CAPITAL UNIVERSITY OF SCIENCE AND TECHNOLOGY, ISLAMABAD

Heat and mass transfer in MHD stagnation point flow of nanofluids in porous medium

by

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A thesis submitted in partial fulfillment for the degree of Mphill Mathematics

in the

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I dedicate this Sincere Effort to my dear **Parents, Sisters** and my elegant **Teachers** who are always source of Inspiration for me and their contributions are uncounted.

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Abstract

A numerical investigation is performed for the MHD stagnation point flow of water based nanofliuds. Heat and mass transfer is investigated for steady, viscous dissipations, Joule heating and thermal radiation in a porous medium. The governing partial differential equations are transformed into an arrangement of the non-linear ordinary differential equations by using the similarity transformation. Utilizing the shooting method, the system of ordinary differential equations has been solved with the help of the computational software MATLAB to compute the numerical results. The numerical solution obtained for the velocity, temperature and concentration profiles has been presented through graphs for different choice of the physical parameters. The numerical values of the skin friction, Nusselt and Sherwood numbers have also been presented and analyzed through tables.

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Abbreviations

ρ	fluid density
μ	viscosity
ν	kinematic viscosity
τ	stress tensor
κ	thermal conductivity
α	thermal diffusivity
τ_w	wall shear stress
η	dimensionless similarity variable
ψ	stream function
θ	dimensionless temperature
ϵ	thermal conductivity parameter

Nomenclature

t	time
p	pressure
V	velocity
F	force
B	magnetic field
k	porosity
a	constant
h	convective heat flux
D	species diffusivity
R	radiation parameter
M	magnetic parameter
P	permeability parameter
γ	chemical reaction parameter
k^*	absorption coefficient
A	temperature dependent viscosity
C_p	specific heat
R_e	Reynolds number
P_r	Prandtl number
N_u	Nusselt number
Sh_x	Sherwood number
Sc	Schmidt number
Ec	Eckert number
C_{fx}	Skin friction coefficient
T	temperature

T_∞	free stream temperature
T_w	wall temperature
U_∞	free stream velocity
(u, v)	velocity components
(x, y)	Cartesian coordinates

Chapter 1

Introduction

Heat and mass transfer is an active research area in fluid dynamics over last few decades. Heat and mass transfer have various applications, for example,geothermal reservoir, drying of porous solids, thermal insulation, enhanced oil recovery and underground species transport. In the latest engineering applications, heat and mass transfer with chemical reaction is of great interest for researchers. Rashad *et al.*[1] discussed the heat and mass exchange. An unsteady MHD flow, heat and mass transfer over a vertical rotating cone in the presence of heat generation/absorption was discussed by Chamka and Mudhalf [2]. The boundary layer flow with heat and mass transfer in a porous medium has great importance in industrial applications such as metal and polymer extrusion, exchangers, chemical processing equipment, etc. A solar-liquid heating collector transfers the solar energy to the internal energy of the transport medium as a kind of heat transfer. By improving the thermophysical properties of the conventional heat transfer fluids, heat transfer performance can be increased.

Fluid is the material which alters continuously by the effect of shear stress. A liquid containing nanometer-sized particles is called nanofluid. Nanoparticles range between 1 and 100nm. In 1995 Choi and Eastman [3] introduced the fundamental vision of incorporating the nano particles within the base fluidto enhance the thermal conductivity. Nano particles are created from different materials such metals (Al, Cu), carbides (SiC), oxides(Al_2O_3) or nonmetals (graphite, carbon nanotubes) and the base fluid is usually a conductive fluid such as water,ethylene glycol.To enhance the thermal conductivity of the base fluids, nanofluids are used e.g, water, ethylene glycol, propylene glycol etc. They have many applications in engineering and biomedical such as cooling, cancer therapy etc. Wong [4] discussed the nanofluid applications. Nanofluid offer many different advantages and applications, for example, fuel cell, nuclear reactor, biomedicine and microelectronics. Khan and Pop [5] explained the work on boundary layer flow of nanofluid through a stretched sheet. Izadi *et al.*[6] numerically discussed the laminar forced convection of alumina-water nanofluid in the annulus. Pak and Cho [7] shows that the Nusselt number and Reynolds number for Al_2O_3water nanofluids increment with an increment value of volume fraction. In any case, the convective heat transfer coefficient for nanofluids at a volume fraction 3 perecent was observed to be 12 percent less than that of pure water while considering a constant normal velocity. Ahmad and Pop [8] and Hamad *at al.*[9] studied of the forced convection of nanofluids over a flat plate.

The area of the magnetic properties of electrically conducting fluids is called Magnetohydrodynamics (MHD). Megantic fluids, liquids, metals, salt, water and electrolytes are the examples of MHD. Hannes Alfen introduced the word MHD. MHD is the sequence of Navier -stokes equations and Maxwell equations ofelectromagnetism is discussed by Chakraborty *et al.* [10]. Shah *et al.*[11] discussed the MHD effects and heat transfer for the UCM fluid along with Joule heating and thermal radiation using the Cattaneo-Christov heat flux model. Hayat *et al.* [12] clarified the mass exchange and MHD flow of an upper convected Maxwell fluid with an extended sheet. Ibrahim and Suneetha [13] studied the effects of Joule heating and viscous dissipation on steady MHD Marangoni convective flow over a flat surface in the presence of radiation. Ellahi [14] considered the MHD flow of non-Newtonian nanofluid ina pipe with variable viscosity and observed that the MHD parameter decreases the fluid motion and speed is bigger than that of the temperature profile. Hiemenz [15] was the first to think about the thick liquid in the neighbour of stagnation point. His work was further explored by another rearchers. Ishak *et al.*[16] discussed the MHD flow through stretched a sheet by using the kaller box method. Mahmoud and Waheed [17] studied the MHD stagnation point flow towards a surface in motion with radiation. Yasin *et al.*[18] explained the stagnation point as well as the heat transfer through a shrinking sheet in a micropolar fluid. In different devices, the effect of viscous dissipation plays an important role in regular convection. It also has a strong gravitational field and geological processes. Hossain [19] discussed the viscous dissipation and Joule heating effect on the flow of an electrically conducting compressible and viscous fluid. Watanabe and Pop [20] highlighted the impact of stress work and viscous dissipation on MHD flow over a flat plate with Joule heating and dissipation through the flow of thick, incompressible and electrically conducting fluid past a semi-infinite impermeable flat. The unsteady free convection boundary layer flow of a nanofluid along a stretching sheet in the presence of magnetic field with thermal radiation was talked about by Khan *et al.*[21]. The chemical reaction can further be classified by consider the heterogeneous and homogenous processes. In the case of the strong compound system, the reaction is heterogenous. In most of the cases of chemical reaction processes, the concentration rate depends upon the species itself as discussed by Magyari and Chamkha *et al.*[22]. Devi and Kandasmy [23] analyzed the impact of homogenous chemical reaction with heat and mass transfer laminar flow along with semi-infinite horizontal plate. Chamkha and Rashad [24] talked about the impact of chemical reaction on MHD flow in the presence of heat generation or absorption of uniform vertical permeable surface. Raptis and Pardikis [25] discussed the heat transfer of a microfluid in the presence of radiation. A numerical study of the MHD flow of Maxwell liquid in the presence of chemical reaction and thermophoresis is reported by Shateyi [26]. Related issues in this area are also explained by Mansour *et al.* [27] and Bhattacharyya[28]. Rana and Bhargava [29] worked out the heat transfer of a nanofluid through a non-linear stretched sheet by using the finite element and finite difference methods. Das [30] explained the impact of chemical reaction with radiation on the heat and mass exchange along the MHD flow. In 2014, Chutia and Deka [31] numerically discussed the heat transfer and steady MHD flow in a rectangular electrically protected pipe in the existence of the attractive field. In energy equation, they considered both the Joule heating and viscous dissipation. The combined effect of Joules heating and viscous dissipation on the mixed convection MHD flow in a vertical channel was noticed by Abo-Eldahab and El-Aziz [32]. Sparrow and Cess [33] examined the Magnetohydrodynamic free convection flow of an electrically conducting fluid along a heated semi-infinite vertical flat plate in the presence of a strong magnetic field. Takhar and Soundalgekar [34] have studied the effects of viscous and Joule heating on the problem posed by Sparrow and Cess [33], using the series expansion method. Mabood *at al.* [35] discussed the MHD flow of heat and mass transfer of nanofluids in the porous medium with radiation, viscous dissipation, chemical reaction.

Thesis contribution:

In this thesis, the findings of Ref [35] have been reproduced and extended by considering the Joule heating, a uniform porous medium and temperature dependant viscosity and thermal radiation. The acquired arrangement of partial differential equations are transformed into non-linear and coupled ordinary differential equations by using a similarity transformation. With the help of shooting technique, numerical solution of the system of ordinary differential equations is achieved and then compared the numerical results obtained by using the Matlab builtin function bvp4c.

Thesis outline:

The thesis is divided into five chapters:

Chapter 2, contains the basic definitions and terminology .

Chapter 3, contains the detailed review of Ref [35]. A numerical study of viscous dissipation, radiation and chemical reaction of MHD stagnation point flow of nanofluids in porous medium. Heat and mass exchange of nanofluid over a flat plate in porous medium is analyzed. The equations of the flow model are solved numerically. The impact of physical parameters concerning with flow model have also been presented through graphs and tables. The result achieved are also compared with the published results of Ref [35] an found excellent agreement between them.

In **Chapter 4**, The heat and mass transfer are analysed for steady, viscous dissipations and Joule heating past a porous medium. This chapter consists mathematical formulation of the extended model, development of numerical solution and results.

Chapter 5 includes the conclusion of the entire research and recommendations for the future work.

All the references are listed in **Bibliography**

Chapter 2

Basic definitions and governing equations

In this chapter, some basic laws, concepts, terminologies and definitions Ref[36] will be explained, which will be helpful in continuing the work for the next chapters.

2.1 Fluid

"A fluid is a substance that deforms continuously under the application of a shear stress no matter how small the stress may be. Thus fluids comprise the liquid and gas (or vapour) phases of the physical forms in which matter exists."

2.2 Fluid mechanics

"Fluid mechanics deals with the behaviour of fluids at rest or in motion. There are two branches of fluid mechanics. "

2.3 Fluid statics

“Fluid static is the part of fluid mechanics, that deals with the fluid and its characteristics at the constant position. ”

2.4 Fluid dynamics

“The branch of fluid mechanics that covers the properties of the fluid in the state of progression from one place to another is called fluid dynamics. ”

2.5 Flow

“It is the deformation of the material under the influence of different forces. If the deformation increase is continuous without any limit then the process is known as flow.”

2.6 Uniform and non-uniform flows

“The flow is said to be uniform if the magnitude and direction of flow velocity are the same at every point and flow is said to be non-uniform if the velocity is not the same at each point of the flow, at a given instant.” Homogenous mixture of base fluid and nanoparticle is termed as the nanofluid.”

2.7 Steady and unsteady flows

“A flow is said to be steady flow in which the fluid properties do not change with time at a specific point, i.e.

$$\frac{\partial \lambda}{\partial t} = 0, \tag{2.1}$$

where λ is any fluid property." "A flow is said to be unsteady flow in which fluid properties change with time. i.e.,

$$\frac{\partial \lambda}{\partial t} \neq 0.'' \tag{2.2}$$

2.8 Laminar and turbulent flows

"A flow is laminar in which the fluid particles move in smooth layers or laminas and a turbulent in which the fluid particles rapidly mix as they move along due to random three-dimensional velocity fluctuations."

2.9 Compressible and incompressible flows

"Flow in which variations in density are negligible is termed as incompressible otherwise it is called compressible. The most common example of compressible flow is the flow of gases, while the flow of liquids may frequently be treated as incompressible." "Mathematically,

$$\frac{D\rho}{Dt} = 0,$$

where ρ denotes the fluid density and $\frac{D}{Dt}$ is the material derivative given by

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla. \tag{2.3}$$

In Eq. (2.3), $\mathbf{V}$ denotes the velocity of the flow and ∇ is the differential operator. In Cartesian coordinate system, ∇ is given as

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}.''$$

2.10 Viscosity

"This is the internal property of a fluid by virtue of which it offers resistance to the flow. Mathematically it is defined as the ratio of the shear stress to the rate of shear strain. i.e,

$$Viscosity = \mu = \frac{shearstress}{rate\ of\ shear\ strain}.$$

In the above definition, μ is the co efficient of viscosity or absolute viscosity or dynamics viscosity or simply viscosity having dimension $[\frac{M}{LT}]$. Water is thin having low viscosity and other hand honey is thick having higher viscosity. Usually liquids have non-zero viscosity. Its unit is $Pa.s$=$\frac{kg}{(s.m)}$. "

2.10.1 Dynamic viscosity

"The property of the fluid that measures the internal resistance of fluid is called dynamic viscosity. This resistance arises from the attractive forces between the molecules of the fluid.

Mathematically, it is written as the ratio of the shear stress to the rate of shear strain and it is denoted by μ.

$$\text{Viscosity}(\mu) = \frac{\text{Shear stress}}{\text{Rate of shear strain}}.$$

In the above expression μ is called the co-efficient of viscosity. This is also known as the absolute viscosity or simply viscosity and its dimension is $[ML^{-1}T^{-1}]$. In system its unit is kg/ms or Pascal-second [Pa.s]."

2.10.2 Kinematic viscosity

"The kinematic viscosity represents the ratio of dynamic viscosity μ to the density of the fluid ρ, it is represented by ν, Mathematically it is written as

$$\nu = \frac{\mu}{\rho}.$$

The dimension of kinematic viscosity is $[L^2T^{-1}]$ and its unit in SI system is m^2/s."

2.11 Nanofluid

"

2.12 Newtonian and non-Newtonian fluid

"The fluid is said to be a Newtonian fluid in which the stress arising from its flow at every point is linearly proportional to the local strain rate. Newtonian fluid behaviour is described by the relation

$$\tau = \mu \frac{du}{dy}.$$

In the above equation, τ is the stress tensor, μ is the viscosity and $\frac{du}{dy}$ is the deformation rate. Fluids are said to be non-Newtonian fluids for which the shear stress is not directly proportional to the deformation rate."

2.13 Generalized continuity equation

"Continuity equation is obtained from the law of conservation of mass which states that mass can neither be created nor be destroyed inside a control volume. The mass inside the fixed control system will not change if we examine a differential control volume system enclosed by a surface fixed in space, Then the equation of continuity can be written as"

$$\text{“}\frac{\partial \rho}{\partial t} + \nabla.(\rho \mathbf{V}) = 0. \tag{2.4}$$

If the density is constant and spatially uniform, in that case Eq. (2.4) become

$$\nabla \cdot \mathbf{V} = 0.''$$

2.14 Generalized Momentum equation

"The equation of generalized linear momentum for the fluid particle is acquired. It is expressed as: the net force $\mathbf{F}$ acting on a fluid particle is equal to the time rate of change of linear momentum. Consider the mass in a system defined by control surface of infinitesimally small dimensions dx, dy and dz. The mass of the system is steady. Newtons second law can be composed as

$$m\frac{D\mathbf{V}}{Dt} = \mathbf{F}.$$

$$\rho\frac{D\mathbf{V}}{Dt} = \nabla.\tau + \rho\mathbf{b},$$

where $\rho\mathbf{b}$ is the net body force, $\nabla.\tau$ is the surface forces and τ is the Cauchy stress tensor."

2.15 Magnetohydrodynamics

"The study of the dynamics of electrically conducting fluids for example plasmas or electrolytes, is known as magnetohydrodynamics (MHD)."

2.16 Stagnation point

"It is a point in a flow field where the fluid velocity is zero. It exists at the surface of objects in the field where fluid is rest by the object. Static pressure is the example of stagnation point."

2.17 Viscous dissipation

"The process in which the work done by fluid is converted into heat is called viscous dissipation."

2.18 Joule heating

"The heat which is produced due to flow of current through conductor is called Joule heating."

2.19 Radiation

"Radiation is the energy transfer due to the release of photons or electromagnetic waves from a surface volume. Radiation doesn't require any medium to transfer heat. The energy produced by radiation is transformed by electromagnetic waves."

2.20 Porosity

"The porosity is the relationship of the volume of void space to the bulk volume of a permeable medium. A permeable medium is often identified by its porosity. The momentum equation with porosity and magnetohydrodynamics is as the following

$$\rho \frac{D\mathbf{V}}{Dt} = \nabla.\tau - \rho\sigma\mathbf{B^2V} - \rho k\mathbf{V}. \tag{2.5}$$

Here k and $\mathbf{B}$ are the porosity and magnetic field of the medium separately."

2.21 Mass transfer

"Mass exchange is the total movement of mass from one place to another."

2.22 Heat transfer

"It is the energy transfer due to the temperature difference. At the point when there is a temperature contrast in a medium or between media, heat transfer must take place. Heat transfer is normally an object from high temperature to a lower temperature."

2.23 Conduction

"Conduction is the process in which heat is transferred through the material between the objects that are in physical contact. For example: picking up a hot cup of tea."

2.24 Convection

"Convection is a mechanism in which heat is transferred through fluids (gases or liquids) from a hot place to a cool place. For example: Macaroni rising and falling in a pot of boiling water."

2.24.1 Forced convection

"Forced convection is a process in which fluid motion is produced by an external source. It is a special type of heat transfer in which fluid moves in order to increase the heat transfer. In other words, a method of heat transfer in which heat transfer is caused by dependent source like a fan and pump etc, is called forced convection. For example: Gas convection heaters have a gas burner to generate the heat, and a fan to force the heated air to circulate around the room."

2.24.2 Natural Convection

"Natural convection is a heat transport process, in which the heat transfer is not caused by an external source, like pump, fan and suction. It happens due to the

FIGURE 2.1: Example of forced convection.

temperature differences which affect the density of the fluid. It is also called free convection. Example: Daily weather."

2.24.3 Mixed convection

"It is a combination of both forced convection and natural convection. For example if fluid is moving upward along the moment of the vertical stretching sheet is forced between while in the same phenomena fluid is freely falling due to the gravity which is forced convection. When these two phenomena appear in the same model then such kind of flow is mixed convection."

2.25 Thermal conductivity

"Thermal conductivity (κ) is the property of a material related to its ability to transfer heat. Mathematically,

$$\kappa = \frac{q\nabla l}{S\nabla T},$$

where q is the heat passing through a surface area S and the effect of a temperature difference ∇T over a distance is ∇l. Here l, S and ∇T all are assumed to be of unit measurement. In system unit of thermal conductivity is $\frac{W}{m.\kappa}$ and its dimension is $[MLT^{-3}\theta^{-1}]$."

2.26 Thermal diffusivity

"Thermal diffusivity is material,s property which identifies the unsteady heat conduction. Mathematically, it can be written as,

$$\alpha = \frac{\kappa}{\rho C_p},$$

where κ, ρ and C_p represents the thermal conductivity of material, the density and the specific heat capacity. In SI system unit and dimension of thermal diffusivity are $m^2 s^{-1}$ and $[LT^{-1}]$ respectively."

2.27 Dimensionless numbers

2.27.1 Reynolds number Re

"It is a dimensionless number which is used to clarify the different flow behaviours like turbulent or laminar flow. It helps to measure the ratio between inertial force and the viscous force. Mathematically,

$$Re = \frac{\frac{\rho U^2}{L}}{\frac{\mu U}{L^2}} \quad \Longrightarrow \quad Re = \frac{LU}{\nu},$$

where U denotes the free stream velocity, L the characteristics length. At low Reynolds number, laminar flow arises where the viscous forces are dominant. At high Reynolds number, turbulent flow arises where the inertial forces are dominant."

2.27.2 Prandtal number (P_r)

"It is the ratio between the momentum diffusivity (ν) and thermal diffusivity (α). Mathematically, it can be defined as"

$$\text{“}P_r = \frac{\nu}{\alpha} = \frac{\mu/\rho}{k/c_p} = \frac{\mu c_p}{k},$$

where μ represents the dynamic viscosity, C_p denotes the specific heat and κ stands for thermal conductivity. The relative thickness of thermal and momentum boundary layer is controlled by Prandtal number. For small P_r, heat distributed rapidly corresponds to the momentum."

2.27.3 Nusselt number (Nu)

"It is the ratio of the convective to the conductive heat transfer to the boundary. Mathematically,

$$Nu = \frac{hL}{\kappa},$$

where h stands for convective heat transfer, L for the characteristics length and κ stands for the thermal conductivity."

2.27.4 Sherwood number (Sh_x)

"It is the nondimensional quantity which show the ratio of the mass transport by convection to the transfer of mass by diffusion. Mathematically:

$$Sh_x = \frac{kL}{D},$$

here L is characteristics lenght, D is the mass diffusivity and k is the mass transfer coefficient.

2.27.5 Skin friction coefficient (C_{fx})

"Skin friction coefficient occurs between the fluid and the solid surface which leads to slow down the motion of the fluid. The skin friction coefficient can be defined as

$$C_f = \frac{2\tau_w}{\rho U^2},$$

where τ_w denotes the wall shear stress, ρ the density and U the free-stream velocity."

2.27.6 Eckert number (Ec)

"It is the dimensionless number used in continuum mechanics. It describes the relation between flows and the boundary layer enthalpy difference and it is used for characterized heat dissipation. Mathematically,

$$Ec = \frac{u^2}{C_p \nabla T},''$$

2.28 Boundary layer flow

"The concept of boundary layer was first introduced by Ludwig Prandtl [37], a German aerodynamicist, in 1904. Prandtl introduced the basic idea of the boundary layer in the motion of a fluid over a surface. Boundary layer is a flow layer of fluid close to the solid region of the wall in contact where the viscosity effects are significants. The flow in this layer is usually laminar. The boundary layer thickness is the measure of the distance apart from the surface. There are two types of boundary layers:

- Hydrodynamic (velocity) boundary layer
- Thermal boundary layer"

2.28.1 Hydrodynamic boundary layer

"A region of a fluid flow where the transition from zero velocity at the solid surface to the free stream velocity at some extent far from the surface in the direction normal to the flow takes place in a very thin layer, is known as the hydrodynamic boundary layer."

2.28.2 Thermal boundary layer

"The heat transfer exchange surface and the free stream a liquid or a gaseous agent for heat transfer. From wall to free stream we come across the change of temperature of heat transfer agent. It increases from wall to the main stream. The surface temperature is assumed to be equal to the temperature of the fluid layer closed to the wall inside the boundary and this temperature is equal to the temperature of the bulk at some point in the fluid."

Chapter 3

Viscous dissipation and chemical reaction for MHD stagnation point flow of nanofluids in porous medium

3.1 Introduction

The numerical study of MHD flow with heat and mass exchange of viscous, incompressible and two-dimensional nanofluid over a flat plate in a uniform permeable medium has been taken under consideration. Using the similarity transformation, the governing PDEs are transformed into the ODEs. The numerical solution for the system of the differential equations is achieved by using the shooting technique. Numerical results for different parameters are found to be in excellent matching with those obtained by the MATLAB built-in function bvp4c. In this chapter, graphs and tables are also discussed to show the importance of different parameters involved in the equation. This chapter provides a detailed review of Ref [35].

3.2 Mathematical modeling

A two dimensional boundary layer flow of a viscous, Newtonian and incompressible nanofluid flow through a flat plate in a porous medium has been considered with main focus on the heat and mass transfer. The geometry of the flow model is shown in Figure 3.1.

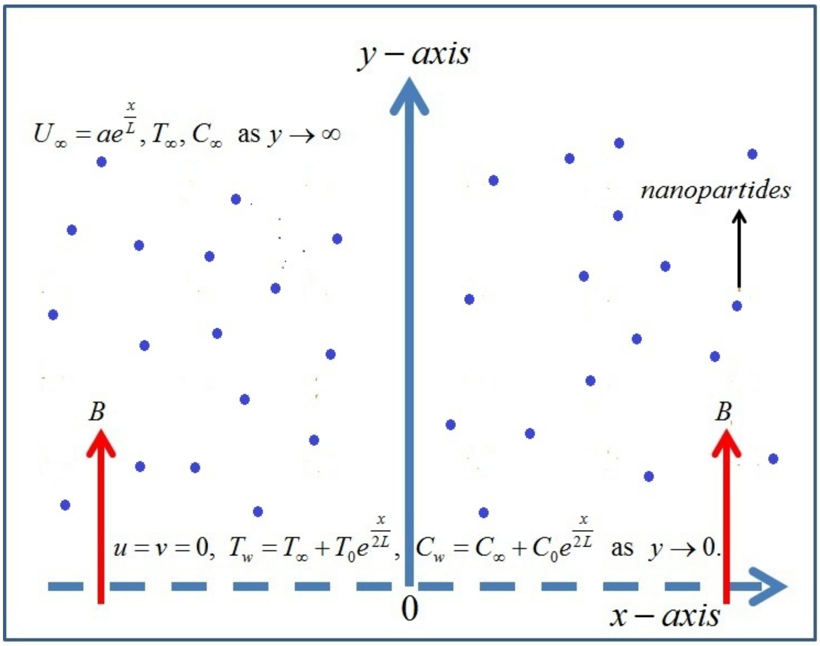

FIGURE 3.1: Geometry for the flow under consideration.

The flow is described by the equation of continuity, equation of momentum and the energy equation as

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{3.1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=U_\infty\frac{dU_\infty}{dx}+\nu_{nf}\frac{\partial^2 u}{\partial y^2}+\frac{\nu_{nf}}{k}(U_\infty-u)+\frac{\sigma B^2}{\rho_{nf}}(U_\infty-u), \tag{3.2}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha_{nf}\frac{\partial^2 T}{\partial y^2}-\frac{1}{(\rho c_p)_{nf}}\frac{\partial q_r}{\partial y}+\frac{\mu_{nf}}{(\rho c_p)_{nf}}\left(\frac{\partial u}{\partial y}\right)^2, \tag{3.3}$$

$$u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}=D\frac{\partial^2 C}{\partial y^2}-K(C-C_\infty). \tag{3.4}$$

Boundary conditions can be written as

$$u = v = 0, T = T_w = T_\infty + T_0 e^{\frac{x}{2L}}, C = C_w = C_\infty + C_0 e^{\frac{x}{2L}} \ at \ y = 0, \tag{3.5}$$

$$u \to U_\infty = a e^{\frac{x}{L}}, T \to T_\infty, C \to C_\infty \ as \ y \to \infty. \tag{3.6}$$

Here σ_s is the electrical conductivity of the base-fluid whereas σ_{nf}, ν_{nf}, ρ_{nf}, α_{nf},k_{nf} are the electric conductivity, the effective viscosity, the effective density, the effective thermal diffusivity, the thermal conductivity of the nanofluid respectively. These quantities are formulated as follows:

$$\alpha_{nf} = \frac{k_{nf}}{(\rho c_p)_{nf}}, \tag{3.7}$$

$$\rho_{nf} = (1-\phi)\rho_f + \phi\rho_s, \tag{3.8}$$

$$\mu_{nf} = \frac{\mu_f}{(1-\phi)^{2.5}}, \tag{3.9}$$

$$(\rho c_p)_{nf} = (1-\phi)(\rho c_p)_f + \phi(\rho c_p)_s, \tag{3.10}$$

$$\frac{k_{nf}}{k_f} = \frac{K_s + 2K_f - 2\phi(K_f - K_s)}{K_s + 2K_f + 2\phi(K_f - K_s)}, \tag{3.11}$$

$$\sigma_{nf} = (1-\phi)\sigma_f + \phi\sigma_s, \tag{3.12}$$

$$\nu_f = \frac{\mu_f}{\rho_f}, \tag{3.13}$$

$$K = K_o e^{\frac{-x}{L}}, \tag{3.14}$$

$$B = B_o e^{\frac{x}{2L}}. \tag{3.15}$$

The radiative heat flux q_r, by using the Rosseland approximation for radiation, can be written as

$$q_r = \frac{-4\sigma^*}{3k^*}\frac{\partial T^4}{\partial y}, \tag{3.16}$$

where σ^* is the Stefan-Boltzman constant and k^* is the absorption coefficient. If temperature contrast is very small, then the temperature difference T^4 might be extended about T_∞ in a Taylor series, as follows:

$$T^4 = T_\infty^4 + \frac{4T_\infty^3}{1!}(T - T_\infty)^1 + \frac{12T_\infty^2}{2!}(T - T_\infty)^2 + \frac{24T_\infty}{3!}(T - T_\infty)^3 + \frac{24}{4!}(T - T_\infty)^4.$$

Disregarding the higher order terms,

$$\begin{aligned} T^4 &= T_\infty^4 + 4T_\infty^3(T - T_\infty) \\ \Rightarrow \quad T^4 &= T_\infty^4 + 4T_\infty^3 T - 4T_\infty^4 \\ \Rightarrow \quad T^4 &= 4T_\infty^3 T - 3T_\infty^4 \\ \Rightarrow \quad \frac{\partial T^4}{\partial y} &= 4T_\infty^3 \frac{\partial T}{\partial y}. \end{aligned} \tag{3.17}$$

Using (3.17) in (3.16) and then differentiating w.r.t y, we get

$$\frac{\partial q_r}{\partial y} = \frac{-16\sigma^* T_\infty^3}{3k^*} \frac{\partial^2 T}{\partial y^2}. \tag{3.18}$$

Then (3.3) gets the following form.

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha_{nf}\frac{\partial^2 T}{\partial y^2} + \frac{16\sigma^* T_\infty^3}{3(\rho c_p)_{nf} k^*}\frac{\partial^2 T}{\partial y^2} + \frac{\mu_{nf}}{(\rho c_p)_{nf}}\left(\frac{\partial u}{\partial y}\right)^2. \tag{3.19}$$

Let ψ be the stream function satisfying the continuity equation in the following sense.

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}. \tag{3.20}$$

For the conversion of the mathematical model (3.1)-(3.4) into the dimensionless form, the following similarity transformation has been introduced.

$$\begin{aligned} \eta &= y\sqrt{\frac{a}{2\nu_f L}} e^{\frac{x}{2L}}, \\ \psi &= \sqrt{2aL\nu_f} f(\eta) e^{\frac{x}{2L}}, \\ \theta(\eta) &= \frac{T - T_\infty}{T_w - T_\infty}, \\ h(\eta) &= \frac{C - C_\infty}{C_w - C_\infty}. \end{aligned}$$

The detailed procedure for the conversion of (3.1)-(3.4) has been described in the upcoming discussion.

- $$\begin{aligned} u &= \frac{\partial \psi}{\partial y} \\ &= \frac{\partial \psi}{\partial \eta}\frac{\partial \eta}{\partial y} \\ &= \sqrt{2a\nu_f L}e^{\frac{x}{2L}}f'(\eta)\sqrt{\frac{a}{2\nu_f L}}e^{\frac{x}{2L}} \\ &= \sqrt{\frac{2a^2L\nu_f}{2\nu_f L}}e^{\frac{2x}{2L}}f'(\eta) = ae^{\frac{x}{L}}f'(\eta). \end{aligned}$$
- $$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial}{\partial x}\left(ae^{\frac{x}{L}}f'(\eta)\right) \\ &= a\left(\frac{\partial f'(\eta)}{\partial \eta}.\frac{\partial \eta}{\partial x}.e^{\frac{x}{L}} + f'(\eta).\frac{\partial}{\partial x}(e^{\frac{x}{L}})\right) \\ &= a\left(f''(\eta).y\sqrt{\frac{a}{2\nu_f L}}e^{\frac{x}{2L}}.\frac{1}{2L}e^{\frac{x}{L}} + \frac{1}{L}.f'(\eta)e^{\frac{x}{L}}\right) \\ &= a\left(\frac{\eta f''(\eta)}{2L} + \frac{f'(\eta)}{L}\right)e^{\frac{x}{L}} \\ &= \frac{ae^{\frac{x}{L}}}{2L}\left(\eta f''(\eta) + 2f'(\eta)\right). \end{aligned} \tag{3.21}$$
- $$\begin{aligned} v &= -\frac{\partial \psi}{\partial x} = -\frac{\partial}{\partial x}\left(\sqrt{2aL\nu_f}f(\eta)e^{\frac{x}{2L}}\right) \\ &= -\sqrt{2aL\nu_f}\left(\frac{\partial f(\eta)}{\partial x}e^{\frac{x}{2L}} + f(\eta)\frac{\partial}{\partial x}e^{\frac{x}{2L}}\right) \\ &= -\sqrt{2aL\nu_f}\left(\frac{\partial f(\eta)}{\partial \eta}.\frac{\partial \eta}{\partial x}.e^{\frac{x}{2L}} + f(\eta)e^{\frac{x}{2L}}.\frac{1}{2L}\right) \\ &= -\sqrt{2aL\nu_f}\left(f'(\eta)y\sqrt{\frac{a}{2\nu_f L}}e^{\frac{x}{2L}}e^{\frac{x}{2L}}\frac{1}{2L} + \frac{1}{2L}f(\eta)e^{\frac{x}{2L}}\right) \\ &= -\sqrt{2aL\nu_f}\left(\eta f'(\eta)e^{\frac{x}{2L}}.\frac{1}{2L} + \frac{1}{2L}f(\eta)e^{\frac{x}{2L}}\right) \\ &= -\sqrt{2aL\nu_f}\frac{e^{\frac{x}{2L}}}{2L}\left(\eta f'(\eta) + f(\eta)\right). \end{aligned}$$
- $$\frac{\partial v}{\partial y} = -\sqrt{2aL\nu_f}\left(\frac{\partial}{\partial y}(\eta f'(\eta)) + \frac{\partial f(\eta)}{\partial y}\right)\frac{1}{2L}e^{\frac{x}{2L}}$$

$$= -\sqrt{2aL\nu_f}\left(\frac{\partial \eta}{\partial y}f'(\eta) + \frac{\partial f'(\eta)}{\partial \eta}.\frac{\partial \eta}{\partial y}.\eta + \frac{\partial f(\eta)}{\partial \eta}\frac{\partial \eta}{\partial y}\right)\frac{1}{2L}e^{\frac{x}{2L}}$$
$$= -\frac{\sqrt{2aL\nu_f}}{2L}.\sqrt{\frac{a}{2L\nu_f}}\left(f'(\eta) + \eta f''(\eta) + f'(\eta)\right)e^{\frac{x}{2L}}.e^{\frac{x}{2L}}$$
$$= -\frac{ae^{\frac{x}{L}}}{2L}\left(\eta f''(\eta) + 2f'(\eta)\right). \tag{3.22}$$

Though the continuity equation (3.1) is already satisfied by the choice of the stream function ψ in (3.20), it can again be verified by using (3.21) and (3.22) in it as follows.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{a}{2L}\left(\eta f''(\eta) + 2f'(\eta)\right)e^{\frac{x}{L}} - \frac{a}{2L}\left(\eta f''(\eta) + 2f'(\eta)\right)e^{\frac{x}{L}} = 0.$$

Now we include below the procedure for the conversion of (3.2) into the dimensionless form.

- $$\frac{\partial u}{\partial y} = ae^{\frac{x}{L}}\frac{\partial f'}{\partial \eta}.\frac{\partial \eta}{\partial y}$$
$$= ae^{\frac{x}{L}}f''(\eta).\sqrt{\frac{a}{2Lv_f}}e^{\frac{x}{2L}}$$
$$= a\sqrt{\frac{a}{2L\nu_f}}e^{\frac{3x}{2L}}f''(\eta).$$
- $$u\frac{\partial u}{\partial x} = \left(ae^{\frac{x}{L}}f'(\eta)\right).\frac{a}{2L}\left(\eta f''(\eta) + 2f'(\eta)\right)e^{\frac{x}{L}}$$
$$= \frac{a^2e^{\frac{2x}{L}}}{2L}\left(\eta f'(\eta)f''(\eta) + 2f'^2(\eta)\right). \tag{3.23}$$
- $$v\frac{\partial u}{\partial y} = \left(-\frac{\sqrt{2aL\nu_f}}{2L}e^{\frac{x}{2L}}\eta f'(\eta) + f(\eta)\right).\left(a\sqrt{\frac{a}{2L\nu_f}}e^{\frac{3x}{2L}}f''(\eta)\right)$$
$$= -a\frac{\sqrt{2aL\nu_f}}{2L}.\sqrt{\frac{a}{2L\nu_f}}e^{\frac{x}{2L}}e^{\frac{3x}{2L}}\left(\eta f'(\eta)f''(\eta) + f(\eta)f''(\eta)\right)$$
$$= -\frac{a^2}{2L}e^{\frac{2x}{L}}\left(\eta f''(\eta)f'(\eta) + f''(\eta)f(\eta)\right). \tag{3.24}$$

Using (3.23) and (3.24), the left side of (3.2) becomes.

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{a^2}{2L}e^{\frac{2x}{L}}\left(\eta f''(\eta)f'(\eta) + 2f'^2(\eta)\right) - \frac{a^2}{2L}e^{\frac{2x}{L}}\left(\eta f''(\eta)f'(\eta) + f''(\eta)f(\eta)\right)$$
$$= \frac{a^2e^{\frac{2x}{L}}}{2L}\left(2f'^2(\eta) - f''(\eta)f(\eta)\right).$$

To convert the right side of Eq. (3.2) into the dimensionless form, the following procedure has been followed.

- $$\begin{aligned} U_\infty \frac{dU_\infty}{dx} &= ae^{\frac{x}{L}} \frac{d}{dx}\left(ae^{\frac{x}{L}}\right) \\ &= ae^{\frac{x}{L}} \frac{1}{L} ae^{\frac{x}{L}} \\ &= \frac{a^2 e^{\frac{2x}{L}}}{L}. \end{aligned} \tag{3.25}$$

- $$\begin{aligned} \frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(a\sqrt{\frac{a}{2L\nu_f}} e^{\frac{3x}{2L}} f''(\eta)\right) \\ &= a\sqrt{\frac{a}{2L\nu_f}} e^{\frac{3x}{2L}} \frac{\partial f''}{\partial \eta} . \frac{\partial \eta}{\partial y} = a\sqrt{\frac{a}{2L\nu_f}} e^{\frac{3x}{2L}} f'''(\eta) \sqrt{\frac{a}{2L\nu_f}} e^{\frac{x}{2L}} \\ &= \frac{a^2 e^{\frac{2x}{L}}}{2L\nu_f} f'''(\eta). \end{aligned}$$

- $$\begin{aligned} \nu_{nf} \frac{\partial^2 u}{\partial y^2} &= \frac{\mu_{nf}}{\rho_{nf}}\left(\frac{\partial^2 u}{\partial y^2}\right) \\ &= \frac{\mu_f}{(1-\phi)^{2.5}\big((1-\phi)\rho_f + \phi\rho_s\big)}\left(\frac{a^2 e^{\frac{2x}{L}}}{2\nu_f L} f'''(\eta)\right) \\ &= \frac{\mu_f a^2 e^{\frac{2x}{L}}}{\frac{\mu_f}{\rho_f}.2L(1-\phi)^{2.5}\rho_f\big(1-\phi+\frac{\phi\rho_s}{\rho_f}\big)} f'''(\eta) \qquad \left(\because \ \nu_f = \frac{\mu_f}{\rho_f}\right) \\ &= \frac{a^2 e^{\frac{2x}{L}}}{2L(1-\phi)^{2.5}\big(1-\phi+\frac{\phi\rho_s}{\rho_f}\big)} f'''(\eta). \end{aligned} \tag{3.26}$$

- $$\begin{aligned} \frac{\nu_{nf}}{k}(U_\infty - u) &= \frac{\mu_f}{k_o e^{\frac{-x}{L}}(1-\phi)^{2.5}\Big((1-\phi)\rho_f + \phi\rho_s\Big)}\left(ae^{\frac{x}{L}} - ae^{\frac{x}{L}} f'(\eta)\right) \\ &= \frac{\nu_f \rho_f}{\rho_f k_o (1-\phi)^{2.5}\big(1-\phi+\phi\frac{\rho_s}{\rho_f}\big)} ae^{\frac{x}{L}}.e^{\frac{x}{L}}(1-f'(\eta)) \qquad \left(\because \ \nu_f = \frac{\mu_f}{\rho_f}\right) \\ &= \frac{a\nu_f e^{\frac{2x}{L}}}{k_o(1-\phi)^{2.5}\big(1-\phi+\phi\frac{\rho_s}{\rho_f}\big)}\Big(1-f'(\eta)\Big). \end{aligned} \tag{3.27}$$

- $$\begin{aligned} \frac{\sigma B^2}{\rho_{nf}}(U_\infty - u) &= \frac{\sigma (B_o e^{\frac{x}{2L}})^2}{(1-\phi)\rho_f + \phi\rho_s}\left(ae^{\frac{x}{L}} - ae^{\frac{x}{L}} f'(\eta)\right) \\ &= \frac{\sigma {B_o}^2 (e^{\frac{x}{2L}})^2}{(1-\phi)\rho_f + \phi\rho_s} ae^{\frac{x}{L}}\Big(1-f'(\eta)\Big) \\ &= \frac{\sigma {B_o}^2 a e^{\frac{2x}{L}}}{\rho_f(1-\phi+\frac{\phi\rho_s}{\rho_f})}\Big(1-f'(\eta)\Big). \end{aligned} \tag{3.28}$$

Using (3.25) - (3.28) in the right side of (3.2), we get

$$
\begin{aligned}
&U_\infty \frac{dU_\infty}{dx} + \nu_{nf}\frac{\partial^2 u}{\partial y^2} + \frac{\nu_{nf}}{k}(U_\infty - u) + \frac{\sigma B^2}{\rho_{nf}}(U_\infty - u)\\
&= \frac{a^2 e^{\frac{2x}{L}}}{L} + \frac{a^2 e^{\frac{2x}{L}} f'''(\eta)}{2L(1-\phi)^{2.5}\big(1-\phi+\frac{\phi\rho_s}{\rho_f}\big)}\\
&\quad + \frac{\nu_f}{k_o(1-\phi)^{2.5}\big(1-\phi+\phi\frac{\rho_s}{\rho_f}\big)} a e^{\frac{2x}{L}}\Big(1-f'(\eta)\Big)\\
&\quad + \frac{\sigma {B_o}^2 a e^{\frac{2x}{L}}}{\rho_f(1-\phi+\frac{\phi\rho_s}{\rho_f})}\Big(1-f'(\eta)\Big).
\end{aligned}
$$

Hence the dimensionless form of (3.2) becomes:

$$
\begin{aligned}
&\frac{a^2 e^{\frac{2x}{L}}}{2L}\Big(2f'^2(\eta) - f''(\eta)f(\eta)\Big) = \frac{a^2 e^{\frac{2x}{2L}}}{2L}\Bigg[2 + \frac{1}{(1-\phi)^{2.5}\big(1-\phi+\frac{\phi\rho_s}{\rho_f}\big)} f'''(\eta)\\
&+ \frac{2L\nu_f}{ak_o(1-\phi)^{2.5}\big(1-\phi+\phi\frac{\rho_s}{\rho_f}\big)}\Big(1-f'(\eta)\Big) + \frac{2L\sigma {B_o}^2}{a\rho_f(1-\phi+\frac{\phi\rho_s}{\rho_f})}\Big(1-f'(\eta)\Big)\Bigg]\\
&\Rightarrow\quad 2f'^2(\eta) - f(\eta)f''(\eta) \quad = 2 + \frac{f'''(\eta)}{(1-\phi)^{2.5}(1-\phi+\frac{\phi\rho_s}{\rho_f})}\\
&+ \left(\frac{2L\nu_f}{ak_o(1-\phi)^{2.5}(1-\phi+\frac{\phi\rho_s}{\rho_f})} + \frac{2L\sigma {B_o}^2}{a\rho_f(1-\phi+\frac{\phi\rho_s}{\rho_f})}\right)\Big(1-f'(\eta)\Big)\\
&\Rightarrow\quad -2 + 2f'^2(\eta) - f(\eta)f''(\eta) = \frac{f'''(\eta)}{(1-\phi)^{2.5}(1-\phi+\frac{\phi\rho_s}{\rho_f})}\\
&+ \left(\frac{2L\nu_f}{ak_o(1-\phi)^{2.5}(1-\phi+\frac{\phi\rho_s}{\rho_f})} + \frac{2L\sigma {B_o}^2}{a\rho_f(1-\phi+\frac{\phi\rho_s}{\rho_f})}\right)\Big(1-f'(\eta)\Big)\\
&\Rightarrow\quad \frac{f'''(\eta)}{(1-\phi)^{2.5}(1-\phi+\frac{\phi\rho_s}{\rho_f})} + 2(1-f'^2(\eta)) + f(\eta)f''(\eta) + \Bigg[\frac{1}{(1-\phi)^{2.5}(1-\phi+\frac{\phi\rho_s}{\rho_f})}\\
&.\left(\frac{2L\nu_f}{ak_o} + \frac{2L\sigma {B_o}^2}{a\rho_f}(1-\phi)^{2.5}\right)\Big(1-f'(\eta)\Big)\Bigg] = 0.
\end{aligned}
\tag{3.29}
$$

Now we include below the procedure for the conversion of (3.3) into the dimensionless form.

- $\theta(\eta) = \dfrac{T - T_\infty}{T_w - T_\infty}$

$$\begin{aligned}
\Rightarrow \quad T &= (T_w - T_\infty)\theta(\eta) + T_\infty \\
&= (T_\infty + T_0 e^{\frac{x}{2L}} - T_\infty)\theta(\eta) + T_\infty \\
&= T_0 e^{\frac{x}{2L}} \theta(\eta).
\end{aligned}$$

- $$\begin{aligned}
\frac{\partial T}{\partial x} &= T_0 \left(\theta(\eta) \frac{\partial}{\partial x}(e^{\frac{x}{2L}}) + e^{\frac{x}{2L}} \frac{\partial \theta(\eta)}{\partial \eta} . \frac{\partial \eta}{\partial x} \right) \\
&= \frac{T_0}{2L} \left(e^{\frac{x}{2L}} \theta(\eta) + e^{\frac{x}{2L}} \theta'(\eta) y \sqrt{\frac{a}{2\nu_f L}} e^{\frac{x}{2L}} \right) \\
&= \frac{T_0 e^{\frac{x}{2L}}}{2L} (\theta(\eta) + \eta \theta'(\eta)).
\end{aligned}$$

- $$\begin{aligned}
u \frac{\partial T}{\partial x} &= a e^{\frac{x}{L}} f'(\eta) \frac{T_0 e^{\frac{x}{2L}}}{2L} \Big(\theta(\eta) + \eta \theta'(\eta) \Big) \\
&= a \frac{T_0}{2L} e^{\frac{3x}{2L}} \Big(f'(\eta)(\theta(\eta) + \eta f'(\eta) \theta'(\eta)) \Big). \qquad (3.30)
\end{aligned}$$

- $$\begin{aligned}
\frac{\partial T}{\partial y} &= \frac{\partial}{\partial y} \Big(T_0 e^{\frac{x}{2L}} \theta(\eta) \Big) \\
&= T_0 e^{\frac{x}{2L}} \frac{\partial \theta(\eta)}{\partial \eta} \frac{\partial \eta}{\partial y} \\
&= T_0 e^{\frac{x}{2L}} \theta'(\eta) \sqrt{\frac{a}{2\nu_f L}} e^{\frac{x}{2L}} \\
&= T_0 \frac{a}{2\nu_f L} e^{\frac{x}{L}} \theta'(\eta).
\end{aligned}$$

- $$\begin{aligned}
v \frac{\partial T}{\partial y} &= \left(-\frac{1}{2L} \sqrt{2aL\nu_f} e^{\frac{x}{2L}} (\eta f'(\eta) + f(\eta)) \right) \left(T_0 e^{\frac{x}{L}} \sqrt{\frac{a}{2\nu_f L}} \theta'(\eta) \right) \\
&= -\frac{T_0}{2L} \sqrt{2aL\nu_f} \sqrt{\frac{a}{2aL\nu_f}} e^{\frac{x}{2L}} e^{\frac{x}{L}} (\eta f'(\eta) + f(\eta)) \theta'(\eta) \\
&= -\frac{T_0}{2L} \sqrt{\frac{2a^2 L\nu_f}{2aL\nu_f}} e^{\frac{3x}{2L}} (\eta f'(\eta) \theta'(\eta) + f(\eta) \theta'(\eta)) \\
&= -\frac{a T_0 e^{\frac{3x}{2L}}}{2L} (\eta f'(\eta) \theta'(\eta) + f(\eta) \theta'(\eta)). \qquad (3.31)
\end{aligned}$$

Using (3.30) and (3.31), the left side of (3.3) gets the following form:

$$\begin{aligned} u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y} &= \frac{aT_0e^{\frac{3x}{2L}}}{2L}(f'(\eta)\theta(\eta)+\eta\theta'(\eta)f'(\eta))-\frac{aT_0e^{\frac{3x}{2L}}}{2L}(\eta f'(\eta)\theta'(\eta)+f(\eta)\theta'(\eta)) \\ &= \frac{aT_0e^{\frac{3x}{2L}}}{2L}\Big(f'(\eta)\theta(\eta)-f(\eta)\theta'(\eta)\Big). \end{aligned}$$

To convert the right side of (3.3) into dimensionless form, we proceed as follows.

- $$\begin{aligned} \frac{\partial^2 T}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\partial T}{\partial y}\right)=\frac{\partial}{\partial y}\left(T_0\sqrt{\frac{a}{2\nu_f L}}e^{\frac{x}{L}}\theta'(\eta)\right) \\ &= T_0\sqrt{\frac{a}{2\nu_f L}}e^{\frac{x}{L}}\frac{\partial\theta'}{\partial\eta}\frac{\partial\eta}{\partial y}=T_0\sqrt{\frac{a}{2\nu_f L}}e^{\frac{x}{L}}\theta''(\eta)\sqrt{\frac{a}{2\nu_f L}}e^{\frac{x}{2L}} \\ &= T_0\left(\sqrt{\frac{a}{2\nu_f L}}\right)^2 e^{\frac{x}{L}}e^{\frac{x}{2L}}\theta''(\eta) \\ &= \frac{aT_0}{2\nu_f L}e^{\frac{3x}{2L}}\theta''(\eta). \end{aligned}$$
- $$\alpha_{nf}\frac{\partial^2 T}{\partial y^2} = \frac{k_{nf}}{(\rho c_p)_{nf}}\frac{aT_0}{2\nu_f L}e^{\frac{3x}{2L}}\theta''(\eta). \tag{3.32}$$
- $$\begin{aligned} \frac{1}{(\rho c_p)_{nf}}\frac{\partial q_r}{\partial y} &= \frac{1}{(\rho c_p)_{nf}}.\frac{-16\sigma^* T_\infty^3}{3k^*}.\frac{aT_0e^{\frac{3x}{2L}}}{2v_f L}\theta''(\eta) \\ &= \frac{-16\sigma^* T_\infty^3}{3k^*(\rho c_p)_{nf}}.\frac{aT_0e^{\frac{3x}{2L}}}{2\nu_f L}\theta''(\eta). \end{aligned} \tag{3.33}$$
- $$\frac{\mu_{nf}}{(\rho c_p)_{nf}}\left(\frac{\partial u}{\partial y}\right)^2 = \frac{\mu_{nf}}{(\rho c_p)_{nf}}a^2(e^{\frac{3x}{2L}})^2\frac{a}{2L\nu_f}f''^2(\eta). \tag{3.34}$$

Using (3.32) - (3.34), the dimensionless form of right side (3.3) is as follows.

$$\begin{aligned} &\alpha_{nf}\frac{\partial^2 T}{\partial y^2}-\frac{1}{(\rho c_p)}\frac{\partial q_r}{\partial y}+\frac{\mu_{nf}}{(\rho c_p)_{nf}}\left(\frac{\partial u}{\partial y}\right)^2 \\ &= \frac{k_{nf}T_0ae^{\frac{3x}{2L}}}{2\nu_f L(\rho c_p)_{nf}}\theta''(\eta)+\frac{16\sigma^* T_\infty^3 aT_oe^{\frac{3x}{L}}}{6k^*\nu_f L(\rho c_p)_{nf}}\theta''(\eta) \\ &\quad+\frac{a^3(e^{\frac{3x}{2L}})^2\mu_{nf}}{2L\nu_f(\rho c_p)_{nf}}f''^2(\eta) \\ &= \frac{aT_0e^{\frac{3x}{2L}}}{2L}\left[\frac{k_{nf}}{\nu_f(\rho c_p)_{nf}}\theta''(\eta)+\frac{16\sigma^* T_\infty^3}{3k^*\nu_f(\rho c_p)_{nf}}\theta''(\eta)\right. \\ &\quad\left.+\frac{\mu_{nf}(a^2e^{\frac{3x}{2L}})}{\nu_{nf}T_0(\rho c_p)_{nf}}f''^2(\eta)\right]. \end{aligned}$$

$$
\begin{aligned}
&= \frac{aT_0 e^{\frac{3x}{2L}}}{2L}\Bigg[\left(\frac{k_{nf}}{k_f}.\frac{(\rho c_p)_f}{(\rho c_p)_{nf}}.\frac{k_f}{(\rho_{cp})_f}\frac{1}{\nu_f}\right)\theta''(\eta)\\
&+\left(\frac{(\rho c_p)_f}{(\rho c_p)_{nf}}.\frac{k_f}{(\rho c_p)_f\nu_f}\frac{16\sigma^* T_\infty^3}{3k^* k_f}\right)\theta''(\eta)\\
&+\frac{\mu_{nf}}{\mu_f}.\frac{(\rho c_p)_f}{(\rho c_p)_{nf}}.\frac{\mu_f}{(\rho c_p)_f}\frac{(a^2 e^{\frac{3x}{2L}})}{T_0\nu_f}f''^2(\eta)\Bigg].\\
&= \frac{aT_0 e^{\frac{3x}{2L}}}{2L}\Bigg[\left(\frac{k_{nf}}{k_f}.\frac{1}{\left(1-\phi+\frac{\phi(\rho c_p)_s}{(\rho c_p)_f}\right)}.\frac{\alpha_f}{\nu_f}\right)\theta''(\eta) && \left(\because\ \ \alpha_f=\frac{k_f}{(\rho c_p)_f}\right)\\
&+\left(\frac{1}{\left(1-\phi+\frac{\phi(\rho c_p)_s}{(\rho c_p)_f}\right)}.\frac{\alpha_f}{\nu_f}\frac{16\sigma^* T_\infty^3}{3k^* k_f}\right)\theta''(\eta) && \left(\because\ \ \alpha_f=\frac{k_f}{(\rho c_p)_f}\right)\\
&+\frac{1}{(1-\phi)^{2.5}\left(1-\phi+\frac{\phi(\rho c_p)_s}{(\rho c_p)_f}\right)}.\frac{\mu_f}{\rho_f}.\frac{1}{\nu_f}\frac{(a^2 e^{\frac{3x}{2L}})}{T_0(c_p)_f}f''^2(\eta)\Bigg].\\
&= \frac{aT_0 e^{\frac{3x}{2L}}}{2L}\Bigg[\left(\frac{k_{nf}}{k_f}.\frac{1}{\left(1-\phi+\frac{\phi(\rho c_p)_s}{(\rho c_p)_f}\right)}.\frac{1}{P_r}\right)\theta''(\eta) && \left(\because\ \ Pr=\frac{\nu_f}{\alpha_f}\right)\\
&+\left(\frac{1}{\left(1-\phi+\frac{\phi(\rho c_p)_s}{(\rho c_p)_f}\right)}.\frac{1}{P_r}\frac{16\sigma^* T_\infty^3}{3k^* k_f}\right)\theta''(\eta) && \left(\because\ \ Pr=\frac{\nu_f}{\alpha_f}\right)\\
&+\frac{1}{(1-\phi)^{2.5}(1-\phi+\frac{\phi(\rho c_p)_s}{(\rho c_p)_f})}\frac{(ae^{\frac{x}{L}})^2}{T_w-T_\infty(c_p)_f}f''^2(\eta)\Bigg]. \qquad (3.35)\\
& && \left(\because \mu_f=\rho_f\nu_f, T_0=\frac{T_w-T_\infty}{e^{\frac{x}{2L}}}\right)
\end{aligned}
$$

Therefore the dimensionless form of (3.3) becomes:

$$
\begin{aligned}
\frac{aT_0 e^{\frac{3x}{2L}}}{2L}\Big(f'(\eta)\theta(\eta)-f(\eta)\theta'(\eta)\Big) &= \frac{aT_0 e^{\frac{3x}{2L}}}{2L}\Bigg[\left(\frac{k_{nf}}{k_f}.\frac{1}{1-\phi+\frac{\phi(\rho c_p)_s}{(\rho c_p)_f}}.\frac{1}{P_r}\right)\theta''(\eta)\\
&+\left(\frac{1}{1-\phi+\frac{\phi(\rho c_p)_s}{(\rho c_p)_f}}.\frac{1}{P_r}\frac{16\sigma^* T_\infty^3}{3k^* k_f}\right)\theta''(\eta)\\
&+\frac{(U_\infty)^2 f''^2(\eta)}{(1-\phi)^{2.5}\left(1-\phi+\frac{\phi(\rho c_p)_s}{(\rho c_p)_f}\right)(T_w-T_\infty)(c_p)_f}\Bigg].
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \quad & \Big(f'(\eta)\theta(\eta) - f(\eta)\theta'(\eta)\Big)P_r = \Bigg[\Bigg(\frac{k_{nf}}{k_f}.\frac{1}{1-\phi+\frac{\phi(\rho c_p)_s}{(\rho c_p)_f}}\Bigg)\theta''(\eta) \\
& \qquad + \Bigg(\frac{1}{1-\phi+\frac{\phi(\rho c_p)_s}{(\rho c_p)_f}}.\frac{16\sigma^* T_\infty^3}{3k^* k_f}\Bigg)\theta''(\eta) \\
& \qquad + \frac{P_r}{(1-\phi)^{2.5}\big(1-\phi+\frac{\phi(\rho c_p)_s}{(\rho c_p)_f}\big)}\frac{(U_\infty)^2 f''^2(\eta)}{(T_w-T_\infty)(c_p)_f}\Bigg] \\
\Rightarrow \quad & \Big(f'(\eta)\theta(\eta) - f(\eta)\theta'(\eta)\Big)P_r = \frac{1}{1-\phi+\frac{\phi(\rho c_p)_s}{(\rho c_p)_f}}\Bigg[\Bigg(\frac{k_{nf}}{k_f}\Bigg)\theta''(\eta) \\
& \qquad + \Bigg(\frac{16\sigma^* T_\infty^3}{3k^* k_f}\Bigg)\theta''(\eta) \\
& \qquad + \frac{1}{(1-\phi)^{2.5}}P_r\frac{(U_\infty)^2}{(T_w-T_\infty)(c_p)_f}f''^2(\eta)\Bigg]. \\
\Rightarrow \quad & \Big(f'(\eta)\theta(\eta) - f(\eta)\theta'(\eta)\Big)P_r = \frac{1}{(1-\phi+\frac{\phi(\rho c_p)_s}{(\rho c_p)_f})}\Bigg[\Bigg(\frac{k_{nf}}{k_f}+\frac{16\sigma^* T_\infty^3}{3k^* k_f}\Bigg)\theta''(\eta) \\
& \qquad + \frac{1}{(1-\phi)^{2.5}}P_r\frac{(U_\infty)^2}{(T_w-T_\infty)(c_p)_f}f''^2(\eta)\Bigg]. \\
\Rightarrow \quad & \Bigg(1-\phi+\frac{\phi(\rho c_p)_s}{(\rho c_p)_f}\Bigg)\Big(f'(\eta)\theta(\eta) - f(\eta)\theta'(\eta)\Big)P_r = \Bigg[\Bigg(\frac{k_{nf}}{k_f}+\frac{16\sigma^* T_\infty^3}{3k^* k_f}\Bigg)\theta''(\eta) \\
& + \frac{1}{(1-\phi)^{2.5}}Pr\frac{(U_\infty)^2}{T_w-T_\infty(c_p)_f}f''^2(\eta)\Bigg]. \\
\Rightarrow \quad & \Bigg(\frac{k_{nf}}{k_f}+\frac{16\sigma^* T_\infty^3}{3k^* k_f}\Bigg)\theta''(\eta) + P_r\Bigg(1-\phi+\frac{\phi(\rho c_p)_s}{(\rho c_p)_f}\Bigg)(-f'(\eta)\theta(\eta)+f(\eta)\theta'(\eta)) \\
& + \frac{1}{(1-\phi)^{2.5}}P_r\frac{(U_\infty)^2}{(T_w-T_\infty)(c_p)_f}f''^2(\eta) = 0.
\end{aligned}
$$

Now, we include below the procedure for the conversion (3.4) into the dimensionless form .

$$
\begin{aligned}
\bullet \qquad & h(\eta) = \frac{C-C_\infty}{C_w-C_\infty} \\
\Rightarrow \quad C \quad & = h(\eta)(C_w-C_\infty)+C_\infty \\
& = h(\eta)(C_\infty + C_0 e^{\frac{x}{2L}} - C_\infty) + C_\infty \\
& = C_0 e^{\frac{x}{2L}} h(\eta) + C_\infty.
\end{aligned}
$$

- $$\begin{aligned}\frac{\partial C}{\partial x} &= h(\eta)\frac{\partial}{\partial x}(C_0 e^{\frac{x}{2L}})\\ &= C_0\frac{e^{\frac{x}{2L}}}{2L}h(\eta) + C_0 e^{\frac{x}{2L}}\frac{\partial h(\eta)}{\partial \eta}\frac{\partial \eta}{\partial x}\\ &= C_0\frac{e^{\frac{x}{2L}}}{2L}h(\eta) + C_0 e^{\frac{x}{2L}}h'(\eta)y\sqrt{\frac{a}{2L\nu_f}}\frac{e^{\frac{x}{2L}}}{2L}\\ &= \frac{C_0}{2L}e^{\frac{x}{2L}}\Big(h(\eta)+\eta h'(\eta)\Big).\end{aligned}$$
- $$\begin{aligned}u\frac{\partial C}{\partial x} &= af'(\eta)e^{\frac{x}{L}}\frac{C_0}{2L}e^{\frac{x}{2L}}(h(\eta)+\eta h'(\eta))\\ &= \frac{ae^{\frac{3x}{2L}}}{2L}C_0(f'(\eta)h(\eta)+\eta h'(\eta)f'(\eta)). \qquad (3.36)\end{aligned}$$
- $$\begin{aligned}\frac{\partial C}{\partial y} &= C_0 e^{\frac{x}{2L}}\frac{\partial h(\eta)}{\partial \eta}\frac{\partial \eta}{\partial y}\\ &= C_0 e^{\frac{x}{2L}}h'(\eta)\sqrt{\frac{a}{2L\nu_f}}e^{\frac{x}{2L}}\\ &= C_0\sqrt{\frac{a}{2L\nu_f}}e^{\frac{x}{L}}h'(\eta).\end{aligned}$$
- $$\begin{aligned}v\frac{\partial C}{\partial y} &= -\sqrt{2aL\nu_f}\frac{e^{\frac{x}{2L}}}{2L}\Big(\eta f'(\eta)+f(\eta)\Big)C_0\sqrt{\frac{a}{2L\nu_f}}e^{\frac{x}{L}}h'(\eta)\\ &= -C_0\frac{ae^{\frac{3x}{2L}}}{2L}(\eta h'(\eta)f'(\eta)+h'(\eta)f(\eta)). \qquad (3.37)\end{aligned}$$

Using (3.39)and (3.40) in the left side of (3.4), we get

$$\begin{aligned}u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y} &= \frac{C_0 ae^{\frac{3x}{2L}}}{2L}(f'(\eta)h(\eta)+\eta h'(\eta)f'(\eta)-\eta h'(\eta)f'(\eta)-h'(\eta)f(\eta))\\ &= \frac{ae^{\frac{3x}{2L}}}{2L}C_0(f'(\eta)h(\eta)-h'(\eta)f(\eta))\\ &= \frac{ae^{\frac{3x}{2L}}}{2L}C_0(f'(\eta)h(\eta)-h'(\eta)f(\eta)). \qquad (3.38)\end{aligned}$$

To convert the right side of (3.4) into the dimensionless form, we proceed as follows.

- $$\begin{aligned}\frac{\partial^2 C}{\partial y^2} &= \frac{\partial}{\partial y}\bigg(\sqrt{\frac{a}{2aL\nu_f}}C_0 e^{\frac{x}{L}}h'(\eta)\bigg)\\ &= \bigg(C_0 e^{\frac{x}{L}}\sqrt{\frac{a}{2L\nu_f}}\frac{\partial h'(\eta)}{\partial \eta}\frac{\partial \eta}{\partial y}\bigg)\\ &= C_0 e^{\frac{x}{L}}\sqrt{\frac{a}{2aL\nu_f}}h''(\eta)\sqrt{\frac{a}{2L\nu_f}}e^{\frac{x}{2L}}\end{aligned}$$

$$= C_0 e^{\frac{x}{L}} e^{\frac{x}{2L}} \frac{a}{2L\nu_f} h''(\eta)$$

$$= C_0 \frac{a e^{\frac{3x}{2L}}}{2L\nu_f} h''(\eta). \tag{3.39}$$

- $$h(\eta) = \frac{C - C_\infty}{C_w - C_\infty}$$

$$\Rightarrow \quad (C - C_\infty) = h(\eta)(C_w - C_\infty)$$

$$\Rightarrow \quad K(C - C_\infty) = K_0 e^{\frac{x}{L}} h(\eta)(C_\infty + C_0 e^{\frac{x}{2L}} - C_\infty)$$

$$= C_0 K_0 e^{\frac{3x}{2L}} h(\eta). \tag{3.40}$$

Using (3.36) and (3.37) in the right side of (3.4), we get

$$D\frac{\partial^2 C}{\partial y^2} - K(C - C_\infty) = DC_0 \frac{a e^{\frac{3x}{2L}}}{2Lv_f} h''(\eta) - C_0 K_0 e^{\frac{3x}{2L}} h(\eta)$$

$$= C_0 e^{\frac{3x}{2L}} \left(D \frac{a}{2L\nu_f} h''(\eta) - K_0 h(\eta) \right). \tag{3.41}$$

Hence the dimensionless form of (3.4) becomes:

$$C_0 \frac{a e^{\frac{3x}{2L}}}{2L} \left(f'(\eta)h(\eta) - h'(\eta)f(\eta) \right) = C_0 e^{\frac{3x}{2L}} \left(\frac{Dah''(\eta)}{2L\nu_f} - K_0 h(\eta) \right)$$

$$\Rightarrow \frac{a}{2L}(f'(\eta)h(\eta) - h'(\eta)f(\eta)) = \frac{aD}{2L\nu_f} h''(\eta) - K_0 h(\eta)$$

$$\Rightarrow f'(\eta)h(\eta) - h'(\eta)f(\eta) = \frac{D}{\nu_f} h''(\eta) - \frac{2LK_0}{a} h(\eta)$$

$$\Rightarrow \frac{\nu_f}{D}(f'(\eta)h(\eta) - h'(\eta)f(\eta)) = h''(\eta) - \frac{2LK_0\nu_f}{aD} h(\eta)$$

$$\Rightarrow \frac{\nu_f}{D}(f'(\eta)h(\eta) - h'(\eta)f(\eta) + \frac{2LK_0\nu_f}{a} h(\eta)) = h''(\eta)$$

$$\Rightarrow \; h''(\eta) + \frac{\nu_f}{D}(f(\eta)h'(\eta) - f'(\eta)h(\eta) - \frac{2LK_0\nu_f}{a} h(\eta)) = 0.$$

The final dimensionless form of the governing model, is:

$$\frac{1}{(1-\phi)^{2.5}(1-\phi+\phi\frac{\rho_s}{\rho_f})}f''' + ff'' + 2(1-f'^2)$$
$$+\frac{1}{(1-\phi)^{2.5}(1-\phi+\phi\frac{\rho_s}{\rho_f})}(P+(1-\phi)^{2.5}M)(1-f') = 0, \tag{3.42}$$
$$\left(\frac{k_{nf}}{k_f}+R\right)\theta'' + P_r\left(1-\phi+\phi\frac{(\rho c_p)_s}{(\rho c_p)_f}\right)(f\theta' - f'\theta) + \frac{1}{(1-\phi)^{2.5}}P_r Ecf''^2 = 0, \tag{3.43}$$
$$h'' + Sc(fh' - f'h - \gamma h) = 0. \tag{3.44}$$

The associated boundary conditions (3.5) - (3.6) get the form:

$$f = 0, f' = 0,\ \theta = 1, h = 1, at\ \eta = 0. \tag{3.45}$$
$$f' \to 1, \theta \to 0, h \to 0\ as\ \eta \to \infty. \tag{3.46}$$

Different parameters used in the above equations have the following formulations:

$$P = \frac{2L\nu_f}{ak_0},\ M = \frac{2\sigma B_0^2 L}{a\rho_f},\ R = \frac{16\sigma^* T_\infty^3}{3kk_f},\ Sc = \frac{\nu_f}{D},$$
$$\gamma = \frac{2LK_0}{a}, P_r = \frac{\nu_f}{\alpha_f},\ Ec = \frac{U_\infty^2}{(c_p)_f(T_w - T_\infty)}. \tag{3.47}$$

The skin friction coefficient, is defined as:

$$C_{fx} = \frac{2\tau_w}{\rho_f U_\infty^2} \tag{3.48}$$
$$\bullet\ \frac{\partial u}{\partial y} = a\sqrt{\frac{a}{2L\nu_f}}e^{\frac{x}{L}}e^{\frac{x}{2L}}f''(\eta).$$
$$\Rightarrow \left(\frac{\partial u}{\partial y}\right)\bigg|_{y=0} = a\sqrt{\frac{a}{2L\nu_f}}e^{\frac{x}{L}}e^{\frac{x}{2L}}f''(0)$$
$$\bullet\ \tau_w = \mu_{nf}\left(\frac{\partial u}{\partial y}\right)\bigg|_{y=0}.$$
$$= \frac{\mu_f}{(1-\phi)^{2.5}}a\sqrt{\frac{a}{2L\nu_f}}e^{\frac{x}{L}}e^{\frac{x}{2L}}f''(0). \tag{3.49}$$

Using (3.49) in equation (3.48) ,we get the following form.

$$
\begin{aligned}
C_{fx} &= \frac{2}{\rho_f U_\infty^2}\frac{\mu_f}{(1-\phi)^{2.5}} a\sqrt{\frac{a}{2L\nu_f}} e^{\frac{x}{L}} e^{\frac{x}{2L}} f''(0) \\
&= \frac{2}{U_\infty^2}\frac{\nu_f}{(1-\phi)^{2.5}} a\sqrt{\frac{a}{2L\nu_f}} e^{\frac{x}{L}} e^{\frac{x}{2L}} f''(0) \quad \therefore \nu_f = \frac{\mu_f}{\rho_f} \\
&= \sqrt{\frac{2\nu_f a}{L}}\frac{1}{(U_\infty)^2} U_\infty e^{\frac{x}{2L}} \frac{f''(0)}{(1-\phi)^{2.5}} \quad \therefore U_\infty = a e^{\frac{x}{L}} \\
&= \sqrt{\frac{2\nu_f}{L}}\frac{1}{U_\infty}(e^{\frac{x}{L}})^{\frac{1}{2}} \frac{f''(0)}{(1-\phi)^{2.5}} \\
&= \sqrt{\frac{2\nu_f}{LU_\infty}}(e^{\frac{x}{L}})^{\frac{1}{2}} \frac{f''(0)}{(1-\phi)^{2.5}} \\
&= \frac{\sqrt{\frac{x}{L}}}{\sqrt{\frac{xU_\infty}{2\nu_f}}} . \frac{f''(0)}{(1-\phi)^{2.5}} \\
&= \frac{\sqrt{\frac{x}{L}}}{\sqrt{\frac{Re_x}{2}}} . \frac{f''(0)}{(1-\phi)^{2.5}} \\
\Rightarrow \frac{C_{fx}\sqrt{\frac{Re_x}{2}}}{\sqrt{\frac{x}{L}}} &= \frac{f''(0)}{(1-\phi)^{2.5}},
\end{aligned}
$$

The local Nusselt number is defined as:

$$
Nu_x = \frac{-xq_w}{k_f(T_w - T_\infty)} \tag{3.50}
$$

$$
\begin{aligned}
\bullet\ q_w &= -k_{nf}\left(\frac{\partial T}{\partial y}\right)\bigg|_{y=0} \\
&= -k_{nf} T_0 \frac{a}{2\nu_f L} e^{\frac{x}{L}} \theta'(0)
\end{aligned} \tag{3.51}
$$

Using (3.51) in (3.50), we get the following form.

$$
\begin{aligned}
Nu_x &= -\frac{x k_{nf}(T_w - T_\infty)\frac{a}{2\nu_f L} e^{\frac{x}{L}}\theta'(0)}{k_f e^{\frac{x}{2L}}(T_w - T_\infty)} \quad \because T_0 = \frac{(T_w - T_\infty)}{e^{\frac{x}{2L}}} \\
&= -\frac{x k_{nf}\frac{a}{2\nu_f L}(e^{\frac{x}{L}})^{\frac{1}{2}}\theta'(0)}{k_f}
\end{aligned} \tag{3.52}
$$

$$
\begin{aligned}
&= -\frac{xk_{nf}\frac{xae^{\frac{x}{L}}}{2x\nu_f L}\theta'(0)}{k_f} \\
&= -\sqrt{\frac{x}{2L}}\frac{k_{nf}}{k_f}\sqrt{Re_x}\theta'(0) \\
&= -\sqrt{\frac{x}{2L}}\frac{k_{nf}}{k_f}\sqrt{Re_x}\theta'(0) \because Re_x = \frac{xU_\infty}{\nu_f} \\
\Rightarrow &\sqrt{\frac{2L}{x}}\sqrt{\frac{1}{Re_x}}Nu_x = -\frac{k_{nf}}{k_f}\theta'(0),
\end{aligned}
$$

The local sherwood number is defined as:

$$
Sh_x = \frac{xq_m}{D(C_w - C_\infty)} \tag{3.53}
$$

$$
\bullet\, q_m = -D\left(\frac{\partial C}{\partial y}\right)\Big|y=0
$$

$$
= -DC_0\sqrt{\frac{a}{2L\nu_f}}e^{\frac{x}{L}}h'(0) \tag{3.54}
$$

Using (3.54) in (3.53) we get:

$$
\begin{aligned}
Sh_x &= \frac{-xDC_0\sqrt{\frac{a}{2L\nu_f}}e^{\frac{x}{L}}h'(0)}{D(C_w - C_\infty)} \\
&= \frac{-xC_0\sqrt{\frac{a}{2L\nu_f}}e^{\frac{x}{L}}h'(0)}{(C_w - C_\infty)} \\
&= \frac{-xC_0\sqrt{\frac{a}{2L\nu_f}}e^{\frac{x}{L}}h'(0)}{(C_w - C_w + C_0e^{\frac{x}{2L}})} \quad \therefore C_\infty = C_w - C_0e^{\frac{x}{2L}} \\
&= \frac{-x\sqrt{\frac{a}{2L\nu_f}}e^{\frac{x}{L}}h'(0)}{-e^{\frac{x}{2L}}} \\
&= \frac{x\sqrt{\frac{a}{2L\nu_f}}e^{\frac{x}{L}}h'(0)}{(e^{\frac{x}{L}})^{\frac{1}{2}}} \\
&= x\sqrt{\frac{a}{2L\nu_f}}e^{\frac{x}{L}}h'(0)(e^{\frac{x}{L}})^{\frac{-1}{2}} \\
&= x\sqrt{\frac{a}{2L\nu_f}}h'(0)(e^{\frac{x}{L}})^{\frac{1}{2}} \\
&= x\sqrt{\frac{ae^{\frac{x}{L}}}{2L\nu_f}}h'(0)
\end{aligned}
$$

$$
\begin{aligned}
&= x\sqrt{\frac{xU_\infty}{x2L\nu_f}}h'(0) \\
&= \left(\sqrt{\frac{xU_\infty}{\nu_f}}\right)\left(x\sqrt{\frac{1}{x2L}}h'(0)\right) \\
&= \sqrt{Re_x}\sqrt{\frac{x}{2L}}h'(0) \\
&\Rightarrow Sh_x\sqrt{\frac{1}{Re_x}}\sqrt{\frac{2L}{x}} = h'(0),
\end{aligned}
$$

where τ_w is the skin fraction, q_w the heat flux from the sheet, and Re_x represents the local Reynolds numbers defined as $Re_x = \frac{xU_\infty}{v_f}$.

3.3 Solution methodology

In order to solve the system of ordinary differential equations(3.42)-(3.44) the shooting method has been used . Let us use the notations:

$$f = y_1, \theta = y_4, h = y_6.$$

Further denote

$$f' = y_1' \text{ by } y_2, f'' = y_2{}' \text{ by } y_3, \theta' = y_4' \text{ by } y_5 \text{ and } h' = y_6{}' \text{ by } y_7.$$

For simplification, the following notations have been opted.

$$
\begin{cases}
(1-\phi)^{2.5}(1-\phi+\phi\frac{\rho_s}{\rho_f}) = b_1, \\
P + (1-\phi)^{2.5}M = b_2, \\
\frac{P_r\left(1-\phi+\phi\left(\frac{\phi(\rho c_p)_s}{(\rho c_p)_f}\right)\right)}{\left(\frac{k_{nf}}{k_f}+\mathrm{R}\right)} = b_3 \\
\frac{(1-\phi)^{2.5}}{\mathrm{P}_r\mathrm{Ec}}\left(\frac{k_{nf}}{k_f} + R\right) = b_4
\end{cases}
$$

The system of equations (3.41)-(3.43), can now be written in the form of following first ODEs

$$
\begin{aligned}
y_1' &= y_2, & y_1(0) &= 0,\\
y_2' &= y_3, & y_2(0) &= 0,\\
y_3' &= -b_1(y_1y_3 + 2(1-y_2^2)) - b_2(1-y_2), & y_3(0) &= s,\\
y_4' &= y_5, & y_4(0) &= 1,\\
y_5' &= b_3(y_2y_4 - y_1y_5) - b_4y_3^2, & y_5(0) &= t,\\
y_6' &= y_7, & y_6(0) &= 1,\\
y_7' &= Sc(\gamma y_6 + y_2y_6 - y_1y_7), & y_7(0) &= w.
\end{aligned}
$$

The above initial value problem will be solved numerically by the RK-4 method. To get the approximate solution, the domain of the problem has been taken as $[0,\ \eta_\infty]$ instead of $[0,\ \infty]$, where η_∞ is an appropriate finite positive real number. In the above system of equations, the missing conditions s, t and w are to be chosen such that

$$y_2(\eta_\infty, s, t, w) = 1,\ \ y_4(\eta_\infty, s, t, w) = 0,\ \ y_6(\eta_\infty, s, t, w) = 0.$$

To solve the above system of algebraic equations, we use the Newton's method which has the following iterative scheme:

$$\begin{pmatrix} s^{(k+1)} \\ t^{(k+1)} \\ w^{(k+1)} \end{pmatrix} = \begin{pmatrix} s^{(k)} \\ t^{(k)} \\ w^{(k)} \end{pmatrix} - \begin{pmatrix} \frac{\partial y_2}{\partial s} & \frac{\partial y_2}{\partial t} & \frac{\partial y_2}{\partial w} \\ \frac{\partial y_4}{\partial s} & \frac{\partial y_4}{\partial t} & \frac{\partial y_4}{\partial w} \\ \frac{\partial y_6}{\partial s} & \frac{\partial y_6}{\partial t} & \frac{\partial y_6}{\partial w} \end{pmatrix}^{-1}_{(s^{(k)},t^{(k)},w^{(k)})} \begin{pmatrix} y_2^{(k)} - 1 \\ y_4^{(k)} \\ y_6^{(k)} \end{pmatrix}_{(s^{(k)},t^{(k)},w^{(k)})}.$$

For further need, the following notations have been introduced.

$$
\begin{aligned}
\frac{\partial y_1}{\partial s} &= y_8, \frac{\partial y_2}{\partial s} = y_9, ..., \frac{\partial y_7}{\partial s} = y_{14},\\
\frac{\partial y_1}{\partial t} &= y_{15}, \frac{\partial y_2}{\partial t} = y_{16}, ..., \frac{\partial y_7}{\partial t} = y_{21},\\
\frac{\partial y_1}{\partial w} &= y_{22}, \frac{\partial y_2}{\partial w} = y_{23}, ..., \frac{\partial y_7}{\partial w} = y_{28}.
\end{aligned}
$$

As a result of these these new notations, the Newton's iterative scheme gets the form:

$$\begin{pmatrix} s^{(k+1)} \\ t^{(k+1)} \\ w^{(k+1)} \end{pmatrix} = \begin{pmatrix} s^{(k)} - 1 \\ t^{(k)} \\ w^{(k)} \end{pmatrix} - \begin{pmatrix} y_9 & y_{16} & y_{23} \\ y_{11} & y_{18} & y_{25} \\ y_{13} & y_{20} & y_{27} \end{pmatrix}^{-1}_{(s^{(k)},t^{(k)},w^{(k)})} \begin{pmatrix} y_2^{(k)} - 1 \\ y_4^{(k)} \\ y_6^{(k)} \end{pmatrix}_{(s^{(k)},t^{(k)},w^{(k)})} . \tag{3.55}$$

Now differentiate the above system of seven first order ODEs with respect to each of the variables s, t and w to have another system of twenty one ODEs. Writing all these twenty eight ODEs together, we have the the following IVP:

$$\begin{aligned}
y_1' &= y_2, & y_1(0) &= 0,\\
y_2' &= y_3, & y_2(0) &= 0,\\
y_3' &= -b_1(y_1y_3 + 2(1-y_2^2)) - b_2(1-y_2), & y_3(0) &= s,\\
y_4' &= y_5, & y_4(0) &= 1,\\
y_5' &= b_3(y_2y_4 - y_1y_5) - b_4y_3^2, & y_5(0) &= t,\\
y_6' &= y_7, & y_6(0) &= 1,\\
y_7' &= Sc(\gamma y_6 + y_2y_6 - y_1y_7), & y_7(0) &= w,\\
y_8' &= y_9, & y_8(0) &= 0,\\
y_9' &= y_{10}, & y_9(0) &= 0,\\
y_{10}' &= -b_1(y_1y_{10} + y_8y_3 - 4y_2y_9) + b_2y_9, & y_{10}(0) &= 1,\\
y_{11}' &= y_{12}, & y_{11}(0) &= 0,\\
y_{12}' &= b_3(y_2y_{11} + y_9y_4 - y_1y_{12} - y_8y_5) - 2b_4y_3y_{10}, & y_{12}(0) &= 0,\\
y_{13}' &= y_{14}, & y_{13}(0) &= 0,\\
y_{14}' &= Sc(\gamma y_{13} + y_2y_{13} + y_6y_9 - y_{21}y_7 - y_1y_{14}), & y_{14}(0) &= 0,\\
y_{15}' &= y_{16}, & y_{15}(0) &= 0,\\
y_{16}' &= y_{17}, & y_{16}(0) &= 0,\\
y_{17}' &= -b_1(y_1y_{17} + y_{15}y_3 - 4y_2y_{16}) + b_2y_{16}, & y_{17}(0) &= 0,\\
y_{18}' &= y_{19}, & y_{18}(0) &= 0,\\
y_{19}' &= b_3(y_2y_{18} + y_{16}y_4 - y_1y_{19} - y_{15}y_5) - 2b_4y_3y_{17}, & y_{19}(0) &= 1,\\
y_{20}' &= y_{21}, & y_{20}(0) &= 0,\\
y_{21}' &= Sc(\gamma y_{20} + y_2y_{20} + y_6y_{16} - y_1y_{21} - y_7y_{15}), & y_{21}(0) &= 0,\\
y_{22}' &= y_{23}, & y_{22}(0) &= 0,\\
y_{23}' &= y_{24}, & y_{23}(0) &= 0,\\
y_{24}' &= -b_1(+y_1y_{24} + y_{22}y_3 - 4y_2y_{23}) + b_2y_{23}, & y_{24}(0) &= 0,\\
y_{25}' &= y_{26}, & y_{25}(0) &= 0,\\
y_{26}' &= b_3(y_2y_{25} + y_{23}y_4 - y_1y_{26} - y_{22}y_5) - 2b_4y_3y_{24}, & y_{26}(0) &= 0,\\
y_{27}' &= y_{28} & y_{27}(0) &= 0,\\
y_{28}' &= Sc(\gamma y_{27} + y_2y_{27} + y_{23}y_6 - y_1y_{28} - y_{22}y_7), & y_{28}(0) &= 1.
\end{aligned}$$

The fourth order Runge-Kutta method is used to solve the above system of twenty eight equations with initial guesses s, t, w. These guesses are updated by the Newton's scheme (3.51). The iterative process is repeated until the following criteria is met:

$$max|y_2(\eta_\infty - 1)|, max|y_4(\eta_\infty - 0)|, max|y_6(\eta_\infty - 0)| < \epsilon,$$

where $\epsilon > 0$ is the tolerance. For all the calculations in this chapter, we have set $\epsilon = 10^{-6}$.

3.4 Results with discussion

In this section, the numerical results have been displayed in the form of graphs and tables. For numerical calculations, different physical properties of water, copper and alumina are considered. The effects of various parameters such as nanoparticles volume fraction ϕ, Prandtal number P_r, chemical reaction γ, permeability parameter P, Magnetic parameter M, on velocity f', temperature θ and concentration h have been analyzed. Figures 4.2 and 4.3 demonstrate the impact of the volume fraction and the magnetic parameter M on the velocity . In these figures, we observe that the velocity increases with an increase in the volume fraction of nanoparticles. These figures show that the hydrodynamic boundary layer of Al_2O_3- water is thick as compared with that of Cu-water.

Figures 3.4 and 4.5 display the effect of the solid volume fraction of nanoparticles on the temperature profile. In these figure, we observe that if we increase the volume fraction ϕ, the temperature profile has also an increasing trend. Hence the thickness of the thermal boundary layer increases. Figures 4.6 and 4.7 show the impact of the volume fraction ϕ together with the magnetic parameter M on the concentration profile h. A decreasing behaviour is found in the dimensionless concentration h for both Cu-water and Al_2O_3 -water. In these figure, one can see that the concentration distribution decreases if there is an increase in the volume fraction ϕ. Figures

4.8 and 4.9 show the impact of the permeability parameter P together with ϕ on the dimensionless velocity for both Cu-water and Al_2O_3-water. Fluid velocity increases with the increasing values of both the permeability parameter P and ϕ . Figure 3.10 and 3.11 show the impact of the volume fraction ϕ and permeability parameters P of nano-particles on the dimensionless temperature θ. When the porosity increases the thermal boundary layer is reduced. It is clearly observed that the temperature profile increases by increasing the volume fraction in the state of increased porosity. If we increase the value of the permeability parameter effect, the temperature distribution is also increased. Figures 4.12 and 4.13 show the effect of ϕ on the dimensionless temperature θ of the water based fluid with or without radiation. In these figures, temperature is increased by increasing the values of the thermal radiation radiation increased. It happens because the thermal radiation increases the thermal diffusion.

Figures 3.14 and 3.15 show the impact of the viscous dissipation together with ϕ on the temperature profile. When the value of the viscous dissipation is increased, the fluid region is allowed to store the energy. As a result of dissipation due to fractional heating, heat is generated. From this figure, we examine that the value of the thermal boundary thickness increases with increasing values of ϕ and it will eventually increase the temperature. Figures 4.16 and 4.17 show the impact of the chemical reaction together with ϕ on the dimensionless concentration. In these figures, it also observed that when the chemical reaction increases, the concentration profiles decreases and the increasing values of the volume fraction have small impact on the dimensionless concentration. It is clear that in these figures velocities $f'(\eta)$ and temperature $\theta^{(\eta)}$ posses the same increasing behaviour for $M = 0$, 10 and the concentration $h(\eta)$ shows the decreasing behaviour.

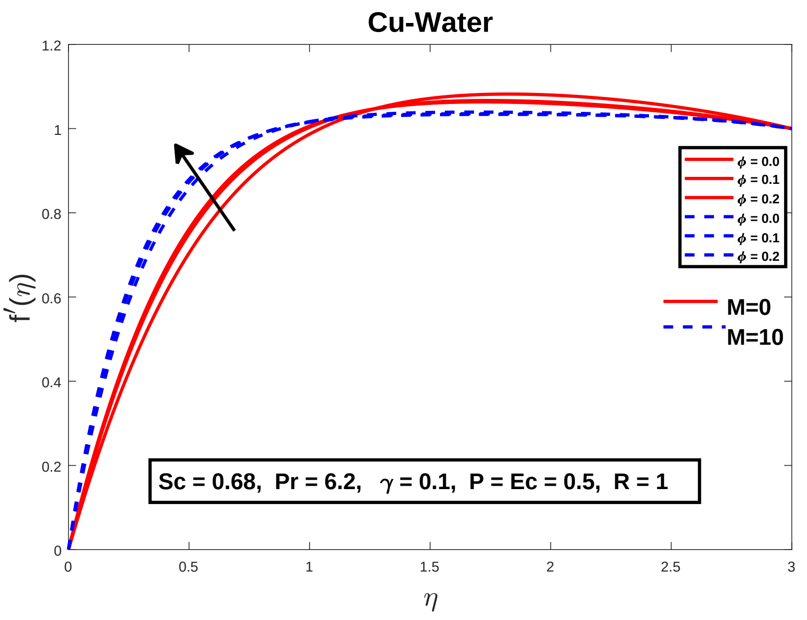

FIGURE 3.2: Impact of ϕ on the dimensionless velocity f' for $Cu - water$
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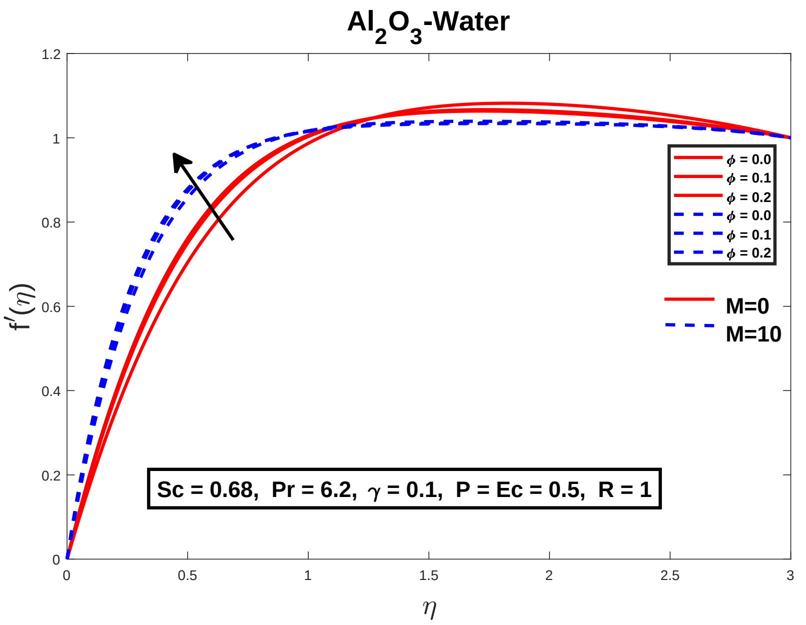

FIGURE 3.3: Impact of ϕ on the dimensionless velocity f' for Al_2O_3
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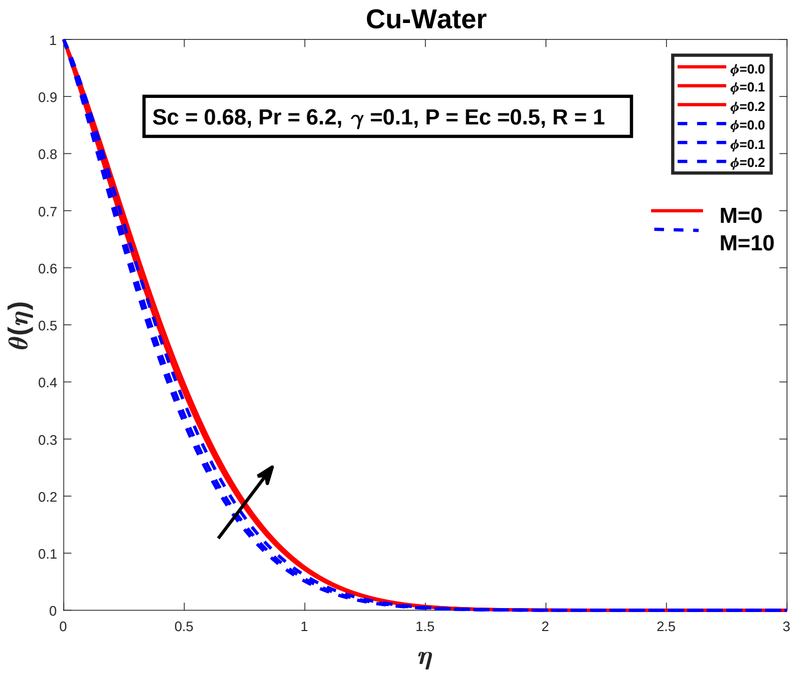

FIGURE 3.4: Impact of ϕ on the dimensionless tempertaure θ for $Cu - water$
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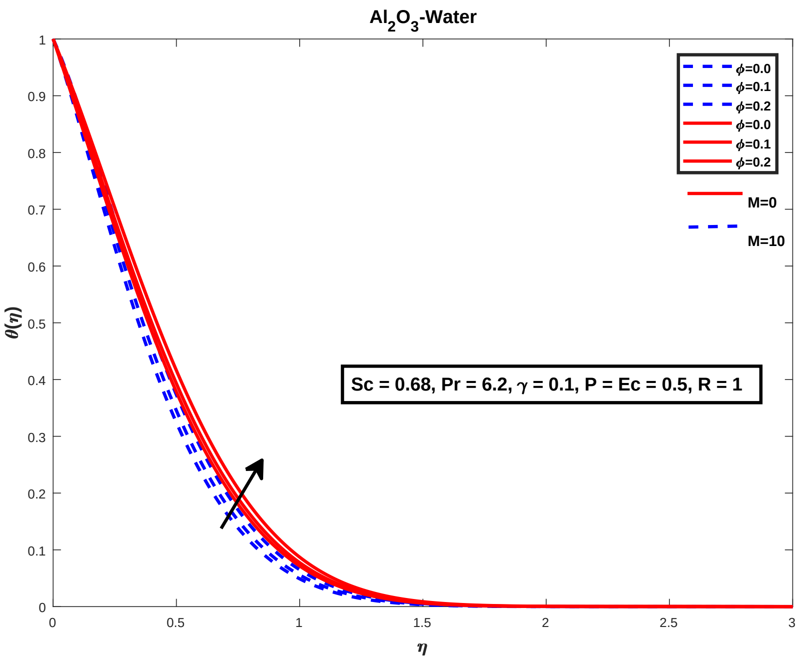

FIGURE 3.5: Influence of ϕ on the dimensionless temperature θ for Al_2O_3

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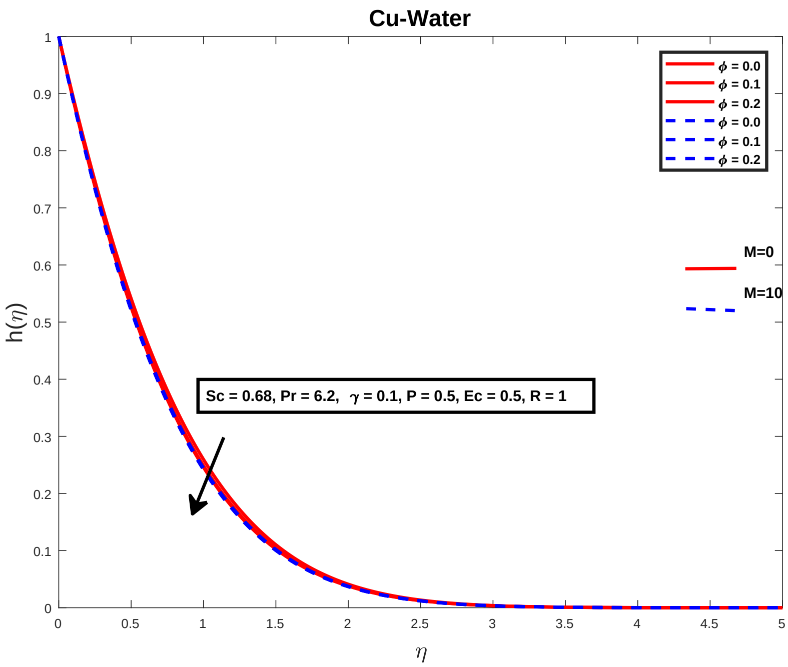

FIGURE 3.6: Impact of ϕ on the dimensionless concentration h for Cu-water

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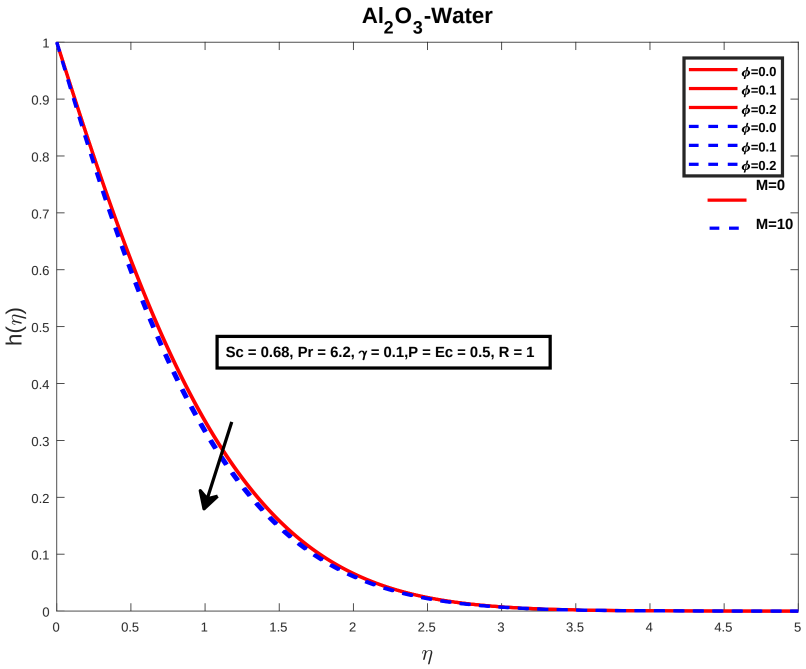

FIGURE 3.7: Impact of ϕ on the dimensionless concentration h for Al_2O_3

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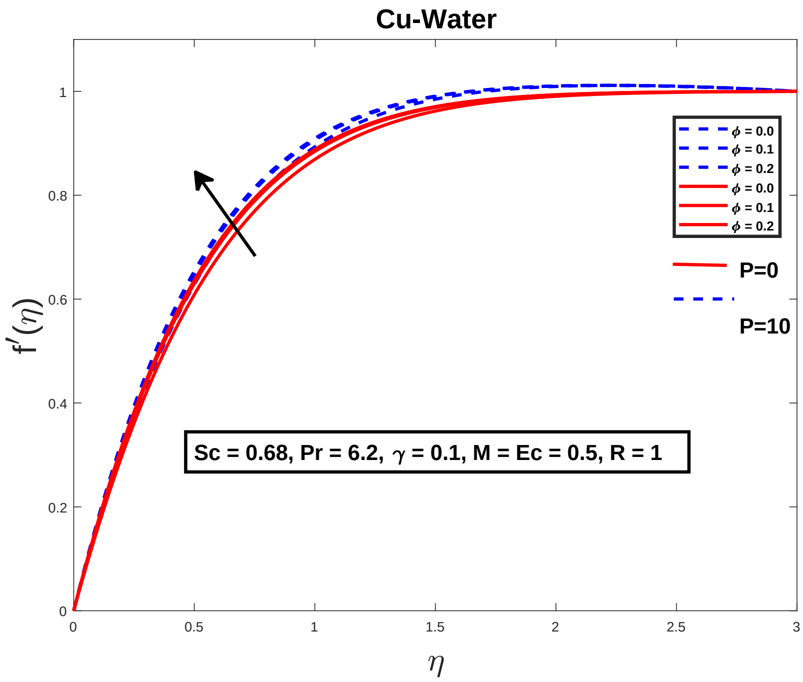

FIGURE 3.8: Impact of ϕ on the dimensionless velocity f' forCu-water

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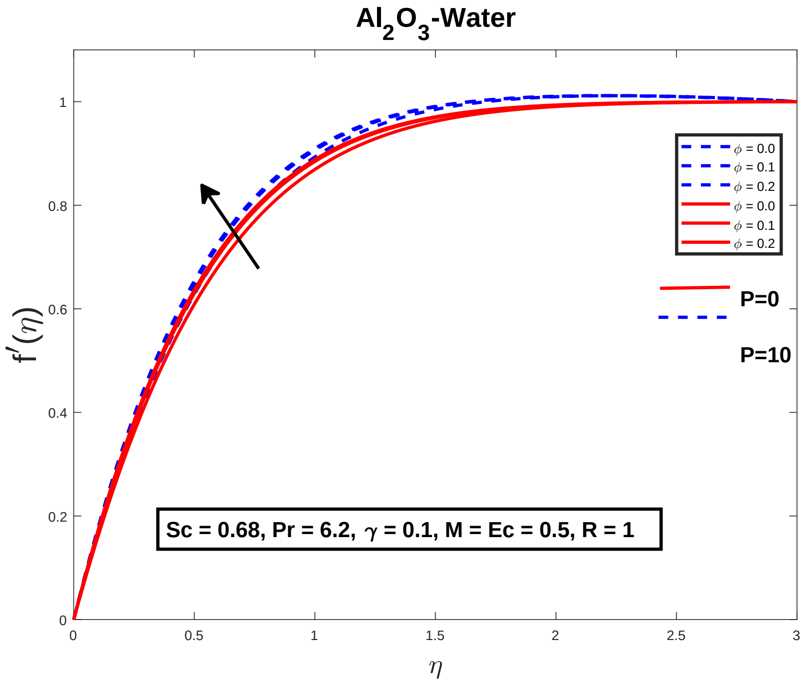

FIGURE 3.9: Impact of ϕ on the dimensionless velocity f' for Al_2O_3

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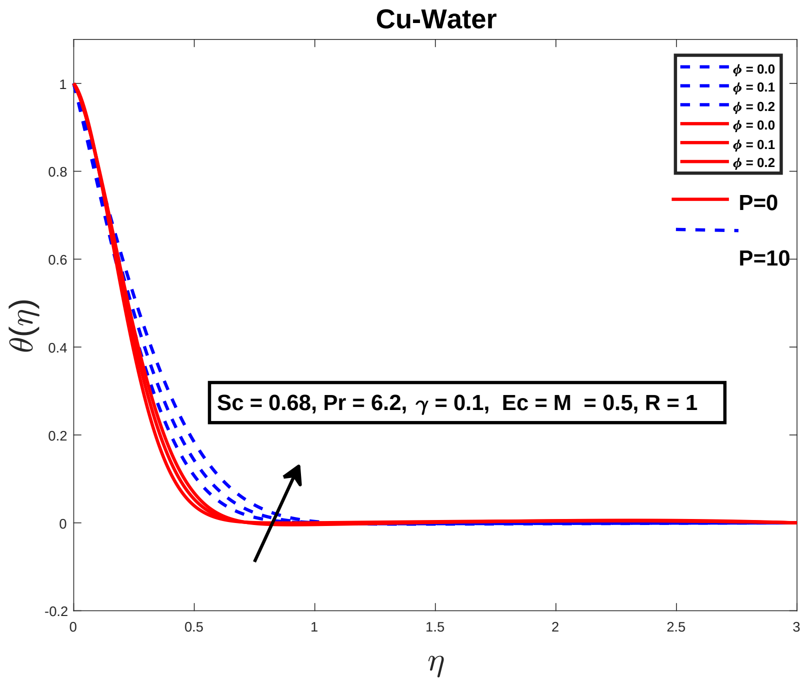

FIGURE 3.10: Impact of ϕ on the dimensionless temperature θ for Cu-water

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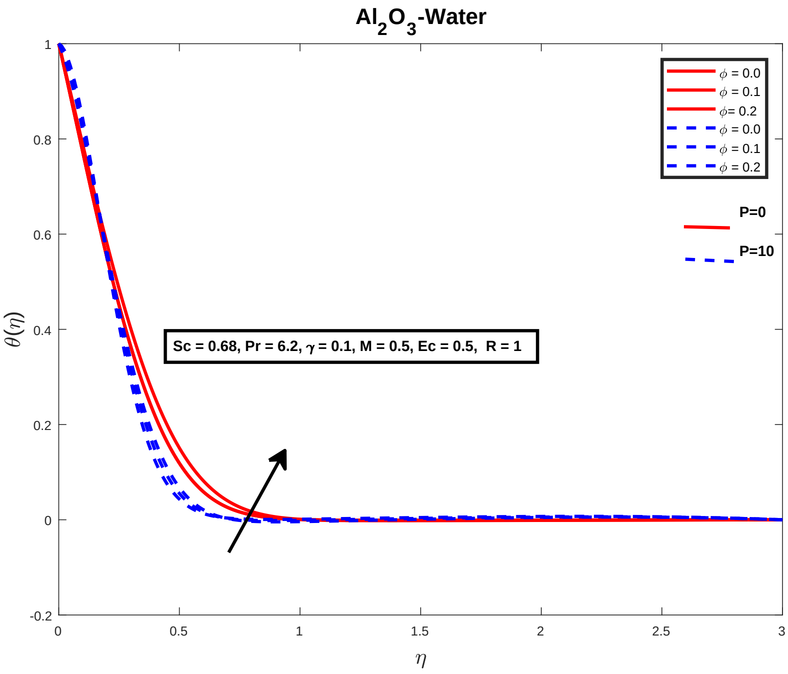

FIGURE 3.11: Impact of ϕ on the dimensionless temperature θ for Al_2O_3

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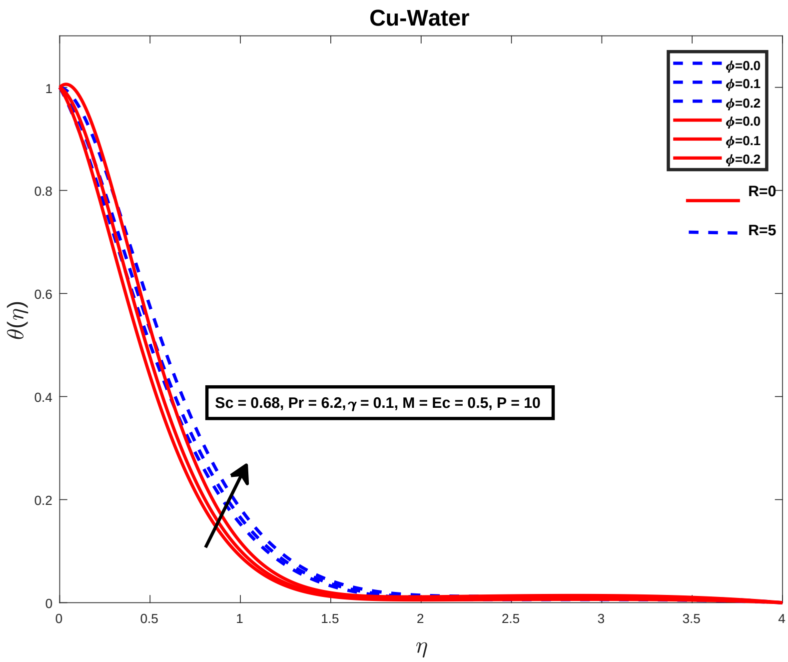

FIGURE 3.12: Impact of ϕ on the dimensionless temperature θ for Cu-water

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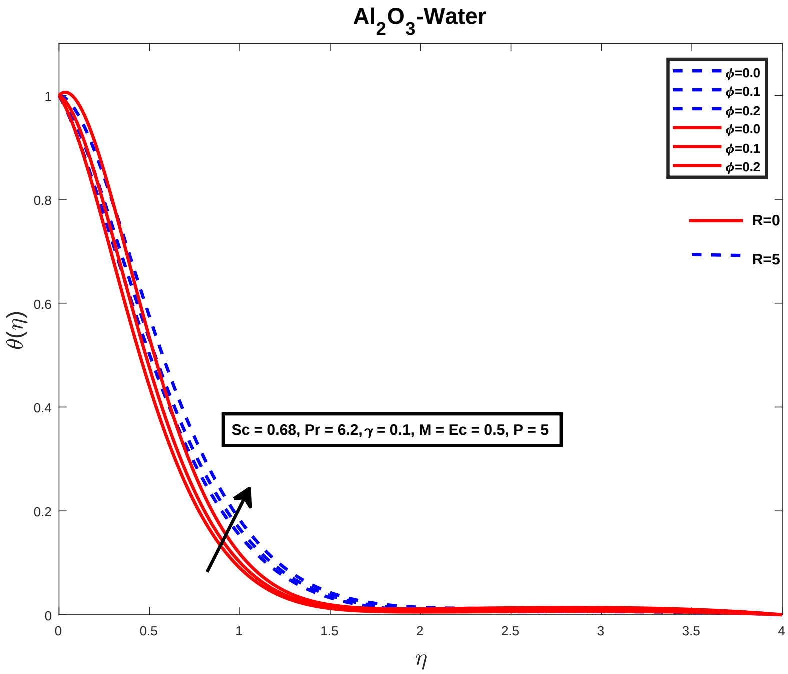

FIGURE 3.13: Impact of ϕ on the dimensionless temperature θ for Al_2O_3

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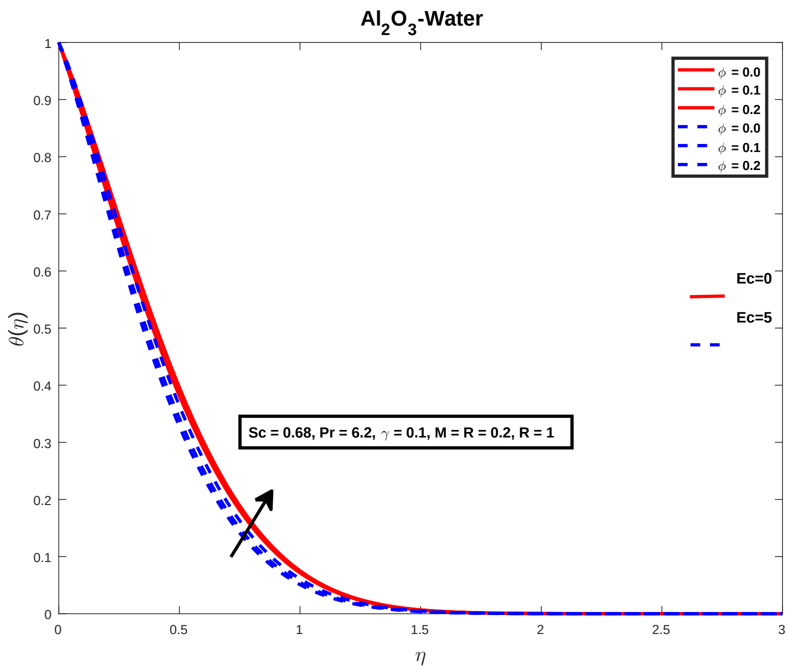

FIGURE 3.14: Impact of ϕ and Ec on the dimensionless temperature θ for Cu-water

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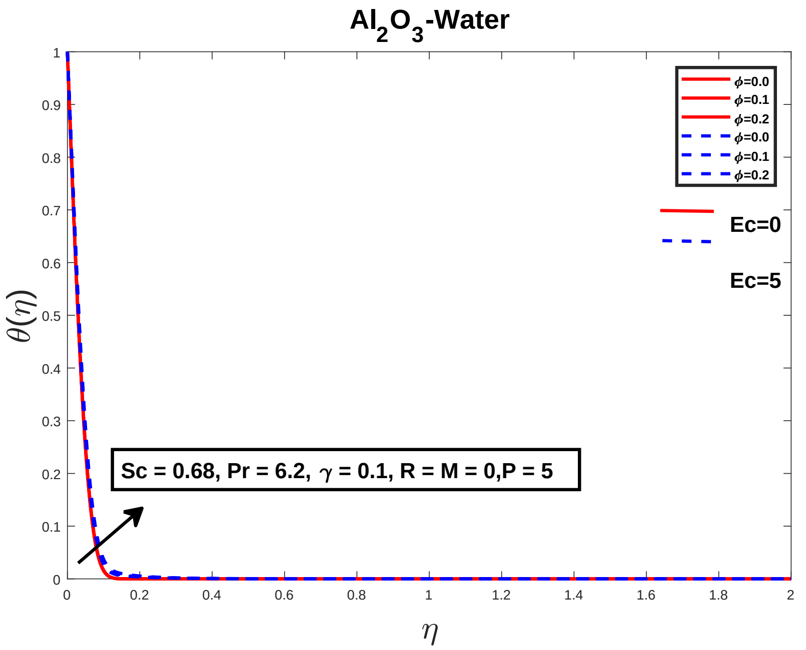

FIGURE 3.15: Impact of ϕ and Ec on the dimensionless temperature θ for Al_2O_3

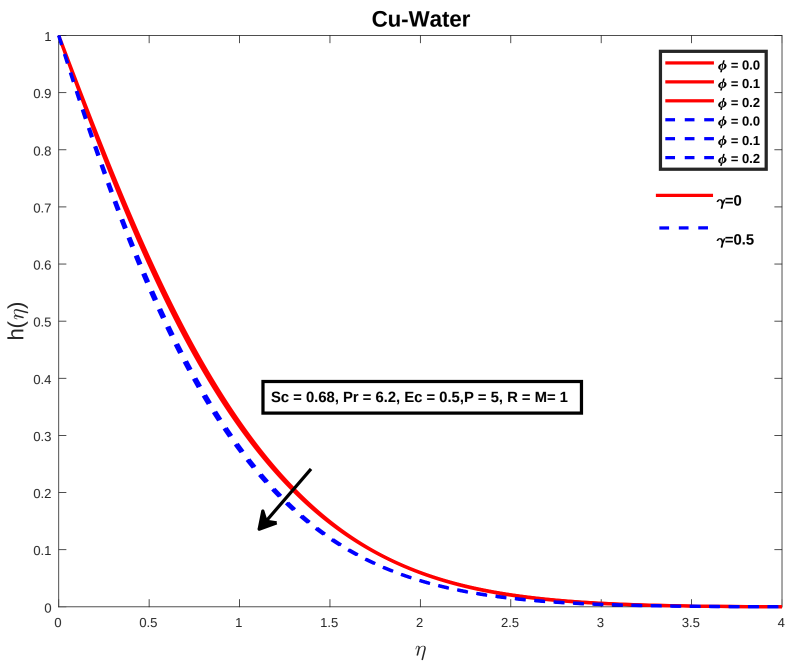

FIGURE 3.16: Impact of ϕ on the dimensionless concentration h for Cu-water

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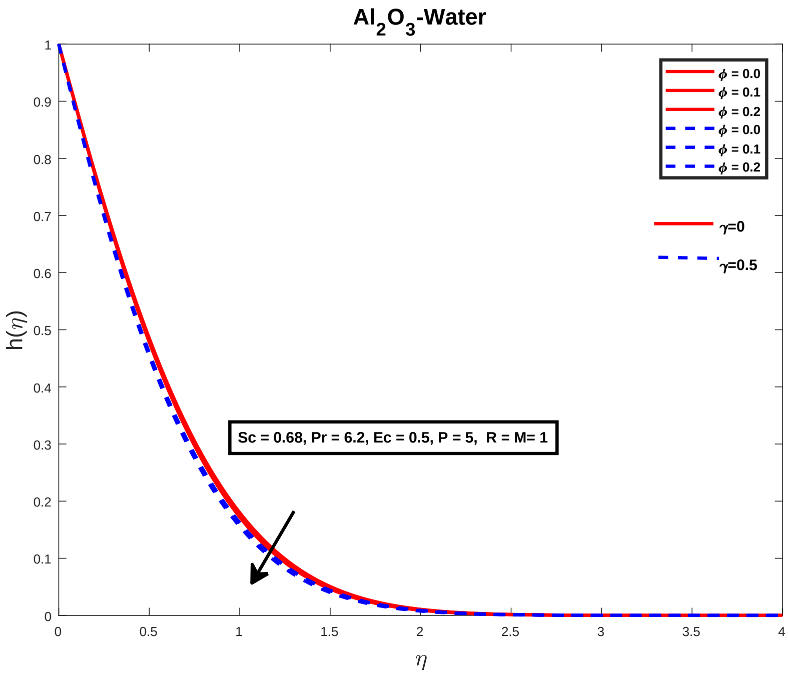

FIGURE 3.17: Impact of ϕ on the dimensionless concentration h for Al_2O_3

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TABLE 3.1: Physical properties of water and nano-particles

F*luid*	$\rho(kg/m^3)$	$c_p(J/k_gK)$	$K(W/mk)$	$\beta\times10^5(K^{-1})$
Pure water	997.1	4179	0.613	21
Copper	8933	385	401	1.67
Alumina	3970	765	40	0.85

TABLE 3.2: Comparison of skin friction $f''(0)$ with Ref [35] when $(P = M = R = Ec)$=0, Pr= 6.2 and Sc= 0.68

$\boldsymbol{\phi}$	$\boldsymbol{Cu-water}$	Present Result		$\boldsymbol{Al_2O_3}$	Present Result	
	Ref[35]	Shooting method	bvp4c	Ref[35]	Shooting method	bvp4c
0.0	1.6871	1.6871	1.6871	1.6871	1.6871	1.6871
0.1	2.5793	2.5793	2.5793	2.1982	2.1982	2.1982
0.2	3.5902	3.5902	2.8174	2.8174	2.8174	

The three physical parameters C_{fx} (local Skin-friction coefficient), Nu_x (local Nusselt number) are of great interest for engineers. Table 3.1 demonstrates the properties fluid water and nanoparticles. In table 3.2 and 3.3, we compare our results for the values of local Skin-friction coefficient) $f''(0)$ and local Nusselt number) $\theta'(0)$ with those of the previous reported value by F.Mabood *et al.* Ref [35]. Table 3.2 and 3.3

TABLE 3.3: Comparison results of Nusselt $\theta'(0)$ with Ref[35] when $(P = M = R = Ec)$=0, Pr=6.2 andSc=0.68

$\boldsymbol{\phi}$	$\boldsymbol{Cu - water}$	Present Result		$\boldsymbol{Al_2O_3}$	Present Result	
	Ref[35]	**S**hooting method	bvp4c	Ref[35]	**S**hooting method	bvp4c
0.0	1.7148	1.7148	1.7148	1.7148	1.7242	1.7242
0.1	2.1357	2.1357	2.1357	2.0231	2.0231	2.0231
0.2	2.5400	2.5400	2.5400	2.3343	2.3343	2.3343

show that the skin friction and local Nusselt number increase by the increasing the values of ϕ. The results are observed to be in a very good assention. By increasing the permeability parameter as well as the Cu-water and Al_2o_3-water nanoparticles increase the skin-friction co-efficient.

Chapter 4

Numerical results of Joule heating in nanofluids

4.1 Introduction

In this chapter we extend the model of Ref [35]by considering the additional effect of Joule heating . Heat and mass transfer are analyzed for steady, viscous dissipations, chemical reaction and Joule heating past a porous medium. By using a suitable similarity transformation, the nonlinear partial differential equations of momentum and heat are converted into a system of ordinary differential equations. Numerical solutions are acquired by using the shooting method. The impact of different physical parameter values is discussed and the results are found to be in excellent agreement with those of the Matlab bvp4c-built in code. The numerical computed effect of different parameters on the dimensionless velocity, temperature and concentration are calculated and presented in the form of graphs and tables.

4.2 Problem formulation

Consider a laminar, two-dimensional and time independent flow of a fluid with heat transfer past a flat permeable plate through a porous medium. The geometry of the flow model is shown in Figure 4.1.

Assume that the fluid under discussion be taken as viscous, incompressible electrically

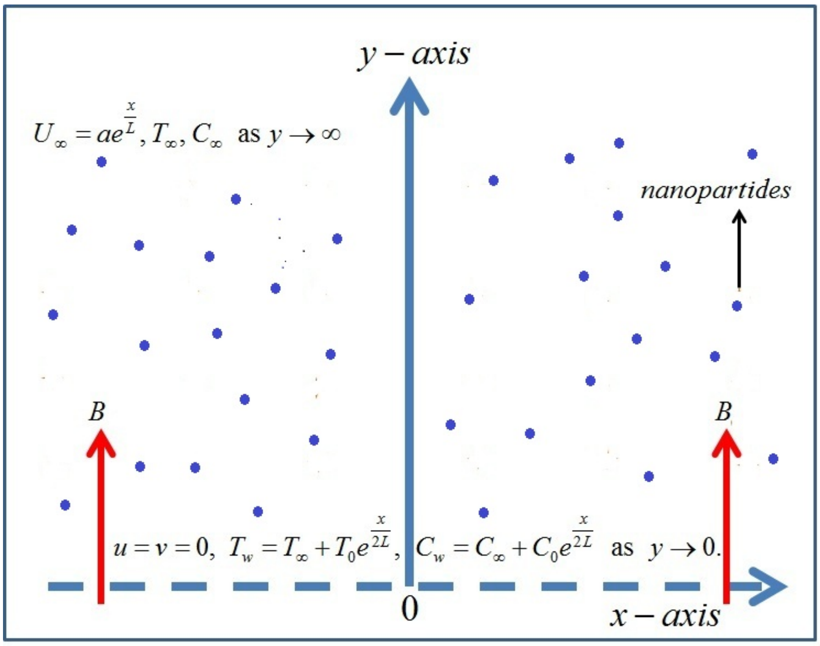

FIGURE 4.1: Geometry for the flow under consideration.

conducting and radiating over a porous medium. The equation of continuity, equation of momentum , energy equation and the concentration equation describing the two dimensional flow are given as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{4.1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U_\infty \frac{dU_\infty}{dx} + \nu_{nf}\frac{\partial^2 u}{\partial y^2} + \frac{\nu_{nf}}{k}(U_\infty - u) + \frac{\sigma B^2}{(\rho c_p)_{nf}}(U_\infty - u), \tag{4.2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha_{nf}\frac{\partial^2 T}{\partial y^2} - \frac{1}{(\rho c_p)_{nf}}\frac{\partial q_r}{\partial y} + \frac{\mu_{nf}}{(\rho c_p)_{nf}}\left(\frac{\partial u}{\partial y}\right)^2 + \frac{\sigma B^2}{(\rho c_p)nf}u^2, \tag{4.3}$$

$$u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} = D\frac{\partial^2 C}{\partial y^2} - K(C - C_\infty). \tag{4.4}$$

The associated boundary conditions for the above system of equations are:

$$u = v = 0, T = T_w = T_\infty + T_0 e^{\frac{x}{2L}}, C = C_w = C_\infty + C_0 e^{\frac{x}{2L}} \ at \ y = 0, \tag{4.5}$$

$$u \to U_\infty = a e^{\frac{x}{L}}, T \to T_\infty, C \to C_\infty \ as \ y \to \infty. \tag{4.6}$$

Here σ_s is the electrical conductivity of the base-fluid whereas σ_{nf}, ν_{nf}, ρ_{nf}, α_{nf}, k_{nf} are the electric conductivity, the effective viscosity, the effective density, the effective thermal diffusivity, the thermal conductivity of the nanofluid, respectively. Different relations of interest have been formulated as:

$$\alpha_{nf} = \frac{k_{nf}}{(\rho c_p)_{nf}}, \tag{4.7}$$

$$\rho_{nf} = (1-\phi)\rho_f + \phi\rho_s, \tag{4.8}$$

$$\mu_{nf} = \frac{\mu_f}{(1-\phi)^{2.5}}, \tag{4.9}$$

$$(\rho c_p)_{nf} = (1-\phi)(\rho c_p)_f + \phi(\rho c_p)_s, \tag{4.10}$$

$$\frac{K_{nf}}{K_f} = \frac{K_s + 2K_f - 2\phi(K_f - K_s)}{K_s + 2K_f + 2\phi(K_f - K_s)}, \tag{4.11}$$

$$\sigma_{nf} = (1-\phi)\sigma_f + \phi\sigma_s, \tag{4.12}$$

$$\nu_f = \frac{\mu_f}{\rho_f}, \tag{4.13}$$

$$K = K_o e^{\frac{-x}{L}}, \tag{4.14}$$

$$B = B_o e^{\frac{x}{2L}}. \tag{4.15}$$

The radiative heat flux q_r by using the Rosseland approximation for radiation, can be written as

$$q_r = \frac{-4\sigma^*}{3k^*}\frac{\partial T^4}{\partial y}, \tag{4.16}$$

where σ^* is the Stefan-Boltzman constant and k^* is the absorption coefficient. If the temperature difference is very small, then the temperature variety T^4 might be extended about T_∞ in a Taylor series, as follows:

$$T^4 = T_\infty^4 + \frac{4T_\infty^3}{1!}(T-T_\infty)^1 + \frac{12T_\infty^2}{2!}(T-T_\infty)^2 + \frac{24T_\infty}{3!}(T-T_\infty)^3 + \frac{24}{4!}(T-T_\infty)^4.$$

Disregarding the higher order terms,

$$\begin{aligned} T^4 &= T_\infty^4 + 4T_\infty^3(T-T_\infty) \\ \Rightarrow\ T^4 &= T_\infty^4 + 4T_\infty^3 T - 4T_\infty^4 \\ \Rightarrow\ T^4 &= 4T_\infty^3 T - 3T_\infty^4 \\ \Rightarrow\ \frac{\partial T^4}{\partial y} &= 4T_\infty^3 \frac{\partial T}{\partial y}. \end{aligned} \tag{4.17}$$

Using (4.17) in (4.16) and then differentiating w.r.t y, we get

$$\frac{\partial q_r}{\partial y} = \frac{-16\sigma^* T_\infty^3}{3k^*}\frac{\partial^2 T}{\partial y^2}. \tag{4.18}$$

Then (4.3) gets the following form.

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha_{nf}\frac{\partial^2 T}{\partial y^2} + \frac{16\sigma^* T_\infty^3}{3(\rho c_p)_{nf}k^*}\frac{\partial^2 T}{\partial y^2} + \frac{\mu_{nf}}{(\rho c_p)_{nf}}\left(\frac{\partial u}{\partial y}\right)^2. \tag{4.19}$$

Let ψ be the stream function satisfying the continuity equation in the following sense.

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}. \tag{4.20}$$

For the conversion of the mathematical model (3.1)-(3.4) into the dimensionless form, the following similarity transformation has been introduced.

$$\eta = y\sqrt{\frac{a}{2v_f L}}e^{\frac{x}{2L}}, \tag{4.21}$$

$$\psi = y\sqrt{2aLv_f}f(\eta)e^{\frac{x}{2L}}, \tag{4.22}$$

$$\theta(\eta) = \frac{T-T_\infty}{T_w-T_\infty}, \tag{4.23}$$

$$h(\eta) = \frac{C-C_\infty}{C_w-C_\infty}. \tag{4.24}$$

The detailed procedure for the conversion of (4.1)-(4.4) has been described in the upcoming discussion.

- $$\begin{aligned} u &= \frac{\partial \psi}{\partial y} \\ &= \frac{\partial \psi}{\partial \eta}\frac{\partial \eta}{\partial y} \\ &= \sqrt{2a\nu_f L}e^{\frac{x}{2L}}f'(\eta)\sqrt{\frac{a}{2\nu_f L}}e^{\frac{x}{2L}} \\ &= \sqrt{\frac{2a^2L\nu_f}{2\nu_f L}}e^{\frac{2x}{2L}}f'(\eta) = ae^{\frac{x}{L}}f'(\eta). \end{aligned}$$
- $$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial}{\partial x}\bigg(ae^{\frac{x}{L}}f'(\eta)\bigg) \\ &= a\bigg(\frac{\partial f'(\eta)}{\partial \eta}.\frac{\partial \eta}{\partial x}.e^{\frac{x}{L}} + f'(\eta).\frac{\partial}{\partial x}(e^{\frac{x}{L}})\bigg) \\ &= a\bigg(f''(\eta).y\sqrt{\frac{a}{2\nu_f L}}e^{\frac{x}{2L}}.\frac{1}{2L}e^{\frac{x}{L}} + \frac{1}{L}.f'(\eta)e^{\frac{x}{L}}\bigg) \\ &= a\bigg(\frac{\eta f''(\eta)}{2L} + \frac{f'(\eta)}{L}\bigg)e^{\frac{x}{L}} \\ &= \frac{a}{2L}\bigg(\eta f''(\eta) + 2f'(\eta)\bigg)e^{\frac{x}{L}} \end{aligned} \tag{4.25}$$
- $$\begin{aligned} v &= -\frac{\partial \psi}{\partial x} = -\frac{\partial}{\partial x}\bigg(\sqrt{2aL\nu_f}f(\eta)e^{\frac{x}{2L}}\bigg) \\ &= -\sqrt{2aL\nu_f}\bigg(\frac{\partial f(\eta)}{\partial x}e^{\frac{x}{2L}} + f(\eta)\frac{\partial}{\partial x}e^{\frac{x}{2L}}\bigg) \\ &= -\sqrt{2aL\nu_f}\bigg(\frac{\partial f(\eta)}{\partial \eta}.\frac{\partial \eta}{\partial x}.e^{\frac{x}{2L}} + f(\eta)e^{\frac{x}{2L}}.\frac{1}{2L}\bigg) \\ &= -\sqrt{2aL\nu_f}\bigg(f'(\eta)y\sqrt{\frac{a}{2\nu_f L}}e^{\frac{x}{2L}}e^{\frac{x}{2L}}\frac{1}{2L} + \frac{1}{2L}f(\eta)e^{\frac{x}{2L}}\bigg) \end{aligned} \tag{4.26}$$

$$= -\sqrt{2aL\nu_f}\left(\eta f'(\eta)e^{\frac{x}{2L}}.\frac{1}{2L} + \frac{1}{2L}f(\eta)e^{\frac{x}{2L}}\right)$$

$$= -\sqrt{2aL\nu_f}\frac{e^{\frac{x}{2L}}}{2L}\left(\eta f'(\eta) + f(\eta)\right).$$

- $$\frac{\partial v}{\partial y} = -\sqrt{2aL\nu_f}\left(\frac{\partial}{\partial y}(\eta f'(\eta)) + \frac{\partial f(\eta)}{\partial y}\right)\frac{1}{2L}e^{\frac{x}{2L}}$$

$$= -\sqrt{2aL\nu_f}\left(\frac{\partial \eta}{\partial y}f'(\eta) + \frac{\partial f'(\eta)}{\partial \eta}.\frac{\partial \eta}{\partial y}.\eta + \frac{\partial f(\eta)}{\partial \eta}\frac{\partial \eta}{\partial y}\right)\frac{1}{2L}e^{\frac{x}{2L}}$$

$$= -\frac{\sqrt{2aL\nu_f}}{2L}.\sqrt{\frac{a}{2L\nu_f}}\left(f'(\eta) + \eta f''(\eta) + f'(\eta)\right)e^{\frac{x}{2L}}.e^{\frac{x}{2L}}$$

$$= -\frac{a}{2L}\left(\eta f''(\eta) + 2f'(\eta)\right)e^{\frac{x}{L}}. \tag{4.27}$$

Though the continuity equation (4.1) is already satisfied by the choice of the stream function ψ in (4.20), it can again be verified by using (4.21) and (4.22) in it as follows.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{a}{2L}\left(\eta f''(\eta) + 2f'(\eta)\right)e^{\frac{x}{L}} - \frac{a}{2L}\left(\eta f''(\eta) + 2f'(\eta)\right)e^{\frac{x}{L}} = 0.$$

Now we include below the procedure for conversion of (4.2) into the dimensionless form.

- $$\frac{\partial u}{\partial y} = ae^{\frac{x}{L}}\frac{\partial f'}{\partial \eta}.\frac{\partial \eta}{\partial y}$$

$$= ae^{\frac{x}{L}}f''(\eta).\sqrt{\frac{a}{2Lv_f}}e^{\frac{x}{2L}}$$

$$= ae^{\frac{3x}{2L}}f''(\eta).\sqrt{\frac{a}{2L\nu_f}}.$$

- $$u\frac{\partial u}{\partial x} = \left(ae^{\frac{x}{L}}f'(\eta)\right).\frac{a}{2L}\left(\eta f''(\eta) + 2f'(\eta)\right)e^{\frac{x}{L}}$$

$$= \frac{a^2}{2L}\left(\eta f'(\eta)f''(\eta) + 2f'^2(\eta)\right)e^{\frac{2x}{L}}. \tag{4.28}$$

- $$v\frac{\partial u}{\partial y} = -\frac{\sqrt{2aL\nu_f}}{2L}\left(\eta f'(\eta) + f(\eta)\right)e^{\frac{x}{2L}}.\left(af''(\eta)\sqrt{\frac{a}{2L\nu_f}}e^{\frac{3x}{2L}}\right).$$

$$= -a\frac{\sqrt{2aL\nu_f}}{2L}.\sqrt{\frac{a}{2L\nu_f}}\left(\eta f'(\eta)f''(\eta) + f(\eta)f''(\eta)\right)e^{\frac{x}{2L}}e^{\frac{3x}{2L}}$$

$$= -\frac{a^2}{2L}e^{\frac{2x}{L}}\left(\eta f''(\eta)f'(\eta) + f''(\eta)f(\eta)\right). \tag{4.29}$$

Using (4.27) and (4.28), the left side of (4.2) becomes.

$$\begin{aligned} u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} &= \frac{a^2}{2L}e^{\frac{2x}{L}}\Big(\eta f''(\eta)f'(\eta)+2f'^2(\eta)\Big)-\frac{a^2}{2L}e^{\frac{2x}{L}}\Big(\eta f''(\eta)f'(\eta)+f''(\eta)f(\eta)\Big)\\ &= \frac{a^2e^{\frac{2x}{L}}}{2L}\Big(2f'^2(\eta)-f''(\eta)f(\eta)\Big). \end{aligned}$$

To convert the right side of (4.2) into the dimensionless form, the following procedure has been followed.

- $$\begin{aligned} U_\infty\frac{dU_\infty}{dx} &= ae^{\frac{x}{L}}\frac{d}{dx}\Big(ae^{\frac{x}{L}}\Big)\\ &= ae^{\frac{x}{L}}\frac{1}{L}ae^{\frac{x}{L}}\\ &= \frac{a^2e^{\frac{2x}{L}}}{L}. \end{aligned} \tag{4.30}$$
- $$\begin{aligned} \frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial y}\Big(\frac{\partial u}{\partial y}\Big)=\frac{\partial}{\partial y}\Big(a\sqrt{\frac{a}{2L\nu_f}}e^{\frac{3x}{2L}}f''(\eta)\Big)\\ &= a\sqrt{\frac{a}{2L\nu_f}}e^{\frac{3x}{2L}}\frac{\partial f''}{\partial\eta}.\frac{\partial\eta}{\partial y}=a\sqrt{\frac{a}{2L\nu_f}}e^{\frac{3x}{2L}}f'''(\eta)\sqrt{\frac{a}{2L\nu_f}}e^{\frac{x}{2L}}\\ &= \frac{a^2e^{\frac{2x}{L}}}{2L\nu_f}f'''(\eta). \end{aligned}$$
- $$\begin{aligned} \nu_{nf}\frac{\partial^2 u}{\partial y^2} &= \frac{\mu_{nf}}{\rho_{nf}}\Big(\frac{\partial^2 u}{\partial y^2}\Big)\\ &= \frac{\mu_f}{(1-\phi)^{2.5}\big((1-\phi)\rho_f+\phi\rho_s\big)}\Big(\frac{a^2}{2\nu_f L}f'''(\eta)e^{\frac{2x}{L}}\Big)\\ &= \frac{\mu_f a^2e^{\frac{2x}{L}}}{\frac{\mu_f}{\rho_f}.2L(1-\phi)^{2.5}\rho_f\big(1-\phi+\frac{\phi\rho_s}{\rho_f}\big)}f'''(\eta) \qquad \because\Big(\nu_f=\frac{\mu_f}{\rho_f}\Big)\\ &= \frac{a^2e^{\frac{2x}{L}}}{2L(1-\phi)^{2.5}\big(1-\phi+\frac{\phi\rho_s}{\rho_f}\big)}f'''(\eta). \end{aligned}$$

- $$\frac{\nu_{nf}}{k}(U_\infty - u) = \frac{\mu_f}{k_o e^{\frac{-x}{L}}(1-\phi)^{2.5}\Big((1-\phi)\rho_f + \phi\rho_s\Big)}\Big(ae^{\frac{x}{L}} - ae^{\frac{x}{L}}f'(\eta)\Big)$$
$$= \frac{\nu_f \rho_f}{\rho_f k_o (1-\phi)^{2.5}\big(1-\phi+\phi\frac{\rho_s}{\rho_f}\big)} ae^{\frac{x}{L}}.e^{\frac{x}{L}}(1-f'(\eta)) \qquad \because \Big(\nu_f = \frac{\mu_f}{\rho_f}\Big)$$
$$= \frac{\nu_f}{k_o (1-\phi)^{2.5}\big(1-\phi+\phi\frac{\rho_s}{\rho_f}\big)} ae^{\frac{2x}{L}}\Big(1-f'(\eta)\Big). \tag{4.31}$$

- $$\frac{\sigma B^2}{\rho_{nf}}(U_\infty - u) = \frac{\sigma(B_o e^{\frac{x}{2L}})^2}{(1-\phi)\rho_f + \phi\rho_s}\Big(ae^{\frac{x}{L}} - ae^{\frac{x}{L}}f'(\eta)\Big)$$
$$= \frac{\sigma {B_o}^2 (e^{\frac{x}{2L}})^2}{(1-\phi)\rho_f + \phi\rho_s} ae^{\frac{x}{L}}\Big(1-f'(\eta)\Big)$$
$$= \frac{\sigma {B_o}^2 a e^{\frac{2x}{L}}}{\rho_f(1-\phi+\frac{\phi\rho_s}{\rho_f})}\Big(1-f'(\eta)\Big). \tag{4.32}$$

Using (4.29) - (4.31) in the right side of Eq.(4.2), we get

$$\begin{aligned}
&U_\infty \frac{dU_\infty}{dx} + \nu_{nf}\frac{\partial^2 u}{\partial y^2} + \frac{\nu_{nf}}{k}(U_\infty - u) + \frac{\sigma B^2}{\rho_{nf}}(U_\infty - u) \\
&= \frac{a^2 e^{\frac{2x}{L}}}{L} + \frac{a^2 e^{\frac{2x}{L}} f'''(\eta)}{2L(1-\phi)^{2.5}\big(1-\phi+\frac{\phi\rho_s}{\rho_f}\big)} \\
&\quad + \frac{\nu_f}{k_o(1-\phi)^{2.5}\big(1-\phi+\phi\frac{\rho_s}{\rho_f}\big)} ae^{\frac{2x}{L}}\Big(1-f'(\eta)\Big) \\
&\quad + \frac{\sigma {B_o}^2 a e^{\frac{2x}{L}}}{\rho_f(1-\phi+\frac{\phi\rho_s}{\rho_f})}\Big(1-f'(\eta)\Big)
\end{aligned}$$

Hence the dimensionless form of (4.2) becomes.

$$\frac{a^2e^{\frac{2x}{L}}}{2L}\left(2f'^2(\eta)-f''(\eta)f(\eta)\right)=\frac{a^2e^{\frac{2x}{2L}}}{2L}\left[2+\frac{1}{(1-\phi)^{2.5}\left(1-\phi+\frac{\phi\rho_s}{\rho_f}\right)}f'''(\eta)\right.$$
$$\left.+\frac{2L\nu_f}{ak_o(1-\phi)^{2.5}\left(1-\phi+\phi\frac{\rho_s}{\rho_f}\right)}\left(1-f'(\eta)\right)+\frac{2L\sigma {B_o}^2}{a\rho_f(1-\phi+\frac{\phi\rho_s}{\rho_f})}\left(1-f'(\eta)\right)\right]$$
$$\Rightarrow\quad 2f'^2(\eta)-f(\eta)f''(\eta)\quad=2+\frac{f'''(\eta)}{(1-\phi)^{2.5}(1-\phi+\frac{\phi\rho_s}{\rho_f})}$$
$$+\left(\frac{2L\nu_f}{ak_o(1-\phi)^{2.5}(1-\phi+\frac{\phi\rho_s}{\rho_f})}+\frac{2L\sigma {B_o}^2}{a\rho_f(1-\phi+\frac{\phi\rho_s}{\rho_f})}\right)\left(1-f'(\eta)\right)$$
$$\Rightarrow\quad -2+2f'^2(\eta)-f(\eta)f''(\eta)=\frac{f'''(\eta)}{(1-\phi)^{2.5}(1-\phi+\frac{\phi\rho_s}{\rho_f})}$$
$$+\left(\frac{2L\nu_f}{ak_o(1-\phi)^{2.5}(1-\phi+\frac{\phi\rho_s}{\rho_f})}+\frac{2L\sigma {B_o}^2}{a\rho_f(1-\phi+\frac{\phi\rho_s}{\rho_f})}\right)\left(1-f'(\eta)\right)$$
$$\Rightarrow\quad \frac{f'''(\eta)}{(1-\phi)^{2.5}(1-\phi+\frac{\phi\rho_s}{\rho_f})}+2(1-f'^2(\eta))+f(\eta)f''(\eta)+\frac{1}{(1-\phi)^{2.5}(1-\phi+\frac{\phi\rho_s}{\rho_f})}$$
$$\left(\frac{2L\nu_f}{ak_o}+\frac{2L\sigma {B_o}^2}{a\rho_f}(1-\phi)^{2.5}\right)\left(1-f'(\eta)\right)=0 \tag{4.33}$$

Now we include below the procedure for conversion of (4.3) into the dimensionless form.

- $$\theta(\eta)=\frac{T-T_\infty}{T_w-T_\infty}$$
$$\begin{aligned}\Rightarrow\quad T&=(T_w-T_\infty)\theta(\eta)+T_\infty\\&=(T_\infty+T_0e^{\frac{x}{2L}}-T_\infty)\theta(\eta)+T_\infty\\&=(T_0e^{\frac{x}{2L}})\theta(\eta)\end{aligned}$$
- $$\begin{aligned}\frac{\partial T}{\partial x}&=T_0\left(\frac{\partial}{\partial x}(e^{\frac{x}{2L}})\theta(\eta)+e^{\frac{x}{2L}}\frac{\partial\theta(\eta)}{\partial\eta}.\frac{\partial\eta}{\partial x}\right)\\&=\frac{T_0}{2L}\left(e^{\frac{x}{2L}}\theta(\eta)+e^{\frac{x}{2L}}\theta'(\eta)y\sqrt{\frac{a}{2\nu_fL}}e^{\frac{x}{2L}}\right)\\&=\frac{T_0e^{\frac{x}{2L}}}{2L}(\theta(\eta)+\eta\theta'(\eta))\end{aligned}$$

- $$u\frac{\partial T}{\partial x} = ae^{\frac{x}{L}}f'(\eta)\frac{T_0e^{\frac{x}{2L}}}{2L}\Bigg(\theta(\eta)+\eta\theta'(\eta)\Bigg)$$
$$= a\frac{T_0}{2L}e^{\frac{3x}{2L}}\Bigg(f'(\eta)(\theta(\eta)+\eta f'(\eta)\theta'(\eta))\Bigg) \tag{4.34}$$

- $$\frac{\partial T}{\partial y} = \frac{\partial}{\partial y}\Bigg((T_0e^{\frac{x}{2L}})\theta(\eta)\Bigg)$$
$$= (T_0e^{\frac{x}{2L}})\frac{\partial\theta(\eta)}{\partial\eta}\frac{\partial\eta}{\partial y}$$
$$= (T_0e^{\frac{x}{2L}})\theta'(\eta)\sqrt{\frac{a}{2\nu_fL}}e^{\frac{x}{2L}}$$
$$= T_0e^{\frac{x}{L}}\sqrt{\frac{a}{2\nu_fL}}\theta'(\eta)$$

- $$v\frac{\partial T}{\partial y} = \Bigg(-\frac{1}{2L}\sqrt{2aL\nu_f}(\eta f'(\eta)+f(\eta))e^{\frac{x}{2L}}\Bigg)\Bigg(T_0e^{\frac{x}{L}}\theta'(\eta)\sqrt{\frac{a}{2\nu_fL}}\Bigg)$$
$$= -\frac{1}{2L}\sqrt{2aL\nu_f}\sqrt{\frac{a}{2aL\nu_f}}(\eta f'(\eta)+f(\eta))\theta'(\eta)e^{\frac{x}{2L}}e^{\frac{x}{L}}$$
$$= -\frac{1}{2L}\sqrt{\frac{2a^2L\nu_f}{2aL\nu_f}}T_oe^{\frac{3x}{2L}}(\eta f'(\eta)\theta'(\eta)+f(\eta)\theta'(\eta))$$
$$= -\frac{aT_0e^{\frac{3x}{2L}}}{2L}(\eta f'(\eta)\theta'(\eta)+f(\eta)\theta'(\eta)) \tag{4.35}$$

Using (4.33) and (4.34) the left side of (4.3) gets the following form:

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y} = \frac{aT_0e^{\frac{3x}{2L}}}{2L}(f'(\eta)\theta(\eta)+\eta\theta'(\eta)f'(\eta)) - \frac{aT_0e^{\frac{3x}{2L}}}{2L}(f'(\eta)\eta\theta'(\eta)+f(\eta)\theta'(\eta))$$
$$= \frac{aT_0e^{\frac{3x}{2L}}}{2L}\Bigg(f'(\eta)\theta(\eta)-f(\eta)\theta'(\eta)\Bigg)$$

To convert the right side of (4.3) into dimensionless form we proceed as follows.

- $$\frac{\partial^2T}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial T}{\partial y}\right) = \frac{\partial}{\partial y}\left(T_0e^{\frac{x}{L}}\sqrt{\frac{a}{2\nu_fL}}\theta'(\eta)\right)$$
$$= T_0e^{\frac{x}{L}}\sqrt{\frac{a}{2\nu_fL}}\frac{\partial\theta'}{\partial\eta}\frac{\partial\eta}{\partial y} = T_0e^{\frac{x}{L}}\sqrt{\frac{a}{2\nu_fL}}\theta''(\eta)\sqrt{\frac{a}{2\nu_fL}}e^{\frac{x}{2L}}$$
$$= T_0e^{\frac{x}{L}}e^{\frac{x}{2L}}\left(\sqrt{\frac{a}{2\nu_fL}}\right)^2\theta''(\eta)$$

$$= T_0 e^{\frac{3x}{2L}} \frac{a}{2\nu_f L}\theta''(\eta)$$

- $$\alpha_{nf}\frac{\partial^2 T}{\partial y^2} = \frac{k_{nf}}{(\rho c_p)_{nf}} T_0 e^{\frac{3x}{2L}} \frac{a}{2\nu_f L}\theta''(\eta) \tag{4.36}$$
- $$\frac{1}{(\rho c_p)_{nf}}\frac{\partial q_r}{\partial y} = \frac{1}{(\rho c_p)_{nf}}.\frac{-16\sigma^* T_\infty^3}{3k^*}\frac{aT_0 e^{\frac{3x}{L}}}{2\upsilon_f L}\theta''(\eta)$$
$$= \frac{-16\sigma^* T_\infty^3}{3k^*(\rho c_p)_{nf}}\frac{aT_0 e^{\frac{3x}{L}}}{2\nu_f L}\theta''(\eta) \tag{4.37}$$
- $$\frac{\mu_{nf}}{(\rho c_p)_{nf}}\left(\frac{\partial u}{\partial y}\right)^2 = \frac{\mu_{nf}}{(\rho c_p)_{nf}} a^2 (e^{\frac{3x}{2L}})^2 f''^2(\eta).\frac{a}{2L\nu_f} \tag{4.38}$$
- $$\frac{\sigma B^2}{(\rho c_p)_{nf}} u^2 = \frac{\sigma (B_0 e^{\frac{x}{2L}})^2}{(\rho c_p)_{nf}}\left(a e^{\frac{x}{L}} f'(\eta)\right)^2 \tag{4.39}$$

Using (4.35) - (4.38) the dimensionless form of right side (4.3) is as follows.

$$\begin{aligned}
&\alpha_{nf}\frac{\partial^2 T}{\partial y^2} - \frac{1}{(\rho c_p)}\frac{\partial q_r}{\partial y} + \frac{\mu_{nf}}{(\rho c_p)_{nf}}\left(\frac{\partial u}{\partial y}\right)^2 + \frac{\sigma B^2}{(\rho c_p)_{nf}} u^2 \\
&= \frac{k_{nf} T_0 a e^{\frac{3x}{2L}}}{2\nu_f L (\rho c_p)_{nf}}\theta''(\eta) + \frac{16\sigma^* T_\infty^3 a T_o e^{\frac{3x}{L}}}{3k^* 2\nu_f L (\rho c_p)_{nf}}\theta''(\eta) \\
&\quad + \frac{a^3 (e^{\frac{3x}{2L}})^2 \mu_{nf}}{2L\nu_f (\rho c_p)_{nf}} f''^2(\eta) + \frac{\sigma (B_0 e^{\frac{x}{2L}})^2}{(\rho c_p)_{nf}}\left(a e^{\frac{x}{L}} f'(\eta)\right)^2 \\
&= \frac{aT_0 e^{\frac{3x}{2L}}}{2L}\left[\frac{k_{nf}}{\nu_f (\rho c_p)_{nf}}\theta'' + \frac{16\sigma^* T_\infty^3}{\nu_f (\rho c_p)_{nf} 3k^*}\theta''(\eta)\right. \\
&\quad \left. + \frac{\mu_f (a^3 e^{\frac{3x}{L}})}{\nu_f T_0 ((\rho c_p)_{nf})} f''^2(\eta) + \frac{2L\sigma (B_0 e^{\frac{x}{2L}})^2}{aT_0 e^{\frac{3x}{2L}} (\rho c_p)_{nf}} (U_\infty)^2 f'^2(\eta)\right]. \\
&= \frac{aT_0 e^{\frac{3x}{2L}}}{2L}\left[\left(\frac{k_{nf}}{k_f}.\frac{(\rho c_p)_f}{(\rho c_p)_{nf}}.\frac{k_f}{(\rho_{cp})_f}\frac{1}{\nu_f}\right)\theta''(\eta)\right. \\
&\quad + \left(\frac{(\rho c_p)_f}{(\rho c_p)_{nf}}.\frac{k_f}{(\rho c_p)_f \nu_f}\frac{16\sigma^* T_\infty^3}{3k^* k_f}\right)\theta''(\eta) \\
&\quad \left. + \frac{\mu_{nf}}{\mu_f}.\frac{(\rho c_p)_f}{(\rho c_p)_{nf}}.\frac{\mu_f}{(\rho c_p)_f}\frac{(a^2 e^{\frac{3x}{2L}})}{T_0 \nu_f} f''^2(\eta) + \frac{2L\sigma (B_0 e^{\frac{x}{2L}})^2}{aT_0 e^{\frac{3x}{2L}}}\frac{(\rho c_p)_f}{(\rho c_p)_{nf}}.\frac{1}{(\rho c_p)_f}(U_\infty)^2 f'^2(\eta)\right].
\end{aligned} \tag{4.40}$$

$$= \frac{aT_0 e^{\frac{3x}{2L}}}{2L}\Bigg[\Bigg(\frac{k_{nf}}{k_f}.\frac{1}{(1-\phi+\frac{\phi(\rho c_p)_s}{(\rho c_p)_f})}.\frac{\alpha_f}{\nu_f}\Bigg)\theta''(\eta) \qquad \Bigg(\because \quad \alpha_f = \frac{k_f}{(\rho c_p)_f}\Bigg)$$

$$+ \Bigg(\frac{1}{(1-\phi+\frac{\phi(\rho c_p)_s}{(\rho c_p)_f})}.\frac{k_f}{(\rho c_p)_f \nu_f}\frac{16\sigma^* T_\infty^3}{3k^* k_f}\Bigg)\theta''(\eta) \qquad \Bigg(\because \quad \alpha_f = \frac{k_f}{(\rho c_p)_f}\Bigg)$$

$$+ \frac{1}{(1-\phi)^{2.5}(1-\phi+\frac{\phi(\rho c_p)_s}{(\rho c_p)_f})}.\frac{\mu_f}{\rho_f}.\frac{1}{\nu_f}\frac{(a^2 e^{\frac{3x}{2L}})}{T_0 (c_p)_f} f''^2(\eta)$$

$$+ \frac{2L\sigma(B_0 e^{\frac{x}{2L}})^2}{aT_0 e^{\frac{3x}{2L}}}\frac{1}{(1-\phi+\frac{\phi(\rho c_p)_s}{(\rho c_p)_f})}.\frac{1}{(\rho c_p)_f}(U_\infty)^2 f'^2(\eta)\Bigg].$$

$$= \frac{aT_0 e^{\frac{3x}{2L}}}{2L}\Bigg[\Bigg(\frac{k_{nf}}{k_f}.\frac{1}{(1-\phi+\frac{\phi(\rho c_p)_s}{(\rho c_p)_f})}.\frac{1}{P_r}\Bigg)\theta''(\eta) \qquad \Bigg(\because \quad Pr = \frac{\nu_f}{\alpha_f}\Bigg)$$

$$+ \Bigg(\frac{1}{(1-\phi+\frac{\phi(\rho c_p)_s}{(\rho c_p)_f})}.\frac{1}{P_r}\frac{16\sigma^* T_\infty^3}{3k^* k_f}\Bigg)\theta''(\eta) \qquad \Bigg(\because \quad Pr = \frac{\nu_f}{\alpha_f}\Bigg)$$

$$+ \frac{1}{(1-\phi)^{2.5}(1-\phi+\frac{\phi(\rho c_p)_s}{(\rho c_p)_f})}\frac{(ae^{\frac{x}{L}})^2}{(T_w - T_\infty(c_p)_f)} f''^2(\eta)$$

$$+ \frac{2L\sigma e^{\frac{x}{2L}}(B_0 e^{\frac{x}{2L}})^2}{a(T_w - T_\infty)e^{\frac{3x}{2L}}}\frac{1}{(1-\phi+\frac{\phi(\rho c_p)_s}{(\rho c_p)_f})}.\frac{1}{(\rho c_p)_{nf}}(U_\infty)^2 f'^2(\eta)\Bigg]. \tag{4.41}$$

$$\Bigg(\because \mu_f = \rho_f \nu_f, T_0 = \frac{T_w - T_\infty}{e^{\frac{x}{2L}}}\Bigg)$$

Therefore the dimensionless form of (4.3) becomes:

$$\frac{aT_0 e^{\frac{3x}{2L}}}{2L}\Bigg(f'(\eta)\theta(\eta) - f(\eta)\theta'(\eta)\Bigg) = \frac{aT_0 e^{\frac{3x}{2L}}}{2L}\Bigg[\Bigg(\frac{k_{nf}}{k_f}.\frac{1}{(1-\phi+\frac{\phi(\rho c_p)_s}{(\rho c_p)_f})}.\frac{1}{P_r}\Bigg)\theta''(\eta)$$

$$+ \Bigg(\frac{1}{(1-\phi+\frac{\phi(\rho c_p)_s}{(\rho c_p)_f})}.\frac{1}{P_r}\frac{16\sigma^* T_\infty^3}{3k^* k_f}\Bigg)\theta''(\eta)$$

$$+ \frac{1}{(1-\phi)^{2.5}(1-\phi+\frac{\phi(\rho c_p)_s}{(\rho c_p)_f})}\frac{(U_\infty)^2 f''^2(\eta)}{T_w - T_\infty(c_p)_f}$$

$$+ \frac{2L\sigma(B_0)^2}{aP_r(T_w - T_\infty)}\frac{(U_\infty)^2 f'^2(\eta)}{(1-\phi+\frac{\phi(\rho c_p)_s}{(\rho c_p)_f})(\rho c_p)_f}\Bigg].$$

$$\begin{aligned}
\Rightarrow \quad \Big(f'(\eta)\theta(\eta)-f(\eta)\theta'(\eta)\Big)P_r &= \Bigg[\Bigg(\frac{k_{nf}}{k_f}.\frac{1}{(1-\phi+\frac{\phi(\rho c_p)_s}{(\rho c_p)_f})}\Bigg)\theta''(\eta)\\
&+\Bigg(\frac{1}{(1-\phi+\frac{\phi(\rho c_p)_s}{(\rho c_p)_f})}\frac{16\sigma^* T_\infty^3}{3k^*k_f}\Bigg)\theta''(\eta)\\
&+\frac{P_r}{(1-\phi)^{2.5}\big(1-\phi+\frac{\phi(\rho c_p)_s}{(\rho c_p)_f}\big)}\frac{(U_\infty)^2 f''^2(\eta)}{(T_w-T_\infty)(c_p)_f}\Bigg]\\
&+\frac{2L\sigma(B_0)^2}{a(T_w-T_\infty)}\frac{1}{(1-\phi+\frac{\phi(\rho c_p)_s}{(\rho c_p)_f})}.\frac{P_r(U_\infty)^2 f'^2(\eta)}{(\rho c_p)_f}\Bigg]\\
\Rightarrow \quad \Big(f'(\eta)\theta(\eta)-f(\eta)\theta'(\eta)\Big)P_r &= \frac{1}{(1-\phi+\frac{\phi(\rho c_p)_s}{(\rho c_p)_f})}\Bigg[\Bigg(\frac{k_{nf}}{k_f}\Bigg)\theta''(\eta)+\Bigg(\frac{16\sigma^* T_\infty^3}{3k^*k_f}\Bigg)\theta''(\eta)\\
&+\frac{P_r}{(1-\phi)^{2.5}}\frac{(a^2e^{\frac{2x}{L}})}{T_w-T_\infty(c_p)_f}f''^2(\eta)\\
&+\frac{2L\sigma(B_0)^2}{a(T_w-T_\infty)}\frac{P_r}{\rho_f(c_p)_f}(U_\infty)^2 f'^2(\eta)\Bigg].\\
\Rightarrow \quad \Big(f'(\eta)\theta(\eta)-f(\eta)\theta'(\eta)\Big)P_r &= \frac{1}{(1-\phi+\frac{\phi(\rho c_p)_s}{(\rho c_p)_f})}\Bigg[\Bigg(\frac{k_{nf}}{k_f}+\frac{16\sigma^* T_\infty^3}{3k^*k_f}\Bigg)\theta''(\eta)\\
&+\frac{P_r}{(1-\phi)^{2.5}}\frac{(U_\infty)^2}{(T_w-T_\infty)(c_p)_f}f''^2(\eta)\\
&+\frac{2L\sigma(B_0)^2}{a\rho_f}.P_r\frac{(U_\infty)^2}{(T_w-T_\infty)(c_p)_f}f'^2(\eta)\Bigg].\\
\Rightarrow \Big(f'(\eta)\theta(\eta)-f(\eta)\theta'(\eta)\Big)P_r(1-\phi+\frac{\phi(\rho c_p)_s}{(\rho c_p)_f}) &= \Bigg[\Bigg(\frac{k_{nf}}{k_f}+\frac{16\sigma^* T_\infty^3}{3k^*k_f}\Bigg)\theta''(\eta)\\
&+\frac{1}{(1-\phi)^{2.5}}(\eta)P_r\frac{(U_\infty)^2}{(T_w-T_\infty)(c_p)_f}f''^2(\eta)\\
&+\frac{2L\sigma(B_0)^2}{a\rho_f}P_r\frac{(U_\infty)^2}{(T_w-T_\infty)(c_p)_f}f'^2(\eta)\Bigg].
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow\Bigg(\frac{k_{nf}}{k_f}+\frac{16\sigma^* T_\infty^3}{3k^*k_f}\Bigg)\theta''(\eta)+P_r(1-\phi+\frac{\phi(\rho c_p)_s}{(\rho c_p)_f})(-f'(\eta)\theta(\eta)+f(\eta)\theta'(\eta))\\
&\quad+\frac{1}{(1-\phi)^{2.5}}P_r\frac{(U_\infty)^2}{(T_w-T_\infty)(c_p)_f}f''^2(\eta)+\frac{2L\sigma(B_0)^2}{a\rho_f}P_r\frac{(U_\infty)^2}{(T_w-T_\infty)(c_p)_f}f'^2(\eta)=0.
\end{aligned}$$

Now we include below the procedure for conversion equation (4.4) into the dimensionless form .

- $$h(\eta) = \frac{C - C_\infty}{C_w - C_\infty}$$

$$\Rightarrow \quad C = h(\eta)(C_w - C_\infty) + C_\infty$$
$$= h(\eta)(C_\infty + C_0 e^{\frac{x}{2L}} - C_\infty) + C_\infty$$
$$= C_0 e^{\frac{x}{2L}} h(\eta) + C_\infty.$$

- $$\frac{\partial C}{\partial x} = h(\eta)\frac{\partial}{\partial x} C_0 e^{\frac{x}{2L}}$$

$$= C_0 \frac{e^{\frac{x}{2L}}}{2L} h(\eta) + C_0 e^{\frac{x}{2L}} \frac{\partial h(\eta)}{\partial \eta}\frac{\partial \eta}{\partial y}$$
$$= C_0 \frac{e^{\frac{x}{2L}}}{2L} h(\eta) + C_0 e^{\frac{x}{2L}} h'(\eta) y \sqrt{\frac{a}{2L\nu_f}} \frac{e^{\frac{x}{2L}}}{2L}$$
$$= \frac{C_0}{2L} e^{\frac{x}{2L}} \Big(h(\eta) + \eta h'(\eta) \Big).$$

- $$u\frac{\partial C}{\partial x} = a f'(\eta) e^{\frac{x}{L}} \frac{C_0}{2L} e^{\frac{x}{2L}} (h(\eta) + \eta h'(\eta))$$

$$= \frac{a e^{\frac{3x}{2L}}}{2L} C_0 (f'(\eta) h(\eta) + \eta h'(\eta) f'(\eta)). \quad (4.42)$$

- $$\frac{\partial C}{\partial y} = C_0 e^{\frac{x}{2L}} \frac{\partial h(\eta)}{\partial \eta}\frac{\partial \eta}{\partial y}$$

$$= C_0 e^{\frac{x}{2L}} h'(\eta) \sqrt{\frac{a}{2L\nu_f}} e^{\frac{x}{2L}}$$
$$= C_0 e^{\frac{x}{L}} h'(\eta) \sqrt{\frac{a}{2L\nu_f}}.$$

- $$v\frac{\partial C}{\partial y} = -\sqrt{2aL\nu_f}\frac{e^{\frac{x}{2L}}}{2L} \Big(\eta f'(\eta) + f(\eta) \Big) C_0 e^{\frac{x}{L}} h'(\eta) \sqrt{\frac{a}{2L\nu_f}}$$

$$= -C_0 \frac{a e^{\frac{3x}{2L}}}{2L} (\eta h'(\eta) f'(\eta) + h'(\eta) f(\eta)). \quad (4.43)$$

Using Eqs (4.41)and (4.42) in left side of (4.4)

$$u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} = \frac{C_0 a e^{\frac{3x}{2L}}}{2L} (f'(\eta)h(\eta) + \eta h'(\eta) f'(\eta) - \eta h'(\eta) f'(\eta) - h'(\eta) f(\eta))$$
$$= \frac{a e^{\frac{3x}{2L}}}{2L} C_0 (f'(\eta) h(\eta) - h'(\eta) f(\eta))$$
$$= \frac{a e^{\frac{3x}{2L}}}{2L} C_0 (f'(\eta) h(\eta) - h'(\eta) f(\eta)). \quad (4.44)$$

To convert the right side of (4.4) into dimensionless form we proceed as follows.

- $$\begin{aligned}\frac{\partial^2 C}{\partial y^2} &= \frac{\partial}{\partial y}\left(\sqrt{\frac{a}{2aL\nu_f}}C_0 e^{\frac{x}{L}} h'(\eta)\right)\\ &= \left(C_0 e^{\frac{x}{L}}\sqrt{\frac{a}{2L\nu_f}}\frac{\partial h'(\eta)}{\partial \eta}\frac{\partial \eta}{\partial y}\right)\\ &= C_0 e^{\frac{x}{L}}\sqrt{\frac{a}{2aL\nu_f}}h''(\eta)\sqrt{\frac{a}{2L\nu_f}}e^{\frac{x}{2L}}\\ &= C_0 e^{\frac{x}{L}}e^{\frac{x}{2L}}\frac{a}{2L\nu_f}h''(\eta)\\ &= C_0\frac{ae^{\frac{3x}{2L}}}{2L\nu_f}h''(\eta). \qquad (4.45)\end{aligned}$$
- $$\begin{aligned} h(\eta) &= \frac{C - C_\infty}{C_w - C_\infty}\\ \Rightarrow \quad (C - C_\infty) &= h(\eta)(C_w - C_\infty)\\ \Rightarrow \quad K(C - C_\infty) &= K_0 e^{\frac{x}{L}} h(\eta)(C_\infty + C_0 e^{\frac{x}{2L}} - C_\infty)\\ &= C_0 K_0 e^{\frac{3x}{2L}} h(\eta). \qquad (4.46)\end{aligned}$$

Using (4.44) and (4.45) in the right side of (4.4),we get

$$\begin{aligned} D\frac{\partial^2 C}{\partial y^2} - K(C - C_\infty) &= DC_0\frac{ae^{\frac{3x}{2L}}}{2Lv_f}h''(\eta) - h(\eta)C_0K_0e^{\frac{3x}{2L}}\\ &= C_0 e^{\frac{3x}{2L}}\left(D\frac{a}{2L\nu_f}h''(\eta) - K_0 h(\eta)\right) \qquad (4.47)\end{aligned}$$

Hence dimensionless form of equation (4.4) becomes:

$$\begin{aligned} &C_0\frac{ae^{\frac{3x}{2L}}}{2L}\left(f'(\eta)h(\eta) - h'(\eta)f(\eta)\right) = C_0 e^{\frac{3x}{2L}}\left(\frac{Dah''(\eta)}{2L\nu_f} - K_0 h(\eta)\right)\\ &\Rightarrow \frac{a}{2L}(f'(\eta)h(\eta) - h'(\eta)f(\eta)) = \frac{aD}{2L\nu_f}h''(\eta) - K_0 h(\eta)\\ &\Rightarrow f'(\eta)h(\eta) - h'(\eta)f(\eta) = \frac{D}{\nu_f}h''(\eta) - \frac{2LK_0}{a}h(\eta)\\ &\Rightarrow \frac{\nu_f}{D}(f'(\eta)h(\eta) - h'(\eta)f(\eta)) = h''(\eta) - \frac{2LK_0\nu_f}{aD}h(\eta)\\ &\Rightarrow \frac{\nu_f}{D}(f'(\eta)h(\eta) - h'(\eta)f(\eta) + \frac{2LK_0\nu_f}{a}h(\eta)) = h''(\eta)\\ &\Rightarrow \ h''(\eta) + \frac{\nu_f}{D}(f(\eta)h'(\eta) - f'(\eta)h(\eta) - \frac{2LK_0\nu_f}{a}h(\eta)) = 0.\end{aligned}$$

The required ODEs are:

$$\frac{1}{(1-\phi)^{2.5}(1-\phi+\phi\frac{\rho_s}{\rho_f})}f''' + ff'' + 2(1-f'^2)$$
$$+\frac{1}{(1-\phi)^{2.5}(1-\phi+\phi\frac{\rho_s}{\rho_f})}(P+(1-\phi)^{2.5}M)(1-f') = 0, \tag{4.48}$$
$$(\frac{K_nf}{K_f}+R)\theta'' + P_r(1-\phi+\phi\frac{(\rho c_p)s}{(\rho c_p)_f}(f\theta'-f'\theta) \tag{4.49}$$
$$+\frac{1}{(1-\phi)^{2.5}}Ecf''^2 + P_rMEcf'^2 = 0,$$
$$h'' + Sc(fh' - f'h - \gamma h) = 0. \tag{4.50}$$

The boundary conditions (4.5)-(4.6) are:

$$f = 0, f' = 0, \theta = 1, h = 1, at\ \eta = 0, \tag{4.51}$$
$$f' \to 1, \theta \to 0, h \to 0as\ \eta \to \infty. \tag{4.52}$$

Different parameters used in the above equations have the following formulations:

$$P = \frac{2L\upsilon_f}{ak_0}, M = \frac{2\sigma_f B_0^2 L}{a\rho_f}, R = \frac{16\sigma_1 T_\infty^3}{3K}, Sc = \frac{\upsilon_f}{D},$$
$$\gamma = \frac{2LK_0}{a}, P_r = \frac{\upsilon_f}{\alpha_f}, Ec = \frac{{U_\infty}^2}{(c_p)_f(T_w - T_\infty)}, J = \frac{2\sigma_f B_0^2 L}{a\rho_f}\frac{\upsilon_f}{\alpha_f}\frac{{U_\infty}^2}{(c_p)_f(T_w - T_\infty)}. \tag{4.53}$$

The skin friction coefficient, is defined as:

$$C_{fx} = \frac{2\tau_w}{\rho_f U_\infty^2}. \tag{4.54}$$

- $\frac{\partial u}{\partial y} = a\sqrt{\frac{a}{2L\nu_f}}e^{\frac{x}{L}}e^{\frac{x}{2L}}f''(\eta)$

$$\Rightarrow \left(\frac{\partial u}{\partial y}\right)\bigg|_{y=0} = a\sqrt{\frac{a}{2L\nu_f}}e^{\frac{x}{L}}e^{\frac{x}{2L}}f''(0).$$

- $\tau_w = \mu_{nf}\left(\frac{\partial u}{\partial y}\right)\bigg|_{y=0}$

$$= \frac{\mu_f}{(1-\phi)^{2.5}}a\sqrt{\frac{a}{2L\nu_f}}e^{\frac{x}{L}}e^{\frac{x}{2L}}f''(0). \tag{4.55}$$

Using (4.55) in equation (4.54) ,we get the following form.

$$\begin{aligned}
C_{fx} &= \frac{2}{\rho_f U_\infty^2}\frac{\mu_f}{(1-\phi)^{2.5}} a\sqrt{\frac{a}{2L\nu_f}} e^{\frac{x}{L}} e^{\frac{x}{2L}} f''(0) \\
&= \frac{2}{U_\infty^2}\frac{\nu_f}{(1-\phi)^{2.5}} a\sqrt{\frac{a}{2L\nu_f}} e^{\frac{x}{L}} e^{\frac{x}{2L}} f''(0) \quad \therefore \nu_f = \frac{\mu_f}{\rho_f} \\
&= \sqrt{\frac{2\nu_f a}{L}}\frac{1}{(U_\infty)^2} U_\infty e^{\frac{x}{2L}} \frac{f''(0)}{(1-\phi)^{2.5}} \quad \therefore U_\infty = a e^{\frac{x}{L}} \\
&= \sqrt{\frac{2\nu_f}{L}}\frac{1}{U_\infty}(e^{\frac{x}{L}})^{\frac{1}{2}} \frac{f''(0)}{(1-\phi)^{2.5}} \\
&= \sqrt{\frac{2\nu_f}{LU_\infty}}(e^{\frac{x}{L}})^{\frac{1}{2}} \frac{f''(0)}{(1-\phi)^{2.5}} \\
&= \frac{\sqrt{\frac{x}{L}}}{\sqrt{\frac{xU_\infty}{2\nu_f}}} . \frac{f''(0)}{(1-\phi)^{2.5}} \\
&= \frac{\sqrt{\frac{x}{L}}}{\sqrt{\frac{Re_x}{2}}} . \frac{f''(0)}{(1-\phi)^{2.5}} \\
\Rightarrow \frac{C_{fx}\sqrt{\frac{Re_x}{2}}}{\sqrt{\frac{x}{L}}} &= \frac{f''(0)}{(1-\phi)^{2.5}},
\end{aligned}$$

The local Nusselt number is defined as:

$$Nu_x = \frac{-xq_w}{k_f(T_w - T_\infty)} \tag{4.56}$$

$$\begin{aligned}
\bullet\ q_w &= -k_{nf}\left(\frac{\partial T}{\partial y}\right)\bigg|_{y=0} \\
&= -k_{nf} T_0 \frac{a}{2\nu_f L} e^{\frac{x}{L}} \theta'(0)
\end{aligned} \tag{4.57}$$

Using (4.57) in (4.56), we get the following form.

$$\begin{aligned}
Nu_x &= -\frac{x k_{nf}(T_w - T_\infty)\frac{a}{2\nu_f L} e^{\frac{x}{L}} \theta'(0)}{k_f e^{\frac{x}{2L}} (T_w - T_\infty)} \qquad \because T_0 = \frac{(T_w - T_\infty)}{e^{\frac{x}{2L}}} \\
&= -\frac{x k_{nf}\frac{a}{2\nu_f L}(e^{\frac{x}{L}})^{\frac{1}{2}} \theta'(0)}{k_f}
\end{aligned} \tag{4.58}$$

$$
\begin{aligned}
&= -\frac{xk_{nf}\frac{xae^{\frac{x}{L}}}{2x\nu_f L}\theta'(0)}{k_f} \\
&= -\frac{k_{nf}}{k_f}\left(\sqrt{\frac{x}{2L}}\sqrt{\frac{xU_\infty}{\nu_f}}\right) \\
&= -\sqrt{\frac{x}{2L}}\frac{k_{nf}}{k_f}\sqrt{Re_x}\theta'(0) \because Re_x = \frac{xU_\infty}{\nu_f} \\
\Rightarrow & \sqrt{\frac{2L}{x}}\sqrt{\frac{1}{Re_x}}Nu_x = -\frac{k_{nf}}{k_f}\theta'(0),
\end{aligned}
$$

The local sherwood number is defined as:

$$
Sh_x = \frac{xq_m}{D(C_w - C_\infty)} \tag{4.59}
$$

$$
\bullet\ q_m = -D\left(\frac{\partial C}{\partial y}\right)\bigg|y{=}0
$$

$$
= -DC_0\sqrt{\frac{a}{2L\nu_f}}e^{\frac{x}{L}}h'(0) \tag{4.60}
$$

Using (4.60) in (4.59) we get:

$$
\begin{aligned}
Sh_x &= \frac{-xDC_0\sqrt{\frac{a}{2L\nu_f}}e^{\frac{x}{L}}h'(0)}{D(C_w - C_\infty)} \\
&= \frac{-xC_0\sqrt{\frac{a}{2L\nu_f}}e^{\frac{x}{L}}h'(0)}{(C_w - C_\infty)} \\
&= \frac{-xC_0\sqrt{\frac{a}{2L\nu_f}}e^{\frac{x}{L}}h'(0)}{(C_w - C_w + C_0e^{\frac{x}{2L}})} \quad \therefore C_\infty = C_w - C_0e^{\frac{x}{2L}} \\
&= \frac{x\sqrt{\frac{a}{2L\nu_f}}e^{\frac{x}{L}}h'(0)}{-e^{\frac{x}{2L}}} \\
&= \frac{x\sqrt{\frac{a}{2L\nu_f}}e^{\frac{x}{L}}h'(0)}{(e^{\frac{x}{L}})^{\frac{1}{2}}} \\
&= x\sqrt{\frac{a}{2L\nu_f}}e^{\frac{x}{L}}h'(0)(e^{\frac{x}{L}})^{\frac{-1}{2}} \\
&= x\sqrt{\frac{a}{2L\nu_f}}h'(0)(e^{\frac{x}{L}})^{\frac{1}{2}} \\
&= x\sqrt{\frac{ae^{\frac{x}{L}}}{2L\nu_f}}h'(0)
\end{aligned}
$$

$$= -x\sqrt{\frac{xU_\infty}{x2L\nu_f}}h'(0)$$
$$= -\left(\sqrt{\frac{xU_\infty}{\nu_f}}\right)\left(x\sqrt{\frac{1}{x2L}}h'(0)\right)$$
$$= -\sqrt{Re_x}\sqrt{\frac{x}{2L}}h'(0)$$
$$\Rightarrow Sh_x\sqrt{\frac{1}{Re_x}}\sqrt{\frac{2L}{x}} = -h'(0),$$

Where τ_w is the skin fraction q_w is the heat flux from the sheet. and Re_x,represent the local Reynolds numbers as $Re_x = \frac{xU_\infty}{v_f}$

4.3 Numerical solution

The shooting method requires the initial guess for $f_3(\eta)$,$f_5(\eta)$ and $f_7(\eta)$ at $\eta = 0$, and through Newton's method we vary each guess until we obtain an appropriate solution for our problem. To check accuracy of the obtained numerical results by Shooting method we compare them by the numerical results acquired by Matlab bvp4c built in function and found them in excellent agreement. The analytic solution of the system of equations with corresponding boundary conditions Eqs.(**??**) - (**??**) cannot be found because they are non linear and coupled. The governing partial differential equation are transformed into ordinary differential equation using similarity transformations. The numerical solution for the system of differential equation are developed using Runge-kutta scheme along with shooting technique. In order to solve the system of Ordinary differential equations with boundary conditions equations. Let us use the notation

$$f = y_1, \theta = y_4, h = y_6.$$

Further denote

$$f' = y_1' \text{ by } y_2, f'' = {y_2}' \text{ by } y_3, \theta' \text{ by } y_5, h' = {y_6}' \text{ by } y_7.$$

For simplification, the following notations have been opted.

$$
\begin{cases}
(1-\phi)^{2.5}(1-\phi+\phi\frac{\rho_s}{\rho_f}) = b_1, \\
P + (1-\phi)^{2.5}M = b_2, \\
\frac{P_r\left(1-\phi+\phi\left(\frac{\phi(\rho cp)_s}{(\rho cp)_f}\right)\right)}{\left(\frac{k_{nf}}{k_f}+\mathrm{R}\right)} = b_3, \\
\frac{(1-\phi)^{2.5}}{\mathrm{P}_r\mathrm{Ec}}\left(\frac{k_{nf}}{k_f}+R\right) = b_4, \\
MP_rE_c = b_5.
\end{cases}
$$

The system of equations 4.41)-(4.43), can now be written in the form of following first ODEs

$$
\begin{aligned}
y_1' &= y_2, & y_1(0) &= 0, \\
y_2' &= y_3, & y_2(0) &= 0, \\
y_3' &= -b_1(y_1y_3 + 2(1-y_2^2)) - b_2(1-y_2), & y_3(0) &= s, \\
y_4' &= y_5, & y_4(0) &= 1, \\
y_5' &= b_3(y_2y_4 - y_1y_5) - b_4y_3^2 + b_5y_2^2, & y_5(0) &= t, \\
y_6' &= y_7, & y_6(0) &= 1, \\
y_7' &= Sc(\gamma y_6 + y_2y_6 - y_1y_7), & y_7(0) &= w.
\end{aligned}
$$

In the above system of equations the missing conditions s, t and w are to be chosen such that

$y_3(\eta_\infty, s, t, w) = 0, \quad y_5(\eta_\infty, s, t, w) = 0, \quad y_7(\eta_\infty, s, t, w) = 0.$

To solve the system of algebraic equations we use the Newton,s method which has the following iterative scheme:

$$\begin{pmatrix} s^{(k+1)} \\ t^{(k+1)} \\ w^{(k+1)} \end{pmatrix} = \begin{pmatrix} s^{(k)} \\ t^{(k)} \\ w^{(k)} \end{pmatrix} - \begin{pmatrix} \frac{\partial y_2}{\partial s} & \frac{\partial y_2}{\partial t} & \frac{\partial y_2}{\partial w} \\ \frac{\partial y_4}{\partial s} & \frac{\partial y_4}{\partial t} & \frac{\partial y_4}{\partial w} \\ \frac{\partial y_6}{\partial s} & \frac{\partial y_6}{\partial t} & \frac{\partial y_6}{\partial w} \end{pmatrix}^{-1}_{(s^{(k)},t^{(k)},w^{(k)})} \begin{pmatrix} y_2^{(k)} \\ y_4(k) \\ y_6^{(k)} \end{pmatrix}_{(s^{(k)},t^{(k)},w^{(k)})}.$$

Now use the following notations:

$$\begin{aligned} \frac{\partial y_1}{\partial s} &= y_8, \frac{\partial y_2}{\partial s} = y_9, ..., \frac{\partial y_7}{\partial s} = y_{14}, \\ \frac{\partial y_1}{\partial t} &= y_{15}, \frac{\partial y_2}{\partial t} = y_{16}, ..., \frac{\partial y_7}{\partial t} = y_{21}, \\ \frac{\partial y_1}{\partial w} &= y_{22}, \frac{\partial y_2}{\partial w} = y_{23}, ..., \frac{\partial y_7}{\partial w} = y_{28}. \end{aligned}$$

As a result of these these new notations, the Newton's iterative scheme gets the form:

$$\begin{pmatrix} s^{(k+1)} \\ t^{(k+1)} \\ w^{(k+1)} \end{pmatrix} = \begin{pmatrix} s^{(k)} \\ t^{(k)} \\ w^{(k)} \end{pmatrix} - \begin{pmatrix} y_9 & y_{16} & y_{23} \\ y_{11} & y_{18} & y_{25} \\ y_{13} & y_{20} & y_{27} \end{pmatrix}^{-1}_{(s^{(k)},t^{(k)},w^{(k)})} \begin{pmatrix} s^{(k)} \\ t^{(k)} \\ w^{(k)} \end{pmatrix}_{(s^{(k)},t^{(k)},w^{(k)})}.$$

Now differentiate the above system of seven first order ODEs with respect to each of the variables s, t and w to have another system of twenty one ODEs. Writing all these twenty eight ODEs together, we have the the following IVP:

$$\begin{aligned}
y_1' &= y_2, & y_1(0) &= 0,\\
y_2' &= y_3, & y_2(0) &= 0,\\
y_3' &= -b_1(y_1y_3 + 2(1-y_2^2)) - b_2(1-y_2), & y_3(0) &= s,\\
y_4' &= y_5, & y_4(0) &= 1,\\
y_5' &= b_3(y_2y_4 - y_1y_5) - b_4y_3^2 + b_5{y_2}^2, & y_5(0) &= t,\\
y_6' &= y_7, & y_6(0) &= 1,\\
y_7' &= Sc(\gamma y_6 + y_2y_6 - y_1y_7), & y_7(0) &= w,\\
y_8' &= y_9, & y_8(0) &= 0,\\
y_9' &= y_{10}, & y_9(0) &= 0,\\
y_{10}' &= -b_1(y_1y_{10} + y_8y_3 - 4(y_2y_9)) + b_2y_9) & y_{10}(0) &= 1,\\
y_{11}' &= y_{12}, & y_{11}(0) &= 0,\\
y_{12}' &= b_3(y_2y_{11} + y_9y_4 - y_1y_{12} - y_8y_5) - 2b_4y_3y_{10} + 2b_5y_2y_9, & y_{12}(0) &= 0,\\
y_{13}' &= y_{14}, & y_{13}(0) &= 0,\\
y_{14}' &= Sc(\gamma y_{13} + y_2y_{13} + y_6y_9 - y_{21}y_7 - y_{14}), & y_{14}(0) &= 0,\\
y_{15}' &= y_{16}, & y_{15}(0) &= 0,
\end{aligned}$$

$$
\begin{aligned}
&y'_{16} = y_{17}, && y_{16}(0) = 0,\\
&y'_{17} = -b_1(y_1y_{17} + y_{15}y_3 - 4(y_2y_{16})) + b_2y_{16}, && y_{17}(0) = 0,\\
&y'_{18} = y_{19}, && y_{18}(0) = 0,\\
&y'_{19} = b_3(y_2y_{18} + y_{16}y_4 - y_1y_{19} - y_{15}y_5) - 2b_4y_3y_{17} + 2b_5y_2y_{16}, && y_{19}(0) = 1,\\
&y'_{20} = y_{21}, && y_{20}(0) = 0,\\
&y'_{21} = Sc(\gamma y_{20} + y_2y_{20} + y_6y_{16} - y_1y_{21} - y_7y_{15}), && y_{21}(0) = 0,\\
&y'_{22} = y_{23}, && y_{22}(0) = 0,\\
&y'_{23} = y_{24}, && y_{23}(0) = 0,\\
&y'_{24} = -b_1(+y_1y_{24} + y_{22}y_3 - 4y_2y_{23}) + b_2y_{23}, && y_{24}(0) = 0,\\
&y'_{25} = y_{26}, && y_{25}(0) = 0,\\
&y'_{26} = b_3(y_2y_{25} + y_{23}y_4 - y_1y_{26} - y_{22}y_5) - 2b_4y_3y_{24} + 2b_5y_2y_{23}, && y_{26}(0) = 0,\\
&y'_{27} = y'_{27}, && y_{27}(0) = 0,\\
&y'_{28} = Sc(\gamma y_{27} + y_2y_{27} + y_{23}y_6 - y_1y_{28} - y_{22}y_7), && y_{28}(0) = 1.
\end{aligned}
$$

The fourth order Runge-Kutta method is used to solve the above system of twenty eight equations with initial guesses s, t, w. These guesses are updated by the Newton's scheme (3.51). The iterative process is repeated until the following criteria is met:

$$max|y_2(\eta_\infty - 1)|, max|y_4(\eta_\infty - 0)|, max|y_6(\eta_\infty - 0)| < \epsilon,$$

where $\epsilon > 0$ is the tolerance. For all the calculations in this chapter, we have set $\epsilon = 10^{-6}$.

TABLE 4.1: Physical properties of water and nano particles

F*luid*	$\rho(kg/m^3)$	$c_p(J/k_gK)$	$K(W/mk)$	$\beta \times 10^5(K^{-1})$
Pure water	997.1	4179	0.61	21
Copper	8933	385	401	1.67
Alumina	3970	765	40	0.85

TABLE 4.2: Numerical results of skin-friction coefficient $f''(0)$, Nusselt number $-\theta'(0$ and Sherwood number $-h'(0)$ for (Cu-water)

Parameters						Shooting method			Matlabbvp4c		
ϕ	P	M	R	E_c	γ	f$''$(0)	-θ'(0)	-h'(0)	f$''$(0)	-θ'(0)	-h'(0)
0.0	0.1	1	0.1	0.1	0.1	2.0136	1.5437	0.7409	2.0136	1.5437	0.7409
0.1						3.6861	1.7001	0.7934	3.4335	1.7001	0.7934
0.2						5.4265	1.7363	0.8066	4.6418	1.7363	0.8066
0.1	0.2					3.7351	1.7076	0.7972	3.7351	1.7076	0.7972
	0.3					3.7840	1.7149	0.8008	4.3001	8.4017	1.0444
	0.4					3.0693	1.0181	0.8432	3.0693	1.0181	0.8432
	0.9					3.3736	1.0483	0.8679	3.3736	1.0483	0.8679
		0				3.2472	1.0483	0.8682	3.2472	1.0483	0.8682
		2				3.6115	1.0586	0.8767	3.6115	1.0586	0.8767
		3				3.7822	1.0632	0.8805	3.7822	1.0632	0.8805
		5				4.1044	1.0716	0.8875	4.1044	1.0716	0.8875
			0.2			3.4335	1.0924	0.9047	3.4335	1.0924	0.9047
			0.3			3.4335	1.0924	0.9360	3.4335	1.1302	0.9360
			0.5			3.4335	1.0924	0.9966	3.4335	1.2033	0.9966
				0.3		2.2991	6.9198	0.8601	2.2991	6.9198	0.8601
				-0.3		2.2989	6.9195	0.8601	2.2989	6.9195	0.8601
						3.4335	1.0536	0.9966	3.4335	1.2033	0.9966
					0.2	3.4335	1.0536	0.8726	3.4335	1.0536	0.8726
					0.3	3.4335	1.0536	0.8726	3.4335	1.0536	0.8726

In table 4.1, shows the physical properties of the fluid water and nanoparticles. In table 4.2, the numerical analysis of various physical parameters C_{fx},Nu_x and Sh_x under discussion is displayed. Table 4.2, it shows that by increasing the values of ϕ skin friction, Nusselt number and Sherwood number is also increasing. By increasing the permeability parameter P it also increase the values $f''(0)$, $\theta'(0)$ and $h'(0)$. If In this table 4.2 demonstrate that by increasing the magnetic parameter M the values of skin friction, Nusselt number and Sherwood number is also increasing. By increasing radiation parameter R the nusselt number decrease and the Sherwood number increase. Due to increase in chemical reaction parameter γ skin friction, the Nusselt number have no effect and only Sherwood number is decreased. By increasing Eckert number Ec then the values of skin friction, Nusselt number are same and Sherwood number values decrease.

4.4 Graphical representation

The numerical results are displayed in the form of graphs and tables. In this section for numerical calculation physical properties of water, copper and alumina are considered. The computations for various values is the volume fraction ϕ, the Prandtl number P_r, chemical reaction γ, permeability parameter P, Magnetic parameter M, and hence the effect of these parameters on heat and mass transfer are discussed.Figures 4.2 and 4.3 demonstrate the impact of the volume fraction and the magnetic parameter M on the velocity . In these figures, we observe that the velocity increases with an increase in the volume fraction of nanoparticles. These figures show that the hydrodynamic boundary layer of Al_2O_3- water is thick as compared with that of Cu-water.

Figures 4.4 and **??** show the effect of Eckert number on temperature. If we increase the value of Eckert number temperature profile increase Hence the thickness of the thermal boundary layer increases. Figures 4.6 and 4.7 show the impact of the volume fraction ϕ together with the magnetic parameter M on the concentration profile h. A decreasing behaviour is found in the dimensionless concentration h for both Cu-water and Al_2O_3 -water. In these figure, one can see that the concentration distribution decreases if there is an increase in the volume fraction ϕ.

Figures 4.8 and 4.9 show the impact of the permeability parameter P together with ϕ on the dimensionless velocity for both Cu-water and Al_2O_3-water. Fluid velocity increases with the increasing values of both the permeability parameter P and ϕ . Figure 4.10 and 4.11 show the impact of magnetic parameter on temperature. If the increase the value of magnetic parameter temperature profile increase. These figures show that the hydrodynamic boundary layer of Al_2O_3- water is thick as compared with that of Cu-water.

Figures 4.12 and 4.13 show the effect of ϕ on the dimensionless temperature θ of the water based fluid with or without radiation. In these figures, temperature is increased by increasing the values of the thermal radiation radiation increased. It happens because the thermal radiation increases the thermal diffusion.

Figures 3.14 and 3.15 show the impact of the viscous dissipation together with ϕ on the temperature profile. When the value of the viscous dissipation is increased, the fluid region is allowed to store the energy. As a result of dissipation due to fractional heating, heat is generated. From this figure, we examine that the value of the thermal boundary thickness increases with increasing values of ϕ and it will eventually increase the temperature. Figures 4.16 and 4.17 show the impact of the chemical reaction together with ϕ on the dimensionless concentration. In these figures, it also observed that when the chemical reaction increases, the concentration profiles decreases and the increasing values of the volume fraction have small impact on the dimensionless concentration. It is clear that in these figures velocities $f'(\eta)$ and temperature $\theta^{(}\eta)$ posses the same increasing behaviour for M and the concentration $h(\eta)$ shows the decreasing behaviour.

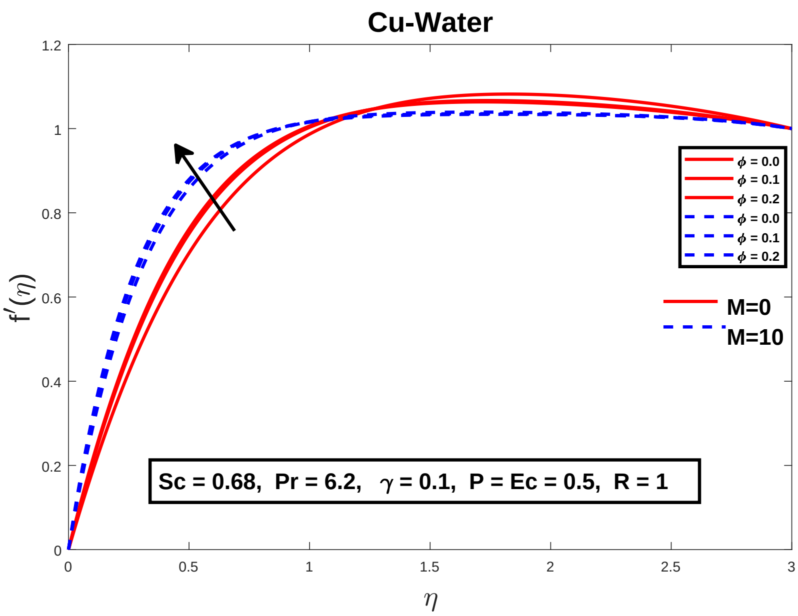

FIGURE 4.2: Impact of ϕ and M on the dimensionless velocity f' for $Cu - water$

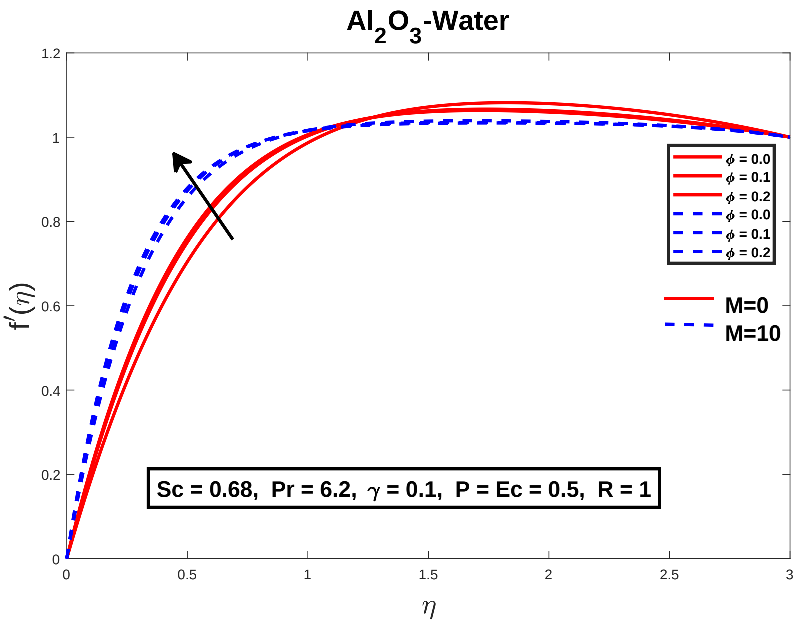

FIGURE 4.3: Impact of ϕ and M on the dimensionless temperature θ for Al_2O_3-water

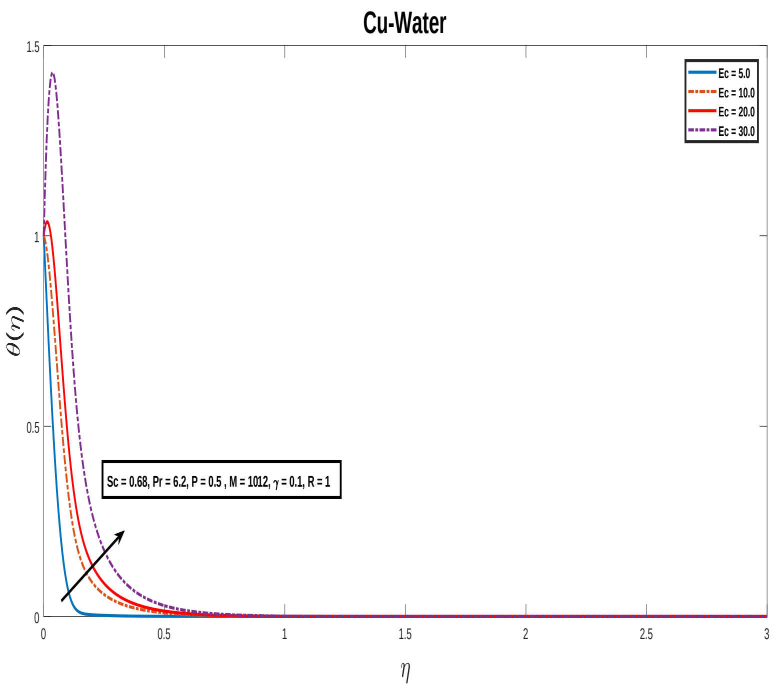

FIGURE 4.4: Impact of Ec on the dimensionless temperature θ for Cu-water

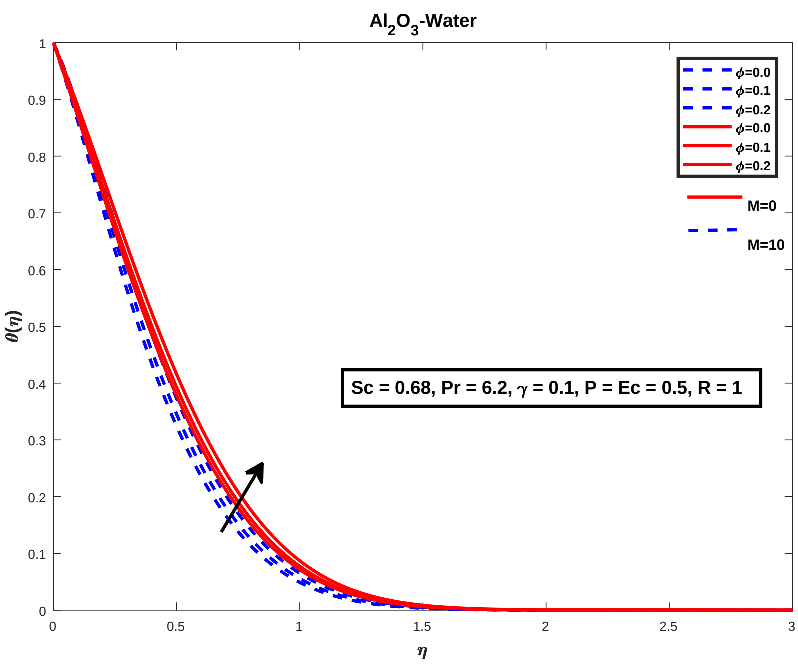

FIGURE 4.5: Influence of ϕ on the dimensionless temperature θ for Al_2O_3

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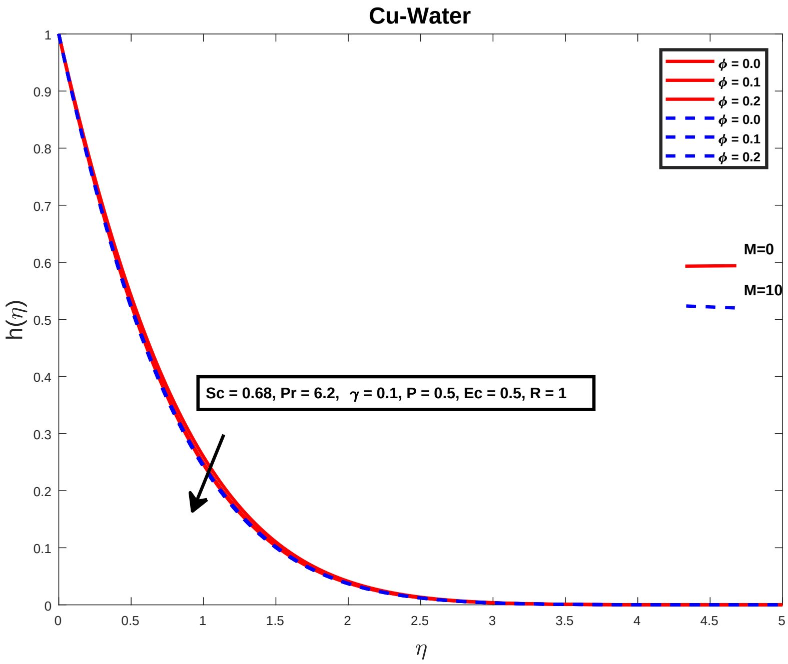

FIGURE 4.6: Impact of ϕ and M on the dimensionless concentration h for Cu-water

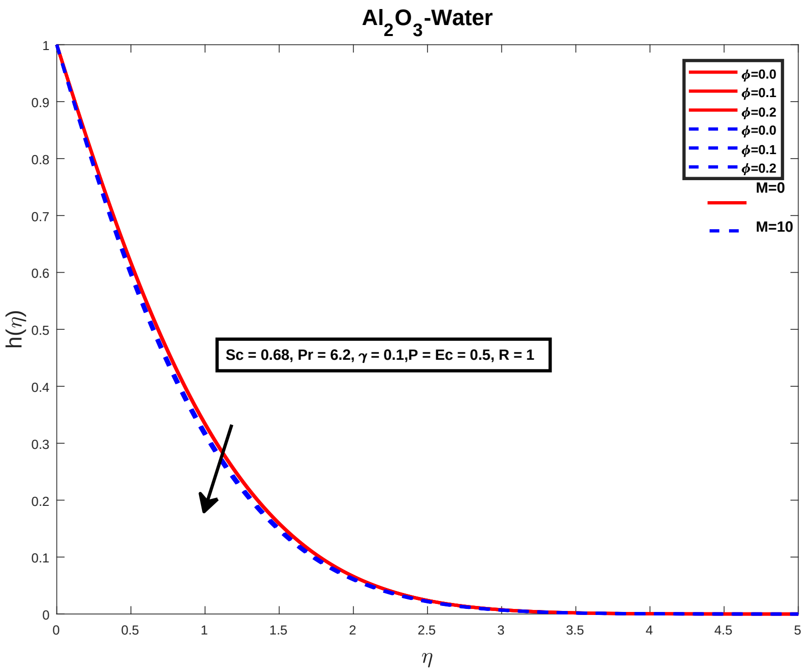

FIGURE 4.7: Impact of ϕ and M on the dimensionless temperature θ for $Al_2O_3-water$

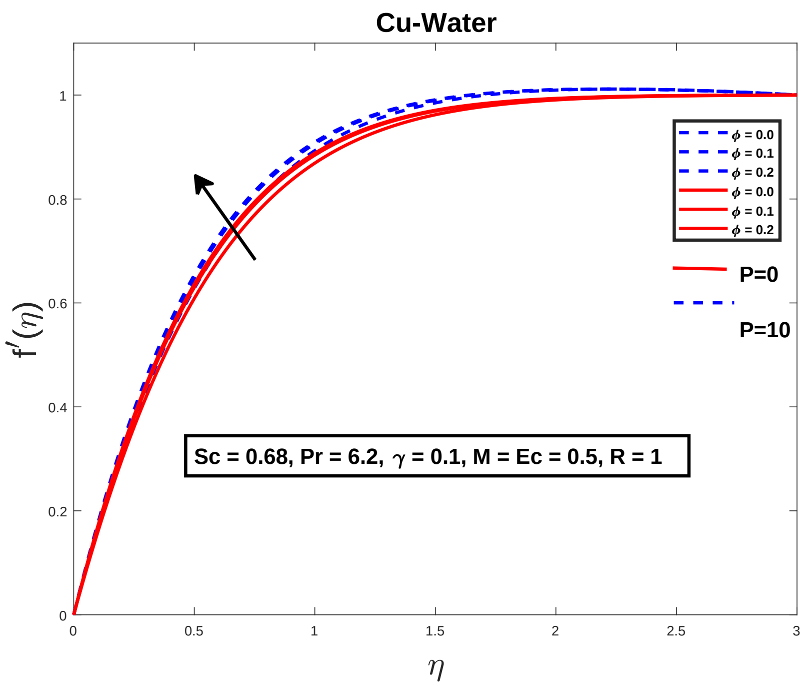

FIGURE 4.8: Impact of ϕ and P on the dimensionless velocity f' for Cu-water

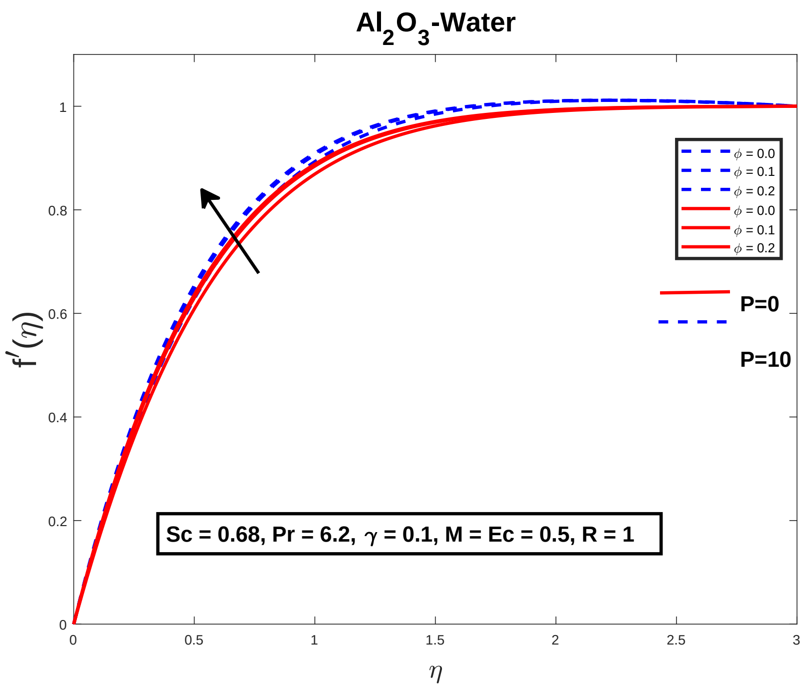

FIGURE 4.9: Impact of ϕ and P on the dimensionless velocity f' for Al_2O_3-water

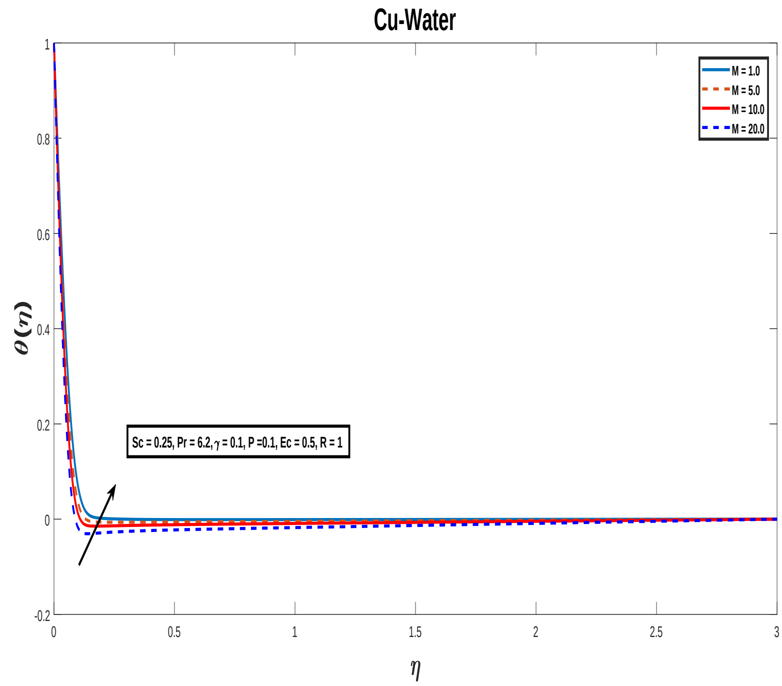

FIGURE 4.10: Impact of M on the dimensionless temperature θ for Cu-water

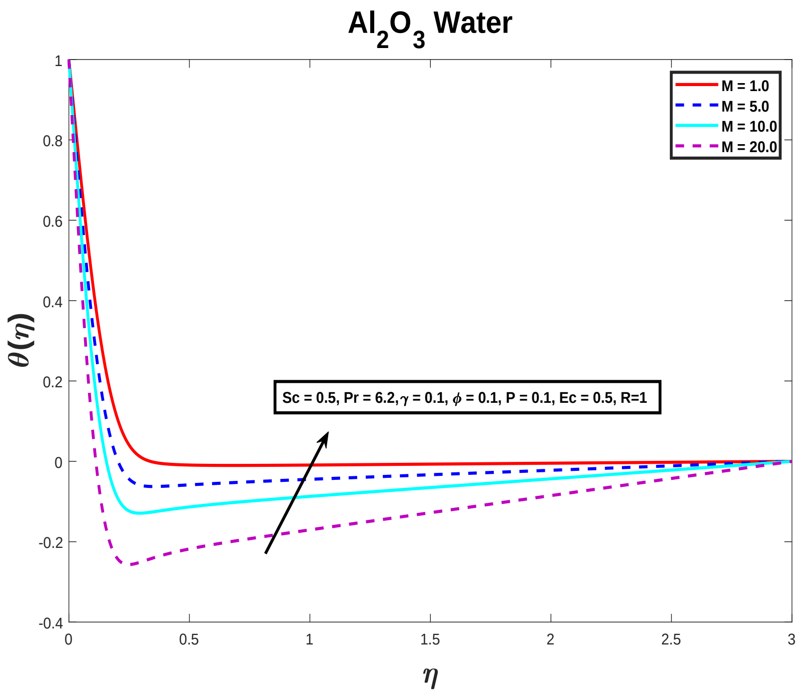

FIGURE 4.11: Impact of M on the dimensionless temperature θ for Al_20_3-water

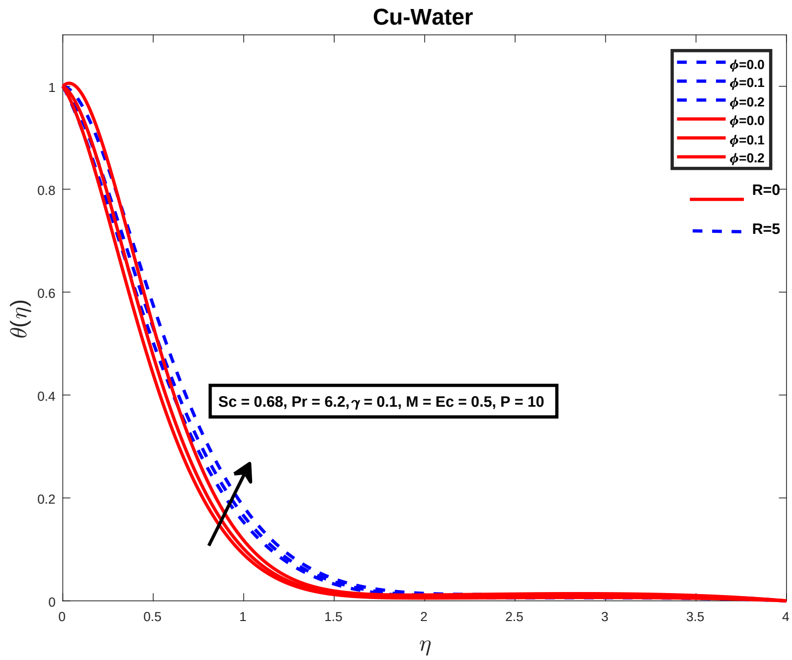

FIGURE 4.12: Impact of ϕ and R on the dimensionless temperature θ for Cu-water

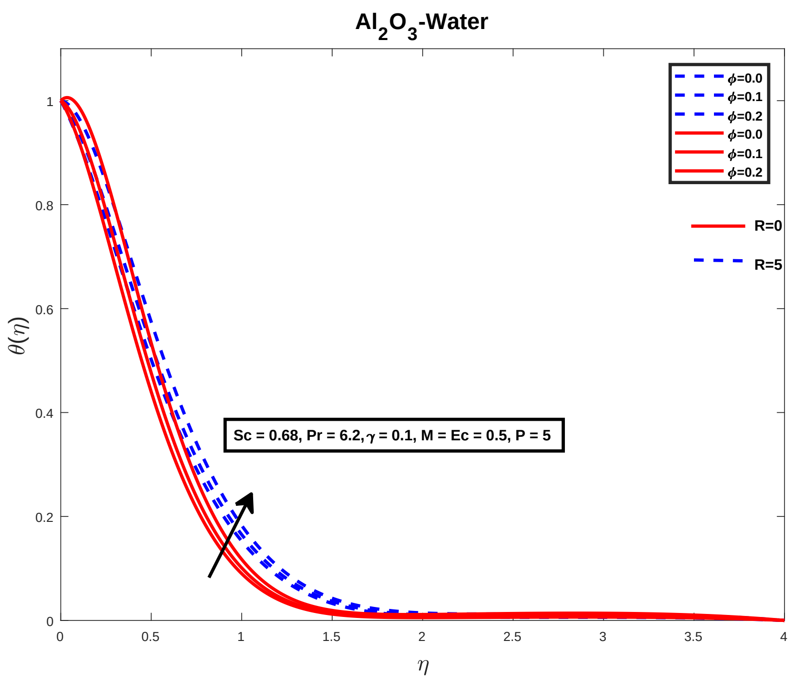

FIGURE 4.13: Impact of ϕ and R on the dimensionless temperature θ for Al_2O_3-water

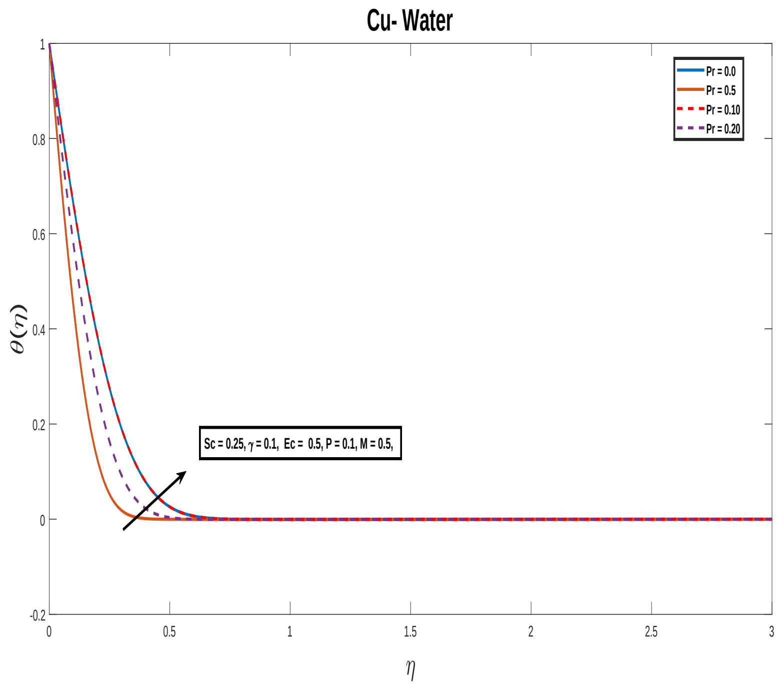

FIGURE 4.14: Impact of ϕ and Pr on the dimensionless temperature θ for Cu-water

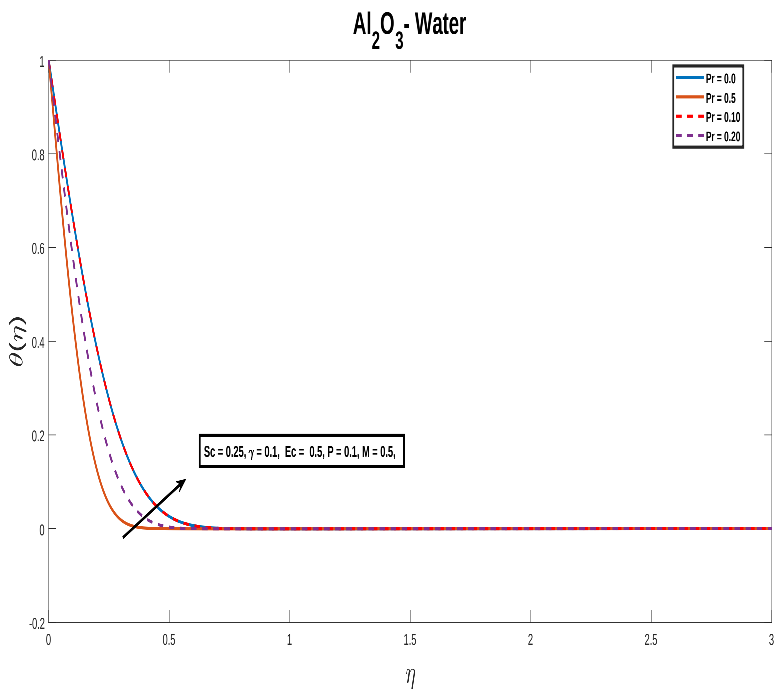

FIGURE 4.15: Impact of ϕ and Pr on the dimensionless temperature θ for Al_2O_3-water

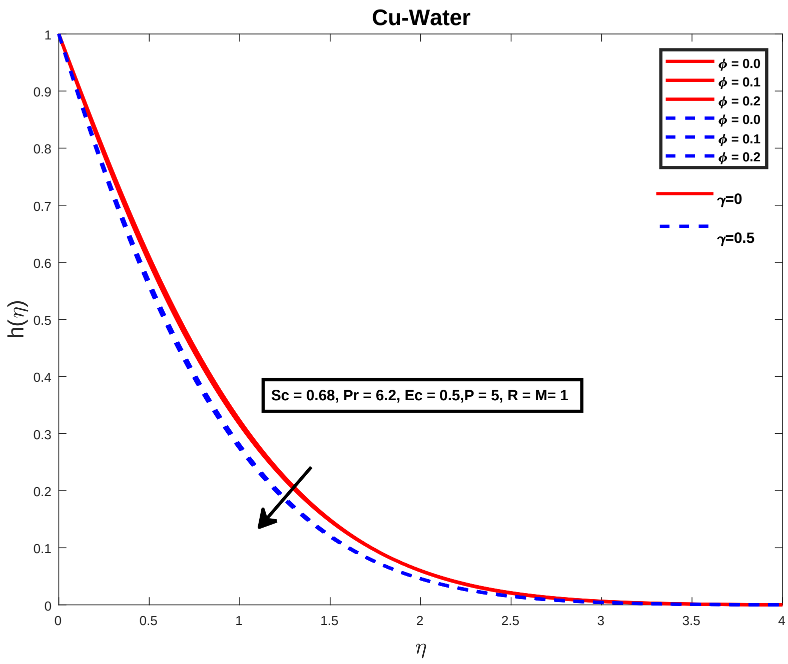

FIGURE 4.16: Impact of ϕ and γ on the dimensionless concentration h for Cu-water

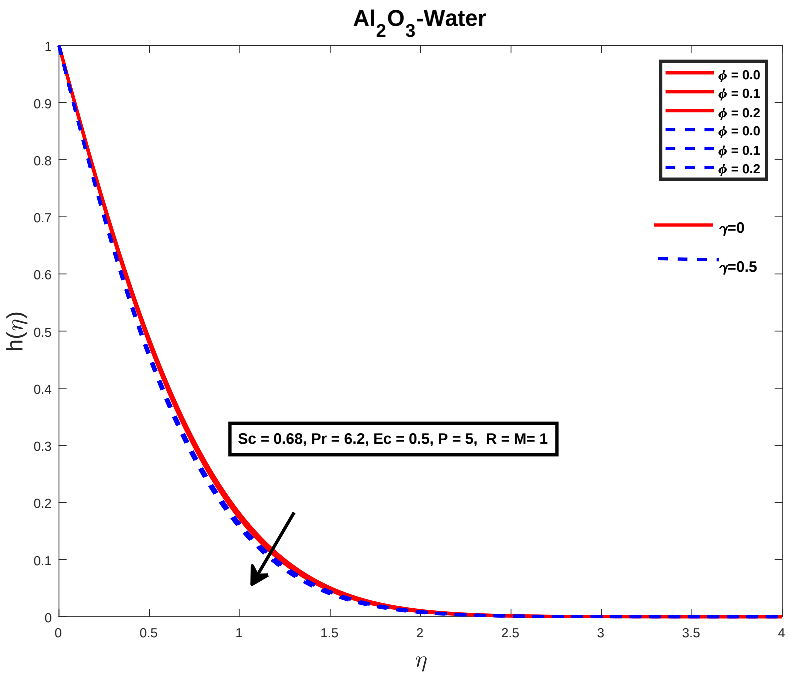

FIGURE 4.17: Impact of ϕ and γ on the dimensionless concentration h for Al_2O_3-water

Chapter 5

Conclusion

In this thesis, we have presented the numerical analysis of the MHD flow of the water-based nanofluids. Heat and mass transfer are analyzed for steady, viscous dissipations and Joule heating past a porous medium. We studied the MHD stagnation point flow of viscous, incompressible and two-dimensional fluid past a permeable flat plate with variable thermal conductivity κ and fluid viscosity μ in a uniform porous medium. Further, we talk about the effect of radiation, chemical reaction, viscous dissipation and Joule heating effect. The governing nonlinear partial differential equations of momentum, energy and mass transfer are changed into the ODEs by utilizing a proper similarity transformation. By using the shooting method, numerical solution of these modeled ordinary differential equations is obtained. A numerical comparison of the solution computed by the shooting with that computed the MATLAB built-in function bvp4c for different parameters has shown an excellent agreement . Distinctive physical parameters are examined, w.r.t The dimensionless velocity, temperature and concentration profile are examined both graphically and in the tabular form for different values of different parameters involved. On the basis of the analysis of solution, we conclude the following findings.

- An increment in the velocity profile is observed for an increment in volume fraction ϕ.
- The temperature profile θ increases with an increase in the volume fraction

ϕ.

• The concentration profile h decreases with an increment in the volume fraction ϕ.

• The skin friction is found to increase by increasing the values of the of permeability parameter P for both Cu and Al_2O_3 nanofluids.

• Looking the effect of volume fraction and the chemical reaction, the mass transfer rate is observed to
rise.

• It also observed that the chemical reaction parameter γ increases with a decrease in concentration profile.

• The temperature field θ reduces due to the a solid volume fraction ϕ.

• For increasing values of the permeability parameter Pr skin friction coefficient C_{fx} shows an increasing behaviour.

• The temperature field θ increase with water based copper and Aluminia due to increase the Ec and R .

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