} = \frac{\kappa + p - 1}{\kappa + p + 1} e^{-\kappa + p} < e^{-1}.$$

So all assumptions in Theorem (3.2.2) and Theorem (3.2.3) are satisfied. Therefore, Γ has a FP. It is important to note that without considering the graph on X, the contraction condition is not satisfied. In fact, by taking $\rho = 0$ and $\sigma = \omega_1$, $H\left(\Gamma(\rho),\ \Gamma(\sigma)\right) = 1$ and $\mathsf{d}(\rho,\ \sigma) = 1$, we get

$$\gamma + \mathcal{F}(H(\Gamma(\rho),\ \Gamma(\sigma))) > \mathcal{F}(\mathsf{d}(\rho,\ \sigma)) \quad \forall\ \mathcal{F} \in \mathcal{J} \text{ and } \gamma > 0.$$

Chapter 4

Fixed Point Results in b-metric Spaces via Graph Structure

In this chapter several FP results in bMS endowed with graph G are presented. These results are generalization of the work of Acar et al. [21]. Some notions used by Acar et al. are defined in the setting of bMS then some FP results are established in the new framework.

4.1 On Multivalued G-Contraction

In this section FP results via graph structure will be proved in bMS. First we will define some terms that will be useful in bMS.

Definition 4.1.1. [37] "A function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is called a b-comparison function (bCF) (with $s \geqslant 1$) if Φ is monotonically increasing and there exist $k_0 \in \mathbb{N},\ \alpha \in (0,1)$ and a convergent series of non-negative terms $\sum_{k=1}^{\infty} v_k$ s.t

$$s^{k+1}\phi^{k+1}(t) \leq \alpha_s^k \phi^k(t) + v_k \quad \text{for } k \geqslant k_0 \text{ and any } t \in \mathbb{R}."$$

Remark 4.1.2. It is evident that when $s = 1$, the notion of a bCF simplifies to that of a (c)-CF.

Lemma 4.1.3. If $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a bCF as stated in [38], then the following conditions hold:

(a) A series $\sum_{k=0}^{\infty} s^k \Phi^k(r)$ is converges to any $r \in \mathbb{R}_+$.

(b) The function $p_b : \mathbb{R}^+ \to \mathbb{R}^+$ is defined as $p_b(t) = \sum_{k=0}^{\infty} b^k \Phi^k(t)$ for $t \in \mathbb{R}^+$, is increasing and continuous at 0.

Definition 4.1.4. Let $(X,\ \mathsf{d}_b)$ be a complete bMS equipped with digraph G. Then $g : X \to CB(X)$ is called rational multivalued G-contraction of type-I if

$$sH(g\psi, g\theta) \leq \Phi(N(\psi, \theta)), \quad \forall\ (\psi,\ \theta) \in \mathsf{E}(G), \tag{4.1}$$

where Φ is (b)-CF and,

$$N(\psi,\ \theta) = \max\left\{\mathsf{d}_b(\psi,\ \theta),\ \frac{D(\psi,\ g\psi) + D(\theta,\ g\theta)}{2},\ \frac{D(\psi,\ g\theta) + D(\theta,\ g\psi)}{2s}\right.$$
$$\left.\frac{D(\psi,\ g\psi)D(\theta,\ g\theta)}{\mathsf{d}_b(\psi,\ \theta)},\ \frac{D(\theta,\ g\theta)[1 + D(\psi,\ g\psi)]}{1 + \mathsf{d}_b(\psi,\ \theta)}\right\}.$$

Theorem 4.1.5. Let $(X,\ \mathsf{d}_b)$ be a complete bMS and $g : X \to CB(X)$ is USC and a *WGP* mapping. Then g has a FP, if it satisfies the following conditions:

(a) g is rational multivalued G-contraction of type-I;

(b) $N_g = \{\psi \in X : (\psi,\ v) \in \mathsf{E}(G) \text{ for } v \in g\psi\}$ is non-empty.

Then g has a FP.

Proof. Let $\psi_0 \in N_g$, where $N_g \neq \phi$. So, there is $\psi_1 \in g(\psi_0)$ s.t $(\psi_0, \psi_1) \in \mathsf{E}(G)$. Now, by condition (a) for ψ_0 and ψ_1,

$$N(\psi_0,\ \psi_1) = \max\left\{\mathsf{d}_b(\psi_0,\ \psi_1),\ \frac{D(\psi_0,\ g\psi_0) + D(\psi_1,\ g\psi_1)}{2},\right.$$
$$\frac{D(\psi_0,\ g\psi_1) + D(\psi_1,\ g\psi_0)}{2s},\ \frac{D(\psi_0,\ g\psi_0),\ D(\psi_1,\ g\psi_1)}{\mathsf{d}_b(\psi_0,\ \psi_1)},$$
$$\left.\frac{D(\psi_1,\ g\psi_1)(1 + D(\psi_0,\ g\psi_0))}{(1 + \mathsf{d}_b(\psi_0,\ \psi_1))}\right\}$$

$$\leq \max\left\{\mathsf{d}_b(\psi_0,\ \psi_1),\ \frac{\mathsf{d}_b(\psi_0,\ \psi_1)+\mathsf{d}_b(\psi_1,\ \psi_2)}{2}, \frac{\mathsf{d}_b(\psi_0,\ \psi_2)}{2s},\ \mathsf{d}_b(\psi_1,\ \psi_2)\right\}$$
$$\leq \max\left\{\mathsf{d}_b(\psi_0,\ \psi_1),\ \frac{\mathsf{d}_b(\psi_0,\ \psi_1)+\mathsf{d}_b(\psi_1,\ \psi_2)}{2},\ \mathsf{d}_b(\psi_1,\ \psi_2)\right\}$$
$$\leq \max\left\{\mathsf{d}_b(\psi_0,\ \psi_1),\ \mathsf{d}_b(\psi_1,\ \psi_2)\right\}$$

Also by condition (a)

$$sD(\psi_1,\ g\psi_1) \leq sH(g\psi_0,\ g\psi_1) \leq \Phi(N(\psi_0,\ \psi_1)),$$

Therefore,

$$sD(\psi_{1,}\ , g\psi_1) \leq \Phi(\max\{\mathsf{d}_b(\psi_0,\ \psi_1),\ \mathsf{d}_b(\psi_1,\ \psi_2)\}) \quad \leq \Phi(\mathsf{d}_b(\psi_0,\ \psi_1)).$$

Let $\varrho > 1$ be an arbitrary constant. So by Lemma (4.1.3) there exist $\psi_2 \in g\psi_1$, s.t

$$\mathsf{d}_b(\psi_1,\ \psi_2) \leq \sqrt{\varrho}H(g\psi_0,\ g\psi_1).$$

As $sH(g\psi_0,\ g\psi_1) \leq \Phi(\mathsf{d}_b(\psi_0,\ \psi_1))$ so,

$$s\mathsf{d}_b(\psi_1,\ \psi_2) \leq \sqrt{\varrho}sH(g\psi_0,\ g\psi_1) \leq \varrho\Phi(\mathsf{d}_b(\psi_0,\ \psi_1)).$$

Due to the strictly increasing nature of the function Φ, we can conclude that

$$0 < \Phi(s\mathsf{d}_b(\psi_1,\ \psi_2)) < \Phi(\varrho\Phi(\mathsf{d}_b(\psi_0,\ \psi_1))).$$

Set $\varrho_1 = \frac{\Phi(\varrho(\mathsf{d}_b(\psi_0,\ \psi_1)))}{\Phi(s\mathsf{d}_b(\psi_1,\ \psi_2))} > 1$.

Since $(\psi_0,\ \psi_1) \in \mathsf{E}(G)$, $\psi_1 \in g\psi_0$ and $\psi_2 \in g\psi_1$, using *WGP* property, $(\psi_1,\ \psi_2) \in \mathsf{E}(G)$ then,

$$N(\psi_1,\ \psi_2) = \max\left\{\mathsf{d}_b(\psi_1,\ \psi_2),\ \frac{D(\psi_1,\ g\psi_1)+D(\psi_2,\ g\psi_2)}{2}, \frac{D(\psi_1, g\psi_2)+D(\psi_2,\ g\psi_1)}{2s},\ \frac{D(\psi_1,\ g\psi_1)\,D(\psi_2,\ g\psi_2)}{\mathsf{d}_b(\psi_1,\ \psi_2)}, \frac{D(\psi_2,\ g\psi_2)(1+D(\psi_1,\ g\psi_1))}{(1+\mathsf{d}_b(\psi_1,\ \psi_2))}\right\}$$

$$\leq \max\left\{\mathsf{d}_b(\psi_1,\ \psi_2),\ \frac{\mathsf{d}_b(\psi_1,\ \psi_2)+\mathsf{d}_b(\psi_2,\ \psi_3)}{2},\ \frac{\mathsf{d}_b(\psi_1,\ \psi_3)}{2s},\ \mathsf{d}_b(\psi_2,\ \psi_3)\right\}$$
$$\leq \max\left\{\mathsf{d}_b(\psi_1,\ \psi_2),\ \frac{\mathsf{d}_b(\psi_1,\ \psi_2)+\mathsf{d}_b(\psi_2,\psi_3)}{2},\ \mathsf{d}_b(\psi_2,\ \psi_3)\right\}$$
$$\leq \max\left\{\mathsf{d}_b(\psi_1,\ \psi_2),\ \mathsf{d}_b(\psi_2,\ \psi_3)\right\}.$$

As by definition of Type-I contraction,

$$sD(\psi_2,\ g\psi_2) \leq sH(g\psi_1,\ g\psi_2) \leq \Phi(N(\psi_1,\ \psi_2)). \tag{4.2}$$

So, from (4.2)

$$sD(\psi_2,\ g\psi_2) \leq \Phi(\max\{\mathsf{d}_b(\psi_1,\ \psi_2),\ \mathsf{d}_b(\psi_2,\ \psi_3)\}) \leq \Phi(\mathsf{d}_b(\psi_1,\ \psi_2)).$$

As $\varrho_1 > 1$, so there exist $\psi_3 \in g\psi_2$, s.t

$$s\mathsf{d}_b(\psi_2,\ \psi_3) \leq \sqrt{\varrho_1}\ sH(g\psi_1,\ g\psi_2) \leq \varrho_1\Phi(\mathsf{d}_b(\psi_1,\ \psi_2)) = \Phi(\varrho\Phi(\mathsf{d}_b(\psi_0,\ \psi_1))).$$

Since Φ is strictly increasing. set $\varrho_2 = \dfrac{\Phi^2(\varrho\Phi(\mathsf{d}_b(\psi_1,\ \psi_1))}{\Phi(s\mathsf{d}_b(\psi_2,\ \psi_3))} > 1$. Next, proceeding similarly to generate a sequence $\{\psi_q\}$ in X s.t $\psi_{q+1} \in g\psi_q$ and $(\psi_q, \psi_{q+1}) \in \mathsf{E}(G)$, and

$$s\mathsf{d}_b(\psi_q,\ \psi_{q+1}) \leq \Phi^q(\varrho\Phi(\mathsf{d}_b(\psi_0,\ \psi_1))).$$

To prove $\{\psi_q\}$ is a CS, take $p,\ q \in \mathbb{N}$ with $p > 1$,

$$\begin{aligned}
\mathsf{d}_b(\psi_q,\ \psi_p) &\leq \sum_{j=q}^{m-1} s^{j+1-q}\ \mathsf{d}_b(\psi_j,\ \psi_{j+1}) \\
&\leq \sum_{j=0}^{\infty} s^{j-q}\Phi^j(\varrho\Phi(\mathsf{d}_b(\psi_0,\ \psi_1))) \\
&\leq \sum_{j=0}^{\infty} s^q s^{j-q}\Phi^j(\varrho\Phi(\mathsf{d}_b(\psi_0,\ \psi_1))) \\
&\leq \sum_{j=0}^{\infty} s^j\Phi^j(\varrho\Phi(\mathsf{d}_b(\psi_0,\ \psi_1))).
\end{aligned}$$

As Φ is a bCF, the series on the R.H.S converges. Hence, as q and p tend to infinity, the distance $\mathsf{d}_b(\psi_q, \psi_p)$ approaches zero. In other words, the sequence $\{\psi_q\}$ is a CS in (X, d_b) which is a complete bMS. Consequently, $\{\psi_q\}$ converges to some element $\mu \in X$. As g is USC so by Lemma (3.1.2) $\mu \in g\mu$. So g possesses a FP. □

Definition 4.1.6. Let $(X,\ \mathsf{d}_b)$ be a complete bMS with digraph G. Then $g : X \to CB(X)$ is called rational multivalued G-contraction of type-II if

$$sH(g\psi, g\theta) \leq \Phi(N(\psi, \theta)), \quad \forall\ (\psi,\ \theta) \in \mathsf{E}(G), \tag{4.3}$$

where Φ is (b)-CF and,

$$N(\psi,\ \theta) = \max\left\{\mathsf{d}_b(\psi,\ \theta),\ \frac{D(\psi,\ g\psi) + D(\theta,\ g\theta)}{2},\right.$$
$$\left.\frac{D(\psi,\ g\theta) + D(\theta,\ g\psi)}{2s},\ \frac{D(\psi,\ g\psi)\ D(\theta,\ g\theta)}{1 + H(g\psi,\ g\theta)}\right\}.$$

Theorem 4.1.7. Consider a complete bMS denoted as (X, d_b), G be the digraph defined on (X, d_b). Consider the multivalued mapping $g : X \to CB(X)$ satisfying the following conditions:

(a) g is a rational multivalued G-contraction of type-II;

(b) $N_g = \{\psi \in X; (\psi,\ u) \in \mathsf{E}(G) \text{ for } u \in g\psi\}$ is non-empty;

(c) The (P)-property is satisfied;

(d) g is WGP mapping.

Then g has a FP.

Proof. Take $\psi_0 \in N_g$. There is an element $\psi \in g\psi_0$ s.t $(\psi_0, \psi_1) \in \mathsf{E}(G)$. As a result condition (a) can be used both for ψ_0 and ψ_1. Then by definition of Type-II,

$$sD(\psi_1,\ g\psi_1) \leq sH(g\psi_0,\ g\psi_1) \leq \Phi(N(\psi_0,\ \psi_1)). \tag{4.4}$$

Now

$$N(\psi_0,\ \psi_1) = \max\left\{\mathsf{d}_b(\psi_0,\ \psi_1),\ \frac{D(\psi_0,\ g\psi_0)+D(\psi_1,\ g\psi_1)}{2},\right.$$
$$\left.\frac{D(\psi_0,\ g\psi_1)+D(\psi_1,\ g\psi_0)}{2s},\ \frac{D(\psi_0,\ g\psi_0)\ D(\psi_1,\ g\psi_1)}{1+H(g\psi_0,\ g\psi_1)}\right\}$$
$$\leq \max\left\{\mathsf{d}_b(\psi_0,\ \psi_1),\ \frac{\mathsf{d}_b(\psi_0,\ \psi_1)+D(\psi_1,\ g\psi_1)}{2},\right.$$
$$\left.\frac{D(\psi_0,\ g\psi_0)\ D(\psi_1,\ g\psi_1)}{1+D(\psi_1,\ g\psi_1)}\right\}$$
$$\leq \max\left\{\mathsf{d}_b(\psi_0,\ \psi_1),\ \frac{\mathsf{d}_b(\psi_0,\ \psi_1)+D(\psi_1,\ g\psi_1)}{2},\ D(\psi_1,\ g\psi_1)\right\}$$
$$\leq \max\{\mathsf{d}_b(\psi_0,\ \psi_1),\ D(\psi_1,\ g\psi_1\})$$

So , Let $\varrho > 1$, is an arbitrary constant. Therefore, there exists $\psi_2 \in g\psi_1$ s.t

$$sD(\psi_1,\ g\psi_1) \leq \Phi(\max\{\mathsf{d}_b(\psi_0,\ \psi_1),\ D(\psi_1,\ g\psi_1)) \leq \Phi(\mathsf{d}_b(\psi_0,\ \psi_1)).$$

$$s\mathsf{d}_b(\psi_1,\ \psi_2) \leq s\sqrt{\varrho}H(g\psi_0,\ g\psi_1) \leq \varrho\Phi(\mathsf{d}_b(\psi_i,\ \psi_1)).$$

Due to the strictly increasing nature of Φ, it follows that

$$0 < \Phi(s\mathsf{d}_b(\psi_1,\ \psi_2)) < \Phi(\varrho\Phi(\mathsf{d}_{\mathrm{b}}(\psi_0,\ \psi_1))).$$

Take $\varrho_1 = \dfrac{\Phi(\varrho\Phi(\mathsf{d}_b(\psi_0,\ \psi_1)))}{\Phi(s\mathsf{d}_b(\psi_1,\ \psi_2))} > 1$. In view of $(\psi_0,\ \psi_1) \in E(G)$, $\psi_1 \in g\psi_0$, $\psi_2 \in g\psi_1$, and using *WGP* property $(\psi_1,\ \psi_2) \in \mathsf{E}(G)$, then

$$N(\psi_1,\ \psi_2) = \max\left\{\mathsf{d}_b(\psi_1,\ \psi_2),\ \frac{D(\psi_1,\ g\psi_1)+D(\psi_2,\ g\psi_2)}{2},\right.$$
$$\left.\frac{D(\psi_1,\ g\psi_2)+D(\psi_2,\ g\psi_1)}{2s},\ \frac{D(\psi_1,\ g\psi_1)\ D(\psi_2,\ g\psi_2)}{1+H(g\psi_1,\ g\psi_2)}\right\}$$
$$\leq \max\left\{\mathsf{d}_b(\psi_1,\ \psi_2),\ \frac{\mathsf{d}_b(\psi_1,\ \psi_2)+D(\psi_2,\ g\psi_2)}{2},\right.$$
$$\left.\frac{D(\psi_1,\ g\psi_1)\ D(\psi_2,\ g\psi_2)}{D(\psi_2,\ g\psi_2)}\right\}$$
$$\leq \max\left\{\mathsf{d}_b(\psi_1,\ \psi_2),\ \frac{\mathsf{d}_b(\psi_1,\ \psi_2)+D(\psi_2,\ g\psi_2)}{2},\ D(\psi_2,\ g\psi_2)\right\}$$
$$\leq \max\{\mathsf{d}_b(\psi_1,\ \psi_2),\ D(\psi_2,\ g\psi_2)\}.$$

Also by condition (a)

$$sD\left(\psi_2,\ g\psi_2\right) \leq sH\left(g\psi_1,\ g\psi_2\right) \leq \Phi\left(N\left(\psi_1,\ \psi_2\right)\right), \tag{4.5}$$

Now,

$$\begin{aligned} sD\left(\psi_2,\ g\psi_2\right) &\leq sH\left(g\psi_1,\ g\psi_2\right) \\ &\leq \Phi\left(\max\left\{\mathsf{d}_b\left(\psi_1,\ \psi_2\right),\ D\left(\psi_2,\ g\psi_2\right)\right\}\right) \\ &\leq \Phi\left(\mathsf{d}_b\left(\psi_1,\ \psi_2\right)\right). \end{aligned}$$

There exist $\psi_3 \in g\psi_2$, s.t

$$s\mathsf{d}_b\left(\psi_2,\ \psi_3\right) \leq \sqrt{\varrho_1} sH\left(g\psi_1, g\psi_2\right) < \varrho_1 \Phi\left(\mathsf{d}_b\left(\psi_1,\ \psi_2\right)\right) = \Phi\left(\varrho\Phi\left(\mathsf{d}_b\left(\psi_0,\ \psi_1\right)\right)\right),$$

since Φ is strictly increasing

$$\Rightarrow 0 < \Phi\left(s\mathsf{d}_b\left(\psi_2,\ \psi_3\right)\right) < \Phi^2\left(\varrho\Phi\left(\mathsf{d}_b\left(\psi_0,\ \psi_1\right)\right)\right).$$

Set $\varrho_2 = \dfrac{\Phi^2\left(\varrho\Phi\left(\mathsf{d}_b\left(\psi_0,\ \psi_1\right)\right)\right)}{\Phi\left(\mathsf{d}_b\left(\psi_2,\ \psi_3\right)\right)} > 1.$

Now a sequence $\{\psi_q\}$ in X s.t $\psi_{q+1} \in g\psi_q$ and $(\psi_q, \psi_{q+1}) \in \mathsf{E}(G)$ can be constructed and

$$s\mathsf{d}_b\left(\psi_q,\ \psi_{q+1}\right) \leq \Phi^q\left(\varrho\Phi\left(\mathsf{d}_b\left(\psi_0,\ \psi_1\right)\right)\right).$$

To show that $\{\psi_q\}$ is a CS, let $p,\ q \in \mathbb{N}$ with $p > q$, by using generalized form of triangular inequality in bMS,

$$\begin{aligned} \mathsf{d}_b\left(\psi_q,\ \psi_p\right) &\leq \sum_{j=q}^{p-1} s^{j+1-q}\ \mathsf{d}_b\left(\psi_j,\ \psi_{j+1}\right) \\ &\leq \sum_{j=q}^{p-1} s^{j+1-q}\Phi^j\left(\varrho\Phi\left(\mathsf{d}_b\left(\psi_0,\ \psi_1\right)\right)\right) \\ &\leq \sum_{j=q}^{p-1} s^{q-1}s^{j+1-q}\Phi^j\left(\varrho\Phi\left(\mathsf{d}_b\left(\psi_0,\ \psi_1\right)\right)\right) \\ &\leq \sum_{j=0}^{\infty} s^j\Phi^j\left(\varrho\Phi\left(\mathsf{d}_b\left(\psi_0,\ \psi_1\right)\right)\right). \end{aligned}$$

Given that Φ is a bCF, the series on the R.H.S converges, so $\mathsf{d}_b(\psi_q,\ \psi_p) \to 0$ as both p and q tend to infinity. This implies that the sequence $\{\psi_q\}$ is a CS in (X, d_b), which is a complete space. Therefore, $\{\psi_q\}$ converges to $\mu \in X$, that is $\lim_{q\to\infty} \psi_q = \mu$.

Using the (P)-property there is a subsequence $\{\psi_{q_k}\}$ of $\{\psi_q\}$ in which $(\psi_{q_k}, \mu) \in \mathsf{E}(G)$ for every $k \in \mathbb{N}$. Let's assume that $D(\mu,\ g\mu) > 0$. As $\lim_{q\to\infty} D\left(\psi_{q_k},\ \psi_{q_{k+1}}\right) = 0$ and $\lim_{q\to\infty} D\left(\psi_{q_k},\ \mu\right) = 0$, $\exists\ \ q_0 \in \mathbb{N}$ s.t for $q_k > q_0$,

$$D\left(\psi_{q_k},\ \psi_{q_{k+1}}\right) < \frac{1}{3} D(\mu,\ g\mu) \tag{4.6}$$

and there exists a natural number q_1 s.t $q_k > q_1$,

$$D\left(\psi_{q_k},\ \mu\right) < \frac{1}{3} D(\mu,\ g\mu). \tag{4.7}$$

If we take $q_k > \max\{q_0,\ q_1\}$, then by (4.6) and (4.7)

$$\begin{aligned} sD\left(\psi_{q_{k+1}},\ g\mu\right) &\leq sH\left(g\psi_{q_k},\ g\mu\right) \\ &\leq \Phi\left(N\left(\psi_{q_k},\ \mu\right)\right) \end{aligned}$$

$$\begin{aligned} &\leq \Phi\Bigg(\max\Bigg\{\mathsf{d}_b\left(\psi_{q_k},\ \mu\right),\ \frac{D\left(\psi_{q_k},\ g\psi_{q_k}\right) + D(\mu,\ g\mu)}{2}, \\ &\qquad \frac{D\left(\psi_{q_k},\ g\mu\right) + D\left(\mu,\ g\psi_{q_k}\right)}{2s},\ \frac{D\left(\psi_{q_k} g\psi_{q_1}\right)\ D(\mu,\ g\mu)}{1 + H\left(g\left(\psi_{q_k},\ g\mu\right)\right.}\Bigg\} \\ &\leq \Phi\Bigg(\max\Bigg\{\frac{D(\mu,\ g\mu)}{3},\ \frac{D(\mu,\ g\mu) + D(\mu,\ g\mu)}{3}, \\ &\qquad \frac{D\left(\psi_{q_k},\ g\mu\right) + D\left(\mu,\ g\psi_{q_k}\right)}{2s},\ \frac{\frac{1}{3} D(\mu,\ g\mu)\ D(\mu,\ g\mu)}{D\left(\psi_{qk+1},\ g\mu\right)}\Bigg\}\Bigg). \end{aligned}$$

Let $k \to \infty$, then $sD(\mu,\ g\mu) \leq \Phi(D(\mu,\ g\mu)) < D(\mu,\ g\mu)$, which is a contradiction. So $D(\mu, g(\mu) = 0$ and since $g\mu$ is closed, then $\mu\ \in\ g\mu$. Hence, g has a FP. □

Theorem 4.1.8. Suppose (X, d_b) be a complete bMS, additionally, there is a digraph G defined on X. Let $g : X \to K(X)$ be a multivalued mapping. Assume that g is USC and WGP mapping. Suppose that the following conditions are satisfied

(a) there is a bCF Φ s.t

$$sH(g\psi,\ g\theta) \leq \Phi(N(\psi,\ \theta)),\quad \forall\ (\psi,\ \theta) \in \mathsf{E}(G),$$

where $N(\psi,\ \theta)$ is same as in Theorem (4.1.7).

(b) N_g is non-empty.

Then, g admits a FP.

Proof. Choose $\psi_0 \in N_g$. There is $\psi_1 \in g\psi_0$ s.t $(\psi_0, \psi_1) \in \mathsf{E}(G)$. Consequently, by usnig condition (a) for ψ_0 and ψ_1. Then,

$$\begin{aligned} sD\left(\psi_1,\ g\psi_1\right) &\leq sH\left(g\psi_0,\ g\psi_1\right) \\ &\leq \Phi\left(N\left(\psi_0,\ \psi_1\right)\right) \\ &= \Phi\left(\max\left\{\mathsf{d}_b\left(\psi_0,\ \psi_1\right),\ \frac{D\left(\psi_0,\ g\psi_0\right)+D\left(\psi_1,\ g\psi_1\right)}{2},\right.\right. \\ &\left.\frac{D\left(\psi_0,\ g\psi_1\right)+D\left(\psi_1,\ g\psi_0\right)}{2s},\ \frac{D\left(\psi_0,\ g\psi_0\right)D\left(\psi_1,\ g\psi_1\right)}{1+H\left(g\psi_0,\ g\psi_1\right)}\right\} \\ &\leq \Phi\left(\max\left\{\mathsf{d}_b\left(\psi_0,\ \psi_1\right),\ \frac{\mathsf{d}_b\left(\psi_0,\ \psi_1\right)+\mathsf{d}_b\left(\psi_1\psi_2\right)}{2}\right.\right. \\ &\qquad \left.\frac{\mathsf{d}_b\left(\psi_0,\ \psi_2\right)}{2s},\ \frac{\mathsf{d}_b\left(\psi_0,\ \psi_1\right)\ D\left(\psi_1,\ g\psi_1\right)}{D\left(\psi_1,\ g\psi_1\right)}\right\} \\ &\leq \Phi\left(\max\left\{\mathsf{d}_b\left(\psi_0,\ \psi_1\right),\frac{\mathsf{d}_b\left(\psi_0,\ \psi_1\right)+\mathsf{d}_b\left(\psi_1,\psi_2\right)}{2},\mathsf{d}_b\left(\psi_1,\ \psi_2\right)\right\}\right) \\ &\leq \Phi\left(\max\left\{\mathsf{d}_b\left(\psi_0,\ \psi_1\right)\right\}\right). \end{aligned}$$

Given the compactness of $g\psi_1$, there is an element ψ_2 in $g\psi_1$ s.t $\mathsf{d}_b\left(\psi_1,\ \psi_2\right) = D\left(\psi_1,\ g\psi_1\right)$, so

$$s\mathsf{d}_b\left(\psi_1,\ \psi_2\right) \leq \Phi\left(\mathsf{d}_b\left(\psi_0,\ \psi_1\right)\right),$$

since $(\psi_0,\ \psi_1) \in \mathsf{E}(G),\ \psi_1 \in g\psi_0$ and $\psi_2 \in g\psi_1$, using the WGP property, we get $(\psi_1,\ \psi_2) \in \mathsf{E}(G)$. Then similarly applying the same procedure as above it can be written as

$$\begin{aligned} sD(\psi_2,\ g\psi_2) &\leq sH(g\psi_1,\ g\psi_2) \\ &\leq \Phi(N(\psi_1,\ \psi_2)) \\ &\leq \Phi(\mathsf{d}_b(\psi_1,\ \psi_2)), \end{aligned}$$

since $g\psi_2$ is compact, again $\exists\ \psi_3 \in g\psi_2$ s.t $\mathsf{d}_b(\psi_2,\ \psi_3) = D(\psi_2,\ g\psi_2)$. Therefore,

$$s\mathsf{d}_b(\psi_2,\ \psi_3) \leq \Phi(\mathsf{d}_b(\psi_1,\ \psi_2)).$$

By repeatedly applying this procedure, we generate a sequence $\{\psi_q\}$ in X s.t ψ_{q+1} belongs to $g\psi_q$ and (ψ_q, ψ_{q+1}) is an element of $\mathsf{E}(G)$

$$\begin{aligned} s\mathsf{d}_b(\psi_q,\ \psi_{q+1}) &\leq \Phi(\mathsf{d}_b(\psi_{q-1},\ \psi_q)) \\ &\leq \Phi^2(\mathsf{d}_b(\psi_{q-2},\ \psi_{q-1})) \\ &\ \ \vdots \\ &\leq \Phi^q(\mathsf{d}_b(\psi_0,\ \psi_1)). \end{aligned}$$

Now we will show that $\{\psi_q\}$ is a CS. Let $p,\ q \in \mathbb{N}$ with $p > q$.

Consider

$$\begin{aligned} \mathsf{d}_b(\psi_q,\ \psi_p) &\leq \sum_{j=q}^{p-1} s^{j+1-q}\ \mathsf{d}_b(\psi_j,\ \psi_{j-1}) \\ &\leq \sum_{j=q}^{p-1} s^{j-q}\Phi^j(\mathsf{d}_b(\psi_0,\ \psi_1)) \\ &\leq \sum_{j=q}^{p-1} s^q\ s^{j-q}\Phi^j(\mathsf{d}_b(\psi_0,\ \psi_1)) \\ &= \sum_{j=q}^{p-1} s^j\Phi^j(\mathsf{d}_b(\psi_0,\ \psi_1)) \\ &\leq \sum_{j=0}^{\infty} s^j\Phi^j(\mathsf{d}_b(\psi_0,\ \psi_1)). \end{aligned}$$

Given that Φ is a bCF, then series on the R.H.S converges. As a result, $\mathsf{d}_b(\psi_q, \psi_p) \to 0$ as $q, p \to \infty$. In other words, the sequence $\{\psi_q\}$ is a CS in the complete bMS. Therefore, $\{\psi_q\}$ converges to a certain point $\mu \in X$. As g is USC and by the Lemma (3.1.2), we can conclude that $\mu \in g\mu$. This implies that g has a FP. □

4.2 $\mathcal{F}$-Contraction

Let G be a diagraph on a MS X and $\Gamma : X \to CB(X)$ be a mapping. Define

$$\Gamma_G \equiv \{(\psi,\ \theta) \in \mathsf{E}(G) : H(\Gamma\psi,\ \Gamma\theta) > 0\},$$
$$X_\Gamma = \{\psi \in X : (\psi,\ \theta) \in \mathsf{E}(G) \text{ for some } \theta \in \Gamma\psi\},$$

and

$$L(\psi,\ \theta) = \max\left\{ \begin{array}{c} \mathsf{d}_b(\psi,\ \theta),\ D(\psi,\ \Gamma\psi),\ D(\theta,\ \Gamma\theta), \\ \dfrac{D(\psi,\ \Gamma\psi)\, D(\psi,\ \Gamma\theta) + D(\theta,\ \Gamma\theta)\, D(\theta,\ \Gamma\psi)}{\max\{D(\psi,\ \Gamma\theta),\ D(\theta,\ \Gamma\psi)\}}, \end{array} \right\}.$$

Now, Γ is a multivalued $\mathcal{F}$-contraction if $\exists\ \mathcal{F} \in \mathcal{J}$ and $\gamma > 0$ s.t
$\forall\ \max\{D(\psi, \Gamma\theta), D(\theta, \Gamma\psi)\} \neq 0$,

$$\gamma + \mathcal{F}(sH(\Gamma\psi,\ \Gamma\theta)) \leq \mathcal{F}(L(\psi,\ \theta)).$$

for $\psi,\ \theta \in X$ with $(\psi,\ \theta) \in \Gamma_G$.

Theorem 4.2.1. If we have a complete bMS (X, d_b) with a digraph G and a multivalued $\mathcal{F}$-contraction $\Gamma : X \to K(X)$, then if the set X_Γ is not empty, we can conclude that Γ has a FP.

Proof. If every $\psi \in X$ satisfies $D(\psi, \Gamma\psi) > 0$ and ψ_0 belongs to X_Γ, then there exists $\psi_1 \in \Gamma\psi_0$ s.t (ψ_0, ψ_1) is an element of $\mathsf{E}(G)$

$$0 < D(\psi_1,\ \Gamma\psi_1) \leq H(\Gamma\psi_0,\ \Gamma\psi_1).$$

By the $F-$contractive condition, it can be written as

$$\begin{aligned}\mathcal{F}\left(sD\left(\psi_1,\ \Gamma\psi_1\right)\right) &\leq \mathcal{F}\left(sH\left(\Gamma\psi_0,\ \Gamma\psi_1\right)\right)\\ &\leq \mathcal{F}\left(L\left(\psi_0,\ \psi_1\right)\right)-\gamma\\ &= F\left(\max\left\{\mathsf{d}_b\left(\psi_0,\ \psi_1\right),\ D\left(\psi_0,\ \Gamma\psi_0\right),\ D\left(\psi_1,\ \Gamma\psi_1\right),\right.\right.\\ &\left.\left.\frac{D\left(\psi_0,\ \Gamma\psi_0\right)D\left(\psi_0,\ \psi_1\right)+D\left(\psi_1,\ \Gamma\psi_1\right),\ D\left(\psi_1,\ \Gamma\psi_0,\ \right)}{\max\left\{D\left(\psi_0,\ \Gamma\psi_1\right),\ D\left(\psi_1,\ \Gamma\psi_0\right)\right\}}\right\}\right)-\gamma\\ &\leq \mathcal{F}\left(\max\left\{\mathsf{d}_b\left(\psi_0,\ \psi_1\right),\ \mathsf{d}_b\left(\psi_1,\ \psi_2\right)\right\}-\gamma.\end{aligned}$$

Because $\Gamma\psi_1$ is compact, $\exists\ \psi_2\in\Gamma\psi_1$ s.t $\mathsf{d}_b\left(\psi_1,\ \psi_2\right)=D\left(\psi_1,\ \Gamma\psi_1\right)$, so we have,

$$\mathcal{F}\left(s\mathsf{d}_b\left(\psi_1,\ \psi_2\right)\right)\leq\mathcal{F}\left(\mathsf{d}_b\left(\psi_0,\ \psi_1\right)\right)-\gamma. \tag{4.8}$$

Since $(\psi_0,\ \psi_1)\in\mathsf{E}(G)$, $\psi_1\in\Gamma\psi_0$ and $\psi_2\in\Gamma\psi_1$, by the *WGP* property $(\psi_1,\ \psi_2)\in\mathsf{E}(G)$ Considering $0<D\left(\psi_2,\ \Gamma\psi_2\right)\leq H\left(\Gamma\psi_1,\ \Gamma\psi_2\right)$, we get $(\psi_1,\ \psi_2)\in\Gamma\psi_2$, then

$$\mathcal{F}\left(sD\left(\psi_2,\ \Gamma\psi_2\right)\right)\leq\mathcal{F}\left(sH\left(\Gamma\psi_1,\ \Gamma\psi_2\right)\right)<\mathcal{F}\left(\mathsf{d}_b\left(\psi_1,\ \psi_2\right)\right)-\gamma$$

Due to Compactness of $\Gamma\psi_2$ there is $\psi_3\in\Gamma\psi_2$, s.t $\mathsf{d}_b\left(\psi_2,\ \psi_3\right)=D\left(\psi_2,\ \Gamma\psi_2\right)$, so we have

$$\mathcal{F}\left(s\mathsf{d}_b\left(\psi_2,\ \psi_3\right)\right)\leq\mathcal{F}\left(\mathsf{d}_b\left(\psi_1,\ \psi_2\right)\right)-\gamma. \tag{4.9}$$

Similarly

$$\mathcal{F}\left(s\mathsf{d}_b\left(\psi_3,\ \psi_4\right)\right)\leq\mathcal{F}\left(\mathsf{d}_b\left(\psi_2,\ \psi_3\right)\right)-\gamma. \tag{4.10}$$

A sequence is generated by continuation of the above procedure $\{\psi_q\}$ within X,

$$\mathcal{F}\left(s\mathsf{d}_b\left(\psi_q,\ \psi_{q+1}\right)\right)\leq\mathcal{F}\left(\mathsf{d}_b\left(\psi_{q-1},\ \psi_q\right)\right)-\gamma\quad\forall\ q\in\mathbb{N}, \tag{4.11}$$

where $\psi_{q+1}\in\Gamma\psi_q$, $(\psi_q,\ \psi_{q+1})\in\Gamma_G$. Let $\mathsf{d}_b\left(\psi_q,\ \psi_{q+1}\right)$ is denoted by τ_q. It follows that τ_q is greater than zero and the sequence $\{\tau_q\}$ exhibits a monotonically decreasing pattern of real numbers. Consequently, there exists a non-negative number ω s.t $\lim_{p\to\infty} \tau_q = \omega$. Now, (4.11) can be written as

$$\mathcal{F}\left(s\tau_q\right) \leq \mathcal{F}\left(\tau_{q-1}\right) - \gamma \ \forall\ q \in \mathbb{N}$$

and some $\gamma > 0$. Then by (2.1)

$$\gamma + \mathcal{F}\left(s^q\tau_q\right) \leq \mathcal{F}\left(s^{q-1}\tau_{q-1}\right), \quad \forall\ q \in \mathbb{N}.$$

Hence by induction

$$\mathcal{F}\left(s^q\tau_q\right) \leq \mathcal{F}\left(s^{q-1}\tau_{q-1}\right) - \gamma \leq \cdots \leq \mathcal{F}\left(\tau_0\right) - q\gamma. \tag{4.12}$$

As q approaches infinity, we obtain $\lim_{q\to\infty} \mathcal{F}\left(s^q\tau_q\right) = -\infty$. By F_3, there exists a value k within the range of $(0,\ 1)$ s.t the expression $\lim_{n\to\infty} s^q\tau_q = 0$ holds. Then $\lim_{q\to\infty} \left(s^q\tau_q\right) \mathcal{F}\left(s^q\tau_q\right) = 0$. Multiplication of (4.12) by $\left(s^q\tau_q\right)^k$ yields

$$0 \leq \left(s^q\tau_q\right) \mathcal{F}\left(s^q u_q\right) + q\gamma \left(s^q\tau_q\right)^k \leq \left(s^q\tau_q\right)^k \mathcal{F}\left(\tau_0\right). \tag{4.13}$$

$$\lim_{q\to\infty} q\left(s^q\tau_q\right)^k = 0. \tag{4.14}$$

There exists a natural no q s.t $q\left(x^q\tau_q\right)^k \leq 1\ \forall\ q \geqslant q_1$. Then $s^q\tau_q \leq \dfrac{1}{n^{1/k}} \quad \forall\ q \geqslant q_1$.

$$\begin{aligned}
\mathsf{d}_b\left(\psi_q,\ \psi_p\right) &\leq \sum_{j=q}^{p-1} s^{j+1-q}\mathsf{d}_b\left(\psi_j,\ \psi_{j+1}\right)\\
&= \sum_{j=q}^{p-1} s^{q-1}s^{j+1-q}\tau_j\\
&= \sum_{j=q}^{\infty} s^j\tau_j\\
&\leq \sum_{j=1}^{\infty} \left(\frac{1}{j^{1/k}}\right).
\end{aligned}$$

where, p and q as natural numbers where $p > q \geq q_1$,and it is given that k is in the interval $(0, 1)$, the series $\sum_{j=1}^{\infty} \frac{1}{(j)^{1/k}}$ converges. As a result, $\mathsf{d}_b(\psi_q, \psi_p)$ tends to 0 as q and p approach infinity. This implies that the sequence $\{\psi_q\}$ is a CS in (X, d_b) which is a complete bMS. Consequently, $\{\psi_q\}$ converges to a certain point $\mu \in X$. Using the upper semi-continuity of the operator Γ and Lemma (3.1.2), we can conclude that μ belongs to $\Gamma\mu$. This leads to our initial assumption so Γ possesses a FP. □

Theorem 4.2.2. Let (X, d_b) be a complete bMS that is equipped with a digraph G. Let $\Gamma : X \to K(X)$ be a multivalued mapping satisfying $\mathcal{F}$-contraction properties, where $\mathcal{F} \in \mathcal{J}*$. If X_Γ is non-empty, then Γ has a FP.

Proof. Suppose, there is no FP of Γ, for every $\psi \in X$ then $D(\psi,\ \Gamma\psi) > 0$. Let $\psi_0 \in X_\Gamma$ and there exists $\psi_1 \in \Gamma\psi_0$ s.t $(\psi_0,\ \psi_1) \in \mathsf{E}(G)$. Consequently, we obtain

$$0 < D\left(\psi_1,\ \Gamma\psi_1\right) \leq H\left(\Gamma\psi_0,\ \Gamma\psi_1\right).$$

Thus $(\psi_0,\ \psi_1) \in \Gamma_G$. so from (F4)

$$\begin{aligned}\mathcal{F}\left(sD\left(\psi_1,\ \Gamma\psi_1\right)\right) &= \mathcal{F}\left(\inf\left\{\mathsf{d}_b\left(\psi_1,\ \nu\right) : \nu \in \Gamma\psi_1\right\}\right) \\ &= \inf\left\{\mathcal{F}\left(s\mathsf{d}_b\left(\psi_1,\ \nu\right) : \nu \in \Gamma\psi_1\right)\right\}.\end{aligned}$$

Furthermore,

$$\inf \mathcal{F}\left(s\mathsf{d}_b\left(\psi_1,\ \nu\right) : \nu \in \Gamma\psi_1\right) < \mathcal{F}\left(\mathsf{d}_b\left(\psi_0,\ \psi_1\right)\right) - \frac{\gamma}{2}.$$

Therefore, there exists $\psi_2 \in \Gamma\psi_1$ s.t

$$\mathcal{F}\left(s\mathsf{d}_b\left(\psi_1,\ \psi_2\right)\right) \leq \mathcal{F}\left(\mathsf{d}_b\left(\psi_0,\ \psi_1\right)\right) - \frac{\gamma}{2}. \tag{4.15}$$

As $(\psi_0,\ \psi_1)$ belongs to the set $\mathsf{E}(G)$ and ψ_1 is an element of $\Gamma\psi_0$, while ψ_2 is an element of $\Gamma\psi_1$, according to the *WGP* property, it can be stated that $(\psi_1,\ \psi_2)$ is an element of $\mathsf{E}(G)$. So $0 < D\left(\psi_2,\ \Gamma\psi_2\right) \leq H\left(\Gamma\psi_1,\ \Gamma\psi_2\right)$.

Again by using (F4).

$$\mathcal{F}\left(sD\left(\psi_2,\ \Gamma\psi_2\right)\right)=\mathcal{F}\left(\inf\left(s\mathsf{d}_b\left(\psi_2,\ v\right):\nu\in\Gamma\psi_2\right\}\right)$$

$$=\inf\left\{\mathcal{F}\left(s\mathsf{d}_b\left(\psi_1,\ \nu\right):v\in\Gamma\psi_2\right)\right\}.$$

This implies

$$\inf\left\{\mathcal{F}\left(s\mathsf{d}_b\left(\psi_2,\ v\right):\nu\in\Gamma\psi_2\right)\right\}<\mathcal{F}\left(\mathsf{d}_b\left(\psi_1,\ \psi_2\right)\right)-\frac{\gamma}{2}.$$

Then there is $\psi_3\in\Gamma\psi_2$ so that

$$\mathcal{F}\left(s\mathsf{d}_b\left(\psi_2,\ \psi_3\right)\right)\leq\mathcal{F}\left(\mathsf{d}_b\left(\psi_1,\ \psi_2\right)\right)-\frac{\gamma}{2}. \tag{4.16}$$

Similarly, we construe a sequence $\{\psi_q\}$ in ψ s.t $\psi_{q+1}\in\Gamma\psi_q$, *thus* $(\psi_q,\ \psi_{q+1})\in\Gamma_G$ s.t

$$\mathcal{F}\left(s\mathsf{d}_b\left(\psi_q,\ \psi_{q+1}\right)\right)\leq\mathcal{F}\left(\mathsf{d}_b\left(\psi_{q-1},\ \psi_q\right)\right)-\frac{\gamma}{2}.$$

Let $\tau_q=\mathsf{d}(\psi_q,\psi_{q+1})$. In this case, τ_q is greater than zero and the sequence $\{\tau_q\}$ forms a decreasing sequence. Thus, there exists a non-negative value ω s.t the limit of τ_q as q approaches infinity is ω. By (F_4)

$$\mathcal{F}\left(s\tau_q\right)\leq\mathcal{F}\left(\tau_{q-1}\right)-\frac{\gamma}{2},\quad\forall\ q\in\mathbb{N}\text{ and }\gamma>0,$$

then

$$\frac{\gamma}{2}+\mathcal{F}\left(s^q\tau_q\right)\leq\mathcal{F}\left(s^{q-1}\tau_{q-1}\right),\ \forall\ q\in\mathbb{N}\text{ and }\gamma>0.$$

Hence by induction

$$\mathcal{F}\left(s^q\tau_q\right)\leq\mathcal{F}\left(s^{q-1}\tau_{q-1}\right)-\frac{\gamma}{2}\leq\ldots\leq\mathcal{F}\left(\tau_0\right)-\frac{q\gamma}{2}. \tag{4.17}$$

So, when q tends to infinity, $\lim_{q\to\infty}\mathcal{F}\left(s^q\tau_q\right)=-\infty$. There exists a value k in the interval $(0,1)$ s.t it satisfies condition (F_3), $\lim_{q\to\infty} (s^q\tau_q)^k \mathcal{F}(s^q\tau_q) = 0$. Multiplication of (4.17) by $(s^q\tau_q)^k$ yields

$$0 \leq \left(s^q\tau_q^k\right)^k \mathcal{F}(s^q\tau_q) + \frac{q\gamma}{2}(s^q\tau_q)^k \leq (s^q\tau_q)^k \mathcal{F}(\tau_0). \tag{4.18}$$

As q tends to infinity, we obtain

$$\lim_{q\to\infty} q\,(s^q\tau_q)^k = 0, \tag{4.19}$$

from (4.18) there is $q \in \mathbb{N}$ s.t $n\,(s^q\tau_q)^k \leq 1\ \forall\ q \geqslant q_1$. Then

$$s^q\tau_q \leq \frac{1}{q^{1/k}}\ \forall\ q \geqslant q_1. \tag{4.20}$$

We now assert that the sequence $\{\psi_q\}$ is a CS. To prove this, consider p and q as natural numbers where $p > q \geq q_1$, then

$$\begin{aligned}
\mathsf{d}_b(\psi_q,\ \psi_p) &\leq \sum_{j=q}^{p-1} s^{j+1-q}\mathsf{d}_b(\psi_j,\ \psi_{j+1})\\
&= \sum_{j=q}^{m-1} s^{q-1}s^{j+1-q}\tau_j\\
&= \sum_{j=q}^{\infty} s^j\tau_j\\
&\leq \sum_{j=1}^{\infty}\left(\frac{1}{j}\right)^{1/k}
\end{aligned}$$

since $k \in (0,\ 1)$, so the series $\sum_{j=1}^{\infty}\left(\frac{1}{j^{1/k}}\right)$ converges. Then $\mathsf{d}_b(\psi_q,\ \psi_p) \to 0$ as $p,\ q \to \infty$. Hence $\{\psi_q\}$ is a CS in complete bMS. Hence $\{\psi_q\}$ is convergent to some $\mu \in X$. By the property of the upper semi-continuity of Γ, we encounter a contradiction to our initial assumption, so Γ has a FP. □

Theorem 4.2.3. Consider (X, d_b) a complete bMS, that is equipped with a digraph G, where the following conditions hold:

for any $\{\psi_q\}$ in X, if $\{\psi_q\}$ converges to ψ and $(\psi_q, \psi_{q+1}) \in \mathsf{E}(G)$, then there exists a subsequence $\{\psi_{q_k}\}$ s.t (ψ_{q_k}, ψ) is an element of $\mathsf{E}(G)$.

Let $\Gamma : X \to K(X)$ be a multivalued mapping (with $\mathcal{F} \in \mathcal{J}*$) satisfying $\mathcal{F}$-contraction properties. If we consider a *WGP* mapping Γ with a non-empty set X_Γ and if $\mathcal{F}$ is a continuous function, then it can be deduced that Γ possesses a FP.

Proof. Suppose that Γ has no FP. Now, proceeding similarly as in Theorem (4.2.1) there is CS $\{\psi_q\}$ converges to some $\mu \in X$. According to the given property, there exists a subsequence $\{\mu q_k\}$ of $\{\psi_q\}$ s.t $(\psi_{q_k}, \mu) \in \mathsf{E}(G)$ for every $k \in \mathbb{N}$. Since $\lim_{q\to\infty} \psi_q = \mu$ and $D(\mu, \Gamma\mu) > 0$, there does not exist a natural number s.t

$$D\left(\psi_{q_{k+1}},\ \Gamma_\mu\right) = 0.$$

Therefore for each $q_k \geqslant q_0$

$$H\left(\Gamma\psi_{q_k},\ \Gamma\mu\right) > 0.$$

Therefore, for all $q_k \geq q_0$, we have $(\psi_{q_k}, \mu) \in \Gamma_G$. By utilizing condition (F_1),

$$\begin{aligned}
\mathcal{F}\left(sD\left(\psi_{q_{k+1}},\ \Gamma\mu\right)\right) &\leq \mathcal{F}\left(sH\left(\Gamma\psi_{q_k},\ \Gamma\mu\right)\right) - \gamma \\
&\leq \mathcal{F}\left(L\left(\psi_{q_k},\ \mu\right)\right) - \gamma \\
&= \mathcal{F}\Bigg(\max\Bigg\{\mathsf{d}_b\left(\psi_{q_k},\ \mu\right),\ D\left(\psi_{q_k},\ \Gamma\psi_{q_k}\right),\ sD(\mu,\ \Gamma\mu), \\
&\qquad \frac{D\left(\psi_{q_k},\ \Gamma\psi_{q_k}\right) D\left(\psi_{q_k},\ \Gamma\mu\right) + D(\mu,\ \Gamma\mu)\mathsf{d}_b\left(\mu,\ \Gamma\psi_{q_k}\right)}{\max\left\{D\left(\psi_{q_k},\ \Gamma\mu\right),\ D\left(\mu,\ \Gamma\psi_{q_k}\right)\right\}}\Bigg\}\Bigg) \\
&\leq \mathcal{F}\Bigg(\max\Bigg\{\mathsf{d}_b\left(\psi_{q_k},\ \mu\right),\ \mathsf{d}_b\left(\psi_{q_k},\ \psi_{q_{k+1}}\right),\ sD(\mu,\ \Gamma\mu), \\
&\qquad \frac{\mathsf{d}_b\left(\psi_{q_k},\ \psi_{q_{k+1}}\right) D\left(\psi_{q_k},\ \Gamma\mu\right) + D(\mu,\ \Gamma\mu)\mathsf{d}_b\left(\mu,\ \psi_{q_{k+1}}\right)}{\max\left\{D\left(\psi_{q_k},\ \tau\mu\right),\ D\left(\mu,\ \Gamma\psi_{q_k}\right)\right\}}\Bigg\}\Bigg)
\end{aligned}$$

$\forall\ q_k \geqslant q_0$. By allowing k to approach infinity and due to the continuity of $\mathcal{F}$, we conclude that

$$\mathcal{F}(sD(\mu,\ \Gamma\mu)) \leq \mathcal{F}(sD(\mu,\ \Gamma\mu)) - \gamma$$

that is

$$\gamma + \mathcal{F}(sD(\mu,\ \Gamma\mu)) \leq \mathcal{F}(sD(\mu,\ \Gamma\mu)).$$

This is a contradiction, implying that Γ possesses a FP. $\square$

Corollary 4.2.4. Suppose $(X,\ \mathsf{d}_b)$ is a complete bMS with a digraph G and a mapping $\Gamma : X \to K(X)$. Suppose that there are $\mathcal{F} \in \mathcal{J}$ and $\gamma > 0$ s.t

$$\gamma + \mathcal{F}(H(\Gamma v,\ \Gamma\psi)) \leq \mathcal{F}(\mathsf{d}_b(v,\ \psi))$$

$\forall\ v,\ \psi \in X$ with $(v,\ \psi) \in \Gamma_G$. If Γ is both USC and a *WGP* mapping and the set X_Γ is non-empty, then Γ has a FP.

Corollary 4.2.5. Let $(X,\ \mathsf{d}_b)$ be a complete bMS endowed with a digraph G, and $\Gamma : X \to CB(X)$ be a mapping. Let $\mathcal{F} \in \mathcal{J}_*$ and γ a positive constant s.t

$$\gamma + \mathcal{F}(H(\Gamma v,\ \Gamma\psi)) \leq \mathcal{F}(\mathsf{d}_b(v,\ \psi))$$

for $v, \psi \in X$, with $(v,\ \psi) \in \Gamma_G$ Let Γ is USC and a *WGP* and the set X_Γ is non-empty, then Γ possesses a FP.

Example 4.2.6. Let $X = \left[0, \frac{2}{3}\right] \cup \{1\}$ and the $\mathsf{d}_b(\rho, \sigma) = |\rho - \sigma|^2$ for all $\rho, \sigma \in X$. Then $(X,\ \mathsf{d}_b)$ is a complete bMS with $s = 2$.
Now, define a mapping $T : X \to CB(X)$ by:

$$T(\rho) = \begin{cases} \{0, \frac{1}{3}, \frac{5}{12}\} & , \text{ if } \rho = 1 \\ \left\{\frac{\rho}{4}\right\} & , \text{ if } \rho \in \left[0, \frac{2}{3}\right] \end{cases}$$

and a graph on X by $V(G) = X$ and

$$E(G) = \left\{(\rho,\ \omega) \mid \rho, \sigma \in \left[0, \frac{2}{3}\right]\right\} \cup \left\{(1, 0), \left(1, \frac{1}{3}\right), \left(1, \frac{5}{12}\right)\right\}$$

Then T is USC and a WGP) mapping. To show that T is a multivalued F-Khan contraction with $k \in \left[\frac{1}{16}, \frac{1}{4}\right]$, where $\mathcal{F}(\rho) = \ln \rho$ and $\gamma = \ln 2$. Let $(\rho,\ \kappa) \in E(G)$ such that $T(\rho) \neq T(\kappa)$.
Then,
$H(T\rho, T\sigma) = \mathsf{d}_b\left(\frac{\rho}{4}, \frac{\sigma}{4}\right)$

$L(\rho,\ \sigma) = \mathsf{d}_b\,(\rho,\ \sigma)$.

Now consider the following cases:

Case-1. $\rho, \sigma \in \left[0, \frac{2}{3}\right]$. Then,

$L(\rho,\ \sigma) = \mathsf{d}_b\,(\rho,\ \sigma)$, and for $k \in \left[\frac{1}{16}, \frac{1}{4}\right]$

$$H\,(T\rho, T\sigma) = \mathsf{d}_b \left(\frac{\rho}{4}, \frac{\sigma}{4}\right) \leq k\mathsf{d}_b\,(\rho,\ \sigma) \leq \frac{1}{4}\,\mathsf{d}_b\,(\rho,\ \sigma) = \frac{1}{4}\,L\,(\rho,\ \sigma)\,.$$

Hence $4H\,(T\rho, T\sigma) \leq L\,(\rho,\ \sigma)$, so we have $\ln 2 + \ln\left(2H\,(T\rho, T\sigma)\right) \leq \ln\left(L\,(\rho,\ \sigma)\right)$.

$\implies \ln 2 + \mathcal{F}\left(2H\,(T\rho, T\sigma)\right) \leq \mathcal{F}\left(L\,(\rho,\ \sigma)\right)$

Case-2. $\rho = 1, \sigma \in 0, \frac{1}{3}, \frac{5}{12}$. So

$$H\,(T\rho, T\sigma) = \mathsf{d}_b \left(\frac{\rho}{4}, \frac{\sigma}{4}\right) \leq k\mathsf{d}_b\,(\rho,\ \sigma) \leq \frac{1}{4}\,\mathsf{d}_b\,(\rho,\ \sigma) = \frac{1}{4}\,L\,(\rho,\ \sigma)\,.$$

Hence $4H\,(T\rho, T\sigma) \leq L\,(\rho,\ \sigma)$, so we have $\ln 2 + \ln\left(2H\,(T\rho, T\sigma)\right) \leq \ln\left(L\,(\rho,\ \sigma)\right)$.

$\implies \ln 2 + \mathcal{F}\left(2H\,(T\rho, T\sigma)\right) \leq \mathcal{F}\left(L\,(\rho,\ \sigma)\right)$

So all assumptions in Theorem (3.2.2) (or Theorem (3.2.3)) are satisfied. Therefore, T has a FP. It is important to note that without considering the graph on X, the contractive condition is not satisfied. In fact, by taking $\rho =$ and $\sigma = 1$, $H\,(T(\rho),\ T(\sigma)) = 0$ and $d(\rho,\ \sigma) = 0$, then from

$$\gamma + \mathcal{F}(H(T(\rho),\ T(\sigma))) < \mathcal{F}(d(\rho,\ \sigma)) \quad \forall \mathcal{F} \in \mathcal{J} \text{ and } \gamma > 0.$$

we get $\gamma < 0$, which is a contradiction.

Chapter 5

Conclusion

In this thesis the work of Acar et al. on "New Fixed Point Results via Graph Structure" is examined and elaborated to represent the complete analysis of this article. This research aimed mainly to extend the above results in the setting of b-metric spaces. For this purpose, the notion of rational-type multivalued G-contractions and $\mathcal{F}$-contractions in b-metric spaces are established. Moreover some fixed point theorems are established in the setting of b-metric space. Our results might be beneficial in determining fixed points in perception of b -metric spaces.

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