CAPITAL UNIVERSITY OF SCIENCE AND TECHNOLOGY, ISLAMABAD

# Non-Linear Control Techniques for Stabilization of Nonholonomic Systems

by

Naveed Hussain

A thesis submitted in partial fulfillment for the degree of Master of Science

in the

Faculty of Engineering
Department of Electrical Engineering

2022

Copyright © 2022 by Naveed Hussain

All rights reserved. No part of this thesis may be reproduced, distributed, or transmitted in any form or by any means, including photocopying, recording, or other electronic or mechanical methods, by any information storage and retrieval system without the prior written permission of the author.

*Dedicated to my teachers who enlightened my soul and directed me towards the right path. Dedicated to my parents who always prayed for me and my friends who have encouraged and supported me in every situation.*

# CERTIFICATE OF APPROVAL

## Non-Linear Control Techniques for Stabilization of Nonholonomic Systems

by

Naveed Hussain

(MEE181018)

### THESIS EXAMINING COMMITTEE

| S. No. | Examiner | Name | Organization |
|---|---|---|---|
| (a) | External Examiner | Dr. Suheel Abdullah Malik | IIU, Islamabad |
| (b) | Internal Examiner | Dr. Aamer Iqbal Bhatti | CUST, Islamabad |
| (c) | Supervisor | Dr. Fazal ur Rehman | CUST, Islamabad |

Dr. Fazal ur Rehman
Thesis Supervisor
September, 2022

Dr. Noor Muhammad Khan
Head
Dept. of Electrical Engineering
September, 2022

Dr. Imtiaz Ahmed Taj
Dean
Faculty of Engineering
September, 2022

# *Author's Declaration*

I, **Naveed Hussain** hereby state that my MS thesis titled "**Non-Linear Control Techniques for Stabilization of Nonholonomic Systems**" is my own work and has not been submitted previously by me for taking any degree from Capital University of Science and Technology, Islamabad or anywhere else in the country/abroad.

At any time if my statement is found to be incorrect even after my graduation, the University has the right to withdraw my MS Degree.

**(Naveed Hussain)**

Registration No: MEE181018

# *Plagiarism Undertaking*

I solemnly declare that research work presented in this thesis titled "**Non-Linear Control Techniques for Stabilization of Nonholonomic Systems**" is solely my research work with no significant contribution from any other person. Small contribution/help wherever taken has been dully acknowledged and that complete thesis has been written by me.

I understand the zero tolerance policy of the HEC and Capital University of Science and Technology towards plagiarism. Therefore, I as an author of the above titled thesis declare that no portion of my thesis has been plagiarized and any material used as reference is properly referred/cited.

I undertake that if I am found guilty of any formal plagiarism in the above titled thesis even after award of MS Degree, the University reserves the right to withdraw/revoke my MS degree and that HEC and the University have the right to publish my name on the HEC/University website on which names of students are placed who submitted plagiarized work.

**(Naveed Hussain)**

Registration No: MEE181018

# *Acknowledgement*

First of all, I am thankful to Almighty Allah who blessed me with knowledge, strength, courage and patience during my studies. I am also very grateful to my supervisor Dr. Fazal ur rehman who monitored the progress of this thesis very closely and provided insights at every stage and corrected the direction where ever necessary.

I would like to pay huge debt of gratitude to my dearest family members: my father, my mother and my siblings who always supported me during thick and thin and encouraged me to stay motivated through out my degree program in order to reach my destination.

I would also like to express my thanks to everyone that helped me to achieve my goals. I also want to express my special appreciation to Dr. Noor Muhammad Khan and Dr.Imtiaz Ahmad Taj for increasing my knowledge and helping me to grasp the basic concepts.

I feel honor to express my feeling of appreciation for the support I have received from the members of Research Group of control system and especially Yaseen, Nadir Mehmood, Sidra Ghayour and Wadood Alam who helped me in my research work and documentation of the thesis

**(Naveed Hussain)**

# *Abstract*

Many real-world systems exhibit velocity-dependent and/or acceleration dependent constraints in their mathematical models. If the constraints are non-integrable then such systems are known as nonholonomic systems. Rolling Wheel, Rolling Sphere, unmanned aerial vehicles (UAVs), robots, underwater vehicles and vertical and landing systems are the examples of nonholonomic systems. Smooth static feedback controller cannot stabilize these systems, requires time-varying or discontinuous state-feedback control. In this research, we are considering higher-order nonholonomic systems that can be transformed into chained or power form which are canonical representations of these mechanical systems. The importance of stabilization problem of perturbed nonholonomic systems is further magnified by the variety of real-world day-to-day applications.

This research presents the solution to the stabilization problems for a selected class of perturbed higher-order nonholonomic mechanical systems. The methodologies are based on backstepping control and adaptive sliding mode control (ASMC). For the perturbed nonholonomic system, the original system is transformed into perturbed chained form. The stabilizing controller for the transformed system is constructed. The compensate controller and the adaptive laws are derived in such a way that derivative of a Lyapunov function becomes strictly negative. The validity of the proposed controllers is verified by simulation higher-order nonholonomic systems in MATLAB.

# Contents

**Author's Declaration** **iv**

**Plagiarism Undertaking** **v**

**Acknowledgement** **vi**

**Abstract** **vii**

**List of Figures** **x**

**Abbreviations** **xii**

**Symbols** **xiii**

**1 Introduction** **1**
1.1 Background . . . . . . . . . . 1
1.2 Motivation . . . . . . . . . . 3
1.3 Research Objectives . . . . . . . . . . 4
1.4 Outline of the Thesis . . . . . . . . . . 4

**2 Literature Survey** **6**
2.1 Control Problem of Nonholonomic Systems . . . . . . . . . . 6
2.2 Control Strategies for Nonholonomic Systems . . . . . . . . . . 8
2.2.1 Time-Varying Continuous Controls . . . . . . . . . . 8
2.2.2 Discontinuous Feedback Control . . . . . . . . . . 9
2.2.3 Hybrid Control . . . . . . . . . . 10
2.3 Types / Orders of Nonholonomic Systems . . . . . . . . . . 10
2.3.1 First-Order Nonholonomic Systems . . . . . . . . . . 11
2.3.1.1 Four Wheel Car . . . . . . . . . . 11
2.3.2 Second-Order Nonholonomic Systems . . . . . . . . . . 11
2.3.2.1 Underactuated Surface Vessel . . . . . . . . . . 12
2.3.3 Third-Order Nonholonomic Systems . . . . . . . . . . 13
2.3.3.1 PPR Manipulator . . . . . . . . . . 14
2.4 Sliding Mode Control . . . . . . . . . . 15
2.4.1 Sliding Phase . . . . . . . . . . 15

2.4.2 Chattering Phenomenon . . . 16
2.5 Summary . . . 17

**3 Nonholonomic Systems in Chained Form 18**
3.1 Second-Order Nonholonomic Systems . . . 18
3.2 Second-Order Chained Form Systems . . . 19
3.3 The Stabilization Problem . . . 20
3.4 3-DOF Manipulator with a Free Link . . . 21
3.4.1 Conversion into Second-Order Chained Form . . . 22
3.5 Planar PPR Manipulator . . . 23
3.5.1 Conversion into Second-Order Chained Form . . . 24

**4 Backstepping Control Technique 27**
4.1 Problem Statement . . . 27
4.2 Backstepping . . . 27
4.2.1 Simulation Results and Discussion . . . 33
4.2.1.1 Planar PPR Manipulator . . . 33
4.2.1.2 Simulation 1: . . . 33
4.2.1.3 Simulation 2: . . . 35
4.2.1.4 Simulation 3: . . . 37

**5 Adaptive Sliding Mode Control Technique 40**
5.1 Introduction . . . 40
5.1.1 Proposed Algorithm 1 . . . 40
5.1.2 Simulation Results and Discussion . . . 44
5.1.2.1 3-DOF Manipulator with a Free Link . . . 44
5.1.2.2 Simulation 1: . . . 44
5.1.2.3 Simulation 2 . . . 47
5.1.2.4 Simulation 3 . . . 50
5.1.3 Proposed Algorithm 2: With Disturbance . . . 53
5.1.4 Simulation Results and Discussion . . . 56
5.1.4.1 3-DOF Manipulator with a Free Link . . . 56
5.1.4.2 Simulation 1: . . . 56

**6 Conclusion and Future Work 61**
6.1 Conclusion . . . 61
6.2 Future Research Directions . . . 62

**Bibliography 63**

# List of Figures


2.1 Kinematic Model of a Four-Wheel Car . . . . . . . . . . . . . . . . . . 12
2.2 Underactuated Surface Vessel . . . . . . . . . . . . . . . . . . . . . . 13
2.3 Planar PPR Manipulator . . . . . . . . . . . . . . . . . . . . . . . . . 15
2.4 Sliding Phase, Reaching Phase and Sliding Surface . . . . . . . . . 16
2.5 Chattering of control input along sliding surface . . . . . . . . . . . 17

3.1 (a) 3-DOF Manipulator with a Free Joint. (b) Model of Passive Link. 21
3.2 (a) Planar PPR Manipulator. . . . . . . . . . . . . . . . . . . . . . . 23

4.1 Backstepping control method . . . . . . . . . . . . . . . . . . . . . . 32
4.2 Stabilization of Planar PPR Manipulator, (a) Positions, (b) Velocities . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 33
4.3 Stabilization of Planar PPR Manipulator, (a) Control input $u_1 = v_1$, (b) Control input $u_2 = v_2$ . . . . . . . . . . . . . . . . . . . . . . 34
4.4 Stabilization of Planar PPR Manipulator, (a) Positions, (b) Velocities 35
4.5 Stabilization of Planar PPR Manipulator, (a) Control input $u_1 = v_1$, (b) Control input $u_2 = v_2$ . . . . . . . . . . . . . . . . . . . . . . 36
4.6 Stabilization of Planar PPR Manipulator, (a) Positions, (b) Velocities 37
4.7 Stabilization of Planar PPR Manipulator, (a) Control input $u_1 = v_1$, (b) Control input $u_2 = v_2$ . . . . . . . . . . . . . . . . . . . . . . 38

5.1 Backstepping control method . . . . . . . . . . . . . . . . . . . . . . 43
5.2 Stabilization of 3-DOF Manipulator with a Free Link, (a) Positions, (b) Velocities . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 44
5.3 Stabilization of 3-DOF Manipulator with a Free Link, (a) Sliding surface $s_1$, (b) Sliding surface $s_2$ . . . . . . . . . . . . . . . . . . . . 45
5.4 Stabilization of 3-DOF Manipulator with a Free Link, (a) Control input $u_1 = v_1$, (b) Control input $u_2 = v_2$ . . . . . . . . . . . . . . . 46
5.5 Stabilization of 3-DOF Manipulator with a Free Link, (a) Positions, (b) Velocities . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 47
5.6 Stabilization of 3-DOF Manipulator with a Free Link, (a) Sliding surface $s_1$, (b) Sliding surface $s_2$ . . . . . . . . . . . . . . . . . . . . 48
5.7 Stabilization of 3-DOF Manipulator with a Free Link, (a) Control input $u_1 = v_1$, (b) Control input $u_2 = v_2$ . . . . . . . . . . . . . . . 49
5.8 Stabilization of 3-DOF Manipulator with a Free Link, (a) Positions, (b) Velocities . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 50
5.9 Stabilization of 3-DOF Manipulator with a Free Link, (a) Sliding surface $s_1$, (b) Sliding surface $s_2$ . . . . . . . . . . . . . . . . . . . . 51

5.10 Stabilization of 3-DOF Manipulator with a Free Link, (a) Control input $u_1 = v_1$, (b) Control input $u_2 = v_2$ . . . . . . . . . . . . . . . . 52
5.11 Stabilization of 3-DOF Manipulator with a Free Link, (a) Positions, (b) Velocities . . . . . . . . . . . . . . . . . . . . . . . . . . . . 56
5.12 Stabilization of 3-DOF Manipulator with a Free, (a) Sliding surface $s_1$, (b) Sliding surface $s_2$, (b) Sliding surface $s_3$ . . . . . . . . . . . 57
5.13 Stabilization of 3-DOF Manipulator with a Free Link, (a),(b,(c) Represents $d_1$,$d_3$,$d_5$ are the injected external disturbances in model. $\hat{d}_1$, $\hat{d}_1$, $\hat{d}_1$ are the estimations of disturbances. . . . . . . . . . . . . . 58
5.14 Stabilization of 3-DOF Manipulator with a Free Link, (a) Control input $u_1 = v_1$, (b) Control input $u_2 = v_2$ . . . . . . . . . . . . . . . . 59

# Abbreviations

| | |
|---|---|
| **AUV** | Autonomous Underwater Vehicles |
| **DC** | Direct Current |
| **DOF** | Degree of Freedom |
| **HOSMC** | Higher Order Sliding Mode Control |
| **IMU** | Inertial Measuring Unit |
| **ISMC** | Integral Sliding Mode Control |
| **PID** | Propotionsl Integral Derivative |
| **RP** | Reaching Phase |
| **SS** | Sliding Surface |
| **SM** | Sliding Manifold |
| **UAV** | Unmanned Aerial Vehicles |
| **UMS** | Underactuated Mechanical Systems |
| **VTOL** | Vertical Take-off and Landing |

# Symbols

| | |
|---|---|
| $\psi$ | orientation of vessel |
| $w_z$ | angular velocity |
| $p_1$ | actuated configuration variables |
| $G_q$ | gravity term |
| $C(q, \dot{q})$ | centrifugal and coriolis term |
| $D(q)$ | inertia term |
| $\omega$ | angular velocity |
| $u$ | control input |
| $\theta$ | angular displacement |

# Chapter 1

# Introduction

This chapter discusses a brief summary of Nonholonomic Systems. Furher, this chapter also answers why stabilizing the Perturbed and Higher-Order Nonholonomic Systems in a conical form is important. The introduction is accompanied by research motivation for stabilizing perturbed and higher-order nonholonomic systems in a canonical form under the heading of research motivation.Moreover this section is about the brief overview of this dissertation.

## 1.1 Background

The design of Robust Stabilizing Control techniques for Nonholonomic Systems has gained great significance over the past couple of decades. Mechanical systems can be classified into two type of systems based on the nature of constraints i.e. 1) Holonomic systems and 2) Nonholonomic Systems. The word ”Holonomic” is derived from two Greek words holos which means whole and nomos which means Law [1]. In mechanics, Holonomic systems are mechanical systems working under constraints, which limit the entire configuration of the mechanical system. The fundamental difference between a Holonomic and Nonholonomic constraint (usually provided in kind of velocity, acceleration or higher-order time derivative of acceleration) is that the Holonomic constraint is integrable, whereas the Nonholonomic constraint is non-integrable. A typical case of a Holonomic constraint is

the fixed length of a simple pendulum; whereas, the rolling ball and rolling disk without side slip are examples of systems having Nonholonomic constraints [2].
In Nonholonomic systems, the presence of non-integrable constraints makes the control problem of the systems a lot more challenging. Due to the wide range of applications of Nonholonomic systems, attention has been paid to design feedback controllers for such systems. These systems are widely been used in several industries which includes transportation, robotics, security , space exploration and inspection [3, 4]. Although these nonlinear systems are controllable; however, the Nonholonomic systems don't satisfy the necessary Brockett's condition for smooth stabilization [5] . Consequently, the well entrenched smooth nonlinear control methodologies are not directly applicable to the control problem of these mechanical systems. Moreover, real-world systems often operate alongside input/model uncertainties and noise disturbances. The effect of these uncertainties and disturbances upon the overall dynamics of the system is considered during the controller design, since uncertainties or disturbances can degrade the system's performance or can even cause system instability if not paid due consideration. Thus, the problem of stabilizing Nonholonomic systems while dealing with the input/model uncertainties is becoming a significant area of research.
Nonholonomic systems can be further classified as:

- First order nonholonomic systems
- Second order nonholonomic systems
- Higher order nonholonomic systems

The constraints in first-order systems include the constraints of position and velocity i.e. $\psi(q, \dot{q}) = 0$. The example of first-order systems includes Wheeled vehicles and robots. The constraints of second-order systems include position, velocity and acceleration $\psi(q, \dot{q}, \ddot{q}) = 0$. Examples of second-order Nonholonomic systems include space robots, spacecrafts, underwater vehicles, surface vessels and under-actuated manipulators [6]. Similarly, the higher-order nonholonomic systems specifically the third-order systems have constraints of position, velocity, acceleration and jerk which can be symbolically illustrated as follows

$\psi(q, \dot{q}, \ddot{q}, \dddot{q}) = 0$. The one of the real world illustrations of such systems include the dynamics of PPR (Prismatic-Prismatic-Revolute) manipulator movement working under the jerk constraint.

In the design of a control system, it is more effective to transform the system to some Canonical form via input or state variable transformation. The Chained form is one of the widely used Canonical form which is discussed in [7]. In [3], it is presented that Nonholonomic systems can be (locally or globally) transformed into the Chained form under an appropriate coordinate transformation system. Many nonholonomic systems having first-order constraints can be (globally or locally) transformed into first-order canonical form. Similarly, the second-order Canonical form plays the same role for the second-order systems as the first order Canonical form system [8]. Through this transformation, the dynamics of the system get significantly simplified and, therefore, it becomes easier to design the control laws.

## 1.2 Motivation

Before the market arrival Nonholonomic systems and their practical use, their scope was limited to numerous scientific and academic terminologies. However, nowadays, a vast selection of these systems is available and being deployed in various areas owing to their significance in practical usage. Nonholonomic system poses new control problems which require the fundamental non-linear approach to address them. The linear approximation of these systems around an arbitrary equilibrium point may not be controllable, and the feedback linearization technique may fail to transform the system for linear control problem. However, under certain conditions, the feedback stabilization and tracking problems are solved by time-varying control or the discontinuous feedback control strategy.

Nonholonomic systems provide a fantastic platform for research and education related areas. The kinematic constraints which are called "Nonholonomic constraints" present in these systems make the control design a complicated problem. When we discuss Nonholonomic systems, some renowned examples include; a Two-Wheel Car Model, a Front-Wheel Car Model, a Vehicle with Trailer Model, and a

Firetruck Model. These mobility of these robots make them highly susceptible to external disturbances e.g. slippery floor or dusty air. So the design of control laws for such systems in the clear presence of external disturbances or uncertainties is an essential but difficult task.

## 1.3 Research Objectives

In the light of motivation provided above for carrying this research and the problems that arise during the stabilization of Nonholonomic systems, This work aims to investigate a novel and more efficient feedback control design methodology for the Nonholonomic system stabilization [5]. Moreover, the simultaneous existence of uncertainties and constraints in Nonholonomic systems make the controller design for its stabilization a lot more difficult. Therefore the research methodology should certainly be general and be applicable to a wide range of Nonholonomic systems instead of limiting its scope to a specific system. Meanwhile, in the absence of smooth static time-invariant feedback law, the principal objective is to develop such techniques that are based on discontinuous feedback control laws. That's why, this work has focused on discontinuous control law based on Adaptive Sliding Mode Control and nonlinear control based on Integral Backstepping Control.

## 1.4 Outline of the Thesis

After having discussed about the Nonholonomic systems; below is given a breif account of organization of this desertation:

- **Chapter 2:**
  This chapter "Literature Review" entails a detailed and in depth exploration into the Literature or previous works related to the stabilization problem of Nonholonomic systems and discusses the previous control techniques applied to nonholonomic systems.

- **Chapter 3:**

  The third chapter is about how to convert a second order Nonholonomic system to first order chain form.

- **Chapter 4:**

  In this chapter a brief discussion is provided about the design of Integral Backstepping Control and its application to Nonholonomic systems.

- **Chapter 5:**

  This chapter is all about results and analysis where two methods of adaptive sliding mode control are separately designed and applied to Nonholonomic systems. First method is applied to systems with an assumption that system is unperturbed. In other method the system is assumed to be perturbed in which external disturbances are estimated by using basis functions.The results obtained through both of these methods are compared and based comparison and analysis is drawn which is concluded in chapter 6.

- **Chapter 6:**

  Chapter 6 concludes all the discussion provided in the previous chapter. It also suggests the future suggestions which needs to be considered while designing control laws for Nonholonomic systems which can lead towards more elaborated research development in the field.

# Chapter 2

# Literature Survey

This chapter entails a detailed and in depth exploration into the Literature or previous work which has been done related to the stabilization of Nonholonomic systems and discusses the previously implemented control techniques on nonholonomic systems. It mainly reviews the control perspective, dynamic modeling and different analytical tools and control techniques that have been designed over the past years for Nonholonomic systems.

## 2.1 Control Problem of Nonholonomic Systems

The issue of control systems subjected to non-integrable constraints has attracted the sight of control community since the 1980's. Initial research in the field was centered around systems with non-integrable kinematic relations. For example, classical first-order nonholonomic systems which include systems with rolling constraints and systems involving symmetries which lead to non-integrable conserved angular momentum. The widely studied examples were mobile robots, wheeled vehicles, robot manipulation and space robots. These studies covered the problems of controllability, motion planning, feedback stabilization and tracking control. The motion planning problem was basically investigated in [9] which discusses how to achieve controllability of car-like robot with just one Nonholonomic constraint. In [10], Nonholonomic systems as an example of inherently nonlinear control systems were identified and an overall process of constructing a piece-wise analytic state feedback was presented. The Controllability proof for a multibody mobile robot using tools from differential geometry is discussed in [11]. A Nonholonomic motion planning approach using geometrical phases was illustrated in [7]. In [8], it had been suggested to stabilize the system in regards to a trajectory rather than a point. Using a non smooth and time-varying feedback controller, global asymptotic stability for any desired configuration was achieved in [12]. Other notable developments include the research of controllability, motion planning [13], stabilization [14] and tracking control [6] of classical first-order Nonholonomic systems. In [15], the ideas presented in [10] were further extended to systems which satisfy the second-order Nonholonomic constraints. Similarly, some under-actuated systems have also been qualified as second-order Nonholonomic systems based on their configuration. The UMS are such systems which have fewer actuators aas compared to DOF. The under-actuated robot manipulators, autonomous under-water vehicles, underactuated surface vessels, the planar vertical take-Off and landing aircraft and under-actuated space vehicles are types of UMS belonging to this class [16]. The key difference is that the second-order Nonholonomic systems include drift terminologies which make the control of these systems a lot more difficult. Whereas, generally speaking, the second-order Nonholonomic systems also don't satisfy the Brockett's Necessary Condition (like that of first-order Nonholonomic systems) [9]. Many approaches have been identified for overcoming this problem which may be classified into the Time-varying but Continuous Feedback Control [3, 17], the Discontinuous Control [18] and the Hybrid Control [19].

Recently, substantial effort on the dynamics formulation for higher-order nonholonomic systems has been witnessed. Representative work in this field of higher-order nonholonomic systems include the investigation by Nielsen, Mangeron, Tzenoff, Appell, Deleanu and Gibbs. Subsequently, modern kinds of differential equations of nonholonomic systems with higher-order constraints were derived. A particular constraint called program constraint is just a demand imposed on a system by design. These program and material constraints are then a part of a unified formulation providing a theoretical framework for the research of robot performance under constrained environments. [20] provides an example of a higher-order Nonholonomic system i.e. a Planar Prismatic-Prismatic-Revolute Robotic Manipulator under the subjection of a jerk constraint. Jerk is a novel case of third-order constraint and is identified as fast changing actuator force in the domain of robot manipulators. Furthermore, jerk is characterized as the triple time derivative of distance. Excessive quantity of jerk results in early wear and tear of the actuators. It produces resonant vibrations in the robotic body and thus makes accurate tracking much more challenging. Specific studies on humans, demonstrate that human brain also realizes a variant of minimum-jerk while grasping actions are now being planned for the arms [21].

## 2.2 Control Strategies for Nonholonomic Systems

The closed-loop approaches have received wider acknowledgment owing to that these provide stabilization of second- and higher-order nonholonomic systems. It includes time-varying Continuous Control, Discontinuous Control as well as Hybrid Control.

### 2.2.1 Time-Varying Continuous Controls

The two approaches for crafting time-varying continuous control suggested in literature are periodic and a-periodic feedback control. The periodic method was proposed by [20] which is based on the power form. Whereas, the a-periodic time-varying feedback control was investigated in [22] and output feedback control based on time varying delay in [21].
The benefits of periodic and aperiodic time-varying continuous methodologies include the input control and state variables are all smooth, asymptotically converging with no oscillations (or sometime damped oscillations, which is damped with time). However, their disadvantage includes that the input depends upon suitable

tuning of parameters which depends on the system's initial states. Initial states must be in convergence region of the control input.

### 2.2.2 Discontinuous Feedback Control

The discontinuous feedback control avoids the issue of designing a single continuous control by allowing the design of the time-varying counterpart. The primary idea in discontinuous control is to change the control law when system's states try to move away from stable manifold. These feedback controllers for stabilizing Nonholonomic systems can further be categorized into piece-wise continuous as well as the Sliding Mode Controllers.

The $\alpha$-process, as discussed in [23], is usually a prevalent methodology of discontinuous controls system design. After applying a state transformation, a stabilized linear system is obtained after which it becomes possible to choose a linear control law to assign stable eigenvalues. Hence, the system can be recognized as globally exponentially stable. Simultaneously, its discrepancy is that the linear control can not be defined for the entire state space.

So, to overcome this challenge, it is recommended to first move the system away from singularity by employing open-loop controls on an calculated time $t_s$ and afterwards switching it on the linear feedback control law [23]. In [24], a formula is acquired from a piece-wise control Lyapunov function for acquiring global stabilization. While no general method exists for designing the control-Lyapunov function which can satisfy conditions of [24]. A piece-wise continuous stabilization of some specific models are mentioned in [24].

In [25] a discontinuous controller is proposed for the higher-order canonical form system (special from in which systems state model represented), which takes two input variables. Exponential convergence is achieved, towards a point, to become a stabilized system which means that the system trajectories converge exponentially towards a single point. However, considering the controller and closed-loop system to be stabilized at a single point does not imply the stability property in the sense of Lyapunov function.

The Sliding mode control (SMC) proposed in [26, 27], is utilized to develop discontinuous time-invariant feedback laws. This discontinuous control enforces the system to go along a well-defined stable manifold towards equilibrium. SMC construction was used for a certain class of higher-dimensional classical and dynamic types of Nonholonomic systems [28]. However, the key disadvantage of using SMC is that it hinders its widespread use due to the chattering phenomenon.

### 2.2.3 Hybrid Control

The Hybrid Control (HC) takes the advantage of both discrete-time and continuous-time control. Effective features of discrete and continuous control are used. The working of HC system depends on variation between different continuous-time controllers at discrete-time instants. The controller switching instant is altered online during the operation of the controller. The soft computing methodologies for Nonholonomic systems based on Neural Networks is discussed in [29, 30] and based on SMC , are discussed in [31].

## 2.3 Types / Orders of Nonholonomic Systems

The phenomenon of nonholonomy, in mechanical systems, can occur because of the following two reasons:

- If a rolling body moves over another body or a rolling body moves over plane without any slippage.
- if a multi-body system, conservation of momentum is maintained with under-actuated control.

The Nonholonomic systems can further be classified on the basis of order of system or constraints on systems as first order systems, second order systems and higher-order systems. Systems with constraints of first, second or even third-order have been mathematically modeled and frequently reported in the literature. The first order Nonholonomic Systems (FONHS) has position and velocity constraints, while second-order Nonholonomic systems (SONHS) have position, velocity and acceleration constraints and third-order Nonholonomic systems (TONHS) have position, velocity, acceleration and jerk constraints.

### 2.3.1 First-Order Nonholonomic Systems

In kinematic systems, state variable $x$ represents to configuration $q$ and the quantity of velocity constraints is given by $k = n - m > 1$, i.e. it is the entire dimension of configuration space excluding the control space.

#### 2.3.1.1 Four Wheel Car

Simple first-order model of a four wheel car is shown in Figure. (2.1). If $u_1$ and $u_2$ are the driving and steering velocities respectively then the mathematical model for four wheel car system is presented by the following set of equations:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} cos(\theta) \\ sin(\theta) \\ \frac{1}{l}tan(\phi) \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u_2 \tag{2.1}$$

### 2.3.2 Second-Order Nonholonomic Systems

The Under-actuated Mechanical Systems (UMS) can lead towards second-order constraints (position, velocity and acceleration constraints, these systems have less number of actuators than configuration variables). Let us consider an underactuated mechanical system with $q$ as the set of generalized coordinates. The

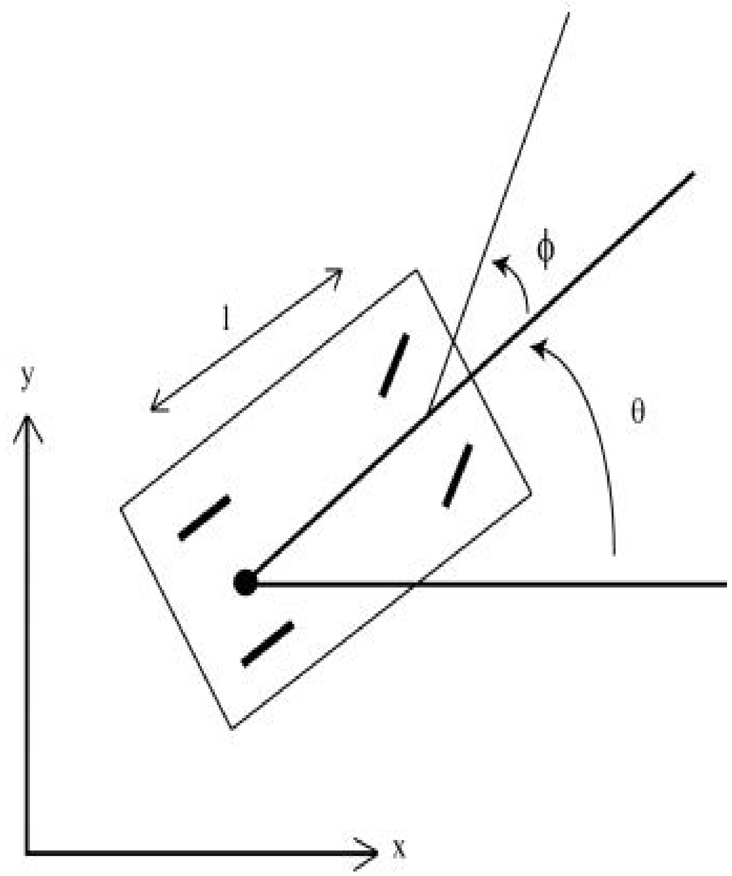



FIGURE 2.1: Kinematic Model of a Four-Wheel Car

equation of motion for the UMS is given below:

$$\begin{aligned} M_{11}(q_1q_2)\ddot{q}_1 + M_{12}(q_1q_2)\ddot{q}_2 + F_1(q,\dot{q}) &= B(q)u \\ M_{21}(q_1q_2)\ddot{q}_1 + M_{22}(q_1q_2)\ddot{q}_2 + F_2(q,\dot{q}) &= 0 \end{aligned} \tag{2.2}$$

Equation (2.2) defines $n - m$ number of relations which involve the generalized coordinates, first and second-order derivatives. If these $n - m$ equations are not integrable, then these equation can be regarded as Nonholonomic constraints of second order systems.

#### 2.3.2.1 Underactuated Surface Vessel

Vessel is one of the good example and benchmark second order nonholonomic system. The vessel shown in Figure (2.2) is an under actuated mechanical system with zero velocity constraint and damping factor. The relationship of earth-fixed

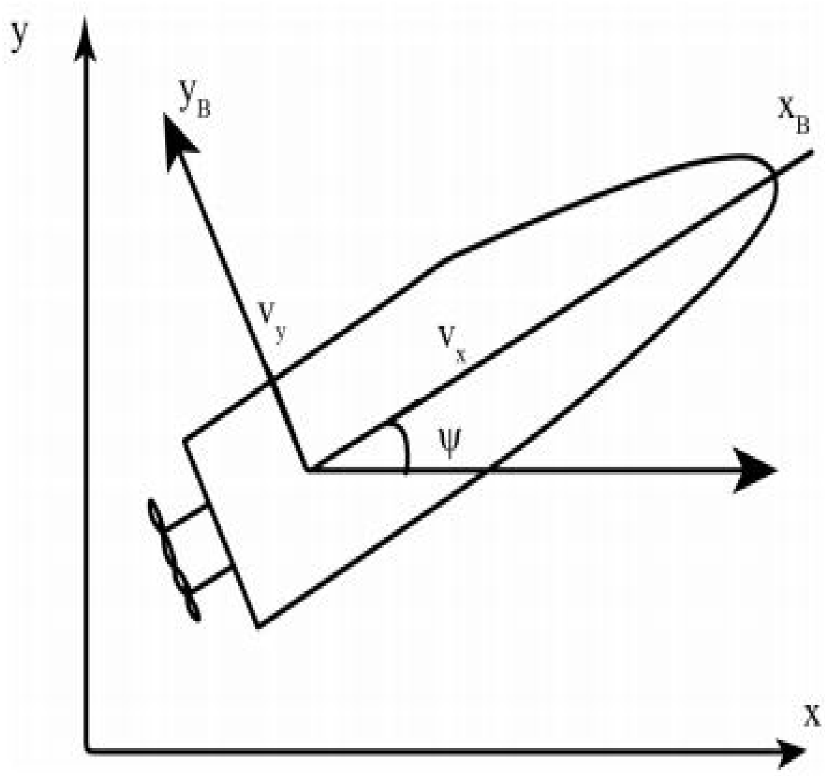



FIGURE 2.2: Underactuated Surface Vessel

(inertial frame) and the body frame is specified model. The kinematic model of vessel is represented in equation (2.3).

$$
\begin{aligned}
\dot{x} &= v_x cos\psi - v_z sin\psi \\
\dot{z} &= v_s in\psi + v_z cos\psi \\
\dot{\psi} &= w_z
\end{aligned}
\tag{2.3}
$$

In kinematic model of vessel which represents in (2.3), the state $x$ represent inertial position of center of mass, $z$ also represent inertial position of center of mass, $\psi$ is the orientation of vessel, whereas the linear velocity is expressed by $v_x, v_z$ and angular velocity is expressed by $w_z$.

### 2.3.3 Third-Order Nonholonomic Systems

Since the start of the $21^{st}$ century, considerable efforts have been made directly into formulation of theory regarding control of higher-order Nonholonomic constraints

[32]. The constraints, defined as, the program constraints, occur by imposing specific conditions on the allowable trajectories. For example, second nonholonomic constraints and third-order nonholonomic constraints occur by imposing torsion and curvature constraints on robot trajectories. By following [32, 33], systems with higher order nonholonomic constraints could be written by using the following models after suitable input transformation and state transformations [34]:

$$\begin{aligned} p_1^{(q)} &= u \\ p_2^{(q)} &= J(p, \dot{p}, ...., p^{(q-1)})u + R(p, \dot{p}, ...., p^{(q-1)}) \end{aligned} \tag{2.4}$$

The (2.4) (which represents the input and state transformation) $p_1 \in R^m$, $m \geq 2$ provides the actuated configuration variables, whereas $u \in R^m$ represents the modified control, and $u \in R^{n-m}$ represents the configuration variables where control is achieved through system coupling.

#### 2.3.3.1 PPR Manipulator

The Planar Prismatic-Prismatic-Revolute Robot Manipulator (PPRM) moving on a horizontal plane such that the gravity term could be avoided is shown in Figure (2.3).

$$\dddot{x}\, sin\phi - \dddot{y}\, cos\phi = 0 \tag{2.5}$$

or

$$\dddot{y}\, sin\phi - \dddot{x}\, cos\phi = 0 \tag{2.6}$$

or

$$A^T(p, \dot{p}, \ddot{p})\, \dddot{p} = 0 \tag{2.7}$$

or

$$A^T(\ddot{p}, \dot{p}, p)\, \dddot{p} = 0 \tag{2.8}$$

where:

$A^T(p, \dot{p}, \ddot{p}) = [sin\phi \quad - cos\phi \quad 0]$

or

$A^T(\ddot{p}, p, \dot{p}) = [0 \quad sin\phi \quad - cos\phi]$ or $A^T(p, \ddot{p}, \dot{p}) = [sin\phi \quad 0 \quad - cos\phi]$

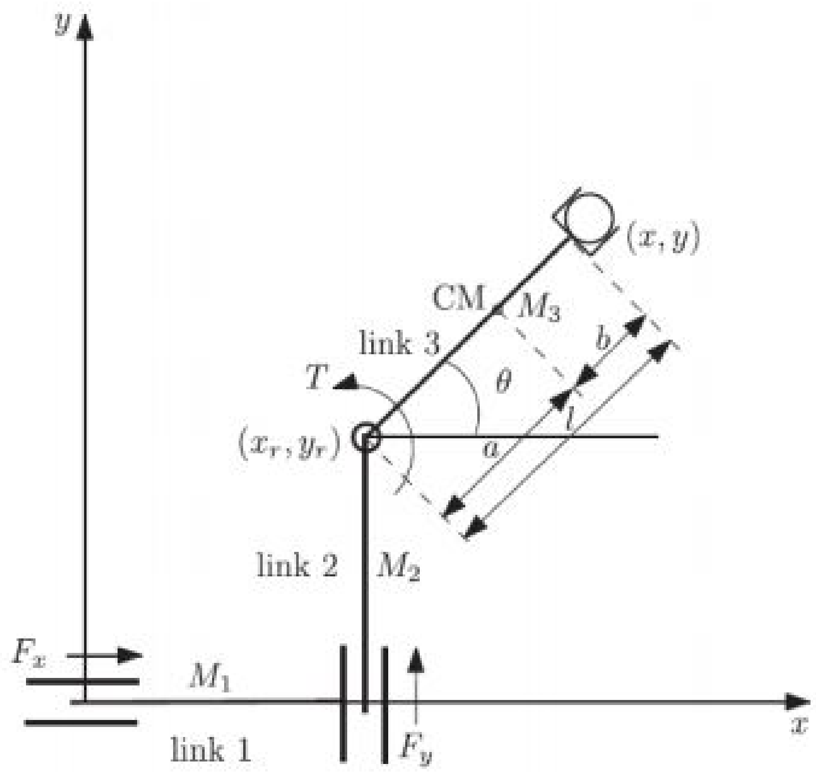



FIGURE 2.3: Planar PPR Manipulator

## 2.4 Sliding Mode Control

Sliding Mode Control (SMC) is a nonlinear control strategy with the ability of robustness against uncertainties in the system. SMC is regarded as a discontinuous control technique due to the discontinuity in controller. Its structure is pretty much easy for design and application. In addition to being a control technique, it is also employed for the disturbance estimation and disturbance rejection. SMC is actually a variable structure control system design procedure. The basics of sliding mode control is discussed in [35, 36]. The sliding plane and reaching phase are briefly discussed in [37].

### 2.4.1 Sliding Phase

Sliding phase is the very first step while designing the sliding mode control. In order to employ SMC, initially, a sliding surface (SS) design is required. The sliding surface can be linear or non linear. The sliding surface is also termed

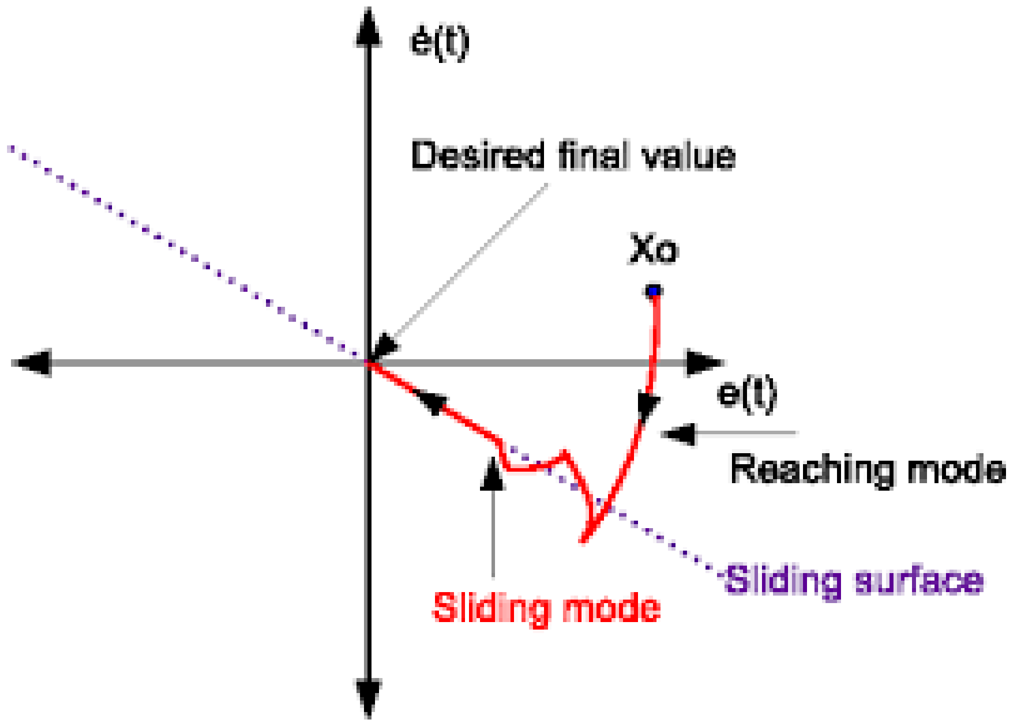



FIGURE 2.4: Sliding Phase, Reaching Phase and Sliding Surface

as switching surface, sliding manifold or hyper plane. Having defined the hyper plane, the aforementioned two phases come into place in specific order. Reaching phase (RP) is responsible for attraction of system states from an initial condition along switching surface. When reaching phase is attained, the system lies upon the sliding surface, then sliding mode (SM) into place, the system's states slide towards the origin (which is equilibrium position) utilizing a discontinuous control action (which also ensures robustness, but having chattering nature). Fig. 2.4 shows the reaching phase (RP) or sliding mode (SM) and sliding surface (SS) or sliding manifold. In 2.4 also shows a chattering nature of sliding mode control.

### 2.4.2 Chattering Phenomenon

The phenomenon of Chattering occurs due to discontinuous control term which is very unnatural and undesirable. It causes undesired effects in mechanical and electro-mechanical parts. This phenomenon leads to undesirable oscillations (some cause the break of mechanical parts or controller failure) that affect the performance of the control in particular and the system in general as shown in Fig. 2.5. The effect of chattering can be avoided by using various solutions that have been

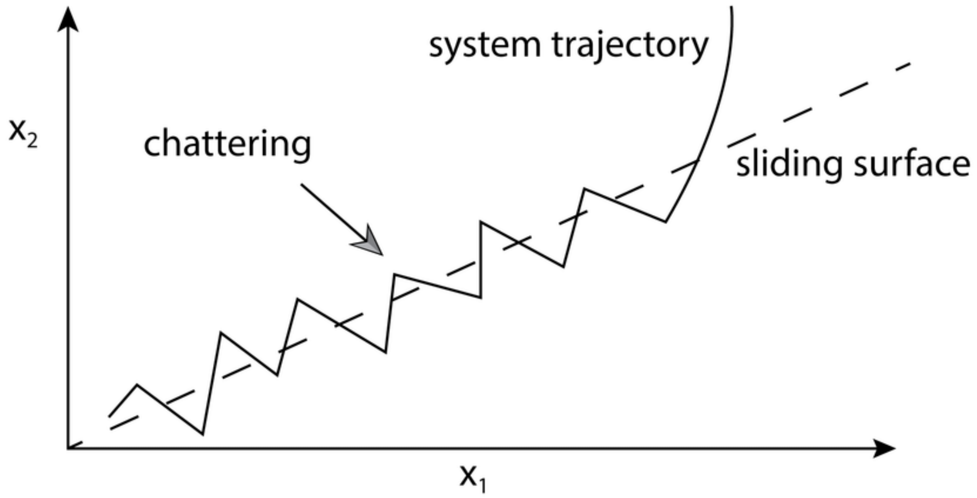



Figure 2.5: Chattering of control input along sliding surface

proposed in literature. Based on literature review, it is found that Dynamic Sliding Mode Control (DSMC) and High Order Sliding Mode (HOSM) control techniques are useful in reducing the affect of chattering.

## 2.5 Summary

This chapter has provided literature review on Nonholonomic systems. Alongside, different control techniques and its achievements are mentioned which were previously been utilized in order to stabilize the Nonholonpmic systems. The mathematical model and types of nonholonomic systems are discussed. The basics of sliding mode control is also discussed in this chapter.

# Chapter 3

# Nonholonomic Systems in Chained Form

This chapter discusses novel solutions to the problem of stabilizing the Nonholonomic systems that are expressed in canonical chained form. The methodologies are based on adaptive ISMC. Firstly, the chained form system is transformed into a Special Structure which comprises of nominal portion and various unknown variables which come through input transformation. Then the transformed system is stabilized by using ISMC control and the unknown terms are computed by using adaptive techniques. The controller for the transformed system consists of nominal control and compensator control. The proposed method is tested on second-order Nonholonomic systems. Subsequently, perturbations are included in the control input and robust stabilizing algorithm is designed in order to overcome the uncertainties. The performance of the proposed methodologies is verified through simulations.

## 3.1 Second-Order Nonholonomic Systems

During the last couple of decades, most of the publications in literature on nnholonomic systems had been on mechanical systems with the first-order nonholonomic constraints. Only recently, the under-actuated systems with second-order nonholonomic constraints got the attention of researchers in control system engineering society. In [38], the researchers discovered a class of underactuated mechanical system with second-order Nonholonomic constraints. The under-actuated robots, ball and beam system, autonomous underwater Vehicles (AUVs), inverted pendulum, underactuated surface vessels and the PVTOL aircraft are few examples of under-actuated systems belonging to this class [38]. The main difference between the two types is that the second-order Nonholonomic systems include drift terms which make the control of these systems much more difficult. Whereas, in general, the second-order Nonholonomic systems also do not satisfy Brockett's Necessary Condition for smooth time-invariant systems which is also similar to the first-order Nonholonomic systems [5].

## 3.2 Second-Order Chained Form Systems

A second-order chained form system can be defined through the following set of equations:

$$\begin{aligned} \ddot{x}_1 &= v_1 \\ \ddot{x}_3 &= v_2 \\ \ddot{x}_5 &= x_3 v_1 \end{aligned} \tag{3.1}$$

In equation (3.1) represents the model in second order dynamic form, In first order form can be written as:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= v_1 \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= v_2 \\ \dot{x}_5 &= x_6 \\ \dot{x}_6 &= x_3 v_1 \end{aligned} \tag{3.2}$$

This second-order chained form plays the exact same role for the second-order Nonholonomic systems as that of the simple chained form system for the first-order Nonholonomic systems. Thus, the dynamics of second-order chained form system are considerably simplified and, therefore, simple to control. It is well known from equation (3.1) that a class of UMS could be transformed to second-order canonical form by constructing input and coordinate transformations [39]. Systems which can be transformed to the second-order chained form include: an under-actuated planar horizontal 3-link serial-drive PPR manipulator (PPR means two prismatic and one revolute joint), an under-actuated planar horizontal PPR manipulator with spring-coupled third link, an under-actuated planar horizontal 3-link serial-drive RRR manipulator, a manipulator driven by end-effector forces, an under-actuated planar horizontal parallel drive RRR manipulator with any two joints unactuated, a planar rigid body having an unactuated DOF, an under-actuated surface vessel and son and so forth [6].

## 3.3 The Stabilization Problem

Taking a general second-order Nonholonomic system into consideration which can be described as under:

$$D(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = F(q)u \tag{3.3}$$

where:

$q \in R^n$ represents the configuration vector.

$D(q) \in R^{n\times n}$ represents the positive definite inertia matrix.

$C(\dot{q}, q)q \in R^n$ represent the Centrifugal and Coriolis terms respectively.

$G(q)$ is the gravity term.

If we assume $F(q) = [I_m \; 0]^T$ and $u \in R^n$ the actuator input vector then the desired set point can be given as follows; $x_{des} = \begin{bmatrix} q_{des} \\ \dot{q}_{des} \end{bmatrix}$. The design feedback controller or control input $u(q, \dot{q})$ is designed such that the desired set point $x_{des}$ is an attractive set for system provided in equation (3.3).

## 3.4 3-DOF Manipulator with a Free Link

Figure3.1 shows a 3-DOF manipulator in horizontal plane. The manipulator has first two prismatic joints that are active. These joints control the third unactuated joint. It is assumed that the third free joint is a revolute around the vertical axis. Assign the coordinate frames $\Sigma_B$, $\Sigma_L$, $\theta$, $(x, y)$ as given in [40]. The generalize coordinates representing the manipulator configuration are given as $(x,\ y,\ \theta)$. Therefore, the equations of motion according to the third link are given as under as was discussed in [40].

$$\begin{aligned} f_x &= m\ddot{x} - md\ddot{\theta}S_i(\theta) - md\dot{\theta}^2 C_s(\theta) \\ f_y &= m\ddot{y} + md\ddot{\theta}C_s(\theta) - md\dot{\theta}^2 S_i(\theta) \\ 0 &= -md\ddot{x}S_i(\theta) + md\ddot{y}C_s(\theta) + (I + md^2)\ddot{\theta} \end{aligned} \tag{3.4}$$

where:

$S_i = sin$, $C_s = cos$, and $I$ are the moment of inertia of third link around G, m is the mass of third link, $[f_x,\ f_y]$ is the translation force and $d$ is the distance $|OG|$ between the center of gravity and the joint. By denoting $\gamma = d + I/md$, the system constraint becomes:

$$-\ddot{x}S_i(\theta) + \ddot{y}C_s(\theta) + \gamma\ddot{\theta} = 0 \tag{3.5}$$

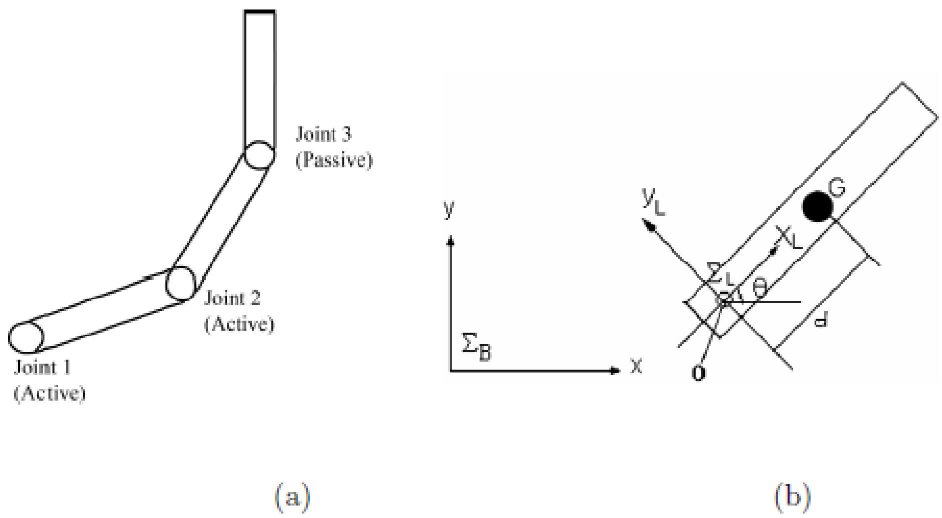



FIGURE 3.1: (a) 3-DOF Manipulator with a Free Joint. (b) Model of Passive Link.

### 3.4.1 Conversion into Second-Order Chained Form

Let $\ddot{x} = \alpha_1$ and $\ddot{y} = \alpha_2$. Then, (3.5) can be written as:

$$\begin{aligned} \ddot{x} &= \alpha_1 \\ \ddot{y} &= \alpha_2 \\ \ddot{\theta} &= \frac{1}{\gamma} S_i(\theta)\alpha_1 - \frac{1}{\gamma} C_s(\theta)\alpha_2 \end{aligned} \tag{3.6}$$

Using the input transformation;

$$\begin{aligned} \alpha_1 &= \tau_1 \\ \alpha_2 &= t_n(\theta)\tau_1 - \gamma S_e(\theta)\tau_2 \end{aligned} \tag{3.7}$$

where $S_e = sec$ and $t_n = tan$, the system provided in equation (3.6) can be transformed as:

$$\begin{aligned} \ddot{x} &= \tau_1 \\ \ddot{\theta} &= \tau_2 \\ \ddot{y} &= t_n(\theta)\tau_1 - S_e(\theta)\tau_2 \end{aligned} \tag{3.8}$$

Using another transformation;

$$\begin{aligned} x_1 &= x + \gamma cos\theta \\ x_3 &= tan\theta \\ x_5 &= y + \gamma sin\theta \\ v_1 &= \tau_1 - \gamma sin\theta\tau_2 - \gamma cos\theta\dot{\theta}^2 \\ v_2 &= sec^2\theta\tau_2 + 2sec^2\theta tan\theta\dot{\theta}^2 \end{aligned} \tag{3.9}$$

Applying the transformation provided in equation (3.9) and applying the transformation provided in equation (3.8), then equation(3.8) can be written in second-order chained form as follows:

$$\begin{aligned} \ddot{x}_1 &= v_1 \\ \ddot{x}_3 &= v_2 \\ \ddot{x}_5 &= x_3 v_1 \end{aligned} \tag{3.10}$$

## 3.5 Planar PPR Manipulator

Let us consider a PPR robot as shown in Fig. (3.2). For simplicity, we have considered that the robot is moving in a horizontal plane. All the joints of the robot are passive and the input forces are applied on the end effector only. Let $d_3$ be the distance between mass center of 3rd link and the joint axis, $l_3$ the length of third link, $m_i$ the mass of the $i^{th}$ link and $I_3$ the central moment of inertia. The dynamical model of the robot can be given as:

$$\begin{aligned} M(q_\gamma)\ddot{q}_\gamma + H(q_\gamma, \dot{q}_\gamma) &= J^T(q_\gamma)F \\ M(q_3)\ddot{q}_3 + H(q_3, \dot{q}_3) &= J^T(q_3)F \end{aligned} \tag{3.11}$$

Equation (3.11) can be rewritten as:

$$\begin{aligned} &a_1\ddot{q}_1 + a_4cos(q_3)\ddot{q}_3 - a_4\dot{q}_3^2sin(q_3) = F_y \\ &a_2\ddot{q}_2 - a_4sin(q_3)\ddot{q}_3 - a_4\dot{q}_3^2cos(q_3) = F_x \\ &a_4cos(q_3)\ddot{q}_1 - a_4sin(q_3)\ddot{q}_2 - a_3\ddot{q}_3 = -l_3sin(q_3)F_x + l_3cos(q_3)F_y \end{aligned} \tag{3.12}$$

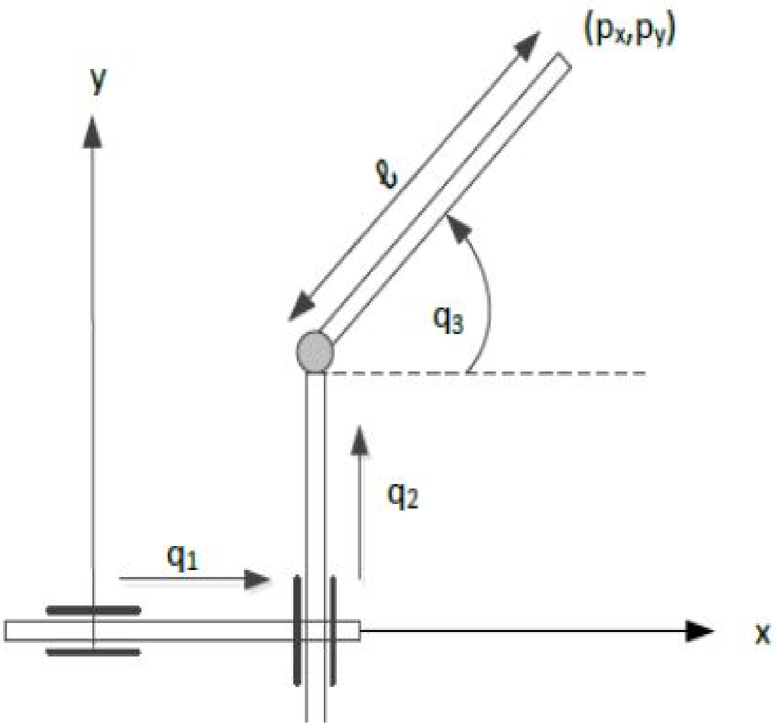



FIGURE 3.2: (a) Planar PPR Manipulator.

Hence,

$$\begin{aligned} &a_1\ddot{q}_1 + a_4cos(q_3)\ddot{q}_3 - a_4\dot{q}_3^2sin(q_3) = F_y \\ &a_2\ddot{q}_2 - a_4sin(q_3)\ddot{q}_3 - a_4\dot{q}_3^2cos(q_3) = F_x \\ &a_4cos(q_3)\ddot{q}_1 - a_4sin(q_3)\ddot{q}_2 - a_3\ddot{q}_3 = -l_3sin(q_3)F_x + l_3cos(q_3)F_y \end{aligned} \tag{3.13}$$

### 3.5.1 Conversion into Second-Order Chained Form

In this section convert of system to second order chained form is shown. Rearrange the equation (3.12a) to get:

$$\ddot{q}_1 = -\frac{a_4}{a_1}cosq_3\ddot{q}_3 + \frac{a_4}{a_1}sinq_3\dot{q}_3^2 + \frac{1}{a_1}F_y \tag{3.14}$$

Substitute (3.14) in (3.12) to get:

$$\begin{aligned} \ddot{q}_2 =& -\frac{1}{r_2}(a_4^2 - a_1a_3)a_4cosq_3\dot{q}_3^2 + \frac{1}{r_2}(l_3a_1a_4sin^2q_3 + a_4^2cos^2q_3 - a_1a_3)F_x \\ &+ \frac{1}{r_2}(a_1 - a_4l_3)a_1cosq_1sinq_3F_y \\ \ddot{q}_3 =& \frac{1}{r_1}(a_1 - a_2)a_4^2cosq_3sinq_1\dot{q}_3^2 + \frac{1}{r_1}(a_1l_2 - a_4)a_2sinq_3F_x + \frac{1}{r_1}(a_2 - a_1l_3) \\ &a_2cosq_3F_y \end{aligned} \tag{3.15}$$

Hence,

$$\begin{aligned} \ddot{q}_2 =& -\frac{1}{r_2}(a_4^2 - a_1a_3)a_4cosq_3\dot{q}_3^2 + \frac{1}{r_2}(l_3a_1a_4sin^2q_3 + a_4^2cos^2q_3 - a_1a_3)F_x \\ &+ \frac{1}{r_2}(a_1 - a_4l_3)a_1cosq_1sinq_3F_y \\ \ddot{q}_3 =& \frac{1}{r_1}(a_1 - a_2)a_4^2cosq_3sinq_1\dot{q}_3^2 + \frac{1}{r_1}(a_1l_2 - a_4)a_2sinq_3F_x + \frac{1}{r_1}(a_2 - a_1l_3) \\ &a_2cosq_3F_y \end{aligned} \tag{3.16}$$

Rearranging the above equations gives;

$$\begin{aligned} a_4cosq_3\ddot{q}_1 - a_4sinq_3\ddot{q}_2 + a_1\ddot{q}_3 =& - l_3sinq_3(a_2\ddot{q}_2 - a_4sinq_3\ddot{q}_3 - a_4\dot{q}_3^2cosq_1) \\ &+ l_3cosq_3(a_1\ddot{q}_1 + a_4cosq_3 - a_4\dot{q}_3^2)sinq_1 \\ \ddot{q}_1(a_4 - a_1l_3cosq_1) =& \ddot{q}_2(a_4 - a_2l_3)sinq_3 + \ddot{q}_3(a_1l_3 - a_3) \end{aligned}$$

$$\ddot{q}_1 = \ddot{q}_2 \frac{a_4 - a_2 l_3}{a_4 - a_1 l_3} tanq_3 + \ddot{q}_3 \frac{a_4 l_3 - a_1}{a_4 - a_1 l_3} secq_3 \tag{3.17}$$

$$\begin{aligned} a_4 cosq_3 \ddot{q}_1 - a_4 sinq_3 \ddot{q}_2 + a_1 \ddot{q}_3 =& - l_3 sinq_3 (a_2 \ddot{q}_2 - a_4 sinq_3 \ddot{q}_3 - a_4 \dot{q}_3^2 cosq_1) \\ & + l_3 cosq_3 (a_1 \ddot{q}_1 + a_4 cosq_3 - a_4 \dot{q}_3^2) sinq_1 \\ \ddot{q}_1 (a_4 - a_1 l_3 cosq_1) =& \ddot{q}_2 (a_4 - a_2 l_3) sinq_3 + \ddot{q}_3 (a_1 l_3 - a_3) \\ \ddot{q}_1 =& \ddot{q}_2 \frac{a_4 - a_2 l_3}{a_4 - a_1 l_3} tanq_3 + \ddot{q}_3 \frac{a_4 l_3 - a_1}{a_4 - a_1 l_3} secq_3 \end{aligned} \tag{3.18}$$

By choosing,

$$\begin{aligned} v_1 =& \frac{1}{r_2}(a_4^2 - a_1 a_3) a_4 cosq_3 \dot{q}_3^2 + \frac{1}{r_2}(l_3 a_1 a_4 sin^2 q_1 + a_4^2 cos^2 q_3 - a_1 a_3) F_x \\ & \frac{1}{r_2}(a_4 - a_1 l_3) a_1 cosq_3 sinq_1 F_y \\ v_2 =& \frac{1}{r_1}(a_2 - a_1) a_4^2 cosq_3 sinq_3 \dot{q}_3^2 + \frac{1}{r_1}(a_2 l_3 - a_4) a_1 sinq_3 F_x + \frac{1}{r_1}(a_4 - a_1 l_3) \\ & a_2 cosq_3 F_y \end{aligned} \tag{3.19}$$

Above Equations reduce to the following form:

$$\begin{aligned} \ddot{q}_2 &= v_1 \\ \ddot{q}_3 &= v_2 \\ \ddot{q}_1 &= a_5 tanq_3 v_1 + a_6 secq_3 v_2 \end{aligned} \tag{3.20}$$

where $a_5 = \frac{a_4 - a_2 l_3}{a_4 - a_1 l_3}$ and $a_6 = \frac{a_4 l_3 - a_1}{a_4 - a_1 l_3}$ Using the input and state transformations:

$$\begin{aligned} x_1 &= q_2 - \frac{a_5}{a_6}(cosq_3 - 1) \\ x_3 &= a_5 tanq_3 \\ x_5 &= q_1 - a_6 sinq_3 \\ u_1 &= v_1 + \frac{a_5}{a_6} sinq_3 v_2 + \frac{a_6}{a_5} cosq_3 \dot{q}_3^2 \\ u_2 &= a_5 sec^2 q_3 v_2 + 2 a_5 sec^2 q_3 tanq_3 \dot{q}_3^2 \end{aligned} \tag{3.21}$$

the system (3.20) can be converted to the following form form:

$$\begin{aligned} \ddot{x}_1 &= u_1 \\ \ddot{x}_3 &= u_2 \end{aligned} \tag{3.22}$$

$$\ddot{x}_5 = x_2 u_1$$

and can be written as:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u_1 \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= u_2 \\ \dot{x}_5 &= x_6 \\ \dot{x}_5 &= x_3 u_1 \end{aligned} \tag{3.23}$$

# Chapter 4

# Backstepping Control Technique

This chapter presents a proposed control strategy which is based on Integral Backstepping Control Technique.

## 4.1 Problem Statement

For a given point $x_{des}$, design a control strategy which pushes the $x \to x_{des}$ as time $t \to \infty$.

## 4.2 Backstepping

The second-order chained form system (3.1) can be written in state-space form as:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = v_1$$
$$\dot{x}_3 = x_4$$
$$\dot{x}_4 = v_2$$
$$\dot{x}_5 = x_6$$
$$\dot{x}_6 = x_3 v_1$$

$$\dot{x}_6 = x_3 v_1 \tag{4.1}$$

**Step 1:**

First two equations of state space model as given in equation (4.1) are written as follows:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= v_1 \end{aligned} \tag{4.2}$$

Lyapunov function can be defined as:

$$V_{e1} = \frac{1}{2} e_1^2 \tag{4.3}$$

$e_1$ is the error between $x_1$ and its desired value.

$$e_1 = x_{1d} - x_1 \tag{4.4}$$

Taking derivative of Lyapunov function;

$$\begin{aligned} \dot{V}_{e1} &= \dot{e}_1 e_1 \\ \dot{V}_{e1} &= e_1(\dot{x}_{1d} - \dot{x}_1) \\ \dot{V}_{e1} &= e_1(\dot{x}_{1d} - x_2) \end{aligned} \tag{4.5}$$

**Krasovskii-LaSalle principle:**

If derivative of Lyapunov function is negative semi-definite then the system is guaranteed to be a stable system [41]. Stabilization of $e_1$ is achieved via virtual control $x_2$.

$$x_{2d} = \dot{x}_{1d} + k_1 e_1 \tag{4.6}$$

$e_2$ can be defined as:

$$\begin{aligned} e_2 &= x_2 - x_{2d} \\ e_2 &= x_2 - (\dot{x}_{1d} + k_1 e_1) \\ e_2 &= x_2 - \dot{x}_{1d} - k_1 e_1 \\ e_2 + k_1 e_1 &= x_2 - \dot{x}_{1d} \end{aligned} \tag{4.7}$$

Augmented Lyapunov function can be written as:

$$
\begin{aligned}
V_{e1,e2} &= \frac{1}{2}e_1^2 + \frac{1}{2}e_2^2 \\
\dot{V}_{e1,e2} &= e_1\dot{e}_1 + e_2\dot{e}_2 \\
\dot{V}_{e1,e2} &= e_1(\dot{x}_{1d} - x_2) + e_2(\dot{x}_2 - \ddot{x}_{1d} - k_1\dot{e}_1) \\
\dot{V}_{e1,e2} &= e_1(\dot{x}_{1d} - x_2) + e_2(v_1 - \ddot{x}_{1d} - k_1\dot{e}_1) \\
\dot{V}_{e1,e2} &= e_1(-e_2 - k_1e_1) + e_2(v_1 - \ddot{x}_{1d} - k_1(\dot{x}_{1d} - \dot{x}_1)) \\
\dot{V}_{e1,e2} &= e_1(-e_2 - k_1e_1) + e_2(v_1 - \ddot{x}_{1d} - k_1(\dot{x}1d - x_2)) \\
\dot{V}_{e1,e2} &= e_1(-e_2 - k_1e_1) + e_2(v_1 - \ddot{x}_{1d} + k_1e_2) \\
\dot{V}_{e1,e2} &= -k_1e_1^2 + e_2(-e_1 + v_1 - \ddot{x}_{1d} + k_1e_2)
\end{aligned}
\tag{4.8}
$$

Choose $v_1 = e_1 + \ddot{x}_{1d} - k_1e_2 - k_2e_2$. Putt $v_1$ in Eq. (4.8) to get:

$$
\dot{V}_{e1,e2} = -k_1e_1^2 - k_2e_2^2 \leq 0 \tag{4.9}
$$

**Step 2:**

Third and fourth equations of state space model (4.1):

$$
\begin{aligned}
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= v_2
\end{aligned}
\tag{4.10}
$$

Integral Lyapunov function can be defined as:

$$
V_{e3} = \frac{1}{2}e_3^2 \tag{4.11}
$$

$e_3$ is the error between $x_3$ and its desired value.

$$
e_3 = x_{3d} - x_3 \tag{4.12}
$$

Take derivative of Lyapunov function.

$$
\begin{aligned}
\dot{V}_{e3} &= \dot{e}_3e_3 \\
\dot{V}_{e3} &= e_3(\dot{x}_{3d} - \dot{x}_3)
\end{aligned}
\tag{4.13}
$$

$$\dot{V}_{e3} = e_3(\dot{x}_{3d} - x_2) \tag{4.14}$$

$$x_{4d} = \dot{x}_{3d} + k_3 e_3 \tag{4.15}$$

$e_4$ can be defined as:

$$\begin{aligned} e_4 &= x_4 - x_{4d} \\ e_4 &= x_4 - \dot{x}_{3d} - k_3 e_3 \\ e_4 + k_3 e_3 &= x_4 - \dot{x}_{3d} \end{aligned} \tag{4.16}$$

Augmented Lyapunov function can be written as:

$$\begin{aligned} V_{e3,e4} &= \frac{1}{2}e_3^2 + \frac{1}{2}e_4^2 \\ \dot{V}_{e3,e4} &= e_3\dot{e}_3 + e_4\dot{e}_4 \\ \dot{V}_{e3,e4} &= e_3(\dot{x}_{3d} - x_4) + e_4(\dot{x}_4 - \ddot{x}_{3d} - k_3\dot{e}_3) \\ \dot{V}_{e3,e4} &= e_3(\dot{x}_{3d} - x_4) + e_4(v_2 - \ddot{x}_{3d} - k_3\dot{e}_3) \\ \dot{V}_{e3,e4} &= e_3(-e_4 - k_3e_3) + e_4(v_2 - \ddot{x}_{3d} - k_3(\dot{x}_{3d} - \dot{x}_3)) \\ \dot{V}_{e3,e4} &= e_3(-e_4 - k_3e_3) + e_4(v_2 - \ddot{x}_{3d} - k_3(\dot{x}_{3d} - x_4)) \\ \dot{V}_{e3,e4} &= e_3(-e_4 - k_3e_3) + e_4(v_2 - \ddot{x}_{3d} + k_3e_4) \\ \dot{V}_{e3,e4} &= -k_3e_3^2 + e_4(-e_3 + v_2 - \ddot{x}_{3d} + k_3e_4) \end{aligned} \tag{4.17}$$

Choose $v_2 = e_3 + \ddot{x}_{3d} - k_3e_4 - k_4e_4$. Putt $v_2$ in Eq. (4.17) to get:

$$\dot{V}_{e3,e4} = -k_3e_3^2 - k_4e_4^2 \leq 0 \tag{4.18}$$

**Step 3:**

Fifth and sixth equation of state space model (4.1):

$$\begin{aligned} \dot{x}_5 &= x_6 \\ \dot{x}_6 &= x_3v_1 + w - \hat{w} - \tilde{w} \end{aligned} \tag{4.19}$$

Integral Lyapunov function can be defined as:

$$V_{e5} = \frac{1}{2}e_5^2 \tag{4.20}$$

$e_5$ is the error between $x_5$ and its desired value.

$$e_5 = x_{5d} - x_5 \tag{4.21}$$

Take derivative of Lyapunov function.

$$\begin{aligned}\dot{V}_{e5} &= \dot{e}_5 e_5 + \dot{\tilde{\omega}}\tilde{\omega} \\ \dot{V}_{e5} &= e_5(\dot{x}_{5d} - \dot{x}_5) \\ \dot{V}_{e5} &= e_5(\dot{x}_{5d} - x_6)\end{aligned} \tag{4.22}$$

$$x_{6d} = \dot{x}_{5d} + 2k_5 e_5 \tag{4.23}$$

$e_6$ can be defined as:

$$\begin{aligned}e_6 &= x_6 - x_{6d} \\ e_6 &= x_6 - \dot{x}_{5d} - k_5 e_5 \\ e_6 + k_5 e_5 &= x_6 - \dot{x}_{5d}\end{aligned} \tag{4.24}$$

Augmented Lyapunov function can be written as:

$$\begin{aligned}V_{e5,e6} &= \frac{1}{2}e_5^2 + \frac{1}{2}e_6^2 + \frac{1}{2}\tilde{\omega}^2 \\ \dot{V}_{e5,e6} &= e_5\dot{e}_5 + e_6\dot{e}_6 + \dot{\tilde{\omega}}\tilde{\omega} \\ \dot{V}_{e5,e6} &= e_5(\dot{x}_{5d} - x_6 + k_5e_5) + e_6(\dot{x}_6 - \ddot{x}_{5d} - k_5\dot{e}_5) + \dot{\tilde{\omega}}\tilde{\omega} \\ \dot{V}_{e5,e6} &= e_5(\dot{x}_{5d} - x_6 + k_5e_5) + e_6(x_3v_1 + w - \hat{w} - \tilde{w} - \ddot{x}_{5d} - k_5\dot{e}_5) + \dot{\tilde{\omega}}\tilde{\omega} \\ \dot{V}_{e5,e6} &= e_5(-e_6 - k_5e_5) + e_6(x_3v_1 + w - \hat{w} - \tilde{w} - \ddot{x}_{5d} - k_5(\dot{x}_{5d} - \dot{x}_5)) + \dot{\tilde{\omega}}\tilde{\omega} \\ \dot{V}_{e5,e6} &= e_5(-e_6 - k_5e_5) + e_6(x_3v_1 + w - \hat{w} - \tilde{w} - \ddot{x}_{5d} - k_5(\dot{x}_{5d} - x_6)) + \dot{\tilde{\omega}}\tilde{\omega} \\ \dot{V}_{e5,e6} &= e_5(-e_6 - k_5e_5) + e_6(x_3v_1 + w - \hat{w} - \tilde{w} - \ddot{x}_{5d} + k_5e_6) + \dot{\tilde{\omega}}\tilde{\omega} \\ \dot{V}_{e5,e6} &= -k_5e_5^2 + e_6(-e_5 + x_3v_1 + w - \hat{w} - \ddot{x}_{5d} + k_5e_6) + \dot{\tilde{\omega}}\tilde{\omega} - \tilde{\omega}e_6 \\ \dot{V}_{e5,e6} &= -k_5e_5^2 + e_6(-e_5 + x_3v_1 + w - \hat{w} - \ddot{x}_{5d} + k_5e_6) + \tilde{\omega}(\dot{\tilde{\omega}} - e_6)\end{aligned} \tag{4.25}$$

Choose

$\omega = e_5 - x_3v_1 + \hat{w} + \ddot{x}_{5d} - k_5e_6 - k_6e_6$

$\dot{\tilde{\omega}} = e_6 - k_7\tilde{\omega}$

$\dot{\hat{\omega}} = -\dot{\tilde{\omega}} \qquad (\omega = \hat{\omega} + \tilde{\omega}$ and $\omega$=constant)

Putt $\omega$ in Eq. (4.25) to get:

$$\dot{V}_{e5,e6} = -k_5 e_5^2 - k_6 e_6^2 - k_7 \tilde{\omega}^2 \leq 0 \tag{4.26}$$

**Step 4:**

Overall lyapunov function can be written as:

$$V = V_{e_1,e_2} + V_{e_3,e_4} + V_{e_5,e_6} \tag{4.27}$$

Take derivative of lyapunov function.

$$\begin{aligned} \dot{V} &= \dot{V}e_1, e_2 + \dot{V}e_3, e_4 + \dot{V}_{e_5,e_6} \\ \dot{V} &= -k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 - k_4 e_4^2 - k_5 e_5^2 - k_6 e_6^2 - k_7 \tilde{\omega}^2 \leq 0 \\ \dot{V} &\leq 0 \end{aligned} \tag{4.28}$$

General diagram of backstepping control is shown in Figure (4.1)

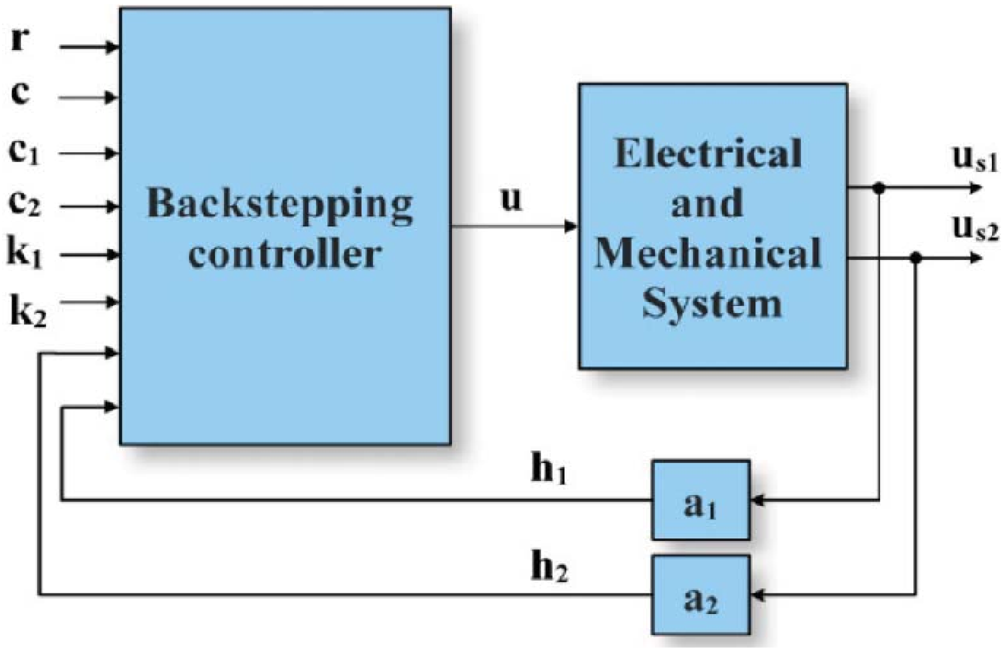


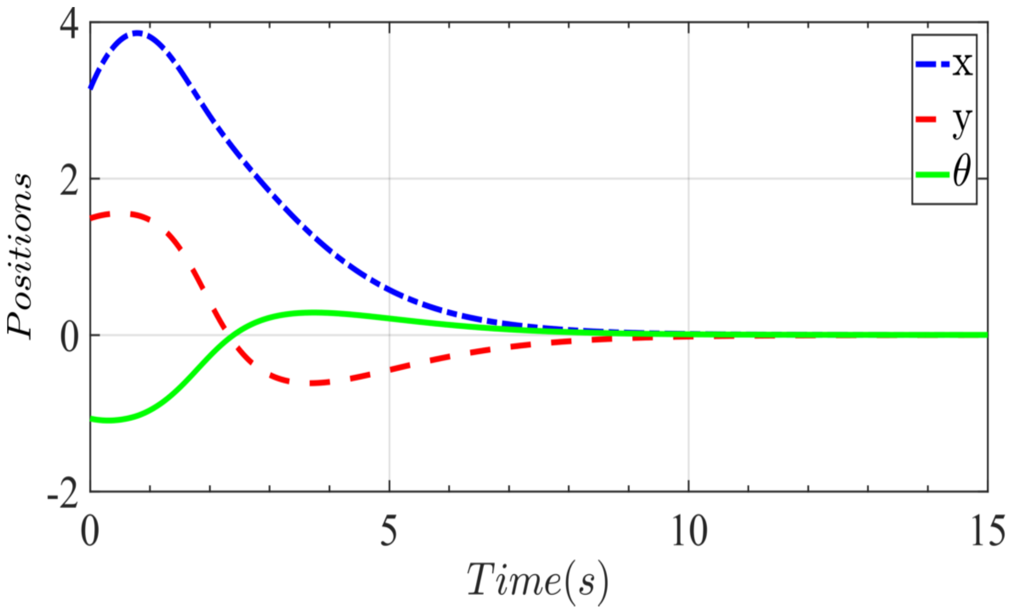

FIGURE 4.1: Backstepping control method

Figure (4.1) show the block diagram of backtepping feedback control technique (nonlinear control technique).

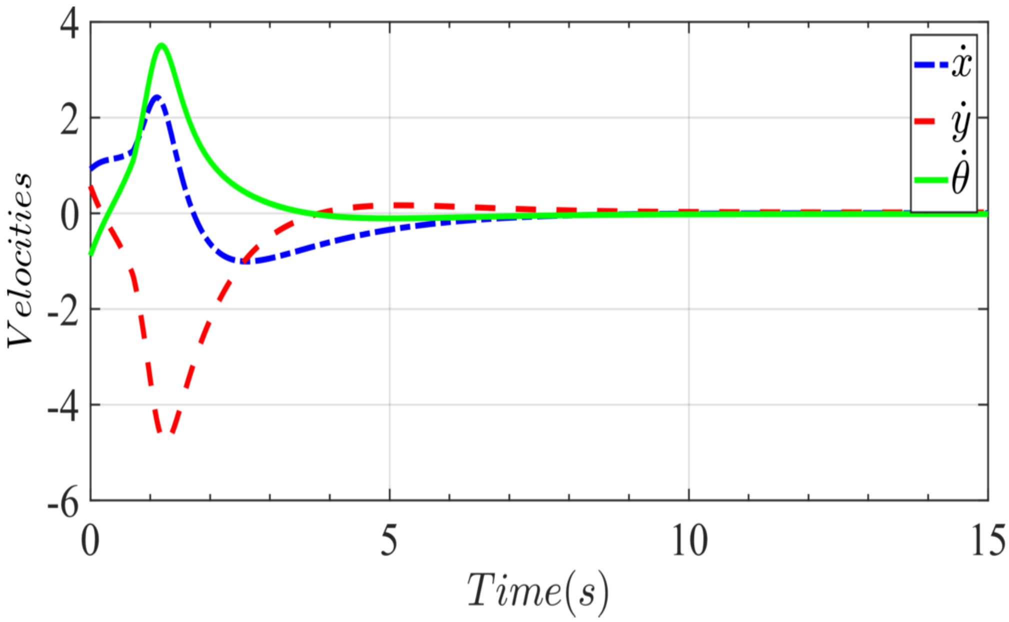
### 4.2.1 Simulation Results and Discussion

#### 4.2.1.1 Planar PPR Manipulator

#### 4.2.1.2 Simulation 1:

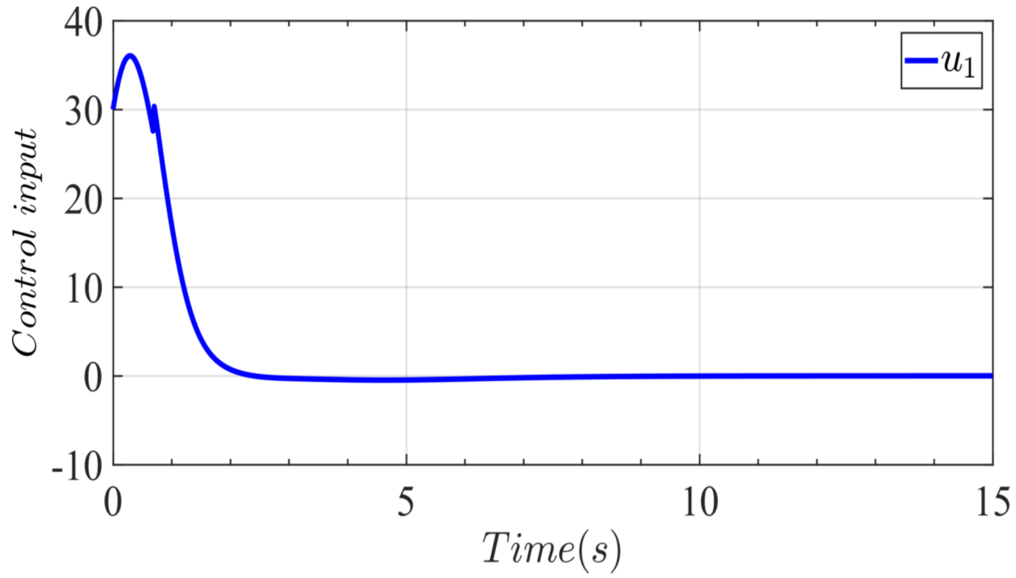



(a)

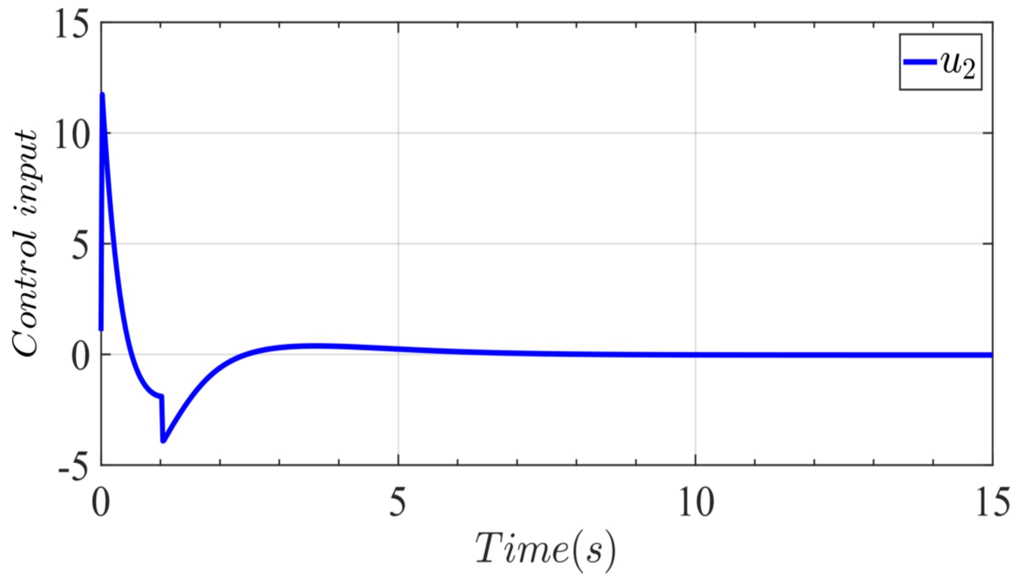



(b)

FIGURE 4.2: Stabilization of Planar PPR Manipulator, (a) Positions, (b) Velocities




(a)

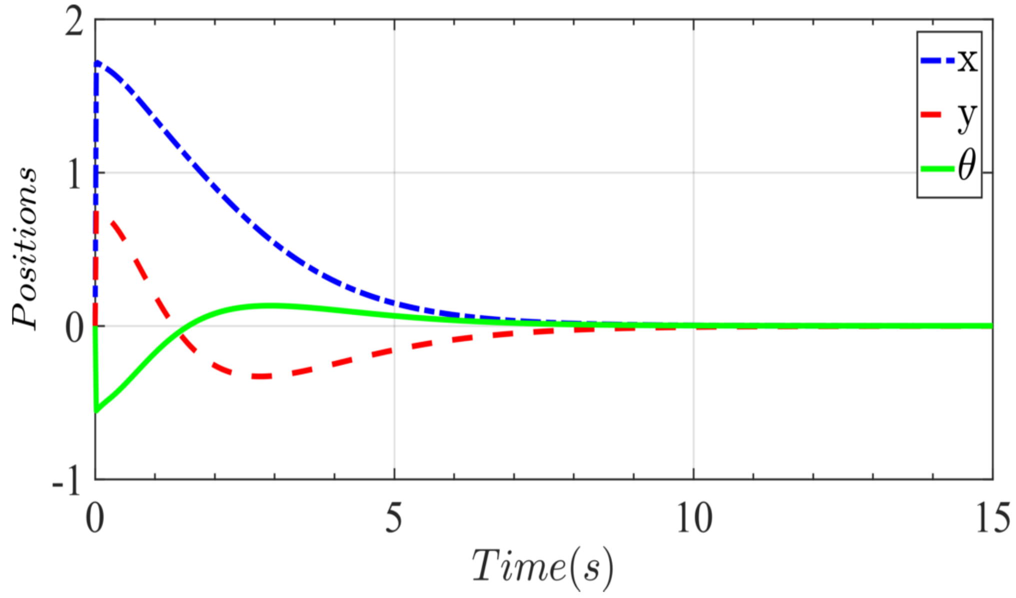



(b)

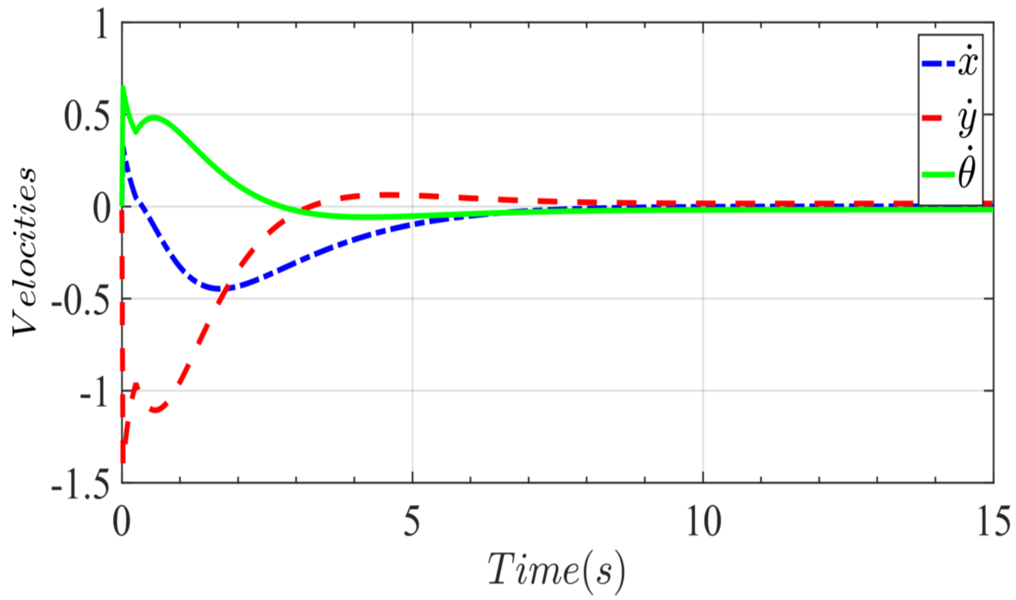
FIGURE 4.3: Stabilization of Planar PPR Manipulator, (a) Control input $u_1 = v_1$, (b) Control input $u_2 = v_2$

**For Initial Condition 1:** Fig. 4.2 represents a closed loop response of Planar PPR Manipulator with backstepping control $u_1$ and $u_2$ in Figure (4.3). The controller parameters are chosen as $k_1 = 1$, $k_2 = 2.1$, $k_3 = 2$, $k_4 = 2.7$, $k_5 = 1.2$, $k_6 = 1$, $k_7 = 1.2$. The controller stabilizes the system from initial condition $[x(0), y(0), \theta(0), \dot{x}(0), \dot{y}(0), \dot{\theta}(0) = \pi, 1.8, -1.2, 1, 0.5, -0.5]^T$ to stable equilibrium position $[0, 0, 0, 0, 0, 0]^T$ in less than 8 seconds. The smooth control effort is shown in Figure (4.3). This control strategy is not robust and simulation is carried out in thee absence of external disturbances.

#### 4.2.1.3 Simulation 2:

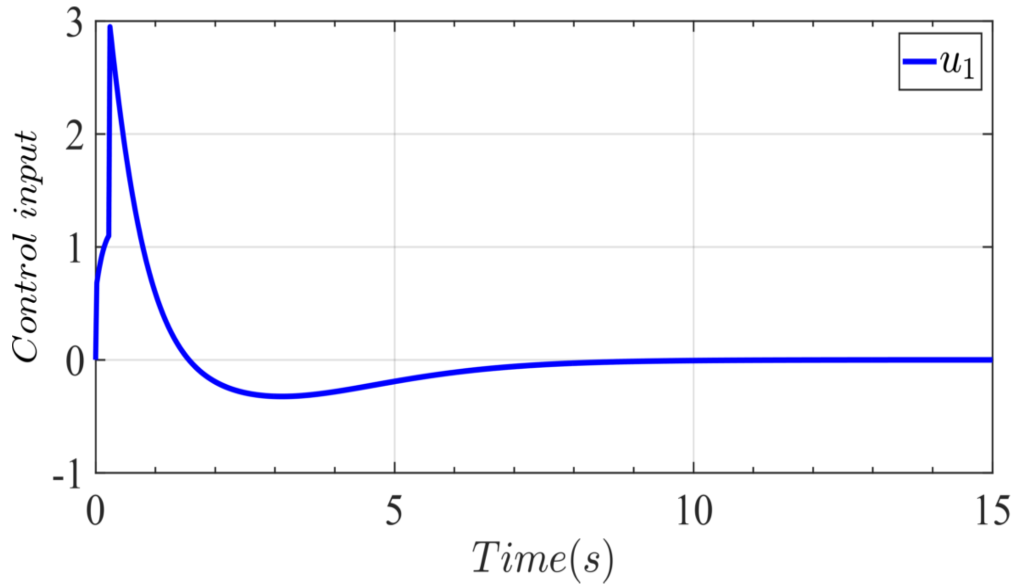



(a)

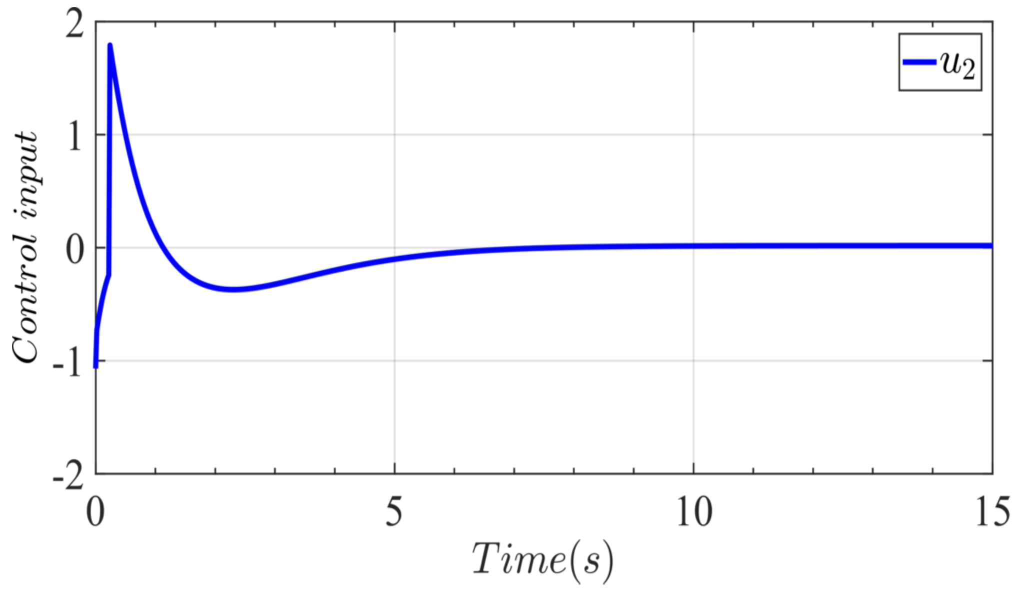



(b)

FIGURE 4.4: Stabilization of Planar PPR Manipulator, (a) Positions, (b) Velocities




(a)




(b)

FIGURE 4.5: Stabilization of Planar PPR Manipulator, (a) Control input $u_1 = v_1$, (b) Control input $u_2 = v_2$

**For Initial Condition 2:** In simulation 2, results are simulated with different initial conditions as compare to simulation 1. Figure 4.4 represents a closed loop response of Planar PPR Manipulator with Backstepping Control $u_1$ and $u_2$ in Figure 4.5 respectively. The controller parameters are chosen as $k_1 = 1.3,\ k_2 = 2.2,$

$k_3 = 1$, $k_4 = 3$, $k_5 = 1$, $k_6 = 1.8$, $k_7 = 1.7$. The controller stabilizes the system from initial condition $[x(0), y(0), \theta(0), \dot{x}(0), \dot{y}(0), \dot{\theta}(0) = 0, 0, 0, 0.3, 0, 0]^T$ to stable equilibrium position $[0, 0, 0, 0, 0, 0]^T$ in less than 7 seconds. The smooth control effort is shown in Figure 4.5. This control strategy is not robust and simulation is carried in the absence of external disturbances.

#### 4.2.1.4 Simulation 3:

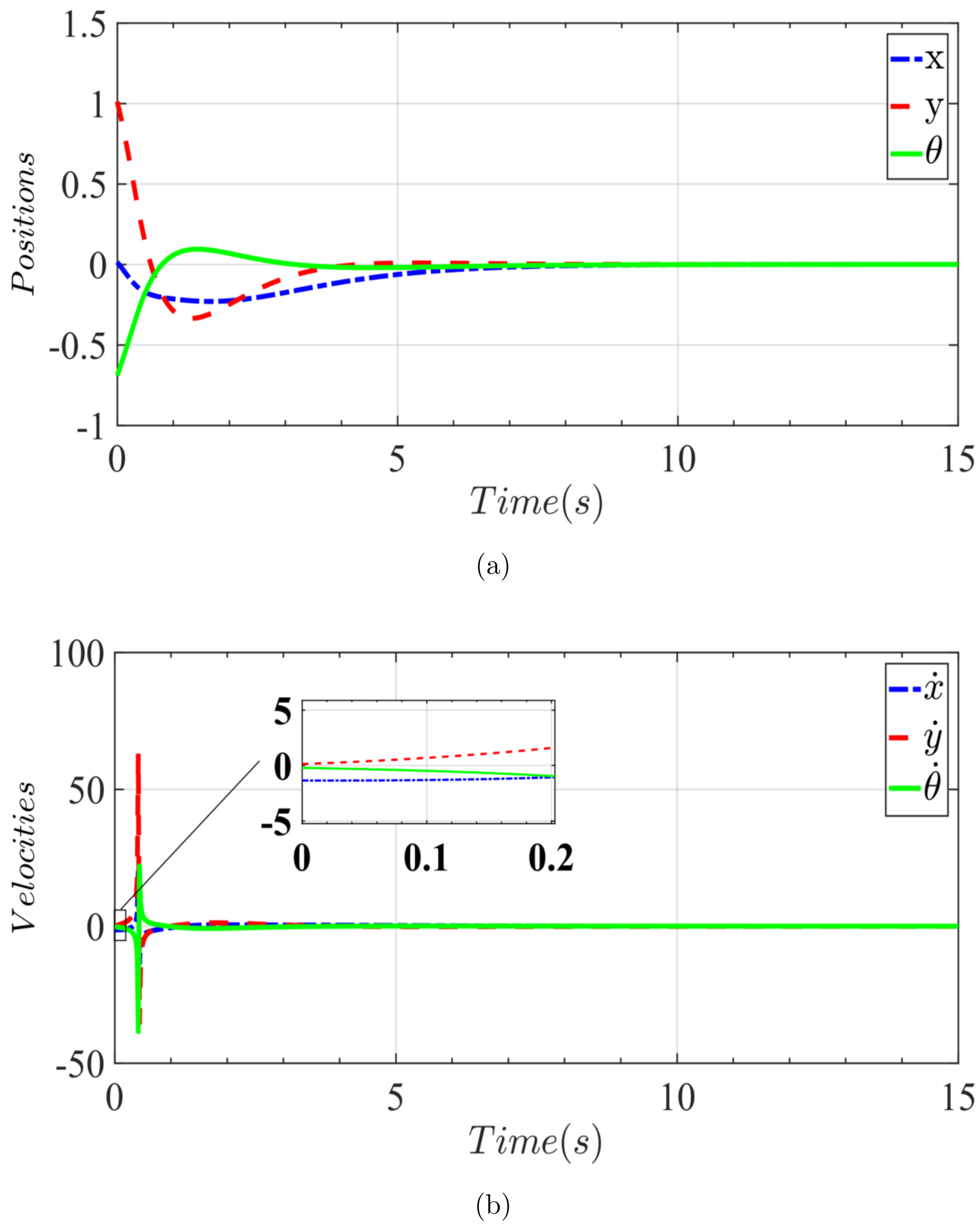
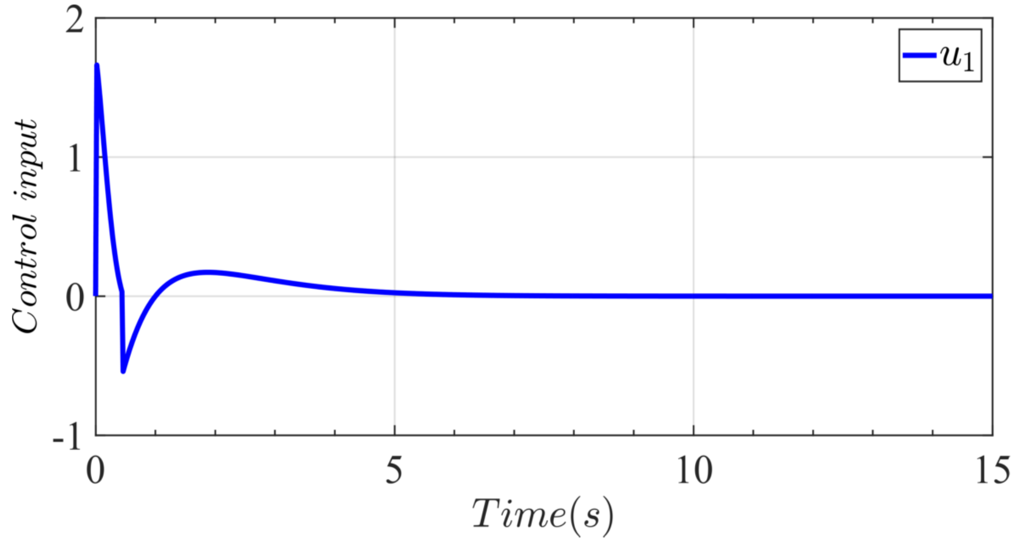
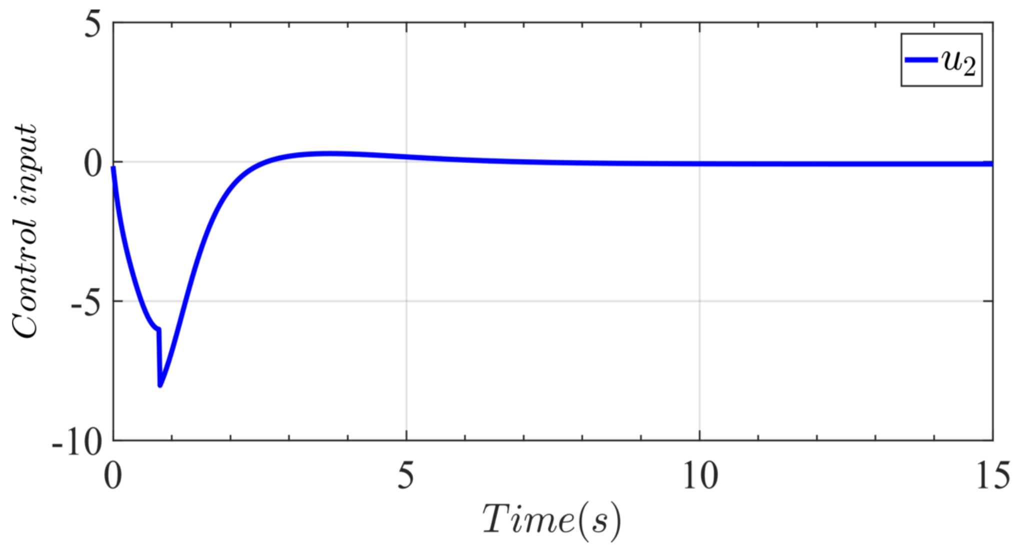



FIGURE 4.6: Stabilization of Planar PPR Manipulator, (a) Positions, (b) Velocities




(a)




(b)

FIGURE 4.7: Stabilization of Planar PPR Manipulator, (a) Control input $u_1 = v_1$, (b) Control input $u_2 = v_2$

**For Initial Condition 3:** In simulation 3, results are simulated with different initial conditions as compared to simulation 1 and simulation 2. Figure 4.6 represents a closed loop response of Planar PPR Manipulator with Backstepping Control $u_1$ and $u_2$ in Figure 4.7. The controller parameters are chosen as $k_1 = 1.4,\ \ k_2 = 1.2$, $k_3 = 2,\ k_4 = 1.5,\ k_5 = 1.2,\ k_6 = 2.8,\ k_7 = 2$. The controller stabilizes the system from initial condition $[x(0), y(0), \theta(0), \dot{x}(0), \dot{y}(0), \dot{\theta}(0) = 0, 1, -0.6, -1, 0.1, 0]^T$ to stable equilibrium position $[0, 0, 0, 0, 0, 0]^T$ in less than 6 seconds. The smooth control effort is shown in Figure 4.7. This control strategy is not robust and simulation is carried out in the absence of external disturbances.

# Chapter 5

# Adaptive Sliding Mode Control Technique

## 5.1 Introduction

The Sliding Mode Control (SMC) has attracted the interest of researchers quite late as compared to the other control techniques discussed in the this thesis. Due to its simplicity, fast response and robustness to external noise it has become an epicenter in the field control design [41, 42]. These attributes of SMC just rely on the design of the switching surface. The two phases of the SMC would be the reaching phase and the sliding phase. During the reaching phase, the system trajectories approache towards sliding surface. Whereas in sliding phase the system trajectories are forced by controller to slide on the sliding surface towards its origin. The system response is dependent upon the parameters of the surface and remains insensitive to parameter variations and external disturbances.

### 5.1.1 Proposed Algorithm 1

The algorithm for Sliding Mode Control (SMC) can be designed in the following series of steps:

**Step 1:**

The second-order chained form system (3.1) is written in state-space form as follows;

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= v_1 \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= v_2 \\ \dot{x}_5 &= x_6 \\ \dot{x}_6 &= x_3 v_1 \end{aligned} \tag{5.1}$$

**Step 2:**

Define $v_1 = x_3$, the system (5.1) is written as:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= v_2 \\ \dot{x}_5 &= x_6 \\ \dot{x}_6 &= x_3^2 + w - \hat{w} - \tilde{w} \end{aligned} \tag{5.2}$$

Equation (5.2) can be divided into two sub systems represented by equation (5.3) and equation (5.4).

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= v_2 \end{aligned} \tag{5.3}$$

and subsystem 2 is:

$$\begin{aligned} \dot{x}_5 &= x_6 \\ \dot{x}_6 &= x_3^2 + w - \hat{w} - \tilde{w} \end{aligned} \tag{5.4}$$

**Step 3:**

The step 3 defines the sliding surface for subsystem provided in equation (5.3):
$s_1 = x_1 + c_1x_2 + c_2x_3 + x_4$

$$s = (\frac{d}{dt} + 1)^{n-1}x_1 \tag{5.5}$$

The coefficients $c_1$ and $c_2$ can be found by using (5.5) for Hurwitz Sliding Surface as given below:

$$\begin{aligned} s_1 &= x_1 + 3x_2 + 3x_3 + x_4 \\ \dot{s_1} &= x_2 + 3x_3 + 3x_4 + v_2 \end{aligned} \tag{5.6}$$

Defining a Lyapunov function:

$$\begin{aligned} V_1 &= \frac{1}{2}s_1^2 \\ \dot{V_1} &= s_1\dot{s_1} \\ \dot{V_1} &= s_1(x_2 + 3x_3 + 3x_4 + v_2) \end{aligned} \tag{5.7}$$

Choose $v_2 = -x_2 - 3x_3 - 3x_4 - k_1s_1 - k_2sign(s_1)$ to get:

$$\dot{V_1} = -k_1s_1^2 - k_2|s_1| < 0 \tag{5.8}$$

**Step 4:**

The step 4 includes defining a sliding surface for the subsystem of equation (5.4):

$$s_2 = x_5 + x_6 \tag{5.9}$$

Now take a nominal system:

$$\begin{aligned} s_2 &= x_5 + x_6 \\ \dot{s_2} &= x_6 + x_3^2 + w \end{aligned} \tag{5.10}$$

and define a Lyapunov function:

$$V_2 = \frac{1}{2}s_2^2 + \frac{1}{2}\tilde{w}^2 \tag{5.11}$$

$$\begin{aligned}\dot{V}_2 &= s_2\dot{s}_2 + \tilde{w}\dot{\tilde{w}} \\ \dot{V}_2 &= s_2(x_6 + x_3^2 + w - \hat{w} - \tilde{w}) + \tilde{w}\dot{\tilde{w}} \\ \dot{V}_2 &= s_2(x_6 + x_3^2 + w - \hat{w}) + \tilde{w}(\dot{\tilde{w}} - s_2)\end{aligned} \tag{5.12}$$

Choose $w = -x_6 - x_3^2 + \hat{\omega} - k_3 s_2 - k_4 sign(s_2)$ and $\dot{\tilde{w}} = s_2 - k_5\tilde{w}$ to get:

$$\dot{V}_2 = -k_3 s_2^2 - k_4|s_2| - k_5\tilde{w}^2 < 0 \tag{5.13}$$

**Step 5:** This step constitutes defining an overall Lyapunov for system expressed in equation (5.2)

$$\begin{aligned}V &= \frac{1}{2}V_1^2 + \frac{1}{2}V_2^2 \\ \dot{V} &= V_1\dot{V}_1 + V_2\dot{V}_2 \\ \dot{V} &= -k_1 s_1^2 - k_2|s_1| - k_3 s_2^2 - k_4|s_2| - k_5\tilde{w}^2 < 0\end{aligned} \tag{5.14}$$

General diagram of backstepping control is shown in Figure (5.1) which is also shown in previous chapter.

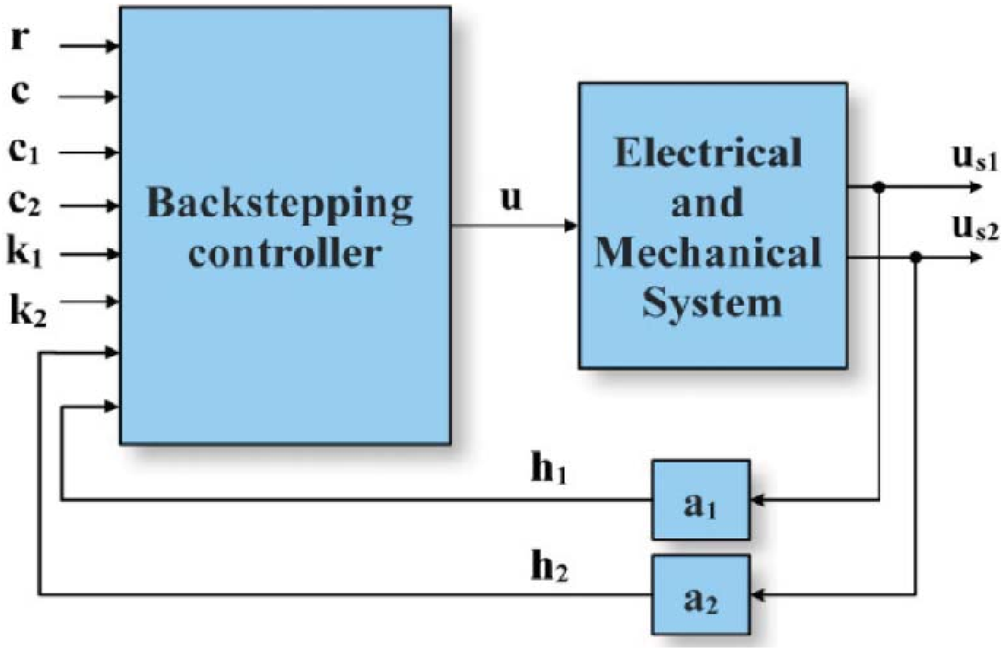


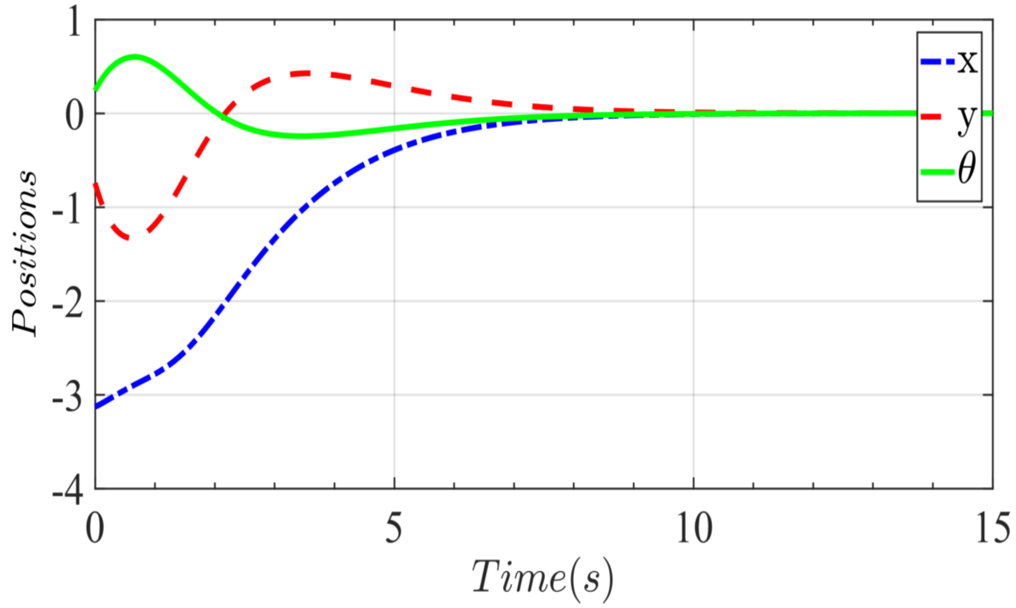

FIGURE 5.1: Backstepping control method

Figure (5.1) show the block diagram of backtepping feedback control technique which is basically nonlinear control technique.

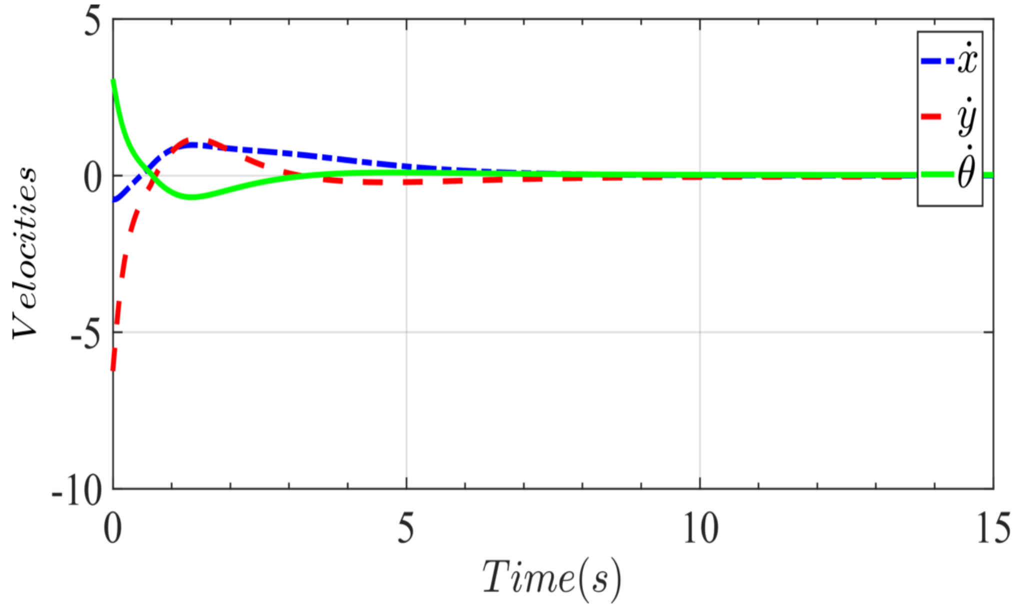
### 5.1.2 Simulation Results and Discussion

#### 5.1.2.1 3-DOF Manipulator with a Free Link

#### 5.1.2.2 Simulation 1:

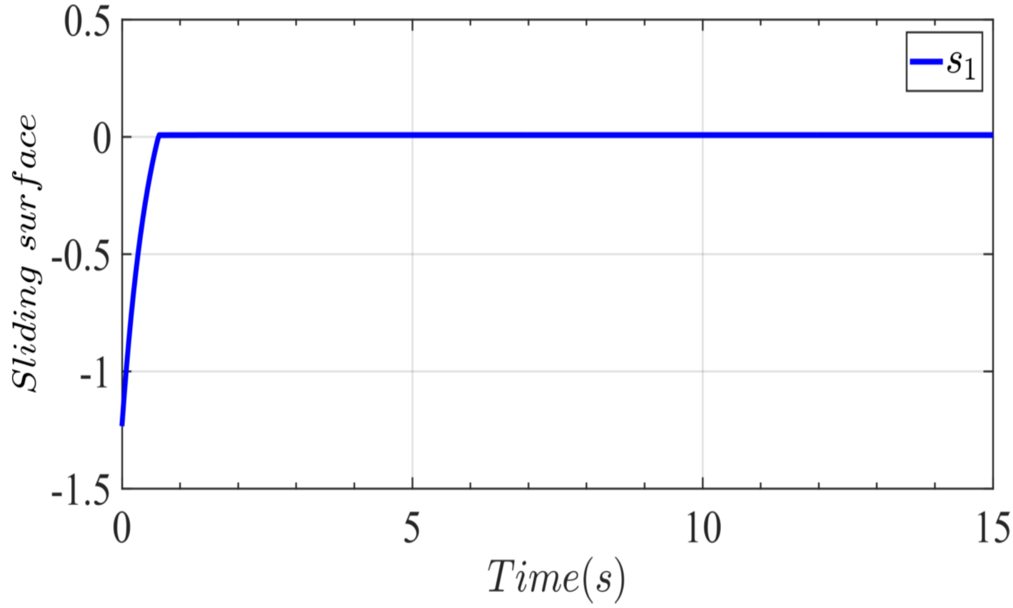



(a)

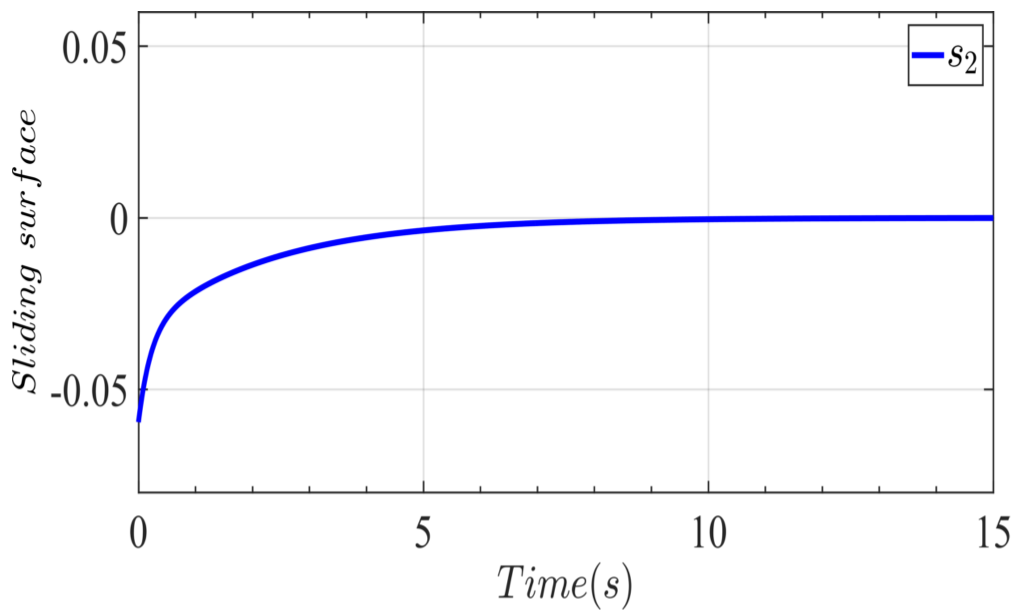



(b)

FIGURE 5.2: Stabilization of 3-DOF Manipulator with a Free Link, (a) Positions, (b) Velocities



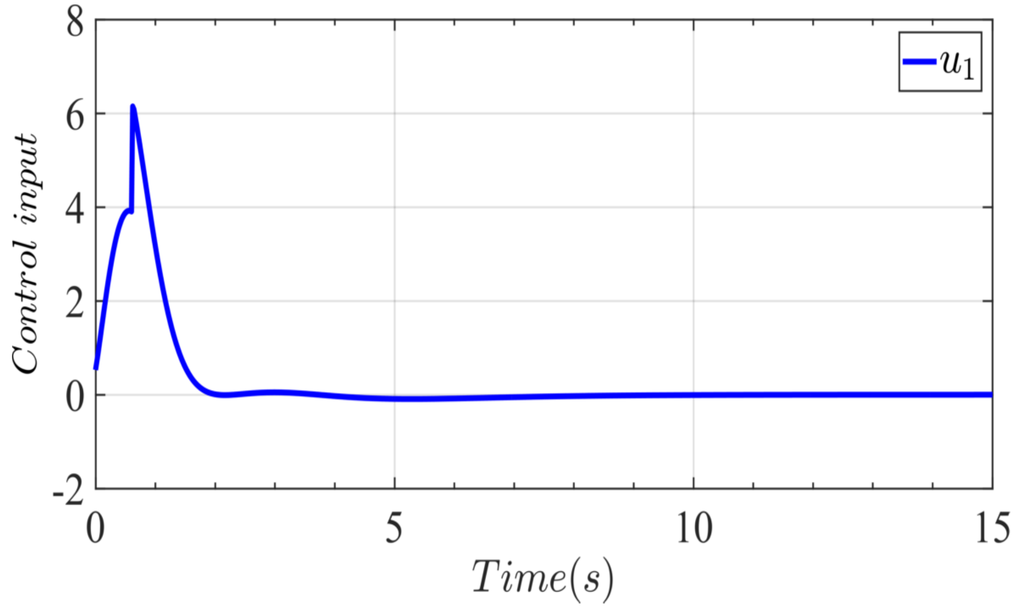

(a)



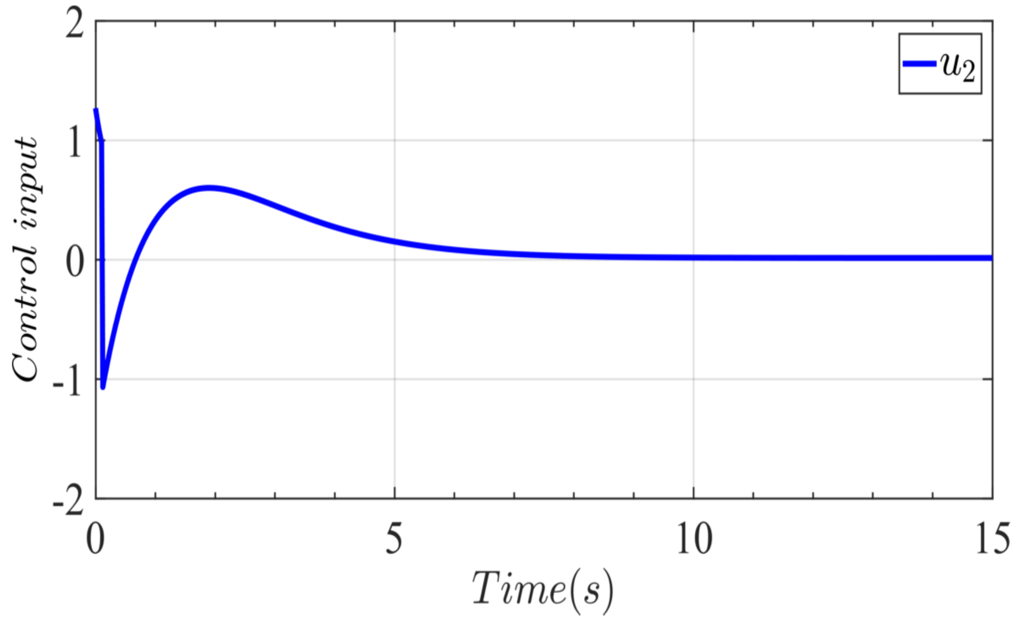

(b)

FIGURE 5.3: Stabilization of 3-DOF Manipulator with a Free Link, (a) Sliding surface $s_1$, (b) Sliding surface $s_2$

**For Initial Condition 1:** Figure 5.2 represents a closed loop response of 3-DOF Manipulator with a Free Link with adaptive sliding mode control $u_1$ and $u_2$ in Figure 5.4. The controller parameters are chosen as $c_1 = 3$, $c_2 = 3$, $k_1 = 1.3$, $k_2 = 2$, $k_3 = 1$, $k_4 = 1.6$, $k_5 = 1.5$. The controller stabilizes the system from initial conditions $[x(0), y(0), \theta(0), \dot{x}(0), \dot{y}(0), \dot{\theta}(0) = -\pi, -0.8, 0.2, -0.8, -6, 2.5]^T$ to stable equilibrium position $[0, 0, 0, 0, 0, 0]^T$ in less than 7 seconds. The sliding surfaces $s_1$ and $s_2$ shown in Figure 5.3 becomes 0 in about 5 seconds. This control strategy is robust but simulation is carried out in the absence of external disturbances.



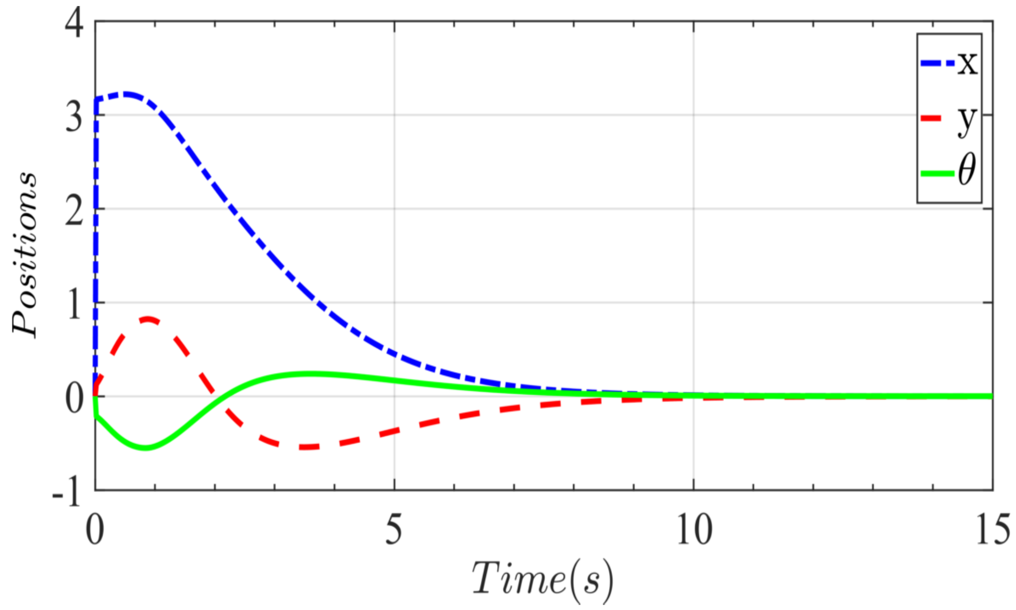

(a)



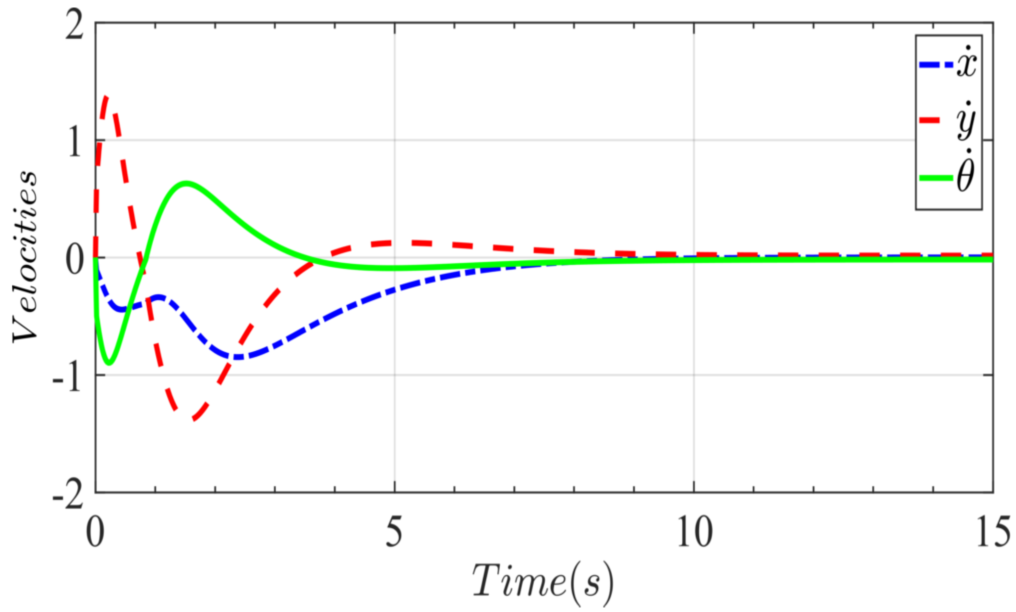

(b)

FIGURE 5.4: Stabilization of 3-DOF Manipulator with a Free Link, (a) Control input $u_1 = v_1$, (b) Control input $u_2 = v_2$

#### 5.1.2.3 Simulation 2



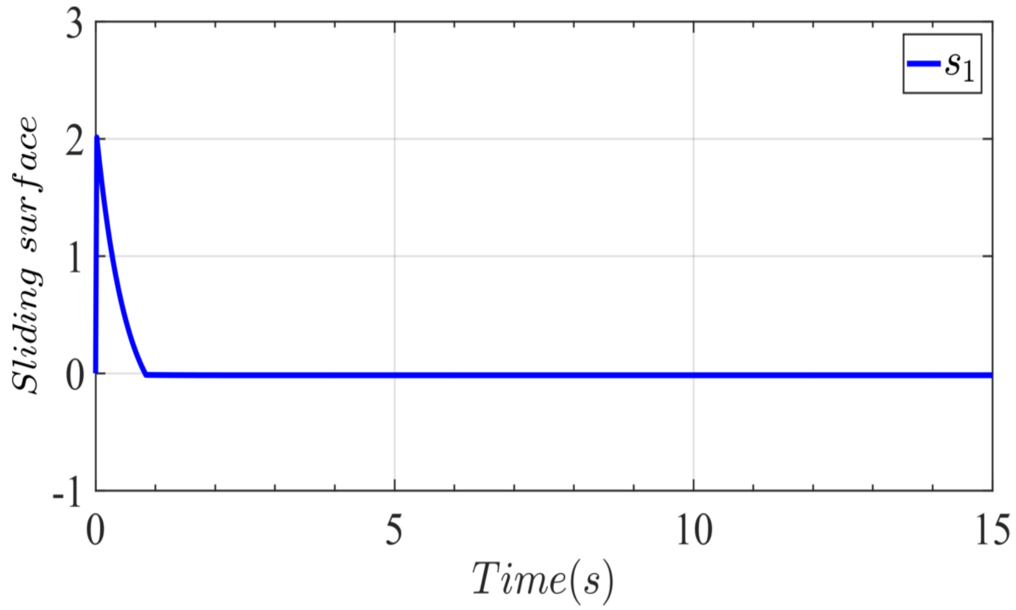

(a)



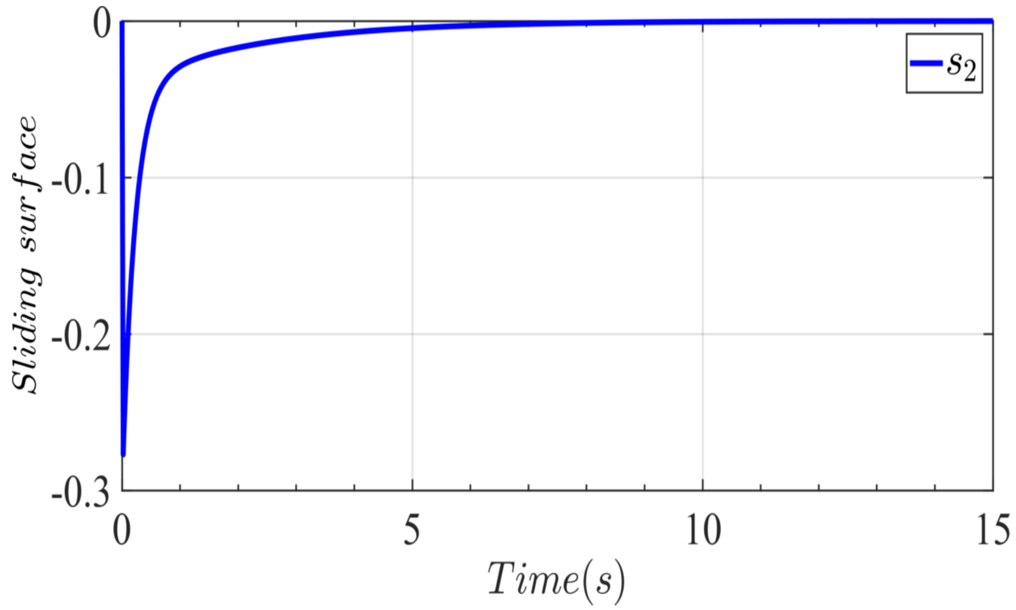

(b)

FIGURE 5.5: Stabilization of 3-DOF Manipulator with a Free Link,
(a) Positions, (b) Velocities

Figure (5.5) represents the trajectories of positions and velocities of 3-DOF manipulator with a free link.



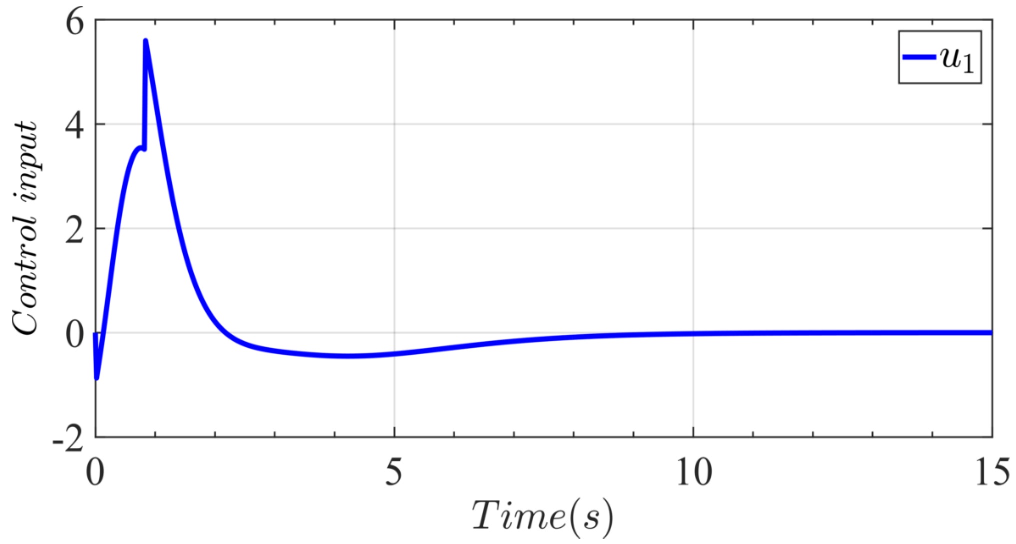

(a)



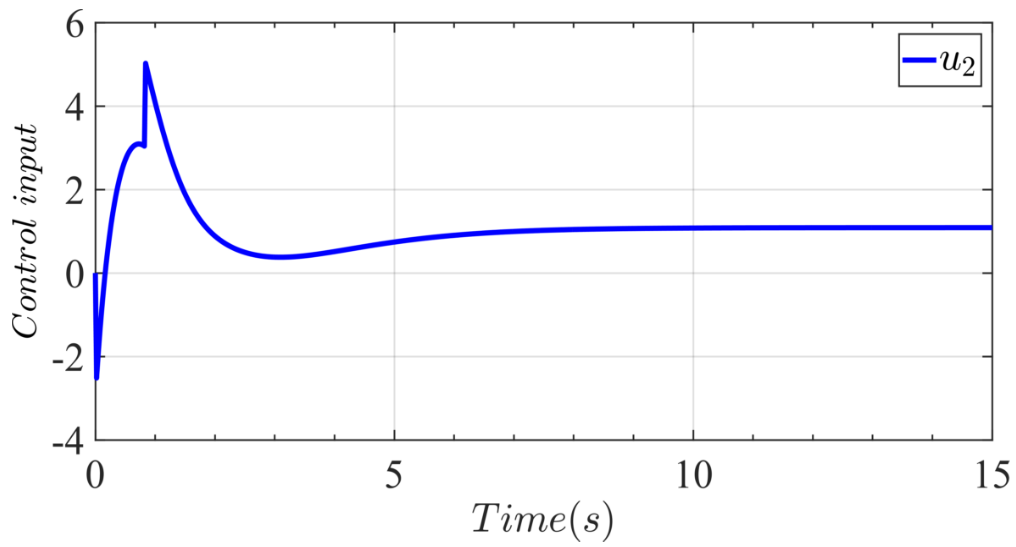

(b)

FIGURE 5.6: Stabilization of 3-DOF Manipulator with a Free Link, (a) Sliding surface $s_1$, (b) Sliding surface $s_2$

Figure (5.6) represents the trajectories of sliding surfaces (linear sliding surface) of 3-DOF manipulator with a free link. In figure (5.6a) and (5.6b) sliding surfaces converge towards equilibrium position in finite time, which ensure finite time convergence of sliding surfaces.




(a)



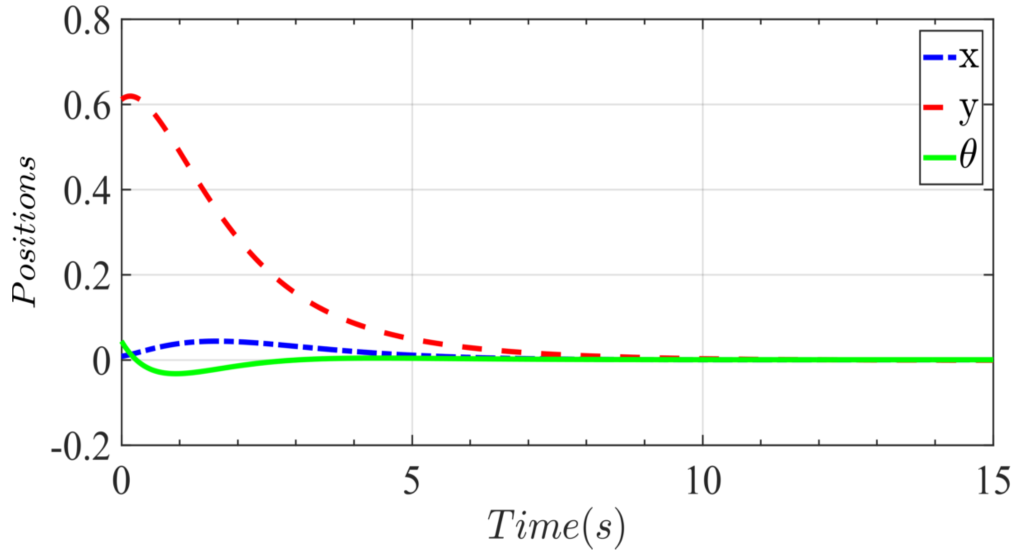

(b)

FIGURE 5.7: Stabilization of 3-DOF Manipulator with a Free Link, (a) Control input $u_1 = v_1$, (b) Control input $u_2 = v_2$

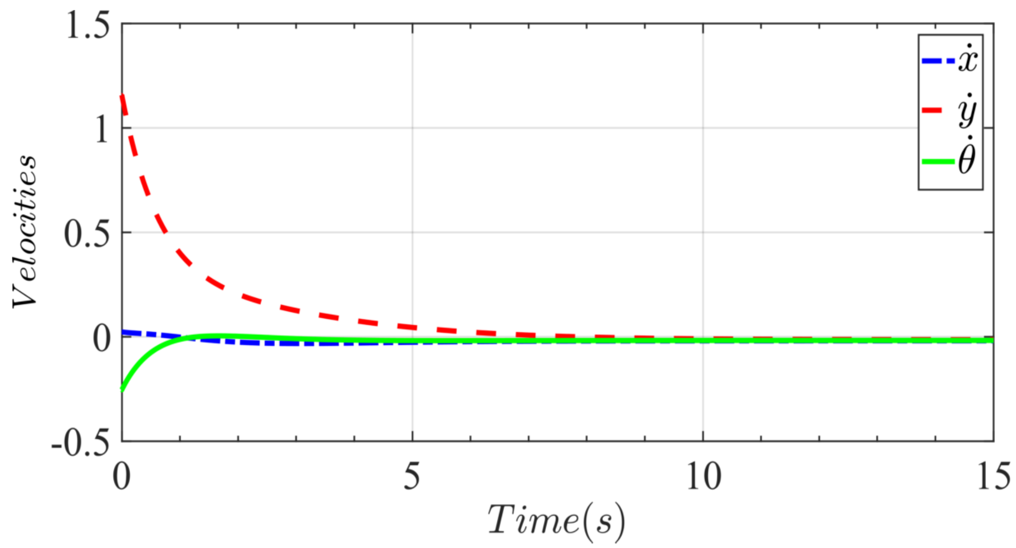
**For Initial Condition 2:** In simulation 2, results are simulated with different initial conditions as compare to simulation 1. Figure 5.5 represents a closed loop response of 3-DOF Manipulator with a Free Link with adaptive sliding mode control $u_1$ and $u_2$ in Figure 5.7. The controller parameters are chosen as $c_1 = 3$, $c_2 = 3$, $k_1 = 1$, $k_2 = 2.2$, $k_3 = 1.7$, $k_4 = 1$, $k_5 = 1.1$. The controller stabilizes the system from initial condition $[x(0), y(0), \theta(0), \dot{x}(0), \dot{y}(0), \dot{\theta}(0) = 0, 0, 0, 0, 0, 0]^T$

to stable a equilibrium position $[0, 0, 0, 0, 0, 0]^T$ in less than 8 seconds. The sliding surfaces $s_1$ and $s_2$ are shown in Figure 5.6 which become 0 in about 5 seconds. This control strategy is robust but simulation is carried out in the absence of external disturbances.

#### 5.1.2.4 Simulation 3

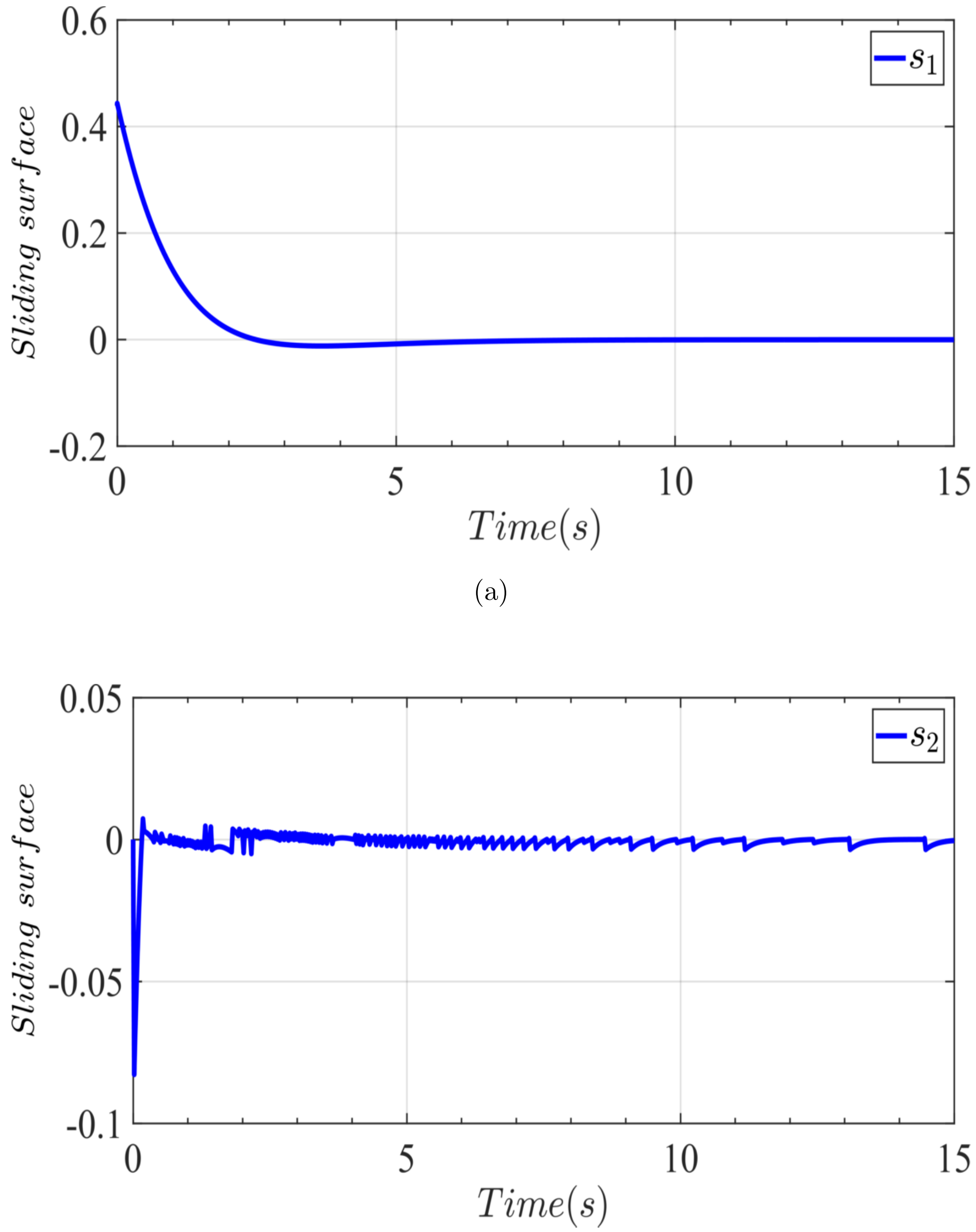



(a)




(b)

FIGURE 5.8: Stabilization of 3-DOF Manipulator with a Free Link, (a) Positions, (b) Velocities



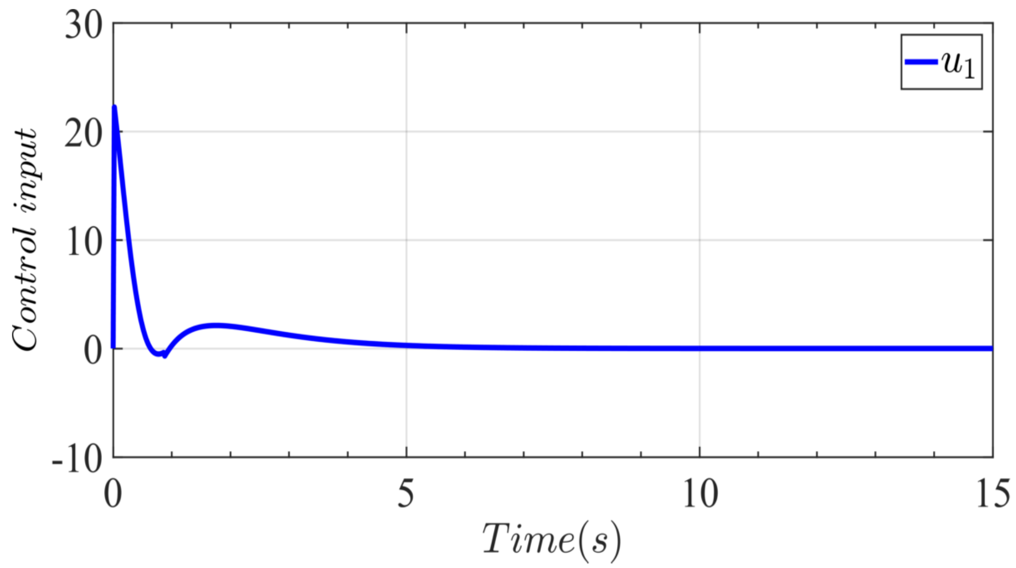

(a)



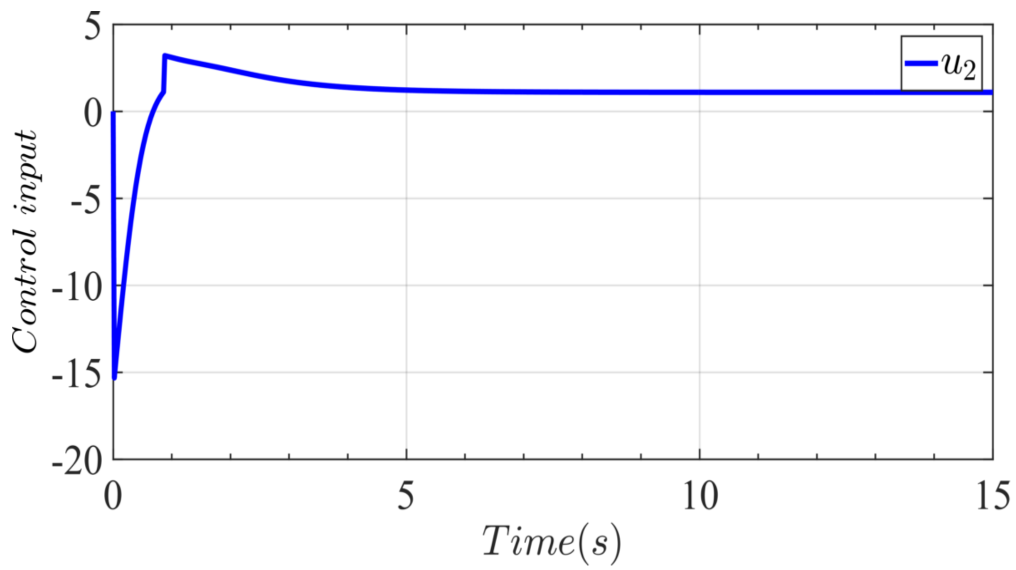
(b)

FIGURE 5.9: Stabilization of 3-DOF Manipulator with a Free Link, (a) Sliding surface $s_1$, (b) Sliding surface $s_2$

**For Initial Condition 3:** In simulation 2, results are simulated with different initial conditions as compared to simulation 1 and simulation 2. Figure 5.8 represents a closed loop response of 3-DOF Manipulator with a Free Link with adaptive sliding mode control $u_1$ and $u_2$ in Figure 5.10. The controller parameters are chosen as $c_1 = 3$, $c_2 = 3$, $k_1 = 1$, $k_2 = 1.2$, $k_3 = 2.7$, $k_4 = 1.4$, $k_5 = 1$. The controller stabilizes the system from initial condition $[x(0), y(0), \theta(0), \dot{x}(0), \dot{y}(0), \dot{\theta}(0)$ =0, 0.01, 0.6, 0, 1.2, -0.25$]^T$ to a stable equilibrium position $[0, 0, 0, 0, 0, 0]^T$ in less than 6 seconds. The sliding surfaces $s_1$ and $s_2$ shown in Figure 5.9 becomes 0 in about 3 seconds. This control strategy is robust but simulation is carried out in the absence of external disturbances.




(a)




(b)

FIGURE 5.10: Stabilization of 3-DOF Manipulator with a Free Link, (a) Control input $u_1 = v_1$, (b) Control input $u_2 = v_2$

### 5.1.3 Proposed Algorithm 2: With Disturbance

The second-order chained form system can be written in state-space form as presented in 5.1:

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= v_1\\ \dot{x}_3 &= x_4\\ \dot{x}_4 &= v_2\\ \dot{x}_5 &= x_6\\ \dot{x}_6 &= x_3 v_1\end{aligned} \tag{5.15}$$

**Step 1:**

The system (5.15) with external disturbances can be written as:

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= v_1 + d_1(x,t)\\ \dot{x}_3 &= x_4\\ \dot{x}_4 &= v_2 + d_2(x,t)\\ \dot{x}_5 &= x_6\\ \dot{x}_6 &= x_3 v_1 + d_3(x,t)\end{aligned} \tag{5.16}$$

which can be written as:

$$\begin{aligned}\ddot{x}_1 &= v_1 + d_1(x,t)\\ \ddot{x}_3 &= v_2 + d_3(x,t)\\ \ddot{x}_5 &= x_3 v_1 + d_5(x,t)\end{aligned} \tag{5.17}$$

In system (5.17) $d_1(x,t)$, $d_3(x,t)$ and $d_5(x,t)$ are estimated using basis function concept.

$$\begin{aligned}\ddot{x}_1 &= v_1 + \hat{k}_{ij} sin(nt) + \hat{k}_{ij} cos(nt) + \tilde{k}_{ij} sin(nt) + \tilde{k}_{ij} cos(nt)\\ \ddot{x}_3 &= v_2 + \hat{k}_{iz} sin(mt) + \hat{k}_{iz} cos(mt) + \tilde{k}_{iz} sin(mt) + \tilde{k}_{iz} cos(mt)\\ \ddot{x}_5 &= x_3 v_1 + w - \hat{w} - \tilde{w} + \hat{k}_{ih} sin(qt) + \hat{k}_{ih} cos(qt) + \tilde{k}_{ih} sin(qt) + \tilde{k}_{ih} cos(qt)\end{aligned} \tag{5.18}$$

In system (5.18):

$ij = 11, 12, 13, 14, 15.......$

$iz = 21, 22, 23, 24, 25.......$

$ih = 31, 32, 33, 34, 35.......$

$(n, m, q) = 10, 20, 30, 40, 50, 60.......$

**Step 2:**

Define a sliding manifolds

$$\begin{aligned} s_1 &= x_1 + c\dot{x}_1 \\ s_2 &= x_3 + c\dot{x}_3 \\ s_3 &= x_5 + c\dot{x}_5 \end{aligned} \tag{5.19}$$

Take derivative of sliding surface to find control inputs. Choose $c = 1$

$$\begin{aligned} s_1 &= x_1 + c\dot{x}_1 \\ \dot{s}_1 &= \dot{x}_1 + \ddot{x}_1 \\ \dot{s}_1 &= x_2 + v_1 + \hat{k}_{ij}sin(nt) + \hat{k}_{ij}cos(nt) \end{aligned} \tag{5.20}$$

similarly sliding surface $s_2$.

$$\dot{s}_2 = x_4 + v_2 + \hat{k}_{iz}sin(mt) + \hat{k}_{iz}cos(mt) \tag{5.21}$$

and sliding surface $s_3$.

$$\dot{s}_3 = x_6 + x_3v_1 + w - \hat{w} + \hat{k}_{ih}sin(qt) + \hat{k}_{ih}cos(qt) \tag{5.22}$$

**Step 3:**

Define a lyapunov function

$$\begin{aligned} V_1 &= \frac{1}{2}s_1^2 + \frac{1}{2}\tilde{k}_{ij}^2 \\ \dot{V}_1 &= s_1\dot{s}_1 + \tilde{k}_{ij}\dot{\tilde{k}}_{ij} \\ \dot{V}_1 &= s_1(x_2 + v_1 + \hat{k}_{ij}sin(nt) + \hat{k}_{ij}cos(nt) + \tilde{k}_{ij}sin(nt) + \tilde{k}_{ij}cos(nt)) + \tilde{k}_{ij}\dot{\tilde{k}}_{ij} \\ \dot{V}_1 &= s_1(x_2 + v_1 + \hat{k}_{ij}sin(nt) + \hat{k}_{ij}cos(nt)) + s_1\tilde{k}_{ij}[sin(nt) + cos(nt)] + \tilde{k}_{ij}\dot{\tilde{k}}_{ij} \\ \dot{V}_1 &= s_1(x_2 + v_1 + \hat{k}_{ij}sin(nt) + \hat{k}_{ij}cos(nt)) + \tilde{k}_{ij}[s_1(sin(nt) + cos(nt)) + \dot{\tilde{k}}_{ij})] \end{aligned} \tag{5.23}$$

Choose

$\dot{\tilde{k}}_{ij} = -s_1(sin(nt) + cos(nt)) - M\tilde{k}_{ij}$

$v_1 = -x_2 - \hat{k}_{ij}sin(nt) - \hat{k}_{ij}cos(nt) - ks_1 - ksign(s_1)$

Putt in Eq. (5.23) to get:

$$\dot{V}_1 = -ks_1^2 - k|s_1| - M\tilde{k}_{ij}^2 \leq 0 \tag{5.24}$$

Similarly for $s_2$ we get.

$$\dot{V}_2 = -ks_2^2 - k|s_2| - M\tilde{k}_{iz}^2 \leq 0 \tag{5.25}$$

Define a lyapunov function for $s_3$

$$\begin{aligned}
V_3 =& s_3^2 + \frac{1}{2}\tilde{\omega}^2 + \frac{1}{2}\tilde{k}_{ij}^2 \\
V_3 =& s_3^2 + \tilde{\omega}\dot{\tilde{\omega}} + \frac{1}{2}\tilde{k}_{ij}^2 \\
\dot{V}_3 =& s_3\dot{s}_3 + \tilde{\omega}\dot{\tilde{\omega}} + \tilde{k}_{ij}\dot{\tilde{k}}_{ij} \\
\dot{V}_3 =& s_3\dot{s}_3 + \tilde{\omega}\dot{\tilde{\omega}} + \tilde{k}_{ij}\dot{\tilde{k}}_{ij} \\
\dot{V}_3 =& s_3(x_6 + x_3v_1 + w - \hat{w} - \tilde{w} + \hat{k}_{ih}sin(qt) + \hat{k}_{ih}cos(qt) + \tilde{k}_{ih}sin(qt) + \tilde{k}_{ih}cos(qt)) \\
& + \tilde{k}_{ij}\dot{\tilde{k}}_{ij} + \tilde{\omega}\dot{\tilde{\omega}} \\
\dot{V}_3 =& s_3(x_6 + x_3v_1 + w - \hat{w} - \tilde{w} + \hat{k}_{ih}sin(qt) + \hat{k}_{ih}cos(qt)) + \tilde{\omega}[\dot{\tilde{\omega}} - s_3] \\
& + \tilde{k}_{ih}[s_3(sin(qt) + cos(qt)) + \dot{\tilde{k}}_{ih}]
\end{aligned} \tag{5.26}$$

choose $\dot{\tilde{k}}_{ih} = -s_3(sin(qt) + cos(qt)) - M\tilde{k}_{ih}$

$\dot{\tilde{\omega}} = s_3 - k\tilde{\omega}$

$w = -x_6 - x_3v_1 + \hat{w} - \hat{k}_{ih}sin(qt) - \hat{k}_{ih}cos(qt) - ks_3 - ksign(s_3)$

Putt in Eq. (5.26) to get:

$$\dot{V}_3 = -ks_3^2 - k|s_3| - M\tilde{k}_{ih}^2 - k\tilde{\omega}^2 \tag{5.27}$$

Overall lyapunov function and its derivative will becomes:

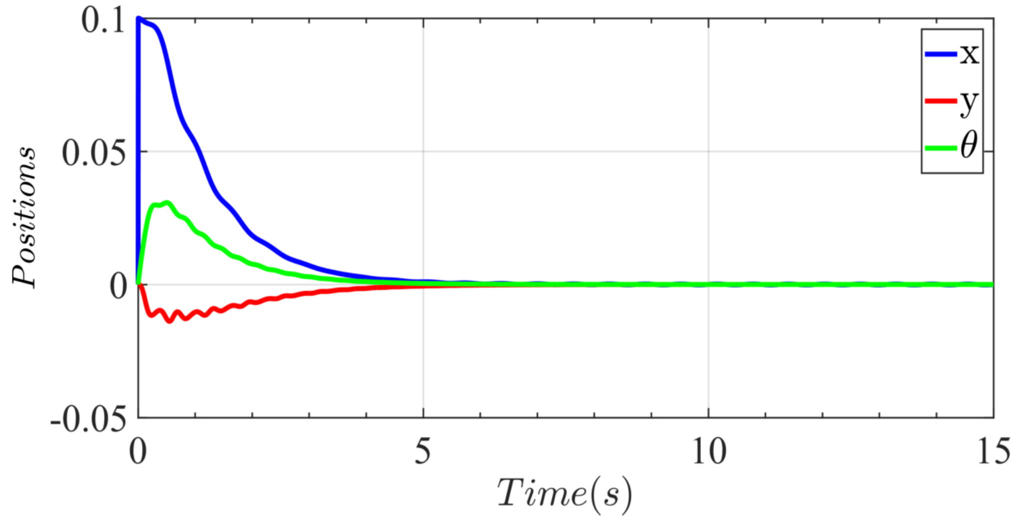
$$V = V_1 + V_2 + V_3 \tag{5.28}$$

$$\dot{V} = -ks_1^2 - k|s_1| - M\tilde{k}_{ij}^2 - ks_2^2 - k|s_2| - M\tilde{k}_{iz}^2 - ks_3^2 - k|s_3| - M\tilde{k}_{ih}^2 - k\tilde{\omega}^2 \leq 0 \tag{5.29}$$

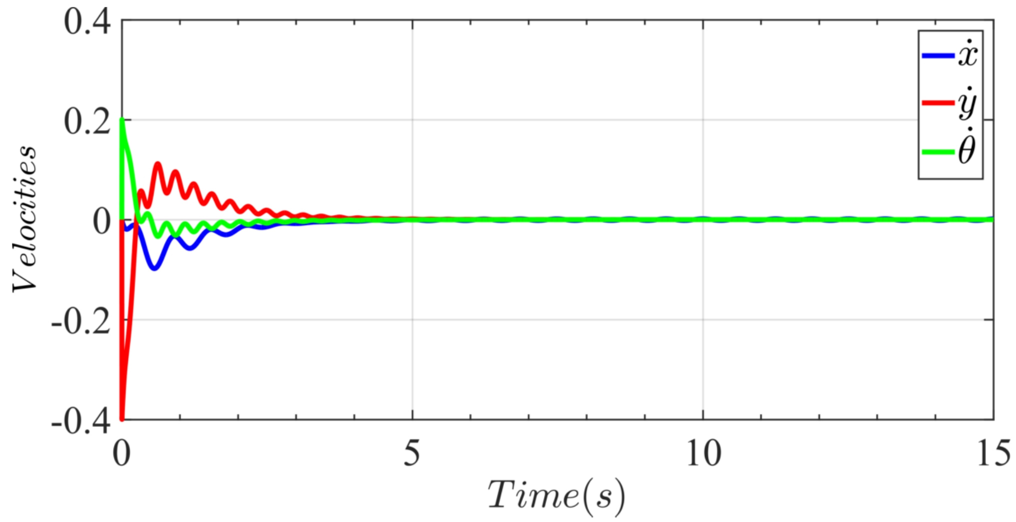
### 5.1.4 Simulation Results and Discussion

#### 5.1.4.1 3-DOF Manipulator with a Free Link

#### 5.1.4.2 Simulation 1:

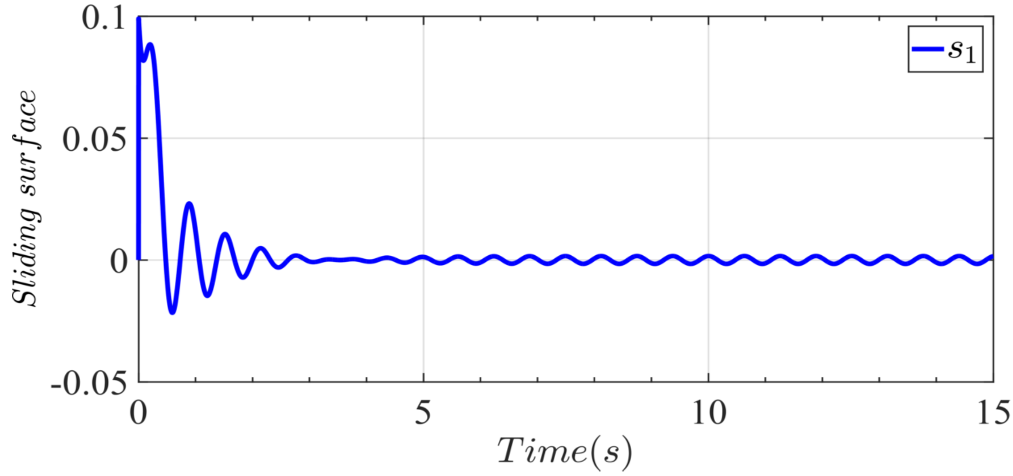



(a)

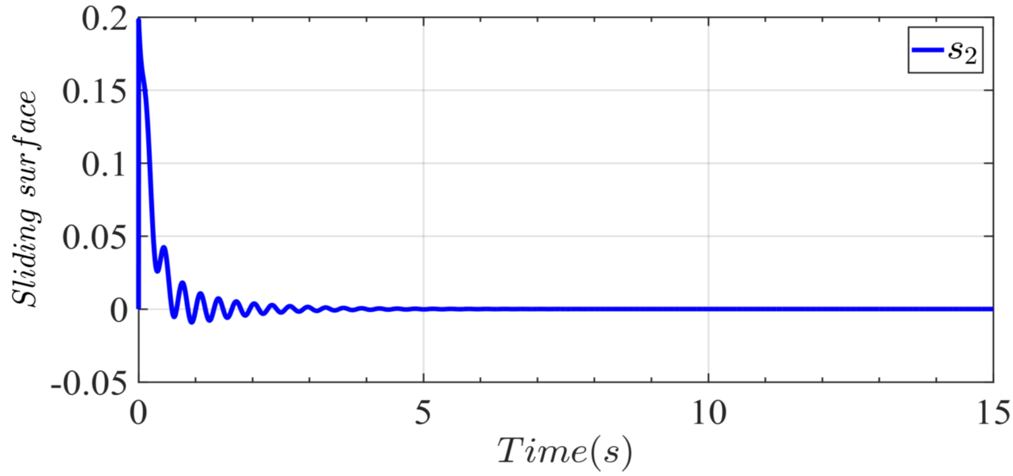



(b)

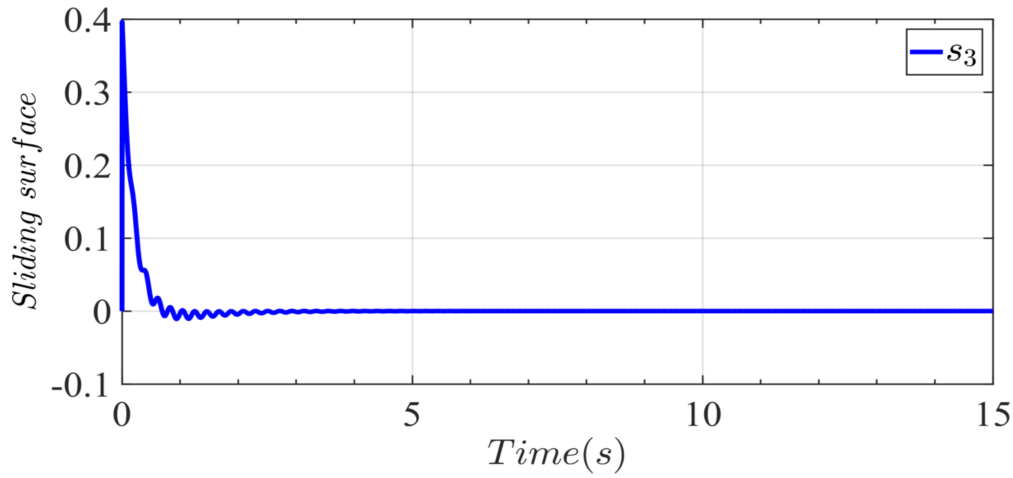
FIGURE 5.11: Stabilization of 3-DOF Manipulator with a Free Link, (a) Positions, (b) Velocities




(a)

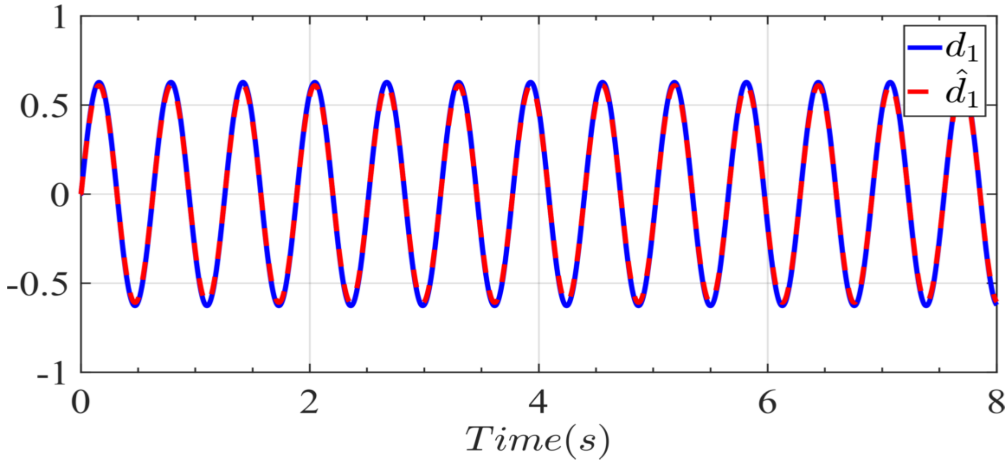



(b)

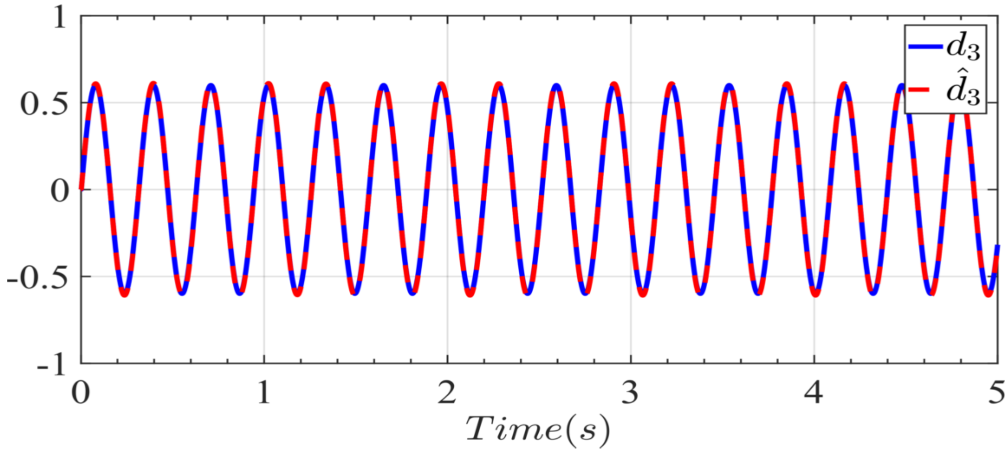



(c)

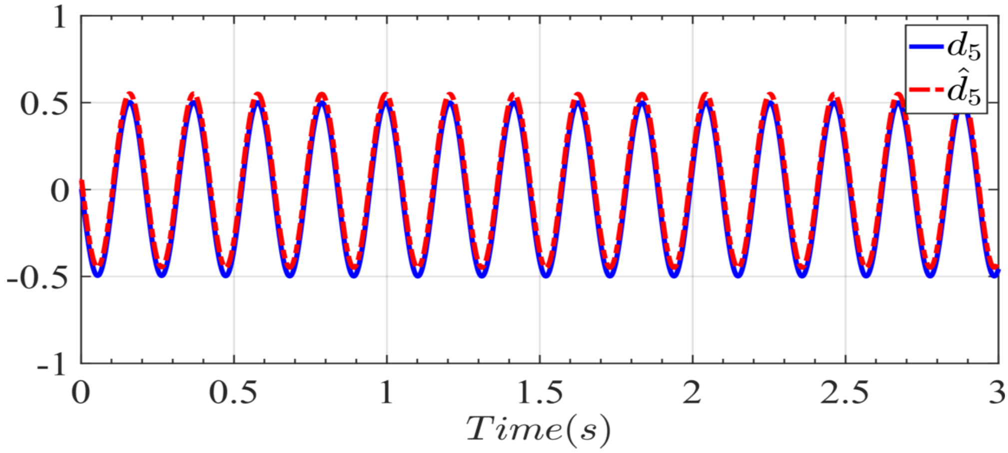
FIGURE 5.12: Stabilization of 3-DOF Manipulator with a Free,
(a) Sliding surface $s_1$, (b) Sliding surface $s_2$, (b) Sliding surface $s_3$




(a)

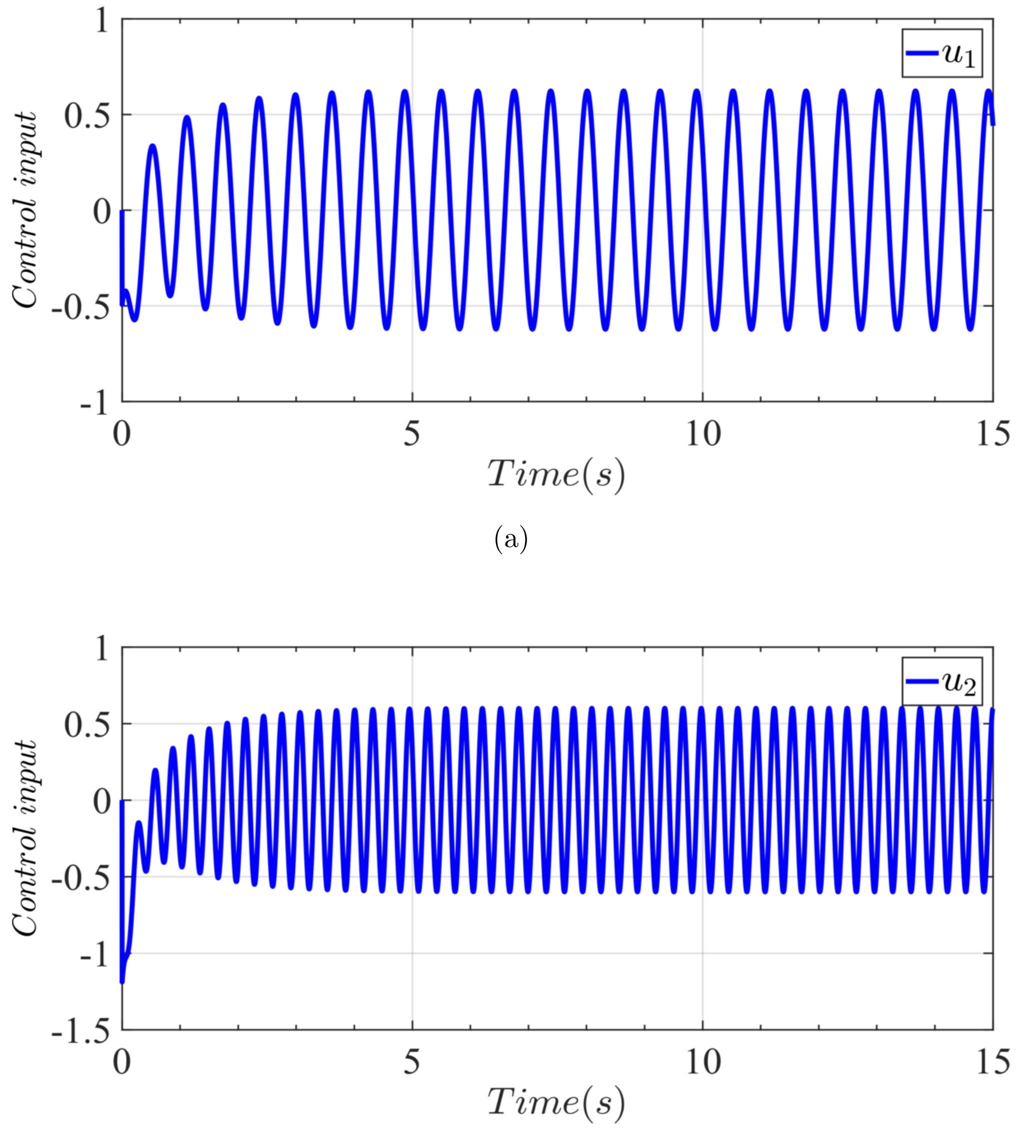



(b)




(c)

FIGURE 5.13: Stabilization of 3-DOF Manipulator with a Free Link, (a),(b,(c) Represents $d_1$,$d_3$,$d_5$ are the injected external disturbances in model. $\hat{d}_1$, $\hat{d}_1$, $\hat{d}_1$ are the estimations of disturbances.




(a)

(b)

FIGURE 5.14: Stabilization of 3-DOF Manipulator with a Free Link, (a) Control input $u_1 = v_1$, (b) Control input $u_2 = v_2$

**For Initial Condition 1:** Figure 5.11 represents a closed loop response of 3-DOF Manipulator with a Free Link while adaptive sliding mode control $u_1$ and $u_2$ is shown in Figure 5.14. The controller parameters are chosen as $c = 1$, $k = 1.5$, $M = 2$. The controller stabilizes the system from initial conditions $[x(0), y(0), \theta(0), \dot{x}(0), \dot{y}(0), \dot{\theta}(0) = 0, 0, 0, 0, 0, 0]^T$ to a stable equilibrium position $[0, 0, 0, 0, 0, 0]^T$ in less than 5 seconds. The sliding surfaces $s_1$, $s_2$ and $s_3$ shown in Figure 5.12 reach the value of 0 in about 5 seconds. This control strategy is robust and results are carried out in the presence of external disturbances. The injected disturbance in $x_2$ is $d_1(x,t) = 0.61sin(10t) + 0.1x_1^2$, in $x_4$ is $d_2(x,t) = 0.6sin(20t) + x_2^2sin(50t)$ and in $x_6$ is $d_3(x,t) = 0.5sin(30t) - 0.3x_1^2cos(2t)$.

# Chapter 6

# Conclusion and Future Work

This research work has provided a brief discussion of nonlinear control techniques which have been previously proposed for Nonholonomic systems on the basis of Backstepping and Adaptive sliding Mode Control. The proposed techniques provide a solution to the control problem of Nonholonomic systems.

## 6.1 Conclusion

During the past decades, the interest of researchers in control system society on nonholonomic systems has greatly increased. This system has a variety of applications in the field of education, defense, robotics and industry. This research work presents stabilization of nonholonomic systems without and with the presence of external disturbances.

The proposed methodologies are based on adaptive sliding mode control and Backstepping Control techniques. In sliding mode control, linear sliding surfaces are defined and controllers are designed such that to force the system states/trajectories reach towards the sliding surfaces and than towards origin which enables to achieve the system stability. In Backstepping control approach, controllers are designed to get the zero error between the actual and desired trajectories. In this strategy forcing the error function towards zero (equilibrium position) which ensure the system stability.

## 6.2 Future Research Directions

Based on this research work, certain key areas need to be focused for further research:

1. Extension of the proposed techniques to third order Nonholonomic systems.
2. Application of Observers.
3. Application sliding mode observation techniques.
4. Practical implementation of the proposed algorithms.

# Bibliography


[1] Z.-P. Jiang and H. Nijmeijer, "A recursive technique for tracking control of nonholonomic systems in chained form," *IEEE Transactions on Automatic control*, vol. 44, no. 2, pp. 265–279, 1999.

[2] P. Guha, "Nonholonomic deformation of coupled and supersymmetric kdv equations and euler–poincare–suslov method," *Reviews in Mathematical Physics*, vol. 27, no. 04, p. 1550011, 2015.

[3] Y. Liu and H. Yu, "A survey of underactuated mechanical systems," *IET Control Theory and Applications*, vol. 7, no. 7, pp. 921–935, 2013.

[4] C. Yang, Y. Qiu, F. Chen, and L. Xiang, "Distributed control for coupled nonholonomic mobile robots under the event-triggered and self-triggered frameworks," *Asian Journal of Control*, vol. 19, no. 3, pp. 900–917, 2017.

[5] R. W. Brockett *et al.*, "Asymptotic stability and feedback stabilization," *Differential geometric control theory*, vol. 27, no. 1, pp. 181–191, 1983.

[6] N. Aneke, H. Nijmeijer, and A. De Jager, "Tracking control of second-order chained form systems by cascaded backstepping," *International Journal of Robust and Nonlinear Control: IFAC-Affiliated Journal*, vol. 13, no. 2, pp. 95–115, 2003.

[7] M. Reyhanoglu and E. Al-Regib, "Nonholonomic motion planning for wheeled mobile systems using geometric phases," in *Proceedings of 1994 9th IEEE International Symposium on Intelligent Control*. IEEE, 1994, pp. 135–140.

[8] G. Walsh, D. Tilbury, S. Sastry, R. Murray, and J.-P. Laumond, "Stabilization of trajectories for systems with nonholonomic constraints," *IEEE Transactions on Automatic Control*, vol. 39, no. 1, pp. 216–222, 1994.

[9] P. Laumond, "Feasible trajectories for mobile robots with kinematic and environment constraints," *IEEE Transactions on Automatic control*, vol. 35, no. 9, pp. 1342–1351, 1996.

[10] A. M. Bloch, M. Reyhanoglu, and N. H. McClamroch, "Control and stabilization of nonholonomic dynamic systems," *IEEE Transactions on Automatic control*, vol. 37, no. 11, pp. 1746–1757, 1992.

[11] J.-P. Laumond, "Controllability of a multibody mobile robot," *IEEE Transactions on Robotics and Automation*, vol. 9, no. 6, pp. 755–763, 1993.

[12] O. J. Sordalen and O. Egeland, "Exponential stabilization of nonholonomic chained systems," *IEEE transactions on automatic control*, vol. 40, no. 1, pp. 35–49, 1995.

[13] R. M. Murray and S. S. Sastry, "Nonholonomic motion planning: Steering using sinusoids," *IEEE transactions on Automatic Control*, vol. 38, no. 5, pp. 700–716, 1993.

[14] Z.-P. Jiang, E. Lefeber, and H. Nijmeijer, "Saturated stabilization and tracking of a nonholonomic mobile robot," *Systems and Control Letters*, vol. 42, no. 5, pp. 327–332, 2001.

[15] M. Reyhanoglu, A. van der Schaft, N. H. McClamroch, and I. Kolmanovsky, "Dynamics and control of a class of underactuated mechanical systems," *IEEE Transactions on Automatic Control*, vol. 44, no. 9, pp. 1663–1671, 1999.

[16] M. Reyhanoglu and J. R. Hervas, "Nonlinear dynamics and control of space vehicles with multiple fuel slosh modes," *Control Engineering Practice*, vol. 20, no. 9, pp. 912–918, 2012.

[17] H. Yuan and Z. Qu, "Smooth time-varying pure feedback control for chained non-holonomic systems with exponential convergent rate," *IET control theory and applications*, vol. 4, no. 7, pp. 1235–1244, 2010.

[18] F. Rehman, "Feedback stabilization of nonholonomic control systems with drift," *Asian Journal of Control*, vol. 6, no. 1, pp. 97–103, 2004.

[19] C. Jammazi, "Continuous and discontinuous homogeneous feedbacks finite-time partially stabilizing controllable multichained systems," *SIAM Journal on Control and Optimization*, vol. 52, no. 1, pp. 520–544, 2014.

[20] J.-B. Pomet, "Explicit design of time-varying stabilizing control laws for a class of controllable systems without drift," *Systems and control letters*, vol. 18, no. 2, pp. 147–158, 1992.

[21] Z.-G. Liu, Y.-Q. Wu, and Z.-Y. Sun, "Output feedback control for a class of high-order nonholonomic systems with complicated nonlinearity and time-varying delay," *Journal of the Franklin Institute*, vol. 354, no. 11, pp. 4289–4310, 2017.

[22] Y.-P. Tian and K.-C. Cao, "Time-varying linear controllers for exponential tracking of non-holonomic systems in chained form," *International Journal of Robust and Nonlinear Control: IFAC-Affiliated Journal*, vol. 17, no. 7, pp. 631–647, 2007.

[23] A. Astolfi, "Discontinuous control of nonholonomic systems," *Systems and control letters*, vol. 27, no. 1, pp. 37–45, 1996.

[24] G. Lafferriere, "A general strategy for computing steering controls of systems without drift," *Systems and control letters*, vol. 37, no. 1, pp. 131–143, 1999.

[25] M.-C. Laiou and A. Astolfi, "Quasi-smooth control of chained systems," in *Proceedings of the 1999 American Control Conference (Cat. No. 99CH36251)*, vol. 6. IEEE, 1999, pp. 3940–3944.

[26] J. Guldner and V. I. Utkin, "Stabilization of non-holonomic mobile robots using lyapunov functions for navigation and sliding mode control," in *Proceedings of 1994 33rd IEEE Conference on Decision and Control*, vol. 3. IEEE, 1994, pp. 2967–2972.

[27] J.-y. Zhai and Z.-b. Song, "Adaptive sliding mode trajectory tracking control for wheeled mobile robots," *International Journal of Control*, vol. 92, no. 10, pp. 2255–2262, 2019.

[28] A. Bloch and S. Drakunov, "Stabilization of a nonholonomic system via sliding modes," in *Proceedings of 1994 33rd IEEE Conference on Decision and Control*, vol. 3. IEEE, 1994, pp. 2961–2963.

[29] S. Khesrani, A. Hassam, O. Boutalbi, and M. Boubezoula, "Motion planning and control of nonholonomic mobile robot using flatness and fuzzy logic concepts," *International Journal of Dynamics and Control*, vol. 196, no. 9, pp. 1–12, 2021.

[30] M. H. Falsafi, K. Alipour, and B. Tarvirdizadeh, "Tracking-error fuzzy-based control for nonholonomic wheeled robots," *Arabian Journal for Science and Engineering*, vol. 44, no. 2, pp. 881–892, 2019.

[31] V. T. Yen, W. Y. Nan, and P. Van Cuong, "Robust adaptive sliding mode neural networks control for industrial robot manipulators," *International Journal of Control, Automation and Systems*, vol. 17, no. 3, pp. 783–792, 2019.

[32] E. Jarzebowska, "On derivation of motion equations for systems with nonholonomic high-order program constraints," *Multibody System Dynamics*, vol. 7, no. 3, pp. 307–329, 2002.

[33] I. Kovacic, "On the field method in non-holonomic mechanics," *Acta Mechanica Sinica*, vol. 21, no. 2, pp. 192–196, 2005.

[34] J. R. Hervas and M. Reyhanoglu, "Control and stabilization of a third-order nonholonomic system," *SIAM Journal on Control and Optimization*, vol. 53, no. 2, pp. 17–22, 2013.

[35] V. Utkin, "Variable structure systems with sliding modes," *IEEE Transactions on Automatic control*, vol. 22, no. 2, pp. 212–222, 1977.

[36] A. J. Koshkouei, K. J. Burnham, and A. S. Zinober, "Dynamic sliding mode control design," *IEE Proceedings-Control Theory and Applications*, vol. 152, no. 4, pp. 392–396, 2005.

[37] A. Polyakov and L. Fridman, "Stability notions and lyapunov functions for sliding mode control systems," *Journal of the Franklin Institute*, vol. 351, no. 4, pp. 1831–1865, 2014.

[38] M. Reyhanoglu and A. van der Schaft, "Nh mcclamroch and i. kolmanovsky,"," *Dynamics and control of a class of underactuated mechanical systems," IEEE Transactions on Automatic Control*, vol. 44, no. 9, pp. 1663–1671, 1999.

[39] W. Xu and B. L. Ma, "Stabilization of second-order nonholonomic systems in canonical chained form," *Robotics and Autonomous Systems*, vol. 34, no. 4, pp. 223–233, 2001.

[40] R. M. Murray and S. S. Sastry, "Steering nonholonomic systems in chained form," *International Journal of Robust and Nonlinear Control: IFAC-Affiliated Journal*, vol. 23, no. 4, pp. 157–165, 1997.

[41] G. He, C. Zhang, W. Sun, and Z. Geng, "Stabilizing the second-order nonholonomic systems with chained form by finite-time stabilizing controllers," *Robotica*, vol. 34, no. 10, pp. 2344–2367, 2016.

[42] C. Song, S. Fei, J. Cao, and C. Huang, "Robust synchronization of fractional-order uncertain chaotic systems based on output feedback sliding mode control," *Mathematics*, vol. 7, no. 7, p. 599, 2019.