CAPITAL UNIVERSITY OF SCIENCE AND TECHNOLOGY
ISLAMABAD

Fixed point theorems for some contractions in Rectangular b-metric space

by

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A thesis submitted in partial fulfillment for the degree of Master of Philosophy

in the

Faculty of Computing
Department of Mathematics

March 2017

Declaration of Authorship

I, Mr Nadeem Abbas, declare that this thesis titled, Fixed point theorems for some contractions in Rectangular b-metric space' and the work presented in it are my own. I confirm that:

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- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
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Pure mathematics is, in its way, the poetry of logical ideas.

Albert Einstein

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Abstract

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The concept of rectangular b-metric space is introduced as a generalization of metric space, rectangular metric space and b-metric space. In this thesis, we obtained b-rectangular metric variant of fixed point results for some contractions. We have also proved some fixed point results for some contractive type conditions in the context of rectangular b-metric space. A particular example is also given in the support of our established result regarding some contraction. Our result generalizes some known results in fixed point theory.

Acknowledgements

The time leading to this MS dissertation have been for me an exciting period of discovery and growth, made possible by the help and support of several people who accompanied me on my journey. I wish to gratefully acknowledge their essential contribution to the results presented here.

First of all, I would like to thank my supervisor, Dr. Rashid Ali. I count myself as being the most fortunate to be able to work under the supervision of such a nice and kind person. The same acknowledgement should also go to Dr. Muhammad Sagheer, head of the Mathematics department (CUST), I deeply appreciate the insight I gained through my association with him. I would like to thank my class-fellows for their help in learning MiKTeX / Latex. Thanks are also due to my cousin Mr Sajjad Mahmood and my senior fellow Mr Muhammad Anwar for providing some help and requested material which was very helpful.

I acknowledge some other people, to whom I requested for some help on their works but didn't get positive response.

Finally, this thesis might not exist at all without the love and support of my family. I am grateful to my parents, who have given all the love and care and brought me up in this stage, always helping me to differentiate between what is right and what is wrong. My mother sacrifices for me more than any other in my whole life. special thanks to my mother once again.

Nadeem Abbas
MMT 143006

Contents

Dedicated to

my beloved sweet

parents

who made my life much more purposeful and pleasant.

Chapter 1

Introduction

The theory of fixed point provides very productive and constructive tools in present-time mathematics and may also assessed as a key topic of nonlinear analysis. In the last 50 years, the theory of fixed point has become the most growing and interesting field of research for almost every mathematicians. The origination of this theory, which date to the later part of the 19th century, rest in the use of unbroken and sequential estimation to built the uniqueness as well as existence of results, especially to the differential equations. This method is related with so many recognized mathematicians alike Banach [23], Lipschitz [14], Fredrick [14] and Picard [22]. One of the major field of theory of fixed point was theory of metric fixed point and we acknowledge that the more valuable content in the growth of non linear analysis is fixed point theory. Historically the starting line in this field was well-defined by the creation of "Banach's Fixed Point Theorem" familiar as "Banach contraction principle" or "BCP" in short.
It states that every contraction on a complete metric space has a unique fixed point. More precisely, if (X, d) is a complete metric space and $T\colon X \to X$ is a self map on X and $\alpha \in [0, 1)$ is such that

$$d(Tx, Ty) \leq \alpha d(x, y)$$

for all $x, y \in X$ then T has a unique fixed point. It is well established that "BCP" is basic result in the innovation of theory of fixed point, have pre-owned and expanded in non-identical ways and so many different kinds of fixed point theorems were put into effect. The BCP is prolonging, refining and extending in two typical directions

1. By expanding and developing of contraction conditions on the mapping T.
2. By spreading the structure of the spaces on which T is defined.

In above first quoted way, the BCP has widespreaded in a lot of other aspects. Actually, the massive quantity of literature wetbacking the generalizations and extensions of that amazing theorem. Banach [18] established that a contraction mapping possessed a unique fixed point in complete metric space. Presic S.B [61] In 1965 and shortly in 1968 Kannan [45] refined this contraction mapping principle, more extention came from A Meir and Emmett Keeler [54] in 1969. After this Dass B.K. Gupta [28] and also Kolmogorov A.N. and S.V. Fomin [46] made more extention through rational expression in Banach contraction principle, then Dolhare U.P [29] extended this remarkable Banach contraction mapping principle. The contraction mapping is weakened in some generalization, (see [9, 23, 44, 54, 62]) and the weakened topology in some other generalization (see [14, 15, 20, 30, 36] and others.

In above second quoted way, The "BCP" is also extended by Nadler (see [59]) from single valued to set valued contraction maps, the metric spaces were widespreaded by changing the axioms of metric such as, partial metric spaces [55], cone metric spaces [41], G-metric spaces [56], 2-metric spaces [39] and many others. These generalized metric spaces frequently arises to be measurable and the contraction conditions conserved the true changes, particularly, the fixed point results may be proven on few generalized metric spaces from questionless outcome in usual metric spaces. In the present days, few new classes of metric spaces were brought in and different types of spaces raised as mixtures of the preceding such as partial cone metric spaces [58], metric like spaces [49], quasi partial metric spaces [35], cone rectangular metric spaces [8], quasi b-metric spaces [66], m-metric spaces [7], quasi metric like spaces [65], partial rectangular metric spaces [64] and others. The weaken axioms are main reason in the construction of these metric spaces.

The usual metric spaces have also been generalized in well known spaces as mentioned above, some of these spaces are b-metric spaces [10], rectangular and rectangular b-metric spaces [8]. The idea of b-metric spaces was introduced by Czerwik [22] after the generalization of metric space. On the other hand, Bakhtin in [10] also generalized the metric spaces. After that, many papers have been published having fixed point conclusion in b-metric spaces for single and multi-valued functions (see [14, 20]). After that many other papers have also been published in the field of this theory with single as well as multivalued mappings in b-metric spaces (for example [11–13, 22, 24–26]) and the references therein.

On the other hand, Branciari [14] has introduced the notion of rectangular b-metric space(RMS) by exchanging the sum of three terms expression in right hand side of triangular instead of two terms. After that more results involving fixed point with different contractive mappings in rectangular metric spaces came into view (see [6, 8, 21, 30–33, 35, 53]).

In this thesis, we used the idea of the rectangular b-metric space, not necessarily Hausdorff which generalizes the idea of the b-metric spaces. Particularly we reveiwed the results presented in [68] in the setting of b-metric space. We then extended these results in the setting of rectangular b-metric spaces. Some of these results like Kannan type theorem as well as BCP are also extended in rectangular b-metric spaces. We have also proved some other contraction mappings in rectangular b-metric spaces. We have also constructed some examples which show that our generalizations are genuine.

The rest of the thesis is organized as follows.

- In Chapter 2, we throw light on basic definitions of abstract spaces like rectangular b-metric spaces and presented some examples which satisfy the properties of above spaces.
- In Chapter 3, we reviewed comprehensively some fixed point results like Ciric's and Generalized Contraction in b-metric spaces presented in [68].
- In Chapter 4, we established new fixed point theorems by extending the results of "BCP" and Kannan type theorem in rectangular b-metric space. In the last section, we conclude our thesis.

Chapter 2

preliminaries

The concept of distance between the points allow us to define more general concept of a metric which move us to the metric space. The idea of the space arrived from fundamental concept of abstract set X whose elements satisfy certain axioms.
In this chapter, we need to recall some basic definitions, lemmas, theorems and necessary results from existing literature. The following definitions, lemmas and theorems are related to our main research. Through out, $\mathbb{R}$ stands for set of real numbers, $\mathbb{R}^+$ stands for set of positive real numbers, $\mathbb{R}_0^+$ stands for set of positive real numbers including 0 and $\mathbb{N}$ stands for set of natural numbers.

2.1 Metric spaces

In the present section, we have stated all basic concepts, definitions with examples for understanding the structure of metric spaces.

Definition 2.1.1. [40]**(Metric space)**
"Let X be a non-empty set, a mapping $d\colon X \times X \to [0, +\infty)$ is called metric on X for all $x, y, z \in X$, if the following axioms satisfied

M1. $d(x, y) = 0 \Leftrightarrow x = y$

M2. $d(x, y) = d(y, x)$

M3. $d(x, z) \leq d(x, y) + d(y, z)$.

and the pair (X, d) is called a metric space."

Example 2.1.2. $\mathbb{R}$ is the real line and define the distance by

$$d(x, y) = |x - y|.$$

then d is metric on $\mathbb{R}$ and the pair $(\mathbb{R}, d)$ is a usual metric space.

Example 2.1.3. $\mathbb{R}^2$ is the plane and the usual distance as

$$d((x_1, y_1), (x_2, y_2)) = [(x_1 - x_2)^2 + (y_1 - y_2)^2]^{\frac{1}{2}}.$$

then sometimes, it is said to be 2-metric d_2.

We are going to give the concept and definition of Cauchy sequence, convergence sequence in metric space and the completeness of the metric space. Following definitions are taken from [40].

Definition 2.1.4. (Cauchy sequence)
"A sequence $\{x_n\}$ in metric space (X, d) is called Cauchy sequence if for every $\varepsilon > 0$ there exist a positive integer N such that for $m, n \geqslant N$, we have

$$d(x_m, x_n) < \varepsilon."$$

Definition 2.1.5. (Convergence sequence)
"A sequence $\{x_n\}$ in metric space (X, d) is called convergent sequence if for every $\varepsilon > 0$ and $n \geqslant N$ we have

$$d(x_n, x) < \varepsilon.$$

where x is called the limit of the sequence $\{x_n\}$."

Definition 2.1.6. (Complete metric space)
"A metric space (X, d) is said to be complete if every Cauchy sequence in X converges to a point of X."

Example 2.1.7. The space $\mathbb{R}$ of real numbers and the space $\mathbb{C}$ of complex numbers (with the metric given by the absolute value) are complete, and so is Euclidean space $\mathbb{R}^n$ with the usual distance metric.

2.2 b-metric spaces

In this section, we have presented some basic concepts and definitions regarding the b-metric spaces. We have also given some examples to understand the structure of b-metric spaces. Czerwike [24] defined a b-metric space as follows:

Definition 2.2.1. [24](**b-metric space**)
"Let X be a non-empty set and if $b \geq 1$ be any real number then a mapping $d_b \colon X \times X \to [0, +\infty)$ is said to be b-metric if for all $x, y, z \in X$ if the following axioms are satisfied

B1. $d_b(x, y) = 0 \Leftrightarrow x = y$

B2. $d_b(x, y) = d_b(y, x)$

B3. $d_b(x, z) \leq b[d_b(x, y) + d_b(y, z)]$.

The pair (X, d_b) is then called b-metric space."

Remark 2.2.2. Let (X, d_b) is b-metric space. Generally the b-metric d_b is not continues. The following example illustrates this fact.

Example 2.2.3. [52]
"Let $X = \mathbb{N} \cup \{\infty\}$ and let $d_b \colon X \times X \to \{0, +\infty\}$ is defined by

$$d_b(m, n) = \begin{cases} 0 & \text{if } m = n, \\ |\frac{1}{m} - \frac{1}{n}| & \text{if one of } m, n \text{ is even and the other is even or } \infty. \\ 5, & \text{if one of } m, n \text{ is odd and the other is odd or } \infty, \\ 2, & \text{otherwise.} \end{cases}$$

It can be checked that for all $m, n, p \in X$, we have

$$d_b(m, p) \leq \frac{5}{2}[d_b(m, n) + d_b(n, p)].$$

Thus (X, d_b) is a b-metric space with $b = 5/2$. Let $x_n = 2n$ for each $n \in \mathbb{N}$, then

$$d_b(2n, \infty) = \frac{1}{2n} \to 0 \quad \text{as} \quad n \to \infty$$

that is, $x_n \to \infty$, but $d_b(x_n, 1) = 2 \nrightarrow 5 = d_b(\infty, 1)$ as $n \to \infty$."

Example 2.2.4. [5] "Let $X = \{0, 1, 2\}$. Define a mapping $d_b \colon X \times X \to (0, +\infty)$ as following
$d_b(0, 0) = d_b(1, 1) = d_b(2, 2) = 0$.
$d_b(0, 1) = d_b(1, 0) = d_b(1, 2) = d_b(2, 1) = 1$.
$d_b(0, 2) = d_b(2, 0) = m \geqslant 2$ for $b = \dfrac{m}{2}$ where $m \geqslant 2$.
the function defined above is a b-metric space but not a metric for $m > 2$ because if we take $m > 2$ then the coefficient $b > 1$ which contradict the third axiom of metric space."

We are going to give the concept and definition of Cauchy sequence, convergence sequence in b-metric space and the completeness of the b-metric space. These definitions are taken from [43].

Definition 2.2.5. (Cauchy sequence)
"A sequence $\{x_n\}$ in b-metric space (X, d_b) is called Cauchy sequence if for every $\varepsilon > 0$ there exist a positive integer N such that for $m, n \geqslant N$, we have

$$d_b(x_m, x_n) < \varepsilon."$$

Definition 2.2.6. (Convergence sequence)
"A sequence $\{x_n\}$ in b-metric space (X, d_b) is called convergent sequence if for every $\varepsilon > 0$ and $n \geqslant N$ we have

$$d_b(x_n, x) < \varepsilon.$$

where x is called the limit point of the sequence $\{x_n\}$."

Definition 2.2.7. (Complete b-metric space)
"A b-metric space (X, d_b) is said to be complete if every Cauchy sequence in X converges to a point of X."

2.3 Rectangular b-metric space

Here, we have given some basic concepts and definitions for the rectangular b-metric spaces. We have given few examples for understanding the structure of rectangular b-metric spaces.

Definition 2.3.1. [38]**(Rectangular metric space)**
"Let X be a non-empty set, a mapping $d_b \colon X \times X \to [0, +\infty)$ is said to be rectangular metric if for every $x, y, z \in X$ and $u, v \in X \setminus \{x, z\}$ satisfies the following axioms

RM1. $d_b(x, y) = 0 \Leftrightarrow x = y$.

RM2. $d_b(x, y) = d_b(y, x)$

RM3. $d_b(x, z) \leq d_b(x, u) + d_b(u, v) + d_b(v, z)$.

The pair (X, d_b) is called Rectangular metric space."

Example 2.3.2. Let $X = \{\frac{1}{n} \colon n \in \mathbb{R}^+\} \cup \{0\}$ define $d_r \colon X \times X \to \mathbb{R}$ by

$$d_r(x,y) = \begin{cases} 0 & \text{if } x = y; \\ \frac{1}{n}, & \text{if } \{x,y\} = \{0, \frac{1}{n}\}; \\ 1, & \text{if } x \neq y, x, y \in X \setminus \{0\} \end{cases}$$

then, (X, d_r) is a rectangular metric space but not a metric space as

$$d_r(x,y) = 1 > \frac{2}{n} = d_r(x,0) + d_r(0,y)$$

for $n > 2$ and all distinct $x, y \in X$.

Definition 2.3.3. [38]**(Rectangular b-metric space)**
"Let X be non empty set and if $b \geq 1$ is a real number then a mapping $d_b \colon X \times X \to [0, +\infty)$ is said to be rectangular b-metric for every $x, z \in X$ and $u, v \in X \setminus \{x, z\}$ satisfy the following axioms

RbM1. $d_b(x,y) = 0 \Leftrightarrow x = y$.

RbM2. $d_b(x,y) = d_b(y,x)$

RbM3. $d_b(x,z) \leq b[d_b(x,u) + d_b(u,v) + d_b(v,z)]$.

The pair (X, d_b) is then called a rectangular b-metric space."

Example 2.3.4. [38]
"Let $X = \mathbb{N}$, define $d_b \colon X \times X \to X$ such that $d_b(x,y) = d_b(y,x)$ for all $x, y \in X$ and

$$d_b(x,y) = \begin{cases} 0 & \text{if } x = y, \\ 10\alpha, & \text{if } x = 1, y = 2, \\ \alpha, & \text{if } x \in \{1,2\} \text{ and } y \in \{3\}, \\ 2\alpha, & \text{if } x \in \{1,2,3\} \text{ and } y \in \{4\}, \\ 3\alpha, & \text{if } x \text{ or } y \notin \{1,2,3,4\} \text{ and } x \neq y, \end{cases}$$

where $\alpha > 0$ is constant. then (X, d_b) is rectangular b- metric space with coefficient $b = 2 > 1$, but (X, d_b) is not rectangular metric space. as

$$d_b(1,2) = 10\alpha > 5\alpha = d_b(1,3) + d_b(3,4) + d_b(4,2)."$$

Example 2.3.5. [38]
"Let $X = \mathbb{N}$, define $d_b \colon X \times X \to X$ by

$$d_b(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 4\alpha, & \text{if } x, y \in \{1, 2\} \text{ and } x \neq y, \\ \alpha, & \text{if } x \text{ or } y \notin \{1, 2\} \text{ and } x \neq y, \end{cases}$$

where $\alpha > 0$ is constant. then (X, d_b) is rectangular b- metric space with coefficient $b = \frac{4}{3} > 1$, but (X, d_b) is not rectangular metric space as

$$d_b(1, 2) = 4\alpha > 3\alpha = d_b(1, 3) + d_b(3, 4) + d_b(4, 2)."$$

We are going to give the concept and definition of Cauchy sequence, convergence sequence in rectangular b-metric space and the completeness of the rectangular b-metric space. Following definitions are taken from [38].

Definition 2.3.6. (Cauchy sequence)
"A sequence $\{x_n\}$ in rectangular b-metric space (X, d_b) is called Cauchy sequence if for every $\varepsilon > 0$ there exist a positive integer N such that for $m, n \geqslant N$, we have

$$d_b(x_m, x_n) < \varepsilon."$$

Definition 2.3.7. (Convergence sequence)
"A sequence $\{x_n\}$ in rectangular b-metric space (X, d_b) is called convergent sequence if for every $\varepsilon > 0$ and $n \geqslant N$ we have

$$d_b(x_n, x) < \varepsilon.$$

where x is called the limit point of the sequence $\{x_n\}$."

Definition 2.3.8. (Complete rectangular b-metric space)
"A rectangular b-metric space (X, d_b) is said to be complete if every Cauchy sequence in X converges to a point of X."

2.4 Fixed point theory and Contractions

In the following section, we have presented the definition of fixed point as well as various types of contractions. We have also given some examples to understand these concepts.

Definition 2.4.1. (Fixed point)
Consider a metric space X and a mapping $F\colon X \to X$. Then any $x \in X$ is said to be a fixed point of F if $F(x) = x$.

Example 2.4.2. Consider $X = \mathbb{R}$ and $F\colon X \to X$ is a mapping define as

$$F(x) = 2x + 1$$

then F has a unique fixed point $x = -1$ in X.

The following definition and example are taken from [70].

Definition 2.4.3. [70]**(lipschitzian mapping)**
"Suppose that X is a metric space and F is a mapping from X to X. The mapping F is called a Lipschitz mapping if there exists a constant $k \geqslant 0$ such that

$$d(F(x), F(y)) \leq kd(x, y)$$

for all $x, y \in X$. The infimum over all such constants k is called the Lipschitz constant."

Example 2.4.4. [70]
"Suppose that $F\colon \mathbb{R} \to \mathbb{R}$ is continuously differentiabe and $|F'(x)| \leq k$ for every $x \in \mathbb{R}$ then according to the mean value theorem

$$|F(x) - F(y)| = |F'(\xi)||x - y| \leq k|x - y|$$

for all $x, y \in \mathbb{R}$. This shows that F is a Lipschitz mapping."

Definition 2.4.5. [42]**(Contraction)**
"Let X be a metric space, a mapping $F\colon X \to X$ is called contraction if there exists $k < 1$ such that for any $x, y \in X$,

$$d(Fx, Fy) \leq kd(x, y).$$

This contraction is also known as Banach contraction."

Example 2.4.6. [70]
"Consider the metric space $(\mathbb{R}, d)$ where d is Euclidean distance metric, that is

$$d(x, y) = |x - y|.$$

The function $f\colon \mathbb{R} \to \mathbb{R}$ where $f(x) = \frac{x}{a+b}$ is a contraction if $a > 1$."

Theorem 2.4.7. [49]**(Banach's Contraction Principle)**
"Let (X, d) be a complete metric space, $T \colon X \to X$ be a contraction mapping. Then T has a unique fixed point x_0, and for each $x \in X$, we have

$$\lim_{n\to\infty} T^n(x) = x_0,$$

Moreover, for each $x \in X$, we have

$$d(T^n(x), x_0) \leq \frac{k^n}{1-k} d(T(x), x).\text{''}$$

Here, we have presented various types of contraction for understanding and these contractions were very useful for our work.

Definition 2.4.8. [19]**(Generalized contraction)**
"Let (X, d) be a metric space, a mapping $F \colon X \to X$ is said to be generalized contraction if and only if for every $x, y \in X$, there exist c_1, c_2, c_3, c_4 such that

$$\sup\{c_1 + c_2 + c_3 + 2c_4 \colon x, y \in X\} < 1$$

and

$$d(Fx, Fy) \leq c_1.d(x, y) + c_2.d(x, Fx) + c_3.d(y, Fy) + c_4.[d(x, Fy) + d(y, Fx)]\text{''}$$

Definition 2.4.9. [47]**(Kannan type contraction)**
"Let (X, d) be a metric space, a mapping $T \colon X \to X$ is said to be a Kannan type mapping if there exist $o < \lambda < 1$ such that, for all $x, y \in X$, the following inequality is satisfied

$$d(Fx, Fy) \leq \frac{\lambda}{2}[d(x, Fx) + d(y, Fy)]\text{''}$$

Definition 2.4.10. [63]**(Ciric's type contraction)**
"Let (X, d) be a metric space, a mapping $F \colon X \to X$ is said to be Cric's type contraction if and only if for all $x, y \in X$, there exist $h < 1$ and

$$d(Fx, Fy) \leq h.\max\{d(x, y), d(x, Fx), d(y, Fy), \frac{d(x, Fy) + d(y, Fx)}{2}\}\text{''}$$

Definition 2.4.11. [19]**(Quasi contraction)**
"Let (X, d) be a metric space, a mapping $F \colon X \to X$ is said to be quasi contraction if and only if for all $x, y \in X$, there exist $h < 1$ and

$$d(Fx, Fy) \leq h.\max\{d(x, y), d(x, Fx), d(y, Fy), d(x, Fy) + d(y, Fx)\}\text{''}$$

Definition 2.4.12. [16]**(Weak contraction)**
"Let (X, d) be a metric space, a self mapping $F\colon X \to X$ is said to be weak contraction if there exist a constant $\alpha \in (0, 1)$ and some $\beta \geq 0$ such that

$$d(Fx, Fy) \leq \alpha.d(x, y) + \beta.d(y, Fx) \tag{2.1}$$

for all $x, y \in X$.
Due to the symmetry of distance, it includes the following

$$d(Fx, Fy) \leq \alpha.d(x, y) + \beta.d(x, Fy) \tag{2.2}$$

for all $x, y \in X$."

The following theorem is very useful for the review of some other theorems.

Theorem 2.4.13. [50]
"Let (X, d_b) be a complete b- metric space with $b \geqslant 1$, a mapping $F\colon X \to X$ be a contraction with $\alpha \in [0, 1)$ and $b\alpha < 1$ then F in X has a unique fixed point."

Chapter 3

Fixed point theorems in b-metric space

In the present chapter, I have reviewed and underrstood the fixed point theorems and also a related lemma for generalized and Ciric's type contraction in b-metric space. We also reviewed few results involving rational contractive type conditions. These results are presented in paper [68] by Muhammad Sarwar. From now on, by a b-metric we mean a continuous b-metric d_b.

3.1 Results in b-metric space

The following lemma is useful for the results we presented in this section.

Lemma 3.1.1. Consider a b-metric space (X, d_b) and a sequence $\{x_n\}$ in b-metric space as

$$d_b(x_n, x_{n+1}) \leq s.d_b(x_{n-1}, x_n) \tag{3.1}$$

here as n= 0, 1, 2, $\ldots$, $0 \leq bs < 1$, $s \in [0, 1)$ and $b \geqslant 1$ then $\{x_n\}$ in X is Cauchy sequence.

Proof : Consider for $n, m \in \mathbb{N}$ and $m > n$, we get

$$\begin{aligned} d_b(x_n, x_m) &\leq b.[d_b(x_n, x_{n+1}) + d_b(x_{n+1}, x_m)] \\ &\leq b.d_b(x_n, x_{n+1}) + b^2.[d_b(x_{n+1}, x_{n+2}) + d_b(x_{n+2}, x_m)] \\ &\leq b.d_b(x_n, x_{n+1}) + b^2.d_b(x_{n+1}, x_{n+2}) + b^3.d_b(x_{n+2}, x_{n+3}) \\ &+ \ldots + b^m.d_b(x_{n+m-1}, x_m)] \end{aligned}$$

As using (3.1) we get

$$\begin{aligned} d_b(x_n, x_m) &\leq bs^n.d_b(x_0, x_1) + b^2 s^{n+1}.d_b(x_0, x_1) + b^3 s^{n+2}.d_b(x_0, x_1) \\ &+ \ldots + b^m s^{n+m-1}.d_b(x_0, x_1) \\ &\leq [1 + bs + (bs)^2 + \ldots + (bs)^{m-1}].bs^n d_b(x_0, x_1) \\ &\leq bs^n(\frac{1-(bs)^m}{1-bs}).d_b(x_0, x_1) \end{aligned}$$

Since $0 \leq bs < 1 \Rightarrow s < \frac{1}{b}$ and $b \geqslant 1$ then we get

$$\lim_{m,n\to\infty} d_b(x_n, x_m) = 0.$$

Thus $\{x_n\}$ is a Cauchy sequence in X.

Theorem 3.1.2. Assume that (X, d_b) be a complete b-metric space with the continues b-metric d_b and $b \geqslant 1$. Consider $F\colon X \to X$ be a self mapping satisfies the conditions given below

$$d_b(Fx, Fy) \leq \alpha.d_b(x, y) + \beta.d_b(x, Fx) + \gamma.d_b(y, Fy) + \mu.[d_b(x, Fy) + d_b(y, Fx)] \quad (3.2)$$

for every $x, y \in X$, where $\alpha, \beta, \gamma, \mu \geqslant 0$ with

$$b\alpha + b\beta + \gamma + (b^2 + b)\mu < 1 \quad (3.3)$$

then F has a unique fixed point in X.

Proof : Assume x_0 be any element in X, then $\{x_n\}$ in X is a sequence define by the rule

$$x_0, x_1 = Fx_0, x_2 = Fx_1, \ldots, x_{n+1} = Fx_n,$$

Suppose

$$d_b(x_n, x_{n+1}) = d_b(Fx_{n-1}, Fx_n)$$

Using (3.2), we get

$$\begin{aligned} d_b(x_n, x_{n+1}) &\leq \alpha.d_b(x_{n-1}, x_n) + \beta.d_b(x_{n-1}, x_n) + \gamma.d_b(x_n, x_{n+1}) \\ &+ \mu.[d_b(x_{n-1}, x_{n+1}) + d_b(x_n, x_n)] \\ &\leq \alpha.d_b(x_{n-1}, x_n) + \beta.d_b(x_{n-1}, x_n) + \gamma.d_b(x_n, x_{n+1}) \\ &+ \mu.b[d_b(x_{n-1}, x_n) + d_b(x_n, x_{n+1})] \end{aligned}$$

So

$$\begin{aligned} d_b(x_n, x_{n+1}) &\leq \frac{(\alpha + \beta + b\mu)}{(1 - (\gamma + b\mu))} d_b(x_{n-1}, x_n) \\ &\leq \eta d_b(x_{n-1}, x_n) \end{aligned}$$

Where

$$\eta = \frac{(\alpha + \beta + b\mu)}{(1 - (\gamma + b\mu))}$$

we can see from (3.3) that $\eta < \frac{1}{b}$.
Using Lemma 3.1.1, $\{x_n\}$ is clearly a Cauchy sequence. There is an element $u \in X$ such that

$$\lim_{n \to \infty} x_n = u.$$

we will verify that u is a fixed point in X, for this we assume

$$\begin{aligned} d_b(Fu, Fx_n) &\leq \alpha.d_b(u, x_n) + \beta.d_b(u, Fu) + \gamma.d_b(x_n, Fx_n) \\ &+ \mu.[d_b(u, Fx_n) + d_b(x_n, Fu)] \\ &\leq \alpha.d_b(u, x_n) + \beta.d_b(u, Fu) + \gamma.d_b(x_n, x_{n+1}) \\ &+ \mu.[d_b(u, x_{n+1}) + d_b(x_n, Fu)] \end{aligned}$$

As $n \to \infty$, we get

$$\begin{aligned} d_b(Fu, u) &\leq \alpha.d_b(u, u) + \beta.d_b(u, Fu) + \gamma.d_b(u, u) + \mu.[d_b(u, u) + d_b(u, Fu)] \\ &\leq (\beta + \mu).d_b(Fu, u) \end{aligned}$$

The above inequality is possible only if $d_b(Fu, u) = 0$ then $Fu = u$.
Thus it is verified u is a fixed point.
uniqueness : We will prove u is unique fixed point, consider that u, v as $u \neq v$ the fixed points of F then

$$\begin{aligned} d_b(u, v) &= d_b(Fu, Fv) \\ &\leq \alpha.d_b(u, v) + \beta.d_b(u, Fu) + \gamma.d_b(v, Fv) + \mu.[d_b(u, Fv) + d_b(v, Fu)] \\ &= (\alpha + 2\mu).d_b(u, v) \end{aligned}$$

As we know u, v of F are two fixed points so finally by (3.3), the above inequality is possible only if $d_b(u, v) = 0$ then $u = v$.
Thus u of F in X is a unique fixed point.

The above theorem yield the following corollary which is given below.

Corollary 3.1.3. Assume (X, d_b) is a complete b-metric space for any coefficient $b \geqslant 1$. Consider F is a self mapping satisfies the conditions given below

$$d_b(Fx, Fy) \leq \alpha.d_b(x, y) + \beta.d_b(x, Fx) + \gamma.d_b(y, Fy) \tag{3.4}$$

for every $x, y \in X$, where $\alpha, \beta, \gamma \geqslant 0$ for

$$b\alpha + b\beta + \gamma < 1 \tag{3.5}$$

then F has a unique fixed point in X.

Proof : Assume x_0 in X be an arbitrary element, then the sequence $\{x_n\}$ in X define by the rule

$$x_0, x_1 = Fx_0, x_2 = Fx_1, \ldots, x_{n+1} = Fx_n$$

Suppose

$$d_b(x_n, x_{n+1}) = d_b(Fx_{n-1}, Fx_n)$$

Using (3.8), we get

$$\begin{aligned} d_b(x_n, x_{n+1}) &= d_b(Fx_{n-1}, Fx_n) \leq \alpha.d_b(x_{n-1}, x_n) + \beta.d_b(x_{n-1}, x_n) + \gamma.d_b(x_n, x_{n+1}) \\ &\leq \alpha.d_b(x_{n-1}, x_n) + \beta.d_b(x_{n-1}, x_n) + \gamma.d_b(x_n, x_{n+1}) \end{aligned}$$

So

$$\begin{aligned} d_b(x_n, x_{n+1}) &\leq \frac{(\alpha + \beta)}{1 - \gamma}.d_b(x_{n-1}, x_n) \\ &\leq \eta.d_b(x_{n-1}, x_n) \end{aligned}$$

Where

$$\eta = \frac{(\alpha + \beta)}{1 - \gamma}$$

We can see from (3.9) that $\eta < \frac{1}{b}$.

Using Lemma 3.1.1, $\{x_n\}$ is clearly a Cauchy sequence, now an element $u \in X$ such that

$$\lim_{n \to \infty} x_n = u.$$

We have to verified u as a fixed point, for this we assume

$$\begin{aligned} d_b(Fu, Fx_n) &\leq \alpha.d_b(u, x_n) + \beta.d_b(u, Fu) + \gamma.d_b(x_n, Fx_n) \\ &\leq \alpha.d_b(u, x_n) + \beta.d_b(u, Fu) + \gamma.d_b(x_n, x_{n+1}) \end{aligned}$$

As $n \to \infty$, we get

$$\begin{aligned} d_b(Fu, u) &\leq \alpha.d_b(u, u) + \beta.d_b(u, Fu) + \gamma.d_b(u, u) \\ &\leq \beta.d_b(u, Fu) \end{aligned}$$

The above inequality is possible only if $d_b(Fu, u) = 0$ so $Fu = u$.
Thus it is verified u is a fixed point.
$Uniqueness$: We will verify u as a unique fixed point in X. Consider the fixed points u, v as $u \neq v$, we get

$$\begin{aligned} d_b(u, v) &= d_b(Fu, Fv) \\ &\leq \alpha.d_b(u, v) + \beta.d_b(u, Fu) + \gamma.d_b(v, Fv) \\ &= \alpha.d_b(u, v) \end{aligned}$$

As we know u, v of F are fixed points so finally by using (3.9), the above inequality is possible only if $b(u, v) = 0$ then $u = v$.
Thus it is verified that u be a unique fixed point in X.

The above corollary yield the following two corollaries given below.

Corollary 3.1.4. Assume (X, d_b) as a complete b-metric space for constant $b \geqslant 1$. Consider the self mapping F satisfies the conditions given below

$$d_b(Fx, Fy) \leq \alpha.d_b(x, y) + \beta.d_b(x, Fx) \tag{3.6}$$

with every $x, y \in X$, also $\alpha, \beta \geqslant 0$ for

$$b\alpha + b\beta < 1 \tag{3.7}$$

then the mapping F has a unique fixed point in X.
$Proof$: We replace $\gamma = 0$ in above corollary then we can easily get our required result.

Corollary 3.1.5. Assume (X, d_b) as a complete b-metric space for any coefficient$b \geqslant 1$. Consider the self mapping F satisfies the conditions given below

$$d_b(Fx, Fy) \leq \alpha.d_b(x, y) \tag{3.8}$$

for every $x, y \in X$, with $\alpha \geqslant 0$ for

$$b\alpha < 1 \tag{3.9}$$

then the mapping F has a unique fixed point in X.
$Proof$: We replace $\beta = 0$ in above corollary then we can easily get our required result.

Theorem 3.1.6. Assume (X, d_b) as a complete b-metric space for any $b \geqslant 1$. Consider the self mapping F satisfies the conditions given below

$$d_b(Fx, Fy) \leq \beta.d_b(x, y) + \mu.\frac{d_b(x, Fx).d_b(x, Fy) + d_b(y, Fy).d_b(y, Fx)}{d_b(x, Fy).d_b(y, Fx)} \tag{3.10}$$

for every $x, y \in X$ with $\beta, \mu \geqslant 0$, $d_b(x, Fy).d_b(y, Fx) \neq 0$ with $b(\beta + \mu) < 1$, then F has a unique fixed point in X.

Proof : Assume x_0 in X is any element then the sequence $\{x_n\}$ is define by the rule

$$x_0, x_1 = Fx_0, x_2 = Fx_1, \ldots, x_{n+1} = Fx_n$$

we will verify the sequence $\{x_n\}$ as a Cauchy in X, we consider

$$d_b(x_n, x_{n+1}) = d_b(Fx_{n-1}, Fx_n)$$

Using (3.10), we get

$$\begin{aligned} d_b(x_n, x_{n+1}) &\leq \beta.d_b(x_{n-1}, x_n) \\ &+ \mu.\frac{d_b(x_{n-1}, Fx_{n-1}).d_b(x_{n-1}, Fx_n) + d_b(x_n, Fx_n).d_b(x_n, Fx_{n-1})}{d_b(x_{n-1}, Fx_n) + d_b(x_n, Fx_{n-1})} \\ &\leq \beta.d_b(x_{n-1}, x_n) \\ &+ \mu.\frac{d_b(x_{n-1}, x_n).d_b(x_{n-1}, x_{n+1}) + d_b(x_n, x_{n+1}).d_b(x_n, x_n)}{d_b(x_{n-1}, x_n) + d_b(x_n, x_n)} \\ &\leq (\beta + \mu).d_b(x_{n-1}, x_n) \end{aligned}$$

Since $(\beta + \mu) < \frac{1}{b}$, therefore by using Lemma 3.1.1, it is clear that $\{x_n\}$ is a Cauchy sequence, now an element $u \in X$ such as

$$\lim_{n \to \infty} x_n = u.$$

We will have to verify u is a fixed point in X, for this consider

$$\begin{aligned} d_b(Fx_n, Fu) &\leq \beta.d_b(x_n, u) + \mu.\frac{d_b(x_n, Fx_n).d_b(x_n, Fu) + d_b(u, Fu).d_b(u, Fx_n)}{d_b(x_n, Fu) + d_b(u, Fx_n)} \\ &\leq \beta.d_b(x_n, u) + \mu.\frac{d_b(x_n, x_{n+1}).d_b(x_n, Fu) + d_b(u, Fu).d_b(u, x_{n+1})}{d_b(x_n, Fu) + d_b(u, x_{n+1})} \end{aligned}$$

As $n \to \infty$, we get

$$\leq \beta.d_b(x_n, u) + \mu.\frac{d_b(x_n, x_{n+1}).d_b(x_n, Fu) + d_b(u, Fu).d_b(u, x_{n+1})}{d_b(x_n, Fu) + d_b(u, x_{n+1})} \longrightarrow 0$$

So

$$d_b(u, Fu) = 0 \textit{ then } Fu = u.$$

so it is clearly verified u is a fixed point in X.
Uniqueness : We will have to verify u as a unique fixed point, for this we consider the fixed points u, v as $u \neq v$, we get

$$\begin{aligned} d_b(u,v) &= d_b(Fu,Fv) \\ &\leq \beta.d_b(u,v) + \mu.\frac{d_b(u,Fu).d_b(u,Fv) + d_b(v,Fv).d_b(v,Fu)}{d_b(u,Fv) + d_b(v,Fu)} \end{aligned}$$

As we know u, v are fixed points in X, so we have

$$\begin{aligned} d_b(u,v) &= d_b(Fu,Fv) \\ &\leq \beta.d_b(u,v) + \mu.\frac{d_b(u,u).d_b(u,v) + d_b(v,v).d_b(v,u)}{d_b(u,v) + d_b(v,u)} \\ &= \beta.d_b(u,v) \end{aligned}$$

The above inequality is possible only if $d_b(u,v) = 0 \rightarrow u = v$
Thus it is verified u as a unique fixed point in X.

Now we are going to review a theorem with rational type contraction in two terms, the previous theorem was in one term rational type contraction.

Theorem 3.1.7. Assume (X, d_b) as a complete b-metric space for any constant $b \geqslant 1$. Consider the self mapping F satisfies the conditions given below

$$d_b(Fx,Fy) \leq \alpha.d_b(x,y) + \beta.\frac{d_b(y,Fy)[1 + d_b(x,Fx)]}{1 + d_b(x,y)} + \gamma.\frac{d_b(y,Fy) + d_b(y,Fx)}{1 + d_b(y,Fy).d_b(y,Fx)} \quad (3.11)$$

for each $x, y \in X$, with $\alpha, \beta, \gamma \geqslant 0$ and also $b\alpha + \beta + \gamma < 1$ then F has a unique fixed point in X.
Proof : Assume x_0 be an element in X then the sequence $\{x_n\}$ in X is define by the rule

$$x_0, x_1 = Fx_0, x_2 = Fx_1, \ldots, x_{n+1} = Fx_n$$

we will verify the sequence $\{x_n\}$ as a Cauchy in X, we consider

$$d_b(x_n, x_{n+1}) = d_b(Fx_{n-1}, Fx_n)$$

Using (3.11), we get

$$\begin{aligned} d_b(x_n,x_{n+1}) &\leq \alpha.d_b(x_{n-1},x_n) + \beta.\frac{d_b(x_n,x_{n+1})[1 + d_b(x_{n-1},x_n)]}{1 + d_b(x_{n-1},x_n)} \\ &+ \gamma.\frac{d_b(x_n,x_{n+1}) + d_b(x_n,x_n)}{1 + d_b(x_n,x_{n+1}).d_b(x_n,x_n)} \end{aligned}$$

$$\leq \quad \alpha.d_b(x_{n-1}, x_n) + \beta.\frac{d_b(x_n, x_{n+1})[1 + d_b(x_{n-1}, x_n)]}{1 + d_b(x_{n-1}, x_n)} + \gamma.d_b(x_n, x_{n+1})$$

Therefore

$$\begin{aligned} d_b(x_n, x_{n+1}) &\leq \frac{\alpha}{1 - (\beta + \gamma)}.d_b(x_{n-1}, x_n) \\ &= k.d_b(x_{n-1}, x_n) \end{aligned}$$

Where $k = \frac{\alpha}{1-(\beta+\gamma)}$ with $k < \frac{1}{b}$, because $b\alpha + \beta + \gamma < 1$ Similarly, we have

$$d_b(x_n, x_{n+1}) \leq k^2.d_b(x_{n-2}, x_{n-1})$$

Continuing this same process, we get

$$d_b(x_n, x_{n+1}) \leq k^n.d_b(x_0, x_1)$$

Since $0 \leq k < 1$ so $k^n \to 0$ as $n \to \infty$, therefore by using Lemma 3.1.1, it is clear that $\{x_n\}$ is a Cauchy sequence, now an element $u \in X$ such as

$$\lim_{n\to\infty} x_n = u.$$

Now we have to verify u as a fixed point in X, for this we assume

$$d_b(Fx_n, Fu) \leq \alpha.d_b(x_n, u) + \beta.\frac{d_b(u, Fu)[1 + d_b(x_n, Fx_n]}{1 + d_b(x_n, u)} + \gamma.\frac{d_b(u, Fu) + d_b(u, Fx_n)}{1 + d_b(u, Fu).d_b(u, Fx_n)} \tag{3.12}$$

From construction, it is clear that $Fx_n = x_{n+1}$ and also a Cauchy sequence $\{x_n\}$ converges to u.
Therefore we takes limit $n \to \infty$ then (3.12) become

$$d_b(u, Fu) \leq (\beta + \gamma).d_b(u, Fu)$$

which is possible only if $d_b(u, Fu) = 0$, so $Fu = u$.
Thus it is verified that u is a fixed point in X.
Uniqueness. We will verify u as a unique fixed point, for this we consider u, v as $u \neq v$ the fixed points so we have

$$\begin{aligned} d_b(u, v) &= d_b(Fu, Fv) \\ &\leq \alpha.d_b(u, v) + \beta.\frac{d_b(v, Fv)[1 + d_b(u, Fu)]}{1 + d_b(u, v)} + \gamma.\frac{d_b(v, Fv) + d_b(v, Fu)}{1 + d_b(v, Fu).d_b(v, Fu)} \end{aligned}$$

so the above inequality become

$$d_b(u,v) \quad \leq \quad (\alpha+\lambda).d_b(u,v)$$

above inequality is possible only if $d_b(u,v)=0$ then $u=v$
Thus it is verified u is a unique fixed point in X.

Now we have reviewed a theorem which has Ciric's type contraction.

Theorem 3.1.8. Assume (X,d_b) as a complete b-metric space for any $b \geqslant 1$. Consider F be a self mapping satisfies the conditions given below

$$d_b(Fx,Fy) \leq k.\max\{d_b(x,y), d_b(x,Fx), d_b(y,Fy), \frac{1}{2b}[d_b(x,Fy)+d_b(y,Fx)]\} \quad (3.13)$$

for all $x,y \in X$ with $k \in [0,1)$ and as $bk \leq 1$ then F has a unique fixed point in X.
Proof : Assume x_0 in X be an element then the sequence $\{x_n\}$ is define by the rule

$$x_0, x_1 = Fx_0, x_2 = Fx_1, \ldots, x_{n+1} = Fx_n$$

we will verify the sequence $\{x_n\}$ as a Cauchy in X, now we consider

$$d_b(x_n,x_{n+1}) = d_b(Fx_{n-1},Fx_n)$$

Using (3.13), to get

$$\begin{aligned} d_b(x_n,x_{n+1}) & \\ \leq & \quad k.\max\{d_b(x_{n-1},x_n), d_b(x_{n-1},Fx_{n-1}), d_b(x_n,Fx_n), \frac{1}{2b}[d_b(x_{n-1},Fx_n) \\ + & \quad d_b(x_n,Fx_{n-1})]\} \end{aligned}$$

$$\begin{aligned} = & \quad k.\max\{d_b(x_{n-1},x_n), d_b(x_{n-1},x_n), d_b(x_n,x_{n+1}), \frac{1}{2b}[d_b(x_{n-1},x_{n+1})+d_b(x_n,x_n)]\} \\ \leq & \quad k.\max\{d_b(x_{n-1},x_n), d_b(x_{n-1},x_n), d_b(x_n,x_{n+1}), \frac{1}{2}[d_b(x_{n-1},x_n)+d_b(x_n,x_{n+1})]\} \end{aligned}$$

$$= k.\max\{d_b(x_{n-1},x_n), d_b(x_n,x_{n+1}), \frac{1}{2}[d_b(x_{n-1},x_n)+d_b(x_n,x_{n+1})]\} \quad (3.14)$$

If

$$d_b(x_{n-1},x_n) \quad < \quad d_b(x_n,x_{n+1}).$$

Then

$$d_b(x_{n-1}, x_n) \ < \ \frac{1}{2}[d_b(x_{n-1}, x_n) + d_b(x_n, x_{n+1})] < d_b(x_n, x_{n+1})$$

Now applying (3.14), we get

$$d_b(x_n, x_{n+1}) \ \leq \ k.d_b(x_n, x_{n+1})$$

Which is not possible because $k < 1$, so we ignore this term $b(x_n, x_{n+1})$, thus (3.14) become

$$d_b(x_n, x_{n+1}) \ \leq \ k.d_b(x_{n-1}, x_n)$$

therefore by using Lemma 3.1.1, it is clearly verified that $\{x_n\}$ is a Cauchy sequence, there is an element $u \in X$ such that

$$\lim_{n\to\infty} x_n = u.$$

Now we will verify u as a fixed point in X, for that consider

$$\begin{aligned} d_b(Fu, Fx_n) \ &\leq \ k.\max\{d_b(u, x_n), d_b(u, Fu), d_b(x_n, Fx_n), \frac{1}{2b}[d_b(u, Fx_n) \\ &+ \ d_b(x_n, Fu)]\} \\ &\leq \ k.\max\{d_b(u, x_n), d_b(u, Fu), d_b(x_n, x_{n+1}), \frac{1}{2b}[d_b(u, x_{n+1}) \\ &+ \ d_b(x_n, Fu)]\} \end{aligned}$$

As $n \to \infty$, to get

$$\begin{aligned} d_b(Fu, u) \ &\leq \ k.\max\{d_b(u, Fu), \frac{1}{2b}d_b(u, Fu)\} \\ &\leq \ k.d_b(Fu, u) \end{aligned}$$

the above inequality is possible only if

$$d_b(Fu, u) = 0 \ then \ Fu = u.$$

Thus it is verified u as a fixed point in X.
$Uniqueness$: We will verify u as a unique fixed point in X, for that we consider u, v as $u \neq v$ the fixed points so we have

$$\begin{aligned} d_b(u, v) \ &= \ d_b(Fu, Fv) \\ &\leq \ k.\max\{d_b(u, v), d_b(u, Fu), d_b(v, Fv), \frac{1}{2b}[d_b(u, Fv) + d_b(v, Fu)\}] \end{aligned}$$

since u, v are the fixed points of F, so finally we get

$$d_b(u,v) \quad \leq \quad k.d_b(u,v)$$

The above inequality is possible only if

$$d_b(u,v) = 0 \; then \; u = v.$$

Thus it is clearly verified that u is a unique fixed point in X.

The corollary given below is produced from the above theorem by using $b = 1$.

Corollary 3.1.9. Assume (X, d_b) be a complete b-metric space with coefficient, also the self mapping $F\colon X \to X$ satisfies the conditions given below

$$d_b(Fx, Fy) \leq k.\max\{d_b(x,y), d_b(x,Fx), d_b(y,Fy), \frac{1}{2}[d_b(x,Fy) + d_b(y,Fx)]\} \qquad (3.15)$$

for every $x, y \in X$ with $k \in [0,1)$ then F has a unique fixed point in X.
$Proof$: Replace $b = 1$ in above theorem then we can easily get the required result.

Example 3.1.10. "Assume that if $X = [0,1]$ with $b(x,y) = |x-y|^2$ along coefficient $b = 2$ is a b-metric for every $x, y \in X$.
Define a mapping F by $Fx = \frac{2}{3}$ if $x \in [0,1)$ and $F(1) = 0$ then F will fulfill every conditions of the theorem (3.4) given above for $k \in [\frac{4}{9}, \frac{1}{2})$ with $x = \frac{2}{3}$ is its fixed point which is unique in X."

Chapter 4

Fixed point theorems in rectangular b-metric space

In the present chapter, we proved some fixed point results with different types of contraction in Rectangular b-metric space which are reviewed already in Chapter 3. We also extended the Banach contraction principle (BCP) in Rectangular b- metric space. We also proved the Kannan type theorem in rectangular b-metric space. We have included some examples which confirm and show that our generalizations are genuine. First, we have extended the lemma in rectangular b-metric space which is already proved in b-metric space and we also used the result of this lemma in our next theorems which are Banach contraction principle and Kannan type fixed point theorem. Here throughout d_b is a continuous rectangular b-metric. From now on, by a rectangular b-metric we mean a continuous rectangular b-metric d_b.

4.1 Main results in RbMS

The following lemma is the generalization of the lemma which is already reviewed in previous chapter. This lemma is very useful for the generalizations of our upcoming theorems and results.

Lemma 4.1.1. Consider a rectangular b-metric space (X, d_b) with the constant $b \geqslant 1$ and also assume a sequence $\{x_n\}$ in rectangular b-metric space such that

$$d_b(x_n, x_{n+1}) \leq \alpha.d_b(x_{n-1}, x_n) \tag{4.1}$$

with $n = 1, 2, 3, \ldots$, $0 \leq b\alpha < 1$, $\alpha \in [0, 1)$ and b is defined in rectangular b-metric space then $\{x_n\}$ is a Cauchy sequence in X.

Proof : Assume if $n, m \in \mathbb{N}$ with $m > n$, we have

$$\begin{aligned} d_b(x_n, x_m) &\leq b[d_b(x_n, x_{n+1}) + d_b(x_{n+1}, x_{n+2}) + d_b(x_{n+2}, x_m)] \\ &\leq b[d_b(x_n, x_{n+1}) + d_b(x_{n+1}, x_{n+2})] + b^2[d_b(x_{n+2}, x_{n+3}) \\ &+ d_b(x_{n+3}, x_{n+4}) + d_b(x_{n+4}, x_m)] \\ &\leq b[d_b(x_n, x_{n+1}) + d_b(x_{n+1}, x_{n+2})] + b^2[d_b(x_{n+2}, x_{n+3}) \\ &+ d_b(x_{n+3}, x_{n+4})] + b^3[d_b(x_{n+4}, x_{n+5}) + d_b(x_{n+5}, x_{n+6})] \\ &+ \ldots + b^m[d_b(x_{n+m-2}, x_m)] \end{aligned}$$

Now using (4.1) and repeating the same process, we have

$$\begin{aligned} d_b(x_n, x_m) &\leq b[\alpha^n + \alpha^{n+1}]d_b(x_0, x_1) + b^2[\alpha^{n+2} + \alpha^{n+3}]d_b(x_0, x_1) \\ &+ b^3[\alpha^{n+4} + \alpha^{n+5}]d_b(x_0, x_1) + \ldots + b^m\alpha^{n+m-2}d_b(x_0, x_1) \\ &\leq [1 + b\alpha^2 + b^2\alpha^4 + b^3\alpha^6 + \ldots + b^m\alpha^{2m}][b\alpha^n d_b(x_0, x_1) + b\alpha^{n+1}d_b(x_0, x_1)] \\ &\leq [\frac{(1-(b\alpha^2)^m}{1-b\alpha^2}].(1+\alpha)b\alpha^n d_b(x_0, x_1) \end{aligned}$$

Since $b\alpha^2 < 1$ therefore we takes $m, n \to \infty$, we get

$$\lim_{m,n\to\infty} d_b(x_n, x_m) = 0.$$

Thus $\{x_n\}$ in rectangular b-metric space X is a Cauchy sequence.

Our next aim is to extend the Banach contraction principle (BCP) in rectangular b-metric space.

Theorem 4.1.2. Consider a complete rectangular b-metric space (X, d_b) for any constant $b > 1$, a self mapping $F\colon X \to X$ satisfies the conditions given below

$$d_b(Fx, Fy) \leq \alpha.d_b(x, y) \tag{4.2}$$

for every $x, y \in X$, for $\alpha \in [0, \frac{1}{b}]$ then F has a unique fixed point in X.
Proof : Assume that $x_0 \in X$ be an element, a sequence $\{x_n\}$ in X is define as

$$x_{n+1} = Fx_n$$

for each $n \geqslant 0$, we will prove $\{x_n\}$ in X as a Cauchy sequence. If $x_n = x_{n+1}$ then x_n in X is a fixed point. Now consider $x_n \neq x_{n+1}$ for every $n \geqslant 0$.

Setting $d_b(x_n, x_{n+1}) = d_{bn}$, it follows from (4.1)

$$\begin{aligned} d_b(x_n, x_{n+1}) &= d_b(Fx_{n-1}, Fx_n) \\ &\leq \alpha.d_b(x_{n-1}, x_n) \\ d_{bn} &\leq \alpha.d_{b(n-1)} \end{aligned}$$

If we repeat this process as shown above, we get

$$d_{bn} \leq \alpha^n.d_{b0} \tag{4.3}$$

Also, we can assume that x_0 is a point of F but not periodic. We have no doubt if $x_0 = x_n$ then by using (4.2), for any $n \geqslant 2$, we have

$$\begin{aligned} d_b(x_0, Fx_0) &= d_b(x_n, Fx_n) \\ d_b(x_0, x_1) &= d_b(x_n, x_{n+1}) \\ d_{b0} &= d_{bn} \\ &\leq \alpha^n.d_{b0} \end{aligned}$$

This is a contradiction. Therefore, we must have $d_{b_0} = 0$, that is, $x_0 = x_1$, and so x_0 is a fixed point. Therefore we will consider $x_m \neq x_n$ with $n, m \in \mathbb{N}$. Again setting $d_b(x_n, x_{n+2}) = d^*_{bn}$ and now by using (4.1) for any $n \in N$, we get

$$\begin{aligned} d_b(x_n, x_{n+2}) &= d_b(Fx_{n-1}, Fx_{n+1}) \\ &\leq \alpha.d_b(x_{n-1}, x_{n+1}) \\ d^*_{bn} &\leq \alpha.d^*_{b(n-1)} \end{aligned}$$

If we repeat this process, we get

$$d_b(x_n, x_{n+2}) \leq \alpha^n.d^*_{b0} \tag{4.4}$$

For the sequence $\{x_n\}$, we consider $d_b(x_n, x_{n+p})$ in two cases. If p is odd say $2m+1$ then by using (4.2) we get

$$\begin{aligned} d_b(x_n, x_{n+2m+1}) &\leq b[d_b(x_n, x_{n+1}) + d_b(x_{n+1}, x_{n+2}) + d_b(x_{n+2}, x_{n+2m+1})] \\ &\leq b[d_{bn} + d_{b(n+1)}] + b^2[d_b(x_{n+2} + x_{n+3}) + d_b(x_{n+3} + x_{n+4}) \\ &+ d_b(x_{n+4} + x_{n+2m+1})] \\ &\leq b[d_{bn} + d_{b(n+1)}] + b^2[d_{b(n+2)} + d_{b(n+3)}] + b^3[d_{b(n+4)} + d_{b(n+5)}] \\ &+ \ldots + b^m d_{b(n+2m)} \\ &\leq b[\alpha^n d_{b0} + \alpha^{n+1} d_{b0}] + b^2[\alpha^{n+2} d_{b0} + \alpha^{n+3} d_{b0}] \\ &+ b^3[\alpha^{n+4} d_{b0} + \alpha^{n+5} d_{b0}] + \ldots + b^m \alpha^{n+2m} d_{b0} \\ &\leq b\alpha^n[1 + b\alpha^2 + b^2\alpha^4 + \ldots]d_{b0} + b\alpha^{n+1}[1 + b\alpha^2 + b^2\alpha^4 + \ldots]d_{b0} \\ &= \frac{1+\alpha}{1-b\alpha^2} b\alpha^n d_{b0} \end{aligned}$$

where $b\alpha^2 < 1$ therefore

$$d(x_n, x_{n+2m+1}) \leq \frac{1+\alpha}{1-b\alpha^2} b\alpha^n d_{b0}$$

If p is even, we can say $2m$ then by using (4.2) and (4.3), we get

$$\begin{aligned} d_b(x_n, x_{n+2m}) &\leq b[d_b(x_n, x_{n+1}) + d_b(x_{n+1}, x_{n+2}) + d_b(x_{n+2}, x_{n+2m})] \\ &\leq b[d_{bn} + d_{b(n+1)}] + b^2[d_b(x_{n+2} + x_{n+3}) + d_b(x_{n+3} + x_{n+4}) \\ &+ d_b(x_{n+4} + x_{n+2m})] \\ &\leq b[d_{bn} + d_{b(n+1)}] + b^2[d_{b(n+2)} + d_{b(n+3)}] + b^3[d_{b(n+4)} + d_{b(n+5)}] \\ &+ \ldots + b^{m-1}[d_{b(2m-4)} + d_{b(2m-3)}] + b^{m-1}[x_{n+2m-2} + x_{n+2m}] \\ &\leq b[\alpha^n d_{b0} + \alpha^{n+1} d_{b0}] + b^2[\alpha^{n+2} d_{b0} + \alpha^{n+3} d_{b0}] \\ &+ b^3[\alpha^{n+4} d_{b0} + \alpha^{n+5} d_{b0}] + \ldots + b^{m-1}[\alpha^{2m-4} d_{b0} + \alpha^{2m-3} d_{b0}] \\ &+ b^{m-1}\alpha^{n+2m-2} d_{b0}^* \\ &\leq b\alpha^n[1 + b\alpha^2 + b^2\alpha^4 + \ldots]d_{b0} + b\alpha^{n+1}[1 + b\alpha^2 + b^2\alpha^4 + \ldots]d_{b0} \\ &+ b^{m-1}\alpha^{n+2m-2} d_{b0}^* \end{aligned}$$

Therefore

$$\begin{aligned} d_b(x_n, x_{n+2m}) &\leq \frac{1+\alpha}{1-b\alpha^2} b\alpha^n d_{b0} + b^{m-1}\alpha^{n+2m-2} d_{b0}^* \\ &\leq \frac{1+\alpha}{1-b\alpha^2} b\alpha^n d_{b0} + (b\alpha)^{2m}\alpha^{n-2} d_{b0}^* \\ &\leq \frac{1+\alpha}{1-b\alpha^2} b\alpha^n d_{b0} + \alpha^{n-2} d_{b0}^* \end{aligned}$$

Therefore

$$d_b(x_n, x_{n+2m}) \leq \frac{1+\alpha}{1-b\alpha^2} b\alpha^n d_{b0} + \alpha^{n-2} d_{b0}^* \tag{4.5}$$

It follows from (4.4) and (4.5) that

$$\lim_{n\to\infty} d_b(x_n, x_{n+p}) = 0 \tag{4.6}$$

for all $p > 0$, Thus it is verified $\{x_n\}$ in X is a Cauchy sequence. we will prove (X, d_b) is a complete space then there is $u \in X$ as

$$\lim_{n\to\infty} x_n = u. \tag{4.7}$$

We will prove u in X is a fixed point by taking any $n \in \mathbb{N}$ then we have

$$\begin{aligned} d_b(u, Fu) &\leq b[d_b(u, x_n) + d_b(x_n, x_{n+1}) + d_b(x_{n+1}, Fu)] \\ &= b[d_b(u, x_n) + d_{bn} + d_b(Fx_n, Fu] \\ &\leq b[d_b(u, x_n) + d_{bn} + \alpha.d_b(x_n, u)] \end{aligned}$$

Now by using (4.6) and (4.7), we have

$$d_b(u, Fu) = 0 \to Fu = u.$$

Hence it is clearly verified u as a fixed point in X.
$Uniquness$: Assume that there are u, v as $v \neq u$, if we follows (4.1) we get

$$d_b(u, v) = d_b(Fu, Fv) \leq \alpha.d_b(u, v) < d_b(u, v)$$

which is contradiction, so we must have

$$d_b(u, v) = 0 \to u = v.$$

Thus it is clearly verified u as a unique fixed point in X.

Example 4.1.3. Assume that if $X = A \cup B$, where $A = \{\frac{1}{n} : n \in \{5, 4, 3, 2\}\}$, $B = [1, 2]$, $d_b \colon X \times X \to [0, \infty)$ is define as $d_b(x, y) = d_b(y, x)$ for every $y, x \in X$

$$\begin{cases} d_b(\frac{1}{2}, \frac{1}{3}) = d_b(\frac{1}{4}, \frac{1}{5}) = 0.03 \\ d_b(\frac{1}{2}, \frac{1}{5}) = d_b(\frac{1}{3}, \frac{1}{4}) = 0.02 \\ d_b(\frac{1}{2}, \frac{1}{4}) = d_b(\frac{1}{5}, \frac{1}{3}) = 0.6 \\ d_b(x, y) = |x - y|^2 & \text{otherwise} \end{cases}$$

Then (X, d_b) is rectangular b-metric space for any constant $b = 4 > 1$, here (X, d_b) is not metric as well as rectangular metric space. Assume a mapping $F \colon X \to X$ is define by

$$Fx = \begin{cases} \frac{1}{4} & \text{if } x \in A \\ \frac{1}{5} & \text{if } x \in B \end{cases}$$

then the mapping F satisfies all the conditions related to BCP and also $x = \frac{1}{4}$ is a unique fixed point.

Next target is to extend the theorem which is kannan type contraction in rectangular b-metric space.

Theorem 4.1.4. Assume that a complete rectangular b-metric space (X, d_b) for constant $b > 1$, $F\colon X \to X$ be a self mapping satisfies the conditions given below

$$d_b(Fx, Fy) \leq \alpha[d_b(x, Fx) + d_b(y, Fy)] \tag{4.8}$$

for every $x, y \in X$ with $\alpha \in [0, \frac{1}{b+1}]$ then the mapping F has a unique fixed point in X.
Proof : Assume that x_0 is an element in X, a sequence $\{x_n\}$ is define as

$$x_{n+1} = Fx_n$$

for each $n \geqslant 0$. First, we shall prove $\{x_n\}$ is a Cauchy sequence. If $x_n = x_{n+1}$ then it is clearly verified x_n is a fixed point, assume $x_{n+1} \neq x_n$ for every $n \geqslant 0$.
Setting $d_b(x_n, x_{n+1}) = d_{bn}$, it follows from (4.8)

$$\begin{aligned} d_b(x_n, x_{n+1}) &= d_b(Fx_{n-1}, Fx_n) \\ &\leq \alpha[d_b(x_{n-1}, Fx_{n-1}) + d_b(x_n, Fx_n)] \\ d_b(x_n, x_{n+1}) &\leq \alpha[d_b(x_{n-1}, x_n) + d_b(x_n, x_{n+1})] \\ d_{bn} &\leq \alpha[d_{b(n-1)} + d_{bn}] \\ d_{bn} &\leq \frac{\alpha}{1-\alpha}.d_{b(n-1)} \\ &= \mu.d_{b(n-1)} \end{aligned}$$

where $\mu = \frac{\alpha}{1-\alpha} < \frac{1}{b}$ as $\alpha < \frac{1}{b+1}$. If we repeat this process then we get

$$d_{bn} \leq \mu^n.d_{b0} \tag{4.9}$$

Also, we can assume that x_0 is a point of F but not period then we have no doubt if $x_0 = x_n$ then by using (4.9) for every $n \geqslant 2$ we get

$$\begin{aligned} d_b(x_0, Fx_0) &= d_b(x_n, Fx_n) \\ d_b(x_0, x_1) &= d_b(x_n, x_{n+1}) \\ d_{b0} &= d_{bn} \end{aligned}$$

Hence

$$d_{b0} \quad \leq \quad \mu^n.d_{b0}$$

This is a contradiction. Therefore we must have $d_{b_0} = 0$, that is $x_0 = x_1$ and so x_0 in X is a fixed point.
Now we consider $x_n \neq x_m$ for $n, m \in \mathbb{N}$. Again setting $d_b(x_n, x_{n+2}) = d^*_{bn}$ and by using (4.8) and (4.9) for any $n \in \mathbb{N}$ we get

$$\begin{aligned}
d_b(x_n, x_{n+2}) &= d_b(Fx_{n-1}, Fx_{n+1}) \\
&\leq \alpha[d_b(x_{n-1}, Fx_{n-1}) + d_b(x_{n+1}, Fx_{n+1})] \\
&= \alpha[d_b(x_{n-1}, x_n) + d_b(x_{n+1}, x_{n+2})] \\
&= \alpha[d_{b(n-1)} + d_{b(n+1)}] \\
&\leq \alpha[\mu^{n-1} d_{b0} + \mu^{n+1} d_{b0}] \\
&= \alpha\mu^{n-1}[1 + \mu^2] d_{b0} \\
&= \beta\mu^{n-1} d_{b0}
\end{aligned}$$

Therefore

$$d_b(x_n, x_{n+2}) \leq \beta\mu^{n-1} d_{b0} \tag{4.10}$$

where $\beta = \alpha[1 + \mu^2] > 0$, for the sequence $\{x_n\}$ we consider $d_b(x_n, x_{n+p})$ in two cases. If p is odd say $2m + 1$ then by using (4.9) we get

$$\begin{aligned}
&d_b(x_n, x_{n+2m+1}) \\
&\leq b[d_b(x_n, x_{n+1}) + d_b(x_{n+1}, x_{n+2}) + d_b(x_{n+2}, x_{n+2m+1})] \\
&\leq b[d_{bn} + d_{b(n+1)}] + b^2[d_b(x_{n+2} + x_{n+3}) + d_b(x_{n+3} + x_{n+4}) \\
&+ d_b(x_{n+4} + x_{n+2m+1})] \\
&\leq b[d_{bn} + d_{b(n+1)}] + b^2[d_{b(n+2)} + d_{b(n+3)}] + b^3[d_{b(n+4)} + d_{b(n+5)}] \\
&+ \ldots + b^m d_{b(n+2m)} \\
&\leq b[\mu^n d_{b0} + \mu^{n+1} d_{b0}] + b^2[\mu^{n+2} d_{b0} + \mu^{n+3} d_{b0}] \\
&+ b^3[\mu^{n+4} d_{b0} + \mu^{n+5} d_{b0}] + \ldots + b^m \mu^{n+2m} d_{b0} \\
&\leq b\mu^n[1 + b\mu^2 + b^2\mu^4 + \ldots] d_{b0} + b\mu^{n+1}[1 + b\mu^2 + b^2\mu^4 + \ldots] d_{b0} \\
&\leq \frac{1 + \mu}{1 - b\mu^2} b\mu^n d_{b0}
\end{aligned}$$

Therefore

$$d(x_n, x_{n+2m+1}) \leq \frac{1 + \mu}{1 - b\mu^2}.b\mu^n d_{b0} \tag{4.11}$$

If p is even, we can say $2m$ then by using (4.9) and (4.10) we get

$$
\begin{aligned}
& d_b(x_n, x_{n+2m}) \\
&\quad \leq b[d_b(x_n, x_{n+1}) + d_b(x_{n+1}, x_{n+2}) + d_b(x_{n+2}, x_{n+2m})] \\
&\quad \leq b[d_{bn} + d_{b(n+1)}] + b^2[d_b(x_{n+2} + x_{n+3}) + d_b(x_{n+3} + x_{n+4}) \\
&\quad + d_b(x_{n+4} + x_{n+2m})] \\
&\quad \leq b[d_{bn} + d_{b(n+1)}] + b^2[d_{b(n+2)} + d_{b(n+3)}] + b^3[d_{b(n+4)} + d_{b(n+5)}] \\
&\quad + \ldots + b^{m-1}[d_{b(2m-4)} + d_{b(2m-3)}] + b^{m-1}[x_{n+2m-2} + x_{n+2m}] \\
&\quad \leq b[\mu^n d_{b0} + \mu^{n+1} d_{b0}] + b^2[\mu^{n+2} d_{b0} + \mu^{n+3} d_{b0}] + b^3[\mu^{n+4} d_{b0} + \mu^{n+5} d_{b0}] \\
&\quad + \ldots + b^{m-1}[\mu^{2m-4} d_{b0} + \mu^{2m-3} d_{b0}] + b^{m-1}\mu^{n+2m-2} d^*_{b0} \\
&\quad \leq b\mu^n[1 + b\mu^2 + b^2\mu^4 + \ldots]d_{b0} + b\mu^{n+1}[1 + b\mu^2 + b^2\mu^4 + \ldots]d_{b0} \\
&\quad + b^{m-1}\mu^{n+2m-2} d^*_{b0}
\end{aligned}
$$

That is

$$
\begin{aligned}
d_b(x_n, x_{n+2m}) &\leq \frac{1+\mu}{1-b\mu^2} b\mu^n d_{b0} + b^{m-1}\mu^{n+2m-2} d^*_{b0} \\
&< \frac{1+\mu}{1-b\mu^2} b\mu^n d_{b0} + (b\mu)^{2m}\mu^{n-2} d^*_{b0} \\
&\leq \frac{1+\mu}{1-b\mu^2} b\mu^n d_{b0} + \mu^{n-2} d^*_{b0}
\end{aligned}
$$

Therefore

$$
d_b(x_n, x_{n+2m}) \leq \frac{1+\mu}{1-b\mu^2} b\mu^n d_{b0} + \mu^{n-2} d^*_{b0} \tag{4.12}
$$

It follows from (4.11) and (4.12) that

$$
\lim_{n\to\infty} d_b(x_n, x_{n+p}) = 0 \tag{4.13}
$$

for all $p > 0$. Thus it is clearly verified $\{x_n\}$ is a Cauchy sequence. We will prove (X, d_b) is complete space by taking $u \in X$ as

$$
\lim_{n\to\infty} x_n = u. \tag{4.14}
$$

we will prove u in X is a fixed point by taking any $n \in \mathbb{N}$ then we have

$$
\begin{aligned}
d_b(u, Fu) &\leq b[d_b(u, x_n) + d_b(x_n, x_{n+1}) + d_b(x_{n+1}, Fu)] \\
&= b[d_b(u, x_n) + d_{bn} + d_b(Fx_n, Fu] \\
&\leq b[d_b(u, x_n) + d_{bn} + \alpha.\{d_b(x_n, Fx_n) + d_b(u, Fu)\}] \\
&= b[d_b(u, x_n) + d_{bn} + \alpha.\{d_b(x_n, x_{n+1}) + d_b(u, Fu)\}] \\
(1 - b\alpha)d_b(u, Fu) &\leq b[d_b(u, x_n) + \mu^n d_{b0} + \alpha.d_b(x_n, x_{n+1})]
\end{aligned}
$$

Now by using (4.13) and (4.14) and we also know that $\alpha < \frac{1}{b+1}$ then the above inequality become

$$d_b(u, Fu) = 0 \to Fu = u.$$

Hence it is clearly verified u in X is a fixed point.
Uniquness : Assume that there are two fixed points u, v in X such that $u \neq v$, now if we follows (4.8) we get

$$d_b(u, v) = d_b(Fu, Fv) \quad \leq \quad \alpha[d_b(u, Fu) + d_b(v, Fv)] = \alpha[d_b(u, u) + d_b(v, v)] = 0$$

Thus

$$d_b(u, v) = 0 \to u = v.$$

Hence it is verified u is a unique fixed point in X.

Remark 4.1.5. In this chapter, we have the following on the basis of discussion contained:

1. The defined of open balls in b-metric, rectangular and rectangular b-metric spaces are not necessarily open set.
2. The open balls collections in b-metric space, rectangular metric space and rectangular b-metric space do not necessarily form a basis for a topology.
3. the b-metric, rectangular and rectangular b-metric spaces are not necessarily Housdorff.

Conclusion

We concluded this thesis as follows:

- We have thrown some light on generalization of b-metric space and used the idea of rectangular b-metric space.
- We have reviewed some fixed point results in the setting of b-metric space.
- We have generalized some results presented in [68] by using the concept of rectangular b-metric space.
- We have also generalized and extended the Banach contraction principle in rectangular b-metric space.
- We have also generalized and extended the Kannan type theorem in rectangular b-metric space.

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